ON A SERIES INVOLVING EULER'S FUNCTION

An Undergraduate Research Scholars Thesis

by

WILLIAM FRENDREISS

Submitted to the LAUNCH: Undergraduate Research office at
Texas A&M University
in partial fulfillment of the requirements for the designation as an

UNDERGRADUATE RESEARCH SCHOLAR

Approved by
Faculty Research Advisor: Dr. Matthew P. Young

May 2023

Major: Applied Mathematics - Mathematics

RESEARCH COMPLIANCE CERTIFICATION

Research activities involving the use of human subjects, vertebrate animals, and/or biohazards must be reviewed and approved by the appropriate Texas A&M University regulatory research committee (i.e., IRB, IACUC, IBC) before the activity can commence. This requirement applies to activities conducted at Texas A&M and to activities conducted at non-Texas A&M facilities or institutions. In both cases, students are responsible for working with the relevant Texas A&M research compliance program to ensure and document that all Texas A&M compliance obligations are met before the study begins.

I, William Frendreiss, certify that all research compliance requirements related to this Undergraduate Research Scholars thesis have been addressed with my Faculty Research Advisor prior to the collection of any data used in this final thesis submission.

This project did not require approval from the Texas A&M University Research Compliance & Biosafety office.

TABLE OF CONTENTS

ABSTRACT

On a Series Involving Euler's Function

William Frendreiss
Department of Mathematics
Texas A&M University

Faculty Research Advisor: Dr. Matthew P. Young
Department of Mathematics
Texas A&M University

The goal of this thesis is to provide an in-depth analysis and discussion of an equivalence to the Riemann Hypothesis (RH) proven by Jean-Louis Nicolas. Nicolas' proof relates RH to an inequality of Euler's totient function φ, and establishes a number-theoretic equivalence to RH. If Nicolas' criterion holds for all primorial numbers, then RH is true. If not, then RH is false. This proof is given an original translation into English from French and annotated, with small corrections to computations and commentary when deemed necessary. His work is then extended by relating the equivalence to the convergence of an infinite series which is shown to converge to 1/2. Using this series and the related partial sum, consequences of the truth or falsehood of RH are explored in the context of Nicolas' criterion. We assume both the truth and falsehood of RH, and in doing so underscore the extreme difficulty of this problem as well as the delicacy of the inequalities involved. Also provided are multiple programs which computationally verify expectations regarding different quantities from the analytic results section. Optimization of these programs are discussed as well as difficulties. These programs produce plots of the behavior of consequential arithmetic-valued functions, which are included in Chapter 4.

The research results were limited by the nature of the problem. None of the analysis on the convergence criterion yielded a contradiction to an established result or conjecture, assuming either RH true or false. However, RH is known to be one of the most difficult problems in modern mathematics and significant progress was largely outside the scope of this thesis. The hope is that this research renews interest into Nicolas' criterion specifically and arithmetic inequalities equivalent to RH in general.

ACKNOWLEDGMENTS

Contributors

I would like to thank my faculty advisor Dr. Matthew P. Young for his guidance, support, and encouragement throughout the research process. I would also like to express gratitude to the entire mathematics department and Texas A&M University as a whole for giving me a home.

I would also like to thank all my friends and peers at the Mathematics Department for helping to make my college experience memorable and fruitful.

Finally, thank you to my mother and father for their support and aid in helping me pursue my dreams.

Funding Sources

I received no funding from outside sources, but was able to access library databases using A&M's access to such journals.

NOMENCLATURE

RH	Riemann Hypothesis
PNT	Prime Number Theorem
p_k	The kth prime number; $p_1 = 2,\ p_2 = 3,\ \ldots$
N_k	The kth primorial number; $N_1 = 2,\ N_2 = 6,\ \ldots$
$\log(x)$	The natural logarithm; $\log_e(x)$ where $e = 2.71828\ldots$ as usual
$\varphi(n)$	Euler's totient function, $\varphi(n) = \#\{1 \le k < n;\ \gcd(k, n) = 1\}$
γ	The Euler-Mascheroni constant $\gamma \approx 0.57722$
$\zeta(s)$	The Riemann Zeta-Function
$\Gamma(z)$	Gamma function, the analytic extension of the factorial; $\Gamma(z) = (z-1)!$
$\pi(x)$	Prime-counting function, $\pi(n) = \#\{0 < p \le n;\ p\,\text{prime}\}$
$\mathrm{li}(x)$	Logarithmic integral $\int_0^x \frac{dt}{\log t}$
$\Lambda(x)$	Von Mangoldt Function $\Lambda(n) = \log(p)$ if n is a prime power, 0 otherwise
$\theta(x)$	First Chebyshev function $\sum_{\substack{p \le n \\ p\,\text{prime}}} \log(p)$
$\psi(x)$	Second Chebyshev function, $\psi(x) := \sum_{n \le x} \Lambda(x)$
$\sigma(n)$	Divisor-sum function $\sigma(n) = \sum_{d \mid n} d$

1. INTRODUCTION

This thesis investigates the relationship between the Riemann Hypothesis and the primorial numbers through the lens of analytic number theory. We begin with a discussion of the Riemann Hypothesis and the genesis of analytic number theory.

1.1 The Riemann Hypothesis

Since the Middle Ages, the class of infinite series of the type

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

for an integer k were studied by mathematicians, mostly out of curiosity. The case $k = 1$, the so-called *harmonic series*, was proven to diverge (albeit slowly) by Nicole Oresme in the mid-14th century [KS06]. For $k > 1$ though, the series can be shown to converge by the integral test that most students learn in elementary calculus. Interest in was spread widely, and Pietro Mengoli posed the question of the value of the series at $k = 2$ in the middle of the 17th century [Ayo74]. Famously, Leonhard Euler proved in 1734 that the series converges to $\frac{\pi^2}{6}$ [Ayo74], solving the so-called "Basel Problem." Euler also famously related sums of this type to classical number theory directly, by proving the Euler Product Formula

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_{p\, prime} \left(1 - \frac{1}{p^k}\right)^{-1}. \tag{1}$$

1.1.1 Riemann's Memoir

In 1859, Bernhard Riemann published his groundbreaking paper on analytic number theory *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the number of primes less than a given quantity). The novelty in his paper was treating k in the infinite sum not necessarily as a real number, but as a complex number $s = \sigma + it$. Riemann, an accomplished geometer and contributor to the field of complex analysis, only wrote this one paper on number theory. Nonetheless, it has become one of the most influential and famous manuscripts in the field and opened up countless avenues of study within the field of analytic number theory.

Following Riemann's convention, define by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

the *Riemann Zeta Function* for the region $\text{Re}(s) = \sigma > 1$. Euler's product formula (1) still holds, and we may write

$$\zeta(s) = \prod_{p\, prime} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{3}$$

converging on the same region as the sum. Using his expertise in complex analysis, Riemann applied the principle of analytic continuation to meromorphically extend $\zeta(s)$ to $\mathbb{C} - \{1\}$. The functional equation describes the relation between the region $\sigma < 0$ and $\sigma > 1$:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-1/2(1-s)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{4}$$

[Dav00, p. 59] for $\sigma < 0$.

While not immediately obvious, (4) shows that $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$; these are the so-called *trivial zeros*. The region $0 < \sigma < 1$ is difficult to describe, but the symmetry in (4) shows that zeros of the function in this region are reflected about the line $\sigma = \frac{1}{2}$. Riemann, after showing that this region (called the *critical strip* by many modern sources) contains infinitely many zeros, made his famous conjecture based upon the recognition of this inherent symmetry:

Conjecture 1.1.1 (Riemann Hypothesis)**.** *The zeros of the Riemann Zeta Function with real part between* 0 *and 1 have real part* $\frac{1}{2}$.

Countless mathematicians in the early 20th century took major efforts to prove the Riemann Hypothesis (RH). However, advances in number theory still have not overcome it. While seemingly a simple conjecture, and one that intuitively makes sense, RH has truly proven to be one of the most difficult challenges in modern mathematics.

1.1.2 Connection to the Distribution of Primes

Riemann's paper became very famous for showing how the Riemann Zeta Function could be used to prove results about prime numbers analytically. In fact, it was used in the proof of the Prime Number Theorem (PNT) at the turn of the century:

Theorem 1.1.2 (Prime Number Theorem). *Let $\pi(x)$ denote the number of primes less than or equal to x. Then*

$$\pi(x) \sim li(x) \sim \frac{x}{\log x}, \tag{5}$$

where $li(x)$ is the logarithmic integral $\int_0^x \frac{dt}{\log t}$.

The PNT was conjectured by Gauss, and evaded proof for a hundred years; it was only until Charles de la Vallée Poussin and Jacques Hadamard indepently used the fact that $\zeta(s)$ has no zeros on the line $\sigma = 1$ to complete a proof.

While this advancement was very important, the true power of the Riemann Hypothesis is in its direct relationship with the distribution of prime numbers. We define several important arithmetic functions for our thesis. The *Von Mangoldt Function* is defined as

$$\Lambda(n) := \begin{cases} \log n & \text{if } n = p^k \text{ for some prime } p \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

[Dav00, p. 55]. Using this, we define the *first Chebyshev function* [RS62, p. 64] and *modified second Chebyshev function* [Dav00, p. 104] in the usual way as

$$\theta(x) := \log\left(\prod_{p\le x} p\right) = \sum_{p \le x} \log p \qquad \text{and} \tag{7}$$

$$\psi_0(x) := \begin{cases} \sum_{n\le x} \Lambda(n) - \frac{1}{2}\Lambda(x) & \text{if } x \text{ is a prime power;} \\ \sum_{n \le x} \Lambda(n) & \text{otherwise} \end{cases}, \tag{8}$$

respectively. It is important to note that Rosser and Schoenfeld developed a great deal of literature for these functions in their highly cited paper [RS62] that lists and proves a plethora of inequalities.

In his original 1859 paper [Rie59], Riemann put forth several conjectures as well as his functional equation. One of these ideas was an explicit connection between the distribution of primes and his new interpretation of $\zeta(s)$ as a function over the complex numbers; Specifically, he proposed that there was an explicit formula for $\pi(x) - \text{li}(x)$ in terms of the zeros of his zeta function. While his formulation was complex and will not be discussed here, it follows from a later

(1895) result due to Von Mangoldt which is much easier to write:

$$\psi_0(x) - x = -\sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - x^{-2}\right), \tag{9}$$

It is important to note that for practical purposes, a truncated version of (9) is often used in conjunction with a small error term. Specifically, we may stop counting the zeta zeros when we reach an imaginary part greater than a number T, writing

$$\psi_0(x) - x = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - x^{-2}\right) + R(x,T) \tag{10}$$

[Dav00, p. 109], where

$$R(x,T) = O\left(\frac{x\log^2(xT)}{T} + \log x\right)$$

The results in (9) (now known as the *exact explicit formula*) and (10) (the *approximate explicit formula*) directly connect the analytic properties of the Riemann Zeta Function to the distribution of prime numbers. The Riemann Hypothesis plays a key part in estimating the sum in (9). In fact, one can show through manipulation that RH gives the error term on the order of $O(\sqrt{x}\log x)$ [Dav00, p. 113] in the difference $\pi(x) - \text{li}(x)$. This is the best possible error term, meaning that if RH is true, we can adequately estimate the error term. It is for this reason that verifying the Riemann Hypothesis has become one of the most important ventures in modern mathematics, let alone number theory.

1.2 Equivalences

The extraordinary difficulty of the Riemann Hypothesis is apparent to anyone who has worked in analytic number theory. Initial attempts were made by Hardy, who showed that infinitely many zeros lie on the line [Har14] but failed to show that every zero does. A similar result by Selberg took a statistical route, showing that a positive proportion of the zeros lie on the line [Sel42]. Recent developments have improved this proportion, but overall no substantial progress has been made towards the Riemann Hypothesis.

As such, a major research question is that of equivalents to RH. Robin proved that RH is equivalent to

$$\sigma(n) < e^{\gamma} n \log\log n$$

being true for all $n \geq 5040$ [Rob84], where $\sigma(n)$ is the divisor-sum function. Lagarias altered Robin's criterion to add in a connection to the harmonic numbers, partial sums of the harmonic series. He showed that the above is equivalent to

$$\sigma(n) < H_n + e^{H_n} \log H_n$$

[Lag02], where H_n is the nth harmonic number $\sum_{1 \leq k \leq n} \frac{1}{k}$. These equivalences have sparked a great deal of literature, see Banks et al. [BHMN09], Briggs [Bri06], and Wójtowicz [W0́7] in particular. Many of the statements on these class of functions focus heavily on the proving the inequalities for various 'special' classes of numbers based on the property of 'abundancy' of divisors. Even the great Ramanujan studied so-called 'colossally abundant' numbers and studied their properties [Ram97] as related to the divisor-count function.

1.3 Nicolas' Criterion

The central object of this thesis is a similar result to Robin and Lagarias' results as discussed in the previous section, provided by Jean-Louis Nicolas in [Nic83]. His paper focuses on a special class of numbers with an optimal ratio of the well-known totient function. Using the notation of Nicolas, we define as follows:

Definition 1.3.1. Let p_k be the kth prime number, with $p_1 = 2$. We define the kth *primorial* number as

$$N_k := \prod_{1 \leq n \leq k} p_n.$$

Remark. Note that $\log N_k = \theta(p_k)$. This will be useful for proofs in Chapter 3, especially the proof of theorem 3.3.1.

The primorials are unique for their number of unique prime factors, and it is plain that they have a greater number of distinct prime factors than any integer less than it. Recall the definition of *Euler's totient function* as

$$\varphi(n) := \#\{0 < k < n | \operatorname{gcd}(k,n) = 1\}.$$

Now this function is multiplicative ($\varphi(mn) = \varphi(m) \cdot \varphi(n)$) if $\operatorname{gcd}(m,n) = 1$. Furthermore, by definition, $\varphi(p) = p - 1$ for all primes p. Since primes are clearly coprime to one another, iterative logic naturally implies that

$$\frac{N_k}{\varphi\left(N_k\right)} = \frac{\prod_{i=1}^{k} p_i}{\prod_{i=1}^{k}(p_i - 1)} = \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)^{-1}. \tag{11}$$

In 1874, Mertens [Mer74] proved his 3 famous theorems, the third of which is of particular importance to this thesis:

$$\lim_{n\to\infty} \log n \prod_{p \le n}\left(1 - \frac{1}{p}\right) = e^{-\gamma}. \tag{12}$$

In a similar result, Landau [Lan53] showed that

$$\liminf_{n\to\infty} \frac{\varphi(n) \log\log n}{n} = e^{-\gamma} \tag{13}$$

which of course can be reformulated as

$$\limsup_{n\to\infty} \frac{n}{\varphi(n) \log\log n} = \left(e^{-\gamma}\right)^{-1} = e^{\gamma}. \tag{14}$$

In view of (11), (12), and (14), one can see the historical interest in certain quantities involving a primorial integer and its totient. Critically, the totient function is relatively small at primorial numbers, and so these ratios are extremal in that case. This should provide the intuition for the main preliminary result of this thesis, dubbed **Nicolas' Criterion**:

Theorem 1.3.2 ([Nic83], Théorème 2)**.** *If the Riemann Hypothesis is true, then*

$$\frac{N_k}{\varphi\left(N_k\right)} > e^{\gamma} \log\log N_k \tag{15}$$

for all $k \in \mathbb{N}$*. If RH is false, then the inequality in* (15) *switches signs infinitely many times.*

Theorem 1.3.2 was proven in the early 1980s by Jean-Louis Nicolas, and has become a well-studied result in analytic number theory. Banks et al [BHMN09] provided a connection between Nicolas' criterion and Robin's criterion [Rob84] through numbers representable by a sum of two squares. Elsewhere, Akbary and Francis [AF20] equivocate Nicolas' criterion to an equivalence to the Generalized Riemann Hypothesis of a Dedekind Zeta Function of a class of cyclotomic fields. In a unique approach, Planat, Solé, and Omar [PSO11] connect Nicolas' criterion and the Riemann Hypothesis to properties of quantum systems. Namely, they use Nicolas' result in multiple proofs to show that RH is equivalent to a similar inequality incorporating temperature as a variable in the Bost-Connes quantum dynamical system. Even Nicolas himself later revisited his original paper [Nic83] and proved that

$$\limsup_{n\to\infty}\left(\frac{n}{\varphi(n)} - e^{\gamma}\log\log n\right)\sqrt{\log n} = e^{\gamma}(2+B),$$

where B is a constant based on a sum over the nontrivial zeta zeros [Nic12]; he also provided several more equivalents to the Riemann Hypothesis based on his computations regarding this limit.

In Chapter 2, we review and give notes on Nicolas' 1982 proof of Theorem 1.3.2. We provide a translation from the original French and give an expanded algebraic manipulation for heightened clarity. In Chapter 3 we provide original work and analysis of Nicolas' Criterion. We prove two results independent of Nicolas' Criterion, and then their properties are explored in the context of the truth or falsehood of the Riemann Hypothesis. Therein the difficulty of the Riemann Hypothesis as a problem is underscored. We end with a discussion of the design of the algorithms and computational difficulties in Chapter 4 before moving on to the conclusion. We also provide original plots of computations performed.

2. NICOLAS' PROOF

We provide here an English translation and rewriting of Nicolas' proof using the original complex analysis techniques. Where we believe there are small errors, we have added notes.

Jean-Louis Nicolas' original paper [Nic83] was written in French and was translated into English for the purposes of this thesis. The author used preexisting knowledge of the language, mathematical nomenclature similarities, and context clues to piece together most of the tranlation. However, where small checks were necessary, Google Translate was used for an occasional unknown word or phrase.

2.1 Definitions

We define the following, which we will use frequently throughout the proof. The definitions for p_k and N_k were already given. The notation used below is the same as used by Nicolas in his original proof for ease of reference.

$$f(x) := e^{\gamma} \log(\theta(x)) \prod_{p \le x} \left(1 - \frac{1}{p}\right)$$

$$S(x) := \theta(x) - x$$

$$R(x) := \psi(x) - x$$

$$K(x) := \int_x^{\infty} \frac{S(t)}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt$$

$$J(x) := \int_x^{\infty} \frac{R(t)}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt$$

$$F_{\rho}(x) := \int_x^{\infty} t^{\rho-2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt$$

Remark. We know from the Prime Number Theorem that $S(t) = O(t/\log t)$ and this ensures the convergence of the integral defining $K(x)$. BY similar logic we know $J(x)$ converges.

2.2 Estimating $f(x)$

Lemma 2.2.1. *For all $x \ge 121$, we have*

$$K(x) - \frac{S^2(x)}{x^2 \log x} \le \log f(x) \le K(x) + \frac{1}{2(x-1)}. \tag{16}$$

Proof. From [RS62, Theorem 4 & Theorem 18], $\theta(s) \geq 4x/5$ for $x \geq 121$. Additionally,

$$-\left(\frac{d^2}{dt^2}\right)(\log\log t) = \frac{1+\log t}{t^2\log^2 t}$$

is a decreasing function in t for $t > 1$, and $t \mapsto t\frac{t+1-a}{(t-a)^2}$ is decreasing for $t > a > 0$. Using $t = \frac{4x}{5}$, $a = \log\frac{5}{4}$ and $x \geq 121$ in the negative second derivative,

$$\begin{aligned}\frac{\log(4x/5)+1}{(4x/5)^2\log^2(4x/5)} &\leq \frac{\log(x/\sqrt{2})+1}{(x/\sqrt{2})^2\log^2(x/\sqrt{2})}\\ &= \frac{2(\log(x/\sqrt{2})+1)}{x^2(\log x - 1/2\log(2))^2}\\ &< \frac{2\log x}{x^2(\log x)^2} = \frac{2}{x^2\log x}.\end{aligned}$$

Using Taylor's Theorem on $\log\log(\theta(x)-x)$ about x,

$$\begin{aligned}\log\log(\theta(x)) &= \log\log x + \frac{1}{x\log x}(\theta(x)-x) + \frac{-\log x - 1}{x^2\log^2 x}(\theta(x)-x)^2 + \cdots\\ &= \log\log x + \frac{1}{x\log x}S(x) - \frac{1}{x^2\log x}S^2(x) - \frac{1}{x^2\log^2 x}S^2(x) \pm \cdots.\end{aligned}$$

By truncating the alternating series, we see that for $x \geq 3 > e$,

$$\log\log(\theta(x)) \leq \log\log x + \frac{S(x)}{x\log x}, \tag{17}$$

as well as

$$\log\log(\theta(x)) \geq \log\log x + \frac{S(x)}{x\log x} - \frac{1}{x^2\log x}S^2(x) \tag{18}$$

when the second derivative is positive and decreasing on $x \geq 121$. It follows that

$$\sum_{p\leq x}\frac{1}{p} = \int_{2^-}^{x}\frac{d(\theta(t))}{t\log t}\,dt = \frac{\theta(x)}{x\log x} + \int_2^x \frac{\theta(t)(\log t+1)}{t^2\log^2 t}\,dt$$

via the definition of $\theta(x)$ and then integration by parts. Using $\theta(x) = x + S(x)$,

$$\begin{aligned}\sum_{p\leq x}\frac{1}{p} &= \frac{\theta(x)-x+x}{x\log x} + \int_2^x \frac{(\theta(t)-t)(\log t+1)}{t^2\log^2 t} + t\frac{\log t+1}{t^2\log^2 t}\,dt\\ &= \frac{S(x)}{x\log x} + \frac{1}{\log x} + \int_2^x \frac{1}{t\log t}\,dt + \int_2^x \frac{1}{t\log^2 t}\,dt + \int_2^x \frac{S(t)(\log t+1)}{t^2\log^2 t}\,dt\\ &= \frac{S(x)}{x\log x} + \frac{1}{\log x} + (\log\log x - \log\log 2) + \left(-\frac{1}{\log x} + \frac{1}{\log 2}\right) + \int_2^x \frac{S(t)(\log t+1)}{t^2\log^2 t}\,dt\\ &= \frac{S(x)}{x\log x} + \log\log x - \log\log 2 + \frac{1}{\log 2} + \int_2^x \frac{S(t)(\log t+1)}{t^2\log^2 t}\,dt.\end{aligned}$$

Writing $B_1 := -\log\log 2 + \frac{1}{\log 2} + \int_2^\infty \frac{S(t)(\log t+1)}{t^2 \log^2 t}\, dt$,

$$\sum_{p\le x} \frac{1}{p} = \frac{S(x)}{x\log x} + \log\log x + B_1 - \int_x^\infty \frac{S(t)(\log t+1)}{t^2\log^2 t}\, dt = \frac{S(x)}{x\log x} + \log\log x + B_1 - K(x). \tag{19}$$

By comparing (19) to Mertens' Second Theorem [Mer74], $\sum_{p\le x}\frac{1}{p} = \log\log x + B_1 + o(1)$, we have

$$B_1 = \gamma + \sum_p \left(\log(1-1/p) + 1/p\right).$$

We now define

$$U(x) := \log\log\theta(x) + \sum_{p\le x}\left(-\frac{1}{p}\right) + B_1 \tag{20}$$

and

$$u(x) := \sum_{p\le x}\left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right) + \gamma - B_1 = \sum_{p>x} -\log\left(1-\frac{1}{p}\right) - \frac{1}{p}.$$

Notice that

$$U(x) + u(x) = \log\log(\theta(x)) + \gamma + \sum_{p\le x}\log\left(1-\frac{1}{p}\right) = \log f(x).$$

Bounding $u(x)$, we have

$$\begin{aligned}
u(x) &= \sum_{p>x} -\log\left(1-\frac{1}{p}\right) - \frac{1}{p} \\
&= \sum_{p>x}\left(\sum_{n=1}^\infty \frac{1}{np^n} - \frac{1}{p}\right) \\
&= \sum_{p>x}\sum_{n=2}^\infty \frac{1}{np^n} \\
&\le \sum_{p>x}\sum_{n=2}^\infty \frac{1}{2p^n} \\
&= \sum_{p>x}\frac{1}{2p^2}\cdot\frac{1}{1-1/p} = \sum_{p>x}\frac{1}{2p(p-1)} \\
&< \sum_{n>x}\frac{1}{2n(n-1)} < \frac{1}{2(x-1)}.
\end{aligned}$$

Since $U(x) = \frac{S(x)}{x\log x} + K(x)$ from (19), we have the second inequality in (16). □

2.3 If RH is True

Lemma 2.3.1. *Let ρ be a complex number with real part $\frac{1}{2}$. We have*

$$F_\rho(x) = -\frac{1}{\rho-1}\frac{x^{\rho-1}}{\log x} + r_\rho(x) \tag{21}$$

where

$$\left|r_\rho(x)\right| \leq \frac{5}{|\rho-1|\sqrt{x}\log^2(x)} \tag{22}$$

for $x > e^2$.

Proof. Integrating by parts on the definition of F_ρ,

$$\begin{aligned}
F_\rho(x) &= \int_x^\infty t^{\rho-2}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt \\
&= \frac{1}{\rho-1} t^{\rho-1}\frac{1+\log t}{\log^2(t)}\bigg|_x^\infty - \int_x^\infty t^{\rho-1}\frac{\frac{1}{t}\log^2 t - (1+\log t)\frac{2}{t}\log t}{\log^4(t)}\, dt \\
&= -\frac{1}{\rho-1}x^{\rho-1}\frac{1+\log x}{\log^2(x)} + \int_x^\infty t^{\rho-2}\left(\frac{1}{\log^2 t} + \frac{2}{\log^3 t}\right) dt \\
&= -\frac{1}{\rho-1}\frac{x^{\rho-1}}{\log(x)} \overbrace{- \frac{1}{\rho-1}\frac{x^{\rho-1}}{\log^2(x)} + \int_x^\infty t^{\rho-2}\left(\frac{1}{\log^2 t} + \frac{2}{\log^3 t}\right) dt}^{r_\rho(x)}\,.
\end{aligned}$$

For $t \geq x > e^2$, we have $\log^3 t \geq 2\log^2 x$ and

$$\left|r_\rho(x)\right| \leq \frac{1}{|\rho-1|\sqrt{x}\log^2 x} + \frac{2}{|\rho-1|\log^2 x}\int_x^\infty t^{-3/2}\, dt$$

by the triangle inequality for integrals. The integral on the right is $\frac{2}{\sqrt{x}}$, yielding the desired bound.

∎

Proposition 2.3.2. *Under the Riemann Hypothesis, we have for $x \geq 55 > e^4$,*

$$J(x) \leq \frac{0.1}{\sqrt{x}\log x}.$$

Proof. By integrating the exact explicit formula (9),

$$\psi_1(x) = \int_0^x \psi(t)\, dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'}{\zeta}(0) + \frac{\zeta'}{\zeta}(-1) - \sum_{i=1}^\infty \frac{x^{1-2i}}{2i(2i-1)}. \tag{23}$$

The series $\sum_\rho \frac{1}{|\rho(\rho+1)|}$ is convergent, say with sum A. Define, for all $t > 0$

$$g(t) := \frac{1}{t^2}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right).$$

This yields

$$\begin{aligned} g'(t) &= \frac{-2}{t^3}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) + \frac{1}{t^2}\left(\frac{-1}{\log^2 t} + \frac{-2}{\log^3 t}\right) \\ &= -\frac{1}{t^3}\left(\frac{2}{\log t} + \frac{3}{\log^2 t} + \frac{2}{\log^3 t}\right). \end{aligned}$$

The series $\sum_\rho g'(t)t^{\rho+1}/\rho(\rho+1)$ is absolutely convergent and under the Riemann Hypothesis, the partial sums are $O\left(|g'(t)|t^{3/2}\right)$ in absolute value, which is integrable on $[x,\infty]$ for $x > 1$.

Due to the absolute convergence,

$$\int_0^\infty \left(\sum_\rho g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\right) dt = \sum_\rho \int_0^\infty g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\,dt. \tag{24}$$

From (23), define

$$g_1(x) := x\frac{\zeta'}{\zeta}(0) - \frac{\zeta'}{\zeta}(-1) + \sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}.$$

For $x > 1$,

$$g_1'(x) = \log 2\pi + \frac{1}{2}\log(1 - 1/x^2),$$

and for $x \geq 2$, we have

$$0 \leq g_1'(x) \leq \log 2\pi. \tag{25}$$

The left side of (24) becomes

$$\int_x^\infty g'(t)\left(\frac{t^2}{2} - \psi_1(t) - g_1(t)\right) dt = g(x)\left(\psi_1(x) - \frac{x^2}{2} - g_1(x)\right) + J(x) + J_1(x),$$

via integration by parts ($g(t) \approx 1/t\log t$) and using

$$J_1(x) := \int_x^\infty g(t)g_1'(t)\,dt. \tag{26}$$

The right side of (24) becomes

$$\sum_\rho \left(-g(x)\frac{x^{\rho+1}}{\rho(\rho+1)} - \int_x^\infty\right)$$

via integration by parts, which can be simplified to

$$g(x)\left(\psi_1(x)-\frac{x^2}{2}+g_1(x)\right)-\sum_\rho \frac{1}{\rho}F_\rho(x).$$

Comparing both sides then yields

$$J(x)=-\sum_\rho \frac{1}{\rho}F_\rho(x)-J_1(x); \tag{27}$$

using (25) and (26), for $x\geq 2$,

$$0\leq J_1(x)\leq \log 2\pi\int_x^\infty d\left(\frac{1}{\log t}\right)=\frac{\log 2\pi}{x\log x}. \tag{28}$$

So, lemma 2.3.1 gives us

$$\begin{aligned}-\sum_\rho \frac{1}{\rho}F_\rho(x)&=\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)\log x}-\sum_\rho \frac{r_\rho(x)}{\rho}\\ &=\frac{1}{\sqrt{x}\log x}\sum_\rho \frac{x^{i\Im\rho}}{\rho(\rho-1)}-\sum_\rho \frac{r_\rho(x)}{\rho}.\end{aligned}$$

The first series is convergent;

$$\sum_\rho \frac{1}{\rho(1-\rho)}=2+\gamma-\log\pi-2\log 2\leq 0.047.$$

We obtain then, using the bound on $|r_\rho|$ from lemma 2.3.1, that

$$\left|\sum_\rho \frac{1}{\rho}F_\rho(x)\right|\leq \frac{0.047}{\sqrt{x}\log x}\left(1+\frac{5}{\log x}\right)\leq \frac{0.1}{\sqrt{x}\log x}$$

for all $x\geq e^4$. □

Note. We believe that in the last line, although Nicolas' claims the inequality holds for all $x\geq e^4$, that we may only obtain the desired bound for $x\geq 84.265>e^4$. However, in the proof of theorem 2.3.3 citing proposition 2.3.2, we only need $x\geq 121$, and so the argument is not substantially impacted.

Theorem 2.3.3. *If the Riemann Hypothesis is true, then*

$$\frac{N_k}{\varphi(N_k)}>e^\gamma\log\log N_k \tag{29}$$

for all $k\geq 1$.

Proof. For all $x\geq 121$,

$$\theta(x)\leq\psi(x)-0.98\sqrt{x}.$$

[RS62, eq. 3.37]. Note that this inequality can be easily obtained using a worse coefficient since $\psi(x) - \theta(x) \geq \theta(\sqrt{x})$. In actuality, we can replace 0.98 with 0.998 [RS75, p. 265].

So $S(x) \leq R(x) - 0.98\sqrt{x}$ for $x \geq 121$, and via lemma 2.2.1,

$$\begin{aligned}
\log f(x) &\leq K(x) + \frac{1}{2(x-1)} \\
&= \int_x^\infty \frac{S(t)}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt + \frac{1}{2(x-1)} \\
&\leq \int_x^\infty \frac{R(t) - 0.98\sqrt{t}}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt + \frac{1}{2(x-1)} \\
&= J(x) - 0.98 F_{1/2}(x) + \frac{1}{2(x-1)}. \qquad (30)
\end{aligned}$$

Via lemma 2.3.1,

$$F_{1/2}(x) = \frac{2}{\sqrt{x}\log x} + r_{1/2}(x),$$

and for $x \geq e^{10}$, $|r_{1/2}(x)| \leq 1/(\sqrt{x}\log x)$ which leads to

$$-0.98 F_{1/2}(x) + \frac{1}{2(x-1)} \leq -\frac{0.9}{\sqrt{x}\log x} \qquad (31)$$

for $x \geq 3000$.

Note. Here, Nicolas originally used e^8, which we have changed to e^{10} due to a perceived typographic error as explained in the last note regarding the end of proposition 2.3.2.

Via equation (30) and proposition 2.3.2,

$$\log f(x) \leq -\frac{0.8}{\sqrt{x}\log x} < 0.$$

Hence $f(x) < 1$ for large x, completing the proof when accounting for Rosser and Schoenfeld's calculations for low x. □

2.4 If RH is False

We will prove that $f(x)$ is not **always** greater than or equal to 1 above any given x_0. By rearranging in the definition of $f(x)$, this implies the other half of Nicolas' criterion, the first half coming from theorem 2.3.3. With (30) and (31), it suffices to show that $J(x)$ is not always positive. For this, we use Landau's lemma:

Lemma 2.4.1. *Let $h : [1, \infty] \to \mathbb{R}$ be a piecewise continuous function. Consider the Mellin Transform of h, defined by*

$$H(s) := \int_1^\infty \frac{h(x)}{x^s}\, dx.$$

If σ_0 is the abcissa of convergence of $H(s)$, then we know $H(s)$ is convergent for $\Re(s) > \sigma_0$. If there exists an $x_0 \geq 1$ such that $h(x)$ has constant sign for $x \geq x_0$, then $H(s)$ cannot be analytically continued to a neighborhood of $s = \sigma_0$.

Theorem 2.4.2. *$J(x)$ switches signs infinitely many times.*

Proof. Let $h(x)$ be $J(x)$ for $x \geq 2$, and $h(x) = 0$ for $x \leq 2$. For $\Re(s) > 1$,

$$G(s) = \int_2^\infty \frac{J(x)}{x^s}\, dx = \int_2^\infty \frac{1}{x^s} \int_x^\infty \frac{R(t)}{t^2} \cdot \frac{1 + \log t}{\log^2 t}\, dt\, dx.$$

Via the Prime Number Theorem, $R(t) = O\left(\frac{t}{\log t}\right)$; the integrals are absolutely convergent. Fubini's Theorem tells us that

$$\begin{aligned} G(s) &= \int_2^\infty \frac{R(t)}{t^2} \cdot \frac{1 + \log t}{\log^2 t} \int_2^t \frac{dx}{x^s}\, dt \\ &= \int_2^\infty \frac{R(t)}{t^2} \cdot \frac{1 + \log t}{\log^2 t} \cdot \left. \frac{x^{1-s}}{1-s} \right|_2^t dt \\ &= \frac{1}{1-s} \int_2^\infty \frac{R(t)}{t^{1+s}} \cdot \frac{1 + \log t}{\log^2 t}\, dt - \frac{2^{1-s}}{1-s} J(2) \\ &= \frac{1}{s-1} \left(2^{1-s} J(2) - \int_2^\infty \frac{R(t)}{t^{1+s}} \cdot \frac{1 + \log t}{\log^2 t}\, dt \right). \end{aligned} \tag{32}$$

At $s = 1$, the expression in the parentheses is $2^0 J(2) - J(2) = 0$. Via a classic result,

$$\int_2^\infty \frac{\psi(t)}{t^{s+1}}\, dt = -\frac{1}{s} \cdot \frac{\zeta'}{\zeta}(s)$$

for $\Re(s) > 1$. We deduce

$$\begin{aligned} \int_2^\infty \frac{R(t)}{t^{s+1}}\, dt = \int_2^\infty \frac{\psi(t) - t}{t^{s+1}}\, dt &= \int_2^\infty \frac{\psi(t)}{t^{s+1}}\, dt - \int_2^\infty \frac{1}{t^s}\, dt \\ &= \int_2^\infty \frac{\psi(t)}{t^{s+1}}\, dt - \int_1^\infty \frac{1}{t^s}\, dt + \int_1^2 \frac{1}{t^s}\, dt. \end{aligned}$$

The third integral (which Nicolas calls $E_1(s)$) is an entire function. For $\Re(s) > 1$,

$$\int_2^\infty \frac{R(t)}{t^{s+1}}\, dt = -\frac{1}{s} \cdot \frac{\zeta'}{\zeta}(s) - \int_1^\infty \frac{1}{t^s}\, dt + E_1(s). \tag{33}$$

We use the identity

$$(1 - 2^{1-s})\,\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

to see that ζ has no real zeros in the critical strip; otherwise the right side would be zero, which it is not, as it is always a convergent alternating series that is lower bounded by $1 - \frac{1}{2^s} > 0$ for $s > 0$. Since the first zero of ζ on the critical line has imaginary part $> 14 =: \delta$, we know that there are no zeros of ζ in the region

$$W = \{s : \Re(s) > 1\} \cup \{s : 0 < \Re(s) \leq 1,\ |\Im(s)| < \delta\},$$

which is plotted below:

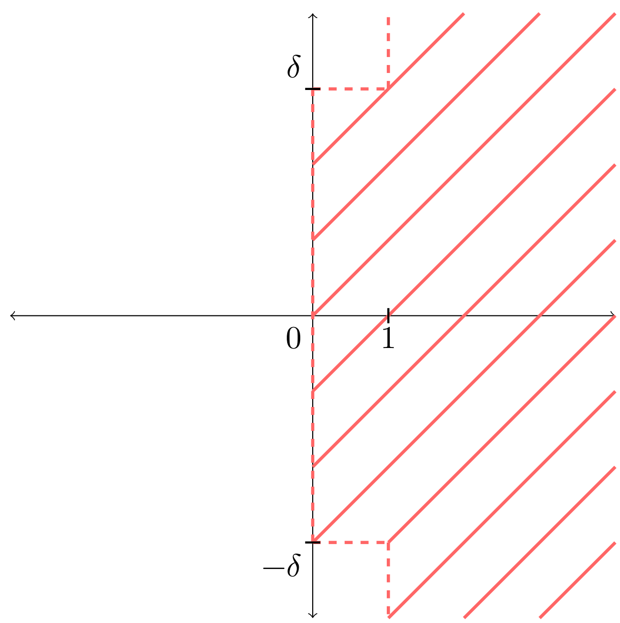

Figure 1: The region W.

The right side of (33) is holomorphic in a neighborhood of $s = 1$ and in W, and so admits a primitive $G_1(s)$ and second primitive $G_2(s)$.

We then have

$$\int_2^{\infty} \frac{R(t)}{t^{1+s}} \cdot \frac{1}{\log t}\, dt = -G_1(s) + \lambda$$

and

$$\int_2^\infty \frac{R(t)}{t^{1+s}} \cdot \frac{1}{\log^2 t}\,dt = G_2(s) + \lambda s + \mu$$

for $\Re(s) > 1$ where λ, μ are constants. Equation (32) also gives us that

$$G(s) = \frac{1}{s-1}\left(G_1(s) - G_2(s) + E_2(s)\right), \tag{34}$$

where $E_2(s)$ is an entire function in s. The function in parentheses is a holomorphic function in the region W, with a zero at $s = 1$ going back to equation (32). As such the whole function G can be holomorphically extended for $0 \leq s < 1$, as $G(s)$ is a Mellin transform that is analytic on a half-plane.

If $J(x)$ has a constant sign for $x > x_0$ sufficiently large, the contrapositive to Landau's Lemma tells us that the abcissa of comvergence is less than or equal to 0, and that $G(s)$ is able to be holomorphically extended to $\Re(s) > 0$.

But this cannot be; from (34) it follows that $G_1(s)-G_2(s)$ is holomorphic for the right half-plane, as is $G_1''(s) - G_2''(s)$. In a zero ρ of ζ with multiplicity m, by using the analytic continuation of the integral in (33) as $\frac{1}{s-1}$,

$$G_2''(s) \sim -\frac{\zeta'}{\zeta}(s) \sim -\frac{1}{\rho} \cdot \frac{m}{s-\rho}$$

and

$$G_1''(s) \sim G_2^{(3)}(s) \sim \frac{m}{\rho(s-\rho)^2},$$

which has a pole of order 2 at $s = \rho$. Hence, by contradiction, our assumption was false; no such x_0 may exist. □

Remark. We did not use the Riemann Hypothesis here, and having $f(x)$ switch signs infinitely many times does not contradict the results of the previous section. If RH is true, we proved that $f(x) < 1$ for all $x > 0$. What we proved in Theorem 2.4.2 was that $J(x)$ changed sign infinitely many times, and therefore by equation (30), $f(x)$ cannot be eternally ≥ 1, which is in total accordance with the previous section. In the coming theorem, we show that the falsehood of RH implies that it cannot either be eternally < 1.

Theorem 2.4.3. *If RH is false, for all* $0 < b < 1/2$, $J(x) = \Omega_{\pm}(x^{-b})$, *whence* $\limsup x^b J(x) > 0$ *and* $\liminf x^b J(x) < 0$.

Proof. Suppose that there is a zero of the Riemann zeta function $\rho = \beta + i\gamma$, where $\beta > 1 - b$ and b satisfies $1 - \Theta < b < 1/2$.

We will show that $J(x) \pm x^{-b}$ does not have a constant sign as $x \to \infty$. For this, we calculate the Mellin Transform

$$\int_2^{\infty} \frac{J(x) - x^{-b}}{x^s}\, dx = G(s) - \frac{1}{s - 1 + b} + E_3(s)$$

where $E_3(s)$ is an entire function, as it is a definite integral between 1 and 2 dependent on s as an exponent. The right side is a holomorphic function in $W \cap \{s;\, \Re(s) > 1 - b\}$. If $J(x) - x^{-b}$ has constant sign, the abcissa of convergence of the Mellin transform is $\leq 1 - b < \beta$, which is impossible because $G(s)$ has a singularity at $\beta + i\gamma$. The proof for $J(x) + x^{-b}$ is very similar and therefore omitted. □

2.5 Proof of Theorem 1.3.2

Proof. By Prop 2.2.1, and (30) and (31), for $x \geq 3000$, we have $\log f(x) \leq J(x)$, and Lemma 2.3.1 gives us that $\liminf x^b \log f(x) < 0$.

It remains to prove that $\log f(x) = \Omega_{+}(x^{-b})$. For this, consider (16). We will study $K(x) - S^2(x)/x^2 \log x$. Consider [RS62, Theorem 13], that

$$\psi(x) - \theta(x) < 1.42620\sqrt{x}$$

for all $x > 0$. It follows that

$$J(x) - \frac{3}{2}F_{1/2}(x) \leq K(x) \leq J(x).$$

The estimation of $F_{1/2}$ provided by lemma 2.3.1 and theorem 2.4.2 indicates that $K(x) = \Omega_{\pm}(x^{-b})$ for all b such that $1 - \Omega < b < 1/2$.

Consider now the function $y(x) = K(x) - x^{-b}$. It is differentiable for all $x \neq p$ where p is prime, and we have

$$y'(x) = -\frac{S(x)}{x^2}\left(\frac{1}{\log x} + \frac{1}{\log^2 x}\right) + \frac{b}{x^{b+1}}.$$

This derivative can change signs at $x = p$, and the study of the equation $(x - a)(\log x + 1) + bx^{1-b} \log^2 x = 0$ (which has no roots in $[1, +\infty)$) for $a > 1$ yields that $y'(x)$ vanishes once in the interval $[p, p']$ where p and p' are consecutive primes greater than 2.

The set of x for which the sign of y' changes is countable; index it as a sequence $\{x_j\}_j$. If RH is false, $S(x) = \Omega_{\pm}\left(x^{(\Theta+1-b)/2}\right)$ via [Ing64, Chapter V], so the sequence is infinite.

At a point x_i, y' changes sign and so also

$$S(x) - bx^{1-b}\frac{\log^2 x}{\log x + 1}.$$

This means that the quantity is either 0 or $\log x_i$. In any case, $S(x_i) = O\left(x_i^{1-b} \log x_i\right)$ and $\frac{S^2(x_i)}{x_i^2 \log x_i} = O\left(\frac{\log x_i}{x_i^{2b}}\right)$. Now there are infinitely many values of i such that $y(x_i) > 0$. Effectively, if the function y were to be negative at all its local extrema, it would be negative everywhere, and for all x, $K(x) \leq x^{-b}$ and we see that $K(x) = \Omega_+(x^{b'})$, where $b' < b$.

This implies that $\log f(x) = \Omega_+(x^{-b})$, as desired. $\square$

3. ANALYSIS

3.1 Preliminaries

The following original lemma will form a basis for later work by transferring a question about a single primorial number into a relation between subsequent prime numbers.

Lemma 3.1.1. *Let N_k be the kth primorial number, and p_k the kth prime. Recall Euler's totient function $\varphi(n)$ which counts the natural numbers under n coprime to n. Then*

$$\frac{1}{2} = \frac{\varphi\left(N_{m+1}\right)}{N_{m+1}} + \sum_{k=1}^{m} \frac{\varphi\left(N_k\right)}{N_{k+1}} \tag{35}$$

for all $m \in \mathbb{N}$.

Proof. We induct on m. For $m = 1$, indeed

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} = \frac{(3-1)(2-1)}{2 \cdot 3} + \frac{1}{2 \cdot 3} = \frac{\varphi\left(N_2\right)}{N_2} + \sum_{k=1}^{1} \frac{\varphi\left(N_k\right)}{N_{k+1}}.$$

So suppose (35) holds for m. Then

$$\begin{aligned}
\frac{1}{2} &= \frac{\varphi\left(N_{m+1}\right)}{N_{m+1}} + \sum_{k=1}^{m} \frac{\varphi\left(N_k\right)}{N_{k+1}} \\
&= \frac{\varphi\left(N_{m+1}\right)}{N_{m+1}} \left(\frac{p_{m+2}-1}{p_{m+2}} + \frac{1}{p_{m+2}}\right) + \sum_{k=1}^{m} \frac{\varphi\left(N_k\right)}{N_{k+1}} \\
&= \frac{\varphi\left(N_{m+1}\right)}{N_{m+1}} \left(\frac{\varphi\left(p_{m+2}\right)}{p_{m+2}} + \frac{1}{p_{m+2}}\right) + \sum_{k=1}^{m} \frac{\varphi\left(N_k\right)}{N_{k+1}}.
\end{aligned}$$

Since N_{m+1} and p_{m+2} are coprime by definition, the totient function is multiplicative, and

$$\begin{aligned}
\frac{1}{2} &= \frac{\varphi\left(N_{m+2}\right)}{N_{m+2}} + \frac{\varphi\left(N_{m+1}\right)}{N_{m+2}} + \sum_{k=1}^{m} \frac{\varphi\left(N_k\right)}{N_{k+1}} \\
&= \frac{\varphi\left(N_{m+2}\right)}{N_{m+2}} + \sum_{k=1}^{m+1} \frac{\varphi\left(N_k\right)}{N_{k+1}},
\end{aligned}$$

as desired. □

Corollary 3.1.2.

$$\frac{1}{2} = \sum_{k=1}^{\infty} \frac{\varphi\left(N_k\right)}{N_{k+1}} \tag{36}$$

Proof. Take $m \to \infty$ in lemma 3.1.1. The tail term $\varphi(N_m)/N_m$ tends to zero;

$$\lim_{m\to\infty} \varphi(N_m)/N_m = \prod_{\text{p prime}} \left(1 - \frac{1}{p}\right),$$

a product which diverges to 0 since the sum of prime reciprocals diverges. □

3.2 RH True

As Nicolas' Criterion describes the two cases of RH true versus RH false, we will consider both cases. In the case of RH true, our aim to is provide mathematical (and later computational) evidence for the Riemann Hypothesis using elementary computations and verification. In assuming RH false, we do not aim to provide a contradiction; rather, we strive to underscore the difficulty of the Riemann Hypothesis by showing how there is *no* contradiction to established results when applying Nicolas' Criterion to certain situations.

Regardless of the difficulty of the problem's concrete solution, a large portion of the mathematical community is of the belief that the Riemann Hypothesis is probably true. Whole books have been written assuming RH, and many known results were once formulated as consequences of RH; the evidence is quite strong. Our goal is to provide more in the section that follows.

Rearranging (15), under RH we have

$$e^{\gamma} \frac{\varphi(N_k)}{N_k} < \frac{1}{\log\log N_k}$$

for all $k \geq 2$ *. Dividing by p_{k+1} on both sides, we obtain

$$e^{\gamma} \frac{\varphi(N_k)}{N_{k+1}} < \frac{1}{p_{k+1} \log\log N_k}.$$

In view of corollary 3.1.2, we sum over k on btoh sides to obtain

$$\frac{e^{\gamma}}{3} < \sum_{k=2}^{\infty} \frac{1}{p_{k+1} \log\log N_k}. \tag{37}$$

It is important to recognize that verifying (37) is not logically equivalent to verification of Nicolas' Criterion and hence RH. While the truth of the Riemann Hypothesis would indeed imply that the above holds, the converse is not true. We expect low values of k to satisfy Nicolas' criterion with particular strength and be mostly responsible for overtaking the constant.

*The rearrangement does not hold for $k = 1$ as it does for (15) because $\log\log N_1 = \log\log 2 < 0$. Of course, multiplication by a negative necessitates reversal of an inequality.

Now note that we have said nothing of the convergence of the series on the right. The divergence of the series would surely supply the evidence we desire, but we now show that the series converges and so some computational checks are needed to verify (37).

Theorem 3.2.1. *The series on the right side of* (37) *converges.*

Proof. We will prove this using the Prime Number Theorem (theorem 1.1.2), which was proven independently of RH at the turn of the 20th century.

Via PNT, $\pi(x) \sim \operatorname{li}(x)$. We can integrate by parts and easily see

$$\begin{aligned}\int_2^x \frac{dt}{\log t} &= \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t} \\ &= \frac{x}{\log x} - \frac{2}{\log 2} + \frac{x}{\log^2 x} - \frac{2}{\log^2 2} + \int_2^x \frac{2}{\log^3 t}\,dt \\ &= \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).\end{aligned}$$

The kth prime occurs when $\pi(x) = k$ for the first time. By the Prime Number Theorem and the equation above, $k \sim p_k/\log p_k$, which is to say $p_k \sim k \log p_k$.

Since $\theta(x) \sim x$ [RS62, (2.3)], $\log\log N_k \sim \log p_k$. The series is then on the order of convergence of

$$\sum_{k=2}^{\infty} \frac{1}{(k+1)\log(p_{k+1})\log(p_k)}.$$

But $p_k > k$ for all $k \in \mathbb{N}$; this follows from the existence of 4 as the first composite number. As such, the series is bounded above by

$$\sum_{k=2}^{\infty} \frac{1}{k\log^2(k)},$$

which converges via the integral test:

$$\int_2^\infty \frac{dx}{x\log^2 x} = \int_{\log 2}^\infty \frac{du}{u^2} = \left.\frac{-1}{u}\right|_{u=\log 2}^{\infty} = \frac{1}{\log 2} < \infty.$$

Hence the series in (37) converges, albeit rather slowly. □

In the next chapter, we use this theorem to motivate discussion of this series and computations of partial sums for some specific values of k. While divergence of the series would trivially verify (37), we show that only very small values of k are needed for the partial sum on the right to overtake the constant on the left ($k \geq 12$ specifically).

3.3 RH False

If the Riemann Hypothesis is untrue (however unlikely that may be), theorem 1.3.2 implies that there are infinitely many k such that

$$\begin{aligned} \frac{\varphi\left(N_k\right)}{N_k} &< e^{-\gamma}\frac{1}{\log\log N_k}; \\ \frac{\varphi\left(N_{k+1}\right)}{N_{k+1}} &> e^{-\gamma}\frac{1}{\log\log N_{k+1}} \end{aligned} \tag{38}$$

(inequality goes from true-false) and *also* infinitely many ℓ such that

$$\begin{aligned} \frac{\varphi\left(N_\ell\right)}{N_\ell} &> e^{-\gamma}\frac{1}{\log\log N_\ell}; \\ \frac{\varphi\left(N_{\ell+1}\right)}{N_{\ell+1}} &< e^{-\gamma}\frac{1}{\log\log N_{\ell+1}} \end{aligned} \tag{39}$$

(inequality goes from false-true). Under lemma 3.1.1, we can replace the left side of these with some finite sums; (38) becomes

$$\begin{aligned} \frac{1}{2} - \sum_{n=1}^{k-1}\frac{\varphi(N_n)}{N_{n+1}} &< e^{-\gamma}\frac{1}{\log\log N_k}; \\ \frac{1}{2} - \sum_{n=1}^{k}\frac{\varphi(N_n)}{N_{n+1}} &> e^{-\gamma}\frac{1}{\log\log N_{k+1}}. \end{aligned}$$

We can condense this into one line, as the left side of the second line is always less than the left side of the first line (each summand is positive). This leads us to

$$e^{-\gamma}\frac{1}{\log\log N_{k+1}} < \frac{1}{2} - \sum_{n=1}^{k}\frac{\varphi(N_n)}{N_{n+1}} < \frac{1}{2} - \sum_{n=1}^{k-1}\frac{\varphi(N_n)}{N_{n+1}} < e^{-\gamma}\frac{1}{\log\log N_k}.$$

Consider any chain of inequalities $0 < a < b < c < d$ for $a, b, c, d \in \mathbb{R}$. In such a progression of positive real numbers, $c - b < d - a$. This implies

$$\frac{\varphi(N_k)}{N_{k+1}} < e^{-\gamma}\left(\frac{1}{\log\log N_k} - \frac{1}{\log\log N_{k+1}}\right) \tag{40}$$

Similarly, (39) becomes

$$\frac{\varphi(N_\ell)}{N_{\ell+1}} > e^{-\gamma}\left(\frac{1}{\log\log N_\ell} - \frac{1}{\log\log N_{\ell+1}}\right) \tag{41}$$

under Lemma 3.1.1 and a similar rearrangement process.

Equations (40) and (41) were derived in the hopes of showing that the falsehood of Nicolas' criterion may lead to a new arithmetic implication for the falsehood of RH. We derived the above equations by applying lemma 3.1.1 to the specific inequality switching points predicted by Nicolas' criterion.

We will now show that (unfortunately) neither (40) nor (41) contradicts the Prime Number Theorem, insofar as the current literature can demonstrate. Rearranging, we obtain

$$e^\gamma \frac{\varphi(N_k)}{N_k} < \frac{p_{k+1}}{\log\log N_k} - \frac{p_{k+1}}{\log\log N_{k+1}}$$
$$e^\gamma \frac{\varphi(N_\ell)}{N_\ell} > \frac{p_{\ell+1}}{\log\log N_\ell} - \frac{p_{\ell+1}}{\log\log N_{\ell+1}}.$$

From (11), this becomes

$$e^\gamma \prod_{n=1}^{k}\left(1-\frac{1}{p_n}\right) < \frac{p_{k+1}}{\log\log N_k} - \frac{p_{k+1}}{\log\log N_{k+1}}$$
$$e^\gamma \prod_{n=1}^{\ell}\left(1-\frac{1}{p_n}\right) > \frac{p_{\ell+1}}{\log\log N_\ell} - \frac{p_{\ell+1}}{\log\log N_{\ell+1}}.$$

Via (12) (Mertens' Third Theorem), the left side is asymptotic to $1/\log p_k$ and $1/\log p_\ell$. As such, the limiting behavior of the right side must also follow this trend; we can now forgo the definitions of k and ℓ as specific numbers such that (38) and (39) hold respectively, and consider the inequalities above in the context of all natural numbers:

Theorem 3.3.1. *Unconditionally, we have*

$$\lim_{k\to\infty} p_{k+1}\log p_k\left(\frac{1}{\log\log N_k} - \frac{1}{\log\log N_{k+1}}\right) = 1. \tag{42}$$

Remark. We provide code and plots illustrating this limit in the following chapter. Note that this theorem is not contingent upon either the truth or falsehood of RH. Ideally, one wishes to show that by assuming that RH is false, something strange happens. This theorem shows that one usually obtains things you would expect to see in regards to patterns in arithmetic functions.

Proof. Via (7),

$$\begin{aligned} p_{k+1}\log p_k\left(\frac{1}{\log\log N_k}-\frac{1}{\log\log N_{k+1}}\right) &= p_{k+1}\log p_k\left(\frac{1}{\log(\theta(p_k))}-\frac{1}{\log(\theta(p_{k+1}))}\right)\\ &= p_{k+1}\log p_k\frac{\log(\theta(p_{k+1}))-\log(\theta(p_k))}{\log(\theta(p_k))\log(\theta(p_{k+1}))}\\ &= \frac{p_{k+1}\log p_k}{\log(\theta(p_k))\log(\theta(p_{k+1}))}\log\left(1+\frac{\log p_{k+1}}{\theta(p_k)}\right). \end{aligned}$$

Via the Taylor Series expansion for $\log(1+x)$ about $x=0$, $\log(1+1/x)\approx 1/x$ as $x\to\infty$. The Prime Number Theorem implies that $\theta(p_k)\sim p_k$ and also that $p_k/\log p_k\sim k$ (see the proof of theorem 3.2.1). In the limit, we can combine these facts to observe

$$p_{k+1}\log p_k\left(\frac{1}{\log\log N_k}-\frac{1}{\log\log N_{k+1}}\right)\sim\frac{p_{k+1}}{\log(p_{k+1})}\log\left(1+\frac{\log p_{k+1}}{p_k}\right)\sim\frac{p_{k+1}}{p_k}.$$

A final application of a corollary of the prime number theorem yields the desired result. □

We now remark on the meaning of theorem 3.3.1 in terms of the distribution of prime numbers. Firstly, we already remarked on the independence of the theorem from RH. If indeed there are infinitely many k and ℓ such that (40) and (41) hold, neither one may contradict established results or even the Riemann Hypothesis itself. This implies that the process used to obtain these equations from (38) and (39) is not totally reversible; (38) implies (40) and (39) implies (41) but the reverse implications may not be true.

4. COMPUTATIONS

The goal of our computations was to verify, explain, and evidence phenomenon explained in the previous two chapters. We explain here our process for computation and plots design, and explore some behaviors we notice. We computed and plotted values of 3 arithmetic-valued functions based on the expressions in (37), (42), and (43).

4.1 Computational Strategy

All of our computations were done on a 2019 MacBook Air with 1.6 GHz Dual-Core Intel Core i5 processor. We used in Python 3.10.6 and the `matplotlib, sympy`, and `numpy` libraries. The `numpy` and `sympy` libraries were used to retrieve prime numbers and natural constants such as Euler's constant $\gamma \approx 0.577215665$. As for the prime numbers, the `sympy` function `prime(i)` yields the ith prime number with `prime(1)` $= 2$. Initially, we defined a `primorial(k)` function recursively, and then input it into our formulas as follows:

Algorithm 1 Initial Computational Strategy

```
N ← user input
while k ≤ N do
    if k == 1 then                                          ▷ base case for recursion
        primorial(k) = 2
        EulerPhiPrimorial(k) = 1
    else
        primorial(k) = primorial(k − 1) × prime(k)
        EulerPhiPrimorial(k) = EulerPhiPrimorial(k − 1) × (prime(k) - 1)
    end if

    loglog = np.log(np.log(primorial(k)))
    —perform desired computation depending on expression—
end while
```

Due to the large size of the primorial numbers, algorithm 1 worked extremely slowly and was unable to compute k values even as low as 100. The philosophy behind improving this is that while the primorial produces an extremely large value, we did not actually need to compute the primorial itself; rather, we needed to compute a rational expression that was 1) low in magnitude, and 2) changed little between different values of k. As such, we realized that our general strategy could be modified to accommodate much larger values of k.

We would define a new value called `OurQuotient(k)` or `our_sum(k)` that incrementally changed by multiplying by a small ratio or incrementally adding a small summand. Similarly, the value of `loglog` was changed by using logarithm rules; namely,

$$\log\log N_{k+1} = \log\left(\log N_k + \log p_{k+1}\right).$$

With this is mind, we developed a new computational strategy.

Algorithm 2 Later Computational Strategy

```
N ← user input
while k ≤ N do
    if k == 1 then                                    ▷ base case for recursion
        logp = np.log(2)
    else
        logp = logp + np.log(prime(k − 1))
        summand/factor = —perform desired computation based on expression—
    end if

    loglog = np.log(logp)
    new value = old value +/× summand/factor
end while
```

Using this process, we were able to compute for very high values of k in minutes. Given the machinery used, we do not believe that our computations needed to be or could be streamlined much more than this. For the purposes of plotting, we selected $N = 10,000$ as the upper limit.

The `matplotlib` library was used to seamlessly create plots of different functions and inequalities using large lists of data.

4.2 The Nicolas Criterion Inequality

Recall theorem 1.3.2, which states that the Riemann Hypothesis is equivalent to

$$\frac{N_k}{\varphi\left(N_k\right)} > e^{\gamma} \log\log N_k$$

holding for all $k \in \mathbb{N}$. Rearranging, the inequality becomes

$$e^{-\gamma} \frac{N_k}{\varphi\left(N_k\right) \log\log N_k} > 1. \tag{43}$$

Denote by $A(k)$ the expression on the left. Using the general process outlined by algorithm 2, we plotted this value for $k = 2 \ldots 10,000$, as well as the horizontal line $y = 1$ for comparison.

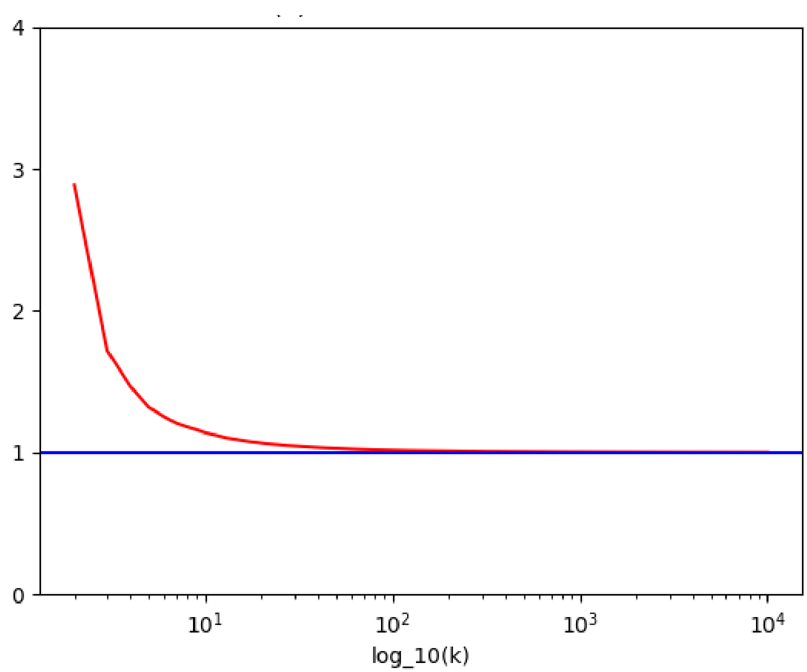

Figure 2: Plot of the Equation (43) Expression for $k = 1 \ldots 10^4$

Based on initial observations of this graph, the expression seems to be decreasing at a near-negative exponential rate; given the logarithmic scale, however, the true rate of decrease is much slower in k. Note that if the red line dips below the blue line, the inequality switches direction and the Riemann Hypothesis must be false. We do not observe this here and would be unlikely to observe it even for large values of k.

It is important to discuss monotonicity of this expression. Since the limit of $A(k)$ is 1 by equation (13), if the expression is indeed monotone, then this would verify Nicolas' criterion.

4.3 RH True (Equation (37))

Next, we turn to the infinite series from (37). We computed the partial sums for k up to $10,000$. We proved this series converged in theorem 3.2.1 and showed in (37) that if RH is true then the value of the series is greater than $\frac{e^{\gamma}}{3}$. Our purpose was to both verify the inequality in (37) itself (independent of assuming RH) as well as obtain a sense of the convergence rate of the series.

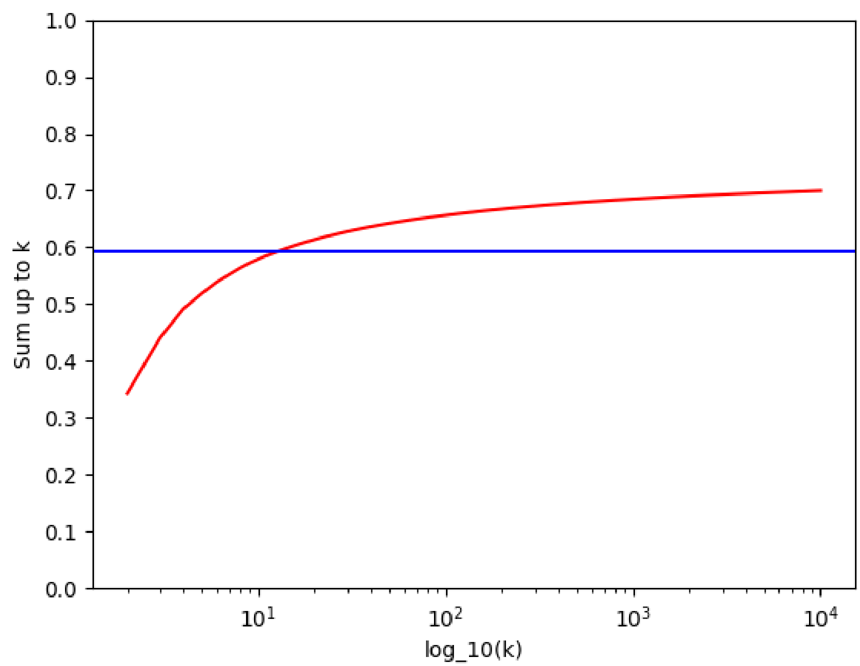

Figure 3: Plot of the Partial Sums in (37) for $k = 1 \ldots 10^4$

First, note that the series overtakes the required constant value at a low $k = 11$. This does nothing to prove the Riemann Hypothesis but does provide weak evidence for it.

The rate of convergence is extremely slow, and has a concave down curve pattern on a logarithmic scale. At $k = 10,000$, the partial sums reach ≈ 0.6997652.

4.4 RH False (Equation (42))

In section 3.3, we showed that we can use the falsehood of the Riemann Hypothesis to show

$$\lim_{k\to\infty} p_{k+1}\log p_k \left(\frac{1}{\log\log N_k} - \frac{1}{\log\log N_{k+1}}\right) = 1,$$

which is equation (42). We then proved that this result is actually independent of the Riemann Hypothesis. The proof relies on some analysis and repeated application of the Prime Number Theorem to transfer arithmetic functions into continuous growth rates.

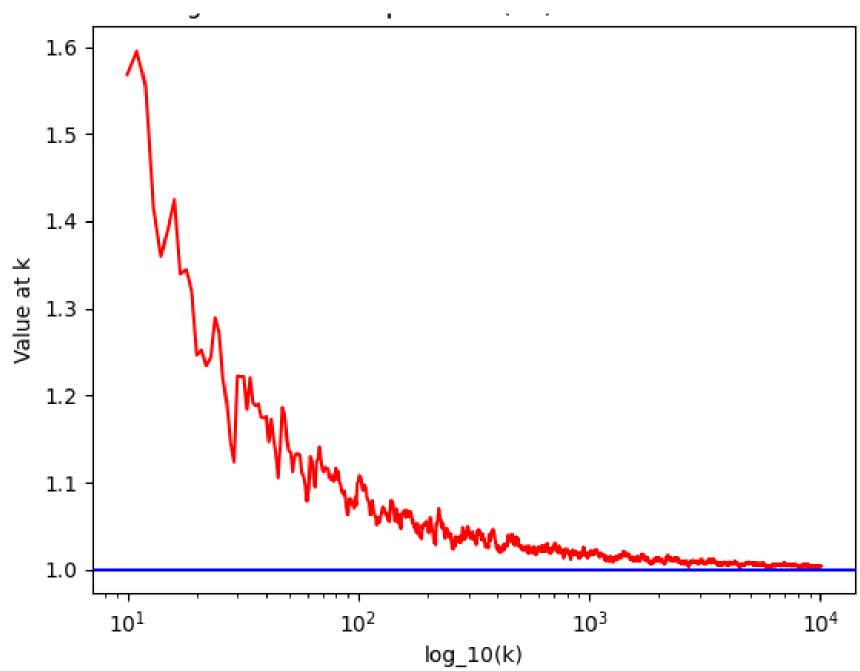

Figure 4: Plot of the Equation (42) Expression for $k = 1 \ldots 10^4$

The main observation here is the jagged nature of the graph. The expression is clearly not monotonic, but does converge to 1 as proven in theorem 3.3.1. We can see this behavior reflected as the red plot approaches the blue line as k increases. One thing to note is how the expression does not seem to dip below 1 for low values of k.

5. CONCLUSION

Studying the Riemann Zeta-Function gives the mathematical community insights into the most basic building blocks of integers, the prime numbers. The power and consequences of the Riemann Hypothesis have placed it among the most pressing unsolved problems in mathematics. Indeed, many would argue it is the *singular* most important unsolved problem. Studying the Riemann Hypothesis through incremental, complex analysis methods is the most common method, and Nicolas' proof is no different.

Our translation and expansion of Nicolas' original proof, while retaining much of the main notation, is aimed at adding clarity to Nicolas' work. Much of the literature surrounding the Riemann Hypothesis focuses on incremental improvements to the complex analytic properties of the zeta function, and this is reflected in Nicolas' original proof. Common techniques of the residue theorem and integral transform identities can be seen in equation (23) and lemma 2.4.1, respectively. In terms of errata and corrections, our review of Nicolas' work found no substantial issues, but did notice an issue in the last inequality of proposition 2.3.2; namely we believe the lower bound of x for which it holds is slightly higher.

This thesis presents a novel approach to the study of Nicolas' criterion for the Riemann Hypothesis through the use of lemma 3.1.1 and the corresponding corollary 3.1.2. In particular, corollary 3.1.2 seems to be a new identity and did not appear in the author's review of current literature. We then synthesized our original work in lemma 3.1.1 with Nicolas' Criterion in theorem 1.3.2. The analysis performed underscores the difficulty of analyzing the Riemann Hypothesis. In particular, in subsection 3.3, assuming the falsehood of RH allowed led us to equation (42). But we were able to prove this unconditionally in the proof of theorem 3.3.1. The precision of RH here is visited, in that even the tightest of bounds may not be contradicted by RH.

Our computations were performed with the intent of exploring patterns in arithmetic functions and identities related to Nicolas' proof. We discussed how we originally designed our algorithms, then how we were able to improve them when they failed to perform well within an allotted time frame. We eventually reduced our runtime to a few minutes with the cutoff of $k = 10,000$ for calculations. Our plots of these values showcase interesting trends in certain arithmetic expressions.

An important clarification is the difference between mathematics and the natural sciences in that rigorous, total proof is necessary for usage of a statement as fact. In other disciplines, it is common for proposed theories to be mainstream accepted and used in engineering, education, and research purposes. In mathematics, one need only look to the case of Skewes' Number to see why this does not hold. The quantity $\pi(x) - \mathrm{li}(x)$ is negative for all x up to very high values [Ske33], so it was conjectured that it must remain so. However, Littlewood proved that it actually switches signs infinitely many times [Lit14]; the lower bound for the first crossing was taken to be less than the gargantuan $10^{10^{10^{34}}}$ by Skewes [Ske33]. To this day, no one has found the first crossing from $\pi(x) - \mathrm{li}(x) < 0$ to $\pi(x) - \mathrm{li}(x) > 0$. The difference switches signs infinitely many times but the first time it does is so large that one can easily see why mathematicians believed it never happened at all.

It is in view of this that we view our own computations, performed only up to $k = 10,000$, as an exploratory and expository exercise. Intuition gained from observing figures 2, 3, and 4 is valuable for noticing patterns and formulating conjectures, but provides no rigorous proof of any unanswered questions.

REFERENCES

[AF20] Amir Akbary and Forrest J. Francis. Euler's function on products of primes in a fixed arithmetic progression. *Math. Comp.*, 89(322):993–1026, 2020.

[Ayo74] Raymond Ayoub. Euler and the zeta function. *Amer. Math. Monthly*, 81:1067–1086, 1974.

[BHMN09] William D. Banks, Derrick N. Hart, Pieter Moree, and C. Wesley Nevans. The Nicolas and Robin inequalities with sums of two squares. *Monatsh. Math.*, 157(4):303–322, 2009.

[Bri06] Keith Briggs. Abundant numbers and the Riemann hypothesis. *Experiment. Math.*, 15(2):251–256, 2006.

[Dav00] Harold Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by Hugh L. Montgomery.

[Har14] G. H. Hardy. Sur les zeros de la fonction $\zeta(s)$. *Comp. Rend. Acad. Sci.*, 158:1012–1014, 1914.

[Ing64] A. E. Ingham. *The distribution of prime numbers*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 30. Stechert-Hafner, Inc., New York, 1964.

[KS06] Steven J. Kifowit and Terra A. Stamps. The harmonic series diverges again and again. *AMATYC Review*, 27(2):31–43, 2006.

[Lag02] Jeffrey C. Lagarias. An elementary problem equivalent to the Riemann hypothesis. *Amer. Math. Monthly*, 109(6):534–543, 2002.

[Lan53] Edmund Landau. *Handbuch der Lehre von der Verteilung der Primzahlen. 2 Bände*. Chelsea Publishing Co., New York, 1953. 2d ed, With an appendix by Paul T. Bateman.

[Lit14] J. E. Littlewood. Sur la distribution des nombres premiers. *Comp. Rend. Acad. Sci*, 158:1869–1872, 1914.

[Mer74] Franz Mertens. Ein Beitrag zur analytischen Zahlentheorie. *J. Reine Angew. Math.*, 78:46–62, 1874.

[Nic83] Jean-Louis Nicolas. Petites valeurs de la fonction d'Euler. *J. Number Theory*, 17(3):375–388, 1983.

[Nic12] Jean-Louis Nicolas. Small values of the Euler function and the Riemann hypothesis. *Acta Arith.*, 155(3):311–321, 2012.

[PSO11] Michel Planat, Patrick Solé, and Sami Omar. Riemann hypothesis and quantum mechanics. *J. Phys. A*, 44(14):145203, 9, 2011.

[Ram97] Srinivasa Ramanujan. Highly composite numbers. *Ramanujan J.*, 1(2):119–153, 1997. Annotated and with a foreword by Jean-Louis Nicolas and Guy Robin.

[Rie59] Bernhard Riemann. On the number of prime numbers less than a given quantity. *Monatsberichte der Berliner Akademie*, 1859. Translated by David R. Wilkins.

[Rob84] G. Robin. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. Pures Appl. (9)*, 63(2):187–213, 1984.

[RS62] J. Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[RS75] J. Barkley Rosser and Lowell Schoenfeld. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. *Math. Comp.*, 29:243–269, 1975.

[Sel42] Atle Selberg. On the zeros of Riemann's zeta-function. *Skr. Norske Vid.-Akad. Oslo I*, 1942(10):59, 1942.

[Ske33] S. Skewes. On the difference $\pi(x) - \mathrm{li}\,(x)$ (I). *J. London Math. Soc.*, 8(4):277–283, 1933.

[Ẃ07] Marek Wójtowicz. Robin's inequality and the Riemann hypothesis. *Proc. Japan Acad. Ser. A Math. Sci.*, 83(4):47–49, 2007.

APPENDIX A: CODE

Computations for Section 4.2

Code A.1: Computing and Plotting (43)

```
import numpy as np
import matplotlib.pyplot as plt
import csv
from sympy import prime

def writeToTable(N): # creates a table with k in one column, and our quotient in the next column
    our_file = "NicolasCriterionTable.csv"
    with open(our_file, 'w') as file:
        writefile = csv.writer(file)
        for i in range(1, N + 1):
            if (i == 1):
                logp = np.log(2)
                loglog = np.log(logp)
                denominator = 1 / (1 * loglog * np.e ** (np.euler_gamma))
                ourQuotient = 2 * denominator
            else:
                logp = logp + np.log(prime(i))
                loglog = np.log(logp)
                old_denominator = 1 / denominator

                primes_fraction = (prime(i)) / (prime(i) - 1) # integers become too big to calculate fraction all at once- better to multiply separately
                denominator = 1 / (loglog * np.e ** (np.euler_gamma))

```

```
                ourQuotient = ourQuotient * primes_fraction * old_denominator
    * denominator

            writefile.writerow([i, ourQuotient])

    file.close()

def plotter(N):
    our_file = "NicolasCriterionTable.csv"
    x_data = [i for i in range(2, N + 1)]
    y_data = []

    with open(our_file, 'r') as file:
        readfile = csv.reader(file, delimiter=',')
        counter = 0
        for row in readfile:
            counter = counter + 1
            if (counter >= N + 1):
                exit
            else:
                y_data.append(float(row[1]))


    file.close()
    del y_data[0]

    plt.title(f"A(k) for k from 2 to {N}")
    plt.plot(x_data, y_data, color="r", marker="None")
    plt.axhline(y = 1, color = 'b', linestyle = '-')
    plt.ylabel("Nicolas Criterion Quotient of k")
    plt.xlabel("log_10(k)")
    plt.xscale("log")
    plt.yticks([0,1,2,3,4])
```

```
    plt.show()

# writeToTable(10000)
plotter(10000)
```

Computations for Section 4.3

Code A.2: Computing and Plotting (37)

```
import numpy as np
import matplotlib.pyplot as plt
import csv
from sympy import prime

constant  = np.e ** (np.euler_gamma) / 3

def infinite_summer(N): # creates a table with k in one column, and the sum of the series up to k in the next
    our_file = "NicolasCriterionSeries.csv"
    with open(our_file, 'w') as file:
        writefile = csv.writer(file)

        for i in range(2, N + 1):
            if (i == 2):
                our_sum = 0
                i = 2
                logp = np.log(6)
                loglog = np.log(logp)
                summand = 1 / (prime(i + 1) * loglog)



            else:
                oldloglog = loglog
```

```
                logp = logp + np.log(prime(i))
                loglog = np.log(logp)
                primfrac = prime(i - 1) / prime(i)

                summand = (summand * oldloglog * primfrac) / loglog

            our_sum = our_sum + summand
            writefile.writerow([i, our_sum])

    file.close()



def plotter(N):
    our_file = "NicolasCriterionSeries.csv"
    x_data = [i for i in range(2, N + 1)]
    y_data = []

    with open(our_file, 'r') as file:
        readfile = csv.reader(file, delimiter=',')
        counter = 1
        for row in readfile:
            counter = counter + 1
            if (counter >= N + 1):
                exit
            else:
                y_data.append(float(row[1]))


    file.close()
    # del y_data[0]

    plt.title(f"kth Partial Sum for k from 2 to {N}")
```

```
    plt.plot(x_data, y_data, color="r", marker="None")
    plt.axhline(y = constant, color = 'b', linestyle = '-')
    plt.ylabel("Sum up to k")
    plt.xlabel("log_10(k)")
    plt.xscale("log")
    plt.yticks([0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1])
    plt.show()

# infinite_summer(10000)
plotter(10000)
```

Computations for Section 4.4

Code A.3: Computing and Plotting (42)

```
import numpy as np
import matplotlib.pyplot as plt
import csv
from sympy import prime

constant  = np.e ** (np.euler_gamma) / 3

def limiter(N): # creates a table with k in one column, and the limit to be
    investigated in the next column
    our_file = "PrimeNumberLimit.csv"
    with open(our_file, 'w') as file:
        writefile = csv.writer(file)

        for i in range(1, N + 1):
            if (i == 1):
                outside = prime(2) * np.log(prime(1))
                logp = np.log(prime(1))
                loglog = np.log(np.log(prime(2)))
                oldloglog = np.log(np.log(prime(1)))
```

```
            else:
                outside = prime(i + 1) * np.log(prime(i))
                oldloglog = loglog
                logp = logp + np.log(prime(i + 1))
                loglog = np.log(logp)

            expression = outside * (1/oldloglog - 1/loglog)
            writefile.writerow([i, expression])

    file.close()



def plotter(N):
    our_file = "PrimeNumberLimit.csv"
    x_data = [i for i in range(10, N + 1)]
    y_data = []

    with open(our_file, 'r') as file:
        readfile = csv.reader(file, delimiter=',')
        counter = 0
        for row in readfile:
            counter = counter + 1
            if (counter >= N + 1 or counter < 10):
                exit
            else:
                y_data.append(float(row[1]))


    file.close()
    # del y_data[0]
```

```

    plt.title(f"Convergence of the Expression (41) for k from 10 to {N}")
    plt.plot(x_data, y_data, color="r", marker="None")
    plt.axhline(y = 1, color = 'b', linestyle = '-')
    plt.ylabel("Value at k")
    plt.xlabel("log_10(k)")
    plt.xscale("log")
    plt.yticks([1,1.1,1.2,1.3,1.4,1.5,1.6])
    plt.show()

# limiter(10000)
plotter(10000)
```

Compiling into a Single Table

Code A.4: Compiling all codes into table B.1

```
import csv

x = [i for i in range(1,10001)]
y = [x, [],["N/A"],[]]

def first_column():
    our_file = "NicolasCriterionTable.csv"
    with open(our_file, 'r') as file:
        readfile = csv.reader(file, delimiter=',')
        for row in readfile:
            y[1].append(row[1])

    file.close()

def second_column():
    our_file = "NicolasCriterionSeries.csv"
    with open(our_file, 'r') as file:
```

```
        readfile = csv.reader(file, delimiter=',')
        for row in readfile:
            y[2].append(row[1])

    file.close()

def third_column():
    our_file = "PrimeNumberLimit.csv"
    with open(our_file, 'r') as file:
        readfile = csv.reader(file, delimiter=',')
        for row in readfile:
            y[3].append(row[1])

    file.close()

def writer():
    our_file = "Combined_Table.csv"
    with open(our_file, 'w') as file:
        writefile = csv.writer(file)
        for i in range(10000):
            writefile.writerow([y[0][i], y[1][i], y[2][i], y[3][i]])

    file.close()

def main():
    first_column()
    second_column()
    third_column()
    writer()

main()
```

APPENDIX B: TABLES

For all of our computations, we computed large data sets up to $k = 10,000$. However, for the purposes of this thesis, the plots given suffice to show specific and general patterns. Special values of k are filtered and provided below on a semi-logarithmic scale.

Table B.1: Various Computed Ratios for Special Values of k

k	**Expression in** (43)	**Expression in** (37)	**Expression in** (42)
10	1.1402406262386682	0.5790491875667372	1.5684549205643685
20	1.0662325172364193	0.6135723330609899	1.2462282451157856
30	1.046765498419316	0.6280968654702005	1.222392650295349
40	1.0353257175752242	0.6364076023824977	1.175775916494441
50	1.0288911204706348	0.6421178223965565	1.1360580756034977
60	1.0248448189011035	0.6463779750602363	1.078503511363428
70	1.021771235035628	0.6497008437641283	1.1145636914343355
80	1.0192813330384016	0.6523889165883965	1.1163431627083857
90	1.017421265385679	0.6546453539672944	1.0627006115006792
100	1.016083538058004	0.656587025629315	1.1009558159263888
200	1.00916846163601	0.6675658453338982	1.0494203204650374
300	1.0067638839867616	0.6727777449579299	1.0480572589637984
400	1.0053872364632739	0.6760754871162583	1.0241228213810183
500	1.0045424989546716	0.678441150548779	1.0276821000815306
600	1.0040195921244364	0.6802652163312078	1.0253911748627176
700	1.0035750472287517	0.6817304362319196	1.0241276658385197
800	1.0032262675787578	0.6829489725706738	1.019850459114365
900	1.0029764083761066	0.6839889829885671	1.0136312343914724
1000	1.0027560178265813	0.6848904817240918	1.0173328009313347
2000	1.0016823077302714	0.690243400796409	1.011219149877978
3000	1.0012730263850966	0.6929756528131495	1.009521251234111
4000	1.001039476967349	0.694763395850174	1.0063679438360345
5000	1.000891444705131	0.6960731191121894	1.007839587005128
6000	1.0007884821706678	0.6970970934575601	1.003476206884238
7000	1.0007127067854389	0.6979324406965518	1.0065000672059805
8000	1.0006487941997149	0.698634356199228	1.0043296128855104
9000	1.0005987554810265	0.6992377455758454	1.0036854862163047
10000	1.0005637964079455	0.6997652012840407	1.003878619123719
∞ (expected/proven)	1	$\infty > x > e^\gamma/3$	1