ON THE STRUCTURE OF A CLASS OF OPERATORS

A Dissertation

by

SAMI M. HAMID

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2005

Major Subject: Mathematics

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ABSTRACT

On the Structure of a Class of Operators. (May 2005)

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In this dissertation we study certain classes of operators on a separable, complex, infinite dimensional Hilbert space $\mathcal{H}$, specifically from the point of view of properties of the hyperlattice (i.e., lattice of hyperinvariant subspaces) for such operators. We show that every (BCP)-operator in C_{00} is hyperquasisimilar to a quasidiagonal (BCP)-operator in C_{00}. Moreover we show that there exists a fixed block diagonal (BCP)-operator B_u with the property that if every compact perturbation $B_u + K$ of B_u in (BCP) and C_{00} with $\|K\| < \varepsilon$ has a nontrivial hyperinvariant subspace, then every nonscalar operator on $\mathcal{H}$ has a nontrivial hyperinvariant subspace. This shows that the study of the structure of the hyperlattice of an arbitrary operator on Hilbert space is essentially equivalent to the study of the hyperlattice structure of some much smaller, special classes of operators, and it is these on which we concentrate.

Moreover, we study some special subclasses $(\mathcal{B}_\theta)$ and $(\mathcal{S}_\theta)$ of the class of invertible (BCP)-operators with a view of obtaining some insight into the problem of determining the structure of operators in these classes.

To Kalthoum and Mahmoud

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

In this dissertation we will study certain classes of operators on a separable, complex, infinite dimensional Hilbert space, specifically from the point of view of properties of the hyperlattice (i.e., lattice of hyperinvariant subspaces) for such operators. This study has been largely motivated by the very recent sequence of papers [18], [24], [17], and [8], from which it results that the study of the structure of the hyperlattice of an arbitrary operator on Hilbert space is essentially equivalent to the study of the hyperlattice structure of some much smaller, special classes of operators, and it is these on which we concentrate.

The dissertation is organized as follows. This chapter is devoted largely to the definitions and notation of various concepts that we shall use. Chapter II consists of the statement of a body of results from the theory of dual algebras of various authors that bear directly on our study, while Chapter III is essentially a version of [24] (in the creation of which the author played a significant role). Finally, in Chapters IV and V, we consider some special subclasses of (BCP)-operators (defined below) to which the above-mentioned sequence of four papers naturally leads.

In what follows, $\mathcal{H}$ will be a fixed separable, infinite dimensional, complex, Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. As usual, we reserve the symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ for the sets of integers, positive integers, nonnegative integers, complex numbers, open unit disc in $\mathbb{C}$, and unit circle in $\mathbb{C}$, respectively. For each $0 \leq \theta < 1$, we shall consistently write, $\mathbf{A}_\theta$ for the annulus $\mathbf{A}_\theta := \{\zeta \in \mathbb{C} : \theta \leq |\zeta| \leq 1\}$. If $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ we denote by $\mathcal{S}^-$ the norm-closure of $\mathcal{S}$

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and the set of all scalar multiples of $1_{\mathcal{H}}$ will be written as $\mathbb{C}1_{\mathcal{H}}$. We also denote the null space and the range of an operator T by $\ker(T)$ and $\operatorname{ran}(T)$, respectively.

An operator T in $\mathcal{L}(\mathcal{H})$ is *compact* if the norm-closure of $\{Tx : \|x\| \leq 1\}$, the image of the unit ball under T, is a compact subset of $\mathcal{H}$, and we shall denote by $\mathbb{K}(\mathcal{H})$ (or simply by $\mathbb{K}$), the two sided norm closed ideal of $\mathcal{L}(\mathcal{H})$ consisting of all compact operators on $\mathcal{H}$ and by π the quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}$.

For an operator T in $\mathcal{L}(\mathcal{H})$, the spectrum of T [resp., left spectrum, right spectrum] is denoted by $\sigma(T)$ [resp., $\sigma_l(T)$, $\sigma_r(T)$]. Moreover, we write $\sigma_p(T)$ for the point spectrum of T (i.e., the set of eigenvalues of T). The essential (i.e., Calkin) spectrum [resp., left essential spectrum, right essential spectrum] of T is the set of all λ in $\mathbb{C}$ such that $\pi(T - \lambda 1_{\mathcal{H}})$ is not invertible [resp., not left invertible, not right invertible] in $\mathcal{L}(\mathcal{H})/\mathbb{K}$. The essential [resp., left essential, right essential] spectrum of T will be denoted by $\sigma_e(T)$ [resp., $\sigma_{le}(T)$, $\sigma_{re}(T)$].

A subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is said to be *invariant* under an operator T in $\mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$, and T is said to have a *nontrivial invariant subspace* (n.i.s.) if there is a subspace $\mathcal{M}$ different from (0) and $\mathcal{H}$ invariant for T. The invariant subspace problem is the question whether every operator in $\mathcal{L}(\mathcal{H})$ has a n.i.s. The lattice of all invariant subspaces of T will be written, as usual, as $\operatorname{Lat}(T)$.

If $\mathcal{C}$ is any subset of $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{C}'$ the *commutant* of $\mathcal{C}$, i.e., $\mathcal{C}' = \{T \in \mathcal{L}(\mathcal{H}) : ST = TS$ for every S in $\mathcal{C}\}$. Recall that a subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is said to be a *nontrivial hyperinvariant subspace* (n.h.s.) for a fixed operator T in $\mathcal{L}(\mathcal{H})$ if $(0) \neq \mathcal{M} \neq \mathcal{H}$ and $S\mathcal{M} \subset \mathcal{M}$ for each S in $\{T\}'$, and that the complete lattice of all hyperinvariant subspaces of T (including (0) and $\mathcal{H}$) is denoted by $\operatorname{Hlat}(T)$. This lattice will frequently be called *the hyperlattice of* T, and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are any two complete lattices, we write $\mathcal{L}_1 \equiv \mathcal{L}_2$ to signify that there is an order preserving isomorphism of one onto the other. The (open) *hyperinvariant subspace problem* (for operators on Hilbert space) is the question whether every operator T in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ has a n.h.s.

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called a *semi-Fredholm* operator on $\mathcal{H}$ (notation: $T \in \mathcal{SF}(\mathcal{H})$) if T has closed range and either $\dim(\ker T) < \aleph_0$ or $\dim(\ker T^*) < \aleph_0$. The map $i : \mathcal{SF}(\mathcal{H}) \to \mathbb{Z} \cup \{+\infty, -\infty\}$, called the *Fredholm index*, is defined by setting $i(T) := \dim(\ker(T)) - \dim(\ker(T^*))$. The set of all operators T in $\mathcal{SF}(\mathcal{H})$ such that $i(T)$ is finite is called the set of *Fredholm* operators on $\mathcal{H}$ and denoted by $\mathcal{F}(\mathcal{H})$. It is well known that i is a norm-continuous function on $\mathcal{SF}(\mathcal{H})$ (where $\mathbb{Z}\cup\{+\infty, -\infty\}$ is given its discrete topology) and thus is constant on open connected subsets of $\mathcal{SF}(\mathcal{H})$.

In what follows we will be concerned with some particular domains (i.e., open, connected sets) $G \subset \mathbb{D}$. Such a domain will be called a *circular subdomain of* $\mathbb{D}$ or, more simply, a *circular domain*, if there exist a finite number of disjoint closed discs $D(\gamma_j, r_j) = \{\zeta \in \mathbb{C} : |\zeta - \gamma_j| \le r_j\} \subset \mathbb{D}$, $j = 1, \cdots, k$, such that $G = \mathbb{D}\backslash \bigcup_{j=1}^{k} D(\gamma_j, r_j)$. (Note that $\mathbb{D}$ is a circular subdomain of itself corresponding to the case $k = 0$ and that all of the annuli $\mathbf{A}_\theta$ defined above are also circular domains.) Recall that the (dual) algebra $H^\infty(G)$ consists of all bounded holomorphic functions on G in the supremum norm. If G is a circular subdomain of $\mathbb{D}$ and $\Lambda \subset G$, then Λ is called a *dominating subset of* G if

$$\sup\{|u(\lambda)| : \lambda \in \Lambda\} = \|u\|_\infty := \sup\{|u(\lambda)| : \lambda \in G\}, \qquad u \in H^\infty(G).$$

For any circular domain G, the algebra of all complex valued rational functions with poles off G^- will be denoted by R^G, and if $T \in \mathcal{L}(\mathcal{H})$ and satisfies $\sigma(T) \subset G^-$, then R^G_T will denote the subalgebra

$$R_T^G = \{r(T) : r \in R^G\}$$

of $\mathcal{L}(\mathcal{H})$. Recall next that a *dual algebra* (or *dual subalgebra of* $\mathcal{L}(\mathcal{H})$) is a weak* closed, unital, subalgebra of $\mathcal{L}(\mathcal{H})$, and if G is any circular domain and T in $\mathcal{L}(\mathcal{H})$ satisfies $\sigma(T) \subset G^-$, then the dual algebra $\mathcal{A}_T^G$ is, by definition, the weak* closure of the algebra R_T^G. An operator T in $\mathcal{L}(\mathcal{H})$ (necessarily satisfying $\sigma(T) \subset G^-$) will be said to belong to the class $\mathbb{A}^G$ if there exists a weak*-continuous, surjective, isometric, algebra isomorphism $\Phi_T : H^\infty(G) \to \mathcal{A}_T^G$, and if $T \in \mathbb{A}^G$ and the dual algebra $\mathcal{A}_T^G$ has property $(\mathbb{A}_{m,n})$ for some cardinal numbers $1 \leq m, n \leq \aleph_0$, as defined in [6], then we say that $T \in \mathbb{A}_{m,n}^G$. If $m = n$, we write simply $\mathbb{A}_n^G$ for $\mathbb{A}_{n,n}^G$.

If $K \geq 1$, a closed set $C \subset \mathbb{C}$ will be called a $K-$*spectral set* for an operator T in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T) \subset C$ if $\|r(T)\| \leq K \sup_{\xi \in C} |r(\xi)|$ for every $r \in R^C$ (the algebra of rational functions with poles off C), and a $1-$spectral set for T is called, more simply, a *spectral set* for T. For $n \in \mathbb{N}$, we denote by $M_n(R^C)$ the algebra of all $n \times n$ matrices with entries from R^C with the norm $\|(r_{ij})\|_\infty = \sup_{\xi \in C} \|(r_{ij}(\xi)\|$ where this last norm is the canonical (operator) norm on $M_n = \mathbb{C}^{n,n}$. If C is a $K-$spectral set $(K \geq 1)$ for some $T \in \mathcal{L}(\mathcal{H})$ (with $\sigma(T) \subset C$) and the inequality $\|(r_{ij}(T))\| \leq K \|(r_{ij})\|_\infty$ persists for every $r \in M_n(R^C)$ and every $n \in \mathbb{N}$, then C is called a *complete* $K-$spectral set for T (here the norm $\|(r_{ij}(T))\|$ is simply the operator norm on $\mathcal{L}(\mathcal{H}^{(n)})$.) If $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) \subset C$, a closed subset of $\mathbb{C}$, and there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N \in \mathcal{L}(\mathcal{K})$ such that $\sigma(N) \subset \partial C$ and such that

$$r(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}, \qquad r \in R^C,$$

then N is called a $\partial C-$*dilation* of T.

A contraction T in $\mathcal{L}(\mathcal{H})$ is called *completely nonunitary* (c.n.u.) if it has no nontrivial reducing subspace on which it acts as a unitary operator. Recall that a c.n.u. contraction T in $\mathcal{L}(\mathcal{H})$ is called a (BCP)-operator if $\mathbb{D}\cap\sigma_e(T)$ is a dominating set for $\mathbb{D}$.

A sequence of operators $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}(\mathcal{H})$ is said to converge to T_0 in the strong operator topology (SOT) if $\|T_n x - T_0 x\| \to 0$ for every $x\in\mathcal{H}$, and we will write $T_n \xrightarrow{SOT} T_0$ to indicate this convergence and $T_n \xrightarrow{*-SOT} T_0$ to mean that $T_n \xrightarrow{SOT} T_0$ and $T_n^* \xrightarrow{SOT} T_0^*$. The class $C_{00}(\mathcal{H})$ consists of the set of all c.n.u. contractions T in $\mathcal{L}(\mathcal{H})$ such that both sequences $\{T^n\}_{n\in\mathbb{N}}$ and $\{(T^*)^n\}_{n\in\mathbb{N}}$ converge to zero in the SOT.

The normed ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$ will be written as $\mathcal{C}_1(\mathcal{H})$ and the corresponding trace-norm denoted by $\|\cdot\|_1$. The duality between $\mathcal{L}(\mathcal{H})$ and $\mathcal{C}_1(\mathcal{H})$ is implemented by the bilinear functional

$$\langle T, L\rangle = \operatorname{trace}(TL) = \sum_{i=1}^{\infty}(TLe_i, e_i), \qquad T\in\mathcal{L}(\mathcal{H}), \quad L\in\mathcal{C}_1(\mathcal{H}),$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$.

If x and y are vectors in $\mathcal{H}$, then the rank-one operator $x\otimes y$, defined as usual by

$$(x\otimes y)(u) = (u, y)\,x, \qquad u\in\mathcal{H}$$

belongs to $\mathcal{C}_1(\mathcal{H})$ and satisfies

$$\operatorname{trace}(x\otimes y) = (x, y)$$

and

$$\|x\otimes y\|_1 = \|x\otimes y\| = \|x\|\,\|y\|\,.$$

Moreover, if $L\in\mathcal{C}_1(\mathcal{H})$, then $L = \sum\limits_{i=1}^{\infty} x_i\otimes y_i$ for some summable sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ (with convergence in the norm $\|\cdot\|_1$).

For any ordinal number n satisfying $1\le n\le\omega$ (the smallest infinite ordinal), we denote by $\mathcal{H}^{(n)}$ the direct sum of n copies of $\mathcal{H}$ (i.e., $\mathcal{H}^{(n)} = \oplus_{0\le k<n}\mathcal{H}_k$ with $\mathcal{H}_k = \mathcal{H}$

for every k), and $T^{(n)}$ denotes the direct sum (ampliation) of n copies of T acting on $\mathcal{H}^{(n)}$ in the usual fashion.

As is well known, operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are said to be *similar* (notation: $T_1 \approx T_2$) if there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XT_1 = T_2X$. Similar operators have isomorphic lattices of invariant and hyperinvariant subspaces. Recall that operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are said to be *quasisimilar* (notation: $T_1 \sim T_2$) if there exist quasiaffinities X and Y in $\mathcal{L}(\mathcal{H})$ (i.e., $\ker X = \ker X^* = \ker Y = \ker Y^* = (0)$) such that $T_1X = XT_2$ and $YT_1 = T_2Y$. (Observe also that in this case we have $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$.) In [28], Hoover proved that quasisimilarity preserves the existence of nontrivial hyperinvariant subspaces and in [25], Herrero has shown that quasisimilarity does not preserve the full hyperlattice.

Recall next from [21] that an operator T in $\mathcal{L}(\mathcal{H})$ is *quasidiagonal* ($T \in (QD)(\mathcal{H})$) if there exists an increasing sequence $\{P_n\}_{n\in\mathbb{N}}$ of finite rank projections such that $P_n \xrightarrow{SOT} 1_{\mathcal{H}}$ and $\|TP_n - P_nT\| \to 0$ and T is *block diagonal* (notation: $T \in (BD)(\mathcal{H})$) if T is unitarily equivalent to a countably infinite (orthogonal) direct sum of operators each of which acts on a (nonzero) finite dimensional space. If, in addition, each of the direct summands T_n satisfies $\|T_n\| < 1$, then T will be called a *strictly norm decreasing* block diagonal operator (since $\|Tx\| < \|x\|$ for every nonzero x in $\mathcal{H}$). An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* [21] (notation: $T \in (QT)(\mathcal{H})$) if T can be written as $T = T_t + K$ where the matrix $(\tau_{ij})_{i,j\in\mathbb{N}}$ for T_t with respect to some ordered orthonormal basis for $\mathcal{H}$ is in the upper triangular form (i.e., $\tau_{ij} = 0$ whenever $i > j$). Moreover, if both $T \in (QT)$ and $T^* \in (QT)$, then T is called *biquasitriangular* (notation: $T \in (BQT)(\mathcal{H})$).

CHAPTER II

SOME RESULTS ABOUT THE DUAL ALGEBRAS $\mathcal{A}_T^G$

In this chapter, we set forth some results (mostly from [16], [10], [15], and [9]) that will be useful toward the end of Chapter III.

Theorem 2.1 [10]. *Suppose G is a circular domain in $\mathbb{C}$, $T \in \mathcal{L}(\mathcal{H})$, $\partial G \subset \sigma(T)$, and G^- is a spectral set for T. (or, equivalently, $T \in \mathbb{A}^G$). Then the algebra $\mathcal{A}_T^G$ has a nontrivial invariant subspace.*

We now turn to some results from [15]. Let $G = \mathbb{D} \backslash \bigcup_{j=1}^{k} D(\gamma_j, r_j)$, and suppose G^- ($\supset \sigma(T)$) is a spectral set for T. Then the operators T_j defined by

$$T_j = r_j(T - \gamma_j)^{-1}, \qquad j = 1, \cdots, k,$$

are all contractions, and we follow [15] in saying that $T \in C_{0\cdot}^G$ if $T^n \xrightarrow{SOT} 0$ and $(T_j)^n \xrightarrow{SOT} 0$ for $j = 1, \cdots, k$. We also set $C_{\cdot 0}^G = \{T^* : T \in C_{0\cdot}^G\}$ and $C_{00}^G = C_{0\cdot}^G \cap C_{\cdot 0}^G$. One of the main results from [15] is as follows.

Theorem 2.2. *The following inclusions are valid, where $G \subset \mathbb{D}$ is an arbitrary circular domain*

(**a**) $\mathbb{A}^G \cap C_{0\cdot}^G \subset \mathbb{A}_{\aleph_0,1}^G$,

(**b**) $\mathbb{A}^G \cap C_{\cdot 0}^G \subset \mathbb{A}_{1,\aleph_0}^G$, *and*

(**c**) $\mathbb{A}^G \cap C_{00}^G \subset \mathbb{A}_{\aleph_0}^G$.

Here is another nice result from [15].

Theorem 2.3. *Suppose G is some circular domain and $T \in \mathbb{A}_{1,\aleph_0}^G$. Then the dual algebra $\mathcal{A}_T^G$ is reflexive.*

A final result from [15] that will be useful later is this.

Theorem 2.4. *Suppose G is some circular domain and $T \in \mathbb{A}^G_{1,\aleph_0}$. Then the set of vectors x in $\mathcal{H}$ generating a $G-$analytic invariant subspace for $\mathcal{A}^G_T$ is dense in $\mathcal{H}$. (To say that $\mathcal{M}$ is a $G-$analytically invariant subspace for $\mathcal{A}^G_T$ means that $\mathcal{M} \in \mathrm{Lat}(\mathcal{A}^G_T)$ and that there exists a nontrivial conjugate analytic map $e : \lambda \to e_\lambda$ from G into $\mathcal{M}$ such that $(T|_{\mathcal{M}} - \lambda 1_{\mathcal{H}})^* e_\lambda \equiv 0$ on G.)*

We turn now to a listing of some results from [4] and [9].

Theorem 2.5. *Let G be a circular domain, and denote by $A^2(G)$ the Bergman space associated with G. (In other words, $A^2(G)$ consists of all functions u holomorphic in G such that $u \in L^2(G,\mu)$, where μ is a planar Lebesgue measure on G.) Moreover, let M_ξ be multiplication by the position function on $A^2(G)$, i.e.,*

$$(M_\xi(u))(\xi) = \xi u(\xi), \qquad A^2(G), \xi \in G.$$

Then $M_\xi \in \mathbb{A}^G_{\aleph_0}$.

Theorem 2.6 [9]. *Let G be a circular domain, let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be a dominating subset of G, and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Then the (diagonal) normal operator N_Λ in $\mathcal{L}(\mathcal{H})$ defined by $N_\Lambda e_n = \lambda_n e_n$, $n \in \mathbb{N}$, belongs to $\mathbb{A}^G_{\aleph_0}$.*

Perhaps the best theorem in [9] is this next one.

Theorem 2.7. *Let G be a circular subdomain of $\mathbb{D}$, and let $\{\gamma_n\}_{n\in\mathbb{N}}$ be any sequence of (not necessarily distinct) points in G. Moreover, let $T \in \mathbb{A}^G_{\aleph_0}(\mathcal{H})$ (so, in particular, $\mathcal{A}^G_T$ is isometrically isomorphic to $H^\infty(G)$ via a weak* homeomorphism). Then there exists a decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{P}$ and an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{N}$*

such that, relative to this decomposition of $\mathcal{H}$, we have, matricially, that

$$u(T) = \begin{bmatrix} A_{11}^{(u)} & A_{12}^{(u)} & A_{13}^{(u)} \\ 0 & u(D) & A_{23}^{(u)} \\ 0 & 0 & A_{33}^{(u)} \end{bmatrix}, \qquad u \in H^\infty(G),$$

where D is the (diagonal) normal operator defined by $Df_n = \gamma_n f_n$, $n \in \mathbb{N}$.

CHAPTER III

A NEW STRUCTURE THEOREM ABOUT (BCP)-OPERATORS

The class of (BCP)-operators, introduced in [14], played an important role in the highly successful theory of dual algebras of operators, and is a subset of the larger class $\mathbb{A}_{\aleph_0}$ (see, e.g., [7] for more information about the theory of dual algebras). It is well known that operators in $\mathbb{A}_{\aleph_0}$ have several good properties. For instance, every direct sum of strict contractions can be realized, up to unitary equivalence, as a compression to some semi-invariant subspace of an arbitrary operator in $\mathbb{A}_{\aleph_0}$ [6]. Moreover, the lattice $\operatorname{Lat}(T)$ of invariant subspaces of any operator T in $\mathbb{A}_{\aleph_0}$ is known to be so large that it contains a sublattice isomorphic to the lattice of all subspaces of $\mathcal{H}$ [6, Theorem 4.8]. Thus, in what follows we will proceed to study the structure theory of (BCP)-operators from various viewpoints. We begin by collecting, from the vast theory of dual algebras, some known results.

Thus, in what follows we will proceed to study the structure theory of (BCP)-operators from various viewpoints. We begin by collecting, from the vast theory of dual algebras, some known results.

Theorem 3.1 [7]. *Let T be any* (BCP)-*operator in $\mathcal{L}(\mathcal{H})$ and let X be any operator in $\mathcal{L}(\mathcal{H})$ such that X is a* (*finite or infinite*) *direct sum of operators each having norm less than one. Then there exists a decomposition of $\mathcal{H}$ as $\mathcal{H} = \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}$ such that the matrix for T relative to this decomposition has the form*

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & X & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix}.$$

Theorem 3.2 [3]. *Let T be an arbitrary* (BCP)-*operator in* $\mathcal{L}(\mathcal{H})$. *Then there exists a family* $\{\mathcal{K}_n\}_{n\in\mathbb{N}}$ *of proper nonzero cyclic invariant subspaces for T such that for all* $m \in \mathbb{N}$, $\mathcal{K}_m \cap \bigvee_{n\in\mathbb{N}\setminus\{m\}} \mathcal{K}_n = (0)$.

Theorem 3.3 [5]. *Every* (BCP)-*operator in* $\mathcal{L}(\mathcal{H})$ *is reflexive.*

Next, we include some definitions and results that are pertinent to our study. A quasiaffinity Q will be said to have the *hereditary property with respect to an operator* $T \in \mathcal{L}(\mathcal{H})$ if $Q \in \{T\}'$ and $(Q\mathcal{M})^- = \mathcal{M}$ for every $\mathcal{M} \in \mathrm{Hlat}(T)$, and if $T_1 \sim T_2$ and there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is *hyperquasisimilar* to T_2 (notation: $T_1 \overset{h}{\sim} T_2$). The important result that makes the relation $\overset{h}{\sim}$ worth studying was proved in [17], and says that if $T_1 \overset{h}{\sim} T_2$, then $\mathrm{Hlat}(T_1) \equiv \mathrm{Hlat}(T_2)$. The following lemma is important in the proof of a needed corollary. Recall that the *numerical range* of an operator $T \in \mathcal{L}(\mathcal{H})$ is defined to be the subset of $\mathbb{C}$ given by $W(T) := \{\langle Tx, x\rangle\,,\ x \in \mathcal{H},\ \|x\| = 1\}$.

Lemma 3.4 [17]. *Suppose* $Q \in \mathcal{L}(\mathcal{H})$ *is a quasiaffinity and* $0 \notin W(Q)$. *Then Q has the hereditary property with respect to every T in* $\mathcal{L}(\mathcal{H})$ *such that* $Q \in \{T\}'$.

Proof. For $Q \in \{T\}'$ and $\mathcal{M} \in \mathrm{Hlat}(T)$ such that $(Q\mathcal{M})^- \neq \mathcal{M}$, there exists a unit vector x in $\mathcal{M} \ominus (Q\mathcal{M})^-$ and $\langle Qx, x\rangle = 0$. □

Corollary 3.5 [17]. *Suppose* $Q \in \mathcal{L}(\mathcal{H})$ *is a quasiaffinity and there exists* $0 \leq \theta < 2\pi$ *such that* $R = \mathrm{Re}(e^{i\theta}Q)$ *is positive definite (i.e.,* $\langle Rx, x\rangle > 0$ *for every* $x \neq 0$ *in* $\mathcal{H}$*). Then Q has the hereditary property with respect to every T in* $\mathcal{L}(\mathcal{H})$ *for which* $Q \in \{T\}'$.

Proof. If $\langle Qx, x\rangle = 0$, then $\langle Rx, x\rangle = \left\langle \frac{1}{2}(e^{i\theta}Q + e^{-i\theta}Q^*)x, x\right\rangle = 0$ so $x = 0$. □

The main result from [17] that we shall need, in addition to those already mentioned, is the following.

Theorem 3.6 [17]. *Suppose* $\{S_n\}_{n\in\mathbb{N}}$ *and* $\{T_n\}_{n\in\mathbb{N}}$ *are bounded sequences of operators in* $\mathcal{L}(\mathcal{H})$ *with* $\widehat{S} := \oplus_{n\in\mathbb{N}} S_n$ *and* $\widehat{T} := \oplus_{n\in\mathbb{N}} T_n$*. Suppose, moreover, that* $\{X_n\}_{n\in\mathbb{N}}$ *is a sequence of invertible operators such that*

$$X_n^{-1} S_n X_n = T_n, \qquad n \in \mathbb{N}.$$

Then $\widehat{S} \overset{h}{\sim} \widehat{T}$ *and consequently* $\mathrm{Hlat}(\widehat{S}) \equiv \mathrm{Hlat}(\widehat{T})$.

Proof. As is well known, $\widehat{X} := \oplus_{n\in\mathbb{N}} X_n / \|X_n\|$ and $\widehat{Y} := \oplus_{n\in\mathbb{N}} (X_n)^{-1} / \|(X_n)^{-1}\|$ belong to $\mathcal{L}(\mathcal{H}^{(\omega)})$ and satisfy $\widehat{S}\widehat{X} = \widehat{X}\widehat{T}$, $\widehat{Y}\widehat{S} = \widehat{T}\widehat{Y}$. Moreover

$$\widehat{X}\widehat{Y} = \oplus_{n\in\mathbb{N}} 1/(\|X_n\| \left\|(X_n)^{-1}\right\|) = \widehat{Y}\widehat{X}$$

is a positive definite operator, and the fact that $\widehat{X}\widehat{Y}$ and $\widehat{Y}\widehat{X}$ have the appropriate hereditary properties is immediate from Corollary 3.5. □

Recall also that it is known from [21] that $(QD)(\mathcal{H}) = (BD)(\mathcal{H}) + \mathbb{K}(\mathcal{H})$ and that if $T \in (QD)(\mathcal{H})$ and $\varepsilon > 0$ are given, then there exist $B_\varepsilon \in (BD)(\mathcal{H})$ and $K_\varepsilon \in \mathbb{K}(\mathcal{H})$ such that $T = B_\varepsilon + K_\varepsilon$ and $||K_\varepsilon|| < \varepsilon$, which utilizes the concept of block diagonal operators.

This next class of operators is somewhat less interesting from the stand point of n.h.s., since all operators in this class have a good supply of n.h.s. (but, in this connection see [8]).

Definition 3.7. An operator T in $\mathcal{L}(\mathcal{H})$ such that there exists a nonzero polynomial p satisfying $p(T) = 0$ is called an *algebraic operator*, and hereafter the set of all algebraic operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $(\mathcal{A})$ (or $(\mathcal{A})(\mathcal{H})$ if necessary to avoid confusion).

The following result is a combination of a well known theorem of Halmos [22] and some easy matricial calculations.

Theorem 3.8. *Suppose* $T \in (\mathcal{A})$ *and* p *is a monic polynomial of minimal degree such that* $p(T) = 0$. *If* $p(z)$ *has the factorization*

$$p(z) = (z - \lambda_1)^{q_1} \dots (z - \lambda_k)^{q_k}$$

where $\lambda_1, \dots, \lambda_k$ *are the distinct zeros of* p, *then* $\sigma(T) = \{\lambda_1, \dots, \lambda_k\}$ *and* T *is similar to an operator* T_1 *of the form*

$$T_1 = (\lambda_1 + N_1) \oplus \dots \oplus (\lambda_k + N_k)$$

where $N_1, \dots, N_k$ *are nilpotent operators.*

As an easy consequence of this result one gets the following well known fact.

Corollary 3.9. *With the notation as above*, $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(N_1) \oplus \dots \oplus \mathrm{Hlat}(N_k)$.

Thus the study of the hyperlattice of an arbitrary algebraic operator reduces quickly to the study of the hyperlattices of a finite number of nilpotent operators. This is not a trivial subject, and it was recently shown in [8], for example, that there exists a nilpotent operator N in $\mathcal{L}(\mathcal{H})$ with $N^3 = 0$ whose hyperlattice is infinite (unlike the known situation for a nilpotent operator acting on a finite dimensional Hilbert space, where the hyperlattice is necessarily finite). But, on the other hand, a nonzero nilpotent operator N in $\mathcal{L}(\mathcal{H})$ with index of nilpotency k does have some obvious hyperinvariant subspaces, namely

$$\ker N, \ker N^2, \dots, \ker N^{k-1}, \tag{1}$$

and also

$$(\operatorname{ran} N)^-, (\operatorname{ran} N^2)^-, \ldots, (\operatorname{ran} N^{k-1})^- \tag{2}$$

together with the lattice generated by the subspaces in (1) and (2) (for example, $\ker N^2 \cap (\operatorname{ran} N)^-$, etc.)

Thus one may say that the hyperlattice structure of an algebraic operator in $\mathcal{L}(\mathcal{H})$ is moderately, if not completely, well understood, and consequently, in the remainder of this dissertation, no further attention will be given to the class $(\mathcal{A})(\mathcal{H})$.

In the remainder of this chapter (written as part of the article [24]), we will establish a new structure theorem for (BCP)-operators which will play a role in later chapters, namely, the following.

Theorem 3.10. *Suppose $T \in (\mathrm{BCP})(\mathcal{H})$ and B is an arbitrary strictly norm decreasing block diagonal operator. Then for every $\varepsilon > 0$ there exist c.n.u. contractions $T_0 = T_0(\varepsilon)$ and $K_i = K_i(\varepsilon)$, $i = 1, 2$, satisfying:*
(**a**) *$K_i \in \mathcal{C}_1(\mathcal{H})$ and $\|K_i\|_1 < \varepsilon$ for $i = 1, 2$,*
(**b**) *$T^{(\omega)} \overset{h}{\sim} \widehat{T}$, where $\widehat{T} \in (\mathrm{BCP})(\mathcal{H}^{(2)})$ is the 2×2 operator matrix*

$$\begin{bmatrix} T_0 & K_1 \\ K_2 & B \end{bmatrix} \tag{3}$$

acting on $\mathcal{H}^{(2)}$ in the usual fashion,
(**c**) *$\sigma_{le}(\widehat{T}) \supset \sigma_{le}(T)$, $\sigma_{re}(\widehat{T}) \supset \sigma_{re}(T)$, and $\sigma(\widehat{T}) \supset \sigma(T)$, and*
(**d**) *if $T \in C_{00}(\mathcal{H})$, then also $\widehat{T} \in C_{00}(\mathcal{H}^{(2)})$.*

The proof of Theorem 3.10 will be made easier by first establishing some needed lemmas.

Lemma 3.11. *Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are complex Hilbert spaces and $\widehat{T} \in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{K}_2)$*

is a c.n.u. contraction defined matricially by

$$\widehat{T} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

Then for every $0 < s < 1$, *the operator* $\widehat{T}_s$ *defined matricially by*

$$\widehat{T}_s = \begin{bmatrix} A & sB \\ 0 & C \end{bmatrix}$$

is also a c.n.u. contraction.

Proof. Let $x \oplus y$ be an arbitrary vector in $\mathcal{K}_1 \oplus \mathcal{K}_2$. It is easy to see that the inequality $||\widehat{T}(x \oplus y)|| \leq ||x \oplus y||$ is equivalent to the inequality

$$\langle (1 - A^*A)x, x \rangle + \langle (1 - C^*C)y, y \rangle \geq \|By\|^2 + 2\,\mathrm{Re}\,\langle B^*Ax, y \rangle\,. \tag{4}$$

Now fix an arbitrary s such that $0 < s < 1$ and choose $\theta = \theta(x, y) \in [0, 2\pi)$ satisfying $\mathrm{Re}\, e^{-i\theta} \langle B^*Ax, y \rangle = |\langle B^*Ax, y \rangle|$. Then inequality (4) implies that

$$\begin{aligned} \langle (1 - A^*A)x, x \rangle + \langle (1 - C^*C)y, y \rangle &\geq \|By\|^2 + 2\,|\langle B^*Ax, y \rangle| \\ &\geq s^2\,\|By\|^2 + 2s\,|\langle B^*Ax, y \rangle| \qquad (5) \\ &\geq s^2\,\|By\|^2 + 2s\,\mathrm{Re}\,\langle B^*Ax, y \rangle\,, \end{aligned}$$

where $x \oplus y \in \mathcal{K}_1 \oplus \mathcal{K}_2$, which proves that $\widehat{T}_s$ is a contraction.

Next, suppose that $\mathcal{M} \subset \mathcal{K}_1 \oplus \mathcal{K}_2$ is an invariant (equivalently, reducing) subspace for $\widehat{T}_s$ such that $\widehat{T}_s|_{\mathcal{M}}$ is a unitary operator. Let $P_{\mathcal{K}_2} \in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ be the projection with range the subspace $(0) \oplus \mathcal{K}_2$. If there exists $x_0 \oplus y_0 \in \mathcal{M}$ with $By_0 \neq 0$, then the inequality (5) becomes strict, and thus

$$||\widehat{T}_s(x_0 \oplus y_0)|| < ||\widehat{T}(x_0 \oplus y_0)|| \leq ||x_0 \oplus y_0||,$$

which contradicts the fact that $\widehat{T}_s|_{\mathcal{M}}$ is unitary. Thus $P_{\mathcal{K}_2}\mathcal{M} \subset (0) \oplus \ker B$ which implies that $\mathcal{M} \in \mathrm{Lat}(A \oplus C)$ and that $(A \oplus C)|_{\mathcal{M}} = \widehat{T}_s|_{\mathcal{M}}$ is a unitary operator. Since $\widehat{T}$ is c.n.u., so are A, C, and $A \oplus C$, and therefore $\mathcal{M} = (0)$ which proves that $\widehat{T}_s$ is completely nonunitary as desired. □

Lemma 3.12. *Suppose that $\widehat{T} \in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3)$ is given matricially as*

$$\widehat{T} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

and $\widehat{T}$ is a c.n.u. contraction. Then for every $0 < s < 1$, the operator

$$\widehat{T}_s = \begin{bmatrix} A_{11} & sA_{12} & s^2 A_{13} \\ 0 & A_{22} & sA_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

is also a c.n.u. contraction.

Proof. Fix an arbitrary s such that $0 < s < 1$ and apply Lemma 3.11 twice; first to give that

$$\left[\begin{array}{c|cc} A_{11} & sA_{12} & sA_{13} \\ \hline 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{array}\right]$$

is a c.n.u. contraction, and then to give that

$$\left[\begin{array}{cc|c} A_{11} & sA_{12} & s^2 A_{13} \\ 0 & A_{22} & sA_{23} \\ \hline 0 & 0 & A_{33} \end{array}\right]$$

is a c.n.u. contraction. □

Proof of Theorem 3.10. Let B be any fixed strictly norm decreasing operator in $(BD)(\mathcal{H})$. Then, by definition, there exist a sequence of finite dimensional Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ (with $\dim \mathcal{H}_n := k_n \in \mathbb{N}$), a sequence $\{B_n \in \mathcal{L}(\mathcal{H}_n)\}_{n\in\mathbb{N}}$, and a Hilbert space isomorphism φ of $\mathcal{H}$ onto $\oplus_{n\in\mathbb{N}}\mathcal{H}_n$, such that $\varphi B \varphi^{-1} = \oplus_{n\in\mathbb{N}} B_n$ and $\|B_n\| < 1$ for every n. Fix an arbitrary $T \in (\mathrm{BCP})(\mathcal{H})$. One knows from [6, Theorem 4.8] that for each $n \in \mathbb{N}$ we may choose a Hilbert space isomorphism φ_n mapping $\mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}_n \oplus \mathcal{H}$ such that

$$T_n^{'} = \varphi_n T \varphi_n^{-1} = \begin{bmatrix} T_{11}^{(n)} & T_{12}^{(n)} & T_{13}^{(n)} \\ 0 & B_n & T_{23}^{(n)} \\ 0 & 0 & T_{33}^{(n)} \end{bmatrix}, \qquad n \in \mathbb{N}. \tag{6}$$

Notice that the ranks of $T_{12}^{(n)}$ and $T_{23}^{(n)}$ are bounded above by k_n. Now let $\varepsilon > 0$ be arbitrarily given, and let $\{r_n\}_{n\in\mathbb{N}}$ be a monotone decreasing sequence of positive real numbers such that $\sum r_n < \varepsilon$. Now for each $n \in \mathbb{N}$, define $s_n = r_n/k_n$ and let $S_n \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}_n \oplus \mathcal{H})$ be defined by $S_n = s_n 1_{\mathcal{H}} \oplus 1_{\mathcal{H}_n} \oplus s_n^{-1} 1_{\mathcal{H}}$. It follows that S_n is an invertible operator and a short calculation gives

$$T_n^{''} = S_n T_n^{'} S_n^{-1} = \begin{bmatrix} T_{11}^{(n)} & s_n T_{12}^{(n)} & s_n^2 T_{13}^{(n)} \\ 0 & B_n & s_n T_{23}^{(n)} \\ 0 & 0 & T_{33}^{(n)} \end{bmatrix}, \qquad n \in \mathbb{N}, \tag{7}$$

and by Lemma 3.12, $T_n^{''}$ is a c.n.u. contraction. Moreover, for each $n \in \mathbb{N}$, $T_n^{''}$ is obviously unitarily equivalent to the c.n.u. contraction

$$T_n^{'''} = \begin{bmatrix} T_{11}^{(n)} & s_n^2 T_{13}^{(n)} & s_n T_{12}^{(n)} \\ 0 & T_{33}^{(n)} & 0 \\ 0 & s_n T_{23}^{(n)} & B_n \end{bmatrix}. \tag{8}$$

Since, by construction, T is similar to each $T_n^{'''}$, one knows that for each $n \in \mathbb{N}$,

$\sigma(T_n^{'''})$, $\sigma_{le}(T_n^{'''})$ and $\sigma_{re}(T_n^{'''})$ coincide with the corresponding parts of the spectrum of T, and if $T \in C_{00}$, then it follows that $T_n^{'''} \in C_{00}$ also. Moreover, one knows (cf., e.g., [34, Prop. 9.7]) that $T^{(\aleph_0)} \sim \oplus_{n\in\mathbb{N}} T_n^{'''} \in \mathcal{L}(\oplus_{n\in\mathbb{N}}(\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}_n))$. Furthermore, by reordering this direct sum of Hilbert spaces as

$$\mathcal{M} = (\oplus_{n\in\mathbb{N}}(\mathcal{H}\oplus\mathcal{H})) \oplus (\oplus_{n\in\mathbb{N}}\mathcal{H}_n),$$

we see that $\underset{n\in\mathbb{N}}{\oplus} T_n^{'''}$ is unitarily equivalent to the operator $M \in \mathcal{L}(\mathcal{M})$ that is given matricially as

$$M = \begin{bmatrix} \oplus_{n\in\mathbb{N}} R_n & \widehat{K}_1 \\ \widehat{K}_2 & \oplus_{n\in\mathbb{N}} B_n \end{bmatrix}, \tag{9}$$

where

(**A**) $R_n \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ is also defined matricially as

$$R_n = \begin{bmatrix} T_{11}^{(n)} & s_n^2 T_{13}^{(n)} \\ 0 & T_{33}^{(n)} \end{bmatrix}, \tag{10}$$

(**B**) $\widehat{K}_1 : \oplus_{n\in\mathbb{N}}\mathcal{H}_n \to (\mathcal{H}\oplus\mathcal{H})^{(\aleph_0)}$ is defined at an arbitrary vector $\oplus_{n\in\mathbb{N}} x_n$ in $\oplus_{n\in\mathbb{N}}\mathcal{H}_n$ by $\widehat{K}_1(\oplus x_n) = \oplus(s_n T_{12}^{(n)} x_n \oplus 0_{\mathcal{H}})$, and

(**C**) $\widehat{K}_2 : (\mathcal{H}\oplus\mathcal{H})^{(\aleph_0)} \to \oplus_{n\in\mathbb{N}}\mathcal{H}_n$ is defined at any $\oplus_{n\in\mathbb{N}}(v_n \oplus y_n)$ in $(\mathcal{H}\oplus\mathcal{H})^{(\aleph_0)}$ by $\widehat{K}_2(\oplus_{n\in\mathbb{N}}(v_n\oplus y_n)) = \oplus_{n\in\mathbb{N}} s_n T_{23}^{(n)} y_n$.

Thus M, being unitarily equivalent to $\oplus_{n\in\mathbb{N}} T_n^{'''}$, is a c.n.u. contraction satisfying $M \overset{h}{\sim} T$, $\sigma_{le}(M) \supset \sigma_{le}(T_n^{'''}) = \sigma_{le}(T)$, $\sigma_{re}(M) \supset \sigma_{re}(T)$, and $\sigma(M) \supset \sigma(T)$. Moreover, if $T \in C_{00}$ then $M \in C_{00}$ also.

Next, define $K_1 = \psi^{-1}\widehat{K}_1\varphi$, $K_2 = \varphi^{-1}\widehat{K}_2\psi$, $T_0 = \psi^{-1}(\oplus_{n\in\mathbb{N}} R_n)\psi$, and $\widehat{T} = (\psi\oplus\varphi)^{-1} M (\psi\oplus\varphi)$. where ψ is some Hilbert space isomorphism of $\mathcal{H}$ onto $\mathcal{H}^{(\aleph_0)}$.

It is obvious that $\widehat{T}$ is given matricially by (3), and from above we know that $T \overset{h}{\sim} \widehat{T}$ and that $\widehat{T}$ has properties (**b**), (**c**), and (**d**) in the statement of the theorem.

Thus the proof can be completed by showing that $K_1,\ K_2 \in \mathcal{C}_1(\mathcal{H})$ and satisfy

$$\|K_1\|_1\,,\ \|K_2\|_1 < \varepsilon. \tag{11}$$

Obviously, (11) is equivalent to $\left\|(K_i^*K_i)^{1/2}\right\|_1 < \varepsilon$, $i = 1, 2$, and since φ and ψ are Hilbert space isomorphisms, it suffices to show that for $i = 1, 2$, $||(\widehat{K}_i^*\widehat{K}_i)^{1/2}||_1 < \varepsilon$. Moreover, the above definitions together with a simple calculation show that

$$\begin{aligned}(\widehat{K}_1^*\widehat{K}_1)^{1/2} &= \oplus_{n\in\mathbb{N}} s_n[(T_{12}^{(n)})^*T_{12}^{(n)}]^{1/2},\\ (\widehat{K}_2^*\widehat{K}_2)^{1/2} &= 0_{\mathcal{H}} \oplus \left(\oplus_{n\in\mathbb{N}} s_n[(T_{23}^{(n)})^*T_{23}^{(n)}]^{1/2}\right).\end{aligned}$$

Thus

$$\begin{aligned}||(\widehat{K}_1^*\widehat{K}_1)^{1/2}||_1 &= \sum\nolimits_{n\in\mathbb{N}} s_n||((T_{12}^{(n)})^*T_{12}^{(n)})^{1/2}||_1\\ &= \sum\nolimits_{n\in\mathbb{N}} s_n \operatorname{tr}((T_{12}^{(n)})^*T_{12}^{(n)})^{1/2}\\ &\le \sum\nolimits_{n\in\mathbb{N}} s_n k_n < \varepsilon,\end{aligned}$$

since for each $n \in \mathbb{N}$, $((T_{12}^{(n)})^*T_{12}^{(n)})^{1/2} \in \mathcal{L}(\mathcal{H}_n)$ and is a contraction, and therefore must have trace at most $\dim \mathcal{H}_n = k_n$. A similar calculation to the one above shows that

$$||(\widehat{K}_2^*\widehat{K}_2)^{1/2}||_1 < \varepsilon,$$

which completes the proof. □

The next important result that we shall need is Voiculescu's representation theorem from [35]. For $T \in \mathcal{L}(\mathcal{H})$ we will write $C^*(T)$ and $C^*(\pi(T))$ for the unital C^*-algebras generated by T (and $1_{\mathcal{H}}$) and $\pi(T)$ (and $1_{\mathcal{L}(\mathcal{H})/\mathbb{K}}$), respectively.

Theorem 3.13 (Voiculescu). *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\tilde{\rho}$ be a unital C^*-algebra homomorphism from $C^*(\pi(T))$ into $\mathcal{L}(\mathcal{H})$. Then there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of* *unitary operators from* $\mathcal{H}$ *to* $\mathcal{H} \oplus \mathcal{H}$ *such that*

(**I**) $U_n A U_n^* - A \oplus \tilde{\rho}(\pi(A)) \in \mathbb{K}(\mathcal{H} \oplus \mathcal{H}), \qquad A \in C^*(T),\ n \in \mathbb{N}$ *and*

(**II**) $\|U_n A U_n^* - A \oplus \tilde{\rho}(\pi(A))\| \to 0, \qquad A \in C^*(T).$

The next preparatory lemmas will be needed to enable us to apply Theorem 3.13 to obtain the desired conclusions. The following lemma is elementary and thus needs no proof.

Lemma 3.14. *Let* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *be a unital* C^**-algebra, and let* $P \in \mathcal{L}(\mathcal{H})$ *be a projection in* $\mathcal{A}'$*. Then the map* φ *defined by* $\varphi(A) = PAP|_{\operatorname{ran} P}$ *is a unital* C^**-algebra homomorphism of* $\mathcal{A}$ into $P\mathcal{A}P|_{\operatorname{ran} P}$ (*with* $\varphi(1_{\mathcal{H}}) = 1_{\operatorname{ran} P}$).

Now we give a complete proof of the following lemma which is similar to lemmas used without proof in [20] and [26].

Lemma 3.15. *Let* $T = \oplus_{n\in\mathbb{N}} T_n \in \mathcal{L}(\mathcal{H}^{(\aleph_0)})$*, and suppose that* $\{T_n\} \subset \mathcal{L}(\mathcal{H})$ *is a sequence of contractions that converges* $*-SOT$ *to a nonzero contraction* S*. Then,*

(**a**) *there exists a unital* C^**-algebra homomorphism* ρ *of* $C^*(T)$ *into* $C^*(S)$ (*i.e.,* $\rho(1_{\mathcal{H}^{(\aleph_0)}}) = 1_{\mathcal{H}}$) *such that* $\rho(T) = S$*, and*

(**b**) $C^*(T) \cap \mathbb{K}(\mathcal{H}^{(\aleph_0)}) \subset \ker \rho$*. Hence*

(**c**) *there exists a unital* C^**-algebra homomorphism* $\tilde{\rho}$ *of* $C^*(\pi(T))$ *such that* $\rho = \tilde{\rho} \circ \pi$*, and therefore* $S = \tilde{\rho}(\pi(T))$*.*

Proof. One knows that if $\{A_n\}$ and $\{B_n\}$ are sequences in $\mathcal{L}(\mathcal{H})$ that converge in the SOT to A_0 and B_0, respectively, then the sequence $\{A_n B_n\}$ converges in the SOT to $A_0 B_0$. Using this fact together with the hypothesis, we see easily that if $p(x, y)$ is any polynomial in the noncommuting variables x and y, we may define

$$\rho(p(T, T^*)) = \rho(\oplus_{n\in\mathbb{N}} p(T_n, T_n^*)) := \mathrm{SOT} - \lim_n p(T_n, T_n^*) = p(S, S^*), \qquad (12)$$

and it is obvious that ρ, so defined, is a contractive $*$-homomorphism. Moreover, since $1_{\mathcal{H}^{(\aleph_0)}} = \oplus_{n\in\mathbb{N}} 1_{\mathcal{H}}$, we clearly have $\rho(1_{\mathcal{H}^{(\aleph_0)}}) = 1_{\mathcal{H}}$, so ρ is unital. Thus ρ extends by continuity to a C^*-algebra homomorphism of $C^*(T)$ into $C^*(S)$.

With respect to (**b**), let $A \in C^*(T) \cap \mathbb{K}(\mathcal{H}^{(\aleph_0)})$ and set $\rho(A) = B$. It is obvious that A must have the form $A = \oplus_{n\in\mathbb{N}} A_n$, and since A is compact it follows easily that each $A_n \in \mathbb{K}(\mathcal{H})$ and that $\|A_n\| \to 0$. Thus, if $\eta > 0$ and $p(x, y)$ is a polynomial such that $\|p(T, T^*) - A\| < \eta$, then for n sufficiently large we have $\|p(T_n, T_n^*)\| \leq \eta$, so from (12) we get that $\|p(S, S^*)\| \leq \eta$. Since $\|p(S, S^*) - B\| \leq \|p(T, T^*) - A\| \leq \eta$, this shows that $B = 0$. That (**c**) is valid is now just an application of the standard result about factoring through quotient spaces. $\square$

In Definition 3.16, we construct a specific block diagonal operator, which, in the terminology of [26], is called a *universal* block diagonal operator.

Definition 3.16. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and set, for each $j \in \mathbb{N}$, $\mathcal{M}_j = \bigvee\{e_1, e_2, \cdots, e_j\}$. Let $\mathbb{N}$ be partitioned as $\mathbb{N} = \cup_{j\in\mathbb{N}} \mathbb{P}_j$, where for each $j \in \mathbb{N}$, $\mathbb{P}_j$ is an infinite set, and define $\mathcal{K} = \oplus_{m\in\mathbb{N}} \mathcal{K}_m$, where $\mathcal{K}_m = \mathcal{M}_j$ for each $m \in \mathbb{P}_j$. Moreover, for each $j \in \mathbb{N}$, let $\mathcal{D}_j$ be a countable set of strict contractions norm-dense in the unit ball of $\mathcal{L}(\mathcal{M}_j)$, and enumerate the elements of $\mathcal{D}_j$ as $\{B_k\}_{k\in\mathbb{P}_j}$. Now, define

$$B_u := \oplus_{k\in\mathbb{N}} B_k \in \mathcal{L}(\mathcal{K}). \tag{13}$$

It is clear that B_u is a C_{00}, strictly norm decreasing, block diagonal (BCP)-operator in $\mathcal{L}(\mathcal{K})$ whose point spectrum $\sigma_p(B_u)$ is dense in $\mathbb{D}$, such that $\sigma_{le}(B_u) = \mathbb{D}^-$. In the next lemma we establish the universality of B_u in the sense that if S is any contraction in $\mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$ is given, then there exist operators $U : \mathcal{K} \to \mathcal{H} \oplus \mathcal{H}$

and $K \in \mathbb{K}(\mathcal{K})$ with U unitary and $\|K\| < \varepsilon$ such that $U(B_u + K)U^* = B_u \oplus S$.

Proposition 3.17. *Let B_u be the operator in $(BD)(\mathcal{K})$ constructed in Definition 3.16, and let S be any nonzero contraction in $\mathcal{L}(\mathcal{H})$. Then there exist unital C^*-algebra homomorphisms $\rho : C^*(B_u) \to C^*(S)$ and $\tilde{\rho} : C^*(\pi(B_u)) \to C^*(S)$ such that $\rho = \tilde{\rho} \circ \pi$ and $\rho(B_u) = \tilde{\rho}(\pi(B_u)) = S$.*

Proof. With the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and the subspaces $\mathcal{M}_j \subset \mathcal{H}$ as in Definition 3.16, let P_j be the projection in $\mathcal{L}(\mathcal{H})$ with $\operatorname{ran} P_j = \mathcal{M}_j$, (so $P_j \xrightarrow{SOT} 1_{\mathcal{H}}$), and define $S_j := P_j S P_j \in \mathcal{L}(\mathcal{H})$. Clearly $S_j \xrightarrow{SOT} S$ and since $S_j^* = P_j S^* P_j$, we get also $S_j^* \xrightarrow{SOT} S^*$. Moreover, as a consequence of the way B_u was constructed, for each $j \in \mathbb{N}$ there exists some $m_j \in \mathbb{P}_j$ such that $B_{m_j} \in \mathcal{D}_j$ and

$$||B_{m_j} - S_j|_{\mathcal{M}_j}||_{\mathcal{M}_j} < 1/2^j, \qquad j \in \mathbb{N}. \tag{14}$$

For each $j \in \mathbb{N}$ define now $\tilde{B}_{m_j} \in \mathcal{L}(\mathcal{H})$ by $\tilde{B}_{m_j} = B_{m_j} \oplus 0_{\mathcal{H}\ominus\mathcal{M}_j}$, and define also

$$\tilde{B} := \oplus_{j\in\mathbb{N}} \tilde{B}_{m_j} \in \mathcal{L}(\mathcal{H}^{(\aleph_0)}).$$

Clearly,

$$||\tilde{B}_{m_j} - S_j||_{\mathcal{H}} < 1/2^j, \qquad j \in \mathbb{N},$$

and since $S_j \xrightarrow{*-SOT} S$, $\tilde{B}_{m_j} \xrightarrow{*-SOT} S$ also. Moreover, there exists a natural unital (surjective) C^*-algebra isomorphism $\phi : C^*(\tilde{B}) \to C^*(QB_uQ)$ where $Q \in \mathcal{L}(\mathcal{K})$ is the projection of $\mathcal{K}$ onto the subspace $\oplus_{j\in\mathbb{N}}\mathcal{K}_{m_j}$. Thus, by Lemma 3.14, to construct a unital C^*-algebra homomorphism ρ of $C^*(B_u)$ into $C^*(S)$ such that $\rho(B_u) = S$, it suffices to construct a unital C^*-algebra homomorphism ψ of $C^*(\tilde{B})$ into $C^*(S)$ such that $\psi(\tilde{B}) = S$, and since $\tilde{B}_{m_j} \xrightarrow{*-SOT} S$, the existence of ψ is immediate from Lemma 3.15. □

The following corollary of Theorem 3.13 and Proposition 3.17, which should probably be credited to Herrero [26], is quite interesting in itself.

Corollary 3.18. *Let B_u be the operator in $(BD)(\mathcal{H}) \cap C_{00}$ constructed in Definition 3.16, let $S \in \mathcal{L}(\mathcal{H})$ be any contraction, and let $\varepsilon > 0$ be given. Then there exist operators $U : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ and $K \in \mathbb{K}(\mathcal{H})$ with U unitary and $\|K\| < \varepsilon$ such that $U(B_u + K)U^* = B_u \oplus S$.*

In this section we first show that every (BCP)-operator is hyperquasisimilar to a quasidiagonal (BCP)-operator, and then we apply this result, together with Corollary 3.20, to obtain a further reduction in the hyperinvariant subspace problem for operators on Hilbert space.

Theorem 3.19. *Suppose $T \in (\mathrm{BCP})(\mathcal{H})$, B_u is as in Definition 3.16, and $\varepsilon > 0$ is given. Then there exists $\widehat{T} \in (\mathrm{BCP}) \cap (QD)$ satisfying*

(**I**) $T \overset{h}{\sim} \widehat{T}$, so T *has a n.h.s. if and only if* $\widehat{T}$ *does,*

(**II**) $\widehat{T} = (T_0 \oplus B_u) + J$, *where T_0 is a c.n.u. contraction and $J \in \mathcal{C}_1(\mathcal{H} \oplus \mathcal{H})$ satisfies $\|J\|_1 < \varepsilon$,*

(**III**) $\sigma_{le}(\widehat{T}) \supset \sigma_{le}(T)$, $\sigma_{re}(\widehat{T}) \supset \sigma_{re}(T)$, *and* $\sigma(\widehat{T}) \supset \sigma(T)$, *and*

(**IV**) *if $T \in C_{00}$, then $\widehat{T} \in C_{00}$ also.*

Proof. Let B_u be the strictly norm decreasing operator in $(BD)(\mathcal{H})$ defined in Definition 3.16. (with $\mathcal{H}$ replacing $\mathcal{K}$). Then, by Theorem 3.10 (with $B = B_u$), there exists an operator $\widehat{T} \in (\mathrm{BCP})(\mathcal{H} \oplus \mathcal{H})$ such that conclusions (**a**)-(**d**) of that theorem are valid. In particular, from (**a**) we have that $\widehat{T} = (T_0 \oplus B_u) + J$, where $J \in \mathcal{C}_1(\mathcal{H} \oplus \mathcal{H})$ with $\|J\|_1 < \varepsilon$, and since (**III**) and (**IV**) are immediate from (**c**) and (**d**), it suffices to show that $T_0 \oplus B_u$ is quasidiagonal. But this follows immediately from Corollary 3.18 and the fact that $(QD) = (BD) + \mathbb{K}$. □

The following is our further reduction of the hyperinvariant subspace problem.

Theorem 3.20. *Let B_u be the operator constructed in Definition 3.16, and let $\varepsilon > 0$ be given. Then*

(**A**) $B_u \in (BD) \cap (\mathrm{BCP}) \cap C_{00}(\mathcal{H})$ *and satisfies* $\sigma(B_u) = \sigma_{le}(B_u) = \mathbb{D}^-$,

(**B**) B_u *has point spectrum dense in* $\mathbb{D}$*, and thus has at least* $\aleph_0$ *different (and "disjoint") n.h.s., and*

(**C**) *if every* C_{00}*, quasidiagonal,* (BCP)*-operator of the form* $B_u + K$ *has a n.h.s., where* $K \in \mathbb{K}(\mathcal{H})$ *and satisfies* $\|K\| < \varepsilon$*, then every operator in* $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ *has a n.h.s.*

Proof. Since every operator in the unit ball of operators on a finite dimensional Hilbert space is the limit of a sequence of direct summands of the operator B_u, elementary spectral theory shows that $\sigma(B_u) = \sigma_{le}(B_u) = \mathbb{D}^-$ which proves (**A**), and (**B**) is obvious. To establish (**C**), it suffices to fix an arbitrary (BCP)-operator T_1 in C_{00} with $\sigma_{le}(T_1) = \mathbb{D}^-$ and to show that T_1 has a n.h.s. under the hypotheses in (**C**). With T_1 as indicated, we conclude from Theorem 3.19 that $T_1 \overset{h}{\sim} \widehat{T}_1 = (T_0 \oplus B_u) + J$ where $\widehat{T}_1$ has properties (**I**)-(**IV**) of that theorem (with $\|J\|_1 < \varepsilon/2$). Thus T_1 has a n.h.s. if and only if $\widehat{T}_1$ does, by [18, Proposition 2.4], and moreover, by Corollary 3.18 (with $S = T_0$), we know that there exist operators U and $K \in \mathbb{K}(\mathcal{H})$ with U unitary and $\|K\| < \varepsilon/2$ such that $U(B_u + K)U^* = T_0 \oplus B_u$. Thus

$$U(B_u + K + U^*JU)U^* = (T_0 \oplus B_u) + J = \widehat{T}_1,$$

and $K + U^*JU \in \mathbb{K}(\mathcal{H})$ and satisfies $\|K + U^*JU\| < \varepsilon$. Since U is unitary, $\widehat{T}_1$ has a n.h.s. if and only if $B_u + K + U^*JU$ does, and the proof is complete. □

The most definitive result in this direction was eventually obtained in [17] and

[8], and goes as follows.

Theorem 3.21 [8, Theorem 4.2]. *Let B_u be a fixed universal block diagonal operator as defined above, let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})\backslash(\mathcal{A})$, and let ε be an arbitrary positive number. Then there exists a compact operator $K = K(T, \varepsilon) \in \mathbb{K}$ such that*:

(**1**) $\|K\| < \varepsilon$,

(**2**) *$B_u + K$ is quasidiagonal,*

(**3**) $\sigma(B_u + K) = \sigma_{le}(B_u + K) = \mathbb{D}^-$,

(**4**) *$B_u + K$ is a C_{00},* (BCP)-*operator, and*

(**5**) $\text{Hlat}(T) \equiv \text{Hlat}(B_u + K)$.

On the other hand, using the techniques and results from this chapter, as well as [17], this other definitive result was obtained in [8].

Theorem 3.22. *Let $0 \leq \theta < 1$ be given, and suppose $T \in \mathcal{L}(\mathcal{H})\backslash(\mathcal{A})$. Then there exists a C_{00},* (BCP)-*operator $\widehat{T}$ such that $\sigma(\widehat{T}) = \sigma_{le}(\widehat{T}) = \mathbf{A}_\theta$, $\theta(\widehat{T}^{-1}) \in C_{00} \cap (\text{BCP})$ whenever $\theta > 0$, and* $\text{Hlat}(T) \equiv \text{Hlat}(\widehat{T})$.

This result clearly demonstrates that it is of considerable interest to determine as many structure theorems about the class of (BCP)-operators satisfying the conclusions of Theorem 3.22 when $\theta > 0$ as possible, and we do this next, using results from Chapter II. (In Chapter IV we will investigate some special subclasses of this class of operators with a view of obtaining some insight into the problem of determining the structure of operators in these classes.)

First, to shorten the hypothesis in the results to follow, we make the following definition.

Definition 3.23. For every $\theta \in (0,1)$, we denote by $(\mathcal{B}_\theta(\mathcal{H}))$ or, more simply, by $\mathcal{B}_\theta$, the set of all (invertible, c.n.u.) contractions $B \in \mathcal{L}(\mathcal{H})$ such that $B \in C_{00} \cap$ (BCP), $\theta B^{-1} \in C_{00} \cap$ (BCP), $\sigma(B) = \sigma_{le}(B)$, and $\sigma(B) \cap \mathbf{A}_\theta^\circ$ is a dominating subset of $\mathbf{A}_\theta^\circ$ (so also $\sigma(\theta B^{-1}) = \sigma_{le}(\theta B^{-1})$ and $\sigma(\theta B^{-1}) \cap \mathbf{A}_\theta^\circ$ is a dominating subset of $\mathbf{A}_\theta^\circ$).

We observe immediately that the (BCP)-operators $\widehat{T}$ appearing in Theorem 3.22 (the hyperlattices of which are universal for hyperlattices of operators in $\mathcal{L}(\mathcal{H})\backslash(\mathcal{A})$) belong to the class $(\mathcal{B}_\theta)$ for an arbitrary $\theta \in (0,1)$, chosen in advance.

Proposition 3.24. *For every $\theta \in (0,1)$ and for every $B \in (\mathcal{B}_\theta)$, there exists $K = K(T,\theta) > 0$ such that $\mathbf{A}_\theta$ is a complete $K-$spectral set for B.*

Proof. Since $\|T\| = 1$ and $\|T^{-1}\| = \theta$, the result follows immediately from [33, Theorem 9.8]. □

As a consequence of this, [33, Corollary 8.12], and [1], we get this next result.

Proposition 3.25. *For every $\theta \in (0,1)$ and for every $B \in (\mathcal{B}_\theta)$, B is similar to an operator B_1 for which the annulus $\mathbf{A}_\theta$ is a (complete) spectral set. Consequently, B_1 has a normal $\partial\mathbf{A}_\theta-$dilation.*

Note that if B and B_1 are as above, then $\mathrm{Hlat}(B) \equiv \mathrm{Hlat}(B_1)$ so we may replace B by B_1 without loss of generality when dealing with questions concerning hyperinvariant subspaces.

Putting together Propositions 3.24 and 3.25, we obtain the desired structure theorem for operators in the classes $(\mathcal{B}_\theta)$.

Theorem 3.26. *For every $\theta \in (0,1)$ and for every $B \in (\mathcal{B}_\theta)$, B is similar to an operator $B_1 \in (\mathcal{B}_\theta)$ such that $B_1 \in \mathbb{A}_{\aleph_0}^{\mathbf{A}_\theta}$ and $\mathcal{A}_{T_1}^{\mathbf{A}_\theta}$ is reflexive.*

Proof. Let B_1 be obtained from B as in Proposition 3.25. Since $\sigma(B_1) = \sigma(B)$,

$\sigma_{le}(B_1) = \sigma_{le}(B)$, and B_1 has a $\partial \mathbf{A}_\theta-$dilation, it is clear that $B_1 \in (\mathcal{B}_\theta)$ with B. Moreover, since $B \in C_{00}^{\mathbf{A}_\theta}$ and B_1 is similar to B, we have $B_1 \in C_{00}^{\mathbf{A}_\theta}$ too. Finally, that B_1 has an isometric $H^\infty(\mathbf{A}_\theta)$ (and weak*-homeomorphic) functional calculus follows from [16, Theorem 7.3 (and proof)], so $B_1 \in \mathbb{A}^{\mathbf{A}_\theta}$ by definition. Moreover, it is known from Theorem 2.2 that $\mathbb{A}^{\mathbf{A}_\theta} \cap C_{00}^{\mathbf{A}_\theta} \subset \mathbb{A}_{\aleph_0}^{\mathbf{A}_\theta}$, so the proof is complete by Theorem 2.3. □

The following applications to the classes $(\mathcal{B}_\theta)$ come from Theorem 2.7 (and from [9]).

Theorem 3.27. *For every $\theta \in (0,1)$ every $B \in (\mathcal{B}_\theta)$ and every sequence $\{\gamma_n\}_{n\in\mathbb{N}_0}$ (of not necessarily distinct points) $\subset \mathbf{A}_\theta$, then B is similar to an operator $B_1 \in (\mathcal{B}_\theta) \cap \mathbb{A}_{\aleph_0}^{\mathbf{A}_\theta}$ and there exist a decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{P}$ and an orthonormal basis $\{f_n\}_{n\in\mathbb{N}_0}$ for $\mathcal{N}$ such that for every rational function $r \in R^{\mathbf{A}_\theta}$, the matrix for $r(B_1)$ has the form*

$$r(B_1) = \begin{bmatrix} r(B_{11}) & * & * \\ 0 & r(D) & * \\ 0 & 0 & r(B_{33}) \end{bmatrix}$$

where D is the diagonal normal operator in $\mathcal{L}(\mathcal{N})$ defined by $Df_n = \gamma_n f_n$, $n \in \mathbb{N}$.

Theorem 3.28. *Suppose $\theta \in (0,1)$, $B \in (\mathcal{B}_\theta)$, and X satisfies $\sigma(X) \subset (\mathbf{A}_\theta)^\circ$. Then B is similar to an operator $B_1 \in (\mathcal{B}_\theta) \cap \mathbb{A}_{\aleph_0}^{\mathbf{A}_\theta}$ with the property that there exist a decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{P}$ and an operator $X_1 \in \mathcal{L}(\mathcal{N})$ such that for every rational function $r \in R^{\mathbf{A}_\theta}$, the matrix*

$$r(B_1) = \begin{bmatrix} r(B_{11}) & * & * \\ 0 & r(X_1) & * \\ 0 & 0 & r(B_{33}) \end{bmatrix}$$

where X_1 *is similar to* X.

Theorems 3.26, 3.27, and 3.28 are completely new, and it is hoped that they will be useful tools in solving problems concerning hyperinvariant subspace lattices of operators in $\mathcal{L}(\mathcal{H})$.

CHAPTER IV

A CERTAIN CLASS OF (BCP)-OPERATORS

In this chapter, we construct, for every $\theta \in (0,1)$, a certain easily described subclass $(\mathcal{S}_\theta)$ of $(\mathcal{B}_\theta)$ which may prove to be very useful in resolving problems concerning hyperlattices of operators in $(\mathcal{B}_\theta)$. Much of the chapter is devoted to showing that, indeed, $(\mathcal{S}_\theta) \subset (\mathcal{B}_\theta)$ for every $\theta \in (0,1)$.

Given an orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$ of $\mathcal{H}$, the unique operator $S \in \mathcal{L}(\mathcal{H})$ such that $Se_n = e_{n+1}$ for $n \in \mathbb{N}_0$ is called a *unilateral shift of multiplicity one*. A trivial computation shows that $S^*e_0 = 0$ and $S^*e_{n+1} = e_n$ for $n \in \mathbb{N}$. If α is any cardinal number less than or equal to $\aleph_0$ then a unilateral shift of multiplicity α, denoted by $S^{(\alpha)}$, is the direct sum of α copies of the unilateral shift S of multiplicity one.

In the remaining part of this chapter, for an arbitrary but fixed $0 < \theta < 1$, we will be studying a class of operators in $\mathcal{L}(\mathcal{H})$ defined as follows.

Definition 4.1. Fix $0 < \theta < 1$. We say $T \in (\mathcal{S}_\theta)$ if and only if T is unitarily equivalent to an operator in $\mathcal{L}(\mathcal{H}^{(3)})$ of the form

$$\begin{bmatrix} S_1P & S_2 & 0 \\ 0 & 0 & S_2^* \\ 0 & 0 & PS_1^* \end{bmatrix}, \tag{15}$$

where S_1 and S_2 are (forward, unweighted) unilateral shifts of infinite multiplicity in $\mathcal{L}(\mathcal{H})$ such that $(\operatorname{ran} S_1)^\perp = (\operatorname{ran} S_2)$ and P is a positive semidefinite operator such that

(1) $\theta,\, 1 \notin \sigma_p(P)$,

(2) $\sigma(P) = [\theta, 1](= \sigma_{le}(P))$, and

(**3**) $PS_1 = S_1P$.

The following lemma follows immediately from the definition of the classes $(\mathcal{S}_\theta)$ and thus needs no proof.

Lemma 4.2. *If* $T \in (\mathcal{S}_\theta)$ *for some* $0 < \theta < 1$ (*and is thus unitarily equivalent to an operator matrix as in* (15)), *then the following equations are valid, where* $Q_i := S_iS_i^*$, $i = 1, 2$:

(**a**) $S_1^*S_1 = 1_{\mathcal{H}}$,

(**b**) $S_2^*S_2 = 1_{\mathcal{H}}$,

(**c**) $S_2^*S_1 = 0$,

(**d**) $S_1^*S_2 = 0$,

(**e**) $Q_i = Q_i^* = Q_i^2$, $i = 1, 2$, *and* $Q_1 + Q_2 = 1_{\mathcal{H}}$, *so* $Q_1Q_2 = Q_2Q_1 = 0$,

(**f**) $S_1^*Q_2 = 0$,

(**g**) $S_2^*Q_1 = 0$,

(**h**) $S_1^*Q_1 = S_1^*$,

(**i**) $S_2^*Q_2 = S_2^*$,

(**j**) $PQ_1 = Q_1P$,

(**k**) $PQ_2 = Q_2P$.

We shall now establish various facts about the operators in the classes $(\mathcal{S}_\theta)$ which will enable us to eventually show that every such operator T is a C_{00}, (BCP)-operator with $\sigma(T) = \sigma_{le}(T) = \mathbf{A}_\theta$ and $\theta(\widehat{T}^{-1}) \in C_{00} \cap \text{(BCP)}$ (for a given $0 < \theta < 1$), and thus that $(\mathcal{S}_\theta)$ is a subset of the class of operators which arise in Theorem 3.22. We will obtain information below which may eventually lead to the existence of n.h.s. for operators in the class(es) $(\mathcal{S}_\theta)$.

Proposition 4.3. *Let $W \in \mathcal{L}(\mathcal{H}^{(3)})$ be given by*

$$W = \begin{bmatrix} 0 & 0 & 1_{\mathcal{H}} \\ 0 & 1_{\mathcal{H}} & 0 \\ 1_{\mathcal{H}} & 0 & 0 \end{bmatrix}.$$

Then $W = W^*$ and $WTW = WTW^* = T^*$, so T is unitarily equivalent to T^*. Moreover, the operator T^nW is self-adjoint for all $n \in \mathbb{N}_0$.

Proof. These facts follow trivially from the matricial calculation that gives $WTW = T^*$. □

Some additional easy matricial calculations based on Lemma 4.2 yield immediately this next result

Proposition 4.4. *If T is the operator matrix in $\mathcal{L}(\mathcal{H}^{(3)})$ given in* (15) *satisfying* (**1**), (**2**), *and* (**3**) *in Definition 4.1, then the following equations hold*:

$$T^2 = \begin{bmatrix} (S_1P)^2 & (S_1P)S_2 & Q_2 \\ 0 & 0 & S_2^*(PS_1^*) \\ 0 & 0 & (PS_1^*)^2 \end{bmatrix}, \tag{16}$$

$$T^3 = \begin{bmatrix} (S_1P)^3 & (S_1P)^2S_2 & P(S_1Q_2 + Q_2S_1^*) \\ 0 & 0 & S_2^*(PS_1^*)^2 \\ 0 & 0 & (PS_1^*)^3 \end{bmatrix}, \tag{17}$$

and by induction, it is easy to see that for $n \in \mathbb{N}\backslash\{1, 2\}$,

$$T^n = \begin{bmatrix} (S_1P)^n & (S_1P)^{n-1}S_2 & \sum_{i=0}^{n-2}(S_1P)^{n-2-i}(Q_2)(PS_1^*)^i \\ 0 & 0 & S_2^*(PS_1^*)^{n-1} \\ 0 & 0 & (PS_1^*)^n \end{bmatrix}, \tag{18}$$

and

$$T^{*n} = \begin{bmatrix} (PS_1^*)^n & 0 & 0 \\ S_2^*(PS_1^*)^{n-1} & 0 & 0 \\ \sum_{i=0}^{n-2}(S_1P)^{n-2-i}(Q_2)(PS_1^*)^i & (S_1P)^{n-1}S_2 & (S_1P)^n \end{bmatrix}, \tag{19}$$

where any nonzero operator to the power 0 *is defined to be* $1_{\mathcal{H}}$.

Proposition 4.5. *The polar decomposition of the operator matrix* T *in* (15) *is* $T = UR$, *where* $U \in \mathcal{L}(\mathcal{H}^{(3)})$ *is the unitary operator*

$$U = \begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & S_2^* \\ 0 & 0 & S_1^* \end{bmatrix}, \tag{20}$$

and

$$R = (T^*T)^{1/2} = \begin{bmatrix} P & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & Q_2 + PQ_1 \end{bmatrix}. \tag{21}$$

Proof. Easy computations show that $T^*T = R^2$, $T = UR$, and U is unitary. □

Proposition 4.6. *The operator* $S_1P(= PS_1)$ *appearing in Definition 4.1 is unitarily equivalent to an operator* $S \otimes P_1 \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, *where* S *is a unilateral shift of multiplicity one and* P_1 *is positive semidefinite and satisfies* $\sigma(P_1) = \sigma_{le}(P_1) = [\theta, 1] = \sigma(P)$.

Proof. Since S_1 is a unilateral shift of infinite multiplicity, it is clear that S_1 is unitarily equivalent to an operator $S \otimes 1_{\mathcal{H}}$ where S is as above. Moreover, since $PS_1 = S_1P$, this unitary equivalence carries P onto an operator $1_{\mathcal{H}} \otimes P_1$ where P_1 has the properties described above, and the result follows. □

Proposition 4.7. *With the notation as in Proposition 4.6,* $\sigma_{le}(S_1P) = \sigma_{le}(S \otimes P_1) = \mathbf{A}_\theta$.

Proof. Let $e^{i\theta} \in \sigma_{le}(S)$ and $r \in \sigma_{le}(P_1)$. Then there exist orthonormal sets $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that

$$\left\|(S - e^{i\theta})e_n\right\| \to 0 \text{ and } \|(P_1 - r)f_n\| \to 0 \text{ as } n \to \infty.$$

Now

$$\begin{aligned} S \otimes P_1 - e^{i\theta} r(1_{\mathcal{H}} \otimes 1_{\mathcal{H}}) &= S \otimes P_1 - e^{i\theta} 1_{\mathcal{H}} \otimes P_1 + e^{i\theta} 1_{\mathcal{H}} \otimes P_1 - e^{i\theta} 1_{\mathcal{H}} \otimes r1_{\mathcal{H}} \\ &= (S - e^{i\theta} 1_{\mathcal{H}}) \otimes P_1 + e^{i\theta}(1_{\mathcal{H}} \otimes P_1 - 1_{\mathcal{H}} \otimes r1_{\mathcal{H}}) \\ &= (S - e^{i\theta} 1_{\mathcal{H}}) \otimes P_1 + e^{i\theta}(1_{\mathcal{H}} \otimes P_1 - r1_{\mathcal{H}}). \end{aligned}$$

Next,

$$\begin{aligned} & (S \otimes P_1 - e^{i\theta} r(1_{\mathcal{H}} \otimes 1_{\mathcal{H}}))(e_n \otimes f_n) \\ = \; & ((S - e^{i\theta} 1_{\mathcal{H}}) \otimes P_1)(e_n \otimes f_n) + e^{i\theta}(1_{\mathcal{H}} \otimes P_1 - r1_{\mathcal{H}})(e_n \otimes f_n) \\ = \; & (S - e^{i\theta} 1_{\mathcal{H}})e_n \otimes P_1 f_n + e^{i\theta}(e_n \otimes (P_1 - r1_{\mathcal{H}})f_n). \end{aligned}$$

But since, as $n \to \infty$,

$$\left\|(S - e^{i\theta} 1_{\mathcal{H}})e_n \otimes P_1 f_n\right\| \leq \left\|(S - e^{i\theta} 1_{\mathcal{H}})e_n\right\| \to 0,$$

and

$$\left\|e^{i\theta}(e_n \otimes (P_1 - r1_{\mathcal{H}})f_n)\right\| \leq \|(P_1 - r1_{\mathcal{H}})f_n\| \to 0,$$

we conclude that $\left\|(S \otimes P_1 - e^{i\theta} r(1_{\mathcal{H}} \otimes 1_{\mathcal{H}}))(e_n \otimes f_n)\right\| \to 0$, which proves that $\mathbf{A}_\theta \subset \sigma_{le}(S_1 P)$. Moreover, an easy calculation shows that $S_1 P$ is bounded below by θ, and thus (since $\|S_1 P\| = 1$) $\sigma_{le}(S_1 P) \subset \sigma_l(S_1 P) \subset \mathbf{A}_\theta$. $\square$

Proposition 4.8. *The operator* $S_1 P \in C_{00}$.

Proof. By the spectral theorem for Hermitian operators, there exists a unique spectral measure E whose support is $\sigma(P)$ such that $P = \int_{\sigma(P)} \lambda dE$. Now we choose

$n_0 \in \mathbb{N}$ such that $\theta < 1/n_0$ and partition the interval $[\theta, 1)$ into subintervals $[\theta, 1/n_0) \cup [1/n_0, 1/(n_0+1)) \cup \cdots$. Next, define $\mathcal{H}_0 := E([\theta, 1/n_0))\mathcal{H}$ and $\mathcal{H}_j := E([1/(n_0+j-1), 1/(n_0+j))\mathcal{H}$ for every $j \in \mathbb{N}$. Each $\mathcal{H}_j$ reduces P, so we define $P|_{\mathcal{H}_j} = P_j$. Moreover, since (by definition) 1 is not an eigenvalue of P, $\mathcal{H} = \oplus_{j\in\mathbb{N}_0}\mathcal{H}_j$ and $P = \oplus_{j\in\mathbb{N}_0}P_j$. Since $\|P_j\| < 1$ for $j \in \mathbb{N}_0$ and $S_1P = PS_1$, the result follows. □

Recall from [32] that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *centered* if the doubly infinite sequence

$$\{\cdots, T^nT^{*n}, \cdots, T^2T^{*2}, TT^*, T^*T, T^{*2}T^2, \cdots, T^{*n}T^n, \cdots\}$$

consists of mutually commuting operators. One knows that, despite the fact that centered operators have been in play since 1974, the invariant subspace problem for such operators remains unsolved.

Notation 4.9. We adopt Halmos' notation by writing, for any $A, B \in \mathcal{L}(\mathcal{H})$, $A \leftrightarrow B$ to mean that $AB = BA$.

We will need the following obvious lemma which needs no proof.

Lemma 4.10. *Suppose $A, B, S \in \mathcal{L}(\mathcal{H})$ with S invertible. Then $A \leftrightarrow B$ if and only if $SAS^{-1} \leftrightarrow SBS^{-1}$.*

We also need the following result, which is an easy consequence of [32, Lemma 3.1].

Proposition 4.11. *If $T \in \mathcal{L}(\mathcal{H})$ and is a quasiaffinity with polar decomposition $T = UR$ (so U is unitary), then T is centered if and only if the sequence $\{U^{*n}R^2U^n\}_{n\in\mathbb{Z}}$ consists of mutually commuting operators, which happens if and only if the sequence*

$\{T^{*n}T^n\}_{n\in\mathbb{N}}$ *consists of mutually commuting operators.*

Proof. The first statement is exactly [32, Lemma 3.1], where it also proved that

$$T^nT^{*n} = ((U^nR^2(U^*)^n)(U^{n-1}R^2(U^*)^{n-1})\cdots(UR^2U^*)), \qquad n \in \mathbb{N},$$

and

$$T^{*n}T^n = ((U^*)^{n-1}R^2(U^{n-1})((U^*)^{n-2}R^2(U^{n-2}))\cdots(U^*RU)P^2, \qquad n \in \mathbb{N},$$

and the result follows easily by repeated application of Lemma 4.10. □

We now return to a discussion of the class $(\mathcal{S}_\theta)$ of operators as in Definition 4.1 (with, as usual, $0 < \theta < 1$ fixed but arbitrary).

To motivate Theorem 4.12 to follow, we observe by using (15), (16), and (17), and making some elementary matricial calculations that

$$T^*T = \begin{bmatrix} P^2 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & Q_2 + P^2Q_1 \end{bmatrix}, \tag{22}$$

$$TT^* = \begin{bmatrix} Q_2 + P^2Q_1 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & P^2 \end{bmatrix}, \tag{23}$$

and

$$T^{*2}T^2 = \begin{bmatrix} P^4 & 0 & 0 \\ 0 & S_2^*P^2S_2 & 0 \\ 0 & 0 & P^4S_1Q_1S_1^* + P^2S_1Q_2S_1^* + Q_2 \end{bmatrix}. \tag{24}$$

Since the above operator matrices in $\mathcal{L}(\mathcal{H}^{(3)})$ are diagonal, then they commute with one another if and only if their corresponding diagonal elements commute. It follows immediately then from (22), (23) and (24) and Lemma 4.2 that $TT^* \leftrightarrow T^*T$ and $T^{*2}T^2 \leftrightarrow T^*T$. This is a special case, of course, of Theorem 4.12 below.

Theorem 4.12. *Every operator in the class $(\mathcal{S}_\theta)$ is a centered operator.*

Proof. If T is given matricially by (15), then, by induction, one can show that

$$T^{*n}T^n = \begin{bmatrix} P^{2n} & 0 & 0 \\ 0 & P^{2n-2} & 0 \\ 0 & 0 & Z_n \end{bmatrix}, \qquad n \in \mathbb{N}\backslash\{1\}, \tag{25}$$

where $Z_n = P^{2(n-2)}Q_2 + P^{2n}(S_1)^n(S_1^*)^n + \sum_{k=1}^{n-1} P^{2k}S_1^kQ_2(S_1^*)^k$.

Since S_1 is a unilateral shift of infinite multiplicity (by definition), there exists an infinite dimensional subspace $\mathcal{K} \subset \mathcal{H}$ and a unitary operator $V : \mathcal{H} \to \mathcal{K}^{(\omega)}$ (where ω is the first infinite ordinal number) such that $VS_1V^* \in \mathcal{L}(\mathcal{K}^{(\omega)})$ is the shift $S^{(\omega)}$ in $\mathcal{L}(\mathcal{K}^{(\omega)})$ defined by $S^{(\omega)}(h_1, h_2, \cdots) = (0, h_1, h_2, \cdots)$. Since $P \leftrightarrow S_1$, $VPV^* \leftrightarrow S^{(\omega)}$, and an easy calculation shows that $VPV^*(h_1, h_2, \cdots) = (P_1h_1, P_1h_2, \cdots)$, where P_1 is some positive semidefinite operator in $\mathcal{L}(\mathcal{H})$ satisfying $\sigma(P_1) = \sigma_{le}(P_1) = [\theta, 1] = \sigma(P)$. Similarly, we obtain that $VQ_1V^*(h_1, h_2, \cdots) = (0, h_2, h_3, \cdots)$ and that $VQ_2V^*(h_1, h_2, \cdots) = (h_1, 0, 0, \cdots)$.

It follows that

$$\begin{aligned} & VZ_nV^*(h_1, h_2, \cdots, h_n, \cdots) \\ = \ & (P_1^{2(n-2)}h_1, P_1^2h_2, P_1^4h_3, \cdots, P_1^{2(n-1)}h_n, P_1^{2n}h_{n+1}, P_1^{2n}h_{n+2}, \cdots) \end{aligned}$$

which shows that $\{VZ_nV^*\}_{n\in\mathbb{N}\backslash\{1\}}$ and thus $\{Z_n\}_{n\in\mathbb{N}\backslash\{1\}}$ is a commutative family of operators and consequently $\{T^{*n}T^n\}_{n\in\mathbb{N}}$ consists of mutually commuting operators. Thus it follows from Proposition 4.11 and (25) that T is a centered operator. □

Recall from [23] that an operator T in $\mathcal{L}(\mathcal{H})$ whose polar decomposition is $T =$

VQ is called *quasinormal* if $V \leftrightarrow Q$. (The structure of such operators was completely determined by Arlen Brown in [11].)

Proposition 4.13. *No operator in* $\bigcup_{\theta>0}(\mathcal{S}_\theta)$ *is either quasinormal, hyponormal, or essentially normal.*

Proof. Let $T \in (\mathcal{S}_\theta)$ for some $\theta > 0$, and without loss of generality, let $T = UR$ where U and R are as in (20) and (21). Then T^*T and TT^* are given by (22) and (23), respectively. A simple matricial calculation shows that U (given by (20)) does not commute with R, so T is not quasinormal. Moreover, from (22) and (23), one obtains

$$T^*T - TT^* = \begin{bmatrix} (P^2-1)Q_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1-P^2)Q_2 \end{bmatrix},$$

and since $\sigma_e((P^2-1)Q_2) = [\theta-1, 0]$, T is neither hyponormal nor essentially normal.□

Proposition 4.14. *For every operator* T *in* $(\mathcal{S}_\theta)$, $T \in C_{00}$ *and* $\theta T^{-1} \in C_{00}$.

Proof. Let T be the operator matrix in $\mathcal{L}(\mathcal{H}^{(3)})$ given by (15). We first show that $T_n \xrightarrow{SOT} 0$ and thus by Proposition 4.3, it will follow that $T \in C_{00}$. Note that by (19) and Proposition 4.8,

$$\lim_{n\to\infty} \|T^n(x,y,0)\| = \lim_{n\to\infty} \left\|(S_1P)^n x + (S_1P)^{n-1}S_2 y\right\| = 0, \quad x, y \in \mathcal{H}.$$

Moreover, a routine calculation gives that

$$\begin{aligned} T^n(0,0,w) &= (\sum_{i=0}^{n-2}(S_1P)^{n-2-i}(Q_2)(PS_1^*)^i w,\, S_2^*(PS_1^*)^{n-1}w,\, (PS_1^*)^n w) \\ &= (x_n(w), y_n(w), z_n(w)), \qquad n \in \mathbb{N}, w \in \mathcal{H}. \end{aligned}$$

Clearly $\{y_n(w)\} \to 0$ and $\{z_n(w)\} \to 0$ for all $w \in \mathcal{H}$. Finally, for any fixed k,

$n_0 \in \mathbb{N}$,

$$\left\|T^{k+n_0}w\right\| \leq \left\|T^k(x_{n_0}(w),0,0)\right\| + \|y_{n_0}(w)\| + \|z_{n_0}(w)\|,$$

and that $T \in C_{00}$ follows from what was shown above upon taking n_0 sufficiently large and letting $k \to \infty$. The argument that $\theta T^{-1} \in C_{00}$ is almost exactly the same, and thus is omitted. $\square$

Proposition 4.15. *The unitary operator U given by* (20) *is a bilateral shift of infinite multiplicity.*

Proof. As is well-known, it suffices to exhibit an infinite dimensional wandering subspace $\mathcal{M} \subset \mathcal{H}^{(3)}$ such that $\bigvee_{n\in\mathbb{Z}} U^n\mathcal{M} = \mathcal{H}^{(3)}$. Define $\mathcal{M} = (0)\oplus\mathcal{H}\oplus(0)$. Then

$$U^n\mathcal{M} = S_1^{n-1}S_2\mathcal{H}\oplus(0)\oplus(0) \text{ for all } n \in \mathbb{N}.$$

Clearly, $\mathcal{M}$ is orthogonal to $U^n\mathcal{M}$ for all $n \in \mathbb{N}$ which shows that $\mathcal{M}$ is an (infinite dimensional) wandering subspace for U. Moreover, $S_1^{m-1}S_2\mathcal{H}$ is orthogonal to $S_1^{n-1}S_2\mathcal{H}$ for $m, n \in \mathbb{N}$ and $m \neq n$. We show first that $\bigvee_{n\in\mathbb{N}} S_1^{n-1}S_2\mathcal{H} = \mathcal{H}$. Since S_1 is a unilateral shift, one knows that $\mathcal{H}\ominus S_1\mathcal{H} = S_2\mathcal{H}$ is a wandering subspace for S_1 and that

$$\bigvee_{k\in\mathbb{N}} S_1^{k-1}(\mathcal{H}\ominus S_1\mathcal{H}) = \mathcal{H},$$

which shows that $\bigvee_{n\in\mathbb{N}} U^{n-1}\mathcal{M} = \mathcal{H}\oplus\mathcal{H}\oplus(0)$. A similar argument shows that $\bigvee_{k\in\mathbb{N}} U^k\mathcal{M} = (0)\oplus\mathcal{H}\oplus\mathcal{H}$, and thus that $\bigvee_{n\in\mathbb{Z}} U^n\mathcal{M} = \mathcal{H}^{(3)}$. $\square$

The above proof was kindly pointed out to us by Professor Ciprian Foias.

Theorem 4.16. *For every operator T in $(\mathcal{S}_\theta)$, T is a* (BCP)*-operator satisfying $\sigma(T) = \sigma_{le}(T) = \mathbf{A}_\theta$.*

Proof. We know from (21) that $\|T\| = 1$ and from Proposition 4.14 that T is a C_{00}-operator (and therefore completely nonunitary). Moreover, since the matrix in

(15) is in upper triangular form with $(1,1)$ entry $PS_1 = S_1P$, we have $\sigma_{le}(S_1P) \subset \sigma_{le}(T)$, and from Proposition 4.7 we know that $\mathbf{A}_\theta = \sigma_{le}(S_1P)$. Thus $\mathbf{A}_\theta \subset \sigma_{le}(T)$, and to complete the proof it suffices to show (in case $0 < \theta$) that $\|T^{-1}\| = 1/\theta$, which automatically gives that $\sigma(T) \subset \sigma_{le}(T) \subset \mathbf{A}_\theta$. But from Proposition 4.5 we have $T^{-1} = R^{-1}U^*$, so from (21) we get $\|T^{-1}\| = \|P^{-1}\| = 1/\theta$. □

Proposition 4.17. *The operator* T *in* $\mathcal{L}(\mathcal{H}^{(3)})$, *given matricially by* (15), *satisfies* $\sigma_p(T) = \sigma_p(T^*) = \emptyset$.

Proof. By Theorem 4.16, we know that $\sigma(T) = \mathbf{A}_\theta$. Thus suppose $\lambda \in \mathbf{A}_\theta$ and $(x, y, z) \in \mathcal{H}^{(3)}$ satisfies

$$T(x, y, z) = \lambda(x, y, z). \tag{26}$$

We will show that $x = y = z = 0$, and thus that $\sigma_p(T) = \emptyset$. Since by Proposition 4.3, T is unitarily equivalent to T^*, this will also show that $\sigma_p(T^*) = \emptyset$. From (15) and (23) we obtain immediately the following system of simultaneous equations:

$$S_1Px + S_2y = \lambda x, \tag{27}$$

$$S_2^*z = \lambda y, \tag{28}$$

$$S_1^*Pz = \lambda z \tag{29}$$

From (28) we obtain that $y = \lambda^{-1}S_2^*z$, and by substitution in (27), we get that

$$(S_1P - \lambda 1_{\mathcal{H}})x = -\lambda^{-1}Q_2z. \tag{30}$$

We next employ the unitary operator $V : \mathcal{H} \to \mathcal{K}^{(\omega)}$ from the proof of Theorem 4.12 to write

$$Vx = (x_1, \cdots, x_n, \cdots) \qquad \text{and} \qquad Vz = (z_1, \cdots, z_n, \cdots),$$

then, using the characterization of VPV^*, VQ_2V^*, and VS_1V^* given in that proof together with (29), we obtain easily that

$$(z_1, \cdots, z_n, \cdots) = Vz = \lambda^{-1}(VS_1^*V^*)(VPV^*)z = \lambda^{-1}(P_1z_2, P_1z_3, \cdots),$$

and hence,

$$\{z_n\}_{n\in\mathbb{N}} = (\lambda P_1^{-1})^{n-1}z_1. \tag{31}$$

Moreover, by using (30) we get that

$$-\lambda^{-1}(z_1, 0, 0, \cdots) = (-\lambda x_1, P_1x_1 - \lambda x_2, P_1x_2 - \lambda x_3, \cdots),$$

and hence,

$$\{x_n\}_{n\in\mathbb{N}} = \lambda^{-2}(\lambda^{-1}P_1)^{n-1}z_1. \tag{32}$$

Thus,

$$\begin{aligned} \|z\|^2 &= \|Vz\|^2 = \sum\nolimits_{n\in\mathbb{N}} \|z_n\|^2 = \sum\nolimits_{n\in\mathbb{N}} \left\|(\lambda P_1^{-1})^{n-1}z_1\right\|^2 < \infty, \\ \|x\|^2 &= \|Vx\|^2 = \sum\nolimits_{n\in\mathbb{N}} \|x_n\|^2 = \lambda^{-4}\sum\nolimits_{n\in\mathbb{N}} \left\|(\lambda^{-1}P_1)^{n-1}z_1\right\|^2 < \infty. \end{aligned}$$

Hence, we have that $\|z_1\|^2 = \left\|(\lambda P_1^{-1})^n z_1\right\| \|(\lambda^{-1}P_1)^n z_1\| \to 0$ as $n \to \infty$ since $\|\lambda^{-1}P_1\| \leq 1$, which gives, via (31) and (32) that $x = y = z = 0$, as desired. □

The next step in our program of obtaining as much information as possible about operators in the class(es) $(\mathcal{S}_\theta)$ is to investigate the commutant of such an operator.

Proposition 4.18. *Let T be an arbitrary operator in the class $(\mathcal{S}_\theta)$ given matricially by* (15) *and let $T' \in \mathcal{L}(\mathcal{H}^{(3)})$ be arbitrary in $\{T\}'$, with*

$$T' = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix}. \tag{33}$$

Then the following equations obtain:

$$S_1PA + S_2D = AS_1P, \tag{34}$$

$$S_1PB + S_2E = AS_2, \tag{35}$$

$$S_1PC + S_2F = BS_2^* + CPS_1^*, \tag{36}$$

$$S_2^*G = DS_1P, \tag{37}$$

$$S_2^*H = DS_2, \tag{38}$$

$$S_2^*K = ES_2^* + FPS_1^*, \tag{39}$$

$$PS_1^*G = GS_1P, \tag{40}$$

$$PS_1^*H = GS_2, \tag{41}$$

and

$$PS_1^*K = HS_2^* + KPS_1^*. \tag{42}$$

Proof. These equations result immediately from (15), (33), and the equation $TT' = T'T$. □

We now list, for future use, some additional equations that result from (**a**)-(**k**) of Lemma 4.2 and (34) − (42):

Left multiplication of (34) by S_1^* gives $$PA = S_1^*AS_1P \tag{43}$$

Left multiplication of (43) by S_1 gives $$S_1PA = Q_1AS_1P \tag{44}$$

Left multiplication of (34) by S_2^* gives $$D = S_2^*AS_1P \tag{45}$$

Left multiplication of (45) by S_2 gives $$S_2D = Q_2AS_1P \tag{46}$$

Left multiplication of (35) by S_1^* gives $$PB = S_1^*AS_2 \tag{47}$$

Left multiplication of (47) by S_1 gives $$S_1PB = Q_1AS_2 \tag{48}$$

Right multiplication of (35) by S_2^* gives $$S_1PBS_2^* + S_2ES_2^* = AQ_2 \tag{49}$$

Left multiplication of (49) by S_2^* gives $$ES_2^* = S_2^* A Q_2 \tag{50}$$

Left multiplication of (35) by S_2^* gives $$E = S_2^* A S_2 \tag{51}$$

Left multiplication of (51) by S_2 gives $$S_2 E = Q_2 A S_2 \tag{52}$$

Right multiplication of (52) by S_2^* gives $$S_2 E S_2^* = Q_2 A Q_2 \tag{53}$$

Left multiplication of (36) by S_2^* gives $$F = S_2^* B S_2^* + S_2^* C P S_1^* \tag{54}$$

Left multiplication of (36) by S_1^* gives $$PC = S_1^* B S_2^* + S_1^* C P S_1^* \tag{55}$$

Right multiplication of (36) by S_1 gives $$S_1 P C S_1 + S_2 F S_1 = CP \tag{56}$$

Right multiplication of (36) by S_2 gives $$S_1 P C S_2 + S_2 F S_2 = B \tag{57}$$

Right multiplication of (54) by S_1 gives $$F S_1 = S_2^* C P \tag{58}$$

Right multiplication of (54) by S_2 gives $$F S_2 = S_2^* B \tag{59}$$

Left multiplication of (55) by S_1 gives $$S_1 P C = Q_1 B S_2^* + Q_1 C P S_1^* \tag{60}$$

Right multiplication of (60) by S_2 gives $$S_1 P C S_2 = Q_1 B \tag{61}$$

Right multiplication of (60) by S_1 gives $$S_1 P C S_1 = Q_1 C P \tag{62}$$

Right multiplication of (55) by S_1 gives $$P C S_1 = S_1^* C P \tag{63}$$

Right multiplication of (56) by S_1^* gives $$S_1 P C Q_1 + S_2 F Q_1 = C P S_1^* \tag{64}$$

Left multiplication of (64) by S_2^* gives $$F Q_1 = S_2^* C P S_1^* \tag{65}$$

Left multiplication of (57) by S_1^* gives $$P C S_2 = S_1^* B \tag{66}$$

Right multiplication of (57) by S_2^* gives $$S_1 P C Q_2 + S_2 F Q_2 = B S_2^* \tag{67}$$

Left multiplication of (37) by S_2 gives $$Q_2 G = S_2 D S_1 P \tag{68}$$

Left multiplication of (38) by S_2 gives $$Q_2 H = S_2 D S_2 \tag{69}$$

Right multiplication of (38) by S_2^* gives $$S_2^* H S_2^* = D Q_2 \tag{70}$$

Left multiplication of (39) by S_2 gives $$Q_2 K = S_2 E S_2^* + S_2 F P S_1^* \tag{71}$$

Right multiplication of (71) by S_1 gives $$Q_2 K S_1 = S_2 F P \tag{72}$$

Right multiplication of (71) by S_2 gives $$Q_2 K S_2 = S_2 E \tag{73}$$

Left multiplication of (71) by S_2^* gives $$S_2^* K = E S_2^* + F P S_1^* \tag{74}$$

Right multiplication of (39) by S_1 gives $$S_2^* K S_1 = F P \tag{75}$$

Right multiplication of (39) by S_2 gives $$S_2^*KS_2 = E \tag{76}$$

Right multiplication of (41) by S_2^* gives $$PS_1^*HS_2^* = GQ_2 \tag{77}$$

Right multiplication of (42) by S_1 gives $$PS_1^*KS_1 = KP \tag{78}$$

Right multiplication of (42) by S_2 gives $$PS_1^*KS_2 = H \tag{79}$$

By doing some additional matricial calculations and using the properties of S_1, S_2, and P from Lemma 4.2, we obtain the following.

Proposition 4.19. *Let $0 < \theta < 1$ be arbitrary but fixed, let T, given matricially by* (15), *be arbitrary in* $(\mathcal{S}_\theta)$, *and let T', given by* (33), *be arbitrary in $\{T\}'$. Then*

$$T' = \begin{bmatrix} A & P^{-1}S_1^*AS_2 & C \\ S_2^*AS_1P & S_2^*AS_2 & S_2^*KS_1P^{-1} \\ G & PS_1^*KS_2 & K \end{bmatrix}. \tag{80}$$

Proof. First, by equations (51) and (79) we obtain that $E = S_2^*AS_2$ and $H = PS_1^*KS_2$, respectively. Now right multiplication of (74) by S_1P^{-1} gives

$$F = S_2^*KS_1P^{-1}. \tag{81}$$

Finally, left multiplication of (47) by P^{-1} gives

$$B = P^{-1}S_1^*AS_2, \tag{82}$$

and by equation (45) we obtain that $D = S_2^*AS_1P$, which gives (80) as desired. □

Theorem 4.20. *Suppose T, given matricially by* (15), *belongs to* $(\mathcal{S}_\theta)$. *Then the linear map $\Phi : \{T\}' \to \mathcal{L}(\mathcal{H})$ defined by $\Phi(T') = A$ for every $T' \in \{T\}'$, where T' is given by* (33), *is injective.*

Proof. If $A = 0$, then (82), (45) and (51) yield that $B = D = E = 0$, respectively. Next, by (37), together with the fact that $D = 0$, we obtain $S_2^*G = 0$ which implies after taking adjoints and right multiplication by S_2 that

$$G^*Q_2 = 0. \tag{83}$$

It follows easily from equation (40) that

$$P^n(S_1^*)^nG = G(S_1)^nP^n, \qquad \forall n \in \mathbb{N}, \tag{84}$$

and left multiplication of (84) by Q_2, we obtain $Q_2(S_1^*)^nG = 0 \ \forall n \in \mathbb{N}_0$, and by taking adjoints, this yields

$$G^*(S_1)^nQ_2 = 0, \qquad \forall n \in \mathbb{N}_0. \tag{85}$$

A simple computation shows that $Q_1 = \sum_{j=1}^{\infty} S_1^j(1-Q_1)(S_1^*)^j$ and therefore,

$$G^*Q_1 = \sum_{j=1}^{\infty} G^*S_1^j(1-Q_1)(S_1^*)^j = 0, \tag{86}$$

it follows then from (83) and (86) that $G^* = 0$.

With $D = 0$, (38) implies that $Q_2H = 0$ and since $G = 0$, then (41) yields that $Q_1H = 0$ and therefore, we conclude that $H = 0$.

Next, by (36), together with the fact that $B = 0$, we obtain

$$S_1PC + S_2F = CPS_1^*. \tag{87}$$

Multiplication of (87) by S_2^* from the left and by S_2 from the right, we obtain that

$$FS_2 = 0. \tag{88}$$

Right multiplication of (87) by S_2 together with (88) yields that

$$S_1PCS_2 = 0 \tag{89}$$

and right multiplication of (89) by S_1^* implies that

$$CS_2 = 0 \tag{90}$$

which yields after right multiplication of (90) by S_2^*, that

$$CQ_2 = 0. \tag{91}$$

Furthermore, multiplication of (87) by S_1^* from the left and by S_1 from the right, we obtain that $PCS_1 = S_1^* CP$, which implies that

$$CS_1 P^{-1} = P^{-1} S_1^* C. \tag{92}$$

It follows easily from equation (92) that $C(S_1)^n P^{-n} = P^{-n}(S_1^*)^n C$ $\forall n \in \mathbb{N}$, and since $P \leftrightarrow S_1^*$ we obtain

$$C(S_1)^n P^{-n} = (S_1^*)^n P^{-n} C, \qquad \forall n \in \mathbb{N}. \tag{93}$$

Now right multiplication of (93) by Q_2, we obtain

$$C(S_1)^n Q_2 = 0 \qquad \forall n \in \mathbb{N}. \tag{94}$$

and therefore,

$$CQ_1 = \sum_{j=1}^{\infty} CS_1^j Q_2 (S_1^*)^j = 0, \tag{95}$$

it follows then from (91) and (95) that $C = 0$.

Using the fact that $B = 0$ and $C = 0$, it follows easily from (88) that $S_2 F = 0$, which in return implies that $F = 0$.

Finally, using (39), together with that facts that $E = 0$ and $F = 0$, we obtain that $S_2^* K = 0$ which implies that

$$Q_2 K = 0. \tag{96}$$

Moreover, right multiplication of (42) by S_2 together with $H = 0$, yields that

$$S_1^* K S_2 = 0, \tag{97}$$

and then multiplication of (97) by S_1 from the left and by S_2^* from the right, we obtain that

$$Q_1 K = 0,$$

which together with (96) imply that $K = 0$. □

To see how the above formulas might eventually be used to provide a n.h.s. for an operator in the class(es) $(\mathcal{S}_\theta)$, we point out the following.

Proposition 4.21. *Suppose T, given matricially by* (15), *is in* $(\mathcal{S}_\theta)$, *and suppose, for example* (*to take one instance out of nine*), *that all of the operators in the linear manifold*

$$\mathcal{A} = \{A(T') : T' \in \{T\}'\}$$

have a common nontrivial invariant subspace $\mathcal{M} \subset \mathcal{H}$. Then T has a n.h.s.

Proof. It suffices, of course, to find nonzero vectors x_0 and y_0 in $\mathcal{H}^{(3)}$ such that $\{\langle T'x_0, y_0\rangle = 0 : T' \in \{T\}'\}$. Choose $\tilde{x}_0 \in \mathcal{M}$ and $\tilde{y}_0 \in \mathcal{H} \ominus \mathcal{M}$ and define $x_0 = \tilde{x}_0 \oplus 0 \oplus 0$, $y_0 = \tilde{y}_0 \oplus 0 \oplus 0$, and compute, using (80). □

CHAPTER V

TOWARD A CANONICAL FORM FOR (BCP)-OPERATORS

As is well known, operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are said to be *equivalent* (notation: $T_1 \overset{e}{\sim} T_2$) if there exist invertible operators X, $Y \in \mathcal{L}(\mathcal{H})$ such that $T_2 = XT_1Y$. Of course, $\overset{e}{\sim}$ is an equivalence relation on $\mathcal{L}(\mathcal{H})$, and one knows from linear algebra that in the finite dimensional case a complete set of invariants for $\overset{e}{\sim}$ is the *rank* of an operator. On the other hand, when $\mathcal{H}$ is (as herein) a Hilbert space of (orthogonal) dimension $\aleph_0$, a complete set of invariants for $\overset{e}{\sim}$ was given very early in [31], a particular case of which we describe below. But first we need a bit of additional terminology and notation.

For each T in $\mathcal{L}(\mathcal{H})$, we write $\operatorname{coker}(T)$ for the kernel of T^*, and we define the following cardinal numbers: $c(T) := \dim \operatorname{coker}(T)$, $k(T) := \dim \ker(T)$, and $r(T) := \dim(\operatorname{ran} T)^- = \operatorname{rank}(T)$. Obviously, $0 \le c(T)$, $k(T)$, $r(T) \le \aleph_0$, and $k(T) + r(T) = c(T) + r(T) = \aleph_0$ for all T in $\mathcal{L}(\mathcal{H})$. For operators in $\mathcal{L}(\mathcal{H})$ with closed range, Köthe's theorem from [31] reads as follows.

Theorem 5.1. *Suppose T_1 and T_2 are operators in $\mathcal{L}(\mathcal{H})$ with closed range. Then $T_1 \overset{e}{\sim} T_2$ if and only if $c(T_1) = c(T_2)$, $k(T_1) = k(T_2)$, and $r(T_1) = r(T_2)$.*

A proof of this theorem that filled a modest gap in [31] was given by L. Williams in [36], and an entirely different, and very illuminating, approach to the study of $\overset{e}{\sim}$ was given by P. Fillmore and J. Williams in [19].

It seems, however that the following interesting and useful canonical form under $\overset{e}{\sim}$, which is available for operators in $\mathcal{L}(\mathcal{H})$ with closed range is new.

Corollary 5.2 (canonical form under $\overset{e}{\sim}$). *Let the cardinal numbers $c(\cdot)$, $k(\cdot)$ and*

$r(\cdot)$ be as defined above and let S be a unilateral shift of multiplicity one. Then for every $T \in \mathcal{L}(\mathcal{H})$ with closed range,

(i) *if $c(T) = k(T)$, then $T \overset{e}{\sim} 0_{k(T)} \oplus 1_{r(T)}$ (a projection),*

(ii) *if $k(T) < c(T)$ (which implies that $k(T)$ is finite), then $T \overset{e}{\sim} 0_{k(T)} \oplus S^{(c(T)-k(T))}$, and*

(iii) *if $c(T) < k(T)$ (which implies that $c(T)$ is finite), then $T \overset{e}{\sim} 0_{c(T)} \oplus (S^{(k(T)-c(T))})^*$.*

To prove Corollary 5.2, we must first establish the following proposition.

Proposition 5.3. *Let α be any cardinal number satisfying $0 \leq \alpha \leq \aleph_0$ and let $\mathcal{M}$ be a Hilbert space of dimension α. Then $S \overset{e}{\sim} 1_{\mathcal{M}} \oplus S$.*

Proof. Let S shift the orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$ as above, and define $\mathcal{K}_1 := \bigvee_{n=0}^{\infty} \{e_{2n}\}$, $\mathcal{K}_2 := \bigvee_{n=0}^{\infty} \{e_{2n+1}\}$. By identifying both orthonormal bases $\{e_{2n}\}_{n\in\mathbb{N}_0}$ of $\mathcal{K}_1$ and $\{e_{2n+1}\}_{n\in\mathbb{N}_0}$ of $\mathcal{K}_2$ with some orthonormal basis $\{f_n\}_{n\in\mathbb{N}_0}$ of $\mathcal{H}$ in the obvious way ($e_{2n+1} \leftrightarrow f_n \leftrightarrow e_{2n}$), we get that S is unitarily equivalent to the 2×2 block matrix $\begin{bmatrix} 0 & S \\ 1_{\mathcal{H}} & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, i.e., there exists a unitary operator $U : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ such that $USU^* = \begin{bmatrix} 0 & S \\ 1_{\mathcal{H}} & 0 \end{bmatrix}$. Define $W = \begin{bmatrix} 0 & 1_{\mathcal{H}} \\ 1_{\mathcal{H}} & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, and note that $(USU^*)W = S \oplus 1_{\mathcal{H}}$, and since U and U^*W are invertible operators, we obtain that $S \overset{e}{\sim} 1_{\mathcal{H}} \oplus S$. Consequently, since $1_{\mathcal{H}}$ is unitarily equivalent to $1_{\mathcal{M}} \oplus 1_{\mathcal{H}}$, we get $S \overset{e}{\sim} 1_{\mathcal{M}} \oplus 1_{\mathcal{H}} \oplus S \overset{e}{\sim} 1_{\mathcal{M}} \oplus S$, as desired. □

Proof of Corollary 5.2. Since by hypothesis, $\operatorname{ran} T$ is closed, one knows that T maps $(\ker T)^{\perp}$ onto $\operatorname{ran} T$ in an invertible fashion.

Case (**i**): write $T = UP$, the polar decomposition of T. The operator P is positive semidefinite and U is a partial isometry with initial space $(\ker T)^{\perp} = (\ker P)^{\perp}$ and final space $\operatorname{ran} T$. Of course, $(\ker T)^\perp$ is a reducing subspace for P and the above invertibility implies that $P|_{(\ker T)^\perp}$ is invertible. Thus, the operator $\widetilde{P} := P \oplus 1_{\ker T}$ is invertible in $\mathcal{L}(\mathcal{H})$. By hypothesis, $c(T) = k(T)(= \dim(\operatorname{ran} T)^\perp)$, so there exists a partial isometry $\widetilde{U} \in \mathcal{L}(\mathcal{H})$ with initial space $\ker T$ and final space $(\operatorname{ran} T)^\perp$. It is clear that $(U+\widetilde{U})$ is a unitary operator in $\mathcal{L}(\mathcal{H})$ and $(U+\widetilde{U})^{-1}T\widetilde{P}^{-1} = (U^*+\widetilde{U}^*)U = 0_{\ker T} \oplus 1_{(\ker T)^\perp}$. Thus T is equivalent to a projection, and case (i) is established.

Case (**ii**): since $\ker T$ is finite dimensional, there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$ for $\mathcal{H}$ such that $\ker T = \bigvee\{e_0, \cdots, e_r\}$ and $(\ker T)^\perp = \bigvee_{n=1}^{\infty} = \{e_{n+r}\}$. Now write $(\ker T)^\perp = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\dim \mathcal{M}_1 = c(T)$ and $\dim \mathcal{M}_2 = r(T)$ (this can be done since $\dim(\ker T)^\perp = \aleph_0$). Define $Z \in \mathcal{L}(\mathcal{H})$ such that Z maps $\ker T^*$ onto $\mathcal{M}_1 \oplus \ker T$ isometrically and Z maps $\operatorname{ran} T$ onto $\mathcal{M}_2$ isometrically. Clearly Z is a unitary operator. Observe that $\ker ZT = \ker T$ and that $ZT((\ker T)^\perp) = \mathcal{M}_2 \subset (\ker T)^\perp$. In other words, $(\ker T)^\perp$ is a reducing subspace for ZT and $ZT|_{(\ker T)^\perp}$ is bounded below. Consequently, if we write the polar decomposition of ZT as $ZT = VQ$, then $Q = 0_{\ker T} \oplus Q_1$, where $Q_1 \in \mathcal{L}((\ker T)^\perp)$ is invertible and $V = 0_{\ker T} \oplus V_1$, where $V_1 \in \mathcal{L}((\ker T)^\perp)$ is an isometry. The operator $\widetilde{Q} := Q_1 \oplus 1_{\ker T}$ is invertible, and thus $ZT\widetilde{Q}^{-1} = 0_{\ker T} \oplus V_1$. Thus we obtain that $T \overset{e}{\sim} 0_{\ker T} \oplus V_1$. Now by von Neumans theorem, one may write $V_1 = W \oplus S$ where S is a unilateral shift with multiplicity $c(T) - k(T)$ since $\dim\ker V_1^* = c(T) - k(T)$ and W is a unitary operator. Obviously $0_{\ker T} \oplus V_1 \overset{e}{\sim} 0_{\ker T} \oplus 1_{\operatorname{range} W} \oplus S$. By Proposition 5.3 and the transitivity of $\overset{e}{\sim}$ we conclude that $T \overset{e}{\sim} 0_{\ker T} \oplus S$, which establishes case (**ii**).

Case (**iii**) follows from case (ii) by taking adjoints, so we say no more about it.□

The use we will make of Corollary 5.2 is conveyed by this next elementary proposition.

Proposition 5.4. *Suppose* A, B, B', $C \in \mathcal{L}(\mathcal{H})$ *and* $B \overset{e}{\sim} B'$ *via invertible operators*

X and Y satisfying $B' = XBY^{-1}$. Then the operator

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathcal{L}(\mathcal{H})$$

is similar to

$$T' = \begin{bmatrix} XAX^{-1} & B' \\ 0 & YCY^{-1} \end{bmatrix}.$$

Proof. This is an immediate consequence of the matricial calculation $T' = (X \oplus Y)T(X^{-1} \oplus Y^{-1})$. □

Using these results, we are able to move closer to a canonical form for the (BCP)-operators appearing in Theorem 3.22.

Theorem 5.5. *Suppose $0 < \theta < 1$ and T is a C_{00},* (BCP)*-operator in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T) = \sigma_{le}(T) = \mathbf{A}_\theta$ and such that $\|T^{-1}\| = 1/\theta$. Then T is unitarily equivalent to an operator matrix $\widetilde{T} \in \mathcal{L}(\mathcal{H}^{(3)})$ of the form*

$$\widetilde{T} = \begin{bmatrix} V_1Q_1 & V_2Q_2 & \widetilde{T}_{13} \\ 0 & 0 & Q_3V_3^* \\ 0 & 0 & Q_4V_4^* \end{bmatrix} \tag{98}$$

where

(**a**) *Q_i is a positive definite invertible operator such that $\sigma(Q_i) \subset [\theta, 1]$, $i = 1, \cdots, 4$,*

(**b**) *V_i is an isometry, $i = 1, \cdots, 4$, and*

(**c**) $\operatorname{ran} V_1 \cap \operatorname{ran} V_2 = (0)$, $\operatorname{ran} V_3 \cap \operatorname{ran} V_4 = (0)$.

Proof. By Theorem 3.1, we know that T is unitarily equivalent to an operator matrix $\widetilde{T}$ in $\mathcal{L}(\mathcal{H}^{(3)})$ of the form

$$\widetilde{T} = \begin{bmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} & \widetilde{T}_{13} \\ 0 & 0 & \widetilde{T}_{23} \\ 0 & 0 & \widetilde{T}_{33} \end{bmatrix}.$$

Moreover, since $\|T^{-1}\| = \|(T^*)^{-1}\| = 1/\theta$, we get immediately by consideration of vectors of the form $x \oplus 0 \oplus 0$ and $0 \oplus y \oplus 0$ that $\widetilde{T}_{11}$, $\widetilde{T}_{12}$, $\widetilde{T}_{23}^*$, and $\widetilde{T}_{33}^*$ are bounded below exactly by θ. Now write the polar decompositions

$$\begin{aligned} \widetilde{T}_{11} &= V_1 Q_1, \\ \widetilde{T}_{12} &= V_2 Q_2, \\ \widetilde{T}_{23}^* &= V_3 Q_3, \\ \widetilde{T}_{33}^* &= V_4 Q_4, \end{aligned}$$

where of course, $Q_1, \cdots, Q_4$ must be invertible positive definite operators, and since the lower bound of each Q_i, $i = 1, \cdots, 4$, is exactly θ, we have (**a**). Furthermore, it is clear that V_1, V_2, V_3^*, and V_4^* must be isometries, which gives (**b**). Finally, suppose that $0 \neq x \in \operatorname{ran} V_1 \cap \operatorname{ran} V_2$. Then since $\operatorname{ran} V_1 = \operatorname{ran} \widetilde{T}_{11}$ and $\operatorname{ran} V_2 = \operatorname{ran} \widetilde{T}_{12}$ there exist vectors y and z in $\mathcal{H}$ such that $\widetilde{T}_{11} y = \widetilde{T}_{12} z = x$, and an easy calculation shows that $\widetilde{T}(y \oplus -z \oplus 0) = 0$, which is impossible since $\widetilde{T}$ is invertible. Similarly, one shows that $\operatorname{ran} V_3 \cap \operatorname{ran} V_4 = (0)$, which gives (**c**). □

Recall that the angle $\Theta(\mathcal{M}, \mathcal{N})$ between two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is defined by

$$\cos \Theta(\mathcal{M}, \mathcal{N}) = \sup_{x \in \mathcal{M},\, y \in \mathcal{N}} \frac{|\langle x, y \rangle|}{\|x\| \, \|y\|}$$

and thus that $\Theta(\mathcal{M}, \mathcal{N}) > 0$ if and only if there do not exist sequences of unit vectors $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ and $\{y_n\}_{n \in \mathbb{N}} \subset \mathcal{N}$ such that $\langle x_n, y_n \rangle \to -1$.

Proposition 5.6. *Suppose T is as in the statement of Theorem 5.5, and $\widetilde{T}$ (which is* *unitarily equivalent to T) is given by* (98). *Then* $\Theta(\operatorname{ran} V_1, \operatorname{ran} V_2) > 0$ *and, similarly,* $\Theta(\operatorname{ran} V_3, \operatorname{ran} V_4) > 0$.

Proof. Suppose, to the contrary, that $\Theta(\operatorname{ran} V_1, \operatorname{ran} V_2) = 0$. Then, as mentioned above, there exist sequences of unit vectors $\{\widetilde{T}_{11} y_n\}_{n\in\mathbb{N}} \subset \operatorname{ran} \widetilde{T}_{11} = \operatorname{ran} V_1$ and $\{\widetilde{T}_{12} z_n\}_{n\in\mathbb{N}} \subset \operatorname{ran} \widetilde{T}_{12} = \operatorname{ran} V_2$ such that $\left\langle \widetilde{T}_{11} y_n, \widetilde{T}_{12} z_n \right\rangle \to -1$. Thus,

$$\widetilde{T}(y_n \oplus z_n \oplus 0) = (\widetilde{T}_{11} y_n + \widetilde{T}_{12} z_n) \oplus 0 \oplus 0,$$

and

$$\begin{aligned} ||\widetilde{T}(y_n \oplus z_n \oplus 0)^T||^2 &= ||\widetilde{T}_{11} y_n||^2 + ||\widetilde{T}_{12} z_n||^2 + 2 \operatorname{Re} \left\langle \widetilde{T}_{11} y_n, \widetilde{T}_{12} z_n \right\rangle \\ &= 2\left(1 + \operatorname{Re} \left\langle \widetilde{T}_{11} y_n, \widetilde{T}_{12} z_n \right\rangle\right) \to 0. \end{aligned}$$

But since T is invertible and bounded below by θ, we know that

$$\begin{aligned} ||\widetilde{T}(y_n \oplus z_n \oplus 0)^T||^2 &\geq \theta^2(||y_n||^2 + ||z_n||^2) \\ &\geq \theta^2(||\widetilde{T}_{11} y_n||^2 + ||\widetilde{T}_{12} z_n||^2) = 2\theta^2 > 0, \end{aligned}$$

since $\widetilde{T}_{11}$ and $\widetilde{T}_{12}$ are obviously contractions, which proves the first assumption, and the proof of the second is obtained by applying the above argument to $\widetilde{T}^*$. □

Corollary 5.7. *If T and $\widetilde{T}$ are as in the statement of Theorem 5.5, then T is similar to the operator matrix*

$$\begin{bmatrix} V_1 Q_1 & V_2 & \widetilde{T}_{13} \\ 0 & 0 & (Q_2^{-1} Q_3) V_3^* \\ 0 & 0 & Q_4 V_4^* \end{bmatrix}. \tag{99}$$

Proof. This follows immediately from Proposition 5.4. □

Remark 5.8. The interested reader will note the resemblance of the matrix in (99) to the matrices of the operators in the classes $(\mathcal{S}_\theta)$, $\theta > 0$. Since the matrix in (99) comes from that of an arbitrary (BCP)-operator $T \in C_{00}$ such that $\sigma(T) = \sigma_{le}(T) = \mathbf{A}_\theta$ and $||T^{-1}|| = 1/\theta$, the existence of Theorem 3.22 justifies the interest shown in the structure of the classes $(\mathcal{S}_\theta)$, $\theta > 0$, exhibited in Chapter IV.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* [21] (notation: $T \in (QT)$) if T can be written as $T = T_t + K$ where the matrix $(\tau_{ij})_{i,j\in\mathbb{N}}$ for T_t with respect to some ordered orthonormal basis for $\mathcal{H}$ is in the upper triangular form (i.e., $\tau_{ij} = 0$ whenever $i > j$). Moreover, if both $T \in (QT)$ and $T^* \in (QT)$, then T is called *biquasitriangular* (notation: $T \in (BQT)$).

The famous theorem of Apostol-Foias-Voiculescu from [2] characterizing biquasitriangular operators is the following

Theorem 5.9 (Apostol-Foias-Voiculescu). *An operator T in $\mathcal{L}(\mathcal{H})$ is biquasitriangular if and only if for every $\lambda \in \mathbb{C}$ such that $T - \lambda 1_\mathcal{H}$ is a semi-Fredholm operator, the Fredholm index $i(T - \lambda 1_\mathcal{H}) = 0$.*

Moreover, a consequence of the beautiful and deep Brown-Douglas-Fillmore Theory [13] is that $(BQT) \cap (EN) = (N + K)$, where (EN) denotes the family of all essentially normal operators in $\mathcal{L}(\mathcal{H})$ and $(N + K) = \{T \in \mathcal{L}(\mathcal{H}) : T$ can be written as $T = N + K$ for some normal operator N and $K \in \mathbb{K}\}$.

Since it is elementary linear algebra that every block-diagonal operator is biquasitriangular, it follows immediately that every quasidiagonal operator is also biquasitriangular, and thus one obtains the following well-known result.

Theorem 5.10. (Brown-Douglas-Fillmore). $(QD) \cap (EN) = (N + K)$.

This fact, together with Theorem 3.20, makes it of interest to explore the question: what can be said about the structure of (BCP)-operators in $(N + K)$?

This leads to the following easy result

Theorem 5.11. *Suppose* $T \in (\mathrm{BCP})(\mathcal{H}) \cap (N+K) \cap C_{00}$, $\theta \in [0,1)$ *and* $\sigma(T) = \sigma_e(T) = \mathbf{A}_\theta$. *Then there exist a normal*, (BCP)-*operator* $N \in C_{00}(\mathcal{H})$ *satisfying* $\sigma(N) = \sigma_{le}(N) = \mathbf{A}_\theta$ *and a* $K \in \mathbb{K}(\mathcal{H})$ *such that* $T = N + K$.

Proof. Recall first from [34] that operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are called *compalent* if there exist a unitary operator $U \in \mathcal{L}(\mathcal{H})$ and a $K \in \mathbb{K}(\mathcal{H})$ such that $UT_1U^* + K = T_2$, and recall also from the (BDF)-theory [13] that, since operators in $(N+K)$ cannot have spectral pictures containing a nonzero Fredholm index, a complete set of compalence invariants for operators in $(N+K)$ is the essential spectrum. Now let N_1 be any normal C_{00}-contraction such that $\sigma(N_1) = \sigma_e(N_1) = \mathbf{A}_\theta$ (so $N \in (\mathrm{BCP})$). (For example, one may take N_1 to be M_z (multiplication by z) on $L^2(\mathbf{A}_\theta, \mu)$ where μ is planar Lebesgue measure on $\mathbf{A}_\theta$.) Then $\sigma_e(T) = \mathbf{A}_\theta = \sigma_e(N_1)$, so T is compalent to N_1, and thus there exist a unitary U and a $K \in \mathbb{K}$ such that $T = UN_1U^* + K$. Upon defining $N = UN_1U^*$, we see that the proof is complete. □

CHAPTER VI

CONCLUSION

This dissertation is an outgrowth of a research project, initiated by Professors Ciprian Foias and Carl Pearcy, which is concerned with the reduction of questions regarding the hyperinvariant subspace lattice of an arbitrary nonalgebraic operator in $\mathcal{L}(\mathcal{H})$, to the corresponding questions about the class of (BCP)-operators.

Using techniques and results from Chapter III of this dissertation, as well as [17] and [8], clearly demonstrate that it is of considerable interest to determine as many structure theorems about the class of (BCP)-operators satisfying the conclusions of Theorem 3.22 when $\theta > 0$ as possible. Accordingly, in Chapter IV, we constructed a certain easily described subclasses of invertible (BCP)-operators, with a view of obtaining some insight into the problem of determining the structure of operators in these classes, which may prove to be very useful in resolving problems concerning hyperlattices of operators.

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