PIECEWISE POLYNOMIAL FUNCTIONS ON A PLANAR REGION: BOUNDARY CONSTRAINTS AND POLYHEDRAL SUBDIVISIONS

A Dissertation

by

TERRY LYNN MCDONALD

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2006

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Henry Schenck
Committee Members,	Paulo Lima-Filho
	Peter Stiller
	Nancy Amato
Head of Department	Albert Boggess

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ABSTRACT

Piecewise Polynomial Functions on a Planar Region: Boundary Constraints and Polyhedral Subdivisions. (May 2006)

Terry Lynn McDonald, B.S., The University of Texas at Tyler;
M.S., Texas A&M University
Chair of Advisory Committee: Dr. Henry Schenck

Splines are piecewise polynomial functions of a given order of smoothness r on a triangulated region Δ (or polyhedrally subdivided region $\square$) of $\mathbf{R}^d$. The set of splines of degree at most k forms a vector space $C^r_k(\Delta)$. Moreover, a nice way to study $C^r_k(\Delta)$ is to embed Δ in $\mathbf{R}^{d+1}$, and form the cone $\widehat{\Delta}$ of Δ with the origin. It turns out that the set of splines on $\widehat{\Delta}$ is a graded module $C^r(\widehat{\Delta})$ over the polynomial ring $\mathbf{R}[x_1, \ldots, x_{d+1}]$, and the dimension of $C^r_k(\Delta)$ is the dimension of $C^r(\widehat{\Delta})_k$.

This dissertation follows the works of Billera and Rose, as well as Schenck and Stillman, who each approached the study of splines from the viewpoint of homological and commutative algebra. They both defined chain complexes of modules such that $C^r(\widehat{\Delta})$ appeared as the top homology module.

First, we analyze the effects of gluing planar simplicial complexes. Suppose Δ_1, Δ_2, and $\Delta = \Delta_1 \cup \Delta_2$ are all planar simplicial complexes which triangulate pseudomanifolds. When $\Delta_1 \cap \Delta_2$ is also a planar simplicial complex, we use the Mayer-Vietoris sequence to obtain a natural relationship between the spline modules $C^r(\widehat{\Delta})$, $C^r(\widehat{\Delta_1})$, $C^r(\widehat{\Delta_2})$, and $C^r(\widehat{\Delta_1 \cap \Delta_2})$.

Next, given a simplicial complex Δ, we study splines which also vanish on the boundary of Δ. The set of all such splines is denoted by $C^r(\Delta_b)$. In this case, we will discover a formula relating the Hilbert polynomials of $C^r(\widehat{\Delta_b})$ and $C^r(\widehat{\Delta})$.

Finally, we consider splines which are defined on a polygonally subdivided region $\square$ of the plane. By adding only edges to $\square$ to form a simplicial subdivision Δ, we will be able to find bounds for the dimensions of the vector spaces $C^r_k(\square)$ for $k \gg 0$. In particular, these bounds will be given in terms of the dimensions of the vector spaces $C^r_k(\Delta)$ and geometrical data of both $\square$ and Δ.

This dissertation concludes with some thoughts on future research questions and an appendix describing the Macaulay2 package *SplineCode*, which allows the study of the Hilbert polynomials of the spline modules.

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I would like to begin by thanking my dissertation advisor, Henry Schenck. I would not have wanted to do this with anyone else. His contagious enthusiasm, patience, and constant encouragement all helped to bring me to this point. I would also like to thank the members of my committee: Paulo Lima-Filho, Peter Stiller, and Nancy Amato. They have given me excellent advice along the way, and they helped provide the tools needed to complete this dissertation.

Next, I would like to thank some of my fellow graduate students who were a great support system. Larry Holifield, Vincent Lemoine, Kevin Abbott and Mitch Hitchcock were my closest friends during graduate school. Not only did we spend many hours working together, we also found time to have fun - bowling, fishing, playing disc golf, and watching movies. Also, thanks to Jimmy Kimball, Eddie Fuselier, Jenny Gilmore, and my office mate, Gaik Ambartsoumian, for making the basement of Milner Hall such a great place.

I would like to conclude by thanking the people who mean the most to me, my family. Thanks to my wife, Jennifer, who inspired me with her love, support, and belief in me. Thanks to my parents who have always shown me the value of hard work and honesty. Finally, thanks to my in-laws and extended family for their enduring support.

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CHAPTER I

INTRODUCTION

A. The Problems

Let Δ be a connected, finite d-dimensional complex supported on $|\Delta| \subset \mathbf{R}^d$. The purpose of this dissertation is to explore problems associated to the spaces of piecewise polynomial functions, known as splines, defined on Δ. In particular, splines that are globally C^r functions on Δ. There are many applications which are related to splines. To name a few, they have been used for tasks such as designing font curves, fitting scattered data, and solving partial differential equations. Also, the study of splines is very appealing because splines reach into several areas of mathematics such as commutative and homological algebra, combinatorics, geometry, and topology.

Billera, in [3], was the first to introduce the use of homological algebra as a tool to study splines when he solved a conjecture about the dimension of the space of smooth planar splines. Further explorations in this direction were conducted by Billera and Rose, in [4, 5], Rose, in [11], as well as Schenck and Stillman, in [13, 14]. For other approaches to the study of spline spaces, see Alfeld and Schumaker, in [1, 2], Chui and Wang, in [6], Dahmen, Dress and Micchelli, in [7], Haas, in [9], or Yuzvinsky in [15]. This dissertation will follow in the homological approach.

We place a few restrictions on Δ. First, Δ must be *pure*. That is, all the maximal faces of Δ must be of dimension d. Also, Δ must be *strongly connected*, which means for any two d-faces σ and $\tilde{\sigma}$, there is a sequence of d-faces

$$\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n = \tilde{\sigma}$$

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such that for $i < n$, $\sigma_i \cap \sigma_{i+1}$ has dimension d $-$ 1. Finally, all links of faces in Δ must also be strongly connected, where the *link* of a face τ of Δ is defined to be the set of all faces γ so that $\tau \cup \gamma \in \Delta$ and $\tau \cap \gamma = \emptyset$. These conditions combined are equivalent to saying that Δ and its links are pseudomanifolds (See [3]).

Let $x_1, \ldots, x_d$ be coordinates of $\mathbf{R}^d$. Given $|\Delta| \subset \mathbf{R}^d$, we can form $\widehat{\Delta}$, the join of Δ with the origin in $\mathbf{R}^{d+1}$. Note that $\widehat{\Delta}$ corresponds to embedding Δ (Fig. 1) in the hyperplane $x_{d+1} = 1$, and then forming a new simplicial complex $\widehat{\Delta}$ (Fig. 2) by joining each simplex in Δ to the origin in $\mathbf{R}^{d+1}$.

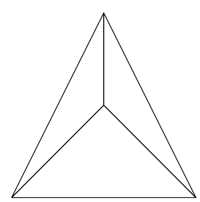

Fig. 1. Let Δ be the above simplicial complex.

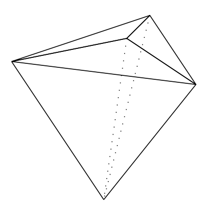

Fig. 2. The associated $\widehat{\Delta}$ for Figure 1.

Now, let $r \geq 0$ be an integer and $R = \mathbf{R}[x_1, \ldots, x_{d+1}]$.

Definition 1. $C^r_k(\Delta) := \{F : |\Delta| \to \mathbf{R} \,|\, F$ is continuously differentiable of order r and $F|_\sigma$ is a polynomial of degree at most k, for all $\sigma \in (\Delta)_d\}$

Definition 2. $C^r(\widehat{\Delta}) := \{F : |\widehat{\Delta}| \to \mathbf{R} \mid F$ is continuously differentiable of order r and $F|_{\widehat{\sigma}} \in R$, for all $\sigma \in (\Delta)_d\}$

The space $C^r(\widehat{\Delta})$ satisfies many nice properties. First, $C^r(\widehat{\Delta})$ is a finitely generated graded R-module. Also, the elements of $C^r(\widehat{\Delta})_k$, the k^{th} degree piece of $C^r(\widehat{\Delta})$, are the homogenizations of the elements of $C^r_k(\Delta)$. Hence, it is sufficient to study $C^r(\widehat{\Delta})$ when trying to answer questions about $C^r_k(\Delta)$. Finally, it has been shown that $C^r(\widehat{\Delta})$ is actually the top homology module of a chain complex $\mathcal{R}/\mathcal{J}$ (to be seen later). Moreover, the homology modules of the chain complex $\mathcal{R}/\mathcal{J}$ are all graded, which allows us to study invariants of these modules: for example, their Hilbert polynomials.

The Hilbert polynomial of $C^r(\widehat{\Delta})$ has been studied by different authors. Billera and Rose, in [4], were able to pinpoint some properties that the Hilbert polynomial and its derivative had to satisfy. A later investigation by Schenck and Stillman, in [13], produced formulas for the Hilbert series for finite planar simplicial complexes where the complex and all its links were pseudomanifolds. These results serve as inspiration for some of the questions discussed in this dissertation.

In all areas of mathematics, it is a common practice to build a new object from already known objects and then try to formulate as much information as possible about the new object in terms of known data. For this dissertation, we will use this common practice on our simplicial complexes. Given two simplicial complexes Δ_1 and Δ_2, we will glue them along a common simplicial subcomplex, denoted $\Delta_1 \cap \Delta_2$, to form a new simplicial complex $\Delta = \Delta_1 \cup \Delta_2$. There are a number of ways to glue, but we will require that the simplicial complexes Δ and $\Delta_1 \cap \Delta_2$ also satisfy the requirements mentioned previously in this chapter. In the planar case, we show that there is a natural relationship between the spline spaces: $C^r(\widehat{\Delta})$, $C^r(\widehat{\Delta_1})$, $C^r(\widehat{\Delta_2})$, and $C^r(\widehat{\Delta_1 \cap \Delta_2})$. There is also a natural relationship in higher dimensions, but it is more complex than the planar case.

Given a simplicial complex Δ, splines are not mandated to satisfy any conditions along the boundary of Δ. Instead, they have only been required to be globally defined C^r functions on the interior of Δ. We will want to see what happens when we require splines to be C^r functions defined throughout the interior and which vanish on the boundary of a simplicial complex. We can extend the previously used chain complexes and techniques to get a handle on these new splines. We want to know how the splines meeting boundary conditions relate to splines that do not satisfy boundary conditions on a given simplicial complex Δ. In the planar case, we find a natural formula for this relationship that is computationally pleasing.

As previously mentioned, splines show up in a wide variety of mathematics, and they have many applications. One of the reasons for this is because splines may be defined on many types of domains. They are not restricted to just being defined on simplicial complexes. In particular, we will sometimes be interested in splines which are defined on a connected polygonal region $\square$ of the plane. As in the simplicial complex case, we will require these regions to be pure, strongly connected, and have faces whose links are strongly connected. The definitions for $C^r_k(\square)$ and $C^r(\widehat{\square})$ will remain the same. Our main goal will be to find bounds on the dimensions of the vector spaces $C^r_k(\square)$ when k is sufficiently large. We will do this by subdividing $\square$ into a simplicial complex Δ and then use homological algebra techniques to relate $C^r(\widehat{\square})$ and $C^r(\widehat{\Delta})$.

The rest of this chapter is organized as follows: the second section will introduce some notation and review two ways that Billera and Rose used to identify the spline space $C^r(\widehat{\Delta})$. The third section will include some additional notation as well as a third way to identify the spline space using the chain complex $\mathcal{R}/\mathcal{J}[\Delta]$. This will prove to be our preferred way of considering the spline space due to some nice properties of this chain complex.

In Chapter II, we examine how gluing simplicial complexes in the plane affects their spline modules. By gluing simplicial complexes Δ_1 and Δ_2 along $\Delta_1 \cap \Delta_2$ to form $\Delta = \Delta_1 \cup \Delta_2$, we will apply the technique of obtaining a Mayer-Vietoris sequence to show a relationship on the Hilbert polynomials of their spline spaces. In particular, we show:

$$HP(C^r(\widehat{\Delta})) = HP(C^r(\widehat{\Delta_1})) + HP(C^r(\widehat{\Delta_2})) - HP(C^r(\widehat{\Delta_1 \cap \Delta_2}))$$

In Chapter III, we will require splines on a given simplicial complex Δ to meet boundary conditions. Then, we show that the space of splines vanishing on $\partial\Delta$, denoted $C^r(\widehat{\Delta_b})$, is related to $C^r(\widehat{\Delta})$ by using the long exact sequence in homology gotten from the short exact sequence of chain complexes in relative simplicial homology. The key to this will be to show that some of the newly created homology modules in the long exact sequence are finite length modules.

In Chapter IV, we examine splines which are defined on a connected polygonal region $\square$ of the plane. We obtain bounds on the dimension of the graded pieces of the spline module $C^r(\widehat{\square})_k$ for $k \gg 0$. To find these bounds, we used a process which has three key steps. First, we find a simplicial subdivision Δ of $\square$ by adding only edges. Second, we build a short exact sequence of chain complexes that relates $C^r(\widehat{\square})$ and $C^r(\widehat{\Delta})$ in the resulting long exact sequence in homology. Finally, we show that the newly introduced homology modules are finite length modules or they have computable Hilbert polynomials.

This dissertation will conclude with two final sections: Chapter V and an Appendix. In Chapter V, we will give a final summary of our conclusions and take a glance at future research questions. In the Appendix, we will look at some more detailed examples using code written for Macaulay2 called *SplineCode* and present the actual code.

Throughout this dissertation, $(\Delta)^0$ denotes the set of interior faces of Δ, $(\Delta)_i$ denotes the i-dimensional faces of Δ, $(\Delta)_i^0$ denotes the set of interior i-dimensional faces of Δ, and $(\partial\Delta)_i$ denotes the i-dimensional boundary faces of Δ. Also, $f^0(\Delta)$, $f_i(\Delta)$, $f_i^0(\Delta)$, and $f_i^\partial(\Delta)$ denote the cardinality of these sets, respectively. Finally, the complexes defined in this chapter depend on an integer $r \geq 0$ even though the notation does not make this explicit.

B. Definitions and Notation

Definition 3. Let R be a ring. A complex $\mathcal{F}$ of R-modules on $(\Delta)^0$ consists of the following:

(a) For each $\sigma \in (\Delta)^0$ an R-module $\mathcal{F}(\sigma)$

(b) For each $i \in \{0, \ldots, d\}$ an R-module homomorphism

$$\partial_i : \bigoplus_{\sigma_i \in (\Delta)_i^0} \mathcal{F}(\sigma_i) \longrightarrow \bigoplus_{\sigma_{i-1} \in (\Delta)_{i-1}^0} \mathcal{F}(\sigma_{i-1})$$

such that $\partial_{i-1} \circ \partial_i = 0$. That is, $\operatorname{image} \partial_i \subseteq \ker \partial_{i-1}$. Note that such a complex is said to be *exact* if $\operatorname{image} \partial_i = \ker \partial_{i-1}$.

Example 1. Let $\mathcal{R}$ be the constant chain complex on $(\Delta)^0$. That is, $\mathcal{R}(\sigma) = R$, for every $\sigma \in (\Delta)^0$, and the maps will be the usual boundary operators.

Example 2. Let $r \geq 0$ be an integer. For each $\sigma \in (\Delta)^0$, let I_σ be the homogeneous ideal of $\widehat{\sigma} \subseteq \mathbf{R}^{d+1}$. For example, if σ is an edge of $\Delta \subset \mathbf{R}^2$ with vertices (x_1, y_1) and (x_2, y_2) then

$$I_\sigma = \langle x(y_1 - y_2) - y(x_1 - x_2) + z(x_1y_2 - x_2y_1) \rangle \subseteq \mathbf{R}[x, y, z]$$

Define a complex $\mathcal{I}[\Delta]$ of ideals on $(\Delta)^0$ by $\mathcal{I}(\sigma) = I_\sigma^{r+1}$ and the quotient complex $\mathcal{R}/\mathcal{I}[\Delta]$ by $\mathcal{R}/\mathcal{I}(\sigma) = R/I_\sigma^{r+1}$.

Given a complex $\mathcal{F}$ of R-modules on $(\Delta)^0$:

$$0 \longrightarrow \bigoplus_{\sigma \in (\Delta)_d} \mathcal{F}(\sigma) \xrightarrow{\partial_d} \bigoplus_{\tau \in (\Delta)^0_{d-1}} \mathcal{F}(\tau) \xrightarrow{\partial_{d-1}} \cdots \bigoplus_{v \in (\Delta)^0_0} \mathcal{F}(v) \longrightarrow 0$$

denote $H_*(\mathcal{F})$ as the homology of this complex. Also recall, if given a short exact sequence of complexes:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

Then there is a resulting long exact sequence in homology:

$$0 \longrightarrow H_d(\mathcal{A}) \longrightarrow H_d(\mathcal{B}) \longrightarrow H_d(\mathcal{C}) \longrightarrow H_{d-1}(\mathcal{A}) \longrightarrow \cdots \longrightarrow H_0(\mathcal{C}) \longrightarrow 0$$

A key fact proved by Billera, in [3], is that the spline module $C^r(\widehat{\Delta})$ is isomorphic to $H_d(\mathcal{R}/\mathcal{I}[\Delta])$.

To finish this section, we should recall a second way to characterize the spline module. Billera and Rose, in [4], used the following exact sequence of graded modules to show that $C^r(\widehat{\Delta})$ is isomorphic to the kernel of a matrix M (also given below):

$$0 \longrightarrow \ker M \longrightarrow R^{f_d} \oplus R^{f^0_{d-1}}(-r-1) \xrightarrow{M} R^{f^0_{d-1}} \longrightarrow \operatorname{coker} M \longrightarrow 0$$

where

$$M = \left(\begin{array}{c|ccc} & l_{\epsilon_1}^{r+1} & & \\ \partial_d & & \ddots & \\ & & & l_{\epsilon_{f^0_{d-1}}}^{r+1} \end{array} \right),$$

∂_d is the simplicial boundary operator on $R^{f_d} \longrightarrow R^{f^0_{d-1}}$, and l_{ϵ_i} is the homogeneous linear form generating I_{ϵ_i} for $\epsilon_i \in (\Delta)^0_{d-1}$.

C. The Complexes $\mathcal{J}[\Delta]$ and $\mathcal{R}/\mathcal{J}[\Delta]$

Let integer $r \geq 0$ be fixed. Define a complex of ideals $\mathcal{J}[\Delta]$ on $(\Delta)^0$ as follows:

$$\begin{array}{ll}
\mathcal{J}(\sigma) = 0 & \text{for } \sigma \in (\Delta)_d \\
\mathcal{J}(\tau) = I_\tau^{r+1} & \text{for } \tau \in (\Delta)^0_{d-1} \\
\mathcal{J}(\xi) = \sum\limits_{\xi \in \tau} I_\tau^{r+1} & \text{for } \xi \in (\Delta)^0_{d-2} \\
\vdots & \vdots \\
\mathcal{J}(v) = \sum\limits_{v \in \tau} I_\tau^{r+1} & \text{for } v \in (\Delta)^0_0
\end{array}$$

Again, let $\mathcal{R}[\Delta]$ be the constant complex defined in Example 1 and $\mathcal{R}/\mathcal{J}[\Delta]$ as the quotient complex of $\mathcal{R}[\Delta]$ and $\mathcal{J}[\Delta]$. Since, $\bigoplus\limits_{\beta \in (\Delta)^0_i} \mathcal{J}(\beta)$ is a submodule of $\bigoplus\limits_{\beta \in (\Delta)^0_i} \mathcal{R}(\beta)$, the usual boundary operator, ∂_i, induces a differential on $\mathcal{J}[\Delta]$ for all i. Similarly, for the quotient $\mathcal{R}/\mathcal{J}[\Delta]$. Hence, the short exact sequence of complexes

$$0 \longrightarrow \mathcal{J}[\Delta] \longrightarrow \mathcal{R}[\Delta] \longrightarrow \mathcal{R}/\mathcal{J}[\Delta] \longrightarrow 0$$

yields a long exact sequence in homology:

$$\cdots \longrightarrow H_{i+1}(\mathcal{R}/\mathcal{J}[\Delta]) \longrightarrow H_i(\mathcal{J}[\Delta]) \longrightarrow H_i(\mathcal{R}[\Delta]) \longrightarrow H_i(\mathcal{R}/\mathcal{J}[\Delta]) \longrightarrow \cdots$$

Notice, the complexes $\mathcal{J}[\Delta]$ and $\mathcal{I}[\Delta]$ agree on the d faces and the $d-1$ faces of Δ. Hence, we know that

$$H_d(\mathcal{R}/\mathcal{J}[\Delta]) = H_d(\mathcal{R}/\mathcal{I}[\Delta])$$

Consequently, the top homology module of the complex $\mathcal{R}/\mathcal{J}[\Delta]$ is isomorphic to $C^r(\widehat{\Delta})$. This gives us a third way to characterize the spline module. However, the lower homology modules of $\mathcal{R}/\mathcal{I}[\Delta]$ and $\mathcal{R}/\mathcal{J}[\Delta]$ do not agree. More importantly, the complex $\mathcal{R}/\mathcal{J}[\Delta]$ and its homology modules are known to have some nice properties. Some of these properties will be noted and used throughout this dissertation.

CHAPTER II

ANALYSIS OF GLUING IN THE PLANAR CASE

A. Introduction

In this chapter, we explore spline spaces using the Mayer-Vietoris sequence. Suppose Δ_1, Δ_2, and $\Delta = \Delta_1 \cup \Delta_2$ are all planar simplicial complexes which triangulate pseudomanifolds. We study the relation between the spline modules on Δ_1, Δ_2, Δ, and $\Delta_1 \cap \Delta_2$ when $\Delta_1 \cap \Delta_2$ also triangulates a pseudomanifold. In this case, there is a natural relationship between the Hilbert polynomials (which measure the dimension of $C^r(\widehat{\Delta})_k$ for $k \gg 0$) of the spline modules.

The main theorem in this chapter states if Δ_1, Δ_2, $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are all finite simplicial complexes triangulating pseudomanifolds, then for $k \gg 0$ we have the following:

$$\dim_{\mathbf{R}} C^r(\widehat{\Delta})_k = \dim_{\mathbf{R}} C^r(\widehat{\Delta_1})_k + \dim_{\mathbf{R}} C^r(\widehat{\Delta_2})_k - \dim_{\mathbf{R}} C^r(\widehat{\Delta_1 \cap \Delta_2})_k$$

Of course, this is a relation on Hilbert polynomials in the language of commutative algebra.

In the next section, we state some basic properties of the homology modules of the chain complex $\mathcal{R}/\mathcal{J}[\Delta]$. In the third section, the Mayer-Vietoris sequence will be presented and applied to our problem. In the final section, the Hilbert polynomial and some of its properties will be presented and used to obtain our first result.

B. The Homology Modules of $\mathcal{R}/\mathcal{J}[\Delta]$

Let $r \geq 0$ be an integer and $R = \mathbf{R}[x, y, z]$. Also, let Δ be a finite, connected simplicial complex embedded in $\mathbf{R}^2$. Define a chain complex $\mathcal{J}[\Delta]$ as follows:

$$\begin{aligned} \mathcal{J}(\sigma) &= 0 && \text{for } \sigma \in (\Delta)_2 \\ \mathcal{J}(\tau) &= I_\tau^{r+1} && \text{for } \tau \in (\Delta)_1^0 \\ \mathcal{J}(v) &= \sum_{v\in\tau} I_\tau^{r+1} && \text{for } v \in (\Delta)_0^0 \end{aligned}$$

Next, the complex $\mathcal{R}/\mathcal{J}[\Delta]$ defined on $(\Delta)^0$ is given by the quotient of the constant complex $\mathcal{R}[\Delta]$ (which computes relative homology with R coefficients) and the complex $\mathcal{J}[\Delta]$ defined above. The maps for this chain complex are again induced from the usual relative simplicial boundary operators.

We recall some results of previous work applying homological algebra to the study of spline theory. In particular, we need those results which describe the homology modules of the chain complex $\mathcal{R}/\mathcal{J}[\Delta]$. The first lemma relates the second homology of $\mathcal{R}/\mathcal{J}[\Delta]$ with the spline space of a simplicial complex. The second lemma describes the homology modules $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ and $H_0(\mathcal{R}/\mathcal{J}[\Delta])$.

Lemma 1. *(See [3]) Let Δ be a connected finite simplicial complex. If $\mathcal{R}/\mathcal{J}[\Delta]$ is the complex defined above, then $C^r(\widehat{\Delta})$ is isomorphic to the module $H_2(\mathcal{R}/\mathcal{J}[\Delta])$.*

Lemma 2. *(See [14]) The R-module $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ has finite length. Moreover, the R-module $H_0(\mathcal{R}/\mathcal{J}[\Delta]) = 0$.*

C. The Mayer-Vietoris Sequence

Suppose that Δ_1 and Δ_2 are two simplicial complexes glued so that $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are also simplicial complexes. Let C_k be the oriented k-simplices and consider the map $\psi : C_k(\Delta_1) \oplus C_k(\Delta_2) \to C_k(\Delta)$ defined as follows: $\psi(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2$. Note that ψ is a surjective map with kernel ψ isomorphic to $C_k(\Delta_1 \cap \Delta_2)$ since $\psi(\sigma, \sigma) = 0$ for $\sigma \in C_k(\Delta_1 \cap \Delta_2)$ (*See* [10]). Hence, there is a short exact sequence of complexes:

$$0 \longrightarrow C_*(\Delta_1 \cap \Delta_2) \longrightarrow C_*(\Delta_1) \oplus C_*(\Delta_2) \longrightarrow C_*(\Delta) \longrightarrow 0$$

Definition 4. The Mayer-Vietoris sequence is the long exact sequence in homology induced from the above short exact sequence of complexes:

$$\cdots \to H_{k+1}(\Delta) \to H_k(\Delta_1 \cap \Delta_2) \to H_k(\Delta_1) \oplus H_k(\Delta_2) \to H_k(\Delta) \to H_{k-1}(\Delta_1 \cap \Delta_2) \to \cdots$$

Example 3. Suppose that Δ_1, Δ_2, $\Delta_1 \cap \Delta_2$, and $\Delta = \Delta_1 \cup \Delta_2$ are all simplicial complexes. The following is a short exact sequence of complexes:

$$0 \longrightarrow \mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2] \longrightarrow \mathcal{R}/\mathcal{J}[\Delta_1] \oplus \mathcal{R}/\mathcal{J}[\Delta_2] \longrightarrow \mathcal{R}/\mathcal{J}[\Delta] \longrightarrow 0$$

Hence, there is a resulting Mayer-Vietoris sequence:

$$\begin{aligned} 0 \to H_2(\mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2]) \to H_2(\mathcal{R}/\mathcal{J}[\Delta_1]) \oplus H_2(\mathcal{R}/\mathcal{J}[\Delta_2]) \to H_2(\mathcal{R}/\mathcal{J}[\Delta]) \to \\ H_1(\mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2]) \to H_1(\mathcal{R}/\mathcal{J}[\Delta_1]) \oplus H_1(\mathcal{R}/\mathcal{J}[\Delta_2]) \to H_1(\mathcal{R}/\mathcal{J}[\Delta]) \to 0 \end{aligned}$$

Now, for an integer $r \geq 0$, we apply Lemma 1 to the Mayer-Vietoris sequence to get the following long exact sequence:

$$\begin{aligned} 0 \to C^r(\widehat{\Delta_1 \cap \Delta_2}) \to C^r(\widehat{\Delta_1}) \oplus C^r(\widehat{\Delta_2}) \to C^r(\widehat{\Delta}) \to H_1(\mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2]) \to \\ H_1(\mathcal{R}/\mathcal{J}[\Delta_1]) \oplus H_1(\mathcal{R}/\mathcal{J}[\Delta_2]) \to H_1(\mathcal{R}/\mathcal{J}[\Delta]) \to 0 \end{aligned}$$

D. The Hilbert Polynomial When $\Delta_1 \cap \Delta_2$ Is a Triangulation

For a finitely generated graded module M over the polynomial ring $R = \mathbf{R}[x, y, z]$ there is a polynomial (the *Hilbert polynomial*) $HP(M, t) \in \mathbf{Z}[t]$ such that $\dim_{\mathbf{R}} M_l = HP(M, l)$ for $l \gg 0$ (See [8], Theorem 1.11). The Hilbert polynomial satisfies some key facts (See [8]).

Lemma 3. *If* $0 \to M \to M_0 \to M_1 \to \cdots \to M_k \to 0$ *is an exact sequence of finitely generated graded modules over* R*, then* $HP(M, t) = \sum_{i=0}^{k} (-1)^i HP(M_i, t)$.

Lemma 4. *If M and N are finitely generated graded R-modules, then* $HP(M \oplus N, t) = HP(M, t) + HP(N, t)$.

Lemma 5. *If M is a finitely generated graded R-module of finite length, then* $HP(M, t) = 0$.

The proofs of Lemma 3 and Lemma 4 are simple exercises and are well known facts for the Hilbert polynomials. The reason that Lemma 5 is true is because a module of finite length vanishes in high degree. Hence, the dimension of the high degree pieces are zero, which causes its Hilbert polynomial to also be zero.

So, by applying Lemmas 3 and 4 to the long exact sequence at the end of the previous section, we obtain the following equation:

$$HP(C^r(\widehat{\Delta})) - HP(C^r(\widehat{\Delta_1})) - HP(C^r(\widehat{\Delta_2})) + HP(C^r(\widehat{\Delta_1 \cap \Delta_2})) =$$
$$HP(H_1(\mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2])) - HP(H_1(\mathcal{R}/\mathcal{J}[\Delta_1])) - HP(H_1(\mathcal{R}/\mathcal{J}[\Delta_2])) + HP(H_1(\mathcal{R}/\mathcal{J}[\Delta]))$$

Finally, by Lemmas 2 and 5, the second line vanishes and we obtain the following:

Theorem 1. *Let $r \geq 0$ be an integer and $R = \mathbf{R}[x, y, z]$. If Δ_1, Δ_2, $\Delta_1 \cap \Delta_2$, and $\Delta = \Delta_1 \cup \Delta_2$ are all simplicial complexes, then*

$$HP(C^r(\widehat{\Delta})) = HP(C^r(\widehat{\Delta_1})) + HP(C^r(\widehat{\Delta_2})) - HP(C^r(\widehat{\Delta_1 \cap \Delta_2}))$$

Example 4. Below, we give two simplicial complexes Δ_1 and Δ_2 along with the simplicial subcomplex $\Delta_1 \cap \Delta_2$ (Fig. 3), on which they are to be glued to form a simplicial complex Δ and Table I which contains various r values demonstrating Theorem 1:

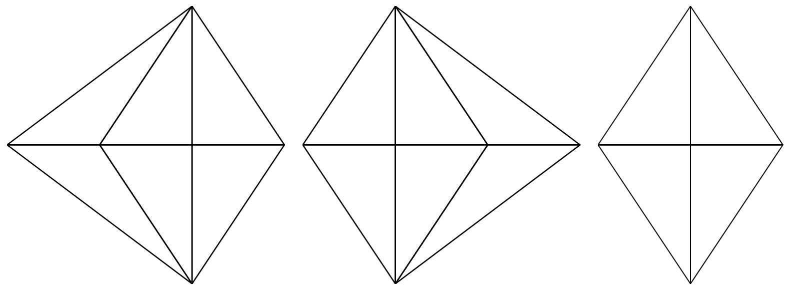

Fig. 3. Simplices Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$, respectively.

Table I. The values of Theorem 1 for the complexes in Fig. 3.

r	$HP(C^r(\widehat{\Delta}))$	$HP(C^r(\widehat{\Delta_1}))$	$HP(C^r(\widehat{\Delta_2}))$	$-HP(C^r(\widehat{\Delta_1 \cap \Delta_2}))$
0	$4t^2 + 2t + 1$	$3t^2 + 2t + 1$	$3t^2 + 2t + 1$	$-2t^2 - 2t - 1$
1	$4t^2 - 8t + 8$	$3t^2 - 5t + 6$	$3t^2 - 5t + 6$	$-2t^2 + 2t - 4$
2	$4t^2 - 18t + 31$	$3t^2 - 12t + 22$	$3t^2 - 12t + 22$	$-2t^2 + 6t - 13$
3	$4t^2 - 28t + 68$	$3t^2 - 19t + 48$	$3t^2 - 19t + 48$	$-2t^2 + 10t - 28$
4	$4t^2 - 38t + 121$	$3t^2 - 26t + 85$	$3t^2 - 26t + 85$	$-2t^2 + 14t - 49$

It is natural to ask what happens in higher dimensions. Results of [12] show that $H_i(\mathcal{R}/\mathcal{J}[\Delta])$ is of dimension $i-2$ for all $1 \leq i < d = \dim(\Delta)$. We will get that $\chi(HP(H_i(\mathcal{R}/\mathcal{J}[\Delta]))) = \chi(HP(H_i(\mathcal{R}/\mathcal{J}[\Delta_1]))) + \chi(HP(H_i(\mathcal{R}/\mathcal{J}[\Delta_2]))) - \chi(HP(H_i(\mathcal{R}/\mathcal{J}[\Delta_1 \cap \Delta_2])))$ but since the lower homology modules may have nonzero Hilbert polynomials (as opposed to the planar case), the Hilbert polynomials themselves satisfy a more complicated relationship.

CHAPTER III

BOUNDARY CONSTRAINTS IN THE PLANAR CASE

A. Introduction

Splines have been defined and studied as globally defined C^r functions across the interior of a simplicial complex Δ; thus far, we have placed no conditions on $\partial\Delta$. This is why the use of simplicial homology relative to the boundary is well suited to the problem. Throughout this chapter, we want to consider splines that are globally defined C^r functions across the interior and which vanish on the boundary of a simplicial complex. We can get a handle on these new splines by forming a new chain complex that involves all the faces of Δ; i.e. both interior and boundary faces:

$$\mathcal{R}/\mathcal{J}[\Delta_b] : \; 0 \longrightarrow \bigoplus_{\sigma \in (\Delta)_2} \mathcal{R}/\mathcal{J}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau \in (\Delta)_1} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{v \in (\Delta)_0} \mathcal{R}/\mathcal{J}(v) \longrightarrow 0$$

Notice that Δ has the same two-dimensional faces as $(\Delta)^0$,but $(\Delta)_1 = (\Delta)_1^0 \cup (\partial\Delta)_1$ and similarly for $(\Delta)_0$. The top homology module of this complex will be denoted by $C^r(\widehat{\Delta_b})$ and is the space of the new type of splines. Our goal is to find a formula relating $HP(C^r(\widehat{\Delta}))$ and $HP(C^r(\widehat{\Delta_b}))$.

When using relative simplicial homology, there is a short exact sequence of complexes. In this case, it takes the following form:

$$
\begin{tikzcd}
\mathcal{R}/\mathcal{J}[\partial\Delta]: & 0 \arrow[r] \arrow[d] & \bigoplus_{\tau\in(\partial\Delta)_1} \mathcal{R}/\mathcal{J}(\tau) \arrow[r] \arrow[d] & \bigoplus_{v\in(\partial\Delta)_0} \mathcal{R}/\mathcal{J}(v) \arrow[d] \\
\mathcal{R}/\mathcal{J}[\Delta_b]: & \bigoplus_{\sigma\in(\Delta)_2} R \arrow[r] \arrow[d] & \bigoplus_{\tau\in(\Delta)_1} \mathcal{R}/\mathcal{J}(\tau) \arrow[r] \arrow[d] & \bigoplus_{v\in(\Delta)_0} \mathcal{R}/\mathcal{J}(v) \arrow[d] \\
\mathcal{R}/\mathcal{J}[\Delta]: & \bigoplus_{\sigma\in(\Delta)_2} R \arrow[r] & \bigoplus_{\tau\in(\Delta)_1^0} \mathcal{R}/\mathcal{J}(\tau) \arrow[r] & \bigoplus_{v\in(\Delta)_0^0} \mathcal{R}/\mathcal{J}(v)
\end{tikzcd}
$$

Thus, we have the following long exact sequence in homology:

$$0 \to C^r(\widehat{\Delta_b}) \to C^r(\widehat{\Delta}) \to H_1(\mathcal{R}/\mathcal{J}[\partial\Delta]) \to H_1(\mathcal{R}/\mathcal{J}[\Delta_b]) \to H_1(\mathcal{R}/\mathcal{J}[\Delta]) \to$$
$$H_0(\mathcal{R}/\mathcal{J}[\partial\Delta]) \to H_0(\mathcal{R}/\mathcal{J}[\Delta_b]) \to 0$$

B. $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$, $H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$, and $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ Are Finite Modules

The previous long exact sequence can be simplified by showing that $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$, $H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$, and $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ are all modules of finite length (by Lemma 2, $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ is a finite length module).

Lemma 6. *The module $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$ is of finite length.*

Proof. The complex $\mathcal{R}/\mathcal{J}[\partial\Delta]$ has the following form:

$$\mathcal{R}/\mathcal{J}[\partial\Delta] : 0 \to \bigoplus_{\tau\in(\partial\Delta)_1} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{v\in(\partial\Delta)_0} \mathcal{R}/\mathcal{J}(v) \to 0$$

In particular, $H_2(\mathcal{R}/\mathcal{J}[\partial\Delta]) = 0$ since $\partial\Delta$ is one-dimensional. Thus, we have the following exact sequence:

$$0 \to H_1(\mathcal{R}/\mathcal{J}[\partial\Delta]) \to \bigoplus_{\tau\in(\partial\Delta)_1} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{v\in(\partial\Delta)_0} \mathcal{R}/\mathcal{J}(v) \to H_0(\mathcal{R}/\mathcal{J}[\partial\Delta]) \to 0$$

Fix $\tilde{v} \in (\partial\Delta)_0$, by a change of coordinates, we may assume $\tilde{v}$ satisfies $\sqrt{J(\tilde{v})} = \langle x, y\rangle$. Let P be a codimension 2 prime ideal. To obtain our result, we will localize at P. Recall that localization is an exact functor and commutes with finite sums. Thus, we have the following exact sequence:

$$0 \to H_1(\mathcal{R}/\mathcal{J}[\partial\Delta])_P \to \bigoplus_{\tau\in(\partial\Delta)_1} (\mathcal{R}/\mathcal{J}(\tau))_P \xrightarrow{\partial_1} \bigoplus_{v\in(\partial\Delta)_0} (\mathcal{R}/\mathcal{J}(v))_P \to H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])_P \to 0$$

Notice, if $\mathcal{J}(v) \nsubseteq P$, then $(\mathcal{R}/\mathcal{J}(v))_P = 0$. Thus, if $P \neq \sqrt{\mathcal{J}(v)}$ for any $v \in (\partial\Delta)_0$, then $\bigoplus_{v\in(\partial\Delta)_0} (\mathcal{R}/\mathcal{J}(v))_P = 0$ which forces $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])_P = 0$.

Suppose $P = \sqrt{\mathcal{J}(\tilde{v})}$. In this case, the exact sequence takes on the form:

$$0 \to H_1(\mathcal{R}/\mathcal{J}[\partial\Delta])_P \to \bigoplus_{\tau\in(\partial\Delta)_1} (\mathcal{R}/\mathcal{J}(\tau))_P \xrightarrow{\partial_1} (\mathcal{R}/\mathcal{J}(\tilde{v}))_P \to H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])_P \to 0$$

Also, note that $(\mathcal{R}/\mathcal{J}(\tau))_P = (R/l_\tau^{r+1})_P \neq 0$ iff $l_\tau \in P$. Since we are working with simplicial complexes, there exist $\tau_1, \ldots, \tau_m \in (\partial\Delta)_1$, with $m \geq 2$ so that $(\mathcal{R}/\mathcal{J}(\tau_i))_P \neq 0$ for $i = 1, \ldots, m$. Since $(\mathcal{R}/\mathcal{J}(v))_P = 0$ for all $v \neq \tilde{v}$, this forces the map $\bigoplus_{i=1}^{m} (\mathcal{R}/\mathcal{J}(\tau_i))_P \xrightarrow{\partial_1} (\mathcal{R}/\mathcal{J}(\tilde{v}))_P$ to be surjective. So, again we have $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])_P = 0$.

Therefore, $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])_P = 0$ for any codimension 2 prime ideal. Hence, $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$ must be supported on the ideal $\langle x, y, z\rangle$, which means that $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$ is a finite length module. (See [8], Cor. 2.17). □

Note that there are two important conclusions that we may draw from Lemma 6. Next, we will present the first corollary but the second corollary will be discussed in the following section.

Corollary 1. *$H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$ is a finite length module.*

Proof. By Lemma 6, $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$ surjects onto $H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$. Since $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$

vanishes in high degree, $H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$ must also. $\square$

Proposition 1. *The module $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ has finite length.*

Proof. To see this, we will be considering the chain complex $\mathcal{R}/\mathcal{J}[\Delta_b]$. Our strategy will be to apply a localization argument at a prime ideal P and show that $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])_P$ vanishes unless P is $\langle x, y, z\rangle$.

First, if we localize at any prime ideal P where $l_\tau \not\subseteq P$ for every $\tau \in (\Delta)_1$, then $\bigoplus_{\tau\in(\Delta)_1} (\mathcal{R}/\mathcal{J}(\tau))_P = \bigoplus_{\tau\in(\Delta)_1} (R/l_\tau^{r+1})_P = 0$. As a consequence, $(\ker \partial_1)_P = 0$ which forces $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])_P = 0$. Thus, we must consider a prime P that contains l_τ for some $\tau \in (\Delta)_1$.

Suppose that $P = \langle l_\epsilon\rangle$ where $\epsilon \in (\Delta)_1$. In this case, $(\mathcal{R}/\mathcal{J}(\tau))_P = 0$ for all $\tau \not\subseteq V(l_\epsilon)$. We also have $(\mathcal{R}/\mathcal{J}(v))_P = 0$ for every $v \in (\Delta)_0$, since $\mathcal{J}(v) \not\subseteq l_\epsilon$. So, the localized complex is of the form:

$$0 \to \bigoplus_{i=1}^{n} R_P \to \bigoplus_{\tau_j\subseteq V(l_\epsilon)} (\mathcal{R}/\mathcal{J}(\epsilon))_P \to 0$$

Hence, $(\ker \partial_1)_P = \bigoplus_{\tau_j\subseteq V(l_\epsilon)} (\mathcal{R}/\mathcal{J}(\epsilon))_P$ and we need to show that ∂_2 is surjective. For this, consider a generator, μ_j of $\bigoplus_{\tau_j\subseteq V(l_\epsilon)} (\mathcal{R}/\mathcal{J}(\epsilon))_P$. That is, $\mu_j = \Big(0, \ldots, 1, \ldots, 0\Big)^t$ with a 1 in the j^{th} position and the rest zeros. Suppose that σ_i is a triangle with boundary edges τ_j, τ_k, and τ_n with $\tau_j \subseteq V(l_\epsilon)$ and $\tau_k, \tau_n \not\subseteq V(l_\epsilon)$. Let $\gamma = \Big(0, \ldots, 1, \ldots, 0\Big)^t$ with a 1 in the i^{th} position and the rest zeros be an element in $\bigoplus_{i=1}^{n} R_P$. Since, $\tau_k, \tau_n \not\subseteq V(l_\epsilon)$, we have that $\partial_2(\gamma) = \Big(0, \ldots, \partial_2(\sigma_i), \ldots, 0\Big)^t = \mu_j \in \bigoplus_{\tau_j\subseteq V(l_\epsilon)} (\mathcal{R}/\mathcal{J}(\epsilon))_P$ which means ∂_2 is surjective. Therefore, $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])_P = 0$.

We finally consider the case when P is a codimension 2 prime ideal. By an earlier observation, $\langle l_\tau\rangle$ must be contained in P for some edge τ of Δ. So, since codimension 2 prime ideals correspond to points, we can assume $P = \langle l_{\epsilon_1}, l_{\epsilon_2}\rangle$. There are two cases to consider. Suppose $V(P) = \tilde{v}$ is a vertex of the simplicial complex Δ. If

$\tilde{v}$ is an interior vertex, then we are done since $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ is of finite length and $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ is isomorphic to $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ in this situation.

So, we assume that $\tilde{v}$ is on the boundary of Δ. Here, we will have $(\mathcal{R}/\mathcal{J}(v))_P = 0$ for all $v \neq \tilde{v}$ and $(\mathcal{R}/\mathcal{J}(\tau))_P = 0$ for τ where $\tilde{v} \not\subseteq V(l_\tau)$. Note, if $\tilde{v} \subseteq V(l_\tau)$ for some edge τ but $\tilde{v} \not\subseteq \tau$, then $H_1(\mathcal{R}/\mathcal{J}[\Delta])_P = 0$ by an argument similar to the last case.

We may suppose that $\tau_1, \ldots, \tau_k$ are all edges which contain $\tilde{v}$. The localized complex will have the following form:

$$0 \to \bigoplus_{i=1}^{k-1} R_P \to \bigoplus_{i=1}^{k} (\mathcal{R}/\mathcal{J}(\tau_i))_P \oplus \bigoplus_{\tilde{v} \not\subseteq \tau} (\mathcal{R}/\mathcal{J}(\tau))_P \to (\mathcal{R}/\mathcal{J}(\tilde{v}))_P \to 0$$

Suppose $\gamma = \Big(g_1, \ldots, g_k\Big)^t \in \bigoplus_{i=1}^{k} (\mathcal{R}/\mathcal{J}(\tau_i))_P$ is an element of $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])_P$ so that $\partial_1(\gamma) = \sum_{i=1}^{k} g_i \in J(\tilde{v})$. γ is only defined up to $\Big(\alpha_1 l_{\tau_1}^{r+1}, \ldots, \alpha_k l_{\tau_k}^{r+1}\Big)^t$, so fix a representative for γ in $\bigoplus_{i=1}^{k} R_P$. There exists a_i, for $i = 1, \ldots, k$ so that $\sum_{i=1}^{k} g_i = \sum_{i=1}^{k} a_i l_{\tau_i}^{r+1}$. Consider $\psi = \Big(-a_1 l_{\tau_1}^{r+1} + a_2 l_{\tau_2}^{r+1}, a_3 l_{\tau_3}^{r+1}, \ldots, a_k l_{\tau_k}^{r+1}\Big)^t \in \bigoplus_{i=1}^{k-1} R_P$ and notice that $\partial_2(\psi) = \Big(a_2 l_{\tau_2}^{r+1}, a_1 l_{\tau_1}^{r+1} + a_3 l_{\tau_3}^{r+1}, a_4 l_{\tau_4}^{r+1}, \ldots a_k l_{\tau_k}^{r+1}, 0\Big)^t = \phi$ and $\partial_1(\phi) = \sum_{i=1}^{k} a_i l_{\tau_i}^{r+1}$. Also, we have the following short exact sequence of complexes with $\gamma - \phi \in \ker \partial_1$:

$$\begin{tikzcd}
\mathcal{J}[\Delta_b]_P : & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^{k} R(-r-1)_P \arrow[r] \arrow[d] & \mathcal{J}(\tilde{v})_P \arrow[d] \\
\mathcal{R}[\Delta_b]_P : & \bigoplus_{i=1}^{k-1} R_P \arrow[r, "\partial_2"] \arrow[d] & \bigoplus_{i=1}^{k} R_P \arrow[r, "\partial_1"] \arrow[d] & R_P \arrow[d] \\
\mathcal{R}/\mathcal{J}[\Delta_b]_P : & \bigoplus_{i=1}^{k-1} R_P \arrow[r] & \bigoplus_{i=1}^{k} (\mathcal{R}/\mathcal{J}(\tau_i))_P \arrow[r] & (R/J(\tilde{v}))_P
\end{tikzcd}$$

Note that $H_1(\mathcal{R}[\Delta_b]) = 0$ since the localized simplicial complex has no holes, and from this we know that $\ker \partial_1 = \operatorname{image} \partial_2$. So, there exists some element $G \in \bigoplus_{i=1}^{k-1} R_P$ with $\partial_2(G) = \gamma - \phi$. This implies that $\gamma = \phi + \partial_2(G) \in \operatorname{image} \partial_2$. Hence, $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])_P = 0$ and this is equivalent to saying that $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ is a finite length module. $\square$

C. The Hilbert Polynomial of $C^r(\widehat{\Delta_b})$

We begin this section with the second corollary of Lemma 6 which gives a formula for $HP(H_1(\mathcal{R}/\mathcal{J}[\partial\Delta]))$. Next, we look at the values of the Hilbert polynomials of the terms in this formula. Finally, we state the result and conclude with an example.

Corollary 2. $HP(H_1(\mathcal{R}/\mathcal{J}[\partial\Delta])) = \sum_{\tau \in (\partial\Delta)_1} HP(\mathcal{R}/\mathcal{J}(\tau)) - \sum_{v \in (\partial\Delta)_0} HP(\mathcal{R}/\mathcal{J}(v))$

This is a key result because the Hilbert polynomials of $\mathcal{R}/\mathcal{J}(\tau)$ and $\mathcal{R}/\mathcal{J}(v)$ are known. First, by definition, $\mathcal{J}(\tau) = I_\tau^{r+1}$, where I_τ is a principal ideal generated by a linear polynomial. Thus, $HP(\mathcal{R}/\mathcal{J}(\tau)) = HP(R) - HP(R(-r-1)) = \binom{t+2}{2} - \binom{t+2-r-1}{2} = (r+1)t - \frac{1}{2}(r^2 - r) + 1$. Second, Schenck and Stillman in [13], found a free resolution for $\mathcal{R}/\mathcal{J}(v)$, which gives a formula for $HP(\mathcal{R}/\mathcal{J}(v))$.

Lemma 7. *Suppose $\mathcal{J}(v)$ is assumed to have the form $\langle (x+a_1y)^{r+1}, \ldots, (x+a_ny)^{r+1} \rangle$ with $a_i \neq a_j$ if $i \neq j$. Let $\alpha(v) = \lfloor \frac{r+1}{n-1} \rfloor$, $s_1(v) = (n-1)\alpha(v) + n - r - 2$, and $s_2(v) = r + 1 - (n-1)\alpha(v)$ then*

$$\begin{aligned} HP(\mathcal{R}/\mathcal{J}(v)) = \tbinom{t+2}{2} - n\tbinom{t+2-r-1}{2} &+ s_1(v)\tbinom{t+2-r-1-\alpha(v)}{2} \\ &+ s_2(v)\tbinom{t+2-r-2-\alpha(v)}{2} \end{aligned}$$

(Note, there is a typo when defining s_2 in Theorem 3.1 of [13], the second + should be a $-$. Nonetheless, the proof is correct.)

Now, we combine the results in this section with those in the previous sections to obtain a formula for comparing the Hilbert polynomials of splines that satisfy boundary conditions with splines having no restrictions on the boundary.

Theorem 2. *Let $r \geq 0$ be an integer and $R = \mathbf{R}[x, y, z]$. If Δ is a connected finite planar simplicial complex, then*

$$HP(C^r(\widehat{\Delta_b})) = HP(C^r(\widehat{\Delta})) - \sum_{\tau \in (\partial\Delta)_1} HP(\mathcal{R}/\mathcal{J}(\tau)) + \sum_{v \in (\partial\Delta)_0} HP(\mathcal{R}/\mathcal{J}(v))$$

Example 5. Below, we give a simplicial complex Δ (Fig. 4) and Table II which contains various r values demonstrating Theorem 2:

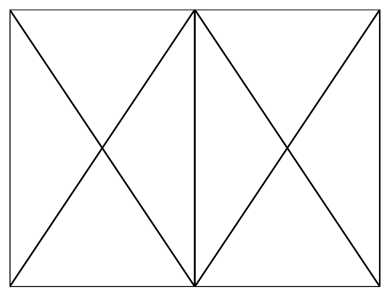

Fig. 4. A simplicial complex Δ to demonstrate Theorem 2.

Table II. The values of Theorem 2 for the complex in Fig. 4.

r	$HP(C^r(\widehat{\Delta_b}))$	$HP(C^r(\widehat{\Delta}))$	$\sum_{\tau \in (\partial\Delta)_1} HP(\mathcal{R}/\mathcal{J}(\tau))$	$\sum_{v \in (\partial\Delta)_0} HP(\mathcal{R}/\mathcal{J}(v))$
0	$4t^2 - 3t + 1$	$4t^2 + 3t + 1$	$6t + 6$	6
1	$4t^2 - 18t + 19$	$4t^2 - 6t + 7$	$12t + 6$	18
2	$4t^2 - 33t + 66$	$4t^2 - 15t + 26$	$18t$	40
3	$4t^2 - 48t + 140$	$4t^2 - 24t + 58$	$24t - 12$	70
4	$4t^2 - 63t + 243$	$4t^2 - 33t + 103$	$30t - 30$	110

The obvious key fact here is that we were able to observe what properties the homology modules $H_0(\mathcal{R}/\mathcal{J}[\partial\Delta])$, $H_0(\mathcal{R}/\mathcal{J}[\Delta_b])$, and $H_1(\mathcal{R}/\mathcal{J}[\Delta_b])$ satisfy. In particular, they are all finite length modules. The other important fact here is that the homology module $H_1(\mathcal{R}/\mathcal{J}[\partial\Delta])$ is not finite length. However, by Corollary 2, we are able to compute its Hilbert polynomial.

CHAPTER IV

ANALYSIS OF SUBDIVIDING POLYGONAL REGIONS OF THE PLANE

A. Introduction

One of the reasons people find splines to be interesting is because of the wide variety of domains on which they may be defined. Consequently, with this variety, the types of questions asked about splines may seem repetitious. However, some questions deserve to be answered regardless of what type of domain the splines are defined on. For example, given any region Δ where splines exist, it is desirable to know the dimensions of the vector spaces $C_k^r(\Delta)$ for $k \gg 0$. In the case when Δ is a triangulation, the following theorem by Alfeld and Schumaker answers this question:

Theorem 3. *Let Δ be a planar simplicial complex. If $k \gg 0$, then*

$$\dim_{\mathbf{R}} C_k^r(\Delta) = \binom{k+2}{2} + \binom{k-r+1}{2} f_1^0 - (\binom{k+2}{2} - \binom{r+2}{2}) f_0^0 + \sigma$$

where $\sigma = \sum \sigma_i$ and $\sigma_i = \frac{1}{2}((1-n(v_i))\alpha(v_i)^2 + (2r-n(v_i)+3)\alpha(v_i))$. In particular, this bound is attained for $k \geq 3r+1$.

For the rest of this chapter, let $\square$ be a connected polygonal region of the plane. We will be focusing on the dimensions of the vector spaces $C_k^r(\square)$ for $k \gg 0$. The ultimate goal is to find a formula for these dimensions similar to the one given by Alfeld and Schumaker in the simplicial complex case. In this dissertation, we begin the search for such a formula by finding bounds for these dimensions. To obtain these bounds, we will be subdividing $\square$ to form a simplicial complex Δ, and we will be using homological machinery to relate the spline modules $C^r(\widehat{\square})$ and $C^r(\widehat{\Delta})$.

At this point, there are two questions to. The first question is, "Why do we want to find a simplicial subdivision Δ of $\square$?" It turns out that the answer to this question gives the upper bounds for the dimensions of the vector spaces $C_k^r(\square)$. In particular, if $\square$ is a polygonal region of the plane and Δ is any subdivision of $\square$, then there is an obvious inclusion $C^r(\widehat{\square}) \hookrightarrow C^r(\widehat{\Delta})$. Therefore, we know $\dim_{\mathbf{R}} C_k^r(\Delta) \geq \dim_{\mathbf{R}} C_k^r(\square)$ for any integer $r \geq 0$ and sufficiently large k. The second question is, "How should we subdivide the region $\square$?" Well, we will add only edges to $\square$ to form a simplicial complex (triangulation) Δ (Fig. 5).

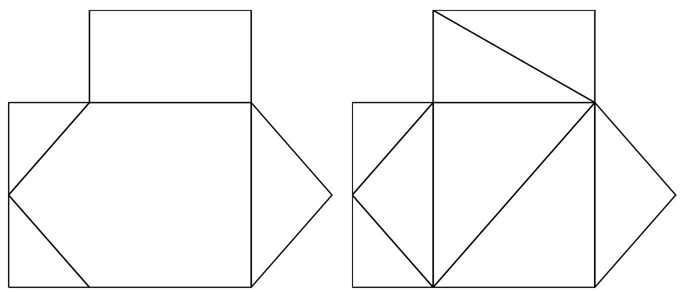

Fig. 5. A polygonal region and one of its simplicial subdivisions.

It is important to note that Δ will not be unique. In fact, for a given polygonal region $\square$, there will be several ways to produce a simplicial complex Δ. However, we must add the same number of edges to $\square$ to get each Δ. Consequently, every such subdivision Δ will have the same number of triangles, $f_2(\Delta)$, and the same number of interior edges, $f_1^0(\Delta)$. For example, if $\square$ is a polygon with n-sides (Fig. 6), then we must add $n-3$ edges to $\square$ to obtain a simplicial complex Δ which will have $n-2$ triangles.

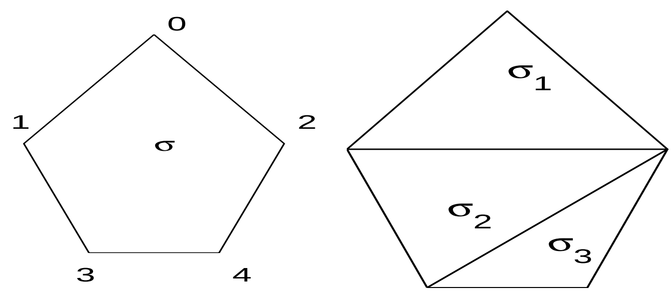

Fig. 6. For a pentagon, σ, we must add 2 edges to obtain a triangulation with three triangles σ_1, σ_2, and σ_3.

B. The Short Exact Sequence $0 \to \mathcal{P} \to \mathcal{R}/\mathcal{J}[\Delta] \to \mathcal{Q} \to 0$

For the rest of this chapter, we will be focusing on finding the lower bounds for the dimensions of the vector spaces $C_k^r(\square)$. In this section, we will be introducing the short exact sequence of chain complexes that will yield the long exact sequence in homology which relates $C^r(\widehat{\square})$ and $C^r(\widehat{\Delta})$.

Let σ be an n-sided polygon (2-face) in the polygonal region $\square$. Label the vertices of σ with $1, \ldots, n$ in the counter-clockwise direction beginning with the left-uppermost vertex (Fig. 7). Consider the map $d_2 : R^{f_2(\square)} \to \bigoplus_{\tau \in (\square)_1^0} \mathcal{R}/\mathcal{J}(\tau)$ which sends σ to the sum of its boundary edges. In particular, if ab denotes the edge connecting the vertices a and b, then $d_2(\sigma) = 12 + 23 + \ldots + (n-1)n + n1$.

Moreover, if ∂_1 is the usual (relative) simplicial boundary operator, then it is easy to verify that $\partial_1 \circ d_2 = 0$. Thus, we have the following chain complex $\mathcal{P}$ of R-modules on $(\square)^0$ whose top homology module is $C^r(\widehat{\square})$:

$$\mathcal{P} : \mathcal{R}^{f_2(\square)} \xrightarrow{d_2} \bigoplus_{\tau \in (\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{v \in (\square)_0^0} \mathcal{R}/\mathcal{J}(v)$$

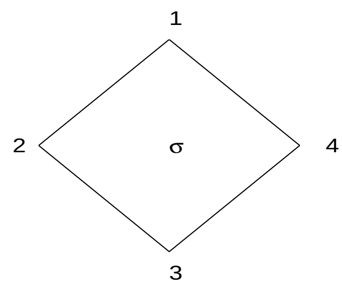

Fig. 7. For the quadrilateral σ, $d_2(\sigma) = 12 + 23 + 34 + 41$.

Again, let σ be an n-sided polygon of the polygonal region $\Box$ and suppose σ gets subdivided into the triangles $\sigma_1, \ldots, \sigma_{n-2}$ of Δ. Consider the map $\theta : R^{f_2(\Box)} \to R^{f_2(\Delta)}$ which sends σ to the sum $\sum_{i=1}^{n-2} \sigma_i$. For example, let the following (Fig. 8) be a subdivision of the quadrilateral σ in Fig. 7,

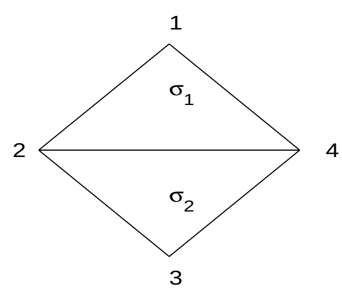

Fig. 8. A subdivided quadrilateral.

then $\theta(\sigma) = \sigma_1 + \sigma_2$.

In the example above, it is easily checked that $i(d_2(\sigma)) = \partial_2(\theta(\sigma))$ where i is an inclusion map and $\partial_2 : R^{f_2(\Delta)} \to \bigoplus_{\tau \in (\Delta)_1^0} \mathcal{R}/\mathcal{J}(\tau)$ is the usual (relative) simplicial boundary map. It turns out that this is true for any polygon. Hence, we know the following is a commutative diagram:

$$\begin{tikzcd}
R^{f_2(\square)} \arrow[r, "d_2"] \arrow[d, "\theta"] & \bigoplus\limits_{\tau\in(\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \arrow[d, hook, "i"] \\
R^{f_2(\Delta)} \arrow[r, "\partial_2"] & \bigoplus\limits_{\tau\in(\Delta)_1^0} \mathcal{R}/\mathcal{J}(\tau)
\end{tikzcd}$$

From this, we conclude that given a polygonally subdivided region $\square$ of the plane and a subdivision Δ of $\square$ obtained by adding only edges, there is a short exact sequence of chain complexes:

$$\begin{tikzcd}
\mathcal{P}: & R^{f_2(\square)} \arrow[r, "d_2"] \arrow[d, "\theta"] & \bigoplus\limits_{\tau\in(\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \arrow[r, "\partial_1"] \arrow[d, hook, "i"] & \bigoplus\limits_{v\in(\square)_0^0} \mathcal{R}/\mathcal{J}(v) \arrow[d, leftrightarrow] \\
\mathcal{R}/\mathcal{J}[\Delta]: & R^{f_2(\Delta)} \arrow[r, "\partial_2"] \arrow[d, two heads] & \bigoplus\limits_{\tau\in(\Delta)_1^0} \mathcal{R}/\mathcal{J}(\tau) \arrow[r, "\partial_1"] \arrow[d, two heads] & \bigoplus\limits_{v\in(\Delta)_0^0} \mathcal{R}/\mathcal{J}(v) \arrow[d] \\
\mathcal{Q}: & R^{f_2(\Delta)-f_2(\square)} \arrow[r, "q"] & \bigoplus\limits_{\tau\in(\Delta)_1^0-(\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{Q}$ is the quotient chain complex and q is defined so that the lower left square commutes. Therefore, we have the following long exact sequence in homology:

$$0 \to C^r(\widehat{\square}) \to C^r(\widehat{\Delta}) \to H_2(\mathcal{Q}) \to H_1(\mathcal{P}) \to H_1(\mathcal{R}/\mathcal{J}[\Delta]) \to H_1(\mathcal{Q}) \to$$
$$H_0(\mathcal{P}) \to H_0(\mathcal{R}/\mathcal{J}[\Delta]) \to 0$$

The next step will be to determine what properties the homology modules of $\mathcal{P}$ and $\mathcal{Q}$ satisfy.

C. The Homology Modules of $\mathcal{P}$ and $\mathcal{Q}$

Recall, by Lemma 2, we know that the zeroth homology module of $\mathcal{R}/\mathcal{J}[\Delta]$ is zero and the first homology module of $\mathcal{R}/\mathcal{J}[\Delta]$ is of finite length. Now, we examine some of the other homology modules in the long exact sequence from the previous section. The goal will be to determine what properties these homology modules satisfy. In particular, we will show that $H_0(\mathcal{P})$ is identically zero and that $H_1(\mathcal{Q})$ is a module of finite length. Moreover, we will observe that there is a nice formula for computing the Hilbert polynomial of $H_2(\mathcal{Q})$. Combined with the fact that the Hilbert polynomial of $C^r(\widehat{\Delta})$ is known, this gives a lower bound for the Hilbert polynomial of $C^r(\widehat{\Box})$.

Lemma 8. *The homology module $H_0(\mathcal{P})$ is zero.*

Proof. Let v be any vertex of $\Box$ and let S be the set of paths in $\Box$ which connect v to the boundary of $\Box$. Define the distance from the vertex v to the boundary of $\Box$, denoted $d(v, \partial\Box)$, to be the minimum length of the paths in S where the length of a path in S is given by the number of edges in the path.

Suppose v_1 is any vertex of $\Box$ satisfying $d(v, \partial\Box) = 1$. If the edge between v_1 and the boundary of $\Box$ is τ_1, then we know that the map $\mathcal{R}/\mathcal{J}(\tau_1) \xrightarrow{\partial_1} \mathcal{R}/\mathcal{J}(v_1)$ is surjective. Thus, no homology can be supported on the vertices that are distance 1 from the boundary.

Next, let v_2 be any vertex of $\Box$ satisfying $d(v_2, \partial\Box) = 2$. Suppose the following path of length 2 (Fig. 9) connects v_2 to the boundary of $\Box$:

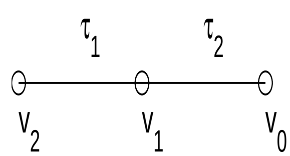

Fig. 9. A path of length 2 connecting v_2 to $\partial\Box$.

In $\mathcal{R}/\mathcal{J}(v_2) \oplus \mathcal{R}/\mathcal{J}(v_1)$, we know that the image of τ_1 (w.r.t. the ∂_1 operator) is

$(0,1)$ and the image of τ_2 is $(1,-1)$. Moreover, we know that $(1,-1)$ is equivalent to $(1,0)$ modulo the image of τ_1. So, we conclude that the map $\mathcal{R}/\mathcal{J}(\tau_2) \oplus \mathcal{R}/\mathcal{J}(\tau_1) \xrightarrow{\partial_1} \mathcal{R}/\mathcal{J}(v_2) \oplus \mathcal{R}/\mathcal{J}(v_1)$ is surjective. That is, no homology is supported on the vertices which are distance 2 from the boundary of $\square$.

Finally, we can repeat this process to conclude that none of the vertices in $\square$ can support any homology. Hence, $H_0(\mathcal{P}) = 0$. $\square$

Corollary 3. *The module $H_1(\mathcal{Q})$ has finite length.*

Proof. We conclude from Lemma 8 that the module $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ surjects onto the module $H_1(\mathcal{Q})$ in the long exact sequence. Moreover, Lemma 2 says $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ is a finite length module. Hence, $H_1(\mathcal{Q})$ must also be a finite length module. $\square$

Lemma 9. $HP(H_2(\mathcal{Q})) = (f_2(\Delta) - f_2(\square))\binom{t+2}{2} - (f_1^0(\Delta) - f_1^0(\square))(\binom{t+2}{2} - \binom{t+2-r-1}{2})$

Proof. The chain complex $\mathcal{Q}$ has the following form:

$$\mathcal{Q} : R^{f_2(\Delta)-f_2(\square)} \xrightarrow{q} \bigoplus_{\tau \in (\Delta)_1^0 - (\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \to 0$$

Consequently, $H_0(\mathcal{Q}) = 0$ and we have the following exact sequence:

$$0 \to H_2(\mathcal{Q}) \to R^{f_2(\Delta)-f_2(\square)} \xrightarrow{q} \bigoplus_{\tau \in (\Delta)_1^0 - (\square)_1^0} \mathcal{R}/\mathcal{J}(\tau) \to H_1(\mathcal{Q}) \to 0$$

Now, by Corollary 3 and the previously discussed properties of the Hilbert polynomial, we obtain the following equality:

$$HP(H_2(\mathcal{Q})) = (f_2(\Delta) - f_2(\square))HP(R) - (f_1^0(\Delta) - f_1^0(\square))HP(R(-r-1))$$

$\square$

D. A Lower Bound for $\dim_{\mathbf{R}} C_k^r(\square)$

In this section, we will produce the lower bounds for the dimensions of the vector spaces $C_k^r(\square)$. Note, these lower bounds will be given in terms of the dimensions of the vector spaces $C_k^r(\Delta)$ as well as geometrical data of both $\square$ and Δ.

Theorem 4. *If $\square$ is a connected polygonal region of $\mathbf{R}^2$ and Δ is a simplicial subdivision of $\square$ obtained by adding only edges to $\square$, then the following inequality holds when k is sufficiently large:*

$$\begin{aligned}\dim_{\mathbf{R}} C_k^r(\square) \geq \dim_{\mathbf{R}} C_k^r(\Delta) &- (f_2(\Delta) - f_2(\square))\tbinom{k+2}{2} \\ &+ (f_1^0(\Delta) - f_1^0(\square))(\tbinom{k+2}{2} - \tbinom{k+2-r-1}{2})\end{aligned}$$

Proof. Recall, we have the following long exact sequence in homology:

$$\begin{aligned}0 \to C^r(\widehat{\square}) \to C^r(\widehat{\Delta}) \to H_2(\mathcal{Q}) \to H_1(\mathcal{P}) \to H_1(\mathcal{R}/\mathcal{J}[\Delta]) \to H_1(\mathcal{Q}) \to \\ H_0(\mathcal{P}) \to H_0(\mathcal{R}/\mathcal{J}[\Delta]) \to 0\end{aligned}$$

Now, by applying the lemmas and corollaries describing the homology modules in this sequence and by using the properties of the Hilbert polynomial, we clearly have the following equality:

$$HP(C^r(\widehat{\square})) = HP(C^r(\widehat{\Delta})) - HP(H_2(\mathcal{Q})) + HP(H_1(\mathcal{P}))$$

Using this equality, the desired inequality follows immediately. $\square$

Example 6. Below, we give a polygonal region $\square$ along with a simplicial subdivision Δ of $\square$ (Fig. 10) and Table III which contains various r values demonstrating Theorem 4:

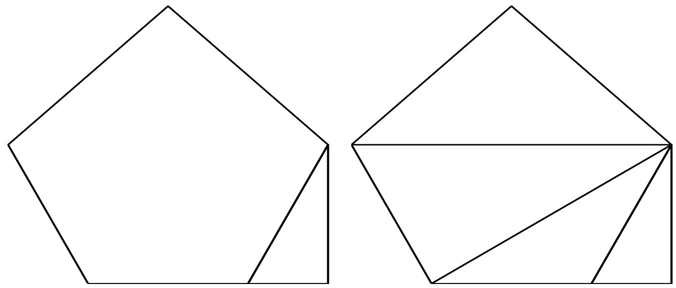

Fig. 10. A polygonal region and one of its simplicial subdivisions.

Table III. The values of Theorem 4 for the complexes in Fig. 10.

r	$\dim_{\mathbf{R}} C_k^r(\Box)$	$\dim_{\mathbf{R}} C_k^r(\Delta)$	$-(f_2(\Delta) - f_2(\Box))\binom{k+2}{2}$	$(f_1^0(\Delta) - f_1^0(\Box)) \quad * \quad (\binom{k+2}{2} - \binom{k+2-r-1}{2})$
0	$k^2 + 2k + 1$	$2k^2 + 3k + 1$	$-k^2 - 3k - 2$	$2k + 2$
1	$k^2 + k + 1$	$2k^2 + 1$	$-k^2 - 3k - 2$	$4k + 2$
2	$k^2 + 2$	$2k^2 - 3k + 4$	$-k^2 - 3k - 2$	$6k$
3	$k^2 - k + 4$	$2k^2 - 6k + 10$	$-k^2 - 3k - 2$	$8k - 4$
4	$k^2 - 2k + 7$	$2k^2 - 9k + 19$	$-k^2 - 3k - 2$	$10k - 10$

The salient fact here is the module $H_2(\mathcal{Q})$ is not of finite length. However, by Lemma 9, we are able to compute its Hilbert polynomial. Moreover, the proof of Theorem 4 suggests there may be a way to obtain a formula for the dimensions of the vector spaces $C_k^r(\Box)$. To follow this suggestion, we will need to examine the homology module $H_1(\mathcal{P})$ to see if we can compute its Hilbert polynomial.

CHAPTER V

SUMMARY AND FUTURE QUESTIONS

In this dissertation, we studied spaces of piecewise polynomial functions (splines) of a prescribed order of smoothness on a triangulated (or polygonally subdivided) region Δ of $\mathbf{R}^2$. By embedding Δ into $\mathbf{R}^3$, and forming the cone $\hat{\Delta}$ of Δ, we were able to ask questions about splines in terms of questions about graded modules over the polynomial ring $\mathbf{R}[x, y, z]$. Thus, we were able to apply methods from homological and commutative algebra.

Following the earlier approaches of Billera and Rose as well as Schenck and Stillman, we used a chain complex of modules where the spline space $C^r(\hat{\Delta})$ appears as the top homology module. Then, by using previous results about the other homology modules in this complex, we were able to answer new questions involving $C^r(\hat{\Delta})$.

We began by gluing two planar simplicial complexes Δ_1 and Δ_2 in a manner so that $\Delta_1 \cap \Delta_2$ and $\Delta = \Delta_1 \cup \Delta_2$ were also planar simplicial complexes. By using a well-known and powerful homological algebra tool, we showed there is a natural relationship among the dimensions of the corresponding spline spaces. In particular, we used Mayer-Vietoris to show for sufficiently large k, the following equality holds:

$$\dim_{\mathbf{R}} C^r(\widehat{\Delta})_k = \dim_{\mathbf{R}} C^r(\widehat{\Delta_1})_k + \dim_{\mathbf{R}} C^r(\widehat{\Delta_2})_k - \dim_{\mathbf{R}} C^r(\widehat{\Delta_1 \cap \Delta_2})_k$$

Next, given a planar simplicial complex Δ, we studied splines that are globally defined C^r functions across the interior of Δ and which vanish on the boundary of Δ. The space of all such splines on Δ we denoted $C^r(\widehat{\Delta_b})$. By defining two chain complexes, $\mathcal{R}/\mathcal{J}[\Delta_b]$ and $\mathcal{R}/\mathcal{J}[\partial\Delta]$, and analyzing their homology modules, we were able to obtain the following result:

$$HP(C^r(\widehat{\Delta_b})) = HP(C^r(\widehat{\Delta})) - \sum_{\tau \in (\partial\Delta)_1} HP(R/J(\tau)) + \sum_{v \in (\partial\Delta)_0} HP(R/J(v))$$

where, on the right-hand side of the equality, τ is a boundary edge, v is a boundary vertex, and each term has a known computation.

Finally, we considered splines defined on a polygonally subdivided region $\square$ of $\mathbf{R}^2$. By adding only edges to $\square$, we were able to obtain a simplicial subdivision Δ. Then, we were able to use Δ to find the following bounds for the dimensions of the vector spaces $C^r_k(\square)$:

$$\begin{aligned}\dim_{\mathbf{R}} C^r_k(\Delta) \geq \dim_{\mathbf{R}} C^r_k(\square) \geq \dim_{\mathbf{R}} C^r_k(\Delta) - (f_2(\Delta) - f_2(\square))\tbinom{k+2}{2} \\ + (f_1^0(\Delta) - f_1^0(\square))(\tbinom{k+2}{2} - \tbinom{k+2-r-1}{2)})\end{aligned}$$

While working on this dissertation, there have been several questions that have arisen as possible future research directions. Some are similar to the questions which have been answered in this dissertation and some are inspired by research done by others.

The first possible direction involves gluing simplicial complexes. Let Δ_1 and Δ_2 be two-dimensional simplicial complexes glued along $\Delta_1 \cap \Delta_2$ to form the two-dimensional complex $\Delta = \Delta_1 \cup \Delta_2$. In Chapter II, we obtained a nice relationship among the spline modules when $\Delta_1 \cap \Delta_2$ is also two-dimensional. It would be interesting to consider the case when $\Delta_1 \cap \Delta_2$ is one-dimensional, and to consider similar cases in higher dimensions.

The second direction combines gluing simplicial complexes with putting boundary constraints on the splines. Again, let Δ_1 and Δ_2 be two-dimensional simplicial complexes glued along $\Delta_1 \cap \Delta_2$ to form the two-dimensional complex $\Delta = \Delta_1 \cup \Delta_2$. It would be nice to obtain results involving the splines which vanish on the boundary of these regions when $\Delta_1 \cap \Delta_2$ is two-dimensional or even one-dimensional. It would be interesting to see how including boundary constraints will change the previous results and techniques. Again, it is natural to consider cases in higher dimensions.

The next direction involves subdividing the simplicial complexes. Given a simplicial complex Δ, let Δ_D be a subdivision of Δ. There is a clear way of seeing that $C^r(\widehat{\Delta}) \hookrightarrow C^r(\widehat{\Delta_D})$. What is the structure of this map? Also, is there is a map in the reverse direction and what structure does it have?

Another possible direction is to relate $C^{r+1}(\widehat{\Delta})$ and $C^r(\widehat{\Delta})$ for a given simplicial complex Δ. There is a clear inclusion $C^{r+1}(\widehat{\Delta}) \hookrightarrow C^r(\widehat{\Delta})$. Also, there is a way to map $C^r(\widehat{\Delta})$ into $C^{r+1}(\widehat{\Delta})$. It would be nice to obtain some results relating these two maps and modules.

The next direction would be to study the freeness of $C^r(\widehat{\Delta_b})$. In [14], Schenck and Stillman showed that $C^r(\widehat{\Delta})$ is free iff $H_1(R/J) = 0$. Are there conditions which determine when the module $C^r(\widehat{\Delta_b})$ is free?

A final direction involves the bounds for the dimensions of the vector spaces $C^r_k(\Box)$ when $\Box$ is a polygonally subdivided region of $\mathbf{R}^2$. It would be interesting to develop a strategy to improve the already existing bounds that were found in Chapter IV of this dissertation. Also, it would be very satisfying to discover a method for obtaining an exact formula for these dimensions. In the case where Δ is a triangulation, such a formula has been discovered by Alfeld and Schumaker in [1].

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APPENDIX A

DOCUMENTATION AND CODE

The *SplineCode* package is primarily designed to build the matrix, introduced by Billera and Rose in [4], associated to the space of splines on a given region. We then use the matrix to find the spline space itself and its Hilbert polynomial.

Example 7. Let Δ be the planar simplicial complex (Fig. 11) given below:

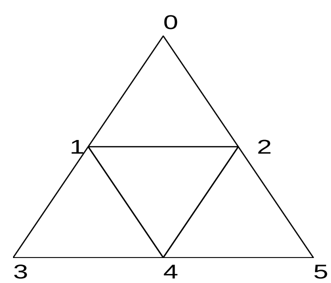

Fig. 11. A sample triangulation.

SplineCode expects three things as input: a list of the triangles, a matrix of vertex locations, and the desired order of smoothness. In Fig. 11, the vertices have been labeled, and there are four triangles. The triangles are denoted by the following list:

$$T = \{\{0, 1, 2\}, \{1, 3, 4\}, \{1, 4, 2\}, \{2, 4, 5\}\}$$

The vertex locations are given in a matrix whose i^{th} column contains the coordinates of the i^{th} vertex. Suppose the vertices are given by the following matrix (which was used to generate the Matlab Fig. 11):

$$V = \begin{bmatrix} 0 & -1 & 1 & -2 & 0 & 2 \\ 2 & 0 & 0 & -2 & -2 & -2 \end{bmatrix}$$

The command to build the matrix is getmat, which expects the three parameters above. The output will be the matrix. After calling getmat, we will be able to ask Macaulay2 questions about the matrix.

```
i1 : load "SplineCode"

--loaded SplineCode

i2 : T = {{0,1,2},{1,3,4},{1,4,2},{2,4,5}}

o2 = {{0, 1, 2}, {1, 3, 4}, {1, 4, 2}, {2, 4, 5}}

o2 : List

i3 : V = transpose matrix{{0,2},{-1,0},{1,0},{-2,-2},{0,-2},{2,-2}}**R

o3 = | 0 -1 1 -2 0  2  |

     | 2 0  0 -2 -2 -2 |

             2       6

o3 : Matrix R  <--- R

i4 : getmat (T,V,0)

o4 = | 1 0  -1 0 2y 0        0        |

     | 0 -1 1  0 0  2x+y+2z 0        |

     | 0 0  -1 1 0  0        2x-y-2z |

             3       7

o4 : Matrix R  <--- R
```

The Macaulay2 command ker will give us the kernel of our matrix (i.e. the spline module $C^0(\widehat{\Delta})$) and the command hilbertPolynomial will give us the Hilbert polynomial of the module:

```
i5 : ker o4
```

```
o5 = image | 1 -2y 0       2x-y-2z |
           | 1 0   2x+y+2z 2x-y-2z |
           | 1 0   0       2x-y-2z |
           | 1 0   0       0       |
           | 0 1   0       0       |
           | 0 0   1       0       |
           | 0 0   0       1       |
                                7
o5 : R-module, submodule of R
i6 : hilbertPolynomial o5
o6 = - 3*P  + 4*P
          1      2
o6 : ProjectiveHilbertPolynomial
i7 : exit
```

Note, Macaulay2 gives the Hilbert polynomial in terms of projective spaces. So, we read P_n as $\binom{n+t}{t}$. Thus, in Example 7, $HP(C^0(\widehat{\Delta})) = 2t^2 + 3t + 1$.

Finally, if the user is strictly interested in finding the Hilbert polynomial of $C^r(\widehat{\Delta})$, then use the same input as before and the command HPCr, which combines the previous steps.

```
i1 : load "SplineCode"
--loaded SplineCode
i2 : T = {{0,1,2},{1,3,4},{1,4,2},{2,4,5}}
o2 = {{0, 1, 2}, {1, 3, 4}, {1, 4, 2}, {2, 4, 5}}
o2 : List
i3 : V = transpose matrix{{0,2},{-1,0},{1,0},{-2,-2},{0,-2},{2,-2}}**R
```

```
o3 = | 0 -1 1 -2 0  2  |
     | 2 0  0 -2 -2 -2 |
             2       6
o3 : Matrix R  <--- R
i4 : HPCr (T,V,0)
- 3*P  + 4*P
     1      2
i5 : exit
```

Example 8. In this example, we again use the triangulation Δ from Fig. 11. However, this time we will be considering splines which vanish on the boundary of Δ. Using the same input as before and the command ngetmat, we can create a matrix which is similar to the Billera-Rose matrix. We can then find the spline module $C^0(\widehat{\Delta})$ and its Hilbert polynomial exactly as we did before with the ker and hilbertPolynomial commands or we can use the HPCrb command which combines the steps.

```
i1 : load "SplineCode"
--loaded SplineCode
i2 : T = {{0,1,2},{1,3,4},{1,4,2},{2,4,5}}
o2 = {{0, 1, 2}, {1, 3, 4}, {1, 4, 2}, {2, 4, 5}}
o2 : List
i3 : V = transpose matrix{{0,2},{-1,0},{1,0},{-2,-2},{0,-2},{2,-2}}**R
o3 = | 0 -1 1 -2 0  2  |
     | 2 0  0 -2 -2 -2 |
             2       6
o3 : Matrix R  <--- R
i4 : HPCrb (T,V,0)
```

```
6*P  - 9*P  + 4*P
   0      1      2
```

Finally, the command compare will actually provide the user with the Hilbert polynomials of $C^r(\widehat{\Delta})$ and $C^r(\widehat{\Delta}_b)$ simultaneously.

```
i5 : compare (T,V,0)

- 3*P  + 4*P
     1      2

6*P  - 9*P  + 4*P
   0      1      2

i6 : exit
```

Example 9. Below, we are given a simplicial complex Δ (Fig. 12), which was built by gluing two planar simplicial complexes Δ_1 and Δ_2 along a planar simplicial complex $\Delta_1 \cap \Delta_2$. In this example, we want to observe that the equality in Theorem 1 holds for various orders of smoothness.

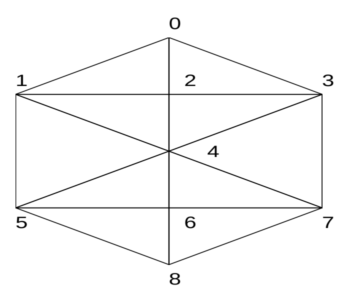

Fig. 12. The resulting simplicial complex Δ after gluing Δ_1 and Δ_2.

SplineCode now expects four pieces of input: a list of the triangles for both Δ_1 and Δ_2, a matrix of vertex locations for Δ, and the desired order of smoothness. We then use the command MV to test the equality.

```
i1 : load "SplineCode"

--loaded SplineCode

i2 : T1 = {{0,1,2},{0,2,3},{1,4,2},{2,4,3},{3,4,7}}

o2 = {{0, 1, 2}, {0, 2, 3}, {1, 4, 2}, {2, 4, 3}, {3, 4, 7}}

o2 : List

i3 : T2 = {{1,4,2},{1,5,4},{2,4,3},{3,4,7},{4,5,6},{4,6,7},{5,8,6},
           {6,8,7}}

o3 = {{1, 4, 2}, {1, 5, 4}, {2, 4, 3}, {3, 4, 7}, {4, 5, 6},
      {4, 6, 7}, {5, 8, 6}, {6, 8, 7}}

o3 : List

i4 : V1 = transpose

matrix{{0,2},{-1,1},{0,1},{1,1},{0,0},{-1,-1},{0,-1},{1,-1},{0,-2}}**R

o4 = | 0 -1 0 1 0 -1 0  1  0  |

     | 2 1  1 1 0 -1 -1 -1 -2 |

             2       9

o4 : Matrix R  <--- R

i5 : MV (T1,T2,V1,0)

o5 = true

i6 :  MV (T1,T2,V1,1)

o6 = true

i7 : MV (T1,T2,V1,3)

o7 = true

i8 :  MV (T1,T2,V1,6)

o8 = true

i9 : exit
```

Example 10. In this last example, we will be given a polygonal region $\square$ (Fig. 13), and we will need to calculate its Hilbert polynomial in order to compare the dimensions of the vector spaces $C^r_k(\square)$ for $k \gg 0$ with those for its simplicial subdivision Δ.

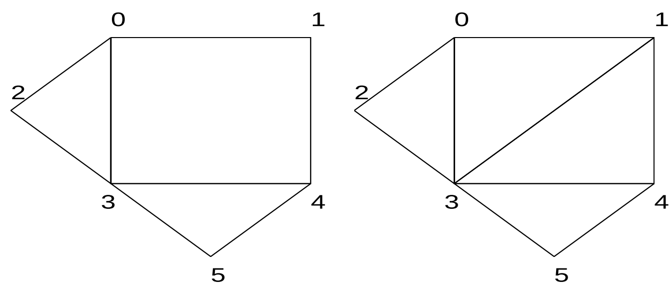

Fig. 13. A polygonal region $\square$ and its simplicial subdivision Δ.

SplineCode again expects three inputs: a list of the maximal simplices, a matrix of vertex locations, and the desired order of smoothness. We use the command getmatpoly to find the matrix associated to the spline space for the region. We then use the commands ker and hilbertPolynomial to find the Hilbert polynomial of the spline module. Again, we have the option of using a single command HPCrpoly which combines the previous steps.

```
i1 : load "SplineCode"

--loaded SplineCode

i2 : P = {{0,2,3},{0,3,4,1},{3,5,4}}

o2 = {{0, 2, 3}, {0, 3, 4, 1}, {3, 5, 4}}

o2 : List

i3 : VP = transpose matrix{{0,2},{2,2},{-1,1},{0,0},{2,0},{1,-1}}**R
```

```
o3 = | 0 2 -1 0 2 1  |
     | 2 2 1  0 0 -1 |

            2       6
o3 : Matrix R  <--- R
i4 : HPCrpoly (P,VP,0)
- 2*P  + 3*P
     1      2
i5 : exit
```

Finally, we close by providing the code for the *SplineCode* package.

```
diag = (f) ->(R = ring f;
              map(R^{(rank target f):1}, R^(rank target f),
              (i,j) -> if i== j then f_(i,0) else 0))
--Takes an n by 1 matrix
--returns a diagonal matrix


remdups = (L)->(T = sort L;
                Out1 = {};
                t=#L-1;
                scan(t, i->if T#i==T#(i+1)
                           then Out1 = append(Out1, T#i));
                Out1)
--Takes a list, possibly with duplicate (but only doubled) entries
--returns the entries which appear twice TH 02/05


remsings = (L)->(T = sort L;
```

```
                Out2 = {};
                t=#L-1;
                scan(t, i->if (T#i=!=T#(i+1) and T#i=!=T#(i-1))
then Out2 = append(Out2, T#i));
if T#t =!=T#(t-1) then Out2=append(Out2, T#t);
                Out2)
--Takes a list, possibly with duplicate (but only doubled) entries.
--returns the entries which appear once TH 06/05

bdpoly=(L)->(M=append(L, L#0);
             N=apply(#L, i->{M#i, M#(i+1)});
             N)
--Takes a polygon L
--returns the edges of L TH 01/06

edges = (D) ->(E = flatten apply(D, i->subsets(i,2)))
--Takes a list of triples(triangles),
--returns all doubles(edges) appearing in triples(triangles)
--possibly with multiplicity TH 02/05

edgespoly = (D) ->(E = flatten apply(D, i->bdpoly(i)))
--Takes a list of polygons
--returns all doubles(edges) appearing in polygons,
--possibly with multiplicity TH 01/06

bdry2 = (L) ->(E=apply(edges(L), i->sort i);
```

```
               intedges = remdups(E); --list of interior edges
               bdedges=remsings(E); --list of boundary edges
               apply(intedges, j->(apply(L, k -> getcolumn(j,k)))))
--Takes an oriented simplicial complex L
-- returns the matrix of the boundary 2 map for relative homology
--(i.e. mod boundary) TH 02/05

nbdry2 = (L) ->(E=unique apply(edges(L), i->sort i);
                  apply(E, j->(apply(L, k -> getcolumn(j,k)))))
--Takes an oriented simplicial complex L
--returns the matrix of the boundary 2 map for simplicial homology
--(i.e. boundary included) TH 06/05

bdry2poly = (L) ->(E=apply(edgespoly(L), i->sort i);
               intedges = remdups(E); --list of interior edges
               bdedges=remsings(E); --list of boundary edges
               apply(intedges, j->(apply(L, k ->
         getcolumnpoly(j,k)))))
--Takes an oriented polygonal region L
-- return the matrix of the "boundary 2" map for relative homology
--(i.e. mod boundary). TH 01/06

getcolumn = (L1,L2)->(s=subsets(L2,2);
                      result = 0;
                      scan(3, l-> if L1==s#l then result =(-1)^l
                           else if L1==sort(s#l)
```

```
then result = (-1)^(l+1));
                      result)
--Takes an edge L1 and a triangle L2
-- returns +1, -1 or 0 depending on orientation
--of edge in bdry of L2 TH 02/05

getcolumnpoly = (L1,L2)->(s=bdpoly(L2);
                      result = 0;
                      scan(#s, l-> if L1==s#l then result =1
                           else if L1==sort(s#l) then result = -1);
                      result)
--Takes an edge L1 and a polygon L2
--returns +1, -1 or 0 depending on orientation
--of edge in bdry of L2 TH 01/06

edgemat = (Elist,Vlocs)->(apply(Elist, i-> det matrix
{{1,(submatrix(Vlocs, ,{i#0}))_(0,0),(submatrix(Vlocs, ,
               {i#0}))_(1,0)},
 {1,(submatrix(Vlocs, ,{i#1}))_(0,0),(submatrix(Vlocs, ,
               {i#1}))_(1,0)},
 {z,x,y}}))
--Takes an edge list and the vertex locations
--returns the matrix with equations of the
--linear forms containing the edges TH 02/05

getmat = (L,V,r)->((matrix bdry2(L)**R)|(diag transpose
```

```
    matrix{edgemat(intedges,V)})^(r+1))
--Takes the triangles, vertex locations and integer r
--returns the associated Billera-Rose matrix for finding
--C^r(Delta), when Delta is a triangulation TH 02/05

ngetmat = (L,V,r)->((matrix nbdry2(L)**R)|(diag transpose
     matrix{edgemat(E,V)})^(r+1))
--Takes the triangles, vertex locations and integer r
--returns the associated Billera-Rose-like matrix for finding
--C^r(Delta_b), when Delta is a triangulation TH 06/05

getmatpoly = (L,V,r)->((matrix bdry2poly(L)**R)|(diag transpose
matrix{edgemat(intedges,V)})^(r+1))
--Takes the polygons, vertex locations and integer r
--returns the associated Billera-Rose matrix for finding
--C^r(Delta), when Delta is a polygonal region TH 01/06

MV = (L1,L2,V,r)->(U = keys (tally (L1|L2));
                 Int = remdups (L1|L2);
                 UU=getmat(U, V,r);
                 II =getmat(Int,V,r);
                 LL1 =getmat(L1,V,r);
                 LL2 =getmat(L2,V,r);
                 h1=hilbertPolynomial ker LL1;
                 h2=hilbertPolynomial ker LL2;
                 hI=hilbertPolynomial ker II;
```

```
                hU=hilbertPolynomial ker UU;
                (h1+h2)==(hU+hI))
--Takes the triangle list of two simplices that are to be glued,
--a single vertex location matrix for the union, and an integer r
--returns a true/false after checking the Hilbert Polynomial
--equation found for glued simplicial complexes
--when the intersection is two-dimensional TH 02/05

compare = (L,V,r)->(L1 = getmat(L,V,r);
                    L2 = ngetmat(L,V,r);
                    print hilbertPolynomial ker L1;
                    print hilbertPolynomial ker L2)
--Takes the triangle list, vertex location matrix, and integer r
--returns the Hilbert Polynomial for C^r(Delta)
--as well as the Hilbert Polynomial for C^r(Delta_b) TH 06/05

HPvert = (m,n)->(H=QQ[k,r,t];
                a=floor((n+1)/(m-1));
                b=(m-1)*a+m-n-2;
                c=n+1-(m-1)*a;
(1/2)*(t+2)*(t+1)-m*(1/2)*(t-n+1)*(t-n)
+c*(1/2)*(t-n-a)*(t-n-a-1)+b*(1/2)*(t-n-a+1)*(t-n-a))
--Takes k = distinct slopes @ v and integer r
--returns HP(R/J(v)) TH 06/05

HPedge = (m,n)->(1/2)*(m+2)*(m+1)-(1/2)*(m-n+1)*(m-n)
```

```
--Takes dimension k and integer smoothness r
--returns HP(R/J(tau)) TH 06/05

HPCr = (L,V,r)->(L3 = getmat(L,V,r);
 print hilbertPolynomial ker L3)
--Takes the triangles, vertex locations and integer r
--returns the Hilbert Polynomial of C^r(Delta)
--when Delta is a triangulation TH 02/05

HPCrb = (L,V,r)->(L4 = ngetmat(L,V,r);
  print hilbertPolynomial ker L4)
--Takes the triangles, vertex locations and integer r
--returns the Hilbert Polynomial of C^r(Delta_b)
--when Delta is a triangulation TH06/05

HPCrpoly = (L,V,r)->(L3 = getmatpoly(L,V,r);
 print hilbertPolynomial ker L3)
--Takes the polygons, vertex locations and integer r
--returns the Hilbert Polynomial of C^r(Delta)
--when Delta is a polygonal region TH 01/06
```

VITA

Terry Lynn McDonald was born in Tyler, Texas on November 13, 1976. He received his Bachelor of Science in mathematics from The University of Texas at Tyler in May, 1999. He was a public school teacher for Tyler Independent School District from August, 1999 through June, 2000. He received his Master of Science in mathematics from Texas A&M University in May, 2002. He received his Ph.D. in mathematics from Texas A&M University in May, 2006. His permanent address is 15683 CR 223, Tyler, TX 75707.