Precalculus

by

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Chapter 0

Review of Algebra

One purpose of this Algebra Review Chapter is to support a "co-requisite" approach to teaching Precalculus. Our goal is to provide instructors with supplemental material linked to the main textbook that can be used to support students who have minor gaps in their pre-college mathematical backgrounds. To that end, we have written a collection of somewhat independent sections designed to review the concepts, skills, and vocabulary that we believe are prerequisite to a rigorous, college-level Precalculus course. This review is not designed to teach the material to students who have never seen it before so the presentation is more succinct and the exercise sets are shorter than those usually found in an Intermediate Algebra or high school Algebra II text.

0.1 Real Numbers and Exponents

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra.

0.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Much of the "real world" can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[1] definition of a real number is given below.

Definition 0.1. A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[2] the real numbers are *constructed* from some of these subsets.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$. The periods of ellipsis '$\ldots$' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.
2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]
4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ where } b \neq 0 \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers is: $\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$
5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$. That is, an irrational number is a real number, which isn't rational. Said differently, $\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$

[a] The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.

Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and because every rational number is a real number the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain:

[1] Math pun intended!

[2] See, for instance, Landau's Foundations of Analysis.

$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above.

There are two primary operations one can perform with real numbers: addition and multiplication. We'll start with the properties of addition.

Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a+b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a+b=b+a$.
- **Associativity:** For all real numbers a, b and c, $a+(b+c)=(a+b)+c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a+0=a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a+(-a)=0$.
- **Definition of Subtraction:** For all real numbers a and b, $a-b=a+(-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception, when defining scientific notation.

Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab=ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc)=(ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1=a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\dfrac{1}{a}$ such that $a\left(\dfrac{1}{a}\right)=1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \dfrac{a}{b} = a\left(\dfrac{1}{b}\right)$.

While most students and some faculty tend to skip over these properties or give them a cursory glance at best, it is important to realize that the properties stated above are what drive the symbolic manipulation in all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as

$(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1+2\cdot 3 = 1+6 = 7$, but $(1+2)\cdot 3 = 3\cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1+2)\cdot 3 = 1\cdot 3+2\cdot 3 = 3+6 = 9$. More generally, we have the following.

The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.
- **Factoring:** $ab+ac = a(b+c)$ and $ac+bc = (a+b)c$.

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), because the commutative property of multiplication gives us one from the other. Also, 'factoring' is really the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2+x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2+x) = 5\cdot 2+5\cdot x = 10+5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x+2x = 5x$? Because $3x+2x = (3+2)x = 5x$.

We continue our review with summaries of other properties of arithmetic. First up are properties of the additive identity 0.

Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both)
  Note: This not only says that $0\cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.
- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0\cdot\left(\dfrac{1}{a}\right) = 0$.
  Note: The quantity $\frac{a}{0}$ is undefined.[a]

[a] The expression $\frac{0}{0}$ is technically an 'indeterminant form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter 3.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2+x-6=0$ is by factoring[3] the left hand side of this equation to get $(x-2)(x+3)=0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x-2=0$ or $x+3=0$ so $x=2$ or $x=-3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 2.

Next up is a review of the arithmetic of 'negatives'. On page 3 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3+(-3)=0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5-3=5+(-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5-3$' and '-3' even though they are mathematically quite different.[4] In the expression '$5-3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3-3=-6$ but that will not work in your calculator. Instead you'd need to type $^{-}3-3$ to get -6 where the first dash comes from the '$+/-$' key and the second dash comes from the subtraction key.

Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a=(-1)a$ and $-(-a)=a$
- **Products of Negatives:** $(-a)(-b)=ab$.
- **Negatives and Products:** $-ab=-(ab)=(-a)b=a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b}=\dfrac{-a}{b}=\dfrac{a}{-b}$ and $\dfrac{-a}{-b}=\dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a+b)=-a-b$ and $-(a-b)=-a+b=b-a$.
- **'Factoring' Negatives:**$-a-b=-(a+b)$ and $b-a=-(a-b)$.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a+b)=(-1)(a+b)=(-1)(a)+(-1)(b)=(-a)+(-b)=-a-b$. Negatives do not 'distribute' across multiplication: $-(2\cdot 3)\neq(-2)\cdot(-3)$. Instead, $-(2\cdot 3)=(-2)\cdot(3)=(2)\cdot(-3)=-6$.

The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$.

[3] Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[4] We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Speaking of fractions, we now review their arithmetic.

Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \frac{a}{1}$ and $\frac{a}{a} = 1$.
- **Fraction Equality:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In particular: $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$

  Note: A common denominator is **not** required to **multiply** fractions!
- **Division**[a] **of Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

  In particular: $1 \div \frac{a}{b} = \frac{b}{a}$ and $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

  Note: A common denominator is **not** required to **divide** fractions!
- **Addition and Subtraction of Fractions:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$.

  Note: A common denominator **is** required to **add or subtract** fractions!
- **Equivalent Fractions:** $\frac{a}{b} = \frac{ad}{bd}$, as $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$

  Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.
- **'Reducing'**[b] **Fractions:** $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$, as $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$.

  In particular, $\frac{ab}{b} = a$ as $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$ and $\frac{b-a}{a-b} = \frac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.

  Note: We may only cancel common **factors** from both numerator and denominator.

[a] The old 'invert and multiply' or 'fraction gymnastics' play.
[b] Or 'Dividing Out' Common Factors - this is really just reading the previous property 'from right to left'.

Students make so many mistakes with fractions that we feel it is necessary to pause the narrative for a moment and offer you the following examples. Please take the time to read these carefully. In the main body of the text we will skip many of the steps shown here and it is your responsibility to understand the arithmetic behind the computations we use throughout the text. We deliberately limited these examples to "nice" numbers (meaning that the numerators and denominators of the fractions are small integers) and will discuss more complicated matters later. In the upcoming example, we will make use of the

Fundamental Theorem of Arithmetic which essentially says that every natural number has a unique prime factorization. Thus 'lowest terms' is clearly defined when reducing the fractions you're about to see.

Example 0.1.1. Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4}+\dfrac{6}{7}$
2. $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{7}{3-5}-\dfrac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\dfrac{12}{5}-\dfrac{7}{24}}{1+\left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)-\left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution.

1. Simplify $\dfrac{1}{4}+\dfrac{6}{7}$.

   It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by identifying the lowest common denominator and then we rewrite the fractions using that new denominator. As 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4\cdot 7=28$.

$$\begin{array}{rcll} \dfrac{1}{4}+\dfrac{6}{7} & = & \dfrac{1}{4}\cdot\dfrac{7}{7}+\dfrac{6}{7}\cdot\dfrac{4}{4} & \text{Equivalent Fractions} \\ & = & \dfrac{7}{28}+\dfrac{24}{28} & \text{Multiplication of Fractions} \\ & = & \dfrac{31}{28} & \text{Addition of Fractions} \end{array}$$

   The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. Simplify $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$.

   We could begin with the subtraction in parentheses, namely $\frac{47}{30}-\frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[5] The lowest

[5]See the remark on page 3 about how we add $1+2+3$.

common denominator[6] for all three fractions is 60.

$$\begin{aligned}
\frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\
&= \frac{5}{12}\cdot\frac{5}{5} - \frac{47}{30}\cdot\frac{2}{2} + \frac{7}{3}\cdot\frac{20}{20} && \text{Equivalent Fractions} \\
&= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\
&= \frac{71}{60} && \text{Addition and Subtraction of Fractions}
\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

3. Simplify $\dfrac{\frac{7}{3-5} - \frac{7}{3-5.21}}{5-5.21}$.

What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction. The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\frac{7}{3-5} - \frac{7}{3-5.21}}{5-5.21} &= \frac{\frac{7}{-2} - \frac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\frac{7}{2} + \frac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\frac{7}{2} - \frac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal in the numerator by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (because $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\frac{7}{2} - \frac{7}{2.21}}{0.21} = \frac{\frac{7}{2} - \frac{7\cdot 100}{2.21\cdot 100}}{0.21} = \frac{\frac{7}{2} - \frac{700}{221}}{0.21}$$

[6]We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

[7]Also called a 'vinculum'.

[8]Try it if you don't believe us.

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' concept and then cancel any factors that we can.

$$\begin{aligned}
\frac{\frac{7}{2}-\frac{700}{221}}{0.21} &= \frac{\frac{7}{2}\cdot\frac{221}{221}-\frac{700}{221}\cdot\frac{2}{2}}{\frac{21}{100}} = \frac{\frac{1547}{442}-\frac{1400}{442}}{\frac{21}{100}} \\
&= \frac{\frac{147}{442}}{\frac{21}{100}} = \frac{147}{442}\cdot\frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42\cdot 350$ and $9282 = 42\cdot 221$.

4. Simplify $\dfrac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)}$.

We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the numerator from the denominator.

$$\frac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' concept. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24\cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property}
\end{aligned}$$

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} && \text{Multiply fractions}\\
&= \frac{\frac{12\cdot 24\cdot \cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot \cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot \cancel{5}\cdot \cancel{24}}{\cancel{5}\cdot \cancel{24}}} && \text{Factor and cancel}\\
&= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)}\\
&= \frac{288-35}{120+84}\\
&= \frac{253}{204}
\end{aligned}$$

$253 = 11\cdot 23$ and $204 = 2\cdot 2\cdot 3\cdot 17$ have no common factors, thus our result is in lowest terms and we are done.

5. Simplify $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$.

This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We begin in the parentheses, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6}\\
&= \frac{(5)(0)+15}{-2}\\
&= \frac{15}{-2}\\
&= -\frac{15}{2} && \text{Properties of Negatives}
\end{aligned}$$

As $15 = 3\cdot 5$ and 2 have no common factors, we are done.

6. Simplify $\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$.

   In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}
\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\
&= \frac{15}{65}-\frac{-48}{65} \\
&= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\
&= \frac{15+48}{65} && \text{Add numerators} \\
&= \frac{63}{65}
\end{aligned}$$

   $63 = 3\cdot 3\cdot 7$ and $65 = 5\cdot 13$ have no common factors, so our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none cause students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2\cdot 2\cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1\cdot(\text{three factors of two}) = 1\cdot(2\cdot 2\cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1\cdot(\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1\div(\text{three factors of two}) = 1\div(2\cdot 2\cdot 2) = \frac{1}{2\cdot 2\cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$. In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a] See the comment regarding '$\frac{0}{0}$' on page 4.

While it is important to state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a+b)^2 \neq a^2 + b^2$. (For example, explore $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols).
2. Evaluate **e**xponents.
3. Evaluate **m**ultiplication and **d**ivision as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

We note that there are many useful mnemonic devices for remembering the order of operations.[a]

[a] Our favorite is '**P**lease **e**ntertain **m**y **d**ear **a**uld **S**asquatch.'

An example of the Order of Operations Agreement is $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Students get into trouble is with expressions like -3^2. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -\left(a^n\right)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ and $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

Example 0.1.2. Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution.

1. Simplify $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$.

   We begin working inside the parentheses, then deal with the exponents, before working through the other operations. As we saw in Example 0.1.1, the division here acts as a grouping symbol, so we save the division to the end.

$$\begin{aligned}
\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} \\
&= \frac{(2)(8)-16}{4} \\
&= \frac{16-16}{4} \\
&= \frac{0}{4} \\
&= 0
\end{aligned}$$

2. Simplify $12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5}$.

   As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5} &= 12(-5)(-2)^{-4}+6(-5)^2(-4)(-2)^{-5} \\
&= 12(-5)\left(\frac{1}{(-2)^4}\right)+6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 12(-5)\left(\frac{1}{16}\right)+6(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-60)\left(\frac{1}{16}\right)+(-600)\left(\frac{1}{-32}\right) \\
&= \frac{-60}{16}+\left(\frac{-600}{-32}\right) \\
&= \frac{-15\cdot\cancel{4}}{4\cdot\cancel{4}}+\frac{-75\cdot\cancel{8}}{-4\cdot\cancel{8}} \\
&= \frac{-15}{4}+\frac{-75}{-4} \\
&= \frac{-15}{4}+\frac{75}{4} \\
&= \frac{-15+75}{4} \\
&= \frac{60}{4} \\
&= 15
\end{aligned}$$

3. Simplify $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$.

   The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction. This gives us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's divide out after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\frac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} = \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} = \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} = 3^{2}\cdot 4^{-2} = 3^{2}\cdot\left(\frac{1}{4^{2}}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) = \frac{9}{16}
\end{aligned}$$

4. Simplify $\dfrac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}}$.

We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 0.1.1. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} &= \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^{2}} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} \\
&= \frac{\left(\frac{24}{5}\right)\cdot 25}{\left(1-\frac{144}{25}\right)\cdot 25} = \frac{\left(\frac{24\cdot 5\cdot\cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\frac{144\cdot\cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Because 120 and 119 have no common factors, we are done. □

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 0.2. Let a be a real number and let n be a natural number. If n is odd, then the **principal $\mathbf{n^{th}}$ root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 0.2 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[9] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' as opposed to a definition because they can be justified using the properties of exponents.

Theorem 0.1. Properties of Radicals: Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 0.1 is based on the definition of the principal n$^{\text{th}}$ root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that because n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with.[10] We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

[9] See Chapter 2.

[10] Otherwise we'd run into an interesting paradox. See Section 0.5.6.

Due to the fact that radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 0.3. Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number.[a]
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

[a] If n is even we need $a \geq 0$.

It would make life really nice if the rational exponents defined in Definition 0.3 had all of the same properties that integer exponents have as listed on page 12 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 0.3, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.

Example 0.1.3. Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$
2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$
3. $(\sqrt[3]{-2} - \sqrt[3]{-54})^2$
4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

Solution.

1. Simplify $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$.

We begin in the numerator and note that the radical here acts a grouping symbol,[11] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} &= \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16+24}}{2(2)} \\
&= \frac{-(-4)-\sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to divide by a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)} \\
&= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}
\end{aligned}$$

Now that the numerator and denominator have no more common factors,[12] we are done.

2. Simplify $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1-\left(\dfrac{\sqrt{3}}{3}\right)^2}$.

Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\frac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1-\left(\dfrac{\sqrt{3}}{3}\right)^2} = \frac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1-\left(\dfrac{(\sqrt{3})^2}{3^2}\right)} = \frac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1-\left(\dfrac{3}{9}\right)}$$

[11] The line extending horizontally from the square root symbol $\sqrt{\ }$ is, you guessed it, another vinculum.

[12] Do you see why we aren't 'dividing out' the remaining 2's?

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Simplify $(\sqrt[3]{-2}-\sqrt[3]{-54})^2$.

   Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[13] we may think of $-2=(-1)(2)$. Because $(-1)^3=-1$, which *is* a perfect cube, we may write $\sqrt[3]{-2}=\sqrt[3]{(-1)(2)}=\sqrt[3]{-1}\sqrt[3]{2}=-\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)}=\sqrt[3]{-27}\sqrt[3]{2}=-3\sqrt[3]{2}$. So,

   $$\sqrt[3]{-2}-\sqrt[3]{-54}=-\sqrt[3]{2}-(-3\sqrt[3]{2})=-\sqrt[3]{2}+3\sqrt[3]{2}.$$

   At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2}=2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

   $$-\sqrt[3]{2}+3\sqrt[3]{2}=(-1)\sqrt[3]{2}+3\sqrt[3]{2}=(-1+3)\sqrt[3]{2}=2\sqrt[3]{2}.$$

   Putting all this together, we get:

   $$\begin{aligned}
   (\sqrt[3]{-2}-\sqrt[3]{-54})^2 &= (-\sqrt[3]{2}+3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
   &= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
   \end{aligned}$$

   There are no perfect integer cubes which are factors of 4 (apart from 1, of course), so we are done.

4. Simplify $2\left(\frac{9}{4}-3\right)^{1/3}+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4}-3\right)^{-2/3}$.

   We start working in the parentheses and get a common denominator to subtract the fractions:

   $$\frac{9}{4}-3=\frac{9}{4}-\frac{3\cdot 4}{1\cdot 4}=\frac{9}{4}-\frac{12}{4}=\frac{-3}{4}$$

[13] Of an integer, that is!

The denominators in the fractional exponents are odd, so we can proceed by using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4}-3\right)^{1/3}+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4}-3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3}+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3}\\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right)+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3}\\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right)+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right)\\
&= \frac{2\cdot(-3)^{1/3}}{4^{1/3}}+\frac{2\cdot 9\cdot 1\cdot 4^{2/3}}{4\cdot 3\cdot(-3)^{2/3}}\\
&= \frac{2\cdot(-3)^{1/3}}{4^{1/3}}+\frac{\cancel{2}\cdot 3\cdot\cancel{3}\cdot 4^{2/3}}{2\cdot\cancel{2}\cdot\cancel{3}\cdot(-3)^{2/3}}\\
&= \frac{2\cdot(-3)^{1/3}}{4^{1/3}}+\frac{3\cdot 4^{2/3}}{2\cdot(-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Due to the fact that $4=2^2$, $4^{1/3}=(2^2)^{1/3}=2^{2/3}$. Similarly, $4^{2/3}=(2^2)^{2/3}=2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3}=\sqrt[3]{-3}=-\sqrt[3]{3}=-3^{1/3}$ and $(-3)^{2/3}=(\sqrt[3]{-3})^2=(-\sqrt[3]{3})^2=(\sqrt[3]{3})^2=3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2\cdot(-3)^{1/3}}{4^{1/3}}+\frac{3\cdot 4^{2/3}}{2\cdot(-3)^{2/3}} &= \frac{2\cdot(-3^{1/3})}{2^{2/3}}+\frac{3\cdot 2^{4/3}}{2\cdot 3^{2/3}}\\
&= \frac{2^1\cdot(-3^{1/3})}{2^{2/3}}+\frac{3^1\cdot 2^{4/3}}{2^1\cdot 3^{2/3}}\\
&= 2^{1-2/3}\cdot(-3^{1/3})+3^{1-2/3}\cdot 2^{4/3-1}\\
&= 2^{1/3}\cdot(-3^{1/3})+3^{1/3}\cdot 2^{1/3}\\
&= -2^{1/3}\cdot 3^{1/3}+3^{1/3}\cdot 2^{1/3}\\
&= 0
\end{aligned}$$

□

We close this section with a note about simplifying. In the preceding examples we used "nice" numbers because we wanted to show as many properties as we could per example. This then begs the question "What happens when the numbers are *not* nice?" Unfortunately, the answer is "Not much simplifying can be done." Take, for example,

$$\frac{\sqrt{7}}{\pi}-\frac{3}{\pi^2}+\frac{4}{\sqrt{11}}=\frac{\pi\sqrt{77}-3\sqrt{11}+4\pi^2}{\pi^2\sqrt{11}}$$

Sadly, that's as good as it gets.

0.1.2 Exercises

In Exercises 1 - 33, perform the indicated operations and simplify.

1. $5-2+3$
2. $5-(2+3)$
3. $\dfrac{2}{3}-\dfrac{4}{7}$
4. $\dfrac{3}{8}+\dfrac{5}{12}$
5. $\dfrac{5-3}{-2-4}$
6. $\dfrac{2(-3)}{3-(-3)}$
7. $\dfrac{2(3)-(4-1)}{2^2+1}$
8. $\dfrac{4-5.8}{2-2.1}$
9. $\dfrac{1-2(-3)}{5(-3)+7}$
10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\dfrac{(-2)^2-(-2)-6}{(-2)^2-4}$
13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$
14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1}-4^{-2}$
19. $\dfrac{1+2^{-3}}{3-4^{-1}}$
20. $\dfrac{3\cdot 5^{100}}{12\cdot 5^{98}}$
21. $\sqrt{3^2+4^2}$
22. $\sqrt{12}-\sqrt{75}$
23. $(-8)^{2/3}-9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3-4)^2+(5-2)^2}$
26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
28. $\dfrac{-12+\sqrt{18}}{21}$
29. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
30. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
31. $2(-5)(-5+1)^{-1}+(-5)^2(-1)(-5+1)^{-2}$
32. $3\sqrt{2(4)+1}+3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1-(-7)}+(-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$
34. Prove the Quotient Rule and Power Rule stated in Theorem 0.1.
35. Discuss with your classmates how you might attempt to simplify the following.

    (a) $\sqrt{\dfrac{1-\sqrt{2}}{1+\sqrt{2}}}$

    (b) $\sqrt[5]{3}-\sqrt[3]{5}$

    (c) $\dfrac{\pi+7}{\pi}$

Section 0.1 Exercise Answers A.1.0

0.2 Simplifying Radicals

In this section we review simplifying expressions and solving equations involving radicals. In addition to the product, quotient and power rules stated in Theorem 0.1 in Section 0.1, we present the following result which states that n^th roots and n^th powers more or less 'undo' each other.[1]

Theorem 0.2. Simplifying n^th powers of n^th roots and n^th roots of n^th powers: Suppose n is a natural number, a is a real number and $\sqrt[n]{a}$ is a real number. Then

- $(\sqrt[n]{a})^n = a$
- if n is odd, $\sqrt[n]{a^n} = a$; if n is even, $\sqrt[n]{a^n} = |a|$.

Because $\sqrt[n]{a}$ is *defined* so that $(\sqrt[n]{a})^n = a$, the first claim in the theorem is just a re-wording of Definition 0.2. The second part of the theorem breaks down along odd/even exponent lines due to how exponents affect negatives. To see this, consider the specific cases of $\sqrt[3]{(-2)^3}$ and $\sqrt[4]{(-2)^4}$.

In the first case, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$, so we have an instance of when $\sqrt[n]{a^n} = a$. The reason that the cube root 'undoes' the third power in $\sqrt[3]{(-2)^3} = -2$ is because the negative is preserved when raised to the third (odd) power. In $\sqrt[4]{(-2)^4}$, the negative 'goes away' when raised to the fourth (even) power: $\sqrt[4]{(-2)^4} = \sqrt[4]{16}$. According to Definition 0.2, the fourth root is defined to give only *non-negative* numbers, so $\sqrt[4]{16} = 2$. Here we have a case where $\sqrt[4]{(-2)^4} = 2 = |-2|$, not -2.

In general, we need the absolute values to simplify $\sqrt[n]{a^n}$ only when n is even because a negative to an even power is always positive. In particular, $\sqrt{x^2} = |x|$, not just 'x' (unless we *know* $x \geq 0$.)[2] We practice these formulas in the following example.

Example 0.2.1. Perform the indicated operations and simplify.

1. $\sqrt{x^2+1}$
2. $\sqrt{t^2-10t+25}$
3. $\sqrt[3]{48x^{14}}$
4. $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$
5. $2x\sqrt[3]{x^2-4}+2\left(\dfrac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x)$
6. $\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2}$

Solution.

1. Simplify $\sqrt{x^2+1}$.

   We told you back on page 16 that roots do not 'distribute' across addition and due to the fact that x^2+1 does not factor over the real numbers, $\sqrt{x^2+1}$ cannot be simplified. It may seem silly to start

[1] See Sections 4.1.2 and 5.1 for a more precise understanding of what we mean here.

[2] This discussion should sound familiar - see the discussion following Definition 0.3 and the discussion following 'Extracting the Square Root' on page 66.

with this example but it is extremely important that you understand what maneuvers are legal and which ones are not.[3]

2. Simplify $\sqrt{t^2-10t+25}$.

   Again we note that $\sqrt{t^2-10t+25} \neq \sqrt{t^2}-\sqrt{10t}+\sqrt{25}$, as radicals do *not* distribute across addition and subtraction.[4] In this case, however, we can factor the radicand and simplify as

   $$\sqrt{t^2-10t+25} = \sqrt{(t-5)^2} = |t-5|$$

   Without knowing more about the value of t, we have no idea if $t-5$ is positive or negative, so $|t-5|$ is our final answer.[5]

3. Simplify $\sqrt[3]{48x^{14}}$.

   To simplify $\sqrt[3]{48x^{14}}$, we need to look for perfect cubes in the radicand. For the cofficient, we have $48 = 8\cdot 6 = 2^3\cdot 6$. To find the largest perfect cube factor in x^{14}, we divide 14 (the exponent on x) by 3 (because we are looking for a perfect *cube*). We get 4 with a remainder of 2. This means $14 = 4\cdot 3+2$, so $x^{14} = x^{4\cdot 3+2} = x^{4\cdot 3}x^2 = (x^4)^3x^2$. Putting this altogether gives:

   $$\begin{array}{rcll} \sqrt[3]{48x^{14}} & = & \sqrt[3]{2^3\cdot 6\cdot (x^4)^3x^2} & \text{Factor out perfect cubes} \\ & = & \sqrt[3]{2^3}\sqrt[3]{(x^4)^3}\sqrt[3]{6x^2} & \text{Rearrange factors, Product Rule of Radicals} \\ & = & 2x^4\sqrt[3]{6x^2} & \end{array}$$

4. Simplify $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$.

   In this example, we are looking for perfect fourth powers in the radicand. In the numerator, r^4, is clearly a perfect fourth power. For the denominator, we take the power on the L, namely 8, and divide by 4 to get 2. This means $L^8 = L^{2\cdot 4} = (L^2)^4$. We get

   $$\begin{array}{rcll} \sqrt[4]{\dfrac{\pi r^4}{L^8}} & = & \dfrac{\sqrt[4]{\pi r^4}}{\sqrt[4]{L^8}} & \text{Quotient Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}\sqrt[4]{r^4}}{\sqrt[4]{(L^2)^4}} & \text{Product Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}|r|}{|L^2|} & \text{Simplify} \end{array}$$

   Without more information about r, we cannot simplify $|r|$ any further. However, we can simplify $|L^2|$. Regardless of the choice of L, $L^2 \geq 0$. Actually, $L^2 > 0$ because L is in the denominator which means

[3] You really do need to understand this otherwise horrible evil will plague your future studies in Math. If you say something totally wrong like $\sqrt{x^2+1} = x+1$ then you may never pass Calculus. PLEASE be careful!

[4] Let $t = 1$ and see what happens to $\sqrt{t^2-10t+25}$ versus $\sqrt{t^2}-\sqrt{10t}+\sqrt{25}$.

[5] In general, $|t-5| \neq |t|-|5|$ and $|t-5| \neq t+5$ so watch what you're doing!

$L \neq 0$. Hence, $|L^2| = L^2$. Our answer simplifies to:

$$\frac{\sqrt[4]{\pi}|r|}{|L^2|} = \frac{|r|\sqrt[4]{\pi}}{L^2}$$

5. Simplify $2x\sqrt[3]{x^2-4} + 2\left(\frac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x)$.

   After a quick division (two of the 2's in the second term) we need to obtain a common denominator. We can view the first term as having a denominator of 1, thus the common denominator is precisely the denominator of the second term, namely $(\sqrt[3]{x^2-4})^2$. With common denominators, we proceed to add the two fractions. Our last step is to factor the numerator to see if there are any opportunities to divide out common factors in the numerator and denominator.

$$\begin{aligned}
2x\sqrt[3]{x^2-4} + 2\left(\frac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x) &= 2x\sqrt[3]{x^2-4} + \cancel{2}\left(\frac{1}{\cancel{2}(\sqrt[3]{x^2-4})^2}\right)(2x) && \text{Reduce} \\
&= 2x\sqrt[3]{x^2-4} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Mutiply} \\
&= (2x\sqrt[3]{x^2-4}) \cdot \frac{(\sqrt[3]{x^2-4})^2}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Equivalent fractions} \\
&= \frac{2x(\sqrt[3]{x^2-4})^3}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Multiply} \\
&= \frac{2x(x^2-4)}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Simplify} \\
&= \frac{2x(x^2-4)+2x}{(\sqrt[3]{x^2-4})^2} && \text{Add} \\
&= \frac{2x(x^2-4+1)}{(\sqrt[3]{x^2-4})^2} && \text{Factor} \\
&= \frac{2x(x^2-3)}{(\sqrt[3]{x^2-4})^2}
\end{aligned}$$

   We cannot reduce this any further, because $x^2 - 3$ is irreducible over the rational numbers.

6. Simplify $\sqrt{(\sqrt{18y} - \sqrt{8y})^2 + (\sqrt{20} - \sqrt{80})^2}$.

   We begin by working inside each set of parentheses, using the product rule for radicals and combining like terms.

$$\begin{aligned}
\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2} &= \sqrt{(\sqrt{9\cdot 2y}-\sqrt{4\cdot 2y})^2+(\sqrt{4\cdot 5}-\sqrt{16\cdot 5})^2} \\
&= \sqrt{(\sqrt{9}\sqrt{2y}-\sqrt{4}\sqrt{2y})^2+(\sqrt{4}\sqrt{5}-\sqrt{16}\sqrt{5})^2} \\
&= \sqrt{(3\sqrt{2y}-2\sqrt{2y})^2+(2\sqrt{5}-4\sqrt{5})^2} \\
&= \sqrt{(\sqrt{2y})^2+(-2\sqrt{5})^2} \\
&= \sqrt{2y+(-2)^2(\sqrt{5})^2} \\
&= \sqrt{2y+4\cdot 5} \\
&= \sqrt{2y+20}
\end{aligned}$$

To see if this simplifies any further, we factor the radicand: $\sqrt{2y+20}=\sqrt{2(y+10)}$. Finding no perfect square factors, we are done. □

0.2.1 Exercises

In Exercises 1 - 13, perform the indicated operations and simplify.

1. $\sqrt{9x^2}$
2. $\sqrt[3]{8t^3}$
3. $\sqrt{50y^6}$
4. $\sqrt{4t^2+4t+1}$
5. $\sqrt{w^2-16w+64}$
6. $\sqrt{(\sqrt{12x}-\sqrt{3x})^2+1}$
7. $\sqrt{\dfrac{c^2-v^2}{c^2}}$
8. $\sqrt[3]{\dfrac{24\pi r^5}{L^3}}$
9. $\sqrt[4]{\dfrac{32\pi\varepsilon^8}{\rho^{12}}}$
10. $\sqrt{x}-\dfrac{x+1}{\sqrt{x}}$
11. $3\sqrt{1-t^2}+3t\left(\dfrac{1}{2\sqrt{1-t^2}}\right)(-2t)$
12. $2\sqrt[3]{1-z}+2z\left(\dfrac{1}{3\left(\sqrt[3]{1-z}\right)^2}\right)(-1)$
13. $\dfrac{3}{\sqrt[3]{2x-1}}+(3x)\left(-\dfrac{1}{3\left(\sqrt[3]{2x-1}\right)^4}\right)(2)$

Section 0.2 Exercise Answers A.1.0

0.3 Factoring Expressions

Now that we have reviewed the basics of polynomial arithmetic it's time to review the basic techniques of factoring polynomial expressions. Our goal is to apply these techniques to help us solve certain specialized classes of non-linear equations. Given that 'factoring' literally means to resolve a product into its factors, it is, in the purest sense, 'undoing' multiplication. If this sounds like division to you then you've been paying attention. Let's start with a numerical example.

Suppose we are asked to factor 16337. We could write $16337 = 16337 \cdot 1$, and while this is technically a factorization of 16337, it's probably not an answer the poser of the question would accept. Usually, when we're asked to factor a natural number, we are being asked to resolve it into to a product of so-called 'prime' numbers.[1] Recall that **prime numbers** are defined as natural numbers whose only (natural number) factors are themselves and 1. They are, in essence, the 'building blocks' of natural numbers as far as multiplication is concerned. Said differently, we can build - via multiplication - any natural number given enough primes.

So how do we find the prime factors of 16337? We start by dividing each of the primes: 2, 3, 5, 7, etc., into 16337 until we get a remainder of 0. Eventually, we determine that $16337 \div 17 = 961$ with a remainder of 0, which means $16337 = 17 \cdot 961$. So factoring and division are indeed closely related - factors of a number are precisely the divisors of that number which produce a zero remainder.[2] We continue our efforts to see if 961 can be factored down further, and we find that $961 = 31 \cdot 31$. Hence, 16337 can be 'completely factored' as $17 \cdot 31^2$. (This factorization is called the **prime factorization** of 16337.)

In factoring natural numbers, our building blocks are prime numbers, so to be completely factored means that every number used in the factorization of a given number is prime. One of the challenges when it comes to factoring polynomial expressions is to explain what it means to be 'completely factored'. In this section, our 'building blocks' for factoring polynomials are 'irreducible' polynomials as defined below.

Definition 0.4. A polynomial is said to be **irreducible** if it cannot be written as the product of polynomials of lower degree.

While Definition 0.4 seems straightforward enough, sometimes a greater level of specificity is required. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. While $x - \sqrt{3}$ and $x + \sqrt{3}$ are perfectly fine polynomials, factoring which requires irrational numbers is usually saved for a more advanced treatment of factoring. For now, we will restrict ourselves to factoring using rational coefficients. So, while the polynomial $x^2 - 3$ can be factored using irrational numbers, it is called irreducible **over the rationals**, because there are no polynomials with *rational* coefficients of smaller degree which can be used to factor it.[3]

Due to the fact that polynomials involve terms, the first step in any factoring strategy involves pulling out factors which are common to all of the terms. For example, in the polynomial $18x^2y^3 - 54x^3y^2 - 12xy^2$, each coefficient is a multiple of 6 so we can begin the factorization as $6(3x^2y^3 - 9x^3y^2 - 2xy^2)$.

[1] As mentioned in Section 0.1, this is possible, in only one way, thanks to the Fundamental Theorem of Arithmetic.

[2] We'll refer back to this when we get to Section 2.3.

[3] If this isn't immediately obvious, don't worry - in some sense, it shouldn't be. We'll talk more about this later.

The remaining coefficients: 3, 9 and 2, have no common factors so 6 was the greatest common factor. What about the variables? Each term contains an x, so we can factor an x from each term. When we do this, we are effectively dividing each term by x which means the exponent on x in each term is reduced by 1: $6x(3xy^3 - 9x^2y^2 - 2y^2)$. Next, we see that each term has a factor of y in it. In fact, each term has at least *two* factors of y in it, as the lowest exponent on y in each term is 2. This means that we can factor y^2 from each term. Again, factoring out y^2 from each term is tantamount to dividing each term by y^2 so the exponent on y in each term is reduced by *two*: $6xy^2(3xy - 9x^2 - 2)$. Just like we checked our division by multiplication in the previous section, we can check our factoring here by multiplication, too. $6xy^2(3xy - 9x^2 - 2) = (6xy^2)(3xy) - (6xy^2)(9x^2) - (6xy^2)(2) = 18x^2y^3 - 54x^3y^2 - 12xy^2 \checkmark$. We summarize how to determine the Greatest Common Factor (G.C.F.) of a polynomial expression below.

Determining the G.C.F. of a Polynomial Expression

- If the coefficients are integers, identify the G.C.F. of the coefficients.

  Note 1: If all of the coefficients are negative, consider the negative as part of the G.C.F..

  Note 2: If the coefficients involve fractions, get a common denominator, combine numerators, reduce to lowest terms and apply this step to the polynomial in the numerator.

- If a variable is common to all of the terms, the G.C.F. contains that variable to the smallest exponent which appears among the terms.

For example, to factor $-\frac{3}{5}z^3 - 6z^2$, we would first get a common denominator and factor as:

$$-\frac{3}{5}z^3 - 6z^2 = \frac{-3z^3 - 30z^2}{5} = \frac{-3z^2(z+10)}{5} = -\frac{3z^2(z+10)}{5} = -\frac{3}{5}z^2(z+10)$$

We now list some common factoring formulas, each of which can be verified by multiplying out the right side of the equation. While they all should look familiar - this is a review section after all - some should look more familiar than others because they are ‘special product’ formulas.

Common Factoring Formulas

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

- **Difference of Two Squares:** $a^2 - b^2 = (a-b)(a+b)$

  Note: In general, the sum of squares, $a^2 + b^2$ is irreducible over the rationals.

- **Sum of Two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  Note: In general, $a^2 - ab + b^2$ is irreducible over the rationals.

- **Difference of Two Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  Note: In general, $a^2 + ab + b^2$ is irreducible over the rationals.

The example on the next page gives us practice with these formulas.

Example 0.3.1. Factor the following polynomials completely over the rationals. That is, write each polynomial as a product polynomials of lowest degree which are irreducible over the rationals.

1. $18x^2 - 48x + 32$
2. $64y^2 - 1$
3. $75t^4 + 30t^3 + 3t^2$
4. $w^4z - wz^4$
5. $81 - 16t^4$
6. $x^6 - 64$

Solution.

1. Factor $18x^2 - 48x + 32$.

   Our first step is to factor out the G.C.F. which in this case is 2. To match what is left with one of the special forms, we rewrite $9x^2 = (3x)^2$ and $16 = 4^2$. We see that we have a perfect square trinomial, because the 'middle' term is $-24x = -2(4)(3x)$.

$$\begin{array}{rcll} 18x^2 - 48x + 32 & = & 2(9x^2 - 24x + 16) & \text{Factor out G.C.F.} \\ & = & 2((3x)^2 - 2(4)(3x) + (4)^2) & \\ & = & 2(3x-4)^2 & \text{Perfect Square Trinomial: } a = 3x,\ b = 4 \end{array}$$

   Our final answer is $2(3x-4)^2$.

   To check our work, we multiply out $2(3x-4)^2$ to show that it equals $18x^2 - 48x + 32$.

2. Factor $64y^2 - 1$.

   For $64y^2 - 1$, we note that the G.C.F. of the terms is just 1, so there is nothing (of substance) to factor out of both terms. Due to the fact that $64y^2 - 1$ is the difference of two terms, one of which is a square, we look to the Difference of Squares Formula for inspiration. Seeing $64y^2 = (8y)^2$ and $1 = 1^2$, we get

$$\begin{array}{rcll} 64y^2 - 1 & = & (8y)^2 - 1^2 & \\ & = & (8y-1)(8y+1) & \text{Difference of Squares, } a = 8y,\ b = 1 \end{array}$$

   As before, we can check our final answer by multiplying out $(8y-1)(8y+1)$ to show that it equals $64y^2 - 1$.

3. Factor $75t^4 + 30t^3 + 3t^2$.

   The G.C.F. of the terms in $75t^4 + 30t^3 + 3t^2$ is $3t^2$, so we factor that out first. We identify what remains as a perfect square trinomial:

$$\begin{array}{rcll} 75t^4 + 30t^3 + 3t^2 & = & 3t^2(25t^2 + 10t + 1) & \text{Factor out G.C.F.} \\ & = & 3t^2((5t)^2 + 2(1)(5t) + 1^2) & \\ & = & 3t^2(5t+1)^2 & \text{Perfect Square Trinomial, } a = 5t,\ b = 1 \end{array}$$

Our final answer is $3t^2(5t+1)^2$, which the reader is invited to check.

4. Factor $w^4z - wz^4$.

   For $w^4z - wz^4$, we identify the G.C.F. as wz and once we factor it out a difference of cubes is revealed:

$$\begin{array}{rcll} w^4z - wz^4 & = & wz(w^3 - z^3) & \text{Factor out G.C.F.} \\ & = & wz(w-z)(w^2+wz+z^2) & \text{Difference of Cubes, } a = w,\ b = z \end{array}$$

   Our final answer is $wz(w-z)(w^2+wz+z^2)$.

   The reader is strongly encouraged to multiply this out to see that it reduces to $w^4z - wz^4$.

5. Factor $81 - 16t^4$.

   The G.C.F. of the terms in $81 - 16t^4$ is just 1 so there is nothing of substance to factor out from both terms. With just a difference of two terms, we are limited to fitting this polynomial into either the Difference of Two Squares or Difference of Two Cubes formula. With the variable here being t^4, and 4 is a multiple of 2, we can think of $t^4 = (t^2)^2$. This means that we can write $16t^4 = (4t^2)^2$ which is a perfect square. (As 4 is not a multiple of 3, we cannot write t^4 as a perfect cube of a polynomial.) Identifying $81 = 9^2$ and $16t^4 = (4t^2)^2$, we apply the Difference of Squares Formula to get:

$$\begin{array}{rcll} 81 - 16t^4 & = & 9^2 - (4t^2)^2 & \\ & = & (9-4t^2)(9+4t^2) & \text{Difference of Squares, } a = 9,\ b = 4t^2 \end{array}$$

   At this point, we have an opportunity to proceed further in the first quantity. Identifying $9 = 3^2$ and $4t^2 = (2t)^2$, we see that we have another difference of squares in the first quantity, which we can reduce. (The sum of two squares in the second quantity <u>cannot</u> be factored over the rationals.)

$$\begin{array}{rcll} 81 - 16t^4 & = & (9-4t^2)(9+4t^2) & \\ & = & (3^2-(2t)^2)(9+4t^2) & \\ & = & (3-2t)(3+2t)(9+4t^2) & \text{Difference of Squares, } a = 3,\ b = 2t \end{array}$$

   As always, the reader is encouraged to multiply out $(3-2t)(3+2t)(9+4t^2)$ to check the result.

6. Factor $x^6 - 64$.

   With a G.C.F. of 1 and just two terms, $x^6 - 64$ is a candidate for both the Difference of Squares and the Difference of Cubes formulas. Notice that we can identify $x^6 = (x^3)^2$ and $64 = 8^2$ (both perfect squares), but also $x^6 = (x^2)^3$ and $64 = 4^3$ (both perfect cubes). If we follow the Difference of Squares approach, we get:

$$\begin{array}{rcll} x^6 - 64 & = & (x^3)^2 - 8^2 & \\ & = & (x^3-8)(x^3+8) & \text{Difference of Squares, } a = x^3 \text{ and } b = 8 \end{array}$$

At this point, we have an opportunity to use both the Difference and Sum of Cubes formulas:

$$\begin{aligned} x^6-64 &= (x^3-2^3)(x^3+2^3) & \\ &= (x-2)(x^2+2x+2^2)(x+2)(x^2-2x+2^2) & \text{Sum / Difference of Cubes, } a=x,\ b=2 \\ &= (x-2)(x+2)(x^2-2x+4)(x^2+2x+4) & \text{Rearrange factors} \end{aligned}$$

From this approach, our final answer is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$.

Following the Difference of Cubes Formula approach, we get

$$\begin{aligned} x^6-64 &= (x^2)^3-4^3 & \\ &= (x^2-4)((x^2)^2+4x^2+4^2) & \text{Difference of Cubes, } a=x^2,\ b=4 \\ &= (x^2-4)(x^4+4x^2+16) & \end{aligned}$$

At this point, we recognize x^2-4 as a difference of two squares:

$$\begin{aligned} x^6-64 &= (x^2-2^2)(x^4+4x^2+16) & \\ &= (x-2)(x+2)(x^4+4x^2+16) & \text{Difference of Squares, } a=x,\ b=2 \end{aligned}$$

Unfortunately, the remaining factor x^4+4x^2+16 is not a perfect square trinomial - the middle term would have to be $8x^2$ for this to work - so our final answer using this approach is $(x-2)(x+2)(x^4+4x^2+16)$. This isn't as factored as our result from the Difference of Squares approach which was $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. While it is true that $x^4+4x^2+16=(x^2-2x+4)(x^2+2x+4)$, there is no 'intuitive' way to motivate this factorization at this point.[4] The moral of the story? When given the option between using the Difference of Squares and Difference of Cubes, start with the Difference of Squares. Our final answer to this problem is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. The reader is strongly encouraged to show that this reduces down to x^6-64 after performing all of the multiplication. □

The formulas on page 27, while useful, can only take us so far. Thus we need to review some additional factoring strategies which should be good friends from back in the day!

Additional Factoring Formulas

- **'un-F.O.I.L.ing':** Given a trinomial Ax^2+Bx+C, try to reverse the F.O.I.L. process.

  That is, find a, b, c and d such that $Ax^2+Bx+C=(ax+b)(cx+d)$.

  Note: This means $ac=A$, $bd=C$ and $B=ad+bc$.

- **Factor by Grouping:** If the expression contains four terms with no common factors among the four terms, try 'factor by grouping':

$$ac+bc+ad+bd=(a+b)c+(a+b)d=(a+b)(c+d)$$

[4] Of course, this begs the question, "How do we know x^2-2x+4 and x^2+2x+4 are irreducible?" (We were told so on page 27, but no reason was given.) Stay tuned! We'll get back to this in due course.

The techniques of 'un-F.O.I.L.ing' and 'factoring by grouping' are difficult to describe in general but should make sense to you with enough practice. Be forewarned - like all 'Rules of Thumb', these strategies work just often enough to be useful, but you can be sure there are exceptions which will defy any advice given here and will require some 'inspiration' to solve. Even though Chapter 2 will give us more powerful factoring methods, we'll find that, in the end, there is no single algorithm for factoring which works for every polynomial. In other words, there will be times when you just have to try something and see what happens.

Example 0.3.2. Factor the following polynomials completely over the integers.[5]

1. $x^2 - x - 6$
2. $2t^2 - 11t + 5$
3. $36 - 11y - 12y^2$
4. $18xy^2 - 54xy - 180x$
5. $2t^3 - 10t^2 + 3t - 15$
6. $x^4 + 4x^2 + 16$

Solution.

1. Factor $x^2 - x - 6$.

   The G.C.F. of the terms $x^2 - x - 6$ is 1 and $x^2 - x - 6$ isn't a perfect square trinomial (Think about why not.) so we try to reverse the F.O.I.L. process and look for integers a, b, c and d such that $(ax+b)(cx+d) = x^2 - x - 6$. To get started, we note that $ac = 1$. Because a and c are meant to be integers, that leaves us with either a and c both being 1, or a and c both being -1. We'll go with $a = c = 1$, as we can factor the negatives into our choices for b and d. This yields $(x+b)(x+d) = x^2 - x - 6$. Next, we use the fact that $bd = -6$. The product is negative so we know that one of b or d is positive and the other is negative. Given b and d are integers, one of b or d is ± 1 and the other is ∓ 6 OR one of b or d is ± 2 and the other is ∓ 3. After some guessing and checking, we find that $x^2 - x - 6 = (x+2)(x-3)$.

2. Factor $2t^2 - 11t + 5$.

   As with the previous example, we check the G.C.F. of the terms in $2t^2 - 11t + 5$, determine it to be 1 and see that the polynomial doesn't fit the pattern for a perfect square trinomial. We now try to find integers a, b, c and d such that $(at+b)(ct+d) = 2t^2 - 11t + 5$. As $ac = 2$, we have that one of a or c is 2, and the other is 1. (Once again, we ignore the negative options.) At this stage, there is nothing really distinguishing a from c so we choose $a = 2$ and $c = 1$. Now we look for b and d such that $(2t+b)(t+d) = 2t^2 - 11t + 5$. We know $bd = 5$ so one of b or d is ± 1 and the other ± 5. Given that bd is positive, b and d must have the same sign. The negative middle term $-11t$ guides us to guess $b = -1$ and $d = -5$ so that we get $(2t-1)(t-5) = 2t^2 - 11t + 5$. We verify our answer by multiplying.[6]

3. Factor $36 - 11y - 12y^2$.

   Once again, we check for a nontrivial G.C.F. and determine if $36 - 11y - 12y^2$ fits the pattern of a perfect square. Twice disappointed, we rewrite $36 - 11y - 12y^2 = -12y^2 - 11y + 36$ for notational

[5] This means that all of the coefficients in the factors will be integers. While it was decided to avoid fractions in this set of examples, don't get complacent, though, because fractions will return with a vengeance soon enough.

[6] That's the 'checking' part of 'guessing and checking'.

convenience. We now look for integers a, b, c and d such that $-12y^2-11y+36=(ay+b)(cy+d)$. Due to the fact that $ac=-12$, we know that one of a or c is ± 1 and the other ∓ 12 OR one of them is ± 2 and the other is ∓ 6 OR one of them is ± 3 while the other is ∓ 4. As their product is -12, however, we know one of them is positive, while the other is negative. To make matters worse, the constant term 36 has its fair share of factors, too. Our answers for b and d lie among the pairs ± 1 and ± 36, ± 2 and ± 18, ± 4 and ± 9, or ± 6. Because we know one of a or c will be negative, we can simplify our choices for b and d and just look at the positive possibilities. After some guessing and checking,[7] we determine $(-3y+4)(4y+9)=-12y^2-11y+36$.

4. Factor $18xy^2-54xy-180x$.

   Given the G.C.F. of the terms in $18xy^2-54xy-180x$ is $18x$, we begin the problem by factoring it out first: $18xy^2-54xy-180x=18x(y^2-3y-10)$. We now focus our attention on $y^2-3y-10$. We can take a and c to both be 1 which yields $(y+b)(y+d)=y^2-3y-10$. Our choices for b and d are among the factor pairs of -10: ± 1 and ∓ 10 or ± 2 and ∓ 5, where one of b or d is positive and the other is negative. We find $(y-5)(y+2)=y^2-3y-10$. Our final answer is $18xy^2-54xy-180x=18x(y-5)(y+2)$.

5. Factor $2t^3-10t^2+3t-15$.

   With $2t^3-10t^2-3t+15$ being four terms, we are pretty much resigned to factoring by grouping. The strategy here is to factor out the G.C.F. from two *pairs* of terms, and see if this reveals a common factor. If we group the first two terms, we can factor out a $2t^2$ to get $2t^3-10t^2=2t^2(t-5)$. We now try to factor something out of the last two terms that will leave us with a factor of $(t-5)$. Sure enough, we can factor out a -3 from both: $-3t+15=-3(t-5)$. Hence, we get

   $$2t^3-10t^2-3t+15=2t^2(t-5)-3(t-5)=(2t^2-3)(t-5)$$

   Now the question becomes can we factor $2t^2-3$ over the integers? This would require integers a,b,c and d such that $(at+b)(ct+d)=2t^2-3$. As a result of $ab=2$ and $cd=-3$, we aren't left with many options - in fact, we really have only four choices: $(2t-1)(t+3)$, $(2t+1)(t-3)$, $(2t-3)(t+1)$ and $(2t+3)(t-1)$. None of these produce $2t^2-3$ - which means it's irreducible over the integers - thus our final answer is $(2t^2-3)(t-5)$.

6. Factor x^4+4x^2+16.

   Our last example, x^4+4x^2+16, is our old friend from Example 0.3.1. As noted there, it is not a perfect square trinomial, so we could try to reverse the F.O.I.L. process. This is complicated by the fact that our highest degree term is x^4, so we would have to look at factorizations of the form $(x+b)(x^3+d)$ as well as $(x^2+b)(x^2+d)$. We leave it to the reader to show that neither of those work. This is an example of where 'trying something' pays off. Even though we've stated that it is not a perfect square trinomial, it's pretty close. Identifying $x^4=(x^2)^2$ and $16=4^2$, we'd have $(x^2+4)^2=x^4+8x^2+16$,

[7] Some of these guesses can be more 'educated' than others. Due to the fact that the middle term is relatively 'small,' we don't expect the 'extreme' factors of 36 and 12 to appear, for instance.

but instead of $8x^2$ as our middle term, we only have $4x^2$. We could add in the extra $4x^2$ we need, but to keep the balance, we'd have to subtract it off. Doing so produces an unexpected opportunity:

$$\begin{aligned}
x^4+4x^2+16 &= x^4+4x^2+16+(4x^2-4x^2) && \text{Adding and subtracting the same term} \\
&= x^4+8x^2+16-4x^2 && \text{Rearranging terms} \\
&= (x^2+4)^2-(2x)^2 && \text{Factoring perfect square trinomial} \\
&= [(x^2+4)-2x][(x^2+4)+2x] && \text{Difference of Squares: } a=(x^2+4),\ b=2x \\
&= (x^2-2x+4)(x^2+2x+4) && \text{Rearraging terms}
\end{aligned}$$

We leave it to the reader to check that neither x^2-2x+4 nor x^2+2x+4 factor over the integers, so we are done. □

0.3.1 Exercises

In Exercises 1 - 30, factor completely over the integers. Check your answer by multiplication.

1. $2x-10x^2$
2. $12t^5-8t^3$
3. $16xy^2-12x^2y$
4. $5(m+3)^2-4(m+3)^3$
5. $(2x-1)(x+3)-4(2x-1)$
6. $t^2(t-5)+t-5$
7. w^2-121
8. $49-4t^2$
9. $81t^4-16$
10. $9z^2-64y^4$
11. $(y+3)^2-4y^2$
12. $(x+h)^3-(x+h)$
13. $y^2-24y+144$
14. $25t^2+10t+1$
15. $12x^3-36x^2+27x$
16. m^4+10m^2+25
17. $27-8x^3$
18. t^6+t^3
19. $x^2-5x-14$
20. $y^2-12y+27$
21. $3t^2+16t+5$
22. $6x^2-23x+20$
23. $35+2m-m^2$
24. $7w-2w^2-3$
25. $3m^3+9m^2-12m$
26. x^4+x^2-20
27. $4(t^2-1)^2+3(t^2-1)-10$
28. $x^3-5x^2-9x+45$
29. $3t^2+t-3-t^3$
30.[8] y^4+5y^2+9

Section 0.3 Exercise Answers A.1.0

[8] $y^4+5y^2+9=(y^4+6y^2+9)-y^2$

0.4 Using Interval Notation

0.4.1 Some Basic Set Theory Notions

We begin this section with the definition of a concept that is central to all of Mathematics.

Definition 0.5. A **set** is a well-defined collection of objects which are called the elements of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1]

In general, there are three ways to describe sets and those methods are listed below.

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to describe the elements the set.
2. **The Roster Method:** Begin with a left brace '{', list each element of the set *only once* and then end with a right brace '}'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x and conditions on that variable.

Let S be the set described *verbally* as the set of letters that make up the word "smolko". A *roster* description of S is $\{$s, m, o, l, k$\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the order of the elements doesn't matter, so $\{$k, l, m, o, s$\}$ is also a roster description of S. A *set-builder* description of S is: $\{x \,|\, x$ is a letter in the word "smolko"$\}$. The way to read this is 'The set of elements x such that x is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '$=$' and write $S = \{$s, m, o, l, k$\}$ or $S = \{x \,|\, x$ is a letter in the word "smolko"$\}$.

Notice that 'm' is in S but many other letters, such as 'q', are not in S. We express these ideas of set inclusion and exclusion mathematically using the symbols m $\in S$ (read 'm is in S') and q $\notin S$ (read 'q is not in S'). More precisely, we have the following.

Definition 0.6. Let A be a set.

- If x is an element of A, then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A, then we write $x \notin A$ which is read 'x is not in A'.

[1]For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous paradox known as Russell's Paradox.

Now let's consider the set $C = \{x \,|\, x \text{ is a consonant in the word "smolko"}\}$. A roster description of C is $C = \{\text{s, m, l, k}\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given at the top of the next page.

Definition 0.7. Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

In our previous example, $C \subseteq S$ yet not vice-versa as 'o' $\in S$ but 'o' $\notin C$. Additionally, the set of vowels $V = \{\text{a, e, i, o, u}\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{\text{s, m, o, l, k, a, e, i, u}\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets as they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{\text{o}\}$, written $S \cap V = \{\text{o}\}$. We formalize these ideas below.

Definition 0.8. Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \,|\, x \in A \text{ AND } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \,|\, x \in A \text{ OR } x \in B \text{ (or both)}\}$

The key words in Definition 0.8 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). Please note that this mathematical use of the word 'or' differs than how we use 'or' in spoken English. In Math, we use the *inclusive or* which allows for the element to be in both sets. At a restaurant if you're asked "Do you want fries or a salad?" you must pick one and only one. This is known as the *exclusive or* and it plays a role in other Math classes. For our purposes it is good enough to say that for an element to belong to the union of two sets it must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{\text{s, m, l, k, a, e, i, o, u}\}$.[2] Their intersection, however, creates a bit of notational awkwardness due to the fact that C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 0.9. The **Empty Set** is the set which contains no elements and is denoted $\emptyset$. That is,

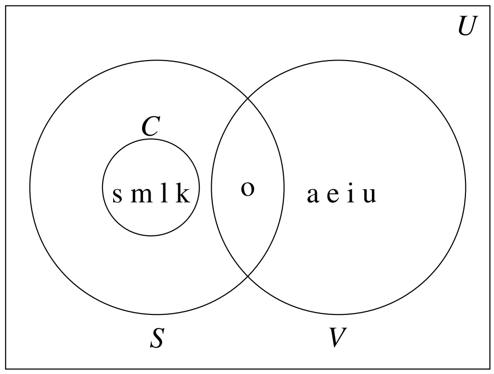
$$\emptyset = \{\} = \{x \,|\, x \neq x\}.$$

As promised, the empty set is the set containing no elements because no matter what x is, $x = x$. Like

[2]Which just so happens to be the same set as $S \cup V$.

the number 0, the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance[4] because saying that $C \cap V = \emptyset$ is unambiguous whereas $\{\}$ looks like a typographical error.

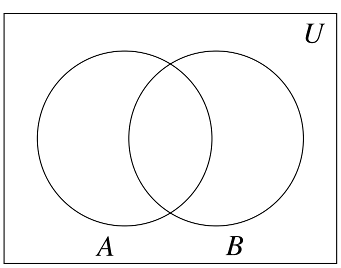
A nice way to visualize the relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn below.



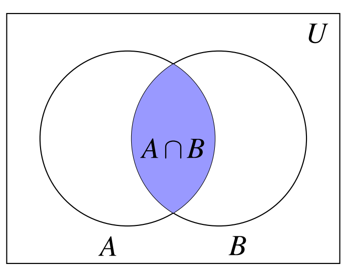
A Venn Diagram for C, S and V.

In the Venn Diagram above we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that because $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

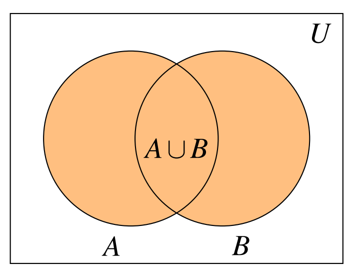
All of these circles lie in a rectangle labeled U for the 'universal' set. A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

[3] Sadly, the full extent of the empty set's role will not be explored in this text.

[4] Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!

The one major limitation of Venn Diagrams is that they become unwieldy if more than four sets need to be drawn simultaneously within the same universal set. This idea is explored in the Exercises.

0.4.2 Special Subsets of Real Numbers

Recall in Section 0.1.1 we defined some special subsets of real numbers.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis '...' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.
2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]
4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,|\, a \in \mathbb{Z} \,\text{and}\, b \in \mathbb{Z} \,\text{where}\, b \neq 0 \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers is: $\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$
5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$. That is, an irrational number is a real number which isn't rational. Said differently,

$$\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$$

[a] The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.

It is time to put all of this together in an example.

Example 0.4.1.

1. Write a roster description for $P = \{2^n \,|\, n \in \mathbb{N}\}$ and $E = \{2n \,|\, n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \,|\, x \in \mathbb{R}\}$.
3. Let $A = \{-117, \frac{4}{5}, 0.202\overline{002}, 0.202002000200002\ldots\}$.
   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?
   (b) Determine $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.
4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

Solution.

1. Write a roster description for $P = \{2^n \,|\, n \in \mathbb{N}\}$ and $E = \{2n \,|\, n \in \mathbb{Z}\}$.

   To construct roster descriptions for each of these sets, we need to list their elements. Starting with the set $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, thus a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '…' indicates that the pattern continues.[5]

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers.[6] A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

2. Write a verbal description for $S = \{x^2 \,|\, x \in \mathbb{R}\}$.

   One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper. What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[7] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. As $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number.[8] This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Because $n \geq 0$, $\sqrt{n}$ is a real number.[9] Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. Let $A = \{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\}$.

   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?

   The set A contains no natural numbers. Clearly $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[10] As for the number

[5] This isn't the most *precise* way to describe this set - it's always dangerous to use '…' because we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[6] This shouldn't be too surprising, as an even integer is *defined* to be an integer multiple of 2.

[7] This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

[8] This means S is a subset of the non-negative real numbers.

[9] This is called the 'square root closed property' of the non-negative real numbers.

[10] So $0.20\overline{2002} = 0.20200220022002\ldots$.

0.202002000200002..., the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, because all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

(b) Determine $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.

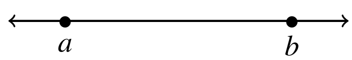
The set $A \cap \mathbb{W} = \{x \,|\, x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in A which are whole numbers. As A contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in A which are integers. -117 is the only integer in A, so $A \cap \mathbb{Z} = \{-117\}$. For the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, thus $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Because every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[11] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. □

0.4.3 The Real Number Line and Interval Notation

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown below.



The real number line with two numbers a and b where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line.[12] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and one irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and infinitely many irrational numbers.

[11]In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

[12]Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness is given a much more precise meaning later in courses like Analysis and Topology.

The root word ‘dense’ here communicates the idea that rationals and irrationals are ‘thoroughly mixed’ into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you’ve done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. (‘Try’ is the operative word, here.)

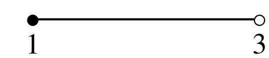
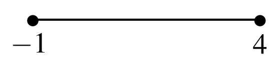
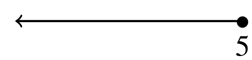
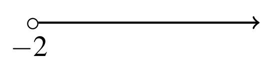
The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). This property is given below.

Law of Trichotomy

If a and b are real numbers then exactly one of the following statements is true:

$a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

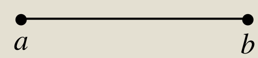
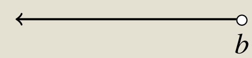
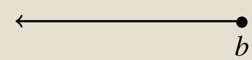
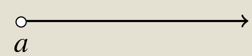
Subset of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1,3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1,4]$	
$\{x \mid x \leq 5\}$	$(-\infty,5]$	
$\{x \mid x > -2\}$	$(-2,\infty)$	

As you can glean from the table, for intervals with finite endpoints we start by writing ‘left endpoint, right endpoint’. We use square brackets, ‘[’ or ‘]’, if the endpoint is included in the interval. This corresponds to a ‘filled-in’ or ‘closed’ dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, ‘(’ or ‘)’ that correspond to an ‘open’ circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. As infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[13]

[13]The importance of understanding interval notation in this book and also in Calculus cannot be overstated so please do yourself a favor and memorize this chart.

Interval Notation

Let a and b be real numbers with $a < b$.

Subset of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a,b)	
$\{x \mid a \leq x < b\}$	$[a,b)$	
$\{x \mid a < x \leq b\}$	$(a,b]$	
$\{x \mid a \leq x \leq b\}$	$[a,b]$	
$\{x \mid x < b\}$	$(-\infty,b)$	
$\{x \mid x \leq b\}$	$(-\infty,b]$	
$\{x \mid x > a\}$	(a,∞)	
$\{x \mid x \geq a\}$	$[a,\infty)$	
$\mathbb{R}$	$(-\infty,\infty)$	

Intervals of the forms (a,b), $(-\infty,b)$ and (a,∞) are said to be **open** intervals. Those of the forms $[a,b]$, $(-\infty,b]$ and $[a,\infty)$ are said to be **closed** intervals.

Unfortunately, the words 'open' and 'closed' are not antonyms here because the empty set $\emptyset$ and the set $(-\infty,\infty)$ are simultaneously open and closed[14] while the intervals $(a,b]$ and $[a,b)$ are neither open nor closed. The inclusion or exclusion of an endpoint might seem like a terribly small thing to fuss about but these sorts of technicalities in the language become important in Calculus so we feel the need to put this

[14] You don't need to worry about that fact until you take an advanced course in Topology.

material in the Precalculus book.

We close this section with an example that ties together some of the concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of union and intersection to describe a variety of sets on the real number line. In many sections of the text to come you will need to be fluent with this notation so take the time to study it deeply now.

Example 0.4.2.

1. Express the following sets of numbers using interval notation.

   (a) $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$

   (b) $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$

   (c) $\{x \,|\, x \neq \pm 3\}$

   (d) $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

Solution.

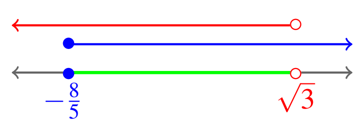
1. (a) Express $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$ using interval notation.

   The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.

$(-\infty, -2] \cup [2, \infty)$

   (b) Express $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$ using interval notation.

   For the set $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$, we need the real numbers less than (to the left of) $\sqrt{3}$ that are simultaneously greater than (to the right of) $-\frac{8}{5}$, including $-\frac{8}{5}$ but excluding $\sqrt{3}$. This yields $\{x \,|\, x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\} = \left[-\frac{8}{5}, \sqrt{3}\right)$.



$\left[-\frac{8}{5}, \sqrt{3}\right)$

   (c) Express $\{x \,|\, x \neq \pm 3\}$ using interval notation.

   For the set $\{x \,|\, x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Refer back to page 2 for a discussion about $x = \pm 3$) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Because the set describes real numbers which come from the first, second *or* third interval, we have $\{x \,|\, x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

$(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

(d) Express $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$ using interval notation.

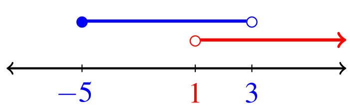
Graphing the set $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1,3]$ along with the single number 5. While we *could* express this single point as $[5,5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. This means that our final answer is written $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\} = (-1,3]\cup\{5\}$.

$(-1,3]\cup\{5\}$

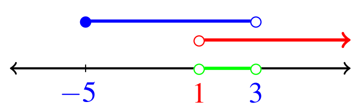
2. Let $A = [-5,3)$ and $B = (1,\infty)$. Find $A\cap B$ and $A\cup B$.

We start by graphing $A = [-5,3)$ and $B = (1,\infty)$ on the number line. To find $A\cap B$, we need to find the numbers common to both A and B; in other words, we need to find the overlap of the two intervals. Clearly, everything between 1 and 3 is in both A and B. However, because 1 is in A but not in B, 1 is not in the intersection. Similarly, because 3 is in B but not in A, it isn't in the intersection either. Hence, $A\cap B = (1,3)$.

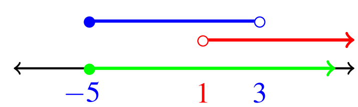
To find $A\cup B$, we need to find the numbers in at least one of A or B. Graphically, we shade A and B along with it. Notice here that even though 1 isn't in B, it is in A, so it's in the union along with all of the other elements of A between -5 and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both A and B together gives us $A\cup B = [-5,\infty)$.



$A = [-5,3)$, $B = (1,\infty)$



$A\cap B = (1,3)$



$A\cup B = [-5,\infty)$ $\square$

0.4.4 Exercises

1. Find a verbal description for $O = \{2n-1 \,|\, n \in \mathbb{N}\}$

2. Find a roster description for $X = \{z^2 \,|\, z \in \mathbb{Z}\}$

3. Let $A = \left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \frac{20}{10}, 117\right\}$

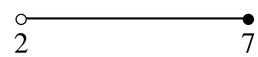
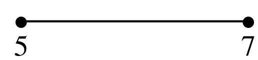
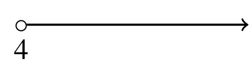
   (a) List the elements of A which are natural numbers.

(b) List the elements of A which are irrational numbers.

(c) Find $A \cap \mathbb{Z}$

(d) Find $A \cap \mathbb{Q}$

4. Fill in the chart below.

Subset of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0,3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3,3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty,9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 5 - 10, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1,5] \cap [0,8)$

6. $(-1,1) \cup [0,6]$

7. $(-\infty,4] \cap (0,\infty)$

8. $(-\infty,0) \cap [1,5]$

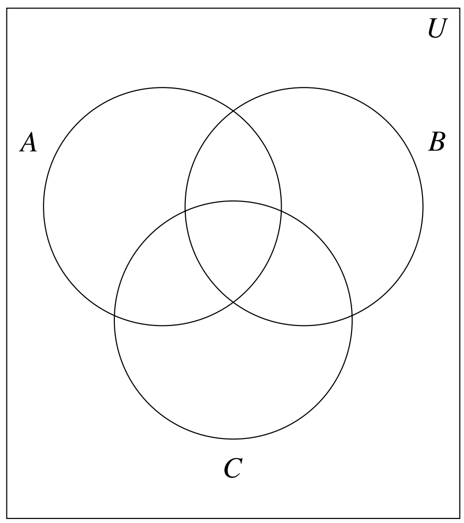
9. $(-\infty,0) \cup [1,5]$

10. $(-\infty,5] \cap [5,8)$

In Exercises 11 - 22, write the set using interval notation.

11. $\{x \mid x \neq 5\}$
12. $\{x \mid x \neq -1\}$
13. $\{x \mid x \neq -3, 4\}$
14. $\{x \mid x \neq 0, 2\}$
15. $\{x \mid x \neq 2, -2\}$
16. $\{x \mid x \neq 0, \pm 4\}$
17. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$
18. $\{x \mid x < 3 \text{ and } x \geq 2\}$
19. $\{x \mid x \leq -3 \text{ or } x > 0\}$
20. $\{x \mid x \leq 2 \text{ and } x > 3\}$
21. $\{x \mid x > 2 \text{ or } x = \pm 1\}$
22. $\{x \mid 3 < x < 13 \text{ and } x \neq 4\}$

For Exercises 23 - 28, use the blank Venn Diagram below with A, B, and C in it as a guide to help you shade the following sets.



23. $A \cup C$
24. $B \cap C$
25. $(A \cup B) \cup C$
26. $(A \cap B) \cap C$
27. $A \cap (B \cup C)$
28. $(A \cap B) \cup (A \cap C)$
29. Explain how your answers to problems 27 and 28 show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.
30. Show that $A \subseteq B$ if and only if $A \cup B = B$.
31. Let $A = \{1,3,5,7,9\}, B = \{2,4,6,8,10\}, C = \{1,6,9\}$ and $D = \{2,7,10\}$. Draw one Venn Diagram that shows all four of these sets. What sort of difficulties do you encounter?

SECTION 0.4 EXERCISE ANSWERS A.1.0

0.5 Solving Equations

In general, equations and inequalities fall into one of three categories: conditional, identity or contradiction, depending on the nature of their solutions. A **conditional** equation or inequality is true for only *certain* real numbers. For example, $2x+1=7$ is true precisely when $x=3$, and $w-3 \leq 4$ is true precisely when $w \leq 7$. An **identity** is an equation or inequality that is true for *all* real numbers. For example, $2x-3=1+x-4+x$ or $2t \leq 2t+3$. A **contradiction** is an equation or inequality that is *never* true. Examples here include $3x-4=3x+7$ and $a-1>a+3$.

As you may recall, solving an equation or inequality means finding all of the values of the variable, if any exist, which make the given equation or inequality true. This often requires us to manipulate the given equation or inequality from its given form to an easier form. For example, if we're asked to solve $3-2(x-3)=7x+3(x+1)$, we get $x=\frac{1}{2}$, but not without a fair amount of algebraic manipulation. In order to obtain the correct answer(s), however, we need to make sure that whatever maneuvers we apply are reversible in order to guarantee that we maintain a chain of **equivalent** equations or inequalities. Two equations or inequalities are called **equivalent** if they have the same solutions. We summarize these 'legal moves' in the box below.

Procedures which Generate Equivalent Equations

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **nonzero** real number.[a]

Procedures which Generate Equivalent Inequalities

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **positive** real number.[b]

[a] Multiplying both sides of an equation by 0 collapses the equation to $0=0$, which doesn't do anybody any good.

[b] Remember that if you multiply both sides of an inequality by a negative real number, the inequality sign is reversed: $3 \leq 4$, but $(-2)(3) \geq (-2)(4)$.

0.5.1 Linear Equations

The first equations we wish to review are **linear** equations as defined below.

Definition 0.10. An equation is said to be **linear** in a variable x if it can be written in the form $ax=b$ where a and b are expressions which do not involve x and $a \neq 0$.

One key point about Definition 0.10 is that the exponent on the unknown 'x' in the equation is 1, that is $x=x^1$. Our main strategy for solving linear equations is summarized below.

Strategy for Solving Linear Equations

In order to solve an equation which is linear in a given variable, say x:

1. Isolate all of the terms containing x on one side of the equation, putting all of the terms not containing x on the other side of the equation.

2. Factor out the x and divide both sides of the equation by its coefficient.

We illustrate this process with a collection of examples below.

Example 0.5.1. Solve the following equations for the indicated variable. Check your answer.

1. Solve for x: $3x-6=7x+4$
2. Solve for t: $3-1.7t=\frac{t}{4}$
3. Solve for a: $\frac{1}{18}(7-4a)+2=\frac{a}{3}-\frac{4-a}{12}$
4. Solve for y: $8y\sqrt{3}+1=7-\sqrt{12}(5-y)$
5. Solve for x: $\frac{3x-1}{2}=x\sqrt{50}+4$
6. Solve for y: $x(4-y)=8y$

Solution.

1. Solve for x: $3x-6=7x+4$.

   The variable we are asked to solve for is x so our first move is to gather all of the terms involving x on one side and put the remaining terms on the other.[1]

$$
\begin{array}{rcll}
3x-6 & = & 7x+4 & \\
(3x-6)-7x+6 & = & (7x+4)-7x+6 & \text{Subtract } 7x\text{, add } 6 \\
3x-7x-6+6 & = & 7x-7x+4+6 & \text{Rearrange terms} \\
-4x & = & 10 & 3x-7x=(3-7)x=-4x \\
\dfrac{-4x}{-4} & = & \dfrac{10}{-4} & \text{Divide by the coefficient of } x \\
x & = & -\dfrac{5}{2} & \text{Reduce to lowest terms}
\end{array}
$$

   To check our answer, we substitute $x=-\frac{5}{2}$ into each side of the **original** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right)-6=-\frac{27}{2}$ and $7\left(-\frac{5}{2}\right)+4=-\frac{27}{2}$.

2. Solve for t: $3-1.7t=\frac{t}{4}$.

[1] In the margin notes, when we speak of operations, e.g., 'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

In our next example, the unknown is t and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$\begin{array}{rcll}
3-1.7t &=& \dfrac{t}{4} & \\
40(3-1.7t) &=& 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3)-40(1.7t) &=& \dfrac{40t}{4} & \text{Distribute} \\
120-68t &=& 10t & \\
(120-68t)+68t &=& 10t+68t & \text{Add } 68t \text{ to both sides} \\
120 &=& 78t & 68t+10t=(68+10)t=78t \\
\dfrac{120}{78} &=& \dfrac{78t}{78} & \text{Divide by the coefficient of } t \\
\dfrac{120}{78} &=& t & \\
\dfrac{20}{13} &=& t & \text{Reduce to lowest terms}
\end{array}$$

To check, we again substitute $t=\frac{20}{13}$ into each side of the original equation. We find that $3-1.7\left(\frac{20}{13}\right)=3-\left(\frac{17}{10}\right)\left(\frac{20}{13}\right)=\frac{5}{13}$ and $\frac{(20/13)}{4}=\frac{20}{13}\cdot\frac{1}{4}=\frac{5}{13}$ as well.

3. Solve for a: $\dfrac{1}{18}(7-4a)+2=\dfrac{a}{3}-\dfrac{4-a}{12}$.

To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$\begin{array}{rcll}
\dfrac{1}{18}(7-4a)+2 &=& \dfrac{a}{3}-\dfrac{4-a}{12} & \\
36\left(\dfrac{1}{18}(7-4a)+2\right) &=& 36\left(\dfrac{a}{3}-\dfrac{4-a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7-4a)+(36)(2) &=& \dfrac{36a}{3}-\dfrac{36(4-a)}{12} & \text{Distribute} \\
2(7-4a)+72 &=& 12a-3(4-a) & \text{Distribute} \\
14-8a+72 &=& 12a-12+3a & \\
86-8a &=& 15a-12 & 12a+3a=(12+3)a=15a \\
(86-8a)+8a+12 &=& (15a-12)+8a+12 & \text{Add } 8a \text{ and } 12 \\
86+12-8a+8a &=& 15a+8a-12+12 & \text{Rearrange terms} \\
98 &=& 23a & 15a+8a=(15+8)a=23a \\
\dfrac{98}{23} &=& \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} &=& a &
\end{array}$$

The check, as usual, involves substituting $a = \frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. Solve for y: $8y\sqrt{3}+1 = 7-\sqrt{12}(5-y)$.

The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case y) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
(8y\sqrt{3}+1)-1-2y\sqrt{3} & = & (7-10\sqrt{3}+2y\sqrt{3})-1-2y\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
8y\sqrt{3}-2y\sqrt{3}+1-1 & = & 7-1-10\sqrt{3}+2y\sqrt{3}-2y\sqrt{3} & \text{Rearrange terms} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
\dfrac{6y\sqrt{3}}{6\sqrt{3}} & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide } 6\sqrt{3} \\
y & = & \dfrac{2\cdot\sqrt{3}\cdot\sqrt{3}-2\cdot 5\cdot\sqrt{3}}{2\cdot 3\cdot\sqrt{3}} & \\
y & = & \dfrac{\cancel{2}\cancel{\sqrt{3}}(\sqrt{3}-5)}{\cancel{2}\cdot 3\cdot\cancel{\sqrt{3}}} & \text{Factor and cancel} \\
y & = & \dfrac{\sqrt{3}-5}{3} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3} = 6-10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3} = 6-10\sqrt{3}$ is in fact linear according to Definition 0.10: the variable is y, the value of A is $6\sqrt{3}$ and $B = 6-10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Solve for x: $\dfrac{3x-1}{2} = x\sqrt{50}+4$.

Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing x on one side and move the other terms to the other, we factor out x to identify its coefficient then divide to get our answer.

$$
\begin{array}{rcll}
\dfrac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\dfrac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=\sqrt{25\cdot 2} \\
2\left(\dfrac{3x-1}{2}\right) & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by 2} \\
\dfrac{\cancel{2}\cdot(3x-1)}{\cancel{2}} & = & 2(5x\sqrt{2})+2\cdot 4 & \text{Distribute} \\
3x-1 & = & 10x\sqrt{2}+8 & \\
(3x-1)-10x\sqrt{2}+1 & = & (10x\sqrt{2}+8)-10x\sqrt{2}+1 & \text{Subtract } 10x\sqrt{2}\text{, add 1} \\
3x-10x\sqrt{2}-1+1 & = & 10x\sqrt{2}-10x\sqrt{2}+8+1 & \text{Rearrange terms} \\
3x-10x\sqrt{2} & = & 9 & \\
(3-10\sqrt{2})x & = & 9 & \text{Factor} \\
\dfrac{(3-10\sqrt{2})x}{3-10\sqrt{2}} & = & \dfrac{9}{3-10\sqrt{2}} & \text{Divide by the coefficient of } x \\
x & = & \dfrac{9}{3-10\sqrt{2}} &
\end{array}
$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\dfrac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. Solve for y: $x(4-y)=8y$.

If we were instructed to solve our last equation for x, we'd be done in one step: divide both sides by $(4-y)$ - assuming $4-y\neq 0$, that is. Alas, we are instructed to solve for y, which means we have some more work to do.

$$
\begin{array}{rcll}
x(4-y) & = & 8y & \\
4x-xy & = & 8y & \text{Distribute} \\
(4x-xy)+xy & = & 8y+xy & \text{Add } xy \\
4x & = & (8+x)y & \text{Factor}
\end{array}
$$

In order to finish the problem, we need to divide both sides of the equation by the coefficient of y which in this case is $8+x$. This expression contains a variable so we need to stipulate that we may perform this division only if $8+x\neq 0$, or, in other words, $x\neq -8$. Hence, we write our solution as:

$$y=\frac{4x}{8+x}, \quad \text{provided } x\neq -8$$

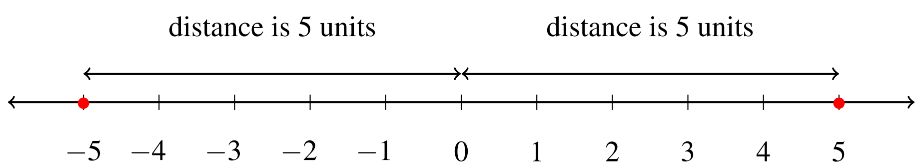
What happens if $x=-8$? Substituting $x=-8$ into the original equation gives $(-8)(4-y)=8y$ or $-32+8y=8y$. This reduces to $-32=0$, which is a contradiction. This means there is no solution when $x = -8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to plug $y = \frac{4x}{8+x}$ into the original equation and show that both sides work out to $\frac{32x}{x+8}$. $\square$

0.5.2 Absolute Value Equations

In this subsection, we review some of the basic concepts involving the absolute value of a real number x. There are a few different ways to define absolute value and in this section we choose the following definition. (Absolute value will be revisited in much greater depth in Section 1.4 where we present what one can think of as the "precise" definition.)

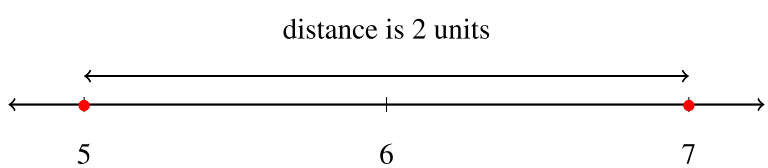
Definition 0.11. Absolute Value as Distance: For every real number x, the **absolute value** of x, denoted $|x|$, is the distance between x and 0 on the number line. More generally, if x and c are real numbers, $|x-c|$ is the distance between the numbers x and c on the number line.

For example, $|5| = 5$ and $|-5| = 5$, because each is 5 units from 0 on the number line:



Graphically why $|-5| = 5$ and $|5| = 5$

Computationally, the absolute value 'makes negative numbers positive', though we need to be a little cautious with this description. While $|-7| = 7$, $|5-7| \neq 5+7$. The absolute value acts as a grouping symbol, so $|5-7| = |-2| = 2$, which makes sense as 5 and 7 are two units away from each other on the number line:



Graphically why $|5-7| = 2$

Next, we list some of the operational properties of absolute value.

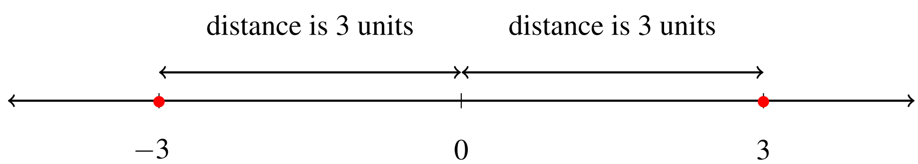
Theorem 0.3. Properties of Absolute Value: Let a and b be real numbers and let n be an integer.[a]

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$

[a]See page 2 if you don't remember what an integer is.

The proof of Theorem 0.3 is difficult, but not impossible, using the distance definition of absolute value or even the 'it makes negatives positive' notion. It is, however, much easier if one uses the "precise" definition given in Section 1.4 so we will revisit the proof then. For now, let's focus on how to solve basic equations and inequalities involving the absolute value.

Thinking of absolute value in terms of distance gives us a geometric way to interpret equations. For example, to solve $|x| = 3$, we are looking for all real numbers x whose distance from 0 is 3 units. If we move three units to the right of 0, we end up at $x = 3$. If we move three units to the left, we end up at $x = -3$. Thus the solutions to $|x| = 3$ are $x = \pm 3$.



The solutions to $|x| = 3$ are $x = \pm 3$.

Thinking this way gives us the following.

Theorem 0.4. Absolute Value Equations: Suppose x, y and c are real numbers.

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.
- $|x| = |y|$ if and only if $x = y$ or $x = -y$. (That is, if two numbers have the same absolute values, they are either the same number or exact opposites of each other.)

Theorem 0.4 is our main tool in solving equations involving the absolute value, as it allows us a way to rewrite such equations as compound linear equations.

Strategy for Solving Equations Involving Absolute Value

In order to solve an equation involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the equation so it has the form $|X| = c$.

2. Apply Theorem 0.4.

The techniques we use to 'isolate the absolute value' are precisely those we used in the previous subsection to isolate the variable when solving linear equations. Time for some practice.

Example 0.5.2. Solve each of the following equations.

1. $|3x-1| = 6$
2. $\dfrac{3-|y+5|}{2} = 1$
3. $3|2t+1| - \sqrt{5} = 0$
4. $4-|5w+3| = 5$
5. $\left|3 - x\sqrt[3]{12}\right| = |4x+1|$
6. $|t-1| - 3|t+1| = 0$

Solution.

1. Solve the equation: $|3x-1| = 6$.

   The equation $|3x-1| = 6$ is of already in the form $|X| = c$, so we know that either $3x-1 = 6$ or $3x-1 = -6$. Solving the former gives us at $x = \frac{7}{3}$ and solving the latter yields $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. Solve the equation: $\dfrac{3-|y+5|}{2} = 1$.

   We begin solving $\frac{3-|y+5|}{2} = 1$ by isolating the absolute value to put it in the form $|X| = c$.

$$\begin{array}{rcll} \dfrac{3-|y+5|}{2} & = & 1 & \\ 3-|y+5| & = & 2 & \text{Multiply by } 2 \\ -|y+5| & = & -1 & \text{Subtract } 3 \\ |y+5| & = & 1 & \text{Divide by } -1 \end{array}$$

   At this point, we have $y+5 = 1$ or $y+5 = -1$, so our solutions are $y = -4$ or $y = -6$. We leave it to the reader to check both answers in the original equation.

3. Solve the equation: $3|2t+1| - \sqrt{5} = 0$.

   As in the previous example, we first isolate the absolute value. Don't let the $\sqrt{5}$ throw you off - it's just another real number, so we treat it as such:

$$\begin{aligned} 3|2t+1|-\sqrt{5} &= 0 \\ 3|2t+1| &= \sqrt{5} && \text{Add } \sqrt{5} \\ |2t+1| &= \frac{\sqrt{5}}{3} && \text{Divide by 3} \end{aligned}$$

From here, we have that $2t+1 = \frac{\sqrt{5}}{3}$ or $2t+1 = -\frac{\sqrt{5}}{3}$. The first equation gives $t = \frac{\sqrt{5}-3}{6}$ while the second gives $t = \frac{-\sqrt{5}-3}{6}$ thus we list our answers as $t = \frac{-3\pm\sqrt{5}}{6}$. The reader should enjoy the challenge of substituting both answers into the original equation and following through the arithmetic to see that both answers work.

4. Solve the equation: $4-|5w+3| = 5$.

   Upon isolating the absolute value in the equation $4-|5w+3| = 5$, we get $|5w+3| = -1$. At this point, we know there cannot be any real solution. By definition, the absolute value is a *distance*, and as such is never negative. We write 'no solution' and carry on.

5. Solve the equation: $\left|3-x\sqrt[3]{12}\right| = |4x+1|$.

   Our next equation already has the absolute value expressions (plural) isolated, so we work from the principle that if $|x| = |y|$, then $x = y$ or $x = -y$. Thus from $\left|3-x\sqrt[3]{12}\right| = |4x+1|$ we get two equations to solve:

   $$3-x\sqrt[3]{12} = 4x+1, \qquad \text{and} \qquad 3-x\sqrt[3]{12} = -(4x+1)$$

   Notice that the right side of the second equation is $-(4x+1)$ and not simply $-4x+1$. Remember, the expression $4x+1$ represents a single real number so in order to negate it we need to negate the *entire* expression $-(4x+1)$. Moving along, when solving $3-x\sqrt[3]{12} = 4x+1$, we obtain $x = \frac{2}{4+\sqrt[3]{12}}$ and the solution to $3-x\sqrt[3]{12} = -(4x+1)$ is $x = \frac{4}{\sqrt[3]{12}-4}$. As usual, the reader is invited to check these answers by substituting them into the original equation.

6. Solve the equation: $|t-1|-3|t+1| = 0$.

   We start by isolating one of the absolute value expressions: $|t-1|-3|t+1| = 0$ gives $|t-1| = 3|t+1|$. While this *resembles* the form $|x| = |y|$, the coefficient 3 in $3|t+1|$ prevents it from being an exact match. Not to worry - because 3 is positive, $3 = |3|$ so

   $$3|t+1| = |3||t+1| = |3(t+1)| = |3t+3|.$$

   Hence, our equation becomes $|t-1| = |3t+3|$ which results in the two equations: $t-1 = 3t+3$ and $t-1 = -(3t+3)$. The first equation gives $t = -2$ and the second gives $t = -\frac{1}{2}$. The reader is encouraged to check both answers in the original equation. $\square$

0.5.3 Solving Equations by Factoring

Many students wonder why they are forced to learn how to factor. Simply put, factoring is our main tool for solving the non-linear equations which arise in many of the applications of Mathematics.[2] We use factoring in conjunction with the Zero Product Property of Real Numbers which was first stated on page 4 and is given here again for reference.

The Zero Product Property of Real Numbers: If a and b are real numbers with $ab = 0$ then either $a = 0$ or $b = 0$ or both.

Consider the equation $6x^2 + 11x = 10$. To see how the Zero Product Property is used to help us solve this equation, we first set the equation equal to zero and then apply the techniques from Example 0.3.2:

$$\begin{array}{rcll} 6x^2+11x & = & 10 & \\ 6x^2+11x-10 & = & 0 & \text{Subtract 10 from both sides} \\ (2x+5)(3x-2) & = & 0 & \text{Factor} \\ 2x+5=0 & \text{or} & 3x-2=0 & \text{Zero Product Property} \\ & & & a = 2x+5, b = 3x-2 \\ x = -\frac{5}{2} & \text{or} & x = \frac{2}{3} & \end{array}$$

The reader should check that both of these solutions satisfy the original equation.

It is critical that you see the importance of setting the expression equal to 0 before factoring. Otherwise, we'd get something silly like:

$$\begin{array}{rcll} 6x^2+11x & = & 10 & \\ x(6x+11) & = & 10 & \text{Factor} \end{array}$$

What we **cannot** deduce from this equation is that $x = 10$ or $6x + 11 = 10$ or that $x = 2$ and $6x + 11 = 5$. (It's wrong and you should feel bad if you do it.) It is precisely because 0 plays such a special role in the arithmetic of real numbers (as the Additive Identity) that we can assume a factor is 0 when the product is 0. No other real number has that ability.

We summarize the **correct** equation solving strategy below.

Strategy for Solving Non-linear Equations

1. Put all of the nonzero terms on one side of the equation so that the other side is 0.
2. Factor.
3. Use the Zero Product Property of Real Numbers and set each factor equal to 0.
4. Solve each of the resulting equations.

[2] Also known as 'story problems' or 'real-world examples'.

Let's finish the subsection with a collection of examples in which we use this strategy.

Example 0.5.3. Solve the following equations.

1. $3x^2 = 35 - 16x$
2. $t = \dfrac{1+4t^2}{4}$
3. $(y-1)^2 = 2(y-1)$
4. $\dfrac{w^4}{3} = \dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4}$
5. $z(z(18z+9)-50) = 25$
6. $x^4 - 8x^2 - 9 = 0$

Solution.

1. Solve the equation: $3x^2 = 35 - 16x$.

   We begin by gathering all of the nonzero terms to one side getting 0 on the other. Then we proceed to factor and apply the Zero Product Property.

$$\begin{array}{rcll} 3x^2 & = & 35-16x & \\ 3x^2+16x-35 & = & 0 & \text{Add } 16x\text{, subtract } 35 \\ (3x-5)(x+7) & = & 0 & \text{Factor} \\ 3x-5=0 & \text{or} & x+7=0 & \text{Zero Product Property} \\ x=\frac{5}{3} & \text{or} & x=-7 & \end{array}$$

   We check our answers by substituting each of them into the original equation. Plugging in $x = \frac{5}{3}$ yields $\frac{25}{3}$ on both sides while $x = -7$ gives 147 on both sides.

2. Solve the equation: $t = \dfrac{1+4t^2}{4}$.

   To solve $t = \frac{1+4t^2}{4}$, we first clear fractions. Then move all of the nonzero terms to one side of the equation, factor and apply the Zero Product Property.

$$\begin{array}{rcll} t & = & \dfrac{1+4t^2}{4} & \\ 4t & = & 1+4t^2 & \text{Clear fractions (multiply by 4)} \\ 0 & = & 1+4t^2-4t & \text{Subtract 4} \\ 0 & = & 4t^2-4t+1 & \text{Rearrange terms} \\ 0 & = & (2t-1)^2 & \text{Factor (Perfect Square Trinomial)} \end{array}$$

   At this point, we get $(2t-1)^2 = (2t-1)(2t-1) = 0$, so, the Zero Product Property gives us $2t-1=0$ in both cases.[3] Our final answer is $t = \frac{1}{2}$, which we invite the reader to check.

[3] More generally, given a positive power p, the only solution to $x^p = 0$ is $x = 0$.

3. Solve the equation: $(y-1)^2 = 2(y-1)$.

   Following the strategy outlined above, the first step to solving $(y-1)^2 = 2(y-1)$ is to gather the nonzero terms on one side of the equation with 0 on the other side and factor.

$$\begin{array}{rcll} (y-1)^2 & = & 2(y-1) & \\ (y-1)^2 - 2(y-1) & = & 0 & \text{Subtract } 2(y-1) \\ (y-1)[(y-1)-2] & = & 0 & \text{Factor out G.C.F.} \\ (y-1)(y-3) & = & 0 & \text{Simplify} \\ y-1=0 & \text{or} & y-3=0 & \\ y=1 & \text{or} & y=3 & \end{array}$$

   Both of these answers are easily checked by substituting them into the original equation.

   An alternative method to solving this equation is to begin by dividing both sides by $(y-1)$ to simplify things outright. As we saw in Example 0.5.1, however, whenever we divide by a variable quantity, we make the explicit assumption that this quantity is nonzero. Thus we must stipulate that $y-1 \neq 0$.

$$\begin{array}{rcll} \dfrac{(y-1)^2}{(y-1)} & = & \dfrac{2(y-1)}{(y-1)} & \text{Divide by } (y-1) \text{ - this assumes } (y-1) \neq 0 \\ y-1 & = & 2 & \\ y & = & 3 & \end{array}$$

   Note that in this approach, we obtain the $y=3$ solution, but we 'lose' the $y=1$ solution. How did that happen? Assuming $y-1 \neq 0$ is equivalent to assuming $y \neq 1$. This is an issue because $y=1$ is a solution to the original equation and it was 'divided out' too early. The moral of the story? If you decide to divide by a variable expression, double check that you aren't excluding any solutions.[4]

4. Solve the equation: $\dfrac{w^4}{3} = \dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4}$

   Proceeding as before, we clear fractions, gather the nonzero terms on one side of the equation, have 0 on the other and factor.

$$\begin{array}{rcll} \dfrac{w^4}{3} & = & \dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4} & \\ 12\left(\dfrac{w^4}{3}\right) & = & 12\left(\dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4}\right) & \text{Multiply by 12} \\ 4w^4 & = & (8w^3-12) - 3(w^2-4) & \text{Distribute} \\ 4w^4 & = & 8w^3 - 12 - 3w^2 + 12 & \text{Distribute} \\ 0 & = & 8w^3 - 12 - 3w^2 + 12 - 4w^4 & \text{Subtract } 4w^4 \\ 0 & = & 8w^3 - 3w^2 - 4w^4 & \text{Gather like terms} \\ 0 & = & w^2(8w - 3 - 4w^2) & \text{Factor out G.C.F.} \end{array}$$

[4] You will see other examples throughout this text where dividing by a variable quantity does more harm than good. Keep this basic one in mind as you move on in your studies - it's a good cautionary tale.

At this point, we apply the Zero Product Property to deduce that $w^2 = 0$ or $8w - 3 - 4w^2 = 0$. From $w^2 = 0$, we get $w = 0$. To solve $8w - 3 - 4w^2 = 0$, we rearrange terms and factor: $-4w^2 + 8w - 3 = (2w-1)(-2w+3) = 0$. Applying the Zero Product Property again, we get $2w - 1 = 0$ (which gives $w = \frac{1}{2}$), or $-2w + 3 = 0$ (which gives $w = \frac{3}{2}$). Our final answers are $w = 0$, $w = \frac{1}{2}$ and $w = \frac{3}{2}$. The reader is encouraged to check each of these answers in the original equation. (You need the practice with fractions!)

5. Solve the equation: $z(z(18z+9)-50) = 25$

   For our next example, we begin by subtracting the 25 from both sides then work out the indicated operations before factoring by grouping.

$$\begin{array}{rcll} z(z(18z+9)-50) & = & 25 & \\ z(z(18z+9)-50)-25 & = & 0 & \text{Subtract 25} \\ z(18z^2+9z-50)-25 & = & 0 & \text{Distribute} \\ 18z^3+9z^2-50z-25 & = & 0 & \text{Distribute} \\ 9z^2(2z+1)-25(2z+1) & = & 0 & \text{Factor} \\ (9z^2-25)(2z+1) & = & 0 & \text{Factor} \end{array}$$

   At this point, we use the Zero Product Property and get $9z^2 - 25 = 0$ or $2z + 1 = 0$. The latter gives $z = -\frac{1}{2}$ whereas the former factors as $(3z-5)(3z+5) = 0$. Applying the Zero Product Property again gives $3z - 5 = 0$ (so $z = \frac{5}{3}$) or $3z + 5 = 0$ (so $z = -\frac{5}{3}$.) Our final answers are $z = -\frac{1}{2}$, $z = \frac{5}{3}$ and $z = -\frac{5}{3}$, each of which is good fun to check.

6. Solve the equation: $x^4 - 8x^2 - 9 = 0$.

   The nonzero terms of the equation $x^4 - 8x^2 - 9 = 0$ are already on one side of the equation so we proceed to factor. This trinomial doesn't fit the pattern of a perfect square so we attempt to reverse the F.O.I.L.ing process. With an x^4 term, we have two possible forms to try: $(ax^2+b)(cx^2+d)$ and $(ax^3+b)(cx+d)$. We leave it to you to show that $(ax^3+b)(cx+d)$ does not work and we show that $(ax^2+b)(cx^2+d)$ does.

   Due to the fact that the coefficient of x^4 is 1, we take $a = c = 1$. The constant term is -9 so we know b and d have opposite signs and our choices are limited to two options: either b and d come from ± 1 and ∓ 9 OR one is 3 while the other is -3. After some trial and error, we get $x^4 - 8x^2 - 9 = (x^2-9)(x^2+1)$. Hence $x^4 - 8x^2 - 9 = 0$ reduces to $(x^2-9)(x^2+1) = 0$. The Zero Product Property tells us that either $x^2 - 9 = 0$ or $x^2 + 1 = 0$. To solve the former, we factor: $(x-3)(x+3) = 0$, so $x - 3 = 0$ (hence, $x = 3$) or $x + 3 = 0$ (hence, $x = -3$). The equation $x^2 + 1 = 0$ has no (real) solution, because for any real number x, x^2 is always 0 or greater. Thus $x^2 + 1$ is always positive. Our final answers are $x = 3$ and $x = -3$. As always, the reader is invited to check both answers in the original equation. □

0.5.4 Solving Radical Equations

Theorem 0.2 allows us to generalize the process of 'Extracting Square Roots' to 'Extracting n$^{\text{th}}$ Roots' which in turn allows us to solve equations[5] of the form $X^n = c$.

Extracting n$^{\text{th}}$ roots:

- If c is a real number and n is odd then the real number solution to $X^n = c$ is $X = \sqrt[n]{c}$.
- If $c \geq 0$ and n is even then the real number solutions to $X^n = c$ are $X = \pm\sqrt[n]{c}$.

Note: If $c < 0$ and n is even then $X^n = c$ has no real number solutions.

Essentially, we solve $X^n = c$ by 'taking the n$^{\text{th}}$ root' of both sides: $\sqrt[n]{X^n} = \sqrt[n]{c}$. Simplifying the left side gives us just X if n is odd or $|X|$ if n is even. In the first case, $X = \sqrt[n]{c}$, and in the second, $X = \pm\sqrt[n]{c}$. Putting this together with the other part of Theorem 0.2, namely $(\sqrt[n]{a})^n = a$, gives us a strategy for solving equations which involve n$^{\text{th}}$ powers and n^{th} roots.

Strategies for Solving Power and Radical Equations

- If the equation involves an n$^{\text{th}}$ power and the variable appears in only one term, isolate the term with the n$^{\text{th}}$ power and extract n$^{\text{th}}$ roots.
- If the equation involves an n$^{\text{th}}$ root and the variable appears in that n$^{\text{th}}$ root, isolate the n$^{\text{th}}$ root and raise both sides of the equation to the n$^{\text{th}}$ power.

Note: When raising both sides of an equation to an *even* power, be sure to check for extraneous solutions.

The note about 'extraneous solutions' can be demonstrated by the basic equation: $\sqrt{x} = -2$. This equation has no solution because, by definition, $\sqrt{x} \geq 0$ for all real numbers x. However, if we square both sides of this equation, we get $(\sqrt{x})^2 = (-2)^2$ or $x = 4$. However, $x = 4$ doesn't check in the original equation, as $\sqrt{4} = 2$, not -2. Once again, the root[6] of all of our problems lies in the fact that a *negative* number to an *even* power results in a *positive* number. In other words, raising both sides of an equation to an even power does *not* produce an equivalent equation, but rather, an equation which may possess *more* solutions than the original. Hence the cautionary remark above about extraneous solutions.

Example 0.5.4. Solve the following equations.

1. $(5x+3)^4 = 16$
2. $1 - \dfrac{(5-2w)^3}{7} = 9$
3. $t + \sqrt{2t+3} = 6$

[5] Well, not entirely. The equation $x^7 = 1$ has seven answers: $x = 1$ and six complex number solutions which we'll find using techniques in Section **??**.

[6] Pun intended!

4. $\sqrt{2} - 3\sqrt[3]{2y+1} = 0$
5. $\sqrt{4x-1} + 2\sqrt{1-2x} = 1$
6. $\sqrt[4]{n^2+2} + n = 0$

For the remaining problems, assume that all of the variables represent positive real numbers.[7]

7. Solve for r: $V = \dfrac{4\pi}{3}(R^3 - r^3)$.
8. Solve for M_1: $\dfrac{r_1}{r_2} = \sqrt{\dfrac{M_2}{M_1}}$
9. Solve for v: $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$. Again, assume that no arithmetic rules are violated.

Solution.

1. Solve the equation: $(5x+3)^4 = 16$.

   In our first equation, the quantity containing x is already isolated, so we extract fourth roots. The exponent is even, so when the roots are extracted we need both the positive and negative roots.

$$\begin{array}{rcll} (5x+3)^4 & = & 16 & \\ 5x+3 & = & \pm\sqrt[4]{16} & \text{Extract fourth roots} \\ 5x+3 & = & \pm 2 & \\ 5x+3=2 & \text{or} & 5x+3=-2 & \\ x = -\dfrac{1}{5} & \text{or} & x=-1 & \end{array}$$

   We leave it to the reader to verify that both of these solutions satisfy the original equation.

2. Solve the equation: $1 - \dfrac{(5-2w)^3}{7} = 9$.

   In this example, we first need to isolate the quantity containing the variable w. Here, third (cube) roots are required and the exponent (index) is odd indicating we do not need the $\pm$:

$$\begin{array}{rcll} 1 - \dfrac{(5-2w)^3}{7} & = & 9 & \\ -\dfrac{(5-2w)^3}{7} & = & 8 & \text{Subtract 1} \\ (5-2w)^3 & = & -56 & \text{Multiply by } -7 \\ 5-2w & = & \sqrt[3]{-56} & \text{Extract cube root} \\ 5-2w & = & \sqrt[3]{(-8)(7)} & \end{array}$$

[7] That is, you needn't worry that you're multiplying or dividing by 0 or that you're forgetting absolute value symbols.

$$\begin{aligned} 5-2w &= \sqrt[3]{-8}\sqrt[3]{7} && \text{Product Rule} \\ 5-2w &= -2\sqrt[3]{7} && \\ -2w &= -5-2\sqrt[3]{7} && \text{Subtract 5} \\ w &= \frac{-5-2\sqrt[3]{7}}{-2} && \text{Divide by } -2 \\ w &= \frac{5+2\sqrt[3]{7}}{2} && \text{Properties of Negatives} \end{aligned}$$

The reader should check the answer because it provides a hearty review of arithmetic.

3. Solve the equation: $t+\sqrt{2t+3}=6$.

   To solve $t+\sqrt{2t+3}=6$, we first isolate the square root, then proceed to square both sides of the equation. In doing so, we run the risk of introducing extraneous solutions therefore checking our answers here is a necessity.

$$\begin{aligned} t+\sqrt{2t+3} &= 6 && \\ \sqrt{2t+3} &= 6-t && \text{Subtract } t \\ (\sqrt{2t+3})^2 &= (6-t)^2 && \text{Square both sides} \\ 2t+3 &= 36-12t+t^2 && \text{F.O.I.L. / Perfect Square Trinomial} \\ 0 &= t^2-14t+33 && \text{Subtract } 2t \text{ and } 3 \\ 0 &= (t-3)(t-11) && \text{Factor} \end{aligned}$$

   From the Zero Product Property, we know either $t-3=0$ (which gives $t=3$) or $t-11=0$ (which gives $t=11$). When checking our answers, we find $t=3$ satisfies the original equation, but $t=11$ does not.[8] So our final answer is $t=3$ only.

4. Solve the equation: $\sqrt{2}-3\sqrt[3]{2y+1}=0$.

   In our next example, we locate the variable (in this case y) beneath a cube root, so we first isolate that root and cube both sides.

$$\begin{aligned} \sqrt{2}-3\sqrt[3]{2y+1} &= 0 && \\ -3\sqrt[3]{2y+1} &= -\sqrt{2} && \text{Subtract } \sqrt{2} \\ \sqrt[3]{2y+1} &= \frac{-\sqrt{2}}{-3} && \text{Divide by } -3 \end{aligned}$$

[8] It is worth noting that when $t=11$ is substituted into the original equation, we get $11+\sqrt{25}=6$. If the $+\sqrt{25}$ were $-\sqrt{25}$, the solution would check. Once again, when squaring both sides of an equation, we lose track of $\pm$, which is what lets extraneous solutions in the door.

$$\begin{array}{rcll}
\sqrt[3]{2y+1} & = & \dfrac{\sqrt{2}}{3} & \text{Properties of Negatives} \\
\left(\sqrt[3]{2y+1}\right)^3 & = & \left(\dfrac{\sqrt{2}}{3}\right)^3 & \text{Cube both sides} \\
2y+1 & = & \dfrac{(\sqrt{2})^3}{3^3} & \\
2y+1 & = & \dfrac{2\sqrt{2}}{27} & \\
2y & = & \dfrac{2\sqrt{2}}{27}-1 & \text{Subtract 1} \\
2y & = & \dfrac{2\sqrt{2}}{27}-\dfrac{27}{27} & \text{Common denominators} \\
2y & = & \dfrac{2\sqrt{2}-27}{27} & \text{Subtract fractions} \\
y & = & \dfrac{2\sqrt{2}-27}{54} & \text{Divide by 2 } \left(\text{multiply by } \frac{1}{2}\right)
\end{array}$$

As we raised both sides to an *odd* power, we don't need to worry about extraneous solutions but we encourage the reader to check the solution just for the fun of it.

5. Solve the equation: $\sqrt{4x-1}+2\sqrt{1-2x}=1$.

In the equation $\sqrt{4x-1}+2\sqrt{1-2x}=1$, we have not one but two square roots. We begin by isolating one of the square roots and squaring both sides.

$$\begin{array}{rcll}
\sqrt{4x-1}+2\sqrt{1-2x} & = & 1 & \\
\sqrt{4x-1} & = & 1-2\sqrt{1-2x} & \text{Subtract } 2\sqrt{1-2x} \text{ from both sides} \\
\left(\sqrt{4x-1}\right)^2 & = & \left(1-2\sqrt{1-2x}\right)^2 & \text{Square both sides} \\
4x-1 & = & 1-4\sqrt{1-2x}+\left(2\sqrt{1-2x}\right)^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
4x-1 & = & 1-4\sqrt{1-2x}+4(1-2x) & \\
4x-1 & = & 1-4\sqrt{1-2x}+4-8x & \text{Distribute} \\
4x-1 & = & 5-8x-4\sqrt{1-2x} & \text{Gather like terms}
\end{array}$$

At this point, we have just one square root so we proceed to isolate it and square both sides a second time.[9]

[9] To avoid complications with fractions, we'll forego dividing by the coefficient of $\sqrt{1-2x}$, namely -4. This is perfectly fine so long as we don't forget to square it when we square both sides of the equation.

$$
\begin{array}{rcll}
4x-1 & = & 5-8x-4\sqrt{1-2x} & \\
12x-6 & = & -4\sqrt{1-2x} & \text{Subtract 5, add } 8x \\
(12x-6)^2 & = & (-4\sqrt{1-2x})^2 & \text{Square both sides} \\
144x^2-144x+36 & = & 16(1-2x) & \\
144x^2-144x+36 & = & 16-32x & \\
144x^2-112x+20 & = & 0 & \text{Subtract 16, add } 32x \\
4(36x^2-28x+5) & = & 0 & \text{Factor} \\
4(2x-1)(18x-5) & = & 0 & \text{Factor some more}
\end{array}
$$

From the Zero Product Property, we know either $2x-1=0$ or $18x-5=0$. The former gives $x=\frac{1}{2}$ while the latter gives us $x=\frac{5}{18}$. As we squared both sides of the equation (twice!), we need to check for extraneous solutions. We determine $x=\frac{5}{18}$ to be extraneous, so our only solution is $x=\frac{1}{2}$.

6. Solve the equation: $\sqrt[4]{n^2+2}+n=0$.

   As usual, our first step in solving $\sqrt[4]{n^2+2}+n=0$ is to isolate the radical. We then proceed to raise both sides to the fourth power to eliminate the fourth root:

$$
\begin{array}{rcll}
\sqrt[4]{n^2+2}+n & = & 0 & \\
\sqrt[4]{n^2+2} & = & -n & \text{Subtract } n \\
(\sqrt[4]{n^2+2})^4 & = & (-n)^4 & \text{Raise both sides to the } 4^{\text{th}} \text{ power} \\
n^2+2 & = & n^4 & \text{Properties of Negatives} \\
0 & = & n^4-n^2-2 & \text{Subtract } n^2 \text{ and } 2 \\
0 & = & (n^2-2)(n^2+1) & \text{Factor - this is a `Quadratic in Disguise'}
\end{array}
$$

   At this point, the Zero Product Property gives either $n^2-2=0$ or $n^2+1=0$. From $n^2-2=0$, we get $n^2=2$, so $n=\pm\sqrt{2}$. From $n^2+1=0$, we get $n^2=-1$, which gives no real solutions.[10] As we raised both sides to an even (the fourth) power, we need to check for extraneous solutions. We determine that $n=-\sqrt{2}$ works but $n=\sqrt{2}$ is extraneous.

7. Solve for r: $V=\dfrac{4\pi}{3}(R^3-r^3)$.

   In this problem, we are asked to solve for r. While there are a lot of letters in this equation[11], r appears

[10] Why is that again?

[11] including a Greek letter, no less!

in only one term: r^3. Our strategy is to isolate r^3 then extract the cube root.

$$
\begin{array}{rcll}
V & = & \dfrac{4\pi}{3}\left(R^3 - r^3\right) & \\
3V & = & 4\pi\left(R^3 - r^3\right) & \text{Multiply by 3 to clear fractions} \\
3V & = & 4\pi R^3 - 4\pi r^3 & \text{Distribute} \\
3V - 4\pi R^3 & = & -4\pi r^3 & \text{Subtract } 4\pi R^3 \\
\dfrac{3V - 4\pi R^3}{-4\pi} & = & r^3 & \text{Divide by } -4\pi \\
\dfrac{4\pi R^3 - 3V}{4\pi} & = & r^3 & \text{Properties of Negatives} \\
\sqrt[3]{\dfrac{4\pi R^3 - 3V}{4\pi}} & = & r & \text{Extract the cube root}
\end{array}
$$

The check is, as always, left to the reader and highly encouraged.

8. Solve the equation: Solve for M_1: $\dfrac{r_1}{r_2} = \sqrt{\dfrac{M_2}{M_1}}$.

   The equation we are asked to solve in this example is from the world of Chemistry and is none other than Graham's Law of Effusion. Subscripts in Mathematics are used to distinguish between variables and have no arithmetic significance. In this example, r_1, r_2, M_1 and M_2 are as different as x, y, z and 117. We are asked to solve for M_1, so we locate M_1 and see it is in the denominator of a fraction which is inside of a square root. We eliminate the square root by squaring both sides and proceed from there.

$$
\begin{array}{rcll}
\dfrac{r_1}{r_2} & = & \sqrt{\dfrac{M_2}{M_1}} & \\
\left(\dfrac{r_1}{r_2}\right)^2 & = & \left(\sqrt{\dfrac{M_2}{M_1}}\right)^2 & \text{Square both sides} \\
\dfrac{r_1^2}{r_2^2} & = & \dfrac{M_2}{M_1} & \\
r_1^2 M_1 & = & M_2 r_2^2 & \text{Multiply by } r_2^2 M_1 \text{ to clear fractions, assume } r_2, M_1 \neq 0 \\
M_1 & = & \dfrac{M_2 r_2^2}{r_1^2} & \text{Divide by } r_1^2\text{, assume } r_1 \neq 0
\end{array}
$$

   As the reader may expect, checking the answer amounts to a good exercise in simplifying rational and radical expressions. The fact that we are assuming all of the variables represent positive real numbers comes in to play, as well.

9. Solve for v: $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

Our last equation to solve comes from Einstein's Special Theory of Relativity and relates the mass of an object to its velocity as it moves.[12] We are asked to solve for v which is located in just one term, namely v^2, which happens to lie in a fraction underneath a square root which is itself a denominator. We have quite a lot of work ahead of us!

$$\begin{array}{rcll}
m & = & \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} & \\
m\sqrt{1-\dfrac{v^2}{c^2}} & = & m_0 & \text{Multiply by } \sqrt{1-\dfrac{v^2}{c^2}} \text{ to clear fractions} \\
\left(m\sqrt{1-\dfrac{v^2}{c^2}}\right)^2 & = & m_0^2 & \text{Square both sides} \\
m^2\left(1-\dfrac{v^2}{c^2}\right) & = & m_0^2 & \text{Properties of Exponents} \\
m^2-\dfrac{m^2v^2}{c^2} & = & m_0^2 & \text{Distribute} \\
-\dfrac{m^2v^2}{c^2} & = & m_0^2-m^2 & \text{Subtract } m^2 \\
m^2v^2 & = & -c^2(m_0^2-m^2) & \text{Multiply by } -c^2 \ (c^2 \neq 0) \\
m^2v^2 & = & -c^2m_0^2+c^2m^2 & \text{Distribute} \\
v^2 & = & \dfrac{c^2m^2-c^2m_0^2}{m^2} & \text{Rearrange terms, divide by } m^2 \ (m^2 \neq 0) \\
v & = & \sqrt{\dfrac{c^2m^2-c^2m_0^2}{m^2}} & \text{Extract Square Roots, } v>0 \text{ so no } \pm \\
v & = & \dfrac{\sqrt{c^2(m^2-m_0^2)}}{\sqrt{m^2}} & \text{Properties of Radicals, factor} \\
v & = & \dfrac{|c|\sqrt{m^2-m_0^2}}{|m|} & \\
v & = & \dfrac{c\sqrt{m^2-m_0^2}}{m} & c>0 \text{ and } m>0 \text{ so } |c|=c \text{ and } |m|=m
\end{array}$$

Checking the answer algebraically would earn the reader great honor and respect on the Algebra battlefield, so it is highly recommended. □

0.5.5 Solving Quadratic Equations

In subsection 0.5.3, we reviewed how to solve basic non-linear equations by factoring. The astute reader should have noticed that all of the equations in that subsection were carefully constructed so that the poly-

[12] See this article on the Lorentz Factor.

nomials could be factored using the integers. To demonstrate just how contrived the equations had to be, we can solve $2x^2+5x-3=0$ by factoring, $(2x-1)(x+3)=0$, from which we obtain $x=\frac{1}{2}$ and $x=-3$. If we change the 5 to a 6 and try to solve $2x^2+6x-3=0$, however, we find that this polynomial doesn't factor over the integers and we are stuck. It turns out that there are two real number solutions to this equation, but they are *irrational* numbers, and the goal of this subsection is to review the techniques which allow us to find these solutions.[13] In this subsection, we focus our attention on **quadratic** equations.

Definition 0.12. An equation is said to be **quadratic** in a variable x if it can be written in the form $ax^2+bx+c=0$ where a, b and c are expressions which do not involve x and $a \neq 0$.

Think of quadratic equations as equations that are one degree up from linear equations - instead of the highest power of x being just $x=x^1$, it's x^2. The simplest class of quadratic equations to solve are the ones in which $b=0$. In that case, we have the following.

Solving Quadratic Equations by Extracting Square Roots

If c is a real number with $c \geq 0$, the solutions to $x^2=c$ are $x=\pm\sqrt{c}$. **Note:** If $c<0$, $x^2=c$ has no real number solutions.

There are a couple different ways to see why Extracting Square Roots works, both of which are demonstrated by solving the equation $x^2=3$. If we follow the procedure outlined in a previous subsection, we subtract 3 from both sides to get $x^2-3=0$ and we now try to factor x^2-3. As mentioned in the remarks following Definition 0.4, we could think of $x^2-3=x^2-(\sqrt{3})^2$ and apply the Difference of Squares formula to factor $x^2-3=(x-\sqrt{3})(x+\sqrt{3})$. We solve $(x-\sqrt{3})(x+\sqrt{3})=0$ by using the Zero Product Property as before by setting each factor equal to zero: $x-\sqrt{3}=0$ and $x+\sqrt{3}-0$. We get the answers $x=\pm\sqrt{3}$. In general, if $c \geq 0$, then $\sqrt{c}$ is a real number, so $x^2-c=x^2-(\sqrt{c})^2=(x-\sqrt{c})(x+\sqrt{c})$. Replacing the '3' with 'c' in the above discussion gives the general result.

Another way to view this result is to visualize 'taking the square root' of both sides: as $x^2=c$, $\sqrt{x^2}=\sqrt{c}$. How do we simplify $\sqrt{x^2}$? We have to exercise a bit of caution here. Note that $\sqrt{(5)^2}$ and $\sqrt{(-5)^2}$ both simplify to $\sqrt{25}=5$. In both cases, $\sqrt{x^2}$ returned a *positive* number, because the negative in -5 was 'squared away' *before* we took the square root. In other words, $\sqrt{x^2}$ is x if x is positive, or, if x is negative, we make x positive - that is, $\sqrt{x^2}=|x|$, the absolute value of x. So from $x^2=3$, we 'take the square root' of both sides of the equation to get $\sqrt{x^2}=\sqrt{3}$. This simplifies to $|x|=\sqrt{3}$, which by Theorem 0.4 is equivalent to $x=\sqrt{3}$ or $x=-\sqrt{3}$. Replacing the '3' in the previous argument with 'c,' gives the general result.

As you might expect, Extracting Square Roots can be applied to more complicated equations. Consider the equation below. We can solve it by Extracting Square Roots provided we first isolate the quantity that is

[13] While our discussion in this subsection departs from factoring, we'll see in Chapter 2 that the same correspondence between factoring and solving equations holds whether or not the polynomial factors over the integers.

being squared :

$$
\begin{aligned}
2\left(x+\frac{3}{2}\right)^2-\frac{15}{2} &= 0 & \\
2\left(x+\frac{3}{2}\right)^2 &= \frac{15}{2} & \text{Add } \frac{15}{2} \\
\left(x+\frac{3}{2}\right)^2 &= \frac{15}{4} & \text{Divide by 2} \\
x+\frac{3}{2} &= \pm\sqrt{\frac{15}{4}} & \text{Extract Square Roots} \\
x+\frac{3}{2} &= \pm\frac{\sqrt{15}}{2} & \text{Property of Radicals} \\
x &= -\frac{3}{2}\pm\frac{\sqrt{15}}{2} & \text{Subtract } \frac{3}{2} \\
x &= -\frac{3\pm\sqrt{15}}{2} & \text{Add fractions}
\end{aligned}
$$

Let's return to the equation $2x^2+6x-3=0$ from the beginning of the section. We leave it to the reader to expand the left side and show that

$$2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}=2x^2+6x-3.$$

In other words, we can solve $2x^2+6x-3=0$ by *transforming* into an equivalent equation. This process, you may recall, is called 'Completing the Square.' We'll revisit Completing the Square in Section 2.1 in more generality and for a different purpose but for now we revisit the steps needed to complete the square to solve a quadratic equation.

Solving Quadratic Equations: Completing the Square

To solve a quadratic equation $ax^2+bx+c=0$ by Completing the Square:

1. Subtract the constant c from both sides.
2. Divide both sides by a, the coefficient of x^2. (Remember: $a \neq 0$.)
3. Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation. (That's half the coefficient of x, squared.)
4. Factor the left hand side of the equation as $\left(x+\frac{b}{2a}\right)^2$.
5. Extract Square Roots.
6. Subtract $\frac{b}{2a}$ from both sides.

To refresh our memories, we apply this method to solve $3x^2 - 24x + 5 = 0$:

$$\begin{array}{rclr}
3x^2 - 24x + 5 & = & 0 & \\
3x^2 - 24x & = & -5 & \text{Subtract } c = 55 \\
x^2 - 8x & = & -\dfrac{5}{3} & \text{Divide by } a = 3 \\
x^2 - 8x + 16 & = & -\dfrac{5}{3} + 16 & \text{Add } \left(\frac{b}{2a}\right)^2 = (-4)^2 = 16 \\
(x-4)^2 & = & \dfrac{43}{3} & \text{Factor: Perfect Square Trinomial} \\
x - 4 & = & \pm\sqrt{\dfrac{43}{3}} & \text{Extract Square Roots} \\
x & = & 4 \pm \sqrt{\dfrac{43}{3}} & \text{Add } 4
\end{array}$$

At this point, we use properties of fractions and radicals to 'rationalize' the denominator:[14]

$$\sqrt{\frac{43}{3}} = \sqrt{\frac{43 \cdot 3}{3 \cdot 3}} = \frac{\sqrt{129}}{\sqrt{9}} = \frac{\sqrt{129}}{3}$$

We can now get a common (integer) denominator which yields:

$$x = 4 \pm \sqrt{\frac{43}{3}} = 4 \pm \frac{\sqrt{129}}{3} = \frac{12 \pm \sqrt{129}}{3}$$

The key to Completing the Square is that the procedure always produces a perfect square trinomial. To see why this works *every single time*, we start with $ax^2 + bx + c = 0$ and follow the procedure:

$$\begin{array}{rclr}
ax^2 + bx + c & = & 0 & \\
ax^2 + bx & = & -c & \text{Subtract } c \\
x^2 + \dfrac{bx}{a} & = & -\dfrac{c}{a} & \text{Divide by } a \neq 0 \\
x^2 + \dfrac{bx}{a} + \left(\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a} + \left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2
\end{array}$$

(Hold onto the line above for a moment.) Here's the heart of the method - we need to show that

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

To show this, we start with the right side of the equation and apply the Perfect Square Formula.

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 \checkmark$$

[14] Recall that this means we want to get a denominator with rational (more specifically, integer) numbers.

With just a few more steps we can solve the general equation $ax^2+bx+c=0$ so let's pick up the story where we left off. (The line on the previous page we told you to hold on to.)

$$\begin{array}{rcll}
x^2+\dfrac{bx}{a}+\left(\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a}+\left(\dfrac{b}{2a}\right)^2 & \\
\left(x+\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a}+\dfrac{b^2}{4a^2} & \text{Factor: Perfect Square Trinomial} \\
\left(x+\dfrac{b}{2a}\right)^2 & = & -\dfrac{4ac}{4a^2}+\dfrac{b^2}{4a^2} & \text{Get a common denominator} \\
\left(x+\dfrac{b}{2a}\right)^2 & = & \dfrac{b^2-4ac}{4a^2} & \text{Add fractions} \\
x+\dfrac{b}{2a} & = & \pm\sqrt{\dfrac{b^2-4ac}{4a^2}} & \text{Extract Square Roots} \\
x+\dfrac{b}{2a} & = & \pm\dfrac{\sqrt{b^2-4ac}}{2a} & \text{Properties of Radicals} \\
x & = & -\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a} & \text{Subtract } \dfrac{b}{2a} \\
x & = & \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} & \text{Add fractions.}
\end{array}$$

Lo and behold, we have derived the legendary **Quadratic Formula**!

Theorem 0.5. Quadratic Formula: The solution(s) to $ax^2+bx+c=0$ with $a\neq 0$ is/are:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We can check our earlier solutions to $2x^2+6x-3=0$ and $3x^2-24x+5=0$ using the Quadratic Formula. For $2x^2+6x-3=0$, we identify $a=2$, $b=6$ and $c=-3$. The quadratic formula gives:

$$x=\frac{-6\pm\sqrt{6^2-4(2)(-3)}}{2(2)}-\frac{-6\pm\sqrt{36+24}}{4}=\frac{-6\pm\sqrt{60}}{4}$$

Using properties of radicals ($\sqrt{60}=2\sqrt{15}$), this reduces to $\frac{2(-3\pm\sqrt{15})}{4}=\frac{-3\pm\sqrt{15}}{2}$. We leave it to the reader to show these two answers are the same as $-\frac{3\pm\sqrt{15}}{2}$, as required.[15]

For $3x^2-24x+5=0$, we identify $a=3$, $b=-24$ and $c=5$. Here, we get:

$$x=\frac{-(-24)\pm\sqrt{(-24)^2-4(3)(5)}}{2(3)}=\frac{24\pm\sqrt{516}}{6}$$

[15] Think about what $-(3\pm\sqrt{15})$ is really telling you.

$\sqrt{516} = 2\sqrt{129}$, so this reduces to $x = \frac{12 \pm \sqrt{129}}{3}$.

It is worth noting that the Quadratic Formula applies to all quadratic equations - even ones we could solve using other techniques. For example, to solve $2x^2 + 5x - 3 = 0$ we identify $a = 2$, $b = 5$ and $c = -3$. Plugging those into the Quadratic Formula yields:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

At this point, we have $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = \frac{-12}{4} = -3$ - the same two answers we obtained factoring. We can also use it to solve $x^2 = 3$, if we wanted to. From $x^2 - 3 = 0$, we have $a = 1$, $b = 0$ and $c = -3$. The Quadratic Formula produces

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)} = \frac{\pm\sqrt{12}}{2} = \pm\frac{2\sqrt{3}}{2} = \pm\sqrt{3}$$

As this last example illustrates, while the Quadratic Formula *can* be used to solve every quadratic equation, that doesn't mean it *should* be used. Many times other methods are more efficient. We now provide a more comprehensive approach to solving Quadratic Equations.

Strategies for Solving Quadratic Equations

- If the variable appears in the squared term only, isolate it and Extract Square Roots.
- Otherwise, put the nonzero terms on one side of the equation so that the other side is 0.
  - Try factoring.
  - If the expression doesn't factor easily, use the Quadratic Formula.

The reader is encouraged to pause for a moment to think about why 'Completing the Square' doesn't appear in our list of strategies despite the fact that we've spent the majority of the subsection so far talking about it. Let's get some practice solving quadratic equations, shall we?

Example 0.5.5. Calculate all real number solutions to the following equations.

1. $3 - (2w-1)^2 = 0$
2. $5x - x(x-3) = 7$
3. $(y-1)^2 = 2 - \dfrac{y+2}{3}$
4. $5(25 - 21x) = \dfrac{59}{4} - 25x^2$
5. $-4.9t^2 + 10t\sqrt{3} + 2 = 0$
6. $2x^2 = 3x^4 - 6$

Solution.

1. Calculate all real number solutions to $3 - (2w-1)^2 = 0$.

As $3-(2w-1)^2=0$ contains a perfect square, we isolate it first then extract square roots:

$$\begin{array}{rcll} 3-(2w-1)^2 & = & 0 & \\ 3 & = & (2w-1)^2 & \text{Add } (2w-1)^2 \\ \pm\sqrt{3} & = & 2w-1 & \text{Extract Square Roots} \\ 1\pm\sqrt{3} & = & 2w & \text{Add } 1 \\ \dfrac{1\pm\sqrt{3}}{2} & = & w & \text{Divide by } 2 \end{array}$$

We find our two answers: $w=\frac{1\pm\sqrt{3}}{2}$. The reader is encouraged to check both answers by substituting each into the original equation.[16]

2. Calculate all real number solutions to $5x-x(x-3)=7$.

   To solve $5x-x(x-3)=7$, we perform the indicated operations and set one side equal to 0.

$$\begin{array}{rcll} 5x-x(x-3) & = & 7 & \\ 5x-x^2+3x & = & 7 & \text{Distribute} \\ -x^2+8x & = & 7 & \text{Gather like terms} \\ -x^2+8x-7 & = & 0 & \text{Subtract } 7 \end{array}$$

   At this point, we attempt to factor and find $-x^2+8x-7=(x-1)(-x+7)$. Using the Zero Product Property, we get $x-1=0$ or $-x+7=0$. Our answers are $x=1$ or $x=7$, which are easily verified.

3. Calculate all real number solutions to $(y-1)^2=2-\dfrac{y+2}{3}$.

   Even though we have a perfect square in $(y-1)^2=2-\frac{y+2}{3}$, Extracting Square Roots won't help matters as we have a y on the other side of the equation. Our strategy here is to perform the indicated operations (and clear the fraction for good measure) and then get 0 on one side of the equation.

$$\begin{array}{rcll} (y-1)^2 & = & 2-\dfrac{y+2}{3} & \\ y^2-2y+1 & = & 2-\dfrac{y+2}{3} & \text{Perfect Square Trinomial} \\ 3(y^2-2y+1) & = & 3\left(2-\dfrac{y+2}{3}\right) & \text{Multiply by } 3 \\ 3y^2-6y+3 & = & 6-3\left(\dfrac{y+2}{3}\right) & \text{Distribute} \\ 3y^2-6y+3 & = & 6-(y+2) & \\ 3y^2-6y+3-6+(y+2) & = & 0 & \text{Subtract 6, Add } (y+2) \\ 3y^2-5y-1 & = & 0 & \end{array}$$

[16]It's excellent practice working with radicals and fractions so we really, *really* want you to take the time to do it.

A cursory attempt at factoring bears no fruit, so we run this through the Quadratic Formula with $a = 3$, $b = -5$ and $c = -1$.

$$\begin{aligned} y &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} \\ y &= \frac{5 \pm \sqrt{25+12}}{6} \\ y &= \frac{5 \pm \sqrt{37}}{6} \end{aligned}$$

37 is prime, so we have no way to reduce $\sqrt{37}$. Thus, our final answers are $y = \frac{5\pm\sqrt{37}}{6}$. The reader is encouraged to supply the details of the challenging verification of the answers.

4. Calculate all real number solutions to $5(25-21x) = \dfrac{59}{4} - 25x^2$.

   We proceed as before; our goal is to gather the nonzero terms on one side of the equation.

$$\begin{aligned} 5(25-21x) &= \frac{59}{4} - 25x^2 & & \\ 125 - 105x &= \frac{59}{4} - 25x^2 & & \text{Distribute} \\ 4(125-105x) &= 4\left(\frac{59}{4} - 25x^2\right) & & \text{Multiply by 4} \\ 500 - 420x &= 59 - 100x^2 & & \text{Distribute} \\ 500 - 420x - 59 + 100x^2 &= 0 & & \text{Subtract 59, Add } 100x^2 \\ 100x^2 - 420x + 441 &= 0 & & \text{Gather like terms} \end{aligned}$$

   With highly composite numbers like 100 and 441, factoring seems inefficient at best,[17] so we apply the Quadratic Formula with $a = 100$, $b = -420$ and $c = 441$:

$$\begin{aligned} x &= \frac{-(-420) \pm \sqrt{(-420)^2 - 4(100)(441)}}{2(100)} \\ &= \frac{420 \pm \sqrt{176400 - 176400}}{200} \\ &= \frac{420 \pm \sqrt{0}}{200} \\ &= \frac{420 \pm 0}{200} \end{aligned}$$

[17] This is actually the Perfect Square Trinomial $(10x-21)^2$.

$$\begin{aligned} &= \frac{420}{200} \\ &= \frac{21}{10} \end{aligned}$$

To our surprise and delight we obtain just one answer, $x = \frac{21}{10}$.

5. Calculate all real number solutions to $-4.9t^2 + 10t\sqrt{3} + 2 = 0$.

   Our next equation $-4.9t^2 + 10t\sqrt{3} + 2 = 0$, already has 0 on one side of the equation, but with coefficients like -4.9 and $10\sqrt{3}$, factoring with integers is not an option. We could make things a *bit* easier by clearing the decimal (by multiplying through by 10) to get $-49t^2 + 100t\sqrt{3} + 20 = 0$ but we simply cannot rid ourselves of the irrational number $\sqrt{3}$. The Quadratic Formula is our only recourse. With $a = -49$, $b = 100\sqrt{3}$ and $c = 20$ we get:

$$\begin{aligned} t &= \frac{-100\sqrt{3} \pm \sqrt{(100\sqrt{3})^2 - 4(-49)(20)}}{2(-49)} \\ &= \frac{-100\sqrt{3} \pm \sqrt{30000 + 3920}}{-98} \\ &= \frac{-100\sqrt{3} \pm \sqrt{33920}}{-98} \\ &= \frac{-100\sqrt{3} \pm 8\sqrt{530}}{-98} \\ &= \frac{2(-50\sqrt{3} \pm 4\sqrt{530})}{2(-49)} \\ &= \frac{-50\sqrt{3} \pm 4\sqrt{530}}{-49} && \text{Reduce} \\ &= \frac{-(-50\sqrt{3} \pm 4\sqrt{530})}{49} && \text{Properties of Negatives} \\ &= \frac{50\sqrt{3} \mp 4\sqrt{530}}{49} && \text{Distribute} \end{aligned}$$

   You'll note that when we 'distributed' the negative in the last step, we changed the '$\pm$' to a '$\mp$.' While this is technically correct, at the end of the day both symbols mean 'plus or minus',[18] so we can write our answers as $t = \frac{50\sqrt{3} \pm 4\sqrt{530}}{49}$. Checking these answers are a true test of arithmetic mettle.

6. Calculate all real number solutions to $2x^2 = 3x^4 - 6$.

[18] There are instances where we need both symbols, however. For example, the Sum and Difference of Cubes Formulas (page 27) can be written as a single formula: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. In this case, all of the 'top' symbols are read to give the sum formula; the 'bottom' symbols give the difference formula.

At first glance, the equation $2x^2 = 3x^4 - 6$ seems misplaced. The highest power of the variable x here is 4, not 2, so this equation isn't a quadratic equation - at least not in terms of the variable x. It is, however, an example of an equation that is 'Quadratic in Disguise'.[19] We introduce a new variable u to help us see the pattern - specifically we let $u = x^2$. Thus $u^2 = (x^2)^2 = x^4$. So in terms of the variable u, the equation $2x^2 = 3x^4 - 6$ is $2u = 3u^2 - 6$. The latter is a quadratic equation, which we can solve using the usual techniques:

$$\begin{array}{rcll} 2u & = & 3u^2 - 6 & \\ 0 & = & 3u^2 - 2u - 6 & \text{Subtract } 2u \end{array}$$

After a few attempts at factoring, we resort to the Quadratic Formula with $a = 3$, $b = -2$ and $c = -6$ to get the following:

$$\begin{array}{rcll} u & = & \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} & \\ & = & \dfrac{2 \pm \sqrt{4+72}}{6} & \\ & = & \dfrac{2 \pm \sqrt{76}}{6} & \\ & = & \dfrac{2 \pm \sqrt{4 \cdot 19}}{6} & \\ & = & \dfrac{2 \pm 2\sqrt{19}}{6} & \text{Properties of Radicals} \\ & = & \dfrac{2(1 \pm \sqrt{19})}{2(3)} & \text{Factor} \\ & = & \dfrac{1 \pm \sqrt{19}}{3} & \text{Reduce} \end{array}$$

We've solved the equation for u, but what we still need to solve the original equation - which means we need to find the corresponding values of x. We stated $u = x^2$, thus we now have we have two equations:

$$x^2 = \frac{1+\sqrt{19}}{3} \quad \text{or} \quad x^2 = \frac{1-\sqrt{19}}{3}$$

We can solve the first equation by extracting square roots to get $x = \pm\sqrt{\frac{1+\sqrt{19}}{3}}$. The second equation, however, has no real number solutions because $\frac{1-\sqrt{19}}{3}$ is a negative number. For our final answers we can rationalize the denominator to get:

$$x = \pm\sqrt{\frac{1+\sqrt{19}}{3}} = \pm\sqrt{\frac{1+\sqrt{19}}{3} \cdot \frac{3}{3}} = \pm\frac{\sqrt{3+3\sqrt{19}}}{3}$$

[19] More formally, **quadratic in form.**

As with the previous exercise, the very challenging check is left to the reader. □

Our last example above, the 'Quadratic in Disguise', hints that the Quadratic Formula is applicable to a wider class of equations than those which are strictly quadratic. Next, we give some general guidelines to recognizing these beasts in the wild.

Identifying Quadratics in Disguise

An equation is a 'Quadratic in Disguise' if it can be written in the form: $ax^{2m} + bx^m + c = 0$. In other words:

- There are exactly three terms, two with variables and one constant term.
- The exponent on the variable in one term is *exactly twice* the variable on the other term.

To transform a Quadratic in Disguise to a quadratic equation, let $u = x^m$ so $u^2 = (x^m)^2 = x^{2m}$. This transforms the equation into $au^2 + bu + c = 0$.

For example, $3x^6 - 2x^3 + 1 = 0$ is a Quadratic in Disguise, because $6 = 2 \cdot 3$. If we let $u = x^3$, we get $u^2 = (x^3)^2 = x^6$, so the equation becomes $3u^2 - 2u + 1 = 0$. However, $3x^6 - 2x^2 + 1 = 0$ is *not* a Quadratic in Disguise, because $6 \neq 2 \cdot 2$. The substitution $u = x^2$ yields $u^2 = (x^2)^2 = x^4$, not x^6 as required. We'll see more instances of 'Quadratics in Disguise' in later sections.

We close this subsection with a review of the **discriminant** of a quadratic equation as defined below.

Definition 0.13. The Discriminant: Given a quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is called the **discriminant** of the equation.

The discriminant is the radicand of the square root in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It *discriminates* between the nature and number of solutions we get from a quadratic equation. The results are summarized below.

Theorem 0.6. Discriminant Theorem: Given a Quadratic Equation $ax^2 + bx + c = 0$, let $D = b^2 - 4ac$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one repeated real number solution.
  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-b \pm 0}{2a}$ reduce to $-\frac{b}{2a}$.
- If $D < 0$, there are no real solutions.

For example, the equation $x^2+x-1=0$ has two real number solutions as a result of the discriminant working out to be $(1)^2-4(1)(-1)=5>0$. This results in a $\pm\sqrt{5}$ in the Quadratic Formula which then generates two different answers. On the other hand, $x^2+x+1=0$ has no real solutions due to the discriminant being $(1)^2-4(1)(1)=-3<0$; which generates a $\pm\sqrt{-3}$ in the Quadratic Formula. The equation $x^2+2x+1=0$ has discriminant $(2)^2-4(1)(1)=0$ so in the Quadratic Formula we get a $\pm\sqrt{0}=0$ thereby generating just one solution. More can be said as well. For example, the discriminant of $6x^2-x-40=0$ is 961. This is a perfect square, $\sqrt{961}=31$, which means our solutions are rational numbers. When our solutions are rational numbers, the quadratic actually factors nicely. In our example $6x^2-x-40=(2x+5)(3x-8)$. Admittedly, if you've already computed the discriminant, you're most of the way done with the problem and probably wouldn't take the time to experiment with factoring the quadratic at this point – but we'll see another use for this analysis of the discriminant in Example 3.1.1.

0.5.6 Complex Numbers

The results of subsection 0.5.5 tell us that the equation $x^2+1=0$ has no real number solutions. However, it *would* have solutions if we could make sense of $\sqrt{-1}$. The **Complex Numbers** do just that - they give us a mechanism for working with $\sqrt{-1}$. As such, the set of complex numbers fill in an algebraic gap left by the set of real numbers.

Here's the basic plan. There is no real number x with $x^2=-1$, due to the fact that for any real number $x^2\geq 0$. However, we could formally extract square roots and write $x=\pm\sqrt{-1}$. We build the complex numbers by relabeling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**.[20] The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i)=6i$, $7i-3i=4i$, $(2-7i)+(3+4i)=5-3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition 0.14. The imaginary unit, i, satisfies the two following properties:

1. $i^2=-1$
2. If c is a real number with $c\geq 0$, then $\sqrt{-c}=i\sqrt{c}$

Property 1 in Definition 0.14 establishes that i does act as a square root[21] of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4}=i\sqrt{4}=i(2)=2i$. However, $\sqrt{-(-4)}\neq i\sqrt{-4}$, otherwise, we'd get

$$2=\sqrt{4}=\sqrt{-(-4)}=i\sqrt{-4}=i(2i)=2i^2=2(-1)=-2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even

[20]Some Technical Mathematics textbooks label it 'j'. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

[21]Note the use of the indefinite article 'a'. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

roots of negative quantities. With Definition 0.14 in place, we can define the set of **complex numbers**.

Definition 0.15. A **complex number** is a number of the form $a+bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

Complex numbers include things you'd normally expect, like $3+2i$ and $\frac{2}{5}-i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers,[22] so 0 and $\pi-\sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 0.14. The next example should help recall how these animals behave.

Example 0.5.6. Perform the indicated operations.

1. $\sqrt{-3}\sqrt{-12}$
2. $\sqrt{(-3)(-12)}$

Solution.

1. Simplify $\sqrt{-3}\sqrt{-12}$.

   We use property 2 of Definition 0.14 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12}=\left(i\sqrt{3}\right)\left(i\sqrt{12}\right)=i^2\sqrt{3\cdot 12}=-\sqrt{36}=-6$.

2. Simplify $\sqrt{(-3)(-12)}$.

   We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)}=\sqrt{36}=6$. □

Example 0.5.7. Determine the complex solutions to the following equations.[23]

1. $\dfrac{2x}{x+1}=x+3$
2. $2t^4=9t^2+5$
3. $z^3+1=0$

Solution.

1. Determine the complex solutions to $\dfrac{2x}{x+1}=x+3$.

[22] To use the language of Section 0.1.1, $\mathbb{R}\subseteq\mathbb{C}$.

[23] Remember, all real numbers are complex numbers, so 'complex solutions' means both real and non-real answers.

Clearing fractions yields a quadratic equation so we then proceed as in Section 0.5.5.

$$\begin{array}{rcll}
\dfrac{2x}{x+1} & = & x+3 & \\
2x & = & (x+3)(x+1) & \text{Multiply by } (x+1) \text{ to clear denominators} \\
2x & = & x^2+x+3x+3 & \text{F.O.I.L.} \\
2x & = & x^2+4x+3 & \text{Gather like terms} \\
0 & = & x^2+2x+3 & \text{Subtract } 2x
\end{array}$$

From here, we apply the Quadratic Formula

$$\begin{array}{rcll}
x & = & \dfrac{-2 \pm \sqrt{2^2-4(1)(3)}}{2(1)} & \text{Quadratic Formula} \\
& = & \dfrac{-2 \pm \sqrt{-8}}{2} & \text{Simplify} \\
& = & \dfrac{-2 \pm i\sqrt{8}}{2} & \text{Definition of } i \\
& = & \dfrac{-2 \pm i2\sqrt{2}}{2} & \text{Product Rule for Radicals} \\
& = & \dfrac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} & \text{Factor and reduce} \\
& = & -1 \pm i\sqrt{2} &
\end{array}$$

We get two answers: $x = -1+i\sqrt{2}$ and its conjugate $x = -1-i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Determine the complex solutions to $2t^4 = 9t^2+5$.

We have a Quadratic in Disguise, because we have three terms and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2). We proceed accordingly.

$$\begin{array}{rcll}
2t^4 & = & 9t^2+5 & \\
2t^4-9t^2-5 & = & 0 & \text{Subtract } 9t^2 \text{ and } 5 \\
(2t^2+1)(t^2-5) & = & 0 & \text{Factor} \\
2t^2+1=0 & \text{or} & t^2=5 & \text{Zero Product Property}
\end{array}$$

From $2t^2+1=0$ we get $2t^2=-1$, or $t^2=-\frac{1}{2}$. We extract square roots as follows:

$$t = \pm\sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2=5$, we get $t=\pm\sqrt{5}$. In total, we have four complex solutions - two real: $t=\pm\sqrt{5}$ and two non-real: $t=\pm\frac{i\sqrt{2}}{2}$.

3. Determine the complex solutions to $z^3+1=0$.

   To find the *real* solutions to $z^3+1=0$, we can subtract the 1 from both sides and extract cube roots: $z^3=-1$, so $z=\sqrt[3]{-1}=-1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{array}{rcll} z^3+1 & = & 0 & \\ (z+1)(z^2-z+1) & = & 0 & \text{Factor (Sum of Two Cubes)} \\ z+1=0 & \text{or} & z^2-z+1=0 & \end{array}$$

   From $z+1=0$, we get our real solution $z=-1$. From $z^2-z+1=0$, we apply the Quadratic Formula to get:

$$z=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

   Thus, we get *three* solutions to $z^3+1=0$ - one real: $z=-1$ and two non-real: $z=\frac{1\pm i\sqrt{3}}{2}$.

   As always, the reader is encouraged to test their algebraic mettle and check these solutions. $\square$

It is no coincidence that the non-real solutions to the equations in Example 0.5.7 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem 0.6 which we state below.

Theorem 0.7. Discriminant Theorem: Given a Quadratic Equation $ax^2+bx+c=0$, where a, b and c are real numbers, let $D=b^2-4ac$ be the discriminant.

- If $D>0$, there are two distinct real number solutions to the equation.
- If $D=0$, there is one (repeated) real number solution.
  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-b\pm 0}{2a}$ reduce to $-\frac{b}{2a}$.
- If $D<0$, there are two non-real solutions which form a complex conjugate pair.

0.5.7 Exercises

In Exercises 1 - 9, solve the given linear equation and check your answer.

1. $3x-4=2-4(x-3)$
2. $\dfrac{3-2t}{4}=7t+1$
3. $\dfrac{2(w-3)}{5}=\dfrac{4}{15}-\dfrac{3w+1}{9}$

4. $-0.02y+1000=0$
5. $\dfrac{49w-14}{7}=3w-(2-4w)$
6. $7-(4-x)=\dfrac{2x-3}{2}$
7. $3t\sqrt{7}+5=0$
8. $\sqrt{50}y=\dfrac{6-\sqrt{8}y}{3}$
9. $4-(2x+1)=\dfrac{x\sqrt{7}}{9}$

In equations 10 - 23, solve each equation for the indicated variable.

10. Solve for y: $3x+2y=4$
11. Solve for x: $3x+2y=4$
12. Solve for C: $F=\dfrac{9}{5}C+32$
13. Solve for x: $p=-2.5x+15$
14. Solve for x: $C=200x+1000$
15. Solve for y: $x=4(y+1)+3$
16. Solve for w: $vw-1=3v$
17. Solve for v: $vw-1=3v$
18. Solve for y: $x(y-3)=2y+1$
19. Solve for π: $C=2\pi r$
20. Solve for V: $PV=nRT$
21. Solve for R: $PV=nRT$
22. Solve for g: $E=mgh$
23. Solve for m: $E=\dfrac{1}{2}mv^2$

In Exercises 24 - 27, the subscripts on the variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat 'P_1' and 'P_2' as two different variables as you would 'x' and 'y.' (The same goes for 'x' and 'x_0,' etc.)

24. Solve for V_2: $P_1V_1=P_2V_2$
25. Solve for t: $x=x_0+at$
26. Solve for x: $y-y_0=m(x-x_0)$
27. Solve for T_1: $q=mc(T_2-T_1)$

In Exercises 28 - 45, solve the equation.

28. $|x|=6$
29. $|3t-1|=10$
30. $|4-w|=7$
31. $4-|y|=3$
32. $2|5m+1|-3=0$
33. $|7x-1|+2=0$
34. $\dfrac{5-|x|}{2}=1$
35. $\dfrac{2}{3}|5-2w|-\dfrac{1}{2}=5$
36. $|3t-\sqrt{2}|+4=6$
37. $\dfrac{|2v+1|-3}{4}=\dfrac{1}{2}-|2v+1|$
38. $|2x+1|=\dfrac{|2x+1|-3}{2}$
39. $\dfrac{|3-2y|+4}{2}=2-|3-2y|$
40. $|3t-2|=|2t+7|$
41. $|3x+1|=|4x|$
42. $|1-\sqrt{2}y|=|y+1|$

43. $|4-x|-|x+2|=0$
44. $|2-5z|=5|z+1|$
45. $\sqrt{3}|w-1|=2|w+1|$

In Exercises 46 - 60, find all real number solutions. Write all answers in exact form. Check your answers.

46. $(7x+3)(x-5)=0$
47. $(2t-1)^2(t+4)=0$
48. $(y^2+4)(3y^2+y-10)=0$
49. $4t=t^2$
50. $y+3=2y^2$
51. $26x=8x^2+21$
52. $16x^4=9x^2$
53. $w(6w+11)=10$
54. $2w^2+5w+2=-3(2w+1)$
55. $x^2(x-3)=16(x-3)$
56. $(2t+1)^3=(2t+1)$
57. $a^4+4=6-a^2$
58. $\dfrac{8t^2}{3}=2t+3$
59. $\dfrac{x^3+x}{2}=\dfrac{x^2+1}{3}$
60. $\dfrac{y^4}{3}-y^2=\dfrac{3}{2}(y^2+3)$

In Exercises 61 - 72, find all real solutions.

61. $(2x+1)^3+8=0$
62. $\dfrac{(1-2y)^4}{3}=27$
63. $\dfrac{1}{1+2t^3}=4$
64. $\sqrt{3x+1}=4$
65. $5-\sqrt[3]{t^2+1}=1$
66. $x+1=\sqrt{3x+7}$
67. $y+\sqrt{3y+10}=-2$
68. $3t+\sqrt{6-9t}=2$
69. $2x-1=\sqrt{x+3}$
70. $w=\sqrt[4]{12-w^2}$
71. $\sqrt{x-2}+\sqrt{x-5}=3$
72. $\sqrt{2x+1}=3+\sqrt{4-x}$

In Exercises 73 - 76, solve each equation for the indicated variable. Assume all quantities represent positive real numbers.

73. Solve for h: $I=\dfrac{bh^3}{12}$.
74. Solve for a: $I_0=\dfrac{5\sqrt{3}a^4}{16}$
75. Solve for g: $T=2\pi\sqrt{\dfrac{L}{g}}$
76. Solve for v: $L=L_0\sqrt{1-\dfrac{v^2}{c^2}}$.

In Exercises 77 - 97, find all real solutions. Check your answers, as directed by your instructor.

77. $3\left(x-\dfrac{1}{2}\right)^2=\dfrac{5}{12}$
78. $4-(5t+3)^2=3$
79. $3(y^2-3)^2-2=10$
80. $x^2+x-1=0$
81. $3w^2=2-w$
82. $y(y+4)=1$
83. $\dfrac{z}{2}=4z^2-1$
84. $0.1v^2+0.2v=0.3$
85. $x^2=x-1$

86. $3-t=2(t+1)^2$
87. $(x-3)^2=x^2+9$
88. $(3y-1)(2y+1)=5y$
89. $w^4+3w^2-1=0$
90. $2x^4+x^2=3$
91. $(2-y)^4=3(2-y)^2+1$
92. $3x^4+6x^2=15x^3$
93. $6p+2=p^2+3p^3$
94. $10v=7v^3-v^5$
95. $y^2-\sqrt{8}y=\sqrt{18}y-1$
96. $x^2\sqrt{3}=x\sqrt{6}+\sqrt{12}$
97. $\dfrac{v^2}{3}=\dfrac{v\sqrt{3}}{2}+1$

In Exercises 98 - 103, find all real solutions. Leave all answers in exact form.

98. $5.54^2+b^2=36$
99. $\pi r^2=37$
100. $54=8r\sqrt{2}+\pi r^2$
101. $-4.9t^2+100t=410$
102. $x^2=1.65(3-x)^2$
103. $(0.5+2A)^2=0.7(0.1-A)^2$

In Exercises 104 - 106, use Theorem 0.4 along with the techniques in this section to find all real solutions to the following.

104. $|x^2-3x|=2$
105. $|2x-x^2|=|2x-1|$
106. $|x^2-x+3|=|4-x^2|$
107. Prove that for every nonzero number p, $x^2+xp+p^2=0$ has no real solutions.
108. Solve for t: $-\dfrac{1}{2}gt^2+vt+h=0$. Assume $g>0$, $v\geq 0$ and $h\geq 0$.

In Exercises 109 - 116, simplify the quantity.

109. $\sqrt{-49}$
110. $\sqrt{-9}$
111. $\sqrt{-25}\sqrt{-4}$
112. $\sqrt{(-25)(-4)}$
113. $\sqrt{-9}\sqrt{-16}$
114. $\sqrt{(-9)(-16)}$
115. $\sqrt{-(-9)}$
116. $-\sqrt{(-9)}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)\cdot i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 117 - 124, use this information to simplify the given power of i.

117. i^5
118. i^6
119. i^7
120. i^8
121. i^{15}
122. i^{26}
123. i^{117}
124. i^{304}

In Exercises 125 - 133, find all complex solutions.

125. $3x^2+6=4x$
126. $15t^2+2t+5=3t(t^2+1)$
127. $3y^2+4=y^4$
128. $\dfrac{2}{1-w}=w$
129. $\dfrac{y}{3}-\dfrac{3}{y}=y$
130. $\dfrac{x^3}{2x-1}=\dfrac{x}{3}$

131. $x = \dfrac{2}{\sqrt{5} - x}$

132. $\dfrac{5y^4 + 1}{y^2 - 1} = 3y^2$

133. $z^4 = 16$

SECTION 0.5 EXERCISE ANSWERS A.1.0

0.6 Basic Inequalities in One Variable

0.6.1 Linear Inequalities

We now turn our attention to linear inequalities. Unlike linear equations which admit at most one solution, the solutions to linear inequalities are generally intervals of real numbers. While the solution strategy for solving linear inequalities is the same as with solving linear equations, we need to remind ourselves that, should we decide to multiply or divide both sides of an inequality by a **negative** number, we need to reverse the direction of the inequality. (See the footnote in the box on page 46.) In the example below, we work not only some 'simple' linear inequalities in the sense there is only one inequality present, but also some 'compound' linear inequalities which require us to revisit the notions of intersection and union.

Example 0.6.1. Solve the following inequalities for the indicated variable.

1. Solve for x: $\dfrac{7-8x}{2} \geq 4x+1$
2. Solve for y: $\dfrac{3}{4} \leq \dfrac{7-y}{2} < 6$
3. Solve for t: $2t-1 \leq 4-t < 6t+1$
4. Solve for x: $5+\sqrt{7}x \leq 4x+1 \leq 8$
5. Solve for w: $2.1-0.01w \leq -3$ or $2.1-0.01w \geq 3$

Solution.

1. Solve for x: $\dfrac{7-8x}{2} \geq 4x+1$.

   We begin by clearing denominators. Then we gather all of the terms containing x to one side of the inequality and put the remaining terms on the other.

$$\begin{array}{rcll}
\dfrac{7-8x}{2} & \geq & 4x+1 & \\
2\left(\dfrac{7-8x}{2}\right) & \geq & 2(4x+1) & \text{Multiply by 2} \\
\dfrac{\cancel{2}(7-8x)}{\cancel{2}} & \geq & 2(4x)+2(1) & \text{Distribute} \\
7-8x & \geq & 8x+2 & \\
(7-8x)+8x-2 & \geq & 8x+2+8x-2 & \text{Add } 8x\text{, subtract 2} \\
7-2-8x+8x & \geq & 8x+8x+2-2 & \text{Rearrange terms} \\
5 & \geq & 16x & 8x+8x=(8+8)x=16x \\
\dfrac{5}{16} & \geq & \dfrac{16x}{16} & \text{Divide by the coefficient of } x \\
\dfrac{5}{16} & \geq & x &
\end{array}$$

We get $\frac{5}{16} \geq x$ or, said differently, $x \leq \frac{5}{16}$. We express this set[1] of real numbers as $\left(-\infty, \frac{5}{16}\right]$. Though not required to do so, we could partially check our answer by substituting $x = \frac{5}{16}$ and a few other values in our solution set ($x = 0$, for instance) to make sure the inequality holds. (It also isn't a bad idea to choose an $x > \frac{5}{16}$, say $x = 1$, to see that the inequality *doesn't* hold there.) The only real way to actually show that our answer works for *all* values in our solution set is to start with $x \leq \frac{5}{16}$ and reverse all of the steps in our solution procedure to prove it is equivalent to our original inequality.

2. Solve for y: $\dfrac{3}{4} \leq \dfrac{7-y}{2} < 6$.

   We have our first example of a 'compound' inequality. The solutions to

$$\frac{3}{4} \leq \frac{7-y}{2} < 6$$

   must satisfy

$$\frac{3}{4} \leq \frac{7-y}{2} \qquad \underline{\text{and}} \qquad \frac{7-y}{2} < 6$$

   One approach is to solve each of these inequalities separately, then intersect their solution sets. While this method works (and will be used later for more complicated problems), our variable y appears only in the middle expression so we can proceed by working both inequalities at once:

$$\begin{array}{rccr}
\dfrac{3}{4} \leq & \dfrac{7-y}{2} & < 6 & \\
4\left(\dfrac{3}{4}\right) \leq & 4\left(\dfrac{7-y}{2}\right) & < 4(6) & \text{Multiply by } 4 \\
\dfrac{\cancel{4} \cdot 3}{\cancel{4}} \leq & \dfrac{\overset{2}{\cancel{4}}(7-y)}{\cancel{2}} & < 24 & \\
3 \leq & 2(7-y) & < 24 & \\
3 \leq & 2(7) - 2y & < 24 & \text{Distrbute} \\
3 \leq & 14 - 2y & < 24 & \\
3 - 14 \leq & (14-2y) - 14 & < 24 - 14 & \text{Subtract } 14 \\
-11 \leq & -2y & < 10 & \\
\dfrac{-11}{-2} \geq & \dfrac{-2y}{-2} & > \dfrac{10}{-2} & \text{Divide by the coefficient of } y \\
 & & & \text{Reverse inequalities} \\
\dfrac{11}{2} \geq & y & > -5 &
\end{array}$$

   Our final answer is $\frac{11}{2} \geq y > -5$, or, said differently, $-5 < y \leq \frac{11}{2}$. In interval notation, this is $\left(-5, \frac{11}{2}\right]$. We could check the reasonableness of our answer as before, and the reader is encouraged to do so.

[1] Using set-builder notation, our 'set' of solutions here is $\{x \,|\, x \leq \frac{5}{16}\}$.

3. Solve for t: $2t-1 \leq 4-t < 6t+1$.

   We have another compound inequality and what distinguishes this one from our previous example is that t appears on both sides of both inequalities. In this case, we need to create two separate inequalities and find all of the real numbers t which satisfy both $2t-1 \leq 4-t$ *and* $4-t < 6t+1$. The first inequality, $2t-1 \leq 4-t$, reduces to $3t \leq 5$ or $t \leq \frac{5}{3}$. The second inequality, $4-t < 6t+1$, becomes $3 < 7t$ which reduces to $t > \frac{3}{7}$. Thus our solution is all real numbers t with $t \leq \frac{5}{3}$ *and* $t > \frac{3}{7}$, or, writing this as a compound inequality, $\frac{3}{7} < t \leq \frac{5}{3}$. Using interval notation,[2] we express our solution as $\left(\frac{3}{7}, \frac{5}{3}\right]$.

4. Solve for x: $5+\sqrt{7}x \leq 4x+1 \leq 8$.

   As before, with this inequality we have no choice but to solve each inequality individually and intersect the solution sets. Starting with the leftmost inequality, we first note that in the term $\sqrt{7}x$, the vinculum of the square root extends over the 7 only, meaning the x is not part of the radicand. In order to avoid confusion, we will write $\sqrt{7}x$ as $x\sqrt{7}$.

$$\begin{array}{rcll} 5+x\sqrt{7} & \leq & 4x+1 & \\ (5+x\sqrt{7})-4x-5 & \leq & (4x+1)-4x-5 & \text{Subtract } 4x \text{ and } 5 \\ x\sqrt{7}-4x+5-5 & \leq & 4x-4x+1-5 & \text{Rearrange terms} \\ x(\sqrt{7}-4) & \leq & -4 & \text{Factor} \end{array}$$

   At this point, we need to exercise a bit of caution because the number $\sqrt{7}-4$ is negative.[3]

   When we divide by it the inequality reverses:

$$\begin{array}{rcll} x(\sqrt{7}-4) & \leq & -4 & \\ \dfrac{x(\sqrt{7}-4)}{\sqrt{7}-4} & \geq & \dfrac{-4}{\sqrt{7}-4} & \begin{array}{r}\text{Divide by the coefficient of } x \\ \text{Reverse inequalities}\end{array} \\ x & \geq & \dfrac{-4}{\sqrt{7}-4} & \\ x & \geq & \dfrac{-4}{-(4-\sqrt{7})} & \\ x & \geq & \dfrac{4}{4-\sqrt{7}} & \end{array}$$

   We're only half done because we still have the rightmost inequality to solve. Fortunately, that one seems rather mundane: $4x+1 \leq 8$ reduces to $x \leq \frac{7}{4}$ without too much incident. Our solution is $x \geq \frac{4}{4-\sqrt{7}}$ *and* $x \leq \frac{7}{4}$. We may be tempted to write $\frac{4}{4-\sqrt{7}} \leq x \leq \frac{7}{4}$ and call it a day but that would be nonsense! To see why, notice that $\sqrt{7}$ is between 2 and 3 so $\frac{4}{4-\sqrt{7}}$ is between $\frac{4}{4-2} = 2$ and $\frac{4}{4-3} = 4$.

[2] If we intersect the solution sets of the two individual inequalities, we get the answer, too: $\left(-\infty, \frac{5}{3}\right] \cap \left(\frac{3}{7}, \infty\right) = \left(\frac{3}{7}, \frac{5}{3}\right]$.

[3] As a result of $4 < 7 < 9$, it stands to reason that $\sqrt{4} < \sqrt{7} < \sqrt{9}$ and thus $2 < \sqrt{7} < 3$.

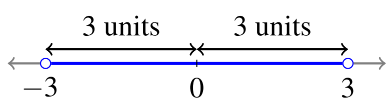
In particular, we get $\frac{4}{4-\sqrt{7}} > 2$. On the other hand, $\frac{7}{4} < 2$. This means that our 'solutions' have to be simultaneously greater than 2 AND less than 2 which is impossible. Therefore, this compound inequality has no solution, which means we did all that work for nothing.

5. Solve for w: $2.1 - 0.01w \leq -3$ or $2.1 - 0.01w \geq 3$.

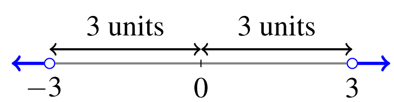
   Our last example is yet another compound inequality but here, instead of the two inequalities being connected with the conjunction '*and*', they are connected with '*or*', which indicates that we need to find the *union* of the results of each. Starting with $2.1 - 0.01w \leq -3$, we get $-0.01w \leq -5.1$, which gives[4] $w \geq 510$. The second inequality, $2.1 - 0.01w \geq 3$, becomes $-0.01w \geq 0.9$, which reduces to $w \leq -90$. Our solution set consists of all real numbers w with $w \geq 510$ *or* $w \leq -90$. In interval notation, this is $(-\infty, -90] \cup [510, \infty)$. □

0.6.2 Absolute Value Inequalities

We now turn our attention to solving some basic inequalities involving the absolute value. Suppose we wished to solve $|x| < 3$. Geometrically, we are looking for all of the real numbers whose distance from 0 is *less* than 3 units. We get $-3 < x < 3$, or in interval notation, $(-3, 3)$. Suppose we are asked to solve $|x| > 3$ instead. Now we want the distance between x and 0 to be *greater* than 3 units. Moving in the positive direction, this means $x > 3$. In the negative direction, this puts $x < -3$. Our solutions would then satisfy $x < -3$ *or* $x > 3$. In interval notation, we express this as $(-\infty, -3) \cup (3, \infty)$.



The solution to $|x| < 3$ is $(-3, 3)$



The solution to $|x| > 3$ is $(-\infty, -3) \cup (3, \infty)$

Generalizing this notion, we get the following:

Theorem 0.8. Inequalities Involving Absolute Value: Let c be a real number.

- If $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- If $c \leq 0$, $|x| < c$ has no solution.
- If $c > 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- If $c \leq 0$, $|x| > c$ is true for all real numbers.

If the inequality we're faced with involves '$\leq$' or '$\geq$,' we can combine the results of Theorem 0.8 with Theorem 0.4 as needed.

[4] Don't forget to change the direction of the inequality!

Strategy for Solving Inequalities Involving Absolute Value

In order to solve an inequality involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the inequality.
2. Apply Theorem 0.8.

Example 0.6.2. Solve the following inequalities.

1. $\left|x-\sqrt[4]{5}\right| > 1$
2. $\dfrac{4-2|2x+1|}{4} \geq -\sqrt{3}$
3. $|2x-1| \leq 3|4-8x|-10$
4. $|2x-1| \leq 3|4-8x|+10$
5. $2 < |x-1| \leq 5$
6. $|10x-5|+|10-5x| \leq 0$

Solution.

1. Solve $\left|x-\sqrt[4]{5}\right| > 1$ for x.

   From Theorem 0.8, $\left|x-\sqrt[4]{5}\right| > 1$ is equivalent to $x-\sqrt[4]{5} < -1$ or $x-\sqrt[4]{5} > 1$. Solving this compound inequality, we get $x < -1+\sqrt[4]{5}$ or $x > 1+\sqrt[4]{5}$. Our answer, in interval notation, is: $\left(-\infty, -1+\sqrt[4]{5}\right) \cup \left(1+\sqrt[4]{5}, \infty\right)$. As with linear inequalities, we can only partially check our answer by selecting values of x both inside and outside of the solution intervals to see which values of x satisfy the original inequality and which do not.

2. Solve $\dfrac{4-2|2x+1|}{4} \geq -\sqrt{3}$ for x.

   Our first step in solving $\frac{4-2|2x+1|}{4} \geq -\sqrt{3}$ is to isolate the absolute value.

$$\begin{array}{rcll}
\dfrac{4-2|2x+1|}{4} & \geq & -\sqrt{3} & \\
4-2|2x+1| & \geq & -4\sqrt{3} & \text{Multiply by } 4 \\
-2|2x+1| & \geq & -4-4\sqrt{3} & \text{Subtract } 4 \\
|2x+1| & \leq & \dfrac{-4-4\sqrt{3}}{-2} & \text{Divide by } -2\text{, reverse the inequality} \\
|2x+1| & \leq & 2+2\sqrt{3} & \text{Reduce}
\end{array}$$

   Due to the fact that we're dealing with '$\leq$' instead of just '$<$,' we can combine Theorems 0.8 and 0.4 to rewrite this last inequality as:[5] $-(2+2\sqrt{3}) \leq 2x+1 \leq 2+2\sqrt{3}$. Subtracting the '1' across

[5]Note the use of parentheses: $-(2+2\sqrt{3})$ as opposed to $-2+2\sqrt{3}$.

both inequalities gives $-3-2\sqrt{3} \leq 2x \leq 1+2\sqrt{3}$, which reduces to $\frac{-3-2\sqrt{3}}{2} \leq x \leq \frac{1+2\sqrt{3}}{2}$. In interval notation this reads as $\left[\frac{-3-2\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right]$.

3. Solve $|2x-1| \leq 3|4-8x| - 10$ for x.

   There are two absolute values in $|2x-1| \leq 3|4-8x| - 10$, so we cannot directly apply Theorem 0.8 here. Notice, however, that $|4-8x| = |(-4)(2x-1)|$. Using this, we get:

$$\begin{array}{rcll} |2x-1| & \leq & 3|4-8x|-10 & \\ |2x-1| & \leq & 3|(-4)(2x-1)|-10 & \text{Factor} \\ |2x-1| & \leq & 3|-4||2x-1|-10 & \text{Product Rule} \\ |2x-1| & \leq & 12|2x-1|-10 & \\ -11|2x-1| & \leq & -10 & \text{Subtract } 12|2x-1| \\ |2x-1| & \geq & \dfrac{10}{11} & \text{Divide by } -11 \text{ and reduce} \end{array}$$

   Now we are allowed to invoke Theorems 0.4 and 0.8 and write the equivalent compound inequality: $2x-1 \leq -\frac{10}{11}$ *or* $2x-1 \geq \frac{10}{11}$. We get $x \leq \frac{1}{22}$ *or* $x \geq \frac{21}{22}$, which when written with interval notation becomes $\left(-\infty, \frac{1}{22}\right] \cup \left[\frac{21}{22}, \infty\right)$.

4. Solve $|2x-1| \leq 3|4-8x| + 10$ for x.

   The inequality $|2x-1| \leq 3|4-8x| + 10$ differs from the previous example in exactly one respect: on the right side of the inequality, we have '$+10$' instead of '-10.' The steps to isolate the absolute value here are identical to those in the previous example, but instead of obtaining $|2x-1| \geq \frac{10}{11}$ as before, we obtain $|2x-1| \geq -\frac{10}{11}$. This latter inequality is *always* true. (Absolute value is, by definition, a distance and hence always 0 or greater.) Thus our solution to this inequality is all real numbers.

5. Solve $2 < |x-1| \leq 5$ for x.

   To solve $2 < |x-1| \leq 5$, we rewrite it as the compound inequality: $2 < |x-1|$ *and* $|x-1| \leq 5$. The first inequality, $2 < |x-1|$, can be re-written as $|x-1| > 2$ so it is equivalent to $x-1 < -2$ *or* $x-1 > 2$. Thus the solution to $2 < |x-1|$ is $x < -1$ or $x > 3$, which in interval notation is $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \leq 5$, we combine the results of Theorems 0.4 and 0.8 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$.

   Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we intersect $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to get our final answer $[-4, -1) \cup (3, 6]$.

6. Solve $|10x-5| + |10-5x| \leq 0$ for x.

   Our first hope when encountering $|10x-5| + |10-5x| \leq 0$ is that we can somehow combine the two absolute value quantities as we'd done in earlier examples. We leave it to the reader to show, however, that no matter what we try to factor out of the absolute value quantities, what remains inside the absolute values will always be different.

At this point, we take a step back and look at the equation in a more general way: we are adding two absolute values together and wanting the result to be less than or equal to 0. The absolute value of anything is always 0 or greater, so there are no solutions to: $|10x-5|+|10-5x|<0$.

Is it possible that $|10x-5|+|10-5x|=0$? Only if there is an x where $|10x-5|=0$ and $|10-5x|=0$ *at the same time*.[6] The first equation holds only when $x=\frac{1}{2}$, while the second holds only when $x=2$. Alas, we have no solution. $\square$

The astute reader will have noticed by now that the authors have done nothing in the way of explaining *why* anyone would ever need to know this stuff. These sections were designed to review skills and concepts that you've already learned. Thus, the deeper applications are in the main body of the text as opposed to here in Chapter 0.

We close this section with an example of how the properties in Theorem 0.3 are used in Calculus. Here, 'ε' is the Greek letter 'epsilon' and it represents a positive real number. Those of you who will be taking Calculus in the future should become *very* familiar with this type of algebraic manipulation.

$$\begin{array}{rcll}
\left|\dfrac{8-4x}{3}\right| & < & \varepsilon & \\
\dfrac{|8-4x|}{|3|} & < & \varepsilon & \text{Quotient Rule} \\
\dfrac{|-4(x-2)|}{3} & < & \varepsilon & \text{Factor} \\
\dfrac{|-4||x-2|}{3} & < & \varepsilon & \text{Product Rule} \\
\dfrac{4|x-2|}{3} & < & \varepsilon & \\
\dfrac{3}{4}\cdot\dfrac{4|x-2|}{3} & < & \dfrac{3}{4}\cdot\varepsilon & \text{Multiply by } \dfrac{3}{4} \\
|x-2| & < & \dfrac{3}{4}\varepsilon &
\end{array}$$

0.6.3 Exercises

In Exercises 1 - 18, solve the given inequality. Write your answer using interval notation.

[6] Do you see why?

1. $3 - 4x \geq 0$
2. $2t - 1 < 3 - (4t - 3)$
3. $\dfrac{7-y}{4} \geq 3y + 1$
4. $0.05R + 1.2 > 0.8 - 0.25R$
5. $7 - (2 - x) \leq x + 3$
6. $\dfrac{10m+1}{5} \geq 2m - \dfrac{1}{2}$
7. $x\sqrt{12} - \sqrt{3} > \sqrt{3}x + \sqrt{27}$
8. $2t - 7 \leq \sqrt[3]{18}t$
9. $117y \geq y\sqrt{2} - 7y\sqrt[4]{8}$
10. $-\dfrac{1}{2} \leq 5x - 3 \leq \dfrac{1}{2}$
11. $-\dfrac{3}{2} \leq \dfrac{4-2t}{10} < \dfrac{7}{6}$
12. $-0.1 \leq \dfrac{5-x}{3} - 2 < 0.1$
13. $2y \leq 3 - y < 7$
14. $3x \geq 4 - x \geq 3$
15. $6 - 5t > \dfrac{4t}{3} \geq t - 2$
16. $2x + 1 \leq -1$ or $2x + 1 \geq 1$
17. $4 - x \leq 0$ or $2x + 7 < x$
18. $\dfrac{5-2x}{3} > x$ or $2x + 5 \geq 1$

In Exercises 19 - 30, solve the inequality. Write your answer using interval notation.

19. $|3x - 5| \leq 4$
20. $|7t + 2| > 10$
21. $|2w + 1| - 5 < 0$
22. $|2 - y| - 4 \geq -3$
23. $|3z + 5| + 2 < 1$
24. $2|7 - v| + 4 > 1$
25. $3 - |x + \sqrt{5}| < -3$
26. $|5t| \leq |t| + 3$
27. $|w - 3| < |3 - w|$
28. $2 \leq |4 - y| < 7$
29. $1 < |2w - 9| \leq 3$
30. $3 > 2|\sqrt{3} - x| > 1$

SECTION 0.6 EXERCISE ANSWERS A.1.0

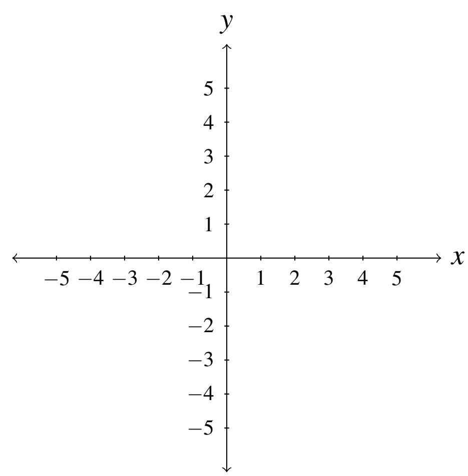
CHAPTER 1

PROPERTIES OF GENERAL FUNCTIONS

1.1 Rectangular Coordinate Plane

1.1.1 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[1] Imagine two real number lines crossing at a right angle at 0 as drawn below.



The horizontal number line is usually called the x**-axis** while the vertical number line is usually called the y**-axis**. As with things in the 'real' world, however, it's best not to get too caught up with labels. Think of x and y as generic label placeholders, in much the same way as the variables x and y are placeholders for real numbers. The letters we choose to identify with the axes depend on the context. For example, if we were plotting the relationship between time and the number of Sasquatch sightings, we might label the horizontal axis as the t-axis (for 'time') and the vertical axis the N-axis (for 'number' of sightings.) As with the usual number line, we imagine these axes extending off indefinitely in both directions.[2]

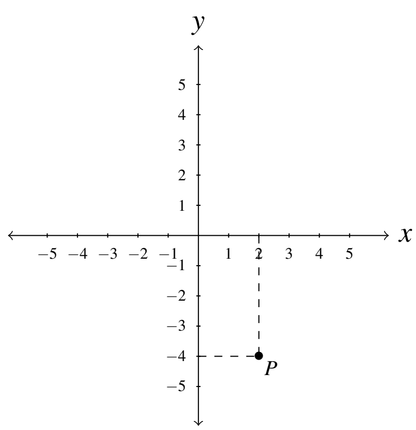
Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves. For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. This process is sometimes called 'projecting' the point P to the x- (respectively y-) axis. We then describe the point P using the **ordered pair** $(2,-4)$. The first number in the ordered pair is called the **abscissa** or x**-coordinate** and the second is called the **ordinate** or y**-coordinate**. Again, the names of the coordinates can vary depending on the context of the application. If, as in the previous paragraph, the horizontal axis represented time and the vertical axis represented the number of Sasquatch sightings,

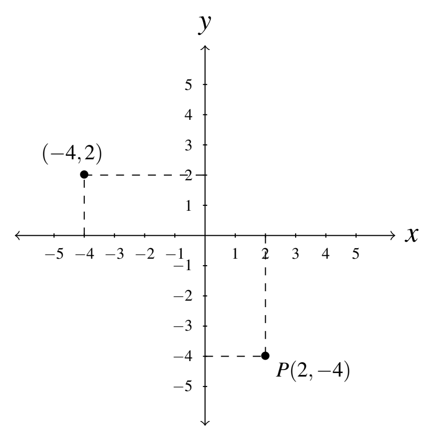
[1] So named in honor of René Descartes.

[2] Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y.

the first coordinate would be called the t-coordinate and the second coordinate would be the N-coordinate. What's important is that we maintain the convention that the abscissa (first coordinate) always corresponds to the horizontal position, while the ordinate (second coordinate) always corresponds to the vertical position. Taken together, the ordered pair $(2,-4)$ comprise the **Cartesian coordinates**[3] of the point P.

In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2,-4)$'. We can think of $(2,-4)$ as instructions on how to reach P from the **origin** $(0,0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important, as are the signs of the numbers in the pair. If we wish to plot the point $(-4,2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x,y) as x and y take values from the real numbers. Below is a summary of some basic, but nonetheless important, facts about Cartesian coordinates.

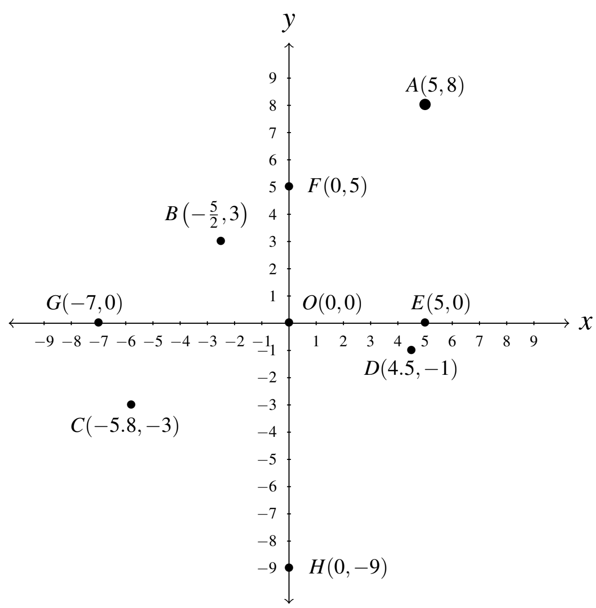
Important Facts about the Cartesian Coordinate Plane

- (a,b) and (c,d) represent the same point in the plane if and only if $a=c$ and $b=d$.
- (x,y) lies on the x-axis if and only if $y=0$.
- (x,y) lies on the y-axis if and only if $x=0$.
- The origin is the point $(0,0)$. It is the only point common to both axes.

[3] Also called the 'rectangular coordinates' of P.

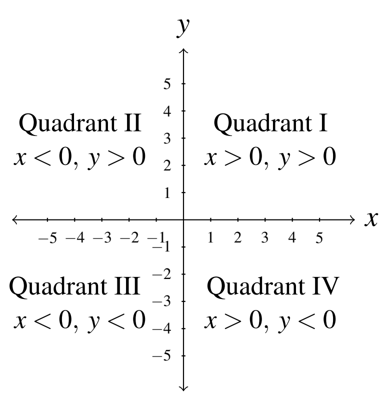
Example 1.1.1. Plot the following points: $A(5,8)$, $B\left(-\frac{5}{2},3\right)$, $C(-5.8,-3)$, $D(4.5,-1)$, $E(5,0)$, $F(0,5)$, $G(-7,0)$, $H(0,-9)$, $O(0,0)$. (The letter O is almost always reserved for the origin.)

Solution. To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



□

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



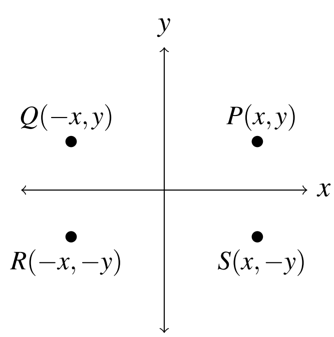
For example, $(1,2)$ lies in Quadrant I, $(-1,2)$ in Quadrant II, $(-1,-2)$ in Quadrant III and $(1,-2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0,4)$ lies on the positive y-axis whereas $(-117,0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.1. Two points (a,b) and (c,d) in the plane are said to be

- **symmetric about the x-axis** if $a = c$ and $b = -d$
- **symmetric about the y-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

Example 1.1.2. Let P be the point $(-2,3)$. State the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

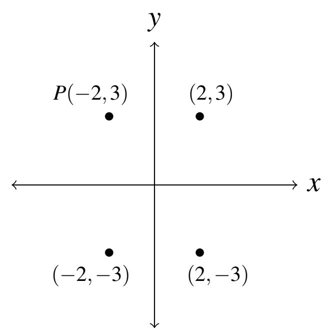
Check your answer by plotting the points.

Solution. The figure after Definition 1.1 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2,3)$.

1. To find the point symmetric to point $P(-2,3)$ about the x-axis, we replace the y-coordinate of 3 with its opposite -3 to get $(-2,-3)$.

2. To find the point symmetric to point $P(-2,3)$ about the y-axis, we replace the x-coordinate of -2 with its opposite $-(-2) = 2$ to get $(2,3)$.

3. To find the point symmetric to point $P(-2,3)$ about the origin, we replace both the x- and y-coordinates with their opposites to get $(2,-3)$.



□

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2,3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2,3)$ across the x-axis would lie at $(-2,-3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2,3)$ across that axis would be $(2,3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2,3)$ to $(-2,-3)$ then to $(2,-3)$, and so we would end up at the point symmetric to $(-2,3)$ about the origin. We summarize and generalize this process below.

Reflections

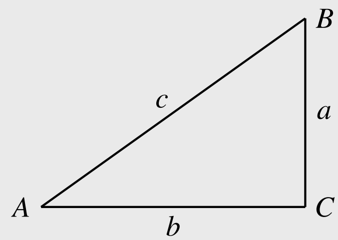
To reflect a point (x,y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.1.2 Distance in the Plane

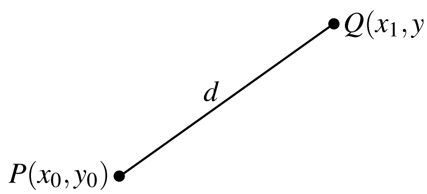
Another fundamental concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Before we can do that, we need to state what we believe is the most important theorem in all of Geometry: The Pythagorean Theorem.

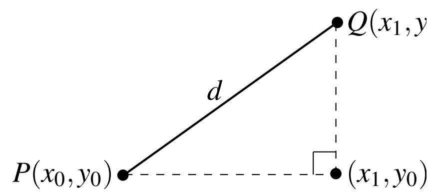
Theorem 1.1. The Pythagorean Theorem: The triangle ABC shown below is a right triangle if and only if $a^2+b^2=c^2$



The theorem actually says two different things. If we know that $a^2+b^2=c^2$ then the angle C must be a right angle. If we know geometrically that C is already a right angle then we have that $a^2+b^2=c^2$. We need the latter statement in the discussion which follows.

Suppose we have two points, $P(x_0,y_0)$ and $Q(x_1,y_1)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.





With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1-x_0|$ and $|y_1-y_0|$ so the Pythagorean Theorem gives us

$$|x_1-x_0|^2+|y_1-y_0|^2=d^2$$

$$(x_1-x_0)^2+(y_1-y_0)^2=d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation 1.1. The Distance Formula: The distance d between the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is:

$$d=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$$

A couple of remarks about Equation 1.1 are in order. First, it is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. Second, distance is a 'length'. So, technically, the number we obtain from the distance formula has some attached units of length. In this text, we'll adopt the convention that the phrase 'units' refers to some generic units of length.[4] Our next example gives us an opportunity to test drive the distance formula as well as brush up on some arithmetic and prerequisite algebra.

Example 1.1.3. Compute and simplify the distance between the following sets of points:

1. $P(-2,3)$ and $Q(1,-3)$
2. $R\left(\frac{1}{2},\frac{2}{3}\right)$ and $S\left(\frac{3}{4},\frac{1}{5}\right)$
3. $T(\sqrt{3},-\sqrt{20})$ and $V(\sqrt{12},\sqrt{5})$
4. $O(0,0)$ and $P(x,y)$.

Solution. In each case, we apply the distance formula, Equation 1.1 with the first point listed taken as (x_0,y_0) and the second point taken as (x_1,y_1).[5]

1. Compute the distance between $P(-2,3)$ and $Q(1,-3)$.

   With $(-2,3)=(x_0,y_0)$ and $(1,-3)=(x_1,y_1)$, we get

$$\begin{array}{rcll} d & = & \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} & \\ & = & \sqrt{(1-(-2))^2+(-3-3)^2} & \\ & = & \sqrt{9+36} & \\ & = & \sqrt{45} & \\ & = & \sqrt{9\cdot 5} & \\ & = & \sqrt{9}\sqrt{5} & \text{For nonnegative numbers, } \sqrt{ab}=\sqrt{a}\sqrt{b}. \\ & = & 3\sqrt{5} & \end{array}$$

   So the distance between P and Q is $3\sqrt{5}$ units.

2. Compute the distance between $R\left(\frac{1}{2},\frac{2}{3}\right)$ and $S\left(\frac{3}{4},\frac{1}{5}\right)$.

[4] As a result, we'll measure area with 'square units,' or units^2 and volume with 'cubic units,' or units^3.

[5] This choice is completely arbitrary. The reader is encouraged to work these examples taking the first point listed as (x_1,y_1) and the second point listed as (x_0,y_0) and verifying the distance works out to be the same. Can you see why the order of the subtraction in Equation 1.1 ultimately doesn't matter?

With $\left(\frac{1}{2}, \frac{2}{3}\right) = (x_0, y_0)$ and $\left(\frac{3}{4}, \frac{1}{5}\right) = (x_1, y_1)$, we get

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{\left(\tfrac{3}{4} - \tfrac{1}{2}\right)^2 + \left(\tfrac{1}{5} - \tfrac{2}{3}\right)^2} && \text{Get common denominators to add and subtract fractions.} \\
&= \sqrt{\left(\tfrac{1}{4}\right)^2 + \left(-\tfrac{7}{15}\right)^2} \\
&= \sqrt{\tfrac{1}{16} + \tfrac{49}{225}} && \text{Because } \left(\tfrac{a}{b}\right)^2 = \tfrac{a^2}{b^2},\ b \neq 0. \\
&= \sqrt{\tfrac{1009}{3600}} \\
&= \tfrac{\sqrt{1009}}{\sqrt{3600}} \\
&= \tfrac{\sqrt{1009}}{60} && \text{For nonnegative numbers, } \sqrt{\tfrac{a}{b}} = \tfrac{\sqrt{a}}{\sqrt{b}},\ b \neq 0.
\end{aligned}$$

So the distance R and S is $\frac{\sqrt{1009}}{60}$ units.

3. Compute the distance between $T(\sqrt{3}, -\sqrt{20})$ and $V(\sqrt{12}, \sqrt{5})$.

   With $(\sqrt{3}, -\sqrt{20}) = (x_0, y_0)$ and $(\sqrt{12}, \sqrt{5}) = (x_1, y_1)$, we get

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{\left(\sqrt{12} - \sqrt{3}\right)^2 + \left(\sqrt{5} - (-\sqrt{20})\right)^2} \\
&= \sqrt{\left(2\sqrt{3} - \sqrt{3}\right)^2 + \left(\sqrt{5} + 2\sqrt{5}\right)^2} && \text{Simplify the radicals to get like terms.} \\
&= \sqrt{\left(\sqrt{3}\right)^2 + \left(3\sqrt{5}\right)^2} \\
&= \sqrt{3 + 9 \cdot 5} && \text{Given } (\sqrt{a})^2 = a \text{ and } (b\sqrt{a})^2 = b^2(\sqrt{a})^2. \\
&= \sqrt{48} \\
&= 4\sqrt{3}
\end{aligned}$$

So the distance between T and V is $4\sqrt{3}$ units.

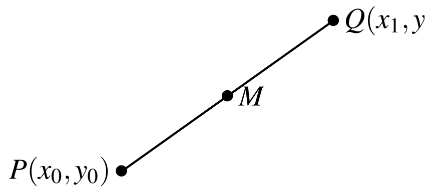
4. Compute the distance between $O(0,0)$ and $P(x,y)$.

   With $(0,0) = (x_0, y_0)$ and $(x,y) = (x_1, y_1)$, we get

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= \sqrt{(x - 0)^2 + (y - 0)^2} \\
&= \sqrt{x^2 + y^2}
\end{aligned}$$

As tempting as it may look, $\sqrt{x^2+y^2}$ does not, in general, reduce to $x+y$ or even $|x|+|y|$. So, in this case, the best we can do is state that the distance between O and P is $\sqrt{x^2+y^2}$ units. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P(x_0,y_0)$ and $Q(x_1,y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.



If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Equation 1.2. The Midpoint Formula: The midpoint M of the line segment connecting $P(x_0,y_0)$ and $Q(x_1,y_1)$ is:

$$M = \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$$

If we let d denote the distance between P and Q, we leave it to the reader to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.4. Determine the midpoint, M, of the line segment connecting the following pairs of points:

1. $P(-2,3)$ and $Q(1,-3)$
2. $R\left(\frac{1}{2},\frac{2}{3}\right)$ and $S\left(\frac{3}{4},\frac{1}{5}\right)$
3. $T(\sqrt{3},-\sqrt{20})$ and $V(\sqrt{12},\sqrt{5})$
4. $O(0,0)$ and $P(x,y)$.

Solution. As with Example 1.1.3, in each case, we apply the midpoint formula, Equation 1.2, with the first point listed taken as (x_0,y_0) and the second point taken as (x_1,y_1).[6] We also note that midpoints are *points*, which means all of our answers should be *ordered pairs*.

1. Determine the midpoint of the line segment connecting $P(-2,3)$ and $Q(1,-3)$.

[6]As in Example 1.1.3, this choice is also completely arbitrary. The reader is encouraged to work these examples taking the first point listed as (x_1,y_1) and the second point listed as (x_0,y_0) and verifying the midpoint works out to be the same. Can you see why the order of the points in Equation 1.2 doesn't matter?

With $(-2,3) = (x_0, y_0)$ and $(1,-3) = (x_1, y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

The midpoint of the line segment connecting P and Q is $\left(-\frac{1}{2}, 0\right)$.

2. Determine the midpoint of the line segment connecting $R\left(\frac{1}{2}, \frac{2}{3}\right)$ and $S\left(\frac{3}{4}, \frac{1}{5}\right)$.

   With $\left(\frac{1}{2}, \frac{2}{3}\right) = (x_0, y_0)$ and $\left(\frac{3}{4}, \frac{1}{5}\right) = (x_1, y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{\frac{1}{2}+\frac{3}{4}}{2}, \frac{\frac{2}{3}+\frac{1}{5}}{2}\right) \\ &= \left(\frac{\left(\frac{1}{2}+\frac{3}{4}\right)\cdot 4}{2\cdot 4}, \frac{\left(\frac{2}{3}+\frac{1}{5}\right)\cdot 15}{2\cdot 15}\right) && \text{Simplify compound fractions.} \\ &= \left(\frac{5}{8}, \frac{13}{30}\right) \end{aligned}$$

   The midpoint of the line segment connecting R and S is $\left(\frac{5}{8}, \frac{13}{30}\right)$.

3. Determine the midpoint of the line segment connecting $T(\sqrt{3}, -\sqrt{20})$ and $V(\sqrt{12}, \sqrt{5})$.

   With $(\sqrt{3}, -\sqrt{20}) = (x_0, y_0)$ and $(\sqrt{12}, \sqrt{5}) = (x_1, y_1)$, we get

$$\begin{aligned} M &= \left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right) \\ &= \left(\frac{\sqrt{3}+\sqrt{12}}{2}, \frac{-\sqrt{20}+\sqrt{5}}{2}\right) \\ &= \left(\frac{\sqrt{3}+2\sqrt{3}}{2}, \frac{-2\sqrt{5}+\sqrt{5}}{2}\right) && \text{Simplify radicals to get like terms.} \\ &= \left(\frac{3\sqrt{3}}{2}, -\frac{\sqrt{5}}{2}\right) \end{aligned}$$

   The midpoint of the line segment connecting T and V is $\left(\frac{3\sqrt{3}}{2}, -\frac{\sqrt{5}}{2}\right)$.

4. Determine the midpoint of the line segment connecting $O(0,0)$ and $P(x,y)$.

   With $(0,0)=(x_0,y_0)$ and $(x,y)=(x_1,y_1)$, we get

$$
\begin{aligned}
M &= \left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)\\
&= \left(\frac{x+0}{2},\frac{y+0}{2}\right)\\
&= \left(\frac{x}{2},\frac{y}{2}\right)
\end{aligned}
$$

   The midpoint of the line segment connecting O and $P$is $\left(\frac{x}{2},\frac{y}{2}\right)$. □

We close with a more abstract application of the Midpoint Formula. We will expand upon this example in Example 1.3.5 in Section 1.3.1.

Example 1.1.5. If $a \neq b$, show that the line $y=x$ equally divides the line segment with endpoints (a,b) and (b,a).

Solution. To prove the claim, we use Equation 1.2 to find the midpoint

$$
\begin{aligned}
M &= \left(\frac{a+b}{2},\frac{b+a}{2}\right)\\
&= \left(\frac{a+b}{2},\frac{a+b}{2}\right)
\end{aligned}
$$

As the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y=x$, as required. □

1.1.3 Exercises

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane.
2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 - 10, find the distance, d, between the two given points and the midpoint M of the line segment which connects the two points.

3. $(1,2)$, $(-3,5)$
4. $(3,-10)$, $(-1,2)$
5. $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$
6. $\left(-\frac{2}{3},\frac{3}{2}\right)$, $\left(\frac{7}{3},2\right)$
7. $\left(\frac{24}{5},\frac{6}{5}\right)$, $\left(-\frac{11}{5},-\frac{19}{5}\right)$.
8. $\left(\sqrt{2},\sqrt{3}\right)$, $\left(-\sqrt{8},-\sqrt{12}\right)$
9. $\left(2\sqrt{45},\sqrt{12}\right)$, $\left(\sqrt{20},\sqrt{27}\right)$.
10. $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
11. Let's assume that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
12. Show that the points A, B, and C below are the vertices of a right triangle.

    $A(-3,2)$, $B(-6,4)$, and $C(1,8)$

Section 1.1 Exercise Answers A.1.1

1.2 Relations and Functions

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected.

In this section, we will study general mappings called *relations*. Then we will turn our focus to a special kind of mapping called *functions*.

Definition 1.2. Given two sets A and B, a **relation** from A to B is a process by which elements of A are matched with (or 'mapped to') elements of B.

1.2.1 Functions as Mappings

The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

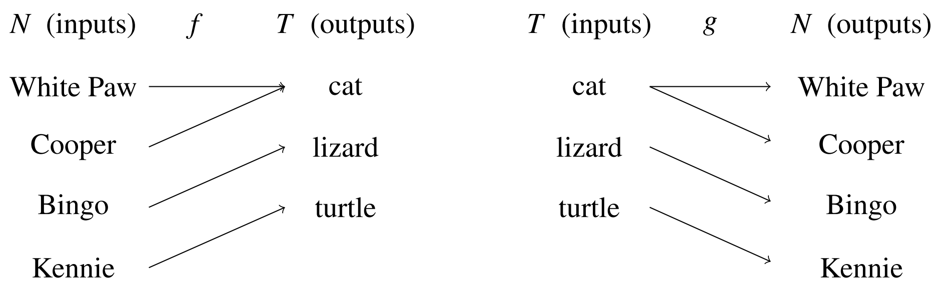
Definition 1.3. Given two sets[a] A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B.

[a] Please refer to Section 0.4 for a review of this terminology.

The grammar here '*from A to B*' is important. Thinking of a function as a process, we can view the elements of the set A as our starting materials, or *inputs* to the process. The function processes these inputs according to some specified rule and the result is a set of *outputs* - elements of the set B. In terms of inputs and outputs, Definition 1.3 says that a function is a process in which each *input* is matched to one and only one *output*.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let N be the set of pet names: $N = \{\text{White Paw, Cooper, Bingo, Kennie}\}$, and let T be the set of pet types: $T = \{\text{cat, lizard, turtle}\}$. Let f be the process that takes each pet's name as the input and returns that pet's type as the output. Let g be the reverse of f: that is, g takes each pet type as the input and returns the names of the pets of that type as the output. Note that both f and g are codifying the *same* given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process f or g is a function and why the other is not, we create **mapping diagrams** for f and g below. In each case, we organize the inputs in a column on the left and the outputs on a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.



The process f is a function because f matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, g matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so g is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

As you might expect, with functions being such an important concept in Mathematics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[1]

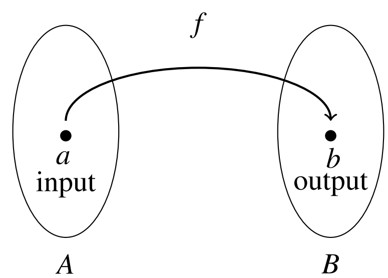
Definition 1.4. Suppose f is a function from A to B.

- If $a \in A$, we write $f(a)$ (read 'f of a') to denote the unique element of B to which f matches a.
  That is, if we view 'a' as the input to f, then '$f(a)$' is the output from f.
- The set A is called the **domain**.
  Said differently, the domain of a function is the set of inputs to the function.
- The set $\{f(a) \,|\, a \in A\}$ is called the **range** of f.
  Said differently, the range of a function is the set of outputs from the function.

Some remarks about Definition 1.4 are in order. First, and most importantly, the notation '$f(a)$' in Definition

[1]Please refer to Section 0.4 for a review of the terminology used in these definitions.

1.4 introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols, to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If f is a function and 'a' is in the domain of f, then '$f(a)$' is the output from f when you input a. The diagram below provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input 'a' and output '$f(a)$'.



In the preceding pet example, the symbol $f(\text{Bingo})$, read 'f of Bingo', is asking what type of pet Bingo is, so $f(\text{Bingo}) = \text{lizard}$. The fact that f is a function means $f(\text{Bingo})$ is unambiguous because f matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation '$g(\text{cat})$' to indicate what pet name g matched to 'cat', we have *two* possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition 1.4 to our pet example, we find that the domain of the function f is N, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \,|\, a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \,|\, a \in A\}$' reads as 'the set of $f(a)$ such that a is in A'. In other words, the range consists of all of the outputs from f - all of the $f(a)$ values - as a varies through each of the elements in the domain A. Note that while every element of the set A is, by definition, an element of the domain of f, not every element of the set B is necessarily part of the range of f.[2]

In our pet example, we can obtain the range of f by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from f as we run through all of the inputs to f. Keep in mind that we list each element of a set only once so the range of f is:[3]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

If we let n denote a generic element of N then $f(n)$ is some element t in T, so we write $t = f(n)$. In this equation, n is called the **independent variable** and t is called the **dependent variable**.[4] Moreover, we say

[2] For purposes of completeness, the set B is called the **codomain** of f. For us, the concepts of domain and range suffice as our codomain will most always be the set of real numbers, $\mathbb{R}$.

[3] If instead of mapping N into T, we could have mapped N into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of f would not have been the entire codomain U.

[4] These adjectives stem from the fact that the value of t *depends* entirely on our (independent) choice of n.

't is a function of n', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name n corresponds to one, and only one, pet type t. Even though f and t are different things,[5] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[6] We will endeavor to point out such ambiguities as we move through the text.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

Example 1.2.1. Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

time (hours after 6 a.m.)	outdoor temperature in degrees Fahrenheit
0	64
2	67
4	75
6	80
8	83
10	83
12	82

1. Explain why the recorded outdoor temperature is a function of the corresponding time.
2. Is time a function of the outdoor temperature? Explain.
3. Let f be the function which matches time to the corresponding recorded outdoor temperature.
   (a) Find and interpret the following:
   - $f(2)$
   - $f(4)$
   - $f(2+4)$
   - $f(2)+f(4)$
   - $f(2)+4$

   (b) Solve and interpret $f(t) = 83$.
   (c) State the range of f. What is lowest recorded temperature of the day? The highest?

Solution.

[5]Specifically, f is a function so it requires and domain, a range and a rule of assignment whereas t is simply the output from f.

[6]In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that 'y is a function of x'.

1. Explain why the recorded outdoor temperature is a function of the corresponding time.

   The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.

2. Is time a function of the outdoor temperature? Explain.

   Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.

3. Find and interpret: $f(2)$, $f(4)$, $f(2+4)$, $f(2)$ + $f(4)$, and $f(2)+4$

   (a)
   - To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.
   - Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.
   - From the table, we find $f(2+4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.
   - Using results from above we see that $f(2)+f(4) = 67+75 = 142$. When adding $f(2)+f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.
   - We compute $f(2)+4 = 67+4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

   (b) Solve and interpret $f(t) = 83$.

   Solving $f(t) = 83$ means finding all of the input (time) values t which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

   (c) State the range of f. What is lowest recorded temperature of the day? The highest?

   The range of f is the set of all of the outputs from f, or in this case, the outside recorded temperatures. Based on the data, we get $\{64,67,75,80,82,83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F. □

A few remarks about Example 1.2.1 are in order. First, note that $f(2+4)$, $f(2)+f(4)$ and $f(2)+4$ all work out to be numerically different, and more importantly, all represent different things.[7] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

[7] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

Next, when solving $f(t) = 83$, the variable 't' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to f produce an output of 83. The choice of the letter 't' here makes sense as the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

Finally, given that the range in this example was a finite set of real numbers, we could identify the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to identify the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

Definition 1.5. Suppose f is a function whose range is a set of real numbers containing m and M.

- The value m is called the **absolute minimum**[a] of f if $m \leq f(x)$ for all x in the domain of f.
  That is, the absolute minimum of f is the smallest output from f, if it exists.
- The value M is called the **absolute maximum**[b] of f if $f(x) \leq M$ for all x in the domain of f.
  That is, the absolute maximum of f is the largest output from f, if it exists.
- Taken together, the values m and M (if they exist) are called the **absolute extrema**[c] of f.

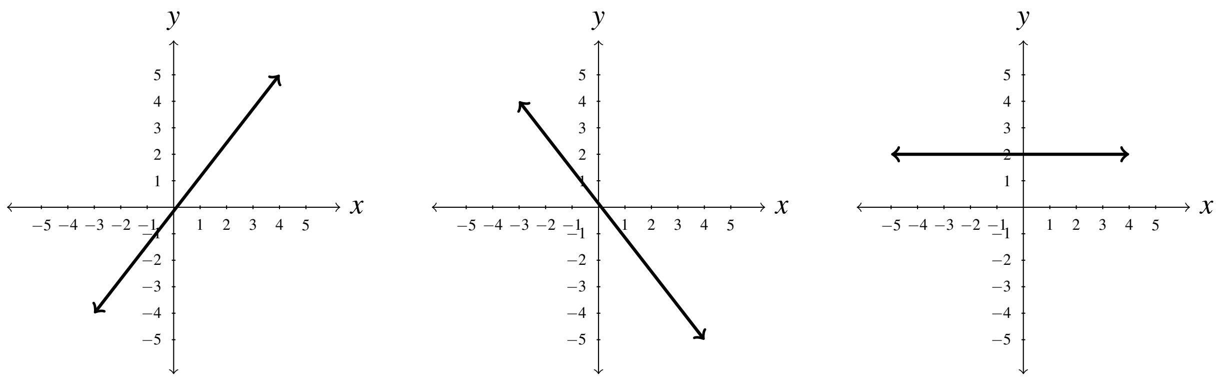
[a] also called the 'global' minimum
[b] also called the 'global' maximum
[c] also called the 'global' extrema or the 'extreme values'

Definition 1.5 is an example where the name of the function, f, is being used almost synonymously with its outputs in that when we speak of 'the minimum and maximum of the *function* f' we are really talking about the absolute minimum and absolute maximum values of the *outputs* $f(x)$ as x varies through the domain of f. Thus, we say that the absolute maximum of f is 83 and the absolute minimum of f is 64 when referring to the highest and lowest recorded temperatures in the previous example.

Definition 1.6. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of an increasing function **rises** as one moves from left to right.
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a decreasing function **falls** as one moves from left to right.
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Also, note that, like Definition 1.5, Definition 1.6 blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of *inputs*. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically below.



Another item of note about functions is the symmetry about the line $x = 0$ (the y-axis). See Definition 1.1 for a review of this concept.) We have that for all x, $(-x, f(-x)) = (-x, f(x))$ on the graph of f, the point symmetric about the y-axis, $(-x, f(x))$ is on the graph, too. An investigation of syemmetry with respect to the origin yields similar results with the major difference being that when a negative number is raised to an odd natural number power the result is still negative.

Definition 1.7.

- A function f is said to be **even** if $f(-x) = f(x)$ for all x in the domain of f.

  NOTE: A function f is even if and only if the graph of $y = f(x)$ is symmetric about the y-axis.

- A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

  NOTE: A function f is odd if and only if the graph of $y = f(x)$ is symmetric about the origin.

1.2.2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

Example 1.2.2.

1. Let f be the function which takes a real number and performs the following sequence of operations:

- Step 1: add 2
- Step 2: multiply the result of Step 1 by 3
- Step 3: subtract 1 from the result of Step 2.

(a) Compute and simplify $f(-5)$.

(b) Identify and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

(a) Compute and simplify the following:

i. $h(-1)$, $h(0)$ and $h(2)$.

ii. $h(2x)$ and $2h(x)$.

iii. $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$.

(b) Solve $h(t) = 0$.

Solution.

1. Compute and simplify $f(-5)$.

(a) We take -5 and follow it through each step:

- Step 1: adding 2 gives us $-5+2 = -3$.
- Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
- Step 3: subtracting 1 from the result of Step 2 produces $-9-1 = -10$.

Hence, $f(-5) = -10$.

(b) Identify and simplify a formula for $f(x)$.

To develop a formula for $f(x)$, we repeat the above process but use the variable 'x' in place of the number -5:

- Step 1: adding 2 gives us the quantity $x+2$.
- Step 2: multiplying the result of Step 1 by 3 yields $(x+2)(3) = 3x+6$.
- Step 3: subtracting 1 from the result of Step 2 produces $(3x+6)-1 = 3x+5$.

Hence, we have codified f using the formula $f(x) = 3x+5$. In other words, the function f matches each real number 'x' with the value of the expression '$3x+5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5)+5 = -10$ as before.

2. Given $h(t) = -t^2 + 3t + 4$, compute and simplify $h(-1)$, $h(0)$ and $h(2)$.

As before, representing the function h as $h(t) = -t^2 + 3t + 4$ means that h matches the real number t with the value of the expression $-t^2 + 3t + 4$.

(a) To find $h(-1)$, we substitute -1 for t in the expression $-t^2+3t+4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

$$\begin{aligned} h(-1) &= -(-1)^2+3(-1)+4 \\ &= -(1)+(-3)+4 \\ &= 0. \end{aligned}$$

Similarly, $h(0)=-(0)^2+3(0)+4=4$, and $h(2)=-(2)^2+3(2)+4=-4+6+4=6$.

(b) Given $h(t)=-t^2+3t+4$, compute and simplify $h(2x)$ and $2h(x)$.

To find $h(2x)$, we substitute $2x$ for t:

$$\begin{aligned} h(2x) &= -(2x)^2+3(2x)+4 \\ &= -(4x^2)+(6x)+4 \\ &= -4x^2+6x+4. \end{aligned}$$

The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting x for t: $h(x)=-x^2+3x+4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8. \end{aligned}$$

(c) Given $h(t)=-t^2+3t+4$, compute and simplify $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$.

To find $h(t+2)$, we substitute the quantity $t+2$ in place of t:

$$\begin{aligned} h(t+2) &= -(t+2)^2+3(t+2)+4 \\ &= -\left(t^2+4t+4\right)+(3t+6)+4 \\ &= -t^2-4t-4+3t+6+4 \\ &= -t^2-t+6. \end{aligned}$$

To find $h(t)+2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t)+2 &= \left(-t^2+3t+4\right)+2 \\ &= -t^2+3t+6. \end{aligned}$$

From our work above, we see that $h(2)=6$ so

$$\begin{aligned} h(t)+h(2) &= \left(-t^2+3t+4\right)+6 \\ &= -t^2+3t+10. \end{aligned}$$

3. Solve $h(t) = 0$.

   We know $h(-1) = 0$ from above, so $t = -1$ should be one of the answers to $h(t) = 0$. In order to see if there are any more, we set $h(t) = -t^2 + 3t + 4 = 0$. Factoring[8] gives $-(t+1)(t-4) = 0$, so we get $t = -1$ (as expected) along with $t = 4$. □

A few remarks about Example 1.2.2 are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the *input* by 2; in the latter, we are multiplying the *output* by 2. The same goes for $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$. The expression $h(t+2)$ calls for adding 2 to the input t and then performing the function h. The expression $h(t)+2$ has us performing the process h first, then adding 2 to the output $h(t)$. Finally, $h(t)+h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example 1.2.1, we see here again the importance paying close attention to syntax.[9]

Let us return for a moment to the function f in Example 1.2.2 which we ultimately represented using the formula $f(x) = 3x+5$. If we introduce the dependent variable y, we get the equation $y = f(x) = 3x+5$, or, more simply $y = 3x+5$. To say that the equation $y = 3x+5$ describes y as a function of x means that for each choice of x, the formula $3x+5$ determines only one associated y-value.

We could turn the tables and ask if the equation $y = 3x+5$ describes x as a function of y. That is, for each value we pick for y, does the equation $y = 3x+5$ produce only one associated x value? One way to proceed is to solve $y = 3x+5$ for x and get $x = \frac{1}{3}(y-5)$. We see that for each choice of y, the expression $\frac{1}{3}(y-5)$ evaluates to just one number, hence, x is a function of y. If we give this function a name, say g, we have $x = g(y) = \frac{1}{3}(y-5)$, where in this equation, y is the independent variable and x is the dependent variable. We explore this idea in the next example.

Example 1.2.3.

1. Consider the equation $x^3 + y^2 = 25$.
   (a) Does this equation represent y as a function of x? Explain.
   (b) Does this equation represent x as a function of y? Explain.
2. Consider the equation $u^4 + t^3u = 16$.
   (a) Does this equation represent t as a function of u? Explain.
   (b) Does this equation represent u as a function of t? Explain.

Solution.

1. (a) Does $x^3 + y^2 = 25$ represent y as a function of x? Explain.

[8] You may need to review Section 0.3.

[9] As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

To say that $x^3 + y^2 = 25$ represents y as a function of x, we need to show that for each x we choose, the equation produces only one associated y-value. To help with this analysis, we solve the equation for y in terms of x.

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ y^2 & = & 25 - x^3 & \\ y & = & \pm\sqrt{25 - x^3} & \text{extract square roots. (See Section 0.2 for a review, if needed.)} \end{array}$$

The presence of the '$\pm$' indicates that there is a good chance that for some x-value, the equation will produce *two* corresponding y-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does *not* represent y as a function of x because $x = 0$ is matched with more than one y-value.

(b) Does $x^3 + y^2 = 25$ represent x as a function of y? Explain.

To see if $x^3 + y^2 = 25$ represents x as a function of y, we solve the equation for x in terms of y:

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ x^3 & = & 25 - y^2 & \\ & = & \sqrt[3]{25 - y^2} & \text{extract cube roots. (See Section 0.2 for a review, if needed.)} \end{array}$$

In this case, each choice of y produces only *one* corresponding value for x, so $x^3 + y^2 = 25$ represents x as a function of y.

2. (a) Does $u^4 + t^3u = 16$ represent t as a function of u? Explain.

To see if $u^4 + t^3u = 16$ represents t as a function of u, we proceed as above and solve for t in terms of u:

$$\begin{array}{rcll} u^4 + t^3u & = & 16 & \\ t^3u & = & 16 - u^4 & \\ t^3 & = & \dfrac{16 - u^4}{u} & \text{assumes } u \neq 0 \\ t & = & \sqrt[3]{\dfrac{16 - u^4}{u}} & \text{extract cube roots.} \end{array}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\frac{16-u^4}{u}}$ will produce just one value of t for each value of u. What if $u = 0$? In that case, the equation $u^4 + t^3u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[10] Hence, $u^4 + t^3u = 16$ represents t as a function of u.

(b) Does $u^4 + t^3u = 16$ represent t as a function of u? Explain.

[10] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3u = 16$.

In order to determine if $u^4 + t^3u = 16$ represents u as a function of t, we could attempt to solve $u^4 + t^3u = 16$ for u in terms of t, but we won't get very far.[11] Instead, we take a different approach and experiment with looking for solutions for u for specific values of t. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one u-value which means $u^4 + t^3u = 16$ does not represent u as a function of t. $\square$

We'll have more to say about using equations to describe functions later in this section. For now, we turn our attention to a geometric way to represent functions.

1.2.3 Geometric Representations of Functions

In this subsection, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section 1.1 as needed.

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function f with domain A, we can represent $f = \{(a, f(a)) \,|\, a \in A\}$. That is, we represent f as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function f which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

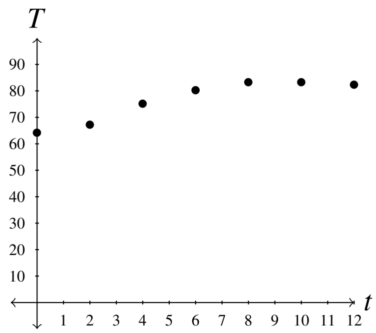
Moving on, we next consider the function f from Example 1.2.1 which relates time to temperature. In this case, $f = \{(0,64), (2,67), (4,75), (6,80), (8,83), (10,83), (12,82)\}$. This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use t to label the horizontal axis. Likewise, we'll use T to label the vertical axis because the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, t and T, respectively.

The plot of these points is called 'the **graph** of f'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable t. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable t and the dependent variable T.

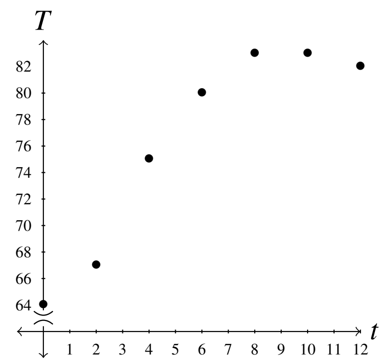
On the next page we present two plots, both of which are graphs of the function f. In both cases, the vertical axis has been scaled in order to save space. In the graph on the left, the same increment on the horizontal axis to measure 1 unit measures 10 units on the vertical axis whereas in the graph on the right, this ratio is 1 : 2. The '$\asymp$' symbol on the vertical axis in the graph on the right is used to indicate a jump in the

[11] Try it for yourself!

vertical labeling. Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of f, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0,64)$, $(8,83)$ and $(10,83)$. More often than not, we will use the graph of a function to help us optimize that function.[12]



The graph of $T = f(t)$.



The graph of $T = f(t)$.

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function f matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of f. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

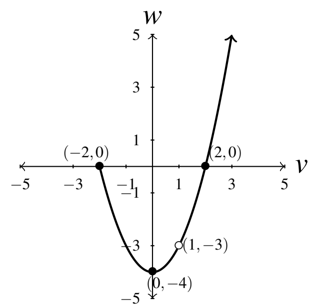
Skippy's temperature function f is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[13] We'll return to this example in Sections 1.3.1 and 2.1 in an attempt to do just that.

In the meantime, our next example involves a function whose domain is (almost) an *interval* of real numbers and whose graph consists of a (mostly) *connected* arc.

[12]One major use of Calculus is to optimize functions analytically - that is, without a graph.

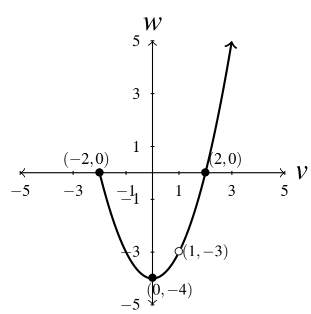
[13]Roughly speaking, a *continuous variable* is a variable which takes on values over an *interval* of real numbers as opposed to values in a discrete list. In this case we would think of time as a 'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A *continuous function* is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.

Example 1.2.4. Consider the graph below.



1. (a) Explain why this graph suggests that w is a function of v, $w = F(v)$.
   (b) Compute $F(0)$ and solve $F(v) = 0$.
   (c) State the domain and range of F using interval notation.[14] Then identify the extrema of F, if any exist.
2. Does this graph suggest v is a function of w? Explain.

Solution. The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of the graph; a gap or 'hole' labeled $(1,-3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.



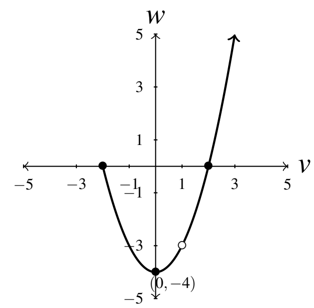
1. (a) Explain why this graph suggests that w is a function of v, $w = F(v)$.

   In order for w to be a function of v, each v-value on the graph must be paired with only one w-value. What if this weren't the case? We'd have at least two points with the *same* v-coordinate with *different* w-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that w is a function of v.

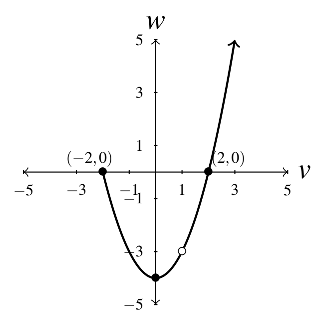
[14] Please consult Section 0.4 for a review of interval notation if need be.

(b) Compute $F(0)$ and solve $F(v) = 0$.

The value $F(0)$ is the output from F when $v = 0$. The points on the graph of F are of the form $(v, F(v))$, thus we are looking for the w-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph (see below), we can be sure $F(0) = -4$.

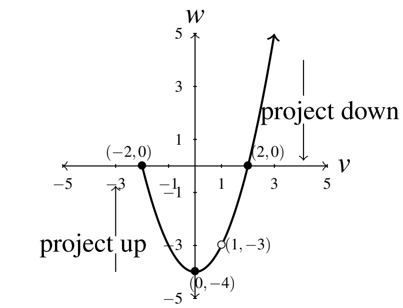


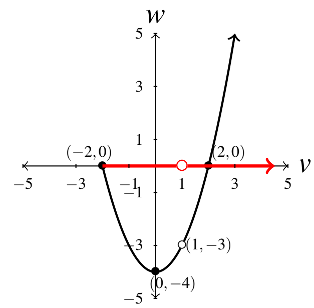
To solve $F(v) = 0$, we are looking for the v-values where the output, or associated w value, is 0. Hence, we are looking for points on the graph with a w-coordinate of 0. We identify two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$.



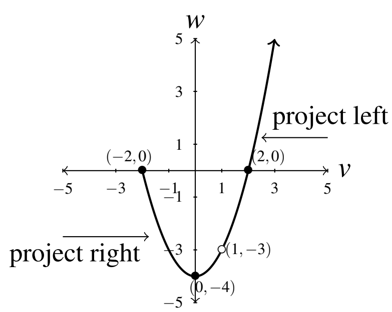
(c) State the domain and range of F using interval notation. Identify the extrema of F, if any exist.

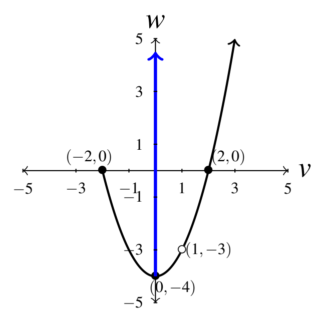
The domain of F is the set of inputs to F. With v as the input here, we need to describe the set of v-values on the graph. We can accomplish this by **projecting** the graph to the v-axis and seeing what part of the v-axis is covered. The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of F. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \mid v \geq -2,\, v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$. Pictures demonstrating the process of projecting the graph to the v-axis are shown below.





To find the range of F, we need to describe the set of outputs - in this case, the w-values on the graph. Here, we project the graph to the w-axis. Vertically, the graph starts at $(0,-4)$ so our range starts at $w=-4$. Note that even though there is a hole at $(1,-3)$, the w-value -3 is covered by what *appears* to be the point $(-1,-3)$ on the graph.[15] The arrow indicates that the graph extends upwards indefinitely so the range of F is $\{w \,|\, w \geq -4\}$ or, in interval notation, $[-4,\infty)$. Regarding extrema, F has a minimum of -4 when $v=0$, but given that the graph extends upwards indefinitely, F has no maximum. Pictures showing the projection of the graph onto the w-axis are given below.





2. Does this graph suggest v is a function of w? Explain.

   Finally, to determine if v is a function of w, we look to see if each w-value is paired with only one v-value on the graph. We have points on the graph, namely $(-2,0)$ and $(2,0)$, that clearly show us that $w=0$ is matched with the *two* v-values $v=2$ and $v=-2$. Hence, v is not a function of w. □

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001,-4.0001)$ instead of $(0,-4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately

[15]For all we know, it could be $(-0.992,-3)$.

made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a w-coordinate of -3.

Before moving on to our next example, it is worth noting that the geometric argument made in Example 1.2.4 to establish that w is a function of v can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' x and y labels on the horizontal and vertical axes, respectively.

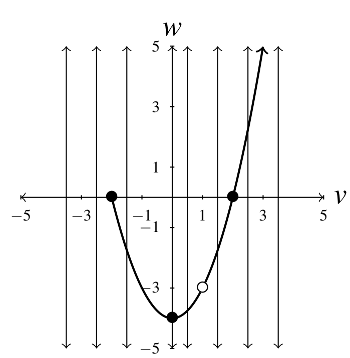
Theorem 1.2. The Vertical Line Test: A graph in the xy-plane[a] represents y as a function of x if and only if no vertical line intersects the graph more than once.

[a]That is, the horizontal axis is labeled with 'x' and the vertical axis is labeled with 'y'.

Let's take a minute to discuss the phrase 'if and only if' used in Theorem 1.2. The statement 'the graph represents y as a function of x *if and only if* no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents y as a function of x *if* no vertical line intersects the graph more than once' and, second, 'the graph represents y as a function of x *only if* no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents y as a function of x. But the question remains: could a graph represent y as a function of x and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the *only* way the graph represents y as a function of x is the case when no vertical line intersects the graph more than once.

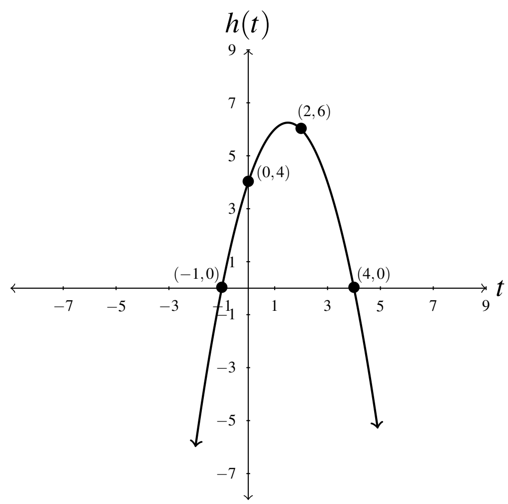
Applying the Vertical Line Test to the graph given in Example 1.2.4, we see below that all of the vertical lines meet the graph at most once (several are shown for illustration) showing w is a function of v. Notice that some of the lines ($v = -3$ and $v = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say *exactly* once so missing the graph altogether is permitted.



There is also a geometric test to determine if the graph above represents v as a function of w. We introduce this aptly-named **Horizontal Line Test** in Exercise 57 and revisit it in Sections 1.2 and 5.1.

Our next example revisits the function h from Example 1.2.2 from a graphical perspective.

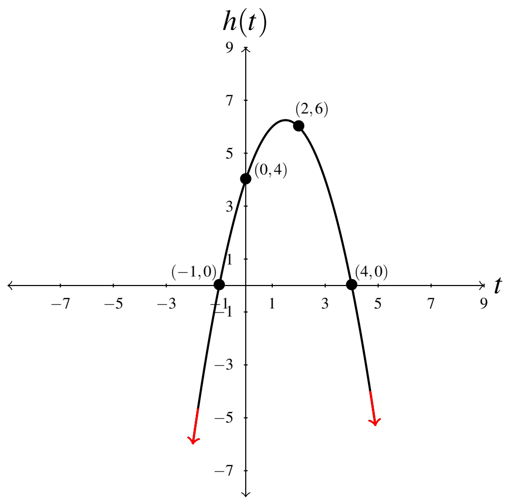
Example 1.2.5. Using the graph of $h(t) = -t^2 + 3t + 4$ below, state the domain, range, any absolute extrema, and the intervals where $h(t)$ is increasing, decreasing, or constant, if any exist.



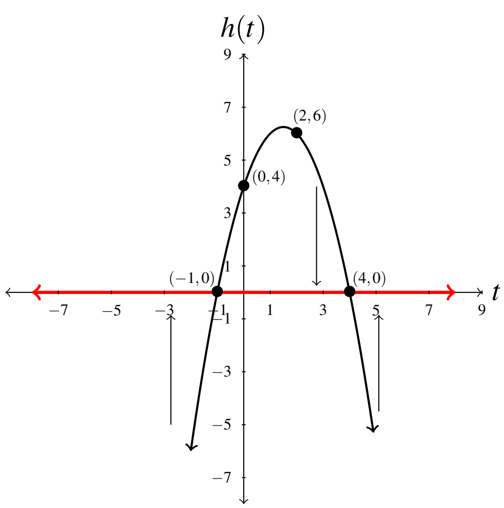
Solution. The dependent variable wasn't specified so we use the default 'y' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example 1.2.2, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us the points $(-1,0)$, $(0,4)$, $(2,6)$ and $(4,0)$, respectively. Using these as a guide, we produce the graph above.[16]

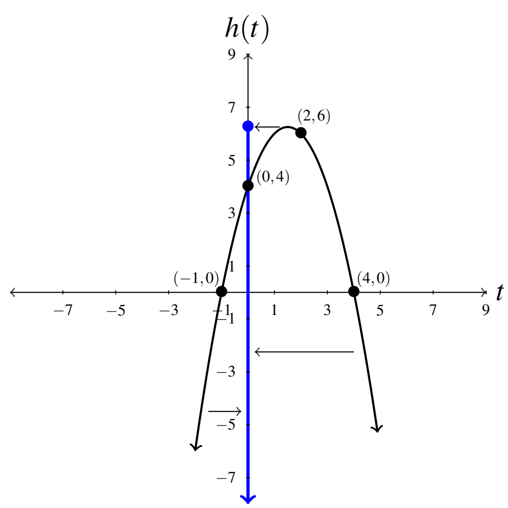
As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable t so the domain of h is all real numbers. However, the graph as presented shows only the behavior of h between roughly $t = -1.5$ and $t = 4.25$. The arrows at the ends of our graph indicate the graph extends downwards indefinitely.

[16] The curve in this example is called a 'parabola'. In Section 2.1, we'll learn how to graph these accurately *by hand*.

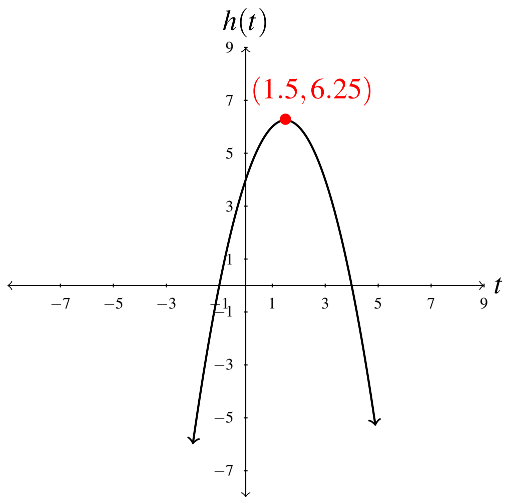


Using projections we note that the domain is $(-\infty,\infty)$ and the range is $(-\infty,6.25]$.

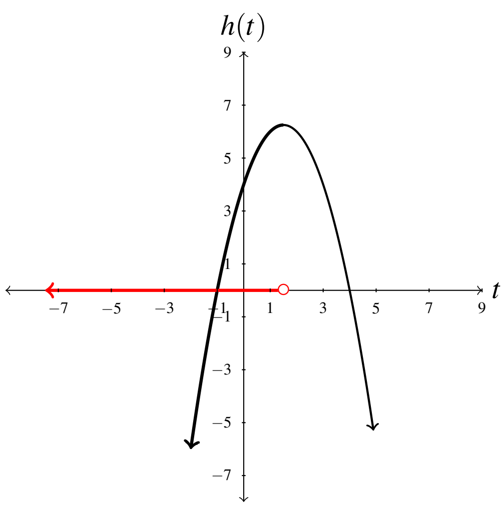




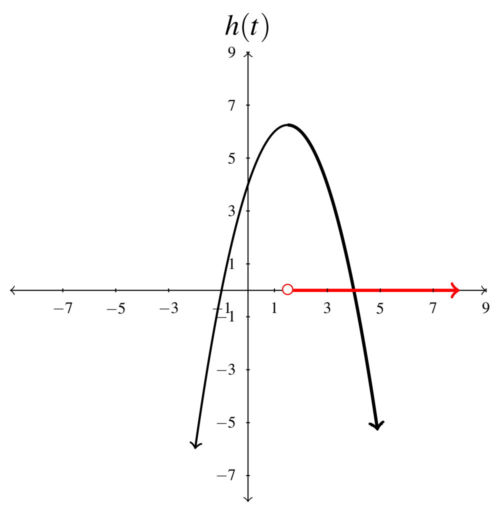
There is no minimum, but the maximum of $h(t)$ is 6.25 and it occurs at $t = 1.5$. The point $(1.5, 6.25)$ is shown on the graph.



$h(t)$ is increasing on the interval $(-\infty, 1.5)$ and $h(t)$ is decreasing on the interval $(1.5, \infty)$. $h(t)$ is not constant on any interval.



Interval $h(t)$ where is increasing

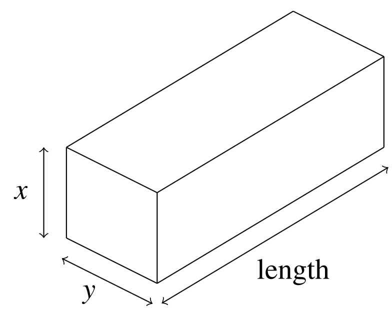


Interval where $h(t)$ is decreasing

□

Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

Example 1.2.6. The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[17] Suppose we wish to ship a rectangular box whose girth forms a square measuring x inches per side as shown below.



It turns out[18] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130-4x)$ for $0 < x \leq 26$.

1. Compute and interpret $V(5)$.
2. Make a table of values and use these to sketch a graph $y = V(x)$.
3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

Solution.

1. Compute and interpret $V(5)$.

   To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130-4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[19]

2. Make a table of values and use these to sketch a graph $y = V(x)$.

   The domain of V is specified by the inequality $0 < x \leq 26$, so we can begin graphing V by sampling V at finitely many x-values in this interval to help us get a sense of the range of V. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130-4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$

[17] See here.

[18] We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise 56a in Section 2.2.

[19] Note that we have $V(5)$ and $25(110)$ in the same string of equality. The first set of parentheses is function notation and directs us to substitute 5 for x in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

because $x = 0$ is not in the domain of V. However, there is nothing stopping us from evaluating $V(x)$ at values x 'very close' to $x = 0$. A table of such values is given below.

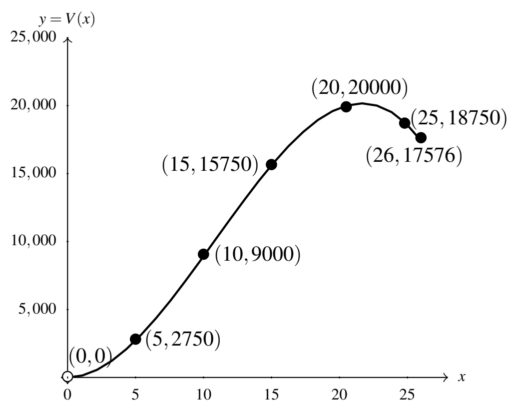
x	$V(x)$
0.1	1.296
0.01	0.012996
0.001	0.000129996
10^{-23}	$\approx 1.3 \times 10^{-44}$

There is no such thing as a 'smallest' positive number,[20] so we will have points on the graph of V to the right of $x = 0$ leading to the point $(0,0)$. We indicate this behavior by putting a hole at $(0,0)$.[21]

Moving forward, we start with $x = 5$ and sample V at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we plot our points $(x, V(x))$ using the domain as a guide to help us set the horizontal bounds (i.e., the bounds on x) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on y). The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$ so we finish the graph by plotting the point $(26, V(26)) = (26, 17576)$.

x	$V(x)$	$(x, V(x))$
≈ 0	≈ 0	hole at $(0,0)$
5	2750	$(5, 2750)$
10	9000	$(10, 9000)$
15	15,750	$(15, 15750)$
20	20,000	$(20, 20000)$
25	18,750	$(25, 18750)$
26	17,576	$(26, 17576)$

Sampling V



The graph of $y = V(x)$

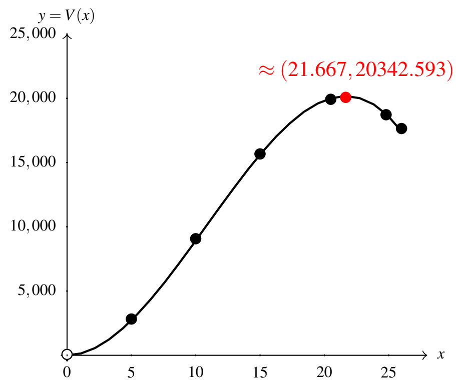
3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

   The largest volume in this case refers to the maximum of V. The biggest y-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by the graphing utility indicates that there are points on the graph of V with y-values (hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and with the use of technology we

[20] If p is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[21] What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

can determine the maximum y-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$. (In Calculus we will learn an algebraic method for computing the exact maximum value of a function.) Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[22]



Finding the maximum volume using the graph of $y = V(x)$.

It is worth noting that while the function V has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all x in its domain,[23] the presence of the hole at $(0,0)$ means that 0 is not in the range of V. Hence, based on our model, we can never make a box with a 'smallest' volume.[24] □

Example 1.2.6 typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of V: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to *analytically* produce *exact* answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

That being said, we'll also encounter problems which we simply *cannot* answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find *approximate* solutions. The most important thing to keep in mind is that while technology may *suggest* a result, it is ultimately Mathematics that *proves* it.

We close this section with a summary of the different ways to represent functions.

[22] We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

[23] said differently, the values of $V(x)$ are **bounded below** by 0.

[24] How realistic is this?

Ways to Represent a Function

Suppose f is a function with domain A. Then f can be represented:

- verbally; that is, by describing how the inputs are matched with their outputs.
- using a mapping diagram.
- as a set of ordered pairs of the form $(\text{input}, \text{output})$: $\{(a, f(a)) \,|\, a \in A\}$.

If f is a function whose domain and range are subsets of real numbers, then f can be represented:

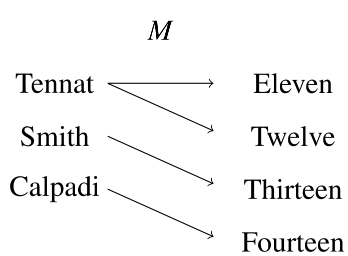
- algebraically as a formula for $f(a)$.
- graphically by plotting the points $\{(a, f(a)) \,|\, a \in A\}$ in the plane.

Note: An important consequence of the last bulleted item is that the point (a,b) is on the graph of $y = f(x)$ if and only if $f(a) = b$.

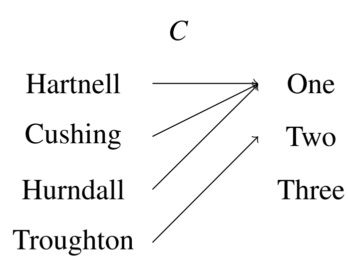
1.2.4 Exercises

In Exercises 1 - 2, determine whether or not the mapping diagram represents a function. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

1.



2.



In Exercises 3 - 4, determine whether or not the data in the given table represents y as a function of x. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

3.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

4.

x	y
0	0
1	1
1	-1
2	2
2	-2
3	3
3	-3

5. Suppose W is the set of words in the English language and we set up a mapping from W into the set of natural numbers $\mathbb{N}$ as follows: word $\rightarrow$ number of letters in the word. Explain why this mapping is a function. What would you need to know to determine the range of the function?

6. Suppose L is the set of last names of all the people who have served or are currently serving as the President of the United States. Consider the mapping from L into $\mathbb{N}$ as follows: last name $\rightarrow$ number of their presidency. For example, Washington $\rightarrow 1$ and Obama $\rightarrow 44$. Is this mapping a function? What if we use full names instead of just last names? (**HINT:** Research Grover Cleveland.)

7. Under what conditions would the time of day be a function of the outdoor temperature?

For the functions f described in Exercises 8 - 13, find $f(2)$ and find and simplify an expression for $f(x)$ that takes a real number x and performs the following three steps in the order given:

8. (1) multiply by 2; (2) add 3; (3) divide by 4.
9. (1) add 3; (2) multiply by 2; (3) divide by 4.
10. (1) divide by 4; (2) add 3; (3) multiply by 2.
11. (1) multiply by 2; (2) add 3; (3) take the square root.
12. (1) add 3; (2) multiply by 2; (3) take the square root.
13. (1) add 3; (2) take the square root; (3) multiply by 2.

In Exercises 14 - 19, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$

- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

14. $f(x)=2x+1$
15. $f(x)=3-4x$
16. $f(x)=2-x^2$
17. $f(x)=x^2-3x+2$
18. $f(x)=6$
19. $f(x)=0$

In Exercises 20 - 25, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a)+f(2)$
- $f\left(\dfrac{2}{a}\right)$
- $\dfrac{f(a)}{2}$
- $f(a+h)$

20. $f(x)=2x-5$
21. $f(t)=5-2t$
22. $f(w)=2w^2-1$
23. $f(q)=3q^2+3q-2$
24. $f(r)=117$
25. $f(z)=\dfrac{z}{2}$

In Exercises 26 - 29, use the given function f to find $f(0)$ and solve $f(x)=0$

26. $f(x)=2x-1$
27. $f(x)=3-\frac{2}{5}x$
28. $f(x)=2x^2-6$
29. $f(x)=x^2-x-12$

In Exercises 30 - 44, determine whether or not the equation represents y as a function of x.

30. $y=x^3-x$
31. $y=\sqrt{x-2}$
32. $x^3y=-4$
33. $x^2-y^2=1$
34. $y=\dfrac{x}{x^2-9}$
35. $x=-6$
36. $x=y^2+4$
37. $y=x^2+4$
38. $x^2+y^2=4$
39. $y=\sqrt{4-x^2}$
40. $x^2-y^2=4$
41. $x^3+y^3=4$
42. $2x+3y=4$
43. $2xy=4$
44. $x^2=y^2$

Exercises 45 - 56 give a set of points in the xy-plane. Determine if y is a function of x. If so, state the domain and range.

45. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

46. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

47. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

48. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

49. $\{(x,y)\,|\,x$ is an odd integer, and y is an even integer$\}$

50. $\{(x,1)\,|\,x$ is an irrational number$\}$

51. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),\ldots\}$

52. $\{\ldots(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$

53. $\{(-2,y)\,|\,-3<y<4\}$

54. $\{(x,3)\,|\,-2\leq x<4\}$

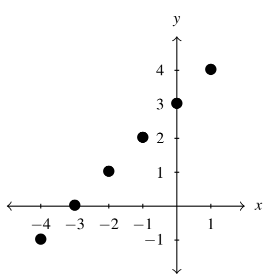
55. $\{\left(x,x^2\right)\,|\,x$ is a real number$\}$

56. $\{\left(x^2,x\right)\,|\,x$ is a real number$\}$

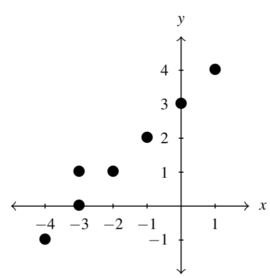
57. The Vertical Line Test is a quick way to determine from a graph if the vertical axis variable is a function of the horizontal axis variable. If we are given a graph and asked to determine if the horizontal axis variable is a function of the vertical axis variable, we can use horizontal lines instead of vertical lines to check. Using Theorem 1.2 as a guide, formulate a 'Horizontal Line Test.' (We'll refer back to this exercise in Section 5.1.)

In Exercises 58 - 61, determine whether or not the graph suggests y is a function of x. For the ones which do, state the domain and range.

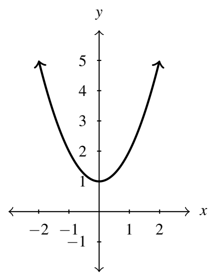
58.



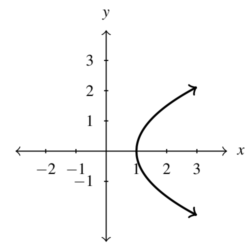
59.



60.



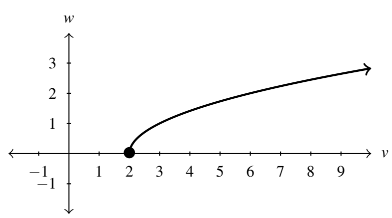
61.



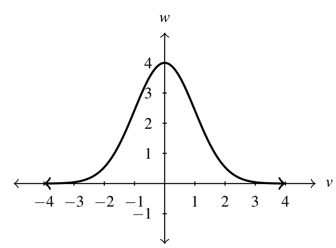
62. Determine which, if any, of the graphs in numbers 58 - 61 represent x as a function of y. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 63 - 66, determine whether or not the graph suggests w is a function of v . For the ones which do, state the domain and range.

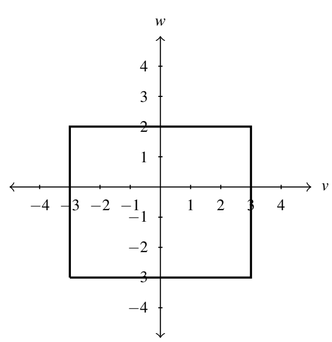
63.



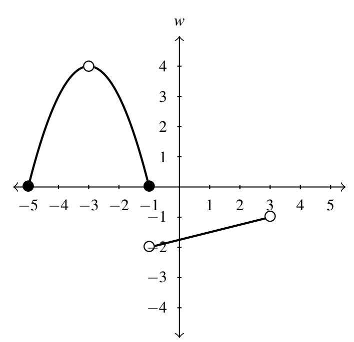
64.



65.



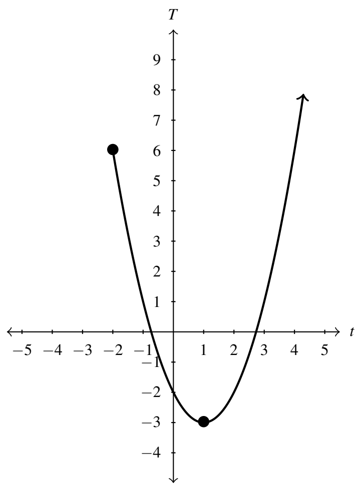
66.



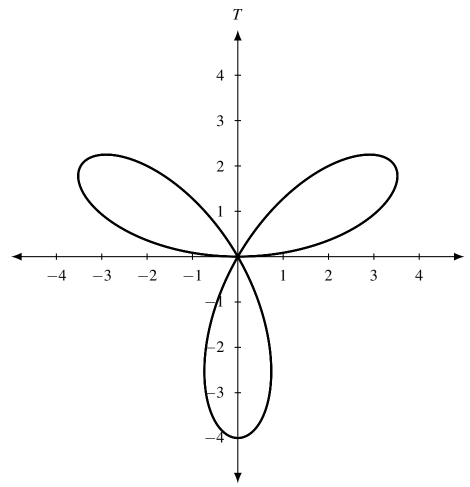
67. Determine which, if any, of the graphs in numbers 63 - 66 represent v as a function of w. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 68 - 71, determine whether or not the graph suggests T is a function of t. For the ones which do, state the domain and range.

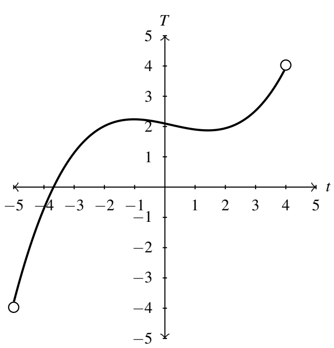
68.



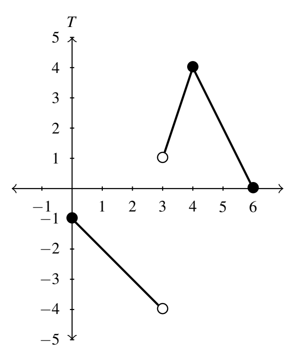
69.



70.



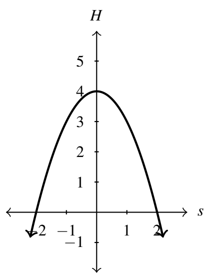
71.



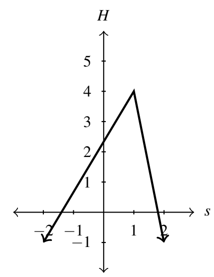
72. Determine which, if any, of the graphs in numbers 68 - 71 represent t as a function of T. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 73 - 76, determine whether or not the graph suggests H is a function of s. For the ones which do, state the domain and range.

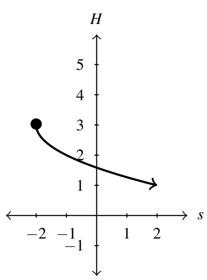
73.



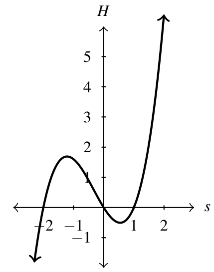
74.



75.



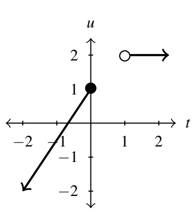
76.



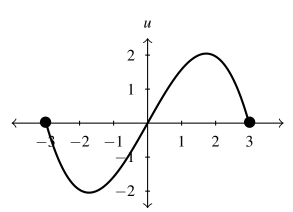
77. Determine which, if any, of the graphs in numbers 73 - 76 represent s as a function of H. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 78 - 81, determine whether or not the graph suggests u is a function of t. For the ones which do, state the domain and range.

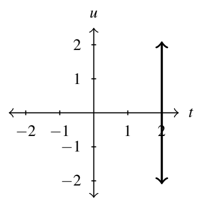
78.



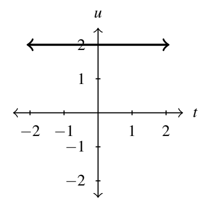
79.



80.

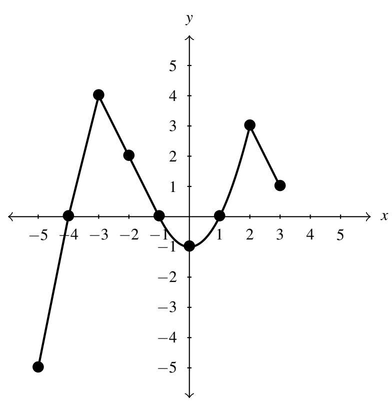


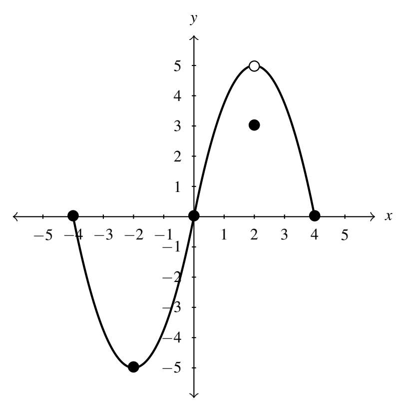
81.



82. Determine which, if any, of the graphs in numbers 78 - 81 represent t as a function of u. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 83 - 92, use the graphs of f and g below to find the indicated values.





83. $f(-2)$

84. $g(-2)$

85. $f(2)$

86. $g(2)$

87. $f(0)$

88. $g(0)$

89. Solve $f(x) = 0$.

90. Solve $g(t) = 0$.

91. State the domain and range of f.

92. State the domain and range of g .

In Exercises 93 - 104, graph each function by making a table and plotting points. Use the independent variable as the horizontal axis label and the default 'y' label for the vertical axis label. State the domain and range of each function.

93. $f(x) = 2 - x$

94. $g(t) = \dfrac{t-2}{3}$

95. $h(s) = s^2 + 1$

96. $f(x) = 4 - x^2$

97. $g(t) = 2$

98. $h(s) = s^3$

99. $f(x) = x(x-1)(x+2)$

100. $g(t) = \sqrt{t-2}$

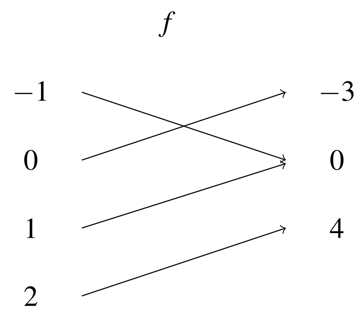
101. $h(s) = \sqrt{5-s}$

102. $f(x) = 3 - 2\sqrt{x+2}$

103. $g(t) = \sqrt[3]{t}$

104. $h(s) = \dfrac{1}{s^2+1}$

105. Consider the function f described below:



(a) State the domain and range of f.

(b) Find $f(0)$ and solve $f(x) = 0$.

(c) Write f as a set of ordered pairs.

(d) Graph f.

106. Let $g = \{(-1,4),(0,2),(2,3),(3,4)\}$

(a) State the domain and range of g.

(b) Create a mapping diagram for g.

(c) Find $g(0)$ and solve $g(x) = 0$.

(d) Graph g.

107. Let $F = \{(t,t^2) \,|\, t \text{ is a real number}\}$. Find $F(4)$ and solve $F(x) = 4$.

HINT: Elements of F are of the form $(x, F(x))$.

108. Let $G = \{(2t, t+5) \,|\, t \text{ is a real number}\}$. Find $G(4)$ and solve $G(x) = 4$.

HINT: Elements of G are of the form $(x, G(x))$.

109. The area enclosed by a square, in square inches, is a function of the length of one of its sides ℓ, when measured in inches. This function is represented by the formula $A(\ell) = \ell^2$ for $\ell > 0$. Find $A(3)$ and solve $A(\ell) = 36$. Interpret your answers to each. Why is ℓ restricted to $\ell > 0$?

110. The area enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This function is represented by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

111. The volume enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides s, when measured in centimeters. This function is represented by the formula $V(s) = s^3$ for $s > 0$. Find $V(5)$ and solve $V(s) = 27$. Interpret your answers to each. Why is s restricted to $s > 0$?

112. The volume enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This function is represented by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

113. The height of an object dropped from the roof of an eight story building is modeled by the function: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

114. The temperature in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

115. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

116. Using data from the Bureau of Transportation Statistics, the average fuel economy in miles per gallon for passenger cars in the US can be modeled by $E(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use a calculator to find $E(0)$, $E(14)$ and $E(28)$. Round your answers to two decimal places and interpret your answers to each.

117. The perimeter of a square, in centimeters, is four times the length of one if its sides, also measured in centimeters. Represent the function P which takes as its input the length of the side of a square in centimeters, s and returns the perimeter of the square in inches, $P(s)$ using a formula.

118. The circumference of a circle, in feet, is π times the diameter of the circle, also measured in feet. Represent the function C which takes as its input the length of the diameter of a circle in feet, D and returns the circumference of a circle in inches, $C(D)$ using a formula.

119. Suppose $A(P)$ gives the amount of money in a retirement account (in dollars) after 30 years as a function of the amount of the monthly payment (in dollars), P.

    (a) What does $A(50)$ mean?

    (b) What is the significance of the solution to the equation $A(P) = 250000$? .

    (c) Explain what each of the following expressions mean: $A(P+50)$, $A(P)+50$, and $A(P)+A(50)$.

120. Suppose $P(t)$ gives the chance of precipitation (in percent) t hours after 8 AM.

    (a) Write an expression which gives the chance of precipitation at noon.

(b) Write an inequality which determines when the chance of precipitation is more than 50%.

121. Explain why the graph in Exercise 63 suggests that not only is v as a function of w but also w is a function of v. Suppose $w = f(v)$ and $v = g(w)$. That is, f is the name of the function which takes v values as inputs and returns w values as outputs and g is the name of the function which does vice-versa. Find the domain and range of g and compare these to the domain and range of f.

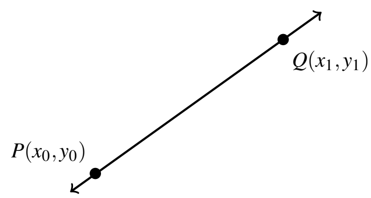
122. Sketch the graph of a function with domain $(-\infty,3)\cup[4,5)$ with range $\{2\}\cup(5,\infty)$.

SECTION 1.2 EXERCISE ANSWERS A.1.1

1.3 Linear Functions

1.3.1 Graphing Lines

In Section 1.1.2, we concerned ourselves with the finite line segment between two points P and Q. Specifically, we found its length (the distance between P and Q) and its midpoint. In this section, our focus will be on the *entire* line, and ways to describe it algebraically. Consider the generic situation below.



To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line as follows. (Read the character Δ as 'change in'.)

Equation 1.3. The **slope** m of the line containing the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x},$$

provided $x_1 \neq x_0$, that is, $\Delta x \neq 0$.

A couple of notes about Equation 1.3 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ (or $\Delta x \neq 0$) ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically when the 'change in x' is 0; the anxious reader can skip along to the next example.

Example 1.3.1. Compute the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

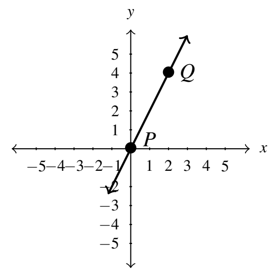
1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

[1] See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

Solution. In each of these examples, we apply the slope formula, Equation 1.3.

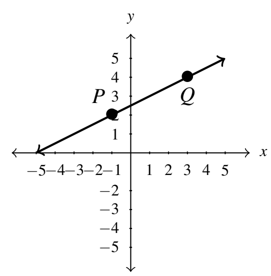
1. Compute the slope of the line containing the points $P(0,0)$ and $Q(2,4)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{4-0}{2-0} = \frac{4}{2} = 2$$



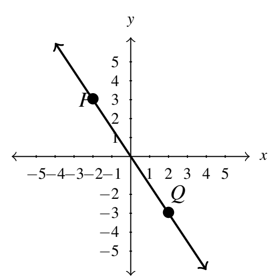
2. Compute the slope of the line containing the points $P(-1,2)$ and $Q(3,4)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$



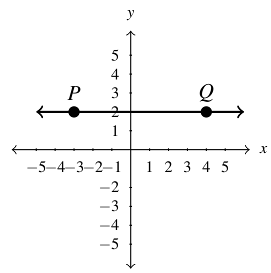
3. Compute the slope of the line containing the points $P(-2,3)$ and $Q(2,-3)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{-3-3}{2-(-2)} = \frac{-6}{4} = -\frac{3}{2}$$



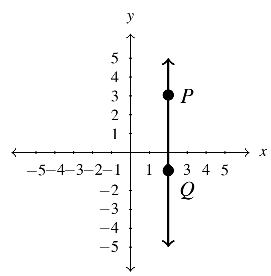
4. Compute the slope of the line containing the points $P(-3,2)$ and $Q(4,2)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{2-2}{4-(-3)} = \frac{0}{7} = 0$$



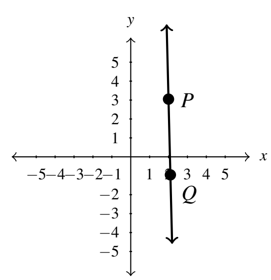
5. Compute the slope of the line containing the points $P(2,3)$ and $Q(2,-1)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{-1-3}{2-2} = \frac{-4}{0}, \text{ which is undefined}$$



6. Compute the slope of the line containing the points $P(2,3)$ and $Q(2.1,-1)$, if it exists. Plot each pair of points and the line containing them.

$$m = \frac{-1-3}{2.1-2} = \frac{-4}{0.1} = -40$$



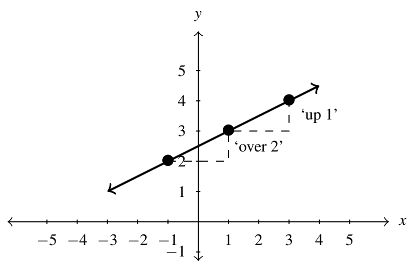
□

A few comments about Example 1.3.1 are in order. First, if the slope is positive then the resulting line is said to be 'increasing', meaning as we move from left to right,[2] the y-values are getting larger. Similarly, if the slope is negative, we say the line is 'decreasing', for as we move from left to right, the y-values are

[2]That is, as we increase the x-values ...

getting smaller. A slope of 0 results in a horizontal line which we say is 'constant', as the y-values here remain unchanged when we move from left to right, and an undefined slope results in a vertical line.[3]

Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, if the slope works out to be $\frac{1}{2}$, we can interpret this as a 'rise' of 1 unit upward for every 'run' of 2 units to the right:

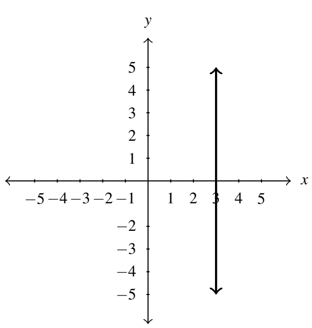


In this way, we may view the slope as 'the **rate of change** of y with respect to x'. From the expression

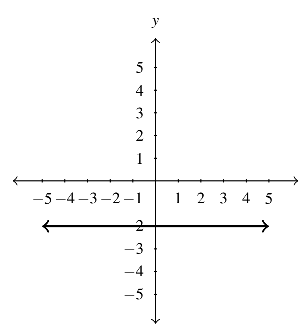
$$m = \frac{\Delta y}{\Delta x}$$

we get $\Delta y = m \cdot \Delta x$ so that the y-values change 'm' times as fast as the x-values. We'll have more to say about this concept when we explore applications of linear functions; presently, we will keep our attention focused on the analytic geometry of lines. To that end, our next task is to find algebraic equations that describe lines and we start with a discussion of vertical and horizontal lines.

Consider the two lines shown below: V (for 'V'ertical Line) and H (for 'H'orizontal Line).



The line V



The line H

All of the points on the line V have an x-coordinate of 3. Conversely, any point with an x-coordinate of 3 lies on the line V. Said differently, the point (x, y) lies on V if and only if $x = 3$. Because of this, we say the equation $x = 3$ *describes* the line V, or, said differently, the *graph* of the equation $x = 3$ is the line V.

[3]Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Turning our attention to H, we note that every point on H has a y-coordinate of -2, and vice-versa. Hence the equation $y = -2$ describes the line H, or the graph of the equation $y = -2$ is H. In general:

Equation 1.4. Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ in the xy-plane is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ in the xy-plane is a **horizontal line** through $(0, b)$.

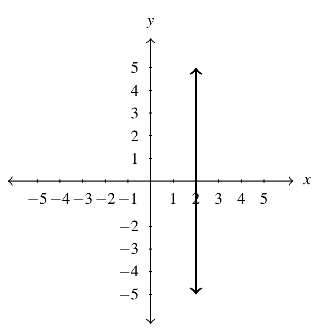
Of course, we may be working on axes which aren't labeled with the 'usual' x's and y's. In this case, we understand Equation 1.4 to say 'horizontal axis label $= a$' describes a *vertical* line through $(a, 0)$ and 'vertical axis label $= b$' describes a *horizontal* line through $(0, b)$.

Example 1.3.2.

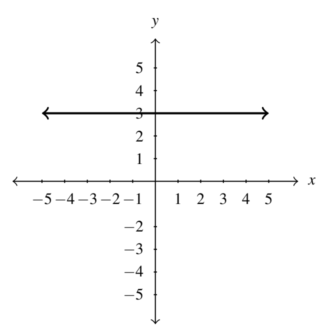
1. Graph the following equations in the xy-plane:

   (a) $y = 3$ (b) $x = -117$

2. Write the equation of each of the given lines.



Line L_1

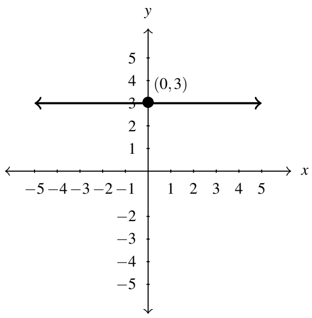


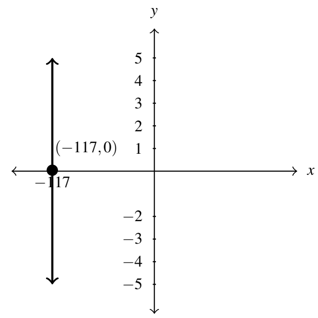
Line L_2

Solution.

1. Graph $y = 3$ and $x = 117$ on the xy-plane.

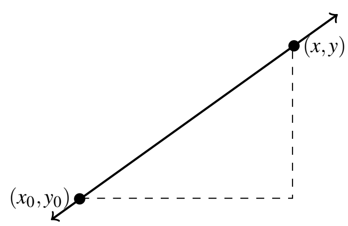
   By now we're familiar with the xy-plane, the graph of $y = 3$ is a horizontal line through $(0, 3)$, shown below on the left and the graph of $x = -117$ is a vertical line through $(-117, 0)$. We scale the x-axis differently than the y-axis to produce the graph below on the right.





2. L_1 is a vertical line through $(2,0)$, and the horizontal axis is labeled with 'x', thus the equation of L_1 is $x = 2$. L_2 is a horizontal line through $(0,3)$ and the vertical axis is labeled as 's', therefore the equation of this line is $s = 3$. $\square$

Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated below.



Equation 1.3 then yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

which is known as the **point-slope form** of a line.

> **Equation 1.5.** The **point-slope form** of the line with slope m containing the point (x_0, y_0) is the equation
>
> $$y - y_0 = m(x - x_0)$$

A few remarks about Equation 1.5 are in order. First, note that if the slope $m = 0$, then the line is horizontal and Equation 1.5 reduces to $y - y_0 = 0$ or $y = y_0$, as prescribed by Equation 1.4.[4] Second, we may need to change the letters in Equation 1.5 from 'x' and 'y' depending on the context, so while Equation 1.5 should be committed to memory, it should be understood that 'x' refers to whichever variable is used to label the horizontal axis, and y refers to whichever variable is used to label the vertical axis. Lastly, while Equation 1.5 is, by far, the easiest way to *construct* the equation of a line given a point and a slope, more often than not, the equation is solved for y and simplified into the form below.

Equation 1.6. The **slope-intercept form** of the line with slope m and y-intercept $(0,b)$ is the equation

$$y = mx + b$$

Equation 1.6 is probably[5] a familiar sight from Intermediate Algebra. You may recall that the 'intercept' in 'slope-intercept' comes from the fact that this line 'intercepts' or crosses the y-axis at the point $(0,b)$.[6] If we set the slope, $m = 0$, we obtain $y = b$, the formula for Horizontal Lines first introduced in Equation 1.4. Hence, any line which has a defined slope m can be represented in both point-slope and slope-intercept forms. The only exceptions are vertical lines which do not have a defined slope. There is one equation - the aptly named 'general form' - which describes every type of line, including vertical lines, and it is presented below.

Equation 1.7. Every line may be represented by an equation of the form $Ax + By = C$, where A, B and C are real numbers for which A and B aren't both zero. This is called *a* **general form** of the line. Some call $Ax + By = C$ the standard form of a line.

Note the indefinite article 'a' in Equation 1.7. The line $y = 5$ is a general form for the horizontal line through $(0,5)$, but so are $3y = 15$ and $0.5y = 2.5$. The reader is left to ponder the use of the definite article 'the' in Equations 1.5 and 1.6. Regardless of *which* form the equation of a line takes, note that the variables involved are all raised to the first power.[7] For instance, there are no $\sqrt{x}$ terms, no y^2 terms or any variables appearing in denominators. Let's look at a few examples.

Example 1.3.3.

1. Graph the following equations in the xy-plane:

   (a) $y = 3x - 1$

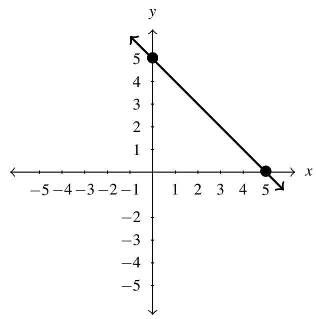
   (b) $2x + 4y = 3$

2. Write the slope-intercept form of the line containing the points $(-1,3)$ and $(2,1)$.

---

[4] Here we have y_0 as the constant whereas in the Equation we used the letter b. The form $y =$ constant is what matters.
[5] Hopefully?
[6] We can verify this algebraically by setting $x = 0$ in the equation $y = mx + b$ and obtaining $y = b$.
[7] Recall, $x = x^1$, $y = y^1$, etc.

3. Write the slope-intercept form of the equation of the line below:

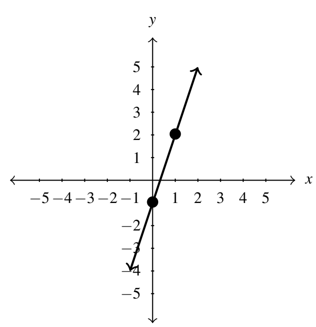


Solution.

1. To graph a line, we need just two points on that line. There are several ways to do this, and we showcase two of them here.

   (a) Graph $y = 3x - 1$ on the xy-plane.

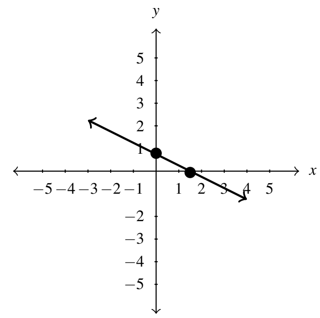
   We recognize that $y = 3x - 1$ is in slope-intercept form, $y = mx + b$, with $m = 3$ and $b = -1$. This immediately gives us one point on the graph – the y-intercept $(0, -1)$. From here, we use the slope $m = 3 = \frac{3}{1}$ and move one unit to the right and three units up, to obtain a second point on the line, $(1, 2)$. Connecting these points gives us the next graph.



   (b) Graph $2x + 4y = 3$ on the xy-plane.

   The equation $2x + 4y = 3$ is a general form of a line. To get two points here, we choose 'convenient' values for one of the variables, and solve for the other variable. Choosing $x = 0$, for example, reduces $2x + 4y = 3$ to $4y = 3$, or $y = \frac{3}{4}$. This means the point $\left(0, \frac{3}{4}\right)$ is on the graph. Choosing $y = 0$ gives $2x = 3$, or $x = \frac{3}{2}$. This gives is a second point on the line, $\left(\frac{3}{2}, 0\right)$.[8] Our graph of $2x + 4y = 3$ is below.

[8] You may recall, that this is the x-intercept of the line.



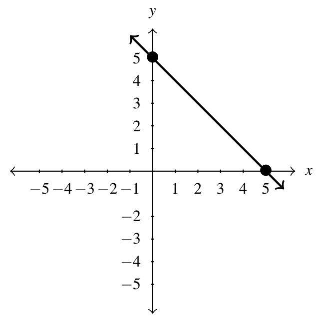
2. Write the slope-intercept form of the line containing the points $(-1,3)$ and $(2,1)$.

We'll assume we're using the familiar (x,y) axis labels and begin by finding the slope of the line using Equation 1.3: $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. Next, we substitute this result, along with one of the given points, into the point-slope equation of the line, Equation 1.5. We have two options for the point (x_0,y_0). We'll use $(-1,3)$ and leave it to the reader to check that using $(2,1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x+1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x - \frac{2}{3} + 3 \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1,3)$ and $(2,1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically by showing that the equation holds true when we substitute $x = -1$ and $y = 3$ and when $x = 2$ and $y = 1$.

3. Write the slope-intercept form of the equation of the line below:



From the graph, we see that the points $(0,5)$ and $(5,0)$ are on the line, so we may proceed as we did in the previous problem. Here, however, we use 't' in place of 'x' and 's' in place of 'y' in accordance to the axis labels given. We find the slope $m = \frac{\Delta s}{\Delta t} = \frac{0-5}{5-0} = -1$. As before, we have two points to choose from to substitute into the point-slope formula, and, as before, we'll select one of them, $(0,5)$ and leave the computations with $(5,0)$ to the reader.

$$\begin{aligned} s - s_0 &= m\,(t - t_0) \\ s - 5 &= (-1)\,(t - 0) \\ s - 5 &= -t \\ s &= -t + 5. \end{aligned}$$

As before we can check this line contains both points $(t,s) = (0,5)$ and $(t,s) = (5,0)$ algebraically. $\square$

While every point on a line holds value and meaning,[9] we've reminded you of certain points, called 'intercepts,' which hold special enough significance to be singled out. Formally, we define these as follows.

Definition 1.8. Given a graph of an equation in the xy-plane:

- A point on a graph which is also on the x-axis is called an **x-intercept** of the graph. To determine the x-intercept(s) of a graph, set $y = 0$ in the equation and solve for x.

  NOTE: x-intercepts always have the form: $(x_0, 0)$.

- A point on a graph which is also on the y-axis is called an **y-intercept** of the graph. To determine the y-intercept(s) of a graph, set $x = 0$ in the equation and solve for y.

  NOTE: y-intercepts always have the form: $(0, y_0)$.

As usual, the labels of the axes in the problem will dictate the labels on the intercepts. If we're working in the vw-plane, for instance, there would be v- and w-intercepts.

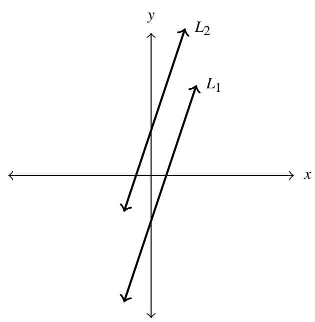
[9] Lines missing points - even one - usually belie some algebraic pathology which we'll discuss in more detail in Chapter 3.

The last little bit of analytic geometry we need to review about lines are the concepts of 'parallel' and 'perpendicular' lines. Parallel lines do not intersect,[10] and hence, parallel lines necessarily have the same slope. Perpendicular lines intersect at a right (90°) angle. The relationship between these slopes is somewhat more complicated, and is summarized below.

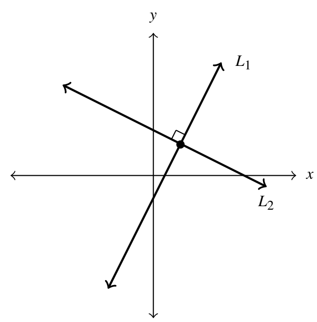
Theorem 1.3. Suppose line L_1 has slope m_1 and line L_2 has slope m_2:

- L_1 and L_2 are parallel (written $L_1 \parallel L_2$) if and only if $m_1 = m_2$.
- If $m_1 \neq 0$ and $m_2 \neq 0$ then L_1 and L_2 are perpendicular (written $L_1 \perp L_2$) if and only if $m_1 m_2 = -1$.

  NOTE: $m_1 m_2 = -1$ is equivalent to $m_2 = -\dfrac{1}{m_1}$, so that perpendicular lines have slopes which are 'opposite reciprocals' of one another.



$L_1 \parallel L_2$, $m_1 = m_2$



$L_1 \perp L_2$, $m_1 m_2 = -1$

A few remarks about Theorem 1.3 are in order. First off, the theorem assumes that the slopes of the lines exist. The reader is encouraged to think about the case when one (or both) of the slopes don't exist. Along those same lines, the reader is encouraged to think about why the stipulations $m_1 \neq 0$ and $m_2 \neq 0$ appear in the statement regarding slopes of perpendicular lines, and what happens in this case as well. (Think geometrically!) In Exercise 41, you'll prove the assertion about the slopes of perpendicular lines. For now, we accept it as true and use it in the following example.

Example 1.3.4. For line $y = 2x - 1$ and the point $(3,4)$, write:

1. the equation of the line parallel to the given line which contains the given point.
2. the equation of the line perpendicular to the given line which contains the given point. Check your answers by graphing them, along with the original line.

Solution.

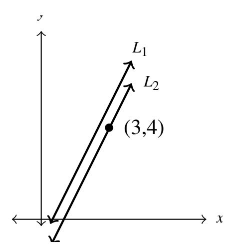
[10] Well, at least in Euclidean Geometry ...

1. For line $y = 2x - 1$ and the point $(3,4)$, write the equation of the line parallel to the given line which contains the given point.

   Seeing as $y = 2x - 1$ is already in slope-intercept form, we have the slope $m = 2$. To find the line parallel to this line containing $(3,4)$, we use the point-slope form with $m = 2$ to get:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 4 &= 2(x - 3) \\ y - 4 &= 2x - 6 \\ y &= 2x - 2 \end{aligned}$$

   Algebraically, we can verify that the slope is indeed 2 and that when $x = 3$ we get $y = 4$. Using a graph centered at the point $(3,4)$, we graph both $y = 2x - 1$ and $y = 2x - 2$ and observe that they appear to be parallel.

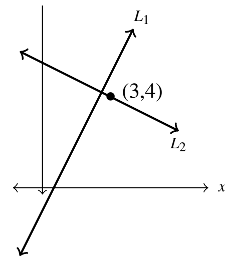


2. For line $y = 2x - 1$ and the point $(3,4)$, write the equation of the line perpendicular to the given line which contains the given point. Check your answers by graphing them, along with the original line.

   To find the line perpendicular to $y = 2x - 1$ containing $(3,4)$, we use the slope $m = -\frac{1}{2}$ in the point-slope formula:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 4 &= -\frac{1}{2}(x - 3) \\ y - 4 &= -\frac{1}{2}x + \frac{3}{2} \\ y &= -\frac{1}{2}x + \frac{11}{2} \end{aligned}$$

   Algebraically, we check that the slope is $m = -\frac{1}{2}$ and when $x = 3$ we get $y = 4$ as required. When checking using a graph, we center the picture at $(3,4)$ and had to 'square' up the graph, to truly observe the perpendicular nature of the lines.

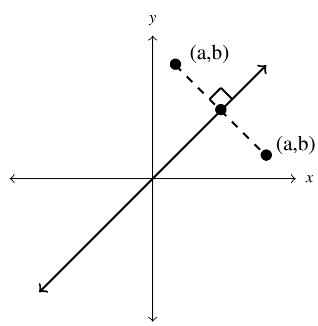


□

Our next example with lines sets up a fourth kind of symmetry which will be revisited in Section 5.1.

Example 1.3.5. Show that the points (a,b) and (b,a) in the xy-plane are symmetric about the line $y=x$.

Solution. If $a=b$ then $(a,b)=(a,a)=(b,a)$ and this point lies on the line $y=x$.[11] To prove the claim for the case when $a \neq b$, we will show that the line $y=x$ is a perpendicular bisector of the line segment with endpoints (a,b) and (b,a), as illustrated below.



To show the 'perpendicular' part, we first note the slope of the line containing (a,b) and (b,a) is

$$m=\frac{a-b}{b-a}=\frac{\cancel{(a-b)}}{-\cancel{(a-b)}}=-1$$

Due to the fact that the slope of $y=x=1x+0$ is $m=1$, we see that the slopes of these two lines are negative reciprocals. Hence, $y=x$ and the line segment with endpoints (a,b) and (b,a) are perpendicular. For the 'bisector' part, we use Equation 1.2 to compute the midpoint of the line segment with endpoints (a,b) and (b,a):

[11] Please ask your instructor if lying on the line counts as being 'symmetric about the line' or not.

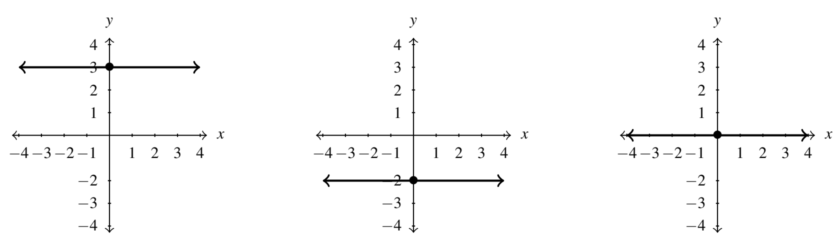
$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

As the x and y coordinates of this point are the same, we determine that the midpoint lies on the line $y = x$. □

1.3.2 Constant Functions

Now that we have defined the concept of a function, we'll spend the rest of Chapter 1 revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to the previous subsection for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise 94) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown below. The Vertical Line Test, Theorem 1.2, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.



$y = 3$ $\qquad$ $y = -2$ $\qquad$ $y = 0$

In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the *same* value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

Definition 1.9. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition 1.9 are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a *family* of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions. Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain $(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[12]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

Example 1.3.6. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \\ 7.25 & \text{if } 6 \leq A < 50 \\ 5.75 & \text{if } A \geq 50 \end{cases}$$

1. Compute and interpret $p(3)$, $p(6)$ and $p(62)$.

2. Explain the pricing structure verbally.

3. Graph $p(A)$.

Solution. The function $p(A)$ described above is an example of a **piecewise-defined** function because the *rule* to determine outputs, not just the value of the output, changes depending on the inputs.

1. Compute and interpret $p(3)$, $p(6)$ and $p(62)$.

   To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.

2. Explain the pricing structure verbally.

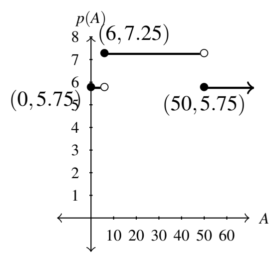
[12] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.

3. Graph $p(A)$.

The independent variable here is specified as A, so we'll label our horizontal axis that way. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[13] at $(6, 5.75)$. Similarly, to graph the second piece, we begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[14] Note the scaling on the horizontal axis compared to the vertical axis.



□

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of** x, denoted by $\lfloor x \rfloor$. In Section 0.1.1 we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[15] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k+1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k+1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 1.3.7. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Compute $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$
2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

[13] See our discussion about holes in graphs in Example 1.2.6 in Section 1.2.

[14] The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.

[15] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count'.

Solution.

1. Compute $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$.

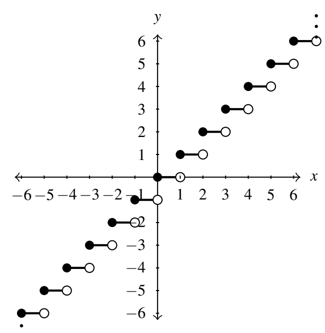
   To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that $-3 \leq -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \leq \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.

2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

   The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k+1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \leq x < k+1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

   A partial description of $\lfloor x \rfloor$ is given alongside a partial graph at the top of the next page. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \leq x < -4 \\ -4 & \text{if } -4 \leq x < -3 \\ -3 & \text{if } -3 \leq x < -2 \\ -2 & \text{if } -2 \leq x < -1 \\ -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } \ \ 0 \leq x < 1 \\ 1 & \text{if } \ \ 1 \leq x < 2 \\ 2 & \text{if } \ \ 2 \leq x < 3 \\ 3 & \text{if } \ \ 3 \leq x < 4 \\ 4 & \text{if } \ \ 4 \leq x < 5 \\ 5 & \text{if } \ \ 5 \leq x < 6 \\ \vdots & \end{cases}$$



$\square$

1.3.3 Linear Functions

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 1.10. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition 1.9, in Definition 1.10, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$.

Recall[16] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition 1.10, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$ and solve $mx + b = 0$ for x. We leave this to the reader in Exercise 79. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to remind you of the definitions of intercepts (1.8) here using function notation.

Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the ***y*-intercept** of the graph of $y = f(x)$.

  That is, $(0, f(0))$ is where the graph meets the y-axis.

- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an ***x*-intercept** of the graph of $y = f(x)$.

  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, the above definition uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are *real numbers*. The x-intercepts are, on the other hand, *points* on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the *zero* of f that corresponds graphically to the *x-intercept* $(3, 0)$.

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and

[16] or see Section 1.3.1

Mathematics cannot be overstated.[17] As you may recall, or quickly read about on page 140, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[18] In this course, it is vital that we regard the slope of a linear function as a rate of change of *function outputs* to *function inputs*. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[19] We'll see the importance of this statement in the upcoming examples.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus.

Example 1.3.8. The cost, in dollars, to produce x PortaBoy game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Compute and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.
2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for \$15,000?

Solution.

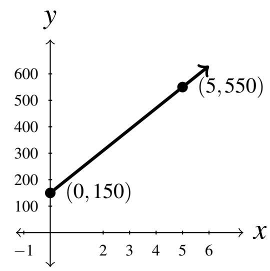
1. Compute and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.

   To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get:

[17] The first half of any introductory Calculus course is about slope.

[18] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

[19] See Exercise 93 for more details.



2. Explain the significance of the restriction on the domain, $x \geq 0$.

   In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[20] so $x \geq 0$.

3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.

   The cost function $C(x) = 80x + 150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:

   $$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{1 \text{ PortaBoy produced}}.$$

   Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. How many PortaBoys can be produced for \$15,000?

   To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$. We would be over budget if we produced 186 PortaBoys, hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare). □

A couple of remarks about Example 1.3.8 are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section 2.1 where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[21]

[20] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

[21] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example 1.2.1 in Section 1.2.

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get

$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$

which agrees with our earlier computation.[22] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:

$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$

In general, we could rewrite $C(x) = C(185) + 80(x - 185)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[23]

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[24] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Equation 1.8. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation 1.8 immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation 1.8 exclusively from this point forward.[25]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation 1.8 as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ (\text{new output}) & = & (\text{known output}) & + & (\text{change in outputs}) \end{array}$$

In other words, changing the *input* by Δx results in changing the *output* by $m\Delta x$. This tracks because

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation 1.8 to good use in the next example.

[22] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

[23] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150 \ldots$

[24] See Section 1.3.1 for a review of this form.

[25] In other words, the slope intercept form of a line is just a special case of the point-slope form.

Example 1.3.9. The local retailer in Example 1.3.8 is trying to mathematically model the relationship between the number of PortaBoy systems sold and the price per system. Suppose 20 systems were sold when the price was \$220 per system, but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

Solution.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

   We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

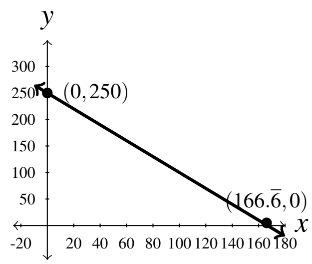
   $$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190-220}{40-20} = \frac{-30}{20} = -1.5$$

   and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

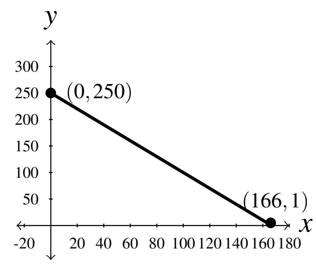
   $$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

   which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = \frac{500}{3} = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[26] The graph on the left is that of the line $y = -1.5x + 250$.

[26]Recall that the bar over the 6 indicates that the decimal repeats. See page 38 for details.



$y = -1.5x + 250$



$y = p(x)$

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[27] Next, we require $p(x) \geq 0$, otherwise we'd be *paying* customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising as our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[28]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed above on the right.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

   The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is *decreasing* at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

   To determine the price which will move 150 PortaBoys, we compute $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. How many systems would sell if the price per system were set at \$150?

   If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, because to sell 67 systems, we'd have to drop the price just under \$150. $\square$

The function p in Example 1.3.9 is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain demand x - that is, how

[27] ignoring returns, that is.

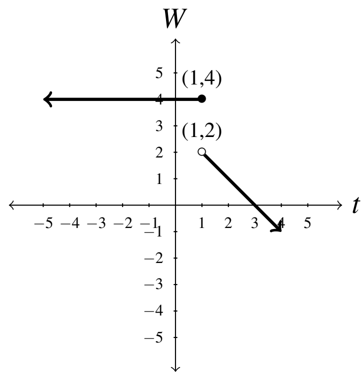
[28] We'll discuss these sorts of connections in greater depth in Section 1.4.

many products will sell.[29] These functions, along with cost functions like the one in Example 1.3.8, will be revisited in Example 2.1.3.

Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

[29] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.

Example 1.3.10. Determine a formula for the function L graphed below.



Solution. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1,4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$. The only 'point' labeled with this part of the graph is the hole at $(1,2)$ and it isn't technically part of the graph, so we will avoid using it.[30] Instead, we infer from the graph two other points: $(2,1)$ and $(3,0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3-2}{0-1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if } t \leq 1 \\ -t+3 & \text{if } t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1,2)$ checks.[31] □

Example 1.3.11. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

Solution. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after

[30] We actually could use the point $(1,2)$ to find the equation of the *line* containing $(1,2)$ and, say $(3,0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[31] Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

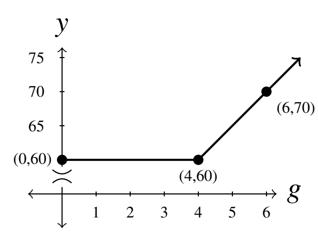
a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is $(\$5)(2) = \$10$. The total cost is the base plus the overages or $\$60 + \$10 = \$70$. In general, if $g > 4$, the expression $(g-4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g-4) = 5g + 40$. Putting this together with our previous work, we get

$$C(g) = \begin{cases} 60 & \text{if } 0 \leq g \leq 4 \\ 5g+40 & \text{if } g > 4 \end{cases}$$

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0,60)$ to $(4,60)$. For $g > 4$, we have the line $y = 5g+40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g+40$ and get $y = 60$. Normally, this would produce a hole at $(4,60)$, but in this case, the point $(4,60)$ is already on the graph from the first piece of the function. Essentially, the point $(4,60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g+40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6,70)$ as our second point. Our graph is below. As with the graphs shown on page 118 from Example 1.2.1, we use '$\asymp$' to denote a break in the vertical axis in order to better display the graph.



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1.3.4 The Average Rate of Change of a Function

As mentioned earlier in the section, the concepts of slope and the more general rates of change are important concepts not just in Mathematics, but also in other fields. Many important phenomena are modeled using non-linear functions, and while the rates of change of these functions are not constant, we can sample the function at two points and compute what is known as an **average rate of change** between them to give some sense as to the function's behavior over that interval.[32]

[32] We are basically pretending that the function is linear on a short interval to see what we can say about its behavior.

Definition 1.11. Let f be a function defined on the interval $[a,b]$. The **average rate of change** of f over $[a,b]$ is defined as:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

Geometrically, the average rate of change is the slope of the line[a] containing $(a, f(a))$ and $(b, f(b))$.

[a]This line is called a *secant* line.

As with Definitions 1.5 and 1.6, the wording in Definition 1.11, while referring to the function f, is really making a statement about its outputs $f(x)$.

If f is increasing over $[a,b]$, then the average rate of change will be positive. Likewise, if f is decreasing or constant, the average rate of change will be negative or 0, respectively. (Think about this for a moment.) However, as the next example demonstrates, the converses of these statements aren't always true.[33]

Example 1.3.12. The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time t, in seconds after lift-off.

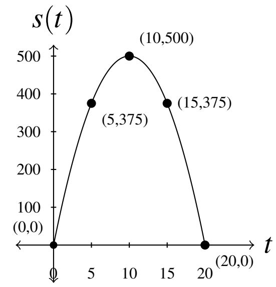
1. Compute $s(0)$, $s(5)$, $s(10)$, $s(15)$ and $s(20)$ and use these points to graph $y = s(t)$.
2. State the range of s and interpret the extrema, if any exist.
3. Compute and interpret the t- and y-intercepts.
4. Determine and interpret the interval(s) over which s is increasing, decreasing or constant.
5. Calculate and interpret the average rate of change of s over the intervals $[0,5]$, $[5,10]$, $[10,20]$ and $[5,15]$.

Solution.

1. Compute $s(0)$, $s(5)$, $s(10)$, $s(15)$ and $s(20)$ and use these points to graph $y = s(t) = -5t^2 + 100t$.

   To find $s(0)$, we substitute $t = 0$ into the formula for $s(t)$: $s(0) = -5(0)^2 + 100(0) = 0$. Similarly, $s(5) = -5(5)^2 + 100(5) = -5(25) + 500 = -125 + 500 = 375$. Continuing, we obtain: $s(10) = 500$, $s(15) = 375$ and $s(20) = 0$. Using these, we construct a a graph to obtain:

[33]For example, the average rate of change over an interval could be positive yet the function could decrease over part of that interval and then increase on a different part.



2. State the range of s and interpret the extrema, if any exist.

   Projecting the graph to the y-axis, we see that the range of s is $[0,500]$ so the minimum of s is 0 and the maximum is 500. This means that the rocket at some point is on the surface of the Moon and reaches its highest altitude of 500 feet above the lunar surface.

3. Compute and interpret the t- and y-intercepts.

   The first intercept we see is $(0,0)$ which is both a t- and a y-intercept. Given that t represents the time after lift-off and $y = s(t)$ represents the height above the Moon's surface, the point $(0,0)$ means that the model rocket was launched ($t = 0$) from the Moon's surface ($s(t) = 0$). The remaining intercept, $(20,0)$, is another t-intercept. This means that 20 seconds after lift-off ($t = 20$), the model rocket returns to the Moon's surface ($s(t) = 0$). Said differently, 20 seconds is the 'time of flight' of the model rocket.

4. Determine and interpret the interval(s) over which s is increasing, decreasing or constant.

   Referring to Definition 1.6, s increases over the interval $[0,10]$, because for those values of t, as we read from left to right, the graph of the function is rising meaning the y values (hence $s(t)$ values) are getting larger. Thus the model rocket is heading upwards for the first 10 seconds of its flight. We find that s decreases over the interval $[10,20]$, indicating once it has reached its highest altitude of 500 feet 10 seconds into the flight, the rocket begins to fall back to the surface of the Moon, landing 20 seconds after lift-off.

5. Calculate and interpret the average rate of change of s over the intervals $[0,5]$, $[5,10]$, $[10,20]$ and $[5,15]$.

   To find the average rate of change of s over the interval $[0,5]$ we compute

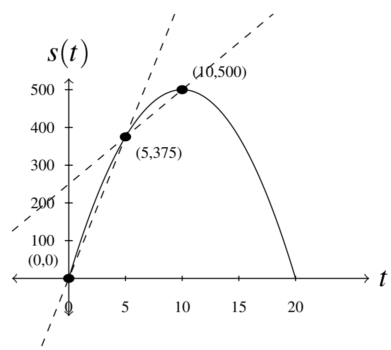
   $$\frac{\Delta[s(t)]}{\Delta t} = \frac{s(5) - s(0)}{5 - 0} = \frac{375 \text{ feet}}{5 \text{ seconds}} = 75 \text{ feet per second.}$$

   In other words, the height is *increasing* at an *average rate* of 75 feet per second during the first 5 seconds of flight. The rate here is called the **average velocity** of the rocket over this interval. Velocity differs from speed in that velocity comes with a direction. In this case, a positive velocity indicates that the rocket is traveling *upwards*, because when s is increasing, the model rocket is climbing higher.

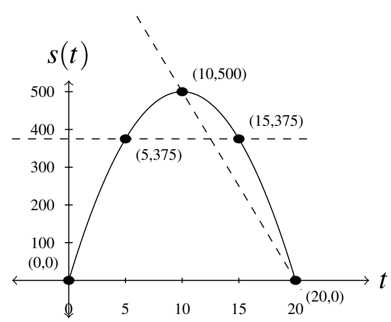
Similarly, the average rate of change of s over the interval $[5,10]$ works out to be 25. This means that the average velocity over the next 5 seconds of the flight has slowed to 25 feet per second. The model rocket is still, on average, traveling upwards, albeit more slowly than before.

Over the interval $[10,20]$, the average rate of change of s works out to be -50. This means that, on average, the rocket is *falling* at a rate of 50 feet per second. The rocket has managed to fall from its highest point 500 feet above the surface of the Moon back to the Moon's surface in 10 seconds so this makes sense. Last, but not least, the average rate of change of s over $[5,15]$ turns out to be 0. This means that the model is the same height above the ground after 5 seconds (375 feet) as it is after 15 seconds.

Geometrically, the average rate of change of a function over an interval can be interpreted as the slope of a secant line. Below is a dotted line containing $(0,0)$ and $(5,375)$ (which has slope 75) along with a dotted line containing the points $(5,375)$ and $(10,500)$ (which has slope 25). Visually, the lines help demonstrate that, while s is increasing over $[0,10]$, the rate of increase is slowing down as t nears 10.



The graph below on the right depicts a dotted line through $(10,500)$ and $(20,0)$ indicating a net decrease over that interval. We also have a horizontal line (0 slope) containing the points $(5,375)$ and $(15,375)$, which shows no net change between those two points, despite the fact that the rocket rose to its maximum height then began its descent during the interval $[5,15]$.



$\square$

An important lesson from the last example is that average rates of change give us a snapshot of what is happening *at the endpoints* of an interval, but not necessarily what happens *over the course* of the interval. Calculus gives us tools to compute slopes *at* points which correspond to *instantaneous* rates of changes. While we don't quite have the machinery to properly express these ideas, we can hint at them in the Exercises. Speaking of exercises ...

1.3.5 Exercises

In Exercises 1 - 10, write both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3,\ P(3,-1)$
2. $m = -2,\ P(-5,8)$
3. $m = -1,\ P(-7,-1)$
4. $m = \frac{2}{3},\ P(-2,1)$
5. $m = -\frac{1}{5},\ P(10,4)$
6. $m = \frac{1}{7},\ P(-1,4)$
7. $m = 0,\ P(3,117)$
8. $m = -\sqrt{2},\ P(0,-3)$
9. $m = -5,\ P(\sqrt{3},2\sqrt{3})$
10. $m = 678,\ P(-1,-12)$

In Exercises 11 - 20, write the slope-intercept form of the line which passes through the given points.

11. $P(0,0),\ Q(-3,5)$
12. $P(-1,-2),\ Q(3,-2)$
13. $P(5,0),\ Q(0,-8)$
14. $P(3,-5),\ Q(7,4)$
15. $P(-1,5),\ Q(7,5)$
16. $P(4,-8),\ Q(5,-8)$
17. $P\left(\frac{1}{2},\frac{3}{4}\right),\ Q\left(\frac{5}{2},-\frac{7}{4}\right)$
18. $P\left(\frac{2}{3},\frac{7}{2}\right),\ Q\left(-\frac{1}{3},\frac{3}{2}\right)$
19. $P\left(\sqrt{2},-\sqrt{2}\right),\ Q\left(-\sqrt{2},\sqrt{2}\right)$
20. $P\left(-\sqrt{3},-1\right),\ Q\left(\sqrt{3},1\right)$

In Exercises 21 - 26, graph the line. Determine the slope, y-intercept and x-intercept, if any exist.

21. $y = 2x - 1$
22. $y = 3 - x$
23. $y = 3$
24. $y = 0$
25. $y = \frac{2}{3}x + \frac{1}{3}$
26. $y = \dfrac{1-x}{2}$
27. Graph $3v + 2w = 6$ on both the vw- and wv-axes. What characteristics to both graphs share? What's different?

28. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

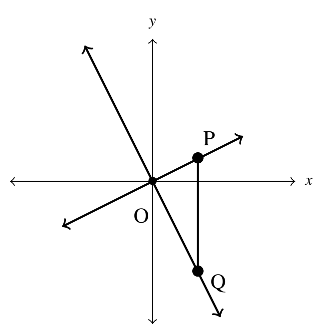
In Exercises 29 - 34, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

29. $y = 3x + 2$, $P(0,0)$
30. $y = -6x + 5$, $P(3,2)$
31. $y = \frac{2}{3}x - 7$, $P(6,0)$
32. $y = \dfrac{4-x}{3}$, $P(1,-1)$
33. $y = 6$, $P(3,-2)$
34. $x = 1$, $P(-5,0)$

In Exercises 35 - 40, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

35. $y = \frac{1}{3}x + 2$, $P(0,0)$
36. $y = -6x + 5$, $P(3,2)$
37. $y = \frac{2}{3}x - 7$, $P(6,0)$
38. $y = \dfrac{4-x}{3}$, $P(1,-1)$
39. $y = 6$, $P(3,-2)$
40. $x = 1$, $P(-5,0)$

41. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0,0)$, $P(1,m_1)$ and $Q(1,m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

In Exercises 42 - 47, graph the function. Then determine the slope and axis intercepts, if any.

42. $f(x) = 2x - 1$

43. $g(t) = 3 - t$

44. $F(w) = 3$

45. $G(s) = 0$

46. $h(t) = \frac{2}{3}t + \frac{1}{3}$

47. $j(w) = \dfrac{1-w}{2}$

In Exercises 48 - 51, graph the function. Then determine the domain, range, and axis intercepts, if any.

48. $f(x) = \begin{cases} 4-x & \text{if} \quad x \leq 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

49. $g(x) = \begin{cases} 2-x & \text{if} \quad x < 2 \\ x-2 & \text{if} \quad x \geq 2 \end{cases}$

50. $F(t) = \begin{cases} -2t-4 & \text{if} \quad t < 0 \\ 3t & \text{if} \quad t \geq 0 \end{cases}$

51. $G(t) = \begin{cases} -3 & \text{if} \quad t < 0 \\ 2t-3 & \text{if} \quad 0 < t < 3 \\ 3 & \text{if} \quad t > 3 \end{cases}$

52. The **unit step function** is defined as $U(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 1. \end{cases}$

(a) Graph $y = U(t)$.

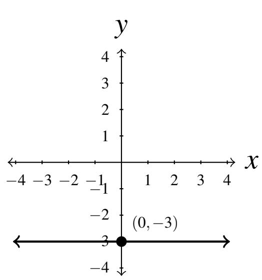
(b) State the domain and range of U.

(c) List the interval(s) over which U is increasing, decreasing, and/or constant.

(d) Write $U(t-2)$ as a piecewise defined function and graph.

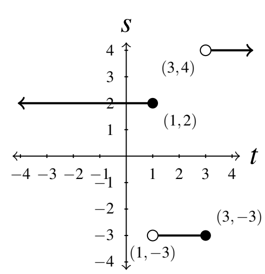
In Exercises 53 - 56, find a formula for the function.

53.



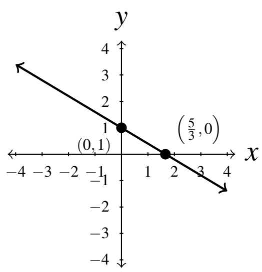
$y = f(x)$

54.



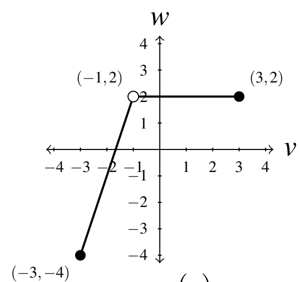
$s = F(t)$

55.



$y = L(x)$

56.



$w = g(v)$

57. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if } \; 1 \leq n \leq 25 \\ 13.50n & \text{if } \; 25 < n \leq 50 \\ 12n & \text{if } \; n > 50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to 57b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

58. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if } \; 1 \leq n \leq 14 \\ 0 & \text{if } \; n \geq 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

59. The cost in dollars $C(m)$ to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if } \; 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } \; m > 1000 \end{cases}$$

(a) How much does it cost to talk 750 minutes per month with this plan?

(b) How much does it cost to talk 20 hours a month with this plan?

(c) Explain the terms of the plan verbally.

60. Jeff can walk comfortably at 3 miles per hour. Find an expression for a linear function $d(t)$ that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

61. Carl can stuff 6 envelopes per *minute*. Find an expression for a linear function $E(t)$ that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.

62. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find an expression for a linear function $C(x)$ which computes the total cost in dollars to deliver x cubic yards of mulch.

63. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find an expression for a linear function $C(t)$ that computes her total daily charges in dollars as a function of the amount of time spent in hours, t at any one given location.

64. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find an expression for a linear function $W(x)$ which computes her total weekly pay in dollars as a function of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

65. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find an expression for a linear function which models the cost of a book in dollars $C(p)$ as a function of the number of pages p. Find and interpret both the slope of the linear function and $C(0)$.

66. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find an expression for a linear function which models the taxi fare $F(m)$ as a function of the number of miles driven, m. Find and interpret both the slope of the linear function and $F(0)$.

67. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

    (a) Find an expression for a linear function $F(T)$ that computes temperature in the Fahrenheit scale as a function of the temperature T given in degrees Celsius. Use this function to convert 20°C into Fahrenheit.

    (b) Find an expression for a linear function $C(T)$ that computes temperature in the Celsius scale as a function of the temperature T given in degrees Fahrenheit. Use this function to convert 110°F into Celsius.

    (c) Is there a temperature T such that $F(T) = C(T)$?

68. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40°F$ outside and only 5 times per hour if it's $70°F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20°F$ outside. What troubles do you encounter when trying to determine a reasonable applied domain?

69. Economic forces have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 1.3.8 with this new cost function.

70. In response to the economic forces in Exercise 69 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Example 1.3.9 with this new data.

71. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost in dollars $C(p)$ of p medium two-topping pizzas delivered during a weekend.

72. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill $T(n)$ of a party of n people who all choose the buffet.

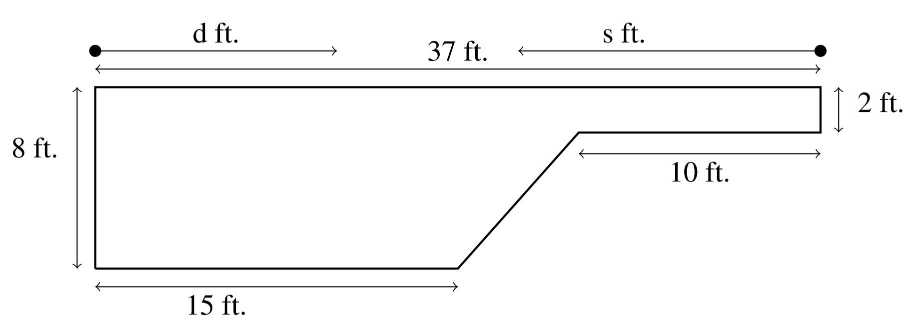
73. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost in dollars $C(m)$ for using m minutes of air time.

    HINT: You may wish to refer to number 59 for inspiration.

74. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price in dollars $P(c)$ of purchasing c crickets.

75. The cross-section of a swimming pool is at the top of the next page. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.
    (b) the distance (in feet) from the edge of the deep end of the pool, s.
    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



76. The function defined by $I(x) = x$ is called the Identity Function. Thinking from a procedural perspective, explain a possible origin of this name.

77. Why must the graph of a function $y = f(x)$ have at most one y-intercept?
    HINT: Consider what would happen graphically if there were more than one ...

78. Why is a discussion of vertical lines omitted when discussing functions?

79. Find a formula for the x-intercept of the graph of $f(x) = mx + b$. Assume $m \neq 0$.

80. Suppose $(c, 0)$ is the x-intercept of a linear function f. Use the point-slope form of a liner function, Equation 1.8 to show $f(x) = m(x - c)$. This is the 'slope x-intercept' form of the linear function.

81. Prove that for all linear functions L with with slope 3, $L(120) = L(100) + 60$.

In Exercises 82 - 87, compute the average rate of change of the function over the specified interval.

82. $f(x) = x^3$, $[-1, 2]$
83. $g(x) = \dfrac{1}{x}$, $[1, 5]$
84. $f(t) = \sqrt{t}$, $[0, 16]$
85. $g(t) = x^2$, $[-3, 3]$
86. $F(s) = \dfrac{s+4}{s-3}$, $[5, 7]$
87. $G(s) = 3s^2 + 2s - 7$, $[-4, 2]$

88. The height of an object dropped from the roof of a building is modeled by: $h(t) = -16t^2 + 64$, for $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground in feet t seconds after the object is dropped. Find and interpret the average rate of change of h over the interval $[0, 2]$.

89. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

90. The temperature $T(t)$ in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

    (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.
    (b) Find and interpret the average rate of change of T over the interval $[4, 8]$.
    (c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.
    (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

91. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

92. Recall from Example 1.3.12 The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds.

    (a) Find and interpret the average rate of change of s over the following intervals:

i. $[14.9, 15]$ ii. $[15, 15.1]$ iii. $[14.99, 15]$ iv. $[15, 15.01]$

(b) What value does the average rate of change appear to be approaching as the interval shrinks closer to the value $t = 15$?

(c) Find the equation of the line containing $(15, 375)$ with slope $m = -50$ and graph it along with s on the same set of axes using a graphing utility. What happens as you zoom in near $(15, 375)$?

93. Show the average rate of change of a function of the form $f(x) = mx + b$ over *any* interval is m.

94. Why doesn't the graph of the vertical line $x = b$ in the xy-plane represent y as a function of x?

95. With help from a graphing utility, graph the following pairs of functions on the same set of axes:[34]

- $f(x) = 2 - x$ and $g(x) = \lfloor 2 - x \rfloor$
- $f(x) = x^2 - 4$ and $g(x) = \lfloor x^2 - 4 \rfloor$
- $f(x) = x^3$ and $g(x) = \lfloor x^3 \rfloor$
- $f(x) = \sqrt{x} - 4$ and $g(x) = \lfloor \sqrt{x} - 4 \rfloor$

Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = \lfloor f(x) \rfloor$ until you can explain how, in general, one would obtain the graph of $y = \lfloor f(x) \rfloor$ given the graph of $y = f(x)$.

96. The Lagrange Interpolate function L for two points (x_0, y_0) and (x_1, y_1) where $x_0 \neq x_1$ is given by:

$$L(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

(a) For each of the following pairs of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

i. $(-1, 3)$, $(2, 3)$
ii. $(-3, -2)$, $(5, -2)$
iii. $(-3, -2)$, $(0, 1)$
iv. $(-1, 5)$, $(2, -1)$

(b) Verify that, in general, $L(x_0) = y_0$ and $L(x_1) = y_1$.

(c) Show the point-slope form of a linear function, Equation 1.8 is equivalent to the formula given for $L(x)$ after making the identifications: $f(x_0) = y_0$ and $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.

SECTION 1.3 EXERCISE ANSWERS A.1.1

[34] See Example 1.3.7 for the definition of $\lfloor x \rfloor$.

1.4 Absolute Value Functions

1.4.1 Graphs of Absolute Value Functions

In Section 1.3.1, we revisited lines in a function context. In this section, we revisit the absolute value in a similar manner, so it may be useful to refresh yourself with the basics in Section 0.5.2. Recall that the absolute value of a real number x, denoted $|x|$, can be defined as the distance from x to 0 on the real number line.[1] This definition is very useful for several applications, and lends itself well to solving equations and inequalities such as $|x-2|+1=5$ or $2|t+1|>4$.

We now wish to explore solving more complicated equations and inequalities, such as $|x-2|+1=x$ and $2|t+1|\geq t+4$. We'll approach these types of problems from a function standpoint and use the interplay between the graphical and analytical representations of these functions to obtain solutions. The key to this section is understanding the absolute value from that function (or procedural) standpoint.

Consider a real number $x\geq 0$ such as $x=0$, $x=\pi$ or $x=117.42$. When computing absolute values, we find $|0|=0$, $|\pi|=\pi$ and $|117.42|=117.42$. In general, if $x\geq 0$, the absolute value function does nothing to change the input, so $|x|=x$. On the other hand, if $x<0$, say $x=-1$, $x=-\sqrt{42}$ or $x=-117.42$, we get $|-1|=1$, $|-\sqrt{42}|=\sqrt{42}$ and $|-117.42|=117.42$. That is, if $x<0$, $|x|$ returns the exact *opposite* of the input x, so $|x|=-x$.

Putting these two observations together, we have the following.

Definition 1.12. The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x\geq 0 \end{cases}$$

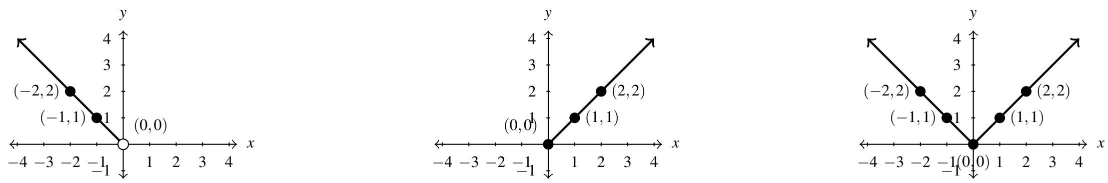
In Definition 1.12, it is *absolutely* essential to read '$-x$' as 'the *opposite* of x' as *opposed* to 'negative x' in order to avoid serious errors later. To see that this description agrees with our previous experience, consider $|117.42|$. Given that $117.42\geq 0$, we use the rule $|x|=x$. Hence, $|117.42|=117.42$. Likewise, $|0|=0$. To compute $|-\sqrt{42}|$, we note that $-\sqrt{42}<0$ we use the rule $|x|=-x$ in this case. We get $|-\sqrt{42}|=-(-\sqrt{42})$ (the opposite of $-\sqrt{42}$), so $|-\sqrt{42}|=-(-\sqrt{42})=\sqrt{42}$.

Another way to view Definition 1.12 is to think of $-x=(-1)x$ and $x=(1)x$. That is, $|x|$ multiplies negative inputs by -1 and non-negative inputs by 1. This viewpoint is especially useful in graphing $f(x)=|x|$. For $x<0$, $|x|=(-1)x$, so the graph of $y=|x|$ is the graph of $y=-x=(-1)x$: a line with slope -1 and y-intercept $(0,0)$. Likewise, for $x\geq 0$, $|x|=x$, so the graph of $y=|x|$ is the graph of $y=x=(1)x$: a line with slope 1 and y-intercept $(0,0)$.

Next, we graph each piece and then put them together. Note that when graphing $f(x)=|x|$ for $x<0$, we

[1] More generally, $|x-c|$ is the distance from x to c on the number line.

have a hole at $(0,0)$ because the inequality $x < 0$ is strict. However, the point $(0,0)$ is included in the graph of $f(x) = |x|$ for $x \geq 0$, so there is no hole in our final graph.



$f(x) = |x|\ x < 0$ $\qquad$ $f(x) = |x|\ x \geq 0$ $\qquad$ $f(x) = |x|$

The graph of $f(x) = |x|$ is a very distinctive '$\vee$' shape and is worth remembering. The point $(0,0)$ on the graph is called the **vertex**. This terminology makes sense from a geometric viewpoint because $(0,0)$ is the point where two lines meet to form an angle. We will also see this term used in Section 2.1 where, more generally, it corresponds to the graphical location of the sole maximum or minimum of a quadratic function.

We put Definition 1.12 to good use in the next example and review the basics of graphing along the way.

Example 1.4.1. For each of the functions below, analytically find the zeros of the function and the axis intercepts of the graph, if any exist. Rewrite the function using Definition 1.12 as a piecewise-defined function and sketch its graph. From the graph, determine the vertex, find the range of the function and any extrema, and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = |x-3|$
2. $g(t) = |t| - 3$
3. $h(u) = |2u-1| - 3$
4. $i(w) = 4 - 2|3w-1|$

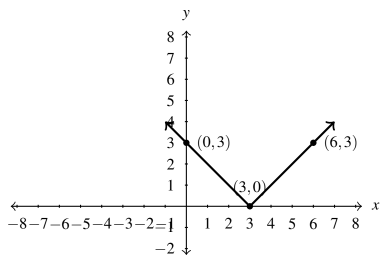
Solution. In what follows below, we will be doing quite a bit of substitution. As we have mentioned before, when substituting one expression in for another, the use of parentheses or other grouping symbols is highly recommended. Also, the dependent variable wasn't specified so we use the default y in each case.

1. Analyze $f(x) = |x-3|$.

   To find the zeros of f, we solve $f(x) = 0$ or $|x-3| = 0$. We get $x = 3$ so the sole x-intercept of the graph of f is $(3,0)$. To find the y-intercept, we compute $f(0) = |0-3| = 3$ and obtain $(0,3)$. Using Definition 1.12 to rewrite the expression for $f(x)$ means that we substitute the expression $x-3$ in for x and simplify. Note that when substituting the $x-3$ in for x, we do so for *every* instance of x – both in the formula (output) as well as the inequality (input).

   $$f(x) = |x-3| = \begin{cases} -(x-3) & \text{if } (x-3) < 0 \\ (x-3) & \text{if } (x-3) \geq 0 \end{cases} \quad \longrightarrow \quad f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3 & \text{if } x \geq 3 \end{cases}$$

   As both pieces of the graph of f are lines, we need just two points for each piece. We already have two points for the graph: $(0,3)$ and $(3,0)$. These two points both lie on the line $y = -x+3$ but the strictness of the inequality means $f(x) = -x+3$ only for $x < 3$, <u>not</u> $x = 3$, so we would have a hole at $(3,0)$ instead of a point there. For $x \geq 3$, $f(x) = x-3$, so the hole we thought we had at $(3,0)$ gets plugged because $f(3) = 3 - 3 = 0$. We need just one more point for $f(x)$ where $x \geq 3$ and choose somewhat arbitrarily $x = 6$. We find $f(6) = |6 - 3| = 3$ so our final point on the graph is $(6,3)$. Now that we have a complete graph,[2] we see that the vertex is $(3,0)$ and the range is $[0,\infty)$. The minimum of f is 0 when $x = 3$ and f has no maximum. Also, f is decreasing over $(-\infty,3]$ and increasing on $[3,\infty)$. The graph is given below.



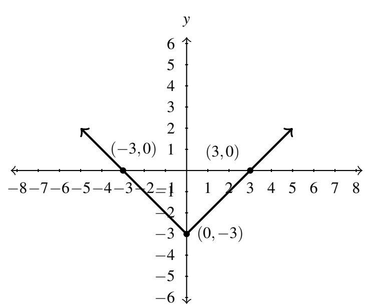
$f(x) = |x - 3|$

2. Analyze $g(t) = |t| - 3$.

   To find the zeros of g, we solve $g(t) = |t| - 3 = 0$ and get $|t| = 3$ or $t = \pm 3$. Hence, the t-intercepts of the graph of g are $(-3,0)$ and $(3,0)$. To find the y-intercept, we compute $g(0) = |0| - 3 = -3$ and get $(0,-3)$. To rewrite $g(t)$ has a piecewise defined function, we first substitute t in for x in Definition 1.12 to get a piecewise definition of $|t|$. This breaks the domain into two pieces: $t < 0$ and $t \geq 0$. For $t < 0$, $|t| = -t$, so $g(t) = |t| - 3 = (-t) - 3 = -t - 3$. Likewise, for $t \geq 0$, $|t| = t$ so $g(t) = |t| - 3 = t - 3$.

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \longrightarrow \quad g(t) = |t| - 3 = \begin{cases} -t - 3 & \text{if } t < 0 \\ t - 3 & \text{if } t \geq 0 \end{cases}$$

   Once again, we have two lines to graph, but in this case we have three points: $(-3,0)$, $(0,-3)$ and $(3,0)$. Both $(-3,0)$ and $(0,-3)$ lie on $y = -t - 3$, but $g(t) = -t - 3$ only for $t < 0$. This would yield a hole at $(0,-3)$, but, just like in the previous example, the hole is plugged thanks to the second piece of the function because $g(0) = 0 - 3 = -3$. We also pick up the second t-intercept, $(3,0)$ and this helps us complete our graph. We see that the vertex is $(0,-3)$ and the range is $[-3,\infty)$. The minimum of g is -3 at $t = 0$ and there is no maximum. Also, g is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. The graph of g is shown below.



$g(t) = |t| - 3$

[2] We know it's complete because we did the Math - no trusting technology on this example!

3. Analyze $h(u) = |2u-1| - 3$.

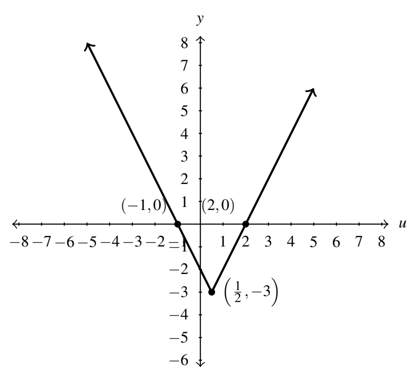
   Solving $h(u) = |2u-1| - 3 = 0$ gives $|2u-1| = 3$ or $2u-1 = \pm 3$. We get two zeros: $u = -1$ and $u = 2$ which correspond to two u-intercepts: $(-1,0)$ and $(2,0)$. We find $h(0) = |2(0)-1| - 3 = -2$ so our y-intercept is $(0,-2)$. To rewrite $h(u)$ as a piecewise defined function, we first rewrite $|2u-1|$ as a piecewise function. Substituting the expression $2u-1$ in for x in Definition 1.12 gives:

$$|2u-1| = \begin{cases} -(2u-1) & \text{if } 2u-1 < 0 \\ 2u-1 & \text{if } 2u-1 \geq 0 \end{cases} \longrightarrow |2u-1| = \begin{cases} -2u+1 & \text{if } u < \frac{1}{2} \\ 2u-1 & \text{if } u \geq \frac{1}{2} \end{cases}$$

   Hence, for $u < \frac{1}{2}$, $|2u-1| = -2u+1$ so $h(u) = |2u-1| - 3 = (-2u+1) - 3 = -2u-2$. Likewise, for $u \geq \frac{1}{2}$, $|2u-1| = 2u-1$ so $h(u) = |2u-1| - 3 = (2u-1) - 3 = 2u-4$.

$$h(u) = |2u-1| - 3 = \begin{cases} (-2u+1) - 3 & \text{if } u < \frac{1}{2} \\ (2u-1) - 3 & \text{if } u \geq \frac{1}{2} \end{cases} \longrightarrow h(u) = \begin{cases} -2u-2 & \text{if } u < \frac{1}{2} \\ 2u-4 & \text{if } u \geq \frac{1}{2} \end{cases}$$

   We have three points to help us graph $y = h(u)$: $(-1,0)$, $(0,-2)$ and $(2,0)$. Unlike in the last two examples, these points do not give us information at the value $u = \frac{1}{2}$ where the rule for $h(u)$ changes. Substituting $u = \frac{1}{2}$ into the expression $-2u-2$ gives -3, so from $h(u) = -2u-2$, $u < \frac{1}{2}$, we get a hole at $\left(\frac{1}{2}, -3\right)$. However, this hole is filled because $h\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 4 = -3$ and this produces the vertex at $\left(\frac{1}{2}, -3\right)$. The range of h is $[-3, \infty)$, with the minimum of h being -3 at $t = \frac{1}{2}$. Moreover, h is decreasing on $\left(-\infty, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, \infty\right)$. The graph of h is given below.



$h(u) = |2u-1| - 3$

4. Analyze $i(w) = 4 - 2|3w-1|$.

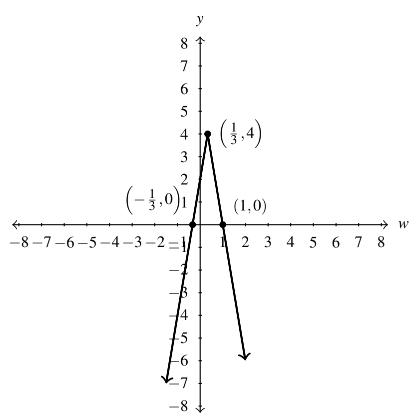
   Solving $i(w) = 4 - 2|3w-1| = 0$ yields $|3w-1| = 2$ or $3w-1 = \pm 2$. This gives two zeros, $w = -\frac{1}{3}$ and $w = 1$, which correspond to two w-intercepts, $\left(-\frac{1}{3}, 0\right)$ and $(1,0)$. Also, $i(0) = 4 - 2|3(0)-1| = 2$, so the y-intercept of the graph is $(0,2)$. As in the previous example, the first step in rewriting $i(w)$ as a piecewise defined function is to rewrite $|3w-1|$ as a piecewise function. Once again, we substitute the expression $3w-1$ in for every occurrence of x in Definition 1.12:

$$|3w-1| = \begin{cases} -(3w-1) & \text{if } 3w-1 < 0 \\ 3w-1 & \text{if } 3w-1 \geq 0 \end{cases} \longrightarrow |3w-1| = \begin{cases} -3w+1 & \text{if } w < \frac{1}{3} \\ 3w-1 & \text{if } w \geq \frac{1}{3} \end{cases}$$

Thus for $w < \frac{1}{3}$, $|3w-1| = -3w+1$, so $i(w) = 4-2|3w-1| = 4-2(-3w+1) = 6w+2$. Likewise, for $w \geq \frac{1}{3}$, $|3w-1| = 3w-1$ so $i(w) = 4-2|3w-1| = 4-2(3w-1) = -6w+6$.

$$i(w) = 4-2|3w-1| = \begin{cases} 4-2(-3w+1) & \text{if } w < \frac{1}{3} \\ 4-2(3w-1) & \text{if } w \geq \frac{1}{3} \end{cases} \longrightarrow \quad i(w) = \begin{cases} 6w+2 & \text{if } w < \frac{1}{3} \\ -6w+6 & \text{if } w \geq \frac{1}{3} \end{cases}$$

As with the previous example, we have three points on the graph of i: $\left(-\frac{1}{3},0\right)$, $(0,2)$ and $(1,0)$, but no information about happens at $w = \frac{1}{3}$. Substituting this value of w into the formula $6w+2$ would produce a hole at $\left(\frac{1}{3},4\right)$. As we've seen several times already, however, $i\left(\frac{1}{3}\right) = 4$ so we don't have a hole at $\left(\frac{1}{3},4\right)$ but, rather, the vertex. From the graph we see that the range of i is $(-\infty,4]$ with the maximum of i being 4 when $w = \frac{1}{3}$. Also, i is increasing over $\left(-\infty,\frac{1}{3}\right]$ and decreasing on $\left[\frac{1}{3},\infty\right)$. Its graph is given below.



$i(w) = 4-2|3w-1|$

□

As we take a step back and look at the graphs produced in Example 1.4.1, some patterns begin to emerge. Indeed, each of the graphs has the common '∨' shape (in the case of the function i it's a '∧') with the vertex located at the x-value where the rule for each function changes from one formula to the other. It turns out that, independent variable labels aside, each and every function in Example 1.4.1 can be rewritten in the form $F(x) = a|x-h|+k$ for real number parameters a, h and k.

Each of the functions from Example 1.4.1 is rewritten in this form below and we record the vertex along with the slopes of the lines in the graph.

- $f(x) = |x-3| = (1)|x-3|+0$: $a=1$, $h=3$, $k=0$; vertex $(3,0)$; slopes ± 1
- $g(t) = |t|-3 = (1)|t-0|+(-3)$: $a=1$, $h=0$, $k=-3$; vertex $(0,-3)$; slopes ± 1
- $h(u) = |2u-1|-3 = 2\left|u-\frac{1}{2}\right|+(-3)$: $a=2$, $h=\frac{1}{2}$, $k=-3$; vertex $\left(\frac{1}{2},-3\right)$; slopes ± 2
- $i(w) = 4-2|3w-1| = -6\left|w-\frac{1}{3}\right|+4$: $a=-6$, $h=\frac{1}{3}$, $k=4$; vertex $\left(\frac{1}{3},4\right)$; slopes ± 6

These specific examples suggest the following theorem.

> **Theorem 1.4.** For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a|x-h|+k$ consists of parts of two lines with slopes $\pm a$ which meet at a vertex (h,k). If $a > 0$, the shape resembles '$\vee$'. If $a < 0$, the shape resembles '$\wedge$'. Moreover, the graph is symmetric about the line $x = h$.

Proof. What separates Mathematics from the other sciences is its ability to actually *prove* patterns like the one stated in the theorem above as opposed to just *verifying* it by working more examples. The proof of Theorem 1.4 uses the exact same concepts as were used in Example 1.4.1, just in a more general context by which we mean using letters as parameters instead of numbers.

The first step is to rewrite $|x-h|$ as a piecewise function.

$$|x-h| = \begin{cases} -(x-h) & \text{if } x-h<0 \\ x-h & \text{if } x-h \geq 0 \end{cases} \quad \longrightarrow \quad |x-h| = \begin{cases} -x+h & \text{if } x<h \\ x-h & \text{if } x \geq h \end{cases}$$

We plug that work into $F(x)$ to rewrite it as a piecewise function. For $x<h$, we have $|x-h| = -x+h$, so

$$F(x) = a|x-h|+k = a(-x+h)+k = -ax+ah+k = -ax+(ah+k)$$

Similarly, for $x \geq h$, we have $|x-h| = x-h$, so

$$F(x) = a|x-h|+k = a(x-h)+k = ax-ah+k = ax+(-ah+k)$$

Hence,

$$F(x) = a|x-h|+k = \begin{cases} a(-x+h)+k & \text{if } x<h \\ a(x-h)+k & \text{if } x \geq h \end{cases} \quad \longrightarrow \quad F(x) = \begin{cases} -ax+(ah+k) & \text{if } x<h \\ ax+(-ah+k) & \text{if } x \geq h \end{cases}$$

All three parameters, a, h and k, are fixed (but arbitrary) real numbers. Thus, for any given choice of a, h and k the numbers $ah+k$ and $-ah+k$ are also just numbers as opposed to variables. This shows that the graph of F is comprised of pieces of two lines, $y = -ax+(ah+k)$ and $y = ax+(-ah+k)$, the former with slope $-a$ and the latter with slope a. Note that substituting $x = h$ into $y = -ax+(ah+k)$ produces $y = -ah+(ah+k) = k$ and substituting $x = h$ into $y = ax+(-ah+k)$ also produces $y = ah+(-ah+k) = k$. This tells us that the two linear pieces meet at the point (h,k).

If $a > 0$ then $-a < 0$ so the line $y = -ax+(ah+k)$, hence F, is decreasing on $(-\infty, h]$. Similarly, the line $y = ax+(-ah+k)$, hence F, is increasing on $[h,\infty)$. This produces a '$\vee$' shape. On the other hand, if $a < 0$ then $-a > 0$ which produces a '$\wedge$' shape because F is increasing on $(-\infty, h]$ followed by decreasing on $[h,\infty)$. (Said another way, $-a > 0$ means that the first linear piece has a positive slope and $a < 0$ means that the second piece has a negative slope.)

To show that the graph is symmetric about the line $x = h$, we need to show that if we move left or right the same distance away from $x = h$, then we get the same y-value on the graph. Suppose we move Δx to the right or left of h. The y-values are the function values so we need to show that $F(a+\Delta x) = F(a-\Delta x)$. Given that

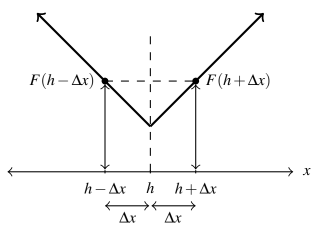
$$F(a+\Delta x) = a|a+\Delta x - a| + k = a|\Delta x| + k$$

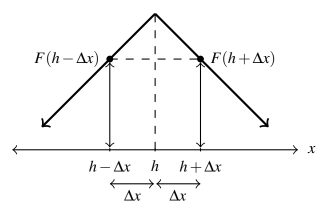
and

$$F(a-\Delta x) = a|a-\Delta x - a| + k = a|-\Delta x| + k = a|\Delta x| + k$$

we see that $F(a+\Delta x) = F(a-\Delta x)$. Thus we have shown that the y-values on the graph on either side of $x = h$ are equal provided we move the same distance away from $x = a$. This completes the proof. $\square$

The line $x = a$ in Theorem 1.4 is called the **axis of symmetry** of the graph of $y = F(x)$. This language is consistent with the basics of symmetry discussed in Section 1.1 and we will build upon our work here in several upcoming sections. For now, we simply present two graphs illustrating the concept of the axis of symmetry below.





While Theorem 1.4 and its proof are specific to the particular family of absolute value functions, there are ideas here that apply to all functions. Thus we wish to take a slight detour away from the main narrative to argue this result again from an even more generalized viewpoint. Our goal is to 'build' the formula $F(x) = a|x-h| + k$ from $f(x) = |x|$ in three stages, each corresponding to the role of one of the parameters a, h and k, and track the geometric changes that go along with each stage. We will revisit all of the ideas described below in complete generality in Section 1.6.

The graph of $f(x) = |x|$ consists of the points $\{(c, |c|) \,|\, c \in \mathbb{R}\}$.[3] Consider $F_1(x) = |x-h|$. The graph of F_1 is the set of points $\{(x, |x-h|) \,|\, x \in \mathbb{R}\}$. If we relabel $x - h = c$, then $x = c + h$, and as x varies through all of the real numbers, so does c and vice-versa.[4]

Hence, we can write $\{(x, |x-h|) \,|\, x \in \mathbb{R}\} = \{(c+h, |c|) \,|\, c \in \mathbb{R}\}$. If we fix a y-coordinate, $|c|$, we see that the corresponding points on the graph of f and F_1, $(c, |c|)$ and $(c+h, |c|)$, respectively, differ only in that one is horizontally shifted by h. In other words, to get the graph of F_1, we simply take the graph of f and shift each point horizontally by adding h to the x-coordinate. Translating the graph in this manner preserves the '$\vee$' shape and symmetry, but moves the vertex from $(0,0)$ to $(h,0)$.

Next, we examine $F_2(x) = a|x-h|$ and compare its graph to that of $F_1(x) = |x-h|$. The graph of F_2 consists of the points $\{(x, a|x-h|) \,|\, x \in \mathbb{R}\}$ whereas the graph of F_1 consists of the points $\{(x, |x-h|) \,|\, x \in \mathbb{R}\}$. The only difference between the points $(x, |x-h|)$ and $(x, a|x-h|)$ is that the y-coordinate of one is a times

[3] See the box on page 129. Also, we use 'c' as our dummy variable to avoid the confusion that would arise by over-using 'x'.

[4] That is, every real number c can be written as $x-h$ for some x, and every real number x can be written as $c+h$ for some c.

the y-coordinate of the other. If $a > 0$, all we are doing is scaling the y-axis by a factor of a. As we've seen when plotting points and graphing functions, the scaling of the y-axis affects only the relative vertical displacement of points[5] and not the overall shape.

If $a < 0$, then in addition to scaling the vertical axis, we are reflecting the points across the x-axis.[6] Such a transformation doesn't change the '$\vee$' shape except for flipping it upside-down to make it a '$\wedge$'. In either case, the vertex $(h,0)$ stays put at $(h,0)$ because the y-value of the vertex is 0 and $a \cdot 0 = 0$ regardless if $a > 0$ or $a < 0$.

Last, we examine the graph of $F(x) = a|x-h|+k$ to see how it relates to the graph of $F_2(x) = a|x-h|$. The graph of F consists of the points $\{(x, a|x-h|+k) \,|\, x \in \mathbb{R}\}$ whereas the graph of F_2 consists of the points $\{(x, a|x-h|) \,|\, x \in \mathbb{R}\}$. The difference between the corresponding points $(x, a|x-h|)$ and $(x, a|x-h|+k)$ is the addition of k in the y-coordinate of the latter. Adding k to each of the y-values translates the graph of F_2 vertically by k units. The basic shape doesn't change but the vertex goes from $(h,0)$ to (h,k).

In summary, the graph of $F(x) = a|x-h|+k$ can be obtained from the graph of $f(x) = |x|$ in three steps: first, add h to each of the x-coordinates; second, multiply each y-coordinate by a; and third, add k to each y-coordinate. Geometrically, these steps mean that we first move the graph left or right, then scale the y-axis by a factor of a (and reflect across the x-axis if $a < 0$), and then move the graph up or down. Throughout all of these *transformations*, the graph maintains its '$\vee$' or '$\wedge$' shape.

Of course, not every function involving absolute values can be written in the form given in Theorem 1.4. A good example of this is $G(x) = |x-2|-x$. However recognizing the ones that can be rewritten will greatly simplify the graphing process. In the next example, we graph four more absolute value functions, two using Theorem 1.4 and two using Definition 1.12.

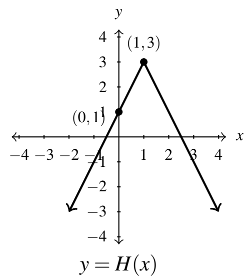
Example 1.4.2.

1. Graph each of the functions below using Theorem 1.4 or by rewriting it as a piecewise defined function using Definition 1.12. Find the zeros, axis-intercepts and the extrema (if any exist) and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $F(x) = |x+3|+2$ (b) $f(t) = \dfrac{4-|5-3t|}{2}$ (c) $G(x) = |x-2|-x$ (d) $g(t) = |t-2|-|t|$

2. Use Theorem 1.4 to write a possible formula for $H(x)$ whose graph is given below:

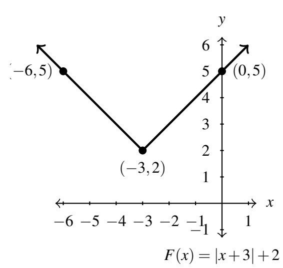
[5] See the discussion following Example 1.2.1 regarding the plot of Skippy's data.
[6] See the box on page 98 in Section 1.1.



Solution.

1. (a) Graph and analyze $F(x) = |x+3|+2$.

   Rewriting $F(x) = |x+3|+2 = (1)|x-(-3)|+2$, we have $F(x)$ in the form stated in Theorem 1.4 with $a = 1$, $h = -3$ and $k = 2$. The vertex is $(-3,2)$ and the graph will be a '$\vee$' shape. Seeing as the vertex is already above the x-axis and the graph opens upwards, there are no x-intercepts on the graph of F, hence there are no zeros.[7] With $F(0) = 5$, the y-intercept is $(0,5)$. To get a third point, we can pick an arbitrary x-value to the left of the vertex or we could use symmetry: three units to the *right* of the vertex the y-value is 5, so the same must be true three units to the *left* of the vertex, at $x = -6$. Sure enough, $F(-6) = |-6+3|+2 = |-3|+2 = 5$. The range of F is $[2,\infty)$ with its minimum of 2 when $x = -3$ and F decreasing on $(-\infty,-3]$ then increasing on $[-3,\infty)$. The graph is below.



   (b) Graph and analyze $f(t) = \dfrac{4-|5-3t|}{2}$.

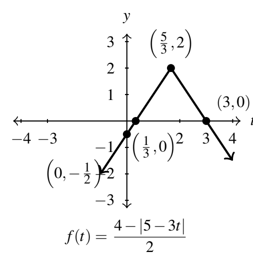
   We see in the formula for $f(t)$ that t appears only once to the first power inside the absolute values, so we proceed to rewrite it in the form $a|t-h|+k$:

$$\begin{aligned} f(t) &= \frac{4-|5-3t|}{2} \\ &= -\frac{|5-3t|}{2} + \frac{4}{2} \end{aligned}$$

[7] Alternatively, setting $|x+3|+2 = 0$ gives $|x+3| = -2$. Absolute values are never negative, thus we have no solution.

$$\begin{aligned} &= \left(-\frac{1}{2}\right)\left|(-3)\left(t-\frac{5}{3}\right)\right|+2 \\ &= \left(-\frac{1}{2}\right)|-3|\left|t-\frac{5}{3}\right|+2 \\ &= -\frac{3}{2}\left|t-\frac{5}{3}\right|+2. \end{aligned}$$

Matching up the constants in the formula $f(t)$ to the parameters of $F(x)$ in Theorem 1.4, we identify $a = -\frac{3}{2}$, $h = \frac{5}{3}$ and $k = 2$. Hence the vertex is $\left(\frac{5}{3}, 2\right)$, and the graph is shaped like '$\wedge$' comprised of pieces of lines with slopes $\pm\frac{3}{2}$. To find the zeros of f, we set $f(t) = 0$. (We can use either expression here.) Solving $-\frac{3}{2}\left|t-\frac{5}{3}\right|+2 = 0$, we get $\left|t-\frac{5}{3}\right| = \frac{4}{3}$, so $t-\frac{5}{3} = \pm\frac{4}{3}$. Hence our zeros are $t = \frac{1}{3}$ and $t = 3$, producing the t-intercepts $\left(\frac{1}{3}, 0\right)$ and $(3,0)$. Using either formula gives $f(0) = -\frac{1}{2}$, so our y-intercept is $\left(0, -\frac{1}{2}\right)$. Plotting the vertex, along with the intercepts, gives us enough information to produce the graph below. The range is $(-\infty, 2]$ with a maximum of 2 at $t = \frac{5}{3}$ and f is increasing on $\left(-\infty, \frac{5}{3}\right]$ then decreasing on $\left[\frac{5}{3}, \infty\right)$.



(c) Graph and analyze $G(x) = |x-2| - x$.

We are unable to apply Theorem 1.4 to $G(x) = |x-2| - x$ because there is an x both inside and outside of the absolute value. We can, however, rewrite the function as a piecewise function using Definition 1.12. Our first step is to rewrite $|x-2|$ as a piecewise function:

$$|x-2| = \begin{cases} -(x-2) & \text{if } x-2 < 0 \\ x-2 & \text{if } x-2 \geq 0 \end{cases} \longrightarrow |x-2| = \begin{cases} -x+2 & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2 \end{cases}$$

Hence, for $x < 2$, $|x-2| = -x+2$ so $G(x) = |x-2| - x = (-x+2) - x = -2x+2$. Likewise, for $x \geq 2$, $|x-2| = x-2$ so $G(x) = |x-2| - x = x-2-x = -2$.

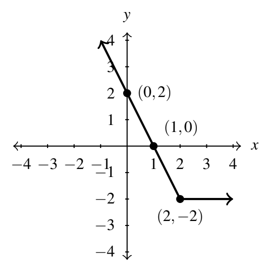
$$G(x) = |x-2| - x = \begin{cases} (-x+2) - x & \text{if } x < 2 \\ (x-2) - x & \text{if } x \geq 2 \end{cases} \longrightarrow G(x) = \begin{cases} -2x+2 & \text{if } x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

To find the zeros of G, we set $G(x) = 0$. Solving $|x-2| - x = 0$ can be problematic, given that x is both inside and outside of the absolute values.[8] We can, however, use the piecewise description

[8] We'll return to this momentarily.

of $G(x)$. With $G(x) = -2x+2$ for $x < 2$, we solve $-2x+2 = 0$ to get $x = 1$. This works because $1 < 2$, so we have $x = 1$ as the zero of G corresponding to the x-intercept $(1,0)$. The other piece of $G(x)$ is $G(x) = -2$ which is never 0. For the y-intercept, we find $G(0) = 2$, and get $(0,2)$.

To graph $y = G(x)$, we have the line $y = -2x+2$ which contains $(0,2)$ and $(1,0)$ and continues to a hole at $(2,-2)$. At this point, $G(x) = -2$ takes over and we have a horizontal line containing $(2,-2)$ extending indefinitely to the right. The range of G is $[-2,\infty)$ with a minimum value of -2 attained for all $x \geq 2$. Moreover, G is decreasing on $(-\infty,2]$ and then constant on $[2,\infty)$. The graph is below.



$G(x) = |x-2| - x$

(d) Graph and analyze $g(t) = |t-2| - |t|$.

Once again we are unable to use Theorem 1.4 because $g(t) = |t-2| - |t|$ has two absolute values with no apparent way to combine them. Thus we proceed by re-writing the function g with two separate applications of Definition 1.12 to remove each instance of the absolute values. To start with we have:

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad |t-2| = \begin{cases} -t+2 & \text{if } t < 2 \\ t-2 & \text{if } t \geq 2 \end{cases}$$

Taken together, these break the domain into *three* pieces: $t < 0$, $0 \leq t < 2$ and $t \geq 2$.

For $t < 0$, $|t| = -t$ and $|t-2| = -t+2$. Therefore $g(t) = |t-2| - |t| = (-t+2) - (-t) = 2$ for $t < 0$.

For $0 \leq t < 2$, $|t| = t$ and $|t-2| = -t+2$, so $g(t) = |t-2| - |t| = (-t+2) - (t) = -2t+2$.

Last, for $t \geq 2$, $|t| = t$ and $|t-2| = t-2$, so $g(t) = |t-2| - |t| = (t-2) - (t) = -2$.

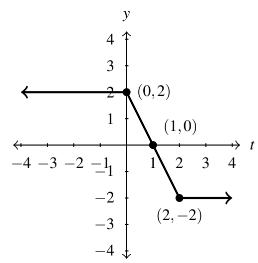
Putting all three parts together yields:

$$g(t) = |t-2| - |t| = \begin{cases} (-t+2) - (-t) & \text{if } t < 0 \\ (-t+2) - (t) & \text{if } 0 \leq t < 2 \\ (t-2) - (t) & \text{if } t \geq 2 \end{cases} = \begin{cases} 2 & \text{if } t < 0 \\ -2t+2 & \text{if } 0 \leq t < 2 \\ -2 & \text{if } t \geq 2 \end{cases}$$

As with the previous example, we'll delay discussing the absolute value algebra needed to find the zeros of g and use the piecewise description instead. To graph g, we have the horizontal line

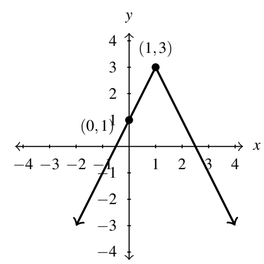
$y = 2$ up to, but not including, the point $(0,2)$. For $0 \le t < 2$, we have the line $y = -2t + 2$ which has a y-intercept at $(0,2)$ (thus picking up where the first part left off) and a t-intercept at $(1,0)$. This piece ends with a hole at $(2,-2)$ which is promptly plugged by the horizontal line $y = -2$ for $t \ge 2$. Hence the only zero of t is $t = 1$.

The range of g is $[-2,2]$ with a minimum of -2 achieved for all $t \ge 2$, and a maximum of 2 for $t \le 0$. We note that g is constant on $(-\infty, 0]$ and $[2,\infty)$, but with different values, and g is decreasing on $[0,2]$. The graph is given below.



$g(t) = |t-2| - |t|$

2. Write a formula for $H(x)$ given the graph below.



$y = H(x)$

We are told to use Theorem 1.4 to find a formula for $H(x)$ so we start with $H(x) = a|x-h| + k$ and look for real numbers a, h and k that make sense. The vertex is labeled as $(1,3)$, meaning $h = 1$ and $k = 3$. Hence we know $H(x) = a|x-1| + 3$, so all that is left for us to find is the value of a. The only other point labeled for us is $(0,1)$, meaning $H(0) = 1$. Substituting $x = 0$ into our formula for $H(x)$ gives: $H(0) = a|0-1| + 3 = a + 3$. Given that $H(0) = 1$, we have $a + 3 = 1$, so $a = -2$. Our final answer is $H(x) = -2|x-1| + 3$. □

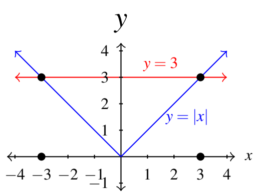
If nothing else, Example 1.4.2 demonstrates the value of *changing forms* of functions and the utility of the interplay between algebraic and graphical descriptions of functions. These themes resonate time and time again in this and later courses in Mathematics.

1.4.2 Graphical Solution Techniques for Equations and Inequalities

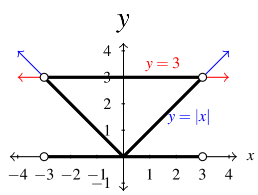
Consider the basic equation and related inequalities: $|x| = 3$, $|x| < 3$ and $|x| > 3$. At some point you learned how to solve these using properties of the absolute value inspired by the distance definition. (If not, see

Section 0.6.2.) While there is nothing wrong with this understanding, we wish to use these problems to motivate powerful graphical techniques which we'll use to solve more complicated equations and inequalities in this section, and in many other sections of the textbook.

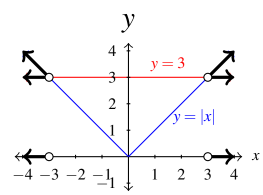
To that end, let's call $f(x) = |x|$ and $g(x) = 3$. If we graph $y = f(x)$ and $y = g(x)$ on the same set of axes then, by looking for x values where $f(x) = g(x)$, we are looking for x-values which have the same y-value on both graphs. That is, the solutions to $f(x) = g(x)$ are the x-coordinates of the *intersection points* of the two graphs. We graph $y = f(x) = |x|$ (the characteristic '$\vee$') along with $y = g(x) = 3$ (the horizontal line) below on the far left. Indeed, the two graphs intersect at $(-3,3)$ and $(3,3)$ so our solutions to $f(x) = g(x)$ are the x-values of these points, $x = \pm 3$.



Graphically solving $|x| = 3$



Graphically solving $|x| < 3$



Graphically solving $|x| \geq 3$

Likewise, if we wish to solve $|x| < 3$, we can view this as a functional inequality $f(x) < g(x)$ which means we are looking for the x-values where the $f(x)$ values are less than the corresponding $g(x)$ values. On the graphs, this means we'd be looking for the x-values where the y-values of $y = f(x)$ are less than, hence *below*, those on the graph of $y = g(x)$. In the middle picture above we see that the graph of f is below the graph of g between $x = -3$ and $x = 3$, so our solution is $-3 < x < 3$, or in interval notation, $(-3,3)$.

Finally, the inequality $|x| > 3$ is equivalent to $f(x) > g(x)$ so we are looking for the x-values where the graph of f is *above* the graph of g.[9] The picture on the far right above shows that this is true for all $x < -3$ or for all $x > 3$. In interval notation, the solution set is $(-\infty,-3) \cup (3,\infty)$.

The methodology and reasoning behind solving the above equation and inequalities extend to any pair of functions f and g, because when graphed on the same set of axes, function outputs are always the dependent variable or the ordinate (second coordinate) of the ordered pairs which comprise the graph. In general:

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions whose domains and ranges are sets of real numbers.

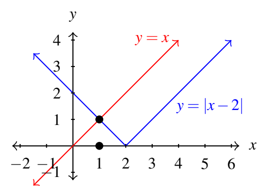
- The solutions to $f(x) = g(x)$ are the x-values where the graphs of f and g intersect.
- The solution to $f(x) < g(x)$ is the set of x-values where the graph of f is *below* the graph of g.
- The solution to $f(x) > g(x)$ is the set of x-values where the graph of f *above* the graph of g.

Let's return to Example 1.4.2 where we were asked to find the zeros of the functions $G(x) = |x-2| - x$ and

[9] Solving $f(x) > g(x)$ is equivalent to solving $g(x) < f(x)$ - that is, finding where the graph of g is below the graph of f.

$g(t) = |t-2| - |t|$. In that Example, instead of tackling the algebra involving the absolute values head on we rewrote each function as a piecewise-defined function and obtained our solutions that way.

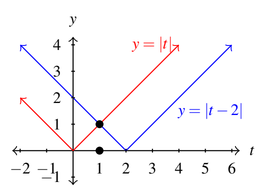
Let's see what this looks like graphically. Note that solving $|x-2| - x = 0$ is equivalent to solving $|x-2| = x$. We graphed $y = |x-2|$ and $y = x$ on the same set of axes on the left of the top of the next page and it appears as if we have just one point of intersection, corresponding to just one solution.



Graphically solving $|x-2| = x$

Indeed, we can *show* that there is just one point of intersection. The graph of $y = |x-2|$ is comprised of parts of two lines, $y = -(x-2)$ and $y = x-2$. The first line has a slope of -1 and the second has slope 1. The line $y = x$ also has a slope 1 meaning it and the 'right half' of $y = |x-2|$ are parallel, so they never intersect. If our graphs are accurate enough, we may even be able to guess that the solution is $x = 1$, which we can verify by substituting $x = 1$ into $|x-2| = x$ and seeing that it checks.

Likewise, solving $|t-2| - |t| = 0$ is equivalent to solving $|t-2| = |t|$. We graphed $y = |t-2|$ and $y = |t|$ and used the same arguments to get the solution $t = 1$ here as well.



Graphically solving $|t-2| = |t|$

There is more to see here. Consider solving $|x-2| - x = 0$ algebraically using the techniques from a previous Algebra course (or Section 0.6.2). Our first step would be to isolate the absolute value quantity: $|x-2| = x$. We then 'drop' the absolute value by paying the price of a '$\pm$': $x-2 = \pm x$. This gives us two equations: $x-2 = x$ and $x-2 = -x$. The first equation, $x-2 = x$ reduces to $-2 = 0$ which has no solution. The second equation, $x-2 = -x$, does have a solution, namely $x = 1$.

How does the algebra tie into the graphs above? Instead of 'dropping' the absolute value and tagging the right hand side with a $\pm$, we can think about the piecewise definition of $|x-2|$ and write $|x-2| = \pm(x-2)$ depending on if $x < 2$ or if $x \geq 2$. That is, $|x-2| = x$ is more precisely equivalent to the two equations: $-(x-2) = x$ which is valid for $x < 2$ or $x-2 = x$ which is valid for $x \geq 2$.

Graphically, the first equation is looking for intersection points between the 'left half' of the '$\vee$' of $y = |x-2|$ and the line $y = x$. Indeed, $-(x-2) = x$ is equivalent to $x-2 = -x$ from which we obtain our solution $x = 1$.

Likewise, the second equation, $x-2=x$ is looking for intersection points of the 'right half' of the '$\vee$' and the line $y=x$, but there is none. The equation $-2=0$ is telling us that for us to have any solutions, the lines $y=x-2$ and $y=x$, which have the same slope, must also have the same y-intercepts: that is, -2 would have to equal 0 and that's just silly.

Similarly, when solving $|t-2|-|t|=0$ or $|t-2|=|t|$, we can use our graphs to prove that the only intersection point is when the 'left half' of $y=|t-2|$ intersects the 'right half' of $y=|t|$ - that is, when $-(t-2)=t$. The moral of the story is this: careful graphs can help us simplify the algebra, because we can narrow down the cases. This is especially useful in solving inequalities, as we'll see in our next example.

Example 1.4.3. Solve the following equations and inequalities.

1. $4-|x|=0.9x-3.6$
2. $|t-3|-|t|=3$
3. $|x+1|\geq\dfrac{x+4}{2}$
4. $2<|t-1|\leq 5$

Solution.

1. Solve $4-|x|=0.9x-3.6$ for x.

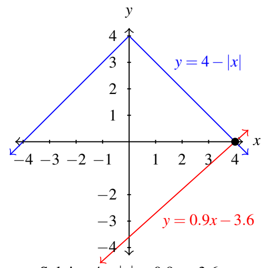
   We begin by graphing $y=4-|x|$ and $y=0.9x-3.6$ to look for intersection points. Using Theorem 1.4, we know that the graph of $y=4-|x|=-|x|+4$ has a vertex at $(0,4)$ and is a '$\wedge$' shape, so there are x-intercepts to find. Solving $4-|x|=0$, we get $|x|=4$, or $x=\pm 4$. Hence, we have two x-intercepts: $(-4,0)$ and $(4,0)$.

   We know from Section 1.3.1 that the graph of $y=0.9x-3.6$ is a line with slope 0.9 and y-intercept $(0,-3.6)$. To find the x-intercept here we solve $0.9x-3.6=0$ and get $x=4$. Hence, $(4,0)$ is an x-intercept here as well, and we have stumbled upon one solution to $4-|x|=0.9x-3.6$, namely $x=4$. The question is if there are any other solutions. Our graph (below on the left) certainly looks as if there is just one intersection point, but we know from Theorem 1.4 that the slopes of the linear parts of $y=4-|x|$ are ± 1. The slope of $y=0.9x-3.6$ is 0.9 and $0.9\neq 1$ so we know that the left hand side of the'$\wedge$' must meet up with the graph of the line because they are not parallel.[10]

   Definition 1.12 tells us that when $x<0$, $|x|=-x$, so $4-|x|=4-(-x)=4+x$. Hence we set about solving $4+x=0.9x-3.6$ and get $x=-76$. Both $x=-76$ and $x=4$ check in our original equation, $4-|x|=0.9x-3.6$, so we have found our two solutions.[11]

[10] See Theorem 1.3.

[11] Our picture shows only one of the solutions. We encourage you to take the time with a graphing utility to get the picture to show both points of intersection.

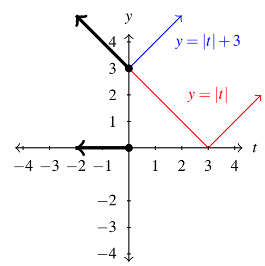


Solving $4 - |x| = 0.9x - 3.6$

2. Solve $|t-3| - |t| = 3$ for t.

   While we could graph $y = |t-3| - |t|$ and $y = 3$ to help us find solutions, we choose to rewrite the equation as $|t-3| = |t| + 3$. This way, we have somewhat easier graphs to deal with, namely $y = |t-3|$ and $y = |t| + 3$. The first graph, $y = |t-3|$, has a vertex at $(3,0)$ and is shaped like a '$\vee$' with slopes ± 1 and a y-intercept of $(0,3)$. The second graph, $y = |t| + 3$, has a vertex at $(0,3)$ and is also shaped like a '$\vee$,' with slopes ± 1, and has no t-intercepts.

   To our surprise and delight, the graphs appear to overlap for $t \leq 0$. Indeed, for $t \leq 0$, $|t-3| = -(t-3) = -t+3$ and $|t| + 3 = -t + 3$. Due to the fact that the formulas are *identical* for these values of t, our solutions are all values of t with $t \leq 0$. Using interval notation, we state our solution as $(-\infty, 0]$. (The other parts of the graphs are non-intersecting parallel lines so we ignored them.)

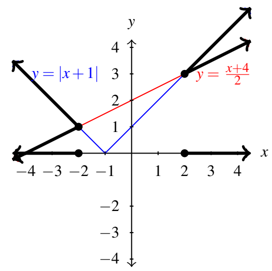


Solving $|t-3| - |t| = 3$

3. Solve $|x+1| \geq \dfrac{x+4}{2}$ for x.

   To solve $|x+1| \geq \frac{x+4}{2}$, we first graph $y = |x+1|$ and $y = \frac{x+4}{2} = \frac{1}{2}x + 2$. The former is '$\vee$' shaped with a vertex at $(-1,0)$ and a y-intercept of $(0,1)$. The latter is a line with y-intercept $(0,2)$, slope $m = \frac{1}{2}$ and x-intercept $(-4,0)$. The picture shows two intersection points. To find these, we solve the equations: $-(x+1) = \frac{x+4}{2}$, obtaining $x = -2$, and $x + 1 = \frac{x+4}{2}$ obtaining $x = 2$.

   Graphically, the inequality $|x+1| \geq \frac{x+4}{2}$ is looking for where the graph of $y = |x+1|$, the '$\vee$,' intersects $(=)$ or is above $(>)$ the line $y = \frac{x+4}{2}$. The graph shows this happening whenever $x \leq -2$ or $x \geq 2$. Using interval notation, our solution is $(-\infty, -2] \cup [2, \infty)$. While we cannot check every single x value individually, choosing test values $x < -2$, $x = 2$, $-2 < x < 2$, $x = 2$, and $x > 2$ to see if the original inequality $|x+1| \geq \frac{x+4}{2}$ holds would help us verify our solution.



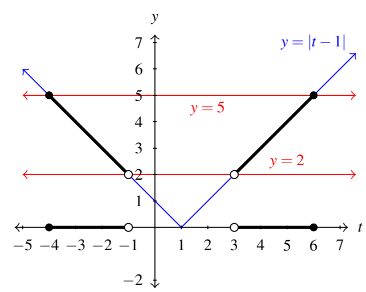
Solving $|x+1| \geq \frac{x+4}{2}$

4. Solve $2 < |t-1| \leq 5$ for t.

   Recall that the inequality $2 < |t-1| \leq 5$ is an example of a 'compound' inequality in that is two inequalities in one.[12] The values of t in the solution set need to satisfy $2 < |t-1|$ *and* $|t-1| \leq 5$. To help us sort through the cases, we graph the horizontal lines $y = 2$ and $y = 5$ along with the '$\vee$' shaped $y = |t-1|$ with vertex $(1,0)$ and y-intercept $(0,1)$.

   Geometrically, we are looking for where $y = |t-1|$ is strictly *above* the line $y = 2$ but *below* (or meets) the line $y = 5$. Solving $|t-1| = 2$ gives $t = -1$ and $t = 3$ whereas solving $|t-1| = 5$ gives $t = -4$ or $t = 6$. Per the graph, we see that $y = |t-1|$ lies between $y = 2$ and $y = 5$ when $-4 \leq t < -1$ and again when $3 < t \leq 6$.

   In interval notation, our solution is $[-4,-1) \cup (3,6]$. As with the previous example, it is impossible to check each and every one of these solutions, but choosing t values both in and around the solution intervals would give us some numerical confidence we have the correct and complete solution.



Solving $2 < |t-1| \leq 5$

$\square$

We will see the interplay of Algebra and Geometry throughout the rest of this course. In the Exercises, do not hesitate to use whatever mix of algebraic and graphical methods you deem necessary to solve the given equation or inequality. Indeed, there is great value in checking your algebraic answers graphically and vice-versa.

[12] See Example 0.6.1 for examples of linear compound inequalities.

One of the classic applications of inequalities involving absolute values is the notion of tolerances.[13] Recall that for real numbers x and c, the quantity $|x-c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x-c| \leq d$ for $d > 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 1.4.4. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution. Let x denote the length of the side of the square piece of particle board so that the area of the board is x^2 square inches. Our tolerance specifies that the area of the board, x^2, needs to be within 0.25 square inches of 576. Mathematically, this translates to $|x^2-576| \leq 0.25$. Rewriting, we get $-0.25 \leq x^2-576 \leq 0.25$, or $575.75 \leq x^2 \leq 576.25$. At this point, we take advantage of the fact that the square root is increasing.[14] Therefore, taking square roots preserves the inequality. When simplifying, we keep in mind that x represents a length and thus $x > 0$.

$$\begin{array}{rcll}
575.75 \leq & x^2 & \leq 576.25 & \\
\sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\
\sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|) \\
\sqrt{575.75} \leq & x & \leq \sqrt{576.25} & (|x| = x \text{ as } x > 0)
\end{array}$$

The side of the piece of particle board must be between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches. This results in a tolerance of (approximately) 0.005 inches of the target length of 24 inches, to ensure that the area is within 0.25 square inches of 576. □

1.4.3 Exercises

In Exercises 1 - 6, graph the function using Theorem 1.4. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

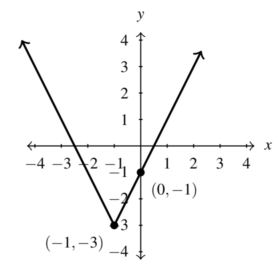
1. $f(x) = |x+4|$
2. $f(x) = |x|+4$
3. $f(x) = |4x|$
4. $g(t) = -3|t|$
5. $g(t) = 3|t+4|-4$
6. $g(t) = \dfrac{1}{3}|2t-1|$

In Exercises 7 - 10, find a formula for each function below in the form $F(x) = a|x-h|+k$.

[13] The underlying concept of Calculus can be phrased in terms of tolerances, so this is well worth your attention.

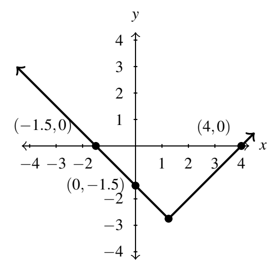
[14] This means that for $a, b \geq 0$, if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

7.



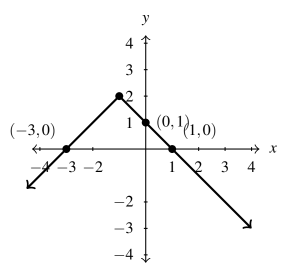
$y = F(x)$

8.



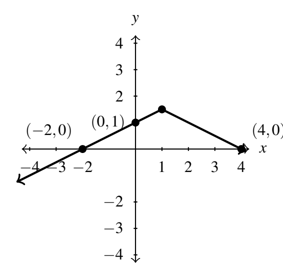
$y = F(x)$

9.



$y = F(x)$

10.

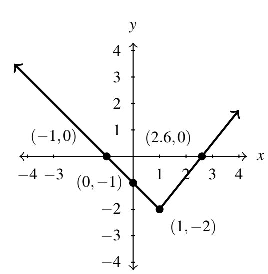


$y = F(x)$

11. Graph the following pairs of functions on the same set of axes:
   - $f(x) = 2 - x$ and $g(x) = |2 - x|$
   - $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$
   - $f(x) = x^3$ and $g(x) = |x^3|$
   - $f(x) = \sqrt{x} - 4$ and $g(x) = |\sqrt{x} - 4|$

   Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = |f(x)|$ until you can explain how, in general, one would obtain the graph of $y = |f(x)|$ given the graph of $y = f(x)$. How does your explanation tie in with with Definition 1.12?

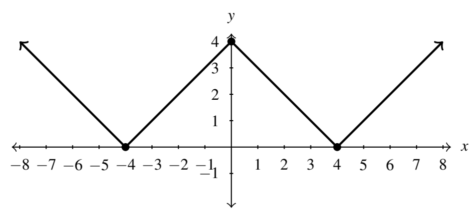
12. Explain the function below cannot be written in the form $F(x) = a|x - h| + k$. Write $F(x)$ as a piecewise-defined linear function.



$y = F(x)$

In Exercises 13 - 18, graph the function by rewriting each function as a piecewise defined function using Definition 1.12. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

13. $f(x) = x + |x| - 3$
14. $f(x) = |x+2| - x$
15. $f(x) = |x+2| - |x|$
16. $g(t) = |t+4| + |t-2|$
17. $g(t) = \dfrac{|t+4|}{t+4}$
18. $g(t) = \dfrac{|2-t|}{2-t}$
19. With the help of your classmates, write an absolute value function whose graph is given below.



In Exercises 20 - 31, solve the equation.

20. $|x| = 6$
21. $|3x-1| = 10$
22. $|4-x| = 7$
23. $4 - |t| = 3$
24. $2|5t+1| - 3 = 0$
25. $|7t-1| + 2 = 0$
26. $\dfrac{5-|w|}{2} = 1$
27. $\frac{2}{3}|5-2w| - \frac{1}{2} = 5$
28. $|w| = w + 3$
29. $|2x-1| = x + 1$
30. $4 - |x| = 2x + 1$
31. $|x-4| = x - 5$

Solve the equations in Exercises 32 - 37 using the property that if $|a| = |b|$ then $a = \pm b$.

32. $|3x-2| = |2x+7|$
33. $|3x+1| = |4x|$
34. $|1-2x| = |x+1|$
35. $|4-t| - |t+2| = 0$
36. $|2-5t| = 5|t+1|$
37. $3|t-1| = 2|t+1|$

In Exercises 38 - 53, solve the inequality. Write your answer using interval notation.

38. $|3x-5| \leq 4$
39. $|7x+2| > 10$
40. $|2t+1| - 5 < 0$
41. $|2-t| - 4 \geq -3$

42. $|3w+5|+2<1$

43. $2|7-w|+4>1$

44. $2 \leq |4-x| < 7$

45. $1 < |2x-9| \leq 3$

46. $|t+3| \geq |6t+9|$

47. $|t-3|-|2t+1|<0$

48. $|1-2x| \geq x+5$

49. $x+5 < |x+5|$

50. $x \geq |x+1|$

51. $|2x+1| \leq 6-x$

52. $t+|2t-3|<2$

53. $|3-t| \geq t-5$

54. Show that if δ is a real number with $\delta > 0$, the solution to $|x-a| < \delta$ is the interval: $(a-\delta, a+\delta)$. That is, an interval centered at a with 'radius' δ.

55. The Triangle Inequality for real numbers states that for all real numbers x and a, $|x+a| \leq |x|+|a|$ and, moreover, $|x+a| = |x|+|a|$ if and only if x and a are both positive, both negative, or one or the other is 0. Graph each pair of functions below on the same pair of axes and use the graphs to verify the triangle inequality in each instance.

    - $f(x) = |x+2|$ and $g(x) = |x|+2$.
    - $f(x) = |x+4|$ and $g(x) = |x|+4$.

SECTION 1.4 EXERCISE ANSWERS A.1.1

1.5 Function Arithmetic

1.5.1 Function Arithmetic

In this section, we begin our study of what can be considered as the *algebra of functions* by defining *function arithmetic.*

Given two real numbers, we have four primary arithmetic operations available to us: addition, subtraction, multiplication, and division (provided we don't divide by 0.) As the functions we study in this text have ranges which are sets of real numbers, it makes sense we can extend these arithmetic notions to functions.

For example, to add two functions means we add their outputs; to subtract two functions, we subtract their outputs, and so on and so forth. More formally, given two functions f and g, we *define* a new function $f+g$ whose rule is determined by adding the outputs of f and g. That is $(f+g)(x) = f(x)+g(x)$. While this looks suspiciously like some kind of distributive property, it is nothing of the sort. The '+' sign in the expression '$f+g$' is part of the *name* of the function we are defining,[1] whereas the plus sign '+' sign in the expression $f(x)+g(x)$ represents real number addition: we are adding the output from f, $f(x)$ with the output from g, $g(x)$ to determine the output from the sum function, $(f+g)(x)$.

Of course, in order to define $(f+g)(x)$ by the formula $(f+g)(x) = f(x)+g(x)$, both $f(x)$ and $g(x)$ need to be defined in the first place; that is, x must be in the domain of f *and* the domain of g. You'll recall[2] this means x must be in the *intersection* of the domains of f and g. We define the following.

Definition 1.13. Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula

$$(f+g)(x) = f(x)+g(x)$$

- The **difference** of f and g, denoted $f-g$, is the function defined by the formula

$$(f-g)(x) = f(x)-g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0.$$

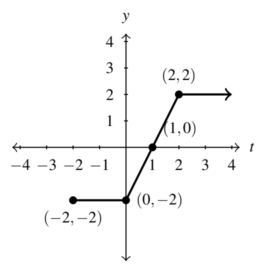
[1] We could have just as easily called this new function $S(x)$ for 'sum' of f and g and defined S by $S(x) = f(x)+g(x)$.

[2] see Section 0.4.

We put these definitions to work for us in the next example.

Example 1.5.1. Consider the following functions:

- $f(x) = 6x^2 - 2x$
- $g(t) = 3 - \dfrac{1}{t}$, $t > 0$
- $h = \{(-3,2), (-2,0.4), (0,\sqrt{2}), (3,-6)\}$
- s whose graph is given below:



1. Compute and simplify the following function values:

   (a) $(f+g)(1)$ (b) $(s-f)(-1)$ (c) $(fg)(2)$ (d) $\left(\dfrac{s}{h}\right)(0)$

   (e) $((s+g)+h)(3)$ (f) $(s+(g+h))(3)$ (g) $\left(\dfrac{f+h}{s}\right)(3)$ (h) $(f(g-h))(-2)$

2. State the domain of each of the following functions:

   (a) hg (b) $\dfrac{f}{s}$

3. Determine expressions for the functions below. State the domain for each.

   (a) $(fg)(x)$ (b) $\left(\dfrac{g}{f}\right)(t)$

Solution.

1. (a) Compute and simplify $(f+g)(1)$.
   By definition, $(f+g)(1) = f(1)+g(1)$. We find $f(1) = 6(1)^2 - 2(1) = 4$ and $g(1) = 3 - \frac{1}{1} = 2$. So we get $(f+g)(1) = 4+2 = 6$.

(b) Compute and simplify $(s-f)(-1)$.

To find $(s-f)(-1) = s(-1) - f(-1)$, we need both $s(-1)$ and $f(-1)$. To get $s(-1)$, we look to the graph of $y = s(t)$ and look for the y-coordinate of the point on the graph with the t-coordinate of -1. While not labeled directly, we infer the point $(-1,-2)$ is on the graph which means $s(-1) = -2$. For $f(-1)$, we compute: $f(-1) = 6(-1)^2 - 2(-1) = 8$. Putting it all together, we get $(s-f)(-1) = (-2)-(8) = -10$.

(c) Compute and simplify $(fg)(2)$.

Because $(fg)(2) = f(2)g(2)$, we first compute $f(2)$ and $g(2)$. We find $f(2) = 6(2)^2 - 2(2) = 20$ and $g(2) = 3 - \frac{1}{2} = \frac{5}{2}$, so $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

(d) Compute and simplify $\left(\frac{s}{h}\right)(0)$.

By definition, $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)}$. As $(0,-2)$ is on the graph of $y = s(t)$, we know $s(0) = -2$. Likewise, the ordered pair $(0,\sqrt{2}) \in h$, so $h(0) = \sqrt{2}$. We get $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

(e) Compute and simplify $((s+g)+h)(3)$.

The expression $((s+g)+h)(3)$ involves *three* functions. Fortunately, they are grouped so that we can apply Definition 1.13 by first considering the sum of the two functions $(s+g)$ and h, then to the sum of the two functions s and g: $((s+g)+h)(3) = (s+g)(3)+h(3) = (s(3)+g(3))+h(3)$. To get $s(3)$, we look to the graph of $y = s(t)$. We infer the point $(3,2)$ is on the graph of s, so $s(3) = 2$. We compute $g(3) = 3 - \frac{1}{3} = \frac{8}{3}$. To find $h(3)$, we note $(3,-6) \in h$, so $h(3) = -6$. Hence, $((s+g)+h)(3) = (s+g)(3)+h(3) = (s(3)+g(3))+h(3) = \left(2+\frac{8}{3}\right)+(-6) = -\frac{4}{3}$.

(f) Compute and simplify $(s+(g+h))(3)$.

The expression $(s+(g+h))(3)$ is very similar to the previous problem, $((s+g)+h)(3)$ except that the g and h are grouped together here instead of the s and g. We proceed as above applying Definition 1.13 twice and find $(s+(g+h))(3) = s(3)+(g+h)(3) = s(3)+(g(3)+h(3))$. Substituting the values for $s(3)$, $g(3)$ and $h(3)$, we get $(s+(g+h))(3) = 2+\left(\frac{8}{3}+(-6)\right) = -\frac{4}{3}$, which, not surprisingly, matches our answer to the previous problem.

(g) Compute and simplify $\left(\frac{f+h}{s}\right)(3)$.

Once again, we find the expression $\left(\frac{f+h}{s}\right)(3)$ has more than two functions involved. As with all fractions, we treat '–' as a grouping symbol and interpret $\left(\frac{f+h}{s}\right)(3) = \frac{(f+h)(3)}{s(3)} = \frac{f(3)+h(3)}{s(3)}$. We compute $f(3) = 6(3)^2 - 2(3) = 48$ and have $h(3) = -6$ and $s(3) = 2$ from above. Hence, $\left(\frac{f+h}{s}\right)(3) = \frac{f(3)+h(3)}{s(3)} = \frac{48+(-6)}{2} = 21$.

(h) Compute and simplify $(f(g-h))(-2)$.

We need to need to exercise caution in parsing $(f(g-h))(-2)$. In this context, f, g, and h are all functions, so we interpret $(f(g-h))$ as the function and -2 as the argument. We view the function $f(g-h)$ as the product of f and the function $g-h$. Hence, $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)]$. We compute $f(-2) = 6(-2)^2 - 2(-2) = 28$, and $g(-2) = 3 - \frac{1}{-2} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$. Because $(-2, 0.4) \in h$, $h(-2) = 0.4$. Putting this altogether, we get $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)] = 28(3.5 - 0.4) = 28(3.1) = 86.8$.

2. (a) State the domain of hg.

   To find the domain of hg, we need to find the real numbers in both the domain of h and the domain of g. The domain of h is $\{-3,-2,0,3\}$ and the domain of g is $\{t \in \mathbb{R} \,|\, t > 0\}$ so the only real number in common here is 3. Hence, the domain of hg is $\{3\}$, which may be small, but it's better than nothing.[3]

   (b) State the domain of $\frac{f}{s}$.

   To find the domain of $\frac{f}{s}$, we first note the domain of f is all real numbers, but that the domain of s, based on the graph, is just $[-2,\infty)$. Moreover, $s(t) = 0$ when $t = 1$, so we must exclude this value from the domain of $\frac{f}{s}$. Hence, we are left with $[-2,1) \cup (1,\infty)$.

3. (a) Determine an expression for $(fg)(x)$. Then state the domain of the function.

   By definition, $(fg)(x) = f(x)g(x)$. We are given $f(x) = 6x^2 - 2x$ and $g(t) = 3 - \frac{1}{t}$ so $g(x) = 3 - \frac{1}{x}$. Hence,

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right) \\
&= 6x^2(3) - 6x^2\left(\frac{1}{x}\right) - 2x(3) + 2x\left(\frac{1}{x}\right) && \text{distribute} \\
&= 18x^2 - 6x - 6x + 2 \\
&= 18x^2 - 12x + 2
\end{aligned}$$

   To find the domain of fg, we note the domain of f is all real numbers, $(-\infty,\infty)$ whereas the domain of g is restricted to $\{t \in \mathbb{R} \,|\, t > 0\} = (0,\infty)$. Hence, the domain of fg is likewise restricted to $(0,\infty)$. Note if we relied solely on the **simplified formula** for $(fg)(x) = 18x^2 - 12x + 2$, we would have obtained the *incorrect* answer for the domains of fg.

   (b) Determine an expression for $\left(\frac{g}{f}\right)(t)$. Then state the domain of the function.

   To find an expression for $\left(\frac{g}{f}\right)(t) = \frac{g(t)}{f(t)}$ we first note $f(t) = 6t^2 - 2t$ and $g(t) = 3 - \frac{1}{t}$. Hence:

[3] Due to the fact that $(hg)(3) = h(3)g(3) = (-6)\left(\frac{8}{3}\right) = -16$, we can write $hg = \{(3,-16)\}$.

$$\begin{aligned}
\left(\frac{g}{f}\right)(t) &= \frac{g(t)}{f(t)} \\
&= \frac{3-\frac{1}{t}}{6t^2-2t} = \frac{3-\frac{1}{t}}{6t^2-2t}\cdot\frac{t}{t} && \text{simplify compound fractions} \\
&= \frac{\left(3-\frac{1}{t}\right)t}{(6t^2-2t)t} = \frac{3t-1}{(6t^2-2t)t} \\
&= \frac{3t-1}{2t^2(3t-1)} = \frac{\cancelto{1}{(3t-1)}}{2t^2\cancel{(3t-1)}} && \text{factor and divide out} \\
&= \frac{1}{2t^2} && \text{provided } t \neq \frac{1}{3}
\end{aligned}$$

Hence, $\left(\frac{g}{f}\right)(t) = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$. To find the domain of $\frac{g}{f}$, a real number must be both in the domain of g, $(0,\infty)$, and the domain of f, $(-\infty,\infty)$ so we start with the set $(0,\infty)$. Additionally, we require $f(t) \neq 0$. Solving $f(t) = 0$ amounts to solving $6t^2 - 2t = 0$ or $2t(3t-1) = 0$. We find $t = 0$ or $t = \frac{1}{3}$ which means we need to exclude these values from the domain. Hence, our final answer for the domain of $\frac{g}{f}$ is $\left(0,\frac{1}{3}\right) \cup \left(\frac{1}{3},\infty\right)$. Note that, once again, using the *simplified formula* for $\left(\frac{g}{f}\right)(t)$ to determine the domain of $\frac{g}{f}$, would have produced erroneous results. □

A few remarks are in order. First, in number 1 parts 1e through 1h, we first encountered combinations of *three* functions despite Definition 1.13 only addressing combinations of *two* functions at a time. It turns out that function arithmetic inherits many of the same properties of real number arithmetic. For example, we showed above that $((s+g)+h)(3) = (s+(g+h))(3)$. In general, given any three functions f, g, and h, $(f+g)+h = f+(g+h)$ that is, function addition is *assocative*. To see this, choose an element x common to the domains of f, g, and h. Then

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x)+h(x) && \text{definiton of } ((f+g)+h)(x) \\
&= (f(x)+g(x))+h(x) && \text{definition of } (f+g)(x) \\
&= f(x)+(g(x)+h(x)) && \text{associative property of real number addition} \\
&= f(x)+(g+h)(x) && \text{definition of } (g+h)(x) \\
&= (f+(g+h))(x) && \text{definition of } (f+(g+h))(x)
\end{aligned}$$

The key step to the argument is that $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$ which is true courtesy of the associative property of real number addition. And just like with real number addition, because function addition is associative, we may write $f+g+h$ instead of $(f+g)+h$ or $f+(g+h)$ even though, when it comes down to computations, we can only add two things together at a time.[4]

[4] Addition is a 'binary' operation - meaning it is defined only on two objects at once. Even though we write $1+2+3=6$, mentally, we add just two of numbers together at any given time to get our answer: for example, $1+2+3=(1+2)+3=3+3=6$.

For completeness, we summarize the properties of function arithmetic in the theorem below. The proofs of the properties all follow along the same lines as the proof of the associative property and are left to the reader. We investigate some additional properties in the exercises.

Theorem 1.5. Suppose f, g and h are functions.

- **Commutative Law of Addition:** $f+g=g+f$
- **Associative Law of Addition:** $(f+g)+h=f+(g+h)$
- **Additive Identity:** The function $Z(x)=0$ satisfies: $f+Z=Z+f=f$ for all functions f.
- **Additive Inverse:** The function $F(x)=-f(x)$ for all x in the domain of f satisfies:

$$f+F=F+f=Z.$$

- **Commutative Law of Multiplication:** $fg=gf$
- **Associative Law of Multiplication:** $(fg)h=f(gh)$
- **Multiplicative Identity:** The function $I(x)=1$ satisfies: $fI=If=f$ for all functions f.
- **Multiplicative Inverse:** If $f(x)\neq 0$ for all x in the domain of f, then $F(x)=\dfrac{1}{f(x)}$ satisfies:

$$fF=Ff=I$$

- **Distributive Law of Multiplication over Addition:** $f(g+h)=fg+fh$

In the next example, we decompose given functions into sums, differences, products and/or quotients of other functions. Note that there are infinitely many different ways to do this, including some trivial ones. For example, suppose we were instructed to decompose $f(x)=x+2$ into a sum or difference of functions. We could write $f=g+h$ where $g(x)=x$ and $h(x)=2$ or we could choose $g(x)=2x+3$ and $h(x)=-x-1$. More simply, we could write $f=g+h$ where $g(x)=x+2$ and $h(x)=0$. We'll call this last decomposition a 'trivial' decomposition. Likewise, if we ask for a decomposition of $f(x)=2x$ as a product, a nontrivial solution would be $f=gh$ where $g(x)=2$ and $h(x)=x$ whereas a trivial solution would be $g(x)=2x$ and $h(x)=1$. In general, non-trivial solutions to decomposition problems avoid using the additive identity, 0, for sums and differences and the multiplicative identity, 1, for products and quotients.

Example 1.5.2.

1. For $f(x)=x^2-2x$, find functions g, h and k to decompose f nontrivially as:

(a) $f = g - h$ (b) $f = g + h$ (c) $f = gh$ (d) $f = g(h - k)$

2. For $F(t) = \dfrac{2t+1}{\sqrt{t^2-1}}$, find functions G, H and K to decompose F nontrivially as:

(a) $F = \dfrac{G}{H}$ (b) $F = GH$ (c) $F = G + H$ (d) $F = \dfrac{G+H}{K}$

Solution.

1. (a) Decompose $f = g - h$.

To decompose $f = g - h$, we need functions g and h so $f(x) = (g-h)(x) = g(x) - h(x)$. Given $f(x) = x^2 - 2x$, one option is to let $g(x) = x^2$ and $h(x) = 2x$. To check, we find $(g-h)(x) = g(x) - h(x) = x^2 - 2x = f(x)$ as required. In addition to checking the formulas match up, we also need to check domains. There isn't much work here as the domains of g and h are all real numbers which combine to give the domain of f which is all real numbers.

(b) Decompose $f = g + h$.

In order to write $f = g + h$, we need $f(x) = (g+h)(x) = g(x) + h(x)$. One way to accomplish this is to write $f(x) = x^2 - 2x = x^2 + (-2x)$ and identify $g(x) = x^2$ and $h(x) = -2x$. To check, $(g+h)(x) = g(x) + h(x) = x^2 - 2x = f(x)$. Again, the domains for both g and h are all real numbers which combine to give f its domain of all real numbers.

(c) Decompose $f = gh$.

To write $f = gh$, we require $f(x) = (gh)(x) = g(x)h(x)$. In other words, we need to factor $f(x)$. We find $f(x) = x^2 - 2x = x(x-2)$, so one choice is to select $g(x) = x$ and $h(x) = x - 2$. Then $(gh)(x) = g(x)h(x) = x(x-2) = x^2 - 2x = f(x)$, as required. As above, the domains of g and h are all real numbers which combine to give f the correct domain of $(-\infty, \infty)$.

(d) Decompose $f = g(h - k)$.

We need to be careful here interpreting the equation $f = g(h-k)$. What we have is an equality of *functions* so the parentheses here *do not* represent function notation here, but, rather function *multiplication*. The way to parse $g(h-k)$, then, is the function g *times* the function $h-k$. Hence, we seek functions g, h, and k so that $f(x) = [g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x))$. From the previous example, we know we can rewrite $f(x) = x(x-2)$, so one option is to set $g(x) = h(x) = x$ and $k(x) = 2$ so that $[g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x)) = x(x-2) = x^2 - 2x = f(x)$, as required. As above, the domain of all constituent functions is $(-\infty, \infty)$ which matches the domain of f.

2. (a) Decompose $F = \frac{G}{H}$.

To write $F = \frac{G}{H}$, we need $G(t)$ and $H(t)$ so $F(t) = \left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)}$. We choose $G(t) = 2t + 1$ and $H(t) = \sqrt{t^2-1}$. Sure enough, $\left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$ as required. When it comes to the domain of F, owing to the square root, we require $t^2-1\geq 0$. We have a denominator as well, therefore we require $\sqrt{t^2-1}\neq 0$. The former requirement is the same restriction on H, and the latter requirement comes from Definition 1.13. Starting with the domain of G, all real numbers, and working through the details, we arrive at the correct domain of F, $(-\infty,-1)\cup(1,\infty)$.

(b) Decompose $F=GH$.

Next, we are asked to find functions G and H so $F(t)=(GH)(t)=G(t)H(t)$. This means we need to rewrite the expression for $F(t)$ as a product. One way to do this is to convert radical notation to exponent notation:

$$F(t)=\frac{2t+1}{\sqrt{t^2-1}}=\frac{2t+1}{(t^2-1)^{\frac{1}{2}}}=(2t+1)\left(t^2-1\right)^{-\frac{1}{2}}.$$

Choosing $G(t)=2t+1$ and $H(t)=\left(t^2-1\right)^{-\frac{1}{2}}$, we see $(GH)(t)=G(t)H(t)=(2t+1)\left(t^2-1\right)^{-\frac{1}{2}}$ as required. The domain restrictions on F stem from the presence of the square root in the denominator - both are addressed when finding the domain of H. Hence, we obtain the correct domain of F as $(-\infty,-1)\cup(1,\infty)$.

(c) Decompose $F=G+H$.

To express F as a sum of functions G and H, we could rewrite

$$F(t)=\frac{2t+1}{\sqrt{t^2-1}}=\frac{2t}{\sqrt{t^2-1}}+\frac{1}{\sqrt{t^2-1}},$$

so that $G(t)=\frac{2t}{\sqrt{t^2-1}}$ and $H(t)=\frac{1}{\sqrt{t^2-1}}$. Indeed, $(G+H)(t)=G(t)+H(t)=\frac{2t}{\sqrt{t^2-1}}+\frac{1}{\sqrt{t^2-1}}=\frac{2t+1}{\sqrt{t^2-1}}=F(t)$, as required. Moreover, the domain restrictions for F are the same for both G and H, so we get agreement on the domain being $(-\infty,-1)\cup(1,\infty)$.

(d) Decompose $F=\frac{G+H}{K}$.

Last, but not least, to write $F=\frac{G+H}{K}$, we require $F(t)=\left(\frac{G+H}{K}\right)(t)=\frac{(G+H)(t)}{K(t)}=\frac{G(t)+H(t)}{K(t)}$. Identifying $G(t)=2t$, $H(t)=1$, and $K(t)=\sqrt{t^2-1}$, we get

$$\left(\frac{G+H}{K}\right)(t)=\frac{(G+H)(t)}{K(t)}=\frac{G(t)+H(t)}{K(t)}=\frac{2t+1}{\sqrt{t^2-1}}=F(t).$$

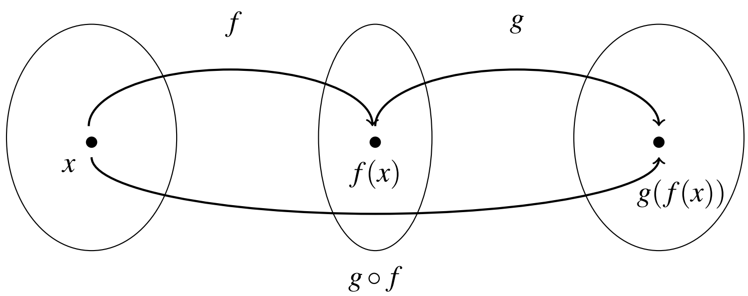
Concerning domains, the domain of both G and H are all real numbers, but the domain of K is restricted to $t^2-1\geq 0$. Coupled with the restriction stated in Definition 1.13 that $K(t)\neq 0$, we recover the domain of F, $(-\infty,-1)\cup(1,\infty)$. □

1.5.2 Function Composition

We just saw how the arithmetic of real numbers carried over into an arithmetic of functions. In this section, we discuss another way to combine functions which is unique to functions and isn't shared with real numbers - function **composition**.

Definition 1.14. Let f and g be functions where the real number x is in the domain of f and the real number $f(x)$ is in the domain of g. The **composite** of g with f, denoted $g \circ f$, and read 'g composed with f' is defined by the formula: $(g \circ f)(x) = g(f(x))$.

To compute $(g \circ f)(x)$, we use the formula given in Defintion 1.14: $(g \circ f)(x) = g(f(x))$. However, from a procedural viewpoint, Defintion 1.14 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have



In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. When evaluating composite function values we present two methods in the example below: the 'inside out' and 'outside in' methods.

Example 1.5.3. Let $f(x) = x^2 - 4x$, $g(t) = 2 - \sqrt{t+3}$, and $h(s) = \dfrac{2s}{s+1}$.

In numbers 1 - 3, compute the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

In numbers 4 - 10, determine and simplify the indicated composite functions. State the domain of each.

4. $(g \circ f)(x)$
5. $(f \circ g)(t)$
6. $(g \circ h)(s)$
7. $(h \circ g)(t)$
8. $(h \circ h)(x)$
9. $(h \circ (g \circ f))(x)$
10. $((h \circ g) \circ f)(x)$

Solution.

1. Compute $(g \circ f)(1)$.

   Using Definition 1.14, $(g \circ f)(1) = g(f(1))$. To start $f(1) = (1)^2 - 4(1) = -3$. Then $g(-3) = 2 - \sqrt{(-3)+3} = 2$, so we have $(g \circ f)(1) = g(f(1)) = g(-3) = 2$.

2. Compute $(f \circ g)(1)$.

   By definition, $(f \circ g)(1) = f(g(1))$. We find $g(1) = 2 - \sqrt{1+3} = 0$, and $f(0) = (0)^2 - 4(0) = 0$, so $(f \circ g)(1) = f(g(1)) = f(0) = 0$.

Comparing this with our answer to the last problem, we see that $(g \circ f)(1) \neq (f \circ g)(1)$ which tells us function composition is not commutative,[5]

3. Compute $(g \circ g)(6)$.

   We note $(g \circ g)(6) = g(g(6))$, and thus we 'iterate' the process g: that is, we apply the process g to 6, then apply the process g again. We find $g(6) = 2 - \sqrt{6+3} = -1$, and $g(-1) = 2 - \sqrt{(-1)+3} = 2 - \sqrt{2}$, so $(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$.

4. Determine and simplify $(g \circ f)(x)$.

   By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

   - *inside out*: We substitute $f(x) = x^2 - 4x$ in for t in the expression $g(t)$ and get

     $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

     $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. The domain of f is all real numbers, allowing us to focus on finding the range elements compatible with g. Owing to the presence of the square root in the formula $g(t) = 2 - \sqrt{t+3}$ we require $t \geq -3$. Hence, we need $f(x) \geq -3$ or $x^2 - 4x \geq -3$. To solve this inequality we rewrite as $x^2 - 4x + 3 \geq 0$ and use a sign diagram. Letting $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$ and obtain

   $$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & 1 & & 3 & \longrightarrow \end{array}$$

   Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

5. Determine and simplify $(f \circ g)(t)$.

   To find $(f \circ g)(t)$, we find $f(g(t))$.

   - *inside out*: We substitute the expression $g(t) = 2 - \sqrt{t+3}$ in for x in the formula $f(x)$ and get

[5] That is, in general, $g \circ f \neq f \circ g$. This shouldn't be too surprising, because, in general, the order of processes matters: adding eggs to a cake batter then baking the cake batter has a much different outcome than baking the cake batter then adding eggs.

$$\begin{aligned}(f\circ g)(t) &= f(g(t)) = f\left(2-\sqrt{t+3}\right)\\ &= \left(2-\sqrt{t+3}\right)^2 - 4\left(2-\sqrt{t+3}\right)\\ &= 4-4\sqrt{t+3}+\left(\sqrt{t+3}\right)^2-8+4\sqrt{t+3}\\ &= 4+t+3-8\\ &= t-1\end{aligned}$$

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned}(f\circ g)(t) &= f(g(t)) = (g(t))^2 - 4\,(g(t))\\ &= \left(2-\sqrt{t+3}\right)^2 - 4\left(2-\sqrt{t+3}\right)\\ &= t-1 && \text{same algebra as before}\end{aligned}$$

Thus we get $(f\circ g)(t) = t-1$. To find the domain of $f\circ g$, we look for the elements t in the domain of g whose outputs, $g(t)$ are in the domain of f. As mentioned previously, the domain of g is limited by the presence of the square root to $\{t\in\mathbb{R}\,|\,t\geq -3\}$, while the domain of f is all real numbers. Hence, the domain of $f\circ g$ is restricted only by the domain of g and is $\{t\in\mathbb{R}\,|\,t\geq -3\}$ or, using interval notation, $[-3,\infty)$. Note that as with Example 1.5.1 of this section, had we used the simplified formula for $(f\circ g)(t) = t-1$ to determine domain, we would have arrived at the incorrect answer.

6. Determine and simplify $(g\circ h)(s)$.

   To find $(g\circ h)(s)$, we compute $g(h(s))$.

   - *inside out*: We substitute $h(s)$ in for t in the expression $g(t)$ to get

$$\begin{aligned}(g\circ h)(s) &= g(h(s)) = g\left(\frac{2s}{s+1}\right)\\ &= 2-\sqrt{\left(\frac{2s}{s+1}\right)+3}\\ &= 2-\sqrt{\frac{2s}{s+1}+\frac{3(s+1)}{s+1}} && \text{get common denominators}\\ &= 2-\sqrt{\frac{5s+3}{s+1}}\end{aligned}$$

   - *outside in*: We use the formula for $g(t)$ first to get

$$(g\circ h)(s) = g(h(s)) = 2-\sqrt{h(s)+3}$$

$$\begin{aligned}
&= 2-\sqrt{\left(\frac{2s}{s+1}\right)+3} \\
&= 2-\sqrt{\frac{5s+3}{s+1}} \qquad \text{get common denominators as before}
\end{aligned}$$

To find the domain of $g \circ h$, we need the elements in the domain of h so that $h(s)$ is in the domain of g. Owing to the $s+1$ in the denominator of the expression $h(s)$, we require $s \neq -1$. Once again, because of the square root in $g(t) = 2-\sqrt{t+3}$, we need $t \geq -3$ or, in this case $h(s) \geq -3$. We rearrange this inequality:

$$\begin{aligned}
\frac{2s}{s+1} &\geq -3 \\
\frac{2s}{s+1}+3 &\geq 0 \\
\frac{5s+3}{s+1} &\geq 0 \qquad \text{get common denominators as before}
\end{aligned}$$

Defining $r(s) = \frac{5s+3}{s+1}$, we see r is undefined at $s=-1$ (a carry over from the domain restriction of h) and $r(s) = 0$ at $s = -\frac{3}{5}$. Our sign diagram is

$$\begin{array}{ccccc}
(+) & \text{¦} & (-) & 0 & (+) \\
\hline
 & -1 & & -\frac{3}{5} &
\end{array}$$

hence our domain is $(-\infty,-1) \cup \left[-\frac{3}{5},\infty\right)$.

7. Determine and simplify $(h \circ g)(t)$.

   We find $(h \circ g)(t)$ by finding $h(g(t))$.

   - *inside out*: We substitute the expression $g(t)$ for s in the formula $h(s)$

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = h\left(2-\sqrt{t+3}\right) \\
&= \frac{2\left(2-\sqrt{t+3}\right)}{\left(2-\sqrt{t+3}\right)+1} \\
&= \frac{4-2\sqrt{t+3}}{3-\sqrt{t+3}}
\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$(h \circ g)(t) = h(g(t)) = \frac{2\left(g(t)\right)}{\left(g(t)\right)+1}$$

$$
\begin{aligned}
&= \frac{2\left(2-\sqrt{t+3}\right)}{\left(2-\sqrt{t+3}\right)+1} \\
&= \frac{4-2\sqrt{t+3}}{3-\sqrt{t+3}}
\end{aligned}
$$

To find the domain of $h \circ g$, we need the elements of the domain of g so that $g(t)$ is in the domain of h. As we've seen already, for t to be in the domain of g, $t \geq -3$. For s to be in the domain of h, $s \neq -1$, so we require $g(t) \neq -1$. Hence, we solve $g(t) = 2 - \sqrt{t+3} = -1$ with the intent of excluding these solutions. Isolating the radical expression gives $\sqrt{t+3} = 3$ or $t = 6$. Sure enough, we check $g(6) = -1$, so we exclude $t = 6$ from the domain of $h \circ g$. Our final answer is $[-3,6) \cup (6,\infty)$.

8. Determine and simplify $(h \circ h)(s)$.

   To find $(h \circ h)(s)$ we find $h(h(s))$:

   - *inside out*: We substitute the expression $h(s)$ for s in the expression $h(s)$ into h to get

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = h\left(\frac{2s}{s+1}\right) \\
&= \frac{2\left(\frac{2s}{s+1}\right)}{\left(\frac{2s}{s+1}\right)+1} \\
&= \frac{\frac{4s}{s+1}}{\frac{2s}{s+1}+1} \cdot \frac{(s+1)}{(s+1)} \\
&= \frac{\frac{4s}{s+1} \cdot (s+1)}{\left(\frac{2s}{s+1}\right) \cdot (s+1) + 1 \cdot (s+1)} \\
&= \frac{\frac{4s}{\cancel{(s+1)}} \cdot \cancelto{1}{(s+1)}}{\frac{2s}{\cancel{(s+1)}} \cdot \cancelto{1}{(s+1)} + s + 1} \\
&= \frac{4s}{3s+1}
\end{aligned}
$$

   - *outside in*: This approach yields

$$\begin{aligned}
(h \circ h)(s) &= h(h(s)) = \frac{2(h(s))}{h(s)+1} \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{4s}{3s+1} && \text{same algebra as before}
\end{aligned}$$

To find the domain of $h \circ h$, we need to find the elements in the domain of h so that the outputs, $h(s)$ are also in the domain of h. The only domain restriction for h comes from the denominator: $s \neq -1$, so in addition to this, we also need $h(s) \neq -1$. To this end, we solve $h(s) = -1$ and exclude the answers. Solving $\frac{2s}{s+1} = -1$ gives $s = -\frac{1}{3}$. The domain of $h \circ h$ is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

9. Determine and simplify $(h \circ (g \circ f))(x)$.

The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and then compose the function h with the result. We know from number 4 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ with domain $(-\infty, 1] \cup [3, \infty)$. We now proceed as usual.

- *inside out*: We substitute the expression $(g \circ f)(x)$ for s in the expression $h(s)$ first to get

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}$$

- *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}$$

To find the domain of $h \circ (g \circ f)$, we need the domain elements of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, so that $(g \circ f)(x)$ is in the domain of h. As we've seen several times already, the only domain restriction for h is $s \neq -1$, so we set $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3} = -1$ and exclude the solutions. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. We solve $x^2 - 4x - 6 = 0$ using the quadratic formula and obtain $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $2 - \sqrt{x^2 - 4x + 3} = -1$ and also belong to the domain of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, and so must be excluded from our final answer.[6] Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

10. Determine and simplify $((h \circ g) \circ f)(x)$.

The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 7, we have

$$(h \circ g)(t) = \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}$$

with domain $[-3, 6) \cup (6, \infty)$.

- *inside out*: We substitute the expression $f(x)$ for t in the expression $(h \circ g)(t)$ to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)\left(x^2 - 4x\right) \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(t)$ first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}} \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

The domain of f is all real numbers, thus the challenge here in computing the domain of $(h \circ g) \circ f$ is to determine the values $f(x)$ which are in the domain of $h \circ g$, $[-3, 6) \cup (6, \infty)$. At first glance, it appears as if we have two (or three!) inequalities to solve: $-3 \leq f(x) < 6$ and $f(x) > 6$. Alternatively, we could solve $f(x) = x^2 - 4x \geq -3$ and exclude the solutions to $f(x) = x^2 - 4x = 6$ which is not only easier from a procedural point of view, but also easier because we've already done both calculations.

[6] We can approximate $\sqrt{10} \approx 3$ so $2 - \sqrt{10} \approx -1$ and $2 + \sqrt{10} \approx 5$.

In number 4, we solved $x^2 - 4x \geq -3$ and obtained the solution $(-\infty, 1] \cup [3, \infty)$ and in number 9, we solved $x^2 - 4x - 6 = 0$ and obtained $x = 2 \pm \sqrt{10}$. Hence, the domain of $(h \circ g) \circ f$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$. $\square$

As previously mentioned, it should be clear from Example 1.5.3 that, in general, $g \circ f \neq f \circ g$, in other words, function composition is not *commutative*. However, numbers 9 and 10 demonstrate the **associative** property of function composition. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. We summarize the important properties of function composition in the theorem below.

Theorem 1.6. Properties of Function Composition: Suppose f, g, and h are functions.

- **Associative Law of Composition:** $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x) = x$ satisfies: $I \circ f = f \circ I = f$ for all functions, f.

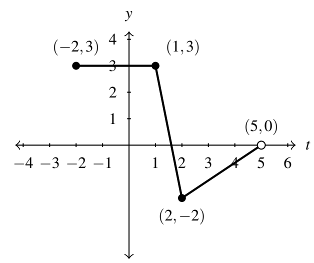
By repeated applications of Definition 1.14, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 1.14. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise 76 in Section 1.3.1. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(I(x)) = f(x)$. This establishes that we have an identity for function composition much in the same way the function $I(x) = 1$ is an identity for function multiplication.

As we know, not all functions are described by formulas, and, moreover, not all functions are described by just *one* formula. The next example applies the concept of function composition to functions represented in various and sundry ways.

Example 1.5.4. Consider the following functions:

- $f(x) = 6x - x^2$
- $g(t) \begin{cases} 2t - 1 & \text{if } -1 \leq t < 3, \\ t^2 & \text{if } t \geq 3. \end{cases}$

- $h = \{(-3,1), (-2,6), (0,-2), (1,5), (3,-1)\}$
- s whose graph is given below:



$y = s(t)$

1. Compute and simplify the following function values:

   (a) $(g \circ f)(2)$ (b) $(h \circ g)(-1)$ (c) $(h \circ s)(-2)$ (d) $(f \circ s)(0)$

2. Find and simplify a formula for $(g \circ f)(x)$.

3. Write $s \circ h$ as a set of ordered pairs.

Solution.

1. (a) Compute and simplify $(g \circ f)(2)$.

   To find $(g \circ f)(2) = g(f(2))$, we first find $f(2) = 6(2) - (2)^2 = 8$. Considering $8 \geq 3$, we use the rule $g(t) = t^2$ so $g(8) = (8)^2 = 64$. Hence, $(g \circ f)(3) = g(f(3)) = g(8) = 64$.

   (b) Compute and simplify $(h \circ g)(-1)$.

   As $(h \circ g)(-1) = h(g(-1))$, we first need $g(-1)$. Given $-1 \leq -1 < 3$, we use the rule $g(t) = 2t - 1$ and find $g(-1) = 2(-1) - 1 = -3$. Next, we need $h(-3)$. As $(-3,1) \in h$, we have that $h(-3) = 1$. Putting this all together, we find $(h \circ g)(-1) = h(g(-1)) = h(-3) = 1$.

   (c) Compute and simplify $(h \circ s)(-2)$.

   To find $(h \circ s)(-2) = h(s(-2))$, we first need $s(-2)$. We see the point $(-2,3)$ is on the graph of s, so $s(-2) = 3$. Next, we see $(3,-1) \in h$, so $h(3) = -1$. Hence, $(h \circ s)(-2) = h(s(-2)) = h(3) = -1$.

   (d) Compute and simplify $(f \circ s)(0)$.

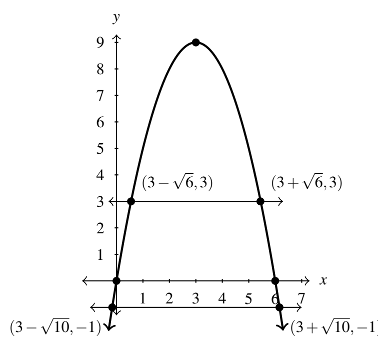
   To find $(f \circ s)(0) = f(s(0))$, we infer from the graph of s that it contains the point $(0,3)$, so $s(0) = 3$. Then $f(3) = 6(3) - (3)^2 = 9$. Thus we have $(f \circ s)(0) = f(s(0)) = f(3) = 9$.

2. Find and simplify a formula for $(g \circ f)(x)$.

   To find a formula for $(g \circ f)(x) = g(f(x))$, we substitute $f(x) = 6x - x^2$ in for t in the formula for $g(t)$:

$$(g \circ f)(x) = g(f(x)) = g(6x - x^2) = \begin{cases} 2(6x - x^2) - 1 & \text{if } -1 \leq 6x - x^2 < 3, \\ (6x - x^2)^2 & \text{if } 6x - x^2 \geq 3. \end{cases}$$

Simplifying each expression, we get $2(6x - x^2) - 1 = -2x^2 + 12x - 1$ for the first piece and $(6x - x^2)^2 = x^4 - 12x^3 + 36x^2$ for the second piece. The real challenge comes in solving the inequalities $-1 \leq 6x - x^2 < 3$ and $6x - x^2 \geq 3$. While we could solve each individually using a sign diagram, a graphical approach works best here. We graph the parabola $y = 6x - x^2$, finding the vertex is $(3, 9)$ with intercepts $(0, 0)$ and $(6, 0)$ along with the horizontal lines $y = -1$ and $y = 3$ below. We determine the intersection points by solving $6x - x^2 = -1$ and $6x - x^2 = 3$. Using the quadratic formula, we find the solutions to each equation are $x = 3 \pm \sqrt{10}$ and $x = 3 \pm \sqrt{6}$, respectively.

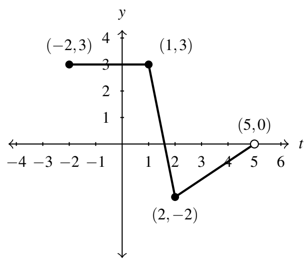


From the graph, we see the parabola $y = 6x - x^2$ is between the lines $y = -1$ and $y = 3$ from $x = 3 - \sqrt{10}$ to $x = 3 - \sqrt{6}$ and again from $x = 3 + \sqrt{6}$ to $x = 3 + \sqrt{10}$. Hence the solution to $-1 \leq 6x - x^2 < 3$ is $[3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}]$. We also note $y = 6x - x^2$ is above the line $y = 3$ for all x between $x = 3 - \sqrt{6}$ and $3 + \sqrt{6}$. Hence, the solution to $6x - x^2 \geq 3$ is $[3 - \sqrt{6}, 3 + \sqrt{6}]$. Hence,

$$(g \circ f)(x) = \begin{cases} -2x^2 + 12x - 1 & \text{if } x \in [3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}], \\ x^4 - 12x^3 + 36x^2 & \text{if } x \in [3 - \sqrt{6}, 3 + \sqrt{6}]. \end{cases}$$

3. Write $s \circ h$ as a set of ordered pairs.

Last but not least, we are tasked with representing $s \circ h$ as a set of ordered pairs. h is described by the discrete set of points, $h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$, so we will find $s \circ h$ point by point. We keep the graph of s handy and construct the table below to help us organize our work.



$y = s(t)$

x	$h(x)$	$s(h(x))$
-3	1	3
-2	6	undefined
0	-2	3
1	5	undefined
3	-1	3

Neither 6 nor 5 are in the domain of s, therefore -2 and 1 are not in the domain of $s \circ h$. Hence, we get $s \circ h = \{(-3,3),(0,3),(3,3)\}$. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates. As with Example 1.5.2, we want to avoid trivial decompositions, which, when it comes to function composition, are those involving the identity function $I(x) = x$ as described in Theorem 1.6.

Example 1.5.5.

1. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

   (a) $F(x) = |3x-1|$ (b) $G(t) = \dfrac{2}{t^2+1}$ (c) $H(s) = \dfrac{\sqrt{s}+1}{\sqrt{s}-1}$

2. For $F(x) = \sqrt{\dfrac{2x-1}{x^2+4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. (a) Write $F(x) = |3x-1|$ as a composition of two or more functions.

   Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 1.14, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the ‘inside’ function and g as being the ‘outside’ function. Looking at $F(x) = |3x-1|$ from an ‘inside versus outside’ perspective, we can think of $3x-1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x-1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, this checks: $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x-1| = F(x)$.

(b) Write $G(t) = \dfrac{2}{t^2+1}$ as a composition of two or more functions.

We attack deconstructing G from an operational approach. Given an input t, the first step is to square t, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of *three* functions: f, g and h. Our first function, f, is the function that squares its input, $f(t) = t^2$. The next function is the function that adds 1 to its input, $g(t) = t+1$. Our last function takes its input and divides it into 2, $h(t) = \frac{2}{t}$. The claim is that $G = h \circ g \circ f$ which checks:

$$(h \circ g \circ f)(t) = h(g(f(t))) = h\left(g\left(t^2\right)\right) = h\left(t^2+1\right) = \frac{2}{t^2+1} = G(x).$$

(c) Write $H(s) = \dfrac{\sqrt{s}+1}{\sqrt{s}-1}$ as a composition of two or more functions.

If we look $H(s) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{s}$ is a simple piece of the larger function. If we define $f(s) = \sqrt{s}$, we have $H(s) = \frac{f(s)+1}{f(s)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(s)$ by looking at what is being done to $f(s)$. We take $g(s) = \frac{s+1}{s-1}$, and check below:

$$(g \circ f)(s) = g(f(s)) = \frac{f(s)+1}{f(s)-1} = \frac{\sqrt{s}+1}{\sqrt{s}-1} = H(s).$$

2. For $F(x) = \sqrt{\dfrac{2x-1}{x^2+4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

To write $F = f \circ \left(\frac{g}{h}\right)$ means

$$F(x) = \sqrt{\frac{2x-1}{x^2+4}} = \left(f \circ \left(\frac{g}{h}\right)\right)(x) = f\left(\left(\frac{g}{h}\right)(x)\right) = f\left(\frac{g(x)}{h(x)}\right).$$

Working from the inside out, we have a rational expression with numerator $g(x)$ and denominator $h(x)$. Looking at the formula for $F(x)$, one choice is $g(x) = 2x-1$ and $h(x) = x^2+4$. Making these identifications, we have

$$F(x) = \sqrt{\frac{2x-1}{x^2+4}} = \sqrt{\frac{g(x)}{h(x)}}.$$

F takes the square root of $\frac{g(x)}{h(x)}$, which tells us that our last function, f, is the function that takes the square root of its input, i.e., $f(x) = \sqrt{x}$. We leave it to the reader to check that, indeed, $F = f \circ \left(\frac{g}{h}\right)$. $\square$

We close this section of a real-world application of function composition.

Example 1.5.6. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to compute the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we conclude $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'intermediary variable' r. $\square$

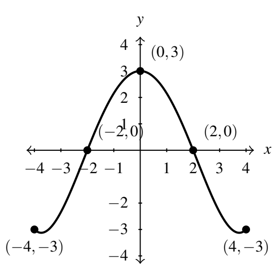
1.5.3 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

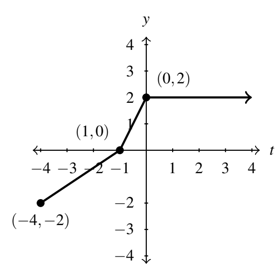
- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1$ and $g(t) = 4-t$
2. $f(x) = x^2$ and $g(t) = -2t+1$
3. $f(x) = x^2 - x$ and $g(t) = 12 - t^2$
4. $f(x) = 2x^3$ and $g(t) = -t^2 - 2t - 3$
5. $f(x) = \sqrt{x+3}$ and $g(t) = 2t - 1$
6. $f(x) = \sqrt{4-x}$ and $g(t) = \sqrt{t+2}$
7. $f(x) = 2x$ and $g(t) = \dfrac{1}{2t+1}$
8. $f(x) = x^2$ and $g(t) = \dfrac{3}{2t-3}$
9. $f(x) = x^2$ and $g(t) = \dfrac{1}{t^2}$
10. $f(x) = x^2 + 1$ and $g(t) = \dfrac{1}{t^2+1}$

Exercises 11 - 20 refer to the functions f and g whose graphs are below.



$y = f(x)$



11. $(f+g)(-4)$
12. $(f+g)(0)$
13. $(f-g)(4)$
14. $(fg)(-4)$
15. $(fg)(-2)$
16. $(fg)(4)$
17. $\left(\frac{f}{g}\right)(0)$
18. $\left(\frac{f}{g}\right)(2)$
19. $\left(\frac{g}{f}\right)(-1)$
20. Find the domains of $f+g$, $f-g$, fg, $\frac{f}{g}$ and $\frac{g}{f}$.

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 42, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

33. $f(x) = 2x+1$ and $g(x) = x-2$
34. $f(x) = 1-4x$ and $g(x) = 2x-1$
35. $f(x) = x^2$ and $g(x) = 3x-1$
36. $f(x) = x^2-x$ and $g(x) = 7x$
37. $f(x) = x^2-4$ and $g(x) = 3x+6$
38. $f(x) = -x^2+x+6$ and $g(x) = x^2-9$
39. $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
40. $f(x) = x-1$ and $g(x) = \frac{1}{x-1}$
41. $f(x) = x$ and $g(x) = \sqrt{x+1}$
42. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 43 - 47, write the given function as a nontrivial decomposition of functions as directed.

43. For $p(z) = 4z - z^3$, find functions f and g so that $p = f - g$.

44. For $p(z) = 4z - z^3$, find functions f and g so that $p = f + g$.

45. For $g(t) = 3t|2t-1|$, find functions f and h so that $g = fh$.

46. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = \dfrac{f}{g}$.

47. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = fg$.

48. Can $f(x) = x$ be decomposed as $f = g - h$ where $g(x) = x + \dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$?

In Exercises 49 - 60, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

49. $f(x) = x^2$, $g(t) = 2t + 1$

50. $f(x) = 4 - x$, $g(t) = 1 - t^2$

51. $f(x) = 4 - 3x$, $g(t) = |t|$

52. $f(x) = |x-1|$, $g(t) = t^2 - 5$

53. $f(x) = 4x + 5$, $g(t) = \sqrt{t}$

54. $f(x) = \sqrt{3-x}$, $g(t) = t^2 + 1$

55. $f(x) = 6 - x - x^2$, $g(t) = t\sqrt{t+10}$

56. $f(x) = \sqrt[3]{x+1}$, $g(t) = 4t^2 - t$

57. $f(x) = \dfrac{3}{1-x}$, $g(t) = \dfrac{4t}{t^2+1}$

58. $f(x) = \dfrac{x}{x+5}$, $g(t) = \dfrac{2}{7-t^2}$

59. $f(x) = \dfrac{2x}{5-x^2}$, $g(t) = \sqrt{4t+1}$

60. $f(x) = \sqrt{2x+5}$, $g(t) = \dfrac{10t}{t^2+1}$

In Exercises 61 - 72, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(t)$
- $(f \circ f)(x)$

61. $f(x) = 2x + 3$, $g(t) = t^2 - 9$

62. $f(x) = x^2 - x + 1$, $g(t) = 3t - 5$

63. $f(x) = x^2 - 4$, $g(t) = |t|$

64. $f(x) = 3x - 5$, $g(t) = \sqrt{t}$

65. $f(x) = |x+1|$, $g(t) = \sqrt{t}$

66. $f(x) = 3 - x^2$, $g(t) = \sqrt{t+1}$

67. $f(x) = |x|$, $g(t) = \sqrt{4-t}$

68. $f(x) = x^2 - x - 1$, $g(t) = \sqrt{t-5}$

69. $f(x) = 3x - 1$, $g(t) = \dfrac{1}{t+3}$

70. $f(x) = \dfrac{3x}{x-1}$, $g(t) = \dfrac{t}{t-3}$

71. $f(x) = \dfrac{x}{2x+1}$, $g(t) = \dfrac{2t+1}{t}$

72. $f(x) = \dfrac{2x}{x^2-4}$, $g(t) = \sqrt{1-t}$

In Exercises 73 - 78, use $f(x) = -2x$, $g(t) = \sqrt{t}$ and $h(s) = |s|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

73. $(h \circ g \circ f)(x)$

74. $(h \circ f \circ g)(t)$

75. $(g \circ f \circ h)(s)$

76. $(g \circ h \circ f)(x)$

77. $(f \circ h \circ g)(t)$

78. $(f \circ g \circ h)(s)$

In Exercises 79 - 91, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Find the following, if it exists.

79. $(f \circ g)(3)$

80. $f(g(-1))$

81. $(f \circ f)(0)$

82. $(f \circ g)(-3)$

83. $(g \circ f)(3)$

84. $g(f(-3))$

85. $(g \circ g)(-2)$

86. $(g \circ f)(-2)$

87. $g(f(g(0)))$

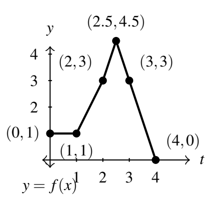
88. $f(f(f(-1)))$

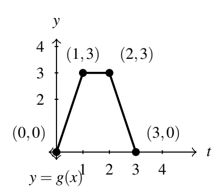
89. $f(f(f(f(f(1)))))$

90. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0)$

91. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 92 - 98, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the following if it exists.





92. $(g \circ f)(1)$

93. $(f \circ g)(3)$

94. $(g \circ f)(2)$

95. $(f \circ g)(0)$

96. $(f \circ f)(4)$

97. $(g \circ g)(1)$

98. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 99 - 108, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

99. $p(x) = (2x+3)^3$

100. $P(x) = \left(x^2 - x + 1\right)^5$

101. $h(t) = \sqrt{2t-1}$

102. $H(t) = |7-3t|$

103. $r(s) = \dfrac{2}{5s+1}$

104. $R(s) = \dfrac{7}{s^2-1}$

105. $q(z) = \dfrac{|z|+1}{|z|-1}$

106. $Q(z) = \dfrac{2z^3+1}{z^3-1}$

107. $v(x) = \dfrac{2x+1}{3-4x}$

108. $w(x) = \dfrac{x^2}{x^4+1}$

109. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

110. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

111. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

112. Let $f(x) = 3x+1$ and let $g(x) = \begin{cases} 2x-1 & \text{if } x \leq 3 \\ 4-x & \text{if } x > 3 \end{cases}$. Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.

113. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

114. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.

    (b) Find and simplify $(R \circ x)(t)$. What does this represent?

    (c) What is the revenue per hour at noon?

SECTION 1.5 EXERCISE ANSWERS A.1.1

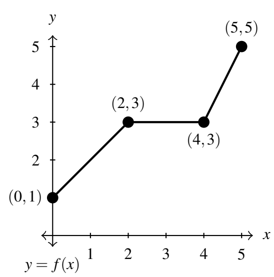
1.6 Transformations

Theorems 1.4, 2.1, 2.2, 3.1, 4.1 and 4.4 all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem 4.4, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx-h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx-h) = (bx-h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx-h)^r) = a(bx-h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$.

In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points' if you will - to track through each transformation.



x	$(x, f(x))$	$f(x)$
0	$(0,1)$	1
2	$(2,3)$	3
4	$(4,3)$	3
5	$(5,5)$	5

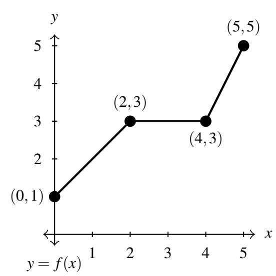
1.6.1 Vertical and Horizontal Shifts

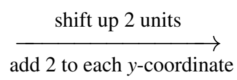
Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

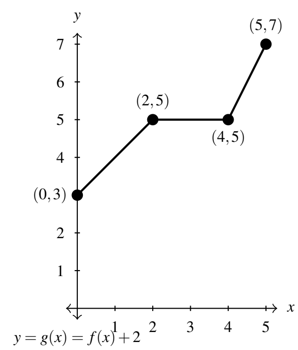
x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0,1)$	1	$1+2=3$	$(0,3)$
2	$(2,3)$	3	$3+2=5$	$(2,5)$
4	$(4,3)$	3	$3+2=5$	$(4,5)$
5	$(5,5)$	5	$5+2=7$	$(5,7)$

In general, if (a,b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a)+2 = b+2$, so the point $(a,b+2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.







You'll note that the domain of f and the domain of g are the same, namely $[0,5]$, but that the range of f is $[1,5]$ while the range of g is $[3,7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1,3]$. In general, we have:

Theorem 1.7. Vertical Shifts. Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem 1.7, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c)+k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem 1.7. In the language of 'inputs' and 'outputs', Theorem 1.7 says adding to the *output* of a function causes the graph to shift *vertically*.

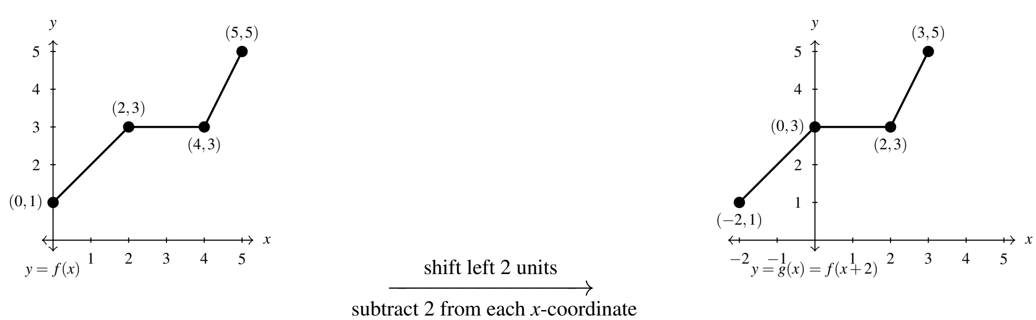
Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x+2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$ which doesn't exist because the domain of f is only $[0,5]$. The same thing happens when we attempt to find $g(5)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0,1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	$(2,3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$g(4) = f(4+2) = f(6) =?$	
5	$(5,5)$	5	$g(5) = f(5+2) = f(7) =?$	

What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $g(-2) = f((-2)+2) = f(0) = 1$ so $(-2,1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x+2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2,1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3,5)$

In summary, the points $(0,1)$, $(2,3)$, $(4,3)$ and $(5,5)$ on the graph of $y = f(x)$ give rise to the points $(-2,1)$, $(0,3)$, $(2,3)$ and $(3,5)$ on the graph of $y = g(x)$, respectively. In general, if (a,b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x+2 = a$ gives $x = a-2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2,b)$ is on the graph of $y = g(x)$. The point $(a-2,b)$ is exactly 2 units to the *left* of the point (a,b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



Note that while the ranges of f and g are the same, the domain of g is $[-2,3]$ whereas the domain of f is

$[0,5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 1.8. Horizontal Shifts. Suppose f is a function and h is a real number.
To graph $F(x) = f(x-h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

To prove Theorem 1.8, we first note the domains of f and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c+h$, due to $F(c+h) = f((c+h)-h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c+h, F(c+h)) = (c+h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c+h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem 1.8 says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems 1.7 and 1.8 present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems 1.7 and 1.8 in the example below.

Example 1.6.1. Use Theorems 1.7 and 1.8 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1,3)$ is on the graph of $y = f(x)$. State a point on the graph of:

   (a) $y = f(x)+5$ (b) $y = f(x+5)$ (c) $f(x-7)+4$

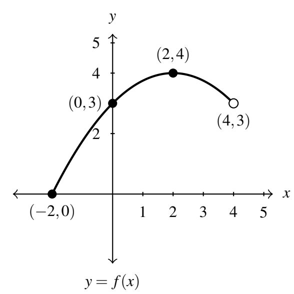
2. Write a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted:

   (a) to the right 4 units. (b) down 2 units.

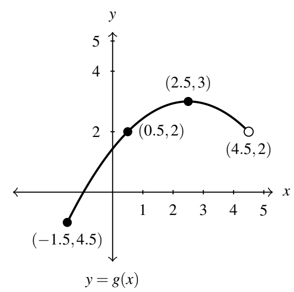
3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \frac{x^{\frac{2}{3}}}{x+2}$.

4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(x-2)$ (b) $F(x) = f(x)+1$ (c) $F(x) = f(x+1)-2$



5. Below is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ from part 4 and vice-versa.



Solution.

1. (a) Suppose $(-1,3)$ is on the graph of $y = f(x)$. State a point on the graph of $y = f(x)+5$.

   To apply Theorem 1.7, we identify $f(x)+5 = f(x)+k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1,3)$ and get $(-1,3+5) = (-1,8)$. To check our answer note that $(-1,3)$ is on the graph of f means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x)+5$, we get $y = f(-1)+5 = 3+5 = 8$. Hence, $(-1,8)$ is on the graph of $f(x)+5$.

   (b) Suppose $(-1,3)$ is on the graph of $y = f(x)$. State a point on the graph of $y = f(x+5)$.

   We note that $f(x+5)$ can be written as $f(x-(-5)) = f(x-h)$ so we apply Theorem 1.8 with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1,3)$ gives $(-1+(-5),3) = (-6,3)$. To check our answer, $(-1,3)$ is on the graph of f, thus $f(-1) = 3$. Substituting $x = -6$ into $y = f(x+5)$ gives $y = f(-6+5) = f(-1) = 3$, proving $(-6,3)$ is on the graph of $y = f(x+5)$.

   (c) Suppose $(-1,3)$ is on the graph of $y = f(x)$. State a point on the graph of $y = f(x-7)+4$.

   Note that the expression $f(x-7)+4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x-7)+4$.

We choose to work from the 'inside' (argument) out and use Theorem 1.8 to first get a point on the graph of $y = f(x-7) = f(x-h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1,3)$ to get $(-1+7,3) = (6,3)$. Hence, $(6,3)$ is a point on the graph of $y = f(x-7)$.

Next, we apply Theorem 1.7 to graph $y = f(x-7)+4$ starting with $y = f(x-7)$. Viewing $f(x-7)+4 = f(x-7)+k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6,3)$ to get $(6,3+4) = (6,7)$.

To check, we note that if we substitute $x = 6$ into $y = f(x-7)+4$, we get $y = f(6-7)+4 = f(-1)+4 = 3+4 = 7$.

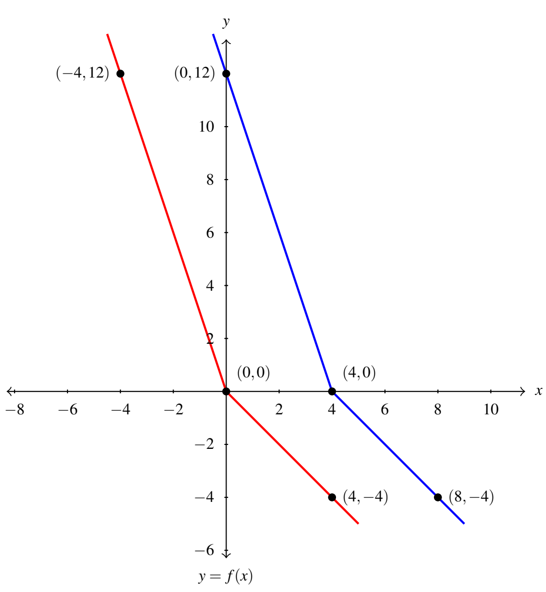
2. Here the independent variable is t instead of x which doesn't affect the geometry in any way as our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (a) Write a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted to the right 4 units.

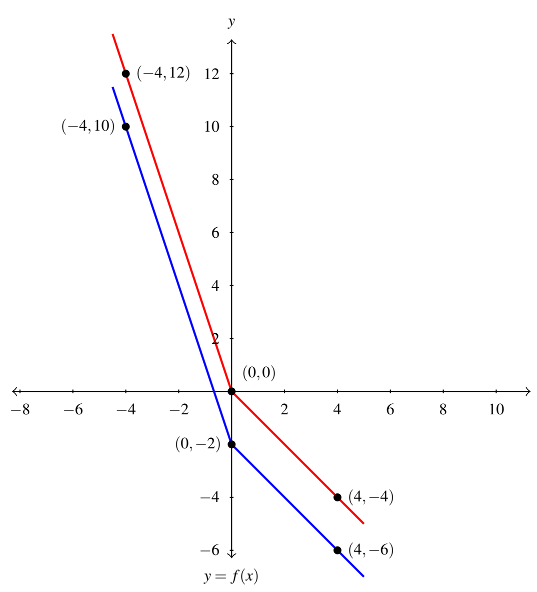
   Per Theorem 1.8, the graph of $g(t) = f(t-4) = |t-4| - 2(t-4) = |t-4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units. Our check is below on the left.

   (b) Write a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted to the down 2 units.

   Per Theorem 1.7, the graph of $g(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units. Our check is below on the right.



$y = |t| - 2t$ and $y = |t-4| - 2t + 8$



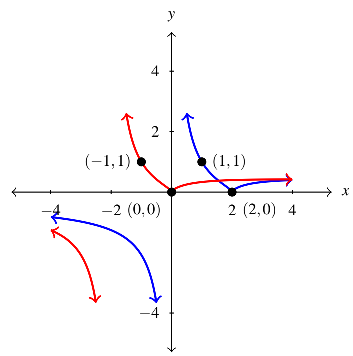
$y = |t| - 2t$ and $y = |t| - 2t - 2$

3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \frac{x^{\frac{2}{3}}}{x+2}$.

   Comparing *formulas*, it appears as if $F(x) = f(x-2)$. We check:

$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

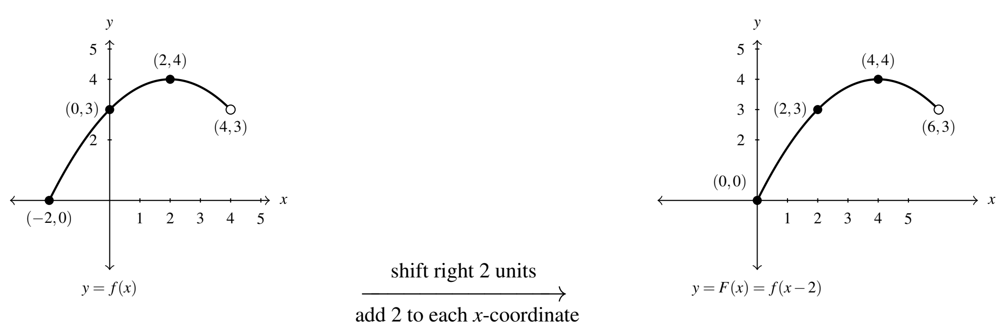
   so, per Theorem 1.8, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. We graph both functions below to confirm our answer.



$y = \dfrac{x^{\frac{2}{3}}}{x+2}$ and $y = \dfrac{(x-2)^{\frac{2}{3}}}{x}$

4. (a) Sketch a graph of $F(x) = f(x-2)$.

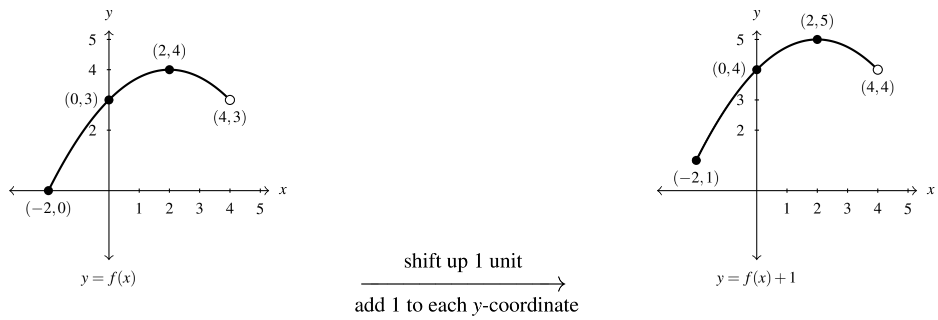
   We recognize $F(x) = f(x-2) = f(x-h)$. With $h = 2$, Theorem 1.8 tells us to add 2 to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units.



   We can check our answer by showing each ordered pair (x,y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0,0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. $(-2,0)$ is on the graph of f, so we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0,0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(b) Sketch a graph of $F(x) = f(x) + 1$.

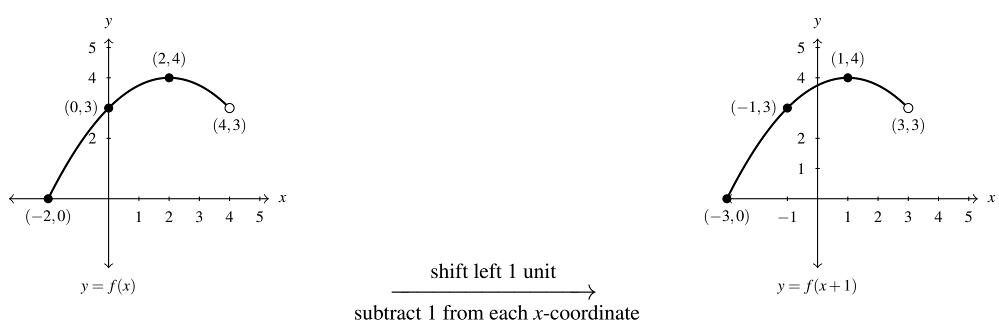
We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem 1.7 tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f.



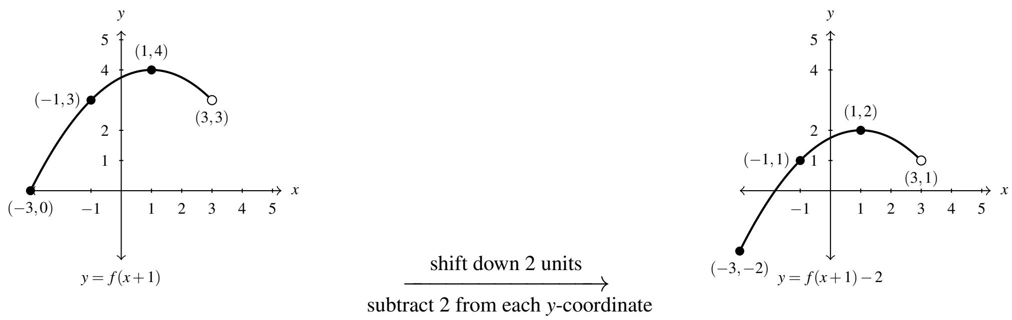
To check our answer, we proceed as above. Starting with the point $(-2,1)$, we substitute $x = -2$ into $y = f(-2)+1$ to get $y = f(-2)+1$. Given $(-2,0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2)+1 = 0+1 = 1$. This proves $(-2,1)$ is on the graph of $y = f(x)+1$. We encourage the reader to check the remaining points in kind.

(c) Sketch a graph of $F(x) = f(x+1) - 2$.

We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x+1) - 2$ from $f(x)$ in stages. First, we apply Theorem 1.8 to graph $y = f(x+1)$ from $y = f(x)$. Rewriting $f(x+1) = f(x-(-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit.



Next, we apply Theorem 1.7 to graph $y = f(x+1) - 2$ starting with the graph of $y = f(x+1)$. Writing $f(x+1) - 2 = f(x+1) + (-2) = f(x+1) + k$, we identify $k = -2$ so Theorem 1.7 instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units.



To check, we start with the point $(-3,-2)$. We find when we substitute $x=-3$ into the equation $y=f(x+1)-2$ we get $y=f(-3+1)-2=f(-2)-2$. Given $(-2,0)$ is on the graph of f, we know $f(-2)=0$, so $y=f(-3+1)-2=f(-2)-2=0-2=-2$. This proves $(-3,-2)$ is on the graph of $y=f(x+1)-2$. We leave the checks of the remaining points to the reader.

5. Write $g(x)$ in terms of $f(x)$ and vice-versa.

   To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems 1.8 and 1.7, $g(x)$ must take the form $g(x)=f(x-h)+k$. Because the horizontal shift is to the *right* 0.5 units, $h=0.5$ and the vertical shift is *down* 1 unit, implies $k=-1$. Hence, we get $g(x)=f(x-0.5)-1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems 1.8 and 1.7 suggest the formula $f(x)=g(x+0.5)+1$. We leave it to the reader to check. □

1.6.2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section 1.1 that to reflect a point (x,y) across the x-axis, we replace y with $-y$. If (x,y) is on the graph of f, then $y=f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y=-f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y=f(-x)$ is the graph of $y=f(x)$ reflected across the y-axis.[1]

[1] We explore impact of symmetry on reflections in Exercise 74.

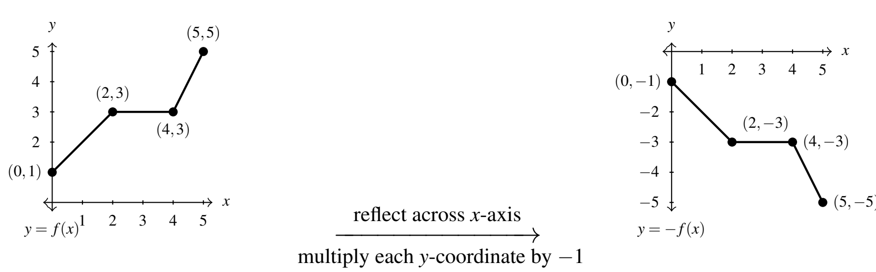
Theorem 1.9. Reflections. Suppose f is a function.
To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the x-axis.
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the y-axis.

The proof of Theorem 1.9 follows in much the same way as the proofs of Theorems 1.7 and 1.8. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see they only difference is the y-coordinates are the exact opposite - indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ because $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

Using the language of inputs and outputs, Theorem 1.9 says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

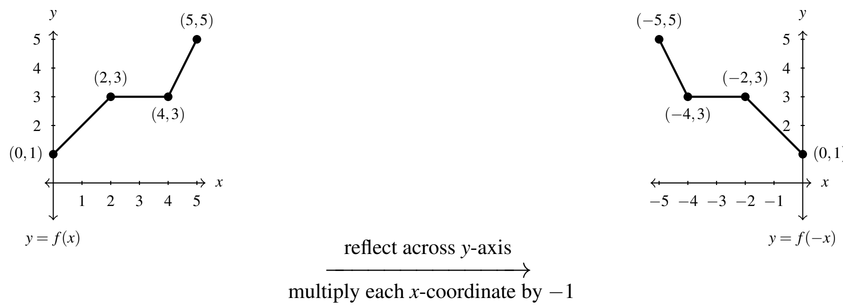
Applying Theorem 1.9 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0,1)$	1	-1	$(0,-1)$
2	$(2,3)$	3	-3	$(2,-3)$
4	$(4,3)$	3	-3	$(4,-3)$
5	$(5,5)$	5	-5	$(5,-5)$



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0,1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2,3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4,3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5,5)$



Example 1.6.2. Use Theorems 1.7, 1.8 and 1.9 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2,-5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

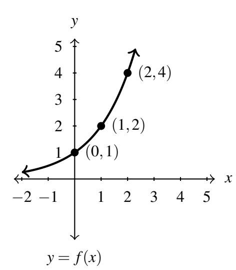
   (a) $y = f(-x)$ (b) $y = -f(x+2)$ (c) $f(8-x)$

2. Write a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

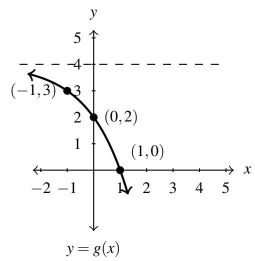
3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

4. Below is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(-x) + 1$ (b) $F(x) = 1 - f(2-x)$



5. Below is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ from part 4 and vice-versa.



NOTE: The x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$ and the line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$.

Solution.

1. (a) Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = f(-x)$.

   To find a point on the graph of $y = f(-x)$, Theorem 1.9 tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

   To check, given $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

   (b) Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = -f(x+2)$.

   To find a point on the graph of $y = -f(x+2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x+2)$.

   First, we find a point on the graph of $y = f(x+2)$. Writing $f(x+2) = f(x-(-2))$, we apply Theorem 1.8 with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2-2, -5) = (0, -5)$.

   Next, we apply Theorem 1.9 to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

   To check, recall $f(2) = -5$, so that when we substitute $x = 0$ into the equation $y = -f(x+2)$, we get $y = -f(0+2) = -f(2) = -(-5) = 5$, as required.

   (c) Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = f(8-x)$.

   Rewriting $f(8-x) = f(-x+8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Both of these transformations affect the x-coordinates of the graph, thus the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

   Recall that $(2, -5)$ is on the graph of f, so we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x+8)$, we need to match up the arguments of $f(-x+8)$ and $f(2)$: $-x+8 = 2$.

To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2,-5)$, shifts the point $(2,-5)$ left 8 units to get the point $(-6,-5)$.

Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6,-5)$ by -1 reflects the point $(-6,-5)$ across the y-axis to $(6,-5)$.

To check we substitute $x = 6$ into $y = f(-x+8)$, and obtain $y = f(-6+8) = f(2) = -5$.

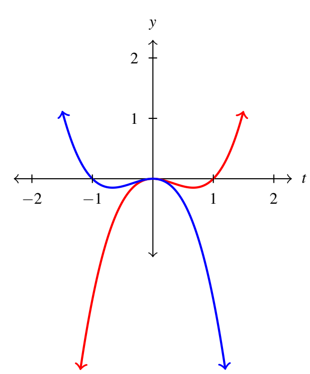
Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem 1.8, this point is on the graph of $y = f(x+8)$.

Next we took a point on the graph of $y = f(x+8)$ and reflected it about the y-axis. Per Theorem 1.9, this put the point on the graph of $y = f(-x+8)$.

In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. Write a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

   In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$. Our confirmation is below.



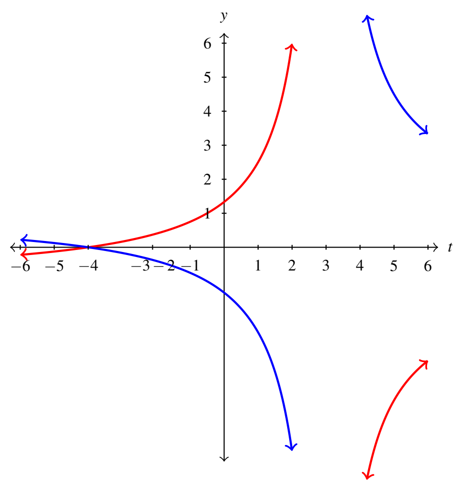
$t = s^3 - s^2$ and $t = -s^3 - s^2$

3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

   Comparing the formulas for $G(t) = \frac{t+4}{t-3}$ and $g(t) = \frac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t-3) = -(3-t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

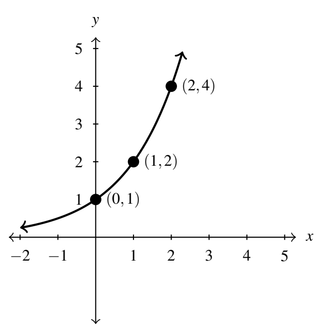
Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis.



$y = \frac{t+4}{3-t}$ and $y = \frac{t+4}{t-3}$

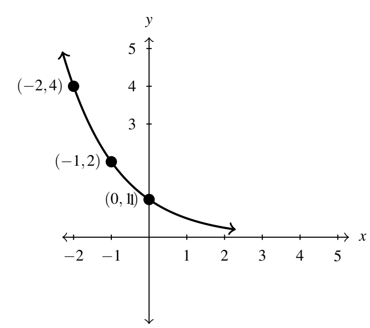
4. (a) Sketch the graph of $F(x) = f(-x) + 1$.

We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem 1.9, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1).



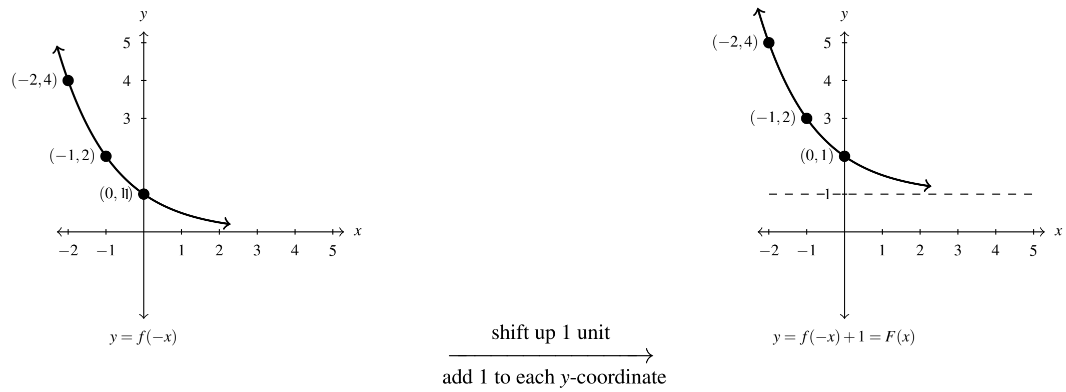
$y = f(x)$

$\xrightarrow[\text{multiply each } x\text{-coordinate by } -1]{\text{reflect across } y\text{-axis}}$



$y = f(-x)$

Next, we use Theorem 1.7 to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$.

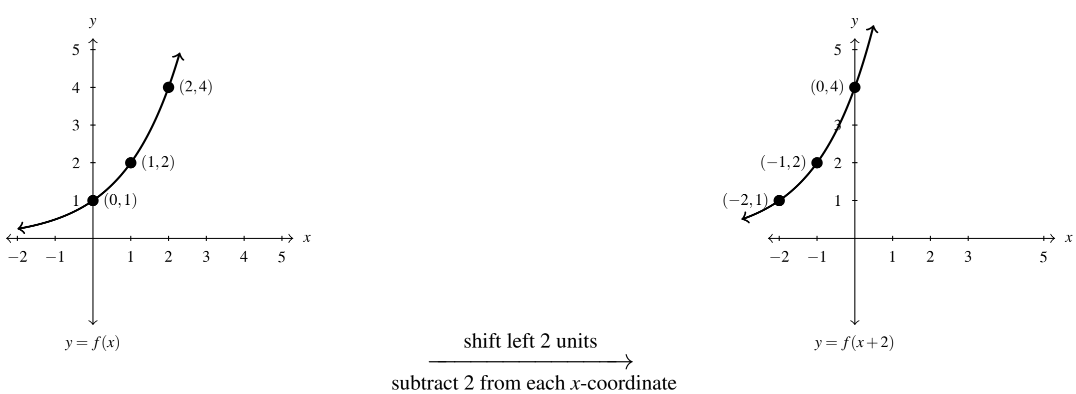


To check our answer, we begin with the point $(0,2)$. Substituting $x = 0$ into $y = f(-x)+1$, we get $y = f(-0)+1 = f(0)+1$. Given the point $(0,1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0)+1 = 1+1 = 2$, so $(0,2)$ is, indeed, on the graph of $y = f(-x)+1$. We leave it to the reader to check the remaining points.

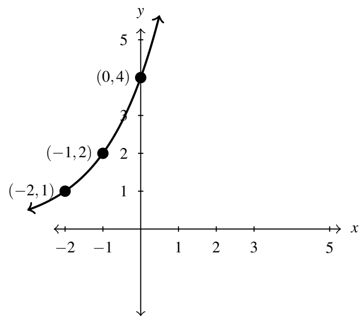
(b) Sketch the graph of $F(x) = 1 - f(2-x)$.

In order to graph $F(x) = 1 - f(2-x)$, we first rewrite as $F(x) = -f(-x+2)+1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem 1.8 to graph $y = f(x+2) = f(x-(-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$.

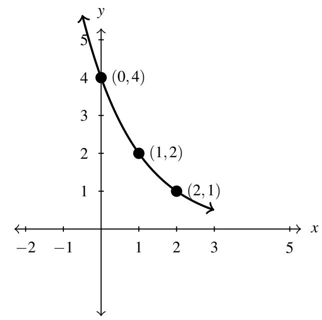


Next, we use Theorem 1.9 to graph $y = f(-x+2)$ starting with the graph of $y = f(x+2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x+2)$ by -1. This reflects the graph of $f(x+2)$ about the y-axis.



$y = f(x+2)$

$\xrightarrow[\text{multiply each } x\text{-coordinate by } -1]{\text{reflect about the } y\text{-axis}}$



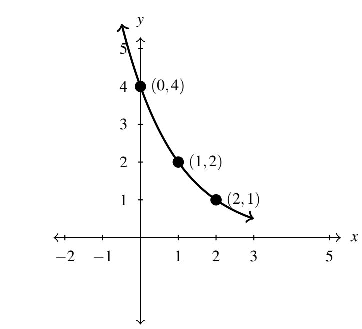
$y = f(-x+2)$

We have the graph of $y = f(-x+2)$ and need to build towards the graph of $y = -f(-x+2)+1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

Once again, we can think algebraically about the problem. We know the point $(0,1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2,1)$ on the graph of $f(-x+2)$. Indeed, when we substitute $x = 2$ into $y = f(-x+2)$, we get $y = f(-2+2) = f(0) = 1$.

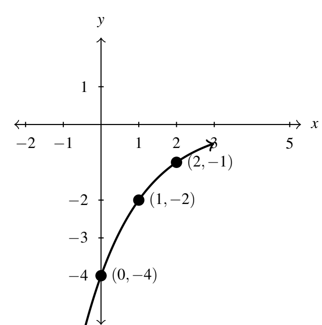
If we substitute $x = 2$ into the formula $y = -f(-x+2)+1$, we get $y = -f(-2+2)+1 = -f(0)+1 = -1(1)+1 = 0$. That is, we first multiply the y-coordinate of $(2,1)$ by -1 and then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed below to obtain the graph of $y = -f(-x+2)$ from $y = f(-x+2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x+2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.



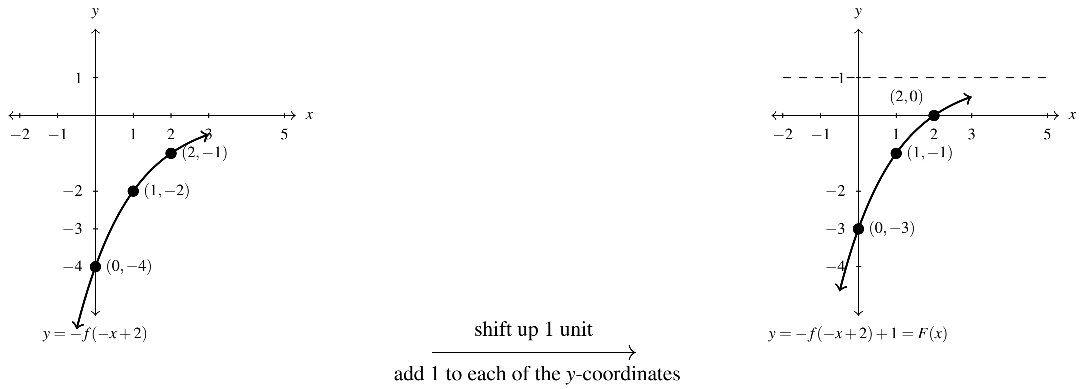
$y = f(-x+2)$

$\xrightarrow[\text{multiply each } y\text{-coordinate by } -1]{\text{reflect about the } x\text{-axis}}$



$y = -f(-x+2)$

Finally, we take care of the vertical shift. Per Theorem 1.7, we graph $y = -f(-x+2)+1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x+2)$. This moves the graph up one unit, including the horizontal asymptote: $y = 0+1 = 1$.



To check, we begin with the point $(2,0)$. Substituting $x = 2$ into $y = 1 - f(2-x)$, we obtain $y = 1 - f(2-2) = 1 - f(0)$. Given $(0,1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2-2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2,0)$ is on the graph of $y = 1 - f(2-x)$, and we recommend the reader check the remaining points.

5. Write $g(x)$ in terms of $f(x)$ and vica-versa.

   With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x) + 4$, the entire graph is *above* the line $y = 4$. Due to the fact that the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0,1)$ on the graph of f must correspond to the point $(-1,3)$ on the graph of g, as these are the points closest to their respective asymptotes.

   Likewise, the points $(1,2)$ and $(2,4)$ on the graph of f must correspond to $(0,2)$ and $(1,0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x+1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

   One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[2] In the same way, when we think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a,b) from its reflection across the x-axis, $(a,-b)$, we simply reflect across the x-axis again: $(a,-(-b)) = (a,b)$. Per Theorem 1.9, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[3]

Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem 1.8, we accomplish this by graphing $y = -g(x-1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x-1) + 4$ and show it matches the graph of $y = f(x)$. $\square$

Some remarks about Example 1.6.2 are in order. In number 1c above, to find a point on the graph of $y = f(-x+8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x+8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x+8 = 2$, which meant we did the exact opposite steps in the opposite order.[4] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in 4b above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

[2]We'll have more to say about this sort of thing in Section 5.1.

[3]To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem 1.9 tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

[4]Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

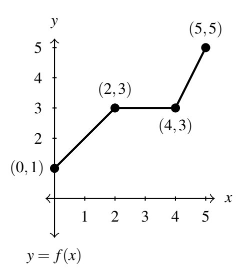
Last but not least, in number 5, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

1.6.3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[5] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

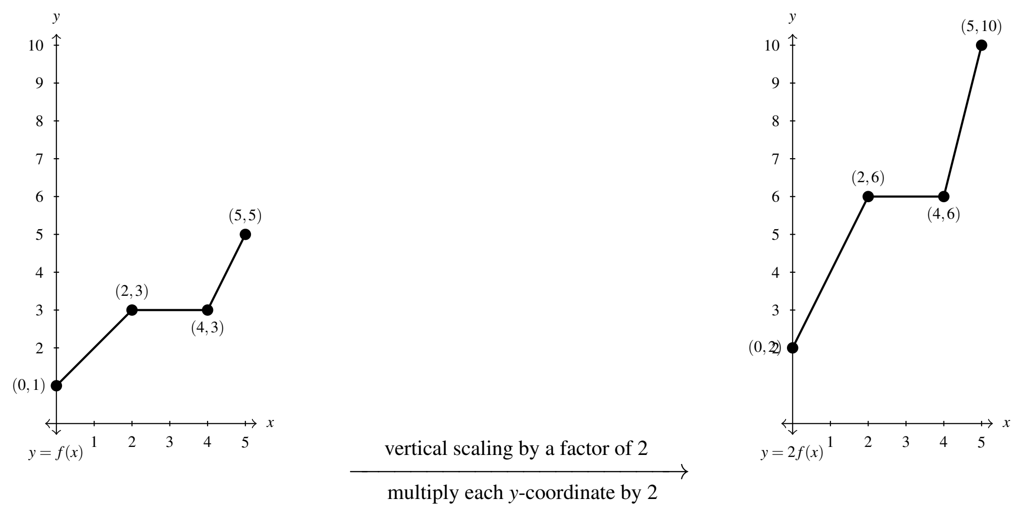
Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.



x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0,1)$	1	2	$(0,2)$
2	$(2,3)$	3	6	$(2,6)$
4	$(4,3)$	3	6	$(4,6)$
5	$(5,5)$	5	10	$(5,10)$

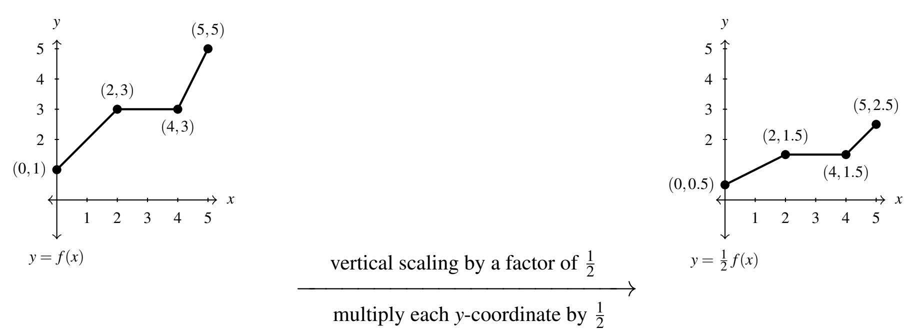
[5] Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

Graphing, we get:



In general, if (a,b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a,2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[6] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[7] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

[6]Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.
[7]Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

Theorem 1.10. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number. To graph $F(x) = a\,f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a] expansion, dilation
[b] compression, contraction

The proof of Theorem 1.10 mimics the proofs of Theorems 1.7 and 1.9. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = a\,f(x)$ is $(c, F(c)) = (c, a\,f(c))$. Comparing the points $(c, f(c))$ and $(c, a\,f(c))$ proves the theorem.

A few remarks about Theorem 1.10 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem 1.10 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

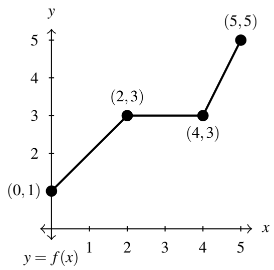
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 1.8, as seen in the table on the left below.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2,3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2\cdot 1) = f(2) = 3$, so that $(1,3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0,1)$	1	$f(2\cdot 0) = f(0) = 1$	$(0,1)$
2	$(2,3)$	3	$f(2\cdot 2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$f(2\cdot 4) = f(8) = ?$	
5	$(5,5)$	5	$f(2\cdot 5) = f(10) = ?$	

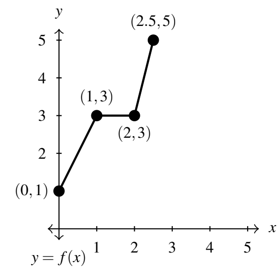
x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2\cdot 0) = f(0) = 1$	$(0,0)$
1	2	$g(1) = f(2\cdot 1) = f(2) = 3$	$(1,3)$
2	4	$g(2) = f(2\cdot 2) = f(4) = 3$	$(2,3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2\cdot\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a,b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2\cdot\frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[8] by a factor of $\frac{1}{2}$.

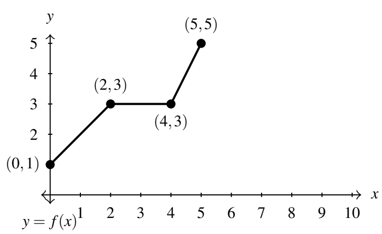


horizontal scaling by a factor of $\frac{1}{2}$

multiply each x-coordinate by $\frac{1}{2}$

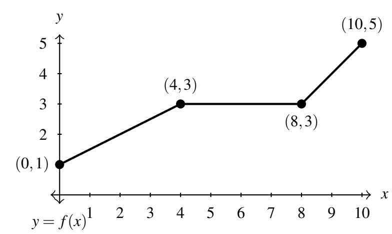


If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[9] by a factor of 2, as demonstrated below.



horizontal scaling by a factor of 2

multiply each x-coordinate by 2



We have the following theorem.

Theorem 1.11. Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number. To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[a] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[b] by a factor of b.

[a] expansion, dilation
[b] compression, contraction

The proof of Theorem 1.11 follows closely the spirit of the proof of Theorems 1.8 and 1.9. If c is an element of the domain of f, then the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ due to the fact that

[8] Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.
[9] Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

$F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem 1.11 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 1.6.3. Use Theorems 1.7, 1.8, 1.9, 1.10 and 1.11 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1,4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

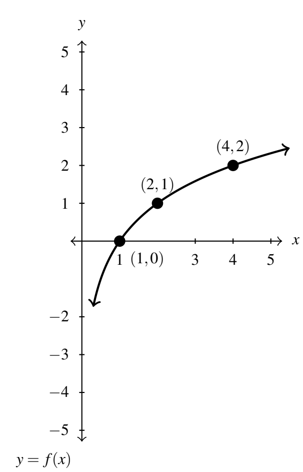
   (a) $y = 3f(x-2)$ (b) $y = f\left(-\frac{1}{2}x\right)$ (c) $f(2x-3)+1$

2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

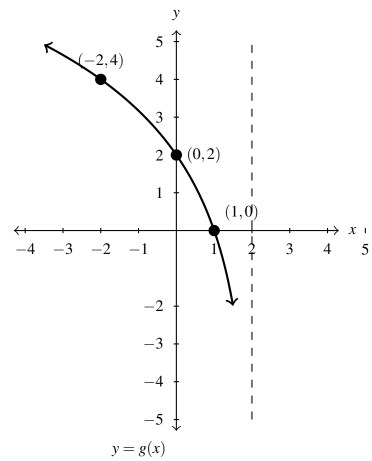
3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

4. Below is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = \dfrac{1-f(x)}{2}$ (b) $F(x) = f\left(\dfrac{1-x}{2}\right)$



5. Below is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ from part 4 and vice-versa.



NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$ and the line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$.

Solution.

1. (a) Suppose $(-1,4)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = 3f(x-2)$.

   As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem 1.8, this shifts all of the points on the graph of $y = f(x)$ right 2 units. Hence, the point $(-1,4)$ on the graph of $y = f(x)$ moves to the point $(-1+2,4) = (1,4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-2)$, we apply Theorem 1.10 with $a = 3$ to the point $(1,4)$ on the graph of $y = f(x-2)$ to get the point $(1,3(4)) = (1,12)$ on the graph of $y = 3f(x-2)$.

   To check, we note that given $(-1,4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

   (b) Suppose $(-1,4)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = f\left(-\frac{1}{2}x\right)$.

   The formula $y = f\left(-\frac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\frac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

   If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\frac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\frac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

   If we think of $-2 = (-1)(2)$, then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2, while multiplying by the '-1' reflects the point across the y-axis.

Applying the horizontal stretch first, we use Theorem 1.11 and start with the point $(-1,4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\frac{1}{2}x\right)$: $(-1(2),4) = (-2,4)$.

Next, we take care of the reflection about the y-axis, using Theorem 1.9 . Starting with $(-2,4)$ on the graph of $y = f\left(\frac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\frac{1}{2}(-x)\right) = f\left(-\frac{1}{2}x\right)$: $((-1)(-2),4) = (2,4)$.

To check, note when $x = 2$ is substituted into $y = f\left(-\frac{1}{2}x\right)$, we get $y = f\left(-\frac{1}{2}(2)\right) = f(-1) = 4$.

Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

Proceeding this way, we start with the point $(-1,4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1),4) = (1,4)$ on the graph of $y = f(-x)$.

Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1),4) = (2,4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$.

In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

(c) Suppose $(-1,4)$ is on the graph of $y = f(x)$. Find a point on the graph of $y = f(2x-3)+1$.

The formula $f(2x-3)+1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Given $f(-1) = 4$, we set $2x-3 = -1$ and solve for x. Our first step is to add 3 to both sides: $2x = (-1)+3 = 2$. Because we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1,4)$ is moved to the point $(2,4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2,4)$ to $(1,4)$.

Hence, the algebra suggests we use Theorem 1.8 first and follow it up with Theorem 1.11. Starting with $(-1,4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1+3,4) = (2,4)$ on the graph of $y = f(x-3)$.

Next, we start with the point $(2,4)$ on the graph of $y = f(x-3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2},4\right) = (1,4)$ on the graph of $y = f(2x-3)$.

Last, but not least, we take care of the vertical shift using Theorem 1.7. Starting with the point $(1,4)$ on the graph of $y = f(2x-3)$, we add 1 to the y-coordinate to get the point $(1,4+1) = (1,5)$ on the graph of $y = f(2x-3)+1$.

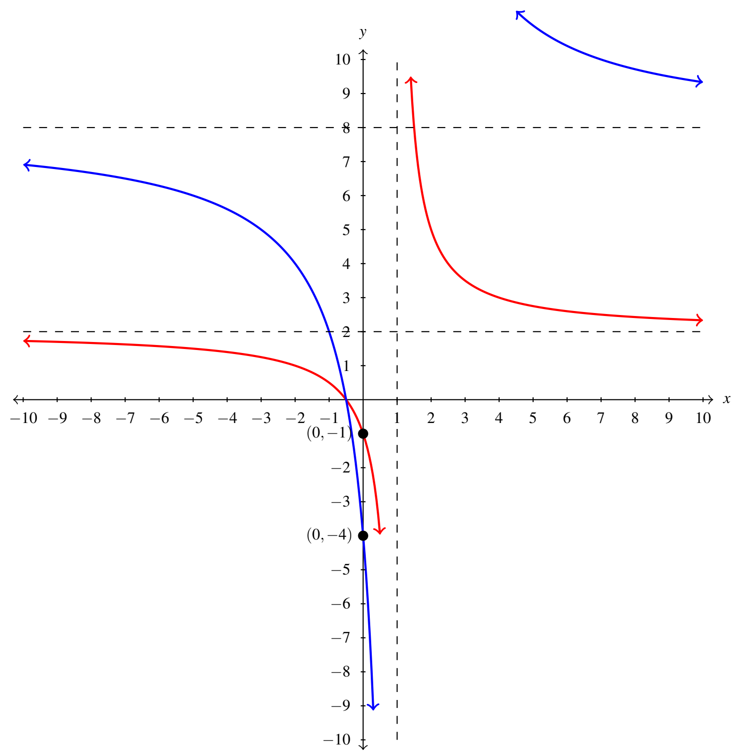
To check, we substitute $x = 1$ into the formula $y = f(2x-3)+1$ and get $y = f(2(1)-3)+1 = f(-1)+1 = 4+1 = 5$, as required.

2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

   To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem 1.10 with $a = 4$ to get

$$G(t) = 4g(t) = 4\left[\frac{2t+1}{t-1}\right] = \frac{4(2t+1)}{t-1} = \frac{8t+4}{t-1}.$$

   We check our answer graphically.



$y = g(t) = \frac{2t+1}{t-1}$ and $y = 4g(t) = \dfrac{8t+4}{t-1}$

3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

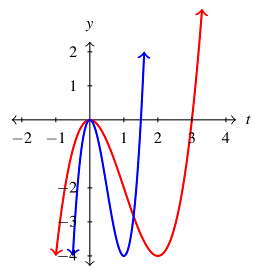
   When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

   We also note that the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, therefore the change is not the result of a vertical scaling (again, why not?)

Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of $s^2 : 3b^2 = 3(2)^2 = 12$.

Hence, we predict the graph of $y = H(s) = (2s)^3 - 3(2s)^2$ to be the same as $y = h(s) = s^3 - 3s^2$ except horizontally compressed by a factor of 2.



$y = h(s) = s^3 - 3s^2$ and $y = H(s) = 8s^3 - 12s^2$

4. (a) Sketch the graph of $F(x) = \frac{1-f(x)}{2}$.

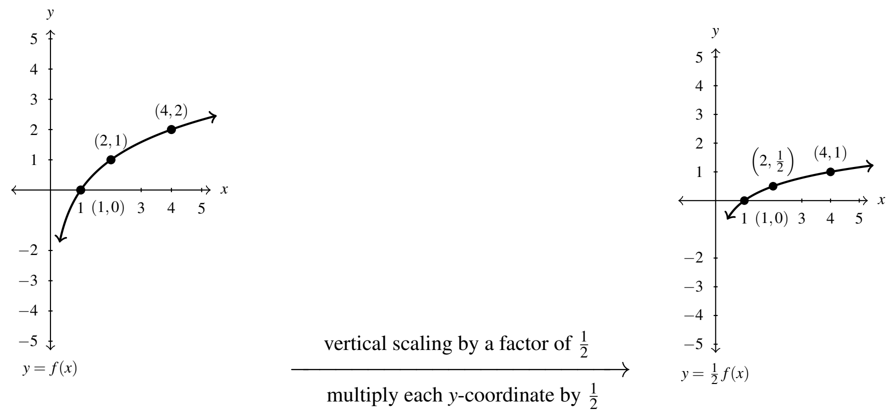
We first rewrite the expression for $F(x) = \frac{1-f(x)}{2} = -\frac{1}{2}f(x) + \frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

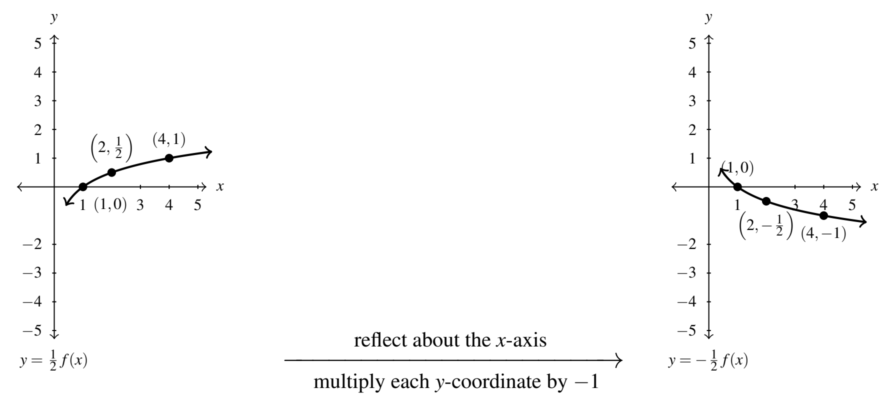
As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1,0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the reflection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

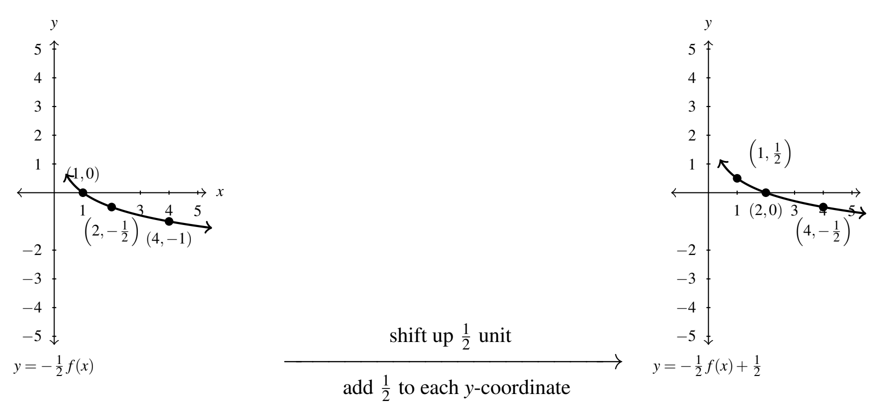
Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem 1.10 to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$.



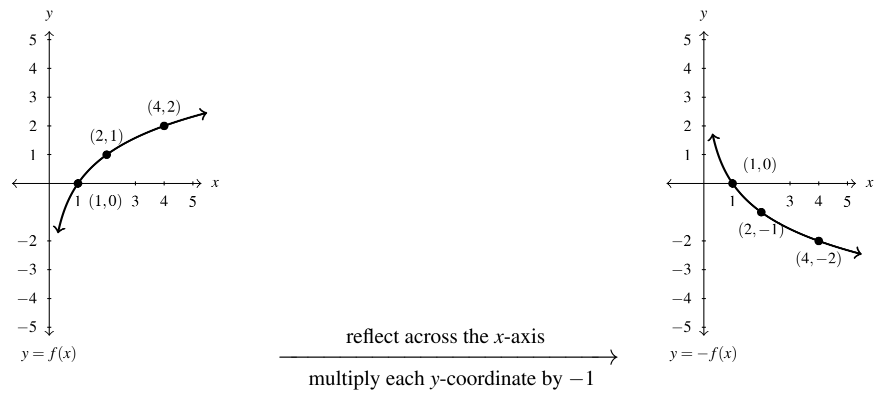
Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1:



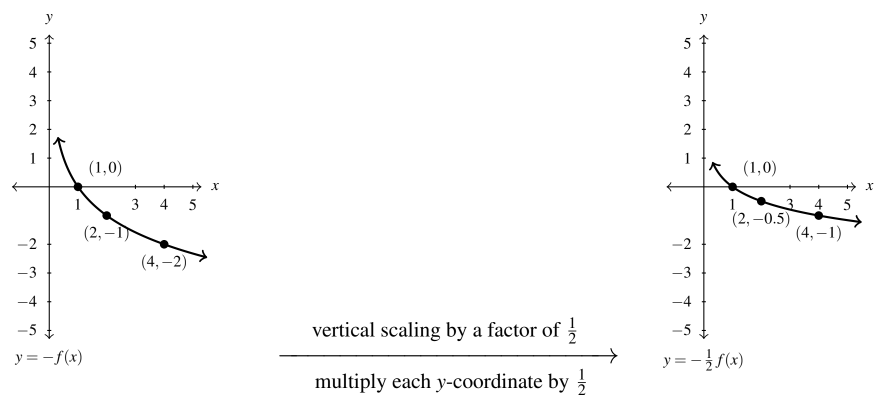
Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.



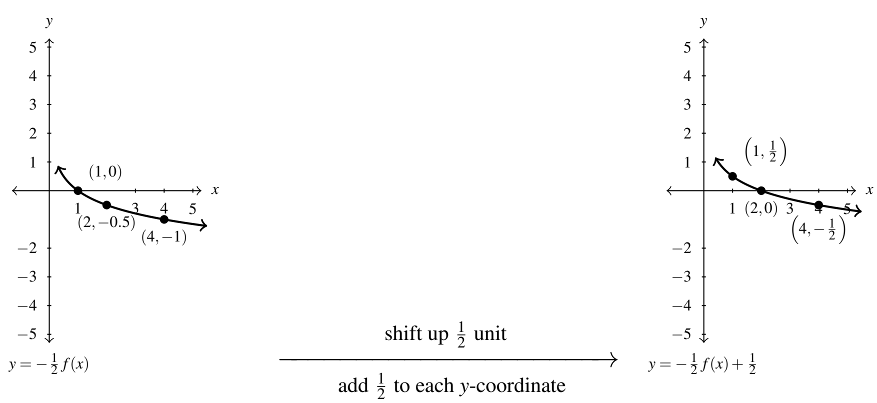
Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first:



Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$:



Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before:



We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1-f(x)}{2}$ to get $y = \frac{1-f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

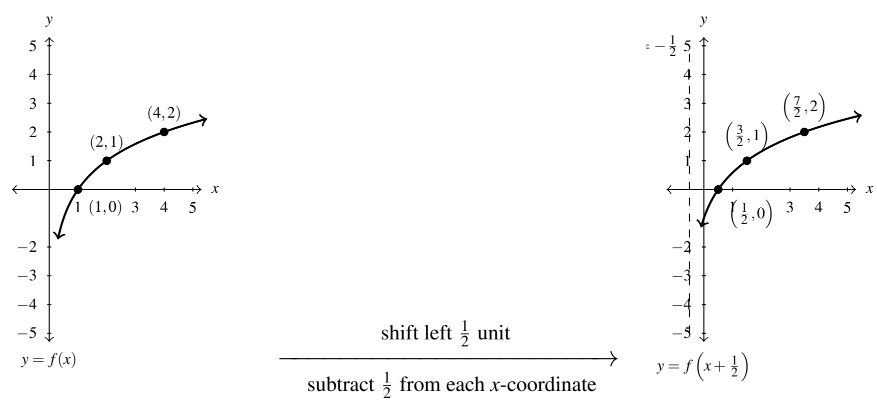
Note that in the preceding example, because none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

(b) Sketch the graph of $F(x) = f\left(\frac{1-x}{2}\right)$.

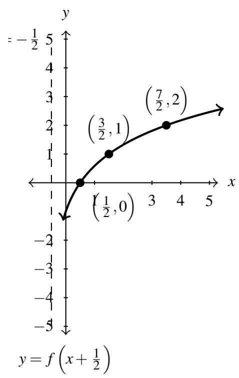
As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$.

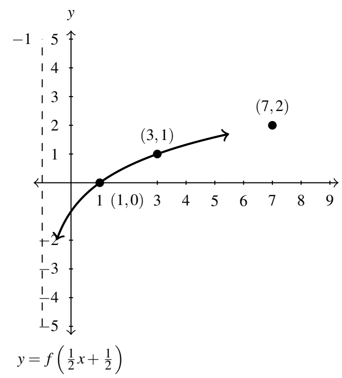


Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$.

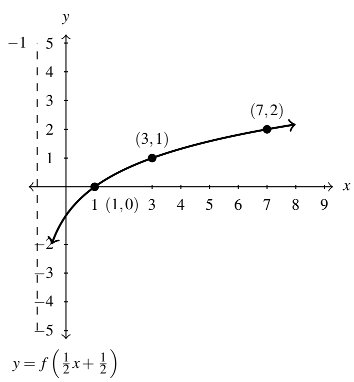


horizontal scaling by a factor of 2

multiply each x-coordinate by 2

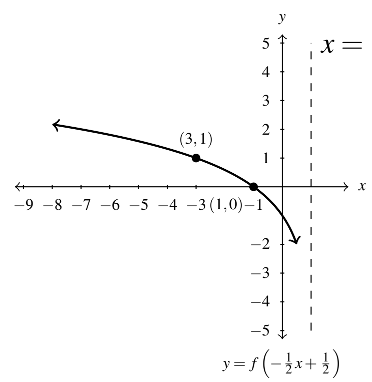


Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$.



reflect about the y-axis

multiply each x-coordinate by -1



To check our answer, we begin with the point $(-1,0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1,0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. Write $g(x)$ in terms of $f(x)$ and vice-versa.

   To write $g(x)$ in terms of $f(x)$, we assume we can find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm a f(\pm bx - h) + k$.

The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

We always shift first and then reflect, so we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x+2)+k$.

Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x+2)$. We leave it to the reader to check the graph of $y = 2f(-x+2)$ matches the graph of $y = g(x)$.

To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

To get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, thus our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

Next, we need to undo the reflection about the y-axis. If the point (a,b) is reflected about the y-axis, we obtain the point $(-a,b)$. To return to the point (a,b), we reflect $(-a,b)$ across the y-axis again: $(-(-a),b) = (a,b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

Our last step is to undo a horizontal shift to the left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$.[10]

We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x+2)$ matches the graph of $y = f(x)$. □

1.6.4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx-h)+k$ without the need of using Theorems 1.7, 1.8, 1.9, 1.10 and 1.11 individually.

Theorem 1.12 is the ultimate generalization of Theorems 1.4, 2.1, 2.2, 3.1, 4.1 and 4.4. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx-h)+k$. Each of these transformations can ultimately be traced back to composing f with linear functions,[11] this fact is verified by showing compositions of linear functions results in a linear function.[12]

[10] To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem 1.8, the graph of $F(x-2) = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$ is the same as the graph of F but shifted 2 units to the right.

[11] See the remarks at the beginning of the section.

[12] See Exercise 72.

Theorem 1.12. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx-h)+k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.

   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.

4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.

   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

Theorem 1.12 can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx-h)$ and $f(c)$, we solve $bx-h=c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + k = af(c+h-h) = af(c)+k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

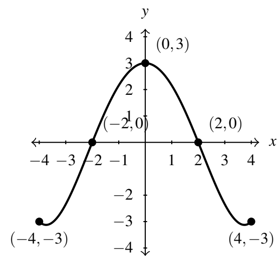
A less precise, but more intuitive way to paraphrase Theorem 1.12 is to think of the quantity $bx-h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a, then add k to find the corresponding y-coordinates on the graph of g.

It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem 1.12 is only one approach. There are various algorithm that can be used. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

We make good use of Theorem 1.12 in the following example.

Example 1.6.4. Below is the complete graph of $y = f(x)$. Use Theorem 1.12 to graph $g(x) = \dfrac{4-3f(1-2x)}{2}$.



$y = f(x)$

Solution. We use Theorem 1.12 to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem 1.12, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal each of the x-coordinates of the key points and solve.

For example, solving $-2x+1=-4$, we first subtract 1 to get $-2x=-5$ then divide by -2 to get $x=\frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in a table below on the left.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x=\frac{5}{2}$, and using the fact that $f(-4)=-3$, we get

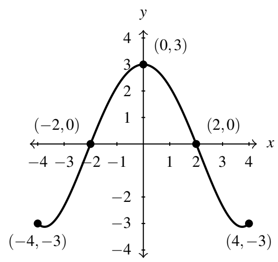
$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below on the right.

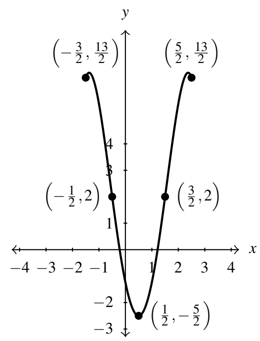
$(c, f(c))$	c	$-2x+1=c$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives



$y = f(x)$



$y = f(x)$

$\square$

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example 1.6.4. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

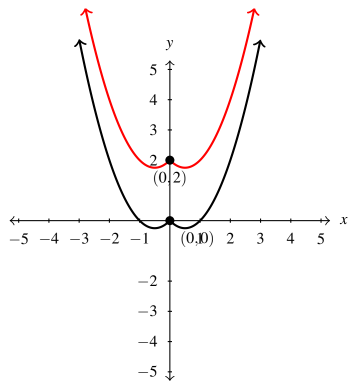
Example 1.6.5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f - which is also the result of applying the second transformation to g_1, and so on.[13]

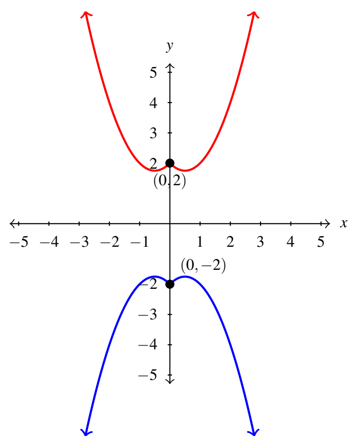
1. Per Theorem 1.7, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$.

[13] So, we can think of $g_0 = f$ and $g_6 = g$.



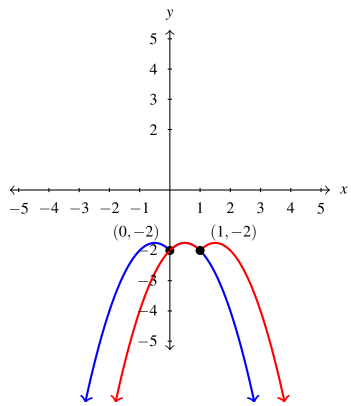
$y = f(x)$ and $y = g_1(x) = f(x) + 2$

2. Per Theorem 1.9, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$.



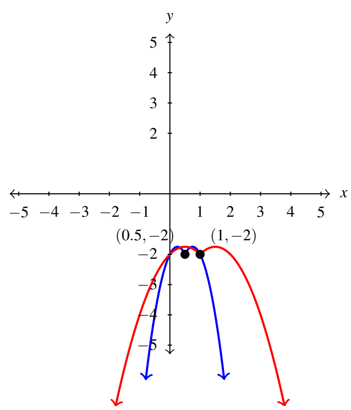
$y = g_1(x)$ and $y = g_2(x) = -g_1(x)$

3. Per Theorem 1.8, $g_3(x) = g_2(x-1) = -(x-1)^2 + |x-1| - 2$.



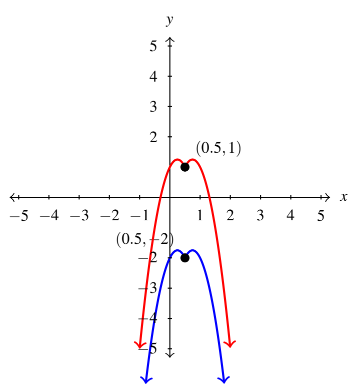
$y = g_2(x)$ and $y = g_3(x) = g_2(x-1)$

4. Per Theorem 1.11, $g_4(x) = g_3(2x) = -(2x-1)^2 + |2x-1| - 2$.



$y = g_3(x)$ and $y = g_4(x) = g_3(2x)$

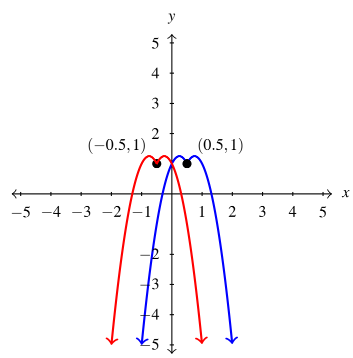
5. Per Theorem 1.7, $g_5(x) = g_4(x) + 3 = -(2x-1)^2 + |2x-1| - 2 + 3 = -(2x-1)^2 + |2x-1| + 1$.



$y = g_4(x)$ and $y = g_5(x) = g_4(x) + 3$

6. Per Theorem 1.9, $g_6(x) = g_5(-x)$:

$$\begin{array}{rcl} g_6(x) & = & g_5(-x) \\ & = & -(2(-x)-1)^2 + |2(-x)-1| + 1 \\ & = & -(-2x-1)^2 + |-2x-1| + 1 \\ & = & -[(-1)(2x+1)]^2 + |[(-1)(2x+1)| + 1 \\ & = & -(-1)^2(2x+1)^2 + |-1||2x+1| + 1 \\ & = & -(2x+1)^2 + |2x+1| + 1 \end{array}$$



$y = g_5(x)$ and $y = g_6(x) = g_5(-x)$

Hence, $g(x) = g_6(x) = -(2x+1)^2 + |2x+1| + 1$. □

It is instructive to show that the expression $g(x)$ in Example 1.6.4 can be written as $g(x) = a\,f(bx-h)+k$.

One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example 1.6.4, but never substitute the formula for $f(x)$ as follows:

1. Per Theorem 1.7, $g_1(x) = f(x) + 2$.

2. Per Theorem 1.9, $g_2(x) = -g_1(x) = -[f(x)+2] = -f(x) - 2$.

3. Per Theorem 1.8, $g_3(x) = g_2(x-1) = -f(x-1) - 2$.

4. Per Theorem 1.11, $g_4(x) = g_3(2x) = -f(2x-1) - 2$.

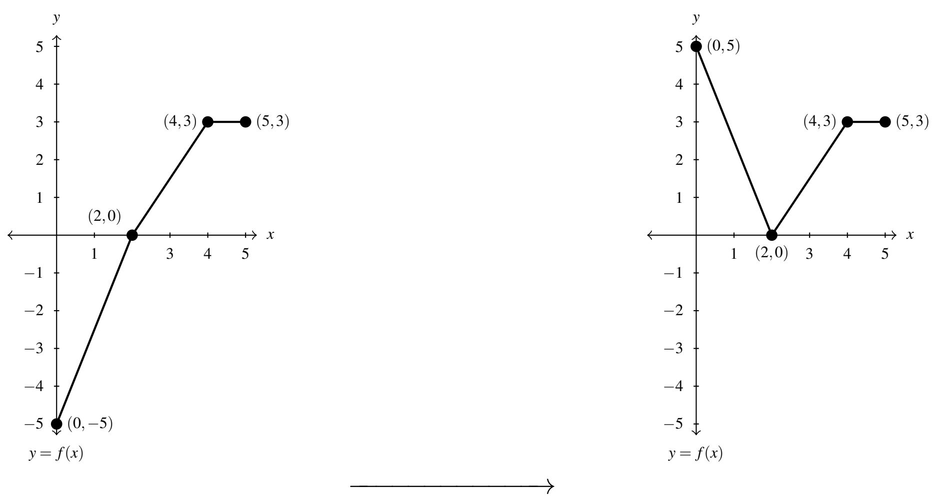
5. Per Theorem 1.7, $g_5(x) = g_4(x) + 3 = -f(2x-1) - 2 + 3 = -f(2x-1) + 1$.

6. Per Theorem 1.9, $g_6(x) = g_5(-x) = -f(2(-x)-1) + 1 = -f(-2x-1) + 1$.

Hence $g(x) = -f(-2x-1)+1$. Note we can show f is even,[14] so $f(-2x-1) = f(-(2x+1)) = f(2x+1)$ and obtain $g(x) = -f(2x+1)+1$.

At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of composing given functions with linear functions. Not all transformations, not even all rigid transformations, fall into these categories.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ below.



In Exercise 76, we explore a non-linear transformation and revisit the pair of functions f and g then.

1.6.5 Exercises

Suppose $(2,-3)$ is on the graph of $y = f(x)$. In Exercises 1 - 18, use Theorem 1.12 to find a point on the graph of the given transformed function.

1. $y = f(x)+3$
2. $y = f(x+3)$
3. $y = f(x)-1$
4. $y = f(x-1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x-3)+1$
10. $y = 2f(x+1)$
11. $y = 10-f(x)$
12. $y = 3f(2x)-1$

[14] Recall this means $f(-x) = f(x)$.

13. $y = \frac{1}{2}f(4-x)$

14. $y = 5f(2x+1)+3$

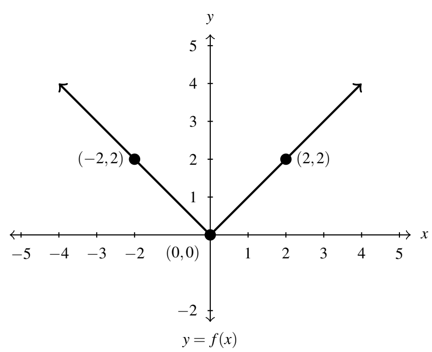
15. $y = 2f(1-x)-1$

16. $y = f\left(\dfrac{7-2x}{4}\right)$

17. $y = \dfrac{f(3x)-1}{2}$

18. $y = \dfrac{4-f(3x-1)}{7}$

The complete graph of $y = f(x)$ is given below. In Exercises 19 - 27, use it and Theorem 1.12 to graph the given transformed function.



19. $y = f(x)+1$

20. $y = f(x)-2$

21. $y = f(x+1)$

22. $y = f(x-2)$

23. $y = 2f(x)$

24. $y = f(2x)$

25. $y = 2-f(x)$

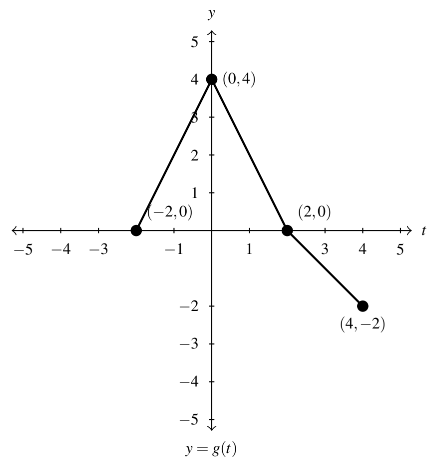
26. $y = f(2-x)$

27. $y = 2-f(2-x)$

28. Some of the answers to Exercises 19 - 27 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

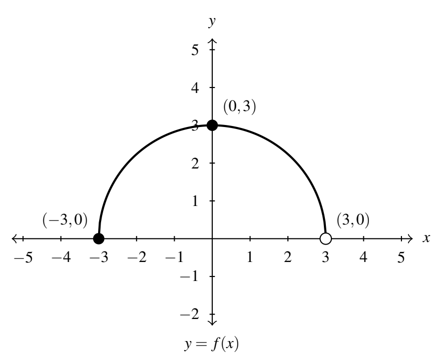
29. The function f used in Exercises 19 - 27 should look familiar. What is $f(x)$? How does this this explain some of the duplication in the answers to Exercises 19 - 27 mentioned in Exercise 28?

The complete graph of $y = g(t)$ is given below. In Exercises 30 - 38, use it and Theorem 1.12 to graph the given transformed function.



30. $y = g(t) - 1$
31. $y = g(t+1)$
32. $y = \frac{1}{2}g(t)$
33. $y = g(2t)$
34. $y = -g(t)$
35. $y = g(-t)$
36. $y = g(t+1) - 1$
37. $y = 1 - g(t)$
38. $y = \frac{1}{2}g(t+1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 39 - 50, use it and Theorem 1.12 to graph the given transformed function.



39. $g(x) = f(x) + 3$
40. $h(x) = f(x) - \frac{1}{2}$
41. $j(x) = f\left(x - \frac{2}{3}\right)$
42. $a(x) = f(x+4)$
43. $b(x) = f(x+1) - 1$
44. $c(x) = \frac{3}{5}f(x)$

45. $d(x) = -2f(x)$

46. $k(x) = f\left(\frac{2}{3}x\right)$

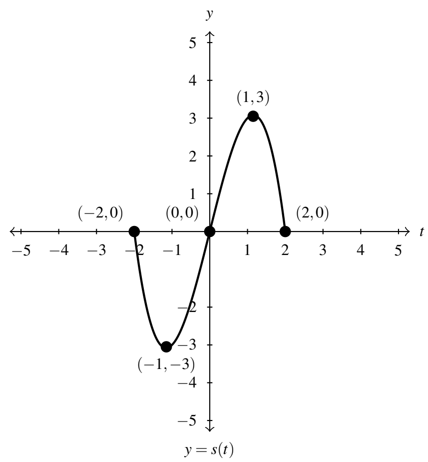
47. $m(x) = -\frac{1}{4}f(3x)$

48. $n(x) = 4f(x-3) - 6$

49. $p(x) = 4 + f(1-2x)$

50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(t)$ is given below.



The purpose of Exercises 51 - 54 is to build up to the graph of $y = \frac{1}{2}S(-t+1)+1$ one step at a time.

51. $y = S_1(t) = S(t+1)$

52. $y = S_2(t) = S_1(-t) = S(-t+1)$

53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$

54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1)+1$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

55. (1) shift right 2 units; (2) shift down 3 units

56. (1) shift down 3 units; (2) shift right 2 units

57. (1) reflect across the x-axis; (2) shift up 1 unit

58. (1) shift up 1 unit; (2) reflect across the x-axis

59. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

60. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

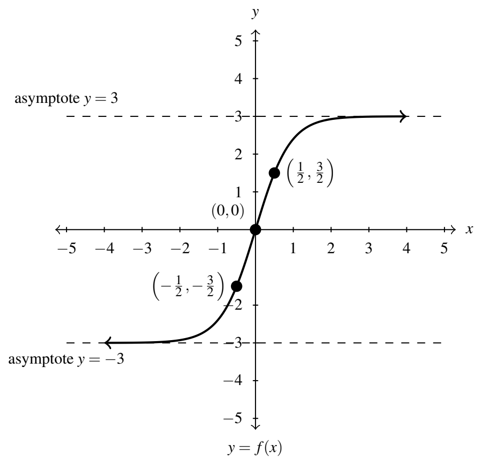
61. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

62. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

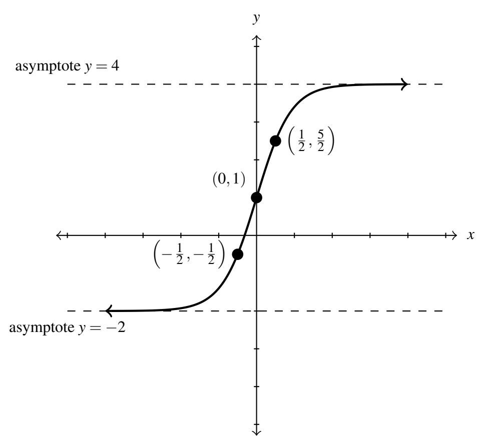
63. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

64. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

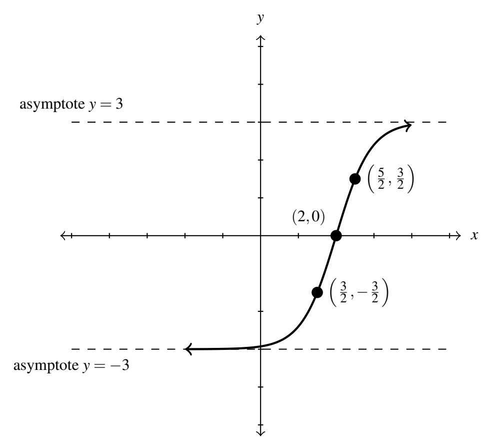
For Exercises 65 - 70, use the given of $y = f(x)$ to write each function in terms of $f(x)$.



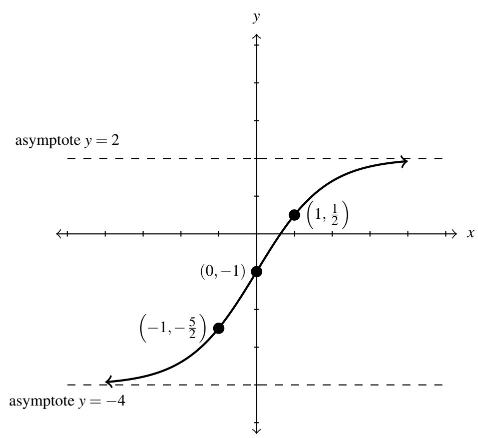
65. $y = g(x)$



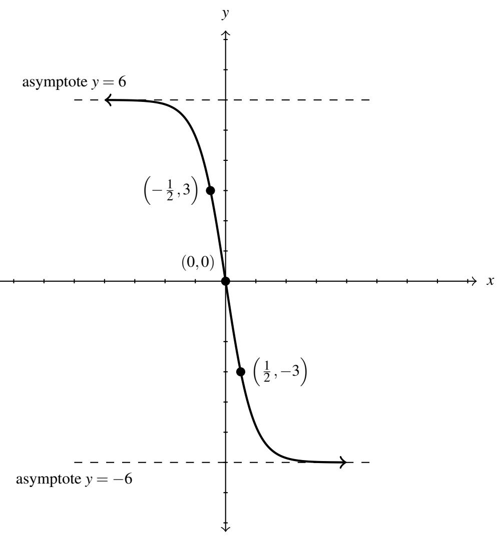
66. $y = h(x)$



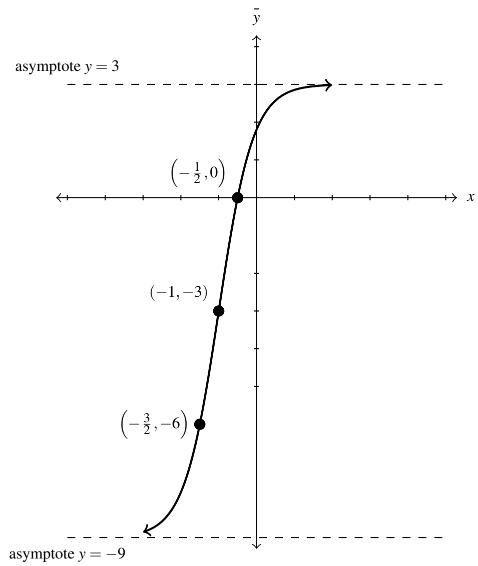
67. $y = p(x)$


y
x
asymptote $y = 2$
$\left(1, \frac{1}{2}\right)$
$(0, -1)$
$\left(-1, -\frac{5}{2}\right)$
asymptote $y = -4$

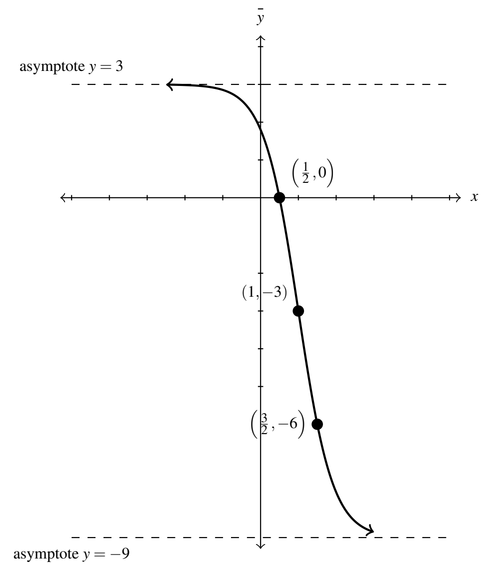
68. $y = q(x)$


y
x
asymptote $y = 6$
$\left(-\frac{1}{2}, 3\right)$
$(0, 0)$
$\left(\frac{1}{2}, -3\right)$
asymptote $y = -6$

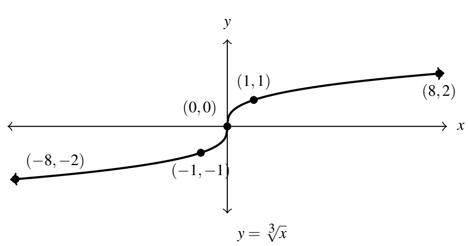
69. $y = r(x)$

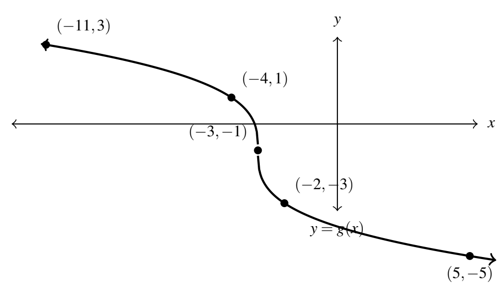


70. $y = s(x)$



71. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.





72. Show that the composition of two linear functions is a linear function. Hence any (finite) sequence of transformations discussed in this section can be combined into the form given in Theorem 1.12.

(HINT: Let $f(x) = ax + b$ and $g(x) = cx + d$. Find $(f \circ g)(x)$.)

73. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, $\sqrt{9x} = 3\sqrt{x}$, so a horizontal compression of $y = \sqrt{x}$ by a factor of 9 results in the same graph as a vertical stretch of $y = \sqrt{x}$ by a factor of 3.

With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$.

What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

74. Discuss the following questions with your classmates.

    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - How would you describe symmetry about the origin in terms of reflections?

75. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

76. This Exercise is a follow-up to Exercise 11 in Section 1.4.

    (a) For each of the following functions, use a graphing utility to compare the graph of $y = f(x)$ with the graphs of $y = |f(x)|$ and $y = f(|x|)$.

    - $f(x) = 3 - x$
    - $f(x) = x^2 - x - 6$
    - $f(x) = \sqrt{x+3} - 1$

    (b) In general, how does the graph of $y = |f(x)|$ compare with that of $y = f(x)$? What about the graph of $y = f(|x|)$ and $y = f(x)$?

    (c) Referring to the functions f and g graphed on page 262, write g in terms of f.

SECTION 1.6 EXERCISE ANSWERS A.1.1

Chapter 2

Polynomial Functions

2.1 Quadratic Functions

2.1.1 Graphs of Quadratic Functions

You may recall studying quadratic equations in a previous Algebra course. If not, you may wish to refer to Section 0.5.5 to revisit this topic. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 2.1. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

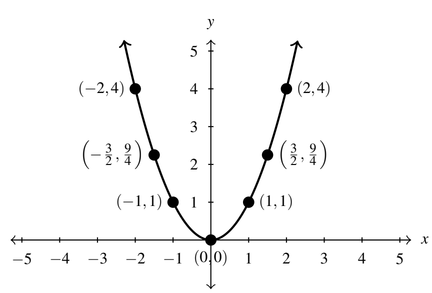
where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

As in Definitions 1.9 and 1.10, the independent variable in Definition 2.1 is x while the values a, b and c are parameters. Note that $a \neq 0$ - otherwise we would have a linear function (see Definition 1.10).

The most basic quadratic function is $f(x) = x^2$, the squaring function, whose graph appears below along with a corresponding table of values. Its shape may look familiar from your previous studies in Algebra – it is called a **parabola**. The point $(0,0)$ is called the **vertex** of the parabola because it is the sole point where the function obtains its extreme value, in this case, a minimum of 0 when $x = 0$.

Indeed, the range of $f(x) = x^2$ appears to be $[0,\infty)$ from the graph. We can substantiate this algebraically because for all x, $f(x) = x^2 \geq 0$. This tells us that the range of f is a subset of $[0,\infty)$. To show that the range of f actually equals $[0,\infty)$, we need to show that every real number c in $[0,\infty)$ is in the range of f. That is, for every $c \geq 0$, we have to show c is an output from f. In other words, we have to show there is a real number x so that $f(x) = x^2 = c$. Choosing $x = \sqrt{c}$, we find $f(x) = f(\sqrt{c}) = (\sqrt{c})^2 = c$, as required.[1]

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



$f(x) = x^2$

[1]This assumes, of course, $\sqrt{c}$ is a real number for all real numbers $c \geq 0$ ...

The techniques we used to graph many of the absolute value functions in Section 1.4 can be applied to quadratic functions, too. In fact, knowing the graph of $f(x) = x^2$ enables us to graph *every* quadratic function, but there's some extra work involved. We start with the following theorem:

Theorem 2.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^2 + k$ is a parabola with vertex (h,k). If $a > 0$, the graph resembles '$\smile$.' If $a < 0$, the graph resembles '$\frown$.' Moreover, the vertical line $x = h$ is the **axis of symmetry** of the graph of $y = F(x)$.

To prove Theorem 2.1 the reader is encouraged to revisit the discussion following the proof of Theorem 1.4, replacing every occurrence of absolute value notation with the squared exponent.[2] Alternatively, the reader can skip ahead and read the statement and proof of Theorem 2.2 in Section 2.2. In the meantime we put Theorem 2.1 to good use in the next example.

Example 2.1.1.

1. Graph the following functions using Theorem 2.1. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

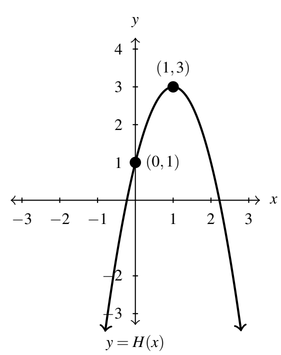
   (a) $f(x) = \dfrac{(x-3)^2}{2}$

   (b) $g(x) = (x+2)^2 - 3$

   (c) $h(t) = -2(t-3)^2 + 1$

   (d) $i(t) = \dfrac{(3-2t)^2 + 1}{2}$

2. Use Theorem 2.1 to write a possible formula for $H(x)$ whose graph is given below:

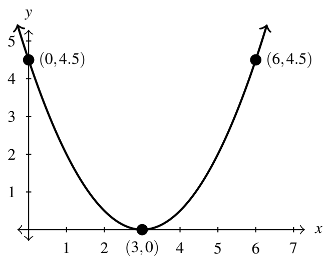


[2] i.e., replace $|x|$ with x^2, $|c|$ with c^2, $|x-h|$ with $(x-h)^2$.

Solution.

1. (a) Graph $f(x) = \dfrac{(x-3)^2}{2}$.

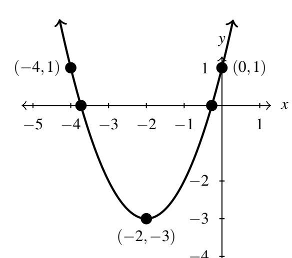
For $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}(x-3)^2 + 0$, we identify $a = \frac{1}{2}$, $h = 3$ and $k = 0$. Thus the vertex is $(3,0)$ and the parabola opens upwards. The only x-intercept is $(3,0)$. Our y-intercept is $\left(0, \frac{9}{2}\right)$, because $f(0) = \frac{1}{2}(0-3)^2 = \frac{9}{2}$. To help us graph the function, it would be nice to have a third point and we'll use symmetry to find it. The y-value three units to the *left* of the vertex is 4.5, so the y-value must be 4.5 three units to the *right* of the vertex as well. Hence, we have our third point: $\left(6, \frac{9}{2}\right)$. From the graph, we identify that the range is $[0,\infty)$ and see that f has the minimum value of 0 at $x = 3$ and no maximum. Also, f is decreasing on $(-\infty, 3]$ and increasing on $[3,\infty)$. The graph is below.



$f(x) = \frac{(x-3)^2}{2}$

(b) Graph $g(x) = (x+2)^2 - 3$.

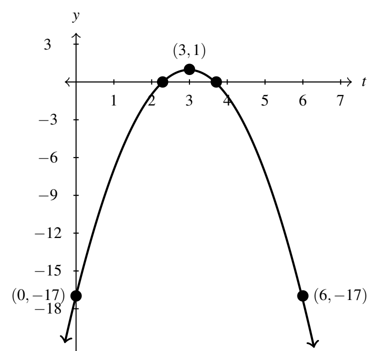
For $g(x) = (x+2)^2 - 3 = (1)(x-(-2))^2 + (-3)$, we identify $a = 1$, $h = -2$ and $k = -3$. This means that the vertex is $(-2,-3)$ and the parabola opens upwards. Thus we have two x-intercepts. To find them, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives us the two zeros of g: $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2-\sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2+\sqrt{3}, 0) \approx (-0.27, 0)$. We find $g(0) = (0+2)^2 - 3 = 1$ so our y-intercept is $(0,1)$. Using symmetry, we get $(-4,1)$ as another point to help us graph. The range of g is $[-3,\infty)$. The minimum of g is -3 at $x = -2$, and g has no maximum. Moreover, g is decreasing on $(-\infty,-2]$ and g is increasing on $[-2,\infty)$. The graph is below on the right.



$g(x) = (x+2)^2 - 3$

(c) Graph $h(t) = -2(t-3)^2 + 1$.

Given $h(t) = -2(t-3)^2 + 1$, we identify $a = -2$, $h = 3$ and $k = 1$. Hence the vertex of the graph is $(3,1)$ and the parabola opens downwards. Solving $h(t) = -2(t-3)^2 + 1 = 0$ gives $(t-3)^2 = \frac{1}{2}$. Extracting square roots[3] gives $t - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[4] we get $t = \frac{6\pm\sqrt{2}}{2}$. Hence, our t-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. To find the y-intercept, we compute $h(0) = -2(0-3)^2 + 1 = -17$. Thus the y-intercept is $(0,-17)$. Using symmetry, we also know $(6,-17)$ is on the graph.



$h(t) = -2(t-3)^2 + 1$

(d) Graph $i(t) = \dfrac{(3-2t)^2 + 1}{2}$.

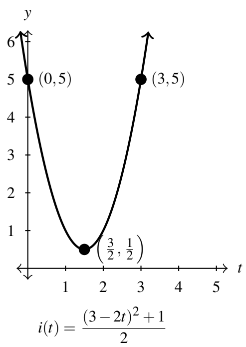
We have some work ahead of us to put $i(t)$ into a form we can use to exploit Theorem 2.1:

$$\begin{aligned} i(t) = \frac{(3-2t)^2+1}{2} &= \tfrac{1}{2}(-2t+3)^2 + \tfrac{1}{2} &&= \tfrac{1}{2}\left[-2\left(t-\tfrac{3}{2}\right)\right]^2 + \tfrac{1}{2} \\ &= \tfrac{1}{2}(-2)^2\left(t-\tfrac{3}{2}\right)^2 + \tfrac{1}{2} &&= 2\left(t-\tfrac{3}{2}\right)^2 + \tfrac{1}{2} \end{aligned}$$

We identify $a = 2$, $h = \frac{3}{2}$ and $k = \frac{1}{2}$. Hence our vertex is $\left(\frac{3}{2}, \frac{1}{2}\right)$ and the parabola opens upwards, meaning there are no t-intercepts. By computing $i(0) = \frac{(3-2(0))^2+1}{2} = 5$, we get $(0,5)$ as the y-intercept. Using symmetry, this means we also have $(3,5)$ on the graph. The range is $\left[\frac{1}{2}, \infty\right)$ with the minimum of i, $\frac{1}{2}$, occurring when $t = \frac{3}{2}$. Also, i is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. The graph is given next.

---

[3] and rationalizing denominators!
[4] and get common denominators!



2. Use Theorem 2.1 to write a possible formula for $H(x)$ whose graph is given.

   We are instructed to use Theorem 2.1, so we know $H(x) = a(x-h)^2 + k$ for some choice of parameters a, h and k. The vertex is $(1,3)$ so we know $h = 1$ and $k = 3$, and hence $H(x) = a(x-1)^2 + 3$. To determine the value of a, we use the fact that the y-intercept, as labeled, is $(0,1)$. This means $H(0) = 1$, or $a(0-1)^2 + 3 = 1$. This reduces to $a + 3 = 1$ or $a = -2$. Our final answer[5] is $H(x) = -2(x-1)^2 + 3$. □

A few remarks about Example 2.1.1 are in order. First note that none of the functions are in the form of Definition 2.1. However, if we took the time to perform the indicated operations and simplify, we'd determine:

- $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}x^2 - 3x + \frac{9}{2}$
- $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$
- $h(t) = -2(t-3)^2 + 1 = -2t^2 + 12t - 17$
- $i(t) = \frac{(3-2t)^2+1}{2} = 2t^2 - 6t + 5$

While the y-intercepts of the graphs of the each of the functions are easier to see when the formulas for the functions are written in the form of Definition 2.1, the vertex is not. For this reason, the form of the functions presented in Theorem 2.1 are given a special name.

Definition 2.2. Standard and General Form of Quadratic Functions:

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$. The standard form is often called the **vertex form**.

[5]The reader is encouraged to compare this example with number 2 of Example 1.4.2.

If we proceed as in the remarks following Example 2.1.1, we can convert any quadratic function given to us in standard form and convert to general form by performing the indicated operation and simplifying:

$$\begin{array}{rcl} f(x) & = & a(x-h)^2+k \\ & = & a\left(x^2-2hx+h^2\right)+k \\ & = & ax^2-2ahx+ah^2+k \\ & = & ax^2+(-2ah)x+(ah^2+k). \end{array}$$

With the identifications $b=-2ah$ and $c=ah^2+k$, we have written $f(x)$ in the form $f(x)=ax^2+bx+c$. Likewise, through a process known as 'completing the square', we can take any quadratic function written in general form and rewrite it in standard form. We briefly review this technique in the following example – for a more thorough review the reader should see Section 0.5.5.

Example 2.1.2. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x)=x^2-4x+3.$
2. $g(t)=6-4t-2t^2$

Solution.

1. Graph $f(x)=x^2-4x+3$. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

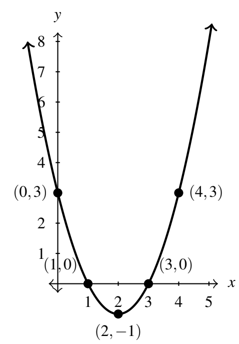
   We follow the procedure for completing the square in Section 0.5.5. The only difference here is instead of the quadratic equation being set to 0, it is equal to $f(x)$. This means when we are finished completing the square, we need to solve for $f(x)$.

$$\begin{array}{rcll} f(x) & = & x^2-4x+3 & \\ f(x)-3 & = & x^2-4x & \text{Subtract 3 from both sides.} \\ f(x)-3+(-2)^2 & = & x^2-4x+(-2)^2 & \text{Add } \left(\frac{1}{2}(-4)\right)^2 \text{ to both sides.} \\ f(x)+1 & = & (x-2)^2 & \text{Factor the perfect square trinomial.} \\ f(x) & = & (x-2)^2-1 & \text{Solve for } f(x). \end{array}$$

   The reader is encouraged to start with $f(x)=(x-2)^2-1$, perform the indicated operations and simplify the result to $f(x)=x^2-4x+3$. From the standard form, $f(x)=(x-2)^2-1$, we see that the vertex is $(2,1)$ and that the parabola opens upwards. To find the zeros of f, we set $f(x)=0$.

   We have two equivalent expressions for $f(x)$ so we could use either the general form or standard form. We solve the former and leave it to the reader to solve the latter to see that we get the same results either way. To solve $x^2-4x+3=0$, we factor: $(x-3)(x-1)=0$ and obtain $x=1$ and $x=3$. We get two x-intercepts, $(1,0)$ and $(3,0)$.

To find the y-intercept, we need $f(0)$. Again, we could use either form of $f(x)$ for this and we choose the general form and find that the y-intercept is $(0,3)$. From symmetry, we know the point $(4,3)$ is also on the graph. We see that the range of f is $[-1,\infty)$ with the minimum -1 at $x=2$. Finally, f is decreasing on $(-\infty,2]$ and increasing from $[2,\infty)$. The graph is below.



$f(x)=x^2-4x+3$

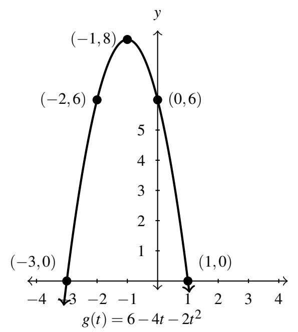
2. Graph $g(t)=6-4t-2t^2$. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

   We first rewrite $g(t)=6-4t-2t^2$ as $g(t)=-2t^2-4t+6$. As with the previous example, once we complete the square, we solve for $g(t)$:

$$\begin{aligned}
g(t) &= -2t^2-4t+6 && \\
g(t)-6 &= -2t^2-4t && \text{Subtract 6 from both sides.}\\
\frac{g(t)-6}{-2} &= \frac{-2t^2-4t}{-2} && \text{Divide both sides by } -2.\\
\frac{g(t)-6}{-2}+(1)^2 &= t^2+2t+(1)^2 && \text{Add } \left(\tfrac{1}{2}(2)\right)^2 \text{ to both sides.}\\
\frac{g(t)-6}{-2}+1 &= (t+1)^2 && \text{Factor the prefect square trinomial.}\\
\frac{g(t)-6}{-2} &= (t+1)^2-1 && \\
g(t)-6 &= -2\left[(t+1)^2-1\right] && \\
g(t) &= -2(t+1)^2+2+6 && \\
g(t) &= -2(t+1)^2+8 &&
\end{aligned}$$

   We can check our answer by expanding $-2(t+1)^2+8$ and show that it simplifies to $-2t^2-4t+6$. From the standard form, we identify the vertex is $(-1,8)$ and that the parabola opens downwards. Setting $g(t)=-2t^2-4t+6=0$, we factor to get $-2(t-1)(t+3)=0$ so $t=-3$ and $t=1$. Hence, our two t-intercepts are $(-3,0)$ and $(1,0)$.

The y-intercept is $(0,6)$, as $g(0)=6$. Using symmetry, we also have the point $(-2,6)$ on the graph. The range is $(-\infty,8]$ with a maximum of 8 when $t=-1$. Finally we note that g is increasing on $(-\infty,-1]$ and decreasing on $[-1,\infty)$.



□

We now generalize the procedure demonstrated in Example 2.1.2. Let $f(x)=ax^2+bx+c$ for $a\neq 0$:

$$\begin{array}{rcll}
f(x) & = & ax^2+bx+c & \\
f(x)-c & = & ax^2+bx & \text{Subtract } c \text{ from both sides.}\\
\dfrac{f(x)-c}{a} & = & \dfrac{ax^2+bx}{a} & \text{Divide both sides by } a\neq 0.\\
\dfrac{f(x)-c}{a} & = & x^2+\dfrac{b}{a}x & \\
\dfrac{f(x)-c}{a}+\left(\dfrac{b}{2a}\right)^2 & = & x^2+\dfrac{b}{a}x+\left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2 \text{ to both sides.}\\
\dfrac{f(x)-c}{a}+\dfrac{b^2}{4a^2} & = & \left(x+\dfrac{b}{2a}\right)^2 & \text{Factor the perfect square trinomial.}\\
\dfrac{f(x)-c}{a} & = & \left(x+\dfrac{b}{2a}\right)^2-\dfrac{b^2}{4a^2} & \text{Solve for } f(x).\\
f(x)-c & = & a\left[\left(x+\dfrac{b}{2a}\right)^2-\dfrac{b^2}{4a^2}\right] & \\
f(x)-c & = & a\left(x+\dfrac{b}{2a}\right)^2-a\dfrac{b^2}{4a^2} & \\
f(x) & = & a\left(x+\dfrac{b}{2a}\right)^2-\dfrac{b^2}{4a}+c & \\
f(x) & = & a\left(x+\dfrac{b}{2a}\right)^2+\dfrac{4ac-b^2}{4a} & \text{Get a common denominator.}
\end{array}$$

By setting $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$, we have written the function in the form $f(x) = a(x-h)^2+k$. This establishes the fact that every quadratic function can be written in standard form. Moreover, writing a quadratic function in standard form allows us to identify the vertex rather quickly, and so our work also shows us that the vertex of $f(x) = ax^2+bx+c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. It is not worth memorizing the expression $\frac{4ac-b^2}{4a}$ due to the fact that we can write this as $f\left(-\frac{b}{2a}\right)$.

We summarize the information detailed above in the following box:

Equation 2.1. Vertex Formulas for Quadratic Functions: Suppose a, b, c, h and k are real numbers where $a \neq 0$.

- If $f(x) = a(x-h)^2+k$, then the vertex of the graph of $y = f(x)$ is the point (h,k).
- If $f(x) = ax^2+bx+c$, then the vertex of the graph of $y = f(x)$ is the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

Completing the square is also the means by which we may derive the celebrated Quadratic Formula, a formula which returns the solutions to $ax^2+bx+c = 0$ for $a \neq 0$. Before we state it here for reference, we wish to encourage the reader to pause a moment and read the derivation if the Quadratic Formula found in Section 0.5.5. The work presented in this section transforms the general form of a quadratic *function* into the standard form whereas the work in Section 0.5.5 finds a formula to solve an *equation*. There is great value in understanding the similarities and differences between the two approaches.

Equation 2.2. The Quadratic Formula: The zeros of the quadratic function $f(x) = ax^2+bx+c$ are:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

It is worth pointing out the symmetry inherent in Equation 2.2. We may rewrite the zeros as:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a},$$

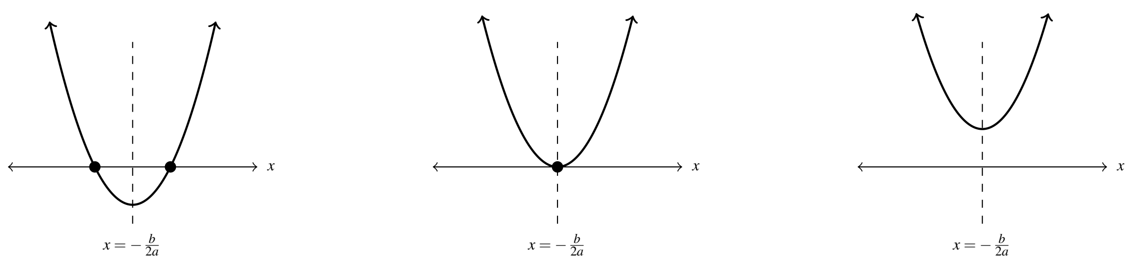
so that, if there are real zeros, they (like the rest of the parabola) are symmetric about the line $x = -\frac{b}{2a}$. Another way to view this symmetry is that the x-coordinate of the vertex is the average of the zeros. We encourage the reader to verify this fact in all of the preceding examples, where applicable.

Next, recall that if the quantity b^2-4ac is strictly negative then we do not have any real zeros. This quantity is called the *discriminant* and is useful in determining the number and nature of solutions to a quadratic equation. We remind the reader of this below.

Equation 2.3. The Discriminant of a Quadratic Function: Given a quadratic function in general form $f(x) = ax^2 + bx + c$, the **discriminant** is the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then f has two unequal (distinct) real zeros.
- If $b^2 - 4ac = 0$ then f has one (repeated) real zero.
- If If $b^2 - 4ac < 0$ then f has two unequal (distinct) non-real zeros.

We'll talk more about what we mean by a 'repeated' zero and how to compute 'non-real' zeros in Chapter 2. For us, the discriminant has the graphical implication that if $b^2 - 4ac > 0$ then we have two x-intercepts; if $b^2 - 4ac = 0$ then we have just one x-intercept, namely, the vertex; and if $b^2 - 4ac < 0$ then we have no x-intercepts because the parabola lies entirely above or below the x-axis. We sketch each of these scenarios below assuming $a > 0$. (The sketches for $a < 0$ are similar.)



$b^2 - 4ac > 0$ $\qquad$ $b^2 - 4ac = 0$ $\qquad$ $b^2 - 4ac < 0$

We now revisit the economic scenario first described in Examples 1.3.8 and 1.3.9 where we were producing and selling PortaBoy game systems. Recall that the cost to produce x PortaBoys is denoted by $C(x)$ and the price-demand function, that is, the price to charge in order to sell x systems is denoted by $p(x)$. We introduce two more related functions below: the **revenue** and **profit** functions.

Definition 2.3. Revenue and Profit: Suppose $C(x)$ represents the cost to produce x units and $p(x)$ is the associated price-demand function. Under the assumption that we are producing the same number of units as are being sold:

- The **revenue** obtained by selling x units is $R(x) = x\,p(x)$.

  That is, revenue $=$ (number of items sold) $\cdot$ (price per item).
- The **profit** made by selling x units is $P(x) = R(x) - C(x)$.

  That is, profit $=$ (revenue) $-$ (cost).

Said differently, the *revenue* is the amount of money *collected* by selling x items whereas the *profit* is how much money is *left over* after the costs are paid.

Example 2.1.3. In Example 1.3.8 the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example 1.3.9 the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Write formulas for the associated revenue and profit functions; include the domain of each.
2. Compute and interpret $P(0)$.
3. Compute and interpret the zeros of P.
4. Graph $y = P(x)$. Find the vertex and axis intercepts.
5. Interpret the vertex of the graph of $y = P(x)$.
6. What should the price per system be in order to maximize profit?
7. Compute and interpret the average rate of change of P over the interval $[0, 57]$.

Solution.

1. Write formulas for the associated revenue and profit functions, including the domain of each.

   The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. The domain of p is restricted to $0 \leq x \leq 166$, and thus the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.

2. Compute and interpret $P(0)$.

   We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. As profit = (revenue) − (cost), this means our costs exceed our revenue by \$150. This makes perfect sense, we don't sell any systems and our revenue is \$0, but our fixed costs (see Example 1.3.8) are \$150.

3. Compute and interpret the zeros of P.

   To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation 2.2. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

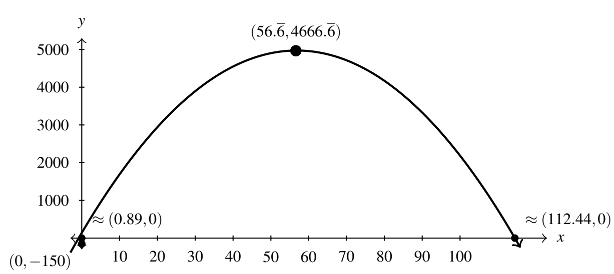
$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)}
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{-170 \pm \sqrt{28000}}{-3} \\
&= \frac{170 \pm 20\sqrt{70}}{3} \\
&\approx 0.89,\ 112.44.
\end{aligned}
$$

Given that profit = (revenue) – (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Graph $y = P(x)$. Find the vertex and axis intercepts.

   Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. As $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation 2.1. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$. Even with this, the intercepts near the origin are crowded.



$y = P(x)$

5. Interpret the vertex of the graph of $y = P(x)$.

   From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. What should the price per system be in order to maximize profit?

   We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. Compute and interpret the average rate of change of P over the interval $[0,57]$.

   To find the average rate of change of P over $[0,57]$, we compute

   $$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

   This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit increased by \$84.50 on average. □

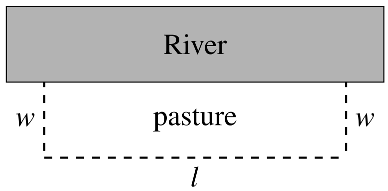
We hope Example 2.1.3 shows the value of using a continuous model to describe a discrete situation. True, we could have 'run the numbers' and computed $P(1)$, $P(2)$, $\ldots$, $P(166)$ to eventually determine the maximum profit, but the vertex formula made much quicker work of the problem.

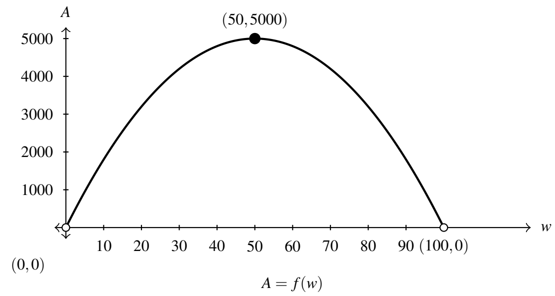
Our next example is classic application of optimizing a quadratic function.

Example 2.1.4. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. We are asked to determine the dimensions of the pasture which would give a maximum area, so we begin by sketching the diagram seen below on the left. We let w denote the width of the pasture and we let ℓ denote the length of the pasture. The units given to us in the statement of the problem are feet, so we assume that w and ℓ are measured in feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Given w and ℓ are both measured in feet, A has units of feet^2, or square feet.

We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$. We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.





Before we go any further, we need to find the applied domain of f so that we know what values of w make sense in this situation.[6] Given that w represents the width of the pasture we need $w > 0$. Likewise, ℓ represents the length of the pasture, so $\ell = 200 - 2w > 0$. Solving this latter inequality yields $w < 100$. Hence, the function we wish to maximize is $f(w) = -2w^2 + 200w$ for $0 < w < 100$. We know two things about the quadratic function f: the graph of $A = f(w)$ is a parabola and (with the coefficient of w^2 being -2) the parabola opens downwards.

This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A = f(50) = -2(50)^2 + 200(50) = 5000$. $w = 50$ lies in the applied domain, $0 < w < 100$, thus we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $\ell = 200 - 2w$ and find $\ell = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A = f(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca. □

The function f in Example 2.1.4 is called the **objective function** for this problem - it's the function we're trying to optimize. In the case above, we were trying to maximize f. The equation $\ell + 2w = 200$ along with the inequalities $w > 0$ and $\ell > 0$ are called the **constraints**. As we saw in this example, and as we'll see again and again, the constraint equation is used to rewrite the objective function in terms of just one of the variables where constraint inequalities, if any, help determine the applied domain.

2.1.2 Exercises

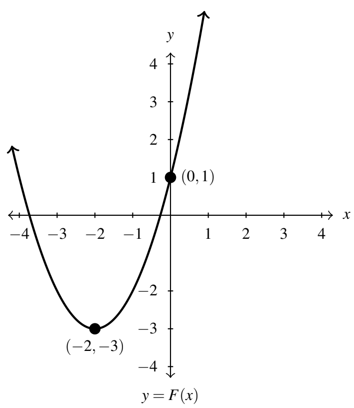
In Exercises 1 - 9, graph the quadratic function. Find the vertex and axis intercepts of each graph, if they exist. State the domain and range, identify the maximum or minimum, and list the intervals over which the function is increasing or decreasing. If the function is given in general form, convert it into standard form; if it is given in standard form, convert it into general form.

[6] Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

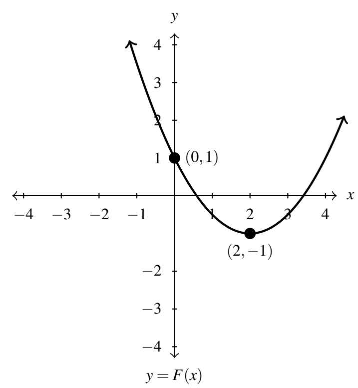
1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $g(t) = -2(t+1)^2 + 4$
5. $g(t) = 2t^2 - 4t - 1$
6. $g(t) = -3t^2 + 4t - 7$
7. $h(s) = s^2 + s + 1$
8. $h(s) = -3s^2 + 5s + 4$
9. $h(s) = s^2 - \dfrac{1}{100}s - 1$

In Exercises 10 - 13, write a formula for each function below in the form $F(x) = a(x-h)^2 + k$.

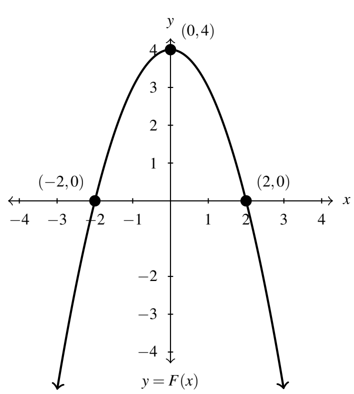
10.



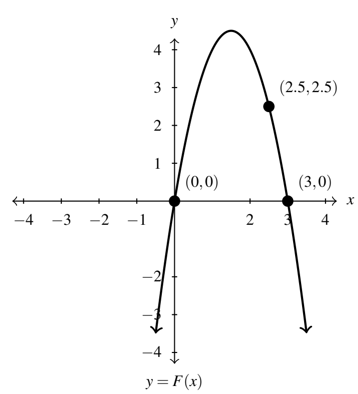
11.



12.



13.



In Exercises 14 - 18, cost and price-demand functions are given. For each scenario,

- Determine the profit function $P(x)$.
- Compute the number of items which need to be sold in order to maximize profit.

- Calculate the maximum profit.
- Calculate the price to charge per item in order to maximize profit.
- Identify and interpret break-even points.

14. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, for $0 \leq x \leq 15$.

15. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, for $0 \leq x \leq 35$.

16. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, for $0 \leq x \leq 30$.

17. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, for $0 \leq x \leq 24$.

18. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, for $0 \leq x \leq 70$.

19. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$ for $0 \leq x \leq 1000$. How many cookies should they bake in order to maximize their profit?

20. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US t years after 1980 can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

21. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

22. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

23. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Due to the fact that one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

24. In the situation of Example 2.1.4, Donnie has a nightmare that one of his alpaca fell into the river. To avoid this, he wants to move his rectangular pasture *away* from the river so that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

25. What is the largest rectangular area one can enclose with 14 inches of string?

26. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[7] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

In Exercises 27 - 32, solve the quadratic equation for the indicated variable.

27. $x^2 - 10y^2 = 0$ for x

28. $y^2 - 4y = x^2 - 4$ for x

29. $x^2 - mx = 1$ for x

30. $y^2 - 3y = 4x$ for y

31. $y^2 - 4y = x^2 - 4$ for y

32. $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)

33. (This is a follow-up to Exercise 96 in Section 1.3.1.) The Lagrange Interpolate function L for three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) where x_0, x_1, and x_2 are three distinct real numbers is given by:

$$L(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

 (a) For each of the following sets of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

 i. $(-1, 1)$, $(1, 1)$, $(2, 4)$

 ii. $(1, 3)$, $(2, 10)$, $(3, 21)$

 iii. $(0, 1)$, $(1, 5)$, $(2, 7)$

 (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, and $L(x_2) = y_2$.

 (c) Find $L(x)$ for the points $(-1, 6)$, $(1, 4)$ and $(3, 2)$. What happens?

 (d) Under what conditions will $L(x)$ produce a quadratic function? Make a conjecture, test some cases, and prove your answer.

SECTION 2.1 EXERCISE ANSWERS A.1.2

[7]The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise 36 in Section 5.7 what shape a free hanging cable makes.

2.2 Properties of Polynomial Functions and Their Graphs

2.2.1 Monomial Functions

Definition 2.4. A **monomial function** is a function of the form

$$f(x) = b \qquad \text{or} \qquad f(x) = ax^n,$$

where a and b are real numbers, $a \neq 0$ and $n \in \mathbb{N}$. The domain of a monomial function is $(-\infty, \infty)$.

Monomial functions, by definition, contain the constant functions along with a two parameter family of functions, $f(x) = ax^n$. We use x as the default independent variable here with a and n as parameters. From Section 0.1.1, we recall that the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, so examples of monomial functions include $f(x) = 2x = 2x^1$, $g(t) = -0.1t^2$, and $H(s) = \sqrt{2}s^{117}$. Note that the function $f(x) = x^0$ is *not* a monomial function. Even though $x^0 = 1$ for all *nonzero* values of x, 0^0 is undefined,[1] and hence $f(x) = x^0$ does *not* have a domain of $(-\infty, \infty)$.[2]

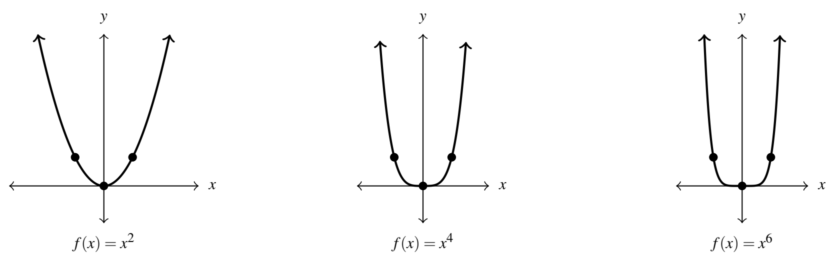
We begin our study of the graphs of polynomial functions by studying graphs of monomial functions. Starting with $f(x) = x^n$ where n is even, we investigate the cases $n = 2$, 4 and 6. Numerically, we see that if $-1 < x < 1$, x^n becomes much smaller as n increases whereas if $x < -1$ or $x > 1$, x^n becomes much larger as n increases. These trends manifest themselves geometrically as the graph 'flattening' for $|x| < 1$ and 'narrowing' for $|x| > 1$ as n increases.[3]

x	x^2	x^4	x^6
-2	4	16	64
-1	1	1	1
-0.5	0.25	0.0625	0.015625
0	0	0	0
0.5	0.25	0.0625	0.015625
1	1	1	1
2	4	16	64

[1] More specifically, 0^0 is an *indeterminate form.* These are studied extensively in Calculus.

[2] This is why we do not describe monomial functions as having the form $f(x) = ax^n$ for any *whole* number n. See Section 0.1.1

[3] Recall that $|x| < 1$ is equivalent to $-1 < x < 1$ and $|x| > 1$ is equivalent to $x < -1$ or $x > 1$. Using absolute values allow us to describe these sets of real numbers more succinctly.



From the graphs, it appears as if the range of each of these functions is $[0,\infty)$. When n is even, $x^n \geq 0$ for all x so the range of $f(x) = x^n$ is contained in $[0,\infty)$. To show that the range of f is all of $[0,\infty)$, we note that the equation $x^n = c$ for $c \geq 0$ has (at least) one solution for every even integer n, namely $x = \sqrt[n]{c}$. (See Section 0.2 for a review of this notation.) Hence, $f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$ which shows that every non-negative real number is in the range of f.[4]

Another item worthy of note is the symmetry about the line $x = 0$ a.k.a the y-axis. (See Definition 1.1 for a review of this concept.) With n being even, $f(-x) = (-x)^n = x^n = f(x)$. At the level of points, we have that for all x, $(-x, f(-x)) = (-x, f(x))$. Hence for every point $(x, f(x))$ on the graph of f, the point symmetric about the y-axis, $(-x, f(x))$ is on the graph, too. We give this sort of symmetry a name honoring its roots here with even-powered monomial functions:

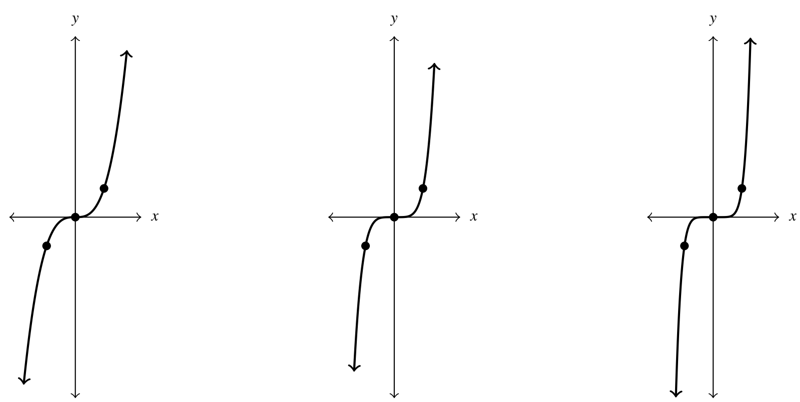
Definition 2.5. A function f is said to be **even** if $f(-x) = f(x)$ for all x in the domain of f.
NOTE: A function f is even if and only if the graph of $y = f(x)$ is symmetric about the y-axis.

An investigation of the odd powered monomial functions ($n \geq 3$) yields similar results with the major difference being that when a negative number is raised to an odd natural number power the result is still negative. Numerically we see that for $|x| > 1$ the values of $|x^n|$ increase as n increases and for $|x| < 1$ the values of $|x^n|$ get closer to 0 as n increases. This translates graphically into a flattening behavior on the interval $(-1,1)$ and a narrowing elsewhere. The graphs are shown on the top of the next page.

The range of these functions appear to be all real numbers, $(-\infty,\infty)$ which is algebraically sound as the equation $x^n = c$ has a solution for every real number,[5] namely $x = \sqrt[n]{c}$. Hence, for every real number c, choose $x = \sqrt[n]{c}$ so that $f(x) = f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$. This shows that every real number is in the range of f.

[4] This should sound familiar - see the comments regarding the range of $f(x) = x^2$ in Section 2.1.
[5] Do you see the importance of n being odd here?

x	x^3	x^5	x^7
-2	-8	-32	-128
-1	-1	-1	-1
-0.5	0.125	-0.03125	-0.0078125
0	0	0	0
0.5	0.125	0.03125	0.0078125
1	1	1	1
2	8	32	128



$f(x) = x^3$ $f(x) = x^5$ $f(x) = x^7$

Here, as a result of n being odd, $f(-x) = (-x)^n = -x^n = -f(x)$. This means that whenever $(x, f(x))$ is on the graph, so is the point symmetric about the origin, $(-x, -f(x))$. (Again, see Definition 1.1.) We generalize this property below. Not surprisingly, we name it in honor of its odd powered heritage:

Definition 2.6. A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
NOTE: A function f is odd if and only if the graph of $y = f(x)$ is symmetric about the origin.

The most important thing to take from the discussion above is the basic shape and common points on the graphs of $y = x^n$ for each of the families when n even and n is odd. While symmetry is nice and should be noted when present, even and odd symmetry are comparatively rare. The point of Definitions 2.5 and 2.6 is to give us the vocabulary to point out the symmetry when appropriate.

Moving on, we take a cue from Theorem 1.4 and prove the following.

Theorem 2.2. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^n + k$ can be obtained from the graph of $f(x) = x^n$ by performing the following operations, in sequence:

1. add h to the x-coordinates of each of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^n$ to $y = (x-h)^n$.

2. multiply the y-coordinates of each of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^n$ to $y = a(x-h)^n$.

3. add k to the y-coordinates of each of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^n$ to $y = a(x-h)^n + k$

Proof. Our goal is to start with the graph of $f(x) = x^n$ and build it up to the graph of $F(x) = a(x-h)^n + k$. We begin by examining $F_1(x) = (x-h)^n$. The graph of $f(x) = x^n$ can be described as the set of points $\{(c, c^n) \,|\, c \in \mathbb{R}\}$.[6] Likewise, the graph of F_1 can be described as the set of points $\{(x, (x-h)^n) \,|\, x \in \mathbb{R}\}$. If we re-label $c = x - h$ so that $x = c + h$, then as x varies through all real numbers so does c.[7] Hence, we can describe the graph of F_1 as $\{(c+h, c^n) \,|\, c \in \mathbb{R}\}$. This means that we can obtain the graph of F_1 from the graph of f by adding h to each of the x-coordinates of the points on the graph of f and that establishes the first step of the theorem.

Next, we consider the graph of $F_2(x) = a(x-h)^n$ as compared to the graph of $F_1(x) = (x-h)^n$. The graph of F_1 is the set of points $\{(x, (x-h)^n \,|\, x \in \mathbb{R}\}$ while the graph of F_2 is the set of points $\{(x, a(x-h)^n) \,|\, x \in \mathbb{R}\}$. The only difference between the points $(x, (x-h)^n)$ and $(x, a(x-h)^n)$ is that the y-coordinate in the latter is a times the y-coordinate of the former.

In other words, to produce the graph of F_2 from the graph of F_1, we take the y-coordinate of each point on the graph of F_1 and multiply it by a to get the corresponding point on the graph of F_2. If $a > 0$, all we are doing is scaling the y-axis by a. If $a < 0$, then, in addition to scaling the y-axis, we are also reflecting each point across the x-axis. In either case, we have established the second step of the theorem.

Last, we compare the graph of $F(x) = a(x-h)^n + k$ to that of $F_2(x) = a(x-h)^n$. Once again, we view the graphs as sets of points in the plane. The graph of F_2 is $\{(x, a(x-h)^n) \,|\, x \in \mathbb{R}\}$ and the graph of F is$\{(x, a(x-h)^n + k) \,|\, x \in \mathbb{R}\}$. Looking at the corresponding points, $(x, a(x-h)^n)$ and $(x, a(x-h)^n + k)$, we see that we can obtain all of the points on the graph of F by adding k to each of the y-coordinates to points on the graph of F_2. This is equivalent to shifting every point vertically by k units which establishes the third and final step in the theorem. $\square$

[6] We are using the dummy variable c here instead of x for reasons that will become apparent shortly.

[7] That is, for a fixed number h every real number c can be written as $x - h$ for some real number x, and every real number x can be written as $c + h$ for some real number c.

This argument should sound familiar. The proof we presented above is more-or-less the same argument we presented after the proof of Theorem 1.4 in Section 1.4 but with '$|\cdot|$' replaced by '$(\cdot)^n$.' Also note that using $n = 2$ in Theorem 2.2 establishes Theorem 2.1 in Section 2.1.

We now use Theorem 2.2 to graph two different "transformed" monomial functions. To provide the reader an opportunity to compare and contrast the graphical behaviors exhibited in the case when n is even versus when n is odd, we graph one of each case.

Example 2.2.1. Use Theorem 2.2 to graph the following functions. Label at least three points on each graph. State the domain and range using interval notation.

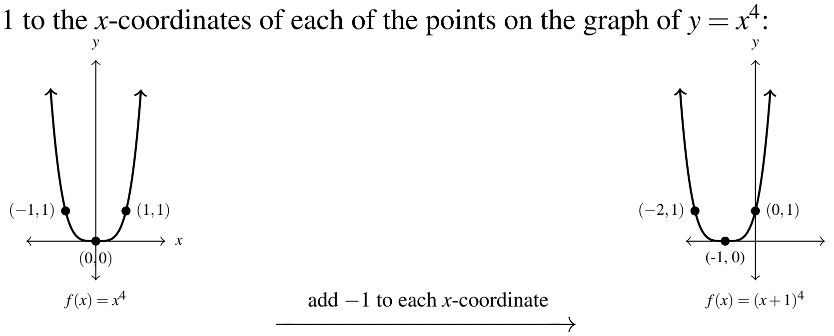
1. $f(x) = -2(x+1)^4 + 3$
2. $g(t) = \dfrac{(2t-1)^3}{5}$

Solution.

1. Graph $f(x) = -2(x+1)^4 + 3$.

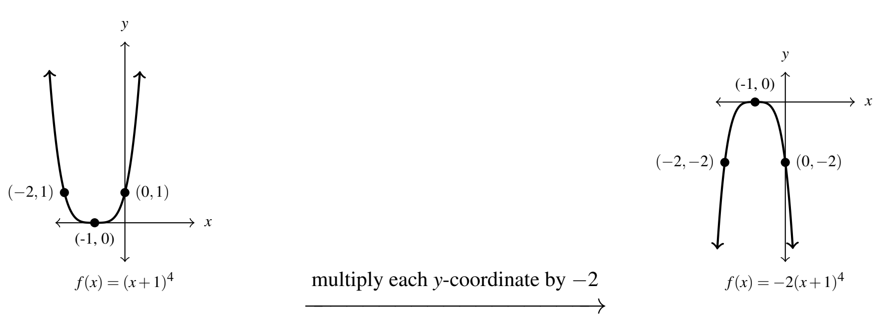
   For $f(x) = -2(x+1)^4 + 3 = -2(x-(-1))^4 + 3$, we identify $n = 4$, $a = -2$, $h = -1$, and $k = 3$. Thus to graph f, we start with $y = x^4$ and perform the following steps, in sequence, tracking the points $(-1,1)$, $(0,0)$ and $(1,1)$ through each step:

   Step 1: add -1 to the x-coordinates of each of the points on the graph of $y = x^4$:



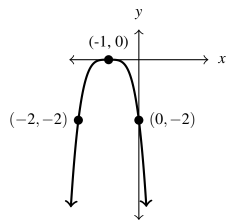
$(-1,1)$, $(0,0)$, $(1,1)$ $\qquad$ $(-2,1)$, $(-1,0)$, $(0,1)$

   Step 2: multiply the y-coordinates of each of the points on the graph of $y = (x+1)^4$ by -2:



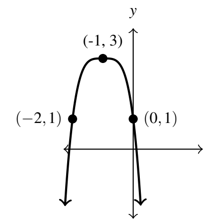
$(-2,1)$, $(-1,0)$, $(0,1)$ $\qquad$ $(-2,-2)$, $(-1,0)$, $(0,-2)$

Step 3: add 3 to the y-coordinates of each of the points on the graph of $y = -2(x+1)^4$:



$f(x) = -2(x+1)^4$

add 3 to each y-coordinate $\longrightarrow$



$f(x) = -2(x+1)^4 + 3$

$(-2,-2)$, $(-1,0)$, $(0,-2)$ $\qquad$ $(-2,1)$, $(-1,3)$, $(0,1)$

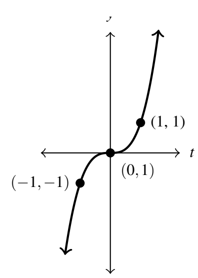
The domain here is $(-\infty,\infty)$ while the range is $(-\infty,3]$.

2. Graph $g(t) = \dfrac{(2t-1)^3}{5}$.

To use Theorem 2.2 to graph $g(t) = \dfrac{(2t-1)^3}{5}$, we must first rewrite the expression for $g(t)$:

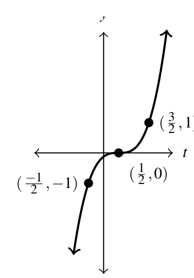
$$g(t) = \frac{(2t-1)^3}{5} = \frac{1}{5}\left(2\left(t-\frac{1}{2}\right)\right)^3 = \frac{1}{5}(2)^3\left(t-\frac{1}{2}\right)^3 = \frac{8}{5}\left(t-\frac{1}{2}\right)^3$$

We identify $n = 3$, $h = \frac{1}{2}$ and $a = \frac{8}{5}$. Hence, we start with the graph of $y = t^3$ and perform the following steps, in sequence, tracking the points $(-1,-1)$, $(0,0)$ and $(1,1)$ through each step:
Step 1: add $\frac{1}{2}$ to each of the t-coordinates of each of the points on the graph of $y = t^3$:



$f(t) = t^3$

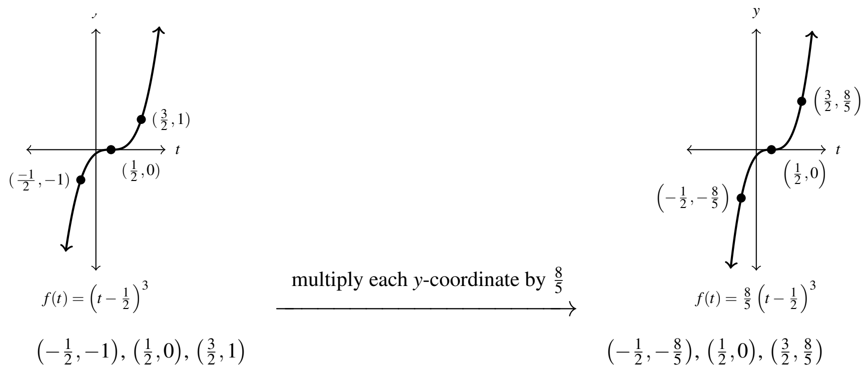
add $\frac{1}{2}$ to each t-coordinate $\longrightarrow$



$f(t) = \left(t-\frac{1}{2}\right)^3$

$(-1,-1)$, $(0,0)$, $(1,1)$ $\qquad$ $\left(-\frac{1}{2},-1\right)$, $\left(\frac{1}{2},0\right)$, $\left(\frac{3}{2},1\right)$

Step 2: multiply each of the y-coordinates of the graph of $y = \left(t-\frac{1}{2}\right)^3$ by $\frac{8}{5}$.

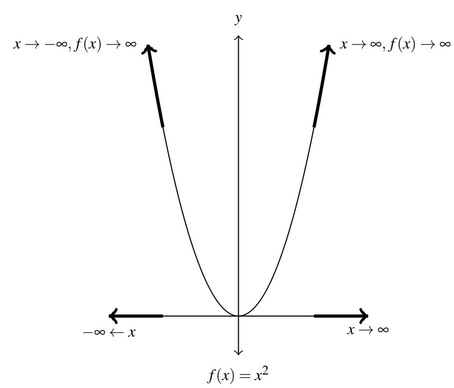


Both the domain and range of g is $(-\infty, \infty)$. □

Example 2.2.1 demonstrates two big ideas in mathematics: first, resolving a complex problem into smaller, simpler steps, and, second, the value of changing form.

Next we wish to focus on the so-called **end behavior** presented in each case.[8] The end behavior of a function is a way to describe what is happening to the outputs from a function as the inputs approach the 'ends' of the domain. Due to the fact that the domain of monomial functions is $(-\infty, \infty)$, we are looking to see what these functions do as their inputs 'approach' $\pm\infty$. The best we can do is sample inputs and outputs and infer general behavior from these observations.[9] The good news is we've wrestled with this concept before. Indeed, every time we add 'arrows' to the graph of a function, we've indicated its end behavior. Let's revisit the graph of $f(x) = x^2$ using the table below.

x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000



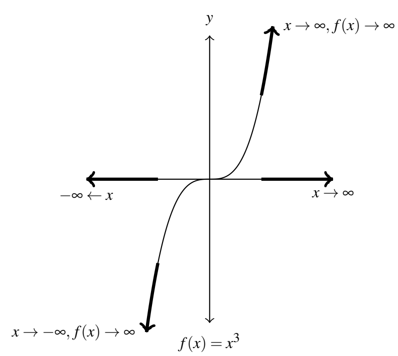
As x takes on smaller and smaller values,[10], we see $f(x)$ takes on larger and larger positive values. The smaller x we use, the larger the $f(x)$ becomes, seemingly without bound.[11] We codify this behavior by

[8]Sometimes called the 'long run' behavior.
[9]and let Calculus students prove our claims.
[10]said differently, negative values that are larger in absolute value
[11]That is, the $f(x)$ values grow larger than any positive number. They are '*unbounded*.'

writing as $x \to -\infty$, $f(x) \to \infty$. Graphically, the farther to the left we travel on the x-axis, the farther up the y-axis the function values travel. This is why we use an ‘arrow’ on the graph in Quadrant II heading upwards to the left. Similarly, we write as $x \to \infty$, $f(x) \to \infty$ because as the x values increase, so do the $f(x)$ values - seemingly without bound. Graphically we indicate this by an arrow on the graph in Quadrant I heading upwards to the right. This behavior holds for all functions $f(x) = x^n$ where $n \geq 2$ is even.

Repeating this investigation for $f(x) = x^3$, we find as $x \to -\infty$, $f(x)$ becomes unbounded in the negative direction, so we write $f(x) \to -\infty$. As $x \to \infty$, $f(x)$ becomes unbounded in the positive direction, so we write $f(x) \to \infty$. This trend holds for all functions $f(x) = x^n$ where n is odd.

x	$f(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000

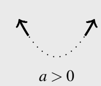


Theorem 2.3 summarizes the end behavior of monomial functions. The results are a consequence of Theorem 2.2 in that the end behavior of a function of the form $y = ax^n$ only differs from that of $y = x^n$ if there is a reflection, that is, if $a < 0$.

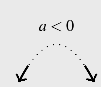
Theorem 2.3. End Behavior of Monomial Functions: Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \in \mathbb{N}$.

- If n is even:

  if $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$:

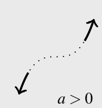
  

  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$:

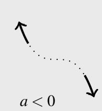
  

- If n is odd:

  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$:

  for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$:

2.2.2 Polynomial Functions

We are now in the position to discuss **polynomial** functions. Simply stated, *poly*nomial functions are sums of *mono*mial functions. The challenge becomes how to describe one of these beasts in general. Up until now, we have used distinct letters to indicate different parameters in our definitions of function families. In other words, we define constant functions as $f(x) = b$, linear functions as $f(x) = mx + b$, and quadratic functions as $f(x) = ax^2 + bx + c$. We even hinted at a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What happens if we wanted to describe a generic polynomial that required, say, 117 different parameters? Our work around is to use subscripted parameters, a_k, that denote the coefficient of x^k. For example, instead of writing a quadratic as $f(x) = ax^2 + bx + c$, we describe it as $f(x) = a_2x^2 + a_1x + a_0$, where a_2, a_1, and a_0 are real numbers and $a_2 \neq 0$. As an added example, consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. and identify $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. This is the notation we use in the following definition.

Definition 2.7. A **polynomial function** is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, ..., a_n are real numbers and $n \in \mathbb{N}$. The domain of a polynomial function is $(-\infty, \infty)$.

As usual, x is used in Definition 2.7 as the independent variable with the a_k each being a parameter. Even though we specify $n \in \mathbb{N}$ so $n \geq 1$, the value of the a_k are unrestricted. Hence, any constant function $f(x) = b$ can be written as $f(x) = 0x + a_0$, and so they are polynomials. Polynomials have an associated vocabulary, and hence, so do polynomial functions.

Definition 2.8.

- Given $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $n \in \mathbb{N}$ and $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term a_nx^n is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.[a]

[a] Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

Again, constant functions are split off in their own separate case Definition 2.8 because of the ambiguity of 0^0. (See the remarks following Definition 2.4.) A consequence of Definition 2.8 is that we can now think of nonzero constant functions as 'zeroth' degree polynomial functions, linear functions as 'first' degree polynomial functions, and quadratic functions as 'second' degree polynomial functions.

Example 2.2.2. Determine the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(t) = 12t - t^3$
3. $H(w) = \dfrac{4-w}{5}$
4. $p(z) = (2z-1)^3(z-2)(3z+2)$

Solution.

1. Determine the degree, leading term, leading coefficient and constant term of $f(x) = 4x^5 - 3x^2 + 2x - 5$.

   There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 2.8, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

2. Determine the degree, leading term, leading coefficient and constant term of $g(t) = 12t - t^3$.

   Two changes here: first, the independent variable is t, not x. Second, the form given in Definition 2.8 specifies the function be written in descending order of the powers of x, or in this case, t. To that end, we re-write $g(t) = 12t - t^3 = -t^3 + 12t$, and see that the degree of g is 3, the leading term is $-t^3$, the leading coefficient is -1 and the constant term is 0.

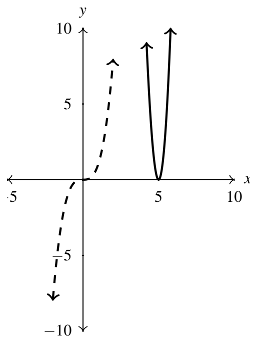
3. Determine the degree, leading term, leading coefficient and constant term of $H(w) = \dfrac{4-w}{5}$.

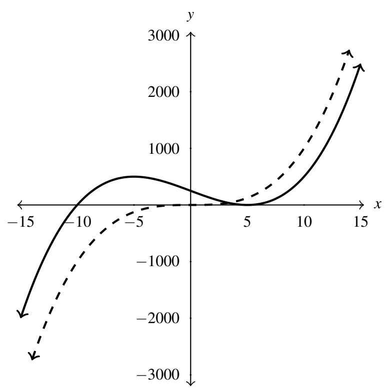
   We need to rewrite the formula for $H(w)$ so that it resembles the form given in Definition 2.8: $H(w) = \frac{4-w}{5} = \frac{4}{5} - \frac{w}{5} = -\frac{1}{5}w + \frac{4}{5}$. We see the degree of H is 1, the leading term is $-\frac{1}{5}w$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

4. Determine the degree, leading term, leading coefficient and constant term of $p(z) = (2z-1)^3(z-2)(3z+2)$.

   It may seem that we have some work ahead of us to get p in the form of Definition 2.8. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2z-1)^3(z-2)(3z+2)$. The leading term of p will be the term which has the highest power of z. The way to get this term is to multiply the terms with the highest power of z from each factor together - in other words, the leading term of $p(z)$ is the product of the leading terms of the *factors* of $p(z)$. Hence, the leading term of p is $(2z)^3(z)(3z) = 24z^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar operation. The constant term of p is obtained by multiplying the constant terms from each of the *factors*: $(-1)^3(-2)(2) = 4$. $\square$

We now turn our attention to graphs of polynomial functions. As polynomial functions are sums of monomial functions, it stands to reason that some of the properties of those graphs carry over to more general polynomials. We first discuss end behavior. Consider $f(x) = x^3 - 75x + 250$. Below are two graphs of $f(x)$ (solid line) along with the graphs of its leading term, $y = x^3$ (dashed line.) On the left is a view 'near' the origin, while on the right is a 'zoomed out' view. Near the origin, the graphs have little in common, but as we look farther out, it becomes that the functions begin to look quite similar.





This observation is borne out numerically as well. Based on the table below, as $x \to \pm\infty$, it certainly appears as if $f(x) \approx g(x)$. One way to think about what is happening numerically is that the leading term x^3 *dominates* the lower order terms $-75x$ and 250 as $x \to \pm\infty$. In other words, x^3 grows so much faster than $-75x$ and 250 that these 'lower order terms' don't contribute anything of significance to the x^3 so $f(x) \approx x^3$. Another way to see this is to rewrite $f(x)$ as[12]

$$f(x) = x^3 - 75x + 250 = x^3 \left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right).$$

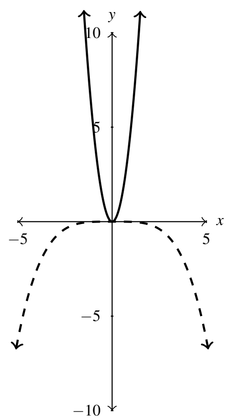
As $x \to \pm\infty$, both $\frac{75}{x^2}$ and $\frac{250}{x^3}$ have constant numerators but denominators that are becoming unbounded. As such, both $\frac{75}{x^2}$ and $\frac{250}{x^3} \to 0$. Therefore, as $x \to \pm\infty$,

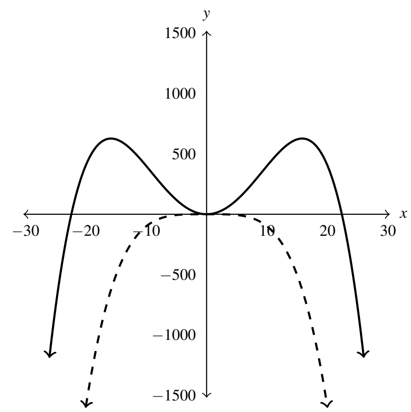
$$f(x) = x^3 - 75x + 250 = x^3 \left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right) \approx x^3(1 + 0 + 0) = x^3.$$

x	$f(x) = x^3 - 75x + 250$	x^3	$-75x$	250	$\frac{75}{x^2}$	$\frac{250}{x^3}$
-1000	$\approx -1 \times 10^9$	-1×10^9	75000	250	7.5×10^{-5}	-2.5×10^{-7}
-100	$\approx -9.9 \times 10^5$	-1×10^6	7500	250	0.0075	-2.5×10^{-4}
-10	0	-1000	750	250	0.75	-0.25
10	500	1000	-750	250	0.75	0.25
100	$\approx 9.9 \times 10^5$	1×10^6	-7500	250	0.0075	2.5×10^{-4}
1000	$\approx 1 \times 10^9$	1×10^9	-75000	250	7.5×10^{-5}	2.5×10^{-7}

[12] We are considering $x \to \pm\infty$, thus we are not concerned with x even being close to 0, so these fractions will all be defined.

Next, consider $g(x) = -0.01x^4 + 5x^2$. Following the logic of the above example, we would expect the end behavior of $y = g(x)$ to mimic that of $y = -0.01x^4$. When we graph $y = g(x)$ (solid line) on the same set of axes as $y = -0.01x^4$ (dashed line), a view near the origin seems to suggest the exact opposite. However, zooming out reveals that the two graphs do share the same end behavior.[13]





Algebraically, for $x \to \pm\infty$, even with the small coefficient of -0.01, $-0.01x^4$ dominates the $5x^2$ term so $g(x) \approx -0.01x^4$. More precisely,

$$g(x) = -0.01x^4 + 5x^2 = x^4\left(-0.01 + \frac{5}{x^2}\right) \approx x^4(-0.01 + 0) = -0.01x^4.$$

The results of these last two examples are generalized below in Theorem 2.4.

Theorem 2.4. End Behavior for Polynomial Functions: The end behavior of polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_nx^n$.
That is, the end behavior of a polynomial function is determined by its leading term.

We argue Theorem 2.4 using an argument similar to ones used above. As $x \to \pm\infty$,

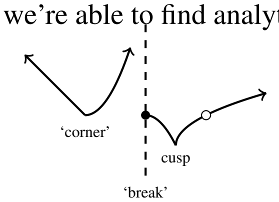
$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \approx x^n(a_n + 0 + \ldots 0) = a_nx^n$$

If this argument looks a little fuzzy, it should. In Calculus, we have the tools necessary to more explicitly state what we mean by ≈ 0. For now, we'll rely on number sense and algebraic intuition.[14]

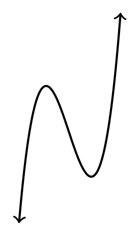
Now that we know how to determine the end behavior of polynomial functions, it's time to investigate what happens 'in between' the ends. First and foremost, polynomial functions are **continuous**. Recall from Section 2.1 that, informally, graphs of continuous functions have no 'breaks' or 'holes' in them.[15] Monomial

[13] Or at least they appear to within the limits of the technology.
[14] Both of which, by the way, can lead one astray, so we must proceed cautiously.
[15] Again, the formal definition of 'continuity' and properties of continuous functions are discussed in Calculus.

functions are continuous (as far as we can tell) and polynomials are sums of monomial functions, so we conclude that polynomial functions are continuous as well. Moreover, the graphs of monomial functions, hence polynomial functions, are **smooth**. Once again, 'smoothness' is a concept defined precisely in Calculus, but for us, functions have no 'corners' or 'sharp turns'. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in the graph on the right. The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial function, even though it is a piecewise-defined function comprised of polynomial functions. Knowing polynomial functions are continuous and smooth gives us an idea of how to 'connect the dots' when sketching the graph from points that we're able to find analytically such as intercepts.

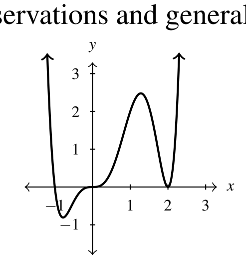


Pathologies not found on graphs of polynomial functions.



The graph of a polynomial function.

Speaking of intercepts, we next focus our attention on the behavior of the graphs of polynomial functions near their zeros. Recall a zero c of a function f is a solution to $f(x) = 0$. Geometrically, the zeros of a function are the x-coordinates of the x-intercepts of the graph of $y = f(x)$. Consider the polynomial function $f(x) = x^3(x-2)^2(x+1)$. To find the zeros of f, we set $f(x) = x^3(x-2)^2(x+1) = 0$. The expression $f(x)$ is already factored, so we set each factor equal to zero.[16] Solving $x^3 = 0$ gives $x = 0$, $(x-2)^2 = 0$ gives $x = 2$, and $x+1 = 0$ gives $x = -1$. Hence, our zeros are $x = -1$, $x = 0$, and $x = 2$. Below, we graph $y = f(x)$ and observe the x-intercepts $(-1,0)$, $(0,0)$ and $(2,0)$. We first note that the graph *crosses* through the x-axis at $(-1,0)$ and $(0,0)$, but the graph *touches* and *rebounds* at $(2,0)$. Moreover, at $(-1,0)$, the graph crosses through the axis is a fairly 'linear' fashion whereas there is a substantial amount of 'flattening' going on near $(0,0)$. Our aim is to explain these observations and generalize them.

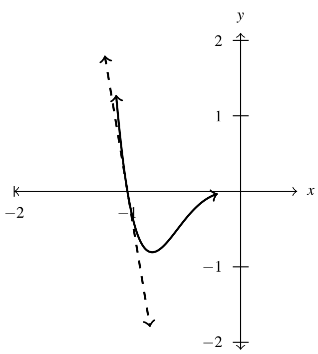


[16]in accordance with the Zero Product Property of the Real Numbers - see Section 0.1.

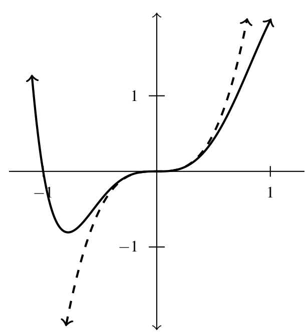
First, let's look at what's happening with the formula $f(x) = x^3(x-2)^2(x+1)$ when $x \approx -1$. We know the x-intercept at $(-1,0)$ is due to the presence of the $(x+1)$ factor in the expression for $f(x)$. So, in this sense, the factor $(x+1)$ is determining a major piece of the behavior of the graph near $x=-1$. For that reason, we focus instead on the other two factors to see what contribution they make. We find when $x \approx -1$, $x^3 \approx (-1)^3 = -1$ and $(x-2)^2 \approx (-1-2)^2 = 9$. Hence, $f(x) = x^3(x-2)^2(x+1) \approx (-1)^3(-1-2)^2(x+1) = -9(x+1)$. Below on the left is a graph of $y = f(x)$ (the solid line) and the graph of $y = -9(x+1)$ (the dashed line.) Sure enough, these graphs approximate one another near $x=-1$.

Likewise, let's look near $x = 0$. The x-intercept $(0,0)$ is a result of the x^3 factor. For $x \approx 0$, $(x-2)^2 \approx (0-2)^2 = 4$ and $(x+1) \approx (0+1) = 1$, so $f(x) = x^3(x-2)^2(x+1) \approx x^3(-2)^2(1) = 4x^3$. Below in the center picture, we have the graph of $y = f(x)$ (again, the solid line) and $y = 4x^3$ (the dashed line) near $x = 0$. Once again, the graphs verify our analysis.

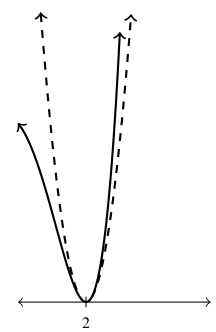
Last, but not least, we analyze f near $x = 2$. Here, the intercept $(2,0)$ is due to the $(x-2)^2$ factor, so we look at the x^3 and $(x+1)$ factors. If $x \approx 2$, $x^3 \approx (2)^3 = 8$ and $(x+1) \approx (2+1) = 3$. Hence, $f(x) = x^3(x-2)^2(x+1) \approx (2)^3(x-2)^2(2+1) = 24(x-2)^2$. Sure enough, as evidenced below on the right, the graphs of $y = f(x)$ and $y = 24(x-2)^2$.



$y = f(x)$ and $y = -9(x+1)$



$y = f(x)$ and $y = 4x^3$



$y = f(x)$ and $y = 24(x-2)^2$

We generalize our observations in Theorem 2.5 below. Like many things we've seen in this text, a more precise statement and proof can be found in a course on Calculus.

Theorem 2.5. Suppose f is a polynomial function and $f(x) = (x-c)^m q(x)$ where $m \in \mathbb{N}$ and $q(c) \neq 0$. Then the the graph of $y = f(x)$ near $(c,0)$ resembles that of $y = q(c)(x-c)^m$.

Let's see how Theorem 2.5 applies to our findings regarding $f(x) = x^3(x-2)^2(x+1)$. For $c = -1$, $(x-c) = (x-(-1)) = (x+1)$. We rewrite $f(x) = x^3(x-2)^2(x+1) = (x-(-1))^1\left[x^3(x-2)^2\right]$ and identify $m = 1$ and $q(x) = x^3(x-2)^2$. We find $q(c) = q(-1) = (-1)^3(-1-2)^2 = -9$ so Theorem 2.5 says that near $(-1,0)$, the graph of $y = f(x)$ resembles $y = q(-1)(x-(-1))^1 = -9(x+1)$. For $c = 0$, $(x-c) = (x-0) = x$ and we can rewrite $f(x) = x^3(x-2)^2(x+1) = (x-0)^3\left[(x-2)^2(x+1)\right]$. We identify $m = 3$ and $q(x) = (x-2)^2(x+1)$. In this case $q(c) = q(0) = (0-2)^2(0+1) = 4$, so Theorem 2.5 guarantees the graph of $y = f(x)$ near $x = 0$

resembles $y = q(0)(x-0)^3 = 4x^3$. Lastly, for $c = 2$, we see $f(x) = (x-2)^2\left[x^3(x+1)\right]$ and we identify $m = 2$ and $q(x) = x^3(x+1)$. We find $q(2) = 2^3(2+1) = 24$, so Theorem 2.5 guarantees the graph of $y = f(x)$ resembles $y = 24(x-2)^2$ near $x = 2$.

As we already mentioned, the formal statement and proof of Theorem 2.5 require Calculus. For now, we can understand the theorem as follows. If we factor a polynomial function as $f(x) = (x-c)^m q(x)$ where $m \geq 1$, then $x = c$ is a zero of f, because $f(c) = (c-c)^m q(c) = 0 \cdot q(c) = 0$. The stipulation that $q(c) \neq 0$ means that we have essentially factored the expression $f(x) = (x-c)^m q(x) = (\text{going to } 0) \cdot (\text{not going to } 0)$. Thinking back to Theorem 2.2, the graph $y = q(c)(x-c)^m$ has an x-intercept at $(c,0)$, a basic overall shape determined by the exponent m, and end behavior determined by the sign of $q(c)$. The fact that *if* $x = c$ is a zero *then* we are guaranteed we can factor $f(x) = (x-c)^m q(x)$ were $q(c) \neq 0$ and, moreover, such a factorization is unique (so that there's only one value of m possible for each zero) is a consequence of two theorems, Theorem 2.8 and The Factor Theorem, Theorem 2.10 which we'll review in the next section. For now, we assume such a factorization is unique in order to define the following.

Definition 2.9. Suppose f is a polynomial function and $m \in \mathbb{N}$. If $f(x) = (x-c)^m q(x)$ where $q(c) \neq 0$, we say $x = c$ is a zero of **multiplicity** m.

So, for $f(x) = x^3(x-2)^2(x+1) = (x-0)^3(x-2)^2(x-(-1))^1$, $x = 0$ is a zero of multiplicity 3, $x = 2$ is a zero of multiplicity 2, and $x = -1$ is a zero of multiplicity 1. Theorems 2.4 and 2.5 give us the following:

Theorem 2.6. The Role of Multiplicity: Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c,0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c,0)$.

Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Compute the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

Theorem 2.7. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

Our next example showcases how all of the above theories can assist in sketching relatively good graphs of polynomial functions without the assistance of technology.

Example 2.2.3. Let $p(x) = (2x-1)(x+1)(1-x^4)$.

1. Find all real zeros of p and state their multiplicities.
2. Describe the behavior of the graph of $y = p(x)$ near each of the x-intercepts.
3. Determine the end behavior and y-intercept of the graph of $y = p(x)$.
4. Sketch $y = p(x)$.

Solution.

1. Compute all real zeros of $p(x) = (2x-1)(x+1)(1-x^4)$ and state their multiplicities.

   To find the zeros of p, we set $p(x) = (2x-1)(x+1)(1-x^4) = 0$. The expression $p(x)$ is already (partially) factored, so we set each factor equal to 0 and solve. From $(2x-1) = 0$, we get $x = \frac{1}{2}$; from $(x+1) = 0$ we get $x = -1$; and from solving $1 - x^4 = 0$ we get $x = \pm 1$. Hence, the zeros are $x = -1$, $x = \frac{1}{2}$, and $x = 1$. In order to determine the multiplicities, we need to factor $p(x)$ as so we can identify the m and $q(x)$ as described in Definition 2.9. The zero $x = -1$ corresponds to the factor $(x+1)$. Notice, however, that writing $p(x) = (x+1)^1\left[(2x-1)(1-x^4)\right]$ with $m = 1$ and $q(x) = (2x-1)(1-x^4)$ does *not* satisfy Definition 2.9 because, $q(-1) = (2(-1)-1)(1-(-1)^4) = 0$. Indeed, we can factor $(1-x^4) = (1-x^2)(1+x^2) = (1-x)(1+x)(x^2+1)$ so that

   $$p(x) = (2x-1)(x+1)(1-x^4) = (2x-1)(x+1)(1-x)(1+x)(x^2+1) = (x+1)^2\left[(2x-1)(1-x)(x^2+1)\right].$$

   Identifying $q(x) = (2x-1)(1-x)(x^2+1)$, we find $q(-1) = (2(-1)-1)(1-(-1))((-1)^2+1) = -12 \neq 0$, which means the multiplicity of $x = -1$ is $m = 2$.

The zero $x = \frac{1}{2}$ came from the factor $(2x-1) = 2(x-\frac{1}{2})$, so we have

$$p(x) = (2x-1)(x+1)^2(1-x)(x^2+1) = (x-\frac{1}{2})^1\left[2(x+1)^2(1-x)(x^2+1)\right].$$

If we identify $q(x) = 2(x+1)^2(1-x)(x^2+1)$, we find $q(\frac{1}{2}) = \frac{45}{16} \neq 0$ so multiplicity here is $m = 1$.

Last but not least, we turn our attention to our last zero, $x = 1$, which we obtained from solving $1-x^4 = 0$. However, from $p(x) = (2x-1)(x+1)^2(1-x)(x^2+1)$, we see the zero $x = 1$ corresponds to the factor $(1-x) = -(x-1)$. We have $p(x) = (x-1)^1\left[-(2x-1)(x+1)^2(x^2+1)\right]$. Identifying $q(x) = -(2x-1)(x+1)^2(x^2+1)$, we see $q(1) = -8$, so the multiplicity $m = 1$ here as well.

2. Describe the behavior of the graph of $y = p(x)$ near each of the x-intercepts.

   From Theorem 2.6, because the multiplicities of $x = \frac{1}{2}$ and $x = 1$ are both *odd*, we know the graph of $y = p(x)$ *crosses* through the x-axis at $(\frac{1}{2}, 0)$ and $(1,0)$. More specifically, due to the fact that the multiplicity for both of these zeros is 1, the graph will look locally linear at these points. Based on our calculations above, near $x = \frac{1}{2}$, the graph will resemble the increasing line $y = \frac{45}{16}(x-\frac{1}{2})$, and near $x = 1$, the graph will resemble the decreasing line $y = -8(x-1)$.

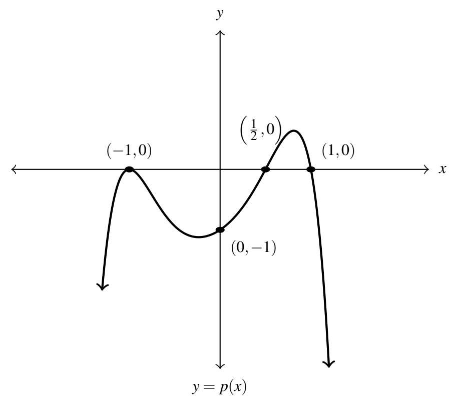
   As the multiplicity of $x = -1$ is *even*, we know the graph of $y = p(x)$ *touches* and *rebounds* at $(-1,0)$. The multiplicity of $x = -1$ is 2, thus the rebound will look locally like a parabola. More specifically, the graph near $x = -1$ will resemble $y = -12(x+1)^2$.

3. Determine the end behavior and y-intercept of the graph of $y = p(x)$.

   Per Theorem 2.4, the end behavior of $y = p(x)$, matches the end behavior of its leading term. As in Example 2.2.2, we multiply the leading terms from each factor together to obtain the leading term for $p(x)$: $p(x) = (2x-1)(x+1)(1-x^4) = (2x)(x)(-x^4) + \ldots = -2x^6 + \ldots$. The degree here, 6, is even and the leading coefficient $-2 < 0$, so we know as $x \to \pm\infty$, $p(x) \to -\infty$. To find the y-intercept, we determine $p(0) = (2(0)-1)(0+1)(1-0^4) = -1$, hence, the y-intercept is $(0,-1)$.

4. Sketch $y = p(x)$.

   From the end behavior, $x \to -\infty$, $p(x) \to -\infty$, we start the graph in Quadrant III and head towards $(-1,0)$. At $(-1,0)$, we 'bounce' off of the x-axis and head towards the y-intercept, $(0,-1)$. We then head towards $\left(\frac{1}{2}, 0\right)$ and cross through the x-axis there. Finally, we head back to the x-axis and cross through at $(1,0)$. Owing to the end behavior $x \to \infty$, $p(x) \to -\infty$, we exit the picture in Quadrant IV. Remember polynomial functions are continuous and smooth, thus we have no holes or gaps in the graph, and all the 'turns' are rounded (no abrupt turns or corners.) We produce something resembling the next graph.



□

A couple of remarks about Example 2.2.3 are in order. First, notice that the factor (x^2+1) was more of a spectator in our discussion of the zeros of p. Indeed, if we set $x^2+1=0$, we have $x^2=-1$ which provides no *real* solutions.[17] That being said, the factor x^2+1 *does* affect the shape of the graph. Next, when connecting up the graph from $(-1,0)$ to $(0,-1)$ to $\left(\frac{1}{2},0\right)$, there really is no way for us to know how low the graph goes, or where the lowest point is between $x=-1$ and $x=\frac{1}{2}$ unless we plot more points. Likewise, we have no idea how high the graph gets between $x=\frac{1}{2}$ and $x=1$. While there are ways to determine these points analytically, more often than not, finding them requires Calculus. As these points do play an important role in many applications, we'll need to discuss them in this course and, when required, we'll use technology to find them. For that reason, we have the following definition:

Definition 2.10. Suppose f is a function with $f(a)=b$.

- We say f has a **local minimum** at the point (a,b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a)=b$ is called 'a local minimum value of f.' That is, b is the minimum $f(x)$ value over an *open interval* containing a. Graphically, no points 'near' a local minimum are lower than (a,b).

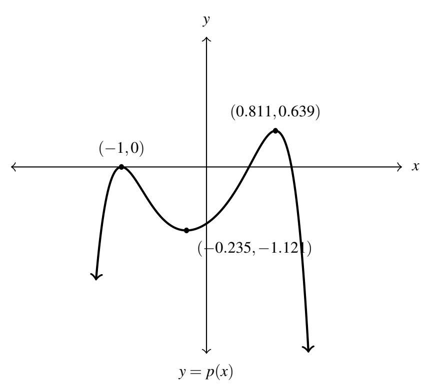
- We say f has a **local maximum** at the point (a,b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a)=b$ is called 'a local maximum value of f.' That is, b is the maximum $f(x)$ value over an *open interval* containing a. Graphically, no points 'near' a local maximum are higher than (a,b).

Taken together, the local maximums and local minimums of a function, if they exist, are called the **local extrema** of the function.

[17] The solutions are $x=\pm i$ - see Section 0.5.6.

Once again, the terminology used in Definition 2.10 blurs the line between the function f and its outputs, $f(x)$. Also, some textbooks use the terms 'relative' minimum and 'relative' maximum instead of the adjective 'local.' Lastly, note the definition of local extrema requires an *open* interval exist in the domain containing a in order for $(a, f(a))$ to be a candidate for a local maximum or local minimum. We'll have more to say about this in later chapters. If our open interval happens to be $(-\infty, \infty)$, then our local extrema are the extrema of f - we'll see an example of this momentarily.

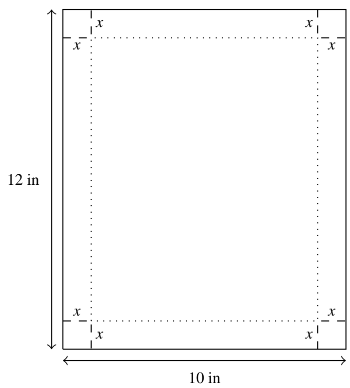
Below we use a graphing utility to graph $y = p(x) = (2x-1)(x+1)(1-x^4)$. We first consider the point $(-1,0)$. Even though there are points on the graph of $y = p(x)$ that are higher than $(-1,0)$, locally, $(-1,0)$ is the top of a hill. To satisfy Definition 2.10, we need to provide an open interval on which $p(-1) = 0$ is the largest, or maximum function value. Note the definition requires us to provide *just one* open interval. One that works is the interval $(-1.5, -0.5)$. We could use any smaller interval or go as large as $\left(-\infty, \frac{1}{2}\right)$ (can you see why?) Next we encounter a 'low' point at approximately $(-0.2353, -1.1211)$. More specifically, for all x in the interval, say, $(-0.5, 0)$, $p(x) \geq -1.1211$, Hence, we have a local minimum at $(-0.2353, -1.1211)$. Lastly, at $(0.811, 0.639)$, we are back to a high point. In fact, 0.639 isn't just a local maximum value, based on the graph, it is *the* maximum of p. Here, we may choose the open interval $(-\infty, \infty)$ as the open interval required by Definition 2.10, because for all x, $p(x) \leq 0.639$. It is important to note that there is no minimum value of p despite there being a local minimum value.[18]

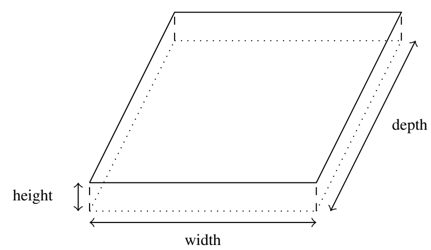


We close this section with a classic application of a third degree polynomial function.

Example 2.2.4. A box with no top is to be fashioned from a 10 inch × 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.

[18] Some books use the adjectives 'global' or 'absolute' when describing the extreme values of a function to distinguish them from their local counterparts.





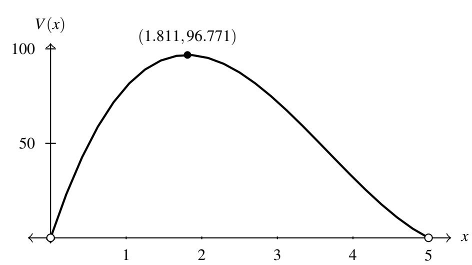
1. Write an expression for $V(x)$, the volume of the box produced by removing squares of edge length x. Include an appropriate domain.

2. Use a graphing utility to help you determine the value of x which produces the box with the largest volume. What is the largest volume? Round your answers to two decimal places.

Solution.

1. From Geometry, we know that Volume $=$ width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10-2x$ inches for the width.[19] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12-2x$ inches remaining. Hence, we get $V(x) = x(10-2x)(12-2x)$. To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and as we are removing $2x$ inches from this dimension, we also require $10-2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graph and technology, we identify a local maximum at approximately $(1.811, 96.771)$. Because the domain of V is restricted to the interval $(0,5)$, the maximum of V is here as well.



[19] There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10-2x$ inches.

This means the maximum volume attainable is approximately 96.77 cubic inches when we remove squares of approximately 1.81 inches per side. □

Notice that there is a very slight, but important, difference between the function $V(x) = x(10-2x)(12-2x)$, $0 < x < 5$ from Example 2.2.4 and the function $p(x) = x(10-2x)(12-2x)$: their domains. The domain of V is restricted to the interval $(0,5)$ while the domain of p is $(-\infty,\infty)$. Indeed, the function V has a maximum of (approximately) 96.771 at (approximately) $x = 1.811$ whereas for the function p, 96.771 is a local maximum value only. We leave it to the reader to verify that V has neither a minimum nor a local minimum.

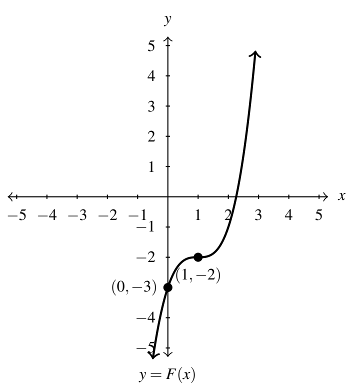
2.2.3 Exercises

In Exercises 1 - 6, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 2.2. Track at least three points of your choice through the transformations. State the domain and range of g.

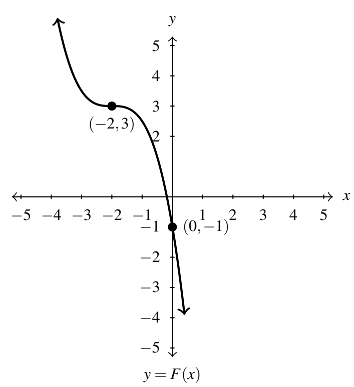
1. $f(x) = x^3$, $F(x) = (x+2)^3 + 1$
2. $f(x) = x^4$, $F(x) = (x+2)^4 + 1$
3. $f(x) = x^4$, $F(x) = 2 - 3(x-1)^4$
4. $f(x) = x^5$, $F(x) = -x^5 - 3$
5. $f(x) = x^5$, $F(x) = (x+1)^5 + 10$
6. $f(x) = x^6$, $F(x) = 8 - x^6$

In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(x-h)^3 + k$.

7.

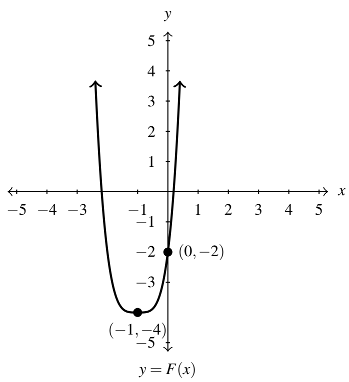


8.

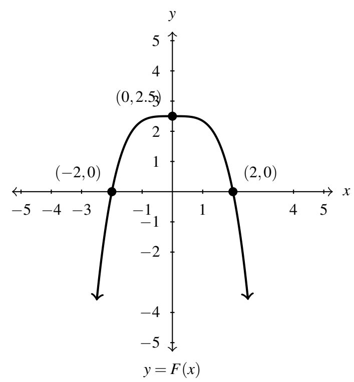


In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a(x-h)^4 + k$.

9.



10.



In Exercises 11 - 20, find the degree, the leading term, the leading coefficient, the constant term and the end behavior of the given polynomial function.

11. $f(x) = 4 - x - 3x^2$

12. $g(x) = 3x^5 - 2x^2 + x + 1$

13. $q(r) = 1 - 16r^4$

14. $Z(b) = 42b - b^3$

15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$

16. $s(t) = -4.9t^2 + v_0 t + s_0$

17. $P(x) = (x-1)(x-2)(x-3)(x-4)$

18. $p(t) = -t^2(3-5t)(t^2+t+4)$

19. $f(x) = -2x^3(x+1)(x+2)^2$

20. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$

In Exercises 21 - 30, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with end behavior to provide a rough sketch of the graph of the polynomial function.

21. $a(x) = x(x+2)^2$

22. $g(t) = t(t+2)^3$

23. $f(z) = -2(z-2)^2(z+1)$

24. $g(x) = (2x+1)^2(x-3)$

25. $F(t) = t^3(t+2)^2$

26. $P(z) = (z-1)(z-2)(z-3)(z-4)$

27. $Q(x) = (x+5)^2(x-3)^4$

28. $h(t) = t^2(t-2)^2(t+2)^2$

29. $H(z) = (3-z)(z^2+1)$

30. $Z(x) = x(42-x^2)$

In Exercises 31 - 45, determine analytically if the following functions are even, odd or neither.

31. $f(x) = 7x$
32. $g(t) = 7t + 2$
33. $p(z) = 7$
34. $F(s) = 3s^2 - 4$
35. $h(t) = 4 - t^2$
36. $g(x) = x^2 - x - 6$
37. $f(x) = 2x^3 - x$
38. $p(z) = -z^5 + 2z^3 - z$
39. $G(t) = t^6 - t^4 + t^2 + 9$
40. $G(s) = s(s^2 - 1)$
41. $f(x) = (x^2 + 1)(x - 1)$
42. $H(t) = (t^2 - 1)(t^4 + t^2 + 3)$
43. $g(t) = t(t - 2)(t + 2)$
44. $P(z) = (2z^5 - 3z)(5z^3 + z)$
45. $f(x) = 0$

46. Suppose $p(x)$ is a polynomial function written in the form of Definition 2.7.
    (a) If the nonzero terms of $p(x)$ consist of even powers of x (or a constant), explain why p is even.
    (b) If the nonzero terms of $p(x)$ consist of odd powers of x, explain why p is odd.
    (c) If $p(x)$ the nonzero terms of $p(x)$ contain at least one odd power of x and one even power of x (or a constant term), then p is neither even nor odd.

47. Use the results of Exercise 46 to determine whether the following functions are even, odd, or neither.
    (a) $p(x) = 3x^4 + x^2 - 1$
    (b) $F(s) = s^3 - 14s$
    (c) $f(t) = 2t^5 - t^2 + 1$
    (d) $g(x) = x^3(x^2 + 1)$

48. Show $f(x) = |x|$ is an even function.

49. Rework Example 2.2.4 assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[20]

50. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[21] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1						
x						
x^2						
x^3						
x^4						
x^5						

[20] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

[21] See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.

51. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[22] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1				
x				
x^2				
x^3				
x^4				
x^5				

In Exercises 52 - 54, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \leq x \leq 10.07$.

52. Graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?

53. Assume the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by the function $C(x) = 200x + 25$ for $x \geq 0$. Find and simplify an expression for the profit function $P(x)$.

    (Remember: Profit = Revenue - Cost.)

54. Graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?

55. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 1.3.8) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.

56. According to US Postal regulations, a rectangular shipping box must satisfy the following inequality: "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.

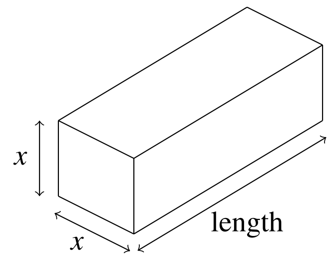
    Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

    (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth $=$ 130 inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

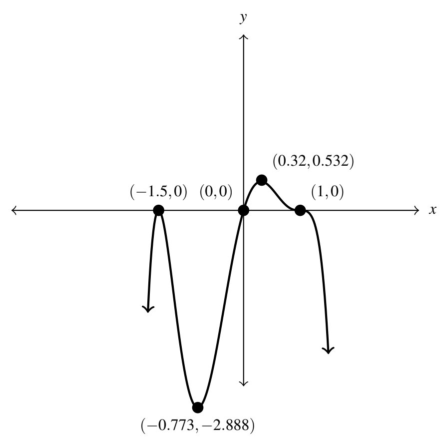
[22] See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts 56a and 56b if the box is shipped using "other services".



57. Below is a graph of a polynomial function $y = p(x)$. Answer the following questions about p based on the graph provided.



(a) Describe the end behavior of $y = p(x)$.

(b) List the real zeros of p along with their respective multiplicities.

(c) List the local minimums and local maximums of the graph of $y = p(x)$.

(d) What can be said about the degree of and leading coefficient $p(x)$?

(e) It turns out that $p(x)$ is a seventh degree polynomial.[23] How can this be?

58. Use the graph of $y = p(x) = (2x-1)(x+1)(1-x^4)$ on page 307 to estimate the largest open interval containing $x = -0.235$ which satisfies the the criteria for 'local minimum' in Definition 2.10.

[23]to be exact, $p(x) = -0.1\,(x+1.5)^2\,(3x)\,(x-1)^3\,(x+5)$.

59. (This is a follow-up to Exercises 96 in Section 1.3.1 and 33 in Section 2.1.) The Lagrange Interpolate function L for four points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) where x_0, x_1, x_2, and x_3 are four distinct real numbers is given by the formula:

$$\begin{aligned} L(x) &= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &\quad + y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}$$

(a) Choose four points with different x-values and construct the Lagrange Interpolate for those points. Verify each of the points lies on the polynomial.

(b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, $L(x_2) = y_2$, and $L(x_3) = y_3$.

(c) Find $L(x)$ for the points $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$. What happens?

(d) Find $L(x)$ for the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$. What happens?

(e) Generalize the formula for $L(x)$ to five points. What's the pattern?

SECTION 2.2 EXERCISE ANSWERS A.1.2

2.3 Real Zeros of Polynomials

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$
\begin{array}{r|l}
 & 32 \\ \cline{2-2}
79 & 2585 \\
 & \underline{-237}\downarrow \\
 & 215 \\
 & \underline{-158} \\
 & 57
\end{array}
$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32) + 57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\quad 3\cdot 10 + 2 \\ \cline{2-2}
7\cdot 10+9 & 2\cdot 10^3 + 5\cdot 10^2 + 8\cdot 10 + 5 \\
 & -\underline{\left(2\cdot 10^3 + 3\cdot 10^2 + 7\cdot 10\right)} \quad \downarrow \\
 & \qquad\qquad 2\cdot 10^2 + 1\cdot 10 + 5 \\
 & \qquad -\underline{\left(1\cdot 10^2 + 5\cdot 10 + 8\right)} \\
 & \qquad\qquad\qquad\quad 5\cdot 10 + 7
\end{array}
$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier.[1] Before we dive into examples, we first state a theorem telling us when we can divide two polynomials, and what to expect when we do so.

Theorem 2.8. Polynomial Division: Suppose $d(x)$ and $p(x)$ are nonzero polynomial functions where the degree of p is greater than or equal to the degree of d. There exist two unique polynomial functions, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

Essentially, Theorem polydivthm tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples.

Example 2.3.1. Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x-2)$
2. $(2t+7) \div (3t-4)$
3. $\left(6y^2 - 1\right) \div (2y+5)$
4. $\dfrac{w^3}{w^2 - \sqrt{2}}$

Solution.

1. Simplify $\left(x^3 + 4x^2 - 5x - 14\right) \div (x-2)$.

   To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x-2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x-2$, by this first term in the quotient to get $x^2(x-2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{r|llll} & x^2 & & & \\ \cline{2-5} x-2 & x^3 & +\,4x^2 & -5x & -14 \\ & -\underline{\left(x^3 - 2x^2\right)} & & \downarrow & \\ & & 6x^2 & -5x & \end{array}$$

   Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x-2) =$

[1] In our opinion - you can judge for yourself.

$6x^2 - 12x$) and subtract.

$$
\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & +\ 6x & & \\ \cline{2-5}
x-2 & x^3 & +\ 4x^2 & -\ 5x & -14 \\
& -\left(x^3\right. & \left.-2x^2\right) & & \downarrow \\ \cline{2-3}
& & 6x^2 & -\ 5x & \downarrow \\
& & -\left(6x^2\right. & \left.-12x\right) & \downarrow \\ \cline{3-4}
& & & 7x & -14
\end{array}
$$

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x - 14$) and subtract.

$$
\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & +\ 6x & +\ 7 & \\ \cline{2-5}
x-2 & x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\
& -\left(x^3\right. & \left.-2x^2\right) & & \\ \cline{2-3}
& & 6x^2 & -\ 5x & \\
& & -\left(6x^2\right. & \left.-12x\right) & \\ \cline{3-4}
& & & 7x & -\ 14 \\
& & & -\ (7x & -14) \\ \cline{4-5}
& & & & 0
\end{array}
$$

In this case, we get a quotient of $x^2 + 6x + 7$ with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2+6x+7\right)+0 = x^3+6x^2+7x-2x^2-12x-14 = x^3+4x^2-5x-14\,\checkmark$$

2. Simplify $(2t+7) \div (3t-4)$.

   To compute $(2t+7) \div (3t-4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t-4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$
\begin{array}{r|rr}
\multicolumn{3}{r}{\dfrac{2}{3}} \\ \cline{2-3}
3t-4 & 2t & +\ 7 \\
& -\left(2t\right. & \left.-\dfrac{8}{3}\right) \\ \cline{2-3}
& & \dfrac{29}{3}
\end{array}
$$

   Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t-4)\left(\frac{2}{3}\right)+\frac{29}{3} = 2t-\frac{8}{3}+\frac{29}{3} = 2t+\frac{21}{3} = 2t+7\,\checkmark$$

3. Simplify $\left(6y^2-1\right) \div (2y+5)$.

When we set-up the tableau for $\left(6y^2-1\right) \div (2y+5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2-1 = 6y^2+0y-1$. We then proceed as before. $\frac{6y^2}{2y} = 3y$, thus $3y$ is the first term in our quotient. We multiply $(2y+5)$ times $3y$ and subtract it from the dividend. We bring down the -1, and repeat.

$$\begin{array}{r|rrr}
\multicolumn{2}{r}{3y} & - & \frac{15}{2} \\ \cline{2-4}
2y+5 & 6y^2 + & 0y & - \;1 \\
& -\left(6y^2 + \right. & \left. 15y\right) & \downarrow \\ \cline{2-3}
& & -15y & - \;1 \\
& & \multicolumn{2}{r}{-\left(-15y - \frac{75}{2}\right)} \\ \cline{3-4}
& & & \frac{73}{2}
\end{array}$$

Our answer is $3y-\frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y-\frac{15}{2}\right)+\frac{73}{2} = 6y^2-15y+15y-\frac{75}{2}+\frac{73}{2} = 6y^2-1 \checkmark$$

4. Simplify $\dfrac{w^3}{w^2-\sqrt{2}}$.

For our last example, we need 'placeholders' for both the divisor $w^2-\sqrt{2} = w^2+0w-\sqrt{2}$ and the dividend $w^3 = w^3+0w^2+0w+0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2+w\sqrt{2}+0 = w\sqrt{2}$.

$$\begin{array}{r|rrrr}
\multicolumn{4}{r}{w} & \\ \cline{2-5}
w^2+0w-\sqrt{2} & w^3 & +0w^2+ & 0w & +0 \\
- & \multicolumn{3}{l}{\left(w^3+0w^2-w\sqrt{2}\right)} & \downarrow \\ \cline{2-4}
& & 0w^2+ & w\sqrt{2} & +0
\end{array}$$

The degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), therefore we are done.[2] Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2-\sqrt{2}\right)w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

□

[2]Because $\frac{0w^2}{w^2} = 0$, we could proceed, write our quotient as $w+0$, and move on... but even pedants have limits.

As you may recall, all of the polynomials in Theorem 2.8 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The word 'unique' here is critical in that it guarantees there is only *one* quotient and remainder for each division problem.[3] The proof of Theorem 2.8 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the section.

Theorem 2.9. The Remainder Theorem: Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $x-c$ the remainder is $p(c)$. Said differently, there is a polynomial function $q(x)$ such that:

$$p(x) = (x-c)q(x) + p(c)$$

The proof of Theorem 2.9 is a direct consequence of Theorem 2.8. Due to the fact that $x-c$ has degree 1, when a polynomial function is divided by $x-c$, the remainder is either 0 or degree 0 (i.e., a nonzero constant.) In either case, $p(x) = (x-c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c-c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one last 'low hanging fruit' to collect which we present below.

Theorem 2.10. The Factor Theorem: Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x-c)$ is a factor of $p(x)$.

Once again, we see the phrase 'if and only if' which means there are really two things being said in The Factor Theorem: if $(x-c)$ is a factor of $p(x)$, then c is a zero of p and the *only* way c is a zero of p is if $(x-c)$ is a factor of $p(x)$. We argue the Factor Theorem as follows: if $(x-c)$ is a factor of $p(x)$, then $p(x) = (x-c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c-c)\,q(c) = 0$, so c is a zero of p. Conversely, suppose c is a zero of p, so $p(c) = 0$. The Remainder Theorem tells us $p(x) = (x-c)q(x) + p(c) = (x-c)q(x) + 0 = (x-c)q(x)$. Hence, $(x-c)$ is a factor of $p(x)$.

We have enough theory to explain why the concept of multiplicity (Definition 2.9) is well-defined. If c is a zero of p, then The Factor Theorem tells us there is a polynomial function q_1 so that $p(x) = (x-c)q_1(x)$. If $q_1(c) = 0$, then we apply the Factor Theorem to q_1 and find a polynomial q_2 so that $q_1(x) = (x-c)q_2(x)$. Hence, we have

$$p(x) = (x-c)q_1(x) = (x-c)(x-c)q_2(x) = (x-c)^2 q_2(x).$$

We now 'rinse and repeat' this process. The degree of p is a finite number, so this process has to end at some point. That is we arrive at a factorization $p(x) = (x-c)^m q(x)$ where $q(c) \neq 0$. Suppose we arrive at a different factorization of p using other methods. That is, we find $p(x) = (x-c)^k Q(x)$, where Q is a polynomial function with $Q(c) \neq 0$. Then we have $(x-c)^m q(x) = (x-c)^k Q(x)$. If $m \neq k$, then either $m < k$ or $m > k$. Assuming the former, then we may divide both sides by $(x-c)^m$ to get: $q(x) = (x-c)^{k-m} Q(x)$. Because $k > m$, $k - m > 0$ and we would have $q(c) = (c-c)^{k-m} Q(c) = 0$, a contradiction as we are assuming

[3] Hence the use of the definite article 'the' when speaking of *the* quotient and *the* remainder.

$q(c) \neq 0$. The assumption that $m > k$ likewise ends in a contradiction. Therefore, we have $m = k$, so $p(x) = (x-c)^m q(x) = (x-c)^m Q(x)$. By the uniqueness guaranteed in Theorem 2.8, we must have that $q(x) = Q(x)$. Hence, we have shown the number m, as well as the quotient polynomial $q(x)$ are unique. The process outlined above, in which we coax out factors of $p(x)$ one at a time until we have all of them serves as a template for our work to come.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomial functions by quantities of the form $x - c$. Fortunately, people like Ruffini and Horner have already blazed this trail. Let's take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +6x & +7 \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -5x & -14 \\
\multicolumn{1}{r}{} & -x^3 & +2x^2 & & \\
\cline{2-3}
\multicolumn{1}{r}{} & & 6x^2 & -5x & \\
\multicolumn{1}{r}{} & & -6x^2 & +12x & \\
\cline{3-4}
\multicolumn{1}{r}{} & & & 7x & -14 \\
\multicolumn{1}{r}{} & & & -7x & +14 \\
\cline{4-5}
\multicolumn{1}{r}{} & & & & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +6x & +7 \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -5x & -14 \\
\multicolumn{1}{r}{} & & 2x^2 & & \\
\cline{2-3}
\multicolumn{1}{r}{} & & 6x^2 & & \\
\multicolumn{1}{r}{} & & & 12x & \\
\cline{3-4}
\multicolumn{1}{r}{} & & & 7x & \\
\multicolumn{1}{r}{} & & & & 14 \\
\cline{4-5}
\multicolumn{1}{r}{} & & & & 0
\end{array}
$$

Let's move terms up a bit and copy the x^3 into the last row.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +6x & +7 \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -5x & -14 \\
\multicolumn{1}{r}{} & & 2x^2 & 12x & 14 \\
\cline{2-5}
\multicolumn{1}{r}{} & x^3 & 6x^2 & 7x & 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results.

2.3.1 Exercises

In Exercises 1 - 12, perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $(5x^2 - 3x + 1) \div (x + 1)$
2. $(3y^2 + 6y - 7) \div (y - 3)$
3. $(6w - 3) \div (2w + 5)$
4. $(2x + 1) \div (3x - 4)$
5. $(t^2 - 4) \div (2t + 1)$
6. $(w^3 - 8) \div (5w - 10)$
7. $(2x^2 - x + 1) \div (3x^2 + 1)$
8. $(4y^4 + 3y^2 + 1) \div (2y^2 - y + 1)$
9. $w^4 \div (w^3 - 2)$
10. $(5t^3 - t + 1) \div (t^2 + 4)$
11. $(t^3 - 4) \div (t - \sqrt[3]{4})$
12. $(x^2 - 2x - 1) \div (x - [1 - \sqrt{2}])$

In Exercises 13 - 26, use long division to perform the following polynomial divisions. Identify the quotient and remainder. Write the dividend, quotient and remainder in the form given in Theorem 2.8.

13. $\left(3x^2 - 2x + 1\right) \div (x - 1)$
14. $\left(x^2 - 5\right) \div (x - 5)$
15. $\left(3 - 4t - 2t^2\right) \div (t + 1)$
16. $\left(4t^2 - 5t + 3\right) \div (t + 3)$
17. $\left(z^3 + 8\right) \div (z + 2)$
18. $\left(4z^3 + 2z - 3\right) \div (z - 3)$
19. $\left(18x^2 - 15x - 25\right) \div \left(x - \frac{5}{3}\right)$
20. $\left(4x^2 - 1\right) \div \left(x - \frac{1}{2}\right)$
21. $\left(2t^3 + t^2 + 2t + 1\right) \div \left(t + \frac{1}{2}\right)$
22. $\left(3t^3 - t + 4\right) \div \left(t - \frac{2}{3}\right)$
23. $\left(2z^3 - 3z + 1\right) \div \left(z - \frac{1}{2}\right)$
24. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) \div \left(z - \frac{3}{2}\right)$
25. $\left(x^4 - 6x^2 + 9\right) \div \left(x - \sqrt{3}\right)$
26. $\left(x^6 - 6x^4 + 12x^2 - 8\right) \div \left(x + \sqrt{2}\right)$

In Exercises 27 - 36, you are given a polynomial function and one of its zeros. Use long division to compute the quotient, then factor the quotient to determine the remaining real zeros, if possible.

27. $x^3 - 6x^2 + 11x - 6, \;\; c = 1$
28. $x^3 - 24x^2 + 192x - 512, \;\; c = 8$
29. $3t^3 + 4t^2 - t - 2, \;\; c = \frac{2}{3}$
30. $2t^3 - 3t^2 - 11t + 6, \;\; c = \frac{1}{2}$
31. $z^3 + 2z^2 - 3z - 6, \;\; c = -2$
32. $2z^3 - z^2 - 10z + 5, \;\; c = \frac{1}{2}$
33. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2
34. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12$, $c = -1$ is a zero of multiplicity 3
35. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324$, $c = -\frac{3}{5}$ is a zero of multiplicity 3
36. $x^2 - 2x - 2, \;\; c = 1 - \sqrt{3}$

SECTION 2.3 EXERCISE ANSWERS A.1.2

Chapter 3

Rational Functions

3.1 Simplifying Rational Expressions

Before we launch into reviewing the basic arithmetic operations of rational expressions, we take a moment to review how to simplify them properly. As with numeric fractions, we 'divide out common *factors*,' not common *terms*. That is, in order to simplify rational expressions, we first *factor* the numerator and denominator. For example:

$$\frac{x^4+5x^3}{x^3-25x} \neq \frac{x^4+5\cancel{x^3}}{\cancel{x^3}-25x}$$

but, rather

$$\begin{aligned}
\frac{x^4+5x^3}{x^3-25x} &= \frac{x^3(x+5)}{x(x^2-25)} && \text{Factor G.C.F.} \\
&= \frac{x^3(x+5)}{x(x-5)(x+5)} && \text{Difference of Squares} \\
&= \frac{\cancelto{x^2}{x^3}\cancel{(x+5)}}{\cancel{x}(x-5)\cancel{(x+5)}} && \text{Divide out the common factors} \\
&= \frac{x^2}{x-5}
\end{aligned}$$

This equivalence holds provided the factors being divided aren't 0. A factor of x and a factor of $x+5$ were divided, thus $x \neq 0$ and $x+5 \neq 0$, so $x \neq -5$. We usually stipulate this as:

$$\frac{x^4+5x^3}{x^3-25x} = \frac{x^2}{x-5}, \qquad \text{provided } x \neq 0,\, x \neq -5$$

While we're talking about common mistakes, please notice that

$$\frac{5}{x^2+9} \neq \frac{5}{x^2} + \frac{5}{9}$$

Just like their numeric counterparts, you don't add algebraic fractions by *adding denominators* of fractions with *common numerators* - it's the other way around:[1]

$$\frac{x^2+9}{5} = \frac{x^2}{5} + \frac{9}{5}$$

It's time to review the basic arithmetic operations with rational expressions.

[1] One of the most common errors students make on college Mathematics placement tests is that they forget how to add algebraic fractions correctly. This places many students into remedial classes even though they are probably ready for college-level Math. We urge you to really study this section with great care so that you don't fall into that trap.

Example 3.1.1. Perform the indicated operations and simplify.

1. $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$
2. $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$
3. $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$
4. $2t^{-3}-(3t)^{-2}$
5. $10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2}$

Solution.

1. Simplify $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$.

   As with numeric fractions, we divide rational expressions by 'inverting and multiplying'. Before we get too carried away however, we factor to see what, if any, factors divide out.

$$\begin{aligned}
\frac{2x^2-5x-3}{x^4-4} \div \frac{x^2-2x-3}{x^5+2x^3} &= \frac{2x^2-5x-3}{x^4-4} \cdot \frac{x^5+2x^3}{x^2-2x-3} && \text{Invert and multiply} \\
&= \frac{(2x^2-5x-3)(x^5+2x^3)}{(x^4-4)(x^2-2x-3)} && \text{Multiply fractions} \\
&= \frac{(2x+1)(x-3)x^3(x^2+2)}{(x^2-2)(x^2+2)(x-3)(x+1)} && \text{Factor} \\
&= \frac{(2x+1)\cancel{(x-3)}x^3\cancel{(x^2+2)}}{(x^2-2)\cancel{(x^2+2)}\cancel{(x-3)}(x+1)} && \text{divide out common factors} \\
&= \frac{x^3(2x+1)}{(x+1)(x^2-2)} && \text{Provided } x \neq 3
\end{aligned}$$

   The '$x \neq 3$' is a result of a factor of $(x-3)$ being divided out as we reduced the expression. We also divided out a factor of (x^2+2). Why is there no stipulation as a result of dividing this factor? Because $x^2+2 \neq 0$ for all real x. (See Section 0.5.6 for details.) At this point, we *could* go ahead and multiply out the numerator and denominator to get

$$\frac{x^3(2x+1)}{(x+1)(x^2-2)} = \frac{2x^4+x^3}{x^3+x^2-2x-2}$$

   but for most of the applications where this kind of algebra is needed (solving equations, for instance), it is best to leave the expressions factored. Your instructor will let you know whether to leave your answer in factored form or not.[2]

[2]Speaking of factoring, do you remember why x^2-2 can't be factored over the integers?

2. Simplify $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$.

   As with numeric fractions we need common denominators in order to subtract. This is already the case here so we proceed by subtracting the numerators.

$$\begin{aligned}
\frac{5}{w^2-9} - \frac{w+2}{w^2-9} &= \frac{5-(w+2)}{w^2-9} && \text{Subtract fractions} \\
&= \frac{5-w-2}{w^2-9} && \text{Distribute} \\
&= \frac{3-w}{w^2-9} && \text{Combine like terms}
\end{aligned}$$

   At this point, we need to determine if we can reduce this expression, so we proceed to factor. It first appears as if we have no common factors among the numerator and denominator until we recall the property of 'factoring negatives' from Page 5: $3-w = -(w-3)$. This yields:

$$\begin{aligned}
\frac{3-w}{w^2-9} &= \frac{-(w-3)}{(w-3)(w+3)} && \text{Factor} \\
&= \frac{-\cancel{(w-3)}}{\cancel{(w-3)}(w+3)} && \text{Divide out the common factors} \\
&= \frac{-1}{w+3} && \text{Provided } w \neq 3
\end{aligned}$$

   The stipulation $w \neq 3$ comes from the division of the $(w-3)$ factor.

3. Simplify $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$.

   In this next example, we are asked to add two rational expressions with *different* denominators. As with numeric fractions, we must first find a *common denominator*. To do so, we start by factoring each of the denominators.

$$\begin{aligned}
\frac{3}{y^2-8y+16} + \frac{y+1}{16y-y^3} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(16-y^2)} && \text{Factor} \\
&= \frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} && \text{Factor some more}
\end{aligned}$$

   To find the common denominator, we examine the factors in the first denominator and note that we need a factor of $(y-4)^2$. We now look at the second denominator to see what other factors we need. We need a factor of y and $(4+y) = (y+4)$. What about $(4-y)$? As mentioned in the last example, we can factor this as: $(4-y) = -(y-4)$. Using properties of negatives, we 'migrate' this negative out to the front of the fraction, turning the addition into subtraction. We find the (least) common denominator to be $(y-4)^2y(y+4)$. We can now proceed to multiply the numerator and denominator of each fraction by whatever factors are missing from their respective denominators to produce equivalent expressions with common denominators.

$$\begin{aligned}
\frac{3}{(y-4)^2}+\frac{y+1}{y(4-y)(4+y)} &= \frac{3}{(y-4)^2}+\frac{y+1}{y(-(y-4))(y+4)} \\
&= \frac{3}{(y-4)^2}-\frac{y+1}{y(y-4)(y+4)} \\
&= \frac{3}{(y-4)^2}\cdot\frac{y(y+4)}{y(y+4)}-\frac{y+1}{y(y-4)(y+4)}\cdot\frac{(y-4)}{(y-4)} && \text{Equivalent Fractions} \\
&= \frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} && \text{Multiply Fractions}
\end{aligned}$$

At this stage, we can subtract numerators and simplify. We'll keep the denominator factored (in case we can reduce down later), but in the numerator, as there are no common factors, we proceed to perform the indicated multiplication and combine like terms.

$$\begin{aligned}
\frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} &= \frac{3y(y+4)-(y+1)(y-4)}{(y-4)^2y(y+4)} && \text{Subtract numerators} \\
&= \frac{3y^2+12y-(y^2-3y-4)}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{3y^2+12y-y^2+3y+4}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{2y^2+15y+4}{y(y+4)(y-4)^2} && \text{Gather like terms}
\end{aligned}$$

We would like to factor the numerator and divide out factors it has in common with the denominator. After a few attempts, it appears as if the numerator doesn't factor, at least over the integers. As a check, we compute the discriminant of $2y^2+15y+4$ and get $15^2-4(2)(4)=193$. This isn't a perfect square so we know that the quadratic equation $2y^2+15y+4=0$ has irrational solutions. This means $2y^2+15y+4$ can't factor over the integers[3] so we are done.

4. Simplify $2t^{-3}-(3t)^{-2}$.

   At first glance, it doesn't seem as if there is anything that can be done with $2t^{-3}-(3t)^{-2}$ because the exponents on the variables are different. However, the exponents are negative, so these are actually rational expressions. In the first term, the -3 exponent applies to the t *only* but in the second term, the exponent -2 applies to *both* the 3 and the t, as indicated by the parentheses. One way to proceed is as follows:

$$\begin{aligned}
2t^{-3}-(3t)^{-2} &= \frac{2}{t^3}-\frac{1}{(3t)^2} \\
&= \frac{2}{t^3}-\frac{1}{9t^2}
\end{aligned}$$

[3] See the remarks following Theorem 0.6.

We see that we are being asked to subtract two rational expressions with different denominators, so we need to find a common denominator. The first fraction contributes a t^3 to the denominator, while the second contributes a factor of 9. Thus our common denominator is $9t^3$, so we are missing a factor of '9' in the first denominator and a factor of 't' in the second.

$$\begin{array}{rcll} \dfrac{2}{t^3} - \dfrac{1}{9t^2} & = & \dfrac{2}{t^3} \cdot \dfrac{9}{9} - \dfrac{1}{9t^2} \cdot \dfrac{t}{t} & \text{Equivalent Fractions} \\ & = & \dfrac{18}{9t^3} - \dfrac{t}{9t^3} & \text{Multiply} \\ & = & \dfrac{18-t}{9t^3} & \text{Subtract} \end{array}$$

We find no common factors among the numerator and denominator, so we are done.

A second way to approach this problem is by factoring. We can extend the concept of the 'Polynomial G.C.F.' to these types of expressions and we can follow the same guidelines as set forth on page 27 to factor out the G.C.F. of these two terms. The key ideas to remember are that we take out each factor with the *smallest* exponent and that factoring is the same as dividing. We first note that $2t^{-3} - (3t)^{-2} = 2t^{-3} - 3^{-2}t^{-2}$ and we see that the smallest power on t is -3. Thus we want to factor out t^{-3} from both terms. It's clear that this will leave 2 in the first term, but what about the second term? As factoring is the same as dividing, we would be dividing the second term by t^{-3} which thanks to the properties of exponents is the same as *multiplying* by $\frac{1}{t^{-3}} = t^3$. The same holds for 3^{-2}. Even though there are no factors of 3 in the first term, we can factor out 3^{-2} by multiplying it by $\frac{1}{3^{-2}} = 3^2 = 9$. We put these ideas together below.

$$\begin{array}{rcll} 2t^{-3} - (3t)^{-2} & = & 2t^{-3} - 3^{-2}t^{-2} & \text{Properties of Exponents} \\ & = & 3^{-2}t^{-3}(2(3)^2 - t^1) & \text{Factor} \\ & = & \dfrac{1}{3^2}\dfrac{1}{t^3}(18-t) & \text{Rewrite} \\ & = & \dfrac{18-t}{9t^3} & \text{Multiply} \end{array}$$

While both ways are valid, one may be more of a natural fit than the other depending on the circumstances and temperament of the student.

5. Simplify $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$.

As with the previous example, we show two different yet equivalent ways to approach simplifying $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$. First up is what we'll call the 'common denominator approach' where we rewrite the negative exponents as fractions and proceed from there.

- *Common Denominator Approach*:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= \frac{10x}{x-3}+\frac{5x^2(-1)}{(x-3)^2} \\
&= \frac{10x}{x-3}\cdot\frac{x-3}{x-3}-\frac{5x^2}{(x-3)^2} && \text{Equivalent Fractions} \\
&= \frac{10x(x-3)}{(x-3)^2}-\frac{5x^2}{(x-3)^2} && \text{Multiply} \\
&= \frac{10x(x-3)-5x^2}{(x-3)^2} && \text{Subtract} \\
&= \frac{5x(2(x-3)-x)}{(x-3)^2} && \text{Factor out G.C.F.} \\
&= \frac{5x(2x-6-x)}{(x-3)^2} && \text{Distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Combine like terms}
\end{aligned}$$

  Both the numerator and the denominator are completely factored with no common factors so we are done.

- *Factoring Approach*: In this case, the G.C.F. is $5x(x-3)^{-2}$. Factoring this out of both terms gives:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= 5x(x-3)^{-2}(2(x-3)^1-x) && \text{Factor} \\
&= \frac{5x}{(x-3)^2}(2x-6-x) && \text{Rewrite, distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Multiply}
\end{aligned}$$

  As expected, we got the same reduced fraction as before. □

3.1.1 Difference Quotients

Recall in Section 1.3.4 the concept of the average rate of change of a function over the interval $[a,b]$ is the slope between the two points $(a,f(a))$ and $(b,f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x}=\frac{f(b)-f(a)}{b-a}.$$

Consider a function f defined over an interval containing x and $x+h$ where $h\neq 0$. The average rate of change of f over the interval $[x,x+h]$ is thus given by the formula:[4]

[4]assuming $h>0$; otherwise, we the interval is $[x+h,x]$. We get the same formula for the difference quotient either way.

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, as it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

In Section 3.2, the average rate of change of position function s can be interpreted as the average velocity (see Definition 3.5.) We can likewise re-cast this definition. After relabeling $t = t_0 + \Delta t$, we get

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0,$$

which measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminant form '$\frac{0}{0}$.' As we will see with rational functions in Section 3.2, when this happens, we can reduce the fraction to lowest terms to see if we have a vertical asymptote or hole in the graph. With this in mind, when we speak of 'simplifying the difference quotient,' we mean to manipulate the expression until the factor of 'h' or 'Δx' divides out from the denominator.

Our next example invites us to simplify three difference quotients, each cast slightly differently. In each case, the bulk of the work involves Intermediate Algebra. We refer the reader to the previous subsection and 0.2 for additional review, if needed.

Example 3.1.2. Compute and simplify the indicated difference quotients for the following functions:

1. For $f(x) = x^2 - x - 2$, compute and simplify:

   (a) $\dfrac{f(3+h) - f(3)}{h}$ (b) $\dfrac{f(x+h) - f(x)}{h}$.

2. For $g(x) = \dfrac{3}{2x+1}$, compute and simplify:

   (a) $\dfrac{g(\Delta x) - g(0)}{\Delta x}$ (b) $\dfrac{g(x+\Delta x) - g(x)}{\Delta x}$.

3. For $r(t) = \sqrt{t}$, compute and simplify:

(a) $\dfrac{r(9+\Delta t) - r(9)}{\Delta t}$ (b) $\dfrac{r(t+\Delta t) - r(t)}{\Delta t}$.

Solution.

1. (a) For $f(x) = x^2 - x - 2$, compute and simplify $\dfrac{f(3+h) - f(3)}{h}$.

For our first difference quotient, we find $f(3+h)$ by substituting the quantity $(3+h)$ in for x:

$$\begin{aligned} f(3+h) &= (3+h)^2 - (3+h) - 2 \\ &= 9 + 6h + h^2 - 3 - h - 2 \\ &= 4 + 5h + h^2 \end{aligned}$$

$f(3) = (3)^2 - 3 - 2 = 4$, so the difference quotient can be rewritten as:

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{(4 + 5h + h^2) - 4}{h} \\ &= \frac{5h + h^2}{h} \\ &= \frac{h(5+h)}{h} && \text{factor} \\ &= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{divide out } h \\ &= 5 + h \end{aligned}$$

(b) For $f(x) = x^2 - x - 2$, compute and simplify $\dfrac{f(x+h) - f(x)}{h}$.

For the second difference quotient, we first find $f(x+h)$. We replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2 - x - h - 2\right) - \left(x^2 - x - 2\right)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \end{aligned}$$

$$\begin{aligned} &= \frac{h(2x+h-1)}{h} && \text{factor} \\ &= \frac{\cancel{h}(2x+h-1)}{\cancel{h}} && \text{divide out } h \\ &= 2x+h-1. \end{aligned}$$

Note if we substitute $x = 3$ into this expression, we obtain $5+h$ which agrees with our answer from the first difference quotient.

2. (a) For $g(x) = \dfrac{3}{2x+1}$, compute and simplify $\dfrac{g(\Delta x) - g(0)}{\Delta x}$.

Rewriting $\Delta x = 0 + \Delta x$, we see the first expression really is a difference quotient:

$$\frac{g(\Delta x) - g(0)}{\Delta x} = \frac{g(0+\Delta x) - g(0)}{\Delta x}.$$

$g(\Delta x) = \frac{3}{2\Delta x+1}$ and $g(0) = \frac{3}{2(0)+1} = 3$, so our difference quotient is:

$$\begin{aligned} \frac{g(0+\Delta x)-g(0)}{\Delta x} &= \frac{\frac{3}{2\Delta x+1} - 3}{\Delta x} \\ &= \frac{\frac{3}{2\Delta x+1} - 3}{\Delta x} \cdot \frac{(2\Delta x+1)}{(2\Delta x+1)} \\ &= \frac{3-3(2\Delta x+1)}{\Delta x(2\Delta x+1)} \\ &= \frac{3-6\Delta x-3}{\Delta x(2\Delta x+1)} \\ &= \frac{-6\Delta x}{\Delta x(2\Delta x+1)} \\ &= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2\Delta x+1)} \\ &= \frac{-6}{2\Delta x+1}. \end{aligned}$$

(b) For $g(x) = \dfrac{3}{2x+1}$, compute and simplify $\dfrac{g(x+\Delta x) - g(x)}{\Delta x}$.

For our next difference quotient, we first find $g(x+\Delta x)$ by replacing every occurrence of x in the formula for $g(x)$ with the quantity $(x+\Delta x)$:

$$\begin{aligned} g(x+\Delta x) &= \frac{3}{2(x+\Delta x)+1} \\ &= \frac{3}{2x+2\Delta x+1}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{g(x+\Delta x)-g(x)}{\Delta x} &= \frac{\dfrac{3}{2x+2\Delta x+1}-\dfrac{3}{2x+1}}{\Delta x} \\ &= \frac{\dfrac{3}{2x+2\Delta x+1}-\dfrac{3}{2x+1}}{\Delta x}\cdot\frac{(2x+2\Delta x+1)(2x+1)}{(2x+2\Delta x+1)(2x+1)} \\ &= \frac{3(2x+1)-3(2x+2\Delta x+1)}{\Delta x(2x+2\Delta x+1)(2x+1)} \\ &= \frac{6x+3-6x-6\Delta x-3}{\Delta x(2x+2\Delta x+1)(2x+1)} \\ &= \frac{-6\Delta x}{\Delta x(2x+2\Delta x+1)(2x+1)} \\ &= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2x+2\Delta x+1)(2x+1)} \\ &= \frac{-6}{(2x+2\Delta x+1)(2x+1)}. \end{aligned}$$

We have managed to divide the factor 'Δx' from the denominator, therefore we are done. Substituting $x=0$ into our final expression gives $\frac{-6}{2\Delta x+1}$, thus checking our previous answer.

3. (a) For $r(t)=\sqrt{t}$, compute and simplify $\dfrac{r(9+\Delta t)-r(9)}{\Delta t}$.

We start with $r(9+\Delta t)=\sqrt{9+\Delta t}$ and $r(9)=\sqrt{9}=3$ and get:

$$\frac{r(9+\Delta t)-r(9)}{\Delta t}=\frac{\sqrt{9+\Delta t}-3}{\Delta t}.$$

In order to divide out the factor 'Δt' from the *denominator*, we set about rationalizing the *numerator* by multiplying both numerator and denominator by the conjugate of the numerator, $\sqrt{9+\Delta t}-3$:

$$\frac{r(9+\Delta t)-r(9)}{\Delta t} = \frac{\sqrt{9+\Delta t}-3}{\Delta t}$$

$$
\begin{aligned}
&= \frac{\left(\sqrt{9+\Delta t}-3\right)}{\Delta t}\cdot\frac{\left(\sqrt{9+\Delta t}+3\right)}{\left(\sqrt{9+\Delta t}+3\right)} && \text{Multiply by the conjugate.}\\
&= \frac{\left(\sqrt{9+\Delta t}\right)^2-(3)^2}{\Delta t\left(\sqrt{9+\Delta t}+3\right)} && \text{Difference of Squares.}\\
&= \frac{(9+\Delta t)-9}{\Delta t\left(\sqrt{9+\Delta t}+3\right)}\\
&= \frac{\Delta t}{\Delta t\left(\sqrt{9+\Delta t}+3\right)}\\
&= \frac{\cancelto{1}{\Delta t}}{\cancel{\Delta t}\left(\sqrt{9+\Delta t}+3\right)}\\
&= \frac{1}{\sqrt{9+\Delta t}+3}
\end{aligned}
$$

(b) For $r(t)=\sqrt{t}$, compute and simplify $\dfrac{r(t+\Delta t)-r(t)}{\Delta t}$.

As one might expect, we use the same strategy to simplify our final different quotient. We have:

$$
\begin{aligned}
\frac{r(t+\Delta t)-r(t)}{\Delta t} &= \frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t}\\
&= \frac{\left(\sqrt{t+\Delta t}-\sqrt{t}\right)}{\Delta t}\cdot\frac{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} && \text{Multiply by the conjugate.}\\
&= \frac{\left(\sqrt{t+\Delta t}\right)^2-\left(\sqrt{t}\right)^2}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} && \text{Difference of Squares.}\\
&= \frac{(t+\Delta t)-t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}\\
&= \frac{\Delta t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}\\
&= \frac{\cancelto{1}{\Delta t}}{\cancel{\Delta t}\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}\\
&= \frac{1}{\sqrt{t+\Delta t}+\sqrt{t}}
\end{aligned}
$$

We have divided the original 'Δt' factor from the denominator, thus we are done. Setting $t = 9$ in this expression, we get $\frac{1}{\sqrt{9+\Delta t}+3}$ which agrees with our previous answer. □

We close this section with an application.

Example 3.1.3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Compute and simplify: $\overline{v}(\Delta t) = \dfrac{s(15+\Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Determine and interpret $\overline{v}(-1)$.
3. Graph $y = \overline{v}(t)$.
4. Describe the behavior of $\overline{v}$ as $\Delta t \to 0$ and interpret.

Solution.

1. Compute and simplify: $\overline{v}(\Delta t) = \dfrac{s(15+\Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.

   To find $\overline{v}(\Delta t)$, we first find $s(15+\Delta t)$:

$$\begin{array}{rcl} s(15+\Delta t) & = & -5(15+\Delta t)^2 + 100(15+\Delta t) \\ & = & -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ & = & -5(\Delta t)^2 - 50\Delta t + 375 \end{array}$$

   $s(15) = -5(15)^2 + 100(15) = 375$, giving us:

$$\begin{array}{rcll} \overline{v}(\Delta t) & = & \dfrac{s(15+\Delta t) - s(15)}{\Delta t} & \\ & = & \dfrac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} & \\ & = & \dfrac{\Delta t(-5\Delta t - 50)}{\Delta t} & \\ & = & \dfrac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} & \\ & = & -5\Delta t - 50 & \Delta t \neq 0 \end{array}$$

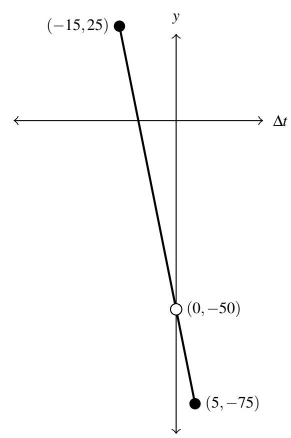
   In addition to the restriction $\Delta t \neq 0$, we also know the domain of s is $0 \leq t \leq 20$. Hence, we also require $0 \leq 15 + \Delta t \leq 20$ or $-15 \leq \Delta t \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$

2. Determine and interpret $\overline{v}(-1)$.

   We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. Graph $y = \overline{v}(t)$.

   The graph of $y = -5\Delta t - 50$ is a line with slope -5 and y-intercept $(0, -50)$. However, as the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, the graph of $\overline{v}$ is a line *segment* from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$.



$y = \overline{v}(\Delta t)$

4. Describe the behavior of $\overline{v}$ as $\Delta t \to 0$ and interpret.

   As $\Delta t \to 0$, $\overline{v}(\Delta t) \to -50$ meaning as we approach $t = 15$, the velocity of the rocket approaches -50 feet per second. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 50 feet per second. $\square$

The reader is invited to compare Example 3.2.3 in Section 3.2 with Example 3.1.3 above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently.

3.1.2 Exercises

In Exercises 1 - 18, perform the indicated operations and simplify.

1. $\dfrac{x^2-9}{x^2} \cdot \dfrac{3x}{x^2-x-6}$

2. $\dfrac{t^2-2t}{t^2+1} \div (3t^2-2t-8)$

3. $\dfrac{4y-y^2}{2y+1} \div \dfrac{y^2-16}{2y^2-5y-3}$

4. $\dfrac{x}{3x-1} - \dfrac{1-x}{3x-1}$

5. $\dfrac{2}{w-1} - \dfrac{w^2+1}{w-1}$

6. $\dfrac{2-y}{3y} - \dfrac{1-y}{3y} + \dfrac{y^2-1}{3y}$

7. $b + \dfrac{1}{b-3} - 2$

8. $\dfrac{2x}{x-4} - \dfrac{1}{2x+1}$

9. $\dfrac{m^2}{m^2-4} + \dfrac{1}{2-m}$

10. $\dfrac{\frac{2}{x}-2}{x-1}$

11. $\dfrac{\frac{3}{2-h}-\frac{3}{2}}{h}$

12. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

13. $3w^{-1}-(3w)^{-1}$

14. $-2y^{-1}+2(3-y)^{-2}$

15. $3(x-2)^{-1}-3x(x-2)^{-2}$

16. $\dfrac{t^{-1}+t^{-2}}{t^{-3}}$

17. $\dfrac{2(3+h)^{-2}-2(3)^{-2}}{h}$

18. $\dfrac{(7-x-h)^{-1}-(7-x)^{-1}}{h}$

In Exercises 19 - 28, find and simplify the difference quotients:

- $\dfrac{f(2+h)-f(2)}{h}$
- $\dfrac{f(x+h)-f(x)}{h}$

19. $f(x)=2x-5$

20. $f(x)=-3x+5$

21. $f(x)=6$

22. $f(x)=3x^2-x$

23. $f(x)=-x^2+2x-1$

24. $f(x)=4x^2$

25. $f(x)=x-x^2$

26. $f(x)=x^3+1$

27. $f(x)=mx+b$ where $m \neq 0$

28. $f(x)=ax^2+bx+c$ where $a \neq 0$

In Exercises 29 - 36, find and simplify the difference quotients:

- $\dfrac{f(-1+\Delta x)-f(-1)}{\Delta x}$
- $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

29. $f(x)=\dfrac{2}{x}$

30. $f(x)=\dfrac{3}{1-x}$

31. $f(x)=\dfrac{1}{x^2}$

32. $f(x)=\dfrac{2}{x+5}$

33. $f(x)=\dfrac{1}{4x-3}$

34. $f(x)=\dfrac{3x}{x+2}$

35. $f(x) = \dfrac{x}{x-9}$

36. $f(x) = \dfrac{x^2}{2x+1}$

In Exercises 37 - 43, find and simplify the difference quotients:

- $\dfrac{g(\Delta t) - g(0)}{\Delta t}$
- $\dfrac{g(t+\Delta t) - g(t)}{\Delta t}$

37. $g(t) = \sqrt{9-t}$

38. $g(t) = \sqrt{2t+1}$

39. $g(t) = \sqrt{-4t+5}$

40. $g(t) = \sqrt{4-t}$

41. $g(t) = \sqrt{at+b}$, where $a \neq 0$.

42. $g(t) = t\sqrt{t}$

43. $g(t) = \sqrt[3]{t}$. **HINT:** $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

44. In this exercise, we explore decomposing a function into its positive and negative parts. Given a function f, we define the **positive part** of f, denoted f_+ and **negative part** of f, denoted f_- by:

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \qquad \text{and} \qquad f_-(x) = \frac{f(x) - |f(x)|}{2}.$$

(a) Graph each of the functions f below along with f_+ and f_-.

- $f(x) = x - 3$
- $f(x) = x^2 - x - 6$
- $f(x) = 4x - x^3$

Why is f_+ called the 'positive part' of f and f_- called the 'negative part' of f?

(b) Show that $f = f_+ + f_-$.

(c) Use Definition 1.12 to rewrite the expressions for $f_+(x)$ and $f_-(x)$ as piecewise defined functions.

SECTION 3.1 EXERCISE ANSWERS A.1.3

3.2 Properties of Rational Functions

If we add, subtract, or multiply polynomial functions, the result is another polynomial function. When we divide polynomial functions, however, we may not get a polynomial function. The result of dividing two polynomials is a **rational function**, so named because rational functions are *ratios* of polynomials.

Definition 3.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.[a]

[a]According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

3.2.1 Laurent Monomial Functions

As with polynomial functions, we begin our study of rational functions with what are, in some sense, the building blocks of rational functions, **Laurent monomial functions**.

Definition 3.2. A **Laurent monomial function** is either a monomial function (see Definition 2.4) or a function of the form $f(x) = \dfrac{a}{x^n} = ax^{-n}$ for $n \in \mathbb{N}$.

Laurent monomial functions are named in honor of Pierre Alphonse Laurent and generalize the notion of 'monomial function' from Chapter 2 to terms with negative exponents. Our study of these functions begins with an analysis of $r(x) = \frac{1}{x} = x^{-1}$, the reciprocal function. The first item worth noting is that $r(0)$ is not defined owing to the presence of x in the denominator. That is, the domain of r is $\{x \in \mathbb{R} \,|\, x \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. Of course excluding 0 from the domain of r serves only to pique our curiosity about the behavior of $r(x)$ when $x \approx 0$. Thinking from a number sense perspective, the closer the denominator of $\frac{1}{x}$ is to 0, the larger the value of the fraction (in absolute value.)[1] So it stands to reason that as x gets closer and closer to 0, the values for $r(x) = \frac{1}{x}$ should grow larger and larger (in absolute value.) This is borne out in the table on the left where it is apparent that for $x \approx 0$, $r(x)$ is becoming unbounded.

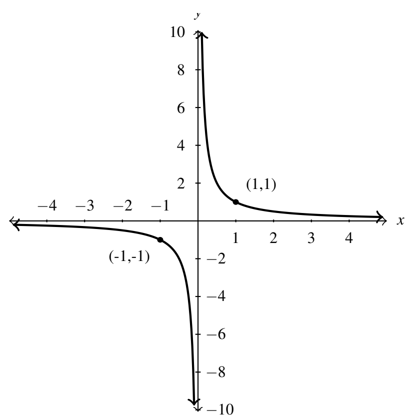
As we investigate the end behavior of r, we find that as $x \to \pm\infty$, $r(x) \approx 0$. Again, number sense agrees here with the data, because as the denominator of $\frac{1}{x}$ becomes unbounded, the value of the fraction should diminish. That being said, we could ask if the graph ever reaches the x-axis. If we attempt to solve $y = r(x) = \frac{1}{x} = 0$. we arrive at the contradiction $1 = 0$ hence, 0 is not in the range of r. Every other real number besides 0 is in the range of r, however. To see this, let $c \neq 0$ be a real number. Then $\frac{1}{c}$ is defined and, moreover, $r\left(\frac{1}{c}\right) = \frac{1}{(1/c)} = c$. This shows c is in the range of r. Hence, the range of r is $\{y \in \mathbb{R} \,|\, y \neq 0\}$ or,

[1]Technically speaking, -1×10^{117} is a 'small' number (because it is very far to the left on the number line.) However, it's absolute value, 1×10^{117} is very large.

using interval notation, $(-\infty, 0) \cup (0, \infty)$.

x	$r(x) = \frac{1}{x}$
-0.01	-100
-0.001	-1000
-0.0001	-10000
-0.00001	-100000
0	undefined
0.00001	100000
0.0001	10000
0.001	1000
0.01	100

x	$r(x) = \frac{1}{x}$
-100000	-0.00001
-10000	-0.0001
-1000	-0.001
-100	-0.01
0	undefined
100	0.01
1000	0.001
10000	0.0001
100000	0.00001



In order to more precisely describe the behavior near 0, we say 'as x approaches 0 *from the left*,' written as $x \to 0^-$, the function values $r(x) \to -\infty$. By 'from the left' we mean we are considering x-values slightly to the *left* of 0 on the number line, such as $x = -0.001$ and $x = -0.0001$ in the table above. If we think of these numbers as all being x-values where $x =$ '$0-$ a little bit', then the '$-$' in the notation '$x \to 0^-$' makes better sense. The notation to describe the $r(x)$ values, $r(x) \to -\infty$, is used here in the same manner as it was in Section 2.2. That is, as $x \to 0^-$, the values $r(x)$ are becoming unbounded in the negative direction.

Similarly, we say 'as x approaches 0 *from the right*,' that is as $x \to 0^+$, $r(x) \to \infty$. Here 'from the right' means we are using x values slightly to the *right* of 0 on the number line: numbers such as $x = 0.001$ which could be described as '$0 +$ a little bit.' For these values of x, the values of $r(x)$ become unbounded (in the positive direction) so we write $r(x) \to \infty$ here.

We can also use this notation to describe the end behavior, but here the numerical roles are reversed. We see as $x \to -\infty$, $r(x) \to 0^-$ and as $x \to \infty$, $r(x) \to 0^+$.

The way we describe what is happening graphically is to say the line $x = 0$ is a **vertical asymptote** of the graph of $y = r(x)$ an the line $y = 0$ is a **horizontal asymptote** of the graph of $y = r(x)$. Roughly speaking, asymptotes are lines which approximate functions as either the inputs or outputs become unbounded. While defined more precisely using the language of Calculus, we will do our best to formally define vertical and horizontal asymptotes below.

Definition 3.3. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

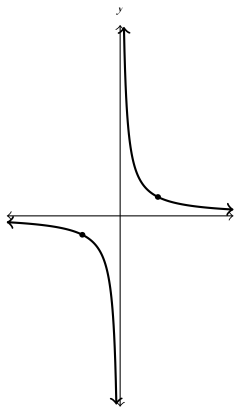
Definition 3.4. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

Note that in Definition 3.4, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so. As we shall see, the graphs of rational functions may, in fact, *cross* their horizontal asymptotes. If this happens, however, it does so only a *finite* number of times (at least in this chapter), and so for each choice of $x \to -\infty$ and $x \to \infty$, $f(x)$ will approach c from either below (in the case $f(x) \to c^-$) or above (in the case $f(x) \to c^+$.) We leave $f(x) \to c$ generic in our definition, however, to allow this concept to apply to less tame specimens in the Precalculus zoo, one that cross horizontal asymptotes an infinite number of times.[2]

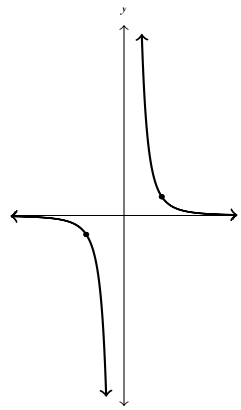
The behaviors illustrated in the graph $r(x) = \frac{1}{x}$ are typical of functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ for natural numbers, n. As with the monomial functions discussed in Section 2.2, the patterns that develop primarily depend on whether n is odd or even. Having thoroughly discussed the graph of $y = \frac{1}{x} = x^{-1}$, we graph it along with $y = \frac{1}{x^3} = x^{-3}$ and $y = \frac{1}{x^5} = x^{-5}$ below. Note the points $(-1,-1)$ and $(1,1)$ are common to all three graphs as are the asymptotes $x = 0$ and $y = 0$. As the n increases, the graphs become steeper for $|x| < 1$ and flatten out more quickly for $|x| > 1$. Both the domain and range in each case appears to be $(-\infty,0)\cup(0,\infty)$. Indeed, owing to the x in the denominator of $f(x) = \frac{1}{x^n}$, $f(0)$, and only $f(0)$, is undefined. Hence the domain is $(-\infty,0)\cup(0,\infty)$. When thinking about the range, note the equation $f(x) = \frac{1}{x^n} = c$ has the solution $x = \sqrt[n]{\frac{1}{c}}$ as long as $c \neq 0$. This means $f\left(\sqrt[n]{\frac{1}{c}}\right) = c$ for every nonzero real number c. If $c = 0$, we are in the same situation as before: $\frac{1}{x^n} = 0$ has no real solution. This establishes the range is $(-\infty,0)\cup(0,\infty)$. Finally, each of the graphs appear to be symmetric about the origin. Indeed, for odd n, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = -x^{-n} = -f(x)$, proving every member of this function family is odd.

x	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^5} = x^{-5}$
-10	-0.1	-0.001	-0.00001
-1	-1	-1	-1
-0.1	-10	-1000	-100000
0	undefined	undefined	undefined
0.1	10	1000	100000
1	1	1	1
10	0.1	0.001	0.00001

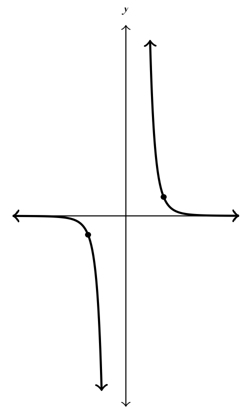
[2] See Exercise 23 in Section 7.3.



$y = \frac{1}{x} = x^{-1}$



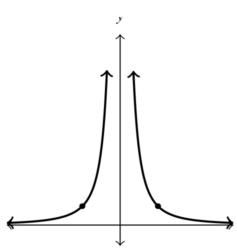
$y = \frac{1}{x^3} = x^{-3}$



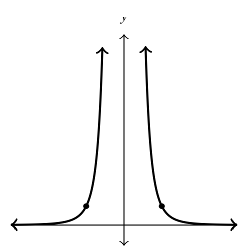
$y = \frac{1}{x^5} = x^{-5}$

We repeat the same experiment with functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ where n is even. $y = \frac{1}{x^2} = x^2$, $y = \frac{1}{x^4} = x^{-4}$ and $y = \frac{1}{x^6} = x^{-6}$. These graphs all share the points $(-1,1)$ and $(1,1)$, and asymptotes $x = 0$ and $y = 0$. The same remarks about the steepness for $|x| < 1$ and the flattening for $|x| > 1$ also apply. For the same reasons as given above, the domain of each of these functions is $(-\infty,0) \cup (0,\infty)$. When it comes to the range, the fact n is even tells us there are solutions to $\frac{1}{x^n} = c$ only if $c > 0$. It follows that the range is $(0,\infty)$ for each of these functions. Concerning symmetry, as n is even, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = x^{-n} = f(x)$, proving each member of this function family is even. Hence, all of the graphs of these functions is symmetric about the y-axis.

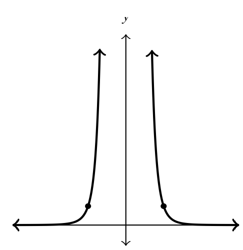
x	$\frac{1}{x^2} = x^{-2}$	$\frac{1}{x^4} = x^{-4}$	$\frac{1}{x^6} = x^{-6}$
-10	0.01	0.0001	1×10^{-6}
-1	1	1	1
-0.1	100	10000	1×10^{6}
0	undefined	undefined	undefined
0.1	100	10000	1×10^{6}
1	1	1	1
10	0.01	0.0001	1×10^{-6}



$y = \frac{1}{x^2} = x^{-2}$



$y = \frac{1}{x^4} = x^{-4}$



$y = \frac{1}{x^5} = x^{-5}$

Not surprisingly, we have an analog to Theorem 2.2 for this family of Laurent monomial functions.

Theorem 3.1. For real numbers a, h, and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x-h)^{-n}$.

   The vertical asymptote moves from $x = 0$ to $x = h$.

2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^{-n}$ to $y = a(x-h)^{-n}$.

3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^{-n}$ to $y = a(x-h)^{-n} + k$.

   The horizontal asymptote moves from $y = 0$ to $y = k$.

The proof of Theorem 3.1 is *identical* to the proof of Theorem 2.2 - just replace x^n with x^{-n}. We nevertheless encourage the reader to work through the details[3] and compare the results of this theorem with Theorems 1.4, 2.1, and 2.2.

We put Theorem 3.1 to good use in the following example.

Example 3.2.1. Use Theorem 3.1 to graph the following. Label at least two points and the asymptotes. State the domain and range using interval notation.

1. $f(x) = (2x-3)^{-2}$
2. $g(t) = \dfrac{2t-1}{t+1}$

Solution.

1. Graph $f(x) = (2x-3)^{-2}$.

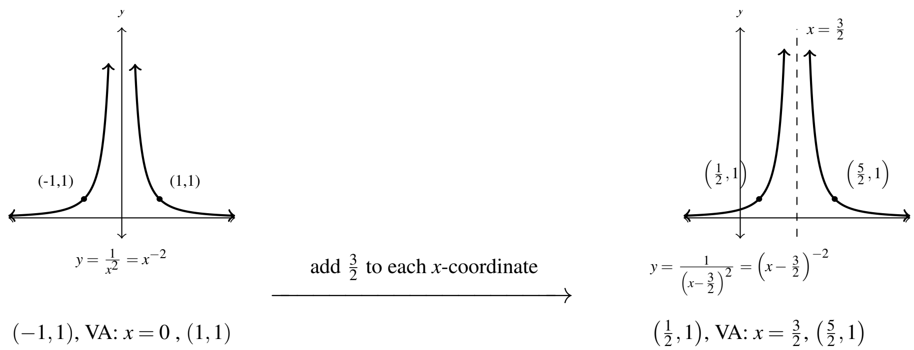
   In order to use Theorem 3.1, we first must put $f(x) = (2x-3)^{-2}$ into the form prescribed by the theorem. To that end, we factor:

$$f(x) = \left(2\left[x - \frac{3}{2}\right]\right)^{-2} = 2^{-2}\left(x - \frac{3}{2}\right)^{-2} = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}$$

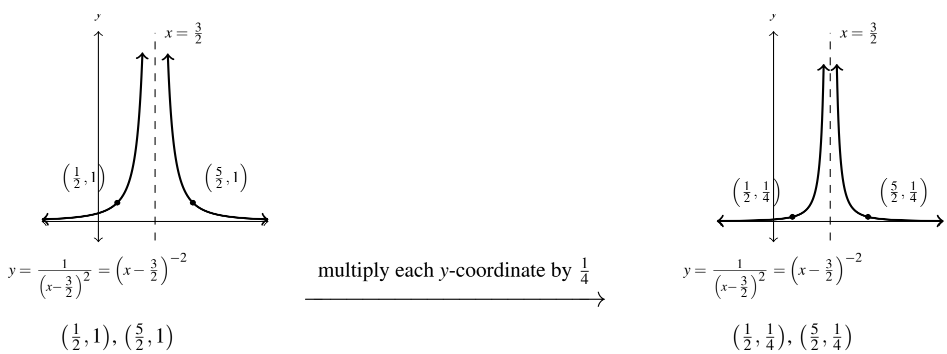
[3] We are, in fact, building on Theorem 1.12 in Section 1.6, so the more you see the forest for the trees, the better off you'll be when the time comes to generalize these moves to all functions.

We identify $n = 2$, $a = \frac{1}{4}$ and $h = \frac{3}{2}$ (and $k = 0$.) Per the theorem, we begin with the graph of $y = x^{-2}$ and track the two points $(-1, 1)$ and $(1, 1)$ along with the vertical and horizontal asymptotes $x = 0$ and $y = 0$, respectively through each step.

Step 1: add $\frac{3}{2}$ to each of the x-coordinates of each of the points on the graph of $y = x^{-2}$. This moves the vertical asymptote from $x = 0$ to $x = \frac{3}{2}$ (which we represent by a dashed line.)



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \left(x - \frac{3}{2}\right)^{-2}$ by $\frac{1}{4}$.



As we did not shift the graph vertically, the horizontal asymptote remains $y = 0$. We can determine the domain and range of f by tracking the changes to the domain and range of our progenitor function, $y = x^{-2}$. We get the domain and range of f is $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and the range of f is $(-\infty, 0) \cup (0, \infty)$.

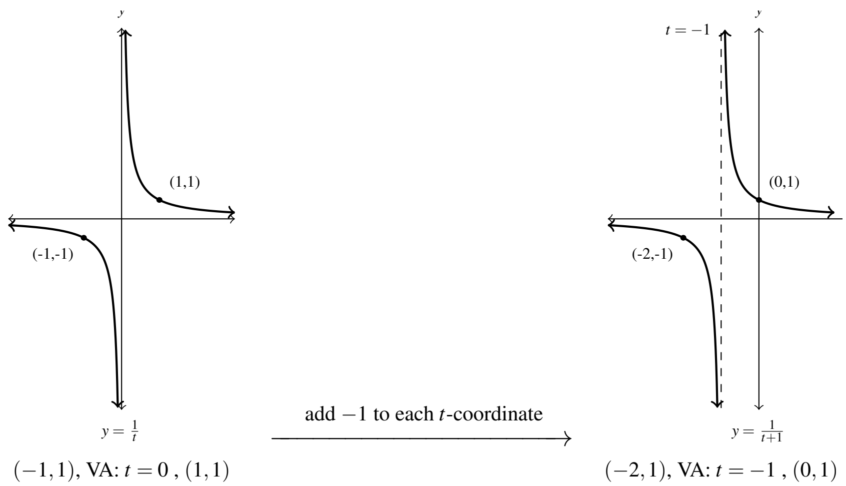
2. Graph $g(t) = \dfrac{2t - 1}{t + 1}$.

   Using long division, we get

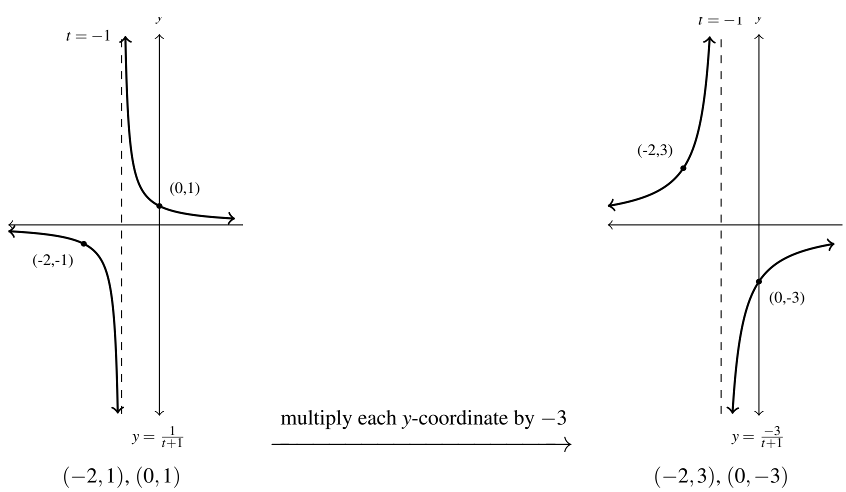
$$g(t) = \frac{2t - 1}{t + 1} = -\frac{3}{t + 1} + 2 = \frac{-3}{(t - (-1))^1} + 2$$

   so we identify $n = 1$, $a = -3$, $h = -1$, and $k = 2$. We start with the graph of $y = \frac{1}{t}$ with points $(-1, -1)$, $(1, 1)$ and asymptotes $t = 0$ and $y = 0$ and track these through each of the steps.

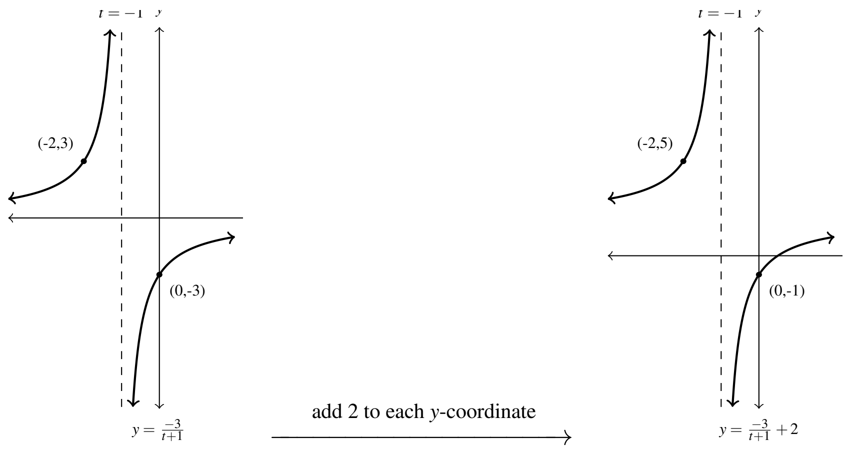
Step 1: Add -1 to each of the t-coordinates of each of the points on the graph of $y = \frac{1}{t}$. This moves the vertical asymptote from $t = 0$ to $t = -1$.



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \frac{1}{t+1}$ by -3.



Step 3: add 2 to each of the y-coordinates of each of the points on the graph of $y = \frac{-3}{t+1}$. This moves the horizontal asymptote from $y = 0$ to $y = 2$.



$(-2,3)$, $(0,-3)$, HA $y=0$ $\qquad$ $(-2,5)$, $(0,-1)$, HA $y=2$

As above, we determine the domain and range of g by tracking the changes in the domain and range of $y=\frac{1}{t}$. We find the domain of g is $(-\infty,-1)\cup(-1,\infty)$ and the range is $(-\infty,2)\cup(2,\infty)$. □

In Example 3.2.1, we once again see the benefit of changing the form of a function to make use of an important result. A natural question to ask is to what extent general rational functions can be rewritten to use Theorem 3.1. In the same way polynomial functions are sums of monomial functions, it turns out, allowing for non-real number coefficients, that every rational function can be written as a sum of (possibly shifted) Laurent monomial functions.

3.2.2 Local Behavior near Excluded Values

We take time now to focus on behaviors of the graphs of rational functions near excluded values. We've already seen examples of one type of behavior: vertical asymptotes. Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[4]

Example 3.2.2. A mathematical model for the population $P(t)$, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t)=\frac{100}{(5-t)^2}$, $0\leq t<5$.

1. Compute and interpret $P(0)$.
2. When will the population reach 100,000?
3. Graph $y=P(t)$.

[4]These functions arise in Differential Equations. The unfortunate name will make sense shortly.

4. Determine and interpret the behavior of P as $t \to 5^-$.

Solution.

1. Compute and interpret $P(0)$.

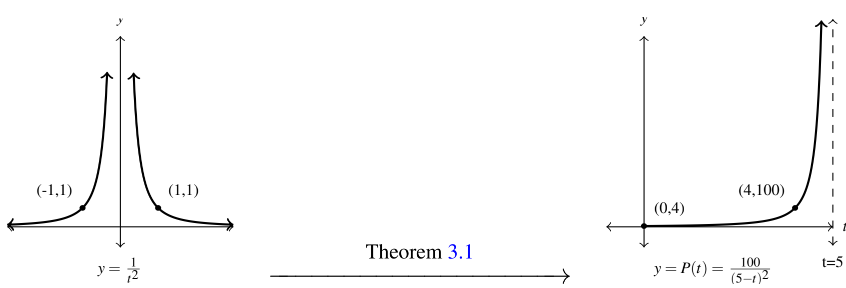
   Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$. Due to the fact that t represents the number of days *after* the bacteria are introduced into the environment, $t = 0$ corresponds to the day the bacteria are introduced. $P(t)$ is measured in *thousands*, thus $P(t) = 4$ means 4000 bacteria are initially introduced into the environment.

2. When will the population reach 100,000?

   To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in *thousands*. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Hence, we need to solve $P(t) = \frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

3. Graph $y = P(t) = \frac{100}{(5-t)^2}$.

   After a slight re-write, we have $P(t) = \frac{100}{(5-t)^2} = \frac{100}{[(-1)(t-5)]^2} = \frac{100}{(t-5)^2}$. Using Theorem 3.1, we start with the graph of $y = \frac{1}{t^2}$ below on the left. After shifting the graph to the right 5 units and stretching it vertically by a factor of 100 (note, the graphs are not to scale!), we restrict the domain to $0 \le t < 5$ to arrive at the graph of $y = P(t)$ below on the right.



$(-1,1)$, VA: $t = 0$, $(1,1)$ $\quad$ $(4,100)$, VA: $t = 5$

4. Determine and interpret the behavior of P as $t \to 5^-$.

   We see that as $t \to 5^-$, $P(t) \to \infty$. This means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ days the environment would collapse. $\square$

Will all values excluded from the domain of a rational function produce vertical asymptotes in the graph? The short answer is 'no.' There are milder interruptions that can occur - holes in the graph - which we explore in our next example. To this end, we formalize the notion of *average velocity* - a concept we first encountered in Example 1.3.12 in Section 1.3.1. In that example, the function $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The function s an example of a **position function** as it provides information about *where* the rocket is at time t. In that example, we interpreted the average rate of change of s over an interval as the average velocity of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. Suppose we have a position function s defined over an interval containing some fixed time t_0. We can define the average velocity as a function of any time t other than t_0:

Definition 3.5. Suppose $s(t)$ gives the position of an object at time t and t_0 is a fixed time in the domain of s. The **average velocity** between time t and time t_0 is given by

$$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0},$$

provided $t \neq t_0$.

It is clear why we must exclude $t = t_0$ from the domain of $\overline{v}$ in Definition 3.5 because otherwise we would have a 0 in the denominator. What is interesting in this case however, is that substituting $t = t_0$ also produces 0 in the *numerator*. (Do you see why?) While '$\frac{0}{0}$' is undefined, it is more precisely called an 'indeterminate form' and is studied extensively in Calculus. We can nevertheless explore this function in the next example.

Example 3.2.3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Identify and simplify an expression for the average velocity of the rocket between times t and 15.
2. Compute and interpret $\overline{v}(14)$.
3. Graph $y = \overline{v}(t)$. Interpret the intercepts.
4. Interpret the behavior of $\overline{v}$ as $t \to 15$.

Solution.

1. Identify and simplify an expression for the average velocity of the rocket between times t and 15.

   Using Definition 3.5 with $t_0 = 15$, we get:

$$\begin{aligned} \overline{v}(t) &= \frac{s(t) - s(15)}{t - 15} \\ &= \frac{(-5t^2 + 100t) - 375}{t - 15} \end{aligned}$$

$$\begin{aligned}
&= \frac{-5(t^2-20t+75)}{t-15} \\
&= \frac{-5(t-15)(t-5)}{t-15} \\
&= \frac{-5\cancel{(t-15)}(t-5)}{\cancel{(t-15)}} \\
&= -5(t-5) = -5t+25 \qquad t \neq 15
\end{aligned}$$

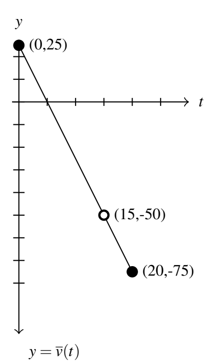
The domain of s is $0 \leq t \leq 20$, thus our final answer is $\overline{v}(t) = -5t+25$, for $t \in [0,15) \cup (15,20]$.

2. Compute and interpret $\overline{v}(14)$.

   We find $\overline{v}(14) = -5(14)+25 = -45$. This means between 14 and 15 seconds after launch, the rocket was traveling, on average a speed 45 feet per second *downwards*, or falling back to the Moon's surface.

3. Graph $y = \overline{v}(t)$. Interpret the intercepts.

   The graph of $\overline{v}(t)$ is a portion of the line $y = -5t+25$. The domain of s is $[0,20]$ and $\overline{v}(t)$ is not defined when $t = 15$, therefore our graph is the line segment starting at $(0,25)$ and ending at $(20,75)$ with a hole at $(15,50)$. The y-intercept is $(0,25)$ which means on average, the rocket is traveling 25 feet per second *upwards*.[5] To get the t-intercept, we set $\overline{v}(t) = -5t+25 = 0$ and obtain $t = 5$. Hence, $\overline{v}(5) = 0$ or the average velocity between times $t = 5$ and $t = 15$ is 0. As you may recall, this is due to the rocket being at the same altitude (375 feet) at both times, hence, $\Delta[s(t)]$ and, hence $\overline{v}(t) = 0$.



4. Interpret the behavior of $\overline{v}$ as $t \to 15$.

   From the graph, we see as $t \to 15$, $\overline{v}(t) \to -50$. (This is also borne out in the numerically in the included tables.) This means as we sample the average velocity between time $t_0 = 15$ and and times closer and closer to 15, the average velocity approaches -50. This value is how we define the **instantaneous velocity** - that is, *at* $t = 15$ seconds, the rocket is falling at a rate of 50 feet per second towards the surface of the Moon.

[5] Note that the rocket has already started its descent at $t = 10$ seconds (see Example 1.3.12 in Section 1.3.1.) However, the rocket is still at a higher altitude at when $t = 15$ than $t = 0$ which produces a positive *average* velocity.

t	$\overline{v}(t)$
14.9	-49.5
14.99	-49.95
14.999	-49.995
15	undefined
15.001	-50.005
15.01	-50.05
15.1	-50.5

□

If nothing else, Example 3.2.3 shows us that just because a value is excluded from the domain of a rational function doesn't mean there will be a vertical asymptote to the graph there. In this case, the factor $(t-15)$ divided out from the denominator, thereby effectively removing the threat of dividing by 0. It turns out, this situation generalizes to the theorem below.

Theorem 3.2. Location of Vertical Asymptotes and Holes:[a] Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros.[b] Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'
[b] In other words, $r(x)$ is in lowest terms.

In English, Theorem 3.2 says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$. Like many properties of rational functions, we owe Theorem 3.2 to Calculus, but that won't stop us from putting Theorem 3.2 to good use in the following example.

Example 3.2.4. For each function below:

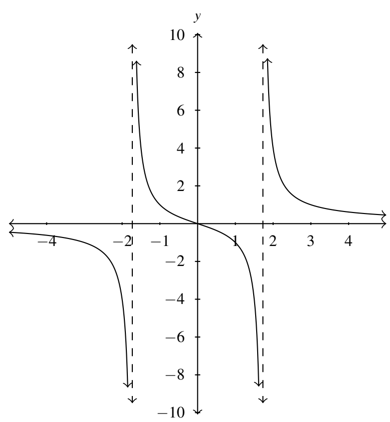
- determine the values excluded from the domain.
- determine whether each excluded value corresponds to a vertical asymptote or hole of the graph.
- verify your answers with a rough sketch of a graph.
- describe the behavior of the graph near each excluded value using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{2x}{x^2-3}$
2. $g(t) = \dfrac{t^2-t-6}{t^2-9}$
3. $h(t) = \dfrac{t^2-t-6}{t^2+9}$
4. $r(t) = \dfrac{t^2-t-6}{t^2+6t+9}$

Solution.

1. Determine the vertical behavior of $f(x) = \dfrac{2x}{x^2-3}$.

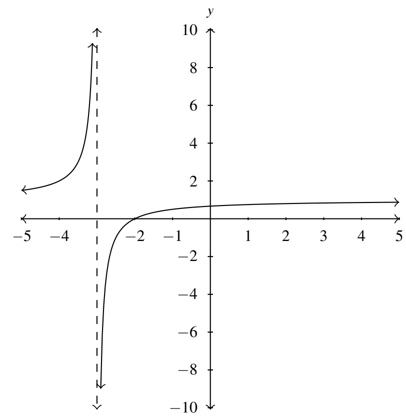
   To use Theorem 3.2, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2-3=0$ and get $x=\pm\sqrt{3}$. Because the expression $f(x)$ is in lowest terms (can you see why?), there is no division of like factors possible, and we conclude that the lines $x=-\sqrt{3}$ and $x=\sqrt{3}$ are vertical asymptotes to the graph of $y=f(x)$. The graph verifies this claim, and from the graph, we see that as $x \to -\sqrt{3}^-$, $f(x) \to -\infty$, as $x \to -\sqrt{3}^+$, $f(x) \to \infty$, as $x \to \sqrt{3}^-$, $f(x) \to -\infty$, and finally as $x \to \sqrt{3}^+$, $f(x) \to \infty$. As a side note, the graph of f appears to be symmetric about the origin. Sure enough, we find: $f(-x) = \frac{2(-x)}{(-x)^2-3} = -\frac{2x}{x^2-3} = -f(x)$, proving f is odd.



$y = f(x) = \frac{3x}{x^2-3}$

2. Determine the vertical behavior of $g(t) = \dfrac{t^2-t-6}{t^2-9}$.

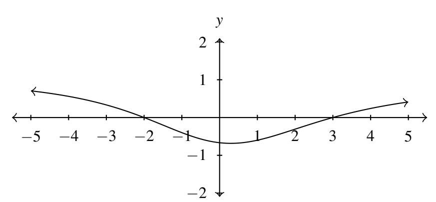
   As above, we find the values excluded from the domain of g by setting the denominator equal to 0. Solving $t^2-9=0$ gives $t=\pm 3$. In lowest terms $g(t) = \frac{t^2-t-6}{t^2-9} = \frac{(t-3)(t+2)}{(t-3)(t+3)} = \frac{t+2}{t+3}$. Because $t=-3$ continues to be a zero of the denominator in the reduced formula, we know the line $t=-3$ is a vertical asymptote to the graph of $y=g(t)$. AS $t=3$ does not produce a '0' in the denominator of the reduced formula, we have a hole at $t=3$. To find the y-coordinate of the hole, we substitute $t=3$ into the reduced formula: $\frac{t+2}{t+3} = \frac{3+2}{3+3} = \frac{5}{6}$ so the hole is at $\left(3, \frac{5}{6}\right)$. Graphing g we can definitely see the vertical asymptote $t=-3$: as $t \to -3^-$, $g(t) \to \infty$ and as $t \to -3^+$, $g(t) \to -\infty$. Near $t=3$, the graph seems to have no interruptions (but we know g is undefined at $t=3$.) As g appears to be increasing on $(-3,\infty)$, we write as $t \to 3^-$, $g(t) \to \frac{5}{6}^-$, and as $t \to 3^+$, $g(t) \to \frac{5}{6}^+$.



$y = g(t) = \frac{t^2-t-6}{t^2-9}$

3. Determine the vertical behavior of $h(t) = \dfrac{t^2-t-6}{t^2+9}$

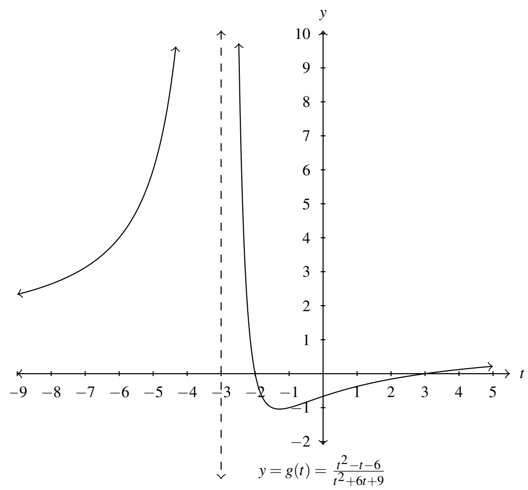
   Setting the denominator of the expression for $h(t)$ to 0 gives $t^2+9=0$, which has no real solutions. Accordingly, the graph of $y = h(t)$ (at least as much as we can discern from the technology) is devoid of both vertical asymptotes and holes.



$y = g(t) = \frac{t^2-t-6}{t^2+9}$

4. Determine the vertical behavior of $r(t) = \dfrac{t^2-t-6}{t^2+6t+9}$.

   Setting the denominator of $r(t)$ to zero gives the equation $t^2+6t+9=0$. We get the (repeated!) solution $t=-3$. Simplifying, we see $r(t) = \frac{t^2-t-6}{t^2+6t+9} = \frac{(t-3)(t+3)}{(t+3)^2} = \frac{t-3}{t+3}$. Because $t=-3$ continues to produce a 0 in the denominator of the reduced function, we know $t=-3$ is a vertical asymptote to the graph. The calculator bears this out, and, moreover, we see that as $t \to -3^-$, $r(t) \to \infty$ and as $t \to -3^+$, $r(t) \to -\infty$.



□

3.2.3 End Behavior

Now that we've thoroughly discussed behavior near values excluded from the domains of rational functions, focus our attention on end behavior. We have already seen one example of this in the form of horizontal asymptotes. Our next example of the section gives us a real-world application of a horizontal asymptote.[6]

Example 3.2.5. The number of students $N(t)$ at local college who have had the flu t months after the semester begins can be modeled by:

$$N(t) = \frac{1500t+50}{3t+1}, \quad t \geq 0.$$

1. Compute and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Use Theorem 3.1 to graph $y = N(t)$.
4. Determine and interpret the behavior of N as $t \to \infty$.

Solution.

1. Compute and interpret $N(0)$.

   Substituting $t = 0$ gives $N(0) = \frac{1500(0)+50}{1+3(0)} = 50$. As t represents the number of months since the beginning of the semester, $t = 0$ describes the state of the flu outbreak at the beginning of the semester. Hence, at the beginning of the semester, 50 students have had the flu.

[6] Though the population below is more accurately modeled with the functions in Chapter 5, we approximate it (using Calculus, of course!) using a rational function.

2. How long will it take until 300 students will have had the flu?

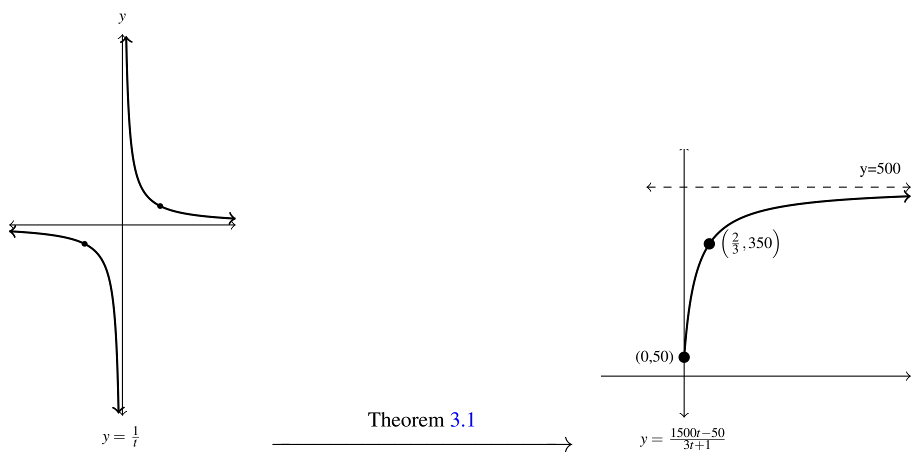
   We set $N(t) = \frac{1500t+50}{3t+1} = 300$ and solve. Clearing denominators gives $1500t + 50 = 300(3t+1)$ from which we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.

3. Use Theorem 3.1 to graph $y = N(t)$.

   To graph $y = N(t)$, we first use long division to rewrite $N(t) = \frac{-450}{3t+1} + 500$. From there, we get

$$N(t) = -\frac{450}{3t+1} + 500 = \frac{-450}{3\left(t+\frac{1}{3}\right)} + 500 = \frac{-150}{t+\frac{1}{3}} + 500$$

   Using Theorem 3.1, we start with the graph of $y = \frac{1}{t}$ below on the left and perform the following steps: shift the graph to the left by $\frac{1}{3}$ units, stretch the graph vertically by a factor of 150, reflect the graph across the t-axis, and finally, shift the graph up 500 units. As the domain of N is $t \geq 0$, we obtain the graph below on the right.



$(-1,1)$, $(1,1)$, HA: $y = 0$

$\left(\frac{2}{3}, 350\right)$, HA: $y = 500$

4. Determine and interpret the behavior of N as $t \to \infty$.

   From the graph, we see as $t \to \infty$, $N(t) \to 500$. (More specifically, 500^{-}.) This means as time goes by, only a total of 500 students will have ever had the flu.

   We determined the horizontal asymptote to the graph of $y = N(t)$ by rewriting $N(t)$ into a form compatible with Theorem 3.1, and while there is nothing wrong with this approach, it will simply not work for general rational functions which cannot be rewritten this way. To that end, we revisit this problem using Theorem 2.4 from Section 2.2. The end behavior of the numerator of $N(t) = \frac{1500t+50}{3t+1}$ is determined by its leading term, $1500t$, and the end behavior of the denominator is likewise determined by its leading term, $3t$. Hence, as $t \to \pm\infty$,

$$N(t) = \frac{1500t+50}{3t+1} \approx \frac{1500t}{3t} = 500.$$

Hence as $t \to \pm\infty$, $y = N(t) \to 500$, producing the horizontal asymptote $y = 500$. □

This same reasoning can be used in general to argue the following theorem.

Theorem 3.3. Location of Horizontal Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a]The use of the definite article will be justified momentarily.

So see why Theorem 3.3 works, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, Theorem 2.4 gives $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively.

If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{ax^n}{bx^n} = \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case.

If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{ax^n}{bx^m} = \frac{a}{bx^{m-n}} \to 0$. As $x \to \pm\infty$, $r(x)$ is more or less a fraction with a constant numerator, a, but a denominator which is unbounded. Hence, $r(x) \to 0$ producing the horizontal asymptote $y = 0$.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^n}{bx^m} = \frac{ax^{n-m}}{b}$, which is a monomial function from Section 2.2. As such, r becomes unbounded as $x \to \pm\infty$.

Note that in the two cases which produce horizontal asymptotes, the behavior of r is identical as $x \to \infty$ and $x \to -\infty$. Hence, if the graph of a rational function has a horizontal asymptote, there is only one.[7]

We put Theorem 3.3 to good use in the following example.

Example 3.2.6. For each function below:

- use Theorem 2.4 to analytically determine the horizontal asymptotes of the graph, if any.

[7]We will (first) encounter functions with more than one horizontal asymptote in Chapter 4.1.

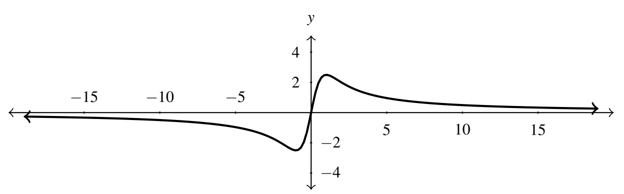
- check your answers using Theorem 3.3 and a graph.
- describe the end behavior of the graph using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $F(s) = \dfrac{5s}{s^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(t) = \dfrac{6t^3-3t+1}{5-2t^3}$
4. $r(x) = 2 - \dfrac{3x^2}{1-x^2}$

Solution.

1. Determine the horizontal behavior of $F(s) = \dfrac{5s}{s^2+1}$.

   Using Theorem 2.4, we get as $s \pm \infty$, $F(s) = \frac{5s}{s^2+1} \approx \frac{5s}{s^2} = \frac{5}{s}$. Hence, as $s \to \infty$, $F(s) \to 0$, so $y = 0$ is a horizontal asymptote to the graph of $y = F(s)$. To check, we note that the degree of the numerator of $F(s)$, 1, is less than the degree of the denominator, 2 so Theorem 3.3 gives $y = 0$ is the horizontal asymptote. Graphically, we see as $s \to \pm\infty$, $F(s) \to 0$. More specifically, as $s \to -\infty$, $F(s) \to 0^-$ and as $s \to \infty$, $F(s) \to 0^+$. As a side note, the graph of F appears to be symmetric about the origin. Indeed, $F(-s) = \frac{5(-s)}{(-s)^2+1} = -\frac{5s}{s^2+1}$ proving F is odd.

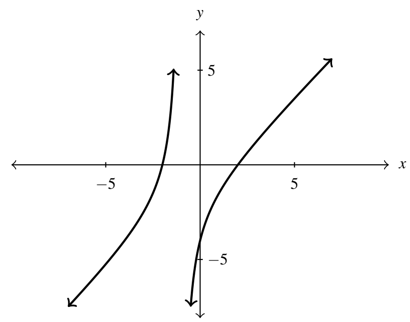


$y = \frac{5s}{s^2+1}$

2. Determine the horizontal behavior of $g(x) = \dfrac{x^2-4}{x+1}$.

   As $x \to \pm\infty$, $g(x) = \frac{x^2-4}{x+1} \approx \frac{x^2}{x} = x$, and while $y = x$ is a line, it is not a horizontal line. Hence, we conclude the graph of $y = g(x)$ has no horizontal asymptotes. Sure enough, Theorem 3.3 supports this as the degree of the numerator of $g(x)$ is 2 which is greater than the degree of the denominator, 1. By, there is no horizontal asymptote. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote.[8]

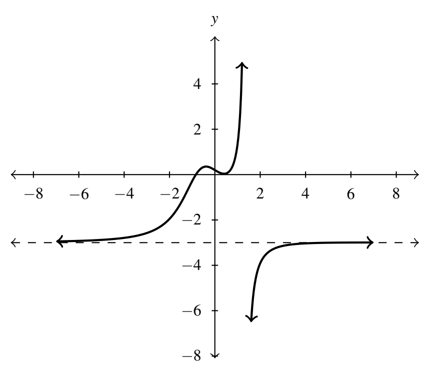
[8] Sit tight! We'll revisit this function and its end behavior shortly.



$y = \frac{x^2-4}{x+1}$

3. Determine the horizontal behavior of $h(t) = \dfrac{6t^3 - 3t + 1}{5 - 2t^3}$.

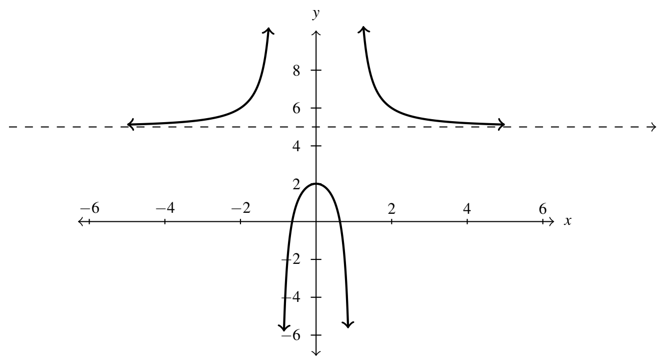
   We have $h(t) = \frac{6t^3-3t+1}{5-2t^3} \approx \frac{6t^3}{-2t^3} = -3$ as $t \to \pm\infty$, indicating a horizontal asymptote $y = -3$. Sure enough, the degrees of the numerator and denominator of $h(t)$ are both three, so Theorem 3.3 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the graph of $y = h(t)$ that as $t \to -\infty$, $h(t) \to -3^+$, and as $t \to \infty$, $h(t) \to -3^-$.



$y = \frac{6t^3-3t+1}{5-2t^3}$

4. Determine the horizontal behavior of $r(x) = 2 - \dfrac{3x^2}{1 - x^2}$.

   If we apply Theorem 2.4 to the $\frac{3x^2}{1-x^2}$ term in the expression for $r(x)$, we find $\frac{3x^2}{1-x^2} \approx \frac{3x^2}{-x^2} = -3$ as $x \to \pm\infty$. It seems reasonable to conclude, then, that $r(x) = 2 - \frac{3x^2}{1-x^2} \approx 2 - (-3) = 5$ so $y = 5$ is our horizontal asymptote. In order to use Theorem 3.3 as stated, however, we need to rewrite the expression $r(x)$ with a single denominator: $r(x) = 2 - \frac{3x^2}{1-x^2} = \frac{2(1-x^2)-3x^2}{1-x^2} = \frac{2-5x^2}{1-x^2}$. Now we apply Theorem 3.3 and note the numerator and denominator have the same degree, we guarantee the horizontal asymptote is $y = \frac{-5}{-1} = 5$. These calculations are borne out graphically where it appears as if as $x \to \pm\infty$, $r(x) \to 5^+$. As a final note, the graph of r appears to be symmetric about the y-axis. We find $r(-x) = 2 - \frac{3(-x)^2}{1-(-x)^2} = 2 - \frac{3x^2}{1-x^2} = r(x)$, proving r is even.



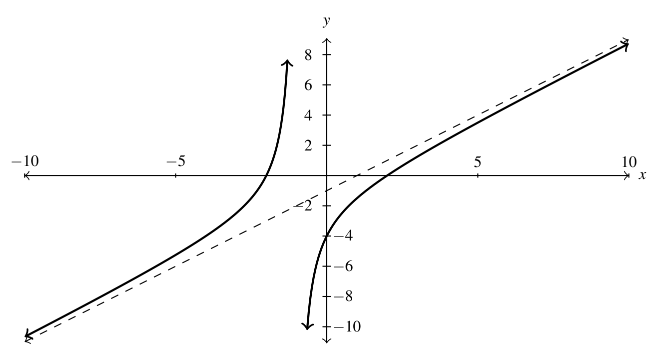
$y = 2 - \frac{3x^2}{1-x^2}$

□

We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example 3.2.6. Performing long division,[9] we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Because the term $\frac{3}{x+1} \to 0$ as $x \to \pm\infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to \pm\infty$. We see this play out both numerically and graphically below. (As usual, the asymptote $y = x - 1$ is denoted by a dashed line.)

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999



$y = \frac{x^2-4}{x+1}$

[9] See the remarks following Theorem 3.3.

The way we symbolize the relationship between the end behavior of $y = g(x)$ with that of the line $y = x-1$ is to write 'as $x \to \pm\infty$, $g(x) \to x-1$' in order to have some notational consistency with what we have done earlier in this section when it comes to end behavior.[10] In this case, we say the line $y = x-1$ is a **slant asymptote**[11] of the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to \infty$ or as $x \to -\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as follows.

Definition 3.6. The line $y = mx+b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx+b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 3.6 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx+b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx+b$' as '$[f(x)-(mx+b)] \to 0$.' In other words, the graph of $y = f(x)$ has the *slant* asymptote $y = mx+b$ if and only if the graph of $y = f(x)-(mx+b)$ has a *horizontal* asymptote $y = 0$. If we wanted to, we could introduce the notations $f(x) \to (mx+b)^+$ to mean $[f(x)-(mx+b)] \to 0^+$ and $f(x) \to (mx+b)^-$ to mean $[f(x)-(mx+b)] \to 0^-$, but these non-standard notations.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \frac{x^2-4}{x+1}$, the degree of the numerator x^2-4 is 2, which is *exactly one more* than the degree if its denominator $x+1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.

Theorem 3.4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is *exactly* one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 3.3 gives us an easy way to see if the graph of a rational function $r(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 3.4 gives us an easy way to check for slant asymptotes. Unlike Theorem 3.3, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 3.4 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it.[12]

Example 3.2.7. For each of the following functions:

[10] Other notations include $g(x) \asymp x-1$ or $g(x) \sim x-1$.

[11] Also called an 'oblique' asymptote in some, ostensibly higher class (and more expensive), texts.

[12] That's OK, though. In the next section, we'll use long division to analyze end behavior and it's worth the effort!

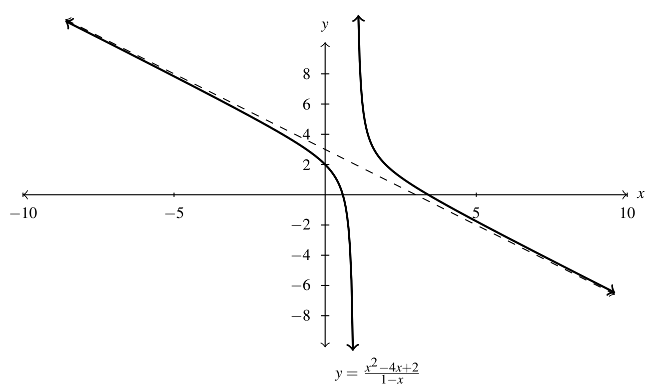
- identify the slant asymptote, if it exists.
- verify your answer using a graph.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{x^2-4x+2}{1-x}$
2. $g(t) = \dfrac{t^2-4}{t-2}$
3. $h(x) = \dfrac{x^3+1}{x^2-4}$
4. $r(t) = 2t-1+\dfrac{4t^3}{1-t^2}$

Solution.

1. Determine any slant asymptotes of $f(x) = \dfrac{x^2-4x+2}{1-x}$.

   The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 3.4 guarantees a slant asymptote exists. To find it, we divide $1-x=-x+1$ into x^2-4x+2 and get a quotient of $-x+3$, so our slant asymptote is $y=-x+3$. We confirm this graphically below.

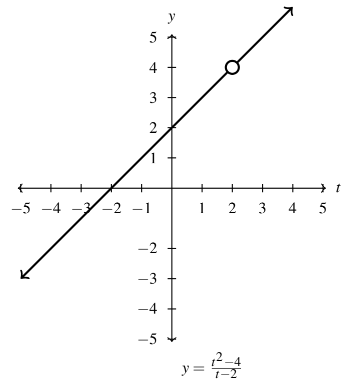
   

2. Determine any slant asymptotes of $g(t) = \dfrac{t^2-4}{t-2}$.

   As with the previous example, the degree of the numerator $g(t) = \frac{t^2-4}{t-2}$ is 2 and the degree of the denominator is 1, so Theorem 3.4 applies. In this case,

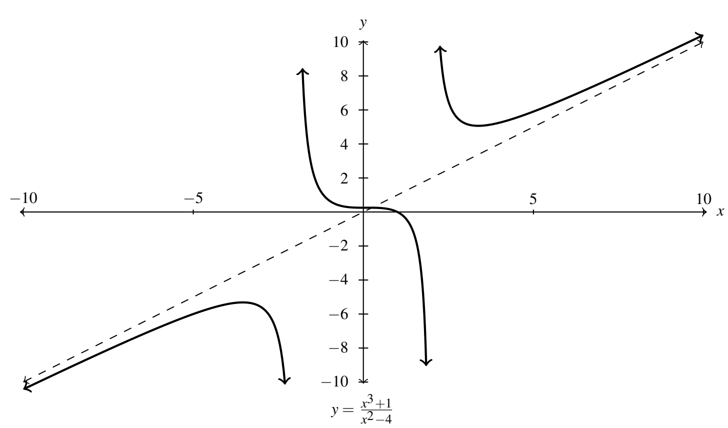
   $$g(t) = \frac{t^2-4}{t-2} = \frac{(t+2)(t-2)}{(t-2)} = \frac{(t+2)\cancel{(t-2)}}{\cancel{(t-2)}^{\,1}} = t+2, \quad t \neq 2$$

   so we have that the slant asymptote, $y=t+2$, is identical to the graph of $y=g(t)$ except at $t=2$ (where the latter has a 'hole' at $(2,4)$.) While the word 'asymptote' has the connotation of 'approaching but not equaling,' Definitions 3.6 and 3.4 allow for these extreme cases.



3. Determine any slant asymptotes of $h(x) = \dfrac{x^3+1}{x^2-4}$.

   For $h(x) = \frac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2, so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right) \div \left(x^2-4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. The graph confirms this. Note the graph of h appears to be symmetric about the origin. We check $h(-x) = \frac{(-x)^3+1}{(-x)^2-4} = \frac{-x^3+1}{x^2-4} = -\frac{x^3-1}{x^2-4}$. However, $-h(x) = -\frac{x^3+1}{x^2-4}$, so it appears as if $h(-x) \neq -h(x)$ for all x. Checking $x = 1$, we find $h(1) = -\frac{2}{3}$ but $h(-1) = 0$ which shows the graph of h, is in fact, *not* symmetric about the origin.

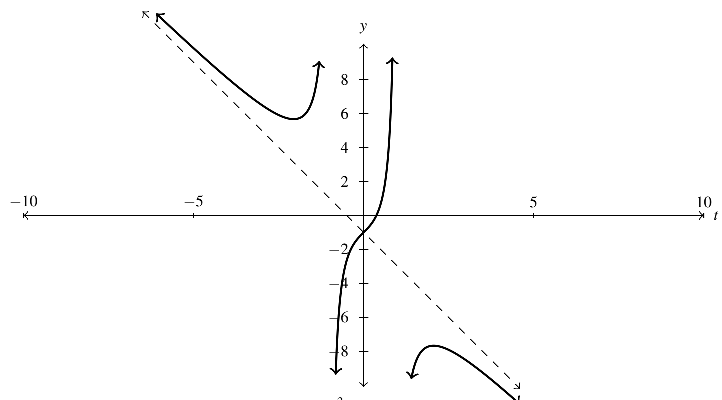


4. Determine any slant asymptotes of $r(t) = 2t - 1 + \dfrac{4t^3}{1-t^2}$.

   For our last example, $r(t) = 2t - 1 + \frac{4t^3}{1-t^2}$, the expression $r(t)$ is not in the form to apply Theorem 3.4 directly. We can, nevertheless, appeal to the spirit of the theorem and use long division to rewrite the term $\frac{4t^3}{1-t^2} = -4t + \frac{4t}{1-t^2}$. We then get:

$$\begin{aligned} r(t) &= 2t - 1 + \frac{4t^3}{1-t^2} \\ &= 2t - 1 - 4t + \frac{4t}{1-t^2} \\ &= -2t - 1 + \frac{4t}{1-t^2} \end{aligned}$$

As $t \to \pm\infty$, Theorem 2.4 gives $\frac{4t}{1-t^2} \approx \frac{4t}{-t^2} = -\frac{4}{t} \to 0$. Hence, as $t \to \pm\infty$, $r(t) \to -2t - 1$, so $y = -2t - 1$ is the slant asymptote to the graph as confirmed by the graph below. from a distance, the graph of r appears to be symmetric about the origin. However, if we look carefully, we see the y-intercept is $(0, -1)$, as borne out by the computation $r(0) = -1$. Hence r cannot be odd. (Do you see why?)



$y = 2t - 1 + \frac{4t^3}{1-t^2}$

□

Our last example gives a real-world application of a slant asymptote. The problem features the concept of **average profit**. The average profit, denoted $\overline{P}(x)$, is the total profit, $P(x)$, divided by the number of items sold, x. In English, the average profit tells us the profit made per item sold. It, along with average cost, is defined below.

Definition 3.7. Let $C(x)$ and $P(x)$ represent the cost and profit to make and sell x items, respectively.

- The **average cost**, $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$.

  NOTE: The average cost is the cost per item produced.

- The **average profit**, $\overline{P}(x) = \dfrac{P(x)}{x}$, $x > 0$.

  NOTE: The average profit is the profit per item sold.

You'll explore average cost (and its relation to variable cost) in Exercise 37. For now, we refer the reader to to Example 2.1.3 in Section 2.1.

Example 3.2.8. Recall the profit (in dollars) when x PortaBoy game systems are produced and sold is given by $P(x) = -1.5x^2 + 170x - 150$, $0 \leq x \leq 166$.

1. Determine and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?
2. Compute and interpret $\overline{P}(50)$.
3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graph.
4. Interpret the slope of the slant asymptote.

Solution.

1. Determine and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?

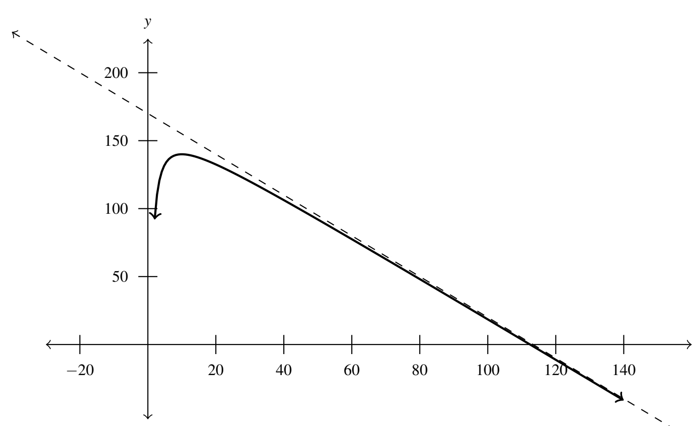
   We find $\overline{P}(x) = \frac{P(x)}{x} = \frac{-1.5x^2+170x-150}{x} = -1.5x + 170 - \frac{150}{x}$. While the domain of P is $[0, 166]$, $x \neq 0$, thus the domain of $\overline{P}$ is $(0, 166]$.

2. Compute and interpret $\overline{P}(50)$.

   We find $\overline{P}(50) = -1.5(50) + 170 - \frac{150}{50} = 92$. This means that when 50 PortaBoy systems are sold, the average profit is \$92 per system.

3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graph.

   Technically, the graph of $y = \overline{P}(x)$ has no slant asymptote because the domain of the function is restricted to $(0, 166]$. That being said, if we were to let $x \to \infty$, the term $\frac{150}{x} \to 0$, so we'd have $\overline{P}(x) \to -1.5x + 170$. This means the slant asymptote would be $y = -1.5x + 170$. We graph $y = \overline{P}(x)$ and $y = -1.5x + 170$.



$y = \overline{P}(x) = \frac{P(x)}{x}$

4. Interpret the slope of the slant asymptote.

   The slope of the slant asymptote $y = -1.5x + 170$ is -1.5. Because, ostensibly $\overline{P}(x) \approx -1.5x + 170$, this means that, as we sell more systems, the average profit is decreasing at about a rate of \$1.50 per system. If the number 1.5 sounds familiar to this problem situation, it should. In Example 1.3.9 in Section 1.3.1, we determined the slope of the demand function to be -1.5. In that situation, the -1.5 meant that in order to sell an additional system, the price had to drop by \$1.50. The fact the average profit is decreasing at more or less this same rate means the loss in profit per system can be attributed to the reduction in price needed to sell each additional system.[13] □

3.2.4 Exercises

(Review of Long Division):[14] In Exercises 1 - 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

[13] We generalize this result in Exercise 38.

[14] For more review, see Section 2.3.

In Exercises 7 - 10, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 3.1. Track at least two points and the asymptotes. State the domain and range using interval notation.

7. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{1}{x-2} + 1$

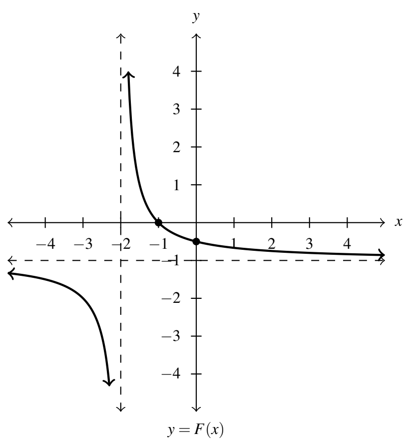
8. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{2x}{x+1}$

9. $f(x) = x^{-1}$, $F(x) = 4x(2x+1)^{-1}$

10. $f(x) = x^{-2}$, $F(x) = -(x-1)^{-2} + 3$

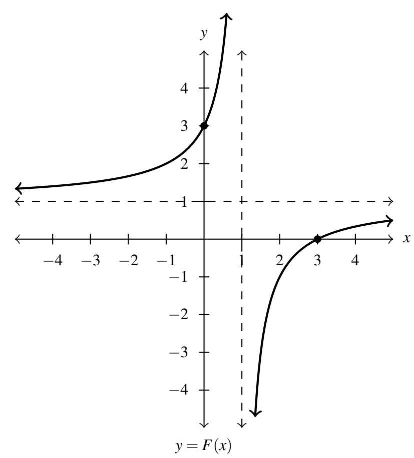
In Exercises 11 - 12, find a formula for each function below in the form $F(x) = \dfrac{a}{x-h} + k$.

11. $y = F(x)$



x-intercept $(-1, 0)$, y-intercept $\left(0, -\frac{1}{2}\right)$

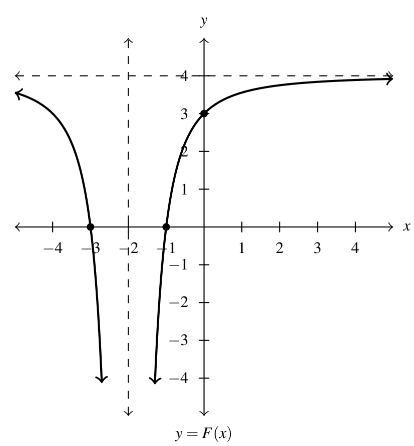
12. $y = F(x)$



x-intercept $(3, 0)$, y-intercept $(0, 3)$

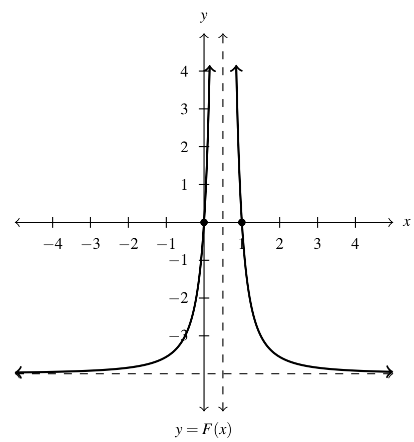
In Exercises 13 - 14, find a formula for each function below in the form $F(x) = \dfrac{a}{(x-h)^2} + k$.

13. $y = F(x)$



x-intercepts $(-3, 0)$, $(-1, 0)$, y-intercept $(0, 3)$

14. $y = F(x)$



x-intercepts $(0,0)$, $(1,0)$, vertical asymptote: $x = \frac{1}{2}$

In Exercises 15 - 32, for the given rational function:

- State the domain.
- Identify any vertical asymptotes of the graph.
- Identify any holes in the graph.
- Determine the horizontal asymptote, if it exists.
- Determine the slant asymptote, if it exists.
- Graph the function and describe the behavior near the asymptotes.

15. $f(x) = \dfrac{x}{3x-6}$

16. $f(x) = \dfrac{3+7x}{5-2x}$

17. $f(x) = \dfrac{x}{x^2+x-12}$

18. $g(t) = \dfrac{t}{t^2+1}$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$

20. $g(t) = \dfrac{t^3+1}{t^2-1}$

21. $r(z) = \dfrac{4z}{z^2+4}$

22. $r(z) = \dfrac{4z}{z^2-4}$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6}$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9}$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1}$

30. $r(z) = \dfrac{z^3}{1-z}$

31. $r(z) = \dfrac{18-2z^2}{z^2-9}$

32. $r(z) = \dfrac{z^3-4z^2-4z-5}{z^2+z+1}$

33. The cost $C(p)$ in dollars to remove $p\%$ of the invasive Ippizuti fish species from Sasquatch Pond is:

$$C(p) = \frac{1770p}{100-p}, \quad 0 \le p < 100$$

(a) Compute and interpret $C(25)$ and $C(95)$.

(b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for \$40000?

34. In the scenario of Example 3.2.3, $s(t) = -5t^2 + 100t$, $0 \le t \le 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. For each of the times t_0 listed below, find and simplify a the formula for the average velocity $\overline{v}(t)$ between t and t_0 (see Definition 3.5) and use $\overline{v}(t)$ to find and interpret the instantaneous velocity of the rocket at $t = t_0$.

(a) $t_0 = 5$

(b) $t_0 = 9$

(c) $t_0 = 10$

(d) $t_0 = 11$

35. The population of Sasquatch in Portage County t years after the year 1803 is modeled by the function

$$P(t) = \frac{150t}{t+15}.$$

Find and interpret the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

36. The cost in dollars, $C(x)$ to make x dOpi media players is $C(x) = 100x + 2000$, $x \geq 0$. You may wish to review the concepts of fixed and variable costs introduced in Example 1.3.8 in Section 1.3.3.

    (a) Write a formula for the average cost $\overline{C}(x)$.

    (b) Compute and interpret $\overline{C}(1)$ and $\overline{C}(100)$.

    (c) How many dOpis need to be produced so that the average cost per dOpi is \$200?

    (d) Interpret the behavior of $\overline{C}(x)$ as $x \to 0^+$.

    (e) Interpret the behavior of $\overline{C}(x)$ as $x \to \infty$.

37. This exercise explores the relationships between fixed cost, variable cost, and average cost. The reader is encouraged to revisit Example 1.3.8 in Section 1.3.3 as needed. Suppose the cost in dollars $C(x)$ to make x items is given by $C(x) = mx + b$ where m and b are positive real numbers.

    (a) Show the fixed cost (the money spent even if no items are made) is b.

    (b) Show the variable cost (the increase in cost per item made) is m.

    (c) Write a formula for the average cost when making x items, $\overline{C}(x)$.

    (d) Show $\overline{C}(x) > m$ for all $x > 0$ and, moreover, $\overline{C}(x) \to m^+$ as $x \to \infty$.

    (e) Interpret $\overline{C}(x) \to m^+$ both geometrically and in terms of fixed, variable, and average costs.

38. Suppose the price-demand function for a particular product is given by $p(x) = mx + b$ where x is the number of items made and sold for $p(x)$ dollars. Here, $m < 0$ and $b > 0$. If the cost (in dollars) to make x of these products is also a linear function $C(x)$, show that the graph of the average profit function $\overline{P}(x)$ has a slant asymptote with slope m and interpret.

39. Electric circuits are built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given below.[15]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power $P(x)$ to resistance x:

$$P(x) = \frac{25x}{(x+3.9)^2}, \quad x \geq 0.$$

(a) Use a graphing utility to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

(b) State and interpret the end behavior of $P(x)$ as $x \to \infty$.

[15] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

40. Let $f(x) = \dfrac{ax^2 - c}{x+3}$. Find values for a and c so the graph of f has a hole at $(-3, 12)$.

41. Let $f(x) = \dfrac{ax^n - 4}{2x^2 + 1}$.

    (a) Determine values for a and n so the graph of $y = f(x)$ has the horizontal asymptote $y = 3$.

    (b) Determine values for a and n so the graph of $y = f(x)$ has the slant asymptote $y = 5x$.

42. Suppose p is a polynomial function and a is a real number. Define $r(x) = \dfrac{p(x) - p(a)}{x - a}$. Use the Factor Theorem, Theorem 2.10, to prove the graph of $y = r(x)$ has a hole at $x = a$.

43. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[16] What trends do you observe? How do your answers manifest themselves graphically? How do you results compare with those of Exercise 51 in Section 2.2?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}				
x^{-2}				
x^{-3}				
x^{-4}				

44. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed.[17] Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

SECTION 3.2 EXERCISE ANSWERS A.1.3

[16] See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.

[17] This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.

3.3 Graphs of Rational Functions

In Section 3.2, we learned about the types of behaviors to expect from graphs of rational functions: vertical asymptotes, holes in graph, horizontal and slant asymptotes. Moreover, Theorems 3.2, 3.3 and 3.4 tell us exactly when and where these behaviors will occur. We used rough sketches extensively in the last section to help us verify results. In this section, we delve more deeply into graphing rational functions with the goal of sketching relatively accurate graphs without the aid of a graphing utility. Your instructor will ultimately communicate the level of detail expected out of you when it comes to producing graphs of rational functions; what we provide here is an attempt to glean as much information about the graph as possible given the analytical tools at our disposal.

One of the standard tools we will use is the sign diagram which was first introduced in Section 1.5.1, and then revisited in Section 2.1. In these sections, to construct a sign diagram for a function f, we first found the zeros of f. The zeros broke the domain of f into a series of intervals. We determined the sign of $f(x)$ over the *entire* interval by finding the sign of $f(x)$ for just *one* test value per interval. The theorem that justified this approach was the Intermediate Value Theorem, which says that *continuous* functions cannot change their sign between two values unless there is a zero between those two values.

This strategy fails in general with rational functions. Indeed, the very first function we studied in Section 3.2, $r(x) = \frac{1}{x}$ changes sign between $x = -1$ and $x = 1$, but there is no zero between these two values - instead, the graph changes sign across a vertical asymptote. We could also well imagine the graph of a rational function having a hole where an x-intercept should be[1] It turns out that with Calculus we can show rational functions are continuous on their domains. What this means for us is when we construct sign diagrams, we need to choose test values on either side of values excluded from the domain in addition to checking around zeros.[2]

Steps for Constructing a Sign Diagram for a Rational Function

Suppose f is a rational function.

1. Determine the domain of f.
2. Identify holes and vertical asymptotes for the graph of f. Indicate any holes in the graph by placing them on the number line with a capital H above them and any vertical asymptotes with a vertical dashed line above them.
3. Determine the zeros of f and place them on the number line with the number 0 above them.
4. Choose a test value in each of the intervals determined in steps 1 and 2.
5. Determine and record the sign of $f(x)$ for each test value in step 3.

[1] Take $f(x) = \frac{x^2}{x}$, for instance.

[2] As excluded values are zeros of the denominator, we can think of this as really just generalizing what we already do.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Determine the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable.
3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.
4. Identify the axis intercepts, if they exist.
5. Analyze the end behavior of r. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph.[a]

[a] It doesn't hurt to check for symmetry at this point, if convenient.

Example 3.3.1. Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2-4}$.

Solution. We follow the six step procedure outlined above.

1. Determine the domain.

   To find the domain, we first find the excluded values. To that end, we solve $x^2 - 4 = 0$ and find $x = \pm 2$. Our domain is $\{x \in \mathbb{R} \mid x \neq \pm 2\}$, or, using interval notation, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2. Reduce $f(x)$ to lowest terms.

   We check if $f(x)$ is in lowest terms by factoring: $f(x) = \frac{3x}{(x-2)(x+2)}$. There are no common factors which means $f(x)$ is already in lowest terms.

3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.

   Per Theorem 3.2, vertical asymptotes and holes in the graph come from values excluded from the domain of f. The two numbers excluded from the domain of f are $x = -2$ and $x = 2$, $f(x)$ didn't reduce, thus Theorem 3.2 tells us $x = -2$ and $x = 2$ are vertical asymptotes of the graph. We can actually go a step further at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,[3] it is good practice for those heading off to Calculus. For the discussion that follows, we use the factored form of $f(x) = \frac{3x}{(x-2)(x+2)}$.

[3] The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

- *The behavior of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use x-values a little less than -2, say -2.1, -2.01 and -2.001. While there is no harm in actually building a table like we did in Section 3.2, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to -6, the quantity $(x-2)$ would be very close to -4, and the factor $(x+2)$ would be very close to 0. More specifically, $(x+2)$ would be a little less than 0, in this case, -0.000001. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value. So, mentally, as $x \to -2^-$,

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)\,(\text{very small }(-))} = \frac{3}{2\,(\text{very small }(-))}$$

  Now, the closer x gets to -2, the smaller $(x+2)$ will become, so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

  The term 'very big $(-)$' means a number with a large absolute value which is negative.[4] What all of this means is that as $x \to -2^-$, $f(x) \to -\infty$.

  Now suppose we wanted to determine the behavior of $f(x)$ as $x \to -2^+$. If we imagine substituting something a little larger than -2 in for x, say -1.999999, we mentally estimate

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small }(+))} = \frac{3}{2\,(\text{very small }(+))} \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$$

  We conclude that as $x \to -2^+$, $f(x) \to \infty$.

- *The behavior of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

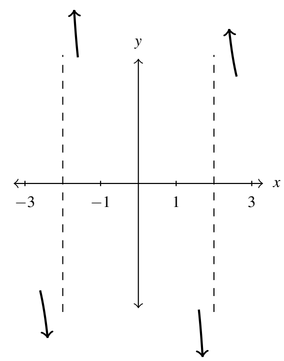
$$f(x) \approx \frac{6}{(\text{very small }(-))\,(4)} = \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

  We conclude that as $x \to 2^-$, $f(x) \to -\infty$.

  Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$ to get $f(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big (+). So as $x \to 2^+$, $f(x) \to \infty$.

We interpret this graphically below.

[4]The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.



behavior near $x = \pm 2$

4. Identify the axis intercepts, if they exist.

   To find the x-intercepts of the graph, we set $y = f(x) = 0$. Solving $\frac{3x}{(x-2)(x+2)} = 0$ results in $3x = 0$, thus $x = 0$. Because $x = 0$ is in our domain, $(0,0)$ is the x-intercept. This is also the y-intercept,[5] as we can quickly verify with $f(0) = \frac{3(0)}{0^2-4} = 0$.

5. Analyze the end behavior of $f(x)$. Find the horizontal or slant asymptote, if one exists.

   Next, we determine the end behavior of the graph of $y = f(x)$. The degree of the numerator is 1, and the degree of the denominator is 2, and so Theorem 3.3 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \frac{3x}{x^2-4}$.

   - *The behavior of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behavior of f as $x \to -\infty$, we would substitute very 'large' negative numbers in for x, say for example, $x = -1$ billion. The numerator $3x$ would then be -3 billion, whereas the denominator $x^2 - 4$ would be $(-1 \text{ billion})^2 - 4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

   $$f(-1 \text{ billion}) \approx \frac{-3\,\text{billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small } (-)$$

     Notice that if we substituted in $x = -1$ trillion, essentially the same kind of division would occur, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit *below* the x-axis as we move to the far left.

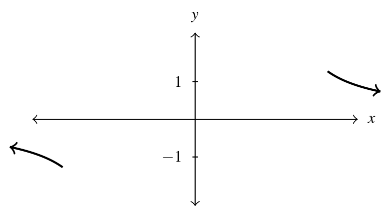
   - *The behavior of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for x and looking at the behavior of $f(x)$. For example, let $x = 1$ billion. Proceeding as before, we get

   $$f(1 \text{ billion}) \approx \frac{3\,\text{billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small } (+)$$

[5] Per Exercise 77, functions can have at most one y-intercept. $(0,0)$ is on the graph, thus it is *the* y-intercept.

The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit *above* the x-axis as we look toward the far right.

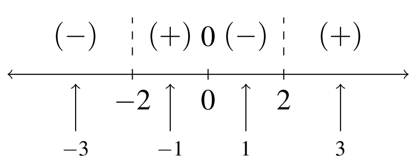
We interpret these findings graphically below.

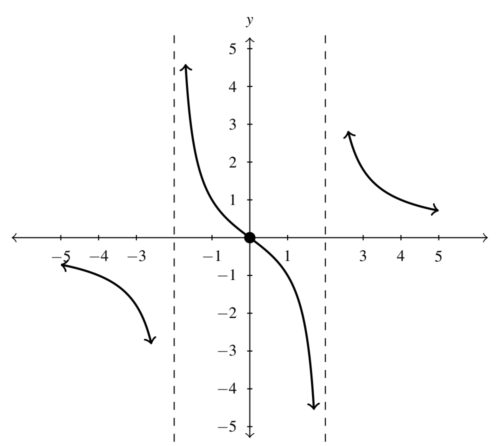


end behavior

6. Use a sign diagram and plot additional points, as needed, to sketch the graph.

   Lastly, we construct a sign diagram for $f(x)$. The x-values excluded from the domain of f are $x = \pm 2$, and the only zero of f is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and we choose the test values[6] $x = -3$, $x = -1$, $x = 1$ and $x = 3$. We find $f(-3)$ is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$ and $f(3)$ is $(+)$. As we begin our sketch, it certainly appears as if the graph could be symmetric about the origin. Taking a moment to check for symmetry, we find $f(-x) = \frac{3(-x)}{(-x)^2-4} = -\frac{3x}{x^2-4} = -f(x)$. Hence, f is odd and the graph of $y = f(x)$ is symmetric about the origin. Putting all of our work together, we get the graph below.





□

[6] In this particular case, we don't need test values because our analysis of the behavior of f near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.

Something important to note about the above example is that while $y = 0$ is the horizontal asymptote, the graph of f actually crosses the x-axis at $(0,0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false,[7] as we shall see again in our next example.

Example 3.3.2. Sketch a detailed graph of $g(t) = \dfrac{2t^2-3t-5}{t^2-t-6}$.

Solution.

1. Determine the domain.

   To find the values excluded from the domain of g, we solve $t^2 - t - 6 = 0$ and find $t = -2$ and $t = 3$. Hence, our domain is $\{t \in \mathbb{R} \,|\, t \neq -23\}$, or using interval notation: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

2. Reduce $f(x)$ to lowest terms.

   To check if $g(t)$ is in lowest terms, we factor: $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$. There is no cancellation, so $g(t)$ is in lowest terms.

3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.

   Due to the fact that $g(t)$ was given to us in lowest terms, we have, once again by Theorem 3.2 vertical asymptotes $t = -2$ and $t = 3$. Keeping in mind $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$, we proceed to our analysis near each of these values.

   - *The behavior of $y = g(t)$ as $t \to -2$:* As $t \to -2^-$, we imagine substituting a number a little bit less than -2. We have

     $$g(t) \approx \frac{(-9)(-1)}{(-5)(\text{very small }(-))} \approx \frac{9}{\text{very small }(+)} \approx \text{very big }(+)$$

     so as $t \to -2^-$, $g(t) \to \infty$.

     On the flip side, as $t \to -2^+$, we get

     $$g(t) \approx \frac{9}{\text{very small }(-)} \approx \text{very big }(-)$$

     so as $t \to -2^+$, $g(t) \to -\infty$.

   - *The behavior of $y = g(t)$ as $t \to 3$:* As $t \to 3^-$, we imagine plugging in a number just shy of 3. We have

     $$g(t) \approx \frac{(1)(4)}{(\text{ very small }(-))(5)} \approx \frac{4}{\text{very small }(-)} \approx \text{very big }(-)$$

     Hence, as $t \to 3^-$, $g(t) \to -\infty$.

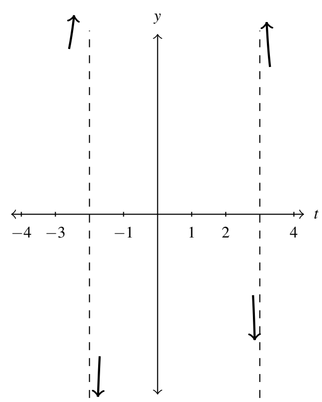
[7] That's why we called it a MYTH!

As $t \to 3^+$, we get

$$g(t) \approx \frac{4}{\text{very small }(+)} \approx \text{very big }(+)$$

so as $t \to 3^+$, $g(t) \to \infty$.

We interpret this analysis graphically below.



behavior near $t = -2$ and $t = 3$

4. Identify the axis intercepts, if they exist.

   To find the t-intercepts we set $y = g(t) = 0$. Using the factored form of $g(t)$ above, we find the zeros to be the solutions of $(2t-5)(t+1) = 0$. We obtain $t = \frac{5}{2}$ and $t = -1$. Both of these numbers are in the domain of g, therefore we have two t-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we find $y = g(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

5. Analyze the end behavior of $f(x)$. Find the horizontal or slant asymptote, if one exists.

   As the degrees of the numerator and denominator of $g(t)$ are the same, we know from Theorem 3.3 that we can find the horizontal asymptote of the graph of g by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behavior which would be lost otherwise. Using long division, we may rewrite $g(t)$ as $g(t) = 2 - \frac{t-7}{t^2-t-6}$. We focus our attention on the term $\frac{t-7}{t^2-t-6}$.

   - *The behavior of $y = g(t)$ as $t \to -\infty$:* If imagine substituting $t = -1$ billion into $\frac{t-7}{t^2-t-6}$, we estimate $\frac{t-7}{t^2-t-6} \approx \frac{-1 \text{ billion}}{1\text{billion}^2} = \frac{-1}{\text{billion}} \approx \text{very small }(-)$.[8] Hence,

$$g(t) = 2 - \frac{t-7}{t^2-t-6} \approx 2 - \text{very small }(-) = 2 + \text{very small }(+)$$

[8]We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the t^2 term dominates the t and constant terms.

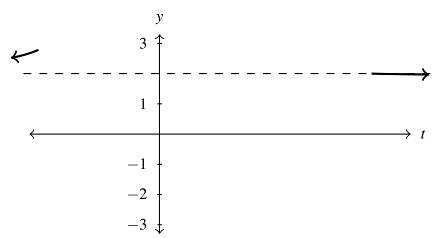
Hence, as $t \to -\infty$, the graph is a little bit *above* the line $y = 2$.

- *The behavior of $y = g(t)$ as $t \to \infty$.* To consider $\frac{t-7}{t^2-t-6}$ as $t \to \infty$, we imagine substituting $t = 1$ billion and, going through the usual mental routine, find

$$\frac{t-7}{t^2-t-6} \approx \text{very small } (+)$$

Hence, $g(t) \approx 2 -$ very small $(+)$, so the graph is just *below* the line $y = 2$ as $t \to \infty$.

We sketch the end behavior below.

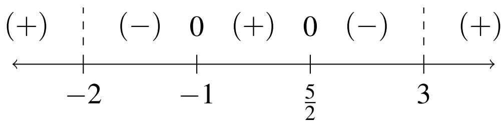


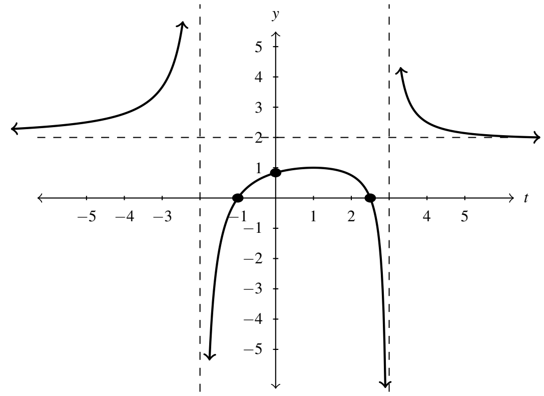
end behavior

6. Use a sign diagram and plot additional points, as needed, to sketch the graph.

Finally we construct our sign diagram. We place an dashed line above $t = -2$ and $t = 3$, and a '0' above $t = \frac{5}{2}$ and $t = -1$. Choosing test values in the test intervals gives us $g(t)$ is $(+)$ on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$ and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$. As we piece together all of the information, it stands to reason the graph must cross the horizontal asymptote at some point after $t = 3$ in order for it to approach $y = 2$ from underneath.[9] To find where $y = g(t)$ intersects $y = 2$, we solve $g(t) = 2 - \frac{t-7}{t^2-t-6} = 2$ and get $t - 7 = 0$, or $t = 7$. Note that $t - 7$ is the remainder when $2t^2 - 3t - 5$ is divided by $t^2 - t - 6$, so it makes sense that for $g(t)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7) = 2$. The location of the t-intercepts alone dashes all hope of the function being even or odd (do you see why?) so we skip the symmetry check in this case.

[9]This subtlety would have been missed had we skipped the long division and subsequent end behavior analysis.





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More can be said about the graph of $y = g(t)$. It stands to reason that g must attain a local minimum at some point past $t = 7$ because the graph of g crosses through $y = 2$ at $(2,7)$ but approaches $y = 2$ from below as $t \to \infty$. Calculus verifies a local minimum at $(13, 1.96)$. We invite the reader to verify this claim using a graphing utility.

Example 3.3.3. Sketch a detailed graph of $h(x) = \dfrac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$.

Solution.

1. Determine the domain.

   Solving $x^2 + 3x + 2 = 0$ gives $x = -2$ and $x = -1$ as our excluded values. Hence, the domain is $\{x \in \mathbb{R} \,|\, x \neq -1, -2\}$ or, using interval notation, $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

2. Reduce $h(x)$ to lowest terms.

   To reduce $h(x)$, we need to factor the numerator and denominator. To factor the numerator, we use the techniques[10] set forth in Section 0.3 and get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} = \frac{(2x+1)(x+1)^{\overset{1}{\cancel{2}}}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

   Note we can use this formula for $h(x)$ in our analysis of the graph of h as long as we are not substituting $x = -1$. To make this exclusion specific, we write $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

[10] Bet you never thought you'd never see *that* stuff again before the Final Exam!

3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.

   From Theorem 3.2, we know that $x = -2$ is a zero of the denominator of the reduced form of $h(x)$, thus we have a vertical asymptote there. As for $x = -1$, the factor $(x+1)$ was divided from the denominator when we reduced $h(x)$, so there will be a hole when $x = -1$. To find the y-coordinate of the hole, we substitute $x = -1$ into $\frac{(2x+1)(x+1)}{x+2}$, per Theorem 3.2 and get 0. Hence, we have a hole on the x-axis at $(-1,0)$. It should make you uncomfortable plugging $x = -1$ into the reduced formula for $h(x)$, especially because we've made such a big deal about the stipulation '$x \neq -1$' that goes along with that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near $x = -1$ which we will show below.

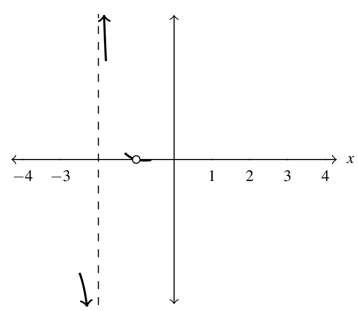
   - *The behavior of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small }(-))} \approx \frac{3}{(\text{very small }(-))} \approx$ very big $(-)$ thus as $x \to -2^-$, $h(x) \to -\infty$.

     On the other side of -2, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big $(+)$, so $h(x) \to \infty$.

   - *The behavior of $y = h(x)$ as $x \to -1$.* As $x \to -1^-$, we imagine plugging in a number a bit less than $x = -1$. We have $h(x) \approx \frac{(-1)(\text{very small }(-))}{1} =$ very small $(+)$. Hence, as $x \to -1^-$, $h(x) \to 0^+$. This means that as $x \to -1^-$, the graph is a bit above the point $(-1,0)$.

     As $x \to -1^+$, we get $h(x) \approx \frac{(-1)(\text{very small }(+))}{1} =$ very small $(-)$. This gives us that as $x \to -1^+$, $h(x) \to 0^-$, so the graph is a little bit lower than $(-1,0)$ here.

   We interpret this graphically below.



behavior near $x = -2$ and $x = -1$

4. Identify the axis intercepts, if they exist.

   To find the x-intercepts, as usual, we set $h(x) = 0$ and solve. Solving $\frac{(2x+1)(x+1)}{x+2} = 0$ yields $x = -\frac{1}{2}$ and $x = -1$. The latter isn't in the domain of h, in fact, we know there is a hole at $(-1,0)$, so we exclude it. Our only x-intercept is $\left(-\frac{1}{2},0\right)$. To find the y-intercept, we set $x = 0$. Due to the fact that $0 \neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0) = \frac{1}{2}$ for a y-intercept of $\left(0,\frac{1}{2}\right)$.

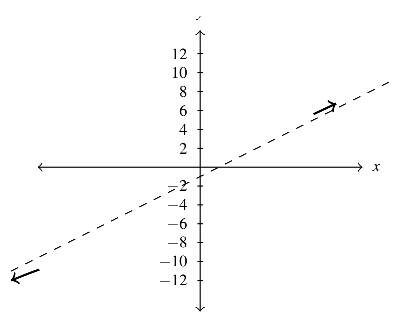
5. Analyze the end behavior of $f(x)$. Find the horizontal or slant asymptote, if one exists.

For end behavior, we note that the degree of the numerator of $h(x)$, $2x^3+5x^2+4x+1$, is 3 and the degree of the denominator, x^2+3x+2, is 2 so by Theorem 3.4, the graph of $y=h(x)$ has a slant asymptote. For $x \to \pm\infty$, we are far enough away from $x=-1$ to use the reduced formula, $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \frac{2x^2+3x+1}{x+2}$, $x \neq -1$, and rewrite $h(x) = 2x-1+\frac{3}{x+2}$, $x \neq -1$. By Theorem 3.4, the slant asymptote is $y = 2x-1$, and to better see *how* the graph approaches the asymptote, we focus our attention on the term generated from the remainder, $\frac{3}{x+2}$.

- *The behavior of $y=h(x)$ as $x \to -\infty$:* Substituting $x=-1$ billion into $\frac{3}{x+2}$, we get the estimate $\frac{3}{-1 \text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x-1+\frac{3}{x+2} \approx 2x-1+$ very small $(-)$. This means the graph of $y=h(x)$ is a little bit *below* the line $y=2x-1$ as $x \to -\infty$.

- *The behavior of $y=h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\frac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x-1+$ very small $(+)$, or that the graph of $y=h(x)$ is a little bit *above* the line $y=2x-1$ as $x \to \infty$.

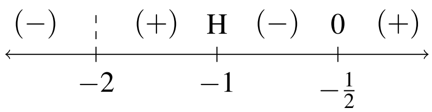
We sketch the end behavior below.

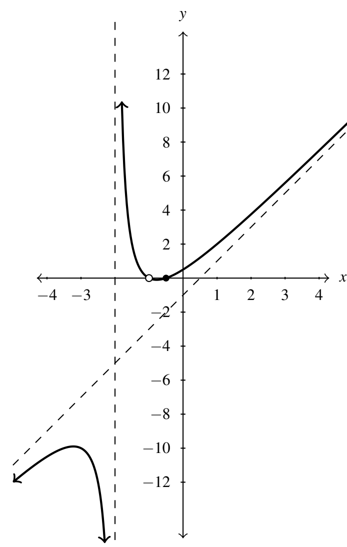


end behavior

6. Use a sign diagram and plot additional points, as needed, to sketch the graph.

   To make our sign diagram, we place a dashed line above $x=-2$ and an 'H' above $x=-1$ and a '0' above $x=-\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2,-1)$ and $\left(-\frac{1}{2},\infty\right)$ and $h(x)$ is $(-)$ on $(-\infty,-2)$ and $\left(-1,-\frac{1}{2}\right)$. Putting all of our work together yields the graph below.





To find if the graph of h ever crosses the slant asymptote, we solve $h(x) = 2x-1+\frac{3}{x+2} = 2x-1$. This results in $\frac{3}{x+2} = 0$, which has no solution.[11] Hence, the graph of h never crosses its slant asymptote.[12] $\square$

Our last graphing example is challenging in that our six step process provides us little information to work with.

[11] Alternatively, the remainder after the long division was $r = 3$ which is never 0.
[12] But rest assured, some graphs do!

Example 3.3.4. Sketch the graph of $r(x) = \dfrac{x^4+1}{x^2+1}$.

Solution.

1. Determine the domain.

   The denominator x^2+1 is never zero which means there are no excluded values. The domain is $\mathbb{R}$, or using interval notation, $(-\infty, \infty)$.

2. Reduce $r(x)$ to lowest terms.

   With no real zeros in the denominator, x^2+1 is an irreducible quadratic. Our only hope of reducing $r(x)$ is if x^2+1 is a factor of x^4+1. Performing long division gives us

   $$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$$

   The remainder is not zero so $r(x)$ is already reduced.

3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.

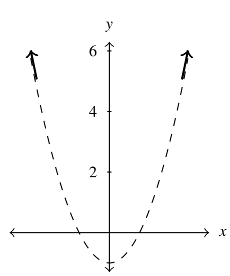
   There are no numbers excluded from the domain of r, so there are no vertical asymptotes or holes in the graph of r.

4. Identify the axis intercepts, if they exist.

   To find the x-intercept, we'd set $r(x) = 0$. As there are no real solutions to $x^4+1 = 0$, we have no x-intercepts. $r(0) = 1$, thus we do get $(0,1)$ as the y-intercept.

5. Analyze the end behavior of $f(x)$. Find the horizontal or slant asymptote, if one exists.

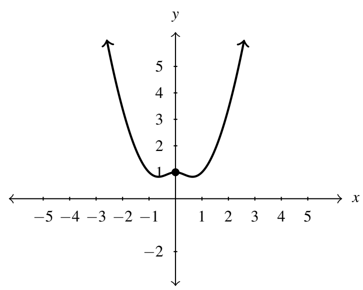
   For end behavior, we note that because the degree of the numerator is exactly *two* more than the degree of the denominator, neither Theorems 3.3 nor 3.4 apply. We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \frac{2}{x^2+1}$, so we focus our attention on the term corresponding to the remainder, $\frac{2}{x^2+1}$ It should be clear that as $x \to \pm\infty$, $\frac{2}{x^2+1} \approx \text{very small } (+)$, which means $r(x) \approx x^2 - 1 + \text{very small } (+)$. So the graph $y = r(x)$ is a little bit *above* the graph of the parabola $y = x^2 - 1$ as $x \to \pm\infty$.



end behavior

6. Use a sign diagram and plot additional points, as needed, to sketch the graph.

   There isn't much work to do for a sign diagram for $r(x)$, because its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty,\infty)$, and we know $r(0)=1$, so we conclude $r(x)$ is $(+)$ for all real numbers. We check for symmetry, and find $r(-x)=\frac{(-x)^4+1}{(-x)^2+1}=\frac{x^4+1}{x^2+1}=r(x)$, so r is even and, hence, the graph is symmetric about the y-axis. It may be tempting at this point to call it quits, reach for a graphing utility, or ask someone who knows Calculus.[13] It turns out, we can do a little bit better. Recall from Section 2.2.1, that when $|x|<1$ but $x\neq 0$, $x^4<x^2$, hence $x^4+1<x^2+1$. This means for $-1<x<0$ and $0<x<1$, $r(x)=\frac{x^4+1}{x^2+1}<1$. Because $r(0)=1$, the graph of $y=r(x)$ must fall to either side before heading off to ∞. This means $(0,1)$ is a local maximum and, moreover, there are at least two local minimums, one on either side of $(0,1)$. We invite the reader to confirm this using a graphing utility.

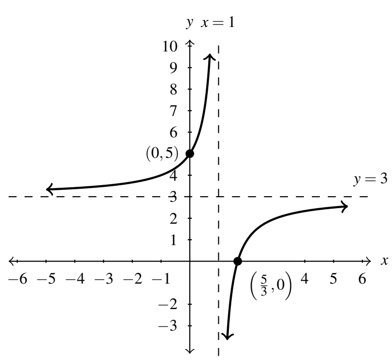


sketch of $y=r(x)$

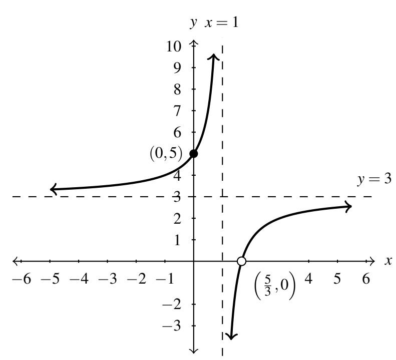
$\square$

Our last example turns the tables and invites us to write formulas for rational function given their graphs.

Example 3.3.5. Write formulas for rational functions $r(x)$ and $F(x)$ given their graphs below:



$y=r(x)$



$y=r(x)$

[13]This is exactly what the authors did in the Third Edition. Special thanks go to Erik Boczko from Ohio University for showing us that, in fact, we could do more with this example algebraically.

Solution. The good news is the graph of r closely resembles the graph of F, so once we know an expression for $r(x)$, we should be able to modify it to obtain $F(x)$. We are told r is a rational function, so we know there are polynomial functions p and q so that $r(x) = \frac{p(x)}{q(x)}$. We can factor $p(x)$ and $q(x)$ completely in terms of their leading coefficients and their zeros. For simplicity's sake, we assume neither p nor q has any non-real zeros.

We focus our attention first on finding an expression for $p(x)$. When finding the x-intercepts, we look for the zeros of r by solving $r(x) = \frac{p(x)}{q(x)} = 0$. This equation quickly reduces to solving $p(x) = 0$. As $\left(\frac{5}{3}, 0\right)$ is an x-intercept of the graph, we know $x = \frac{5}{3}$ is a zero of r, and, hence, a zero of p. Due to the fact that we are shown no other x-intercepts, we assume r, hence p, have no other real zeros (and no non-real zeros by our assumption.) Definition 2.9 gives $p(x) = a\left(x - \frac{5}{3}\right)^m$ where a is the leading coefficient of $p(x)$ and m is the multiplicity of the zero $x = \frac{5}{3}$. The graph of $y = r(x)$ crosses through the x-axis in what appears to be a fairly linear fashion at $\left(\frac{5}{3}, 0\right)$, so it seems reasonable to set $m = 1$. Hence, $p(x) = a\left(x - \frac{5}{3}\right)$.

Next, we focus our attention on finding $q(x)$. Theorem 3.2 says $x = 1$ comes from a factor of $(x-1)$ in the denominator of $r(x)$. This means $(x-1)$ is a factor of $q(x)$. Because there are no other vertical asymptotes or holes in the graph, $x = 1$ is the only real zero, hence (per our assumption) the only zero of q. At this point, we have $q(x) = b(x-1)^m$ where b is the leading coefficient of $q(x)$ and m is the multiplicity of the zero $x = 1$. As the graph of r has the *horizontal* asymptote $y = 3$,Theorem 3.4 tells us two things: first, degree of q must match the degree of p; second, the ratio $\frac{a}{b} = 3$. Hence, the degree of q is 1 so that:

$$\begin{aligned}
r(x) &= \frac{a\left(x - \frac{5}{3}\right)}{b(x-1)} \\
&= \frac{a}{b}\left(\frac{x - \frac{5}{3}}{x-1}\right) \\
&= 3\left(\frac{x - \frac{5}{3}}{x-1}\right) \\
&= \frac{3x-5}{x-1}.
\end{aligned}$$

We have yet to use the y-intercept, $(0,5)$. In this case, we use it as a partial check: $r(0) = \frac{3(0)-5}{0-1} = 5$, as required. We can sketch $y = r(x)$ by hand, to give a better check of our work.

Now it is time to find a formula for $F(x)$. The graphs of r and F look identical except the graph of F has a hole at $\left(\frac{5}{3}, 0\right)$ instead of an x-intercept. Theorem 3.2 tells us this happens because a factor of $\left(x - \frac{5}{3}\right)$ divides out from the denominator when the formula for $F(x)$ is reduced. Hence, we reverse this process and multiply the numerator and denominator of our expression for $r(x)$ by $\left(x - \frac{5}{3}\right)$:

$$\begin{aligned}
F(x) &= r(x) \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)} \\
&= \frac{3x-5}{x-1} \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)}
\end{aligned}$$

$$= \frac{3x^2 - 10x + \frac{25}{3}}{x^2 - \frac{8}{3}x + \frac{5}{3}} \qquad \text{expand}$$

$$= \frac{9x^2 - 30x + 25}{3x^2 - 8x + 5} \qquad \text{multiply by } 1 = \tfrac{3}{3} \text{ to reduce complex fractions.}$$

Again, we can check our answer by applying the six step method to this function or, for a quick verification, we can use a graphing utility.[14] □

Another way to approach Example 3.3.5 is to take a cue from Theorem 3.1. The graph of $y = r(x)$ certainly *appears* to be the result of moving around the graph of $f(x) = \frac{1}{x}$. To that end, suppose $r(x) = \frac{a}{x-k} + k$. As the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 3$, we get $h = 1$ and $k = 3$. At this point, we have $r(x) = \frac{a}{x-1} + 3$. We can determine a by using the y-intercept, $(0,5)$: $r(0) = 5$ gives us $-a + 3 = 5$ so $a = -2$. Hence, $r(x) = \frac{-2}{x-1} + 3$. At this point we could check the x-intercept $\left(\frac{5}{3}, 0\right)$ is on the graph, check our answer using a graphing utility, or even better, get common denominators and write $r(x)$ as a single rational expression to compare with our answer in the above example.

As usual, the authors offer no apologies for what may be construed as 'pedantry' in this section. We feel that the detail presented in this section is necessary to obtain a firm grasp of the concepts presented here and it also serves as an introduction to the methods employed in Calculus. In the end, your instructor will decide how much, if any, of the kinds of details presented here are 'mission critical' to your understanding of Precalculus. Without further delay, we present you with this section's Exercises.

3.3.1 Exercises

In Exercises 1 - 16, use the six-step procedure to graph the rational function. Be sure to draw any asymptotes as dashed lines.

1. $f(x) = \dfrac{4}{x+2}$
2. $f(x) = 5x(6-2x)^{-1}$
3. $g(t) = t^{-2}$
4. $g(t) = \dfrac{1}{t^2 + t - 12}$
5. $r(z) = \dfrac{2z-1}{-2z^2 - 5z + 3}$
6. $r(z) = \dfrac{z}{z^2 + z - 12}$
7. $f(x) = 4x(x^2+4)^{-1}$
8. $f(x) = 4x(x^2-4)^{-1}$
9. $g(t) = \dfrac{t^2 - t - 12}{t^2 + t - 6}$
10. $g(t) = 3 - \dfrac{5t - 25}{t^2 - 9}$

[14] Be warned, however, a graphing utility may not show the hole at $\left(\frac{5}{3}, 0\right)$.

11. $r(z) = \dfrac{z^2 - z - 6}{z + 1}$

12. $r(z) = -z - 2 + \dfrac{6}{3 - z}$

13. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$

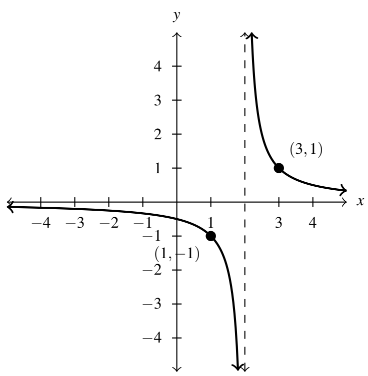
14. $f(x) = \dfrac{5x}{9 - x^2} - x$

15. $g(t) = \dfrac{1}{2}t - 1 + \dfrac{t + 1}{t^2 + 1}$

16. $g(t) = \dfrac{t^2 - 2t + 1}{t^3 + t^2 - 2t}$

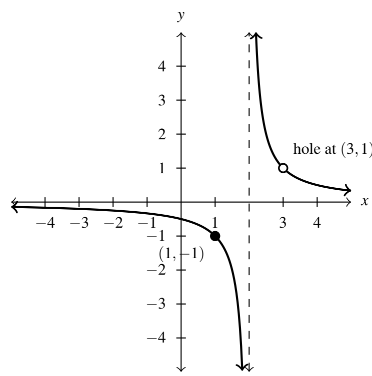
In Exercises 17 - 20, write a possible formula for the function whose graph is given.

17. $y = f(x)$



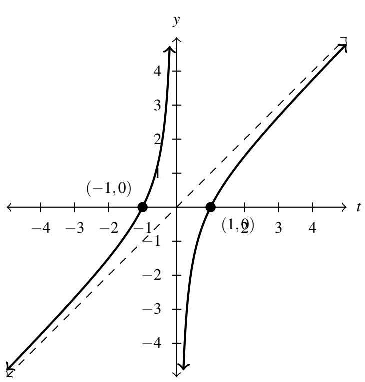
Vertical asymptote: $x = 2$

18. $y = F(x)$



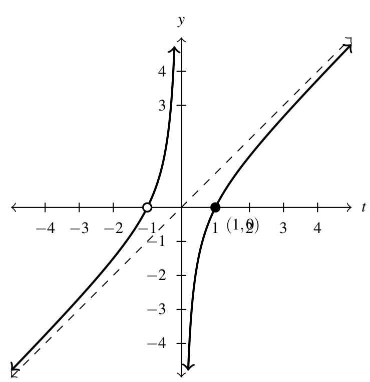
Vertical asymptote: $x = 2$

19. $y = g(t)$



slant asymptote at $y = t$

20. $y = G(t)$



slant asymptote at $y = t$

SECTION 3.3 EXERCISE ANSWERS A.1.3

3.4 Solving Equations Involving Rational Functions

In this section, we review the solving of equations which involve rational expressions. As with equations involving numeric fractions, our first step in solving equations with algebraic fractions is to clear denominators. In doing so, we run the risk of introducing what are known as **extraneous** solutions - 'answers' which don't satisfy the original equation. As we illustrate the techniques used to solve these basic equations, see if you can find the step which creates the problem for us.

Example 3.4.1. Solve the following equations.

1. $1+\frac{1}{x}=x$
2. $\frac{t^3-2t+1}{t-1}=\frac{1}{2}t-1$
3. $\frac{3}{1-w\sqrt{2}}-\frac{1}{2w+5}=0$
4. $3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x)=0$
5. Solve $x=\frac{2y+1}{y-3}$ for y.
6. Solve $\frac{1}{f}=\frac{1}{S_1}+\frac{1}{S_2}$ for S_1.

Solution.

1. Solve $1+\frac{1}{x}=x$ for x.

   Our first step is to clear the fractions by multiplying both sides of the equation by x. In doing so, we are implicitly assuming $x \neq 0$; otherwise, we would have no guarantee that the resulting equation is equivalent to our original equation.[1]

$$\begin{array}{rcll} 1+\frac{1}{x} & = & x & \\ \left(1+\frac{1}{x}\right)x & = & (x)x & \text{Provided } x \neq 0 \\ 1(x)+\frac{1}{x}(x) & = & x^2 & \text{Distribute} \\ x+\frac{x}{x} & = & x^2 & \text{Multiply} \\ x+1 & = & x^2 & \\ 0 & = & x^2-x-1 & \text{Subtract } x\text{, subtract } 1 \\ x & = & \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} & \text{Quadratic Formula} \\ x & = & \frac{1\pm\sqrt{5}}{2} & \text{Simplify} \end{array}$$

[1]See page 46.

We obtain two answers, $x = \frac{1 \pm \sqrt{5}}{2}$. Neither of these are 0 thus neither contradicts our assumption that $x \neq 0$. The reader is invited to check both of these solutions.[2]

2. Solve $\dfrac{t^3 - 2t + 1}{t - 1} = \dfrac{1}{2}t - 1$ for t.

   To solve the equation, we clear denominators. Here, we need to assume $t - 1 \neq 0$, or $t \neq 1$.

$$
\begin{array}{rcll}
\dfrac{t^3 - 2t + 1}{t - 1} & = & \dfrac{1}{2}t - 1 & \\
\left(\dfrac{t^3 - 2t + 1}{t - 1}\right) \cdot 2(t-1) & = & \left(\dfrac{1}{2}t - 1\right) \cdot 2(t-1) & \text{Provided } t \neq 1 \\
\dfrac{(t^3 - 2t + 1)(2\cancel{(t-1)})}{\cancel{(t-1)}} & = & \dfrac{1}{\cancel{2}}t(\cancel{2}(t-1)) - 1(2(t-1)) & \text{Multiply, distribute} \\
2(t^3 - 2t + 1) & = & t^2 - t - 2t + 2 & \text{Distribute} \\
2t^3 - 4t + 2 & = & t^2 - 3t + 2 & \text{Distribute, combine like terms} \\
2t^3 - t^2 - t & = & 0 & \text{Subtract } t^2 \text{, add } 3t \text{, subtract } 2 \\
t(2t^2 - t - 1) & = & 0 & \text{Factor} \\
t = 0 & \text{or} & 2t^2 - t - 1 = 0 & \text{Zero Product Property} \\
t = 0 & \text{or} & (2t+1)(t-1) = 0 & \text{Factor} \\
t = 0 & \text{or} & 2t + 1 = 0 \quad \text{or} \quad t - 1 = 0 & \\
t & = & 0, \ -\dfrac{1}{2} \text{ or } 1 &
\end{array}
$$

   We assumed that $t \neq 1$ in order to clear denominators. Sure enough, the candidate $t = 1$ doesn't check in the original equation because it causes division by 0. In this case, we call $t = 1$ an *extraneous* solution. Note that $t = 1$ *does* work in every equation *after* we clear denominators. In general, multiplying by variable expressions can produce these 'extra' solutions, which is why checking our answers is always encouraged.[3] The other two candidates, $t = 0$ and $t = -\frac{1}{2}$, are solutions.

3. Solve $\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w + 5} = 0$ for w.

   As before, we begin by clearing denominators. Here, we assume $1 - w\sqrt{2} \neq 0$ (so $w \neq \frac{1}{\sqrt{2}}$) and $2w + 5 \neq 0$ (so $w \neq -\frac{5}{2}$).

[2] The check relies on being able to 'rationalize' the denominator - a skill we haven't reviewed yet. Additionally, the positive solution to this equation is the famous Golden Ratio.

[3] Contrast this with what happened in Example 0.5.3 when we divided by a variable and 'lost' a solution.

$$\begin{array}{rcll}
\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5} & = & 0 & \\
\left(\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5}\right)(1-w\sqrt{2})(2w+5) & = & 0(1-w\sqrt{2})(2w+5) & w\neq \dfrac{1}{\sqrt{2}},-\dfrac{5}{2} \\
\dfrac{3\cancel{(1-w\sqrt{2})}(2w+5)}{\cancel{(1-w\sqrt{2})}}-\dfrac{1(1-w\sqrt{2})\cancel{(2w+5)}}{\cancel{(2w+5)}} & = & 0 & \text{Distribute} \\
3(2w+5)-(1-w\sqrt{2}) & = & 0 &
\end{array}$$

The result is a *linear* equation in w so we gather the terms with w on one side of the equation and put everything else on the other. We factor out w and divide by its coefficient.

$$\begin{array}{rcll}
3(2w+5)-(1-w\sqrt{2}) & = & 0 & \\
6w+15-1+w\sqrt{2} & = & 0 & \text{Distribute} \\
6w+w\sqrt{2} & = & -14 & \text{Subtract 14} \\
(6+\sqrt{2})w & = & -14 & \text{Factor} \\
w & = & -\dfrac{14}{6+\sqrt{2}} & \text{Divide by } 6+\sqrt{2}
\end{array}$$

This solution is different than our excluded values, $\frac{1}{\sqrt{2}}$ and $-\frac{5}{2}$, so we keep $w=-\frac{14}{6+\sqrt{2}}$ as our final answer. The reader is invited to check this in the original equation.

4. Solve $3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x)=0$ for x.

   To solve our next equation, we have two approaches to choose from: we could rewrite the quantities with negative exponents as fractions and clear denominators, or we can factor. We showcase each technique below.

   - *Clearing Denominators Approach*: We rewrite the negative exponents as fractions and clear denominators. In this case, we multiply both sides of the equation by $(x^2+4)^2$, which is never 0. (Think about that for a moment.) As a result, we need not exclude any x values from our solution set.

$$\begin{array}{rcll}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
\dfrac{3}{x^2+4}+\dfrac{3x(-1)(2x)}{(x^2+4)^2} & = & 0 & \text{Rewrite} \\
\left(\dfrac{3}{x^2+4}-\dfrac{6x^2}{(x^2+4)^2}\right)(x^2+4)^2 & = & 0(x^2+4)^2 & \text{Multiply} \\
\dfrac{3\cancelto{(x^2+4)}{(x^2+4)^2}}{\cancel{(x^2+4)}}-\dfrac{6x^2\cancel{(x^2+4)^2}}{\cancel{(x^2+4)^2}} & = & 0 & \text{Distribute} \\
3(x^2+4)-6x^2 & = & 0 &
\end{array}$$

$$\begin{array}{rcll} 3x^2+12-6x^2 & = & 0 & \text{Distribute} \\ -3x^2 & = & -12 & \text{Combine like terms, subtract 12} \\ x^2 & = & 4 & \text{Divide by } -3 \\ x & = & \pm\sqrt{4} = \pm 2 & \text{Extract square roots} \end{array}$$

We leave it to the reader to show that both $x=-2$ and $x=2$ satisfy the original equation.

- *Factoring Approach*: Because the equation is already set equal to 0, we're ready to factor. Following the guidelines presented in Example 3.1.1, we factor out $3(x^2+4)^{-2}$ from both terms and look to see if more factoring can be done:

$$\begin{array}{rcll} 3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\ 3(x^2+4)^{-2}((x^2+4)^1+x(-1)(2x)) & = & 0 & \text{Factor} \\ 3(x^2+4)^{-2}(x^2+4-2x^2) & = & 0 & \\ 3(x^2+4)^{-2}(4-x^2) & = & 0 & \text{Gather like terms} \\ 3(x^2+4)^{-2}=0 & \text{or} & 4-x^2=0 & \text{Zero Product Property} \\ \dfrac{3}{x^2+4}=0 & \text{or} & 4=x^2 & \end{array}$$

The first equation yields no solutions (Think about this for a moment.) while the second gives us $x=\pm\sqrt{4}=\pm 2$ as before.

5. Solve $x=\dfrac{2y+1}{y-3}$ for y.

We are asked to solve this equation for y so we begin by clearing fractions with the stipulation that $y-3\neq 0$ or $y\neq 3$. We are left with a linear equation in the variable y. To solve this, we gather the terms containing y on one side of the equation and everything else on the other. Next, we factor out the y and divide by its coefficient, which in this case turns out to be $x-2$. In order to divide by $x-2$, we stipulate $x-2\neq 0$ or, said differently, $x\neq 2$.

$$\begin{array}{rcll} x & = & \dfrac{2y+1}{y-3} & \\ x(y-3) & = & \left(\dfrac{2y+1}{y-3}\right)(y-3) & \text{Provided } y\neq 3 \\ xy-3x & = & \dfrac{(2y+1)\cancel{(y-3)}}{\cancel{(y-3)}} & \text{Distribute, multiply} \\ xy-3x & = & 2y+1 & \\ xy-2y & = & 3x+1 & \text{Add } 3x\text{, subtract } 2y \\ y(x-2) & = & 3x+1 & \text{Factor} \end{array}$$

$$y = \frac{3x+1}{x-2} \qquad \text{Divide provided } x \neq 2$$

We highly encourage the reader to check the answer algebraically to see where the restrictions on x and y come into play.[4]

6. Solve $\frac{1}{f} = \frac{1}{S_1} + \frac{1}{S_2}$ for S_1.

   Our last example comes from physics and the world of photography.[5] We take a moment here to note that while superscripts in Mathematics indicate exponents (powers), subscripts are used primarily to distinguish one or more variables. In this case, S_1 and S_2 are two *different* variables (much like x and y) and we treat them as such. Our first step is to clear denominators by multiplying both sides by fS_1S_2 - provided each is nonzero. We end up with an equation which is linear in S_1 so we proceed as in the previous example.

$$\begin{array}{rcll}
\frac{1}{f} & = & \frac{1}{S_1} + \frac{1}{S_2} & \\
\left(\frac{1}{f}\right)(fS_1S_2) & = & \left(\frac{1}{S_1} + \frac{1}{S_2}\right)(fS_1S_2) & \text{Provided } f \neq 0,\, S_1 \neq 0,\, S_2 \neq 0 \\
\frac{fS_1S_2}{f} & = & \frac{fS_1S_2}{S_1} + \frac{fS_1S_2}{S_2} & \text{Multiply, distribute} \\
\frac{\cancel{f}S_1S_2}{\cancel{f}} & = & \frac{f\cancel{S_1}S_2}{\cancel{S_1}} + \frac{fS_1\cancel{S_2}}{\cancel{S_2}} & \text{Divide out} \\
S_1S_2 & = & fS_2 + fS_1 & \\
S_1S_2 - fS_1 & = & fS_2 & \text{Subtract } fS_1 \\
S_1(S_2 - f) & = & fS_2 & \text{Factor} \\
S_1 & = & \frac{fS_2}{S_2 - f} & \text{Divide provided } S_2 \neq f
\end{array}$$

   As always, the reader is highly encouraged to check the answer.[6] □

[4] It involves simplifying a compound fraction!

[5] See this article on focal length.

[6] . . . and see what the restriction $S_2 \neq f$ means in terms of focusing a camera!

3.4.1 Exercises

In Exercises 1 - 9, determine all real solutions. Be sure to check for extraneous solutions.

1. $\dfrac{x}{5x+4} = 3$
2. $\dfrac{3y-1}{y^2+1} = 1$
3. $\dfrac{1}{w+3} + \dfrac{1}{w-3} = \dfrac{w^2-3}{w^2-9}$
4. $\dfrac{2x+17}{x+1} = x+5$
5. $\dfrac{t^2-2t+1}{t^3+t^2-2t} = 1$
6. $\dfrac{-y^3+4y}{y^2-9} = 4y$
7. $w+\sqrt{3} = \dfrac{3w-w^3}{w-\sqrt{3}}$
8. $\dfrac{2}{x\sqrt{2}-1} - 1 = \dfrac{3}{x\sqrt{2}+1}$
9. $\dfrac{x^2}{(1+x\sqrt{3})^2} = 3$

In Exercises 10 - 12, use Theorem 0.4 along with the techniques in this section to determine all real solutions.

10. $\left|\dfrac{3n}{n-1}\right| = 3$
11. $\left|\dfrac{2x}{x^2-1}\right| = 2$
12. $\left|\dfrac{2t}{4-t^2}\right| = \left|\dfrac{2}{t-2}\right|$

In Exercises 13 - 15, compute all real solutions.

13. $2.41 = \dfrac{0.08}{4\pi R^2}$
14. $\dfrac{x^2}{(2.31-x)^2} = 0.04$
15. $1 - \dfrac{6.75\times 10^{16}}{c^2} = \dfrac{1}{4}$

In Exercises 16 - 21, solve the given equation for the indicated variable.

16. Solve for y: $\dfrac{1-2y}{y+3} = x$
17. Solve for y: $x = 3 - \dfrac{2}{1-y}$
18.[7] Solve for T_2: $\dfrac{V_1}{T_1} = \dfrac{V_2}{T_2}$
19. Solve for t_0: $\dfrac{t_0}{1-t_0t_1} = 2$
20. Solve for x: $\dfrac{1}{x-v_r} + \dfrac{1}{x+v_r} = 5$
21. Solve for R: $P = \dfrac{25R}{(R+4)^2}$

Section 3.4 Exercise Answers A.1.3

[7] Recall: subscripts on variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat quantities like 'V_1' and 'V_2' as two different variables as you would 'x' and 'y.'

CHAPTER 4

ROOT AND POWER FUNCTIONS

4.1 Properties of Root Functions and Their Graphs

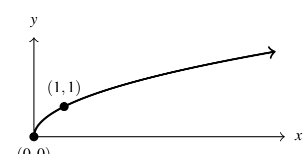
4.1.1 Root Functions

As with polynomial functions and rational functions, we begin our study of functions involving radicals with a special family of functions: the (principal) root functions.

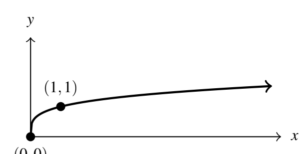
Definition 4.1. Let $n \in \mathbb{N}$ with $n \geq 2$. The n **th (principal) root function** is the function $f(x) = \sqrt[n]{x}$.
NOTE: If n is even, the domain of f is $[0, \infty)$; if n is odd, the domain of f is $(-\infty, \infty)$.

The domain restriction for even indexed roots means that, once again, we are restricting our attention to *real* numbers.[1] We graph a few members of the root function family below, and quickly notice that, as with the monomial, and, more generally, the Laurent monomial functions, the behavior of the root functions depends primarily on whether the root is even or odd.

In addition to having the common domain of $[0, \infty)$, the graphs of $f(x) = \sqrt[n]{x}$ for even indices, n, all share the points $(0,0)$ and $(1,1)$. As n increases, the functions become 'steeper' near the y-axis and 'flatter' as $x \to \infty$. To show $f(x) \to \infty$ as $x \to \infty$, we show, more generally, the range of f is $[0, \infty)$. Indeed, if $c \geq 0$ is a real number, then $f(c^n) = \sqrt[n]{c^n} = c$ so c is in the range of f. Note that f is increasing: that is, if $a < b$, then $f(a) = \sqrt[n]{a} < \sqrt[n]{b} = f(b)$. This property is useful in solving certain types of polynomial inequalities.[2]



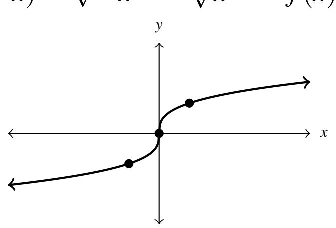
$y = \sqrt{x}$



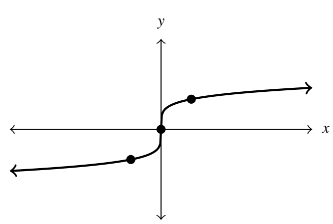
$y = \sqrt[4]{x}$

$y = \sqrt[6]{x}$

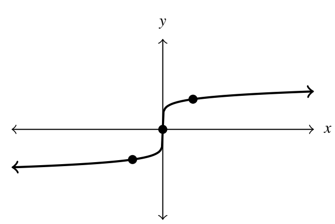
The functions $f(x) = \sqrt[n]{x}$ for odd natural numbers, $n \geq 3$, also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. The range for these functions is $(-\infty, \infty)$ because if c is any real number, $f(c^n) = \sqrt[n]{c^n} = c$, so c is in the range of f. Like the even indexed roots, the odd indexed roots are also increasing. Moreover, these graphs appear to be symmetric about the origin. Sure enough, when n is odd, $f(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -f(x)$ so f is an odd function.



$y = \sqrt[3]{x}$



$y = \sqrt[5]{x}$



$y = \sqrt[7]{x}$

[1] Although we discussed imaginary numbers in Section 1.5, we restrict our attention to real numbers in this section.
[2] See Exercise 13.

At this point, you're probably expecting a theorem like Theorems 1.4, 2.1, 2.2, 3.1 - that is, a theorem which tells us how to obtain the graph of $F(x) = a\sqrt[n]{x-h}+k$ from the graph of $f(x) = \sqrt[n]{x}$ - and you would not be wrong. Here, however, we need to add an extra parameter 'b' to the recipe and discuss functions of the form $F(x) = a\sqrt[n]{bx-h}+k$. The reason is that, with all of the previous function families, we were always able to factor out the coefficient of x. We list some examples of this below, and invite the reader to revisit other examples in the text:

- $F(x) = |6-2x| = |-2x+6| = |-2(x-3)| = |-2||x-3| = 2|x-3|$.
- $F(x) = (2x-1)^2+1 = \left[2\left(x-\frac{1}{2}\right)\right]^2+1 = (2)^2\left(x-\frac{1}{2}\right)^2+1 = 4\left(x-\frac{1}{2}\right)^2+1$.
- $F(x) = \frac{2}{(1-x)^3}-5 = \frac{2}{[(-1)(x-1)]^3}-5 = \frac{2}{(-1)^3(x-1)^3}-5 = \frac{2}{-(x-1)^3}-5 = \frac{-2}{(x-1)^3}-5$.

For a function like $F(x) = \sqrt{4x-12}+1 = \sqrt{4(x-6)}+1 = \sqrt{4}\sqrt{x-3}+1 = 2\sqrt{x-3}+1$, this approach works fine. However, if the coefficient of x is *negative*, for example, $F(x) = \sqrt{1-x} = \sqrt{(-1)(x-1)}$ we get stuck the product rule for radicals doesn't extend to negative quantities when the index is even.[3] Hence we add an extra parameter which means we have an extra step. We state Theorem 4.1 below.

Theorem 4.1. For real numbers a, b, h, and k with $a,b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx-h}+k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x}$ to $y = \sqrt[n]{x-h}$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x-h}$ to $y = \sqrt[n]{bx-h}$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a\sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}+k$.

Proof. As usual, we 'build' the graph of $F(x) = a\sqrt[n]{bx-h}+k$ starting with the graph of $f(x) = \sqrt[n]{x}$ one step at a time. First, we consider the graph of $F_1(x) = \sqrt{x-h}$. A generic point on the graph of F_1 looks like $(x, \sqrt[n]{x-h})$. Note that if n is odd, x can be any real number whereas if n is even $x-h \geq 0$ so $x \geq h$. If we

[3] Because, otherwise, $-1 = i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$, a contradiction.

let $c = x - h$, then $x = c + h$ and we can change (dummy) variables[4] and obtain a new representation of the point: $(c+h, \sqrt[n]{c})$. Note that if n is odd, x and c vary through all real numbers; if n is even, $x \geq h$ and, hence, $c \geq 0$. As a generic point on the graph of $f(x) = \sqrt[n]{x}$ can be represented as $(c, \sqrt[n]{c})$ for applicable values of c, we see that we can obtain every point on the graph of F_1 by adding h to each x-coordinate of the graph of f, establishing step 1 of the theorem.

Proceeding to (the new!) step 2, a point on the graph of $F_2(x) = \sqrt[n]{bx-h}$ has the form $(x, \sqrt[n]{bx-h})$. If n is odd, as usual, x can vary through all real numbers. If n is even, we require $bx - h \geq 0$ or $bx \geq h$. If $b > 0$, this gives $x \geq \frac{h}{b}$. If, on the other hand, $b < 0$, then we have $x \leq \frac{h}{b}$. Let $c = bx$ and thus by assumpting $b \neq 0$, we have $x = \frac{c}{b}$. Once again, we change dummy variables from x to c and describe a generic point on the graph of F_2 as $\left(\frac{c}{b}, \sqrt[n]{c-h}\right)$. If n is odd, x and c can vary through all real numbers. If n is even and $b > 0$, then $x \geq \frac{h}{b}$ and, hence, $c = bx \geq h$; if $b < 0$, then $x \leq \frac{h}{b}$ also gives $c = bx \geq h$. A generic point on the graph of F_1 can be represented as $(c, \sqrt{c-h})$ for applicable values of c, so we see we can obtain every point on the graph of F_2 by dividing every x-coordinate on the graph of F_1 by b, as per step 2 of the theorem.

The proof of steps 3 and 4 of Theorem 4.1 are identical to the proof of Theorem 2.2 (just with $\sqrt[n]{\cdot}$ instead of $(\cdot)^n$) so we invite the reader to work through the details on their own. $\square$

We demonstrate Theorem 4.1 in the following example.

Example 4.1.1. Use Theorem 4.1 to graph the following functions. Label at least three points on the graph. State the domain and range using interval notation.

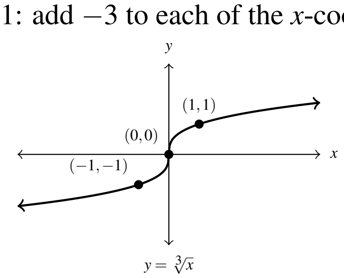
1. $f(x) = 1 - 2\sqrt[3]{x+3}$
2. $g(t) = \dfrac{\sqrt{1-2t}}{4}$

Solution.

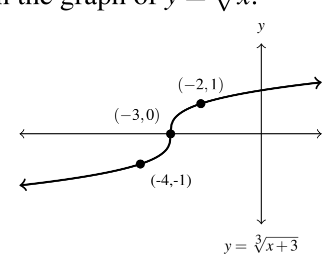
1. Graph $f(x) = 1 - 2\sqrt[3]{x+3}$.

   We begin by rewriting the expression for $f(x)$ in the form prescribed Theorem 4.1:
   $f(x) = -2\sqrt[3]{x-(-3)} + 1$. We identify $n = 3$, $a = -2$, $b = 1$, $h = -3$ and $k = 1$.

   Step 1: add -3 to each of the x-coordinates of each of the points on the graph of $y = \sqrt[3]{x}$:



add -3 to each x-coordinate $\longrightarrow$

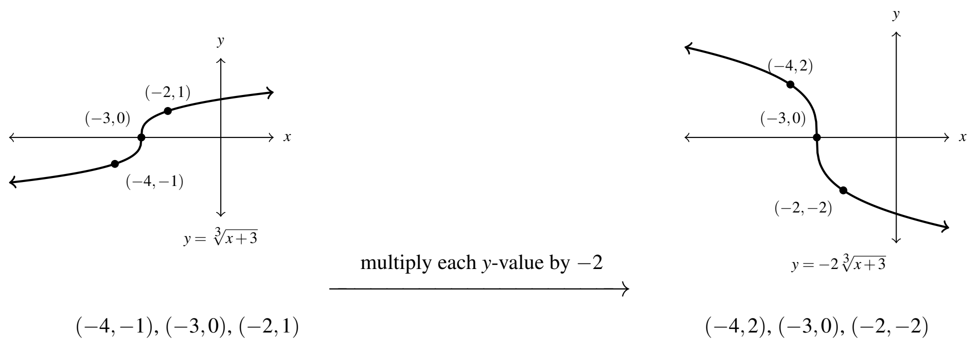


$(-1,1)$, $(0,0)$, $(1,1)$ $\qquad$ $(-4,-1)$, $(-3,0)$, $(-2,1)$

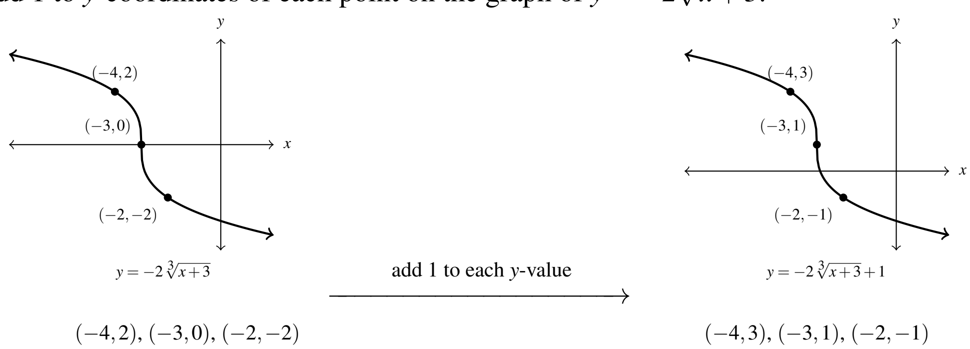
[4]again this is because every real number can be represented as both $x - h$ for some value x and as $c + h$ for some value c.

AS $b = 1$, we can proceed to Step 3 (dividing a real number by 1 results in the same real number.)

Step 3: multiply each of the y-coordinates of each point on the graph of $y = \sqrt[3]{x+3}$ by -2:



Step 4: add 1 to y-coordinates of each point on the graph of $y = -2\sqrt[3]{x+3}$:

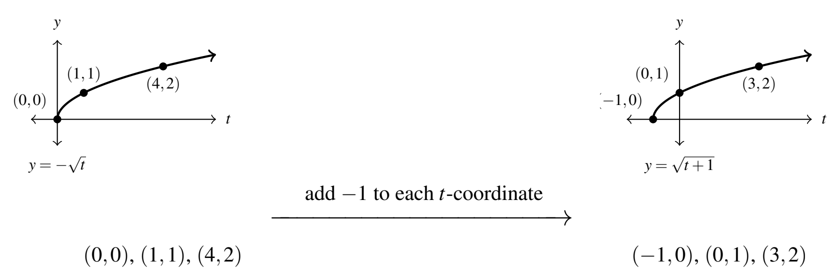


We get the domain and range of f are both $(-\infty, \infty)$.

2. Graph $g(t) = \dfrac{\sqrt{1-2t}}{4}$.

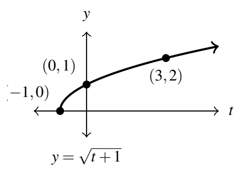
   For $g(t) = \dfrac{\sqrt{1-2t}}{4} = \frac{1}{4}\sqrt{-2t+1}$, we identify $n = 2$, $a = \frac{1}{4}$, $b = -2$, $h = -1$ and $k = 0$. We are asked to label *three* points on the graph, so we track $(4,2)$ along with $(0,0)$ and $(1,1)$.[5]

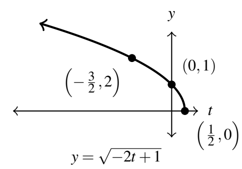
   Step 1: add -1 to each of the t-coordinates of each of the points on the graph of $y = \sqrt{t}$:



[5] As $\sqrt{4} = 2$, we know $(4,2)$ is on the graph of $y = \sqrt{t}$.

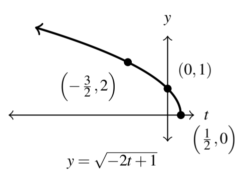
Step 2: divide each of the t-coordinates of each of the points on the graph of $y = \sqrt{t+1}$ by -2:

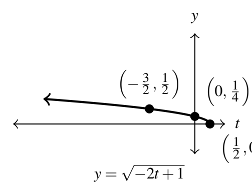




$$(-1,0),\ (0,1),\ (3,2) \xrightarrow{\text{divide each } t\text{-coordinate by } -2:} \left(\tfrac{1}{2},0\right),\ (0,1),\ \left(-\tfrac{3}{2},2\right)$$

Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = \sqrt{-2t+1}$ by $\frac{1}{4}$:





$$\left(\tfrac{1}{2},0\right),\ (0,1),\ \left(-\tfrac{3}{2},2\right) \xrightarrow{\text{multiply each } y\text{-coordinate by } \frac{1}{4}:} \left(\tfrac{1}{2},0\right),\ \left(0,\tfrac{1}{4}\right),\ \left(-\tfrac{3}{2},\tfrac{1}{2}\right)$$

We get the domain is $\left(-\infty, \frac{1}{2}\right]$ and the range is $[0,\infty)$. □

4.1.2 Other Functions involving Radicals

Now that we have some practice with basic root functions, we turn our attention to more general functions involving radicals. In general, Calculus is the best tool with which to study these functions. Nevertheless, we will use what algebra we know in combination with a graphing utility to help us visualize these functions and preview concepts which are studied in greater depth in later courses. In the table below, we summarize some of the properties of radicals from elsewhere in this text we will be using in the coming examples.

Theorem 4.2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.[a]
Simplifying n th powers and n th roots:[b]

- $\left(\sqrt[n]{x}\right)^n = x$.
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$.

Root Functions Preserve Inequality:[c] if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

[a] i.e., if n is odd, x, a, and b can be any real numbers; if, on the other hand n is even, $x \geq 0$, $a \geq 0$, and $b \geq 0$.
[b] a.k.a., 'Inverse Properties.' See Section 5.1.
[c] i.e., root functions are increasing.

Example 4.1.2. For the following functions:

- Analytically:
  - state the domain.
  - identify the axis intercepts.
  - analyze the end behavior.
- Construct a sign diagram for each function using the intercepts and sketch a graph.
- Using technology determine:
  - the range.
  - the local extrema, if they exist.
  - intervals where the function is increasing.
  - intervals where the function is decreasing.

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$
3. $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$
4. $r(t) = t^{-1}\sqrt{16t^4-1}$

Solution.

1. Analyze and graph $f(x) = 3x\sqrt[3]{2-x}$.

   When looking for the domain, we have two thing to watch out for: denominators (which we must make sure aren't 0) and even indexed radicals (whose radicands we must ensure are nonnegative.) Looking at the expression for $f(x)$, we have no denominators nor do we have an even indexed radical, so we are confident the domain is all real numbers, $(-\infty,\infty)$.

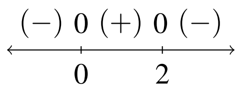
   To find the x-intercepts, we find the zeros of f by solving $f(x) = 3x\sqrt[3]{2-x} = 0$. Using the zero product property, we get $3x = 0$ or $\sqrt[3]{2-x} = 0$. The former gives $x = 0$ and to solve the latter, we cube both sides and get $2-x = 0$ or $x = 2$. Hence, the x-intercepts are $(0,0)$ and $(2,0)$. As $(0,0)$ is also on the y-axis and functions can have at most one y-intercept, we know $(0,0)$ is the only y-intercept.[6] That being said, we can quickly verify $f(0) = 3(0)\sqrt[3]{2-0} = 0$.

   To determine the end behavior, we consider $f(x)$ as $x \to \pm\infty$. Using 'number sense,' [7] we have $f(x) = 3x\sqrt[3]{2-x} = 3x\sqrt[3]{-x+2} \approx (\text{big }(+))\sqrt[3]{\text{big }(-)} = (\text{big }(+))(\text{big }(-)) = \text{big }(-)$, so $f(x) \to -\infty$. As $x \to -\infty$ we get $f(x) = 3x\sqrt[3]{-x+2} \approx (\text{big }(-))\sqrt[3]{\text{big }(+)} = (\text{big }(-))(\text{big }(+)) = \text{big }(-)$, so $f(x) \to -\infty$ here, too.

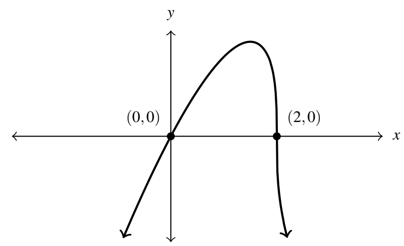
   To create a sign diagram for $f(x)$, we note that the function has zeros $x = 0$ and $x = 2$. For $x < 0$, $f(x) < 0$ or $(-)$, for $0 < x < 2$, $f(x) > 0$ or $(+)$, and for $x > 2$, $f(x) < 0$ or $(-)$. The sign diagram for $f(x)$ is on the left. The graph of f is on the right.

[6] Why is this, again?

[7] remember this means we use the adjective 'big' here to mean large in *absolute value*

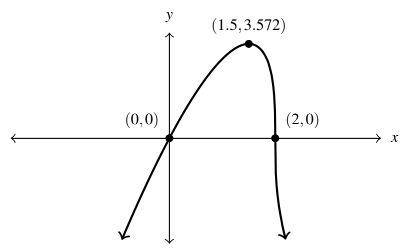


Sign Diagram for $f(x)$



The graph of $y = f(x)$

From the graph and our use of technology, the range is approximately $(-\infty, 3.572]$ with a local maximum (which also happens to be *the* maximum) at $(1.5, 3.572)$. We also see f appears to be increasing on $(-\infty, 1.5)$ and decreasing on $(1.5, \infty)$. It is also worth noting that there appears to be 'unusual steepness' near the x-intercept $(2,0)$. We invite the reader to zoom in on the graph near $(2,0)$ to see that the function appears 'locally vertical.'[8]



The graph of $y = f(x)$ using technology.

2. Analyze and graph $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$.

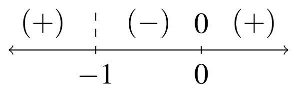
   The index of the radical in the expression for $g(t)$ is odd, so our only concern is the denominator. Setting $t+1 = 0$ gives $t = -1$, which we exclude, so our domain is $\{t \in \mathbb{R} \,|\, t \neq -1\}$ or using interval notation, $(-\infty, -1) \cup (-1, \infty)$. If we take the time to analyze the behavior of g near $t = -1$, we find that as $t \to -1^-$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \approx \sqrt[3]{\frac{-8}{\text{small } (-)}} \approx \sqrt[3]{\text{big}(+)} = \text{big } (+)$. That is, as $t \to -1^-$, $g(t) \to \infty$. Likewise, as $t \to -1^+$, $g(t) \approx \sqrt[3]{\frac{-8}{\text{small } (+)}} \approx \sqrt[3]{\text{big } (-)} = \text{big } (-)$. This suggests as $t \to -1^+$, $g(t) \to -\infty$. This behavior points to a vertical asymptote, $t = -1$.

   To find the t-intercepts of the graph of g, we find the zeros of g by setting $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 0$. Cubing both sides and clearing denominators gives $8t = 0$ or $t = 0$. Hence our t-, and in this case, y- intercept is $(0,0)$.

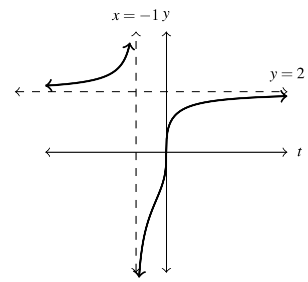
   To determine the end behavior, we note that as $t \to \pm\infty$, $\frac{8t}{t+1} \to \frac{8}{1} = 8$. Hence, it stands to reason that as $t \to \pm\infty$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \to \sqrt[3]{8} = 2$. This suggests the graph of $y = g(t)$ has a horizontal asymptote at $y = 2$.

[8]Of course, the Vertical Line Test prohibits the graph from actually *being* a vertical line. This behavior is more precisely defined and more closely studied in Calculus.

To create a sign diagram for $g(t)$, we note that the function is undefined when $t = -1$ (so we place a dashed line above it) and has a zero $t = 0$. When $t < -1$, $g(t) > 0$ or $(+)$, for $-1 < t < 0$, $g(t) < 0$ or $(-)$, and for $t > 0$, $g(t) > 0$ or $(+)$. On the left is a sign diagram for $g(t)$. The graph of f is on the right.



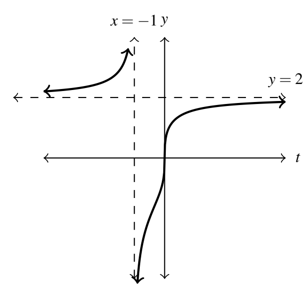
Sign Diagram for $g(t)$



The graph of $y = g(t)$

The graph confirms our suspicions about the asymptotes $t = -1$ and $y = 2$. Moreover, the range appears to be $(-\infty, 2) \cup (2, \infty)$. We could check if the graph ever crosses its horizontal asymptote by attempting to solve $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 2$. Cubing both sides and clearing denominators gives $8t = 8(t+1)$ which results in $0 = 8$, a contradiction. This proves 2 is not in the range, as we had suspected.

Scanning the graph, there appears to be no local extrema, and, moreover, the graph suggests g is increasing on $(-\infty, -1)$ and again on $(-1, \infty)$. As with the previous example, the graph appears locally vertical near its intercept $(0, 0)$.



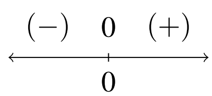
The graph of $y = g(t)$

3. Analyze and graph $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$.

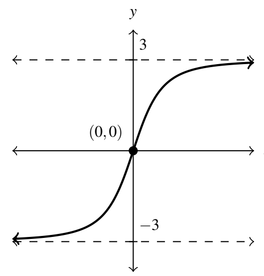
The expression for $h(x) = \frac{3x}{\sqrt{x^2+1}}$ has both a denominator and an even-indexed radical, so we have to be extra cautious here. Fortunately for us, the quantity $x^2 + 1 > 0$ for al real numbers x. Not only does this mean $\sqrt{x^2+1}$ is always defined, it also tells us $\sqrt{x^2+1} > 0$ for all x, too. This means the domain of h is all real numbers, $(-\infty, \infty)$.

Solving for the zeros of h gives only $x = 0$, and we find, once again, $(0, 0)$ is both our lone x- and y-intercept. Moving on to end behavior, as $x \to \pm\infty$, the term x^2 is the dominant term in the radicand in the denominator. As such, $h(x) = \frac{3x}{\sqrt{x^2+1}} \approx \frac{3x}{\sqrt{x^2}} = \frac{3x}{|x|}$. As $x \to \infty$, $|x| = x$ (because $x > 0$), so $h(x) \approx \frac{3x}{x} = 3$, so $h(x) \to 3$. Likewise, as $x \to -\infty$, $|x| = -x$ (because $x < 0$) and hence, $h(x) \approx \frac{3x}{-x} = -3$, so $h(x) \to -3$. This analysis suggests the graph of $y = h(x)$ has not one, but *two* horizontal asymptotes.[9] The graph of h below on the right bears this out.

The domain of h is all real number and the only zero of h is $x = 0$, so the sign diagram for $h(x)$ is fairly straight forward. For $x < 0$, $h(x) < 0$ or $(-)$ and for $x > 0$, $h(x) > 0$ or $(+)$.

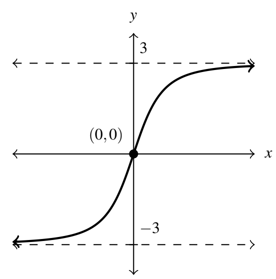


Sign Diagram for $h(x)$



The graph of $y = h(x)$

From the graph, we see the range of h appears to be $(-3,3)$. Attempting to solve $h(x) = \frac{3x}{\sqrt{x^2+1}} = \pm 3$ gives, in either case, $9x^2 = 9(x^2+1)$ which reduces to $0 = 9$, a contradiction. Hence, the graph of $y = h(x)$ never reaches its horizontal asymptotes. Moreover, h appears to be always increasing, with no local extrema or 'unusual' steepness. One last remark: it appears as if the graph of h is symmetric about the origin. We check $h(-x) = \frac{3(-x)}{\sqrt{(-x)^2+1}} = -\frac{3x}{\sqrt{x^2+1}} = -h(x)$ which verifies h is odd.



The graph of $y = h(x)$

4. Analyze and graph $r(t) = t^{-1}\sqrt{16t^4 - 1}$.

   The first thing to note about the expression $r(t) = t^{-1}\sqrt{16t^4-1}$ is that $t^{-1} = \frac{1}{t}$. Hence, we must exclude $t = 0$ from the domain straight away. Next, we have an even-indexed radical expression: $\sqrt{16t^4-1}$. In order for this to return a real number, we require $16t^4 - 1 \geq 0$. Instead of using a sign diagram to solve this, we opt instead to *carefully* use properties of radicals. Isolating t^4, we have $t^4 \geq \frac{1}{16}$. As the root functions are increasing, we can apply the fourth root to both sides and preserve

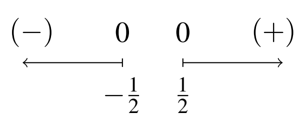
[9] We warned you this was coming ... see the discussion following Theorem 3.3 in Section 3.2.

the inequality: $\sqrt[4]{t^4} \geq \sqrt[4]{\frac{1}{16}}$ which gives[10] $|t| \geq \frac{1}{2}$. Note that $t = 0$ does *not* satisfy this inequality, thus restricting t in this manner takes care of *both* domain issues, so the domain is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

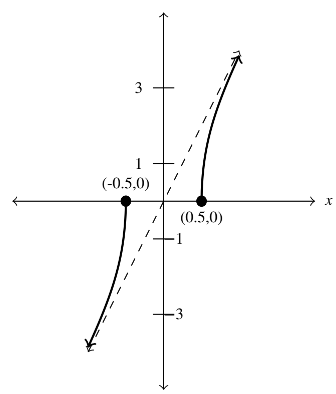
Next, we look for zeros. Setting $r(t) = t^{-1}\sqrt{16t^4 - 1} = \frac{\sqrt{16t^4-1}}{t} = 0$ gives $\sqrt{16t^4 - 1} = 0$. After squaring both sides, we get $16t^4 - 1 = 0$ or $t^4 = \frac{1}{16}$. Extracting fourth roots, we get $t = \pm\frac{1}{2}$. Both of these are (barely!) in the domain of r, so our t intercepts are $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Note, the graph of r has no y-intercept, because $r(0)$ is undefined ($t = 0$ is not in the domain of r).

Concerning end behavior, we note the term $16t^4$ dominates the radicand $\sqrt{16t^4 - 1}$ as $t \to \pm\infty$, hence, $r(t) = \frac{\sqrt{16t^4-1}}{t} \approx \frac{\sqrt{16t^4}}{t} = \frac{4t^2}{t} = 4t$. This suggests the graph of $y = r(t)$ has a slant asymptote with slope 4.[11]

To construct the sign diagram for $r(t)$ we note r has two zeros, $t = \pm\frac{1}{2}$. For $t < \frac{1}{2}$, $r(t) < 0$ or $(-)$ and when $t > \frac{1}{2}$, $r(t) > 0$ or $(+)$. When $-\frac{1}{2} < t < \frac{1}{2}$, r is undefined so we have removed that segment from the diagram, as seen below on the left. The graph of $r(t)$ is on the right.



Sign Diagram for $r(t)$



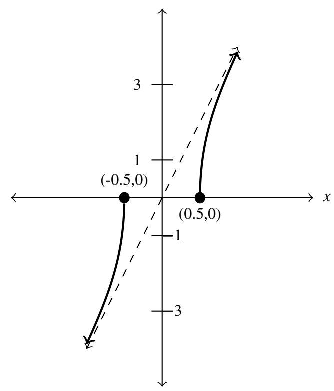
The graph of $y = r(t)$

W see the range appears to be all real numbers, $(-\infty, \infty)$. It appears as if r is increasing on $\left(-\infty, -\frac{1}{2}\right]$ and again on $\left[\frac{1}{2}, \infty\right)$. The graph does appear to be asymptotic to $y = 4t$, and it also appears to be symmetric about the origin. Sure enough, we find $r(-t) = \frac{\sqrt{16(-t)^4-1}}{-t} = -\frac{\sqrt{16t^4-1}}{t} = -r(t)$, proving r is an odd function.

---

[10]Recall: $\sqrt[n]{x^n} = |x|$, not x, if n is even.

[11]Note: this analysis suggests the slant asymptote is $y = 4t + b$, but from this analysis, we cannot determine the value of b. As with slant asymptotes in Section 3.2, we'd need to perform a more detailed analysis which we omit in this case owing to the complexity of the function.



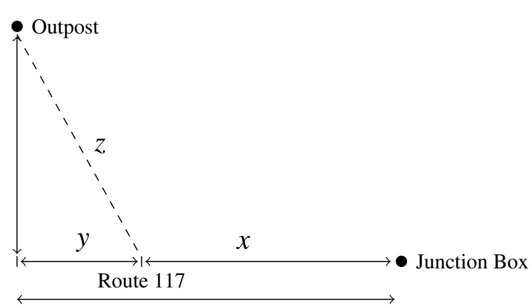
The graph of $y = r(t)$

□

We end this section with a classic application of root functions.

Example 4.1.3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.

1. Write an expression, $C(x)$, which computes the cost of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.
2. Graph $C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?



Solution.

1. Write an expression $C(x)$.

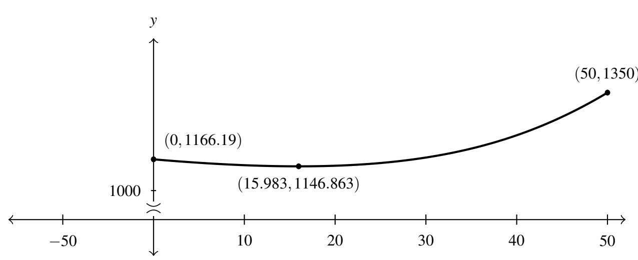
The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. x represents the miles of cable run along Route 117, thus the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $15x+20z$.

Our next goal is to determine z in terms of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2+30^2=z^2$. But we also see $x+y=50$ so that $y=50-x$. Substituting $(50-x)$ in for y we obtain $z^2=(50-x)^2+900$. Solving for z, we obtain $z=\pm\sqrt{(50-x)^2+900}$. Because z represents a distance, we choose $z=\sqrt{(50-x)^2+900}$.

Hence, the cost as a function of x is given by $C(x)=15x+20\sqrt{(50-x)^2+900}$. From the context of the problem, we have $0\leq x\leq 50$.

2. Graph $C(x)$.

We graph $y=C(x)$ below and find our (local) minimum to be at the point $(15.98, 1146.86)$, using technology. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live Sasquatch-Casts? Priceless.



$C(x)=15x+20\sqrt{(50-x^2)+900}$

□

4.1.3 Exercises

In Exercises 1 - 8, given the pair of functions f and F, sketch the graph of $y=F(x)$ by starting with the graph of $y=f(x)$ and using Theorem 4.1. Track at least two points and state the domain and range using interval notation.

1. $f(x)=\sqrt{x}$, $F(x)=\sqrt{x+3}-2$
2. $f(x)=\sqrt{x}$, $F(x)=\sqrt{4-x}-1$
3. $f(x)=\sqrt[3]{x}$, $F(x)=\sqrt[3]{x-1}-2$
4. $f(x)=\sqrt[3]{x}$, $F(x)=-\sqrt[3]{8x+8}+4$

5. $f(x)=\sqrt[4]{x}$, $F(x)=\sqrt[4]{x-1}-2$

6. $f(x)=\sqrt[4]{x}$, $F(x)=-3\sqrt[4]{x-7}+1$

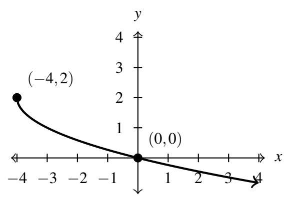
7. $f(x)=\sqrt[5]{x}$, $F(x)=\sqrt[5]{x+2}+3$

8. $f(x)=\sqrt[8]{x}$, $F(x)=\sqrt[8]{-x}-2$

In Exercises 9 - 10, find a formula for each function below in the form $F(x)=a\sqrt{bx-h}+k$.

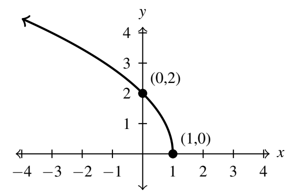
NOTE: There may be more than one solution!

9. $y=F(x)$



$x-$, y-intercept $(0,0)$

10. $y=F(x)$

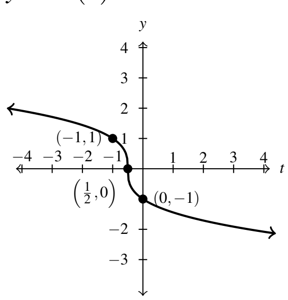


x-intercept $(1,0)$, y-intercept $(0,2)$

In Exercises 11 - 12, find a formula for each function below in the form $F(x)=a\sqrt[3]{bx-h}+k$.

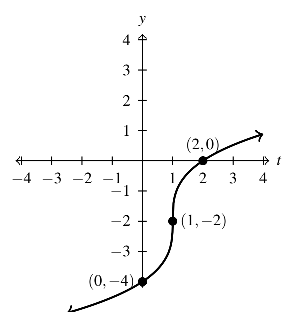
NOTE: There may be more than one solution!

11. $y=F(x)$



x-intercept $(-\frac{1}{2},0)$, y-intercept $(0,-1)$

12. $y=F(x)$



x-intercept $(2,0)$, y-intercept $(0,-4)$

13. Use the fact that the nth root functions are increasing to solve the following polynomial inequalities:

(a) $x^3 \leq 64$ (b) $2-t^5<34$ (c) $\dfrac{(2z+1)^3}{4} \geq 2$

For the following inequalities, remember $\sqrt[n]{x^n}=|x|$ if n is even:

(d) $x^4 \leq 16$ (e) $6-t^6<-58$ (f) $\dfrac{(2z+1)^4}{3} \geq 27$

For each function in Exercises 14 - 21 below

- Analytically:
  - state the domain.
  - determine the axis intercepts.
  - analyze the end behavior.
- Construct a sign diagram for each function using the intercepts and sketch a graph.
- Using technology determine:
  - the range.
  - the local extrema, if they exist.
  - intervals where the function is increasing/decreasing.
  - any ‘unusual steepness’ or ‘local’ verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Comment on any observed symmetry.

14. $f(x)=\sqrt{1-x^2}$
15. $f(x)=\sqrt{x^2-1}$
16. $g(t)=t\sqrt{1-t^2}$
17. $g(t)=t\sqrt{t^2-1}$
18. $f(x)=\sqrt[4]{\dfrac{16x}{x^2-9}}$
19. $f(x)=\dfrac{5x}{\sqrt[3]{x^3+8}}$
20. $g(t)=\sqrt{t(t+5)(t-4)}$
21. $g(t)=\sqrt[3]{t^3+3t^2-6t-8}$

22. Rework Example 4.1.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.

23. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V=\frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S=\pi r\sqrt{r^2+h^2}$. In the following problems, suppose a cone is to have a volume of 100 cubic centimeters.

    (a) Use the formula for volume to find the height as a function of r, $h(r)$.

    (b) Use the formula for surface area along with your answer to 23a to find the surface area as a function of r, $S(r)$.

    (c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

24. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

25. According to Einstein's Theory of Special Relativity, the observed mass of an object is a function of how fast the object is traveling. Specifically, if m_r is the mass of the object at rest, v is the speed of the object and c is the speed of light, then the observed mass of the object $m(v)$ is given by:

$$m(v) = \frac{m_r}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(a) State the applied domain of the function.

(b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

(c) As $v \to c^-$, what happens to $m(x)$?

(d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

26. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

Section 4.1 Exercise Answers A.1.4

4.2 Properties of Power Functions and Their Graphs

Monomial, and, more generally, Laurent monomial functions are specific examples of a much larger class of functions called **power functions**, as defined below.

Definition 4.2. Let a and p be nonzero real numbers. A **power function** is either a constant function or a function of the form $f(x) = ax^p$.

Definition 4.2 broadens our scope of functions to include non-integer exponents such as $f(x) = 2x^{4/3}$, $g(t) = t^{0.4}$ and $h(w) = w^{\sqrt{2}}$. Our primary aim in this section is to ascribe meaning to these quantities.

4.2.1 Rational Number Exponents

The road to real number exponents starts by defining rational number exponents.

Definition 4.3. Let r be a rational number where in lowest terms $r = \frac{m}{n}$ where m is an integer and n is a natural number.[a] If $n = 1$, then $x^r = x^m$. If $n > 1$, then

$$x^r = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m},$$

whenever $\left(\sqrt[n]{x}\right)^m$ is defined.[b]

[a]Recall 'lowest terms' means m and n have no common factors other than 1.
[b]That is, if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

There are quite a few items worthy of note which are consequences of Definition 4.3. First off, if m is an integer, then $x^{\frac{m}{1}} = x^m$. So expressions like $x^{\frac{3}{1}}$ are synonymous with x^3, as we would expect.[1] Second, the definition of $x^{\frac{m}{n}}$ can be taken as just $\left(\sqrt[n]{x}\right)^m$ and shown to be equal to $\sqrt[n]{x^m}$ (or vice-versa) courtesy of properties of radicals. We state both in Definition 4.3 to allow for the reader to choose whichever form is more convenient in a given situation. The critical point to remember is no matter which representation you choose, keep in mind the restrictions if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

Moreover, per this definition, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$, so we may rewrite principal roots as exponents: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[5]{x} = x^{\frac{1}{5}}$. This makes sense from an algebraic standpoint because per Theorem 4.2, $\left(\sqrt[n]{x}\right)^n = x$. Hence if we were to assign an exponent notation to $\sqrt[n]{x}$, say $\sqrt[n]{x} = x^r$, then $\left(\sqrt[n]{x}\right)^n = (x^r)^n = x$. If the properties of exponents are to hold, then, necessarily, $(x^r)^n = x^{rn} = x = x^1$, so $rn = 1$ or $r = \frac{1}{n}$. While this argument helps *motivate* the notation, as we shall see shortly, great care must be exercised in applying exponent properties in these cases. The long and short of this is that root functions as defined in Section 4.1 are all members of the 'power functions' family.

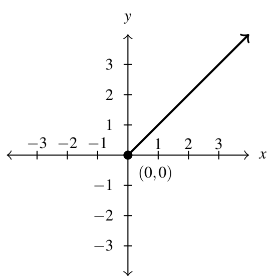
Another important item worthy of note in Definition 4.3 is that it is absolutely essential we express the

[1]Either $n = 1$ is a special case in Definition 4.3 or we need to define what is meant by $\sqrt[1]{x}$. The authors chose the former.

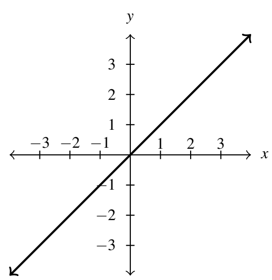
rational number r in *lowest terms* before applying the root-power definition. For example, consider $x^{0.4}$. Expressing r in lowest terms, we get: $r = 0.4 = \frac{4}{10} = \frac{2}{5}$. Hence, $x^{0.4} = x^{2/5} = (\sqrt[5]{x})^2$ or $\sqrt[5]{x^2}$, either of which is defined for all real numbers x. In contrast, consider the equivalence $r = 0.4 = \frac{4}{10}$. Here, the expression $(\sqrt[10]{x})^4$ is defined only for $x \geq 0$ owing to the presence of the even indexed root, $\sqrt[10]{x}$. Hence, $(\sqrt[10]{x})^4 \neq x^{\frac{4}{10}} = x^{\frac{2}{5}}$ unless $x \geq 0$. On the other hand, the expression $\sqrt[10]{x^4}$ *is* defined for all numbers, x, as $x^4 \geq 0$ for all x. In fact, it can be shown that $\sqrt[10]{x^4} = \sqrt[5]{x^2}$ for all real numbers. This means $\sqrt[10]{x^4} = \sqrt[5]{x^2} = x^{\frac{2}{5}} = x^{\frac{4}{10}}$. So, to review, in general we have: $x^{\frac{4}{10}} = \sqrt[10]{x^4}$, but $x^{\frac{4}{10}} \neq (\sqrt[10]{x})^4$ unless $x \geq 0$. Once again the easiest way to avoid confusion here is to *reduce* the exponent to *lowest terms* before converting it to root-power notation.

Likewise, we have to be careful about the properties of exponents when it comes to rational exponents. Consider, for instance, the product rule for integer exponents: $x^m x^n = x^{m+n}$. Consider $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2}+\frac{1}{2}}$. In the first case, $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = \sqrt{x}\sqrt{x} = (\sqrt{x})^2 = x$ only for $x \geq 0$. In the second case, $g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x^1 = x$ for all real numbers x. Even though $f(x) = g(x)$ for $x \geq 0$, f and g are *different functions* as they have *different domains*.

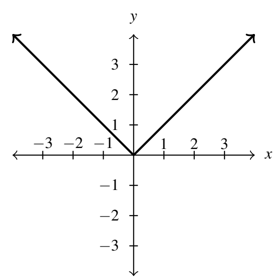
Similarly, the power rule for integer exponents: $(x^n)^m = x^{nm}$ does not hold in general for rational exponents. To see this, consider the three functions: $f(x) = (x^{\frac{1}{2}})^2$, $g(x) = x^{\frac{2}{2}}$, and $h(x) = (x^2)^{\frac{1}{2}}$. In the first case, $f(x) = (x^{\frac{1}{2}})^2 = (\sqrt{x})^2 = x$ for $x \geq 0$ only (this is the same function f above.) In the second case, the rational number $r = \frac{2}{2} = 1$, so $g(x) = x^{\frac{2}{2}} = x^{\frac{1}{1}} = x^1 = x$ for *all* real numbers, x (this is the same function g from above.) In the last case, $h(x) = (x^2)^{\frac{1}{2}} = \sqrt{x^2} = |x|$ for all real numbers, x. Once again, despite $f(x) = g(x) = h(x)$ for all $x \geq 0$, f, g and h and are *three different functions*. We graph f, g, and h below.



$f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = \left(x^{\frac{1}{2}}\right)^2 = x, x \geq 0$



$g(x) = x^{\frac{1}{2}+\frac{1}{2}} = \left(x^{\frac{2}{2}}\right) = x$



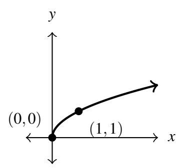
$h(x) = x^{\frac{1}{2}+\frac{1}{2}} = \left(x^{\frac{2}{2}}\right) = x$

In general, the properties of integer exponents *do not extend* to rational exponents *unless* the bases involved represent non-negative real numbers *or* the roots involved are *odd*. We have the following:

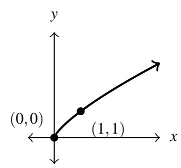
Theorem 4.3. Let r and s are rational numbers. The following properties hold provided none of the computations results in division by 0 and **either** r and s have odd denominators **or** $x \geq 0$ *and* $y \geq 0$:

- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$.
- **Quotient Rules:** $\dfrac{x^r}{x^s} = x^{r-s}$ and $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

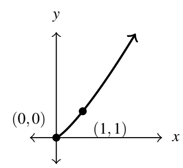
Next, we turn our attention to the graphs of $f(x) = x^r = x^{\frac{m}{n}}$ for varying values of m and n. When n is even, the domain is restricted owing to the presence of the even indexed root to $[0,\infty)$. The range is likewise $[0,\infty)$, a fact left to the reader. All of the functions below are increasing on their domains, and it turns out this is always the case provided $r > 0$. There is, however, a difference in *how* the functions are increasing - and this is the concept of *concavity*. As with many concepts we've encountered so far in the text, concavity is most precisely defined using Calculus terminology, but we can nevertheless get a sense of concavity geometrically. For us, a curve is **concave up** over an interval if it resembles a portion of a '$\smile$' shape. Similarly, a curve is called **concave down** over an interval if resembles part of a '$\frown$' shape. When $0 < r < 1$, the graphs of $f(x) = x^r$ resemble the left half of $\frown$ and so are concave down; when $r > 1$, the graphs resemble the right half of a '$\smile$' and are hence described as 'concave up.'



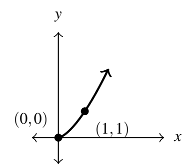
$f(x) = x^{\frac{1}{2}}$



$f(x) = x^{\frac{3}{4}}$

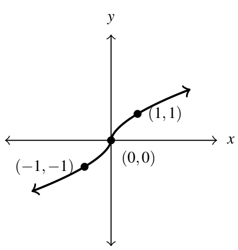


$f(x) = x^{\frac{5}{4}}$

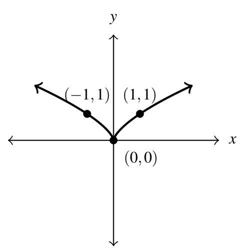


$f(x) = x^{\frac{3}{2}}$

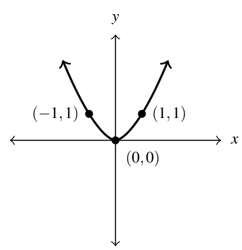
Below we graph several examples of $f(x) = x^r = x^{\frac{m}{n}}$ where n is odd. Here, the domain is $(-\infty,\infty)$ because the index on the root here is odd. Note that when m is even, the graphs appear to be symmetric about the y-axis and the range looks to be $[0,\infty)$. When m is odd, the graphs appear to be symmetric about the origin with range $(-\infty,\infty)$. We leave verification of these facts to the reader. Note here also that for $x \geq 0$, the graphs are down for $0 < r < 1$ and concave up for $r > 1$.



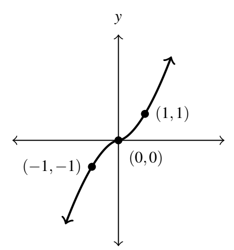
$f(x) = x^{\frac{3}{5}}$



$f(x) = x^{\frac{2}{3}}$

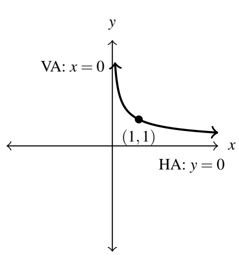


$f(x) = x^{\frac{8}{5}}$

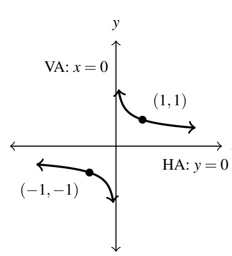


$f(x) = x^{\frac{5}{3}}$

When $r < 0$, we have variables appear in the denominator which open the opportunities for vertical and horizontal asymptotes. Below are graphed two examples



$y = x^{-\frac{1}{2}}$



$y = x^{-\frac{1}{3}}$

Unsurprisingly, Theorem 4.1, which, as stated, applied to root functions, generalizes to all rational powers.

Theorem 4.4. For real numbers a, b, h, and k and rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx-h)^r + k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^r$ to $y = (x-h)^r$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = (x-h)^r$ to $y = (bx-h)^r$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (bx-h)^r$ to $y = a(bx-h)^r$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

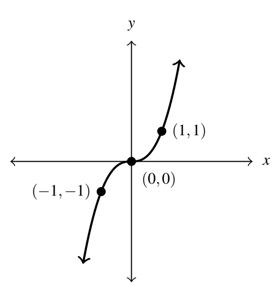
   NOTE: This transforms the graph of $y = a(bx-h)^r$ to $y = a(bx-h)^r + k$.

The proof of Theorem 4.4 is identical to that of Theorem 4.1, and we suggest the reader work through the details. We give Theorem 4.4 a test run in the following example.

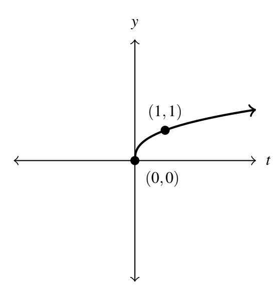
Example 4.2.1. Use the given graphs of f and g below long with Theorem 4.4 to graph F and G. State the domain and range of F and G using interval notation.

1. Graph $F(x) = (2x-1)^{2.6}$.

2. Graph $G(t) = 1 - 2(t+3)^{\frac{3}{8}}$.



$f(x) = x^{2.6}$



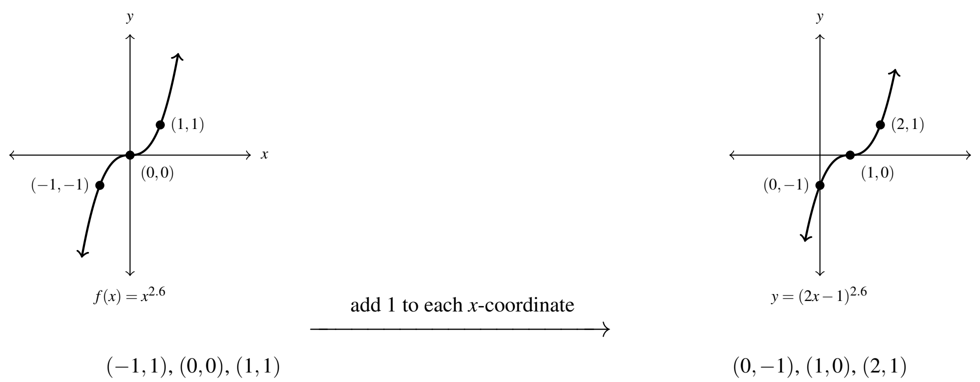
$g(t) = t^{\frac{3}{8}}$

Solution.

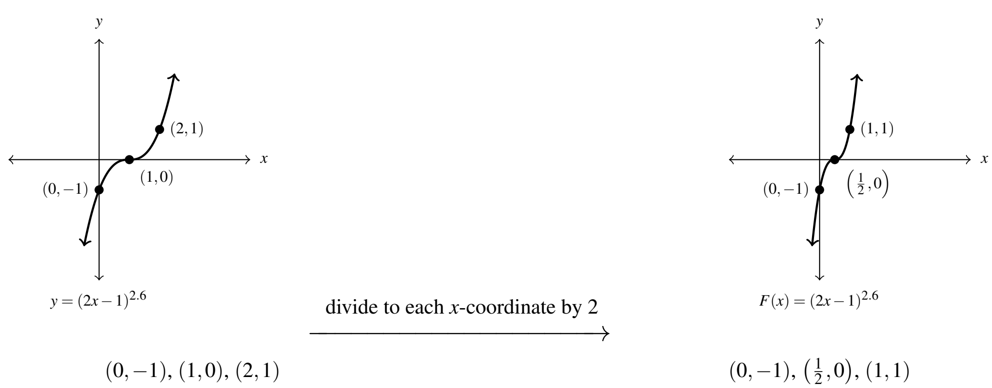
1. Graph $F(x) = (2x-1)^{2.6}$ using the graph of $f(x) = x^{2.6}$ provided.

   The expression $F(x) = (2x-1)^{2.6}$ is given to us in the form prescribed by Theorem 4.4, and we identify $r = 2.6$, $a = 1$, $b = 2$, $h = 1$, and $k = 0$. Even though the graph of $f(x) = x^{2.6}$ is given to us, it's worth taking a moment to reinforce some concepts. In lowest terms, $2.6 = \frac{26}{10} = \frac{13}{5}$, thus it makes sense the domain and range of $f(x) = x^{2.6}$ are both all real numbers and the graph is symmetric about the origin.[2] Moreover, because $2.6 > 1$, the concavity matches what we would expect, too. We proceed as we have several times in the past, beginning with the horizontal shift.

   Step 1: add 1 to each of the x-coordinates of each of the points on the graph of $y = x^{2.6}$:



   Step 2: divide each of the x-coordinates of each of the points on the graph of $y = (x-1)^{2.6}$ by 2:



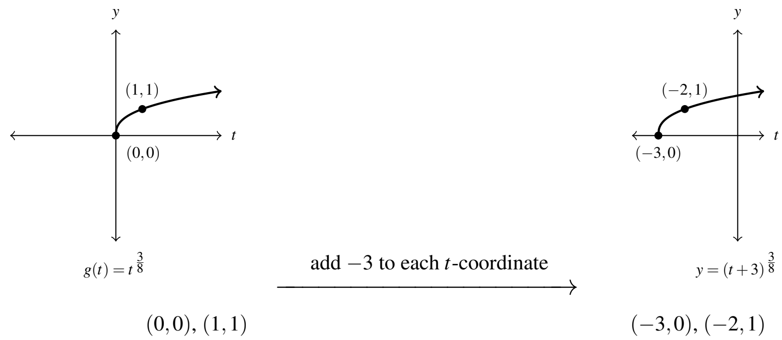
   We get the domain and range here are both $(-\infty, \infty)$.

[2]The domain is all real numbers as the denominator (root) 5 is odd; the range is all real numbers because the numerator (power) 13 is odd. Because both power and root are odd, the function itself is an odd function, hence the symmetry about the origin.

2. Graph $G(t) = 1 - 2(t+3)^{\frac{3}{8}}$ using the graph of $g(t) = t^{\frac{3}{8}}$ provided.

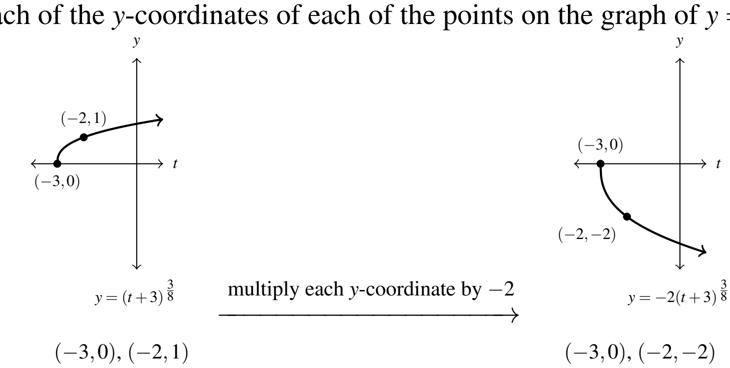
   We first need to rewrite $G(t) = 1 - 2(t+3)^{\frac{3}{8}}$ in the form required by Theorem 4.4: $G(t) = -2(t+3)^{\frac{3}{8}} + 1$. We identify $r = \frac{3}{8}$, $a = -2$, $b = 1$, $h = -3$, and $k = 1$. As $\frac{3}{8}$ is in lowest terms and has an even denominator,8, it makes sense the domain and range of $g(t) = t^{\frac{3}{8}}$ is $[0,\infty)$. Also, because $0 < \frac{3}{8} < 1$, the graph of $y = t^{\frac{3}{8}}$ is concave down, as we would expect. As usual, we start with the horizontal shift.

   Step 1: add -3 to each of the t-coordinates of each of the points on the graph of $y = t^{\frac{3}{8}}$:

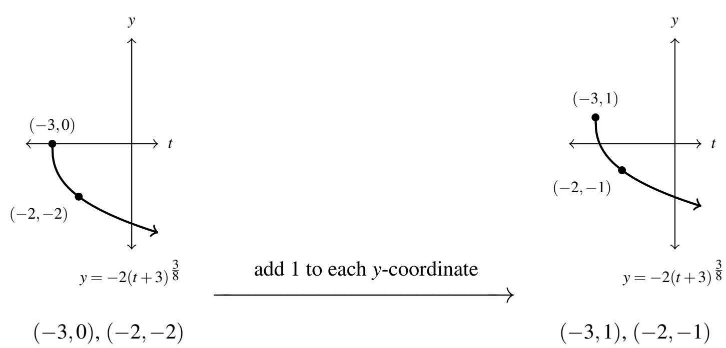


   Step 2: $b = 1$, so we proceed directly to Step 3.

   Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = (t+3)^{\frac{3}{8}}$ by -2:



   Step 4: add 1 to each of the y-coordinates of each of the points on the graph of $y = -2(t+3)^{\frac{3}{8}}$:



   From the graph, we get the domain is $[-3,\infty)$ and the range is $(-\infty,1]$. $\square$

We now turn our attention to more complicated functions involving rational exponents.

Example 4.2.2. For the following functions:

- Analytically:
  - state the domain.
  - identify the axis intercepts.
  - analyze the end behavior.
- Construct a sign diagram for each function using the intercepts and sketch a graph.
- Use technology to determine:
  - the range.
  - the local extrema, if they exist.
  - intervals where the function is increasing.
  - intervals where the function is decreasing.

1. $f(x) = 3x^2(x^3-8)^{-\frac{2}{3}}$
2. $g(t) = \dfrac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$

Solution.

1. Analyze and graph $f(x) = 3x^2(x^3-8)^{-\frac{2}{3}}$.

   We first note that, owing to the negative exponent, the quantity $(x^3-8)^{\frac{2}{3}}$ is in the denominator, alerting us to a potential domain issue. Rewriting $(x^3-8)^{\frac{2}{3}}$ we set about solving $\sqrt[3]{(x^3-8)^2} = 0$. Cubing both sides and extracting square roots gives $x^3-8 = 0$ or $x = 2$. Hence, $x = 2$ is excluded from the domain.[3] The root involved here is odd (3), so the only issue we have is with the denominator, hence our domain is $\{x \in \mathbb{R} \,|\, x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

   While not required to do so, we analyze the behavior of f near $x = 2$. As $x \to 2^-$, $3x^2 \approx 12$ and $x^3-8 \approx$ small $(-)$. Hence, $(x^3-8)^{\frac{2}{3}} = \sqrt[3]{(x^3-8)^2} \approx \sqrt[3]{(\text{small }(-))^2} \approx \sqrt[3]{\text{small }(+)} \approx \text{small}(+)$. As such, $f(x) \approx \frac{12}{\text{small}(+)} \approx$ big $(+)$. We conclude as $x \to 2^-$, $f(x) \to \infty$. As $x \to 2^+$, $3x^2 \approx 12$ and $x^3-8 \approx$ small $(+)$, and we likewise get $f(x) \to \infty$. This analysis suggests $x = 2$ is a vertical asymptote to the graph.

   To find the x-intercepts, we set $f(x) = 3x^2(x^3-8)^{-\frac{2}{3}} = 0$, so that $3x^2 = 0$ or $x = 0$. We get $(0,0)$ is our only x- (and y-)intercept.

   For end behavior, we note that in the denominator the x^3 term dominates the constant term, so as $x \to \pm\infty$,

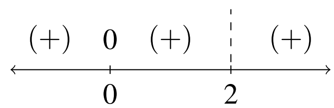
   $$f(x) = 3x^2(x^3-8)^{-\frac{2}{3}} = \frac{3x^2}{(x^3-8)^{\frac{2}{3}}} \approx \frac{3x^2}{(x^3)^{\frac{2}{3}}} = \frac{3x^2}{\sqrt[3]{(x^3)^2}} = \frac{3x^2}{\sqrt[3]{x^6}} = \frac{3x^2}{x^2} = 3.$$

[3] In general if $u^p = 0$ where $p > 0$, then $u = 0$.

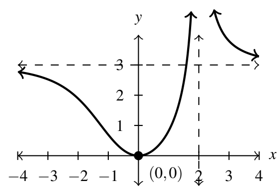
This suggests $y=3$ is a horizontal asymptote to the graph.

For the sign diagram, we note that f has only one zero, $x=0$ and is undefined at $x=2$. For all x values between these two numbers, $f(x)>0$ or $(+)$. Our sign diagram for $f(x)$ is below.

Graphing $y=f(x)$ below bears out our analysis regarding zeros and asymptotes.

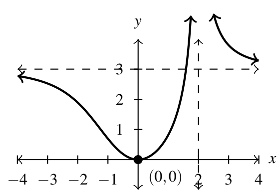


Sign Diagram for $f(x)$



The sketch of $y=f(x)$

The range appears to be $[0,\infty)$, with the graph of $y=f(x)$ crossing its horizontal asymptote between $x=1$ and $x=2$. We see we have a single local minimum at $(0,0)$ with f is decreasing on $(-\infty,0]$ and $(2,\infty)$ and increasing on $[0,2)$.



The graph of $y=f(x)$

2. Analyze and graph $g(t)=\dfrac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$.

   To find the domain of $g(t)=\frac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$, we have two issues to address: the denominator and an even (square) root. Solving $t^2-36=0$ gives two excluded values, $t=\pm 6$. For the numerator, we may rewrite $(t^2-4)^{\frac{3}{2}}=(\sqrt{t^2-4})^3$, so we require $t^2-4\geq 0$, or $t^2\geq 4$. Extracting square roots, we have $\sqrt{t^2}\geq\sqrt{4}$ or $|t|\geq 2$ which means $t\leq -2$ or $t\geq 2$. Taking into account our excluded values $t=\pm 6$, we get the domain of g is $(-\infty,-6)\cup(-6,-2]\cup[2,6)\cup(6,\infty)$.

   Looking near $t=-6$, we note that as $t\to -6$, $(t^2-4)^{\frac{3}{2}}\approx 32^{\frac{3}{2}}=32^{1.5}$, a positive number. As $t\to -6^-$, $t^2-36\approx$ small $(+)$, so $g(t)\approx\frac{32^{1.5}}{\text{small}(+)}\approx$ big $(+)$. This suggests as $t\to -6^-$, $g(t)\to\infty$. On the other hand, as $t\to -6^+$, $t^2-36\approx$ small $(-)$, so $g(t)\approx\frac{32^{1.5}}{\text{small}(-)}\approx$ big $(-)$, suggesting $g(t)\to -\infty$. Similarly, we find as $t\to 6^-$, $g(t)\to -\infty$ and as $t\to 6^+$, $g(t)\to\infty$. This suggests we have two vertical asymptotes to the graph of $y=g(t)$: $t=-6$ and $t=6$.

To find the t-intercepts, we set $g(t) = 0$ and solve $(t^2-4)^{\frac{3}{2}} = 0$. This reduces to $t^2-4 = 0$ or $t = \pm 2$. As these are (just barely!) in the domain of g, we have two t-intercepts, $(-2,0)$ and $(2,0)$. The graph of g has no y-intercepts, because 0 is not in the domain of g, so $g(0)$ is undefined.

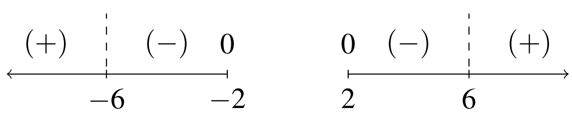
Regarding end behavior, as $t \to \pm\infty$, the t^2 in both numerator and denominator dominate the constant terms, so we have

$$g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36} \approx \frac{\left(t^2\right)^{\frac{3}{2}}}{t^2} = \frac{\left(\sqrt{t^2}\right)^3}{t^2} = \frac{|t|^3}{t^2} = \frac{|t||t|^2}{t^2} = \frac{|t|t^2}{t^2} = |t|.$$

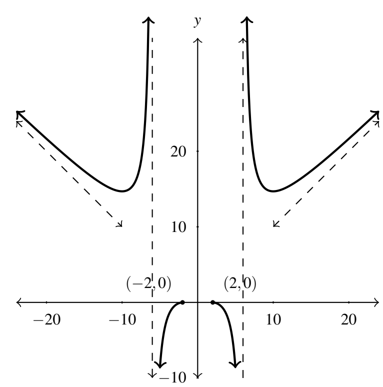
This suggests that as $t \to \infty$, the graph of $y = g(t)$ resembles $y = |t|$. Using the piecewise definition of $|t|$, we have that as $t \to -\infty$, $g(t) \approx -t$ and as $t \to \infty$, $g(t) \approx t$. In other words, the graph of $y = g(t)$ has *two* slant asymptotes with slopes ± 1.

For the sign diagram for $g(t)$, we note that g has zeros $t = \pm 2$ and is undefined at $t = \pm 6$. Moreover, there is a gap in the domain for all values in the interval $(-2,2)$, so we excise that portion of the real number line for our discussion. We find $g(t) > 0$ or $(+)$ on the intervals $(-\infty - 6)$ and $(6,\infty)$ while $g(t) < 0$ or $(-)$ on $(-6,-2)$ and $(2,6)$. Our sign diagram for $g(t)$ is below.

Graphing $y = g(t)$ below verifies our analysis.

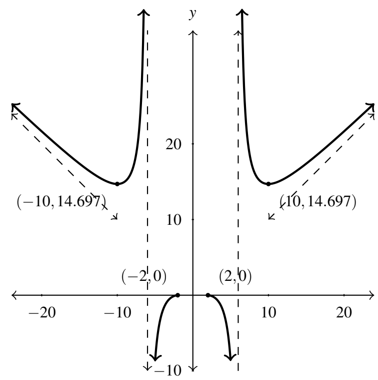


Sign Diagram for $g(t)$



The sketch of $y = g(t)$

From the graph, the range appears to be $(-\infty,0] \cup [14.697,\infty)$. The points $(-10,14.697)$ and $(10,14.697)$ are local minimums. g appears to be decreasing on $(-\infty,-10]$, $[2,6)$, and $(6,10]$. Likewise, g is increasing on $[-10,-6)$, $(-6,-2]$ and $[10,\infty)$. The graph of $y = g(t)$ certainly appears to be symmetric about the y-axis. We leave it to the reader to show g is, indeed, an even function.



The graph of $y = g(t)$

$\square$

4.2.2 Real Number Exponents

We wish now to extend the concept of 'exponent' from rational to all real numbers which means we need to discuss how to interpret an irrational exponent. Once again, the notions presented here are best discussed using the language of Calculus or Analysis, but we nevertheless do what we can with the notions we have.

Consider the wildly famous irrational number π. The number π is defined geometrically as the ratio of the circumference of a circle to that circle's diameter.[4] The reason we use the *symbol* 'π' instead of any numerical expression is that π is an irrational number, and, as such, its decimal representation neither terminates nor repeats. Hence we *approximate* π as $\pi \approx 3.14$ or $\pi \approx 3.14159265$. No matter how many digits we write, however, what we have is a *rational number* approximation of π.

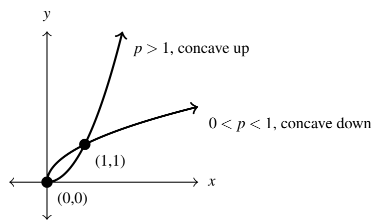
The good news is we can approximate π to any desired accuracy using rational numbers by taking enough digits, so while we'll never 'reach' the *exact* value of π with rational numbers, we can get *as close as we like* to π using rational numbers. That being said, we assume π exists on the real number line, despite the fact the list of digits to pinpoint its location is, in some sense, infinite.

We take this approach when defining the value of a number raised to an irrational exponent. Consider, for instance, 2^{π}. We can compute $2^3 = 8$, $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$, $2^{3.14} = 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.8512$, and so on, so one way to *define* 2^{π} as the unique real number we obtain as the exponents 'approach' π.

It is with this understanding that we present the notion of a 'power function,' as described in Definition 4.2: $f(x) = ax^p$ where a and p are nonzero real number parameters. Here the exponent p is open to any (nonzero) real number. Because of how we define real number exponents, if p is irrational, then $x \geq 0$ to avoid having negatives under even-indexed roots as we go through the approximation process.[5]

[4] This works for each and every circle, by the way, regardless of how large or small the circle is!
[5] or $x > 0$ if p is negative.

In general, real number exponents inherit their properties from rational number exponents. For instance, Theorem 4.3 also holds for all real number exponents and the graphs of power functions inherit their behavior from graphs of rational exponent functions. More specifically, the graphs of functions of the form $f(x) = x^p$ where $p > 0$ all contain the points $(0,0)$ and $(1,1)$. Moreover, these functions are increasing and their graphs are concave down if $0 < p < 1$ and concave up if $p > 1$.

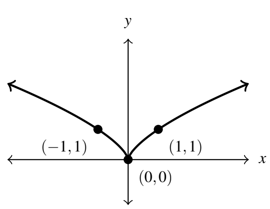


$f(x) = x^p$ for varying values of p

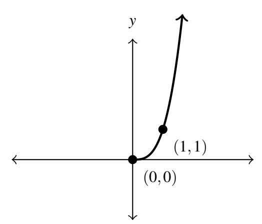
Theorem 4.4 generalizes to real number power functions, so, for instance to graph $F(x) = (x-2)^{\pi}$, one need only start with $y = x^{\pi}$ and shift horizontally two units to the right. (See the Exercises.)

4.2.3 Exercises

In Exercises 1 - 6, use the given graphs along with Theorem 4.4 to graph the given function. Track at least two points and state the domain and range using interval notation.



$f(x) = 2(x)^{\frac{2}{3}}$



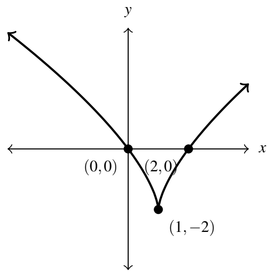
$g(t) = (t)^{\pi}$

1. $F(x) = (x-2)^{\frac{2}{3}} - 1$
2. $G(t) = (t+3)^{\pi} + 1$
3. $F(x) = 3 - x^{\frac{2}{3}}$
4. $G(t) = (1-t)^{\pi} - 2$
5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$
6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1$

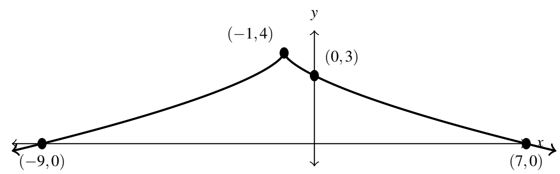
In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(bx-h)^{\frac{2}{3}} + k$.

NOTE: There may be more than one solution!

7. $y = F(x)$



8. $y = F(x)$



For each function in Exercises 9 - 16 below

- Analytically:
  - state the domain.
  - identify the axis intercepts.
  - analyze the end behavior.
- Construct a sign diagram for each function using the intercepts and sketch a graph.
- Use technology to determine:
  - the range.
  - the local extrema, if they exist.
  - intervals where the function is increasing/decreasing.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Comment on any observed symmetry.

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$

10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$

11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$

12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$

13. $f(x) = x^{0.4}(3-x)^{0.6}$

14. $f(x) = x^{0.5}(3-x)^{0.5}$

15. $g(t) = 4t(9-t^2)^{-\sqrt{2}}$

16. $g(t) = 3(t^2+1)^{-\pi}$

17. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[6] What trends do you observe? How do your answers manifest themselves graphically? Compare the results of this exercise with those of Exercise 51 in Section 2.2 and Exercise 43 in Section 3.2

[6] See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
$x^{\frac{1}{2}}$				
$x^{\frac{2}{3}}$				
$x^{-0.23}$				
x^{π}				

18. The National Weather Service uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215\, T_a - 35.75\, V^{0.16} + 0.4275\, T_a\, V^{0.16}$$

where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

(a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Compute the wind chill temperature. Round your answer to two decimal places.

(b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Compute the wind speed. Round your answer to two decimal places.

SECTION 4.2 EXERCISE ANSWERS A.1.4

4.3 Solving Equations Involving Root and Power Functions

In this section, we set about solving equations and inequalities involving power functions. Our first example demonstrates the usual sorts of strategies to employ when solving equations.

Example 4.3.1. Solve the following equations analytically and verify your answers using a graph.

1. $(7-x)^{\frac{3}{2}} = 8$
2. $(2t-1)^{\frac{2}{3}} - 4 = 0$
3. $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$
4. $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$
5. $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$
6. $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$

Solution.

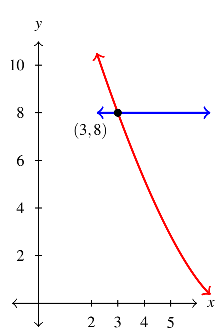
1. Solve $(7-x)^{\frac{3}{2}} = 8$ for x.

   One way to proceed to solve $(7-x)^{\frac{3}{2}} = 8$ is to use Definition 4.3 to rewrite $(7-x)^{\frac{3}{2}}$ as either $(\sqrt{7-x})^3$ or $\sqrt{(7-x)^3}$. We opt for the former given 8 is a perfect cube:

$$\begin{array}{rcll} (7-x)^{\frac{3}{2}} & = & 8 & \\ (\sqrt{7-x})^3 & = & 8 & \text{rewrite using Definition 4.3} \\ \sqrt[3]{(\sqrt{7-x})^3} & = & \sqrt[3]{8} & \text{extract cube roots} \\ \sqrt{7-x} & = & 2 & \sqrt[3]{u^3} = u \end{array}$$

   From $\sqrt{7-x} = 2$, we square both sides and obtain $7-x = 4$, so $x = 3$. We verify our answer analytically by substituting $x = 3$ into the original equation.

   Geometrically, we are looking for where the graph of $f(x) = (7-x)^{\frac{3}{2}}$ intersects the graph of $g(x) = 8$. Below are those graphs and we see the intersection point of both graphs is $(3,8)$, thereby checking our solution $x = 3$.



Checking $(7-x)^{\frac{3}{2}} = 8$

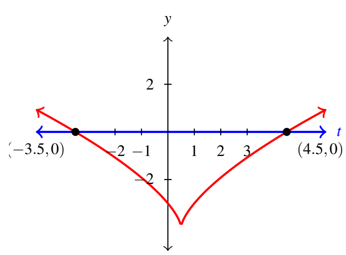
2. Solve $(2t-1)^{\frac{2}{3}}-4=0$ for x.

   Proceeding similarly to the above, to solve $(2t-1)^{\frac{2}{3}}-4=0$, we rewrite $(2t-1)^{\frac{2}{3}}$ as $(\sqrt[3]{2t-1})^2$ and solve:

$$\begin{array}{rcll} (2t-1)^{\frac{2}{3}}-4 & = & 0 & \\ (\sqrt[3]{2t-1})^2-4 & = & 0 & \text{rewrite using Definition 4.3} \\ (\sqrt[3]{2t-1})^2 & = & 4 & \text{isolate the variable term} \\ \sqrt{(\sqrt[3]{2t-1})^2} & = & \sqrt{4} & \text{extract square roots} \\ |\sqrt[3]{2t-1}| & = & 2 & \sqrt{u^2}=|u| \\ \sqrt[3]{2t-1} & = & \pm 2 & \text{for } c>0,\ |u|=c \text{ is equivalent to } u=\pm c. \end{array}$$

   From $\sqrt[3]{2t-1}=2$ we cube both sides and obtain $2t-1=8$, so $t=\frac{9}{2}=4.5$. Similarly, from $\sqrt[3]{2t-1}=-2$, we cube both sides and obtain $2t-1=-8$, so $t=-\frac{7}{2}=-3.5$. Both of these are solutions to the given equation.

   In this case we are looking for where the graph of $f(t)=(2t-1)^{\frac{2}{3}}-4$ intersects the graph of $g(t)=0$ - i.e., the t-intercepts of the graph of g. We find these are $(-3.5,0)$ and $(4.5,0)$, as predicted.



Checking $(2t-1)^{\frac{2}{3}}-4=0$

3. Solve $(x+3)^{0.5}=2(7-x)^{0.5}+1$for x.

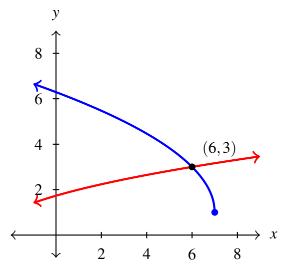
   $0.5=\frac{1}{2}$, so we may rewrite $(x+3)^{0.5}=2(7-x)^{0.5}+1$ as $(x+3)^{\frac{1}{2}}=2(7-x)^{\frac{1}{2}}+1$. Using Definition 4.3, we then have $\sqrt{x+3}=2\sqrt{7-x}+1$. One of the square roots is already isolated, thus we can rid ourselves of it by squaring both sides.

$$\begin{array}{rcll} \sqrt{x+3} & = & 2\sqrt{7-x}+1 & \\ (\sqrt{x+3})^2 & = & (2\sqrt{7-x}+1)^2 & \text{square both sides} \\ x+3 & = & (2\sqrt{7-x})^2+2(2\sqrt{7-x})(1)+1 & (\sqrt{u})^2=u \text{ and } (a+b)^2=a^2+2ab+b^2 \\ x+3 & = & 4(7-x)+4\sqrt{7-x}+1 & (ab)^2=a^2b^2 \text{ and, again, } (\sqrt{u})^2=u \\ x+3 & = & 28-4x+4\sqrt{7-x}+1 & \\ 5x-26 & = & 4\sqrt{7-x} & \text{isolate } \sqrt{7-x} \end{array}$$

   We square both sides *again* and get $(5x-26)^2=(4\sqrt{7-x})^2$ which reduces to $25x^2-260x+676=16(7-x)$. At last, we have a quadratic equation which we can solve by setting to zero and factoring.

We get $25x^2-244x+564=0$, so $(x-6)(25x-94)=0$ so $x=6$ or $x=\frac{94}{25}=3.76$. When we go to check these answers, we verify $x=6$ is a solution, but $x=3.76$ is not. Hence, $x=3.76$ is an 'extraneous' solution.[1]

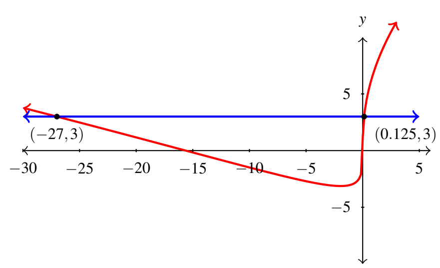
We graph both $f(x)=\sqrt{x+3}$ and $g(x)=2\sqrt{7-x}+1$ below and confirm there is only one intersection point, $(6,3)$.



Checking $(x+3)^{0.5}=2(7-x)^{0.5}+1$

4. Solve $2t^{\frac{2}{3}}+5t^{\frac{1}{3}}=3$ for x.

   While we *could* approach solving $2t^{\frac{2}{3}}+5t^{\frac{1}{3}}=3$ as the previous example, we would encounter cubing binomials[2] which we would prefer to avoid. Instead, we take a step back and notice there are three terms here with the exponent on one term, $t^{\frac{2}{3}}$ exactly twice the exponent on another term, $t^{\frac{1}{3}}$. We have ourselves a 'quadratic in disguise.'[3] To help us see the forest for the trees, we let $u=t^{\frac{1}{3}}$ so that $u^2=t^{\frac{2}{3}}$. (Note that root here, 3, is odd, so we can use the properties of exponents stated in Theorem 4.3.) Hence, in terms of u, the equation $2t^{\frac{2}{3}}+5t^{\frac{1}{3}}=3$ becomes the quadratic $2u^2+5u=3$, or $2u^2+5u-3=0$. Factoring gives $(2u-1)(u+3)=0$ so $u=t^{\frac{1}{3}}=\frac{1}{2}$ or $u=t^{\frac{1}{3}}=-3$. Given $t^{\frac{1}{3}}=\sqrt[3]{t}$, we solve both equations by cubing both sides to get $t=\frac{1}{8}=0.125$ and $t=-27$. Both of these are solutions to our original equation. Looking at the graphs of $f(t)=2t^{\frac{2}{3}}+5t^{\frac{1}{3}}$ and $g(t)=3$, we find two intersection points, $(-27,3)$ and $(0.125,3)$, as required.



Checking $2t^{\frac{2}{3}}+5t^{\frac{1}{3}}=3$

5. Solve $2(3x-1)^{-0.5}=3x(3x-1)^{-1.5}$ for x.

[1] We invite the reader to see at which point in our machinations $x=3.76$ *does* check. Knowing a solution is extraneous is one thing; understanding *how* it came about is another.

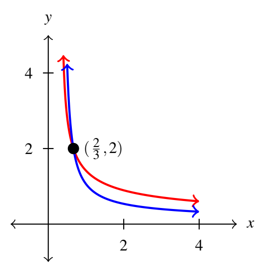
[2] that is, expanding things like $(a+b)^3$.

[3] See Section 0.5.5 or, more recently, Example **??** in Section **??**.

Next we are to solve $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$ which, when written without negative exponents is: $\frac{2}{(3x-1)^{0.5}} = \frac{3x}{(3x-1)^{1.5}}$. The rational exponents here are $0.5 = \frac{1}{2}$ and $1.5 = \frac{3}{2}$, both involve an *even* indexed root (the square root in this case!) which means $3x-1 \geq 0$. Moreover, the $3x-1$ resides in the *denominator* meaning $3x-1 \neq 0$ so our equation is really valid only for values of x where $3x-1 > 0$ or $x > \frac{1}{3}$. Hence, we clear denominators and can apply Theorem 4.3:

$$\begin{aligned}
\frac{2}{(3x-1)^{0.5}} &= \frac{3x}{(3x-1)^{1.5}} \\
\left[\frac{2}{(3x-1)^{0.5}}\right] \cdot (3x-1)^{1.5} &= \left[\frac{3x}{(3x-1)^{1.5}}\right] \cdot (3x-1)^{1.5} \\
2 \cdot \frac{(3x-1)^{1.5}}{(3x-1)^{0.5}} &= 3x \\
2(3x-1)^{1.5-0.5} &= 3x && \text{Theorem 4.3 applies as } 3x-1>0. \\
2(3x-1)^{1} &= 3x
\end{aligned}$$

We get $6x-2 = 3x$, or $x = \frac{2}{3}$. Because $x = \frac{2}{3} > \frac{1}{3}$, we keep it and, sure enough, it is a solution to our original equation. Graphically we see $f(x) = 2(3x-1)^{-0.5}$ intersects $g(x) = 3x(3x-1)^{-1.5}$ at $\left(\frac{2}{3}, 2\right)$.



Checking $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$

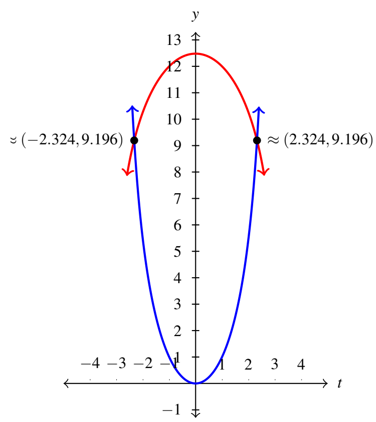
6. Solve $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$ for x.

   Our last equation to solve is $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$, which, when rewritten without negative exponents is: $6(9-t^2)^{\frac{1}{3}} = \frac{4t^2}{(9-t^2)^{\frac{2}{3}}}$. Again, the root here (3) is odd, so we can use the exponent properties listed in Theorem 4.3. We begin by clearing denominators:

$$\begin{aligned}
6(9-t^2)^{\frac{1}{3}} &= \frac{4t^2}{(9-t^2)^{\frac{2}{3}}} \\
6(9-t^2)^{\frac{1}{3}} \cdot (9-t^2)^{\frac{2}{3}} &= \left[\frac{4t^2}{(9-t^2)^{\frac{2}{3}}}\right] \cdot (9-t^2)^{\frac{2}{3}} \\
6(9-t^2)^{\frac{1}{3}+\frac{2}{3}} &= 4t^2 && \text{Theorem 4.3 applies because the root here 3 is odd.} \\
6(9-t^2)^{1} &= 4t^2
\end{aligned}$$

   We get $54-6t^2 = 4t^2$ or $10t^2 = 54$. As fraction $t^2 = \frac{54}{10} = \frac{27}{5}$ so $t = \pm\sqrt{\frac{27}{5}} = \pm 3\sqrt{155} \approx \pm 2.324$. While not the easiest to check analytically, both of these solutions do work in the original equation.

Graphing $f(t) = 6(9-t^2)^{\frac{1}{3}}$ and $g(t) = 4t^2(9-t^2)^{-\frac{2}{3}}$ below, we see the graphs intersect when $t \approx \pm 2.324$.



Checking $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-x^2)^{\frac{2}{3}}$

□

Note that in Example 4.3.1, there are several ways to correctly solve each equation, and we endeavored to demonstrate a variety of methods. For example, for number 1, instead of converting $(7-x)^{\frac{3}{2}}$ to a radical equation, we could use Theorem 4.3. The root here (2) is even, thus we know $7-x \geq 0$ or $x \leq 7$. Hence we may apply exponent properties:

$$\begin{array}{rcll}
(7-x)^{\frac{3}{2}} & = & 8 & \\
\left[(7-x)^{\frac{3}{2}}\right]^{\frac{2}{3}} & = & 8^{\frac{2}{3}} & \text{raise both sides to the } \frac{2}{3} \text{ power} \\
(7-x)^{\frac{3}{2}\cdot\frac{2}{3}} & = & 4 & \text{Theorem 4.3} \\
(7-x)^{1} & = & 4 &
\end{array}$$

from which we get $x = 3$. If we try this same approach to solve number 2, however, we encounter difficulty. From $(2t-1)^{\frac{2}{3}} - 4 = 0$, we get $(2t-1)^{\frac{2}{3}} = 4$.

$$\begin{array}{rcll}
(2t-1)^{\frac{2}{3}} & = & 4 & \\
\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} & = & 4^{\frac{3}{2}} & \text{raise both sides to the } \frac{3}{2} \text{ power}
\end{array}$$

Given the root here (3) is odd, we have no restriction on $2t-1$ but the exponent $\frac{3}{2}$ has an even denominator. Hence, Theorem 4.3 *does not apply*. That is,

$$\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} \neq (2t-1)^{\frac{2}{3}\cdot\frac{3}{2}} = (2t-1)^{1} = (2t-1).$$

Note that if we weren't careful, we'd have $2t-1 = 4^{\frac{3}{2}} = 8$ which gives $t = \frac{9}{2} = 4.5$ only. We'd have missed the solution $t = -3.5$. Truth be told, you *can* simplify $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}}$ - just not using Theorem 4.3. We leave it as an exercise to show $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = |2t-1|$ and, more generally, $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = |x|$.

Our next example is an application of the Cobb Douglas production model of an economy. The Cobb-Douglas model states that the yearly total dollar value of the production output in an economy is a function of *two* variables: labor (the total number of hours worked in a year) and capital (the total dollar value of the physical goods required for manufacturing.) The equation relating the production output level P, labor L and capital K takes the form $P = aL^bK^{1-b}$ where $0 < b < 1$; that is, the production level varies jointly with some power of the labor and capital.

Example 4.3.2. In their original paper *A Theory of Production*[4] Cobb and Douglas modeled the output of the US Economy (using 1899 as a baseline) using the formula $P = 1.01L^{0.75}K^{0.25}$ where P, L, and K were percentages of the 1899 figures for total production, labor, and capital, respectively.

1. For 1910, the recorded labor and capital figures for the US Economy are 144% and 208% of the 1899 figures, respectively. Compute P using these figures and interpret your answer.

2. The recorded production value figure for 1920 is 231% of the 1899 figure. Use this to write K as a function of L, $K = f(L)$. Compute and interpret $f(193)$.

3. Graph $K = f(L)$ and interpret the behavior as $L \to 0^+$ and $L \to \infty$.

Solution.

1. For 1910, the recorded labor and capital figures for the US Economy are 144% and 208% of the 1899 figures, respectively. Compute P using these figures and interpret your answer.

   In this case, $P = 1.01L^{0.75}K^{0.25} = 1.01(144)^{0.75}(208)^{0.25} \approx 159$ which means the dollar value of the total US Production in 1920 was approximately 159% of what it was in 1899.[5]

2. The recorded production value figure for 1920 is 231% of the 1899 figure. Use this to write K as a function of L, $K = f(L)$. Compute and interpret $f(193)$.

   We are given $P = 231 = 1.01L^{0.75}K^{0.25}$, so to write K as a function of L, we need to solve this equation for K. L and K are positive by definition, so we can employ properties of exponents:

[4] available here.

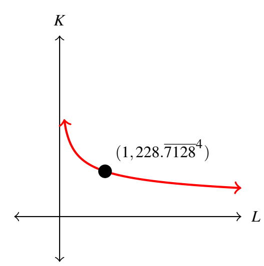
[5] This answer is remarkably accurate. Note: all the dollar values here are recorded in '1880' dollars, per the source article.

$$\begin{array}{rcll}
231 & = & 1.01L^{0.75}K^{0.25} & \\
\dfrac{231}{1.01L^{0.75}} & = & \dfrac{1.01L^{0.75}K^{0.25}}{1.01L^{0.75}} & L>0\text{, hence } L^{0.75} \neq 0. \\
K^{0.25} & = & 228.\overline{7128}L^{-0.75} & \text{rewrite} \\
\left(K^{0.25}\right)^{\frac{1}{0.25}} & = & \left(228.\overline{7128}L^{-0.75}\right)^{\frac{1}{0.25}} & \\
K^{\frac{0.25}{0.25}} & = & (228.\overline{7128})^{\frac{1}{0.25}}L^{-\frac{0.75}{0.25}} & \text{Theorem 4.3} \\
K & = & (228.\overline{7128})^{4}L^{-3} & \text{simplify}
\end{array}$$

Hence, $K = f(L) = (228.\overline{7128})^4L^{-3}$ where $L > 0$. We find $f(193) = (228.\overline{7128})^4(193)^{-3} \approx 381$ meaning that in order to maintain a production level of 231% of 1889 with a labor level at 193% of 1889, the required capital is 381% that of 1889.[6]

3. Graph $K = f(L)$ and interpret the behavior as $L \to 0^+$ and $L \to \infty$.

   The function $f(L)$ is a Laurent Monomial (see Section 3.2) with $n = 3$ and $a = (228.\overline{7128})^4$. As such, as $L \to 0^+$, $f(L) \to \infty$. This means that in order to maintain the given production level, as the available labor diminishes, the capital requirement become unbounded. As $L \to \infty$, we have $f(L) \to 0$ meaning that as the available labor increases, the need for capital diminishes. The graph of f is called an 'isoquant' - meaning 'same quantity.' In this context, the graph displays all combinations of labor and capital, (L,K) which result in the *same* production level, in this case, 231% of what was produced in 1889.



$K = f(L) = (228.\overline{7128})^4L^{-3}$

$\square$

[6]The actual recorded figure is 407.

4.3.1 Exercises

In Exercises 1 - 14, solve the equation.

1. $x+1=(3x+7)^{\frac{1}{2}}$
2. $2x+1=(3-3x)^{\frac{1}{2}}$
3. $t+(3t+10)^{0.5}=-2$
4. $3t+(6-9t)^{0.5}=2$
5. $x^{-1.5}=8$
6. $2x-1=(x+1)^{-0.5}$
7. $t^{\frac{2}{3}}=4$
8. $(t-2)^{\frac{1}{2}}+(t-5)^{\frac{1}{2}}=3$
9. $(2x+1)^{\frac{1}{2}}=3+(4-x)^{\frac{1}{2}}$
10. $5-(4-2x)^{\frac{2}{3}}=1$
11. $2t^{\frac{2}{3}}=6-t^{\frac{1}{3}}$
12. $2t^{\frac{1}{3}}=1-3t^{\frac{2}{3}}$
13. $2x^{1.5}=15x^{0.75}+8$
14. $35x^{-0.75}=x^{-1.5}+216$
15. The Cobb-Douglas production model[7] for the country of Sasquatchia is $P=1.25L^{0.4}K^{0.6}$. Here, P represents the country's production (measured in thousands of Bigfoot Bullion), L represents the total labor (measured in thousands of hours) and K represents the total investment in capital (measured in Bigfoot Bullion.) Let $P=300$ and solve for K as a function of L. If $L=100$, what is K? Interpret each of the quantities in this case.

Section 4.3 Exercise Answers A.1.4

[7] See Example 4.3.2 for more details on these sorts of models.

4.4 Solving Nonlinear Inequalities

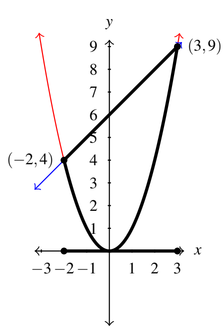
4.4.1 Inequalities involving Quadratic Functions

We now turn our attention to solving inequalities involving quadratic functions. Consider the inequality $x^2 \leq 6$. We could use the fact that the square root is increasing[8] to get: $\sqrt{x^2} \leq \sqrt{6}$, or $|x| \leq \sqrt{6}$. This reduces to $-\sqrt{6} \leq x \leq \sqrt{6}$ or, using interval notation, $[-\sqrt{6}, \sqrt{6}]$. If, however, we had to solve $x^2 \leq x+6$, things are more complicated. One approach is to complete the square:

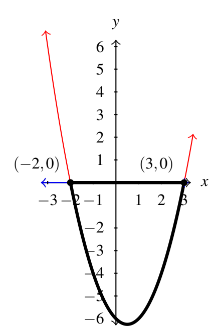
$$\begin{array}{rcl} x^2 & \leq & x+6 \\ x^2 - x & \leq & 6 \\ x^2 - x + \frac{1}{4} & \leq & 6 + \frac{1}{4} \\ \left(x - \frac{1}{2}\right)^2 & \leq & \frac{25}{4} \\ \sqrt{\left(x - \frac{1}{2}\right)^2} & \leq & \sqrt{\frac{25}{4}} \\ \left|x - \frac{1}{2}\right| & \leq & \frac{5}{2} \\ -\frac{5}{2} \leq & x - \frac{1}{2} & \leq \frac{5}{2} \\ -2 & \leq x \leq & 3 \end{array}$$

We get the solution $[-2,3]$. While there is nothing wrong with this approach, we seek methods here that will generalize to higher degree polynomials such as those seen in Chapter 2.

To that end, we look at the inequality $x^2 \leq x+6$ graphically. Identifying $f(x) = x^2$ and $g(x) = x+6$, we graph f and g on the same set of axes below on the left and look for where the graph of f (the parabola) meets or is below the graph of g (the line). There are two points of intersection which we determine by solving $f(x) = g(x)$ or $x^2 = x+6$. As usual, we rewrite this equation as $x^2 - x - 6 = 0$ in order to use the primary tools we've developed to handle these types[9] of quadratic equations: factoring, or failing that, the Quadratic Formula. We find $x^2 - x - 6 = (x+2)(x-3)$ so we get two solutions to $(x+2)(x-3) = 0$, namely $x = -2$ and $x = 3$. Putting these together with the graph, we obtain the same solution: $[-2,3]$.



Solving $x^2 \leq x+6$



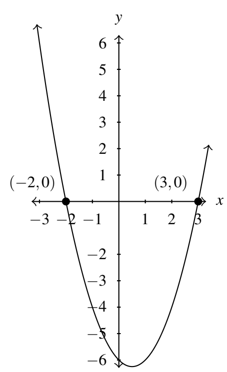
Solving $x^2 - x - 6 \leq 0$

[8] That is, if $a < b$, then $\sqrt{a} < \sqrt{b}$.
[9] Namely ones with a nonzero coefficient of 'x'.

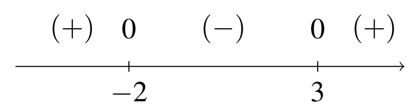
Yet a third way to attack $x^2 \leq x+6$ is to rewrite the inequality as $x^2 - x - 6 \leq 0$. Here, we graph $f(x) = x^2 - x - 6$ to look for where the graph meets or is below the graph of $g(x) = 0$, a.k.a. the x-axis. Doing so requires us to find the zeros of f, that is, solve $f(x) = x^2 - x - 6 = 0$ from which we obtain $x = -2$ and $x = 3$ as before. We find the same solution, $[-2,3]$ as is showcased in the graph at the bottom of the previous page on the right.

One advantage to using this last approach is that we are essentially concerned with one function and its *zeros*. This approach can be generalized to all functions - not just quadratics, so we take the time to develop this method more thoroughly now.

Consider the graph of $f(x) = x^2 - x - 6$ below. The zeros of f are $x = -2$ and $x = 3$ and they divide the domain (the x-axis) into three intervals: $(-\infty,-2)$, $(-2,3)$ and $(3,\infty)$. For every number in $(-\infty,-2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty,-2)$. Similarly, $f(x) < 0$ for all x in $(-2,3)$, and $f(x) > 0$ for all x in $(3,\infty)$. We represent this schematically with the **sign diagram** below.



$f(x) = x^2 - x - 6$



A sign diagram for $f(x) = x^2 - x - 6$

The $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x and the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. For the inequality $f(x) = x^2 - x - 6 \leq 0$, we read from the sign diagram that the solution is $[-2,3]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. While parabolas aren't that bad to graph knowing what we know, our sights are set on more general functions whose graphs are more complicated.

An important property of parabolas is that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis at some point in between. Said differently, if the function is positive at one point and negative at another, the function must have at least one zero in between. This property is a consequence of quadratic functions being **continuous**. A precise definition of 'continuous' requires the language of Calculus, but it suffices for us to know that the graph of a continuous function has no gaps or holes. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval.

The result below applies to all continuous functions defined on an interval of real numbers, but we restrict our attention to quadratic functions for the time being,

Steps for Creating A Sign Diagram for A Quadratic Function

Suppose f is a quadratic function.

1. Compute the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

To use a sign diagram to solve an inequality, we must always remember to compare the function to 0.

Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so some function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

We practice this approach in the following example.

Example 4.4.1. Solve the following inequalities analytically[10] and check your solutions graphically.

1. $2x^2 \leq 3 - x$
2. $t^2 - 2t > 1$
3. $x^2 + 1 \leq 2x$
4. $2t - t^2 \geq |t-1| - 1$

Solution.

1. Solve $2x^2 \leq 3 - x$.

   To solve $2x^2 \leq 3 - x$, we rewrite it as $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$. Factoring gives $(2x+3)(x-1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[11] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$,

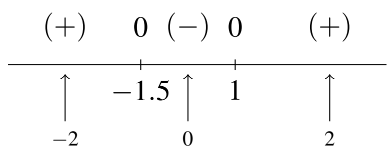
[10] By 'solve analytically' we mean 'algebraically' using a sign diagram.

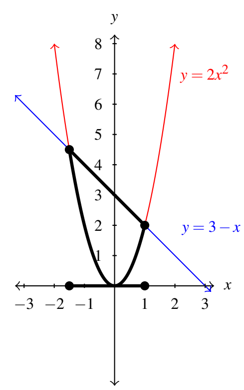
[11] We have to choose *something* in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.

$x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.

We are solving $2x^2 + x - 3 \leq 0$, so we need solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$ above them. The sign diagram shows only one: $\left(-\frac{3}{2}, 1\right)$. Also, we know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). The graphs of g and h are given on the right with the sign chart on the left.

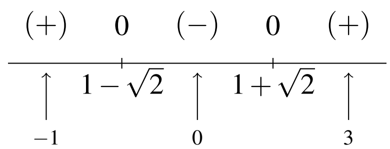


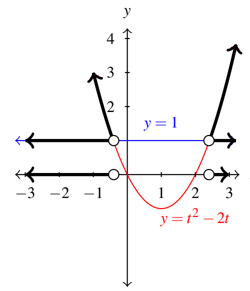


2. Solve $t^2 - 2t > 1$.

   Once again, we re-write $t^2 - 2t > 1$ as $t^2 - 2t - 1 > 0$ and we identify $f(t) = t^2 - 2t - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $t^2 - 2t - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula and find $t = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $t = -1$, $t = 0$ and $t = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $t^2 - 2t - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

   To check the inequality $t^2 - 2t > 1$ graphically, we set $g(t) = t^2 - 2t$ and $h(t) = 1$. We are looking for the t values where the graph of g is above the graph of h. As before we present the graphs on the right and the sign chart on the left.

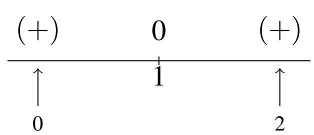


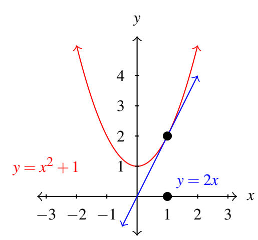


3. Solve $x^2+1 \leq 2x$.

   To solve $x^2+1 \leq 2x$, as before, we solve $x^2-2x+1 \leq 0$. Setting $f(x) = x^2-2x+1 = 0$, we find only one zero of f: $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$. Because we are looking for solutions to $x^2-2x+1 \leq 0$, we are looking for x values where $x^2-2x+1 < 0$ as well as where $x^2-2x+1 = 0$. Looking at our sign diagram, there are no places where $x^2-2x+1 < 0$ (there are no $(-)$), so our solution is only $x = 1$ (where $x^2-2x+1 = 0$). We write this as $\{1\}$.

   Graphically, we solve $x^2+1 \leq 2x$ by graphing $g(x) = x^2+1$ and $h(x) = 2x$. We are looking for the x values where the graph of g is below the graph of h (for $x^2+1 < 2x$) and/or where the two graphs intersect ($x^2+1 = 2x$). Notice that the line and the parabola touch at $(1,2)$, but the parabola is always above the line otherwise.[12]





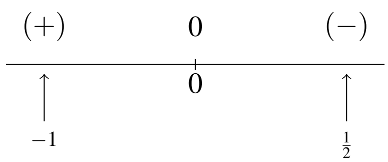
4. Solve $2t-t^2 \geq |t-1|-1$.

   To solve $2t-t^2 \geq |t-1|-1$ analytically we first rewrite the absolute value using cases. For $t < 1$, $|t-1| = -(t-1) = -t+1$, so we get $2t-t^2 \geq (-t+1)-1$ which simplifies to $t^2-3t \leq 0$. Finding

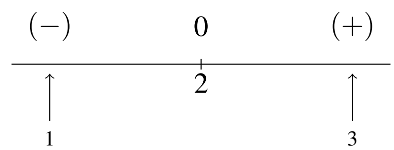
[12]In this case, we say the line $y = 2x$ is **tangent** to $y = x^2+1$ at $(1,2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

the zeros of $f(t) = t^2 - 3t$, we get $t = 0$ and $t = 3$. However, we are concerned only with the portion of the number line where $t < 1$, so the only zero that we deal with is $t = 0$. This divides the interval $t < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $t = -1$ and $t = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Hence, our solution to $t^2 - 3t \leq 0$ for $t < 1$ is $[0, 1)$.

Next, we turn our attention to the case $t \geq 1$. Here, $|t-1| = t-1$, so our original inequality becomes $2t - t^2 \geq (t-1) - 1$, or $t^2 - t - 2 \leq 0$. Setting $g(t) = t^2 - t - 2$, we find the zeros of g to be $t = -1$ and $t = 2$. Of these, only $t = 2$ lies in the region $t \geq 1$, so we ignore $t = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $t = 1$ and $t = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. Hence, our solution to $g(t) = t^2 - t - 2 \leq 0$, in this region is $[1, 2)$.

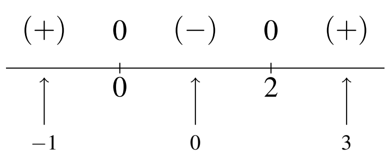


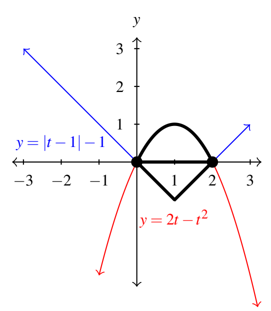
Solving $2t - t^2 \geq |t-1| - 1$ for $t < 1$



Solving $2t - t^2 \geq |t-1| - 1$ for $t \geq 1$

Combining these into one sign diagram, we have that our solution is $[0, 2]$. Graphically, to check $2t - t^2 \geq |t-1| - 1$, we set $h(t) = 2t - t^2$ and $i(t) = |t-1| - 1$ and look for the t values where the graph of h intersects or is above the the graph of i. The combined sign chart is given on the left and the graphs are on the right.





□

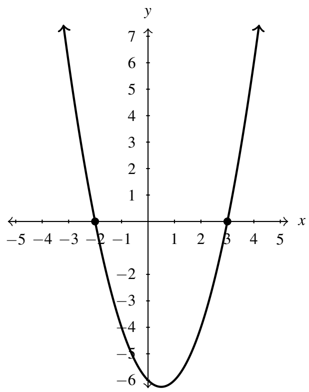
We continue this section with an example that combines quadratic inequalities with piecewise functions.

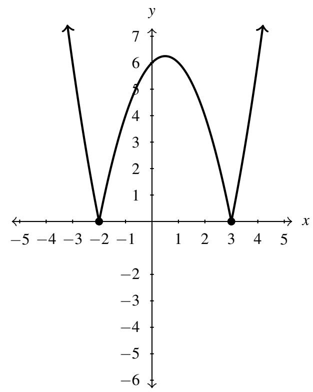
Example 4.4.2. Rewrite $g(x) = |x^2 - x - 6|$ as a piecewise function and graph.

Solution. Using the definition of absolute value, Definition 1.12 and the sign diagram we constructed for $f(x) = x^2 - x - 6$ near the beginning of the subsection, we get:

$$g(x) = |x^2 - x - 6| = \begin{cases} -(x^2 - x - 6) & \text{if } (x^2 - x - 6) < 0, \\ (x^2 - x - 6) & \text{if } (x^2 - x - 6) \geq 0. \end{cases} \longrightarrow \quad g(x) = \begin{cases} -x^2 + x + 6 & \text{if } -2 < x < 3, \\ x^2 - x - 6 & \text{if } x \leq -2 \text{ or } x \geq 3. \end{cases}$$

Going through the usual machinations results on the graph on the right. Compare it to the graph on the left. Notice anything?





If we take a step back and look at the graphs of f and g, we notice that to obtain the graph of g from the graph of f, we reflect a *portion* of the graph of f about the x-axis. In general, if $g(x) = |f(x)|$, then:

$$g(x) = |f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ f(x) & \text{if } f(x) \geq 0. \end{cases}$$

The function g is defined so that when $f(x)$ is negative (i.e., when its graph is below the x-axis), the graph of g is the reflection of the graph of f across the x-axis. This is a general method to graph functions of the form $g(x) = |f(x)|$. Indeed, the graph of $g(x) = |x|$ can be obtained by reflection the portion of the line $f(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.[13] □

4.4.2 Inequalities involving Rational Functions and Applications

In this subsection, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving equations and inequalities.

Example 4.4.3.

1. Solve $\dfrac{x^3 - 2x + 1}{x - 1} = \dfrac{1}{2}x - 1$.
2. Solve $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1$.
3. Verify your solutions to 1 and 2 using a graph.

[13] See Exercise 11 in Section 1.4.

Solution.

1. Solve $\dfrac{x^3-2x+1}{x-1} = \dfrac{1}{2}x-1$.

   To solve the equation, we clear denominators

$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & = & \dfrac{1}{2}x-1 & \\
\left(\dfrac{x^3-2x+1}{x-1}\right)\cdot 2(x-1) & = & \left(\dfrac{1}{2}x-1\right)\cdot 2(x-1) & \\
2x^3-4x+2 & = & x^2-3x+2 & \text{expand} \\
2x^3-x^2-x & = & 0 & \\
x(2x+1)(x-1) & = & 0 & \text{factor} \\
x & = & -\frac{1}{2},\, 0,\, 1 &
\end{array}$$

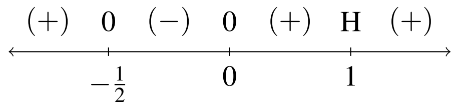
   Due to the fact that we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x=1$ does not satisfy the original equation, so our only solutions are $x=-\frac{1}{2}$ and $x=0$.

2. Solve $\dfrac{x^3-2x+1}{x-1} \geq \dfrac{1}{2}x-1$.

   To solve the inequality, it may be tempting to begin as we did with the equation − namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on x, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 370 in Section 3.3.

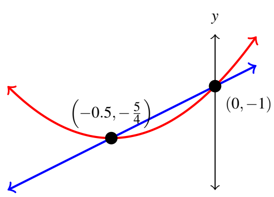
$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & \geq & \dfrac{1}{2}x-1 & \\
\dfrac{x^3-2x+1}{x-1} - \dfrac{1}{2}x+1 & \geq & 0 & \\
\dfrac{2\left(x^3-2x+1\right)}{2(x-1)} - \dfrac{x(x-1)}{2(x-1)} + \dfrac{2(x-1)}{2(x-1)} & \geq & 0 & \text{get a common denominator} \\
\dfrac{2\left(x^3-2x+1\right)-x(x-1)+2(x-1)}{2(x-1)} & \geq & 0 & \\
\dfrac{2x^3-x^2-x}{2x-2} & \geq & 0 & \text{expand}
\end{array}$$

   Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x=1$ which is the solution to $2x-2=0$. The zeros of r are the solutions to $2x^3-x^2-x=0$, which we have already found to be $x=0$, $x=-\frac{1}{2}$ and $x=1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.

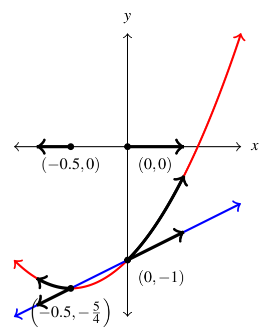


We are interested in where $r(x) \geq 0$. We see $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0,1)$ and $(1,\infty)$. We know $r(x) = 0$ when $x = -\frac{1}{2}$ and $x = 0$. Hence, $r(x) \geq 0$ on $\left(-\infty, -\frac{1}{2}\right] \cup [0,1) \cup (1,\infty)$.

3. To check our answers graphically, let $f(x) = \frac{x^3-2x+1}{x-1}$ and $g(x) = \frac{1}{2}x - 1$. The solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. We graph both f (red) and g (blue) below. We find only two intersection points, $(-0.5,-1.25)$ and $(0,-1)$ which correspond to our solutions $x = -\frac{1}{2}$ and $x = 0$. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above $(>)$ the graph of $g(x)$. From the graph, this *appears* to happen on $\left(-\infty, -\frac{1}{2}\right] \cup [0,\infty)$ which *almost* matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, so it cannot be included in our solution.[14]



$\frac{x^3-2x+1}{x-1} = \frac{1}{2}x - 1$



$\frac{x^3-2x+1}{x-1} \geq \frac{1}{2}x - 1$

□

The important take-away from Example 4.4.3 is not to clear fractions when working with an inequality unless you know for certain the sign of the denominators. We offer another example.

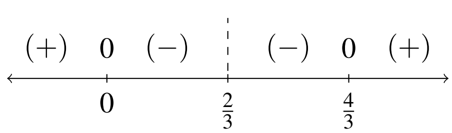
Example 4.4.4. Solve: $2t(3t-2)^{-1} \leq 3t^2(3t-2)^{-2}$. Check your answer using a graph.

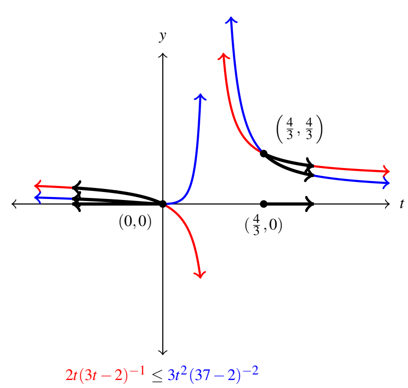
Solution. We begin by rewriting the terms with negative exponents as fractions and gathering all nonzero terms to one side of the inequality:

[14] We invite the reader to show there is a hole in the graph of $y = f(x)$ at $(1,1)$.

$$\begin{aligned}
2t(3t-2)^{-1} &\leq 3t^2(3t-2)^{-2} \\
\frac{2t}{3t-2} &\leq \frac{3t^2}{(3t-2)^2} \\
\frac{2t}{3t-2} - \frac{3t^2}{(3t-2)^2} &\leq 0 \\
\frac{2t(3t-2)}{(3t-2)^2} - \frac{3t^2}{(3t-2)^2} &\leq 0 && \text{get a common denominator} \\
\frac{2t(3t-2)-3t^2}{(3t-2)^2} &\leq 0 \\
\frac{3t^2-4t}{(3t-2)^2} &\leq 0 && \text{expand}
\end{aligned}$$

We define $r(t) = \frac{3t^2-4t}{(3t-2)^2}$ and set about constructing a sign diagram for r. Solving $(3t-2)^2 = 0$ gives $t = \frac{2}{3}$ as our sole excluded value. To find the zeros of r, we set $r(t) = \frac{3t^2-4t}{(3t-2)^2} = 0$ and solve $3t^2 - 4t = 0$. Factoring gives $t(3t-4) = 0$ so our solutions are $t = 0$ and $t = \frac{4}{3}$. After choosing test values, we get the sign diagram below on the left. We are looking for where $r(t) \leq 0$, we are looking for where $r(t)$ is $(-)$ or $r(t) = 0$. Hence, our final answer is $\left[0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{4}{3}\right]$. Below on the right, we graph $f(t) = 2t(3t-1)^{-1}$ and $g(t) = 3t^2(3t-2)^{-2}$. Sure enough, the graph of f intersects the graph of g when $t = 0$ and $t = \frac{4}{3}$. Moreover, the graph of f is below the graph of g everywhere they are defined between these values, in accordance with our algebraic solution.





□

One thing to note about Example 4.4.4 is that the quantity $(3t-2)^2 \geq 0$ for all values of t. Hence, as long as we remember $t = \frac{2}{3}$ is excluded from consideration, we could actually multiply both sides of the inequality in Example 4.4.4 by $(3t-2)^2$ to obtain $2t(3t-2) \leq 3t^2$. We could then solve this (slightly easier) inequality using the methods of Section 2.1 as long as we remember to exclude $t = \frac{2}{3}$ from our solution. Once again, the more you *understand*, the less you have to *memorize*. If you know the '*why*' behind an algorithm instead of just the '*how*,' you will know when you can short-cut it.

Our next example is an application of average cost. Recall from Definition 3.7 if $C(x)$ represents the cost to make x items then the average cost per item is given by $\overline{C}(x) = \frac{C(x)}{x}$, for $x > 0$.

Example 4.4.5. Recall from Example 1.3.8 that the cost, $C(x)$, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Write an expression for the average cost function, $\overline{C}(x)$.
2. Solve $\overline{C}(x) < 100$ and interpret.
3. Determine the behavior of $\overline{C}(x)$ as $x \to \infty$ and interpret.

Solution.

1. Write an expression for the average cost function, $\overline{C}(x)$.

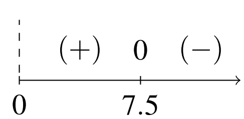
   From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but remember $x = 0$ causes problems for $\overline{C}(x)$, so we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solve $\overline{C}(x) < 100$ and interpret.

   Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \quad \text{common denominator} \\ \frac{150-20x}{x} &< 0 \end{aligned}$$

   If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.



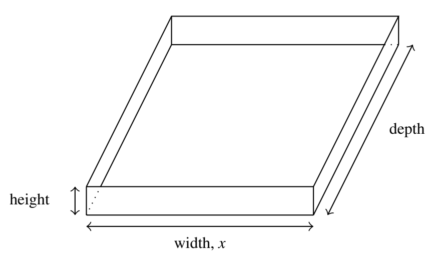
In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Because it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. Determine the behavior of $\overline{C}(x)$ as $x \to \infty$ and interpret.

   When we apply Theorem 3.3 to $\overline{C}(x)$ we find that $y = 80$ is a horizontal asymptote to the graph of $y = \overline{C}(x)$. To more precisely determine the behavior of $\overline{C}(x)$ as $x \to \infty$, we first use long division[15] and rewrite $\overline{C}(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^{+}$, which means $\overline{C}(x) \approx 80 + \text{very small } (+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $\overline{C}(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 1.3.8, we realize \$80 is the variable cost per system – the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively 'zero out' the fixed cost. □

Note that number 2 in Example 4.4.5 is another opportunity to short-cut the standard algorithm and obtain the solution more quickly if we take stock of the situation. The applied domain is $x > 0$, so we can multiply through the inequality $\frac{80x+150}{x} < 100$ by x without worrying about changing the sense of the inequality. This reduces the problem to $80x + 150 < 100x$, a basic linear inequality whose solution is readily seen to be $x > 7.5$. It is absolutely critical here that $x > 0$. Indeed, any time you decide to multiply an inequality by a variable expression, it is necessary to justify why the inequality is preserved. Our next example is another classic 'box with no top' problem. The reader is encouraged to compare and contrast this problem with Example 2.2.4 in Section 2.2.

Example 4.4.6. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let x denote the width of the box, in centimeters as seen below.



1. Explain why the height of the box (in centimeters) is a function of the width x. Call this function h and write an expression for $h(x)$, complete with an appropriate applied domain.

[15] In this case, long division amounts to term-by-term division.

2. Solve $h(x) \geq x$ and interpret.

3. Determine and interpret the behavior of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.

4. Express the surface area of the box as a function of x, $S(x)$ and state the applied domain.

5. Use technology to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

Solution.

1. Explain why the height of the box (in centimeters) is a function of the width x. Call this function h and write an expression for $h(x)$, complete with an appropriate applied domain.

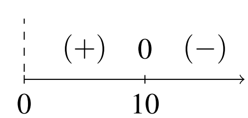
   We are told that the volume of the box is 1000 cubic centimeters and that x represents the width, in centimeters. As x represents a physical dimension of a box, we have that $x > 0$. From geometry, we know volume $=$ width $\times$ height $\times$ depth. The base of the box is a square, so the width and the depth are both x centimeters. Hence, $1000 = x^2(\text{height})$. Solving for the height, we get height $= \frac{1000}{x^2}$. In other words, for each width $x > 0$, we are able to compute *the*[16] corresponding height using the formula $\frac{1000}{x^2}$. Hence, the height is a function of x. Using function notation, we write $h(x) = \frac{1000}{x^2}$. As mentioned before, our only restriction is $x > 0$ so the domain of h is $(0, \infty)$.

2. Solve $h(x) \geq x$ and interpret.

   To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator} \end{aligned}$$

   We consider the left hand side of the inequality as our rational function $r(x)$. We see immediately the only value excluded from the domain of r is 0, but our applied domain is $x > 0$, thus we restrict our attention to the interval $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, or when $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the following:



[16]that is, the one and only one

We see $r(x) > 0$ on $(0,10)$, and because $r(x) = 0$ at $x = 10$, our solution is $(0,10]$. In the context of the problem, $h(x)$ represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.[17]

3. Determine and interpret the behavior of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.

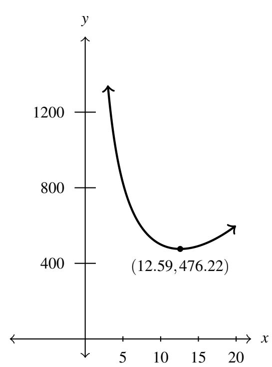
   As $x \to 0^+$, $h(x) = \frac{1000}{x^2} \to \infty$. This means that the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimeters. As $x \to \infty$, we find $h(x) \to 0^+$, which means that in order to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger as the height becomes smaller.

4. Express the surface area of the box as a function of x, $S(x)$ and state the applied domain.

   As a result of the box having no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by $h(x)$. We get the surface area, $S(x) = x^2 + 4xh(x)$. But $h(x) = \frac{1000}{x^2}$, so we have $S(x) = x^2 + 4x\left(\frac{1000}{x^2}\right) = x^2 + \frac{4000}{x}$. The domain of S is the same as h, namely $(0,\infty)$, for the same reasons as above.

5. Use technology to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

   To graph $y = S(x)$, we create a table of values. Doing so, we find a local minimum when $x \approx 12.60$. As far as we can tell,[18] this is the only local extremum, so it is the (absolute) minimum as well. This means that the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.[19]



$y = x^2 + \frac{4000}{x}$

[17] As with the previous example, knowing $x > 0$ means $x^2 > 0$ so we can clear denominators right away and solve $x^3 \leq 1000$, or $x \leq 10$. Coupled with our applied domain, $x > 0$, we would arrive at the same solution, $(0,10]$.

[18] without Calculus, that is...

[19] The y-coordinate here, 476.22 means the minimum surface area possible is 476.22 square centimeters. Minimizing the surface area minimizes the material required to make the box, therein helping to reduce the cost of the box.

□

4.4.3 Inequalities involving Power and Root Functions

Next, we move on to solving inequalities with power functions. As we've seen with other types of non-linear inequalities,[20] an invaluable tool for us is the Sign Diagram.

Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line with a dashed above them.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

As you may recall, sign diagrams compare functions to 0, thus the first step in solving inequalities using a sign diagram is to gather all the nonzero terms one one side of the inequality. We demonstrate this technique in the following example.

Example 4.4.7. Solve the following inequalities. Check your answers graphically.

1. $2 - \sqrt[4]{x+3} \geq 0$
2. $t^{2/3} < t^{4/3} - 6$
3. $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$
4. $(t-4)^{\frac{2}{3}} \geq -\dfrac{2t}{3(t-4)^{\frac{1}{3}}}$

Solution.

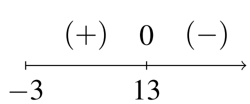
1. Solve $2 - \sqrt[4]{x+3} \geq 0$.

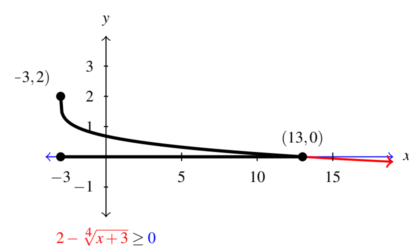
   To solve $2 - \sqrt[4]{x+3} \geq 0$, it is tempting to rewrite this inequality as $2 \geq \sqrt[4]{x+3}$ and rid ourselves of the fourth root by raising both sides of this inequality to the fourth power. While this technique works *sometimes*, it doesn't work *all* the time because raising both sides of an inequality to the fourth (more

[20] see subsections 4.4.1 and 4.4.2

generally, to an even) power does not necessarily preserve inequalities .[21] For that reason, we solve this inequality using a sign diagram as this technique will *always* produce a correct solution.

We already have all the nonzero terms on one side of the inequality, so we let $r(x) = 2 - \sqrt[4]{x+3}$ and proceed to make a sign diagram. Owing to the presence of the fourth root, we know $x+3 \geq 0$ or $x \geq -3$. Hence, we only concern ourselves with the portion of the number line representing $[3, \infty)$. Next, we find the zeros of r by solving $r(x) = 2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$, so $x+3 = 16$ and we get $x = 13$. We find this solution checks in our original equation,[22] and proceed to construct the sign diagram below on the left. As we are looking for where $r(x) = 2 - \sqrt[4]{x+3} \geq 0$, we are looking for the zeros of r along with the intervals over which $r(x)$ is $(+)$. We record our answer as $[-3, 13]$. Below is the graph of $y = 2 - \sqrt[4]{x+3}$, and we can see that, indeed, the graph is above the x-axis ($y = 0$) from $[-3, 13)$ and meets the x-axis at $x = 13$, verifying our answer.



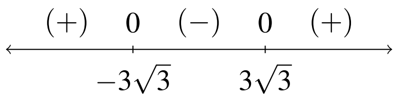


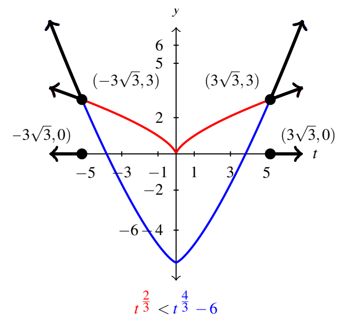
2. Solve $t^{2/3} < t^{4/3} - 6$.

To solve $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$, we first rewrite as $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$. We set $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6$ and note that the denominators in the exponents are 3, so they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(t) = 0$ and attempt to solve $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 = 0$. As a result of there being three terms, and the exponent on one of the variable terms, $t^{\frac{4}{3}}$, is exactly twice that of the other, $t^{\frac{2}{3}}$, we have ourselves a 'quadratic in disguise.' If we let $u = t^{\frac{2}{3}}$, then $u^2 = t^{\frac{4}{3}}$, so in terms of u, we have $u^2 - u - 6 = 0$. Solving we get $u = -2$ or $u = 3$, hence $t^{\frac{2}{3}} = -2$ or $t^{\frac{2}{3}} = 3$. In root-power notation, these are $\sqrt[3]{t^2} = -2$ or $\sqrt[3]{t^2} = 3$. Cubing both sides of these equations results in $t^2 = -8$, which admits no real solution, or $t^2 = 27$, which gives $t = \pm 3\sqrt{3}$. Using these zeros, we construct the sign diagram below on the left. We find $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(t) = t^{\frac{2}{3}}$ and $g(t) = t^{\frac{4}{3}} - 6$. The solution to $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$ corresponds to the inequality $f(t) < g(t)$, which means we are looking for the t values for which the graph of f is *below* the graph of g. On the graph below, we see the graph of f is below the graph of g for $t < -5.196$ and again for $t > 5.196$, which are the grapher's approximations to $\pm 3\sqrt{3}$.

[21] For instance, $-2 \leq 1$ but $(-2)^4 \geq (1)^2$. We invite the reader to see what goes wrong if attempting to solve either of the following inequalities using this method: $-2 \geq \sqrt[4]{x+3}$, which has no solution, or $-2 \leq \sqrt[4]{x+3}$, whose solution is $[-3, \infty)$.

[22] Recall that raising both sides to an even power could produce extraneous solutions, so it is important we check here.





3. Solve $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$.

   To solve $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$, we first gather all the nonzero terms to one side and obtain $3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \leq 0$. Setting $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, because the denominators of the rational exponents are odd, we have no domain concerns owing to even indexed roots. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}}$$

   Setting the denominator equal to zero we get $(2-x)^{\frac{2}{3}} = 0$, which reduces to $2-x=0$, or $x=2$. Hence, the domain of r is $(-\infty,2)\cup(2,\infty)$.

   To find the zeros of r, we set $r(x)=0$, so we set about solving

$$3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} = 0.$$

   Clearing denominators, we get $3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} - x = 0$. As the denominators of the exponents are odd, we may use Theorem 4.3 to simplify this to $3(2-x)^1 - x = 0$, and obtain $6-4x=0$ or $x=\frac{3}{2}$. In order for us to be able to more easily determine the sign of $r(x)$ at the test values, we rewrite $r(x)$ as a single term.[23] There are two schools of thought on how to proceed, so we demonstrate both.

   - *Factoring Approach.* From $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, we note $-\frac{2}{3} < \frac{1}{3}$, so we factor $(2-x)^{-\frac{2}{3}}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-\frac{2}{3}}$ *from* $(2-x)^{\frac{1}{3}}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents.

[23]This also gives us a chance to review some good intermediate algebra!

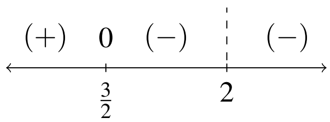
$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{\frac{3}{3}} - x\right] \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{1} - x\right] \\
&= (2-x)^{-\frac{2}{3}}(6-4x) \\
&= (2-x)^{-\frac{2}{3}}(6-4x)
\end{aligned}$$

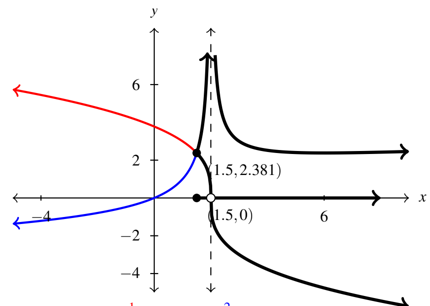
Written without negative exponents, we have $r(x) = \dfrac{6-4x}{(2-x)^{\frac{2}{3}}}$.

- *Common Denominator Approach.* We rewrite

$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{common denominator} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{Theorem 4.3} \\
&= \frac{3(2-x)^{\frac{3}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)-x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{6-4x}{(2-x)^{\frac{2}{3}}}
\end{aligned}$$

Using either approach, we end up with the same, simpler, expression for $r(x)$ and we use that to create our sign diagram as shown below on the left. We find $r(x) \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{\frac{1}{3}}$ and $g(x) = x(2-x)^{-\frac{2}{3}}$. We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x > \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.





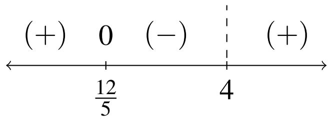
$3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$

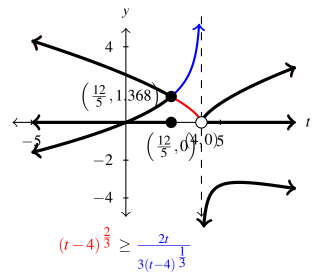
4. Solve $(t-4)^{\frac{2}{3}} \geq -\dfrac{2t}{3(t-4)^{\frac{1}{3}}}$.

While it may be tempting to begin solving our last inequality by clearing denominators, owing to the odd root, the quantity $3(t-4)^{\frac{1}{3}}$ can be both positive and negative for different values of t. This means that if we chose to multiply both sides of our inequality by this quantity, we have no guarantee if the inequality would be preserved. Hence we proceed as usual by gathering all the nonzero terms to one side, and, with the ultimate goal of creating a sign diagram, get common denominators.

$$
\begin{array}{rcll}
(t-4)^{\frac{2}{3}} & \geq & -\dfrac{2t}{3(t-4)^{\frac{1}{3}}} & \\
(t-4)^{\frac{2}{3}} + \dfrac{2t}{3(t-4)^{\frac{1}{3}}} & \geq & 0 & \\
\dfrac{(t-4)^{\frac{2}{3}} \cdot 3(t-4)^{\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \dfrac{2t}{3(t-4)^{\frac{1}{3}}} & \geq & 0 & \text{common denominator} \\
\dfrac{3(t-4)^{\frac{2}{3}+\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \dfrac{2t}{3(t-4)^{\frac{1}{3}}} & \geq & 0 & \text{Theorem 4.3} \\
\dfrac{3(t-4)^{1}}{3(t-4)^{\frac{1}{3}}} + \dfrac{2t}{3(t-4)^{\frac{1}{3}}} & \geq & 0 & \\
\dfrac{3(t-4)+2t}{3(t-4)^{\frac{1}{3}}} & \geq & 0 & \\
\dfrac{5t-12}{3(t-4)^{\frac{1}{3}}} & \geq & 0 &
\end{array}
$$

We identify $r(t)$ as the left hand side of the inequality and see right away we must exclude $t = 4$ from the domain owing to the quantity $(t-4)$ in the denominator. As we have already mentioned, the root here (3) is odd, so we have no domain issues stemming from that. To find the zeros of r, we set $r(t) = 0$ which quickly reduces to solving $5t - 12 = 0$. We get $t = \frac{12}{5}$. From the sign diagram, we find $r(t) \geq 0$ on $\left(-\infty, \frac{5}{12}\right] \cup (4, \infty)$. Graphing $f(t) = (t-4)^{\frac{2}{3}}$ and $g(t) = -\frac{2t}{3(t-4)^{\frac{1}{3}}}$, we see the graph of f is above the graph of g for $t < 2.4$ and again for $t > 4$, with an intersection point at $t = 2.4 = \frac{12}{5}$.





□

Note that in Example 4.4.3 number 3, because $(2-x)^{\frac{2}{3}}$ is always positive for $x \neq 2$ (owing to the squared exponent), we *could* have short-cut the sign diagram, choosing to clear denominators instead:

$$\begin{array}{rcll} 3(2-x)^{\frac{1}{3}} & \leq & x(2-x)^{-\frac{2}{3}} & \\ 3(2-x)^{\frac{1}{3}} & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}} & \\ \left[3(2-x)^{\frac{1}{3}}\right]\left[(2-x)^{\frac{2}{3}}\right] & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}}\left[(2-x)^{\frac{2}{3}}\right] & \text{provided } x \neq 2 \\ 3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} & \leq & x & \\ 3(2-x)^{\frac{1}{3}+\frac{2}{3}} & \leq & x & \\ 3(2-x) & \leq & x & \end{array}$$

Hence, we get $6-3x \leq x$ or $x \geq \frac{3}{2}$, provided $x \neq 2$. This matches our solution $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. If, on the other hand, we tried this same manipulation with number 4, we would clear denominators, assuming $t \neq 4$ to obtain $3(t-4) \geq -2t$ or $t \geq \frac{12}{5}$ which is *not* the correct solution. The moral of the story is the more you understand, the less you need to rely on memorized processes and the more efficient your solution methodologies can become. The sign diagram algorithm is a fail-safe method, but, in some cases, may be far from the most efficient one. It's always best to understand the *why* of a procedure as much as the *how*.

4.4.4 Exercises

In Exercises 1 - 16, solve the inequality. Write your answer using interval notation.

1. $x^2+2x-3 \geq 0$
2. $16x^2+8x+1 > 0$
3. $t^2+9 < 6t$
4. $9t^2+16 \geq 24t$
5. $u^2+4 \leq 4u$
6. $u^2+1 < 0$

7. $3x^2 \leq 11x+4$

8. $x > x^2$

9. $2t^2 - 4t - 1 > 0$

10. $5t+4 \leq 3t^2$

11. $2 \leq |x^2 - 9| < 9$

12. $x^2 \leq |4x-3|$

13. $t^2 + t + 1 \geq 0$

14. $t^2 \geq |t|$

15. $x|x+5| \geq -6$

16. $x|x-3| < 2$

17. The height of an object dropped from the roof of an eight story building is modeled by by the function $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

18. The height $h(t)$ in feet of a model rocket above the ground t seconds after lift-off is given by the function $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

19. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height $h(t)$ in feet of the hammer above the ground t seconds after Jason lets it go is modeled by the function $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

20. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

    (a) What is the applied domain of this function?

    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

    (c) Come up with a scenario in which $s_0 < 0$.

    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

    (e) If the marble is shot from the top of a 25 meter tall tower, when does it hit the ground?

    (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

21. Graph $f(x) = |1 - x^2|$

22. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

23. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0,0)$. Use this to find the point on L closest to $(0,0)$.

(Review of Solving Equations):[24] In Exercises 24 - 29, solve the rational equation. Be sure to check for extraneous solutions.

24. $\dfrac{x}{5x+4}=3$

25. $\dfrac{3x-1}{x^2+1}=1$

26. $\dfrac{1}{t+3}+\dfrac{1}{t-3}=\dfrac{t^2-3}{t^2-9}$

27. $\dfrac{2t+17}{t+1}=t+5$

28. $\dfrac{z^2-2z+1}{z^3+z^2-2z}=1$

29. $\dfrac{4z-z^3}{z^2-9}=4z$

In Exercises 30 - 45, solve the rational inequality. Express your answer using interval notation.

30. $\dfrac{1}{x+2}\geq 0$

31. $\dfrac{5}{x+2}\geq 1$

32. $\dfrac{x}{x^2-1}<0$

33. $\dfrac{4t}{t^2+4}\geq 0$

34. $\dfrac{2t+6}{t^2+t-6}<1$

35. $\dfrac{5}{t-3}+9<\dfrac{20}{t+3}$

36. $\dfrac{6z+6}{2+z-z^2}\leq z+3$

37. $\dfrac{6}{z-1}+1>\dfrac{1}{z+1}$

38. $\dfrac{3z-1}{z^2+1}\leq 1$

39. $(2x+17)(x+1)^{-1}>x+5$

40. $(4x-x^3)(x^2-9)^{-1}\geq 4x$

41. $(x^2+1)^{-1}<0$

42. $(2t-8)(t+1)^{-1}\leq (t^2-8t)(t+1)^{-2}$

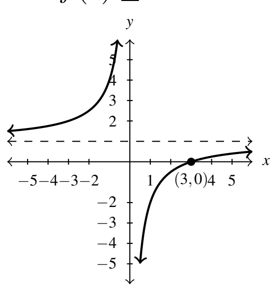
43. $(t-3)(2t+7)(t^2+7t+6)^{-2}\geq (t^2+7t+6)^{-1}$

44. $60z^{-2}+23z^{-1}\geq 7(z-4)^{-1}$

45. $2z+6(z-1)^{-1}\geq 11-8(z+1)^{-1}$

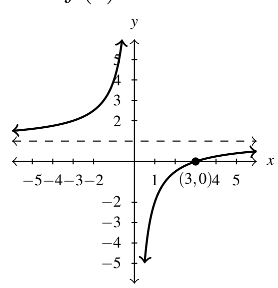
In Exercises 46 - 51, use the the graph of the given rational function to solve the stated inequality.

46. Solve $f(x)\geq 0$.



$y=f(x)$, asymptotes: $x=0$, $y=1$

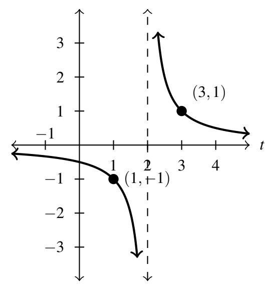
47. Solve $f(x)<1$.



$y=f(x)$, asymptotes: $x=0$, $y=1$

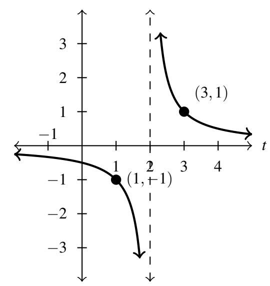
48. Solve $g(t)\geq -1$.

[24] For more review, see Section 3.1.



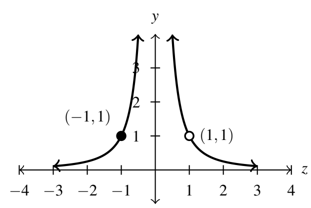
$y = g(t)$, asymptotes: $t = 2$, $y = 0$

49. Solve $-1 \leq g(t) < 1$.



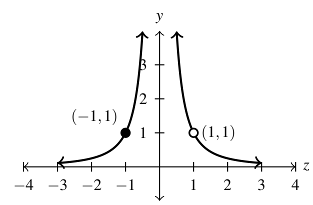
$y = g(t)$, asymptotes: $t = 2$, $y = 0$

50. Solve $r(z) \leq 1$



$y = r(z)$, asymptote: $z = 0$, $y = 0$

51. Solve $r(z) > 0$.



$y = r(z)$, asymptotes: $z = 0$, $y = 0$

52. In Exercise 55 in Section 2.2, the function $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ was used to model the cost (in dollars) to produce x PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost $\overline{C}$. Round your answer to the nearest number of systems.

53. Suppose we are in the same situation as Example 4.4.6. If the volume of the box is to be 500 cubic centimeters, use a graphing utility to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

54. The box for the new Sasquatch-themed cereal, 'Crypt-Os', is to have a volume of 140 cubic inches. For aesthetic reasons, the height of the box needs to be 1.62 times the width of the base of the box.[25] Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.

55. Sally is Skippy's neighbor from Exercise 23 in Section 2.1. Sally also wants to plant a vegetable garden along the side of her home. She doesn't have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn't need fencing along that side.)

[25] 1.62 is a crude approximation of the so-called 'Golden Ratio' $\phi = \frac{1+\sqrt{5}}{2}$.

56. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[26]

    (a) Write an expression for the volume V of the can in terms of the height h and the base radius r.

    (b) Write an expression for the surface area S of the can in terms of the height h and the base radius r. (Hint: The top and bottom of the can are circles of radius r and the side of the can is really just a rectangle that has been bent into a cylinder.)

    (c) Using the fact that $V = 33.6$, write S as a function of r and state its applied domain.

    (d) Use a graph to determine the dimensions of the can which has minimal surface area.

57. A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

58. In Exercise 35 in Section 3.2, the population of Sasquatch in Portage County is modeled by

$$P(t) = \frac{150t}{t+15}, \quad t \geq 0,$$

where $t = 0$ corresponds to the year 1803. According to this model, when were there fewer than 100 Sasquatch in Portage County?

[26] According to www.dictionary.com, there are different values given for this conversion. We use 33.6 in^3 for this problem.

In Exercises 59 - 74, solve the inequalities and check your answer graphically.

59. $10-\sqrt{t-2}\leq 11$

60. $t^{\frac{2}{3}}\leq 4$

61. $\sqrt[3]{x}\leq x$

62. $(2-3x)^{\frac{1}{3}}>3x$

63. $(t^2-1)^{-\frac{1}{2}}\geq 2$

64. $(t^2-1)^{-\frac{1}{3}}\leq 2$

65. $3(x-1)^{\frac{1}{3}}+x(x-1)^{-\frac{2}{3}}\geq 0$

66. $3(x-1)^{\frac{2}{3}}+2x(x-1)^{-\frac{1}{3}}\geq 0$

67. $2(t-2)^{-\frac{1}{3}}-\frac{2}{3}t(t-2)^{-\frac{4}{3}}\leq 0$

68. $-\frac{4}{3}(t-2)^{-\frac{4}{3}}+\frac{8}{9}t(t-2)^{-\frac{7}{3}}\geq 0$

69. $2x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}}+x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}}\geq 0$

70. $\sqrt[3]{x^3+3x^2-6x-8}>x+1$

71. $4(7-t)^{0.75}-3t(7-t)^{-0.25}\leq 0$

72. $4t^{0.75}(t-3)^{-\frac{2}{3}}+9t^{-0.25}(t-3)^{\frac{1}{3}}<0$

73. $x^{-\frac{1}{3}}(x-3)^{-\frac{2}{3}}-x^{-\frac{4}{3}}(x-3)^{-\frac{5}{3}}(x^2-3x+2)\geq 0$

74. $\frac{2}{3}(t+4)^{\frac{3}{5}}(t-2)^{-\frac{1}{3}}+\frac{3}{5}(t+4)^{-\frac{2}{5}}(t-2)^{\frac{2}{3}}\geq 0$

SECTION 4.4 EXERCISE ANSWERS A.1.4

CHAPTER 5

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.1 Inverse Functions

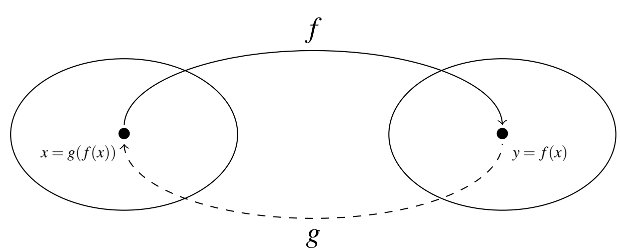
In Section 1.2, we defined functions as processes. In this section, we seek to reverse, or 'undo' those processes. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like baking a cake) are not.

Consider the function $f(x) = 3x + 4$. Starting with a real number input x, we apply two steps in the following sequence: first we multiply the input by 3 and, second, we add 4 to the result.

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes and then we take off the socks. In much the same way, the function g should undo each step of f but in the opposite order. That is, the function g should first *subtract* 4 from the input x then *divide* the result by 3. This leads us to the formula $g(x) = \frac{x-4}{3}$.

Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, we simplify $g(f(x)) = g(3x+4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x-4) + 4 = x$. Using the language of function composition developed in Section 1.5.2, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.[1] Abstractly, we can visualize the relationship between f and g in the diagram below.



The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

[1] At the level of functions, $g \circ f = f \circ g = I$, where I is the identity function as defined as $I(x) = x$ for all real numbers, x.

Definition 5.1. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f **and**
2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

If we abstract one step further, we can express the sentiment in Definition 5.1 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[2] to the domain of f and I_2 is the identity function restricted to the domain of g.

In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 1.6, we can show that function inverses are unique.[3]

Suppose g and h are both inverses of a function f. By Theorem 5.1, the domain of g is equal to the domain of h, because both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.

We summarize the important properties of invertible functions in the following theorem. Apart from introducing notation, each of the results below are immediate consequences of the idea that inverse functions map the outputs from a function f back to their corresponding inputs.

Theorem 5.1. Properties of Inverse Functions: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$

  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a,c) is on the graph of f if and only if (c,a) is on the graph of f^{-1}

  NOTE: This means graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across $y = x$.[a]
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

[a]See Example 1.1.5 in Section 1.1 and Example 1.3.5 in Section 1.3.1.

The notation f^{-1} is an unfortunate choice because you've been programmed since Algebra I to think of this

[2]The identity function I, first introduced in Exercise 76 in Section 1.3.1 and mentioned in Theorem 1.6, has a domain of all real numbers. As the domains of f and g may not be all real numbers, we need the restrictions listed here.

[3]In other words, invertible functions have exactly one inverse.

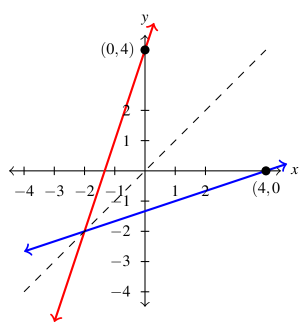
as $\frac{1}{f}$. This is most definitely *not* the case, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Why does this confusing notation persist? As we mentioned in Section 1.5.2, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Before we embark on an example, we demonstrate the pertinent parts of Theorem 5.1 to the inverse pair $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$. Suppose we wanted to solve $3x + 4 = 7$. Going through the usual machinations, we obtain $x = 1$.

If we view this equation as $f(x) = 7$, however, then we are looking for the input x corresponding to the output $f(x) = 7$. This is exactly the question f^{-1} was built to answer. In other words, the solution to $f(x) = 7$ is $x = f^{-1}(7) = 1$. In other words, the formula $f^{-1}(x)$ encodes all of the algebra required to 'undo' what the formula $f(x)$ does to x. More generally, any time you have ever solved an equation, you have really been working through an inverse problem.

We also note the graphs of $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$ are easily seen to be reflections across the line $y = x$, as seen below. In particular, note that the y-intercept $(0,4)$ on the graph of $y = f(x)$ corresponds to the x-intercept on the graph of $y = f^{-1}(x)$. Indeed, the point $(0,4)$ on the graph of $y = f(x)$ can be interpreted as $(0,4) = (0, f(0)) = (f^{-1}(4), 4)$ just as the point $(4,0)$ on the graph of $y = f^{-1}(x)$ can be interpreted as $(4,0) = (4, f^{-1}(4)) = (f(0), 0)$.



Graphs of inverse functions $y = f(x) = 3x + 4$ and $y = f^{-1}(x) = \dfrac{x-4}{3}$.

Example 5.1.1. For each pair of functions f and g below:

1. Verify each pair of functions f and g are inverses: (a) algebraically and (b) graphically.
2. Use the fact f and g are inverses to solve $f(x) = 5$ and $g(x) = -3$

- $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$
- $f(t) = \dfrac{2t}{t+1}$ and $g(t) = \dfrac{t}{2-t}$

Solution.

- **Solution for** $f(x) = \sqrt[3]{x-1} + 2$ **and** $g(x) = (x-2)^3 + 1$.

1. (a) To verify $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$ are inverses, we appeal to Definition 5.1 and show $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for all real numbers, x.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt[3]{x-1}+2) \\ &= [(\sqrt[3]{x-1}+2)-2]^3+1 \\ &= (\sqrt[3]{x-1})^3+1 \\ &= x-1+1 \\ &= x \checkmark \end{aligned}$$

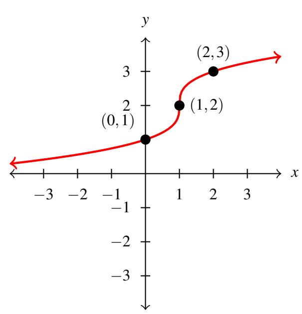
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f((x-2)^3+1) \\ &= \sqrt[3]{[(x-2)^3+1]-1}+2 \\ &= \sqrt[3]{(x-2)^3}+2 \\ &= x-4+4 \\ &= x \checkmark \end{aligned}$$

As the root here, 3, is odd, Theorem 4.2 gives $(\sqrt[3]{x-1})^3 = x-1$ and $\sqrt[3]{(x-2)^3} = x-2$.

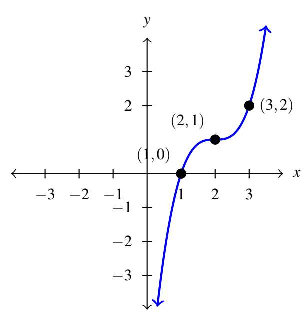
(b) To show f and g are inverses graphically, we graph $y = f(x)$ and $y = g(x)$ on the same set of axes and check to see if they are reflections about the line $y = x$.

The graph of $y = f(x) = \sqrt[3]{x-1} + 2$ appears below on the left courtesy of Theorem 4.1 in Section 4.1. The graph of $y = g(x) = (x-2)^3 + 1$ appears below in the middle thanks to Theorem 2.2 in Section 2.2.

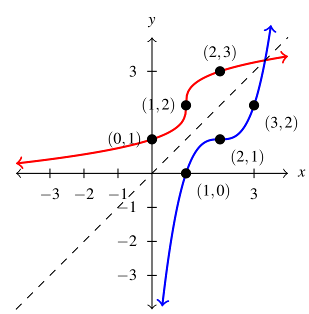
We can immediately see three pairs of corresponding points: $(0,1)$ and $(1,0)$, $(1,2)$ and $(2,1)$, $(2,3)$ and $(3,2)$. When graphed on the same pair of axes, the two graphs certainly appear to be symmetric about the line $y = x$, as required.



$y = f(x) = \sqrt[3]{x-1}+2$



$y = g(x) = (x-2)^3+1$



$y = f(x)$, $y = g(x)$, $y = x$

2. f and g are inverses, so the solution to $f(x) = 5$ is $x = f^{-1}(5) = g(5) = (5-2)^3 + 1 = 28$. To check, we find $f(28) = \sqrt[3]{28-1} + 2 = \sqrt[3]{27} + 2 = 3 + 2 = 5$, as required.

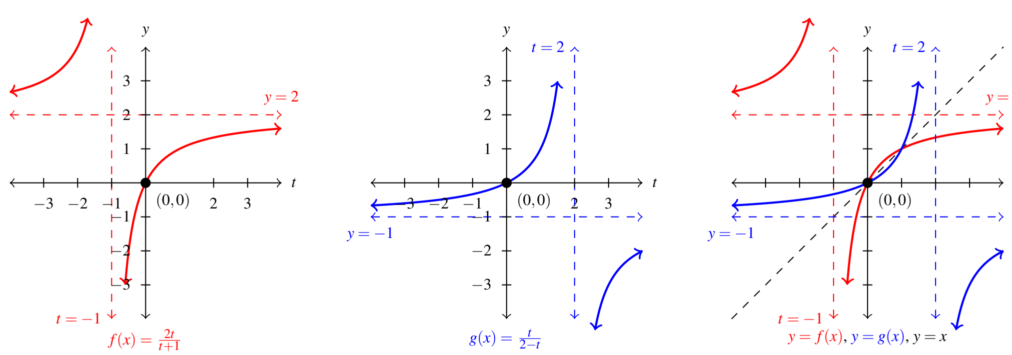
   Likewise, the solution to $g(x) = -3$ is $x = g^{-1}(-3) = f(-3) = \sqrt[3]{(-3)-1} + 2 = 2 - \sqrt[3]{4}$. Once again, to check, we find $g(2 - \sqrt[3]{4}) = (2 - \sqrt[3]{4} - 2)^3 + 1 = (-\sqrt[3]{4})^3 + 1 = -4 + 1 = -3$.

- **Solution for** $f(t) = \dfrac{2t}{t+1}$ **and** $g(t) = \dfrac{t}{2-t}$.

1. (a) Note the domain of f excludes $t=-1$ and the domain of g excludes $t=2$. Hence, when simplifying $(g \circ f)(t)$ and $(f \circ g)(t)$, we tacitly assume $t \neq -1$ and $t \neq 2$, respectively.

$$\begin{aligned}
(g \circ f)(t) &= g(f(t)) \\
&= g\left(\frac{2t}{t+1}\right) \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \cdot \frac{(t+1)}{(t+1)} \\
&= \frac{2t}{2(t+1)-2t} \\
&= \frac{2t}{2t+2-2t} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}
\qquad
\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= f\left(\frac{t}{2-t}\right) \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right)+1} \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right)+1} \cdot \frac{(2-t)}{(2-t)} \\
&= \frac{2t}{t+(1)(2-t)} \\
&= \frac{2t}{t+2-t} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}$$

(b) We graph $y = f(t)$ and $y = g(t)$ using the techniques discussed in Sections 3.2 and 3.3.



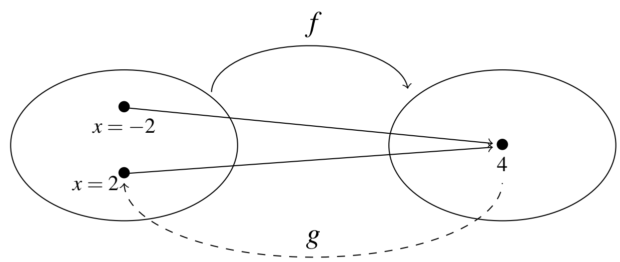
We find the graph of f has a vertical asymptote $t=-1$ and a horizontal asymptote $y=2$. Corresponding to the *vertical* asymptote $t=-1$ on the graph of f, we find the graph of g has a *horizontal* asymptote $y=-1$.

Likewise, the *horizontal* asymptote $y = 2$ on the graph of f corresponds to the *vertical* asymptote $t = 2$ on the graph of g. Both graphs share the intercept $(0,0)$. When graphed together on the same set of axes, the graphs of f and g do appear to be symmetric about the line $y = t$.

2. Don't let the fact that f and g in this case were defined using the independent variable, 't' instead of 'x' deter you in your efforts to solve $f(x) = 5$. Remember that, ultimately, the function f here is the *process* represented by the formula $f(t)$, and is the same process (with the same inverse!) regardless of the letter used as the independent variable. Hence, the solution to $f(x) = 5$ is $x = f^{-1}(1) = g(5)$. We get $g(5) = \frac{5}{2-5} = -\frac{5}{3}$.

   To check, we find $f\left(-\frac{5}{3}\right) = \left(-\frac{10}{3}\right)/\left(-\frac{2}{3}\right) = 5$. Similarly, we solve $g(x) = -3$ by finding $x = g^{-1}(-3) = f(-3) = \frac{-6}{-2} = 3$. Sure enough, we find $g(3) = \frac{3}{2-3} = -3$. □

We now investigate under what circumstances a function is invertible. As a way to motivate the discussion, we consider $f(x) = x^2$. A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. However, $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x unless $x \geq 0$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. That is, g failed to return the input -2 from its output 4. Instead, g matches the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. Schematically:



We see from the diagram that both $f(-2)$ and $f(2)$ are 4, thus it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$. Recall that by definition, a function can match 4 with only *one* number.

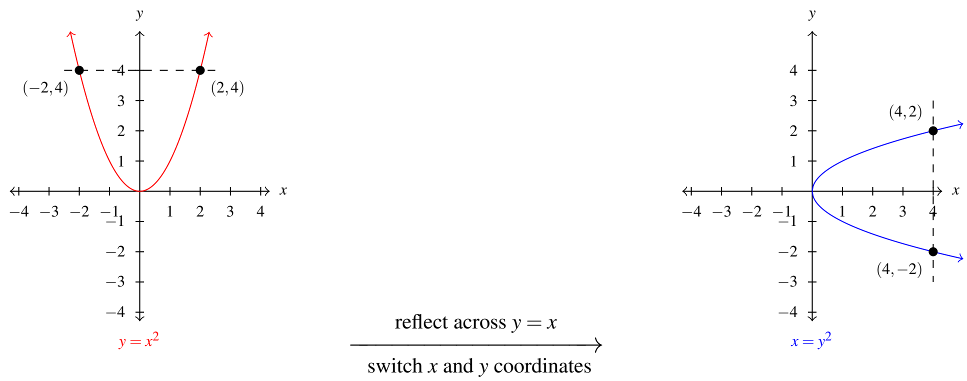
In general, in order for a function to be invertible, each output can come from only *one* input. By definition, a function matches up each input to only *one* output, thus invertible functions have the property that they match one input to one output and vice-versa. We formalize this concept below.

Definition 5.2. A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.

Note that an equivalent way to state Definition 5.2 is that a function is one-to-one if *different* inputs go to *different* outputs. That is, if $a \neq b$, then $f(a) \neq f(b)$.

Before we solidify the connection between invertible functions and one-to-one functions, we take a moment to see what goes wrong graphically when trying to find the inverse of $f(x) = x^2$.

Per Theorem 5.1, the graph of $y = f^{-1}(x)$, if it exists, is obtained from the graph of $y = x^2$ by reflecting $y = x^2$ about the line $y = x$. Procedurally, this is accomplished by interchanging the x and y coordinates of each point on the graph of $y = x^2$. Algebraically, we are swapping the variables 'x' and 'y' which results in the equation $x = y^2$ whose graph is below on the right.



We see immediately the graph of $x = y^2$ fails the Vertical Line Test, Theorem 1.2. In particular, the vertical line $x = 4$ intersects the graph at two points, $(4,-2)$ and $(4,2)$ meaning the relation described by $x = y^2$ matches the x-value 4 with two different y-values, -2 and 2.

Note that the *vertical* line $x = 4$ and the points $(4,\pm 2)$ on the graph of $x = y^2$ correspond to the *horizontal* line $y = 4$ and the points $(\pm 2, 4)$ on the graph of $y = x^2$ which brings us right back to the concept of one-to-one. The fact that both $(-2,4)$ and $(2,4)$ are on the graph of f means $f(-2) = f(2) = 4$. Hence, f takes different inputs, -2 and 2, to the same output, 4, so f is not one-to-one.

Recall the Horizontal Line Test from Exercise 57 in Section 1.2. Applying that result to the graph of f we say the graph of f 'fails' the Horizontal Line Test because the horizontal line $y = 4$ intersects the graph of $y = x^2$ more than once. This means that the equation $y = x^2$ does not represent x as a function of y.

Said differently, the Horizontal Line Test detects when there is at least one y-value (4) which is matched to more than one x-value (± 2). In other words, the Horizontal Line Test can be used to detect whether or not a function is one-to-one.

So, to review, $f(x) = x^2$ is not invertible, not one-to-one, and its graph fails the Horizontal Line Test. It turns out that these three attributes: being invertible, one-to-one, and having a graph that passes the Horizontal Line Test are mathematically equivalent. That is to say if one if these things is true about a function, then they all are; it also means that, as in this case, if one of these things *isn't* true about a function, then *none* of them are. We summarize this result in the following theorem.

Theorem 5.2. Equivalent Conditions for Invertibility: For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a]i.e., no horizontal line intersects the graph more than once.

To prove Theorem 5.2, we first suppose f is invertible. Then there is a function g so that $g(f(x)) = x$ for all x in the domain of f. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. As a result of $g(f(x)) = x$, the equation $g(f(a)) = g(f(b))$ reduces to $a = b$. We've shown that if $f(a) = f(b)$, then $a = b$, proving f is one-to-one.

Next, assume f is one-to-one. Suppose a horizontal line $y = c$ intersects the graph of $y = f(x)$ at the points (a,c) and (b,c). This means $f(a) = c$ and $f(b) = c$ so $f(a) = f(b)$. Because f is one-to-one, means $a = b$ so the points (a,c) and (b,c) are actually one in the same. This establishes that each horizontal line can intersect the graph of f at most once, so the graph of f passes the Horizontal Line Test.

Last, but not least, suppose the graph of f passes the Horizontal Line Test. Let c be a real number in the range of f. Then the horizontal line $y = c$ intersects the graph of $y = f(x)$ just *once*, say at the point $(a,c) = (a, f(a))$. Define the mapping g so that $g(c) = g(f(a)) = a$. The mapping g is a *function* because each horizontal line $y = c$ where c is in the range of f intersects the graph of f only *once*. By construction, we have the domain of g is the range of f and that for all x in the domain of f, $g(f(x)) = x$. We leave it to the reader to show that for all x in the domain of g, $f(g(x)) = x$, too.

Hence, we've shown: first, if f invertible, then f is one-to-one; second, if f is one-to-one, then the graph of f passes the Horizontal Line Test; and third, if f passes the Horizontal Line Test, then f is invertible. Therefore if f is satisfies any one of these three conditions, then f must satisfy the other two.[4]

We put this result to work in the next example.

Example 5.1.2. Determine if the following functions are one-to-one:

(a) analytically using Definition 5.2 and

(b) graphically using the Horizontal Line Test.

For the functions that are one-to-one, graph the inverse.

[4]For example, if we know f is one-to-one, we showed the graph of f passes the HLT which, in turn, guarantees f is invertible.

1. $f(x) = x^2 - 2x + 4$
2. $g(t) = \dfrac{2t}{1-t}$
3. $F = \{(-1,1), (0,2), (1,-3), (2,1)\}$
4. $G = \{(t^3+1, 2t) \,|\, t \text{ is a real number.}\}$

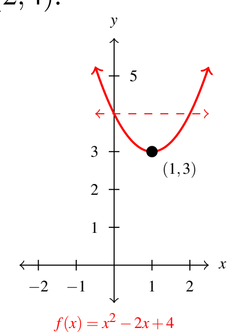
Solution.

1. Determine if $f(x) = x^2 - 2x + 4$ is a one-to-one function.

   (a) To determine whether or not f is one-to-one analytically, we assume $f(a) = f(b)$ and work to see if we can deduce $a = b$. As we work our way through the problem, we encounter a quadratic equation. We rewrite the equation so it equals 0 and factor by grouping. We get $a = b$ as one possibility, but we also get the possibility that $a = 2 - b$. This suggests that f may not be one-to-one. Taking $b = 0$, we get $a = 0$ or $a = 2$. We have two different inputs with the same output as $f(0) = 4$ and $f(2) = 4$, proving f is neither one-to-one nor invertible.

$$\begin{array}{rcl} f(a) & = & f(b) \\ a^2 - 2a + 4 & = & b^2 - 2b + 4 \\ a^2 - 2a & = & b^2 - 2b \\ a^2 - b^2 - 2a + 2b & = & 0 \\ (a+b)(a-b) - 2(a-b) & = & 0 \\ (a-b)((a+b)-2) & = & 0 \\ a - b = 0 & \text{or} & a + b - 2 = 0 \\ a = b & \text{or} & a = 2 - b \end{array}$$

   (b) We note that f is a quadratic function and we graph $y = f(x)$ using the techniques presented in Section 2.1. We see the graph fails the Horizontal Line Test quite often - in particular, crossing the line $y = 4$ at the points $(0,4)$ and $(2,4)$.

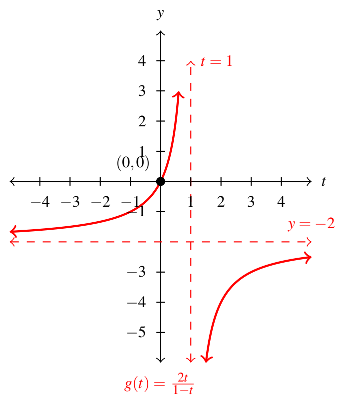


2. Determine if $g(t) = \dfrac{2t}{1-t}$ is a one-to-one function.

   (a) We begin with the assumption that $g(a) = g(b)$ for a, b in the domain of g (That is, we assume $a \neq 1$ and $b \neq 1$.) Through our work, we deduce $a = b$, proving g is one-to-one.

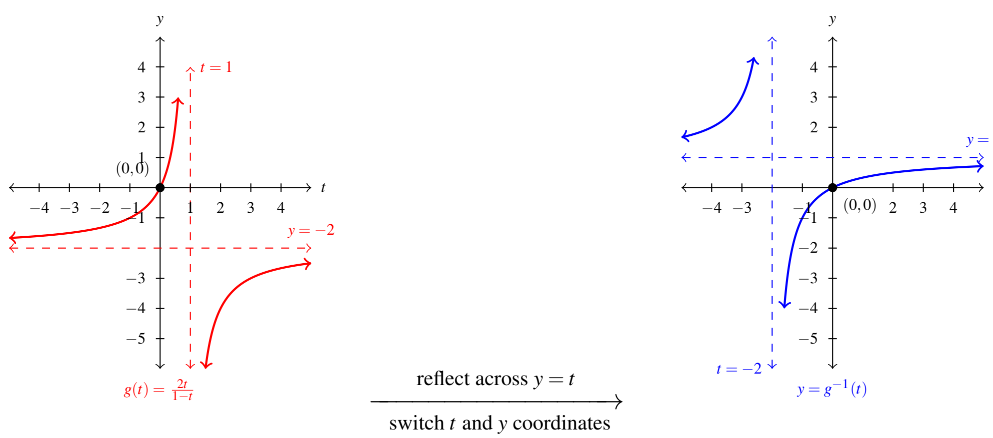
$$\begin{aligned}
g(a) &= g(b)\\
\frac{2a}{1-a} &= \frac{2b}{1-b}\\
2a(1-b) &= 2b(1-a)\\
2a-2ab &= 2b-2ba\\
2a &= 2b\\
a &= b\ \checkmark
\end{aligned}$$

(b) We graph $y = g(t)$ using the procedure outlined in Section 3.3. We find the sole intercept is $(0,0)$ with asymptotes $t = 1$ and $y = -2$. Based on our graph, the graph of g appears to pass the Horizontal Line Test, verifying g is one-to-one.



Because g is one-to-one, g is invertible. Even though we do not have a *formula* for $g^{-1}(t)$, we can nevertheless sketch the graph of $y = g^{-1}(t)$ by reflecting the graph of $y = g(t)$ across $y = t$.

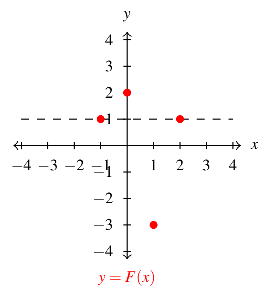
Corresponding to the *vertical* asymptote $t = 1$ on the graph of g, the graph of $y = g^{-1}(t)$ will have a *horizontal* asymptote $y = 1$. Similarly, the *horizontal* asymptote $y = -2$ on the graph of g corresponds to a *vertical* asymptote $t = -2$ on the graph of g^{-1}. The point $(0,0)$ remains unchanged when we switch the t and y coordinates, so it is on both the graph of g and g^{-1}.



3. Determine if $F = \{(-1,1),(0,2),(1,-3),(2,1)\}$ is a one-to-one function.

   (a) The function F is given to us as a set of ordered pairs. Recall each ordered pair is of the form $(a, F(a))$. As $(-1,1)$ and $(2,1)$ are both elements of F, this means $F(-1) = 1$ and $F(2) = 1$. Hence, we have two distinct inputs, -1 and 2 with the same output, 1, thus F is not one-to-one and, hence, not invertible.

   (b) To graph F, we plot the points in F below on the left. We see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.



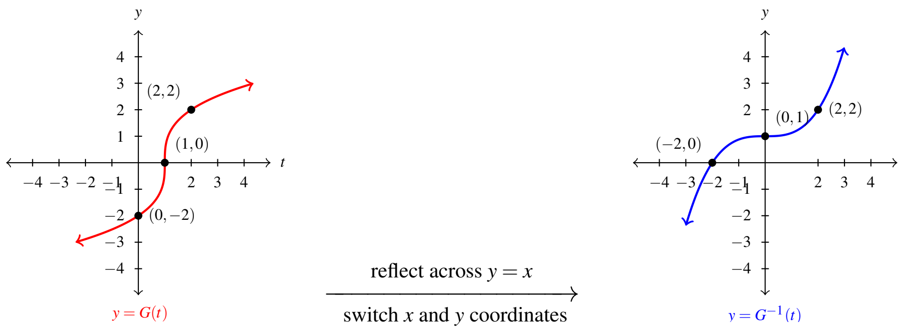
4. Determine if $G = \{(t^3+1, 2t) \,|\, t \text{ is a real number}\}$ is a one-to-one function.

   Like the function F above, the function G is described as a set of ordered pairs. Before we set about determining whether or not G is one-to-one, we take a moment to show G is, in fact, a function. That is, we must show that each real number input to G is matched to only one output.

   We are given $G = \{(t^3+1, 2t) \,|\, t \text{ is a real number}\}$. and we know that when represented in this way, each ordered pair is of the form $(\text{input}, \text{output})$. Hence, the inputs to G are of the form $t^3 + 1$ and the outputs from G are of the form $2t$. To establish G is a function, we must show that each input produces only one output. If it should happen that $a^3 + 1 = b^3 + 1$, then we must show $2a = 2b$. The equation $a^3 + 1 = b^3 + 1$ gives $a^3 = b^3$, or $a = b$. From this it follows that $2a = 2b$ so G is a function.

   (a) To show G is one-to-one, we must show that if two outputs from G are the same, the corresponding inputs must also be the same. That is, we must show that if $2a = 2b$, then $a^3 + 1 = b^3 + 1$. We see almost immediately that if $2a = 2b$ then $a = b$ so $a^3 + 1 = b^3 + 1$ as required. This shows G is one-to-one and, hence, invertible.

   (b) We graph G below on the left by plotting points in the default xy-plane by choosing different values for t. For instance, $t = 0$ corresponds to the point $(0^3+1, 2(0)) = (1,0)$, $t = 1$ corresponds to the point $(1^3+1, 2(1)) = (2,2)$, $t = -1$ corresponds to the point $((-1)^3+1, 2(-1)) = (0,-2)$, etc. Our graph appears to pass the Horizontal Line Test, confirming G is one-to-one. We obtain the graph of G^{-1} below on the right by reflecting the graph of G about the line $y = x$



□

In Example 5.1.2, we showed the functions G and g are invertible and graphed their inverses. While graphs are perfectly fine representations of functions, we have seen where they aren't the most accurate. Ideally, we would like to represent G^{-1} and g^{-1} in the same manner in which G and g are presented to us. The key to doing this is to recall that inverse functions take outputs back to their associated inputs.

Consider $G = \{(t^3+1, 2t) \,|\, t \text{ is a real number}\}$. As mentioned in Example 5.1.2, the ordered pairs which comprise G are in the form $(\text{input}, \text{output})$. Hence to find a compatible description for G^{-1}, we simply interchange the expressions in each of the coordinates to obtain $G^{-1} = \{(2t, t^3+1) \,|\, t \text{ is a real number}\}$.

The function g was defined in terms of a formula, so we would like to find a formula representation for g^{-1}. We apply the same logic as above. Here, the input, represented by the independent variable t, and the output, represented by the dependent variable y, are related by the equation $y = g(t)$. Hence, to exchange inputs and outputs, we interchange the 't' and 'y' variables. Doing so, we obtain the equation $t = g(y)$ which is an *implicit* description for g^{-1}. Solving for y gives an explicit formula for g^{-1}, namely $y = g^{-1}(t)$. We demonstrate this technique below.

$$\begin{array}{rcll}
y & = & g(t) & \\
y & = & \dfrac{2t}{1-t} & \\
t & = & \dfrac{2y}{1-y} & \text{interchange variables: } t \text{ and } y \\
t(1-y) & = & 2y & \\
t - ty & = & 2y & \\
t & = & ty + 2y & \\
t & = & y(t+2) & \text{factor} \\
y & = & \dfrac{t}{t+2} &
\end{array}$$

We claim $g^{-1}(t) = \frac{t}{t+2}$, and leave the algebraic verification of this to the reader.

We generalize this approach below. As always, we resort to the default 'x' and 'y' labels for the independent and dependent variables, respectively.

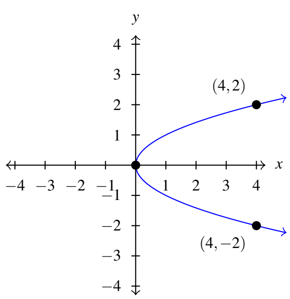
Steps for finding a formula for the Inverse of a One-to-One function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

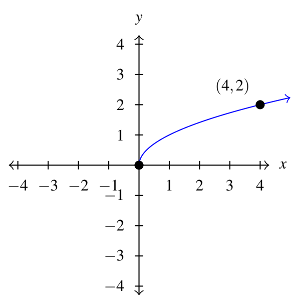
We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible, but our goal here is to see what goes wrong algebraically.

If we attempt to follow the algorithm above to find a formula for $f^{-1}(x)$, we start with the equation $y = x^2$ and interchange the variables 'x' and 'y' to produce the equation $x = y^2$. Solving for y gives $y = \pm\sqrt{x}$. It's this '$\pm$' which is causing the problem for us as this produces *two* y-values for any $x > 0$.

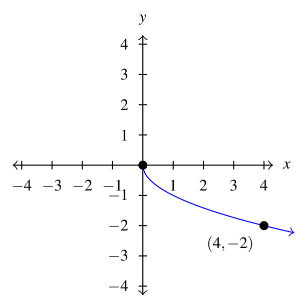
Using the language of Section 1.2, the equation $x = y^2$ implicitly defines *two* functions, $g_1(x) = \sqrt{x}$ and $g_2(x) = -\sqrt{x}$, each of which represents the top and bottom halves, respectively, of the graph of $x = y^2$.



$x = y^2$

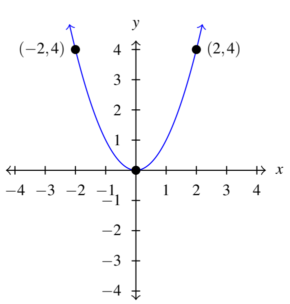


$y = g_1(x) = \sqrt{x}$

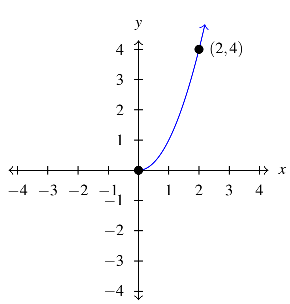


$y = g_2(x) = -\sqrt{x}$

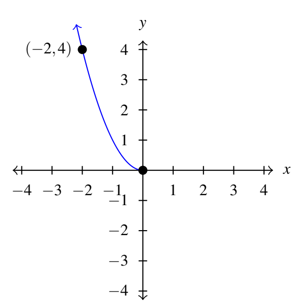
Hence, in some sense, we have two *partial* inverses for $f(x) = x^2$: $g_1(x) = \sqrt{x}$ returns the *positive* inputs from f and $g_2(x) = -\sqrt{x}$ returns the *negative* inputs to f. In order to view each of these functions as strict inverses, however, we need to split f into two parts: $f_1(x) = x^2$ for $x \geq 0$ and $f_2(x) = x^2$ for $x \leq 0$.



$y = f(x) = x^2$



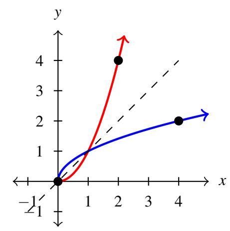
$y = f_1(x) = x^2, x \geq 0$



$y = f_2(x) = x^2, x \leq 0$

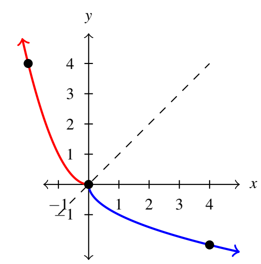
We claim that f_1 and g_1 are an inverse function pair as are f_2 and g_2. Indeed, we find:

$$\begin{aligned}(g_1 \circ f_1)(x) &= g_1(f_1(x)) \\ &= g_1(x^2) \\ &= \sqrt{x^2} \\ &= |x| = x, \text{ as } x \geq 0.\end{aligned} \qquad \begin{aligned}(f_1 \circ g_1)(x) &= f_1(g_1(x)) \\ &= f_1(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$



$= f_1(x) = x^2, x \geq 0$ and $y = g_1(x) = \sqrt{x}$

$$\begin{aligned}(g_2 \circ f_2)(x) &= g_2(f_2(x)) \\ &= g_2(x^2) \\ &= -\sqrt{x^2} \\ &= -|x| \\ &= -(-x) = x, \text{ as } x \leq 0.\end{aligned} \qquad \begin{aligned}(f_2 \circ g_2)(x) &= f_2(g_2(x)) \\ &= f_2(-\sqrt{x}) \\ &= (-\sqrt{x})^2 \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$



$= f_2(x) = x^2, x \leq 0$ and $y = g_2(x) = \sqrt{x}$

Hence, by restricting the domain of f we are able to produce invertible functions. Said differently, because the equation $x = y^2$ implicitly describes a pair of functions, the equation $y = x^2$ implicitly describes a pair of *invertible* functions.

Our next example continues the theme of restricting the domain of a function to find inverse functions.

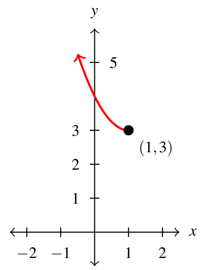
Example 5.1.3. Graph the following functions to show they are one-to-one and determine their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4, \; x \leq 1.$
2. $k(t) = \sqrt{t+2} - 1$

Solution.

1. Graph $j(x) = x^2 - 2x + 4, \; x \leq 1$ to show it is one-to-one and determine its inverse.

   The function j is a restriction of the function f from Example 5.1.2. The domain of j is restricted to $x \leq 1$, therefore we are selecting only the 'left half' of the parabola. Hence, the graph of j, seen below, passes the Horizontal Line Test and thus j is invertible.



$y = j(x)$ for $x \leq 1$

Next, we find an explicit formula for $j^{-1}(x)$ using our standard algorithm.[5]

$$
\begin{array}{rcll}
y & = & j(x) & \\
y & = & x^2 - 2x + 4, \;\; x \leq 1 & \\
x & = & y^2 - 2y + 4, \;\; y \leq 1 & \text{switch } x \text{ and } y \\
0 & = & y^2 - 2y + 4 - x & \\
y & = & \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c = 4 - x \\
y & = & \dfrac{2 \pm \sqrt{4x - 12}}{2} & \\
y & = & \dfrac{2 \pm \sqrt{4(x-3)}}{2} & \\
y & = & \dfrac{2 \pm 2\sqrt{x-3}}{2} & \\
y & = & \dfrac{2\left(1 \pm \sqrt{x-3}\right)}{2} & \\
y & = & 1 \pm \sqrt{x-3} & \\
y & = & 1 - \sqrt{x-3} & \text{due to the fact that } y \leq 1.
\end{array}
$$

Hence, $j^{-1}(x) = 1 - \sqrt{x-3}$.

To check our answer algebraically, we simplify $(j^{-1} \circ j)(x)$ and $(j \circ j^{-1})(x)$ Note the importance of the domain restriction $x \leq 1$ when simplifying $(j^{-1} \circ j)(x)$.

$$
\begin{array}{rcl}
\left(j^{-1} \circ j\right)(x) & = & j^{-1}(j(x)) \\
 & = & j^{-1}\left(x^2 - 2x + 4\right), \;\; x \leq 1 \\
 & = & 1 - \sqrt{(x^2 - 2x + 4) - 3} \\
 & = & 1 - \sqrt{x^2 - 2x + 1}
\end{array}
$$

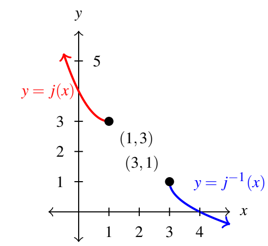
$$
\begin{array}{rcl}
\left(j \circ j^{-1}\right)(x) & = & j\left(j^{-1}(x)\right) \\
 & = & j\left(1 - \sqrt{x-3}\right) \\
 & = & \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
 & = & 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 \\
 & & +2\sqrt{x-3} + 4
\end{array}
$$

[5]Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

$$\begin{aligned} &= 1-\sqrt{(x-1)^2} \\ &= 1-|x-1| \\ &= 1-(-(x-1)) \text{ as } x \leq 1 \\ &= x \checkmark \end{aligned}$$

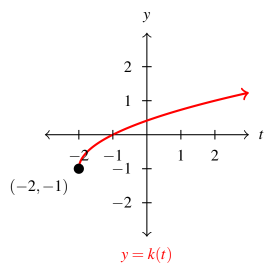
$$\begin{aligned} &= 1+x-3-2+4 \\ &= x \checkmark \end{aligned}$$

We graph both j and j^{-1} on the axes below. They appear to be symmetric about the line $y = x$.



2. Graph $k(t) = \sqrt{t+2} - 1$ to show it is one-to-one and determine its inverse.

   Graphing $y = k(t) = \sqrt{t+2} - 1$, we see k is one-to-one, so we proceed to find an formula for k^{-1}.



$$\begin{aligned} y &= k(t) \\ y &= \sqrt{t+2}-1 \\ t &= \sqrt{y+2}-1 \quad \text{switch } t \text{ and } y \\ t+1 &= \sqrt{y+2} \\ (t+1)^2 &= \left(\sqrt{y+2}\right)^2 \\ t^2+2t+1 &= y+2 \\ y &= t^2+2t-1 \end{aligned}$$

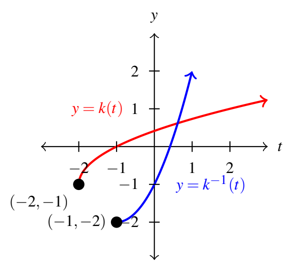
We have $k^{-1}(t) = t^2 + 2t - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted.

Theorem 5.1 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $t \geq -1$.

We now check that this works in our compositions. Note the importance of the domain restriction, $t \geq -1$ when simplifying $(k \circ k^{-1})(t)$.

$$\begin{aligned}
\left(k^{-1} \circ k\right)(t) &= k^{-1}(k(t)) \\
&= k^{-1}\left(\sqrt{t+2}-1\right) \\
&= \left(\sqrt{t+2}-1\right)^2+2\left(\sqrt{t+2}-1\right)-1 \\
&= \left(\sqrt{t+2}\right)^2-2\sqrt{t+2}+1 \\
&\quad +2\sqrt{t+2}-2-1 \\
&= t+2-2 \\
&= t \checkmark
\end{aligned}
\qquad
\begin{aligned}
\left(k \circ k^{-1}\right)(t) &= k\left(t^2+2t-1\right),\ t \geq -1 \\
&= \sqrt{(t^2+2t-1)+2}-1 \\
&= \sqrt{t^2+2t+1}-1 \\
&= \sqrt{(t+1)^2}-1 \\
&= |t+1|-1 \\
&= t+1-1, \text{ as } t \geq -1 \\
&= t \checkmark
\end{aligned}$$

Graphically, everything checks out, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions. Recall in Example 1.3.9 in Section 1.3.1, we modeled the demand for PortaBoy game systems as the price per system, $p(x)$ as a function of the number of systems sold, x. In the following example, we find $p^{-1}(x)$ and interpret what it means.

Example 5.1.4. Recall the price-demand function for PortaBoy game systems is modeled by the formula $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$ where x represents the number of systems sold (the demand) and $p(x)$ is the price per system, in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Compute and interpret $p^{-1}(220)$.
3. Recall from Section 2.1 that the profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Write and interpret $\left(P \circ p^{-1}\right)(x)$.
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 2.1.3.

Solution.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.

   Recall the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0,250)$ to $(166,1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We determine the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1,250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. Compute and interpret $p^{-1}(220)$.

   We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. The function p took as inputs the number of systems sold and returned the price per system as the output, thus p^{-1} takes the price per system as its input and returns the number of systems sold as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold if the price is set at \$220 per system.

3. Write and interpret $\left(P \circ p^{-1}\right)(x)$.

   We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[6] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$.

   To understand what this means, recall that the original profit function P gave us the profit as a function of the number of systems sold. The function p^{-1} gives us the number of systems sold as a function of the price. Hence, when we compute $(P \circ p^{-1})(x) = P(p^{-1}(x))$, we input a price per system, x into the function p^{-1}.

   The number $p^{-1}(x)$ is the number of systems sold at that price. This number is then fed into P to return the profit obtained by selling $p^{-1}(x)$ systems. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the profit (in dollars) as a function of the price per system, x.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 2.1.3.

   We know from Section 2.1 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. We are concerned only with the price per system, so we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 2.1, $x = -\frac{b}{2a} = 165$.

   Hence, the weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 2.1.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

5.1.1 Exercises

In Exercises 1 - 8, verify the given pairs of functions are inverses algebraically and graphically.

[6] It is good review to actually do this!

1. $f(x) = 2x+7$ and $g(x) = \dfrac{x-7}{2}$
2. $f(x) = \dfrac{5-3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.
3. $f(t) = \dfrac{5}{t-1}$ and $g(t) = \dfrac{t+5}{t}$
4. $f(t) = \dfrac{t}{t-1}$ and $g(t) = f(t) = \dfrac{t}{t-1}$
5. $f(x) = \sqrt{4-x}$ and $g(x) = -x^2+4$, $x \geq 0$
6. $f(x) = 1-\sqrt{x+1}$ and $g(x) = x^2-2x$, $x \leq 1$.
7. $f(t) = (t-1)^3+5$ and $g(t) = \sqrt[3]{t-5}+1$
8. $f(t) = -\sqrt[4]{t-2}$ and $g(t) = t^4+2$, $t \leq 0$.

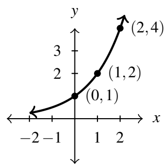
In Exercises 9 - 28, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify the range of the function is the domain of its inverse and vice-versa.

9. $f(x) = 6x-2$
10. $f(x) = 42-x$
11. $g(t) = \dfrac{t-2}{3}+4$
12. $g(t) = 1-\dfrac{4+3t}{5}$
13. $f(x) = \sqrt{3x-1}+5$
14. $f(x) = 2-\sqrt{x-5}$
15. $g(t) = 3\sqrt{t-1}-4$
16. $g(t) = 1-2\sqrt{2t+5}$
17. $f(x) = \sqrt[5]{3x-1}$
18. $f(x) = 3-\sqrt[3]{x-2}$
19. $g(t) = t^2-10t$, $t \geq 5$
20. $g(t) = 3(t+4)^2-5$, $t \leq -4$
21. $f(x) = x^2-6x+5$, $x \leq 3$
22. $f(x) = 4x^2+4x+1$, $x < -1$
23. $g(t) = \dfrac{3}{4-t}$
24. $g(t) = \dfrac{t}{1-3t}$
25. $f(x) = \dfrac{2x-1}{3x+4}$
26. $f(x) = \dfrac{4x+2}{3x-6}$
27. $g(t) = \dfrac{-3t-2}{t+3}$
28. $g(t) = \dfrac{t-2}{2t-1}$
29. Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.
    (a) $F = \{(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3)\}$
    (b) $G = \{(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3),\ldots\}$
    NOTE: The difference between F and G is the '. . . .'
    (c) $P = \{(2t^5,3t-1) \,|\, t \text{ is a real number}\}$
    (d) $Q = \{(n,n^2) \,|\, n \text{ is a } \textit{natural} \text{ number}\}$[7]

In Exercises 30 - 33, explain why each graph represents[8] a one-to-one function and graph its inverse.

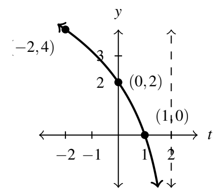
[7]Recall this means $n = 0,1,2,\ldots$.
[8]or, more precisely, *appears* to represent . . .

30. $y = f(x)$



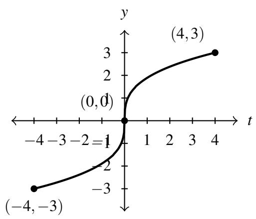
Asymptotes at $y = 0$

31. $y = g(t)$



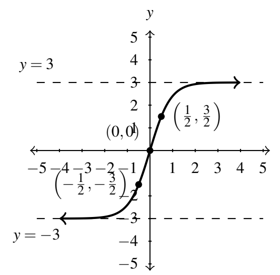
Asymptotes at $t = 2$

32. $y = S(t)$



Domain: $[-4,4]$

33. $y = R(s)$



Asymptotes at $y = \pm 3$

34. The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

    (a) Find $p^{-1}(x)$ and state its domain.

    (b) Compute and interpret $p^{-1}(105)$.

    (c) The profit (in dollars) made from producing and selling x dOpis per week is given by the formula $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

35. Show that the Fahrenheit to Celsius conversion function found in Exercise 67 in Section 1.3.3 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

36. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Use Theorem 5.1 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$. What happens when you attempt to find a formula for $f^{-1}(x)$?

37. Let $f(x) = \dfrac{2x}{x^2 - 1}$.

    (a) Graph $y = f(x)$ using the techniques in Section 3.3. Check your answer using a graphing utility.

    (b) Verify that f is one-to-one on the interval $(-1, 1)$.

    (c) Use the procedure outlined on Page 470 to find the formula for $f^{-1}(x)$ for $-1 < x < 1$.

(d) Because $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

38. The function given in number 4 is an example of a function which is its own inverse.

(a) Algebraically verify every function of the form: $f(x) = \dfrac{ax+b}{cx-a}$ is its own inverse.

What assumptions do you need to make about the values of a, b, and c?

(b) Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

Section 5.1 Exercise Answers A.1.5

5.2 Properties and Graphs of Exponential Functions

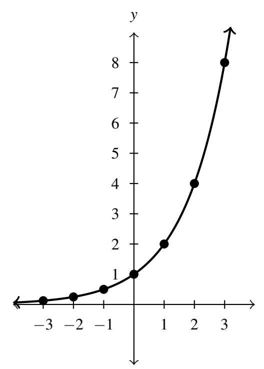
Of all of the functions we study in this text, exponential functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties.

Up to this point, we have dealt with functions which involve terms like x^3, $x^{\frac{3}{2}}$, or x^{π} - in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant.

In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with the time-honored classic, $f(x) = 2^x$.

We make a table of function values, plot enough points until we are more or less confident with the shape of the curve, and connect the dots in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0,1)$
1	$2^1 = 2$	$(1,2)$
2	$2^2 = 4$	$(2,4)$
3	$2^3 = 8$	$(3,8)$



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ are in order. As $x \to -\infty$ and takes on values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$.

In other words, as $x \to -\infty$, $2^x \approx \frac{1}{\text{very big } (+)} \approx$ very small $(+)$ That is, as $x \to -\infty$, $2^x \to 0^+$. This produces the x-axis, $y = 0$ as a horizontal asymptote to the graph as $x \to -\infty$.

On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$.

We note that by 'connecting the dots in a pleasing fashion,' we are implicitly using the fact that $f(x) = 2^x$ is not only defined for all real numbers,[1] but is also *continuous*. Moreover, we are assuming $f(x) = 2^x$ is increasing: that is, if $a < b$, then $2^a < 2^b$. While these facts are true, the proofs of these properties are best left to Calculus. For us, we assume these properties in order to state the domain of f is $(-\infty, \infty)$, the range of f is $(0, \infty)$ and, f is increasing, do f is one-to-one, hence invertible.

[1] See the discussion of real number exponents in Section 4.2.

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ because $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator,[2] then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$.

What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. As we know everything about this function, we ignore this case.

The only other base we exclude is $b = 1$, because the function $f(x) = 1^x = 1$ for all real numbers x. We are now ready for our definition of exponential functions.

Definition 5.3. An **exponential function** is the function of the form

$$f(x) = b^x$$

where b is a real number, $b > 0$, $b \neq 1$. The domain of an exponential function $(-\infty, \infty)$.

NOTE: More specifically, $f(x) = b^x$ is called the '*base b exponential function*.'

We leave it to the reader to verify[3] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$.

What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Per Section 1.6, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis.

$$\xrightarrow[\text{multiply each } x\text{-coordinate by } -1]{\text{reflect across } y\text{-axis}}$$

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is

[2] or, as we defined real number exponents in Section 4.2, if x is an irrational number ...

[3] Meaning, graph some more examples on your own.

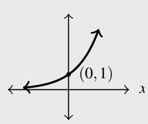
also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g.

We summarize the basic properties of exponential functions in the following theorem.

Theorem 5.3. Properties of Exponential Functions: Suppose $f(x) = b^x$.

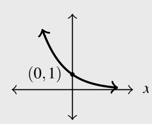
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0,1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = f(x) = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$= f(x) = b^x$, $0 < b <$

[a] Recall that this means the graph of f has no sharp turns or corners.

Exponential functions also inherit the basic properties of exponents from Theorem 4.3. We formalize these below and use them as needed in the coming examples.

Theorem 5.4. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $b^{u+w} = b^u b^w$
- **Quotient Rule:** $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(b^u)^w = b^{uw}$

In addition to base 2 which is important to computer scientists,[4] two other bases are used more often than not in scientific and economic circles. The first is base 10. Base 10 is called the '**common base**' and is important in the study of intensity (sound intensity, earthquake intensity, acidity, etc.)

The second base is an irrational number, e. Like $\sqrt{2}$ or π, the decimal expansion of e neither terminates nor repeats, so we represent this number by the letter 'e.' A decimal approximation of e is $e \approx 2.718$, so the function $f(x) = e^x$ is an increasing exponential function.

The number e is called the '**natural base**' for lots of reasons, one of which is that it 'naturally' arises in the study of growth functions in Calculus. We will more formally discuss the origins of e in Section 5.7.

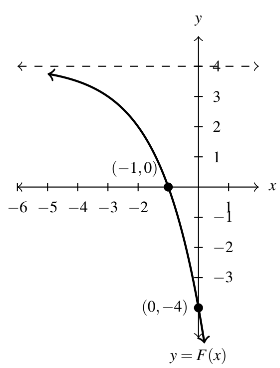
It is time for an example.

Example 5.2.1.

1. Graph the following functions by starting with a basic exponential function and using transformations, Theorem 1.12. Track at least three points and the horizontal asymptote through the transformations.

   (a) $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ (b) $G(t) = 2 - e^{-t}$

2. Write a formula for the graph of the function below. Assume the base of the exponential is 2.



Solution.

1. (a) Graph $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$.

   The base of the exponent in $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ is $\frac{1}{3}$, so we start with the graph of $f(x) = \left(\frac{1}{3}\right)^x$.

   To use Theorem 1.12, we first need to choose some 'control points' on the graph of $f(x) = \left(\frac{1}{3}\right)^x$. Because we are instructed to track three points (and the horizontal asymptote, $y = 0$) through

[4] The digital world is comprised of bytes which take on one of *two* values: 0 or 'off' and 1 or 'on.'

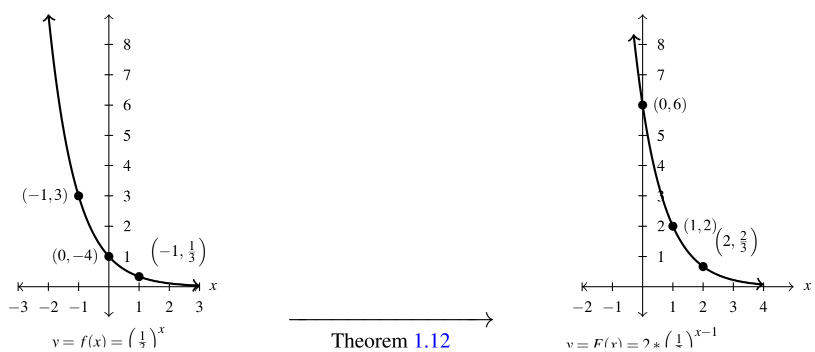
the transformations, we choose the points corresponding to $x=-1$, $x=0$, and $x=1$: $(-1,3)$, $(0,1)$, and $\left(1,\frac{1}{3}\right)$, respectively.

Next, we need determine how to modify $f(x)=\left(\frac{1}{3}\right)^x$ to obtain $F(x)=2\left(\frac{1}{3}\right)^{x-1}$. The key is to recognize the argument, or 'inside' of the function is the exponent and the 'outside' is anything outside the base of $\frac{1}{3}$. Using these principles as a guide, we find $F(x)=2f(x-1)$.

Per Theorem 1.12, we first add 1 to the x-coordinates of the points on the graph of $y=f(x)$, shifting the graph to the right 1 unit. Next, multiply the y-coordinates of each point on this new graph by 2, vertically stretching the graph by a factor of 2.

Looking point by point, we have $(-1,3)\to(0,3)\to(0,6)$, $(0,1)\to(1,1)\to(1,2)$, and $\left(1,\frac{1}{3}\right)\to\left(2,\frac{1}{3}\right)\to\left(2,\frac{2}{3}\right)$. The horizontal asymptote, $y=0$ remains unchanged under the horizontal shift and the vertical stretch because $2\cdot 0=0$.

Below we graph $y=f(x)=\left(\frac{1}{3}\right)^x$ on the left $y=F(x)=2\left(\frac{1}{3}\right)^{x-1}$ on the right.



As always we can check our answer by verifying each of the points $(0,6)$, $(1,2)$, $\left(2,\frac{2}{3}\right)$ is on the graph of $F(x)=2\left(\frac{1}{3}\right)^{x-1}$ by checking $F(0)=6$, $F(1)=2$, and $F(2)=\frac{2}{3}$.

We can check the end behavior as well, that is, as $x\to-\infty$, $F(x)\to\infty$ and as $x\to\infty$, $F(x)\to 0$. We leave these calculations to the reader.

(b) Graph $G(t)=2-e^{-t}$.

The base of the exponential in $G(t)=2-e^{-t}$ is e, so we start with the graph of $g(t)=e^t$.

Note that as e is an irrational number, we will use the approximation $e\approx 2.718$ when *plotting* points. However, when it comes to tracking and labeling said points, we do so with *exact* coordinates, that is, in terms of e.

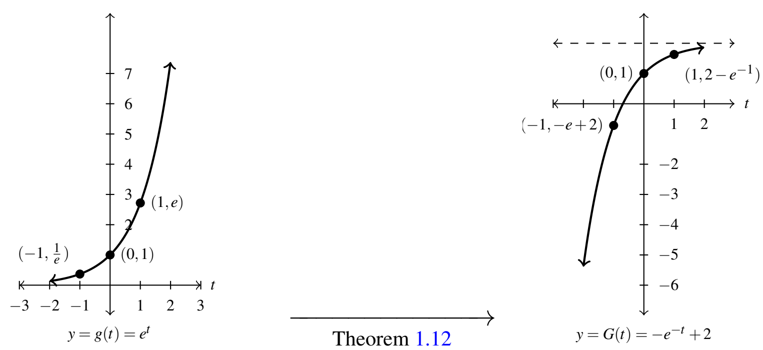
We choose points corresponding to $t=-1$, $t=0$, and $t=1$: $(-1,e^{-1})\approx(-1,0.368)$, $(0,1)$, and $(1,e)\approx(1,2.718)$, respectively.

Next, we need to determine how the formula for $G(t)=2-e^{-t}$ can be obtained from the formula $g(t)=e^t$. Rewriting $G(t)=-e^{-t}+2$, we find $G(t)=-g(-t)+2$.

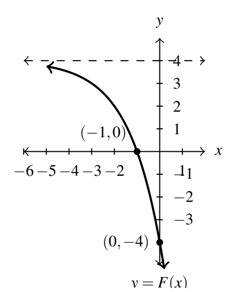
Following Theorem 1.12, we first multiply the t-coordinates of the graph of $y = g(t)$ by -1, effecting a reflection across the y-axis. Next, we multiply each of the y-coordinates by -1 which reflects the graph about the t-axis. Finally, we add 2 to each of the y-coordinates of the graph from the second step which shifts the graph up 2 units.

Tracking points, we have $(-1,e^{-1}) \to (1,e^{-1}) \to (1,-e^{-1}) \to (1,-e^{-1}+2) \approx (1,1.632)$, $(0,1) \to (0,1) \to (0,-1) \to (0,1)$, and $(1,e) \to (-1,e) \to (-1,-e) \to (-1,-e+2) \approx (-1,-0.718)$. The horizontal asymptote is unchanged by the reflections, but is shifted up 2 units $y = 0 \to y = 2$.

We graph $g(t) = e^t$ below on the left and the transformed function $G(t) = -e^{-t} + 2$ below on the right. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking end behavior. We leave these details to the reader.



2. Write a formula for the graph of the function below. Assume the base of the exponential is 2.



We are told to assume the base of the exponential function is 2, thus we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = 2^x$ using Theorem 1.12. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx-h)+k = a \cdot 2^{bx-h} + k$.

Because the horizontal asymptote to the graph of $y = f(x) = 2^x$ is $y = 0$ and the horizontal asymptote to the graph $y = F(x)$ is $y = 4$, we know the vertical shift is 4 units up, so $k = 4$.

Next, looking at how the graph of F approaches the vertical asymptote, it stands to reason the graph of $f(x) = 2^x$ undergoes a reflection across x-axis, meaning $a < 0$. For simplicity, we assume $a = -1$ and see if we can find values for b and h that go along with this choice.

Because $(-1,0)$ and $(0,-4)$ on the graph of $F(x) = -(2)^{bx-h} + 4$, we know $F(-1) = 0$ and $F(0) = -4$. From $F(-1) = 0$, we have $-(2)^{-b-h} + 4 = 0$ or $2^{-b-h} = 4 = 2^2$. Hence, $-b-h = 2$ is one solution.[5]

Next, using $F(0) = -4$, we get $-(2)^{-h} + 4 = -4$ or $2^{-h} = 8 = 2^3$. From this, we have $-h = 3$ so $h = -3$. Putting this together with $-b-h = 2$, we get $-b+3 = 2$ so $b = 1$.

Hence, one solution to the problem is $F(x) = -(2)^{x+3} + 4$. To check our answer, we leave it to the reader verify $F(-1) = 0$, $F(0) = -4$, as $x \to -\infty$, $F(x) \to 4$ and as $x \to \infty$, $F(x) \to -\infty$.

Because we made a simplifying assumption ($a = -1$), we may well wonder if our solution is the *only* solution. Indeed, we started with what amounts to three pieces of information and set out to determine the value of four constants. We leave this for a thoughtful discussion in Exercise 14. □

Our next example showcases an important application of exponential functions: economic depreciation.

Example 5.2.2. The value of a car can be modeled by $V(t) = 25(0.8)^t$, where $t \geq 0$ is number of years the car is owned and $V(t)$ is the value in thousands of dollars.

1. Calculate and interpret $V(0)$, $V(1)$, and $V(2)$.
2. Compute and interpret the average rate of change of V over the intervals $[0,1]$ and $[0,2]$ and $[1,2]$.
3. Determine and interpret $\dfrac{V(1)}{V(0)}$, $\dfrac{V(2)}{V(1)}$ and $\dfrac{V(2)}{V(0)}$.
4. For $t \geq 0$, find and interpret $\dfrac{V(t+1)}{V(t)}$ and $\dfrac{V(t+k)}{V(t)}$.
5. Compute and interpret $\dfrac{V(1)-V(0)}{V(0)}$, $\dfrac{V(2)-V(1)}{V(1)}$, and $\dfrac{V(2)-V(0)}{V(0)}$.
6. For $t \geq 0$, find and interpret $\dfrac{V(t+1)-V(t)}{V(t)}$ and $\dfrac{V(t+k)-V(t)}{V(t)}$.
7. Graph $y = V(t)$ starting with the graph of $y = V(t)$ and using transformations.
8. Interpret the horizontal asymptote of the graph of $y = V(t)$.
9. Using technology and your graph, determine how long it takes for the car to depreciate to (a) one half its original value and (b) one quarter of its original value. Round your answers to the nearest hundredth.

[5] This is the *only* solution. $f(x) = 2^x$, so the equation $2^{-b-h} = 2^2$ is equivalent to the functional equation $f(-b-h) = f(2)$. f is one-to-one, so we know this is true *only* when $-b-h = 2$.

Solution.

1. Calculate and interpret $V(0)$, $V(1)$, and $V(2)$.

   We find $V(0) = 25(0.8)^0 = 25 \cdot 1 = 25$, $V(1) = 25(0.8)^1 = 25 \cdot 0.8 = 20$ and $V(2) = 25(0.8)^2 = 25 \cdot 0.64 = 16$. t represents the number of years the car has been owned, so $t = 0$ corresponds to the purchase price of the car. $V(t)$ returns the value of the car in *thousands* of dollars, so $V(0) = 25$ means the car is worth \$25,000 when first purchased. Likewise, $V(1) = 20$ and $V(2) = 16$ means the car is worth \$20,000 after one year of ownership and \$16,000 after two years, respectively.

2. Compute and interpret the average rate of change of V over the intervals $[0,1]$ and $[0,2]$ and $[1,2]$.

   Recall to find the average rate of change of V over an interval $[a,b]$, we compute: $\frac{V(b)-V(a)}{b-a}$. For the interval $[0,1]$, we find $\frac{V(1)-V(0)}{1-0} = \frac{20-25}{1} = -5$, which means over the course of the first year of ownership, the value of the car depreciated, on average, at a rate of \$5000 per year.

   For the interval $[0,1]$, we compute $\frac{V(2)-V(0)}{2-0} = \frac{16-25}{2} = -4.5$, which means over the course of the first two years of ownership, the car lost, on average, \$4500 per year in value.

   Finally, we find for the interval $[1,2]$, $\frac{V(2)-V(1)}{2-1} = \frac{16-20}{1} = -4$, meaning the car lost, on average, \$4000 in value per year between the first and second years.

   Notice that the car lost more value over the first year (\$5000) than it did the second year (\$4000), and these losses average out to the average yearly loss over the first two years (\$4500 per year.)[6]

3. Determine and interpret $\dfrac{V(1)}{V(0)}$, $\dfrac{V(2)}{V(1)}$ and $\dfrac{V(2)}{V(0)}$.

   We compute: $\frac{V(1)}{V(0)} = \frac{20}{25} = 0.8$, $\frac{V(2)}{V(1)} = \frac{16}{20} = 0.8$, and $\frac{V(2)}{V(0)} = \frac{16}{25} = 0.64$.

   The ratio $\frac{V(1)}{V(0)} = 0.8$ can be rewritten as $V(1) = 0.8V(0)$ which means that the value of the car after 1 year, $V(1)$ is 0.8 times, or 80% the initial value of the car, $V(0)$.

   Similarly, the ratio $\frac{V(2)}{V(1)} = 0.8$ rewritten as $V(2) = 0.8V(1)$ means the value of the car after 2 years, $V(2)$ is 0.8 times, or 80% the value of the car after one year, $V(1)$.

   Finally, the ratio $\frac{V(2)}{V(0)} = 0.64$, or $V(2) = 0.64V(0)$ means the value of the car after 2 years, $V(2)$ is 0.64 times, or 64% of the initial value of the car, $V(0)$.

   Note that this last result tracks with the previous answers. Because $V(1) = 0.8V(0)$ and $V(2) = 0.8V(1)$, we get $V(2) = 0.8V(1) = 0.8(0.8V(0)) = 0.64V(0)$. Also note it is no coincidence that the base of the exponential, 0.8 has shown up in these calculations, as we'll see in the next problem.

[6]It turns out for any function f, the average rate of change over the interval $[x, x+2]$ is the average of the average rates of change of f over $[x, x+1]$ and $[x+1, x+2]$. See Exercise 21.

4. For $t \geq 0$, find and interpret $\dfrac{V(t+1)}{V(t)}$ and $\dfrac{V(t+k)}{V(t)}$.

   Using properties of exponents, we find

$$\frac{V(t+1)}{V(t)} = \frac{25(0.8)^{t+1}}{25(0.8)^t} = (0.8)^{t+1-t} = 0.8$$

   Rewriting, we have $V(t+1) = 0.8V(t)$. This means after one year, the value of the car $V(t+1)$ is only 80% of the value it was a year ago, $V(t)$.

   Similarly, we find

$$\frac{V(t+k)}{V(t)} = \frac{25(0.8)^{t+k}}{25(0.8)^t} = (0.8)^{t+k-t} = (0.8)^k$$

   which, rewritten, says $V(t+k) = V(t)(0.8)^k$. This means in k years' time, the value of the car $V(t+k)$ is only $(0.8)^k$ times what it was worth k years ago, $V(t)$.

   These results shouldn't be too surprising. Verbally, the function $V(t) = 25(0.8)^t$ says to multiply 25 by 0.8 multiplied by itself t times. Therefore, for each additional year, we are multiplying the value of the car by an additional factor of 0.8.

5. Compute and interpret $\dfrac{V(1)-V(0)}{V(0)}$, $\dfrac{V(2)-V(1)}{V(1)}$, and $\dfrac{V(2)-V(0)}{V(0)}$.

   We compute $\frac{V(1)-V(0)}{V(0)} = \frac{20-25}{25} = -0.2$, $\frac{V(2)-V(1)}{V(1)} = \frac{16-20}{20} = -0.2$, and $\frac{V(2)-V(0)}{V(0)} = \frac{16-25}{25} = -0.36$.

   The ratio $\frac{V(1)-V(0)}{V(0)}$ computes the ratio of *difference* in the value of the car after the first year of ownership, $V(1)-V(0)$, to the initial value, $V(0)$. We find this to be -0.2 or a 20% decrease in value. This makes sense as we know from our answer to number 3, the value of the car after 1 year, $V(1)$ is 80% of the initial value, $V(0)$. Indeed:

$$\frac{V(1)-V(0)}{V(0)} = \frac{V(1)}{V(0)} - \frac{V(0)}{V(0)} = \frac{V(1)}{V(0)} - 1,$$

   and because $\frac{V(1)}{V(0)} = 0.8$, we get $\frac{V(1)-V(0)}{V(0)} = 0.8 - 1 = -0.2$.

   Likewise, the ratio $\frac{V(2)-V(1)}{V(1)} = -0.2$ means the value of the car has lost 20% of its value over the course of the second year of ownership.

   Finally, the ratio $\frac{V(2)-V(0)}{V(0)} = -0.36$ means that over the first two years of ownership, the car value has depreciated 36% of its initial purchase price. Again, this tracks with the result of number 3 which tells us that after two years, the car is only worth 64% of its initial purchase price.

6. For $t \geq 0$, find and interpret $\dfrac{V(t+1)-V(t)}{V(t)}$ and $\dfrac{V(t+k)-V(t)}{V(t)}$.

Using properties of fractions and exponents, we get:

$$\frac{V(t+1)-V(t)}{V(t)} = \frac{25(0.8)^{t+1}-25(0.8)^{t}}{25(0.8)^{t}} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} - \frac{25(0.8)^{t}}{25(0.8)^{t}} = 0.8-1 = -0.2,$$

so after one year, the value of the car $V(t+1)$ has lost 20% of the value it was a year ago, $V(t)$.

Similarly, we find:

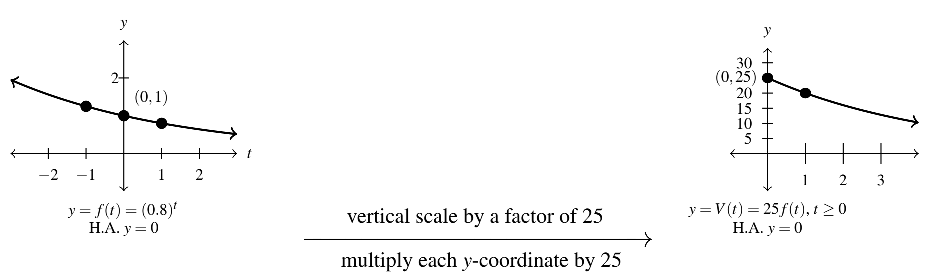
$$\frac{V(t+k)-V(t)}{V(t)} = \frac{25(0.8)^{t+k}-25(0.8)^{t}}{25(0.8)^{t}} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} - \frac{25(0.8)^{t}}{25(0.8)^{t}} = (0.8)^{k}-1,$$

so after k years' time, the value of the car $V(t)$ has decreased by $((0.8)^k - 1) \cdot 100\%$ of the value k years ago, $V(t)$.

7. Graph $y = V(t)$ starting with the graph of $y = V(t)$ and using transformations.

   To graph $y = 25(0.8)^t$, we start with the basic exponential function $f(t) = (0.8)^t$. The base $b = 0.8$ satisfies $0 < b < 1$, therefore the graph of $y = f(t)$ is decreasing. We plot the y-intercept $(0,1)$ and two other points, $(-1, 1.25)$ and $(1, 0.8)$, and label the horizontal asymptote $y = 0$.

   To obtain the graph of $y = 25(0.8)^t = 25f(t)$, we multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) in accordance with Theorem 1.10 to obtain the points $(-1, 31.25)$, $(0,25)$ and $(1,20)$. The horizontal asymptote remains the same, $(25 \cdot 0 = 0.)$ Finally, we restrict the domain to $[0,\infty)$ to fit with the applied domain given to us.



8. Interpret the horizontal asymptote of the graph of $y = V(t)$.

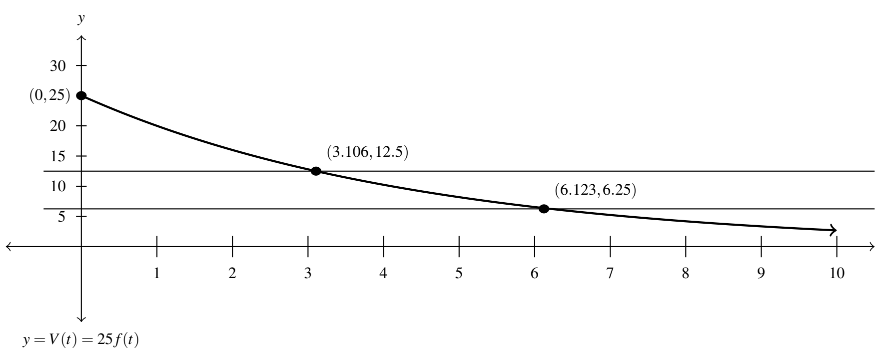
   We see from the graph of V that its horizontal asymptote is $y = 0$. This means as the car gets older, its value diminishes to 0.

9. Using technology and your graph, determine how long it takes for the car to depreciate to (a) one half its original value and (b) one quarter of its original value. Round your answers to the nearest hundredth.

   We know the value of the car, brand new, is \$25,000, so when we are asked to find when the car depreciates to one half and one quarter of this value, we are trying to find when the value of the car dips to \$12,500 and \$6,125, respectively. $V(t)$ is measured in *thousands* of dollars, so we this translates to solving the equations $V(t) = 12.5$ and $V(t) = 6.125$.

Because we have yet to develop any analytic means to solve equations like $25(0.8)^t = 12.5$ (remember t is in the exponent here), we are forced to approximate solutions to this equation numerically or use a graphing utility. Choosing the latter, we graph $y = V(t)$ along with the lines $y = 12.5$ and $y = 6.125$ and look for intersection points.

We find $y = V(t)$ and $y = 12.5$ intersect at (approximately) $(3.10612.5)$ which means the car depreciates to half its initial value in (approximately) 3.11 years. Similarly, we find the car depreciates to one-quarter its initial value after (approximately) 6.23 years.[7]



□

Some remarks about Example 5.2.2 are in order. First the function in the previous example is called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 5.7.

Second, as seen in numbers 3 and 4, $V(t+1) = 0.8V(t)$. That is to say, the function V has a *constant unit multiplier*, in this case, 0.8 because to obtain the function value $V(t+1)$, we *multiply* the function value $V(t)$ by b. It is not coincidence that the multiplier here is the base of the exponential, 0.8.

Indeed, exponential functions of the form $f(x) = a \cdot b^x$ have a constant unit multiplier, b. To see this, note

$$\frac{f(x+1)}{f(x)} = \frac{a \cdot b^{x+1}}{a \cdot b^x} = b^1 = b.$$

Hence $f(x+1) = f(x) \cdot b$. This will prove useful to us in Section 5.7 when making decisions about whether or not a data set represents exponential growth or decay.

[7] It turns out that it takes exactly twice as long for the car to depreciate to one-quarter of its initial value as it takes to depreciate to half its initial value. Can you see why?

We close this section with another important application of exponential functions, Newton's Law of Cooling.

Example 5.2.3. According to Newton's Law of Cooling[8] the temperature of coffee $T(t)$ (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Compute and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Determine and interpret the behavior of $T(t)$ as $t \to \infty$.

Solution.

1. Compute and interpret $T(0)$.

   $T(0) = 70 + 90e^{-0.1(0)} = 160$, thus the temperature of the coffee when it is served is 160°F.

2. Sketch the graph of $y = T(t)$ using transformations.

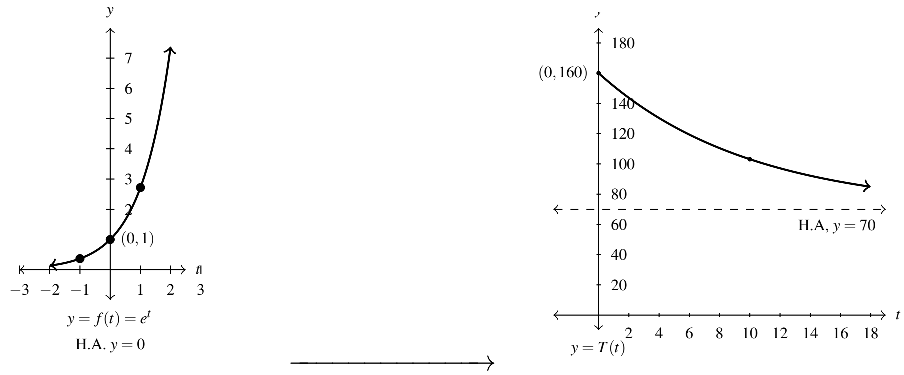
   To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As in Example 5.2.1, we track the points $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, along with the horizontal asymptote $y = 0$ through each of transformations.

   To use Theorem 1.12, we rewrite $T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Following Theorem 1.12, we first divide the t-coordinates of each point on the graph of $y = f(t)$ by -0.1 which results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis.

   Next, we multiply the y-values of the points on this new graph by 90 which effects a vertical stretch by a factor of 90. Last but not least, we add 70 to all of the y-coordinates of the points on this second graph, which shifts the graph upwards 70 units.

   Tracking points, we have $(-1, e^{-1}) \to (10, e^{-1}) \to (10, 90e^{-1}) \to (10, 90e^{-1} + 70) \approx (10, 103.112)$, $(0, 1) \to (0, 1) \to (0, 90) \to (0, 160)$, and $(1, e) \to (-10, e) \to (-10, 90e) \to (-10, 90e + 70) \approx (-10, 314.62)$. The horizontal asymptote $y = 0$ is unaffected by the horizontal expansion, reflection about the y-axis, and the vertical stretch, but the vertical shift moves the horizontal asymptote up 70 units, $y = 0 \to y = 70$. After restricting the domain to $t \geq 0$, we get the graph on the right.

[8]We will discuss this in greater detail in Section 5.7.



Theorem 1.12

3. Determine and interpret the behavior of $T(t)$ as $t \to \infty$.

   We can determine the behavior of $T(t)$ as $t \to \infty$ two ways. First, we can employ the 'number sense' developed in Chapter 3.

   That is, as $t \to \infty$, we get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. As $e > 1$, $e^{\text{very big }(-)} \approx \text{very small }(+)$. The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx \text{very small }(+)$. Hence, $T(t) = 70 + 90e^{-0.1t} \approx 70 + \text{very small }(+) \approx 70$.

   Alternatively, we can look to the graph of $y = T(t)$. We know the horizontal asymptote is $y = 70$ which means as $t \to \infty$, $T(t) \approx 70$.

   In either case, we find that as time goes by, the temperature of the coffee is cooling to 70° Fahrenheit, ostensibly room temperature. $\square$

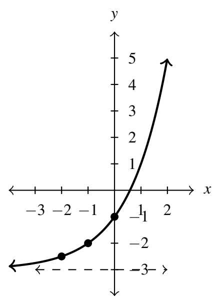
5.2.1 Exercises

In Exercises 1 - 8, sketch the graph of g by starting with the graph of f and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

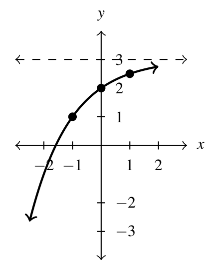
1. $f(x) = 2^x$, $g(x) = 2^x - 1$
2. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$
3. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$
4. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$
5. $f(t) = (0.5)^t$, $g(t) = 100(0.5)^{0.1t}$
6. $f(t) = (1.25)^t$, $g(t) = 1 - (1.25)^{t-2}$
7. $f(x) = e^t$, $g(x) = 8 - e^{-t}$
8. $f(x) = e^t$, $g(x) = 10e^{-0.1t}$

In Exercises, 9 - 12, the graph of an exponential function is given. Find a formula for the function in the form $F(x) = a \cdot 2^{bx-h} + k$.

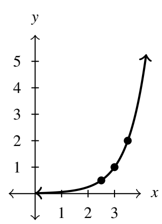
9. Points: $\left(-2,-\frac{5}{2}\right)$, $(-1,-2)$, $(0,-1)$,
Asymptote: $y=-3$.



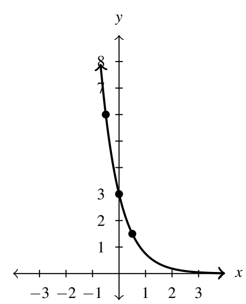
10. Points: $(-1,1)$, $(0,2)$, $\left(1,\frac{5}{2}\right)$,
Asymptote: $y=3$.



11. Points: $\left(\frac{5}{2},\frac{1}{2}\right)$, $(3,1)$, $\left(\frac{7}{2},2\right)$,
Asymptote: $y=0$.



12. Points: $\left(-\frac{1}{2},6\right)$, $(0,3)$, $\left(\frac{1}{2},\frac{3}{2}\right)$,
Asymptote: $y=0$.



13. Find a formula for each graph in Exercises 9 - 12 of the form $G(x)=a\cdot 4^{bx-h}+k$. Did you change your solution methodology? What is the relationship between your answers for $F(x)$ and $G(x)$ for each graph?

14. In Example 5.2.1 number 2, we obtained the solution $F(x)=-2^{x+3}+4$ as one formula for the given graph by making a simplifying assumption that $a=-1$. This exercises explores if there are any other solutions for different choices of a.

    (a) Show $G(x)=-4\cdot 2^{x+1}+4$ also fits the data for the given graph, and use properties of exponents to show $G(x)=F(x)$. (Use the fact that $4=2^2$ ...)

    (b) With help from your classmates, find solutions to Example 5.2.1 number 2 using $a=-8$, $a=-16$ and $a=-\frac{1}{2}$. Show all your solutions can be rewritten as: $F(x)=-2^{x+3}+4$.

    (c) Using properties of exponents and the fact that the range of 2^x is $(0,\infty)$, show that any function of the form $f(x)=-a\cdot 2^{bx-h}+k$ for $a>0$ can be rewritten as $f(x)=-2^c\,2^{bx-h}+k=-2^{bx-h+c}+k$. Relabeling, this means every function of the form $f(x)=-a\cdot 2^{bx-h}+k$ with four parameters (a, b, h, and k) can be rewritten as $f(x)=-2^{bx-H}+k$, a formula with just three parameters: b, H, and k. Conclude that *every* solution to Example 5.2.1 number 2 reduces to $F(x)=-2^{x+3}+4$.

In Exercises 15 - 20, write the given function as a nontrivial decomposition of functions as directed.

15. For $f(x) = e^{-x} + 1$, find functions g and h so that $f = g + h$.

16. For $f(x) = e^{2x} - x$, find functions g and h so that $f = g - h$.

17. For $f(t) = t^2 e^{-t}$, find functions g and h so that $f = gh$.

18. For $r(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, find functions f and g so $r = \dfrac{f}{g}$.

19. For $k(x) = e^{-x^2}$, find functions f and g so that $k = g \circ f$.

20. For $s(x) = \sqrt{e^{2x} - 1}$, find functions f and g so $s = g \circ f$.

21. Show that the average rate of change of a function over the interval $[x, x+2]$ is average of the average rates of change of the function over the intervals $[x, x+1]$ and $[x+1, x+2]$. Can the same be said for the average rate of change of the function over $[x, x+3]$ and the average of the average rates of change over $[x, x+1]$, $[x+1, x+2]$, and $[x+2, x+3]$? Generalize.

22. Which is larger: e^{π} or π^{e}? How do you know? Can you find a proof that doesn't use technology?

SECTION 5.2 EXERCISE ANSWERS A.1.5

5.3 Properties and Graphs of Logarithmic Functions

In Section 5.2, we saw exponential functions $f(x) = b^x$ are one-to-one which means they are invertible. In this section, we explore their inverses, the *logarithmic functions* which are called 'logs' for short.

Definition 5.4. For the exponential function $f(x) = b^x$, $f^{-1}(x) = \log_b(x)$ is called the **base b logarithm function**. We read '$\log_b(x)$' as 'log base b of x.'

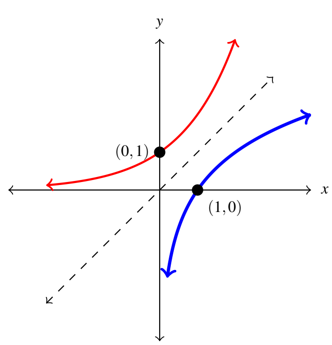
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 5.5.

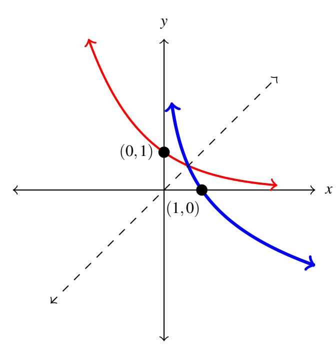
- The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$.
- The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

As logs are defined as the inverses of exponential functions, we can use Theorems 5.1 and 5.3 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0,\infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty,\infty)$.

Moreover, we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, thus we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$. The y-intercept $(0,1)$ on the graph of f corresponds to an x-intercept of $(1,0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$
$y = \log_b(x)$, $b > 1$



$y = b^x$, $0 < b < 1$
$y = \log_b(x)$, $0 < b < 1$

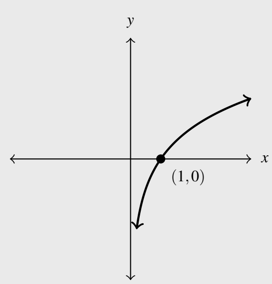
Procedurally, logarithmic functions 'undo' the exponential functions. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$.

More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 5.5. Properties of Logarithmic Functions: Suppose $f(x) = \log_b(x)$, for $b > 0$, and $b \neq 1$.

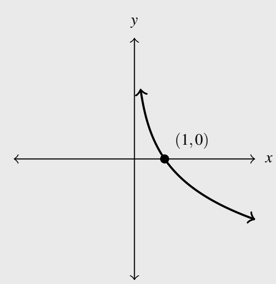
- The domain of f is $(0,\infty)$ and the range of f is $(-\infty,\infty)$.
- $(1,0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b(b^x) = x$ for all real numbers x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 5.5 is a consequence of Theorems 5.1 and 5.3. However, it is worth the reader's time to understand Theorem 5.5 from an exponent perspective.

As an example, we know that the domain of $g(x) = \log_2(x)$ is $(0,\infty)$. Why? Because the range of $f(x) = 2^x$ is $(0,\infty)$. In a way, this says everything, but at the same time, it doesn't.

To really *understand* why the domain of $g(x) = \log_2(x)$ is $(0, \infty)$, consider trying to compute $\log_2(-1)$. We are searching for the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, because all powers of 2 are positive.

While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 5.5 at this level instead of just merely memorizing the facts.

Our first example gives us practice computing logarithms as well as constructing basic graphs.

Example 5.3.1.

1. Simplify the following.

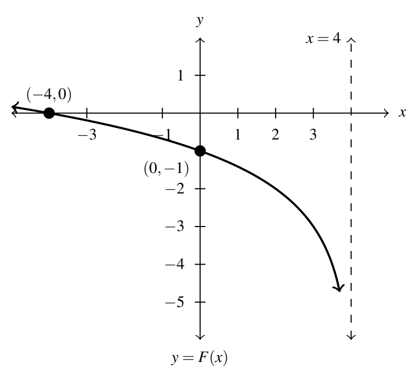
   (a) $\log_3(81)$ (b) $\log_2\left(\dfrac{1}{8}\right)$ (c) $\log_{\sqrt{5}}(25)$ (d) $\ln\left(\sqrt[3]{e^2}\right)$

   (a) $\log(0.001)$ (b) $2^{\log_2(8)}$ (c) $117^{-\log_{117}(6)}$

2. Graph the following functions by starting with a basic logarithmic function and using transformations, Theorem 1.12. Track at least three points and the vertical asymptote through the transformations.

   (a) $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ (b) $G(t) = -\ln(2-t)$

3. Write a formula for the graph of the function below. Assume the base of the logarithm is 2.



Solution.

1. (a) Simplify $\log_3(81)$.

   The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.

(b) Simplify $\log_2\left(\dfrac{1}{8}\right)$.

To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

(c) Simplify $\log_{\sqrt{5}}(25)$.

To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.

(d) Simplify $\ln\left(\sqrt[3]{e^2}\right)$.

First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.

(e) Simplify $\log(0.001)$.

Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.

(f) Simplify $2^{\log_2(8)}$.

We can use Theorem 5.5 directly to simplify $2^{\log_2(8)} = 8$.

We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Because $8 = 2^3$, we have $\log_2(8) = 3$.

We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.

(g) Simplify $117^{-\log_{117}(6)}$.

From Theorem 5.5, we know $117^{\log_{117}(6)} = 6$,[1] but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$ without first using a property of exponents. (Can you see why?)

Rather, we find: $117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$.

2. (a) Graph $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$.

To graph $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ we start with the graph of $f(x) = \log_{\frac{1}{3}}(x)$. and use Theorem 1.12.

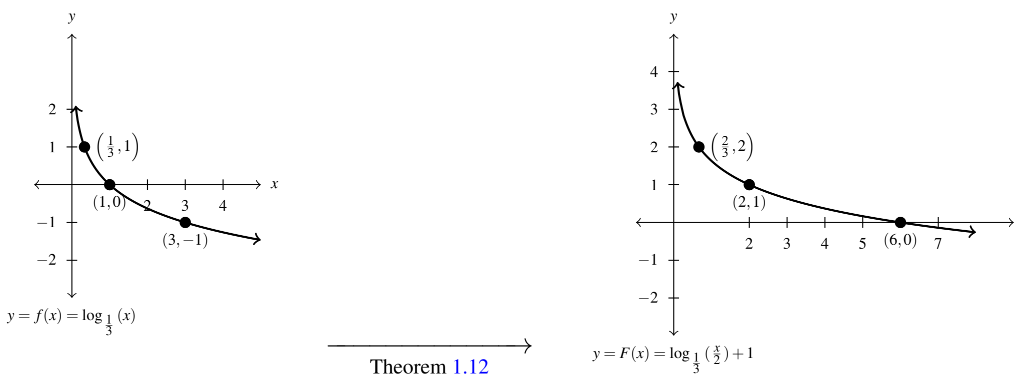
First we choose some 'control points' on the graph of $f(x) = \log_{\frac{1}{3}}(x)$. We are instructed to track three points (and the vertical asymptote, therefore $x = 0$) through the transformations, we choose the points corresponding to powers of $\frac{1}{3}$: $\left(\frac{1}{3}, 1\right)$, $(1, 0)$, and $(3, -1)$, respectively.

[1] It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117, so we get 6.

Next, we note $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1 = f\left(\frac{x}{2}\right) + 1$. Per Theorem 1.12, we first multiply the x-coordinates of the points on the graph of $y = f(x)$ by 2, horizontally expanding the graph by a factor of 2. Next, we add 1 to the y-coordinates of each point on this new graph, vertically shifting the graph up 1.

Looking at each point, we get $\left(\frac{1}{3}, 1\right) \to \left(\frac{2}{3}, 1\right) \to \left(\frac{2}{3}, 2\right)$, $(1,0) \to (2,0) \to (2,1)$, and $(3,-1) \to (6,-1) \to (6,0)$. The horizontal asymptote, $x = 0$ remains unchanged under the horizontal stretch and the vertical shift.

Below we graph $y = f(x) = \log_{\frac{1}{3}}(x)$ on the left and $y = F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ on the right.



As always we can check our answer by verifying each of the points $\left(\frac{2}{3}, 2\right)$, $(2,1)$ and $(6,0)$, is on the graph of $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ by checking $F\left(\frac{2}{3}\right) = 2$, $F(2) = 1$, and $F(6) = 0$. We can check the end behavior as well, that is, as $x \to 0^+$, $F(x) \to \infty$ and as $x \to \infty$, $F(x) \to -\infty$. We leave these calculations to the reader.

(b) Graph $G(t) = -\ln(2-t)$.

Due to the fact that the base of $G(t) = -\ln(2-t)$ is e, we start with the graph of $g(t) = \ln(t)$. e is an irrational number, so we use the approximation $e \approx 2.718$ when plotting points, but label points using exact coordinates in terms of e.

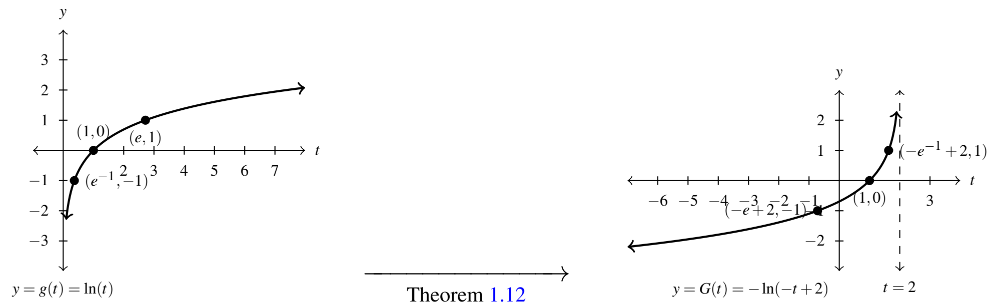
We choose points corresponding to powers of e on the graph of $g(t) = \ln(t)$: $(e^{-1}, -1) \approx (0.368, -1)$, $(1,0)$, and $(e,1) \approx (2.718, 1)$, respectively.

$G(t) = -\ln(2-t) = -\ln(-t+2) = -g(-t+2)$, so Theorem 1.12 instructs us to first subtract 2 from each of the t-coordinates of the points on the graph of $g(t) = \ln(t)$, shifting the graph to the left two units.

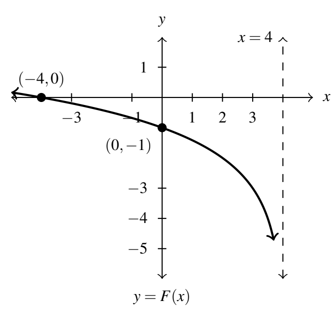
Next, we multiply (divide) the t-coordinates of points on this new graph by -1 which reflects the graph across the y-axis. Lastly, we multiply each of the y-coordinates of this second graph by -1, reflecting it across the t-axis.

Tracking points, we have $(e^{-1},-1) \to (e^{-1}-2,-1) \to (-e^{-1}+2,-1) \to (-e^{-1}+2,1) \approx (1.632,1)$, $(1,0) \to (-1,0) \to (1,0) \to (1,0)$, and $(e,1) \to (e-2,1) \to (-e+2,1) \to (-e+2,-1) \approx (-0.718,-1)$. The vertical asymptote is affected by the horizontal shift and the reflection about the y-axis only: $t=0 \to t=-2 \to t=2$.

We graph $g(t)=\ln(t)$ below on the left and the transformed function $G(t)=-\ln(-t+2)$ below on the right. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y=G(t)$ along with checking the behavior as $t \to -\infty$ and $t \to 2^{-}$.



3. Write a formula for $y=F(x)$, assume the base of the logarithm is 2.



We are told to assume the base of the exponential function is 2. We assume then the function $F(x)$ is the result of the transforming the graph of $f(x)=\log_2(x)$ using Theorem 1.12. This means we are tasked with finding values for a, b, h, and k so that $F(x)=af(bx-h)+k=a\log_2(bx-h)+k$.

Because the vertical asymptote to the graph of $y=f(x)=\log_2(x)$ is $x=0$ and the vertical asymptote to the graph $y=F(x)$ is $x=4$, we know we have a horizontal shift of 4 units. Moreover, the curve approaches the vertical asymptote from the *left*, so we also know we have a reflection about the y-axis and $b<0$ (this is not your base, but instead the coefficient of x.) The recipe in Theorem 1.12 instructs us to perform the horizontal shift *before* the reflection across the y-axis, so we take $h=-4$ and assume for simplicity $b=-1$ so $F(x)=a\log_2(-x+4)+k$.

To determine a and k, we make use of the two points on the graph. $(-4,0)$ is on the graph of F, so $F(-4)=a\log_2(-(-4)+4)+k=0$. This reduces to $a\log_2(8)+k=0$ or $3a+k=0$. Next, we use the point $(0,-1)$ to get $F(0)=a\log_2(-(0)+4)+k=-1$. This reduces to $a\log_2(4)+k=-1$

or $2a+k=-1$. From $3a+k=0$, we get $k=-3a$ which when substituted into $2a+k=-1$ gives $2a+(-3a)=-1$ or $a=1$. Hence, $k=-3a=-3(1)=-3$.

Putting all of this work together we find $F(x)=\log_2(-x+4)-3$. As always, we can check our answer by verifying $F(-4)=0$, $F(0)=-1$, $F(x)\to\infty$ as $x\to-\infty$ and $F(x)\to-\infty$ as $x\to 4^-$. We leave these details to the reader.[2] $\square$

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even indexed radicals. With the introduction of logarithms, we now have another restriction. The argument of the logarithm[3] must be strictly positive, because the domain of $f(x)=\log_b(x)$ is $(0,\infty)$.

Example 5.3.2. State the domain each function analytically and check your answer using a graph.

1. $f(x)=2\log(3-x)-1$
2. $g(x)=\ln\left(\dfrac{x}{x-1}\right)$

Solution.

1. State the domain of $f(x)=2\log(3-x)-1$.

   We set $3-x>0$ to obtain $x<3$, or $(-\infty,3)$.

   To verify our domain, we graph f using transformations. Taking a cue from Theorem 1.12, we rewrite $f(x)=2\log_{10}(-x+3)-1$ and view this function as a transformed version of $h(x)=\log_{10}(x)$.

   To graph $y=\log(x)=\log_{10}(x)$, We select three points to track corresponding to powers of 10: $(0.1,-1)$, $(1,0)$ and $(10,1)$, along with the vertical asymptote $x=0$.

   As $f(x)=2h(-x+3)-1$, Theorem 1.12 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis).

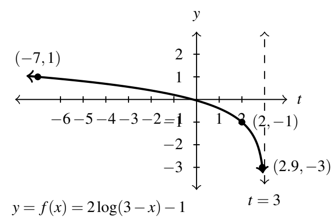
   Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit.

   Tracking points, we find: $(0.1,-1)\to(-2.9,-1)\to(2.9,-1)\to(2.9,-2)\to(2.9,-3)$, $(1,0)\to(-2,0)\to(2,0)\to(2,0)\to(2,-1)$, and $(10,1)\to(7,1)\to(-7,1)\to(-7,2)\to(-7,1)$. The vertical shift and reflection about the y-axis affects the vertical asymptote: $x=0\to x=-3\to x=3$.

   Plotting these three points along with the vertical asymptote produces the graph of f.

[2] As with Exercise 5.2.1 in Section 5.2, we may well wonder if our solution to this problem is the *only* solution because we made a simplifying assumption that $b=-1$. We leave this for a thoughtful discussion in Exercise 40 in Section 5.4.

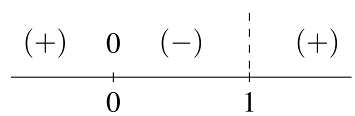
[3] that is, what's 'inside' the log



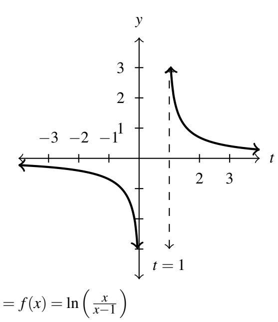
2. State the domain of $g(x) = \ln\left(\dfrac{x}{x-1}\right)$.

   To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$ using a sign diagram.[4]

   If we define $r(x) = \frac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below.



   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ which is the domain of g. The graph below confirms this.



   We can tell from the graph of g that it is not the result of Section 1.6 transformations being applied to the graph $y = \ln(x)$, (do you see why?) so barring a more detailed analysis using Calculus, producing a graph using a graphing utility is the best we could do, for now.

   One thing worthy of note, however, is the end behavior of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter 3 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$. $\square$

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example reviews not only the major topics of this section, but reviews the salient points from Section 5.1.

[4]See Section 3.3 for a review of this process, if needed.

Example 5.3.3. Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check for symmetry about the line $y = x$.
6. Use f or f^{-1} to solve the following equations. Check your answers algebraically.

   (a) $2^{x-1} - 3 = 4$ (b) $\log_2(t+3) + 1 = 0$

Solution.

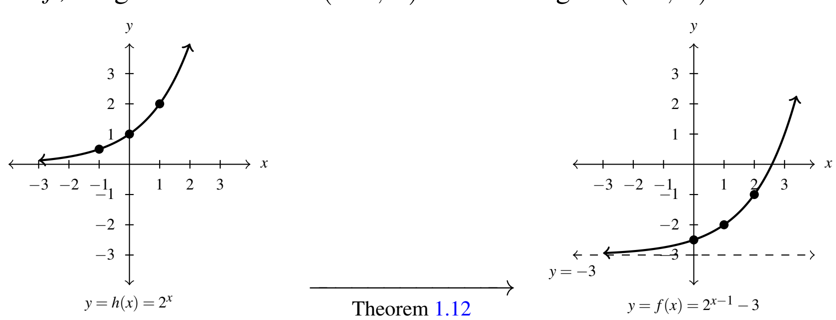
1. Graph f using transformations and state the domain and range of f.

   To graph $f(x) = 2^{x-1} - 3$ using Theorem 1.12, we first identify $g(x) = 2^x$ and note $f(x) = g(x-1) - 3$. Choosing the 'control points' of $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$, we implement the algorithm set forth in Theorem 1.12.

   First, we first add 1 to the x-coordinates of the points on the graph of g which shifts the the graph of g to the right one unit. Next, we subtract 3 from each of the y-coordinates on this new graph, shifting the graph down 3 units to get the graph of f.

   Looking point-by-point, we have $\left(-1, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \to \left(0, -\frac{5}{2}\right)$, $(0,1) \to (1,1) \to (1,-2)$, and, finally, $(1,2) \to (2,2) \to (2,-1)$. The horizontal asymptote is affected only by the vertical shift, $y = 0 \to y = -3$.

   From the graph of f, we get the domain is $(-\infty, \infty)$ and the range is $(-3, \infty)$.



2. Explain why f is invertible and find a formula for $f^{-1}(x)$.

   The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible.

   To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} procedurally.

   Thinking of f as a process, the formula $f(x) = 2^{x-1} - 3$ takes an input x and applies the steps: first subtract 1. Second put the result of the first step as the exponent on 2. Last, subtract 3 from the result of the second step.

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm.

   By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should: first, add 3. Second, take the logarithm base 2 of the result of the first step. Lastly, add 1 to the result of the second step. In symbols, $f^{-1}(x) = \log_2(x+3)+1$.

3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.

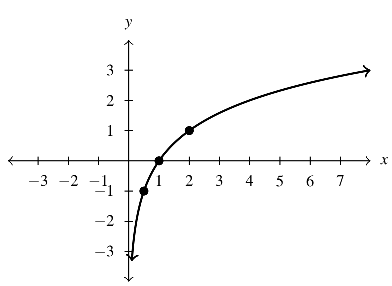
   To graph $f^{-1}(x) = \log_2(x+3)+1$ using Theorem 1.12, we start with $j(x) = \log_2(x)$ and track the points $\left(\frac{1}{2}, -1\right)$, $(1,0)$ and $(2,1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations.

   As $f^{-1}(x) = j(x+3)+1$, we first subtract 3 from each of the x-coordinates of each of the points on the graph of $y = j(x)$ shifting the graph of j to the left three units. We then add 1 to each of the y-coordinates of the points on this new graph, shifting the graph up one unit.

   Tracking points, we get $\left(\frac{1}{2}, -1\right) \to \left(-\frac{5}{2}, -1\right) \to \left(-\frac{5}{2}, 0\right)$, $(1,0) \to (-2,1) \to (-2,2)$, and $(2,1) \to (-1,1) \to (-1,2)$.

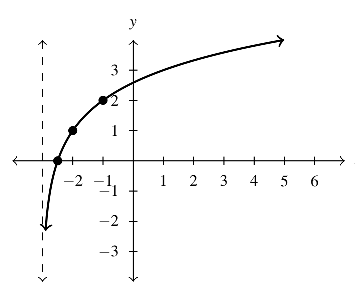
   The vertical asymptote is only affected by the horizontal shift, so we have $x = 0 \to x = -3$.

   From the graph below, we get the domain of f^{-1} is $(-3,\infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty,\infty)$, which matches the domain of f, in accordance with Theorem 5.1.



$y = j(x) = \log_2(x)$

Theorem 1.12



$x = -3$ $\quad$ $y = f^{-1}(x) = \log_2(x+3)+1$

4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

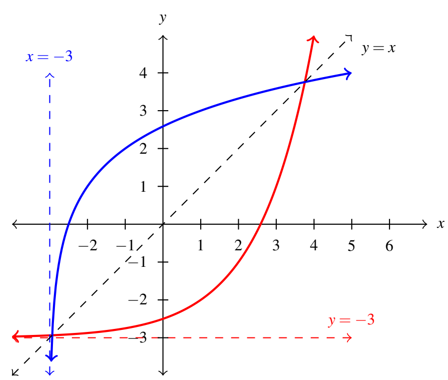
   We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3)+1$ satisfy the composition requirement for inverses. When simplifying $(f^{-1} \circ f)(x)$ we assume x can be any real number while when simplifying $(f \circ f^{-1})(x)$, we restrict our attention to $x > -3$. (Do you see why?)

   Note the use of the inverse properties of exponential and logarithmic functions from Theorem 5.5 when it comes to simplifying expressions of the form $\log_2(2^u)$ and $2^{\log_2(u)}$.

$$\begin{aligned} \left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(2^{x-1}-3\right) \\ &= \log_2\left(\left[2^{x-1}-3\right]+3\right)+1 \\ &= \log_2\left(2^{x-1}\right)+1 \\ &= (x-1)+1 \\ &= x \checkmark \end{aligned} \qquad \begin{aligned} \left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\ &= f\left(\log_2(x+3)+1\right) \\ &= 2^{(\log_2(x+3)+1)-1}-3 \\ &= 2^{\log_2(x+3)}-3 \\ &= (x+3)-3 \\ &= x \checkmark \end{aligned}$$

5. Graph f and f^{-1} on the same set of axes and check for symmetry about the line $y = x$.

   Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and observe the symmetry about the line $y = x$.



6. (a) Use f or f^{-1} to solve $2^{x-1} - 3 = 4$.

   Viewing $2^{x-1} - 3 = 4$ as $f(x) = 4$, we apply f^{-1} to 'undo' f to get $f^{-1}(f(x)) = f^{-1}(4)$, which reduces to $x = f^{-1}(4)$. We have shown (algebraically and graphically!) that $f^{-1}(x) = \log_2(x+3)+1$, we get $x = f^{-1}(4) = \log_2(4+3)+1 = \log_2(7)+1$.

   Alternatively, we know from Theorem 5.1 that $f(x) = 4$ is equivalent to $x = f^{-1}(4)$ directly.

   Note that, by definition, $2^{\log_2(7)} = 7$, thus $2^{(\log_2(7)+1)-1} - 3 = 2^{\log_2(7)} - 3 = 7 - 3 = 4$, as required.

(b) Use f or f^{-1} to solve $\log_2(t+3)+1=0$.

Because we may think of the equation $\log_2(t+3)+1=0$ as $f^{-1}(t)=0$, we can solve this equation by applying f to both sides to get $f(f^{-1}(t))=f(0)$ or $t=2^{0-1}-3=\frac{1}{2}-3=-\frac{5}{2}$.

As a result of $\log_2(2^{-1})=-1$, we get $\log_2\left(-\frac{5}{2}+3\right)+1=\log_2\left(\frac{1}{2}\right)+1=\log_2(2^{-1})-1+1=0$, as required. □

5.3.1 Exercises

In Exercises 1 - 15, use the property: $b^a=c$ if and only if $\log_b(c)=a$ from Theorem 5.5 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3=8$
2. $5^{-3}=\frac{1}{125}$
3. $4^{5/2}=32$
4. $\left(\frac{1}{3}\right)^{-2}=9$
5. $\left(\frac{4}{25}\right)^{-1/2}=\frac{5}{2}$
6. $10^{-3}=0.001$
7. $e^0=1$
8. $\log_5(25)=2$
9. $\log_{25}(5)=\frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right)=-4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right)=-1$
12. $\log(100)=2$
13. $\log(0.1)=-1$
14. $\ln(e)=1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right)=-\frac{1}{2}$

In Exercises 16 - 42, evaluate the expression without using a calculator.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$

37. $\log\left(\sqrt[3]{10^5}\right)$

38. $\ln\left(\frac{1}{\sqrt{e}}\right)$

39. $\log_5\left(3^{\log_3(5)}\right)$

40. $\log\left(e^{\ln(100)}\right)$

41. $\log_2\left(3^{-\log_3(2)}\right)$

42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 - 57, find the domain of the function.

43. $f(x) = \ln(x^2+1)$

44. $f(x) = \log_7(4x+8)$

45. $g(t) = \ln(4t-20)$

46. $g(t) = \log\left(t^2+9t+18\right)$

47. $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$

48. $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$

49. $g(t) = \ln(7-t)+\ln(t-4)$

50. $g(t) = \ln(4t-20)+\ln\left(t^2+9t+18\right)$

51. $f(x) = \log\left(x^2+x+1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $g(t) = \log_9(|t+3|-4)$

54. $g(t) = \ln(\sqrt{t-4}-3)$

55. $f(x) = \dfrac{1}{3-\log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3-x^2+13x-6)$

In Exercises 58 - 65, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

58. $f(x) = \log_2(x)$, $g(x) = \log_2(x+1)$

59. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x)+1$

60. $f(x) = \log_3(x)$, $g(x) = -\log_3(x-2)$

61. $f(x) = \log(x)$, $g(x) = 2\log(x+20)-1$

62. $g(t) = \log_{0.5}(t)$, $g(t) = 10\log_{0.5}\left(\frac{t}{100}\right)$

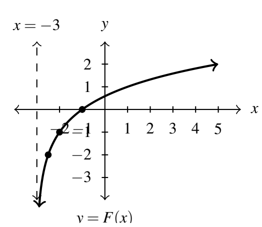
63. $g(t) = \log_{1.25}(t)$, $g(t) = \log_{1.25}(-t+1)+2$

64. $g(t) = \ln(t)$, $g(t) = -\ln(8-t)$

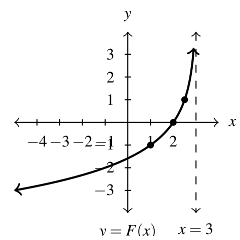
65. $g(t) = \ln(t)$, $g(t) = -10\ln\left(\frac{t}{10}\right)$

In Exercises, 66 - 69, the graph of a logarithmic function is given. Find a formula for the function in the form $F(x) = a\cdot\log_2(bx-h)+k$.

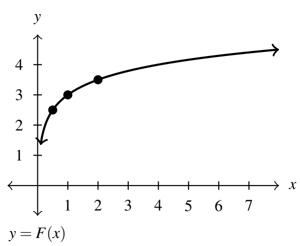
66. Points: $\left(-\frac{5}{2},-2\right)$, $(-2,-1)$, $(-1,0)$,
Asymptote: $x = -3$.



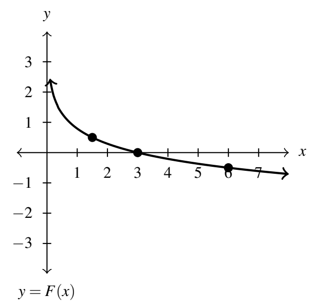
67. Points: $(1,-1)$, $(2,0)$, $\left(\frac{5}{2},1\right)$
Asymptote: $x=3$.



68. Points: $\left(\frac{1}{2},\frac{5}{2}\right)$, $(1,3)$, $\left(2,\frac{7}{2}\right)$,
Asymptote: $x=0$.



69. Points: $\left(6,-\frac{1}{2}\right)$, $(3,0)$, $\left(\frac{3}{2},\frac{1}{2}\right)$,
Asymptote: $x=0$.



70. Find a formula for each graph in Exercises 66 - 69 of the form $G(x)=a\cdot\log_4(bx-h)+k$.

In Exercises 71 - 74, find the inverse of the function from the 'procedural perspective' discussed in Example 5.3.3 and graph the function and its inverse on the same set of axes.

71. $f(x)=3^{x+2}-4$

72. $f(x)=\log_4(x-1)$

73. $g(t)=-2^{-t}+1$

74. $g(t)=5\log(t)-2$

In Exercises 75 - 80, write the given function as a nontrivial decomposition of functions as directed.

75. For $f(x)=\log_2(x+3)+4$, find functions g and h so that $f=g+h$.

76. For $f(x)=\log(2x)-e^{-x}$, find functions g and h so that $f=g-h$.

77. For $f(t)=3t\log(t)$, find functions g and h so that $f=gh$.

78. For $r(x)=\dfrac{\ln(x)}{x}$, find functions f and g so $r=\dfrac{f}{g}$.

79. For $k(t)=\ln(t^2+1)$, find functions f and g so that $k=g\circ f$.

80. For $p(z)=(\ln(z))^2$, find functions f and g so $p=g\circ f$.

(Logarithmic Scales) In Exercises 81 - 83, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

81. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[5] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80,000)$.

(c) Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.

(d) Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

82. While the decibel scale can be used in many disciplines,[6] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound. The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\,\frac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

83. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\mathrm{pH} = -\log[\mathrm{H}^+]$ where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

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[5] Rock-solid, perhaps?

[6] See this webpage for more information.

(a) The hydrogen ion concentration of pure water is $[H^+] = 10^{-7}$. Find its pH.

(b) Find the pH of a solution with $[H^+] = 6.3 \times 10^{-13}$.

(c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

84. Use the definition of logarithm to explain why $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0$, $b \neq 1$.

SECTION 5.3 EXERCISE ANSWERS A.1.5

5.4 Properties of Logarithms

In Section 5.3, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing.

As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Algebra. We first extract two properties from Theorem 5.5 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 5.6. (Inverse Properties of Exponential and Logarithmic Functions)
Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 5.7. (One-to-one Properties of Exponential and Logarithmic Functions) Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

Next, we re-state Theorem 5.4 for reference below.

(Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $b^{u+w} = b^u b^w$
- **Quotient Rule:** $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $\left(b^u\right)^w = b^{uw}$

To each of these properties of listed in Theorem 5.4, there corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 5.8. (Algebraic Properties of Logarithmic Functions) Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $\log_b(u^w) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 5.8 is true. For instance, consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$.

Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$.

By the one-to-one property of b^x, $b^a = b^{c+d}$ gives $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly.

From a purely functional approach, we can see the properties in Theorem 5.8 as an example of how inverse functions interchange the roles of inputs and outputs.

For instance, the Product Rule for exponential functions given in Theorem 5.4, $b^{u+w} = b^u b^w$, says that adding inputs results in multiplying outputs.

Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. The outputs from f are the inputs to f^{-1} and vice-versa, thus f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely one way to interpret the Product Rule for Logarithmic functions: $\log_b(uw) = \log_b(u) + \log_b(w)$.

The reader is encouraged to view the remaining properties listed in Theorem 5.8 similarly.

The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 5.8 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.[1]

Example 5.4.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

[1]Interestingly enough, it is the exact *opposite* process (which we will practice later) that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

1. $\log_2\left(\dfrac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\dfrac{3}{et}\right)^2$
4. $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2-4\right)$

Solution.

1. Expand and simplify $\log_2\left(\dfrac{8}{x}\right)$.

   To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u=8$ and $w=x$ and simplify.

$$\begin{aligned}
\log_2\left(\frac{8}{x}\right) &= \log_2(8)-\log_2(x) && \text{Quotient Rule}\\
&= 3-\log_2(x) && \text{Because } 2^3=8\\
&= -\log_2(x)+3
\end{aligned}$$

2. Expand and simplify $\log_{0.1}\left(10x^2\right)$.

   In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product, and the question becomes which property, Power Rule or Product Rule to use first.

   In order to use the Power Rule, the *entire* quantity inside the log must be raised to the same exponent. The exponent 2 applies only to the x, so we first apply the Product Rule with $u=10$ and $w=x^2$. Once the x^2 is by itself inside the log, we apply the Power Rule with $u=x$ and $w=2$.

$$\begin{aligned}
\log_{0.1}\left(10x^2\right) &= \log_{0.1}(10)+\log_{0.1}\left(x^2\right) && \text{Product Rule}\\
&= \log_{0.1}(10)+2\log_{0.1}(x) && \text{Power Rule}\\
&= -1+2\log_{0.1}(x) && \text{Because } (0.1)^{-1}=10\\
&= 2\log_{0.1}(x)-1
\end{aligned}$$

3. Expand and simplify $\ln\left(\dfrac{3}{et}\right)^2$.

   We have a power, quotient and product occurring in $\ln\left(\frac{3}{et}\right)^2$. This time the exponent 2 applies to the entire quantity inside the logarithm, so we begin with the Power Rule with $u=\frac{3}{et}$ and $w=2$.

   Next, we see the Quotient Rule is applicable, with $u=3$ and $w=et$, so we replace $\ln\left(\frac{3}{et}\right)$ with the quantity $\ln(3)-\ln(et)$.

   Because $\ln\left(\frac{3}{et}\right)$ is being multiplied by 2, the entire quantity $\ln(3)-\ln(et)$ is multiplied by 2.

Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(et)$ with the quantity $\ln(e) + \ln(t)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{aligned}
\ln\left(\frac{3}{et}\right)^2 &= 2\ln\left(\frac{3}{et}\right) && \text{Power Rule} \\
&= 2\left[\ln(3) - \ln(et)\right] && \text{Quotient Rule} \\
&= 2\ln(3) - 2\ln(et) && \\
&= 2\ln(3) - 2\left[\ln(e) + \ln(t)\right] && \text{Product Rule} \\
&= 2\ln(3) - 2\ln(e) - 2\ln(t) && \\
&= 2\ln(3) - 2 - 2\ln(t) && \text{Because } e^1 = e \\
&= -2\ln(t) + 2\ln(3) - 2 &&
\end{aligned}$$

4. Expand and simplify $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$.

   In Theorem 5.8, there is no mention of how to deal with radicals. However, thinking back to Definition 0.3, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule[2], and we keep in mind that the common log is log base 10.

$$\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3} && \\
&= \tfrac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule} \\
&= \tfrac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] && \text{Quotient Rule} \\
&= \tfrac{1}{3}\log\left(100x^2\right) - \tfrac{1}{3}\log\left(yz^5\right) && \\
&= \tfrac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \tfrac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] && \text{Product Rule} \\
&= \tfrac{1}{3}\log(100) + \tfrac{1}{3}\log\left(x^2\right) - \tfrac{1}{3}\log(y) - \tfrac{1}{3}\log\left(z^5\right) && \\
&= \tfrac{1}{3}\log(100) + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Power Rule} \\
&= \tfrac{2}{3} + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Because } 10^2 = 100 \\
&= \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) + \tfrac{2}{3} &&
\end{aligned}$$

5. Expand and simplify $\log_{117}\left(x^2 - 4\right)$.

[2] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, as none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2-4=(x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule. Assuming both $x+2>0$ and $x-2>0$, that is, $x>2$ we expand as follows.

$$\begin{array}{rcll} \log_{117}\left(x^2-4\right) & = & \log_{117}\left[(x+2)(x-2)\right] & \text{Factor} \\ & = & \log_{117}(x+2)+\log_{117}(x-2) & \text{Product Rule} \end{array}$$

□

A couple of remarks about Example 5.4.1 are in order. First, if we take a step back and look at each problem in the foregoing example, a general rule of thumb to determine which log property to apply first when faced with a multi-step problem is to apply the logarithm properties in the 'reverse order of operations.'

For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. The last property of logarithm to apply would the be the power rule applied to $\log_{0.1}(x^2)$.

Second, the equivalence $\log_{117}\left(x^2-4\right)=\log_{117}(x+2)+\log_{117}(x-2)$ is valid only if $x>2$. Indeed, the functions $f(x)=\log_{117}\left(x^2-4\right)$ and $g(x)=\log_{117}(x+2)+\log_{117}(x-2)$ have different domains, and, hence, are different functions.[3] In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so.

One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right)=\log_{117}\left(x^2\right)-\log_{117}(4)$, which simply isn't true, in general. The unwritten[4] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 5.4.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1)-\log_3(x+1)$
2. $\log(x)+2\log(y)-\log(z)$
3. $4\log_2(x)+3$
4. $-\ln(t)-\frac{1}{2}$

Solution. Whereas in Example 5.4.1 we read the properties in Theorem 5.8 from left to right to expand logarithms, in this example we read them from right to left.

1. Write $\log_3(x-1)-\log_3(x+1)$ as a single logarithm.

   The difference of logarithms requires the Quotient Rule: $\log_3(x-1)-\log_3(x+1)=\log_3\left(\frac{x-1}{x+1}\right)$.

[3]We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$.

[4]The authors relish the irony involved in writing what follows.

2. Write $\log(x)+2\log(y)-\log(z)$ as a single logarithm.

   In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms.

   Before we use the product rule to combine $\log(x)+2\log(y)$, we note that we need to apply the Power Rule to rewrite the coefficient 2 as the power on y. We then apply the Product and Quotient Rules as we move from left to right.

$$\begin{array}{rcll} \log(x)+2\log(y)-\log(z) & = & \log(x)+\log\left(y^2\right)-\log(z) & \text{Power Rule} \\ & = & \log\left(xy^2\right)-\log(z) & \text{Product Rule} \\ & = & \log\left(\dfrac{xy^2}{z}\right) & \text{Quotient Rule} \end{array}$$

3. Write $4\log_2(x)+3$ as a single logarithm.

   We begin to rewrite $4\log_2(x)+3$ by applying the Power Rule: $4\log_2(x)=\log_2\left(x^4\right)$.

   In order to continue, we need to rewrite 3 as a logarithm base 2. From Theorem 5.6, we know $3=\log_2\left(2^3\right)$. Rewriting 3 this way paves the way to use the Product Rule.

$$\begin{array}{rcll} 4\log_2(x)+3 & = & \log_2\left(x^4\right)+3 & \text{Power Rule} \\ & = & \log_2\left(x^4\right)+\log_2\left(2^3\right) & \text{Because } 3=\log_2\left(2^3\right) \\ & = & \log_2\left(x^4\right)+\log_2(8) & \\ & = & \log_2\left(8x^4\right) & \text{Product Rule} \end{array}$$

4. Write $-\ln(t)-\frac{1}{2}$ as a single logarithm.

   To get started with $-\ln(t)-\frac{1}{2}$, we rewrite $-\ln(t)$ as $(-1)\ln(t)$. We can then use the Power Rule to obtain $(-1)\ln(t)=\ln\left(t^{-1}\right)$.

   As in the previous problem, in order to continue, we need to rewrite $\frac{1}{2}$ as a natural logarithm. Theorem 5.6 gives us $\frac{1}{2}=\ln\left(e^{1/2}\right)=\ln(\sqrt{e})$. Hence,

$$\begin{array}{rcll} -\ln(t)-\frac{1}{2} & = & (-1)\ln(t)-\frac{1}{2} & \\ & = & \ln\left(t^{-1}\right)-\frac{1}{2} & \text{Power Rule} \\ & = & \ln\left(t^{-1}\right)-\ln\left(e^{1/2}\right) & \text{Because } \frac{1}{2}=\ln\left(e^{1/2}\right) \\ & = & \ln\left(t^{-1}\right)-\ln(\sqrt{e}) & \\ & = & \ln\left(\dfrac{t^{-1}}{\sqrt{e}}\right) & \text{Quotient Rule} \\ & = & \ln\left(\dfrac{1}{t\sqrt{e}}\right) & \end{array}$$

$\square$

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, to rewrite an expression as a single logarithm, we apply log properties following the usual order of operations: first, rewrite coefficients of logs as powers using the Power Rule, then rewrite addition and subtraction using the Product and Quotient Rules, respectively, as written from left to right.

Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1,\infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty,-1)\cup(1,\infty)$. We'll need to keep this in mind in Section 5.6 when such manipulations could result in extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 5.9. (Change of Base Formulas) Let $a,b>0$, $a,b\neq 1$.

- $a^x = b^{x\log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x>0$.

To prove these formulas, consider $b^{x\log_b(a)}$. Using the Power Rule, we can rewrite $x\log_b(a)$ as $\log_b\left(a^x\right)$. Following this with the Inverse Properties in Theorem 5.6, we get

$$b^{x\log_b(a)} = b^{\log_b(a^x)} = a^x.$$

To verify the logarithmic form of the property, we use the Power Rule and an Inverse Property to get:

$$\log_a(x)\cdot\log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x).$$

We get the result by dividing both sides of the equation $\log_a(x)\cdot\log_b(a) = \log_b(x)$ by $\log_b(a)$.

Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$.

While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra, while the exponential form isn't usually introduced until Calculus.

Example 5.4.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

1. Convert 3^2 to an equivalent expression base 10.

   We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$.

2. Convert 2^x to an equivalent expression base e.

   Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$.

3. Convert $\log_4(5)$ to an equivalent expression base e.

   Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this gives the numerical approximation $\frac{\ln(5)}{\ln(4)} \approx 1.16$.

4. Convert $\ln(x)$ to an equivalent expression base 10.

   We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$.

□

What Theorem 5.9 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base, be it 10, 0.42, π, or 117.

5.4.1 Exercises

In Exercises 1 - 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\dfrac{128}{x^2+4}\right)$
3. $\log_5\left(\dfrac{z}{25}\right)^3$
4. $\log(1.23 \times 10^{37})$
5. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2 - 25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}(9x(y^3 - 8))$
9. $\log\left(1000x^3y^5\right)$

10. $\log_3\left(\dfrac{x^2}{81y^4}\right)$

11. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$

12. $\log_6\left(\dfrac{216}{x^3y}\right)^4$

13. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$

14. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 - 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$

17. $\log_2(x)+\log_2(y)-\log_2(z)$

18. $\log_3(x)-2\log_3(y)$

19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$

20. $2\ln(x)-3\ln(y)-4\ln(z)$

21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$

22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$

23. $\log_5(x)-3$

24. $3-\log(x)$

25. $\log_7(x)+\log_7(x-3)-2$

26. $\ln(x)+\frac{1}{2}$

27. $\log_2(x)+\log_4(x)$

28. $\log_2(x)+\log_4(x-1)$

29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 - 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e

31. $\log_3(x+2)$ to base 10

32. $\left(\dfrac{2}{3}\right)^x$ to base e

33. $\log(x^2+1)$ to base e

In Exercises 34 - 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$

35. $\log_5(80)$

36. $\log_6(72)$

37. $\log_4\left(\dfrac{1}{10}\right)$

38. $\log_{\frac{3}{5}}(1000)$

39. $\log_{\frac{2}{3}}(50)$

40. In Example 5.3.1 number 3 in Section 5.3, we obtained the solution $F(x)=\log_2(-x+4)-3$ as one formula for the given graph by making a simplifying assumption that $b=-1$. This exercises explores if there are any other solutions for different choices of b.

    (a) Show $G(x)=\log_2(-2x+8)-4$ also fits the data for the given graph.

    (b) Use properties of logarithms to show $G(x)=\log_2(-2x+8)-4=\log_2(-x+4)-3=F(x)$.

(c) With help from your classmates, find solutions to Example 5.3.1 number 3 in Section 5.3 by assuming $b = -4$ and $b = -8$. In each case, use properties of logarithms to show the solutions reduce to $F(x) = \log_2(-x+4) - 3$.

(d) Using properties of logarithms and the fact that the range of $\log_2(x)$ is all real numbers, show that any function of the form $f(x) = a\log_2(bx-h)+k$ where $a \neq 0$ can be rewritten as:

$$f(x) = a\left(\log_2(bx-h) + \frac{k}{a}\right) = a(\log_2(bx-h)+\log_2(p)) = a\log_2(p(bx-h)) = a\log_2(pbx-ph),$$

where $\frac{k}{a} = \log_2(p)$ for some positive real number p. Relabeling, we get every function of the form $f(x) = a\log_2(bx-h)+k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = a\log_2(Bx-H)$, a formula with just three parameters: a, B, and H.

Show *every* solution to Example 5.3.1 number 3 in Section 5.3 can be written in the form $f(x) = \log_2\left(-\frac{1}{8}x+\frac{1}{2}\right)$ and that, in particular, $F(x) = \log_2(-x+4)-3 = \log_2\left(-\frac{1}{8}x+\frac{1}{2}\right) = f(x)$. Hence, there is really just one solution to Example 5.3.1 number 3 in Section 5.3.

41. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction

$$HA \rightleftharpoons H^+ + A^-.$$

The acid disassociation constant, K_a, is given by

$$K_a = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

where the square brackets denote the concentrations just as they did in Exercise 83 in Section 5.3. The symbol pK_a is defined similarly to pH in that p$K_a = -\log(K_a)$. Using the definition of pH from Exercise 83 and the properties of logarithms, derive the Henderson-Hasselbalch Equation:

$$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

42. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

43. Prove the Quotient Rule and Power Rule for Logarithms.

44. Give numerical examples to show that, in general,

(a) $\log_b(x+y) \neq \log_b(x) + \log_b(y)$

(b) $\log_b(x-y) \neq \log_b(x) - \log_b(y)$

(c) $\log_b\left(\frac{x}{y}\right) \neq \frac{\log_b(x)}{\log_b(y)}$

45. Research the history of logarithms including the origin of the word ‘logarithm’ itself. Why is the abbreviation of natural log ‘ln’ and not ‘nl’?

46. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

SECTION 5.4 EXERCISE ANSWERS A.1.5

5.5 Solving Equations involving Exponential Functions

In this section we will develop techniques for solving equations involving exponential functions. Consider the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 5.7, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution.

Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 5.6 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$.

After all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 5.9, to give us something more calculator friendly. Typically this means we convert our answer to base 10 or base e, and we choose the latter: $\log_2(129) = \frac{\ln(129)}{\ln(2)} \approx 7.011$.

Still another way to obtain this answer is to 'take the natural log' of both sides of the equation. Due to the fact that $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[1]

We then use the Power Rule to write the exponent x as a factor then divide both sides by the constant $\ln(2)$ to obtain our answer.[2]

$$\begin{array}{rcll} 2^x & = & 129 & \\ \ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\ x\ln(2) & = & \ln(129) & \text{Power Rule} \\ x & = & \dfrac{\ln(129)}{\ln(2)} & \end{array}$$

We summarize our two strategies for solving equations featuring exponential functions below.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the **natural log** of both sides of the equation and use the Power Rule.

Example 5.5.1. Solve the following equations. Check your answer using a graphing utility.

[1] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 5.7.

[2] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^{x} = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^{x} = 5^{x} + 6$
6. $\frac{e^{x}-e^{-x}}{2} = 5$

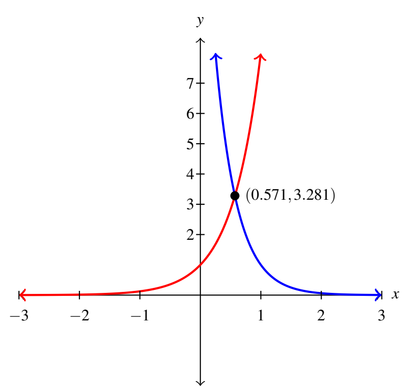
Solution.

1. Solve $2^{3x} = 16^{1-x}$ for x.

   16 is a power of 2, so we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^{4}\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$.

   Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$.

   Graphing $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.571$.

   Checking $2^{3x} = 16^{1-x}$

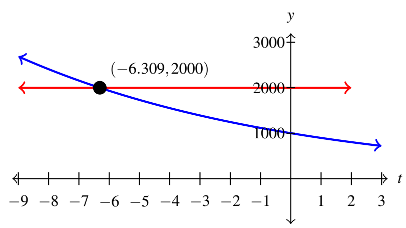
2. Solve $2000 = 1000 \cdot 3^{-0.1t}$ for t.

   We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$.

   As it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$.

   Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ and obtain $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$.

   We see the graphs of $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.309$.



Checking $2000 = 1000 \cdot 3^{-0.1t}$

3. Solve $9 \cdot 3^x = 7^{2x}$ for x.

   We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$.

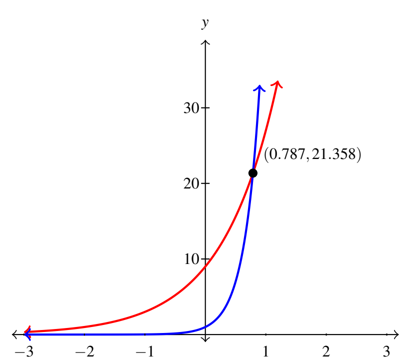
   As it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$.

   The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants.

   The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation (do you see why?) and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{array}{rcll} (x+2)\ln(3) & = & 2x\ln(7) & \\ x\ln(3)+2\ln(3) & = & 2x\ln(7) & \\ 2\ln(3) & = & 2x\ln(7)-x\ln(3) & \\ 2\ln(3) & = & x(2\ln(7)-\ln(3)) & \text{Factor.} \\ x & = & \frac{2\ln(3)}{2\ln(7)-\ln(3)} & \end{array}$$

   We see the graphs of $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ intersect at $x = \frac{2\ln(3)}{2\ln(7)-\ln(3)} \approx 0.787$.



Checking $9 \cdot 3^x = 7^{2x}$

4. Solve $75 = \frac{100}{1+3e^{-2t}}$ for t.

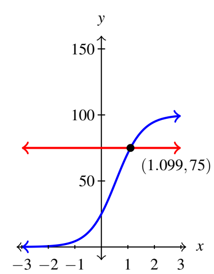
   Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential.

   To that end, we clear denominators and get $75\left(1+3e^{-2t}\right) = 100$, or $75+225e^{-2t} = 100$. We get $225e^{-2t} = 25$, so finally, $e^{-2t} = \frac{1}{9}$.

   Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. As natural log is log base e, $\ln\left(e^{-2t}\right)$ simplifies to be $-2t$. Likewise, we use the Power Rule to rewrite $\ln\left(\frac{1}{9}\right) = -\ln(9)$.

   Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?)

   To check, we see the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$, intersect at $x = \ln(3) \approx 1.099$.



Checking $75 = \frac{100}{1+3e^{-2t}}$

5. Solve $25^x = 5^x + 6$ for x.

   We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$.

   Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs.

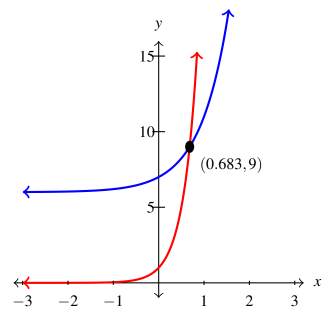
   If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a ‘quadratic in disguise’.

   Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. As $u = 5^x$, we have $5^x = -2$ or $5^x = 3$.

   $5^x = -2$ has no real solution,[3] so we focus on $5^x = 3$. It isn’t convenient to express 3 as a power of 5, thus we take natural logs and get $\ln\left(5^x\right) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$.

   We see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.683$.

[3]Why not?



Checking $25^x = 5^x + 6$

6. Solve $\frac{e^x - e^{-x}}{2} = 5$ for x.

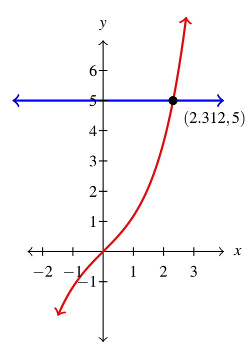
   Clearing the denominator in $\frac{e^x - e^{-x}}{2} = 5$ gives $e^x - e^{-x} = 10$, at which point we pause to consider how to proceed. Rewriting $e^{-x} = \frac{1}{e^x}$, we see we have another denominator to clear: $e^x - \frac{1}{e^x} = 10$.

   Doing so gives $e^{2x} - 1 = 10e^x$, which, once again fits the criteria of being a 'quadratic in disguise.'

   If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$ using the quadratic formula gives $u = 5 \pm \sqrt{26}$.

   From this, we have $e^x = 5 \pm \sqrt{26}$. Because $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$ (why not?) but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$.

   We see the graphs of $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$ intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$.



Checking $\frac{e^x - e^{-x}}{2} = 5$

□

Note that verifying our solutions to the equations in Example 5.5.1 *analytically* holds great educational value, as it reviews many of the properties of logarithms and exponents in tandem.

For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and check:

$$\begin{array}{rcll} 2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\ 2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\ 2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\ 2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\ 2000 & \stackrel{\checkmark}{=} & 2000 & \end{array}$$

We strongly encourage the reader to check the remaining equations analytically as well.

We close this section by finding a function inverse.

Example 5.5.2. The function $f(x) = \dfrac{5e^x}{e^x+1}$ is one-to-one.

1. Write a formula for $f^{-1}(x)$.
2. Solve $\dfrac{5e^x}{e^x+1} = 4$.

Solution.

1. Write a formula for $f^{-1}(x)$.

   We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{array}{rcll} y & = & \dfrac{5e^x}{e^x+1} & \\ x & = & \dfrac{5e^y}{e^y+1} & \text{Switch } x \text{ and } y \\ x\left(e^y+1\right) & = & 5e^y & \\ xe^y + x & = & 5e^y & \\ x & = & 5e^y - xe^y & \\ x & = & e^y(5-x) & \\ e^y & = & \dfrac{x}{5-x} & \\ \ln\left(e^y\right) & = & \ln\left(\dfrac{x}{5-x}\right) & \\ y & = & \ln\left(\dfrac{x}{5-x}\right) & \end{array}$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this, as well as a graphical check, to the reader in Exercise 41.

2. Write a formula for $f^{-1}(x)$.

   We recognize the equation $\frac{5e^x}{e^x+1} = 4$ as $f(x) = 4$. Hence, our solution is $x = f^{-1}(4) = \ln\left(\frac{4}{5-4}\right) = \ln(4)$.

   We can check this fairly quickly algebraically. Using $e^{\ln(4)} = 4$, we find $\frac{5e^{\ln(4)}}{e^{\ln(4)}+1} = \frac{5(4)}{4+1} = \frac{20}{5} = 4$. $\square$

5.5.1 Exercises

In Exercises 1 - 33, solve the equation analytically.

1. $2^{4x} = 8$
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2t} = \frac{1}{2}$
5. $8^t = \frac{1}{128}$
6. $2^{(t^3-t)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-t} = 2$
11. $5^t = -2$
12. $3^{(t-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1-e^{2t}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1t} = 20$
19. $\dfrac{100e^x}{e^x+2} = 50$
20. $\dfrac{5000}{1+2e^{-3t}} = 2500$
21. $\dfrac{150}{1+29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^x = 10$
23. $e^{2x} = 2e^x$
24. $7e^{2t} = 28e^{-6t}$
25. $3^{(x-1)} = 2^x$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2t} - 3e^t - 10 = 0$
29. $e^{2t} = e^t + 6$
30. $4^t + 2^t = 12$
31. $e^x - 3e^{-x} = 2$
32. $e^x + 15e^{-x} = 8$
33. $3^x + 25 \cdot 3^{-x} = 10$

In Exercises 34 - 39, find the domain of the function.

34. $T(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
35. $C(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

36. $s(t) = \sqrt{e^{2t} - 3}$

37. $c(t) = \sqrt[3]{e^{2t} - 3}$

38. $L(x) = \log\left(3 - e^{x}\right)$

39. $\ell(x) = \ln\left(\dfrac{e^{2x}}{e^{x} - 2}\right)$

40. Compute the inverse of $f(x) = \dfrac{e^{x} - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

41. In Example 5.5.2, we found that the inverse of $f(x) = \dfrac{5e^{x}}{e^{x}+1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.

    (a) Algebraically check our answer by verifying: $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) With help of a graphing utility, graph $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes. How does this help to verify our answer?

    (d) Let $g(x) = \dfrac{5x}{x+1}$ and $h(x) = e^{x}$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

SECTION 5.5 EXERCISE ANSWERS A.1.5

5.6 Solving Equations involving Logarithmic Functions

In Section 5.5 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from.

For example, per Theorem 5.7, the *only* solution to $\log_2(x) = \log_2(5)$ is $x = 5$. Now consider $\log_2(x) = 3$. To use Theorem 5.7, we need to rewrite 3 as a logarithm base 2. Theorem 5.6 gives us $3 = \log_2\left(2^3\right) = \log_2(8)$. Hence, $\log_2(x) = 3$ is equivalent to $\log_2(x) = \log_2(8)$ so that $x = 8$.

A second approach to solving $\log_2(x) = 3$ us to apply the corresponding exponential function, $f(x) = 2^x$ to both sides: $2^{\log_2(x)} = 2^3$ so $x = 2^3 = 8$.

A third approach to solving $\log_2(x) = 3$ is to use Theorem 5.6 to rewrite $\log_2(x) = 3$ as $2^3 = x$, so $x = 8$.

In the grand scheme of things, all three approaches we have presented to solve $\log_2(x) = 3$ are mathematically equivalent, so we opt to choose the last approach in our summary below.

Steps for Solving an Equation involving Logarithmic Functions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate arguments.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 5.6.1. Solve the following equations. Check your solutions graphically.

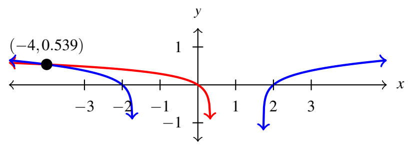
1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(t-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2t) = 1 - \log_7(3-t)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(t+1) = 2\log_2(t)$

Solution.

1. Solve $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$ for x.

   We have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, thus we equate the arguments (what's inside) of the logs to get $1-3x = x^2-3$. Solving $x^2+3x-4 = 0$ gives $x = -4$ and $x = 1$.

   To check these answers using a graph, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$. We see these graphs intersect only at $x = -4$. however.



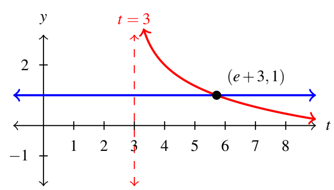
Checking $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$

To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.

2. Solve $2 - \ln(t-3) = 1$ for t.

   To solve $2 - \ln(t-3) = 1$, we first isolate the logarithm and get $\ln(t-3) = 1$. Rewriting $\ln(t-3) = 1$ as an exponential equation, we get is $e^1 = t-3$, so $t = e+3$.

   A graph shows the graphs of $f(t) = 2 - \ln(t-3)$ and $g(t) = 1$ intersect at $t = e+3 \approx 5.718$.



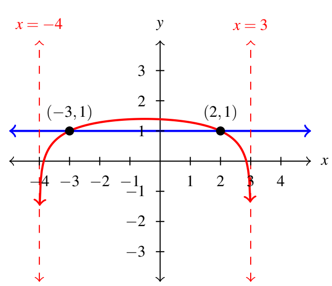
Checking $2 - \ln(t-3) = 1$

3. Solve $\log_6(x+4) + \log_6(3-x) = 1$ for x.

   We start solving $\log_6(x+4) + \log_6(3-x) = 1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right] = 1$.

   Rewriting as an exponential equation gives $6^1 = (x+4)(3-x)$ which reduces to $x^2 + x - 6 = 0$. We get two solutions: $x = -3$ and $x = 2$.

   Using the change of base formula, we graph $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$ and we see the graphs intersect twice, at $x = -3$ and $x = 2$, as required.



Checking $\log_6(x+4) + \log_6(3-x) = 1$

[1] They do, however, represent the same **family** of complex numbers. We refer the reader to a course in Complex Variables.

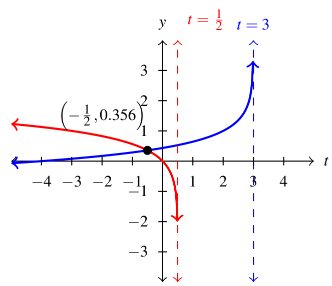
4. Solve $\log_7(1-2t) = 1 - \log_7(3-t)$ for t.

   Taking a cue from the previous problem, we begin solving $\log_7(1-2t) = 1 - \log_7(3-t)$ by first collecting the logarithms on the same side, $\log_7(1-2t) + \log_7(3-t) = 1$, and then using the Product Rule to get $\log_7[(1-2t)(3-t)] = 1$.

   Rewriting as an exponential equation gives $7^1 = (1-2t)(3-t)$ or $7 = 2t^2 - 7t + 3$ which can be rewritten as $2t^2 - 7t - 4 = 0$. Solving, we find $t = -\frac{1}{2}$ and $t = 4$.

   Once again, we use the change of base formula and find the graphs of $y = f(t) = \frac{\ln(1-2t)}{\ln(7)}$ and $y = g(t) = 1 - \frac{\ln(3-t)}{\ln(7)}$ intersect only at $t = -\frac{1}{2}$.

   Checking $t = 4$ in the original equation produces $\log_7(-7) = 1 - \log_7(-1)$, showing $t = 4$ is not in the domain of f nor g.



Checking $\log_7(1-2t) = 1 - \log_7(3-t)$

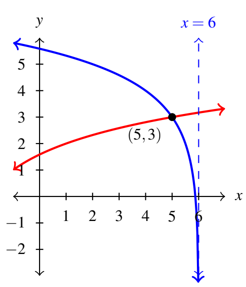
5. Solve $\log_2(x+3) = \log_2(6-x) + 3$ for x.

   Our first step in solving $\log_2(x+3) = \log_2(6-x) + 3$ is to gather the logarithms to one side of the equation: $\log_2(x+3) - \log_2(6-x) = 3$.

   The Quotient Rule gives $\log_2\left(\frac{x+3}{6-x}\right) = 3$ which, as an exponential equation is $2^3 = \frac{x+3}{6-x}$.

   Clearing denominators, we get $8(6-x) = x+3$, which reduces to $x = 5$.

   Using the change of base once again, we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$ and find they intersect at $x = 5$.



Checking $\log_2(x+3) = \log_2(6-x) + 3$

6. Solve $1+2\log_4(t+1) = 2\log_2(t)$ for t.

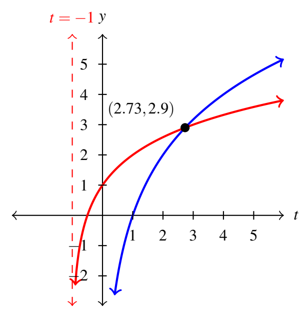
   Our first step in solving $1+2\log_4(t+1) = 2\log_2(t)$ is to gather the logs on one side of the equation. We obtain $1 = 2\log_2(t) - 2\log_4(t+1)$ but find we need a common base to combine the logs.

   As 4 is a power of 2, we use change of base to convert $\log_4(t+1) = \frac{\log_2(t+1)}{\log_2(4)} = \frac{1}{2}\log_2(t+1)$. Hence, our original equation becomes

$$\begin{array}{rcll} 1 & = & 2\log_2(t) - 2\left(\frac{1}{2}\log_2(t+1)\right) & \\ 1 & = & 2\log_2(t) - \log_2(t+1) & \\ 1 & = & \log_2\left(t^2\right) - \log_2(t+1) & \text{Power Rule} \\ 1 & = & \log_2\left(\dfrac{t^2}{t+1}\right) & \text{Quotient Rule} \end{array}$$

   Rewriting $1 = \log_2\left(\frac{t^2}{t+1}\right)$ in exponential form gives $\frac{t^2}{t+1} = 2$ or $t^2 - 2t - 2 = 0$. Using the quadratic formula, we obtain $t = 1 \pm \sqrt{3}$.

   One last time, we use the change of base formula and graph $f(t) = 1 + \frac{2\ln(t+1)}{\ln(4)}$ and $g(t) = \frac{2\ln(t)}{\ln(2)}$. We see the graphs intersect only at $t = 1+\sqrt{3} \approx 2.732$.



   Checking $1+2\log_4(t+1) = 2\log_2(t)$

   Note the solution $t = 1-\sqrt{3} < 0$. Hence if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined, which explains why the graphs intersect only once. □

If nothing else, Example 5.6.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot: any supposed solution which causes the argument of a logarithm to be negative must be discarded.

While identifying extraneous solutions is important, it is equally important to understand which machinations create the opportunity for extraneous solutions to appear. In the case of Example 5.6.1, extraneous

[2]Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

solutions, by and large, result from using the Power, Product, or Quotient Rules. We encourage the reader to take the time to track each extraneous solution found in Example 5.6.1 backwards through the solution process to see at precisely which step it fails to be a solution.

We close this section by finding an inverse of a one-to-one function which involves logarithms.

Example 5.6.2. The function $f(x) = \dfrac{\log(x)}{1-\log(x)}$ is one-to-one.

1. Write a formula for $f^{-1}(x)$ and check your answer graphically.
2. Solve $\dfrac{\log(x)}{1-\log(x)} = 1$

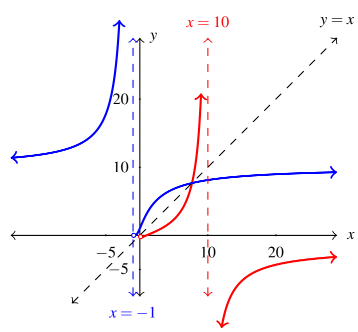
Solution.

1. Write a formula for $f^{-1}(x)$ and check your answer graphically.

   We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{array}{rcll} y & = & f(x) & \\ y & = & \dfrac{\log(x)}{1-\log(x)} & \\ x & = & \dfrac{\log(y)}{1-\log(y)} & \text{Interchange } x \text{ and } y. \\ x(1-\log(y)) & = & \log(y) & \\ x - x\log(y) & = & \log(y) & \\ x & = & x\log(y)+\log(y) & \\ x & = & (x+1)\log(y) & \\ \dfrac{x}{x+1} & = & \log(y) & \\ y & = & 10^{\frac{x}{x+1}} & \text{Rewrite as an exponential equation.} \end{array}$$

   We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same graph produces the required symmetry about $y = x$.



2. Solve $\dfrac{\log(x)}{1-\log(x)} = 1$.

   Recognizing $\frac{\log(x)}{1-\log(x)} = 1$ as $f(x) = 1$, we have $x = f^{-1}(1) = 10^{\frac{1}{1+1}} = 10^{\frac{1}{2}} = \sqrt{10}$.

   To check our answer algebraically, first recall $\log(\sqrt{10}) = \log_{10}(\sqrt{10})$. Next, we know $\sqrt{10} = 10^{\frac{1}{2}}$. Hence, $\log_{10}\left(10^{\frac{1}{2}}\right) = \frac{1}{2} = 0.5$. It follows that $\frac{\log(\sqrt{10})}{1-\log(\sqrt{10})} = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1$, as required. □

5.6.1 Exercises

In Exercises 1 - 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-t^2\right) = \ln(2-t)$
4. $\log_5\left(18-t^2\right) = \log_5(6-t)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(t^2-99\right) = 0$
8. $\log(t^2-3t) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6-3\log_5(2t) = 0$
14. $3\ln(t)-2 = 1-\ln(t)$
15. $\log_3(t-4)+\log_3(t+4) = 2$
16. $\log_5(2t+1)+\log_5(t+2) = 1$
17. $\log_{169}(3x+7)-\log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1)-\ln(x) = 3$
19. $2\log_7(t) = \log_7(2)+\log_7(t+12)$
20. $\log(t)-\log(2) = \log(t+8)-\log(t+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x)+8$
22. $\ln(\ln(x)) = 3$
23. $(\log(t))^2 = 2\log(t)+15$
24. $\ln(t^2) = (\ln(t))^2$

In Exercises 25 - 30, state the domain of the function.

25. $r(x) = \dfrac{x}{1-\ln(x)}$
26. $R(x) = \dfrac{x\ln(x)}{1-\ln(x)}$

27. $s(t) = \sqrt{2 - \log(t)}$

28. $c(t) = (2\ln(t) - 1)^{\frac{2}{3}}$

29. $\ell(t) = \ln(\ln(t))$

30. $L(x) = \log\left(\dfrac{x\ln(x)}{1 - \ln(x)}\right)$

31. Solve $\ln(3 - y) - \ln(y) = 2x + \ln(5)$ for y.

32. In Example 5.6.2 we found the inverse of $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

    (a) Algebraically check our answer by verifying $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) Let $g(x) = \dfrac{x}{1 - x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

33. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

SECTION 5.6 EXERCISE ANSWERS A.1.5

5.7 Applications of Exponential and Logarithmic Functions

As we mentioned in Sections 5.2 and 5.3, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, we will often express our final answers as decimal approximations (after finding exact answers first, of course!)

5.7.1 Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have \$100 to invest at your local bank and they are offering a whopping 5 % annual percentage interest rate. This means that after one year, the bank will pay *you* 5% of that \$100, or $\$100(0.05) = \$5$ in interest, so you now have \$105. This is in accordance with the formula for *simple interest* which you have undoubtedly run across at some point before.

Equation 5.1. Simple Interest:
The amount of interest I accrued at an annual rate r on an investment[a] P after t years is

$$I = Prt$$

The amount in the account after t years, $A(t)$ is given by

$$A(t) = P + I = P + Prt = P(1 + rt)$$

[a]Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[3] Naturally, you withdraw your money and try to invest it at the higher rate there. As six months is one half of a year, that initial \$100 yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest.

You take your \$102.50 off to the competitor and find out that those restrictions which *may* apply actually do apply, so you return to your bank and re-deposit the \$102.50 for the remaining six months of the year.

To your surprise and delight, at the end of the year your statement reads \$105.06, not \$105 as you had expected.[4] Where did those extra six cents come from?

For the first six months of the year, interest was earned on the original principal of \$100, but for the second six months, interest was earned on \$102.50, that is, you earned interest on your interest. This is the basic concept behind **compound interest**.

[3]Some restrictions may apply.
[4]Actually, the final balance should be \$105.0625.

In the previous discussion, we would say that the interest was compounded twice per year, or semiannually.[5] If more money can be earned by earning interest on interest already earned, one wonders what happens if the interest is compounded more often, say every three months - 4 times a year, or 'quarterly.'

In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1+rt) = \$100\left(1+0.05\cdot\frac{1}{4}\right) = \$101.25$. We now invest the \$101.25 for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1+0.05\cdot\frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is \$103.79, and, at last, we obtain \$105.08. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound.

In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let A_k denote the amount in the account after the k^{th} compounding.

Then $A_1 = P\left(1+r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1+\frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1+\frac{r}{n}\right) = \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1+\frac{r}{n}\right)^3$, $A_4 = P\left(1+\frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1+\frac{r}{n}\right)^k$.

As we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

Equation 5.2. Compounded Interest:
If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount in the account after t years, $A(t)$ is given by

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 5.2 becomes $A(t) = 100\left(1+\frac{0.05}{4}\right)^{4t}$ which reduces to $A(t) = 100(1.0125)^{4t}$. To check this new formula against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 5.7.1. Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.

1. Express the amount $A(t)$ in the account as a function of the term of the investment t in years.

2. How much is in the account after 5 years?

3. How long will it take for the initial investment to double?

[5] Using this convention, simple interest after one year is the same as compounding the interest only once per year.

4. Compute and interpret the average rate of change[6] of the amount in the account:

   - from the end of the fourth year to the end of the fifth year
   - from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution.

1. Express the amount $A(t)$ in the account as a function of the term of the investment t in years.

   Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (interest is compounded *monthly*) into Equation 5.2 yields $A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$.

2. How much is in the account after 5 years?

   To find the amount in the account after 5 years, we compute $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately \$2852.92.

3. How long will it take for the initial investment to double?

   Our initial investment is \$2000, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. That is, we need to solve $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$.

   Taking natural logs as in Section 5.5, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. Find and interpret the average rate of change of the amount in the account.

   Recall to find the average rate of change of A over an interval $[a,b]$, we compute $\frac{A(b)-A(a)}{b-a}$.

   - Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year.

     The average rate of change of A from the end of the fourth year to the end of the fifth year is $\frac{A(5)-A(4)}{5-4} \approx 195.63$.
     This means that the value of the investment is increasing at a rate of approximately \$195.63 per year between the end of the fourth and fifth years.

   - Find and interpret the average rate of change of the amount in the account from the end of the thirty-fourth year to the end of the thirty-fifth year.

     Likewise, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$, so the value of the investment is increasing at a rate of approximately \$1648.21 per year during this time.

[6] See Definition 1.11 in Section 1.3.1.

So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[7] □

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[8]

Consider an investment of $1 invested at 100% interest for 1 year compounded n times a year. Equation 5.2 tells us that the amount of money in the account after 1 year is $A = \left(1+\frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing.

We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest.

With Calculus, we can show[9] that as $n \to \infty$, $A = \left(1+\frac{1}{n}\right)^n \to e$, where e is the natural base first presented in Section 5.2. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 5.10. Investing $1 at 100% interest compounded continuously for one year returns $$e$.

Using this definition of e and a little Calculus, we can take Equation 5.2 and produce a formula for continuously compounded interest.

[7]In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

[8]Once you've had a semester of Calculus, you'll be able to fully appreciate this very lame pun.

[9]Or define, depending on your point of view.

Equation 5.3. Continuously Compounded Interest:
If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount in the account after t years, $A(t)$ is given by

$$A(t) = Pe^{rt}$$

If we take the scenario of Example 5.7.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about \$24,035.28, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about \$24,213.18, a difference of less than 1%.

Equations 5.2 and 5.3 both use exponential functions to describe the growth of an investment. It turns out, the same principles which govern compound interest are also used to model short term growth of populations. As with many concepts in this text, these notions are best formalized using the language of Calculus. Nevertheless, we do our best here.

In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time.[10] In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Solving said differential equation gives us the formula below.

Equation 5.4. Uninhibited Growth:
If a population increases according to The Law of Uninhibited Growth, the number of organisms at time t, $N(t)$ is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation:

$$(\textit{ instantaneous rate of change of } N(t) \textit{ at time } t) = kN(t)$$

It is worth taking some time to compare Equations 5.3 and 5.4. In Equation 5.3, we use P to denote the initial investment; in Equation 5.4, we use N_0 to denote the initial population. In Equation 5.3, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 5.4 corresponds to a growth rate as well. While Equations 5.3 and 5.4 look entirely different, they both represent the same mathematical concept.

Example 5.7.2. In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells

[10] The average rate of change of a function over an interval was first introduced in Section 1.3.1. The notion of *instantaneous* rate of change was introduced in the remarks following Example 1.3.12 and revisited in Example 3.2.3.

grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, in thousands, after t days, $N(t)$.

Solution. We begin with $N(t) = N_0 e^{kt}$. Recall $N(t)$ gives the number of cells *in thousands*, so $N_0 = 12$ and $N(t) = 12e^{kt}$.

Next, we need to determine the growth rate k. We know that after one week, the number of cells has grown to five million. As t measures days and the units of $N(t)$ are in thousands, this translates mathematically to $N(7) = 5000$ or $12e^{7k} = 5000$. Solving, we get $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$, so $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$.

Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells. □

Whereas Equations 5.3 and 5.4 model the growth of quantities, we can use equations like them to describe the decline of quantities.

One example we've seen already is Example 5.2.2 in Section 5.2. There, the value of a car decreased from its purchase price of \$25,000 to nothing at all.

Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays.

This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 5.4 with the exception that the rate constant k is negative.

Equation 5.5. Radioactive Decay:
The amount of a radioactive element at time t, $A(t)$ is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$(\textit{ instantaneous rate of change of } A(t) \textit{ at time } t) = kA(t)$$

Example 5.7.3. Iodine-131 is a commonly used radioactive isotope and is used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 5.5, and that the half-life[11] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

Solution. As we start with 5 grams initially, Equation 5.5 gives $A(t) = 5e^{kt}$.

Because the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Mathematically, this translates to $A(8) = 2.5$, or $5e^{8k} = 2.5$. Solving for k, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily.

Hence, our final answer is $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$. □

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 5.2.3 of Section 5.2.

In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where t was measured in minutes. In that situation, we knew the physical limit of the temperature of the coffee was room temperature,[12] and the differential equation which gives rise to our formula for $T(t)$ takes this into account.

Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature *gap* exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly.

Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and as the physics behind warming and cooling is the same, we combine both cases in the equation below.

Equation 5.6. Newton's Law of Cooling (Warming):
The temperature of an object at time t, $T(t)$ is given by the formula

$$T(t) = T_a + (T_0 - T_a)\,e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\textit{ instantaneous rate of change of } T(t) \textit{ at time } t) = k\,(T(t) - T_a)$$

[a]That is, the temperature of the surroundings.

[11]The time it takes for half of the substance to decay.

[12]The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

If we re-examine the situation in Example 5.2.3 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 5.6, $T(t) = 70 + (160-70)e^{-0.1t}$ which reduces to the original formula given in that example. The rate constant $k = 0.1$ in this case indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings.

Note in Equation 5.6 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 5.7.4. A roast initially at 40°F cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, write a formula for the temperature of the roast $T(t)$ as a function of its time in the oven, t, in hours.

2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution.

1. Assuming the temperature of the roast follows Newton's Law of Warming, write a formula for the temperature of the roast $T(t)$ as a function of its time in the oven, t, in hours.

   The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming gives $T(t) = 350 + (40-350)e^{-kt}$, or after some simplification, $T(t) = 350 - 310e^{-kt}$.

   To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is

   $$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

   To find when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. Hence, the roast is done after roughly 3 hours and 15 minutes. □

If we had taken the time to graph $y = T(t)$ in Example 5.7.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion analytically by applying 'number sense'.

As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.

The function T in this situation is sometimes called a **limited** growth model, because the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate.

Our final model, the **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Equation 5.7. Logistic Growth:
If a population behaves according to the assumptions of logistic growth, the number of organisms at time t, $N(t)$ is given by

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population,[a] and C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\textit{ instantaneous rate of change of } N(t) \textit{ at time } t) = kN(t)\,(L - N(t))$$

[a]That is, as $t \to \infty$, $N(t) \to L$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[13]

Example 5.7.5. The number of people $N(t)$, in hundreds, at a local community college who have heard the rumor 'Carl's afraid of Sasquatch' can be modeled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

where $t \geq 0$ is the number of days after April 1, 2016.

1. Compute and interpret $N(0)$.
2. Write and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?

[13]Which can be just as damaging as diseases.

4. Check your answers to 2 and 3 using technology.

Solution.

1. Compute and interpret $N(0)$.

   We find $N(0) = \frac{84}{1+2799e^0} = \frac{84}{2800} = 0.03$. As $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. $t = 0$ corresponds to April 1, 2016, thus we may conclude that on that day, 3 people have heard the rumor.

2. Write and interpret the end behavior of $N(t)$.

   We could simply note that $N(t)$ is written in the form of Equation 5.7, and identify $L = 84$. However, to see better *why* the answer is 84, we proceed analytically.

   As the domain of N is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[14] as $t \to \infty$, we have $1997e^{-t} \to 0^+$ and so $N(t) \approx \frac{84}{1+\text{very small }(+)} \approx 84$.

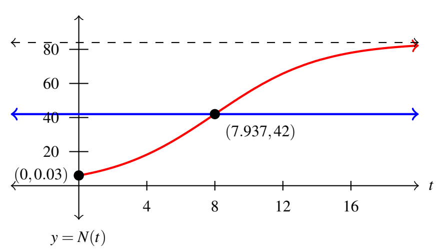
   Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.

3. How long until 4200 people have heard the rumor?

   To compute how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$, so it takes around 8 days until 4200 people have heard the rumor.

4. Check your answers to 2 and 3 using a graph.

   Graphing $y = N(t)$ below, we see $y = 84$ is the horizontal asymptote of the graph, confirming our answer to number 2, and the graph intersects the line $y = 42$ at $t \approx 7.937 \approx \ln(2799)$, which confirms our answer to number 3.



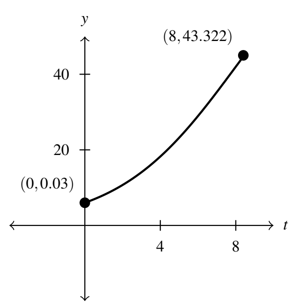
□

If we take the time to analyze the graph of $y = N(t)$ in Example 5.7.5, we can see *graphically* how logistic growth combines features of uninhibited and limited growth.

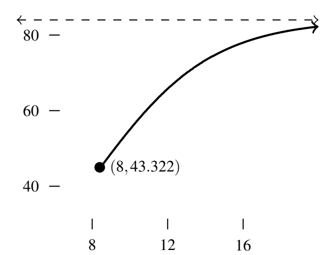
[14] See, for example, Example 5.2.3.

The curve is concave up, rising steeply, then at some point, becomes concave down and begins to level off.[15] The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. Even though the function is still increasing through the inflection point, the *rate* at which it does so begins to decrease.

With Calculus, one can show the point of diminishing returns always occurs at half the limiting population. (In our case, when $N(t) = 42$.) So with that in mind, we present two portions of the graph of $y = N(x)$, one on the interval $[0,8]$, the other on $[8,15]$. The former looks strikingly like uninhibited growth while the latter like limited growth.



$y = N(t)$ for $0 \leq t \leq 8$



$y = N(t)$ for $8 \leq t \leq 15$

5.7.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 81, 82 and 83 of Section 5.3, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 5.7.6. The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[16] password is comprised of letters and numbers only, find the associated information entropy.
2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

[15] We introduced the notion of concavity in Section 4.2.

[16] That is, upper and lower case letters are treated as different characters.

Solution.

1. If a 7 character case-sensitive[17] password is comprised of letters and numbers only, find the associated information entropy.

   There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. The password is to be 7 characters long, so $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

   We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[18] □

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 83 in Section 5.3.

Example 5.7.7. Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[19] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution. We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. We get $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr. □

5.7.3 Exercises

For each of the scenarios given in Exercises 1 - 6,

- Express the amount, A, in the account as a function of the term of the investment t in years.
- To the nearest cent, determine how much is in the account after 5, 10, 30 and 35 years.

[17] That is, upper and lower case letters are treated as different characters.

[18] As there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.

[19] Derived from the Henderson-Hasselbalch Equation. See Exercise 41 in Section 5.4. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

- To the nearest year, determine how long will it take for the initial investment to double.
- Compute and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.

1. \$500 is invested in an account which offers 0.75%, compounded monthly.
2. \$500 is invested in an account which offers 0.75%, compounded continuously.
3. \$1000 is invested in an account which offers 1.25%, compounded monthly.
4. \$1000 is invested in an account which offers 1.25%, compounded continuously.
5. \$5000 is invested in an account which offers 2.125%, compounded monthly.
6. \$5000 is invested in an account which offers 2.125%, compounded continuously.
7. Look back at your answers to Exercises 1 - 6. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.
8. How much money needs to be invested now to obtain \$2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.
9. How much money needs to be invested now to obtain \$5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.
10. On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation 5.2 to answer the following.
    (a) If $P = 2000$ what is $A(8)$?
    (b) Solve the equation $A(t) = 4000$ for t.
    (c) What principal P should be invested so that the account balance is \$2000 is three years?
11. Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.
    (a) If $P = 2000$ what is $A(8)$?
    (b) Solve the equation $A(t) = 4000$ for t.
    (c) What principal P should be invested so that the account balance is \$2000 in three years?
    (d) The Annual Percentage Yield is the <u>simple</u> interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

12. A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?

13. Use Equation 5.2 to show that the time it takes for an investment to double in value does not depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[20] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

In Exercises 14 - 18, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- Find the decay constant k. Round your answer to four decimal places.
- Find a function which gives the amount of isotope A which remains after time t. (Keep the units of A and t the same as the given data.)
- Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

14. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

15. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

16. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

17. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

18. Uranium 235, used for nuclear power, initial amount 1 kg grams, half-life 704 million years.

19. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 14 - 18 to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k. Find a formula, in terms of k only, to determine how long it takes for 90% of a radioactive isotope to decay.

20. In Example 5.2.2 in Section 5.2, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use a change of base formula to rewrite the model in the form $V(t) = 25e^{kt}$.

21. The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modeled by:
$$G(t) = 9743.77e^{0.0514t}$$

[20] Awesome pun!

(a) Find and interpret $G(0)$.

(b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was $\$14,369.1$ billion and the 2010 GDP was $\$14,657.8$ billion.)

22. The diameter D of a tumor, in millimeters, t days after it is detected is given by:

$$D(t) = 15e^{0.0277t}$$

(a) What was the diameter of the tumor when it was originally detected?

(b) How long until the diameter of the tumor doubles?

23. Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

(a) Find the growth constant k. Round your answer to four decimal places.

(b) Find a function which gives the number of bacteria $N(t)$ after t minutes.

(c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

24. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$.

(a) Find the growth constant k. Round your answer to four decimal places.

(b) Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.

(c) What is the doubling time for this strain of yeast?

25. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0 e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

26. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points

to fit a model of the form $N(t) = N_0 e^{kt}$ were $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy? For more on this, see Exercise **??**.

27. The population of Sasquatch in Bigfoot county is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

(a) Find and interpret $P(0)$.

(b) Find the population of Sasquatch in Bigfoot county in 2013 rounded to the nearest Sasquatch.

(c) To the nearest year, when will the population of Sasquatch in Bigfoot county reach 60?

(d) Find and interpret the end behavior of the graph of $y = P(t)$ analytically. Check your answer using a graphing utility.

28. Let $f(x) = \dfrac{10}{1 + e^{-x+1}}$.

(a) From Calculus, we know the inflection point of the graph of $y = f(x)$ is $(1, 5)$. This means the function is increasing the fastest at $x = 1$, or, equivalently, the slope at $(1, 5)$ is the largest anywhere on the graph. Graph $y = f(x)$ using a graphing utility and convince yourself of the reasonableness of this claim.

(b) Find average rate of change of f over each of the intervals below. What do you guess the slope of the curve is at $(1, 5)$? Zoom in on the graph near $(1, 5)$ to check your guess.

- $[0.75, 1]$
- $[0.9, 1]$
- $[0.99, 1]$
- $[1, 1.01]$
- $[1, 1.1]$
- $[1, 1.25]$

29. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

(a) Use Equation 5.5 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

(b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

(c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

(d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

30. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 5.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

31. Use Equation 5.5 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

32. A pork roast[21] was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F.

    Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 5.6),

    (a) Express the temperature T (in °F) as a function of time t (in minutes).

    (b) Find the time at which the roast would have dropped to 140°F had it not been eaten.

33. In reference to Exercise **??** in Section 4.2, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

    Graph this path for $x > 0$ using a graphing utility. Describe the behavior of y as $x \to 0^+$ and interpret this physically.

34. The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine the value of i as $t \to \infty$. (This is called the **steady state** current.)

35. If the voltage in the circuit in Exercise 34 above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if } 0 \leq t < 30 \\ \left(2 - 2e^{-300}\right)e^{-10t+300} & \text{if } t \geq 30 \end{cases}$$

    With the help of a graphing utility, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \leq t < 30$ and $t \geq 30$.

36. In Exercise 26 in Section 2.1, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a catenary and its basic shape is given by $y = \frac{1}{2}\left(e^x + e^{-x}\right)$. Use a graphing utility to graph this function. What are its domain and range? What

[21] This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

is its end behavior? Is it invertible? How do you think it is related to the function given in Exercise 40 in Section 5.5 and the one given in the answer to Exercise 33 in Section 5.6?

When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2}\left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}}\right)$$

where x and y are measured in feet and $-315 \leq x \leq 315$. Find the highest point on the arch.

Section 5.7 Exercise Answers A.1.5

CHAPTER 6

SYSTEMS OF LINEAR/NONLINEAR EQUATIONS

6.1 Solving Systems of Linear Equations

This section combines ideas from Section 0.5 and 1.3.1 so that we can start to solve systems of linear equations. Before we get ahead of ourselves, let's review a few definitions.

Definition 6.1. A **linear equation in two variables** is an equation of the form $a_1x+a_2y=c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

We are using subscripts in Definition 6.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x-\frac{y}{2}=0.1$ is a linear equation in two variables with $a_1=3$, $a_2=-\frac{1}{2}$ and $c=0.1$. We can also consider $x=5$ to be a linear equation in two variables[1] by identifying $a_1=1$, $a_2=0$, and $c=5$.

If a_1 and a_2 are both 0, then depending on c, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c=0$, then we get $0=0$, which is always true. If $c\neq 0$, then we'd have $0\neq 0$, which is never true.) Even though identities and contradictions will a large role to play throughout this Chapter, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2+y=1$, $xy=5$ and $e^{2x}+\ln(y)=1$. The reader should consider why these do not satisfy Definition 6.1.

We know from our work is Sections 1.3.1 that the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example explores the basic techniques for solving these systems. Remember - if we are looking for points in the plane, then both the x and y values are important. This is a key distinction between solving one equation and solving a *system* of equations.

Example 6.1.1. Solve the following systems of equations. Check your answer algebraically and graphically. (Said another way, make sure both x and y are correct!)

1. $\left\{\begin{array}{rcr} 2x-y & = & 1 \\ y & = & 3 \end{array}\right.$

2. $\left\{\begin{array}{rcr} 3x+4y & = & -2 \\ -3x-y & = & 5 \end{array}\right.$

3. $\left\{\begin{array}{rcr} \frac{x}{3}-\frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9}+\frac{y}{3} & = & \frac{1}{2} \end{array}\right.$

4. $\left\{\begin{array}{rcr} 2x-4y & = & 6 \\ 3x-6y & = & 9 \end{array}\right.$

5. $\left\{\begin{array}{rcr} 6x+3y & = & 9 \\ 4x+2y & = & 12 \end{array}\right.$

6. $\left\{\begin{array}{rcr} x-y & = & 0 \\ x+y & = & 2 \\ -2x+y & = & -2 \end{array}\right.$

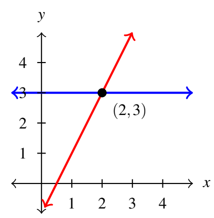
[1] Critics may argue that $x=5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

Solution.

1. Solve the system $\begin{cases} 2x - y &= 1 \\ y &= 3 \end{cases}$.

   Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of x, we **substitute** this value for y into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2,3)$.

   To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2,3)$.



$2x - y = 1$ and $y = 3$

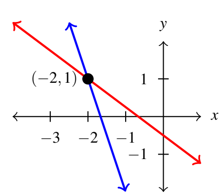
2. Solve the system $\begin{cases} 3x + 4y &= -2 \\ -3x - y &= 5 \end{cases}$.

   To solve the second system, we use the **addition** method to **eliminate** the variable x. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

   This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2,1)$.

   Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2,1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2,1)$.



$3x + 4y = -2$ and $-3x - y = 5$

3. Solve the system $\left\{ \begin{array}{rcr} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$.

The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

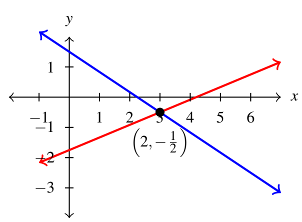
$$\left\{ \begin{array}{rcr} 5x - 12y & = & 21 \\ 4x + 6y & = & 9 \end{array} \right.$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by -5, we will be in a position to eliminate the x term

$$\begin{array}{rrcr} & 20x - 48y & = & 84 \\ + & (-20x - 30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$.

At this point, we have no choice; in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.



$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$

4. Solve the system $\left\{ \begin{array}{rcr} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$.

An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and both sides of the second equation by -2, we are ready to eliminate the x

$$\begin{array}{rrcr} & 6x - 12y & = & 18 \\ + & (-6x + 12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

We eliminated not only the x, but the y as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x, y) satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$.

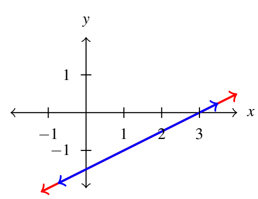
This system has infinitely many solutions and one way to describe the solution set to this system is to use the roster method[2] and write $\{(x,y) \,|\, 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system**.

Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of x, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding y-value of a solution. As we have no restriction on x, it is called a **free variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t, \frac{1}{2}t - \frac{3}{2}\right) \,|\, -\infty < t < \infty\right\}$.[3]

For specific values of t, we can generate solutions. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$; $t = 117$ gives us $(117, 57)$, and while we can check that each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always.

We claim that for any real number t, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too.

Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.



$2x - 4y = 6$ and $3x - 6y = 9$

5. Solve the system $\left\{\begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array}\right.$.

Multiplying both sides of the first equation by 2 and the both sides of the second equation by -3, we

[2] See Section 0.4 for a review of this.

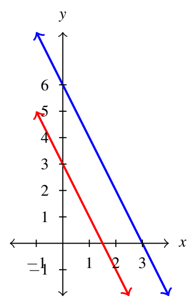
[3] Note that we could have just as easily chosen to solve $2x - 4y = 6$ for x to obtain $x = 2y + 3$. Letting y be the parameter t, we have that for any value of t, $x = 2t + 3$, which gives $\{(2t + 3, t) \,|\, -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

set the stage to eliminate x

$$\begin{array}{rrcr} & 12x+6y & = & 18 \\ + & (-12x-6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

As in the previous example, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0=-18$. This tells us that it is impossible to find a pair (x,y) which satisfies both equations; in other words, the system has no solution.

Graphically, the lines $6x+3y=9$ and $4x+2y=12$ are distinct and parallel, so they do not intersect.



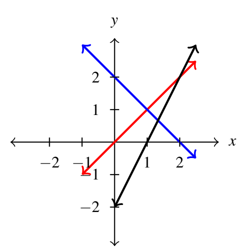
$6x+3y=9$ and $4x+2y=12$

6. Solve the system $\left\{\begin{array}{rcr} x-y & = & 0 \\ x+y & = & 2 \\ -2x+y & = & -2 \end{array}\right.$.

We can begin to solve our last system by adding the first two equations

$$\begin{array}{rrcr} & x-y & = & 0 \\ + & (x+y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x=1$. Substituting this into the first equation gives $1-y=0$ so that $y=1$. We seem to have determined a solution to our system, $(1,1)$. While this checks in the first two equations, when we substitute $x=1$ and $y=1$ into the third equation, we get $-2(1)+(1)=-2$ which simplifies to the contradiction $-1=-2$. Graphing the lines $x-y=0$, $x+y=2$, and $-2x+y=-2$, we see that the first two lines do, in fact, intersect at $(1,1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found. Thus, we say the system has no solution.



$x-y=0$, $x+y=2$, and $-2x+y=-2$ ☐

A few remarks about Example 6.1.1 are in order. Notice that some of the systems of linear equations had solutions while others did not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish between the two different types of behavior among consistent systems. Those which admit free variables are called **dependent** and those with no free variables are called **independent**.[4]

Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 6.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, because we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

We end this section with a story problem. It is an example of a classic "mixture" problem and should be familiar to most readers. The basic goal here is to create two equations: one which represents

$$\text{stuff} + \text{other stuff} = \text{total stuff}$$

and the other which represents

$$\text{value of stuff} + \text{value of other stuff} = \text{value of total stuff}.$$

Example 6.1.2. The Dude-Bros want to create a highly caffeinated, yet still drinkable, fruit punch for their annual "Disturb the Neighbors BBQ and Dance Competition". They plan to add Sasquatch Sweat$^{\text{TM}}$ Energy Drink, which has 100 mg of caffeine per fluid ounce, to Frooty Giggle Delight$^{\text{TM}}$, which has only 3 mg of caffeine per fluid ounce. How much of each component is required to make 5 gallons[9] of a fruit punch that has 80 mg of caffeine per fluid ounce.

Solution. Let S stand for the number of fluid ounces of Sasquatch Sweat$^{\text{TM}}$ Energy Drink and let F be the number of fluid ounces of Frooty Giggle Delight$^{\text{TM}}$ that will be added together. The goal is to make 5

[4] In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5] The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[6] If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7] We need more than two variables to give an example of the latter.

[8] Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

[9] Warning: unit conversion ahead!

gallons and there are 128 fluid ounces per gallon so the first equation is

$$S+F=640.$$

That equation describes “stuff + other stuff = total stuff” measured in fluid ounces. Now we need to consider the value of the stuff - in this case we need to see how much caffeine is being contributed by each component. Each fluid ounce of Sasquatch Sweat$^{\text{TM}}$ contains 100 mg of caffeine so S fluid ounces would contain $100S$ mg of caffeine.

Similarly, the F fluid ounces of Frooty Giggle Delight$^{\text{TM}}$ add $3F$ mg of caffeine to the total mixture. Thus when we go to express “value of stuff + value of other stuff = value of total stuff” we need to figure out how much caffeine is supposed to be in the end product. Well, the goal was 5 gallons of punch that had 80 mg of caffeine per fluid ounce so the Dude-Bros need to end up with $5*128*80=51200$ mg of caffeine when they're done. Hence the second is equation is

$$100S+3F=51200.$$

By turning the first equation into $F=640-S$ and substituting that into the second equation we get

$$100S+3(640-S)=51200$$

which yields $S=\frac{49280}{97}\approx 508.04$ fluid ounces. Back-substituting this value of S into the first equation gives us $F=\frac{12800}{97}\approx 131.96$ fluid ounces.

The reader should take the time to verify that $S=\frac{49280}{97}$ and $F=\frac{12800}{97}$ do indeed satisfy both equations and thus are the solution to the problem. Those are fairly unattractive numbers so we end this example by discussing a way to verify an approximate answer which is *reasonable* without having to fight with fractions. Round S down to 508 and round F up to 132. Clearly $508+132=640$ so the first equation is still satisfied. Notice that $100*508+3*132=51196$ which is really close to 51200. Thus the second equation is nearly satisfied which means the values $S=508$ and $F=132$, while not precise, are reasonable.[10] $\square$

6.1.1 Exercises

In Exercises 1 - 8, solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent. Check your answers both algebraically and graphically.

1. $\left\{\begin{array}{rcl} x+2y & = & 5 \\ x & = & 6 \end{array}\right.$

2. $\left\{\begin{array}{rcr} 2y-3x & = & 1 \\ y & = & -3 \end{array}\right.$

[10] Just be careful here - sometimes “close enough for the Dude-Bros” is not good enough for your Professor!

3. $\left\{ \begin{array}{rcr} \frac{x+2y}{4} & = & -5 \\ \frac{3x-y}{2} & = & 1 \end{array} \right.$

4. $\left\{ \begin{array}{rcr} \frac{2}{3}x - \frac{1}{5}y & = & 3 \\ \frac{1}{2}x + \frac{3}{4}y & = & 1 \end{array} \right.$

5. $\left\{ \begin{array}{rcr} \frac{1}{2}x - \frac{1}{3}y & = & -1 \\ 2y - 3x & = & 6 \end{array} \right.$

6. $\left\{ \begin{array}{rcr} x + 4y & = & 6 \\ \frac{1}{12}x + \frac{1}{3}y & = & \frac{1}{2} \end{array} \right.$

7. $\left\{ \begin{array}{rcr} 3y - \frac{3}{2}x & = & -\frac{15}{2} \\ \frac{1}{2}x - y & = & \frac{3}{2} \end{array} \right.$

8. $\left\{ \begin{array}{rcr} \frac{5}{6}x + \frac{5}{3}y & = & -\frac{7}{3} \\ -\frac{10}{3}x - \frac{20}{3}y & = & 10 \end{array} \right.$

9. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

10. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

11. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

12. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each are needed?

13. This exercise is a follow-up to Example 6.1.2. Work with your classmates to explain why mixing 4 gallons of Sasquatch Sweat™ Energy Drink and 1 gallon of Frooty Giggle Delight™ would also produce a mixture that was "close enough for the Dude-Bros".

Section 6.1 Exercise Answers A.1.6

6.2 Solving Systems of Nonlinear Equations

In this section, we study systems of non-linear equations. In non-linear equations, we can have variables raised to powers other than 1, different variables multiplied together, or variables can occur as arguments of exponential and logarithmic functions.

Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and domain restrictions are once again present.

Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

Example 6.2.1. Solve the following systems of equations. Verify answers algebraically and graphically.

1. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2+9y^2 & = & 36 \end{array}\right.$
2. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2-9y^2 & = & 36 \end{array}\right.$
3. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ y-2x & = & 0 \end{array}\right.$
4. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ y-x^2 & = & 0 \end{array}\right.$

Solution.

1. Solve the system $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2+9y^2 & = & 36 \end{array}\right.$.

   Both equations contain x^2 and y^2 only, so we can use elimination as seen in Section 6.1. We multiply both sides of the first equation by -4 and then add the two equations together.

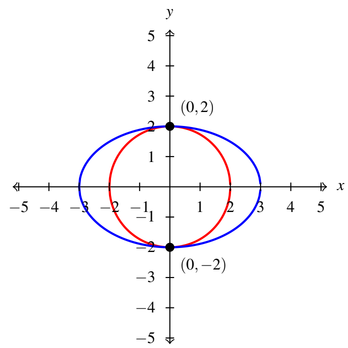
$$\left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2+9y^2 & = & 36 \end{array}\right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 5y^2 & = & 20 \end{array}\right.$$

   From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0,2),(0,-2)\}$.

   To verify these answers algebraically, we would need to show that the pair $(x,y) = (0,2)$ and $(x,y) = (0,-2)$ each satisfy *both* equations. We leave this to the reader.

   To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2+y^2 = 4$ is a circle centered at $(0,0)$ with a radius of 2. To graph $4x^2+9y^2 = 36$, we convert to standard form $\frac{x^2}{9}+\frac{y^2}{4} = 1$ and recognize it as an ellipse centered at $(0,0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4.

We see from the graph that the two curves intersect at their y-intercepts only, $(0,\pm 2)$.



$x^2+y^2=4$ and $4x^2+9y^2=36$

2. Solve the system $\left\{ \begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2-9y^2 & = & 36 \end{array} \right.$.

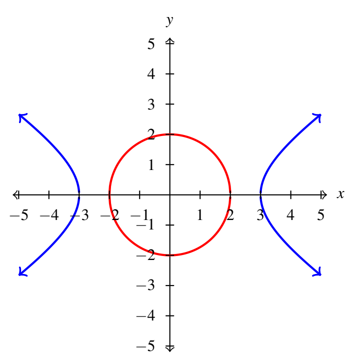
We proceed as before to eliminate one of the variables. Again we will multiply both sides of the first equation by -4 and add the sulting equation to the second.

$$\left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2-9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{array} \right.$$

Because the equation $-13y^2=20$ admits no real solution, the system is inconsistent.

To verify this graphically, we note that $x^2+y^2=4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9}-\frac{y^2}{4}=1$, we find a hyperbola centered at $(0,0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4.

We see that the circle and the hyperbola have no points in common, hence, there are no solutions.



$x^2+y^2=4$ and $4x^2-9y^2=36$

3. Solve the system $\left\{\begin{array}{rcl} x^2+y^2 & = & 4 \\ y-2x & = & 0 \end{array}\right.$.

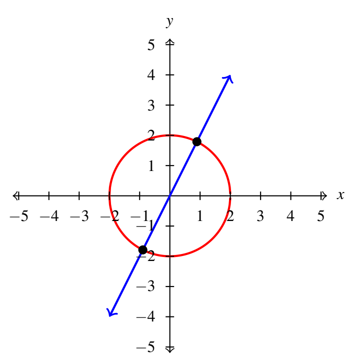
   There are no like terms among the two equations, thus elimination won't work here. Instead, we proceed using substitution.

   From the equation $y-2x=0$, we get $y=2x$. Substituting this into $x^2+y^2=4$ gives $x^2+(2x)^2=4$. Solving, we find $5x^2=4$ or $x=\pm\frac{2\sqrt{5}}{5}$.

   Returning to the equation we used for the substitution, $y=2x$, we find $y=\frac{4\sqrt{5}}{5}$ when $x=\frac{2\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5},\frac{4\sqrt{5}}{5}\right)$ and the other is $\left(-\frac{2\sqrt{5}}{5},-\frac{4\sqrt{5}}{5}\right)$. Hence, our final answer is $\left\{\left(\frac{2\sqrt{5}}{5},\frac{4\sqrt{5}}{5}\right),\left(-\frac{2\sqrt{5}}{5},-\frac{4\sqrt{5}}{5}\right)\right\}$.

   As before, we leave the algebraic check to the reader.

   The graph of $x^2+y^2=4$ is our circle from before and the graph of $y-2x=0$, or $y=2x$ is a line through the origin with slope 2. Even though we cannot easily verify the numerical values of the points of intersection from our sketch, we can be sure there are just two solutions: one in Quadrant I and one in Quadrant III. This observation, combined with our (your) algebraic check gives us confidence our solution is correct.[1]



$x^2+y^2=4$ and $y-2x=0$

4. Solve the system $\left\{\begin{array}{rcl} x^2+y^2 & = & 4 \\ y-x^2 & = & 0 \end{array}\right.$

   While it may be tempting to solve $y-x^2=0$ as $y=x^2$ and substitute, we note that this system is set up for elimination.[2]

$$\left\{\begin{array}{lrcl} (E1) & x^2+y^2 & = & 4 \\ (E2) & -x^2+y & = & 0 \end{array}\right. \xrightarrow[E1+E2]{\text{Replace } E2 \text{ with}} \left\{\begin{array}{lrcl} (E1) & x^2+y^2 & = & 4 \\ (E2) & y^2+y & = & 4 \end{array}\right.$$

[1] Of course, we could check our answers more accurately using a graphing utility.
[2] We encourage the reader to solve the system using substitution to see that you get the same solution.

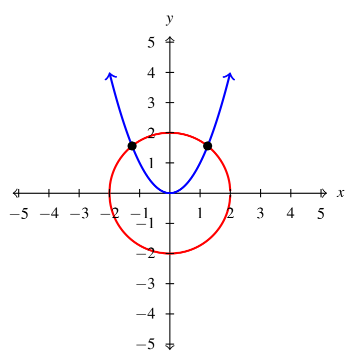
From $y^2+y=4$ we get $y^2+y-4=0$ which gives $y=\frac{-1\pm\sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations first to head off any extraneous solutions we may encounter.

We see that the circle $x^2+y^2=4$ intersects the parabola $y=x^2$ exactly *twice*, and both of these points have a positive y value.

Of the two solutions for y, only $y=\frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this into $y-x^2=0$ and solve for x. We get $x=\pm\sqrt{\frac{-1+\sqrt{17}}{2}}=\pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$.

Our final answer is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2},\frac{-1+\sqrt{17}}{2}\right),\left(-\frac{\sqrt{-2+2\sqrt{17}}}{2},\frac{-1+\sqrt{17}}{2}\right)\right\}$.

Checking these answers algebraically amounts to a true test of anyone's algebraic mettle and as such is left to the reader.



$x^2+y^2=4$ and $y-x^2=0$

$\square$

A couple of remarks about Example 6.2.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'.

Secondly, as we saw with the last problem, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it pays to take advantage when they are.

6.2.1 Exercises

In Exercises 1 - 6, solve the given system of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

1. $\left\{\begin{array}{rcl} x^2-y & = & 4 \\ x^2+y^2 & = & 4 \end{array}\right.$

2. $\left\{\begin{array}{rcl} x^2+y^2 & = & 4 \\ x^2-y & = & 5 \end{array}\right.$

3. $\left\{\begin{array}{rcl} x^2+y^2 & = & 16 \\ 16x^2+4y^2 & = & 64 \end{array}\right.$

4. $\left\{\begin{array}{rcr} x^2+y^2 & = & 16 \\ 9x^2-16y^2 & = & 144 \end{array}\right.$

5. $\left\{\begin{array}{rcr} x^2+y^2 & = & 16 \\ \frac{1}{9}y^2-\frac{1}{16}x^2 & = & 1 \end{array}\right.$

6. $\left\{\begin{array}{rcr} x^2+y^2 & = & 16 \\ x-y & = & 2 \end{array}\right.$

In Exercises 7 - 14, solve the given system of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

7. $\left\{\begin{array}{rcl} x^2-y^2 & = & 1 \\ x^2+4y^2 & = & 4 \end{array}\right.$

8. $\left\{\begin{array}{rcl} \sqrt{x+1}-y & = & 0 \\ x^2+4y^2 & = & 4 \end{array}\right.$

9. $\left\{\begin{array}{rcl} x+2y^2 & = & 2 \\ x^2+4y^2 & = & 4 \end{array}\right.$

10. $\left\{\begin{array}{rcl} (x-2)^2+y^2 & = & 1 \\ x^2+4y^2 & = & 4 \end{array}\right.$

11. $\left\{\begin{array}{rcr} x^2+y^2 & = & 25 \\ y-x & = & 1 \end{array}\right.$

12. $\left\{\begin{array}{rcl} x^2+y^2 & = & 25 \\ x^2+(y-3)^2 & = & 10 \end{array}\right.$

13. $\left\{\begin{array}{rcr} y & = & x^3+8 \\ y & = & 10x-x^2 \end{array}\right.$

14. $\left\{\begin{array}{rcl} x^2-xy & = & 8 \\ y^2-xy & = & 8 \end{array}\right.$

15. A certain bacteria culture follows the Law of Uninbited Growth, Equation 5.4. After 10 minutes, there are 10,000 bacteria. Five minutes later, there are 14,000 bacteria. How many bacteria were present initially? How long before there are 50,000 bacteria?

16. Consider the system of nonlinear equations below

$$\left\{\begin{array}{rcr} \dfrac{4}{x}+\dfrac{3}{y} & = & 1 \\[2ex] \dfrac{3}{x}+\dfrac{2}{y} & = & -1 \end{array}\right.$$

If we let $u=\frac{1}{x}$ and $v=\frac{1}{y}$ then the system becomes

$$\left\{\begin{array}{rcr} 4u+3v & = & 1 \\ 3u+2v & = & -1 \end{array}\right.$$

This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u=-5$ and $v=7$. Thus $x=\frac{1}{u}=-\frac{1}{5}$ and $y=\frac{1}{v}=\frac{1}{7}$.

We say that the original system is **linear in form** because its equations are not linear but a few substitutions reveal a structure that we can treat like a system of linear equations. The system below is linear in form. Make the appropriate substitutions and solve for x and y.

$$\begin{cases} 4x^3 + 3\sqrt{y} & = & 1 \\ 3x^3 + 2\sqrt{y} & = & -1 \end{cases}$$

17. The polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

    (a) Show that p has no real zeros.

    (b) Because p has no real zeros, its factorization must be of the form $(x^2 + ax + b)(x^2 + cx + d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

    (c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns a, b, c and d. Write $p(x)$ in factored form.

SECTION 6.2 EXERCISE ANSWERS A.1.6

CHAPTER 7

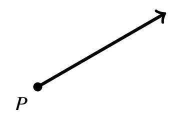
TRIGONOMETRIC FUNCTIONS

7.1 Degree and Radian Measure of Angles

This section serves as a review of the concept of 'angle' and the use of the degree and radian systems to measure angles.

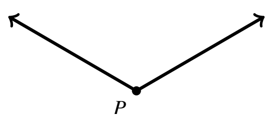
7.1.1 Degree Measure

Recall that a **ray** is usually described as a 'half-line' and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other, as pictured below. The point from which the ray originates is called the **initial point** of the ray.

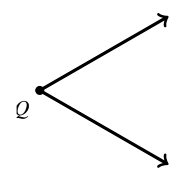


A ray with initial point P.

When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Two examples of what are commonly thought of as angles are

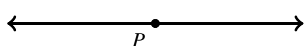


An angle with vertex P.



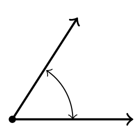
An angle with vertex Q.

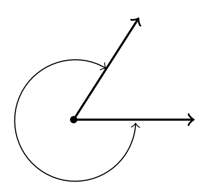
However, the two figures below also depict angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**; in the second, the rays are identical so the 'angle' is indistinguishable from the ray itself.



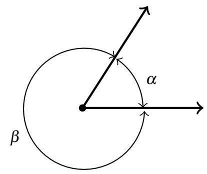
A straight angle.

The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below.

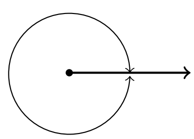




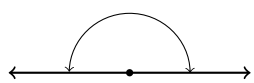
Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.[1] Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have



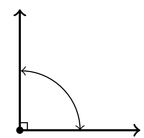
One system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the symbol '°.' One complete revolution as shown below is 360°, and parts of a revolution are measured proportionately.[2] Thus half of a revolution (a straight angle) measures $\frac{1}{2}(360^\circ) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4}(360^\circ) = 90^\circ$ and so on.



One revolution ↔ 360°

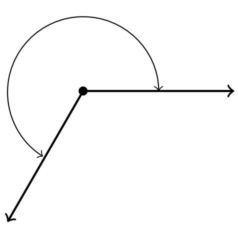


180°

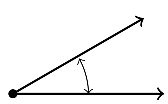


90°

Note that in the above figure, we have used the small square □ to denote a right angle, as is commonplace in Geometry. Recall that if an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of an entire revolution.[3] For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360^\circ) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360^\circ) = 30^\circ$ and an angle which indicates no rotation at all is measured as 0°.



240°



30°

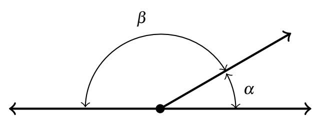
0°

[1]The phrase 'at least' will be justified in short order.

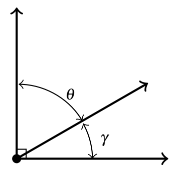
[2]The choice of '360' is most often attributed to the Babylonians.

[3]This is how a protractor is graded.

Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In the diagram below, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.



Supplementary Angles



Complementary Angles

In practice, the distinction between the angle itself and its measure is blurred so that the sentence 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$.' It is now time for an example.

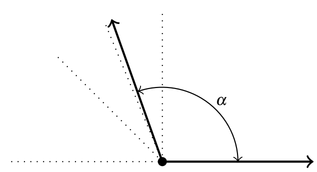
Example 7.1.1. Let $\alpha = 110^\circ$ and $\beta = 37^\circ$.

1. Sketch α and β.
2. Compute a supplementary angle for α.
3. Compute a complementary angle for β.

Solution.

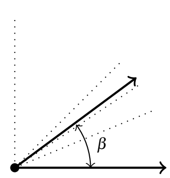
1. Sketch α and β.

   To sketch α, we first note that $90^\circ < \alpha < 180^\circ$. Dividing this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for α, as shown below.



Angle α

   Proceeding similarly for β, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.



Angle β

2. Compute a supplementary angle for α.

   To find a supplementary angle for α, we seek an angle θ so that $\alpha + \theta = 180^\circ$.

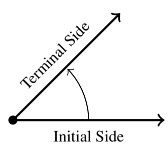
   We get $\theta = 180^\circ - \alpha = 180^\circ - 110^\circ = 70^\circ$.

3. Compute a complementary angle for β.

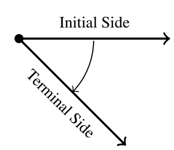
   To find a complementary angle for β, we seek an angle γ so that $\beta + \gamma = 90^\circ$.

   We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ = 53^\circ$. $\square$

Up to this point, we have discussed only angles which measure between 0° and 360°, inclusive. Ultimately, we want to use the arsenal of Algebra which we have stockpiled in Chapters 1 through 6 to not only solve geometric problems involving angles, but also to extend their applicability to other real-world phenomena. A first step in this direction is to extend our notion of 'angle' from merely measuring an extent of rotation to quantities which indicate an amount of rotation along with a **direction**. To that end, we introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

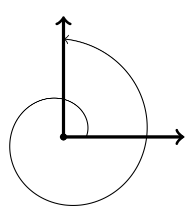


A positive angle, 45°



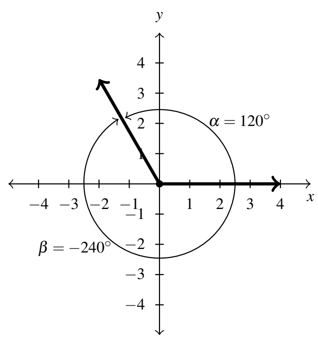
A negative angle, -45°

At this point, we also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the 'first' 360°) then continue with an additional 90° counter-clockwise rotation, as seen below.



450°

To further connect angles with the Algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive horizontal (usually labeled as the x-) axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a 'Quadrant I angle'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side.[4] In the figure below, $\alpha = 120^\circ$ and $\beta = -240^\circ$ are two coterminal Quadrant II angles drawn in standard position. Note that $\alpha = \beta + 360^\circ$, or equivalently, $\beta = \alpha - 360^\circ$. We leave it as an exercise to the reader to verify that coterminal angles always differ by a multiple of 360°.[5] More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360^\circ \cdot k$ where k is an integer.[6]



Two coterminal angles, $\alpha = 120^\circ$ and $\beta = -240^\circ$, in standard position.

Example 7.1.2. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Determine three coterminal angles, at least one of which is positive and one of which is negative.

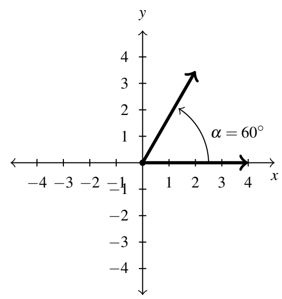
1. $\alpha = 60^\circ$
2. $\beta = -225^\circ$
3. $\gamma = 540^\circ$
4. $\phi = -750^\circ$

Solution.

1. Graph $\alpha = 60^\circ$ in standard position.

   To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$. Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$. Finally, if we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

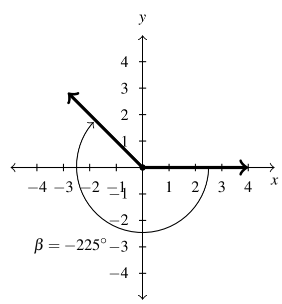
[4]Note that by being in standard position they automatically share the same initial side which is the positive x-axis.
[5]It is worth noting that all of the pathologies of Analytic Trigonometry result from this fact.
[6]Recall that this means $k = 0, \pm 1, \pm 2, \ldots$.



$\alpha = 60°$ in standard position.

2. Graph $\beta = -225°$ in standard position.

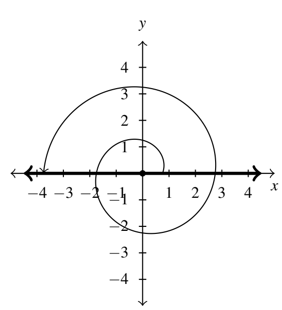
   As a result of $\beta = -225°$ being negative, we start at the positive x-axis and rotate *clockwise* $\frac{225°}{360°} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225° + 360° \cdot k$ for integer values of k. We find $135°$, $-585°$ and $495°$ are all coterminal with $-225°$, when $k = 1$, -2, and 2 respectively



$\beta = -225°$ in standard position.

3. Graph $\gamma = 540°$ in standard position.

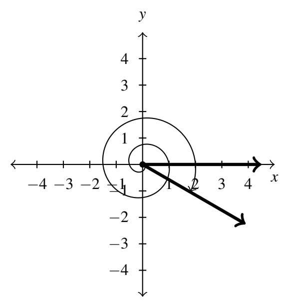
   As $\gamma = 540°$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $360°$, with $180°$, or $\frac{1}{2}$ of a revolution remaining. The terminal side of γ lies on the negative x-axis, so γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540° + 360° \cdot k$, where k is an integer. Working through the arithmetic as before, we find three such angles: $180°$, $-180°$ and $900°$.



$\gamma = 540°$ in standard position.

4. Graph $\phi = -750^\circ$ in standard position.

   The Greek letter ϕ is pronounced 'fee' or 'fie' and because ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°.



$\phi = -750^\circ$ in standard position.

□

7.1.2 Radian Measure

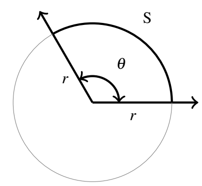
While degrees are the unit of choice for many applications of trigonometry, we introduce here the concept of the **radian measure** of an angle. As we will see, this concept naturally ties angles to real numbers. While the concept may seem foreign at first, we assure the reader that the utility of radian measure in modeling real-world phenomena is well worth the effort. We begin our development with a definition from Geometry.

Definition 7.1. The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

$$\pi = \frac{C}{d}$$

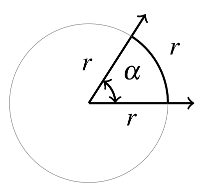
While Definition 7.1 is quite possibly the 'standard' definition of π, the authors would be remiss if we didn't mention that buried in this definition is actually a theorem. As the reader is probably aware, the number π is a mathematical constant - that is, it doesn't matter *which* circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle. While this is indeed true, it is far from obvious and leads to a counter intuitive scenario which is explored in the Exercises. Because the diameter of a circle is twice its radius, we can quickly rearrange the equation in Definition 7.1 to get a formula more useful for our purposes, namely: $2\pi = \frac{C}{r}$. Hence, for any circle, the ratio of its circumference to its radius is 2π.

Suppose we take a *portion* of the circle, and we compare some arc measuring s units in length to the radius. Let θ be the **central angle** subtended by this arc, that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality (similarity) arguments, it stands to reason that the ratio $\frac{s}{r}$ should also be a constant among all circles. It is this ratio, $\frac{s}{r}$, which defines the **radian measure** of an angle.

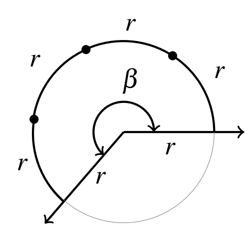


The radian measure of θ is $\dfrac{s}{r}$.

To get a better feel for radian measure, we note that an angle with radian measure 1 means the corresponding arc length s equals the radius of the circle r, that is, $s = r$. When the radian measure is 2, we have $s = 2r$; when the radian measure is 3, $s = 3r$, and so forth. Thus the radian measure of an angle θ tells us how many 'radius lengths' we need to sweep out along the circle to subtend the angle θ.



α has radian measure 1



β has radian measure 4

One revolution sweeps out the circumference $2\pi r$, so one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$, a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered 'pure' numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word 'radians' to denote these dimensionless units as needed. For instance, we say one revolution measures '2π radians,' half of a revolution measures 'π radians,' and so forth.

As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write '$\theta = \frac{\pi}{2}$', we mean θ is an angle which measures $\frac{\pi}{2}$ radians.[7] We extend radian measure to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation.[8] Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

[7]The authors are well aware that we are now identifying radians with real numbers. We will justify this shortly.
[8]This, in turn, endows the subtended arcs with an orientation as well. We address this in short order.

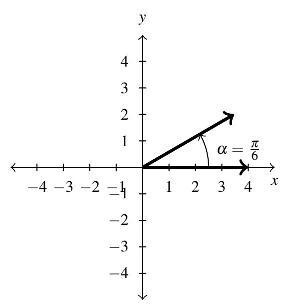
Example 7.1.3. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \dfrac{\pi}{6}$
2. $\beta = -\dfrac{4\pi}{3}$
3. $\gamma = \dfrac{9\pi}{4}$
4. $\phi = -\dfrac{5\pi}{2}$

Solution.

1. Graph and classify $\alpha = \dfrac{\pi}{6}$.

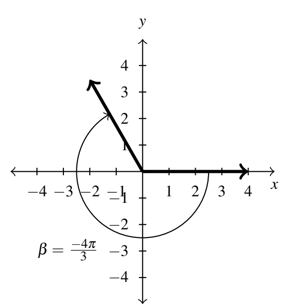
   The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle. Coterminal angles θ are of the form $\theta = \alpha + 2\pi \cdot k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$, thus when $k = 1$, we get $\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Substituting $k = -1$ gives $\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$ and when we let $k = 2$, we get $\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$.



$\alpha = \frac{\pi}{6}$ in standard position.

2. Graph and classify $\beta = -\dfrac{4\pi}{3}$.

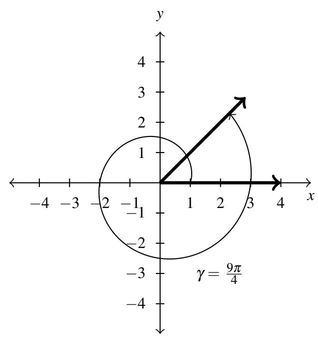
   As $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle. To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles.



$\beta = -\frac{4\pi}{3}$ in standard position.

3. Graph and classify $\gamma = \dfrac{9\pi}{4}$.

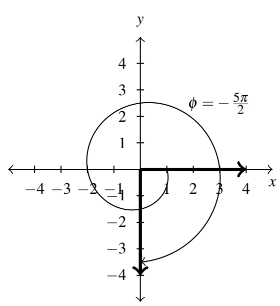
   AS $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a Quadrant I angle. All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where k is an integer. Working through the arithmetic for $k = 1, -1, 2$, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$, respectively.



$\gamma = \frac{9\pi}{4}$ in standard position.

4. Graph and classify $\phi = -\dfrac{5\pi}{2}$.

   To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. The terminal side of ϕ lies on the negative y-axis, thus ϕ is a quadrantal angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers $k = 1, 2, 3$ and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.



$\phi = -\frac{5\pi}{2}$ in standard position.

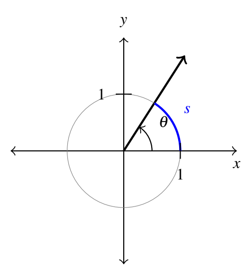
□

It is worth mentioning that we could have plotted the angles in Example 7.1.3 by first converting them to degree measure and following the procedure set forth in Example 7.1.2. While converting back and forth from degrees and radians is certainly a good skill to have, it is best that you learn to 'think in radians' as well as you can 'think in degrees'. The authors would, however, be derelict in our duties if we ignored the basic conversion between these systems altogether. As one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems. For example, to convert 60° to radians we find $60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3}$ radians, or simply $\frac{\pi}{3}$. To convert from radian measure back to degrees, we multiply by the ratio $\frac{180^\circ}{\pi \text{ radian}}$. For example, $-\frac{5\pi}{6}$ radians is equal to $\left(-\frac{5\pi}{6} \text{ radians}\right)\left(\frac{180^\circ}{\pi \text{ radians}}\right) = -150^\circ$.[9] Hence, an angle which measures 1 in radian measure is equal to $\frac{180^\circ}{\pi} \approx 57.2958^\circ$. To summarize:

Equation 7.1. Degree - Radian Conversion:

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$

In light of Example 7.1.3 and Equation 7.1, the reader may well wonder what the allure of radian measure is. The numbers involved are, admittedly, much more complicated than degree measure. The answer lies in how easily angles in radian measure can be identified with real numbers. Consider the Unit Circle, $x^2 + y^2 = 1$, the angle θ in standard position and the corresponding arc measuring s units in length. By definition, and the fact that the Unit Circle has radius 1, the radian measure of θ is $\frac{s}{r} = \frac{s}{1} = s$ so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$. In order to identify real numbers with oriented angles, we essentially 'wrap' the real number line around the Unit Circle and associating to each real number t an *oriented* arc on the Unit Circle with initial point $(1,0)$.

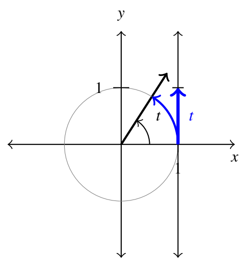


On the Unit Circle, $\theta = 2$

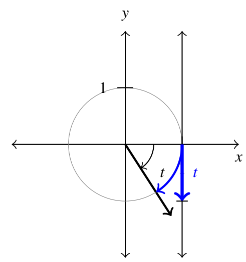
Viewing the vertical line $x = 1$ as another real number line demarcated like the y-axis, given a real number $t > 0$, we 'wrap' the (vertical) interval $[0,t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of t units and therefore the corresponding angle has radian measure equal to t. If $t < 0$, we wrap the interval $[t,0]$ *clockwise* around the Unit Circle. We have defined clockwise rotation as having negative radian measure, therefore the angle determined by this arc has radian measure equal to t. If

[9] Note that the negative sign indicates clockwise rotation in both systems, and so it is carried along accordingly.

$t=0$, we are at the point $(1,0)$ on the x-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number t with the corresponding angle with radian measure t.



Identifying $t>0$ with an angle.



Identifying $t<0$ with an angle.

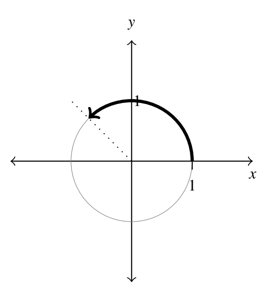
Example 7.1.4. Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t=\dfrac{3\pi}{4}$
2. $t=-2\pi$
3. $t=-2$
4. $t=117$

Solution.

1. Sketch the oriented arc on the Unit Circle corresponding to $t=\dfrac{3\pi}{4}$.

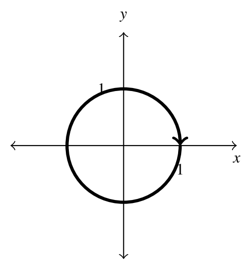
   The arc associated with $t=\frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. As $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1,0)$ proceeds counter-clockwise up to midway through Quadrant II.



$t=\frac{3\pi}{4}$

2. Sketch the oriented arc on the Unit Circle corresponding to $t=-2\pi$.

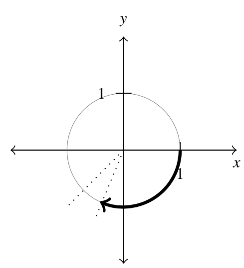
   One revolution is 2π radians and $t=-2\pi$ is negative, so we graph the arc which begins at $(1,0)$ and proceeds *clockwise* for one full revolution.



$t = -2\pi$

3. Sketch the oriented arc on the Unit Circle corresponding to $t = -2$.

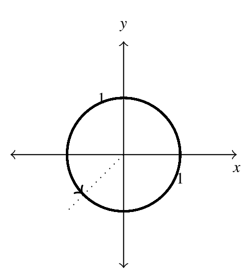
   Like $t = -2\pi$, $t = -2$ is negative, so we begin our arc at $(1,0)$ and proceed clockwise around the unit circle. Because $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find that rotating 2 radians clockwise from the point $(1,0)$ lands us in Quadrant III. To more accurately place the endpoint, we proceed by successively halving the angle measure until we find $\frac{5\pi}{8} \approx 1.96$ which tells us our arc extends just a bit beyond the quarter mark into Quadrant III.



$t = -2$

4. Sketch the oriented arc on the Unit Circle corresponding to $t = 117$.

   Because 117 is positive, the arc corresponding to $t = 117$ begins at $(1,0)$ and proceeds counter-clockwise. As 117 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{117}{2\pi}$ as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of $\frac{5}{8}$ of a revolution to spare. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III.

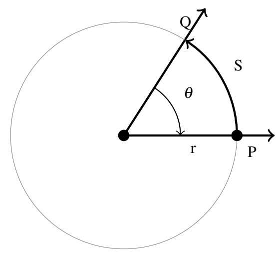


$t = 117$

$\square$

7.1.3 Applications of Radian Measure: Circular Motion

Now that we have paired angles with real numbers via radian measure, a whole world of applications awaits us. Our first excursion into this realm comes by way of circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.



Here s represents a *displacement*, so that $s > 0$ means the object is traveling in a counter-clockwise direction and $s < 0$ indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds as a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation. In Physics, the **average velocity** of the object, denoted $\overline{v}$ and read as 'v-bar', is defined as the average rate of change of the position of the object with respect to time.[10] As a result, we have $\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}$. The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. The contribution of direction in the quantity $\overline{v}$ is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion), so that the quantity $|\overline{v}|$ quantifies how fast the object is moving - it is the **speed** of the object. Measuring θ in radians we have $\theta = \frac{s}{r}$ thus $s = r\theta$ and

$$\overline{v} = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t}$$

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$. If $\overline{\omega}$ is constant throughout the duration of the motion, then it can be shown[11] that the average velocities involved, namely $\overline{v}$ and $\overline{\omega}$, are the same as their instantaneous counterparts, v and ω, respectively. In this case, v is simply called the 'velocity' of the object and ω is called the 'angular velocity.'[12]

If the path of the object were 'uncurled' from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity v is often called the *linear* velocity of the object in order to distinguish it from the *angular* velocity, ω. Putting together the ideas of the previous paragraph, we get the following.

[10] See Definition 3.5 in Section 3.2 for a review of this concept.

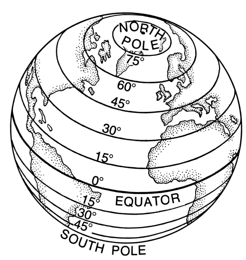
[11] You guessed it, using Calculus . . .

[12] See Example 3.2.3 in Section 3.2 for more of a discussion on instantaneous velocity.

Equation 7.2. Velocity for Circular Motion:
For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by $v = r\omega$.

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units $\text{length} \cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}}$. The supposed contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length} \cdot \text{radians}}{\text{time}}$ reduce to the units $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example 7.1.5. Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle (this is the **parallel of latitude** of the point) as seen in the figure below.[13] It takes the Earth (approximately) 24 hours to rotate, so the object takes 24 hours to complete one revolution along this circle. Lakeland Community College is at 41.628° north latitude, and it can be shown[14] that the radius of the earth at this Latitude is approximately 2960 miles. Find the linear velocity, in miles per hour, of Lakeland Community College as the world turns.



Solution. To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The earth makes one revolution in 24 hours, and one revolution is 2π radians, so $\omega = \frac{2\pi\,\text{radians}}{24\,\text{hours}} = \frac{\pi}{12\,\text{hours}}$. Note that once again, we are identifying angles in radian measure as real numbers so we can drop the 'radian' units as they are dimensionless. Also note that for simplicity's sake, we assume that we are viewing the rotation of the earth as counter-clockwise so $\omega > 0$. Hence, the linear velocity is

$$v = 2960\,\text{miles} \cdot \frac{\pi}{12\,\text{hours}} \approx 775\,\frac{\text{miles}}{\text{hour}}$$

□

It is worth noting that the quantity $\frac{1\,\text{revolution}}{24\,\text{hours}}$ in Example 7.1.5 is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object

[13] Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.
[14] We will discuss how we arrived at this approximation in Example 7.2.5.

makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of Example 7.1.5, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency because they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the 'magnification factor' which relates ω and v. We will have more to say about frequencies and periods in Section 7.3. While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: if an object is moving on a circular path of radius r with a fixed angular velocity (frequency) ω, what is the position of the object at time t? The answer to this question is the very heart of Trigonometry and is answered in the next section.

7.1.4 Exercises

In Exercises 1 - 12, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

1. $30°$
2. $120°$
3. $225°$
4. $330°$
5. $-30°$
6. $-135°$
7. $-240°$
8. $-270°$
9. $405°$
10. $840°$
11. $-510°$
12. $-900°$

In Exercises 13 - 28, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

13. $\frac{\pi}{3}$
14. $\frac{5\pi}{6}$
15. $-\frac{11\pi}{3}$
16. $\frac{5\pi}{4}$
17. $\frac{3\pi}{4}$
18. $-\frac{\pi}{3}$
19. $\frac{7\pi}{2}$
20. $\frac{\pi}{4}$
21. $-\frac{\pi}{2}$
22. $\frac{7\pi}{6}$
23. $-\frac{5\pi}{3}$
24. 3π
25. -2π
26. $-\frac{\pi}{4}$
27. $\frac{15\pi}{4}$
28. $-\frac{13\pi}{6}$

In Exercises 29 - 36, convert the angle from degree measure into radian measure, giving the exact value in terms of π.

29. $0°$
30. $240°$
31. $135°$
32. $-270°$
33. $-315°$
34. $150°$
35. $45°$
36. $-225°$

In Exercises 37 - 44, convert the angle from radian measure into degree measure.

37. π
38. $-\frac{2\pi}{3}$
39. $\frac{7\pi}{6}$
40. $\frac{11\pi}{6}$
41. $\frac{\pi}{3}$
42. $\frac{5\pi}{3}$
43. $-\frac{\pi}{6}$
44. $\frac{\pi}{2}$

In Exercises 45 - 49, sketch the oriented arc on the Unit Circle which corresponds to the given real number.

45. $t = \frac{5\pi}{6}$
46. $t = -\pi$
47. $t = 6$
48. $t = -2$
49. $t = 12$

50. A yo-yo which is 2.25 inches in diameter spins at a rate of 4500 revolutions per minute. How fast is the edge of the yo-yo spinning in miles per hour? Leave your answer in exact form.

51. How many revolutions per minute would the yo-yo in exercise 50 have to complete if the edge of the yo-yo is to be spinning at a rate of 42 miles per hour? Leave your answer in exact form.

52. In the yo-yo trick 'Around the World,' the performer throws the yo-yo so it sweeps out a vertical circle whose radius is the yo-yo string. If the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle, compute the speed of the yo-yo in miles per hour. Leave your answer in exact form.

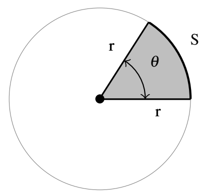
53. A computer hard drive contains a circular disk with diameter 2.5 inches and spins at a rate of 7200 revolutions per minute. Find the linear speed of a point on the edge of the disk in miles per hour.

54. A rock got stuck in the tread of my tire and when I was driving 70 miles per hour, the rock came loose and hit the inside of the wheel well of the car. How fast, in miles per hour, was the rock traveling when it came out of the tread? (The tire has a diameter of 23 inches.)

55. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet. It completes two revolutions in 2 minutes and 7 seconds.[15] Assuming the riders are at the edge of the circle, how fast are they traveling in miles per hour?

[15]Source: Cedar Point's webpage.

56. Consider the circle of radius r pictured below with central angle θ, measured in radians, and subtended arc of length s. Prove that the area of the shaded sector is $A = \frac{1}{2}r^2\theta$.
(Hint: Use the proportion $\frac{A}{\text{area of the circle}} = \frac{s}{\text{circumference of the circle}}$.)



In Exercises 57 - 62, use the result of Exercise 56 to compute the areas of the circular sectors with the given central angles and radii.

57. $\theta = \dfrac{\pi}{6}$, $r = 12$
58. $\theta = \dfrac{5\pi}{4}$, $r = 100$
59. $\theta = 330^\circ$, $r = 9.3$
60. $\theta = \pi$, $r = 1$
61. $\theta = 240^\circ$, $r = 5$
62. $\theta = 1^\circ$, $r = 117$

63. Imagine a rope tied around the Earth at the equator. Show that you need to add only 2π feet of length to the rope in order to lift it one foot above the ground around the entire equator. (You do NOT need to know the radius of the Earth to show this.)

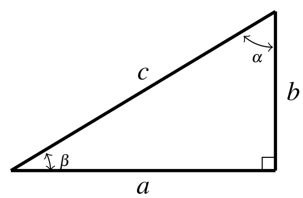
64. With the help of your classmates, look for a proof that π is indeed a constant.

SECTION 7.1 EXERCISE ANSWERS A.1.7

7.2 Sine and Cosine Functions

7.2.1 Right Triangle Defintions

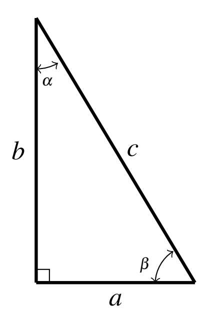
The word 'trigonometry' literally means 'measuring triangles,' so naturally most students' first introduction to trigonometry focuses on triangles. This section focuses on **right triangles**, triangles in which one angle measures 90°. Consider the right triangle below, where, as usual, the small square □ denotes the right angle, the labels 'a,' 'b,' and 'c' denote the lengths of the sides of the triangle, and α and β represent the (measure of) the non-right angles. As you may recall, the side opposite the right angle is called the **hypotenuse** of the right triangle. Also note that because the sum of the measures of all angles in a triangle must add to 180°, we have $\alpha+\beta+90^\circ = 180^\circ$, or $\alpha+\beta = 90^\circ$. Said differently, the non-right angles in a right triangle are *complements*.

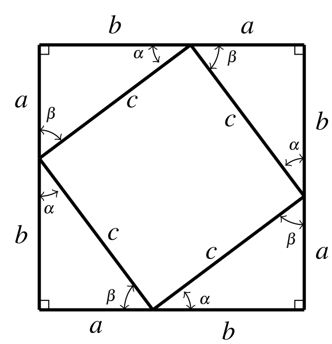


We now state and prove the most famous result about right triangles: **The Pythagorean Theorem**.

Theorem 7.1. (**The Pythagorean Theorem**) The square of the length of the hypotenuse of a right triangle is equal to the sums of the squares of the other two sides. More specifically, if c is the length of the hypotenuse of a right triangle and a and b are the lengths of the other two sides, then $a^2+b^2 = c^2$.

There are several proofs of the Pythagorean Theorem,[1] but the one we choose to reproduce here showcases a nice interplay between algebra and geometry. Consider taking four copies of the right triangle below on the left and arranging them as seen below on the right.





It should be clear that we have produced a large square with a side length of $(a+b)$. What is also true, but may not be obvious, is that the inner quadrilateral is also a square. We can readily see the inner quadrilateral has equal sides of length c. Moreover, because $\alpha+\beta = 90^\circ$, we get the interior angles of the inner quadrilateral are each 90°. Hence, the inner quadrilateral is indeed a square.

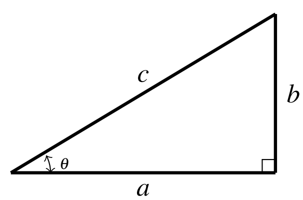
[1] Including one by Mentor, Ohio native President James Garfield.

We finish the proof by computing the area of the large square in two ways. First, we square the length of its side: $(a+b)^2$. Next, we add up the areas of the four triangles, each having area $\frac{1}{2}ab$ along with the area of the inner square, c^2. Equating these to expressions gives: $(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$. As a result of $(a+b)^2 = a^2+2ab+b^2$ and $4\left(\frac{1}{2}ab\right) = 2ab$, we have $a^2+2ab+b^2 = 2ab+c^2$ or $a^2+b^2 = c^2$, as required.

It should be noted that the converse of the Pythagorean Theorem is also true. That is if a, b, and c are the lengths of sides of a triangle and $a^2+b^2=c^2$, then the triangle is a right triangle.[2]

A list of integers (a,b,c) which satisfy the relationship $a^2+b^2=c^2$ is called a **Pythagorean Triple**. Some of the more common triples are: $(3,4,5)$, $(5,12,13)$, $(7,24,25)$, and $(8,15,17)$. We leave it to the reader to verify these integers satisfy the equation $a^2+b^2=c^2$ and suggest committing these triples to memory.

Next, we set about defining characteristic ratios associated with acute angles. Given any acute angle θ, we can imagine θ being an interior angle of a right triangle as seen below.



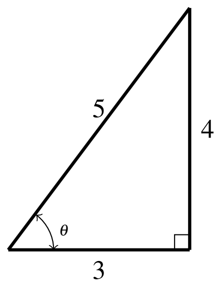
Focusing on the arrangement of the sides of the triangle with respect to the angle θ, we make the following definitions: the side with length a is called the side of the triangle which is **adjacent** to θ and the side with length b is called the side of the triangle **opposite** θ. As usual, the side labeled 'c' (the side opposite the right angle) is the hypotenuse. Using this diagram, we define two important **trigonometric ratios** of θ.

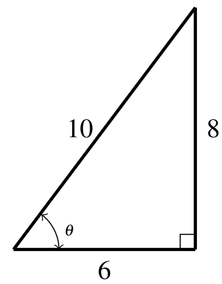
Definition 7.2. Suppose θ is an acute angle residing in a right triangle as depicted above.

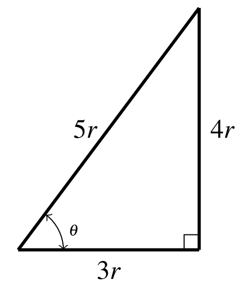
- The **sine** of θ, denoted $\sin(\theta)$ is defined by the ratio: $\sin(\theta) = \dfrac{b}{c}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of hypotenuse'}}$.
- The **cosine** of θ, denoted $\cos(\theta)$ is defined by the ratio: $\cos(\theta) = \dfrac{a}{c}$, or $\dfrac{\text{'length of adjacent'}}{\text{'length of hypotenuse'}}$.

For example, consider the angle θ indicated in the 3-4-5 triangle given. Using Definition 7.2, we get $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$. One may well wonder if these trigonometric ratios we've found for θ change if the triangle containing θ changes. For example, if we scale all the sides of the 3-4-5 triangle on the left by a factor of 2, we produce the **similar triangle** below in the middle.[3] Using this triangle to compute our ratios for θ, we find $\sin(\theta) = \frac{8}{10} = \frac{4}{5}$ and $\cos(\theta) = \frac{6}{10} = \frac{3}{5}$. Note that the scaling factor, here 2, is common to all sides of the triangle, and, hence, divides out of the numerator and denominator when simplifying each of the ratios.

[2]We will prove this in Section 8.5 by generalizing the Pythagorean Theorem to a formula that works for *all* triangles.
[3]That is, a triangle with the same 'shape' - that is, the same angles.

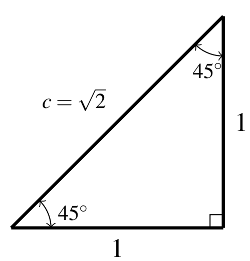






In general, thanks to the Angle Angle Similarity Postulate, any two *right* triangles which contain our angle θ are similar which means there is a positive constant r so that the sides of the triangle are $3r$, $4r$, and $5r$ as seen above on the right. Hence, regardless of the right triangle in which we choose to imagine θ, $\sin(\theta) = \frac{4r}{5r} = \frac{4}{5}$ and $\cos(\theta) = \frac{3r}{5r} = \frac{3}{5}$. Generalizing this same argument to any acute angle θ assures us that the ratios as described in Definition 7.2 are independent of the triangle we use.

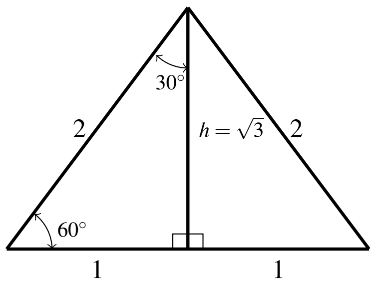
Our next objective is to determine the values of $\sin(\theta)$ and $\cos(\theta)$ for some of the more commonly used angles. We begin with 45°. In a right triangle, if one of the non-right angles measures 45°, then the other measures 45° as well. It follows that the two legs of the triangle must be congruent. We may choose any right triangle containing a 45° angle for our computations, thus we choose the length of one (hence both) of the legs to be 1. The Pythagorean Theorem gives the hypotenuse is: $c^2 = 1^2 + 1^2 = 2$, so $c = \sqrt{2}$. (We take only the positive square root here as c represents the length of the hypotenuse here, so, necessarily $c > 0$.) From this, we obtain the values below, and suggest committing them to memory.



- $\sin(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\cos(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Note that we have 'rationalized' here to avoid the irrational number $\sqrt{2}$ appearing in the denominator. This is a common convention in trigonometry, and we will adhere to it unless extremely inconvenient.

Next, we investigate 60° and 30° angles. Consider the equilateral triangle each of whose sides measures 2 units. Each of its interior angles is necessarily 60°, so if we drop an altitude, we produce two $30^\circ - 60^\circ - 90^\circ$ triangles each having a base measuring 1 unit and a hypotenuse of 2 units. Using the Pythagorean Theorem, we can find the height, h of these triangles: $1^2 + h^2 = 2^2$ so $h^2 = 3$ or $h = \sqrt{3}$. Using these, we can find the values of the trigonometric ratios for both 60° and 30°. Again, we recommend committing these values to memory.

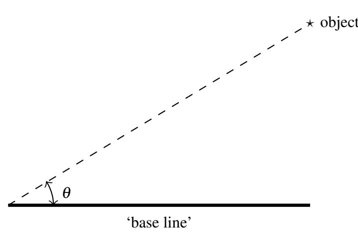


- $\sin(60^\circ) = \frac{\sqrt{3}}{2}$
- $\cos(60^\circ) = \frac{1}{2}$
- $\sin(30^\circ) = \frac{1}{2}$
- $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

Recal 30° and 60° are complements, so the side *adjacent* to the 60° angle is the side *opposite* the 30° and the side *opposite* the 60° angle is the side *adjacent* to the 30° . This sort of 'swapping' is true of all complementary angles and will be generalized in Section 8.2, Theorem 8.6.

Note that the values of the trigonometric ratios we have derived for 30°, 45°, and 60° angles are the *exact* values of these ratios. For these angles, we can conveniently express the exact values of their sines and cosines resorting, at worst, to using square roots. The reader may well wonder if, for instance, we can express the exact value of, say, $\sin(42^\circ)$ in terms of radicals. The answer in this case is 'yes' (see here), but, in general, we will not take the time to pursue such representations.[4] Hence, if a problem requests an 'exact' answer involving $\sin(42^\circ)$, we will leave it written as '$\sin(42^\circ)$' and use a calculator to produce a suitable approximation as the situation warrants.

The angle of inclination (or angle of elevation) of an object refers to the angle whose initial side is some kind of base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. Schematically:

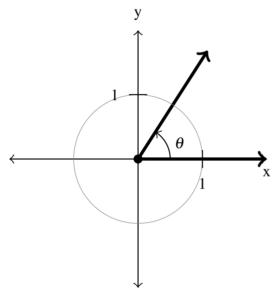


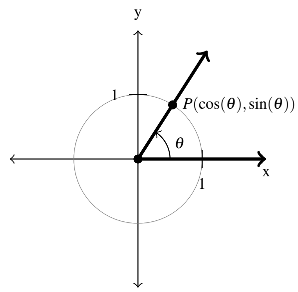
The angle of inclination from the base line to the object is θ

7.2.2 Unit Circle Definitions

In Section 7.1.3, we introduced circular motion and derived a formula which describes the linear velocity of an object moving on a circular path at a constant angular velocity. One of the goals of this section is describe the *position* of such an object. To that end, consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle, as diagrammed below.

[4] We will do a little of this in Section 8.2.





By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. For each angle θ, the terminal side of θ, when graphed in standard position, intersects The Unit Circle only once, so the mapping of θ to P is a function.[5] Because there is only *one* way to describe a point using rectangular coordinates,[6] the mappings of θ to each of the x and y coordinates of P are also functions. We give these functions names in the following definition.

Definition 7.3. Suppose an angle θ is graphed in standard position. Let $P(x,y)$ be the point of intersection of the terminal side of P and the Unit Circle.

- The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$.
- The y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$.[a]

[a]The etymology of the name 'sine' is quite colorful, and the interested reader is invited to research it; the 'co' in 'cosine' is related to the concept of 'co'mplementary angles (see Sections 7.1 and 7.2.1) and is explained in detail in Section 8.2.

You may have already seen definitions for the sine and cosine of an (acute) angle in terms of ratios of sides of a right triangle.[7] While not incorrect, defining sine and cosine using right triangles limits the angles we can study to acute angles only. Definition 7.3, on the other hand, applies to *all* angles. These functions are defined in terms of points on the Unit *Circle*, thus they are called **circular** functions. Rest assured, Definition 7.3 specializes to Definition 7.2 when θ is an acute angle. We will see instances of this fact in the next example.

Example 7.2.1. State the sine and cosine of the following angles.

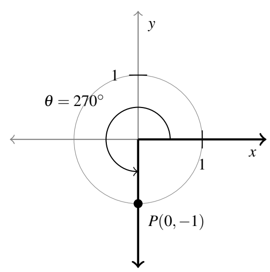
1. $\theta = 270^\circ$
2. $\theta = -\pi$
3. $\theta = 45^\circ$
4. $\theta = \frac{\pi}{6}$
5. $\theta = \frac{5\pi}{6}$

[5]See Definition 1.3 in Section 1.2.
[6]See page 95 in Section 1.1.
[7]For instance, Definition 7.2 in Section 7.2.1.

Solution.

1. State the sine and cosine of $\theta = 270°$.

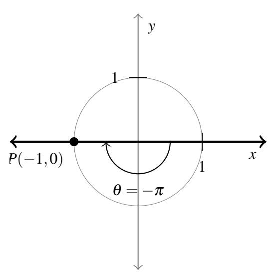
   To find $\cos(270°)$ and $\sin(270°)$, we plot the angle $\theta = 270°$ in standard position and find the point on the terminal side of θ which lies on the Unit Circle. As $270°$ represents $\frac{3}{4}$ of a counter-clockwise revolution, the terminal side of θ lies along the negative y-axis. Hence, the point we seek is $(0,-1)$ so that $\cos(270°) = 0$ and $\sin(270°) = -1$.



Finding $\cos(270°)$ and $\sin(270°)$

2. State the sine and cosine of $\theta = -\pi$.

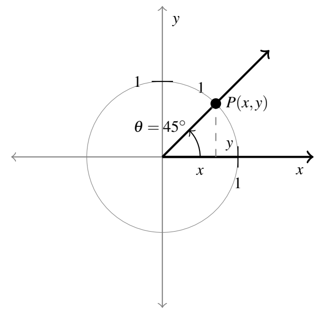
   The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative x-axis. The point on the Unit Circle that lies on the negative x-axis is $(-1,0)$ which means $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.



Finding $\cos(-\pi)$ and $\sin(-\pi)$

3. State the sine and cosine of $\theta = 45°$.

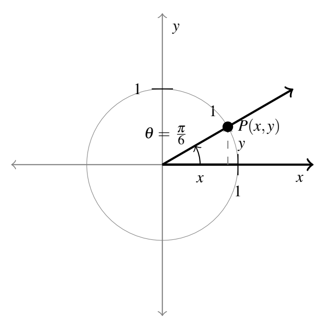
   In this section, we derived values for $\cos(45°)$ and $\sin(45°)$ using Definition 7.2. In order to connect what we know from Section 7.2.1 with what we are asked to find per Definition 7.3, we sketch $\theta = 45°$ in standard position and let $P(x,y)$ denote the point on the terminal side of θ which lies on the Unit Circle. If we drop a perpendicular line segment from P to the x-axis as shown, we obtain a $45° - 45° - 90°$ right triangle whose legs have lengths x and y units with hypotenuse 1. From our work in Section 7.2.1, we obtain the (familiar) values $x = \cos(45°) = \frac{\sqrt{2}}{2}$ and $y = \sin(45°) = \frac{\sqrt{2}}{2}$.



Finding $\cos(45°)$ and $\sin(45°)$

4. State the sine and cosine of $\theta = \frac{\pi}{6}$.

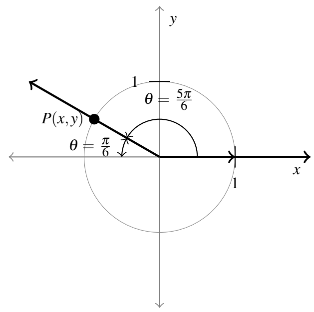
   As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x,y)$ denote the point on the terminal side of θ which lies on the Unit Circle, we drop a perpendicular line segment from P to the x-axis to form a $30° - 60° - 90°$ right triangle. Re-using some of our work from this section, we get $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

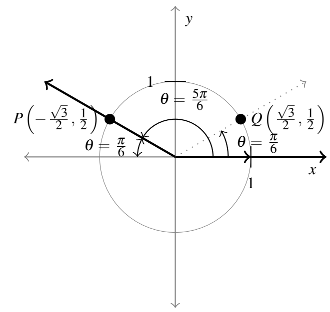


Finding $\cos(\frac{\pi}{6})$ and $\sin(\frac{\pi}{6})$

5. State the sine and cosine of $\theta = \frac{5\pi}{6}$.

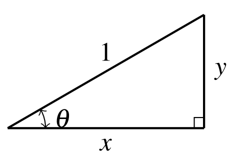
   We plot $\theta = \frac{5\pi}{6}$ in standard position below on the left and, as usual, let $P(x,y)$ denote the point on the terminal side of θ which lies on the Unit Circle. In plotting θ, we find it lies $\frac{\pi}{6}$ radians short of one half revolution. As we've just determined that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, we know the coordinates of the point Q below on the right are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using symmetry, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

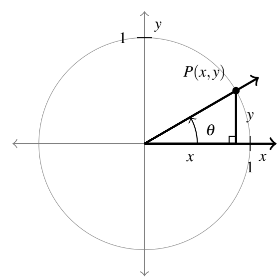




□

A few remarks are in order. First, after having re-used some of our work from this section in a few specific instances, we can reconcile Definition 7.3 with Definition 7.2 in the case θ is an acute angle. We situate θ in a right triangle with hypotenuse length 1, adjacent side length 'x,' and the opposite side length 'y' as seen below on the left. Placing the vertex of θ at the origin and the adjacent side of θ along the x-axis as seen below on the right effectively puts θ in standard position with θ's adjacent side as the initial side of θ and the hypotenuse as the terminal side of θ. The hypotenuse of the triangle has length 1, thus we know the point $P(x,y)$ is on the Unit Circle.[8]



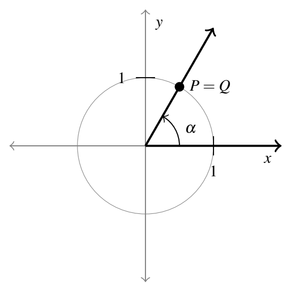


Definition 7.2 gives $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$ which exactly matches Definition 7.3. Hence, in the case of acute angles, the two definitions agree. In other words, the values of the *trigonometric ratios* of acute angles are the same as the corresponding *circular function* values.

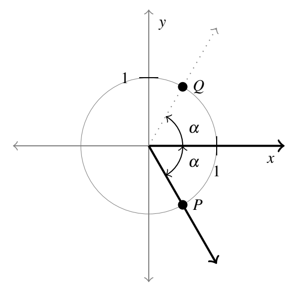
A second important take-away from Example 7.2.1 is use of symmetry in number 5. Indeed, we found the sine and cosine of $\frac{5\pi}{6}$ using the (acute) angle $\frac{\pi}{6}$ 'for reference.' As the Unit Circle is rife with symmetry, we would like to generalize this concept and exploit symmetry whenever possible. To that end, we introduce the notion of **reference angle**.

In general, for a non-quadrantal angle θ, the reference angle for θ (which we'll usually denote α) is the *acute* angle made between the terminal side of θ and the x-axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* x-axis:

[8]Do you see why?

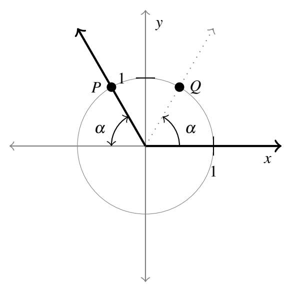


Reference angle α for a Quadrant I angle

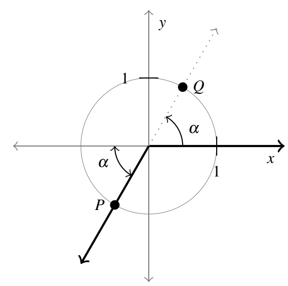


Reference angle α for a Quadrant IV angle

If θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* x-axis:



Reference angle α for a Quadrant II angle



Reference angle α for a Quadrant III angle

If we let P denote the point $(\cos(\theta), \sin(\theta))$, then P lies on the Unit Circle. Due to the fact that the Unit Circle possesses symmetry with respect to the x-axis, y-axis and origin, regardless of where the terminal side of θ lies, there is a point Q symmetric with P which determines θ's reference angle, α. The only difference between the points P and Q are the signs of their coordinates, $\pm$. Hence, we have the following:

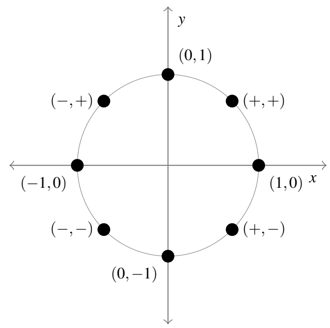
Theorem 7.2. Reference Angle Theorem.
Suppose α is the reference angle for θ. Then:

$$\cos(\theta) = \pm\cos(\alpha) \text{ and } \sin(\theta) = \pm\sin(\alpha),$$

where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

In light of Theorem 7.2, it pays to know the sine and cosine values for certain common Quadrant I angles as well as to keep in mind the signs of the coordinates of points in the given quadrants.

θ (degrees)	θ (radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1



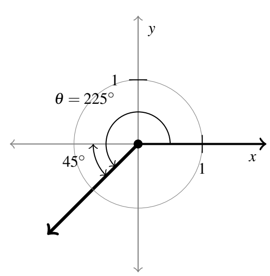
Example 7.2.2. Determine the sine and cosine of the following angles.

1. $\theta = 225^\circ$
2. $\theta = \frac{11\pi}{6}$
3. $\theta = -\frac{5\pi}{4}$
4. $\theta = \frac{7\pi}{3}$

Solution.

1. Determine the sine and cosine of $\theta = 225^\circ$.

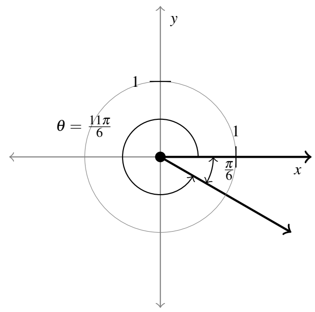
   We begin by plotting $\theta = 225^\circ$ in standard position and find its terminal side overshoots the negative x-axis to land in Quadrant III. Hence, we obtain θ's reference angle α by subtracting: $\alpha = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$. As θ is a Quadrant III angle, both $\cos(\theta) < 0$ and $\sin(\theta) < 0$. The Reference Angle Theorem yields: $\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$ and $\sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$.

   Finding $\cos(225^\circ)$ and $\sin(225^\circ)$

2. Determine the sine and cosine of $\theta = \frac{11\pi}{6}$.

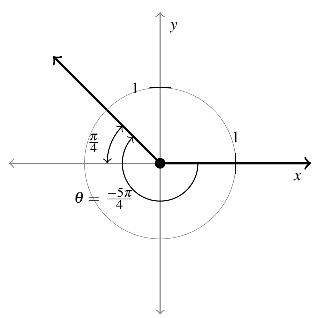
   The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive x-axis. To find θ's reference angle α, we subtract: $\alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. As θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.



Finding $\cos\left(\frac{11\pi}{6}\right)$ and $\sin\left(\frac{11\pi}{6}\right)$

3. Determine the sine and cosine of $\theta = -\frac{5\pi}{4}$.

   To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive x-axis. The terminal side of θ, therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative x-axis. As θ is a Quadrant II angle, $\cos(\theta) < 0$ and $\sin(\theta) > 0$ so the Reference Angle Theorem gives: $\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

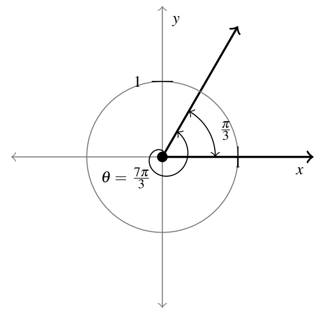


Finding $\cos\left(-\frac{5\pi}{4}\right)$ and $\sin\left(-\frac{5\pi}{4}\right)$

4. Determine the sine and cosine of $\theta = \frac{7\pi}{3}$.

   Given the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Hence, θ and α have the same terminal side,[9] and so $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

[9]Recall we say they are 'coterminal.'



Finding $\cos\left(\frac{7\pi}{3}\right)$ and $\sin\left(\frac{7\pi}{3}\right)$

□

A couple of remarks are in order. First off, the reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.[10]

Also note in number 4 of the last example, the angles $\frac{\pi}{3}$ and $\frac{7\pi}{3}$ are coterminal. As a result, have the same values for sine and cosine. It turns out that we can characterize coterminal angles in this manner, as stated below.

Theorem 7.3. Two angles α and β are coterminal if and only if:

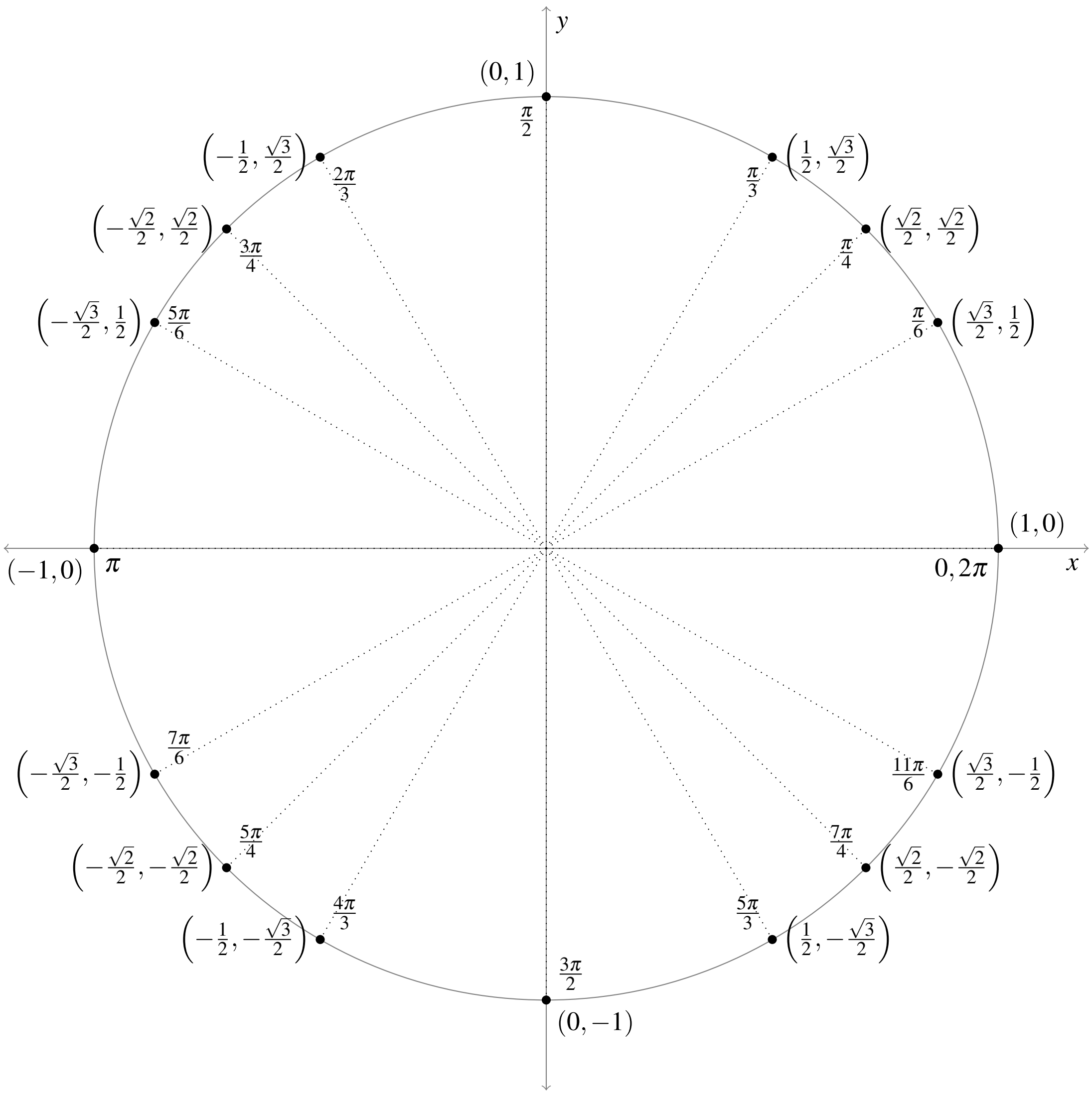
$$\cos(\alpha) = \cos(\beta) \text{ and } \sin(\alpha) = \sin(\beta).$$

Recall the phraseology 'if and only if' means there are two things to argue in Theorem 7.3: first, if α and β are co-terminal, then $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$. This is immediate as coterminal share terminal sides, and, in particular, the (unique) point on the Unit Circle shared by said terminal side. Second, we need to argue that if $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$, then α and β are coterminal.

To prove this second claim, note that when an angle is drawn in standard position, the terminal side of the angle is the ray that starts at the origin and is completely determined by *any* other point on the terminal side. If $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$, then their terminal sides share a point on the Unit Circle, namely $(\cos(\alpha), \sin(\alpha)) = (\cos(\beta), \sin(\beta))$. Hence, α and β are coterminal.

The Reference Angle Theorem in conjunction with the table of sine and cosine values on Page 598 can be used to generate the figure on the next page. We recommend committing it to memory.

[10]For once, we have something convenient about using radian measure in contrast to the abstract theoretical nonsense about using them as a 'natural' way to match oriented angles with real numbers!



Important Points of the Unit Circle

Our next example uses The Reference Angle Theorem in a slightly more sophisticated context.

Example 7.2.3. Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

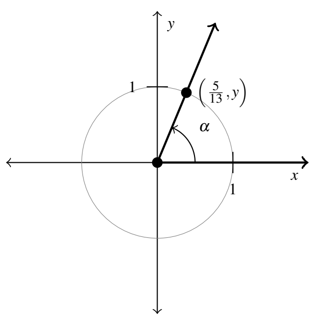
1. Determine $\sin(\alpha)$ and use this to plot α in standard position.
2. State the sine and cosine of the following angles:

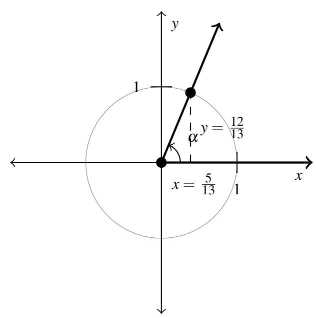
   (a) $\theta = \pi + \alpha$ (b) $\theta = 2\pi - \alpha$ (c) $\theta = 3\pi - \alpha$ (d) $\theta = \frac{\pi}{2} + \alpha$

Solution.

1. Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$. Determine $\sin(\alpha)$ and use this to plot α in standard position.

   Given α is an acute angle, we know $0 < \alpha < \frac{\pi}{2}$, so the terminal side of α lies in Quadrant I. Moreover, because $\cos(\alpha) = \frac{5}{13}$, we know the x-coordinate of the intersection point of the terminal side of α and the Unit Circle is $\frac{5}{13}$. To find $\sin(\alpha)$, we need the y-coordinate. Taking a cue from Example 7.2.1, we drop a perpendicular from the terminal side of α to the x-axis as seen below on the right to form a right triangle with one leg measuring $\frac{5}{13}$ units and hypotenuse with a length of 1 unit.

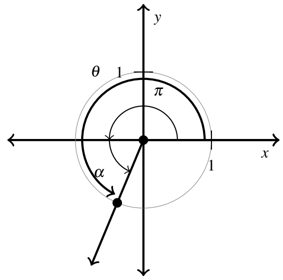
   The Pythagorean Theorem gives $\left(\frac{5}{13}\right)^2 + y^2 = 1^2$ or $y = \frac{12}{13}$. Hence, $\sin(\alpha) = \frac{12}{13}$.

2. Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

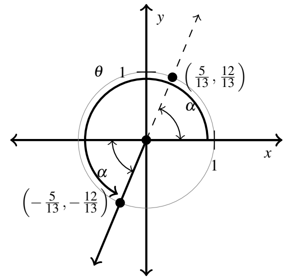
   (a) State the sine and cosine of $\theta = \pi + \alpha$.

   To find the cosine and sine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of π radians followed by a rotation of α radians.[11] We see that α is the reference angle for θ. By The Reference Angle Theorem, $\cos(\theta) = \pm\cos(\alpha) = \pm\frac{5}{13}$ and $\sin(\theta) = \pm\sin(\alpha) = \pm\frac{12}{13}$. As the terminal side of θ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative, so $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

[11] Because $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations given here. You should do this.



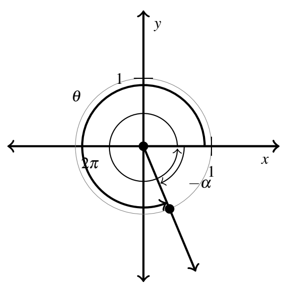
Visualizing $\theta = \pi + \alpha$



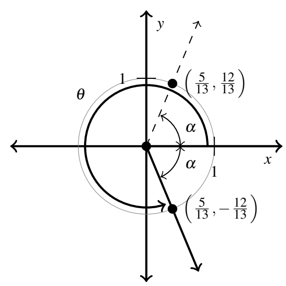
θ has reference angle α

(b) State the sine and cosine of $\theta = 2\pi - \alpha$.

Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a *clockwise* revolution, or 'backing up,' of α radians. Once again, we see that α is θ's reference angle. Because θ is a Quadrant IV angle, we choose the appropriate signs and get: $\cos(\theta) = \frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.



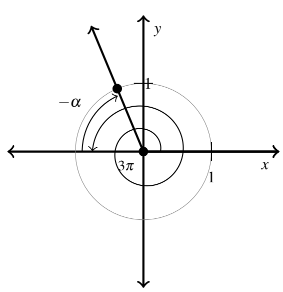
Visualizing $\theta = 2\pi - \alpha$



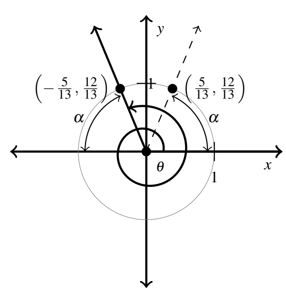
θ has reference angle α

(c) State the sine and cosine of $\theta = 3\pi - \alpha$.

Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so that when we 'back up' α radians, we end up in Quadrant II. As α is the reference angle for θ, The Reference Angle Theorem gives $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = \frac{12}{13}$.



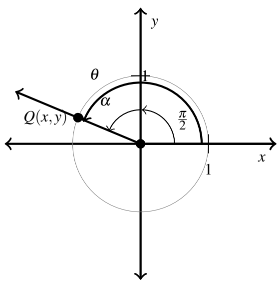
Visualizing $\theta = 3\pi - \alpha$



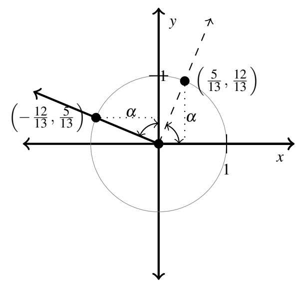
θ has reference angle α

(d) State the sine and cosine of $\theta = \frac{\pi}{2} + \alpha$.

To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is *not* α, so The Reference Angle Theorem is not immediately applicable. (It's important that you see why this is the case. Take a moment to think about this before reading on.) Let $Q(x,y)$ be the point on the terminal side of θ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph α in standard position, we use the fact from Geometry that equal angles subtend equal chords to show that the dotted lines in the figure below are equal. Hence, $x = \cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$.



Visualizing $\theta = \frac{\pi}{2} + \alpha$



Using symmetry to determine $Q(x,y)$

□

A couple of remarks about Example 7.2.3 are in order. First, we note the right triangle we used to find $\sin(\alpha)$ is a scaled 5-12-13 triangle. Recognizing this Pythagorean Triple[12] may have simplified our workflow. Along the same lines, because, the Unit Circle, by definition, is described by the equation $x^2 + y^2 = 1$, we could substitute $x = \frac{5}{13}$ in order to find y. We leave it to the reader to show we get the exact same answer regardless of the approach used.

Our next example turns the tables and makes good use of the Unit Circle values given on page 602 as well as Theorem 7.3 in a different way: instead of giving information about the angle and asking for sine or cosine values, we are given sine or cosine values and asked to produce the corresponding angles. In other words, we solve some rudimentary equations involving sine and cosine.[13]

Example 7.2.4. Determine all angles that satisfy the following equations. Express your answers in radians.[14]

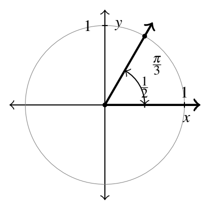
1. $\cos(\theta) = \dfrac{1}{2}$
2. $\sin(\alpha) = -\dfrac{1}{2}$
3. $\cos(\beta) = 0$
4. $\sin(\gamma) = \dfrac{3}{2}$

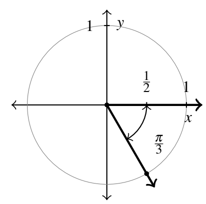
[12]See Section 7.2.1 for more examples of Pythagorean Triples.
[13]We will study equations in more detail in Section 8.3.2.
[14]This ensures we keep building the 'fluency with radians' which is so necessary for later work.

Solution.

1. Determine all angles that satisfy $\cos(\theta) = \dfrac{1}{2}$.

   If $\cos(\theta) = \frac{1}{2}$, then we know the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x = \frac{1}{2}$. This means θ is a Quadrant I or IV angle. Because $\cos(\theta) = \frac{1}{2}$, we know from the values on page 602 that the reference angle is $\frac{\pi}{3}$.

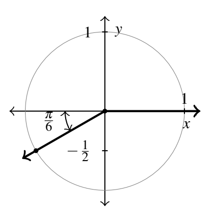
   

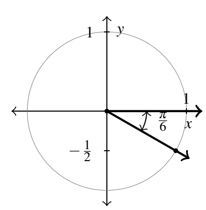
   

   One solution in Quadrant I is $\theta = \frac{\pi}{3}$. Per Theorem 7.3, all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$. Recall from Section 7.1.2, two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π. Hence to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k = 0, \pm 1, \pm 2, \ldots$. Hence, we record our final answer as $\theta = \frac{\pi}{3} + 2\pi k$ for integers k. Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ here is $\frac{5\pi}{3}$, so our answer in this Quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

2. Determine all angles that satisfy $\sin(\alpha) = -\dfrac{1}{2}$.

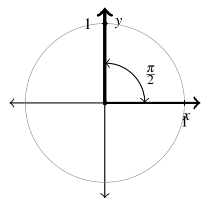
   If $\sin(\alpha) = -\frac{1}{2}$, then when α is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine α is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$. In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π: $\alpha = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\alpha = \frac{11\pi}{6} + 2\pi k$ for integers k.

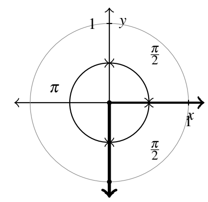
   

3. Determine all angles that satisfy $\cos(\beta) = 0$.

   If $\cos(\beta) = 0$, then the terminal side of β must lie on the line $x = 0$, also known as the y-axis.





While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle (it's not acute), we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\beta = \frac{\pi}{2} + 2\pi k$ and $\beta = \frac{3\pi}{2} + 2\pi k$ for integers, k. While this solution is correct, it can be shortened to $\beta = \frac{\pi}{2} + \pi k$ for integers k. The reader is encouraged to see the geometry using the diagram above on the left.

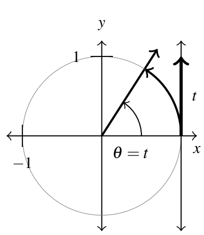
4. Determine all angles that satisfy $\sin(\gamma) = \dfrac{3}{2}$.

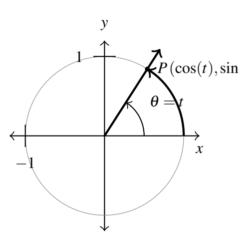
   We are asked to solve $\sin(\gamma) = \frac{3}{2}$. As sine values are y-coordinates on the Unit Circle, $\sin(\gamma)$ can't be any larger than 1. Hence, $\sin(t) = \frac{3}{2}$ has no solutions. □

One of the key items to take from Example 7.2.4 is that, in general, solutions to trigonometric equations consist of infinitely many answers. This is especially important when checking answers to the exercises.

For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 in the last example could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface, this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

It is also worth noting that when asked to solve equations in algebra, we are usually looking for *real number* solutions. Thanks to the identifications made on page 583, we are able to regard the inputs to the sine and cosine functions as real numbers by identifying any real number t with an oriented angle θ measuring $\theta = t$ radians. That is, for each real number t, we associate an oriented arc t units in length with initial point $(1,0)$ and endpoint $P(\cos(t), \sin(t))$.





In practice this means in expressions like '$\cos(\pi)$' and '$\sin(2)$,' the inputs can be thought of as either angles in radian measure or real numbers, whichever is more convenient.

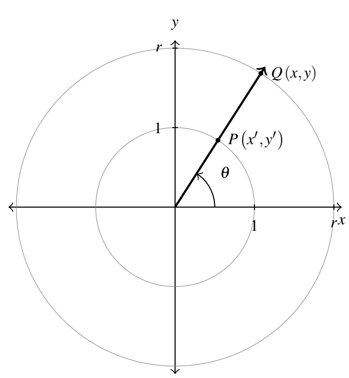
Suppose, as in the Exercises, we are asked to find all *real number* solutions to the equation such as $\sin(t) = -\frac{1}{2}$. The discussion above allows us to find the *real number* solutions to this equation by *thinking* in angles. Indeed, we would solve this equation in the exact way we solved $\sin(\theta) = -\frac{1}{2}$ in Example 7.2.4 number 2. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, k.

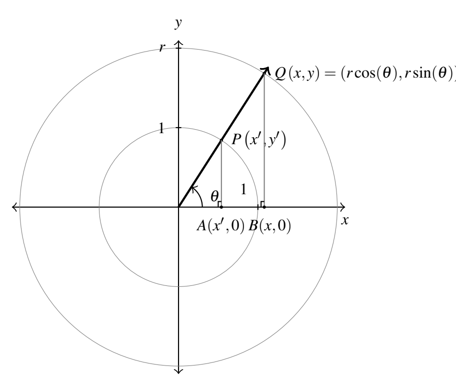
We will study the sine and cosine functions in greater detail in Section 7.3. Until then, keep in mind that any properties of the sine and cosine functions developed in the following sections which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

7.2.3 Beyond the Unit Circle

In Definition 7.3, we define the sine and cosine functions using the *Unit* Circle, $x^2 + y^2 = 1$. It turns out that we can use *any* circle centered at the origin to determine the sine and cosine values of angles. To show this, we essentially recycle the same similarity arguments used in Section 7.2.1 to show the trigonometric ratios described in Definition 7.2 are independent of the choice of right triangle used.[15]

Consider for the moment the *acute* angle θ drawn below in standard position. Let $Q(x,y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two right triangles, ΔOPA and ΔOQB. These triangles are similar,[16] thus it follows that $\frac{x}{x'} = \frac{r}{1} = r$, so $x = rx'$ and, similarly, we find $y = ry'$. Because, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of Q to be $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By reflecting these points through the x-axis, y-axis and origin, we obtain the result for all non-quadrantal angles θ, and we leave it to the reader to verify these formulas hold for the quadrantal angles as well.





[15] Another approach uses transformations. See Exercise 72.

[16] Do you remember why?

Not only can we describe the coordinates of Q in terms of $\cos(\theta)$ and $\sin(\theta)$ but due to the fact that the radius of the circle is $r = \sqrt{x^2+y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of Q. These results are summarized in the following theorem.

Theorem 7.4. If $Q(x,y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2+y^2=r^2$ then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

Note that in the case of the Unit Circle we have $r = \sqrt{x^2+y^2} = 1$, so Theorem 7.4 reduces to our definitions of $\cos(\theta)$ and $\sin(\theta)$ in Definition 7.3. Our next example makes good use of Theorem 7.4.

Example 7.2.5.

1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4,-2)$. Compute $\sin(\theta)$ and $\cos(\theta)$.

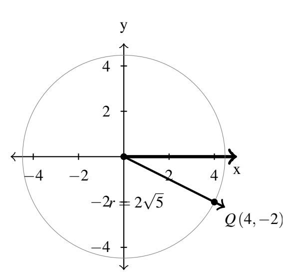
2. Suppose $\frac{\pi}{2} < \theta < \pi$ with $\sin(\theta) = \frac{8}{17}$. Compute $\cos(\theta)$.

3. In Example 7.1.5 in Section 7.1.2, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the spherical radius of the Earth is 3960 miles.

Solution.

1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4,-2)$. Compute $\sin(\theta)$ and $\cos(\theta)$.

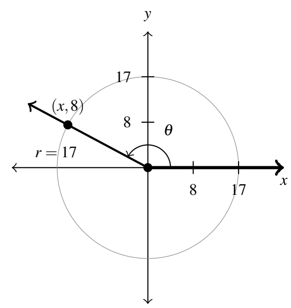
   We are given both the x and y coordinates of a point on the terminal side of this angle, so we can use Theorem 7.4 directly. First, we find $r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+4^2} = \sqrt{20} = 2\sqrt{5}$. This means the point Q lies on a circle of radius $2\sqrt{5}$ units. Hence, $\cos(\theta) = \frac{x}{r} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}$ and $\sin(\theta) = \frac{y}{r} = \frac{-2}{2\sqrt{5}} = -\frac{\sqrt{5}}{5}$.



$Q(4,-2)$ lies on a circle of radius $2\sqrt{5}$

2. Suppose $\frac{\pi}{2} < \theta < \pi$ with $\sin(\theta) = \frac{8}{17}$. Compute $\cos(\theta)$.

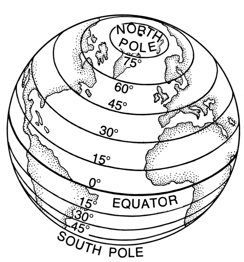
   We are told $\frac{\pi}{2} < \theta < \pi$, so, in particular, θ is a Quadrant II angle. Per Theorem 7.4, $\sin(\theta) = \frac{8}{17} = \frac{y}{r}$ where y is the y-coordinate of the intersection point of the circle $x^2 + y^2 = r^2$ and the terminal side of θ (when plotted in standard position, of course!) For convenience, we choose $r = 17$ so that $y = 8$, and we get the diagram. Given $x^2 + y^2 = r^2$, we get $x^2 + 8^2 = 17^2$. We find $x = \pm 15$, and because θ is a Quadrant II angle, we get $x = -15$. Hence, $\cos(\theta) = -\frac{15}{17}$.

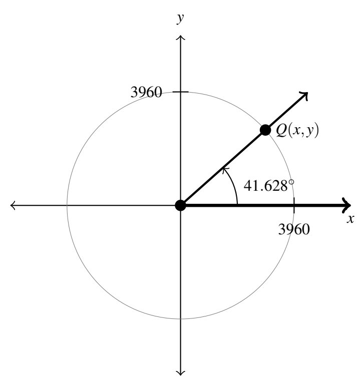


$0 < \theta < \frac{\pi}{2}$ with $\sin(\theta) = \frac{8}{17}$

3. In Example 7.1.5 in Section 7.1.2, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the spherical radius of the Earth is 3960 miles.

   Recall the diagram below on the left indicating the circles which are the parallels of latitude.[17]





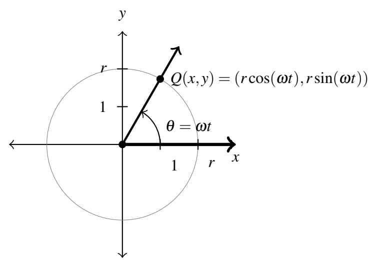
A point on the Earth at 41.628°N.

Assuming the Earth is a sphere of radius 3960 miles, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the x-axis, the value we seek is the x-coordinate of the point $Q(x, y)$ indicated in the figure above on the right. Using Theorem 7.4, we get $x = 3960 \cos(41.628^{\circ}) \approx 2960$. Hence, the radius of the Earth at North Latitude 41.628° is approximately 2960 miles. $\square$

Theorem 7.4 gives us what we need to 'circle back' to the question posed at the the beginning of the section: how to describe the position of an object traveling in a circular path of radius r with constant angular velocity

[17] Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.

ω. Suppose that at time t, the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r,0)$ when $t=0$, the angle θ is in standard position. By definition, $\omega = \frac{\theta}{t}$ which we rewrite as $\theta = \omega t$. According to Theorem 7.4, the location of the object $Q(x,y)$ on the circle is found using the equations $x = r\cos(\theta) = r\cos(\omega t)$ and $y = r\sin(\theta) = r\sin(\omega t)$. Hence, at time t, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$, as seen in the diagram below.



Equations for Circular Motion

We have just argued the following.

Equation 7.3. Suppose an object is traveling in a circular path of radius r centered at the origin with constant angular velocity ω. If $t=0$ corresponds to the point $(r,0)$, then the x and y coordinates of the object are functions of t and are given by $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$. Here, $\omega > 0$ indicates a counter-clockwise direction and $\omega < 0$ indicates a clockwise direction.

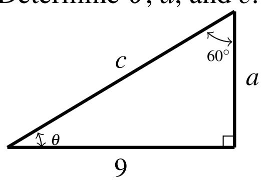
Example 7.2.6. Suppose we are in the situation of Example 7.1.5. Determine the equations of motion of Lakeland Community College as the earth rotates.

Solution. From Example 7.1.5, we take $r = 2960$ miles and and $\omega = \frac{\pi}{12\,\text{hours}}$. Hence, the equations of motion are $x = r\cos(\omega t) = 2960\cos\left(\frac{\pi}{12}t\right)$ and $y = r\sin(\omega t) = 2960\sin\left(\frac{\pi}{12}t\right)$, where x and y are measured in miles and t is measured in hours. □

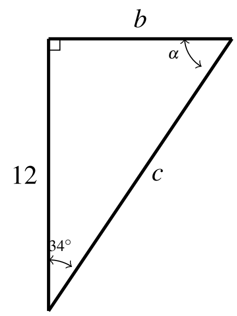
7.2.4 Exercises

In Exercises 1 - 4, compute the requested quantities.

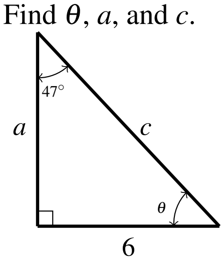
1. Determine θ, a, and c.



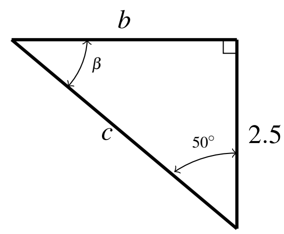
2. Determine α, b, and c.



3. Find θ, a, and c.



4. Find β, b, and c.



In Exercises 5 - 7, answer the following questions assuming θ is an angle in a right triangle.

5. If $\theta = 15^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?
6. If $\theta = 38.2^\circ$ and the side opposite θ has lengh 14, how long is the hypoteneuse?
7. If $\theta = 2.05^\circ$ and the hypotenuse has length 3.98, how long is the side adjacent to θ?

In Exercises 8 - 27, determine the exact value of the cosine and sine of the given angle.

8. $\theta = 0$
9. $\theta = \dfrac{\pi}{4}$
10. $\theta = \dfrac{\pi}{3}$
11. $\theta = \dfrac{\pi}{2}$
12. $\theta = \dfrac{2\pi}{3}$
13. $\theta = \dfrac{3\pi}{4}$
14. $\theta = \pi$
15. $\theta = \dfrac{7\pi}{6}$
16. $\theta = \dfrac{5\pi}{4}$
17. $\theta = \dfrac{4\pi}{3}$
18. $\theta = \dfrac{3\pi}{2}$
19. $\theta = \dfrac{5\pi}{3}$
20. $\theta = \dfrac{7\pi}{4}$
21. $\theta = \dfrac{23\pi}{6}$
22. $\theta = -\dfrac{13\pi}{2}$
23. $\theta = -\dfrac{43\pi}{6}$

24. $\theta = -\frac{3\pi}{4}$
25. $\theta = -\frac{\pi}{6}$
26. $\theta = \frac{10\pi}{3}$
27. $\theta = 117\pi$

In Exercises 28 - 36, compute all of the angles which satisfy the given equation.

28. $\sin(\theta) = \frac{1}{2}$
29. $\cos(\theta) = -\frac{\sqrt{3}}{2}$
30. $\sin(\theta) = 0$
31. $\cos(\theta) = \frac{\sqrt{2}}{2}$
32. $\sin(\theta) = \frac{\sqrt{3}}{2}$
33. $\cos(\theta) = -1$
34. $\sin(\theta) = -1$
35. $\cos(\theta) = \frac{\sqrt{3}}{2}$
36. $\cos(\theta) = -1.001$

In Exercises 37 - 45, solve the equation for t. (See the remarks on page 7.2.2.)

37. $\cos(t) = 0$
38. $\sin(t) = -\frac{\sqrt{2}}{2}$
39. $\cos(t) = 3$
40. $\sin(t) = -\frac{1}{2}$
41. $\cos(t) = \frac{1}{2}$
42. $\sin(t) = -2$
43. $\cos(t) = 1$
44. $\sin(t) = 1$
45. $\cos(t) = -\frac{\sqrt{2}}{2}$

In Exercises 46 - 49, let θ be the angle in standard position whose terminal side contains the given point then compute $\cos(\theta)$ and $\sin(\theta)$.

46. $P(-7,24)$
47. $Q(3,4)$
48. $R(5,-9)$
49. $T(-2,-11)$

In Exercises 50 - 59, use the results developed throughout the section to find the requested value.

50. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?
51. If $\cos(\theta) = \frac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?
52. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?
53. If $\cos(\theta) = -\frac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?
54. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?
55. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?
56. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?

57. If $\cos(\theta) = \dfrac{\sqrt{10}}{10}$ and $2\pi < \theta < \dfrac{5\pi}{2}$, what is $\sin(\theta)$?

58. If $\sin(\theta) = -0.42$ and $\pi < \theta < \dfrac{3\pi}{2}$, what is $\cos(\theta)$?

59. If $\cos(\theta) = -0.98$ and $\dfrac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

In Exercises 60 - 64, write the given function as a nontrivial decomposition of functions as directed.

60. For $f(t) = 3t + \sin(2t)$, find functions g and h so that $f = g + h$.

61. For $f(\theta) = 3\cos(\theta) - \sin(4\theta)$, find functions g and h so that $f = g - h$.

62. For $f(t) = e^{-0.1t}\sin(3t)$, find functions g and h so that $f = gh$.

63. For $r(t) = \dfrac{\sin(t)}{t}$, find functions f and g so $r = \dfrac{f}{g}$.

64. For $r(\theta) = \sqrt{3\cos(\theta)}$, find functions f and g so $r = g \circ f$.

65. For each function $S(t)$ listed below, compute the average rate of change over the indicated interval.[18] What trends do you notice? Be sure your calculator is in radian mode!

$S(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\sin(t)$			
$\sin(2t)$			
$\sin(3t)$			
$\sin(4t)$			

In Exercises 66 - 69, find the equations of motion for the given scenario. Assume that the center of the motion is the origin, the motion is counter-clockwise and that $t = 0$ corresponds to a position along the positive x-axis. (See Equation 7.3 and Example 7.1.5.)

66. A point on the edge of the spinning yo-yo in Exercise 50 from Section 7.1.2.

    Recall: The diameter of the yo-yo is 2.25 inches and it spins at 4500 revolutions per minute.

67. The yo-yo in exercise 52 from Section 7.1.2.

    Recall: The radius of the circle is 28 inches and it completes one revolution in 3 seconds.

68. A point on the edge of the hard drive in Exercise 53 from Section 7.1.2.

    Recall: The diameter of the hard disk is 2.5 inches and it spins at 7200 revolutions per minute.

[18] See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.

69. A passenger on the Big Wheel in Exercise 55 from Section 7.1.2.

    Recall: The diameter is 128 feet and completes 2 revolutions in 2 minutes, 7 seconds.

70. Consider the numbers: 0, 1, 2, 3, 4. Take the square root of each of these numbers, then divide each by 2. The resulting numbers should look hauntingly familiar. (See the values in the table on 598.)

71. On page 608, we see that the sine and cosine functions of *angles* can be considered functions of *real numbers*. With help from your classmates, discuss the domains and ranges of $f(t) = \sin(t)$ and $g(t) = \cos(t)$. Write your answers using interval notation.

72. Another way to establish Theorem 7.4 is to use transformations. Transform the Unit Circle, $x^2 + y^2 = 1$, to $x^2 + y^2 = r^2$ using horizontal and vertical stretches. Show if the coordinates on the Unit Circle are $(\cos(\theta), \sin(\theta))$, then the corresponding coordinates on $x^2 + y^2 = r^2$ are $(r\cos(\theta), r\sin(\theta))$.

73. In the scenario of Equation 7.3, we assumed that at $t = 0$, the object was at the point $(r,0)$. If this is not the case, we can adjust the equations of motion by introducing a 'time delay.' If $t_0 > 0$ is the first time the object passes through the point $(r,0)$, show, with the help of your classmates, the equations of motion are $x = r\cos(\omega(t - t_0))$ and $y = r\sin(\omega(t - t_0))$.

Section 7.2 Exercise Answers A.1.7

7.3 Graphs of Sine and Cosine

On page 608, we discussed how to interpret the sine and cosine of real numbers. To review, we identify a real number t with an oriented angle θ measuring t radians[1] and define $\sin(t) = \sin(\theta)$ and $\cos(t) = \cos(\theta)$. Every real number can be identified with one and only one angle θ this way, therefore the domains of the functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are all real numbers, $(-\infty, \infty)$.

When it comes to range, recall that the sine and cosine of angles are coordinates of points on the Unit Circle and hence, each fall between -1 and 1 inclusive. As the real number line,[2] when wrapped around the Unit Circle completely covers the circle, we can be assured that every point on the Unit Circle corresponds to at least one real number. Putting these two facts together, we conclude the range of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are both $[-1, 1]$. We summarize these two important facts below.

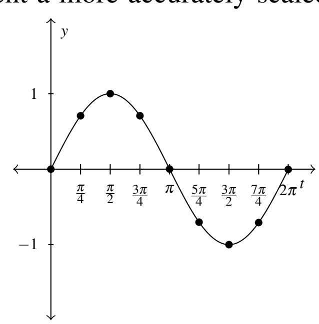
Theorem 7.5. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \sin(t)$
  - has a domain of $(-\infty, \infty)$
  - has a range of $[-1, 1]$
- The function $g(t) = \cos(t)$
  - has a domain of $(-\infty, \infty)$
  - has a range of $[-1, 1]$

Our aim in this section is to become familiar with the graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$. To that end, we begin by making a table and plotting points. We'll start by graphing $f(t) = \sin(t)$ by making a table of values and plotting the corresponding points. We'll keep the independent variable 't' for now and use the default 'y' as our dependent variable.[3] Note in the graph below, on the right, the scale of the horizontal and vertical axis is far from 1:1. (We will present a more accurately scaled graph shortly.)

t	$\sin(t)$	$(t, \sin(t))$
0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$

t	$\sin(t)$	$(t, \sin(t))$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$



Graphing $y = \sin(t)$

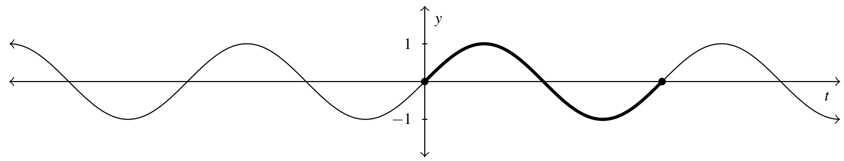
If we plot additional points, we soon find that the graph repeats itself. This shouldn't come as too much of a surprise considering Theorem 7.3. In fact, in light of that theorem, we expect the function to repeat itself

[1] which, you'll recall essentially 'wraps the real number line around the Unit Circle

[2] in particular the interval $[0, 2\pi)$

[3] Keep in mind that we're using 'y' here to denote the *output* from the sine function. It is a coincidence that the y-values on the graph of $y = \sin(t)$ correspond to the y-values on the Unit Circle.

every 2π units. Below is a more accurately scaled graph highlighting the portion we had already graphed above. The graph is often described as having a 'wavelike' nature and is sometimes called a **sine wave** or, more technically, a **sinusoid**.



A more accurately scaled graph of $f(t) = \sin(t)$

Note that by copying the highlighted portion of the graph and pasting it end-to-end, we obtain the entire graph of $f(t) = \sin(t)$. We give this 'repeating' property a name.

Definition 7.4. Periodic Functions:
A function f is said to be **periodic** if there is a real number c so that $f(t+c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t+p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

We have already seen a family of periodic functions in Section 1.3.1: the constant functions. However, despite being periodic a constant function has no period. (We'll leave that odd gem as an exercise for you.)

Returning to $f(t) = \sin(t)$, we see that by Definition 7.4, f is periodic as $\sin(t+2\pi) = \sin(t)$. To determine the period of f, we need to find the smallest real number p so that $f(t+p) = f(t)$ for all real numbers t or, said differently, the smallest positive real number p such that $\sin(t+p) = \sin(t)$ for all real numbers t.

We know that $\sin(t+2\pi) = \sin(t)$ for all real numbers t but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\sin(t+p) = \sin(t)$ for all real numbers t. Then, in particular, $\sin(0+p) = \sin(0) = 0$ so that $\sin(p) = 0$. From this we know p is a multiple of π. Given $\sin\left(\frac{\pi}{2}\right) \neq \sin\left(\frac{\pi}{2}+\pi\right)$, we know $p \neq \pi$. Hence, $p = 2\pi$ so the period of $f(t) = \sin(t)$ is 2π.

Having period 2π essentially means that we can completely understand everything about the function $f(t) = \sin(t)$ by studying *one* interval of length 2π, say $[0, 2\pi]$.[4] For this reason, when graphing sine (and cosine) functions, we typically restrict our attention to graphing these functions over the course of one period to produce one **cycle** of the graph.

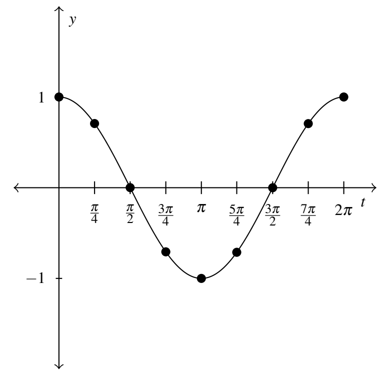
Not surprisingly, the graph of $g(t) = \cos(t)$ exhibits similar behavior as $f(t) = \sin(t)$.[6]

[4] Technically, we should study the interval $[0, 2\pi)$,[5] as whatever happens at $t = 2\pi$ is the same as what happens at $t = 0$. As we will see shortly, $t = 2\pi$ gives us an extra 'check' when we go to graph these functions.

[5] In some advanced texts, the interval of choice is $[-\pi, \pi)$.

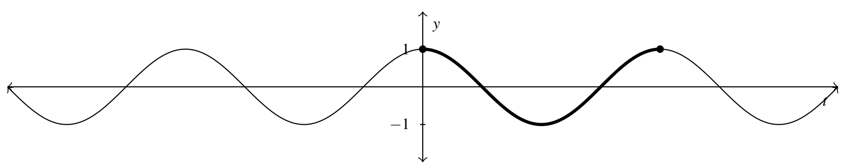
[6] Here note that the dependent variable 'y' represents the outputs from $g(t) = \cos(t)$ which are x-coordinates on the Unit CIrcle.

t	$\cos(t)$	$(t,\cos(t))$
0	1	$(0,1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2},0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4},-\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi,-1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4},-\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2},0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4},\frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi,1)$



Graphing $y=\cos(t)$

Like $f(t)=\sin(t)$, $g(t)=\cos(t)$ is a wavelike curve with period 2π. Moreover, the graphs of the sine and cosine functions have the same shape - differing only in what appears to be a horizontal shift. As we'll prove in Section 8.2, $\sin\left(t+\frac{\pi}{2}\right)=\cos(t)$, which means we can obtain the graph of $y=\cos(t)$ by shifting the graph of $y=\sin(t)$ to the left $\frac{\pi}{2}$ units.[7]



A more accurately scaled graph of $f(t)=\cos(t)$

While arguably the most important property shared by $f(t)=\sin(t)$ and $g(t)=\cos(t)$ is their periodic 'wave-like' nature,[8] their graphs suggest these functions are both continuous and smooth. Recall from Section 2.2 that, like polynomial functions, the graphs of the sine and cosine functions have no jumps, gaps, holes in the graph, vertical asymptotes, corners or cusps.

Moreover, the graphs of both $f(t)=\sin(t)$ and $g(t)=\cos(t)$ meander and never 'settle down' as $t\to\pm\infty$ to any one real number. So even though these functions are 'trapped' (or **bounded**) between -1 and 1, neither graph has any horizontal asymototes.

Lastly, the graphs of $f(t)=\sin(t)$ and $g(t)=\cos(t)$ suggest each enjoy one of the symmetries introduced in Section 2.2. The graph of $y=\sin(t)$ appears to be symmetric about the origin while the graph of $y=\cos(t)$

[7]Hence, we can obtain the graph of $y=\sin(t)$ by shifting the graph of $y=\cos(t)$ to the *right* $\frac{\pi}{2}$ units: $\cos\left(t-\frac{\pi}{2}\right)=\sin(t)$.

[8]this is the reason they are so useful in the Sciences and Engineering

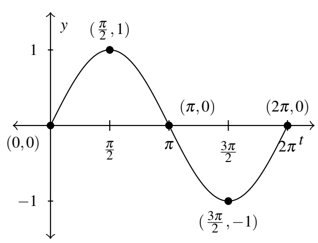
appears to be symmetric about the y-axis. Indeed, as we'll prove in Section 8.2, $f(t) = \sin(t)$ is, in fact, an odd function: [9] that is, $\sin(-t) = -\sin(t)$ and $g(t) = \cos(t)$ is an even function, so $\cos(-t) = \cos(t)$.

We summarize all of these properties in the following result.

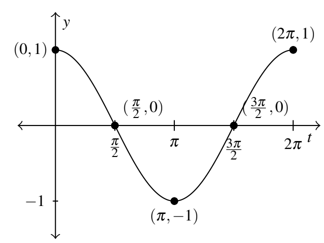
Theorem 7.6. Properties of the Sine and Cosine Functions

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π
- Conversion formulas: $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ and $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$

Now that we know the basic shapes of the graphs of $y = \sin(t)$ and $y = \cos(t)$, we can use the results of Section 1.6 to graph more complicated functions using transformations. As mentioned already, the fact that both of these functions are periodic means we only have to know what happens over the course of one period of the function in order to determine what happens to all points on the graph. To that end, we graph the '**fundamental cycle**', the portion of each graph generated over the interval $[0, 2\pi]$, for each of the sine and cosine functions below.



The 'fundamental cycle' of $y = \sin(t)$.



The 'fundamental cycle' of $y = \cos(t)$.

In working through Section 1.6 , it was very helpful to track 'key points' through the transformations. The 'key points' we've indicated on the graphs above correspond to the quadrantal angles and generate the zeros and the extrema of functions. Due to the fact that the quadrantal angles divide the interval $[0, 2\pi]$ into four equal pieces, we shall refer to these angles henceforth as the 'quarter marks.'

It is worth noting that because the transformations discussed in Section 1.6 are linear,[10] the *relative spacing*

[9] The reader may wish to review Definitions 2.5 and 2.6 as needed.
[10] See the remarks at the beginning of Section 1.6.

of the points before and after the transformations remains the same.[11] In particular, wherever the interval $[0,2\pi]$ is mapped, the quarter marks of the new interval correspond to the quarter marks of $[0,2\pi]$. (Can you see why?) We will exploit this fact in the following example.

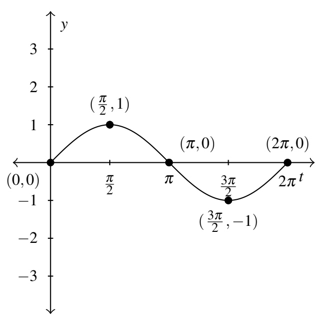
Example 7.3.1. Graph one cycle of the following functions. State the period of each.

1. $f(t) = 3\sin(2t)$
2. $g(t) = 2\cos\left(t+\frac{\pi}{2}\right)+1$

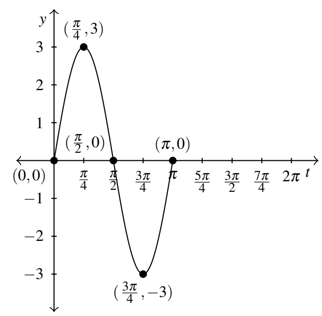
Solution.

1. Graph one cycle of $f(t) = 3\sin(2t)$.

   One way to proceed is to use Theorem 1.12 and follow the procedure outlined there. Starting with the fundamental cycle of $y=\sin(t)$, we divide each t-coordinate by 2 and multiply each y-coordinate by 3 to obtain one cycle of $y=3\sin(2t)$.



The ‘fundamental cycle’ of $y=\sin(t)$.



One cycle of $y=3\sin(2t)$.

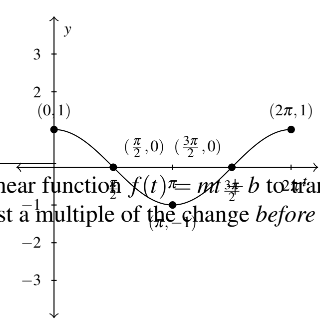
   As a reult of one cycle of $y=f(t)$ being completed over the interval $[0,\pi]$, the period of f is π.

2. Graph one cycle of $g(t) = 2\cos\left(t+\frac{\pi}{2}\right)+1$.

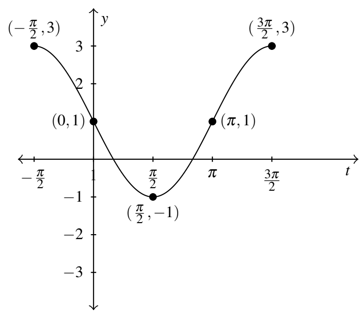
   Starting with the fundamental cycle of $y=\cos(t)$ and using Theorem 1.12, we subtract $\frac{\pi}{2}$ from each of the t-coordinates, then multiply each y-coordinate by 2, and add 1 to each y-coordinate.

   We find one cycle of $y=g(t)$ is completed over the interval $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$, the period is $\frac{3\pi}{2}-\left(-\frac{\pi}{2}\right)=2\pi$.

The ‘fundamental cycle’ of $y=\cos(t)$.



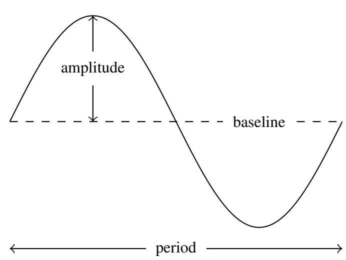
[11] If we use a linear function $f(t)=mt+b$ to transform the inputs, t, then $\Delta[f(t)]=m\Delta t$. That is, the change *after* the transformation, $m\Delta t$, is just a multiple of the change *before* the transformation, Δt.



One cycle of $g(t) = 2\cos\left(t+\frac{\pi}{2}\right)+1$.

□

As previously mentioned, the curves graphed in Example 7.3.1 are examples of sinusoids. A sinusoid is the result of taking the graph of $y = \sin(t)$ or $y = \cos(t)$ and performing any of the transformations mentioned in Section 1.6. We graph one cycle of a generic sinusoid below. Sinusoids can be characterized by four properties: period, phase shift, vertical shift (or 'baseline'), and amplitude.



We have already discussed the period of a sinusoid. If we think of t as measuring time, the period is how long it takes for the sinusoid to complete one cycle and is usually represented by the letter T. The standard period of both $\sin(t)$ and $\cos(t)$ is 2π, but horizontal scalings will change this.

In Example 7.3.1, for instance, the function $f(t) = 3\sin(2t)$ has period π instead of 2π because the graph is horizontally compressed by a factor of 2 as compared to the graph of $y = \sin(t)$. However, the period of $g(t) = 2\cos\left(t+\frac{\pi}{2}\right)+1$ is the same as the period of $\cos(t)$, 2π, because there are no horizontal scalings.

The **phase shift** of the sinusoid is the horizontal shift. Again, thinking of t as time, the phase shift of a sinusoid can be thought of as when the sinusoid 'starts' as compared to $t = 0$. Assuming there are no reflections across the y-axis, we can determine the phase shift of a sinusoid by finding where the value $t = 0$ on the graph of $y = \sin(t)$ or $y = \cos(t)$ is mapped to under the transformations.

For $f(t) = 3\sin(2t)$, the phase shift is '0' because the value $t = 0$ on the graph of $y = \sin(t)$ remains stationary under the transformations. Loosely speaking, this means both $y = \sin(t)$ and $y = 3\sin(2t)$ 'start' at the same time. The phase shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is $-\frac{\pi}{2}$ or '$\frac{\pi}{2}$ to the *left*' because the value $t = 0$ on the graph of $y = \cos(t)$ is mapped to $t = -\frac{\pi}{2}$ on the graph of $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$. Again, loosely speaking, this means $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ starts $\frac{\pi}{2}$ time units *earlier* than $y = \cos(t)$.

The vertical shift of a sinusoid is exactly the same as the vertical shifts in Section 1.6 and determines the new 'baseline' of the sinusoid. Thanks to symmetry, the vertical shift can always be found by averaging the maximum and minimum values of the sinusoid. For $f(t) = 3\sin(2t)$, the vertical shift is 0 whereas the vertical shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is 1 or '1 up.'

The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. Said differently, the amplitude measures how much the curve gets displaced from its 'baseline. ' The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.

In Example 7.3.1, the amplitude of $f(t) = 3\sin(2t)$ is 3, owing to the vertical stretch by a factor of 3 as compared with the graph of $y = \sin(t)$. In the case of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$, the amplitude is 2 due to its vertical stretch as compared with the graph of $y = \cos(t)$. Note that the '+1' here does *not* affect the amplitude of the curve; it merely changes the 'baseline' from $y = 0$ to $y = 1$.

The following theorem shows how these four fundamental quantities relate to the parameters which describe a generic sinusoid. The proof follows from Theorem 1.12 and is left to the reader in Exercise 31.

Theorem 7.7. For $B > 0$, the graphs of

$$S(t) = A\sin(Bt + C) + D \quad \text{and} \quad E(t) = A\cos(Bt + C) + D$$

- have frequency of B
- have period $T = \dfrac{2\pi}{B}$
- have amplitude $|A|$
- have phase shift $-\dfrac{C}{B}$
- have vertical shift or 'baseline' D

We put Theorem 7.7 to good use in the next example.

Example 7.3.2. Use Theorem 7.7 to determine the frequency, period, phase shift, amplitude, and vertical shift of each of the following functions and use this information to graph one cycle of each function.

1. $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1$
2. $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$

Solution.

1. Determine the frequency, period, phase shift, amplitude, and vertical shift of $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1$ and then graph on cycle of the function.

To use Theorem 7.7, we first need to rewrite $f(t)$ in the form prescribed by Theorem 7.7. To that end, we rewrite: $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1 = 3\cos\left(\frac{\pi}{2}t + \left(-\frac{\pi}{2}\right)\right) + 1$.

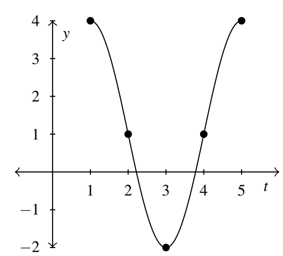
From this, we identify $A = 3$, $B = \frac{\pi}{2}$, $C = -\frac{\pi}{2}$ and $D = 1$. According to Theorem 7.7, the frequency is $B = \frac{\pi}{2}$, the period is $T = \frac{2\pi}{B} = \frac{2\pi}{\pi/2} = 4$, the phase shift is $-\frac{C}{B} = -\frac{-\pi/2}{\pi/2} = 1$ (indicating a shift to the *right* 1 unit), the amplitude is $|A| = |3| = 3$, and the vertical shift is $D = 1$ (indicating a shift *up* 1 unit.)

To graph $y = f(t)$, we know one cycle begins at $t = 1$ (the phase shift.) Because the period is 4, we know the cycle ends 4 units later at $t = 1 + 4 = 5$. If we divide the interval $[1,5]$ into four equal pieces, each piece has length $\frac{4}{4} = 1$. Hence, we to get our quarter marks, we start with $t = 1$ and add 1 unit until we reach the endpoint, $t = 5$. Our new quarter marks are: $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$.

We now substitute these new quarter marks into $f(t)$ to obtain the corresponding y-values on the graph.[12] We connect the dots in a 'wavelike' fashion to produce the graph on the right.

Note that we can (partially) spot-check our answer by noting the average of the maximum and minimum is $\frac{4+(-2)}{2} = 1$ (our vertical shift) and the amplitude, $4 - 1 = 1 - (-2) = 3$ is indeed 3.

t	$f(t)$	$(t, f(t))$
1	4	$(1,4)$
2	1	$(2,1)$
3	-2	$(3,-2)$
4	1	$(4,1)$
5	4	$(5,4)$



One cycle of $y = f(t)$

Thought not asked for, this example provides a nice opportunity to interpret the ordinary frequency: $f = \frac{1}{T} = \frac{1}{4}$. Hence, $\frac{1}{4}$ of the sinusoid is traced out over an interval that is 1 unit long.

2. Determine the frequency, period, phase shift, amplitude, and vertical shift of $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$ and then graph on cycle of the function.

   Turning our attention now to the function g, we first note that the coefficient of t is negative. In order to use Theorem 7.7, we need that coefficient to be positive. Hence, we first use the odd property of the sine function to rewrite $\sin(\pi - 2t)$ so that instead of a coefficient of -2, t has a coefficient of 2. We get $\sin(\pi - 2t) = \sin(-2t + \pi) = \sin(-(2t - \pi)) = -\sin(2t - \pi)$. Hence, $g(t) = -\frac{1}{2}\sin(2t + (-\pi)) + \frac{3}{2}$.

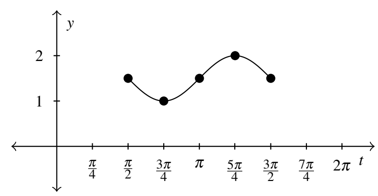
[12]Note when we substitute the quarter marks into $f(t)$, the argument of the cosine function simplifies to the quadrantal angles. That is, when we substitute $t = 1$, the argument of cosine simplifies to 0; when we substitute $t = 2$, the argument simplifies $\frac{\pi}{2}$ and so on. This provides a quick check of our calculations.

We identitfy $A = -\frac{1}{2}$, $B = 2$, $C = -\pi$ and $D = \frac{3}{2}$. The frequency is $B = 2$, the period is $T = \frac{2\pi}{2} = \pi$, the phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$ (indicating a shift *right* $\frac{\pi}{2}$ units), the amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$, and, finally, the vertical shift is *up* $\frac{3}{2}$.

Proceeding as before, we know one cycle of g starts at $t = \frac{\pi}{2}$ and ends at $t = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$. Dividing the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ into four equal pieces gives pieces of length $\frac{\pi}{4}$ units. Hence, to obtain our new quarter marks, we start at $t = \frac{\pi}{2}$ and add $\frac{\pi}{4}$ until we reach $t = \frac{3\pi}{2}$. Our new quarter marks are: $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$, $t = \pi$, $t = \frac{5\pi}{4}$, $t = \frac{3\pi}{2}$. Substituting these values into g gives us the points to plot to produce the graph below on the right.

Again, we can quickly check the vertical shift by averaging the maximum and minimum values: $\frac{2+1}{2} = \frac{3}{2}$ and verify the amplitude: $2 - \frac{3}{2} = \frac{3}{2} - 1 = \frac{1}{2}$.

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{3\pi}{4}$	1	$\left(\frac{3\pi}{4}, 1\right)$
π	$\frac{3}{2}$	$\left(\pi, \frac{3}{2}\right)$
$\frac{5\pi}{4}$	2	$\left(\frac{5\pi}{4}, 2\right)$
$\frac{3\pi}{2}$	$\frac{3}{2}$	$\left(\frac{3\pi}{2}, \frac{3}{2}\right)$



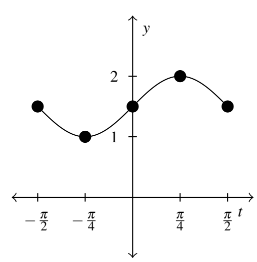
One cycle of $y = g(t)$

□

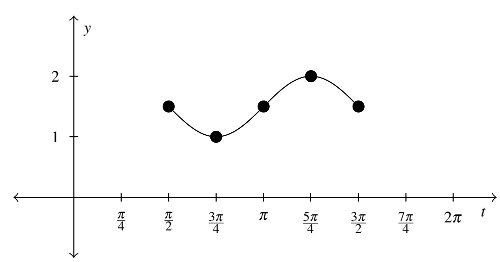
Note that in this section, we have discussed *two* ways to graph sinusoids: using Theorem 1.12 from Section 1.6 and using Theorem 7.7. Both methods will produce one cycle of the resulting sinusoid, but each method may produce a *different* cycle of the same sinusoid.

For example, if we graphed the function $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$ from Example 7.3.2 using Theorem 1.12, we obtain the following:

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{\pi}{4}$	2	$\left(\frac{\pi}{4}, 2\right)$
0	$\frac{3}{2}$	$\left(0, \frac{3}{2}\right)$
$-\frac{\pi}{4}$	1	$\left(-\frac{\pi}{4}, 1\right)$
$-\frac{\pi}{2}$	$\frac{3}{2}$	$\left(-\frac{\pi}{2}, \frac{3}{2}\right)$



One cycle via Theorem ??



One cycle via Theorem ??

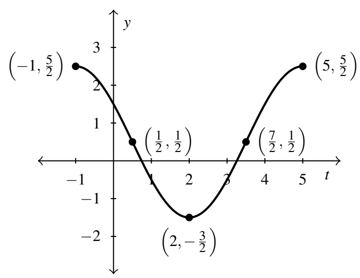
Comparing this result with the one obtained in Example 7.3.2 side by side, we see that one cycle ends right where the other starts. The cause of this discrepancy goes back to using the odd property of sine.

Essentially, the odd property of the sine function converts a reflection across the y-axis into a reflection across the t-axis. (Can you see why?) For this reason, whenever the coefficient of t is negative, Theorems 1.12 and 7.7 will produce different results.

In the Exercises, we assume the problems are worked using Theorem 7.7. If you choose to use Theorems 1.12 instead, your answer may look different than what is provided even though both your answer and the textbook's answer represent *one* cycle of the *same* function.

In the next example, we use Theorem 7.7 to determine the formula of a sinusoid given the graph of one cycle. Note that in some disciplines, sinusoids are written in terms of sines whereas in others, cosines functions are preferred. To cover all bases, we ask for both.

Example 7.3.3. Below is the graph of one complete cycle of a sinusoid $y = f(t)$.



One cycle of $y = f(t)$.

1. Write $f(t)$ in the form $E(t) = A\cos(Bt+C)+D$, for $B > 0$.

2. Write $f(t)$ in the form $S(t) = A\sin(Bt+C)+D$, for $B > 0$.

Solution.

1. Write $f(t)$ in the form $E(t) = A\cos(Bt+C)+D$, for $B > 0$.

   One cycle is graphed over the interval $[-1,5]$, thus its period is $T = 5-(-1) = 6$. According to Theorem 7.7, $6 = T = \frac{2\pi}{B}$, so that $B = \frac{\pi}{3}$. Next, we see that the phase shift is -1, so we have $-\frac{C}{B} = -1$, or $C = B = \frac{\pi}{3}$.

   To find the baseline, we average the maximum and minimum values: $D = \frac{1}{2}\left[\frac{5}{2} + \left(-\frac{3}{2}\right)\right] = \frac{1}{2}(1) = \frac{1}{2}$. To find the amplitude, we subtract the maximum value from the baseline: $A = \frac{5}{2} - \frac{1}{2} = 2$.

   Putting this altogether, we obtain our final answer is $f(t) = 2\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) + \frac{1}{2}$.

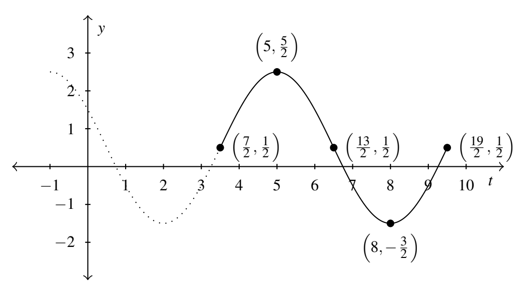
2. Write $f(t)$ in the form $S(t) = A\sin(Bt+C)+D$, for $B > 0$.

Because we have written $f(t)$ in terms of cosines, we can use the conversion from sine to cosine as listed in Theorem 7.6. As $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$, $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\left[\frac{\pi}{3}t + \frac{\pi}{3}\right] + \frac{\pi}{2}\right)$, so $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right)$. Our final answer is $f(t) = 2\sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right) + \frac{1}{2}$.

However, for the sake of completeness, we provide another solution strategy which enables us to write $f(t)$ in terms of sines without starting with our answer from part 1.

Note that we obtain the period, amplitude, and vertical shift as before: $B = \frac{\pi}{3}$, $A = 2$ and $B = \frac{1}{2}$. The trickier part is finding the phase shift.

To that end, we imagine extending the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at $\left(\frac{7}{2}, \frac{1}{2}\right)$. Taking the phase shift to be $\frac{7}{2}$, we get $-\frac{C}{B} = \frac{7}{2}$, or $C = -\frac{7}{2}B = -\frac{7}{2}\left(\frac{\pi}{3}\right) = -\frac{7\pi}{6}$. Hence, our answer is $f(t) = 2\sin\left(\frac{\pi}{3}t - \frac{7\pi}{6}\right) + \frac{1}{2}$.



Extending the graph of $y = f(t)$.

□

Note that each of the answers given in Example 7.3.3 is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at $\left(\frac{1}{2}, \frac{1}{2}\right)$ taking $A = -2$. In this case, the phase shift is $\frac{1}{2}$ so $C = -\frac{\pi}{6}$ for an answer of $f(t) = -2\sin\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) + \frac{1}{2}$. The ultimate check of any solution is to graph the answer and check it matches the given data.

7.3.1 Applications of Sinusoids

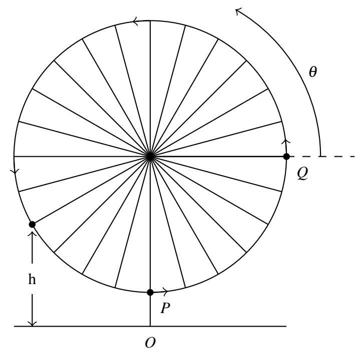
In the same way exponential functions can be used to model a wide variety of phenomena in nature,[13] the sine and cosine functions can be used to model their fair share of natural behaviors. Our first foray into sinusoidal motion revisits circular motion - in particular Equation 7.3 .

Example 7.3.4. Recall from Exercise 55 in Section 7.1.2 that The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes

[13] See Section 5.7.

two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid which describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.

Solution. We sketch the problem situation below and assume a counter-clockwise rotation.[14]



We know from the equations given on page 612 in Section 7.2.3 that the y-coordinate for counter-clockwise motion on a circle of radius r centered at the origin with constant angular velocity (frequency) ω is given by $y = r\sin(\omega t)$. Here, $t = 0$ corresponds to the point $(r,0)$ so that θ, the angle measuring the amount of rotation, is in standard position.

In our case, the diameter of the wheel is 128 feet, so the radius is $r = 64$ feet. Because the wheel completes two revolutions in 2 minutes and 7 seconds (which is 127 seconds) the period $T = \frac{1}{2}(127) = \frac{127}{2}$ seconds. Hence, the frequency is $B = \frac{2\pi}{T} = \frac{4\pi}{127}$ radians per second.

Putting these two pieces of information together, we have that $y = 64\sin\left(\frac{4\pi}{127}t\right)$ describes the y-coordinate on the Giant Wheel after t seconds, assuming it is centered at $(0,0)$ with $t = 0$ corresponding to point Q.

In order to find an expression for h, we take the point O in the figure as the origin. As the base of the Giant Wheel ride is 8 feet above the ground and the Giant Wheel itself has a radius of 64 feet, its center is 72 feet above the ground. To account for this vertical shift upward,[15] we add 72 to our formula for y to obtain the new formula $h = y + 72 = 64\sin\left(\frac{4\pi}{127}t\right) + 72$.

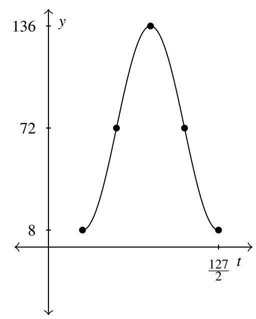
Next, we need to adjust things so that $t = 0$ corresponds to the point P instead of the point Q. This is where the phase comes into play. Geometrically, we need to shift the angle θ in the figure back $\frac{\pi}{2}$ radians.

From the discussion on page 612, we know $\theta = \omega t = \frac{4\pi}{127}t$, so we (temporarily) write the height in terms of θ as $h = 64\sin(\theta) + 72$. Subtracting $\frac{\pi}{2}$ from θ gives $h(t) = 64\sin\left(\theta - \frac{\pi}{2}\right) + 72 = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72$.

We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $\left[0, \frac{127}{2}\right]$ and visualizing the path of a person on the Big Wheel ride over the course of one rotation.

[14]Otherwise, we could just observe the motion of the wheel from the other side.

[15]We are readjusting our 'baseline' from $y = 0$ to $y = 72$.



$\square$

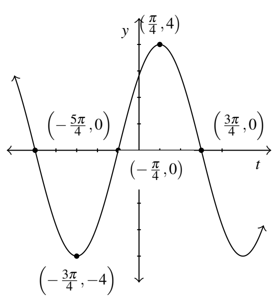
A few remarks about Example 7.3.4 are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Second, the phase shift of our answer works out to be $\frac{\pi/2}{4\pi/127} = \frac{127}{8} = 15.875$. This represents the 'time delay' (in seconds) we introduce by starting the motion at the point P as opposed to the point Q. Said differently, passengers which 'start' at P take 15.875 seconds to 'catch up' to the point Q.

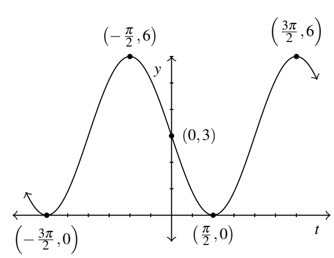
7.3.2 Exercises

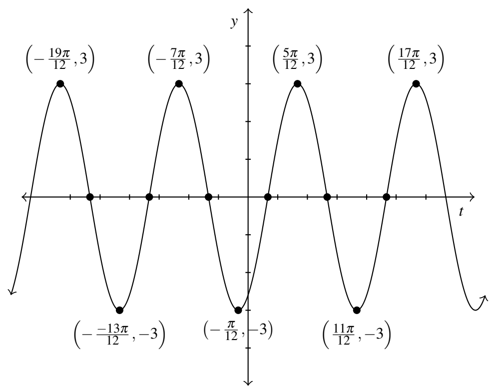
In Exercises 1 - 12, graph one cycle of the given function. State the period, amplitude, phase shift and vertical shift of the function.

1. $f(t) = 3\sin(t)$
2. $g(t) = \sin(3t)$
3. $h(t) = -2\cos(t)$
4. $f(t) = \cos\left(t - \frac{\pi}{2}\right)$
5. $g(t) = -\sin\left(t + \frac{\pi}{3}\right)$
6. $h(t) = \sin(2t - \pi)$
7. $f(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
8. $g(t) = \cos(3t - 2\pi) + 4$
9. $h(t) = \sin\left(-t - \frac{\pi}{4}\right) - 2$
10. $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4t\right) + 1$
11. $g(t) = -\frac{3}{2}\cos\left(2t + \frac{\pi}{3}\right) - \frac{1}{2}$
12. $h(t) = 4\sin(-2\pi t + \pi)$

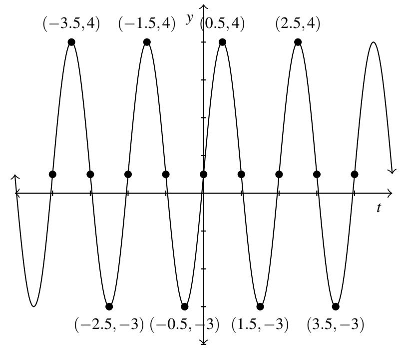
In Exercises 13 - 16, a sinusoid is graphed. Write a formula for the sinusoid in the form $S(t) = A\sin(Bt + C) + B$ and $E(t) = A\cos(Bt + C) + B$. Select B so $B > 0$. Check your answer by graphing.

13. 

14. 



15.



16.

17. Use the graph of $S(t) = 4\sin(t)$ to graph each of the following functions. State the period of each.

    (a) $f(t) = |4\sin(t)|$

    (b) $g(t) = \sqrt{4\sin(t)}$

In Exercises 18 - 23, use a graphing utility to graph each function and discuss the related questions with your classmates.

18. $f(t) = \cos(3t) + \sin(t)$. Is this function periodic? If so, what is the period?

19. $f(t) = \frac{\sin(t)}{t}$. What appears to be the horizontal asymptote of the graph?

20. $f(t) = t\sin(t)$. Graph $y = \pm t$. What do you notice?

21. $f(t) = \sin\left(\frac{1}{t}\right)$. What's happening as $t \to 0$?

22. $f(t) = e^{-0.1t}\left(\cos(2t) + \sin(2t)\right)$. Graph $y = \pm e^{-0.1t}$ on the same set of axes. What do you notice?

23. $f(t) = e^{-0.1t}\left(\cos(2t) + 2\sin(t)\right)$. Graph $y = \pm e^{-0.1t}$ on the same set of axes. What do you notice?

24. Show every constant function f is periodic by explaining why $f(x+117) = f(x)$ for all real numbers x. Then show that f has no period by showing that you cannot find a *smallest* number p such that $f(x+p) = f(x)$ for all real numbers x.

    Said differently, show that $f(x+p) = f(x)$ for all real numbers x for ALL values of $p > 0$, so no smallest value exists to satisfy the definition of 'period'.

25. The sounds we hear are made up of mechanical waves. The note 'A' above the note 'middle C' is a sound wave with ordinary frequency $f = 440$ Hertz $= 440\,\frac{\text{cycles}}{\text{second}}$. Find a sinusoid which models this note, assuming that the amplitude is 1 and the phase shift is 0.

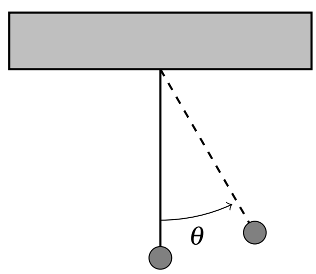
26. The voltage V in an alternating current source has amplitude $220\sqrt{2}$ and ordinary frequency $f = 60$ Hertz. Find a sinusoid which models this voltage. Assume that the phase is 0.

27. The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counter-clockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on to, and off of, the ride. Find a sinsuoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.

28. On page 612 in Section 7.2.3, we found the x-coordinate of counter-clockwise motion on a circle of radius r with angular frequency ω to be $x = r\cos(\omega t)$, where $t = 0$ corresponds to the point $(r, 0)$. Suppose we are in the situation of Exercise 27 above. Find a sinsusoid which models the horizontal *displacement* x of the passenger from the center of the Eye in meters t minutes after they board the Eye. Here we take $x(t) > 0$ to mean the passenger is to the *right* of the center, while $x(t) < 0$ means the passenger is to the *left* of the center.

29. In Exercise 52 in Section 7.1.2, we introduced the yo-yo trick 'Around the World' in which a yo-yo is thrown so it sweeps out a vertical circle. As in that exercise, suppose the yo-yo string is 28 inches and it completes one revolution in 3 seconds. If the closest the yo-yo ever gets to the ground is 2 inches, find a sinsuoid which models the height h of the yo-yo above the ground in inches t seconds after it leaves its lowest point.

30. Consider the pendulum below. Ignoring air resistance, the angular displacement of the pendulum from the vertical position, θ, can be modeled as a sinusoid.[16]



The amplitude of the sinusoid is the same as the initial angular displacement, θ_0, of the pendulum and the period of the motion is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity.

(a) Find a sinusoid which gives the angular displacement θ as a function of time, t. Arrange things so $\theta(0) = \theta_0$.

[16] Provided θ is kept 'small.' Carl remembers the 'Rule of Thumb' as being 20° or less. Check with your friendly neighborhood physicist to make sure.

(b) In Exercise 24 section 4.1, you found the length of the pendulum needed in Jeff's antique Seth-Thomas clock to ensure the period of the pendulum is $\frac{1}{2}$ of a second. Assuming the initial displacement of the pendulum is 15°, find a sinusoid which models the displacement of the pendulum θ as a function of time, t, in seconds.

31. Use Theorem 1.12 to prove Theorem 7.7.

SECTION 7.3 EXERCISE ANSWERS A.1.7

7.4 OTHER TRIGONOMETRIC FUNCTIONS

In subsection 7.2.2, we extended the notion of $\sin(\theta)$ and $\cos(\theta)$ from acute angles to any angles using the coordinate values of points on the Unit Circle. In total, there are six circular functions, as listed below.

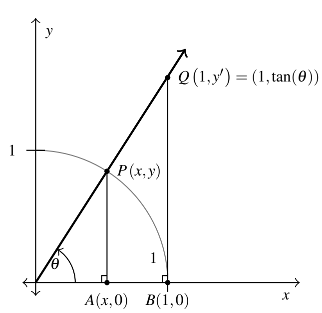
Definition 7.5. The Circular Functions: Suppose an angle θ is graphed in standard position.

Let $P(x,y)$ be the point of intersection of the terminal side of P and the Unit Circle.

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided $y \neq 0$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

While we left the history of the name 'sine' as an interesting research project in Section 7.2.2, we take a slight detour here to explain the origin of the names 'tangent' and 'secant.'

Consider the acute angle θ in standard position sketched in the diagram below.



As usual, $P(x,y)$ denotes the point on the terminal side of θ which lies on the Unit Circle, but we also consider the point $Q(1,y')$, the point on the terminal side of θ which lies on the vertical line $x = 1$.

The word 'tangent' comes from the Latin meaning 'to touch,' and for this reason, the line $x = 1$ is called a *tangent* line to the Unit Circle as it intersects, or 'touches', the circle at only one point, namely $(1,0)$.

Dropping perpendiculars from P and Q creates a pair of similar triangles ΔOPA and ΔOQB. Hence the corresponding sides are proportional. We get $\frac{y'}{y} = \frac{1}{x}$ which gives $y' = \frac{y}{x} = \tan(\theta)$.

We have just shown that for acute angles θ, $\tan(\theta)$ is the y-coordinate of the point on the terminal side of θ which lies on the line $x = 1$ which is *tangent* to the Unit Circle.

The word 'secant' means 'to cut', so a secant line is any line that 'cuts through' a circle at two points.[1] The line containing the terminal side of θ (not just the terminal side itself) is one such secant line as it intersects the Unit Circle in Quadrants I and III.

With the point P lying on the Unit Circle, the length of the hypotenuse of ΔOPA is 1. If we let h denote the length of the hypotenuse of ΔOQB, we have from similar triangles that $\frac{h}{1} = \frac{1}{x}$, or $h = \frac{1}{x} = \sec(\theta)$.

Hence for an acute angle θ, $\sec(\theta)$ is the length of the line segment which lies on the secant line determined by the terminal side of θ and 'cuts off' the tangent line $x = 1$.

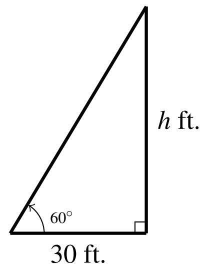
As we mentioned in Definition 7.3, the 'co' in 'cosecant' and 'cotangent' tie back to the concept of 'co'mplementary angles and is explained in detail in Section 8.2.

Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for Calculus which we'll explore in the Exercises.

In Definition 7.2 we introduced the trigonometric ratios for sine and cosine. There are four more trigonometric ratios which are commonly used and they are defined in the same manner the ratios for sine and cosine are defined using the given right triangle. They are listed below.

[1] Compare this with the definition given in Section 1.3.4.

Definition 7.6. Suppose θ is an acute angle residing in the right triangle as depicted below.

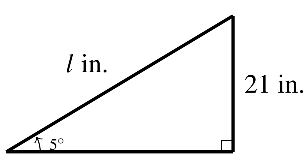


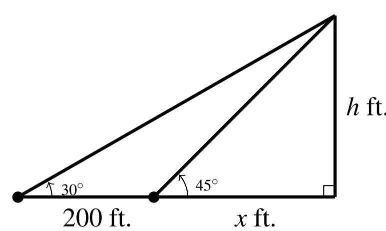
- The **tangent** of θ, denoted $\tan(\theta)$ is defined by the ratio: $\tan(\theta) = \frac{b}{a}$, or $\frac{\text{‘length of opposite’}}{\text{‘length of adjacent’}}$.
- The **cosecant** of θ, denoted $\csc(\theta)$ is defined by the ratio: $\csc(\theta) = \frac{c}{b}$, or $\frac{\text{‘length of hypotenuse’}}{\text{‘length of opposite’}}$.
- The **secant** of θ, denoted $\sec(\theta)$ is defined by the ratio: $\sec(\theta) = \frac{c}{a}$, or $\frac{\text{‘length of hypotenuse’}}{\text{‘length of adjacent’}}$.
- The **cotangent** of θ, denoted $\cot(\theta)$ is defined by the ratio: $\cot(\theta) = \frac{a}{b}$, or $\frac{\text{‘length of adjacent’}}{\text{‘length of opposite’}}$.

We practice these definitions in the following example.

Example 7.4.1. Suppose θ is an acute angle with $\cot(\theta) = 3$. Find the values of the remaining five trigonometric ratios: $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\csc(\theta)$, and $\sec(\theta)$.

Solution. We are given $\cot(\theta) = 3$. So, to proceed, we construct a right triangle in which the length of the side adjacent to θ and the length of the side opposite of θ has a ratio of $3 = \frac{3}{1}$. Note there are infinitely many such right triangles - we have produced two below for reference. We will focus our attention on the triangle below on the left and encourage the reader to work through the details using the triangle below on the right to verify the choice of triangle doesn't matter.





From the diagram, we see immediately $\tan(\theta) = \frac{1}{3}$, but in order to determine the remaining four trigonometric ratios, we need to first compute the value of the hypotenuse. The Pythagorean Theorem gives $1^2 + 3^2 = c^2$

so $c^2 = 10$ or $c = \sqrt{10}$. Rationalizing denominators, we find $\sin(\theta) = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$, $\cos(\theta) = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$, $\csc(\theta) = \frac{\sqrt{10}}{1} = \sqrt{10}$ and $\sec(\theta) = \frac{\sqrt{10}}{3}$. $\square$

7.4.1 Reciprocal and Quotient Identities

Of the six circular functions, only sine and cosine are defined for all angles θ. Given $x = \cos(\theta)$ and $y = \sin(\theta)$ in Definition 7.5, it is customary to rephrase the remaining four circular functions in Definition 7.5 in terms of sine and cosine.

Theorem 7.8. Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

We call the equations listed in Theorem 7.8 **identities** because they are relationships which are true regardless of the values of θ. This is in contrast to **conditional equations** such as $\sin(\theta) = 1$ which are true for only **some** values of θ. We will study identities more extensively in Sections 8.1 and 8.2.

While the Reciprocal and Quotient Identities presented in Theorem 7.8 allow us to always reduce problems involving secant, cosecant, tangent and cotangent to problems involving sine and cosine, it is not always convenient to do so.[2] It is worth taking the time to memorize the tangent and cotangent values of the common angles summarized below.

Tangent and Cotangent Values of Common Angles

[2]As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to cosines and sines. However, when solving for tangent or cotangent, we usually stick with what we're dealt.

θ(degrees)	θ(radians)	$\tan(\theta)$	$\cot(\theta)$
0°	0	0	undefined
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	1	1
60°	$\frac{\pi}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	undefined	0

Coupling Theorem 7.8 with the Reference Angle Theorem, Theorem 7.2, we get the following.

Theorem 7.9. Generalized Reference Angle Theorem.

The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle. More specifically, if α is the reference angle for θ, then:

$$\sin(\theta) = \pm\sin(\alpha), \quad \cos(\theta) = \pm\cos(\alpha), \quad \tan(\theta) = \pm\tan(\alpha)$$

and

$$\sec(\theta) = \pm\sec(\alpha), \quad \csc(\theta) = \pm\csc(\alpha), \quad \cot(\theta) = \pm\cot(\alpha)$$

where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

It is high time for an example.

Example 7.4.2.

1. Compute the exact value of the following, if it exists:

   (a) $\sec(60^\circ)$ (b) $\csc\left(\frac{7\pi}{4}\right)$ (c) $\tan(225^\circ)$ (d) $\cot\left(-\frac{7\pi}{6}\right)$

2. Determine all angles which satisfy the given equation.

   (a) $\sec(\theta) = 2$ (b) $\csc(\theta) = -\sqrt{2}$ (c) $\tan(\theta) = \sqrt{3}$ (d) $\cot(\theta) = -1$.

Solution.

1. (a) Compute the exact value of $\sec(60^\circ)$.

   According to Theorem 7.8, $\sec(60^\circ) = \frac{1}{\cos(60^\circ)}$. Hence, $\sec(60^\circ) = \frac{1}{(1/2)} = 2$.

(b) Compute the exact value of $\csc\left(\frac{7\pi}{4}\right)$.

Recall $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, so $\csc\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.

(c) Compute the exact value of $\tan(225^\circ)$.

We have two ways to proceed to determine $\tan(225^\circ)$. First, we can use Theorem 7.8 and note that $\tan(225^\circ) = \frac{\sin(225^\circ)}{\cos(225^\circ)}$. As $\sin(225^\circ) = \cos(225^\circ) = -\frac{\sqrt{2}}{2}$, we get $\tan(225^\circ) = 1$.

Another way to proceed is to note that 225° has a reference angle of 45°. Per Theorem 7.9, $\tan(225^\circ) = \pm\tan(45^\circ) = \pm 1$. Because 225° is a Quadrant III angle, where both the x and y coordinates of points are both negative, and tangent is defined as the *ratio* of coordinates $\frac{y}{x}$, we know $\tan(225^\circ) > 0$. Hence, $\tan(225^\circ) = 1$.

(d) Compute the exact value of $\cot\left(-\frac{7\pi}{6}\right)$.

As with the previous example, we have two ways to proceed. Using Theorem 7.8, we have $\cot\left(-\frac{7\pi}{6}\right) = \frac{\cos\left(-\frac{7\pi}{6}\right)}{\sin\left(-\frac{7\pi}{6}\right)}$. Because $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2}$, we get $\cot\left(-\frac{7\pi}{6}\right) = -\sqrt{3}$.

Alternatively, we note $-\frac{7\pi}{6}$ is a Quadrant II angle with reference angle $\frac{\pi}{6}$. Hence, Theorem 7.9 tells us $\cot\left(-\frac{7\pi}{6}\right) = \pm\cot\left(\frac{\pi}{6}\right) = \pm\sqrt{3}$. Because $-\frac{7\pi}{6}$ is a Quadrant II angle, where the x and y coordinates have different signs, and cotangent is defined as the ratio of coordinates $\frac{x}{y}$, we know $\cot\left(-\frac{7\pi}{6}\right) < 0$. Hence, $\cot\left(-\frac{7\pi}{6}\right) = -\sqrt{3}$.

2. (a) Determine all angles which satisfy the equation $\sec(\theta) = 2$.

To solve $\sec(\theta) = 2$, we convert to cosines and get $\frac{1}{\cos(\theta)} = 2$ or $\cos(\theta) = \frac{1}{2}$. This is the exact same equation we solved in Example 7.2.4, number 1, so we know the answer is: $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

(b) Determine all angles which satisfy the equation $\csc(\theta) = -\sqrt{2}$.

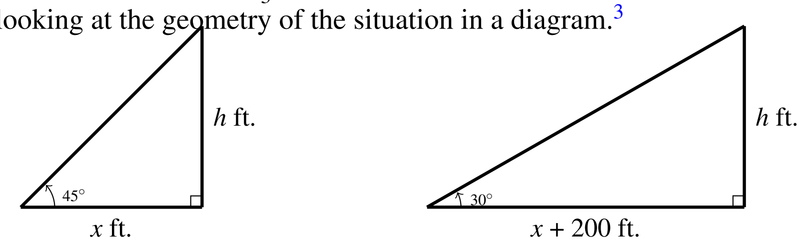
To solve $\csc(\theta) = -\sqrt{2}$, we convert to sines and get $\frac{1}{\sin(\theta)} = -\sqrt{2}$ or $\sin(\theta) = -\frac{\sqrt{2}}{2}$. Using the table of values for $\sin(t)$ found below Theorem 7.5 in Section 7.3 we know $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. Thus the answer is: $\theta = \frac{5\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$ for integers k.

(c) Determine all angles which satisfy the equation $\tan(\theta) = \sqrt{3}$.

From the table of common values, we see $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. According to Theorem 7.9, we know the solutions to $\tan(\theta) = \sqrt{3}$ must, therefore, have a reference angle of $\frac{\pi}{3}$.

To find the quadrants in which our solutions lie, we note that tangent is defined as the ratio $\frac{y}{x}$ of points (x, y) on the Unit Circle. Hence, tangent is positive when x and y have the same sign (i.e., when they are both positive or both negative.) This happens in Quadrants I and III.

In Quadrant I, we get the solutions: $\theta = \frac{\pi}{3} + 2\pi k$ for integers k, and for Quadrant III, we get $\theta = \frac{4\pi}{3} + 2\pi k$ for integers k. While these descriptions of the solutions are correct, they can be combined into one list as $\theta = \frac{\pi}{3} + \pi k$ for integers k. The latter form of the solution is best understood looking at the geometry of the situation in a diagram.[3]

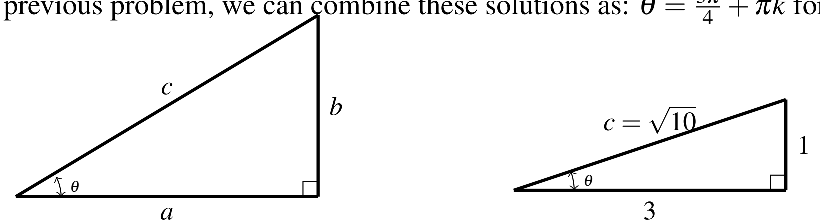


(d) Determine all angles which satisfy the equation $\cot(\theta) = -1$.

From the table of common values, we see that $\frac{\pi}{4}$ has a cotangent of 1, which means the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$.

To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point (x, y) on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then x and y must have different signs (i.e., one positive and one negative.) Hence, our solutions lie in Quadrants II and IV.

Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV, we get $\theta = \frac{7\pi}{4} + 2\pi k$ for integers k. As in the previous problem, we can combine these solutions as: $\theta = \frac{3\pi}{4} + \pi k$ for integers k.

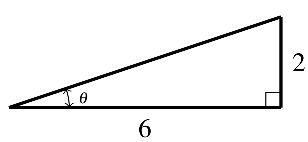


□

A few remarks about Example 7.4.2 are in order. First note that the signs ($\pm$) of secant and cosecant are the same as the signs of cosine and sine, respectively.

On the other hand, as a result of tangent and cotangent being defined in terms of the *ratios* of coordinates x and y, tangent and cotangent are positive in Quadrants I and III (where both x and y have the same sign) and negative in Quadrants II and IV (where x and y have opposite signs.)

The diagram below summarizes which circular functions are positive in which quadrants.

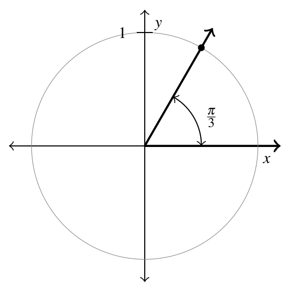


Positive Circular Functions

Also note it is no coincidence that both of our solutions to the equations involving tangent and cotangent in Example 7.4.2 could be simplified to just one list of angles differing by multiples of π.

[3]See Example 7.2.4 number 3 in Section 7.2.2 for another example of this kind of simplification of the solution.

Indeed, any two angles that are π units apart will not only have the same reference angle, but points on their terminal sides on the Unit Circle will be reflections through the origin, as illustrated below.



The period of $\tan(\theta)$ and $\cot(\theta)$ is π.

It follows that the tangent and cotangent of such angles (if defined) will be the same, which means the period of these function is (at most) π.

Using an argument similar to the one we used to establish the period of sine and cosine in Section 7.3, we note that if $\tan(x+p) = \tan(x)$ for all real numbers x, then, in particular, $\tan(p) = \tan(0+p) = \tan(0) = 0$. Hence, p is a multiple of π, and the smallest multiple of π is π itself.

Hence, the period of tangent (and cotangent) is π, and we will see the consequences of this both when solving equations in this section and when graphing these functions in Section 7.5.

As with sine and cosine, the circular functions defined in Definition 7.5 agree with those put forth in Definitions 7.2 and 7.6 in Section 7.2.1 for acute angles situated in right triangles. The argument is identical to the one given in Section 7.2.2 and is left to the reader.

Moreover, Definition 7.5 can be extended to circles of arbitrary radius $r > 0$ using the same similarity arguments in Section 7.2.3 to generalize Definition 7.3 to Theorem 7.4 as summarized below.

Theorem 7.10. Suppose $Q(x,y)$ is the point on the terminal side of an angle θ (plotted in standard position) which lies on the circle of radius r, $x^2+y^2=r^2$. Then:

- $\sin(\theta) = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2+y^2}}$
- $\cos(\theta) = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2+y^2}}$
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2+y^2}}{x}$, provided $x \neq 0$.
- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2+y^2}}{y}$, provided $y \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

We make good use of Theorem 7.10 in the following example.

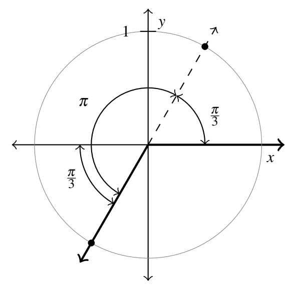
Example 7.4.3. Use Theorem 7.10 to solve the following.

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3,4)$. Calculate the values of the six circular functions of θ.
2. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Calculate the values of the five remaining circular functions of θ.
3. Compute $\sin(\theta)$, where $\sec(\theta) = -\sqrt{5}$ and θ is a Quadrant II angle.
4. Compute $\cos(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

Solution.

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3,4)$. Calculate the values of the six circular functions of θ.

   Given $x=3$ and $y=4$ and $x^2+y^2=r^2$, then $(3)^2+(4)^2=r^2$ so $r^2=25$, or $r=5$. Theorem 7.10 tells us $\sin(\theta)=\frac{4}{5}$, $\cos(\theta)=\frac{3}{5}$, $\tan(\theta)=\frac{4}{3}$, $\sec(\theta)=\frac{5}{3}$, $\csc(\theta)=\frac{5}{4}$, and $\cot(\theta)=\frac{3}{4}$.



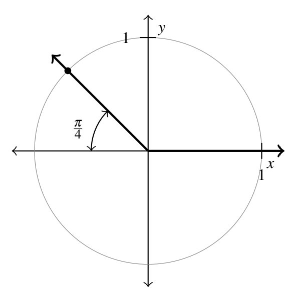
$Q(3,4)$ lies on a circle of radius 5 units.

2. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Calculate the values of the five remaining circular functions of θ.

   In order to use Theorem 7.10, we need to find a point $Q(x,y)$ which lies on the terminal side of θ, when θ is plotted in standard position.

   We have that $\cot(\theta) = -4 = \frac{x}{y}$. θ is a Quadrant IV angle, so we also know $x > 0$ and $y < 0$. Rewriting $-4 = \frac{4}{-1}$, we choose[4] $x = 4$ and $y = -1$ so that $r = \sqrt{x^2+y^2} = \sqrt{(4)^2+(-1)^2} = \sqrt{17}$.

   Applying Theorem 7.10, we find $\sin(\theta) = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$, $\cos(\theta) = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$, $\tan(\theta) = -\frac{1}{4}$, $\sec(\theta) = \frac{\sqrt{17}}{4}$, and $\csc(\theta) = -\sqrt{17}$.



θ is in Quadrant IV with $\cot(\theta) = -4$.

3. Compute $\sin(\theta)$, where $\sec(\theta) = -\sqrt{5}$ and θ is a Quadrant II angle.

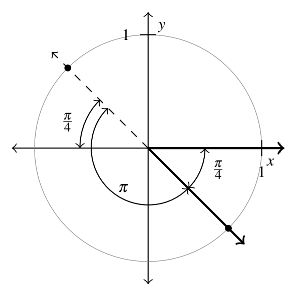
[4]We could have just as easily chosen $x = 8$ and $y = -2$ - just so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$.

To find $\sin(\theta)$ using Theorem 7.10, we need to determine the y-coordinate of a point $Q(x,y)$ on the terminal side of θ, when θ is plotted in standard position, and the corresponding radius r.

Given $\sec(\theta) = \frac{r}{x}$ and $r > 0$, we rewrite $\sec(\theta) = \frac{r}{x} = -\sqrt{5} = \frac{\sqrt{5}}{-1}$ and take $r = \sqrt{5}$ and $x = -1$.

To find y, we substitute $x = -1$ and $r = \sqrt{5}$ into $x^2 + y^2 = r^2$ to get $(-1)^2 + y^2 = (\sqrt{5})^2$. We find $y^2 = 4$ or $y = \pm 2$. We were told θ is a Quadrant II angle, thus we select $y = 2$.

Hence, $\sin(\theta) = \frac{y}{r} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$.



θ is in Quadrant II with $\sec(\theta) = -\sqrt{5}$.

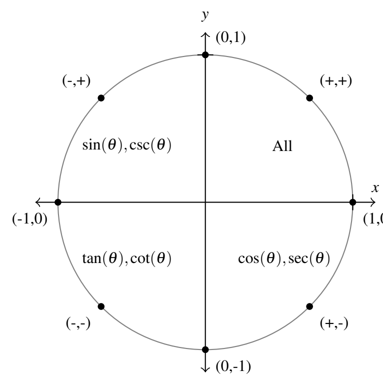
4. Compute $\cos(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

We are told $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$, so we know θ is a Quadrant III angle.

To find $\cos(\theta)$ using Theorem 7.10, we need to find the x-coordinate of a point $Q(x,y)$ on the terminal side of θ, when θ is plotted in standard position, and the corresponding radius, r.

Given $\tan(\theta) = \frac{y}{x}$ and θ is a Quadrant III angle, we rewrite $\tan(\theta) = 3 = \frac{-3}{-1} = \frac{y}{x}$ and choose $x = -1$ and $y = -3$. From $x^2 + y^2 = r^2$, we get $r = \sqrt{10}$.

Hence, $\cos(\theta) = \frac{x}{r} = \frac{-1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$.



θ is in Quadrant III with $\tan(\theta) = 3$.

□

As we did in Section 7.2.3, we may consider $\tan(t)$, $\sec(t)$, $\csc(t)$, and $\cot(t)$ as functions of *real numbers* by associating each real number t with an angle θ measuring t radians as discussed on page 608 and using Definition 7.5, or, more generally, Theorem 7.10.

Alternatively, we could define each of these four functions in terms of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ as demonstrated in Theorem 7.8. For example, we could simply *define* $\sec(t) = \frac{1}{\cos(t)}$, so long as $\cos(t) \neq 0$.

Either way, we have the means to explore these functions in greater detail. Before doing so, we'll need practice with these additional four circular functions courtesy of the Exercises.

We are overdue for an example.

Example 7.4.4.

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland's Armington Clocktower[5] is 60°. Determine the height of the Clocktower to the nearest foot.

2. The Americans with Disabilities Act (ADA) stipulates the incline on an accessibility ramp be 5°. If a ramp is to be built so that it replaces stairs that measure 21 inches tall, how long does the ramp need to be? Round your answer to the nearest inch.

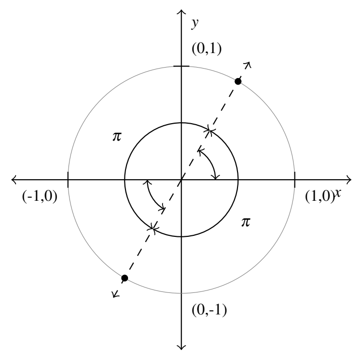
3. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

Solution.

[5] Named in honor of Raymond Q. Armington, Lakeland's Clocktower has been a part of campus since 1972.

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland's Armington Clocktower. Determine the height of the Clocktower to the nearest foot.

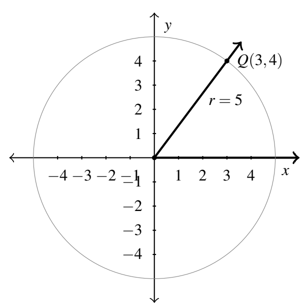
   We can represent the problem situation using a right triangle as shown below on the left. If we let h denote the height of the tower, then we have $\tan(60^\circ) = \frac{h}{30}$. From this we get an exact answer of $h = 30\tan(60^\circ) = 30\sqrt{3}$ feet. Using a calculator, we get the approximation 51.96 which, when rounded to the nearest foot, gives us our answer of 52 feet.



Finding the height of the Clocktower

2. The Americans with Disabilities Act (ADA) stipulates the incline on an accessibility ramp be 5°. If a ramp is to be built so that it replaces stairs that measure 21 inches tall, how long does the ramp need to be? Round your answer to the nearest inch.

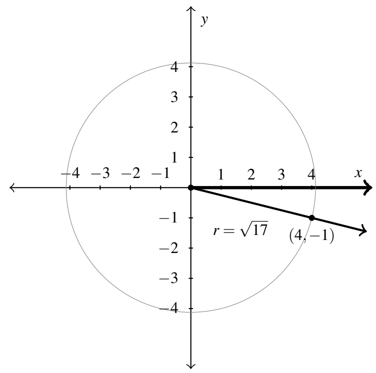
   We diagram the situation below using ℓ to represent the unknown length of the ramp. We have $\sin(5^\circ) = \frac{21}{\ell}$ so that $\ell = \frac{21}{\sin(5^\circ)} \approx 240.95$ inches. Hence, the ramp is 241 inches long.



Finding the length of an accessibility ramp.

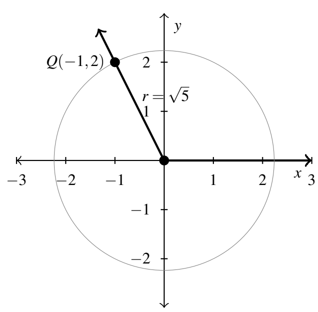
3. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

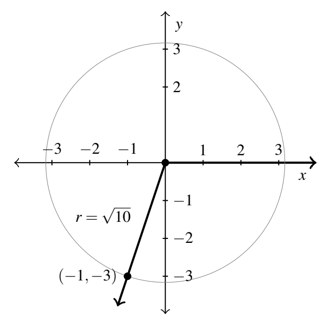
Sketching the problem situation, we find ourselves with two unknowns: the height h of the tree and the distance x from the base of the tree to the first observation point.



Finding the height of a California Redwood

Luckily, we have two right triangles to help us find each unknown, as shown below. From the triangle below on the left, we get $\tan(45^\circ) = \frac{h}{x}$. From the triangle below on the right, we see $\tan(30^\circ) = \frac{h}{x+200}$.





As $\tan(45^\circ) = 1$, the first equation gives $\frac{h}{x} = 1$, or $x = h$. Substituting this into the second equation gives $\frac{h}{h+200} = \tan(30^\circ) = \frac{\sqrt{3}}{3}$. Clearing fractions, we get $3h = (h+200)\sqrt{3}$. The result is a linear equation for h, so we expand the right hand side and gather all the terms involving h to one side.

$$\begin{aligned}
3h &= (h+200)\sqrt{3} \\
3h &= h\sqrt{3} + 200\sqrt{3} \\
3h - h\sqrt{3} &= 200\sqrt{3} \\
(3-\sqrt{3})h &= 200\sqrt{3} \\
h &= \frac{200\sqrt{3}}{3-\sqrt{3}} \approx 273.20
\end{aligned}$$

Hence, the tree is approximately 273 feet tall. □

7.4.2 Exercises

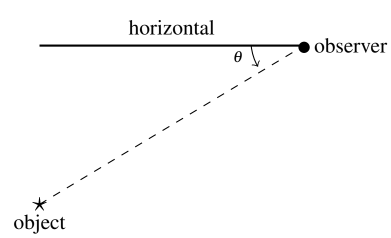
In Exercises 1 - 15, θ is an acute angle. Use the given trigonometric ratio to find the exact values of the remaining trigonometric ratios of θ.

1. $\sin(\theta) = \dfrac{3}{5}$
2. $\tan(\theta) = \dfrac{12}{5}$
3. $\csc(\theta) = \dfrac{25}{24}$
4. $\sec(\theta) = 7$
5. $\csc(\theta) = \dfrac{10\sqrt{91}}{91}$
6. $\cot(\theta) = 23$
7. $\tan(\theta) = 2$
8. $\sec(\theta) = 4$
9. $\cot(\theta) = \sqrt{5}$
10. $\cos(\theta) = \dfrac{1}{3}$
11. $\cot(\theta) = 2$
12. $\csc(\theta) = 5$
13. $\tan(\theta) = \sqrt{10}$
14. $\sec(\theta) = 2\sqrt{5}$
15. $\cos(\theta) = 0.4$

16. A tree standing vertically on level ground casts a 120 foot long shadow. The angle of elevation from the end of the shadow to the top of the tree is 21.4°. Find the height of the tree to the nearest foot.

17. The broadcast tower for radio station WSAZ (Home of "Algebra in the Morning with Carl and Jeff") has two enormous flashing red lights on it: one at the very top and one a few feet below the top. From a point 5000 feet away from the base of the tower on level ground the angle of elevation to the top light is 7.970° and to the second light is 7.125°. Find the distance between the lights to the nearest foot.

18. On page 593 we defined the angle of inclination (also known as the angle of elevation) and in this exercise we introduce a related angle - the angle of depression (also known as the angle of declination). The angle of depression of an object refers to the angle whose initial side is a horizontal line above the object and whose terminal side is the line-of-sight to the object below the horizontal. This is represented schematically below.



The angle of depression from the horizontal to the object is θ

(a) Show that if the horizontal is above and parallel to level ground then the angle of depression (from observer to object) and the angle of inclination (from object to observer) will be congruent because they are alternate interior angles.

(b) From a firetower 200 feet above level ground in the Sasquatch National Forest, a ranger spots a fire off in the distance. The angle of depression to the fire is 2.5°. How far away from the base of the tower is the fire?

(c) The ranger in part 18b sees a Sasquatch running directly from the fire towards the firetower. The ranger takes two sightings. At the first sighting, the angle of depression from the tower to the Sasquatch is 6°. The second sighting, taken just 10 seconds later, gives the the angle of depression as 6.5°. How far did the Saquatch travel in those 10 seconds? Round your answer to the nearest foot. How fast is it running in miles per hour? Round your answer to the nearest mile per hour. If the Sasquatch keeps up this pace, how long will it take for the Sasquatch to reach the firetower from his location at the second sighting? Round your answer to the nearest minute.

19. When I stand 30 feet away from a tree at home, the angle of elevation to the top of the tree is 50° and the angle of depression to the base of the tree is 10°. What is the height of the tree? Round your answer to the nearest foot.

20. From the observation deck of the lighthouse at Sasquatch Point 50 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2° and the second sighting had an angle of depression of 25.9°. How far had the boat traveled between the sightings?

21. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it makes a 43° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?

In Exercises 22 - 41, compute the exact value or state that it is undefined.

22. $\tan\left(\frac{\pi}{4}\right)$	23. $\sec\left(\frac{\pi}{6}\right)$	24. $\csc\left(\frac{5\pi}{6}\right)$	25. $\cot\left(\frac{4\pi}{3}\right)$
26. $\tan\left(-\frac{11\pi}{6}\right)$	27. $\sec\left(-\frac{3\pi}{2}\right)$	28. $\csc\left(-\frac{\pi}{3}\right)$	29. $\cot\left(\frac{13\pi}{2}\right)$
30. $\tan(117\pi)$	31. $\sec\left(-\frac{5\pi}{3}\right)$	32. $\csc(3\pi)$	33. $\cot(-5\pi)$
34. $\tan\left(\frac{31\pi}{2}\right)$	35. $\sec\left(\frac{\pi}{4}\right)$	36. $\csc\left(-\frac{7\pi}{4}\right)$	37. $\cot\left(\frac{7\pi}{6}\right)$

38. $\tan\left(\frac{2\pi}{3}\right)$

39. $\sec(-7\pi)$

40. $\csc\left(\frac{\pi}{2}\right)$

41. $\cot\left(\frac{3\pi}{4}\right)$

In Exercises 42 - 45, use the given the information to determine the quadrant in which the terminal side of the angle lies when plotted in standard position.

42. $\sin(\theta) > 0$ but $\tan(\theta) < 0$.

43. $\cot(\alpha) > 0$ but $\cos(\alpha) < 0$.

44. $\sin(\beta) > 0$ and $\tan(\beta) > 0$.

45. $\cos(\gamma) > 0$ but $\cot(\gamma) < 0$.

In Exercises 46 - 59, use the given the information to compute the exact values of the circular functions of θ.

46. $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II

47. $\tan(\theta) = \frac{12}{5}$ with θ in Quadrant III

48. $\csc(\theta) = \frac{25}{24}$ with θ in Quadrant I

49. $\sec(\theta) = 7$ with θ in Quadrant IV

50. $\csc(\theta) = -\frac{10\sqrt{91}}{91}$ with θ in Quadrant III

51. $\cot(\theta) = -23$ with θ in Quadrant II

52. $\tan(\theta) = -2$ with θ in Quadrant IV.

53. $\sec(\theta) = -4$ with θ in Quadrant II.

54. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.

55. $\cos(\theta) = \frac{1}{3}$ with θ in Quadrant I.

56. $\cot(\theta) = 2$ with $0 < \theta < \frac{\pi}{2}$.

57. $\csc(\theta) = 5$ with $\frac{\pi}{2} < \theta < \pi$.

58. $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \frac{3\pi}{2}$.

59. $\sec(\theta) = 2\sqrt{5}$ with $\frac{3\pi}{2} < \theta < 2\pi$.

In Exercises 60 - 67, use your calculator to approximate the given value to three decimal places. Make sure your calculator is in the proper angle measurement mode!

60. $\csc(78.95^\circ)$

61. $\tan(-2.01)$

62. $\cot(392.994)$

63. $\sec(207^\circ)$

64. $\csc(5.902)$

65. $\tan(39.672^\circ)$

66. $\cot(3^\circ)$

67. $\sec(0.45)$

In Exercises 68 - 82, find all of the angles which satisfy the equation.

68. $\tan(\theta) = \sqrt{3}$

69. $\sec(\theta) = 2$

70. $\csc(\theta) = -1$

71. $\cot(\theta) = \frac{\sqrt{3}}{3}$

72. $\tan(\theta) = 0$

73. $\sec(\theta) = 1$

74. $\csc(\theta) = 2$

75. $\cot(\theta) = 0$

76. $\tan(\theta) = -1$
77. $\sec(\theta) = 0$
78. $\csc(\theta) = -\frac{1}{2}$
79. $\sec(\theta) = -1$
80. $\tan(\theta) = -\sqrt{3}$
81. $\csc(\theta) = -2$
82. $\cot(\theta) = -1$

In Exercises 83 - 90, solve the equation for t. Give exact values.

83. $\cot(t) = 1$
84. $\tan(t) = \frac{\sqrt{3}}{3}$
85. $\sec(t) = -\frac{2\sqrt{3}}{3}$
86. $\csc(t) = 0$
87. $\cot(t) = -\sqrt{3}$
88. $\tan(t) = -\frac{\sqrt{3}}{3}$
89. $\sec(t) = \frac{2\sqrt{3}}{3}$
90. $\csc(t) = \frac{2\sqrt{3}}{3}$

In Exercises 91 - 98, write the given function as a nontrivial decomposition of functions as directed.

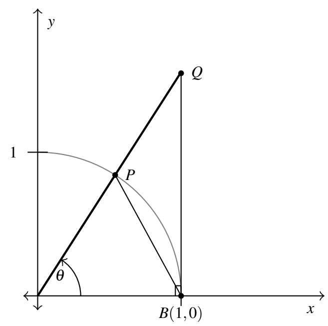
91. For $f(t) = 3t^2 + 2\tan(3t)$, find functions g and h so that $f = g + h$.
92. For $f(\theta) = \sec(\theta) - \tan(\theta)$, find functions g and h so that $f = g - h$.
93. For $f(t) = -\csc(t)\cot(t)$, find functions g and h so that $f = gh$.
94. For $r(t) = \frac{\tan(3t)}{t}$, find functions f and g so $r = \frac{f}{g}$.
95. For $T(\theta) = \tan(4\theta)$, find functions f and g so $T = g \circ f$.
96. For $s(\theta) = \sec^2(\theta)$, find functions f and g so $s = g \circ f$.
97. For $L(x) = \ln(\sin(x))$, find functions f and g so $L = g \circ f$.
98. For $\ell(\theta) = \ln|\sec(\theta) - \tan(\theta)|$, find find functions f, g, and h so $\ell = h \circ (f - g)$.
99. Let $S(t) = \sin(t)$ and $C(t) = \cos(t)$, $F(t) = \tan(t)$, and $G(t) = \cot(t)$. Explain why $F = \frac{S}{C}$ but $F \neq \frac{1}{G}$.

    HINT: Think about domains ...
100. For each function $T(t)$ listed below, compute the average rate of change over the indicated interval.[6] What trends do you notice? Compare your answer with what you discovered in Section 7.2.2 number 65. Be sure your calculator is in radian mode!

$T(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\tan(t)$			
$\tan(2t)$			
$\tan(3t)$			
$\tan(4t)$			

101. We wish to establish the inequality $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \frac{\pi}{2}$. Use the diagram from the beginning of the section, partially reproduced below, to answer the following.

[6]See Definition 1.11 in Section 1.3.4 for a review of this concept, as needed.



(a) Show that triangle OPB has area $\frac{1}{2}\sin(\theta)$ and triangle OQB has area $\frac{1}{2}\tan(\theta)$.

(b) Show that the circular sector OPB with central angle θ has area $\frac{1}{2}\theta$.

(c) Comparing areas, show that $\sin(\theta) < \theta < \tan(\theta)$ for $0 < \theta < \frac{\pi}{2}$.

(d) Use the inequality $\sin(\theta) < \theta$ to show that $\frac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \frac{\pi}{2}$.

(e) Use the inequality $\theta < \tan(\theta)$ to show that $\cos(\theta) < \frac{\sin(\theta)}{\theta}$ for $0 < \theta < \frac{\pi}{2}$. Combine this with the previous part to complete the proof.

102. Show that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ also holds for $-\frac{\pi}{2} < \theta < 0$.

SECTION 7.4 EXERCISE ANSWERS A.1.7

7.5 Graphs of Other Trigonometric Functions

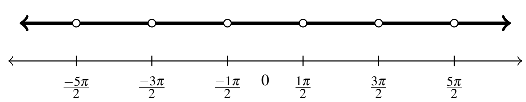
7.5.1 Graphs of the Secant and Cosecant Functions

As mentioned at the end of Section 7.4, one way to proceed with our analysis of the circular functions is to use what we know about the functions $\sin(t)$ and $\cos(t)$ to rewrite the four additional circular functions in terms of sine and cosine with help from Theorem 7.8. We use this approach to analyze $F(t) = \sec(t)$.

Rewriting $F(t) = \sec(t) = \frac{1}{\cos(t)}$, we first note that $F(t)$ is undefined whenever $\cos(t) = 0$. Thanks to Example 7.2.4 number 3, we know $\cos(t) = 0$ whenever $t = \frac{\pi}{2} + \pi k$ for integers k.

This gives us one way to describe the domain of F: $\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. To get a better feel for the set of real numbers we're dealing with, we write out and graph the domain on the number line.

Running through a few values of k, we find some of the values excluded from the domain: $t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$. Using these we can graph the domain on the number line below.



Expressing this set using interval notation is a bit of a challenge, owing to the infinitely many intervals present. As a first attempt, we have: $\ldots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$, where, as usual, the periods of ellipsis indicate the pattern continues indefinitely. Hence, for now, it suffices to know that the domain of $F(t) = \sec(t)$ excludes the odd multiples of $\frac{\pi}{2}$.

To find the range of F, we find it helpful once again to view $F(t) = \sec(t) = \frac{1}{\cos(t)}$. We know the range of $\cos(t)$ is $[-1, 1]$ and $F(t) = \sec(t) = \frac{1}{\cos(t)}$ is undefined when $\cos(t) = 0$, so we split our discussion into two cases: when $0 < \cos(t) \leq 1$ and when $-1 \leq \cos(t) < 0$.

If $0 < \cos(t) \leq 1$, then we can divide the inequality $\cos(t) \leq 1$ by $\cos(t)$ to obtain $\sec(t) = \frac{1}{\cos(t)} \geq 1$. Moreover, we see as $\cos(t) \to 0^+$, $\sec(t) \to \infty$. If, on the other hand, if $-1 \leq \cos(t) < 0$, then dividing by $\cos(t)$ causes a reversal of the inequality so that $\sec(t) = \frac{1}{\cos(t)} \leq -1$. In this case, as $\cos(t) \to 0^-$, $\sec(t) \to -\infty$. As $\cos(t)$ admits all of the values in $[-1, 1]$, the function $F(t) = \sec(t)$ admits all of the values in $(-\infty, -1] \cup [1, \infty)$.

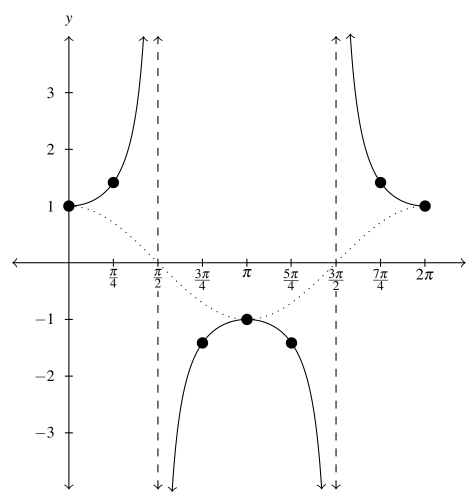
Because $\cos(t)$ is periodic with period 2π, it shouldn't be too surprising to find that $\sec(t)$ is also. Indeed, provided $\sec(\alpha)$ and $\sec(\beta)$ are defined, $\sec(\alpha) = \sec(\beta)$ if and only if $\cos(\alpha) = \cos(\beta)$. Said differently, $\sec(t)$ 'inherits' its period from $\cos(t)$.

We now turn our attention to graphing $F(t) = \sec(t)$. Using the table of values we tabulated when graphing $y = \cos(t)$ in Section 7.3, we can generate points on the graph of $y = \sec(t)$ by taking reciprocals.

Using the techniques developed in Section 3.2, we can more closely analyze the behavior of F near the values excluded from its domain. We find as $t \to \frac{\pi}{2}^-$, $\cos(t) \to 0^+$, so $\sec(t) \to \infty$. Similarly, we get as $t \to \frac{\pi}{2}^+$, $\sec(t) \to -\infty$; as $t \to \frac{3\pi}{2}^-$, $\sec(t) \to -\infty$; and as $t \to \frac{3\pi}{2}^+$, $\sec(t) \to \infty$. This means the lines $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are vertical asymptotes to the graph of $y = \sec(t)$.

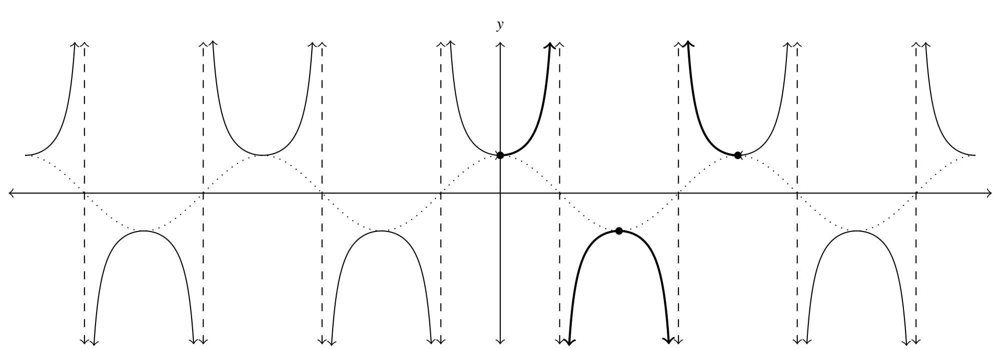
Below on the right we graph a fundamental cycle of $y = \sec(t)$ with the graph of the fundamental cycle of $y = \cos(t)$ dotted for reference.

t	$\cos(t)$	$\sec(t)$	$(t, \sec(t))$
0	1	1	$(0,1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$



The 'fundamental cycle' of $y = \sec(t)$.

To get a graph of the entire secant function, we paste copies of the fundamental cycle end to end to produce the graph below. The graph suggests that $F(t) = \sec(t)$ is even. Indeed, because $\cos(t)$ is even, that is, $\cos(-t) = \cos(t)$, we have $\sec(-t) = \frac{1}{\cos(-t)} = \frac{1}{\cos(t)} = \sec(t)$. Hence, along with its period, the secant function inherits its symmetry from the cosine function.

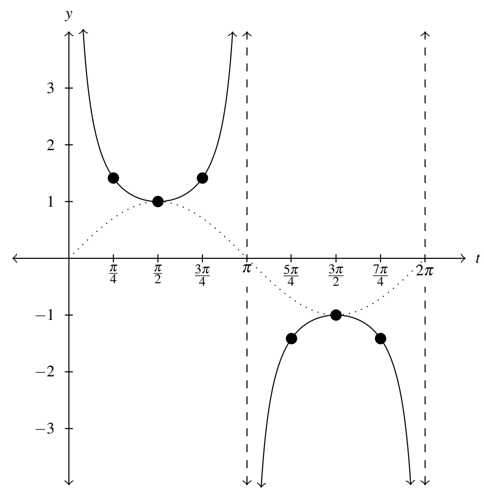


The graph of $y = \sec(t)$.

As one would expect, to graph $G(t) = \csc(t)$ we begin with $y = \sin(t)$ and take reciprocals of the corresponding y-values. Here, we encounter issues at $t = 0$, $t = \pi$, $t = 2\pi$, and, in general, at all whole number multiples of π, so the domain of G is $\{t \,|\, t \neq \pi k, \text{for integers } k\}$. Not surprisingly, these values produce vertical asymptotes.

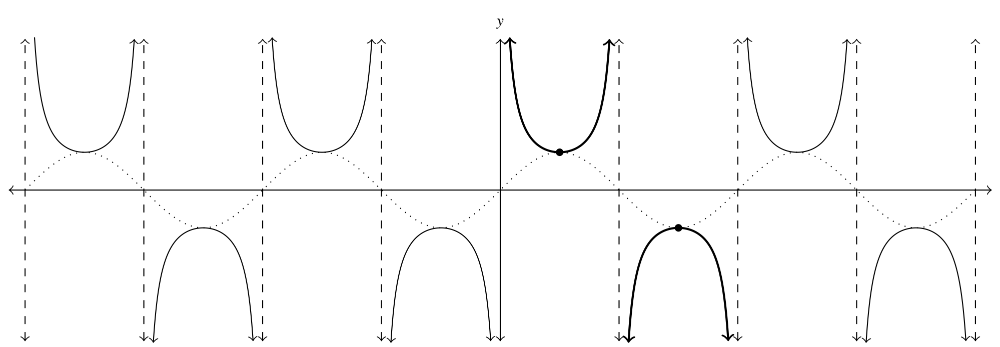
Proceeding as above, we produce the graph of the fundamental cycle of $y = \csc(t)$ below along with the dotted graph of $y = \sin(t)$ for reference.

x	$\sin(x)$	$\csc(x)$	$(x, \csc(x))$
0	0	undefined	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	



The 'fundamental cycle' of $y = \csc(t)$.

Pasting copies of the fundamental period of $y = \csc(t)$ end to end produces the graph below. Due to the fact that the graphs of $y = \sin(t)$ and $y = \cos(t)$ are merely phase shifts of each other, it is not too surprising to find the graphs of $y = \csc(t)$ and $y = \sec(t)$ are as well.



The graph of $y = \csc(t)$.

As with the graph of secant, the graph of cosecant suggests symmetry. Indeed, as the sine function is odd, that is $\sin(-t) = -\sin(t)$, so too is the cosecant function: $\csc(-t) = \frac{1}{\sin(-t)} = -\frac{1}{\sin(t)} = -\csc(t)$. Hence, the graph of $G(t) = \csc(t)$ is symmetric about the origin.

Note that, on the intervals between the vertical asymptotes, both $F(t) = \sec(t)$ and $G(t) = \csc(t)$ are continuous and smooth. In other words, they are continuous and smooth *on their domains*.[1]

The following theorem summarizes the properties of the secant and cosecant functions. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

Theorem 7.11. Properties of the Secant and Cosecant Functions

- The function $F(t) = \sec(t)$
  - has domain $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \ k \text{ is an integer}\right\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(t) = \csc(t)$
  - has domain $\{t \,|\, t \neq \pi k, \ k \text{ is an integer}\}$
  - has range $(-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π

In the next example, we discuss graphing more general secant and cosecant curves. We make heavy use of the fact they are reciprocals of sine and cosine functions and apply what we learned in Section 7.3.

Example 7.5.1. Graph one cycle of the following functions. State the period of each.

1. $f(t) = 1 - 2\sec(2t)$
2. $g(t) = \dfrac{\csc(-\pi t - \pi) - 5}{3}$

[1] Just like the rational functions in Chapter 3 are continuous and smooth on their domains because polynomials are continuous and smooth everywhere, the secant and cosecant functions are continuous and smooth on their domains as a result of the cosine and sine functions are continuous and smooth everywhere.

Solution.

1. Graph one cycle of $f(t) = 1 - 2\sec(2t)$.

   To graph $f(t) = 1 - 2\sec(2t)$, we follow the same procedure as in Example 7.3.2. That is, we use the concept of frequency and phase shift to identify quarter marks, then substitute these values into the function to obtain the corresponding points.

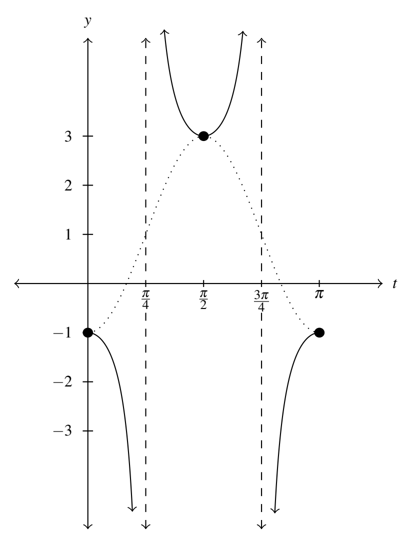
   If we think about a related *cosine* curve, $y = 1 - 2\cos(2t) = -2\cos(2t) + 1$, we know from Section 7.3, that the frequency is $B = 2$, so the period is $T = \frac{2\pi}{2} = \pi$. $C = 0$, so there is no phase shift. Hence, the new quarter marks for this curve are $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$, and $t = \pi$.

   These same t-values are the new quarter marks for $f(t) = 1 - 2\sec(2t)$, because we obtained the fundamental cycle of the secant curve from the fundamental cycle of the cosine curve.

   Substituting these t values $f(t)$, we get the table below on the left. Note that if $f(t)$ exists, we have a point on the graph; otherwise, we have found a vertical asymptote.[2]

   We graph one cycle of $f(t) = 1 - 2\sec(2t)$ below on the right along with the associated cosine curve, $y = 1 - 2\cos(2t)$ which is dotted, and confirm the period is $\pi - 0 = \pi$.

t	$f(t)$	$(t, f(t))$
0	-1	$(0, -1)$
$\frac{\pi}{4}$	undefined	
$\frac{\pi}{2}$	3	$\left(\frac{\pi}{2}, 3\right)$
$\frac{3\pi}{4}$	undefined	
π	-1	$(\pi, -1)$



One cycle of $y = 1 - 2\sec(2t)$.

2. Graph one cycle $g(t) = \dfrac{\csc(-\pi t - \pi) - 5}{3}$.

   As with the previous example, we start graphing $g(t) = \frac{\csc(-\pi t - \pi) - 5}{3}$ by first finding the quarter marks of the associated sine curve: $y = \frac{\sin(-\pi t - \pi) - 5}{3} = \frac{1}{3}\sin(-\pi t - \pi) - \frac{5}{3}$.

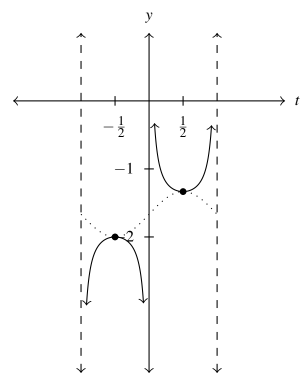
[2]As with the examples in Section 7.3, note that we can partially check our answer because the argument of the secant function should simplify to the 'original' quarter marks - the quadrantal angles.

The coefficient of t is negative, thus we make use of the odd property of sine to rewrite the function as: $y = \frac{1}{3}\sin(-\pi t - \pi) - \frac{5}{3} = \frac{1}{3}\sin(-(\pi t + \pi)) - \frac{5}{3} = -\frac{1}{3}\sin(\pi t + \pi) - \frac{5}{3}$.

We find the frequency is $B = \pi$, so the period is $T = \frac{2\pi}{\pi} = 2$. $C = \pi$, so the phase shift is $-\frac{\pi}{\pi} = -1$. Hence the fundamental cycle $[0, 2\pi]$ is shifted to the interval $[-1, 1]$ with quarter marks $t = -1$, $t = -\frac{1}{2}$, $t = 0$, $t = \frac{1}{2}$ and $t = 1$.

Substituting these t-values into $g(t)$, we generate the graph below on the right confirm the period is $1 - (-1) = 2$. The associated sine curve, $y = \frac{\sin(-\pi t - \pi) - 5}{3}$, is dotted in as a reference.

t	$g(t)$	$(t, g(t))$
-1	undefined	
$-\frac{1}{2}$	-2	$\left(-\frac{1}{2}, -2\right)$
0	undefined	
$\frac{1}{2}$	$-\frac{4}{3}$	$\left(\frac{1}{2}, -\frac{4}{3}\right)$
1	undefined	



One cycle of $g(t) = \dfrac{\csc(-\pi t - \pi) - 5}{3}$.

$\square$

As suggested in Example 7.5.1, the concepts of frequency, period, phase shift, and baseline are alive and well with graphs of the secant and cosecant functions. The secant and cosecant curves are unbounded, therefore we do not have the concept of 'amplitude' for these curves. That being said, the amplitudes of the corresponding cosine and sine curves do play a role here - they measure how wide the gap is between the baseline and the curve.

We gather these observations in the following result whose proof is a consequence of Theorem 7.7 and is relegated to Exercise 19.

Theorem 7.12. For $B > 0$, the graphs of

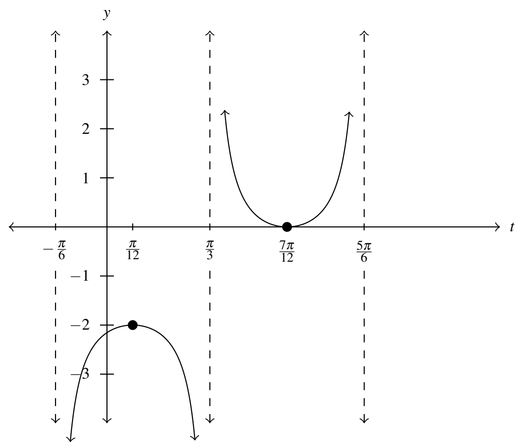
$$F(t) = A\sec(Bt + C) + D \quad \text{and} \quad G(t) = A\csc(Bt + C) + D$$

- have frequency B
- have period $T = \dfrac{2\pi}{B}$
- have phase shift $-\dfrac{C}{B}$
- have 'baseline' D and have a vertical gap $|A|$ units between the the baseline and the graph.[a]

[a]In other words, the range of these functions is $(-\infty, D - |A|] \cup [D + |A|, \infty)$.

We put Theorem 7.12 to good use in the next example.

Example 7.5.2. Below is the graph of one cycle of a secant (cosecant) function, $y = f(t)$.



1. Write $f(t)$ in the form $F(t) = A\sec(Bt+C)+D$ for $B > 0$.

2. Write $f(t)$ in the form $G(t) = A\csc(Bt+C)+D$ for $B > 0$.

Solution.

1. Write $f(t)$ in the form $F(t) = A\sec(Bt+C)+D$ for $B > 0$.

   We first note the period: $T = \frac{5\pi}{6} - \left(-\frac{\pi}{6}\right) = \pi$ and $T = \frac{2\pi}{B} = \pi$, so we get $B = 2$.

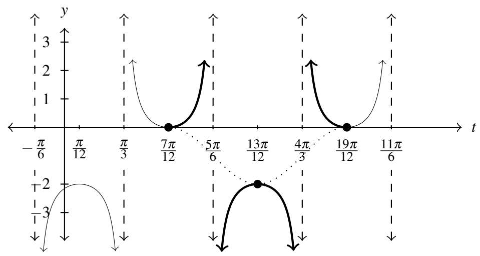
   To find C, we need to first determine the phase shift. Recall that what is graphed here is only one cycle of the function, so by copying and pasting one more cycle, we identify what looks like a fundamental cycle of the secant function to us[3] as highlighted below on the left.

   We get the phase shift is $\frac{7\pi}{12}$ so solving $-\frac{C}{2} = \frac{7\pi}{12}$, we get $C = -\frac{7\pi}{6}$.

   To find the baseline, D, we take a cue from our work in Example 7.3.3 in Section 7.3. We find the average of the local minimums and maximums to be $\frac{-2+0}{2} = -1$, so $D = -1$. As there is a 1 unit gap between the baseline and the graph of the function, we have $A = 1$. Alternatively, we can sketch the corresponding cosine curve (dotted in the figure below) and determine D and A that way.

   We find our final answer to be $f(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$. As usual, we check our answer by graphing.

[3] Assuming $A > 0$, that is.



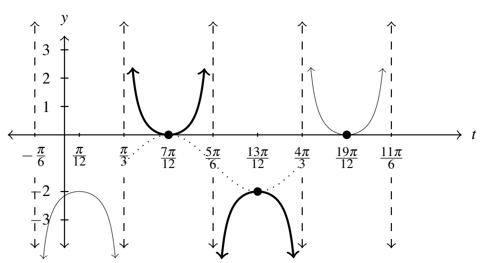
$F(t) = A\sec(Bt+C)+D$, for $B>0$

2. Write $f(t)$ in the form $G(t) = A\csc(Bt+C)+D$ for $B>0$.

   The secant and cosecant curves are phase shifts of each other, therefore we could find a formula for $f(t)$ in terms of cosecants by shifting our formula $F(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$. We leave this to the reader.[4]

   Working 'from scratch,' we would find $T = \pi$, $B = 2$, $D = -1$, and $A = 1$ the same as above.[5] To determine the phase shift, we refer to the figure below.

   The phase shift is $\frac{\pi}{3}$, thus we solve $-\frac{C}{2} = \frac{\pi}{3}$ to get $C = -\frac{2\pi}{3}$. Putting all our work together, we get our final answer: $f(t) = \csc\left(2t - \frac{2\pi}{3}\right) - 1$. Again, our best check here is to graph.



$F(t) = A\csc(Bt+C)+D$, for $B>0$

□

We cannot stress enough that our answers to Example 7.5.2 are one of many. For example, in Exercise 18, we ask you to rework this example choosing $A<0$. It is well worth the time to think about what relationships exist between the different answers, however.

[4]See Exercise 17.

[5]Again, assuming we want $A>0$.

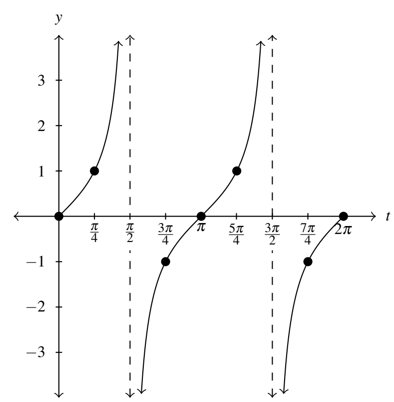
7.5.2 Graphs of the Tangent and Cotangent Functions

Next, we turn our attention to the tangent and cotangent functions. Viewing $J(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$, we find the domain of J excludes all values where $\cos(t) = 0$. Hence, the domain of J is $\{t \,|\, t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. Using this information along with the common values we given in Section 7.4, we create the table of values below.

Investigating the behavior near the values excluded from the domain, we find as $t \to \frac{\pi}{2}^-$, $\sin(t) \to 1^-$ and $\cos(t) \to 0^+$. Hence, $\tan(t) = \frac{\sin(t)}{\cos(t)} \to \infty$ producing a vertical asymptote to the graph at $t = \frac{\pi}{2}$. Similarly, we get that as $t \to \frac{\pi}{2}^+$, $\tan(t) \to -\infty$; as $t \to \frac{3\pi}{2}^-$, $\tan(t) \to \infty$; and as $t \to \frac{3\pi}{2}^+$, $\tan(t) \to -\infty$.

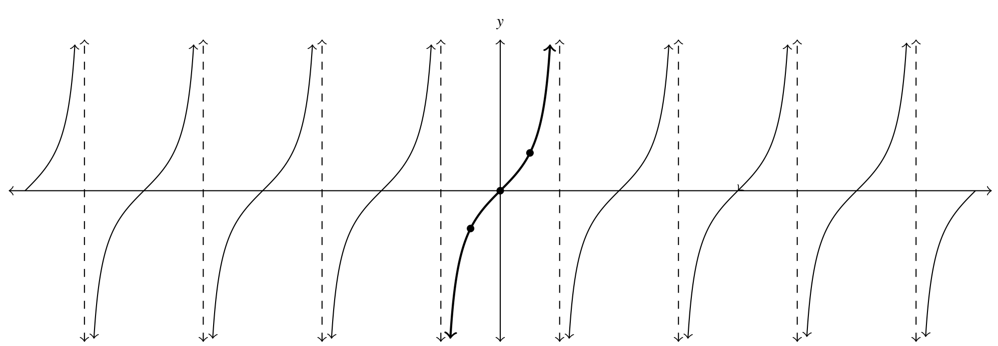
Putting all of this information together, we graph $y = \tan(t)$ over the interval $[0, 2\pi]$ below.

t	$\tan(t)$	$(t, \tan(t))$
0	0	$(0,0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	undefined	
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	undefined	
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	0	$(2\pi, 0)$



The graph of $y = \tan(t)$ over $[0, 2\pi]$.

After the usual 'copy and paste' procedure, we create the graph of $y = \tan(t)$ below:



The graph of $y = \tan(t)$.

The graph of $y = \tan(t)$ suggests symmetry through the origin. Indeed, tangent is odd because sine is odd and cosine is even: $\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$.

We also see the graph suggests the range of $J(t) = \tan(t)$ is all real numbers, $(-\infty, \infty)$. We present one proof of this fact in Exercise 21.

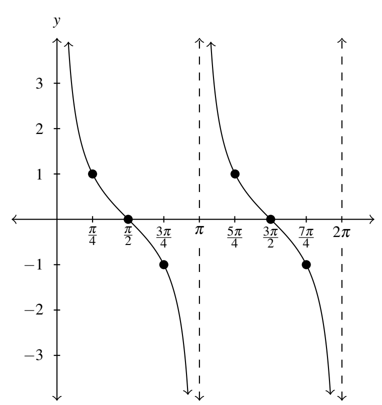
Moreover, as noted in Section 7.4, the period of the tangent function is π, and we see that reflected in the graph. This means we can choose *any* interval of length π to serve as our 'fundamental cycle.'

We choose the cycle traced out over the (open) interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as highlighted above. In addition to the asymptotes at the endpoints $t = \pm\frac{\pi}{2}$, we use the 'quarter marks' $t = \pm\frac{\pi}{4}$ and $t = 0$.

It should be no surprise that $K(t) = \cot(t)$ behaves similarly to $J(t) = \tan(t)$. As $\cot(t) = \frac{\cos(t)}{\sin(t)}$, the domain of K excludes the values where $\sin(t) = 0$: $\{t \,|\, t \neq \pi k, \text{for integers } k\}$.

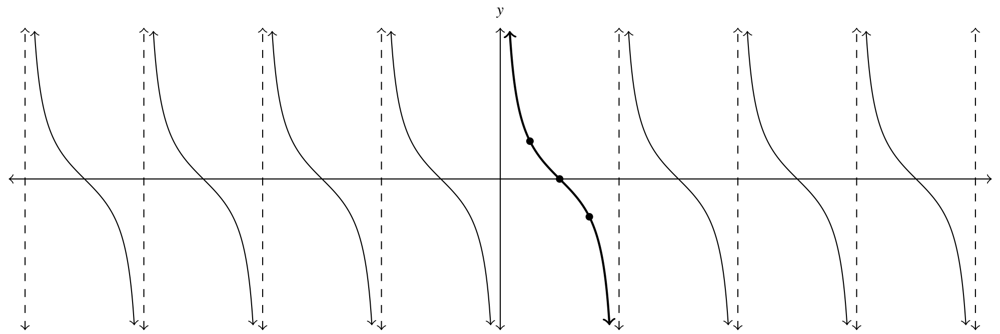
After analyzing the behavior of K near the values excluded from its domain along with plotting points, we graph $y = \cot(t)$ over the interval $[0, 2\pi]$ below on the right.

t	$\cot(t)$	$(t, \cot(t))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	



The graph of $y = \cot(t)$.

As usual, pasting copies end to end produces the graph of $K(t) = \cot(t)$ below.



The graph of $y = \cot(x)$.

As with $J(t) = \tan(t)$, the graph of $K(t) = \cot(t)$ suggests K is odd, a fact we leave to the reader to prove in Exercise 22. Also, we see that the period of cotangent (like tangent) is π and the range is $(-\infty, \infty)$.

We take as one fundamental cycle the graph as traced out over the interval $(0, \pi)$, highlighted above, with quarter marks: $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$ and $t = \pi$.

The properties of the tangent and cotangent functions are summarized below. As with Theorem 7.11, each of the results in Theorem 7.13 can be traced back to properties of the cosine and sine functions and the definition of the tangent and cotangent functions as quotients thereof.

Theorem 7.13. Properties of the Tangent and Cotangent Functions

- The function $J(t) = \tan(t)$
  - has domain $\left\{t \,|\, t \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(t) = \cot(t)$
  - has domain $\{t \,|\, t \neq \pi k,\ k \text{ is an integer}\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π

Unlike the secant and cosecant functions, the tangent and cotangent functions have different periods than sine and cosine. Moreover, in the case of the tangent function, the fundamental cycle we've chosen starts at $-\frac{\pi}{2}$ instead of 0. Nevertheless, we can use the same notions of period and phase shift to graph transformed versions of tangent and cotangent functions, because these results ultimately trace back to applying Theorem 1.12. We state a version of Theorem 7.7 for tangent and cotangent functions below.

Theorem 7.14. For $B > 0$, the functions

$$J(t) = A\tan(Bt + C) + D \quad \text{and} \quad K(t) = A\cot(Bt + C) + D$$

- have frequency B
- have period $T = \dfrac{\pi}{B}$
- have vertical shift or 'baseline' D
- The phase shift for $y = J(t)$ is $-\dfrac{C}{B} - \dfrac{\pi}{2B}$.
- The phase shift for $y = K(t)$ is $-\dfrac{C}{B}$.

The proof of Theorem 7.14 is left to the reader in Exercise 20.

We put Theorem 7.14 to good use in the following example.

Example 7.5.3. Graph one cycle of the following functions. Find the period.

1. $f(t) = 1 - \tan\left(\frac{t}{2} - \pi\right)$.
2. $g(t) = 2\cot(2\pi - \pi t) - 1$.

Solution.

1. Graph one cycle of $f(t) = 1 - \tan\left(\frac{t}{2} - \pi\right)$.

   Rewriting $f(t)$ so it fits the form in Theorem 7.14, we get $f(t) = -\tan\left(\frac{1}{2}t + (-\pi)\right) + 1$.

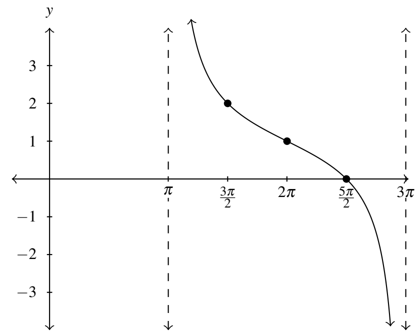
   With $B = \frac{1}{2}$, we find the period $T = \frac{\pi}{1/2} = 2\pi$. As $C = -\pi$, the phase shift is $-\frac{(-\pi)}{1/2} - \frac{\pi}{2(1/2)} = \pi$.

   Hence, one cycle of $f(t)$ starts at $t = \pi$ and finishes at $t = \pi + 2\pi = 3\pi$. Our quarter marks are $\frac{2\pi}{4} = \frac{\pi}{2}$ units apart and are $t = \pi$, $t = \frac{3\pi}{2}$, $t = 2\pi$, $t = \frac{5\pi}{2}$, and, finally, $t = 3\pi$.

   Substituting these t-values into $f(t)$, we find points on the graph and the vertical asymptotes.[6]

[6]Here, as with all tangent functions, we can partially check our new quarter marks by noting the argument of the tangent function simplifies, in each case, to one of the original quarter marks of the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

t	$f(t)$	$(t, f(t))$
π	undefined	
$\frac{3\pi}{2}$	2	$\left(\frac{3\pi}{2}, 2\right)$
2π	1	$(2\pi, 1)$
$\frac{5\pi}{2}$	0	$\left(\frac{5\pi}{2}, 0\right)$
3π	undefined	



One cycle of $y = 1 - \tan\left(\frac{t}{2} - \pi\right)$.

We confirm that the period is $3\pi - \pi = 2\pi$.

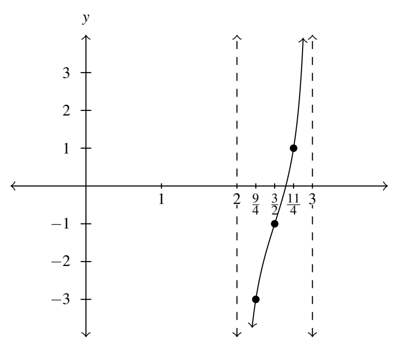
2. Graph one cycle of $g(t) = 2\cot(2\pi - \pi t) - 1$.

   To put $g(t)$ into the form prescribed by Theorem 7.14, we make use of the odd property of cotangent: $g(t) = 2\cot(2\pi - \pi t) - 1 = 2\cot(-[\pi t - 2\pi]) - 1 = -2\cot(\pi t - 2\pi) - 1 = -2\cot(\pi t + (-2\pi)) - 1$.

   We identify $B = \pi$ so the period is $T = \frac{\pi}{\pi} = 1$. Because $C = -2\pi$, the phase shift is $-\frac{-2\pi}{\pi} = 2$. Hence, one cycle of $g(t)$ starts at $t = 2$ and ends at $t = 2 + 1 = 3$.

   Our quarter marks are $\frac{1}{4}$ units apart and are $t = 2$, $t = \frac{9}{4}$, $t = \frac{5}{2}$, $t = \frac{11}{4}$, and $t = 3$. We generate the next graph.

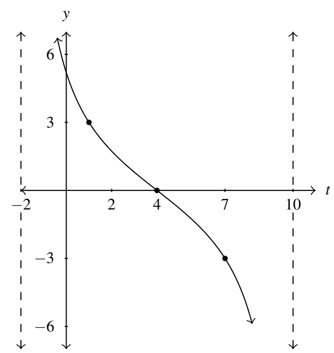
t	$g(t)$	$(t, g(t))$
2	undefined	
$\frac{9}{4}$	-3	$\left(\frac{9}{4}, -3\right)$
$\frac{5}{2}$	-1	$\left(\frac{5}{2}, -1\right)$
$\frac{11}{4}$	1	$\left(\frac{11}{4}, 1\right)$
3	undefined	



One cycle of $y = 2\cot(2\pi - \pi t) - 1$.

We confirm the period is $3 - 2 = 1$. $\square$

Example 7.5.4. Below is the graph of one cycle of a tangent (cotangent) function, $y = f(t)$.



1. Write $f(t)$ in the form $J(t) = A\tan(Bt + C) + D$ for $B > 0$.
2. Write $f(t)$ in the form $K(t) = A\cot(Bt + C) + D$ for $B > 0$.

Solution.

1. Write $f(t)$ in the form $J(t) = A\tan(Bt + C) + D$ for $B > 0$.

   We first find the period $T = 10 - (-2) = 12$. Per Theorem 7.14, we know $\frac{\pi}{B} = 12$, or $B = \frac{\pi}{12}$.

   Next, we look for the phase shift. We notice the cycle graphed for us is decreasing instead of the usual increasing we expect for a standard tangent cycle. When this sort of thing happened in Examples 7.3.3 and 7.5.2, we pasted another cycle of the function and used that to help identify the phase shift in order to keep the value of $A > 0$. Here, no amount of 'copying and pasting' will produce an increasing cycle (do you see why?), so we know $A < 0$ and use -2, as the phase shift.

   The formula given in Theorem 7.14 tells us $-\frac{C}{B} - \frac{\pi}{2B} = -2$ so substituting $B = \frac{\pi}{12}$ gives $C = -\frac{\pi}{3}$.

   Next, we see the baseline here is still the t-axis, so $D = 0$. This means all that's left to find is A. We have already established that $A < 0$ to account for the reflection across the t-axis. Moreover, the y-values of the points off of the baseline are 3 units from the baseline, indicating a vertical stretch by a factor of 3. Hence, $A = -3$ and $f(t) = -3\tan\left(\frac{\pi}{12}t - \frac{\pi}{3}\right)$. As usual, the ultimate check is to graph, which we will leave to the reader.

2. Write $f(t)$ in the form $K(t) = A\cot(Bt + C) + D$ for $B > 0$.

   We find $T = 12$, $B = \frac{\pi}{12}$, and $D = 0$ as above. As the fundamental cycle of cotangent is decreasing, we know $A > 0$ and identify the phase shift as -2.

   Using Theorem 7.14, we know $-\frac{C}{B} = -2$ so substituting $B = \frac{\pi}{12}$, we get $C = \frac{\pi}{6}$. As above, the vertical stretch is by a factor of 3, so we take $A = 3$ for our final answer: $f(t) = 3\cot\left(\frac{\pi}{12}t + \frac{\pi}{6}\right)$. We leave it to the reader to check our answer by graphing. □

Once again, our answers to Example 7.5.4 are one of many, and we invite the reader to think about what all of the solutions would have in common. We close this section with an application.

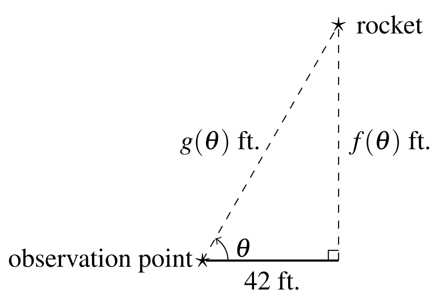
Example 7.5.5. Let θ be the angle of inclination from an observation point on the ground 42 feet away from the launch site of a model rocket. Assuming the rocket is launched directly upwards:

1. Write a formula for $f(\theta)$, the distance from the rocket to the ground (in feet) as a function of θ. Compute and interpret $f\left(\frac{\pi}{3}\right)$.

2. Write a formula for $g(\theta)$, the distance from the rocket to the observation point on the ground (in feet) as a function of θ. Compute and interpret $g\left(\frac{\pi}{3}\right)$.

3. Write and interpret the behavior of $f(\theta)$ and $g(\theta)$ as $\theta \to \frac{\pi}{2}^-$.

Solution. We begin by sketching the scenario below. Given the rocket is launched 'directly upwards,' we may assume the rocket is launched at a 90° angle which provides us with a right triangle.



1. Write a formula for $f(\theta)$, the distance from the rocket to the ground (in feet) as a function of θ. Compute and interpret $f\left(\frac{\pi}{3}\right)$.

   From the remarks preceding Theorem 7.10, we know the definitions of the circular functions agree with those specified for acute angles in right triangles as described in Definition 7.6 in Section 7.2.1. Hence, $\tan(\theta) = \frac{f(\theta)}{42}$, so $f(\theta) = 42\tan(\theta)$.

   We find $f\left(\frac{\pi}{3}\right) = 42\tan\left(\frac{\pi}{3}\right) = 30\sqrt{3}$. This means when the angle of inclination is $\frac{\pi}{3}$ or 60°, the rocket is or $30\sqrt{3} \approx 73$ feet off of the ground.

2. Write a formula for $g(\theta)$, the distance from the rocket to the observation point on the ground (in feet) as a function of θ. Compute and interpret $g\left(\frac{\pi}{3}\right)$.

   Again, working with the triangle, we find $\sec(\theta) = \frac{g(\theta)}{42}$ so that $g(\theta) = 42\sec(\theta)$. We find $g\left(\frac{\pi}{3}\right) = 42\sec\left(\frac{\pi}{3}\right) = 84$, so when the angle of inclination is 60°, the rocket is 84 feet from the observation point on the ground.

3. Write and interpret the behavior of $f(\theta)$ and $g(\theta)$ as $\theta \to \frac{\pi}{2}^-$.

As $\theta \to \frac{\pi}{2}^-$, both $f(\theta) \to \infty$ and $g(\theta) \to \infty$ (a fact we could verify graphically, if needs be.) This means as the angle of inclination approaches $\frac{\pi}{2}$ or 90°, the distances from the rocket to the ground and from to the rocket to the observation point increase without bound. Barring the effects of drift or the curvature of space, this matches our intuition. $\square$

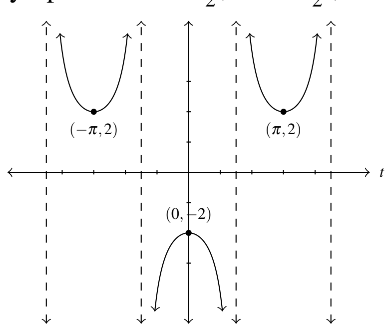
7.5.3 Exercises

In Exercises 1 - 12, graph one cycle of the given function. State the period of the function.

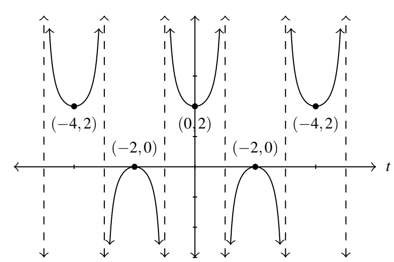
1. $y = \tan\left(t - \dfrac{\pi}{3}\right)$
2. $y = 2\tan\left(\dfrac{1}{4}t\right) - 3$
3. $y = \dfrac{1}{3}\tan(-2t - \pi) + 1$
4. $y = \sec\left(t - \dfrac{\pi}{2}\right)$
5. $y = -\csc\left(t + \dfrac{\pi}{3}\right)$
6. $y = -\dfrac{1}{3}\sec\left(\dfrac{1}{2}t + \dfrac{\pi}{3}\right)$
7. $y = \csc(2t - \pi)$
8. $y = \sec(3t - 2\pi) + 4$
9. $y = \csc\left(-t - \dfrac{\pi}{4}\right) - 2$
10. $y = \cot\left(t + \dfrac{\pi}{6}\right)$
11. $y = -11\cot\left(\dfrac{1}{5}t\right)$
12. $y = \dfrac{1}{3}\cot\left(2t + \dfrac{3\pi}{2}\right) + 1$

In Exercises 13 - 14, the graph of a (co)secant function is given. Write a formula for the function in the form $F(t) = A\sec(Bt + C) + D$ and $G(t) = A\csc(Bt + C) + D$. Select B so $B > 0$. Check your answer by graphing.

13. Asymptotes: $t = \pm\frac{\pi}{2}$, $t = \pm\frac{3\pi}{2}$, $\ldots$

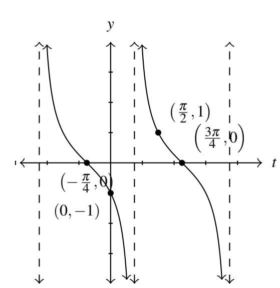


14. Asymptotes: $t = \pm 1$, $t = \pm 3$, $t = \pm 5$, $\ldots$

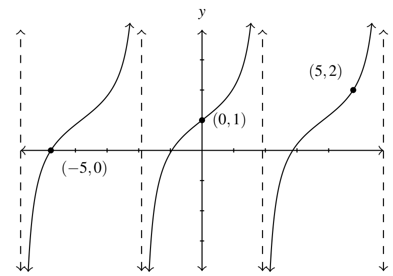


In Exercises 15 - 16, the graph of a (co)tangent function given. Find a formula the function in the form $J(t) = A\tan(Bt + C) + D$ and $K(t) = A\cot(Bt + C) + D$. Select B so $B > 0$. Check your answer by graphing.

15. Asymptotes: $t = -\frac{3\pi}{4}$, $t = \frac{\pi}{4}$, $t = \frac{5\pi}{4}$, $\ldots$



16. Asymptotes: $t = \pm 2$, $t = \pm 6$, $t = \pm 10$, $\ldots$



17. (a) Use the conversion formulas listed in Theorem 7.6 to create conversion formulas between secant and cosecant functions.
    (b) Use a conversion formula to rewrite our first answer to Example 7.5.2, $f(t) = \sec\left(2t - \frac{7\pi}{6}\right) - 1$, in terms of cosecants.

18. Rework Example 7.5.2 and find answers with $A < 0$.

19. Prove Theorem 7.12 using Theorem 7.7.

20. Prove Theorem 7.14 using Theorem 1.12.

21. In this Exercise, we argue the range of the tangent function is $(-\infty, \infty)$. Let M be a fixed, but arbitrary positive real number.
    (a) Show there is an acute angle θ with $\tan(\theta) = M$. (Hint: think right triangles.)
    (b) Using the symmetry of the Unit Circle, explain why there are angles θ with $\tan(\theta) = -M$.
    (c) Determine angles with $\tan(\theta) = 0$.
    (d) Combine the three parts above to conclude the range of the tangent function is $(-\infty, \infty)$.

22. Prove $\cot(t)$ is odd. (Hint: mimic the proof given in the text that $\tan(t)$ is odd.)

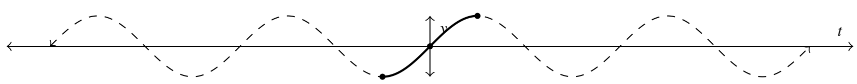
SECTION 7.5 EXERCISE ANSWERS A.1.7

7.6 Inverse Trigonometric Functions

In this section we concern ourselves with finding inverses of the circular (trigonometric) functions.[1] Our immediate problem is that, owing to their periodic nature, none of the six circular functions are one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example 5.1.3 in Section 5.1 to obtain a one-to-one function.

7.6.1 Inverses of Sine and Cosine

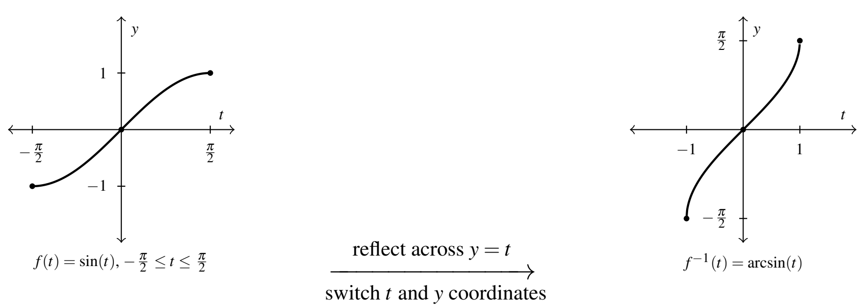
We start with $f(t) = \sin(t)$ and restrict our domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in order to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.



Restricting the domain of $f(t) = \sin(t)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

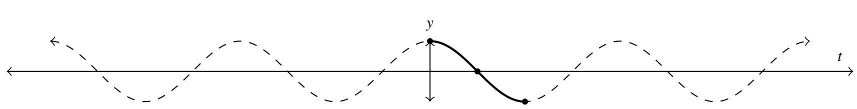
Recall from Section 5.1 that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(t) = \sin^{-1}(t)$ for the inverse of $f(t) = \sin(t)$. The obvious pitfall here is our convention of writing $(\sin(t))^2$ as $\sin^2(t)$, $(\sin(t))^3$ as $\sin^3(t)$ and so on. It is far too easy to confuse $\sin^{-1}(t)$ with $\frac{1}{\sin(t)} = \csc(t)$ so we will not use this notation in our text.[2]

Instead, we use the notation $f^{-1}(t) = \arcsin(t)$, read 'arc-sine of t'. We'll explain the 'arc' in 'arcsine' shortly. For now, we graph $f(t) = \sin(t)$ and $f^{-1}(t) = \arcsin(t)$, where we obtain the latter from the former by reflecting it across the line $y = t$, in accordance with Theorem 5.1.



$f(t) = \sin(t)$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ $\qquad$ $f^{-1}(t) = \arcsin(t)$

Next, we consider $g(t) = \cos(t)$. Here, we select the interval $[0, \pi]$ for our restriction.

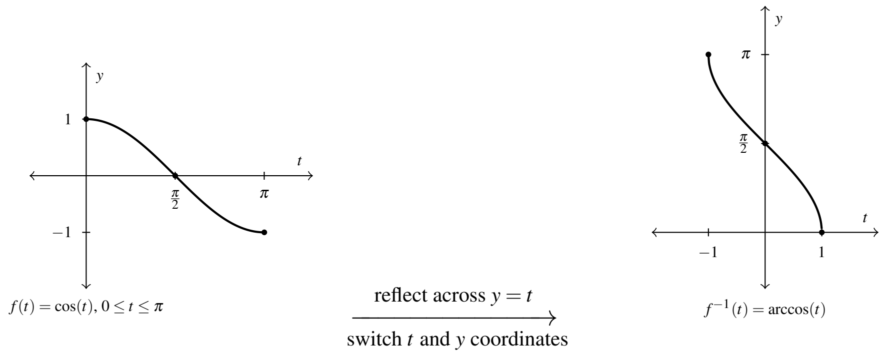


Restricting the domain of $f(t) = \cos(t)$ to $[0, \pi]$.

[1] We have already discussed this concept in Section 7.2.1 as the 'angle finder' in the context of acute angles in right triangles.

[2] But be aware that many books do! As always, be sure to check the context!

Reflecting the across the line $y = t$ produces the graph $y = g^{-1}(t) = \arccos(t)$.



We list some important facts about the arcsine and arccosine functions in the following theorem.[3] Everything in Theorem 7.15 is a direct consequence of Theorem 5.1 as applied to the (restricted) sine and cosine functions, and as such, its proof is left to the reader.

Theorem 7.15. Properties of the Arcsine and Arccosine Functions

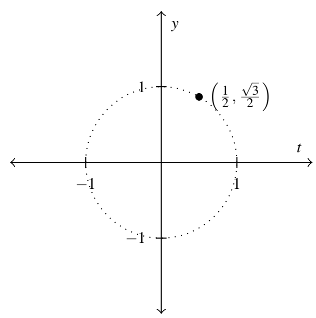
- Properties of $F(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $\sin(t) = x$ and $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$
  - $\sin(\arcsin(x)) = x$ provided $-1 \le x \le 1$
  - $\arcsin(\sin(t)) = t$ provided $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$
  - $F(x) = \arcsin(x)$ is odd
- Properties of $G(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $\cos(t) = x$ and $0 \le t \le \pi$
  - $\cos(\arccos(x)) = x$ provided $-1 \le x \le 1$
  - $\arccos(\cos(t)) = t$ provided $0 \le t \le \pi$

Before moving to an example, we take a moment to understand the 'arc' in 'arcsine.' Consider the figure below which illustrates the specific case of $\arcsin\left(\frac{\sqrt{3}}{2}\right)$.

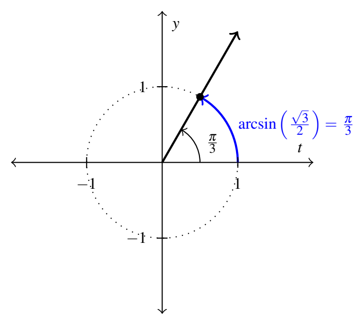
[3]We switch the input variable to the arcsine and arccosine functions to 'x' to avoid confusion with the outputs we label 't.'

By definition, the real number $t = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ satisfies $\sin(t) = \frac{\sqrt{3}}{2}$ with $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. In other words, we are looking for angle measuring t radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of $\frac{\sqrt{3}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

In terms of oriented arcs[4], if we start at $(1,0)$ and travel along the Unit Circle in the positive (counterclockwise) direction for $\frac{\pi}{3}$ units, we will arrive at the point whose y-coordinate is $\frac{\sqrt{3}}{2}$. Hence, the real number $\frac{\pi}{3}$ also corresponds to 'arc' corresponding to the 'sine' that is $\frac{\sqrt{3}}{2}$.



y-value on Unit Circle is $\frac{\sqrt{3}}{2}$.



corresponding oriented arc: $t = \frac{\pi}{3}$

In general, the function $f(t) = \sin(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\sin(\theta)$. The value $\sin(\theta) = \sin(t)$ is the y-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1,0)$.

Hence, we may view the inputs to $f(t) = \sin(t)$ as oriented arcs and the outputs as y-coordinates on the Unit Circle. Therefore, the function f^{-1} reverses this process and takes y-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arcsine.

It is high time for an example.

Example 7.6.1.

1. Determine the exact values of the following.

   (a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

   (b) $\arccos\left(\frac{1}{2}\right)$

   (c) $\arcsin\left(-\frac{1}{2}\right)$

   (d) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

   (e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

   (f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

[4]See page 583 if you need a review of how we associate real numbers with angles in radian measure.

(g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ (h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite the composite function $f(x) = \tan(\arccos(x))$ as algebraic functions of x and state the domain.

Solution. The best way to approach these problems is to remember that $\arcsin(x)$ and $\arccos(x)$ are real numbers which correspond to the radian measure of angles that fall within a certain prescribed range.

1. (a) Determine the exact value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$.

   To find $\arcsin\left(\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

   (b) Determine the exact value of $\arccos\left(\frac{1}{2}\right)$.

   To find $\arccos\left(\frac{1}{2}\right)$, we are looking for the angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{1}{2}$. Our answer is $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (c) Determine the exact value of $\arcsin\left(-\frac{1}{2}\right)$.

   For $\arcsin\left(-\frac{1}{2}\right)$, we are looking for an angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = -\frac{1}{2}$. Hence, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

   Alternatively, we could use the fact that the arcsine function is odd, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right)$. We find $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (d) Determine the exact value of $\arccos\left(-\frac{\sqrt{2}}{2}\right)$.

   For $\arccos\left(-\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between 0 and π with $\cos(t) = -\frac{\sqrt{2}}{2}$. Hence, $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (e) Determine the exact value of $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$.

   As $0 \le \frac{\pi}{6} \le \pi$, we could simply invoke Theorem 7.15 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

   However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine.

   Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. To find $\arccos\left(\frac{\sqrt{3}}{2}\right)$, we need an angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{\sqrt{3}}{2}$. We get $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

   (f) Determine the exact value of $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$.

$\frac{11\pi}{6}$ does not fall between 0 and π, therefore Theorem 7.15 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

(g) Determine the exact value of $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$.

One way to simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ is to use Theorem 7.15 directly. Because $-\frac{3}{5}$ is between -1 and 1, we have that $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$ and we are done.

However, as before, to really understand *why* this cancellation occurs, we let $t = \arccos\left(-\frac{3}{5}\right)$. By definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) Determine the exact value of $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$.

As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(t) = -\frac{3}{5}$ for some angle measuring t radians between 0 and π.

For $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that t corresponds to an angle in Quadrant II.

In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$, and because we know $\cos(t)$, the fastest route is using the Pythagorean Identity, $x^2 + y^2 = 1$ or $\sin^2(t) + \cos^2(t) = 1$.[5]

We get $\sin^2(t) = 1 - \cos^2(t) = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25}$. t corresponds to a Quadrant II angle, thus we choose the positive root, $\sin(t) = \frac{4}{5}$, so $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. Rewrite $f(x) = \tan(\arccos(x))$ as an algebraic function of x and state the domain.

We begin this problem in the same manner we began the previous two problems. We let $t = \arccos(x)$, so our goal is to find a way to express $\tan(\arccos(x)) = \tan(t)$ in terms of x.

By letting $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \le t \le \pi$. One approach[6] to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. As we know $\cos(t)$, we just need to find $\sin(t)$.

Using the Pythagorean Identity, $\sin^2(t) + \cos^2(t) = 1$, we get $\sin^2(t) = 1 - \cos^2(t) = 1 - x^2$ so that $\sin(t) = \pm\sqrt{1-x^2}$. Given $0 \le t \le \pi$, $\sin(t) \ge 0$ and $\sin(t) = \sqrt{1-x^2}$.

Thus, $\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1-x^2}}{x}$, so $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$.

To determine the domain, we harken back to Section 1.5.2. The function $f(x) = \tan(\arccos(x))$ can

[5]The Pythagorean Identities are a direct result of the definition of the six trigonometric functions on a right triangle. We will prove these identities in this example in Section 8.1. See Theorem 8.3

[6]Alternatively, we could use the identity: $1 + \tan^2(t) = \sec^2(t)$. As $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

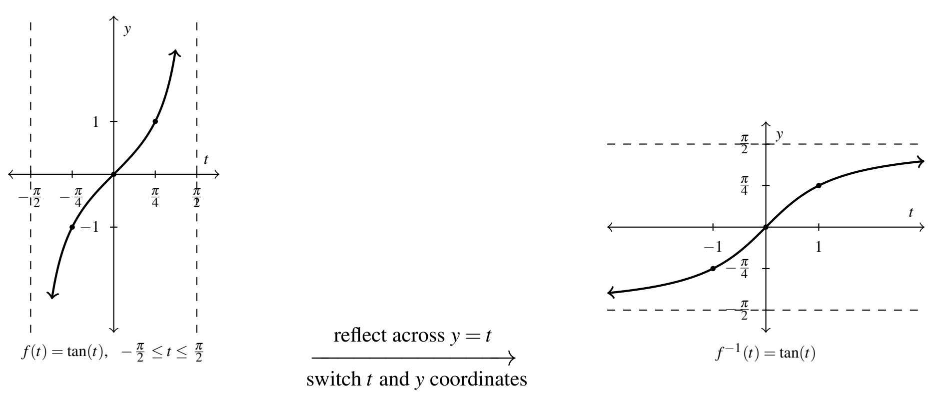
be thought of as a two step process: first, take the arccosine of a number, and second, take the tangent of whatever comes out of the arccosine.

The domain of $\arccos(x)$ is $-1 \leq x \leq 1$, so the domain of f will be some subset of $[-1,1]$. The range of $\arccos(x)$ is $[0,\pi]$, and of these values, only $\frac{\pi}{2}$ will cause a problem for the tangent function. As $\arccos(x) = \frac{\pi}{2}$ happens when $x = \cos\left(\frac{\pi}{2}\right) = 0$, we exclude $x = 0$ from our domain. Hence, the domain of $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is $[-1,0) \cup (0,1]$.

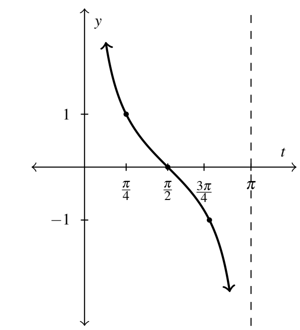
Note that *in this particular case*, we could have obtained the correct domain of f using its algebraic description: $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$. This is not always true, however, as we'll see soon.

7.6.2 Inverses of Tangent and Cotangent

The next pair of functions we wish to discuss are the inverses of tangent and cotangent. First, we restrict $f(t) = \tan(t)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain the arctangent function, $f^{-1}(t) = \arctan(t)$. Among other things, note that the *vertical* asymptotes $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ of the graph of $f(t) = \tan(t)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(t) = \arctan(t)$.

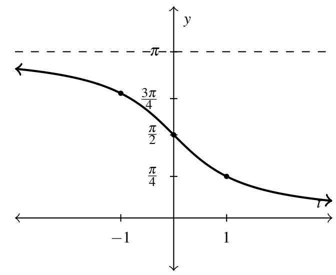


Next, we restrict $g(t) = \cot(t)$ to its fundamental cycle on $(0,\pi)$ to obtain $g^{-1}(t) = \operatorname{arccot}(t)$, the arccotangent function. Once again, the vertical asymptotes $t = 0$ and $t = \pi$ of the graph of $g(t) = \cot(t)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(t) = \operatorname{arccot}(t)$.



$f(t) = \cot(t)$, $0 \leq t \leq \pi$

$$\xrightarrow[\text{switch } t \text{ and } y \text{ coordinates}]{\text{reflect across } y = t}$$



$f^{-1}(t) = \operatorname{arccot}(t)$

Below we summarize the important properties of the arctangent and arccotangent functions.

Theorem 7.16. Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $\tan(t) = x$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(t)) = t$ provided $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $F(x) = \arctan(x)$ is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $\cot(t) = x$ and $0 < t < \pi$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(t)) = t$ provided $0 < t < \pi$

The properties listed in Theorem 7.16 are consequences of the definitions of the arctangent and arccotangent functions along with Theorem 5.1, and its proof is left to the reader.

Example 7.6.2.

1. Determine the exact values of the following.

   (a) $\arctan(\sqrt{3})$

   (b) $\text{arccot}(-\sqrt{3})$

   (c) $\cot(\text{arccot}(-5))$

   (d) $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(2\arctan(x))$

   (b) $\cos(\text{arccot}(2x))$

Solution.

1. (a) Determine the exact value of $\arctan(\sqrt{3})$.

   To find $\arctan(\sqrt{3})$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

   (b) Determine the exact value of $\text{arccot}(-\sqrt{3})$.

   To find $\text{arccot}(-\sqrt{3})$, we need the angle measuring t radians which lies between 0 and π with $\cot(t) = -\sqrt{3}$. Hence, $\text{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

   (c) Determine the exact value of $\cot(\text{arccot}(-5))$.

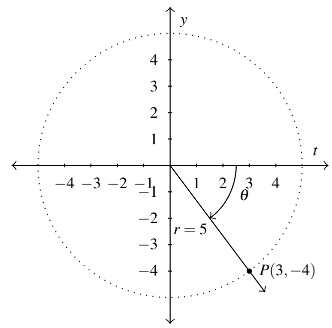
   We can apply Theorem 7.16 directly and obtain $\cot(\text{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case.

   Letting $t = \text{arccot}(-5)$, by definition, $\cot(t) = -5$. Hence, $\cot(\text{arccot}(-5)) = \cot(t) = -5$.

   (d) Determine the exact value of $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$.

   We start simplifying $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$ by letting $t = \arctan\left(-\frac{4}{3}\right)$. By definition, $\tan(t) = -\frac{4}{3}$ for some angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. As $\tan(t) < 0$, we know, in fact, t corresponds to a Quadrant IV angle.

   We are given $\tan(t)$ but wish to know $\sin(t)$. There is no direct identity to marry the two, so we make a quick sketch of the situation below. Because $\tan(t) = -\frac{4}{3} = \frac{-4}{3}$, we take $P(3,-4)$ as a point on the terminal side of $\theta = t = \arctan\left(-\frac{4}{3}\right)$ radians.



$P(3,-4)$ is on the terminal side of θ.

We find $r = \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2} = 5$, so $\sin(t) = -\frac{4}{5}$. Hence, $\sin\left(\arctan\left(-\frac{4}{3}\right)\right) = -\frac{4}{5}$.

2. (a) Rewrite $\tan(2\arctan(x))$ as an algebraic function of x and state the domain.

We proceed as above and let $t = \arctan(x)$. We have $\tan(t) = x$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Our goal is to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x.

From Theorem 8.9, we know $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$. Hence $f(x) = \tan(2\arctan(x)) = \frac{2x}{1-x^2}$.

To find the domain, we once again think of $f(x) = \tan(2\arctan(x))$ as a sequence of steps and work from the inside out.

The first step is to find the arctangent of a real number. The domain of $\arctan(x)$ is all real numbers, so we have no restrictions here and we get out all values $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The next step is to multiply $\arctan(x)$ by 2. There are no restrictions here, either. The range of $\arctan(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, thus the range of $2\arctan(x)$ is $(-\pi, \pi)$.

The last step is to take the tangent of $2\arctan(x)$. As we are taking the tangent of values in the interval $(-\pi, \pi)$, we will run into trouble if $2\arctan(x) = \pm\frac{\pi}{2}$, that is, if $\arctan(x) = \pm\frac{\pi}{4}$. This happens exactly when $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$, so we must exclude $x = \pm 1$ from the domain of f.

Hence, the domain of $f(x) = \tan(2\arctan(x))$ is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. In this example, we could have obtained the correct answer by looking at the algebraic equivalence, $f(x) = \frac{2x}{1-x^2}$.

(b) Rewrite $\cos(\operatorname{arccot}(2x))$ as an algebraic function of x and state the domain.

To get started, we let $t = \operatorname{arccot}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of t, $\cos(\operatorname{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x.

One way to proceed is to rewrite the identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ as $\cos(t) = \cot(t)\sin(t)$ and use the fact that $\cot(t) = 2x$ to find $\sin(t)$ in terms of x. This isn't as hopeless as it might seem, because the Pythagorean Identity $\csc^2(t) = 1+\cot^2(t)$ relates cotangent to cosecant, and $\sin(t) = \frac{1}{\csc(t)}$.

Following this strategy, we get $\csc^2(t) = 1+\cot^2(t) = 1+(2x)^2 = 1+4x^2$ so $\csc(t) = \pm\sqrt{4x^2+1}$. Due to the fact that t is between 0 and π, $\csc(t) > 0$. Hence, $\csc(t) = \sqrt{4x^2+1}$, so $\sin(t) =$

$\frac{1}{\csc(t)} = \frac{1}{\sqrt{4x^2+1}}$.

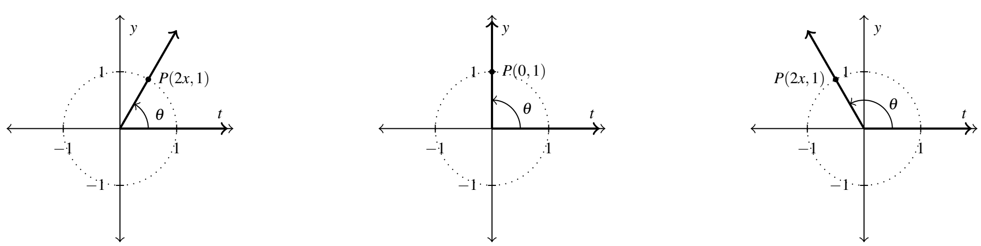
We find $\cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, $g(x) = \cos(\operatorname{arccot}(2x)) = \frac{2x}{\sqrt{4x^2+1}}$.

Viewing $g(x) = \cos(\operatorname{arccot}(2x))$ as a sequence of steps, we see we first double the input x, then take the arccotangent, and, finally, take the cosine. Each of these processes are valid for all real numbers, so the domain of g is $(-\infty,\infty)$. □

The reader may well wonder if there isn't a more direct way to handle Example 7.6.2 number 2b. Indeed, we can take some inspiration from Section 7.4 and imagine an angle θ measuring t radians so that $\cot(\theta) = \cot(t) = 2x$ where $0 < \theta < \pi$.

Thinking of $\cot(\theta)$ as a ratio of coordinates on a circle, we may rewrite $\cot(\theta) = 2x = \frac{2x}{1}$ and we would like to identify a point $P(2x,1)$ on the terminal side of θ.

We need to be careful here. Given $\cot(\theta) = 2x$, $x = \frac{1}{2}\cot(\theta)$, so as θ ranges between 0 and π, x can take on positive, negative or 0 values. We need to argue that the point $P(2x,1)$ lies in the quadrant we expect (as depicted below) in all cases before we delve too far into our analysis.



$0 < \theta < \frac{\pi}{2}, x > 0$. $\theta = \frac{\pi}{2}, x = 0$. $\frac{\pi}{2} < \theta < \pi, x < 0$.

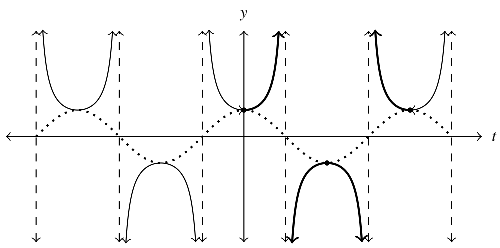
If $0 < \theta < \frac{\pi}{2}$, then $\cot(\theta) > 0$. Hence, $x > 0$ so the point $P(2x,1)$ is in Quadrant I, as required. If $\theta = \frac{\pi}{2}$, then $x = 0$, and our point $P(2x,1) = (0,1)$, as required. If $\frac{\pi}{2} < \theta < \pi$, then $\cot(\theta) < 0$. Hence, $x < 0$, so $P(2x,1)$ is in Quadrant II, as required.

Hence, in all three cases, our formula for the point $P(2x,1)$ determines a point in the same quadrant as the terminal side of θ, as illustrated above.

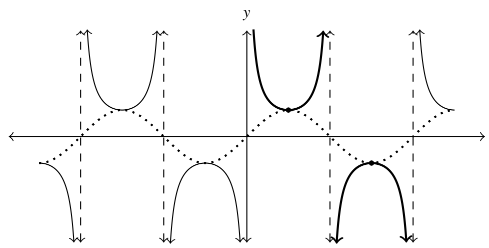
This allows us to use Theorem 7.10 from Section 7.4. We find $r = \sqrt{(2x)^2+1^2} = \sqrt{4x^2+1}$, and hence, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, which agrees with our answer from Example 7.6.2.

It shouldn't surprise the reader that there are some cases where the approach outlined above doesn't go as smoothly (as we'll see in the discussion following Example 7.6.3.)

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection 7.5.1, are given below with the fundamental cycles highlighted.



The graph of $y = \sec(x)$.



The graph of $y = \csc(x)$.

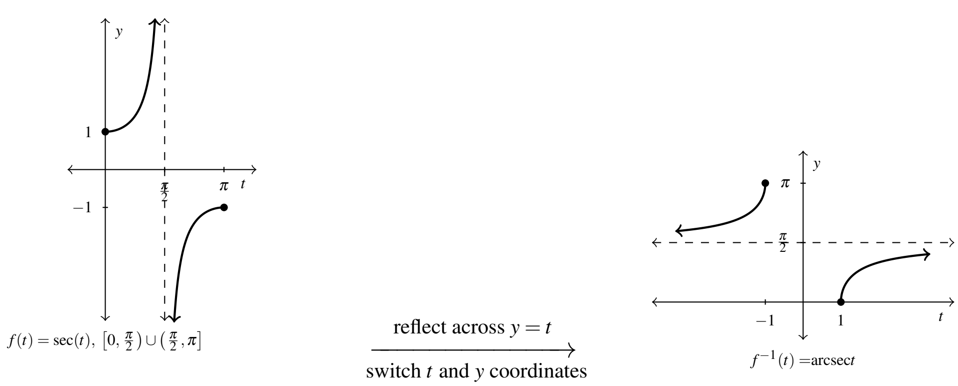
It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant.

Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$.

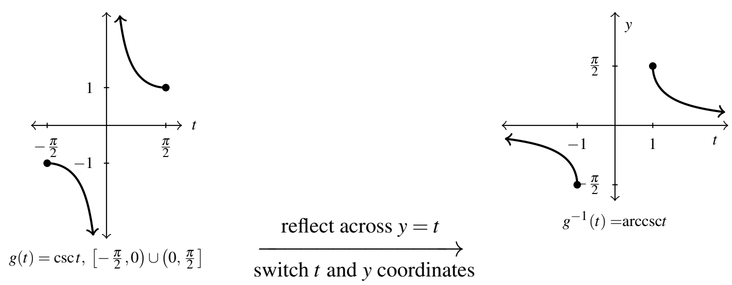
There are two generally accepted ways to make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. We choose to focus on the Trigonometric approach.

7.6.3 Inverses of Secant and Cosecant

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(t) = \sec(t)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



and we restrict $g(t) = \csc(t)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



Note that for both arcsecant and arccosecant, the domain is $(-\infty,-1] \cup [1,\infty)$. Taking a page from Section 0.6.2, we can rewrite this as $\{x \,|\, |x| \geq 1\}$. (This is often done in Calculus textbooks, so we include it here for completeness.)

Using these definitions along with Theorem 5.1, we get the following properties of the arcsecant and arccosecant functions.

Theorem 7.17. Properties of the Arcsecant and Arccosecant Functions[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \to -\infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{+}$; as $x \to \infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec\left(\operatorname{arcsec}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \to -\infty$, $\operatorname{arccsc}(x) \to 0^{-}$; as $x \to \infty$, $\operatorname{arccsc}(x) \to 0^{+}$
  - $\operatorname{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc\left(\operatorname{arccsc}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(t)) = t$ provided $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $G(x) = \operatorname{arccsc}(x)$ is odd

[a] . . . assuming the "Trigonometry Friendly" ranges are used.

The reason the ranges here are called 'Trigonometry Friendly' is specifically because of two properties listed in Theorem 7.17: $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Note: We may also "adjust" the restriction of $f(t) = \sec(t)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and $g(t) = \csc(t)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ to develop 'Calculus Friendly' ranges of $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ for $F(x) = \operatorname{arcsec}(x)$ and $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ for $G(x) = \operatorname{arccsc}(x)$. At this time it is difficult to explain why these choices for the ranges of arcsecant and arccosecant are 'Calculus Friendly.'

These formulas essentially allow us to always convert arcsecants and arccosecants back to arccosines and arcsines, respectively. We see this play out in our next example.

Example 7.6.3.

1. Determine the exact values of the following.

   (a) $\text{arcsec}(2)$ (b) $\text{arccsc}(-2)$ (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$ (d) $\cot(\text{arccsc}(-3))$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $f(x) = \tan(\text{arcsec}(x))$ (b) $g(x) = \cos(\text{arccsc}(4x))$

Solution.

1. (a) Determine the exact value of $\text{arcsec}(2)$.

   Using Theorem 7.17, we have $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Determine the exact value of $\text{arccsc}(-2)$.

   Once again, Theorem 7.17 comes to our aid giving $\text{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (c) Determine the exact value of $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$.

   $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, thus we cannot use the inverse property stated in Theorem 7.17. Hence, we work from the 'inside out.'

   We get: $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \text{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (d) Determine the exact value of $\cot(\text{arccsc}(-3))$.

   We begin simplifying $\cot(\text{arccsc}(-3))$ by letting $t = \text{arccsc}(-3)$. Then, $\csc(t) = -3$. For $\csc(t) < 0$, t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$, so t corresponds to a Quadrant IV angle.

   To find $\cot(\text{arccsc}(-3)) = \cot(t)$, we use the Pythagorean Identity: $\cot^2(t) = \csc^2(t) - 1$. We get $\csc^2(t) = (-3)^2 - 1 = 8$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$.

   As t corresponds to a Quadrant IV angle, $\cot(t) < 0$. Hence, $\cot(\text{arccsc}(-3)) = -2\sqrt{2}$.

2. (a) Rewrite $f(x) = \tan(\text{arcsec}(x))$ as an algebraic function of x and state the domain.

   Proceeding as above, we let $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. We seek a formula for $\tan(\text{arcsec}(x)) = \tan(t)$ in terms of x.

   To relate $\sec(t)$ to $\tan(t)$, we use the Pythagorean Identity: $\tan^2(t) = \sec^2(t) - 1$. Substituting $\sec(t) = x$, we get $\tan^2(t) = \sec^2(t) - 1 = x^2 - 1$, so $\tan(t) = \pm\sqrt{x^2 - 1}$.

   If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$. On the the other hand, if t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a *piecewise defined* function for $\tan(t)$:

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

Now we need to determine what these conditions on t mean for x. For $x = \sec(t)$, when $0 \le t < \frac{\pi}{2}$, $x \ge 1$, and when $\frac{\pi}{2} < t \le \pi$, $x \le -1$,

$$f(x) = \tan(\text{arcsec}(x)) = \begin{cases} \sqrt{x^2-1}, & \text{if } x \ge 1 \\ -\sqrt{x^2-1}, & \text{if } x \le -1 \end{cases}$$

To find the domain of f, we consider $f(x) = \tan(\text{arcsec}(x))$ as a two step process. First, we have the arcsecant function, whose domain is $(\infty, -1] \cup [1, \infty)$.

The range of $\text{arcsec}(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, so taking the tangent of any output from $\text{arcsec}(x)$ is defined. Hence, the domain of f is $(-\infty, -1] \cup [1, \infty)$.

(b) Rewrite $g(x) = \cos(\text{arccsc}(4x))$ as an algebraic function of x and state the domain.

Taking a cue from the previous problem, we start by letting $t = \text{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Our goal is to rewrite $\cos(\text{arccsc}(4x)) = \cos(t)$ in terms of x.

From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use of the Pythagorean Identity: $\cos^2(t) = 1 - \sin^2(t)$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) = 1 - \left(\frac{1}{4x}\right)^2 = 1 - \frac{1}{16x^2}$. Getting a common denominator and extracting square roots, we obtain:

$$\cos(t) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}.$$

As t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \ge 0$, so we choose $\cos(t) = \frac{\sqrt{16-x^2}}{4|x|}$. (The absolute values here are necessary, because x could be negative.) Therefore,

$$g(x) = \cos(\text{arccsc}(4x)) = \frac{\sqrt{16-x^2}}{4|x|}.$$

To find the domain of $g(x) = \cos(\text{arccsc}(4x))$, as usual, we think of g as a series of processes. First, we take the input, x, and multiply it by 4. This can be done to any real number, so we have no restrictions here.

Next, we take the arccosecant of $4x$. Using interval notation, the domain of the arccosecant function is written as: $(-\infty, -1] \cup [1, \infty)$. Hence to take the arccosecant of $4x$, the quantity $4x$ must lie in one of these two intervals.[7] That is, $4x \le -1$ or $4x \ge 1$, so $x \le -\frac{1}{4}$ or $x \ge \frac{1}{4}$.

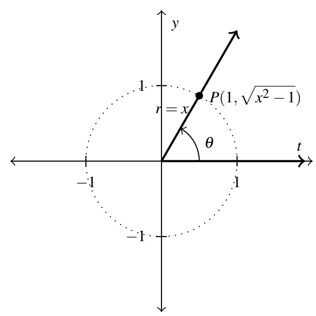
The third and final process coded in $g(x) = \cos(\text{arccsc}(4x))$ is to take the cosine of $\text{arccsc}(4x)$. As the cosine accepts any real number, we have no additional restrictions. Hence, the domain of g is $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. □

As promised in the discussion following Example 7.6.2, in which we used the methods from Section 7.4 to circumvent some onerous identity work, we take some time here to revisit number 2a to see what issues arise when we take a Section 7.4 approach here.

[7]Alternatively, we can write the domain of $\text{arccsc}(x)$ as $|x| \ge 1$, so the domain of $\text{arccsc}(4x)$ is $|4x| \ge 1$.

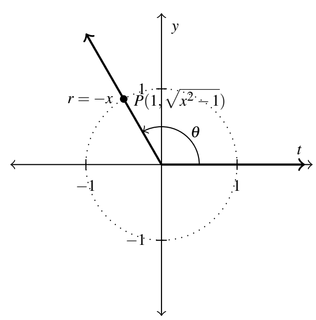
As above, we start rewriting $f(x) = \tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$ so that $\sec(t) = x$ where $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$. We let $\theta = t$ radians and wish to view $\sec(\theta) = \sec(t) = x$ as described in Theorem 7.10: the ratio of the radius of a circle, r centered at the origin, divided by the abscissa[8] of a point on the terminal side of θ which intersects said circle.

If we make the usual identification $\sec(\theta) = x = \frac{x}{1}$, we see that if $0 \leq \theta < \frac{\pi}{2}$, then $x = \sec\theta \geq 1$, so it makes sense to identify the quantity x as the radius of the circle with 1 as the abscissa of the point where the terminal side of θ intersects said circle. To find the associated ordinate (y-coordinate), we have $1^2 + y^2 = x^2$ so $y = \sqrt{x^2-1}$, where we have chosen the positive root as we are in Quadrant I. We sketch out this scenario below.



$0 < \theta < \frac{\pi}{2}, \quad x \geq 1, \quad r = x$

If, however, $\frac{\pi}{2} < t \leq \pi$, then $x = \sec(t) \leq -1$, so we need to rewrite $\sec(\theta) = x = \frac{x}{1} = \frac{-x}{-1}$ in order to keep the radius of the circle, $r = -x > 0$ and the abscissa, $-1 < 0$. From $(-1)^2 + y^2 = (-x)^2$, we still get $y = \sqrt{x^2-1}$, as shown below.



$\frac{\pi}{2} < \theta < \pi, \quad x \leq -1, \quad r = -x$

In the Quadrant I case, when $x \geq 1$, we get $\tan(\theta) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$. In Quadrant II, when $x \leq -1$, we obtain $\tan(\theta) = \frac{\sqrt{x^2-1}}{-1} = -\sqrt{x^2-1}$. Hence, we get the piecewise definition for $f(x)$ as we did in number 2a above: $f(x) = \tan(\text{arcsec}(x)) = \sqrt{x^2-1}$ if $x \geq 1$ and $f(x) = \tan(\text{arcsec}(x)) = -\sqrt{x^2-1}$ if $x \leq -1$.

[8] We'll avoid the label 'x-coordinate' here because as we'll see, the quantity x in this problem is tied to the radius as opposed to the coordinates of points on the terminal side of θ.

The moral of the story here is that you are free to choose whichever route you like to simplify expressions like those found in Example 7.6.3 number 2a. Whether you choose identities or a more geometric route, just be careful to keep in mind which quadrants are in play, which variables represent which quantities, and what signs ($\pm$) each should have.

7.6.4 Exercises

In Exercises 1 - 40, compute the exact value.

1. $\arcsin(-1)$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin(1)$
10. $\arccos(-1)$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\arccos\left(-\frac{1}{2}\right)$
14. $\arccos(0)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos(1)$
19. $\arctan(-\sqrt{3})$
20. $\arctan(-1)$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
22. $\arctan(0)$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
24. $\arctan(1)$
25. $\arctan(\sqrt{3})$
26. $\operatorname{arccot}(-\sqrt{3})$
27. $\operatorname{arccot}(-1)$
28. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
29. $\operatorname{arccot}(0)$
30. $\operatorname{arccot}\left(\frac{\sqrt{3}}{3}\right)$
31. $\operatorname{arccot}(1)$
32. $\operatorname{arccot}(\sqrt{3})$
33. $\operatorname{arcsec}(2)$
34. $\operatorname{arccsc}(2)$
35. $\operatorname{arcsec}(\sqrt{2})$
36. $\operatorname{arccsc}(\sqrt{2})$
37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
39. $\operatorname{arcsec}(1)$
40. $\operatorname{arccsc}(1)$

In Exercises 41 - 48, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 7.6.3.)

41. $\operatorname{arcsec}(-2)$
42. $\operatorname{arcsec}\left(-\sqrt{2}\right)$
43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
44. $\operatorname{arcsec}(-1)$
45. $\operatorname{arccsc}(-2)$
46. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
48. $\operatorname{arccsc}(-1)$

In Exercises 49 - 56, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value. (See Section 7.6.3.)

49. $\operatorname{arcsec}(-2)$
50. $\operatorname{arcsec}\left(-\sqrt{2}\right)$
51. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
52. $\operatorname{arcsec}(-1)$
53. $\operatorname{arccsc}(-2)$
54. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
55. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
56. $\operatorname{arccsc}(-1)$

In Exercises 57 - 86, determine the exact value or state that it is undefined.

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$
59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$
60. $\sin(\arcsin(-0.42))$
61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$
63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$
65. $\cos(\arccos(-0.998))$
66. $\cos(\arccos(\pi))$
67. $\tan(\arctan(-1))$
68. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$
69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
70. $\tan(\arctan(0.965))$
71. $\tan(\arctan(3\pi))$
72. $\cot(\operatorname{arccot}(1))$
73. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right)$
74. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$
75. $\cot(\operatorname{arccot}(-0.001))$
76. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$
77. $\sec(\operatorname{arcsec}(2))$
78. $\sec(\operatorname{arcsec}(-1))$
79. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$
80. $\sec(\operatorname{arcsec}(0.75))$
81. $\sec(\operatorname{arcsec}(117\pi))$
82. $\csc\left(\operatorname{arccsc}\left(\sqrt{2}\right)\right)$
83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$

84. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$

85. $\csc\left(\operatorname{arccsc}(1.0001)\right)$

86. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$

In Exercises 87 - 106, determine the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan\left(\tan(\pi)\right)$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

102. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

104. $\operatorname{arccot}\left(\cot(\pi)\right)$

105. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

106. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

In Exercises 107 - 118, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 7.6.3.)

107. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

108. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

109. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

110. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

111. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

112. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

113. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

114. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

115. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

116. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

117. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

118. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 119 - 130, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value.

119. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

120. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

121. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$
123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$
124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$
125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$
126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$
127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$
128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$
129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$
130. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 131 - 154, compute the exact value or state that it is undefined.

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$
132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$
133. $\sin(\arctan(-2))$
134. $\sin\left(\text{arccot}\left(\sqrt{5}\right)\right)$
135. $\sin(\text{arccsc}(-3))$
136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$
137. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$
138. $\cos(\text{arccot}(3))$
139. $\cos(\text{arcsec}(5))$
140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$
141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$
142. $\tan\left(\text{arcsec}\left(\frac{5}{3}\right)\right)$
143. $\tan(\text{arccot}(12))$
144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$
145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
146. $\cot\left(\text{arccsc}\left(\sqrt{5}\right)\right)$
147. $\cot(\arctan(0.25))$
148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$
149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$
150. $\sec(\arctan(10))$
151. $\sec\left(\text{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$
152. $\csc(\text{arccot}(9))$
153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$
154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

In Exercises 155 - 164, determine the exact value or state that it is undefined.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$
156. $\cos(\text{arcsec}(3)+\arctan(2))$
157. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$
158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$
159. $\sin\left(2\text{arccsc}\left(\frac{13}{5}\right)\right)$
160. $\sin(2\arctan(2))$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

162. $\cos\left(2\text{arcsec}\left(\frac{25}{7}\right)\right)$

163. $\cos\left(2\text{arccot}\left(-\sqrt{5}\right)\right)$

164. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 165 - 184, rewrite each of the following composite functions as algebraic functions of x and state the domain.

165. $f(x) = \sin(\arccos(x))$

166. $f(x) = \cos(\arctan(x))$

167. $f(x) = \tan(\arcsin(x))$

168. $f(x) = \sec(\arctan(x))$

169. $f(x) = \csc(\arccos(x))$

170. $f(x) = \sin(2\arctan(x))$

171. $f(x) = \sin(2\arccos(x))$

172. $f(x) = \cos(2\arctan(x))$

173. $f(x) = \sin(\arccos(2x))$

174. $f(x) = \sin\left(\arccos\left(\frac{x}{5}\right)\right)$

175. $f(x) = \cos\left(\arcsin\left(\frac{x}{2}\right)\right)$

176. $f(x) = \cos(\arctan(3x))$

177. $f(x) = \sin(2\arcsin(7x))$

178. $f(x) = \sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$

179. $f(x) = \cos(2\arcsin(4x))$

180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x))$

181. $f(x) = \sin(\arcsin(x) + \arccos(x))$

182. $f(x) = \cos(\arcsin(x) + \arctan(x))$

183. $f(x) = \tan(2\arcsin(x))$

184. $f(x) = \sin\left(\frac{1}{2}\arctan(x)\right)$

185. If $\theta = \arcsin\left(\frac{x}{2}\right)$, find an expression for $\theta + \sin(2\theta)$ in terms of x.

186. If $\theta = \arctan\left(\frac{x}{7}\right)$, find an expression for $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta)$ in terms of x.

187. If $\theta = \text{arcsec}\left(\frac{x}{4}\right)$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x assuming $x \geq 4$.

In Exercises 188 - 199, find the domain of the given function. Write your answers in interval notation.

188. $f(x) = \arcsin(5x)$

189. $f(x) = \arccos\left(\frac{3x-1}{2}\right)$

190. $f(x) = \arcsin\left(2x^2\right)$

191. $f(x) = \arccos\left(\frac{1}{x^2-4}\right)$

192. $f(x) = \arctan(4x)$

193. $f(x) = \text{arccot}\left(\frac{2x}{x^2-9}\right)$

194. $f(x) = \arctan(\ln(2x-1))$

195. $f(x) = \text{arccot}(\sqrt{2x-1})$

196. $f(x) = \text{arcsec}(12x)$

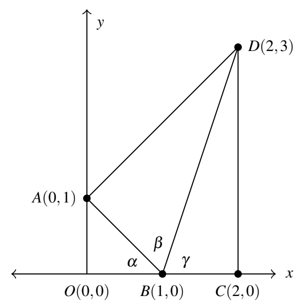
197. $f(x) = \operatorname{arccsc}(x+5)$

198. $f(x) = \operatorname{arcsec}\left(\frac{x^3}{8}\right)$

199. $f(x) = \operatorname{arccsc}\left(e^{2x}\right)$

200. Use the diagram below along with the accompanying questions to show:

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



(a) Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

(b) Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

(c) Use $\triangle BAD$ to show that $\beta = \arctan(2)$

(d) Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

(e) Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

SECTION 7.6 EXERCISE ANSWERS A.1.7

CHAPTER 8

TRIGONOMETRIC PROPERTIES AND IDENTITIES

8.1 Fundamental and Pythagorean Identities

In Section 7.4, we first encountered the concept of an **identity** when discussing Theorem 7.8. Recall that an identity is an equation which is true regardless of the choice of variable. Identities are important in mathematics because they facilitate changing forms.[1]

We take a moment to generalize Theorem 7.8 below.

Theorem 8.1. Reciprocal and Quotient Identities: The following relationships hold for all angles θ provided each side of each equation is defined.

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)}$
- $\tan(\theta) = \dfrac{1}{\cot(\theta)}$

It is important to remember that the equivalences stated in Theorem 8.1 are valid only when *all* quantities described therein are defined. As an example, $\tan(0) = 0$, but $\tan(0) \neq \frac{1}{\cot(0)}$ because $\cot(0)$ is undefined.

When it comes down to it, the Reciprocal and Quotient Identities amount to giving different ratios on the Unit Circle different names. The main focus of this section is on a more algebraic relationship between certain pairs of the circular functions: the **Pythagorean Identities**.

Recall in Definition 7.3, the cosine and sine of an angle is defined as the x and y-coordinate, respectively, of a point on the Unit Circle. The coordinates of all points (x,y) on the Unit Circle satisfy the equation $x^2+y^2=1$, thus we get for all angles θ, $(\cos(\theta))^2+(\sin(\theta))^2=1$. An unfortunate[2] convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

Theorem 8.2. The Pythagorean Identity:
For any angle θ,

$$\cos^2(\theta)+\sin^2(\theta)=1.$$

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the Distance Formula

[1] We've seen the utility of changing form throughout the text, most recently when we completed the square in Chapter 2 to put general quadratic equations into standard form in order to graph them.

[2] This is unfortunate from a 'function notation' perspective. See Section 7.6.

and the equation for a circle are ultimately derived.[3] The word 'Identity' reminds us that, regardless of the angle θ, the equation in Theorem 8.2 is always true.

If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 8.2 can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the $(\pm)$ and completely determine the missing value.[4] We illustrate this approach in the following example.

Example 8.1.1. Use Theorem 8.2 and the given information to compute the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, compute $\cos(\theta)$.
2. If $\pi < t < \frac{3\pi}{2}$ with $\cos(t) = -\frac{\sqrt{5}}{5}$, compute $\sin(t)$.
3. If $\sin(\theta) = 1$, compute $\cos(\theta)$.

Solution.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, compute $\cos(\theta)$.

   When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Given θ is a Quadrant II angle, we know $\cos(\theta) < 0$. Hence, we select $\cos(\theta) = -\frac{4}{5}$.

2. If $\pi < t < \frac{3\pi}{2}$ with $\cos(t) = -\frac{\sqrt{5}}{5}$, compute $\sin(t)$.

   Here we're using the variable t instead θ which usually corresponds to a real number variable instead of an angle. As usual, we associate real numbers t with angles θ measuring t radians,[5] so the Pythagorean Identity works equally well for all real numbers t as it does for all angles θ.

   Substituting $\cos(t) = -\frac{\sqrt{5}}{5}$ into $\cos^2(t) + \sin^2(t) = 1$ gives $\sin(t) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. given $\pi < t < \frac{3\pi}{2}$, we know t corresponds to a Quadrant III angle, so $\sin(t) < 0$. Hence, $\sin(t) = -\frac{2\sqrt{5}}{5}$.

3. If $\sin(\theta) = 1$, compute $\cos(\theta)$.

   When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$. □

The reader is encouraged to compare and contrast the solution strategies demonstrated in Example 8.1.1 with those showcases in Examples 7.2.3 and 7.2.5 in Section 7.2.2.

[3] See Section 1.1 for details.

[4] See the illustration following Example 7.4.2 to refresh yourself which circular functions are positive in which quadrants.

[5] See page 583 if you need a review of how we associate real numbers with angles in radian measure.

As with many tools in mathematics, identities give us a different way to approach and solve problems.[6] As always, the key is to determine which approach makes the most sense (is more efficient, for instance) in the given scenario.

Our next task is to use the Reciprocal and Quotient Identities found in Theorem 8.1 coupled with the Pythagorean Identity found in Theorem 8.2 to derive new Pythagorean-like identities for the remaining four circular functions.

Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$.

If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 8.1 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$.

These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.

Theorem 8.3. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

[6] For example, factoring, completing the square, and the quadratic formula are three different (yet equivalent) ways to solve a quadratic equation. See Section 0.5.5 for a refresher.

As usual, the formulas states in Theorem 8.3 work equally well for (the applicable) angles as well as real numbers.

Example 8.1.2. Use Theorems 8.1 and 8.3 to determine the indicated values.

1. If θ is a Quadrant IV angle with $\sec(\theta) = 3$, determine $\tan(\theta)$.
2. Compute $\csc(t)$ if $\pi < t < \frac{3\pi}{2}$ and $\cot(t) = 2$.
3. If θ is a Quadrant II angle with $\cos(\theta) = -\frac{3}{5}$, determine the exact values of the remaining circular functions.

Solution.

1. If θ is a Quadrant IV angle with $\sec(\theta) = 3$, determine $\tan(\theta)$.

   Per Theorem 8.3, $\tan^2(\theta) = \sec^2(\theta) - 1$. Given $\sec(\theta) = 3$, we have $\tan^2(\theta) = (3)^2 - 1 = 8$, or $\tan(\theta) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Because θ is a Quadrant IV angle, we know $\tan(\theta) < 0$ so $\tan(\theta) = -2\sqrt{2}$.

2. Compute $\csc(t)$ if $\pi < t < \frac{3\pi}{2}$ and $\cot(t) = 2$.

   Again, using Theorem 8.3, we have $\csc^2(t) = 1 + \cot^2(t)$, so we have $\csc^2(t) = 1 + (2)^2 = 5$. This gives $\csc(t) = \pm\sqrt{5}$. Given $\pi < t < \frac{3\pi}{2}$, t corresponds to a Quadrant III angle, so $\csc(t) = -\sqrt{5}$.

3. If θ is a Quadrant II angle with $\cos(\theta) = -\frac{3}{5}$, determine the exact values of the remaining circular functions.

   With five function values to find, we have our work cut out for us. From Theorem 8.1, we know $\sec(\theta) = \frac{1}{\cos(\theta)}$, so we (quickly) get $\sec(\theta) = \frac{1}{-3/5} = -\frac{5}{3}$.

   Next, we go after $\sin(\theta)$ because between $\sin(\theta)$ and $\cos(\theta)$, we can get all of the remaining values courtesy of Theorem 8.1.

   From Theorem 8.3, we have $\sin^2(\theta) = 1 - \cos^2(\theta)$, so $\sin^2(\theta) = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$. Hence, $\sin(\theta) = \pm\frac{4}{5}$ but θ is a Quadrant II angle, so we select $\sin(\theta) = \frac{4}{5}$.

   Back to Theorem 8.1, we get $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{4/5} = \frac{5}{4}$, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{4/5}{-3/5} = -\frac{4}{3}$, and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{-3/5}{4/5} = -\frac{3}{4}$. □

Again, the reader is encouraged to study the solution methodology illustrated in Example 8.1.2 as compared with that employed in Example 7.4.3 in Section 7.4.

Trigonometric identities play an important role in not just Trigonometry, but in Calculus as well. We'll use them in this book to find the values of the circular functions of an angle and solve equations and inequalities.

In Calculus, they are needed to simplify otherwise complicated expressions. In the next example, we make good use of the Theorems 8.1 and 8.3.

Example 8.1.3. Verify the following identities. Assume that all quantities are defined.

1. $\tan(\theta) = \sin(\theta)\sec(\theta)$
2. $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = -1$
3. $\sin^2(x)\cos^3(x) = \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)$
4. $\dfrac{\sec(t)}{1 - \tan(t)} = \dfrac{1}{\cos(t) - \sin(t)}$
5. $6\sec(x)\tan(x) = \dfrac{3}{1 - \sin(x)} - \dfrac{3}{1 + \sin(x)}$
6. $\dfrac{\sin(\theta)}{1 - \cos(\theta)} = \dfrac{1 + \cos(\theta)}{\sin(\theta)}$

Solution. In verifying identities, we typically start with the more complicated side of the equation and use known identities to *transform* it into the other side of the equation.

1. Verify $\tan(\theta) = \sin(\theta)\sec(\theta)$.

   Starting with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:

$$\begin{aligned} \sin(\theta)\sec(\theta) &= \sin(\theta)\frac{1}{\cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta), \end{aligned}$$

   where the last equality is courtesy of Theorem 8.1.

2. Verify $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = -1$.

   Expanding the left hand side, we get: $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = \tan^2(t) - \sec^2(t)$. From Theorem 8.3, we know $\sec^2(t) - \tan^2(t) = 1$, which isn't *quite* what we have. We are off by a negative sign $(-)$, so we factor it out:

$$\begin{aligned} (\tan(t) - \sec(t))(\tan(t) + \sec(t)) &= \tan^2(t) - \sec^2(t) \\ &= (-1)(\sec^2(t) - \tan^2(t)) \\ &= (-1)(1) = -1. \end{aligned}$$

3. Verify $\sin^2(x)\cos^3(x) = \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)$.

   Starting with the right hand side,[7] we notice we have a quantity we can immediately simplify per Theorem 8.3: $1 - \sin^2(x) = \cos^2(x)$. This increases the number of factors of cosine, (which is part of our goal in looking at the left hand side), so we proceed:

$$\begin{aligned} \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x) &= \sin^2(x)\cos^2(x)\cos(x) \\ &= \sin^2(x)\cos^3(x). \end{aligned}$$

[7]We hope by this point a shift of variable to 'x' instead of 'θ' or 't' is a non-issue.

4. Verify $\dfrac{\sec(t)}{1-\tan(t)} = \dfrac{1}{\cos(t)-\sin(t)}$.

While both sides of our next identity contain fractions, the left side affords us more opportunities to use our identities.[8] Substituting $\sec(t) = \frac{1}{\cos(t)}$ and $\tan(t) = \frac{\sin(t)}{\cos(t)}$, we get:

$$\begin{aligned}
\frac{\sec(t)}{1-\tan(t)} &= \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} \\
&= \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} \cdot \frac{\cos(t)}{\cos(t)} \\
&= \frac{\left(\dfrac{1}{\cos(t)}\right)(\cos(t))}{\left(1-\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} \\
&= \frac{1}{(1)(\cos(t)) - \left(\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} \\
&= \frac{1}{\cos(t)-\sin(t)},
\end{aligned}$$

which is exactly what we had set out to show.

5. Verify $6\sec(x)\tan(x) = \dfrac{3}{1-\sin(x)} - \dfrac{3}{1+\sin(x)}$.

Starting with the right hand side, we can get started by obtaining common denominators to add:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{3(1+\sin(x))}{(1-\sin(x))(1+\sin(x))} - \frac{3(1-\sin(x))}{(1+\sin(x))(1-\sin(x))} \\
&= \frac{3+3\sin(x)}{1-\sin^2(x)} - \frac{3-3\sin(x)}{1-\sin^2(x)} \\
&= \frac{(3+3\sin(x))-(3-3\sin(x))}{1-\sin^2(x)} \\
&= \frac{6\sin(x)}{1-\sin^2(x)}
\end{aligned}$$

[8] Or, to put to another way, earn more partial credit if this were an exam question!

At this point, we have at least reduced the number of fractions from two to one, it may not be clear how to proceed. When this happens, it isn't a bad idea to start working with the other side of the identity to get some clues how to proceed.

Using a reciprocal and quotient identity, we find $6\sec(x)\tan(x) = 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{6\sin(x)}{\cos^2(x)}$.

Theorem 8.3 tells us $1-\sin^2(x) = \cos^2(x)$, which means to our surprise and delight, we are much closer to our goal that we may have originally thought:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{6\sin(x)}{1-\sin^2(x)} \\
&= \frac{6\sin(x)}{\cos^2(x)} \\
&= 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= 6\sec(x)\tan(x).
\end{aligned}$$

6. Verify $\dfrac{\sin(\theta)}{1-\cos(\theta)} = \dfrac{1+\cos(\theta)}{\sin(\theta)}$.

It is debatable which side of the identity is more complicated. One thing which stands out is that the denominator on the left hand side is $1-\cos(\theta)$, while the numerator of the right hand side is $1+\cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1+\cos(\theta)$. Theorem 8.3 comes to our aid once more when we simplify $1-\cos^2(\theta) = \sin^2(\theta)$:

$$\begin{aligned}
\frac{\sin(\theta)}{1-\cos(\theta)} &= \frac{\sin(\theta)}{(1-\cos(\theta))} \cdot \frac{(1+\cos(\theta))}{(1+\cos(\theta))} \\
&= \frac{\sin(\theta)(1+\cos(\theta))}{(1-\cos(\theta))(1+\cos(\theta))} \\
&= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)} = \frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)} \\
&= \frac{\cancel{\sin(\theta)}(1+\cos(\theta))}{\cancel{\sin(\theta)}\sin(\theta)} \\
&= \frac{1+\cos(\theta)}{\sin(\theta)}
\end{aligned}$$

$\square$

In Example 8.1.3 number 6 above, we see that multiplying $1-\cos(\theta)$ by $1+\cos(\theta)$ produces a difference of squares that can be simplified to one term using Theorem 8.3.

This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1-\sqrt{2}$ by $1+\sqrt{2}$ or $3-4i$ by $3+4i$. In algebra, these sorts of expressions were called 'conjugates.'[9]

For this reason, the quantities $(1-\cos(\theta))$ and $(1+\cos(\theta))$ are called 'Pythagorean Conjugates.' Below is a list of other common Pythagorean Conjugates.

Pythagorean Conjugates

- $1-\cos(\theta)$ and $1+\cos(\theta)$: $(1-\cos(\theta))(1+\cos(\theta)) = 1-\cos^2(\theta) = \sin^2(\theta)$
- $1-\sin(\theta)$ and $1+\sin(\theta)$: $(1-\sin(\theta))(1+\sin(\theta)) = 1-\sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta)-1$ and $\sec(\theta)+1$: $(\sec(\theta)-1)(\sec(\theta)+1) = \sec^2(\theta)-1 = \tan^2(\theta)$
- $\sec(\theta)-\tan(\theta)$ and $\sec(\theta)+\tan(\theta)$: $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta)) = \sec^2(\theta)-\tan^2(\theta) = 1$
- $\csc(\theta)-1$ and $\csc(\theta)+1$: $(\csc(\theta)-1)(\csc(\theta)+1) = \csc^2(\theta)-1 = \cot^2(\theta)$
- $\csc(\theta)-\cot(\theta)$ and $\csc(\theta)+\cot(\theta)$: $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta)) = \csc^2(\theta)-\cot^2(\theta) = 1$

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics.

Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

[9] See Sections 0.5.6 and 0.2.

Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 8.1 to write functions on one side of the identity in terms of the functions on the other side of the identity.
  Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 8.3 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 8.3.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.
- Try *something*. The more you work with identities, the better you'll get with identities.

8.1.1 Exercises

In Exercises 1 - 11, use the Reciprocal and Quotient Identities (Theorem 8.1) along with the Pythagorean Identities (Theorem 8.3), to compute the value of the circular function requested below. (Compute the exact value unless otherwise indicated.)

1. If $\sin(\theta) = \frac{\sqrt{5}}{5}$, compute $\csc(\theta)$.
2. If $\sec(\theta) = -4$, compute $\cos(\theta)$.
3. If $\tan(t) = 3$, compute $\cot(t)$.
4. If θ is a Quadrant IV angle with $\cos(\theta) = \frac{5}{13}$, compute $\sin(\theta)$.
5. If θ is a Quadrant III angle with $\tan(\theta) = 2$, compute $\sec(\theta)$.
6. If $\frac{\pi}{2} < t < \pi$ with $\cot(t) = -2$, compute $\csc(t)$.
7. If $\sec(\theta) = 3$ and $\sin(\theta) < 0$, compute $\tan(\theta)$.
8. If $\sin(\theta) = -\frac{2}{3}$ but $\tan(\theta) > 0$, compute $\cos(\theta)$.
9. If $0 < t < \frac{\pi}{2}$ and $\sin(t) = 0.42$, compute $\cos(t)$, rounded to four decimal places.

10. If θ is Quadrant IV angle with $\sec(\theta) = 1.17$, compute $\tan(\theta)$, rounded to four decimal places.

11. If $\pi < t < \frac{3\pi}{2}$ with $\cot(t) = 4.2$, compute $\csc(t)$, rounded to four decimal places.

In Exercises 12 - 25, use the Reciprocal and Quotient Identities (Theorem 8.1) along with the Pythagorean Identities (Theorem 8.3), to compute the exact values of the remaining circular functions.

12. $\sin(\theta) = \dfrac{3}{5}$ with θ in Quadrant II

13. $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III

14. $\csc(\theta) = \dfrac{25}{24}$ with θ in Quadrant I

15. $\sec(\theta) = 7$ with θ in Quadrant IV

16. $\csc(\theta) = -\dfrac{10\sqrt{91}}{91}$ with θ in Quadrant III

17. $\cot(\theta) = -23$ with θ in Quadrant II

18. $\tan(\theta) = -2$ with θ in Quadrant IV.

19. $\sec(\theta) = -4$ with θ in Quadrant II.

20. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.

21. $\cos(\theta) = \dfrac{1}{3}$ with θ in Quadrant I.

22. $\cot(t) = 2$ with $0 < t < \dfrac{\pi}{2}$.

23. $\csc(t) = 5$ with $\dfrac{\pi}{2} < t < \pi$.

24. $\tan(t) = \sqrt{10}$ with $\pi < t < \dfrac{3\pi}{2}$.

25. $\sec(t) = 2\sqrt{5}$ with $\dfrac{3\pi}{2} < t < 2\pi$.

26. Skippy claims $\cos(\theta) + \sin(\theta) = 1$ is an identity because when $\theta = 0$, the equation is true. Is Skippy correct? Explain.

In Exercises 27 - 73, verify the identity. Assume that all quantities are defined.

27. $\cos(\theta)\sec(\theta) = 1$

28. $\tan(t)\cos(t) = \sin(t)$

29. $\sin(\theta)\csc(\theta) = 1$

30. $\tan(t)\cot(t) = 1$

31. $\csc(x)\cos(x) = \cot(x)$

32. $\dfrac{\sin(t)}{\cos^2(t)} = \sec(t)\tan(t)$

33. $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

34. $\dfrac{1+\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

35. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \csc(\theta) - \cot(\theta)$

36. $\dfrac{\cos(t)}{1-\sin^2(t)} = \sec(t)$

37. $\dfrac{\sin(x)}{1-\cos^2(x)} = \csc(x)$

38. $\dfrac{\sec(t)}{1+\tan^2(t)} = \cos(t)$

39. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)} = \sin(\theta)$

40. $\dfrac{\tan(x)}{\sec^2(x)-1} = \cot(x)$

41. $\dfrac{\cot(t)}{\csc^2(t)-1} = \tan(t)$

42. $4\cos^2(\theta)+4\sin^2(\theta) = 4$

43. $9-\cos^2(t)-\sin^2(t) = 8$

44. $\tan^3(t) = \tan(t)\sec^2(t)-\tan(t)$

45. $\sin^5(x) = \left(1-\cos^2(x)\right)^2\sin(x)$

46. $\sec^{10}(t) = \left(1+\tan^2(t)\right)^4\sec^2(t)$

47. $\cos^2(x)\tan^3(x) = \tan(x)-\sin(x)\cos(x)$

48. $\sec^4(t)-\sec^2(t) = \tan^2(t)+\tan^4(t)$

49. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1} = \dfrac{1+\sec(\theta)}{1-\sec(\theta)}$

50. $\dfrac{\sin(t)+1}{\sin(t)-1} = \dfrac{1+\csc(t)}{1-\csc(t)}$

51. $\dfrac{1-\cot(x)}{1+\cot(x)} = \dfrac{\tan(x)-1}{\tan(x)+1}$

52. $\dfrac{1-\tan(t)}{1+\tan(t)} = \dfrac{\cos(t)-\sin(t)}{\cos(t)+\sin(t)}$

53. $\tan(\theta)+\cot(\theta) = \sec(\theta)\csc(\theta)$

54. $\csc(t)-\sin(t) = \cot(t)\cos(t)$

55. $\cos(x)-\sec(x) = -\tan(x)\sin(x)$

56. $\cos(x)(\tan(x)+\cot(x)) = \csc(x)$

57. $\sin(t)(\tan(t)+\cot(t)) = \sec(t)$

58. $\dfrac{1}{1-\cos(\theta)}+\dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

59. $\dfrac{1}{\sec(t)+1}+\dfrac{1}{\sec(t)-1} = 2\csc(t)\cot(t)$

60. $\dfrac{1}{\csc(x)+1}+\dfrac{1}{\csc(x)-1} = 2\sec(x)\tan(x)$

61. $\dfrac{1}{\csc(t)-\cot(t)}-\dfrac{1}{\csc(t)+\cot(t)} = 2\cot(t)$

62. $\dfrac{\cos(\theta)}{1-\tan(\theta)}+\dfrac{\sin(\theta)}{1-\cot(\theta)} = \sin(\theta)+\cos(\theta)$

63. $\dfrac{1}{\sec(t)+\tan(t)} = \sec(t)-\tan(t)$

64. $\dfrac{1}{\sec(x)-\tan(x)} = \sec(x)+\tan(x)$

65. $\dfrac{1}{\csc(t)-\cot(t)} = \csc(t)+\cot(t)$

66. $\dfrac{1}{\csc(\theta)+\cot(\theta)} = \csc(\theta)-\cot(\theta)$

67. $\dfrac{1}{1-\sin(x)} = \sec^2(x)+\sec(x)\tan(x)$

68. $\dfrac{1}{1+\sin(t)} = \sec^2(t)-\sec(t)\tan(t)$

69. $\dfrac{1}{1-\cos(\theta)} = \csc^2(\theta)+\csc(\theta)\cot(\theta)$

70. $\dfrac{1}{1+\cos(x)} = \csc^2(x)-\csc(x)\cot(x)$

71. $\dfrac{\cos(t)}{1+\sin(t)} = \dfrac{1-\sin(t)}{\cos(t)}$

72. $\csc(\theta)-\cot(\theta) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

73. $\dfrac{1-\sin(x)}{1+\sin(x)} = (\sec(x)-\tan(x))^2$

In Exercises 74 - 77, verify the identity. You may need to consult Sections 1.4 and 5.4 for a review of the properties of absolute value and logarithms before proceeding.

74. $\ln|\sec(x)| = -\ln|\cos(x)|$

75. $-\ln|\csc(x)| = \ln|\sin(x)|$

76. $-\ln|\sec(x)-\tan(x)| = \ln|\sec(x)+\tan(x)|$

77. $-\ln|\csc(x)+\cot(x)| = \ln|\csc(x)-\cot(x)|$

Section 8.1 Exercise Answers A.1.8

8.2 Other Trigonometric Identities

In Section 8.1, we saw the utility of identities in finding the values of the circular functions of a given angle as well as simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond.

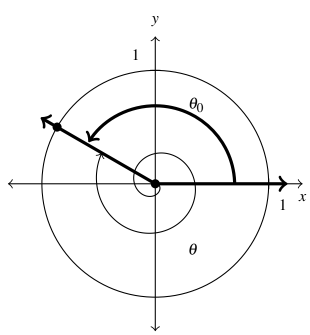
Our first set of identities is the 'Even / Odd' identities. We *observed* the even and odd properties of the circular functions graphically in Sections 7.3 and 7.5. Here, we take the time to *prove* these properties from first principles. We state the theorem below for reference.

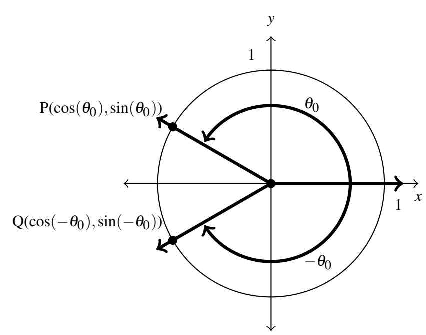
Theorem 8.4. Even / Odd Identities:
For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

We start by proving $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \leq \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive x-axis to the terminal side of θ as pictured below.) θ and θ_0 are coterminal, so $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.





We now consider the angles $-\theta$ and $-\theta_0$. As θ is coterminal with θ_0, there is some integer k such that $\theta = \theta_0 + 2\pi \cdot k$. Hence, $-\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k)$. Because k is an integer, so is $(-k)$, which means $-\theta$ is coterminal with $-\theta_0$. Therefore, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$.

Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.

Because θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the x-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$.

The cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, thus we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required.

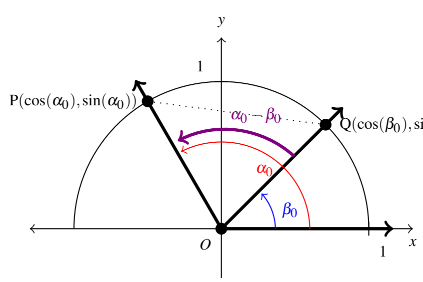
As we saw in Section 7.5, the remaining four circular functions 'inherit' their even/odd nature from sine and cosine courtesy of the Reciprocal and Quotient Identities, Theorem 8.1.

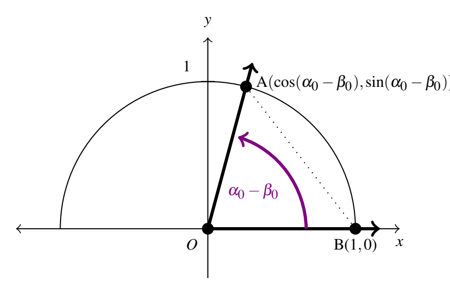
Our next set of identities establish how the cosine function handles sums and differences of angles.

Theorem 8.5. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Because α and α_0 are coterminal, as are β and β_0, it follows that $(\alpha-\beta)$ is coterminal with $(\alpha_0-\beta_0)$. Consider the case below where $\alpha_0 \geq \beta_0$.





Because the angles POQ and AOB are congruent, the distance between P and Q is equal to the distance between A and B.[1] The distance formula, Equation 1.1, yields

$$\sqrt{(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2} \quad = \quad \sqrt{(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 &= \cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)\\ &\quad +\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\\ &= \cos^2(\alpha_0)+\sin^2(\alpha_0)+\cos^2(\beta_0)+\sin^2(\beta_0)\\ &\quad -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

[1] In the picture we've drawn, the <u>tri</u>angles POQ and AOB are congruent, which is even better. However, $\alpha_0-\beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

From the Pythagorean Identities, $\cos^2(\alpha_0)+\sin^2(\alpha_0)=1$ and $\cos^2(\beta_0)+\sin^2(\beta_0)=1$, so

$$(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 \quad = \quad 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)$$

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 &= \cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1+\sin^2(\alpha_0-\beta_0)\\ &= 1+\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)=1$, so that

$$(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 \quad = \quad 2-2\cos(\alpha_0-\beta_0)$$

Putting it all together, we get $2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)=2-2\cos(\alpha_0-\beta_0)$, which simplifies to: $\cos(\alpha_0-\beta_0)=\cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0)$.

Given α and α_0, β and β_0, and $(\alpha-\beta)$ and $(\alpha_0-\beta_0)$ are all coterminal pairs of angles, we have established the identity: $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$.

For the case where $\alpha_0\leq\beta_0$, we can apply the above argument to the angle $\beta_0-\alpha_0$ to obtain the identity $\cos(\beta_0-\alpha_0)=\cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0)$. Using this formula in conjunction with the Even Identity of cosine gives us the result in this case, too:

$$\begin{aligned}\cos(\alpha_0-\beta_0)=\cos(-(\alpha_0-\beta_0))=\cos(\beta_0-\alpha_0) &= \cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0)\\ &= \cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0).\end{aligned}$$

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

We put these newfound identities to good use in the following example.

Example 8.2.1.

1. Compute the exact value of $\cos(15^\circ)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$.
3. Suppose α is a Quadrant I angle with $\sin(\alpha)=\frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta)=4$. Determine the exact value of $\cos(\alpha+\beta)$.

Solution.

1. Compute the exact value of $\cos(15^\circ)$.

   In order to use Theorem 8.5 to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ = 45^\circ - 30^\circ$. We find:

$$\begin{aligned}\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\ &= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

   Using Theorem 8.5 gives:

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) \\ &= (0)(\cos(\theta)) + (1)(\sin(\theta)) \\ &= \sin(\theta).\end{aligned}$$

3. Suppose α is a Quadrant I angle with $\sin(\alpha) = \frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta) = 4$. Determine the exact value of $\cos(\alpha + \beta)$.

   Per Theorem 8.5, we know $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Hence, we need to find the sines and cosines of α and β to complete the problem.

   We are given $\sin(\alpha) = \frac{3}{5}$, so our first task is to find $\cos(\alpha)$. We can quickly get $\cos(\alpha)$ using the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$. We get $\cos(\alpha) = \frac{4}{5}$, choosing the positive root because α is a Quadrant I angle.

   Next, we need the $\sin(\beta)$ and $\cos(\beta)$. $\sec(\beta) = 4$, so we immediately get $\cos(\beta) = \frac{1}{4}$ courtesy of the Reciprocal and Quotient Identities.

   To get $\sin(\beta)$, we employ the Pythagorean Identity: $\sin^2(\beta) = 1 - \cos^2(\beta) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$. Here, as β is a Quadrant IV angle, we get $\sin(\beta) = -\frac{\sqrt{15}}{4}$.

   Finally, we get: $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \left(\frac{4}{5}\right)\left(\frac{1}{4}\right) - \left(\frac{3}{5}\right)\left(-\frac{\sqrt{15}}{4}\right) = \frac{4+3\sqrt{15}}{20}$. $\square$

The identity verified in Example 8.2.1, namely, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, is the first of the celebrated 'cofunction' identities.

From $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we get: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \theta\right]\right) = \cos(\theta)$, which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 8.6. Cofunction Identities: For all applicable angles θ,

- $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sec\left(\dfrac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\csc\left(\dfrac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\cot\left(\dfrac{\pi}{2} - \theta\right) = \tan(\theta)$

The Cofunction Identities enable us to derive the sum and difference formulas for sine. We first convert to sine to cosine and expand:

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) \\
&= \cos\left(\left[\frac{\pi}{2} - \alpha\right] - \beta\right) \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(\beta) + \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta) \\
&= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)
\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha - \beta)$ as $\sin(\alpha + (-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 8.7. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

We try out these new identities in the next example.

Example 8.2.2.

1. Compute the exact value of $\sin\left(\frac{19\pi}{12}\right)$
2. Suppose α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$. Compute the exact value of $\sin(\alpha - \beta)$.
3. Derive a formula for $\tan(\alpha + \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution.

1. Compute the exact value of $\sin\left(\frac{19\pi}{12}\right)$.

   As in Example 8.2.1, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination[2] is $\frac{19\pi}{12} = \frac{4\pi}{3} + \frac{\pi}{4}$. Applying Theorem 8.7, we get

$$\begin{aligned} \sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3} + \frac{\pi}{4}\right) \\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{-\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

2. Suppose α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$. Compute the exact value of $\sin(\alpha - \beta)$.

   In order to find $\sin(\alpha - \beta)$ using Theorem 8.7, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$.

   To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\frac{5}{13}\right)^2 = \frac{144}{169}$. We get $\cos(\alpha) = -\frac{12}{13}$, the negative, here, owing to the fact that α is a Quadrant II angle.

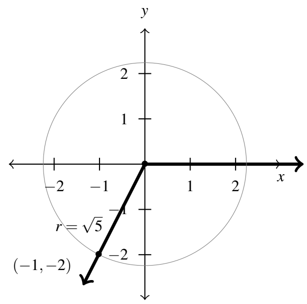
   We now set about finding $\sin(\beta)$ and $\cos(\beta)$. We have several ways to proceed at this point, but as there isn't a direct way to get from $\tan(\beta) = 2$ to either $\sin(\beta)$ or $\cos(\beta)$, we opt for a more geometric approach as presented in Section 7.4.

   Because β is a Quadrant III angle with $\tan(\beta) = 2 = \frac{-2}{-1}$, we know the point $Q(x,y) = (-1,-2)$ is on the terminal side of β as illustrated.[3]

---

[2] It takes some trial and error to find this combination. One alternative is to convert to degrees ...

[3] Note that even though $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, we *cannot* take $\sin(\beta) = -2$ and $\cos(\beta) = -1$. Recall that $\sin(\beta)$ and $\cos(\beta)$ are the y and x coordinates on a *specific* circle, the Unit Circle. As we'll see shortly, $(-1,-2)$ lies on a circle of $\sqrt{5}$, so *not* the Unit Circle.



The terminal side of β contains $Q(-1,-2)$

We find $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$, so per Theorem 7.10, $\sin(\beta) = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ and $\cos(\beta) = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$.

At last, we have $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{29\sqrt{5}}{65}$.

3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

We can start by expanding $\tan(\alpha+\beta)$ using a quotient identity and then the sum formulas

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \end{aligned}$$

As $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}} \\ &= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \end{aligned}$$

$$= \frac{\dfrac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \dfrac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\dfrac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}} - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}$$

$$= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

Naturally, this formula is limited to those cases where all of the tangents are defined. □

The formula developed in Exercise 8.2.2 for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$ and using the odd property of tangent. (The reader is encouraged to fill in the details.) Below we summarize all of the sum and difference formulas.

Theorem 8.8. Sum and Difference Identities: For all applicable angles α and β,

- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\tan(\alpha\pm\beta) = \dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

In the statement of Theorem 8.8, we have combined the cases for the sum '+' and difference '−' of angles into one formula. The convention here is that if you want the formula for the sum '+' of two angles, you use the top sign in the formula; for the difference, '−', use the bottom sign. For example,

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

If we set $\alpha=\beta$ in the sum formulas in Theorem 8.8, we obtain the following 'Double Angle' Identities:

Theorem 8.9. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta)-\sin^2(\theta) \\ 2\cos^2(\theta)-1 \\ 1-2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity. For instance, if we substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ into the first formula for $\cos(2\theta)$, we get $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$.

It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

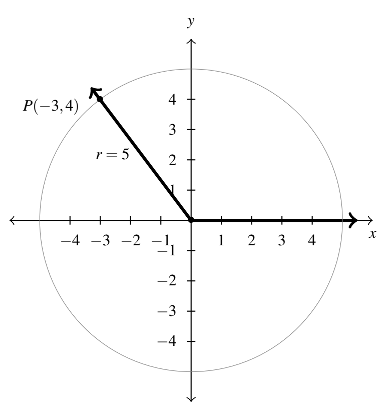
Example 8.2.3.

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Compute $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.
2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.
3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.
4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution.

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Compute $\cos(2\theta)$ and $\sin(2\theta)$.

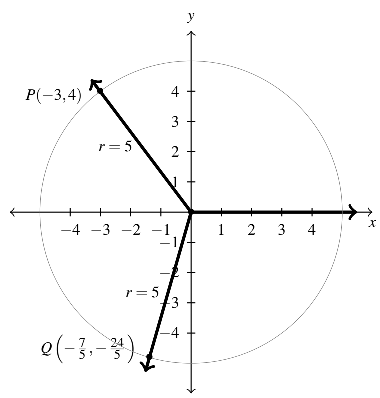
   We sketch the terminal side of θ below. Using Theorem 7.4 from Section 7.2.2 with $x = -3$ and $y = 4$, we find $r = \sqrt{x^2 + y^2} = 5$. Hence, $\cos(\theta) = -\frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$.



Graph of $P(-3,4)$ is on the terminal side of θ

Theorem 8.9 gives us three different formulas to choose from to find $\cos(2\theta)$. Using the first formula,[4] we get: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$. For $\sin(2\theta)$, we get $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$.

Both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III. To see this more clearly, we plot the terminal side of 2θ, along with the terminal side of θ.



Graph of $Q\left(-\frac{7}{5},-\frac{24}{5}\right)$ is on the terminal side of 2θ

Note that in order to find the point $Q(x,y)$ on the terminal side of 2θ of a circle of radius 5, we use Theorem 7.4 again and find $x = r\cos(2\theta) = 5\left(-\frac{7}{25}\right) = -\frac{7}{5}$ and $y = r\sin(2\theta) = 5\left(-\frac{24}{25}\right) = -\frac{24}{5}$.

2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

   If your first reaction to '$\sin(\theta) = x$' is 'No it's not, $\cos(\theta) = x$!' then you have indeed learned something, and we take comfort in that.

   While we have mostly used 'x' to represent the x-coordinate of the point the terminal side of an angle θ, here, 'x' represents the quantity $\sin(\theta)$ and our task is to express $\sin(2\theta)$ in terms of x.

   As a result of $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\cos(\theta)$, what remains is to express $\cos(\theta)$ in terms of x.

   Substituting $\sin(\theta) = x$ into the Pythagorean Identity, we get $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$, or $\cos(\theta) = \pm\sqrt{1-x^2}$. Given $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, and thus $\cos(\theta) = \sqrt{1-x^2}$.

   Our final answer is $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1-x^2}$.

3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

[4] We invite the reader to check this answer using the other two formulas.

We start with the right hand side of the identity and note that $1+\tan^2(\theta) = \sec^2(\theta)$. Next, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} \\
&= \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\dfrac{1}{\cos^2(\theta)}} \\
&= 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) \\
&= 2\sin(\theta)\cos(\theta) = \sin(2\theta).
\end{aligned}$$

4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

   In Theorem 8.9, one of the formulas for $\cos(2\theta)$, namely $\cos(2\theta) = 2\cos^2(\theta) - 1$, expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$.

   Using the sum formula for cosine, we begin with

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta+\theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta).
\end{aligned}$$

   Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. To that end, we substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields:

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)
\end{aligned}$$

   Finally, we exchange $\sin^2(\theta) = 1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}
\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta).
\end{aligned}$$

   Hence, $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. □

In the last problem in Example 8.2.3, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine.

Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 8.10. Power Reduction Formulas:

For all angles θ,

- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

Our next example is a typical application of Theorem 8.10 that you'll likely see in Calculus.

Example 8.2.4. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution. We begin with a straightforward application of Theorem 8.10

$$\begin{aligned} \sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right) \\ &= \frac{1}{4}\left(1-\cos^2(2\theta)\right) \\ &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$\begin{aligned} \sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \\ &= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) \\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta) \\ &= \frac{1}{8} - \frac{1}{8}\cos(4\theta) \end{aligned}$$

□

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the

Power Reduction Formula to $\cos^2\left(\frac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1+\cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent.

We summarize these formulas below.

Theorem 8.11. Half Angle Formulas:
For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\dfrac{\theta}{2}$ lies.

Example 8.2.5.

1. Use a half angle formula to compute the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Determine $\sin\left(\frac{t}{2}\right)$.
3. Use the identity given in number 3 of Example 8.2.3 to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution.

1. Use a half angle formula to compute the exact value of $\cos(15^\circ)$.

   To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and 15° is a Quadrant I angle, so its cosine is positive. Thus we have

$$\begin{aligned}\cos(15^\circ) &= +\sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Back in Example 8.2.1, we found $\cos(15^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$ by using the difference formula for cosine. The reader is encouraged to prove that these two expressions are equal algebraically.

2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Determine $\sin\left(\frac{t}{2}\right)$.

If $-\pi \leq t \leq 0$, then $-\frac{\pi}{2} \leq \frac{t}{2} \leq 0$, which means $\frac{t}{2}$ corresponds to a Quadrant IV angle. Hence, $\sin\left(\frac{t}{2}\right) < 0$, so we choose the negative root formula from Theorem 8.11:

$$\begin{aligned}\sin\left(\frac{t}{2}\right) &= -\sqrt{\frac{1-\cos(t)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5}\end{aligned}$$

3. Use the identity given in number 3 of Example 8.2.3 to derive the identity $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$.

Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 8.2.3 and manipulate it into the identity we are asked to prove.

The identity we are asked to start with is $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$

$$\begin{aligned}\sin\left(2\left(\tfrac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\end{aligned}$$

We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. We substitute $1+\tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$, and continue to manipulate our given identity by converting secants to cosines.

$$\begin{aligned}\sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right)\end{aligned}$$

Finally, we apply a power reduction formula, and then solve for $\tan\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right) \\
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\
\sin(\theta) &= \tan\left(\tfrac{\theta}{2}\right)(1+\cos(\theta)) \\
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)}
\end{aligned}$$

□

Our next batch of identities, the Product to Sum Formulas,[5] are easily verified by expanding each of the right hand sides in accordance with Theorem 8.8 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 8.12. Product to Sum Formulas:
For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Section 8.3.2. These are essentially restatements of the Product to Sum Formulas (by re-labeling the arguments of the sine and cosine functions) and as such, their proofs are left as exercises.

Theorem 8.13. Sum to Product Formulas:
For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\dfrac{\alpha\pm\beta}{2}\right)\cos\left(\dfrac{\alpha\mp\beta}{2}\right)$

[5]These are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

Example 8.2.6.

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

Solution.

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.

   Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{array}{rcl} \cos(2\theta)\cos(6\theta) & = & \frac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right] \\ & = & \frac{1}{2}\cos(-4\theta)+\frac{1}{2}\cos(8\theta) \\ & = & \frac{1}{2}\cos(4\theta)+\frac{1}{2}\cos(8\theta), \end{array}$$

   where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta)=\cos(4\theta)$.

2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

   Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{array}{rcl} \sin(\theta)-\sin(3\theta) & = & 2\sin\left(\dfrac{\theta-3\theta}{2}\right)\cos\left(\dfrac{\theta+3\theta}{2}\right) \\ & = & 2\sin(-\theta)\cos(2\theta) \\ & = & -2\sin(\theta)\cos(2\theta), \end{array}$$

   where the last equality is courtesy of the odd identity for sine, $\sin(-\theta)=-\sin(\theta)$. □

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers.

8.2.1 Sinusoids, Revisited

We first studied sinusoids in Section 7.3. Using the sum formulas for sine and cosine, we can expand the forms given to us in Theorem 7.7:

$$S(t) = A\sin(Bt+C)+D = A\sin(Bt)\cos(C)+A\cos(Bt)\sin(C)+D,$$

and

$$E(t) = A\cos(Bt+C)+D = A\cos(Bt)\cos(C)-A\sin(Bt)\sin(C)+B.$$

As we'll see in the next example, recognizing these 'expanded' forms of sinusoids allows us to graph functions as sinusoids which, at first glance, don't appear to fit the forms of either $E(t)$ or $S(t)$.

Example 8.2.7. Consider the function $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$.

1. Write a formula for $f(t)$ in the form $E(t) = A\cos(Bt+C)+D$ for $B > 0$.
2. Write a formula for $f(t)$ in the form $S(t) = A\sin(Bt+C)+D$ for $B > 0$.

Check your answers analytically using identities.

Solution.

1. Write a formula for $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ in the form $E(t) = A\cos(Bt+C)+D$ for $B > 0$.

   The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. We start by equating $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of $E(t) = A\cos(Bt+C)+D$: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(Bt)\cos(C) - A\sin(Bt)\sin(C) + D$.

   If we take $B = 2$ and $D = 0$, we get: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(2t)\cos(C) - A\sin(2t)\sin(C)$.

   To determine A and C, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation.

   On the left hand side, the coefficient of $\cos(2t)$ is 1, while on the right hand side, it is $A\cos(C)$. As this equation is to hold for all real numbers, we must have that $A\cos(C) = 1$.

   Similarly, we find by equating the coefficients of $\sin(2t)$ that $A\sin(C) = \sqrt{3}$. In conjunction with $A\cos(C) = 1$, we have a system of two (nonlinear) equations and two unknowns.

   As usual, our first task is to reduce this system of two equations and two unknowns to one equation and one unknown. We can temporarily eliminate the dependence on C by using a Pythagorean Identity. From $\cos^2(C) + \sin^2(C) = 1$, we multiply through by A^2 to get $A^2\cos^2(C) + A^2\sin^2(C) = A^2$.

   In our case, $A\cos(C) = 1$ and $A\sin(C) = \sqrt{3}$, hence $A^2 = A^2\cos^2(C) + A^2\sin^2(C) = 1^2 + (\sqrt{3})^2 = 4$ so $A = \pm 2$. We can *choose* $A = 2$, and then find C associated with this choice[6].

   Substituting $A = 2$ into our two equations, $A\cos(C) = 1$ and $A\sin(C) = \sqrt{3}$, we get $2\cos(C) = 1$ and $2\sin(C) = \sqrt{3}$. After some rearrangement, $\cos(C) = \frac{1}{2}$ and $\sin(C) = \frac{\sqrt{3}}{2}$. One such angle C which satisfies this criteria is $C = \frac{\pi}{3}$.

   Hence, one way to write $f(t)$ as a sinusoid is $f(t) = 2\cos\left(2t + \frac{\pi}{3}\right)$.

   We can check our answer using the sum formula for cosine :

$$\begin{aligned} f(t) &= 2\cos\left(2t + \tfrac{\pi}{3}\right) \\ &= 2\left[\cos(2t)\cos\left(\tfrac{\pi}{3}\right) - \sin(2t)\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= 2\left[\cos(2t)\left(\tfrac{1}{2}\right) - \sin(2t)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\ &= \cos(2t) - \sqrt{3}\sin(2t). \end{aligned}$$

[6] Remember choosing $A = -2$ results in a different but equally correct phase shift.

2. Write a formula for $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ in the form $S(t) = A\sin(Bt+C)+D$ for $B > 0$.

   Proceeding as before, we equate $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of of the sinusoid $S(t) = A\sin(Bt+C)+D$ to get: $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(Bt)\cos(C) + A\cos(Bt)\sin(C) + D$.

   Taking $B = 2$ and $D = 0$, we get $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(2t)\cos(C) + A\cos(2t)\sin(C)$. We equate[7] the coefficients of $\cos(2t)$ on either side and get $A\sin(C) = 1$ and $A\cos(C) = -\sqrt{3}$.

   Using $A^2\cos^2(C) + A^2\sin^2(C) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$.

   This means $2\sin(C) = 1$, or $\sin(C) = \frac{1}{2}$, and $2\cos(C) = -\sqrt{3}$, so $\cos(C) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $C = \frac{5\pi}{6}$.

   Hence, we have $f(t) = 2\sin\left(2t + \frac{5\pi}{6}\right)$.

   Checking our work analytically, we have

$$\begin{array}{rcl} f(t) & = & 2\sin\left(2t + \frac{5\pi}{6}\right) \\ & = & 2\left[\sin(2t)\cos\left(\frac{5\pi}{6}\right) + \cos(2t)\sin\left(\frac{5\pi}{6}\right)\right] \\ & = & 2\left[\sin(2t)\left(-\frac{\sqrt{3}}{2}\right) + \cos(2t)\left(\frac{1}{2}\right)\right] \\ & = & \cos(2t) - \sqrt{3}\sin(2t) \end{array}$$

□

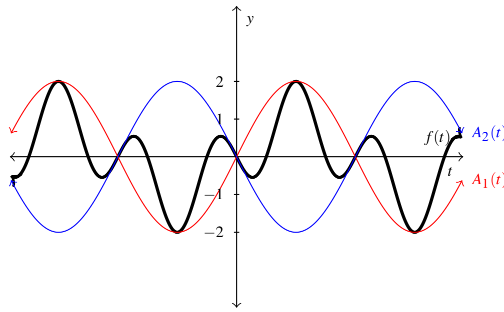
A couple of remarks about Example 8.2.7 are in order. First, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 8.2.7, our final answers would have *looked* different. The reader is encouraged to rework Example 8.2.7 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent.[8]

It is important to note that in order for the technique presented in Example 8.2.7 to fit a function into one of the forms in Theorem 7.7, the *frequencies* of the sine and cosine terms must match. For example, in the Exercises, you'll be asked to write $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$ in the form of $S(t)$ and $C(t)$ above, and because both the sine and cosine terms have frequency 3, this is possible.

However, a function such as $f(t) = \sin(t) - \sin(3t)$ cannot be written in the form of $S(t)$ or $C(t)$. The quickest way to see this is to examine its graph below which is decidedly not a sinusoid. That being said, we can still analyze this curve using identities.

[7] Be careful here!

[8] The general equations to fit a function of the form $f(x) = a\cos(Bx) + b\sin(Bx) + D$ into one of the forms in Theorem 7.7 are explored in Exercise 36.



Using our result from number 2 Example 8.2.6, we may rewrite $f(t) = \sin(t) - \sin(3t) = -2\sin(t)\cos(2t)$. Grouping factors, we can view $f(t) = [-2\sin(t)]\cos(2t) = A(t)\cos(2t)$ as the curve $y = \cos(2t)$ with a *variable* amplitude, $A(t) = -2\sin(t)$.

Overlaying the graphs of $f(t)$ with the (dashed) graphs of $A_1(t) = 2\sin(t)$ and $A_2(t) = -2\sin(t)$, we can see the role these two curves play in the graph of $y = f(t)$. They create a kind of 'wave envelope' for the graph of $y = f(t)$. This is an example of the beats phenomenon. Note that when written as a product of sinusoids, it is always the *lower* frequency factor which creates the 'wave-envelope' of the curve.

Note that in order to rewrite a sum or difference of sine and cosine functions with different frequencies into a product using the sum to product identities, Theorem 8.13, we need the *amplitudes* of each term to be the same. We explore more examples of these functions and this behavior in the Exercises.

8.2.2 Exercises

In Exercises 1 - 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$
2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$
3. $\tan(-x^2 + 1) = -\tan(x^2 - 1)$
4. $\csc(-\theta - 5) = -\csc(\theta + 5)$
5. $\sec(-6x) = \sec(6x)$
6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 - 21, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

7. $\cos(75^\circ)$
8. $\sec(165^\circ)$
9. $\sin(105^\circ)$
10. $\csc(195^\circ)$
11. $\cot(255^\circ)$
12. $\tan(375^\circ)$
13. $\cos\left(\frac{13\pi}{12}\right)$
14. $\sin\left(\frac{11\pi}{12}\right)$
15. $\tan\left(\frac{13\pi}{12}\right)$

16. $\cos\left(\frac{7\pi}{12}\right)$
17. $\tan\left(\frac{17\pi}{12}\right)$
18. $\sin\left(\frac{\pi}{12}\right)$
19. $\cot\left(\frac{11\pi}{12}\right)$
20. $\csc\left(\frac{5\pi}{12}\right)$
21. $\sec\left(-\frac{\pi}{12}\right)$
22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    (a) $\cos(\alpha+\beta)$
    (b) $\sin(\alpha+\beta)$
    (c) $\tan(\alpha+\beta)$
    (d) $\cos(\alpha-\beta)$
    (e) $\sin(\alpha-\beta)$
    (f) $\tan(\alpha-\beta)$
23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    (a) $\cos(\alpha+\beta)$
    (b) $\sin(\alpha+\beta)$
    (c) $\tan(\alpha+\beta)$
    (d) $\cos(\alpha-\beta)$
    (e) $\sin(\alpha-\beta)$
    (f) $\tan(\alpha-\beta)$
24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    (a) $\sin(\alpha+\beta)$
    (b) $\cos(\alpha-\beta)$
    (c) $\tan(\alpha-\beta)$
25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find
    (a) $\csc(\alpha-\beta)$
    (b) $\sec(\alpha+\beta)$
    (c) $\cot(\alpha+\beta)$

In Exercises 26 - 35, use Example 8.2.7 as a guide to show that the function is a sinusoid by rewriting it in the forms $E(t) = A\cos(Bt+C)+D$ and $S(t) = A\sin(Bt+C)+D$ for $B > 0$ and $0 \leq C < 2\pi$.

26. $f(t) = \sqrt{2}\sin(t)+\sqrt{2}\cos(t)+1$
27. $f(t) = 3\sqrt{3}\sin(3t)-3\cos(3t)$
28. $f(t) = -\sin(t)+\cos(t)-2$
29. $f(t) = -\frac{1}{2}\sin(2t)-\frac{\sqrt{3}}{2}\cos(2t)$
30. $f(t) = 2\sqrt{3}\cos(t)-2\sin(t)$
31. $f(t) = \frac{3}{2}\cos(2t)-\frac{3\sqrt{3}}{2}\sin(2t)+6$
32. $f(t) = -\frac{1}{2}\cos(5t)-\frac{\sqrt{3}}{2}\sin(5t)$
33. $f(t) = -6\sqrt{3}\cos(3t)-6\sin(3t)-3$
34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t)-\frac{5\sqrt{2}}{2}\cos(t)$
35. $f(t) = 3\sin\left(\frac{t}{6}\right)-3\sqrt{3}\cos\left(\frac{t}{6}\right)$
36. In Exercises 26 - 35, you should have noticed a relationship between the phases C for the $S(t)$ and $E(t)$. Show that if $f(t) = A\sin(Bt+\alpha)+D$, then $f(t) = A\cos(Bt+\beta)+D$ where $\beta = \alpha - \frac{\pi}{2}$.
37. Let C be an angle measured in radians and let $P(a,b)$ be a point on the terminal side of C when it is drawn in standard position. Use Theorem 7.4 and the sum identity for sine in Theorem 8.7 to show that $f(t) = a\sin(Bt)+b\cos(Bt)+D$ (with $B > 0$) can be rewritten as $f(t) = \sqrt{a^2+b^2}\sin(Bt+C)+D$.

38. Two (seemingly) different formulas to model the hours of daylight are given here, $H(t)$: $H_1(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55$ and $H_2(t) = -8.13\sin\left(\frac{\pi}{6}t - 4.70\right) + 12.5$. Use the difference identities for sine to expand $H_1(t)$ and $H_2(t)$. How different are they?

In Exercises 39 - 53, verify the identity.[9]

39. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(t)$

40. $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(t)$

41. $\cos(\theta - \pi) = -\cos(\theta)$

42. $\sin(\pi - \theta) = \sin(\theta)$

43. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

44. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

45. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

46. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

47. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

48. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

49. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

50. $\dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \dfrac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

51. $\dfrac{\sin(t+h) - \sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h) - 1}{h}\right)$

52. $\dfrac{\cos(t+h) - \cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h) - 1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

53. $\dfrac{\tan(t+h) - \tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1 - \tan(t)\tan(h)}\right)$

In Exercises 54 - 63, use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

54. $\cos(75^\circ)$ (compare with Exercise 7)

55. $\sin(105^\circ)$ (compare with Exercise 9)

56. $\cos(67.5^\circ)$

57. $\sin(157.5^\circ)$

58. $\tan(112.5^\circ)$

59. $\cos\left(\frac{7\pi}{12}\right)$ (compare with Exercise 16)

60. $\sin\left(\frac{\pi}{12}\right)$ (compare with Exercise 18)

61. $\cos\left(\frac{\pi}{8}\right)$

62. $\sin\left(\frac{5\pi}{8}\right)$

63. $\tan\left(\frac{7\pi}{8}\right)$

In Exercises 64 - 73, use the given information about θ to compute the exact values of

[9]Note: numbers 39 and 40 are the conversion formulas stated in Theorem 7.6 in Section 7.3.

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

64. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$
65. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$
66. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
67. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$
68. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$
69. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
70. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$
71. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$
72. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$
73. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

In Exercises 74 - 88, verify the identity. Assume all quantities are defined.

74. $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$
75. $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$
76. $\tan(2t) = \frac{1}{1-\tan(t)} - \frac{1}{1+\tan(t)}$
77. $\csc(2\theta) = \frac{\cot(\theta)+\tan(\theta)}{2}$
78. $8\sin^4(x) = \cos(4x) - 4\cos(2x) + 3$
79. $8\cos^4(x) = \cos(4x) + 4\cos(2x) + 3$
80. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$
81. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$
82. $32\sin^2(t)\cos^4(t) = 2 + \cos(2t) - 2\cos(4t) - \cos(6t)$
83. $32\sin^4(t)\cos^2(t) = 2 - \cos(2t) - 2\cos(4t) + \cos(6t)$
84. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$
85. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 84.)
86. $\sec(2x) = \dfrac{\cos(x)}{\cos(x) + \sin(x)} + \dfrac{\sin(x)}{\cos(x) - \sin(x)}$
87. $\dfrac{1}{\cos(\theta) - \sin(\theta)} + \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$
88. $\dfrac{1}{\cos(\theta) - \sin(\theta)} - \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$
89. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

    (a) $\cos(\theta) = \sqrt{1 - x^2}$ (b) $\sin(2\theta) = 2x\sqrt{1 - x^2}$ (c) $\cos(2\theta) = 1 - 2x^2$

90. Discuss with your classmates how each of the formulas, if any, in Exercise 89 change if we change assume θ is a Quadrant II, III, or IV angle.
91. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

(a) $\cos(\theta) = \dfrac{1}{\sqrt{x^2+1}}$

(b) $\sin(\theta) = \dfrac{x}{\sqrt{x^2+1}}$

(c) $\sin(2\theta) = \dfrac{2x}{x^2+1}$

(d) $\cos(2\theta) = \dfrac{1-x^2}{x^2+1}$

92. Discuss with your classmates how each of the formulas, if any, in Exercise 91 change if we change assume θ is a Quadrant II, III, or IV angle.

93. If $\sin(t) = x$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\tan(t)$ in terms of x.

94. If $\tan(\theta) = x$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sec(\theta)$ in terms of x.

95. If $\sec(\theta) = x$ where θ is a Quadrant II angle, find an expression for $\tan(\theta)$ in terns of x.

96. If $\sin(t) = \frac{x}{2}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\cos(2t)$ in terms of x.

97. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

98. If $\sec(t) = \frac{x}{4}$ for $0 < t < \frac{\pi}{2}$, find an expression for $\ln|\sec(t) + \tan(t)|$ in terms of x.

99. Show that $\cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$ for all θ.

100. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.

101. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

102. In part 4 of Example 8.2.3, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 84, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?

103. In Exercise 80, we has you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

In Exercises 104 - 109, verify the identity by graphing the right and left hand using a graphing utility.

104. $\sin^2(t) + \cos^2(t) = 1$

105. $\sec^2(x) - \tan^2(x) = 1$

106. $\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$

107. $\tan(x+\pi) = \tan(x)$

108. $\sin(2t) = 2\sin(t)\cos(t)$

109. $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$

In Exercises 110 - 115, write the given product as a sum. Note: you may need to use an Even/Odd Identity to match the answer provided.

110. $\cos(3\theta)\cos(5\theta)$
111. $\sin(2t)\sin(7t)$
112. $\sin(9x)\cos(x)$
113. $\cos(2\theta)\cos(6\theta)$
114. $\sin(3t)\sin(2t)$
115. $\cos(x)\sin(3x)$

In Exercises 116 - 121, write the given sum as a product. Note: you may need to use an Even/Odd or Cofunction Identity to match the answer provided.

116. $\cos(3\theta)+\cos(5\theta)$
117. $\sin(2t)-\sin(7t)$
118. $\cos(5x)-\cos(6x)$
119. $\sin(9\theta)-\sin(-\theta)$
120. $\sin(t)+\cos(t)$
121. $\cos(x)-\sin(x)$

In Exercises 122 - 125, using the remarks following Example 8.2.7 on page 723 as a guide, rewrite the given function $f(t)$ as a product of sinusoids. Identify the functions which create the 'wave envelope.' Check your answer by graphing the function along with the 'wave-envelope' using a graphing utility.

122. $f(t)=\cos(3t)+\cos(5t)$
123. $f(t)=3\cos(5t)-3\cos(6t)$
124. $f(t)=\frac{1}{2}\sin(9t)+\frac{1}{2}\sin(t)$
125. $f(t)=\frac{2}{3}\sin(2t)-\frac{2}{3}\sin(7t)$
126. Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.
127. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.
128. Verify the Difference Identities for sine and tangent.
129. Verify the Product to Sum Identities.
130. Verify the Sum to Product Identities.

SECTION 8.2 EXERCISE ANSWERS A.1.8

8.3 Solving Equations involving Trigonometric Functions

8.3.1 Solving Equations Using the Inverse Trigonometric Functions.

In Sections 7.2.2 and 7.4, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ and $\tan(t) = -1$. In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of 'common angles' listed on page 602.

If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or solve $\tan(t) = -2$ for real numbers t, we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations.

A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly, 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know there are answers, but we can't express them using 'friendly' numbers.

To solve $x^2 = 7$, we make use of the square root function (which is an inverse to $f(x) = x^2$ on a restricted domain) and write our answer as $x = \pm\sqrt{7}$. We need the $\pm$ to adjust for the fact that $\sqrt{7}$ is defined to be positive only, but we know we have two solutions, one positive and one negative. Using a calculator, we can certainly *approximate* the values $\pm\sqrt{7}$, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$.

In the same way, we will use the arcsine function (the inverse to the sine function on a restricted domain) to solve $\sin(\theta) = \frac{1}{3}$. However, we will need to adjust for the fact that there is more than one answer to this equation (infinitely many, in fact!) As it turns out, we will be able to express every solution in terms of $\arcsin\left(\frac{1}{3}\right)$, as our next example illustrates.

Example 8.3.1. Solve the following equations.

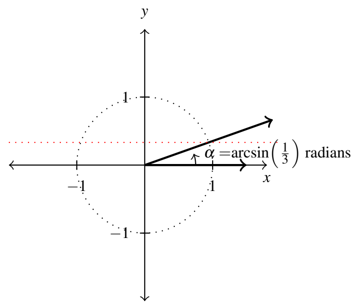
1. Determine all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Determine all real numbers t for which $\tan(t) = -2$.
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

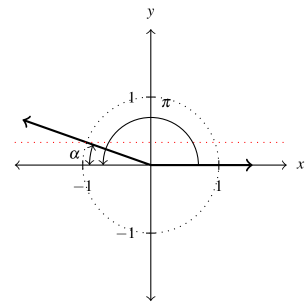
Solution.

1. Determine all angles θ for which $\sin(\theta) = \frac{1}{3}$.

   If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II.

   If we let α denote the acute solution to the equation, then all the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all of the solutions to this equation in Quadrant II as seen next.





Considering $\frac{1}{3}$ isn't the sine of any of the 'common angles' we've encountered, we use the arcsine functions to express our answers. By definition, there exists a real number $t = \arcsin\left(\frac{1}{3}\right)$ such that $\sin(t) = \frac{1}{3}$ with $0 < t < \frac{\pi}{2}$.

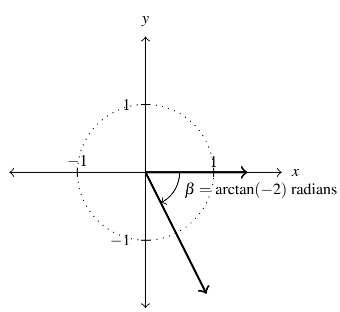
Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians is an acute angle with $\sin(\alpha) = \frac{1}{3}$. Because all of the Quadrant I solutions θ are all coterminal with α, we get $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k.

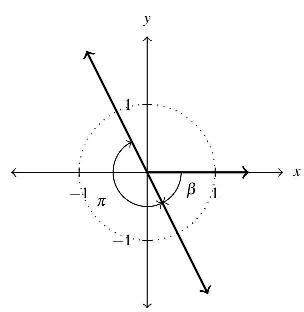
Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for integers k.

2. Determine all real numbers t for which $\tan(t) = -2$

   The real number solutions to $\tan(t) = -2$ correspond to angles θ with $\tan(\theta) = -2$. Due to the fact that tangent is negative only in Quadrants II and IV, we focus our efforts there.

   The real number $t = \arctan(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \arctan(-2)$ radians, then all of the Quadrant IV solutions to $\tan(\theta) = -2$ are coterminal with β.



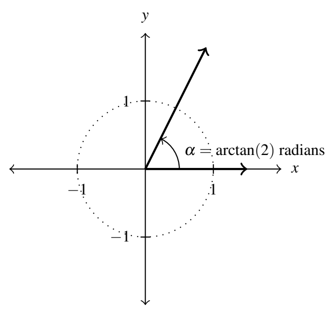


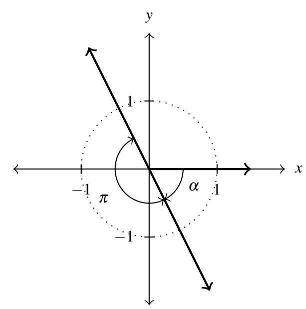
   Moreover, the solutions from Quadrant II differ by exactly π units from the solutions in Quadrant IV (recall, the period of the tangent function is π.) Hence, all of the solutions to $\tan(\theta) = -2$ are of the form $\theta = \beta + \pi k = \arctan(-2) + \pi k$ for some integer k. Switching back to the variable t, we record our final answer to $\tan(t) = -2$ as $t = \arctan(-2) + \pi k$ for integers k.

Another tact we could have taken to solve this problem is to use reference angles. Consider the (angle) equation: $\tan(\theta) = -2$. If we let α be the reference angle for the solutions θ, we know α is an acute angle with $\tan(\alpha) = 2$.

By definition, the real number $t = \arctan(2)$ satisfies $0 < t < \frac{\pi}{2}$ with $\tan(t) = 2$. Hence, the angle $\alpha = \arctan(2)$ radians is the reference angle for our solutions to $\tan(\theta) = -2$.

Adjusting for quadrants, we get our answers to $\tan(\theta) = -2$ are $\theta = -\alpha + \pi k = -\arctan(2) + \pi k$ for integers k. Again, we cosmetically change the variable from θ back to t so our answer to $\tan(t) = -2$ is $t = -\arctan(2) + \pi k$. Thanks to the odd property of arctangent, $\arctan(-2) = -\arctan(2)$ and we see this family of solutions is identical to what we obtained earlier.

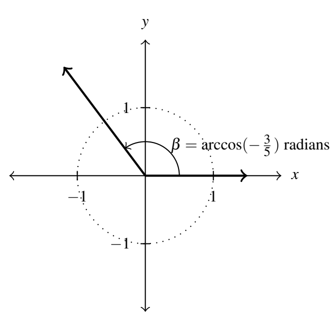


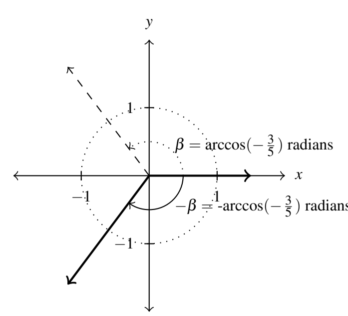


3. Solve $\sec(x) = -\frac{5}{3}$ for x.

   In the last equation, $\sec(x) = -\frac{5}{3}$, we are not told whether or not x represents an angle or a real number. This isn't really much of an issue, as we attack both problems the same way.

   Taking a cue from our work in Section 7.4 and use a Reciprocal Identity to convert the equation $\sec(x) = -\frac{5}{3}$ to $\cos(x) = -\frac{3}{5}$. Thinking geometrically, we are looking for angles θ with $\cos(\theta) = -\frac{3}{5}$. As $\cos(\theta) < 0$, we are looking for solutions in Quadrants II and III. Due to the fact that $-\frac{3}{5}$ isn't the cosine of any of the 'common angles', we'll need to express our solutions in terms of the arccosine function.





The real number $t = \arccos\left(-\frac{3}{5}\right)$ is defined so that $\frac{\pi}{2} < t < \pi$ with $\cos(t) = -\frac{3}{5}$. Hence, the angle $\beta = \arccos\left(-\frac{3}{5}\right)$ radians is a Quadrant II angle which satisfies $\cos(\beta) = -\frac{3}{5}$. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\arccos\left(-\frac{3}{5}\right)$.

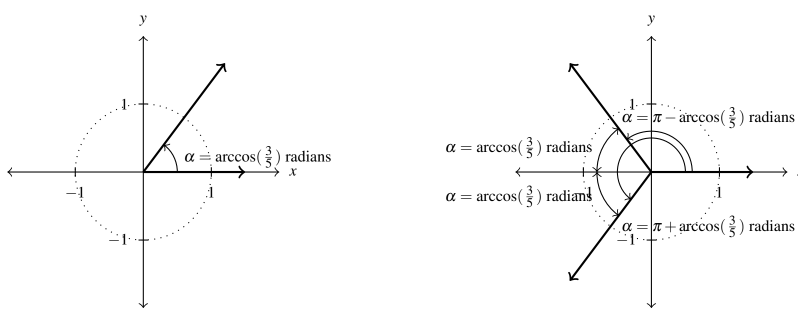
All angle solutions are coterminal with β or $-\beta$, so we get our solutions to $\cos(\theta) = -\frac{3}{5}$ to be $\theta = \beta + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $\theta = -\beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

Switching back to the variable x, we record our final answer to $\sec(x) = -\frac{5}{3}$ as $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

As with the previous problem, we can approach solving $\cos(\theta) = -\frac{3}{5}$ using reference angles. Letting α represent the reference angle for the solutions θ, we know α is an acute angle with $\cos(\alpha) = \frac{3}{5}$.

We know the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, hence $\alpha = \arccos\left(\frac{3}{5}\right)$ radians is the reference angle for the solutions to $\cos(\theta) = -\frac{3}{5}$.

Hence, the Quadrant II solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi - \alpha + 2\pi k = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ while the Quadrant IV solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi + \alpha + 2\pi k = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k.



Shifting back to the variable x, we get our solution to $\sec(x) = -\frac{5}{3}$ are $x = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ or $x = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k.

While these certainly look quite different than what we obtained before, $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k, they are, in fact, equivalent. To show this, we start with $\arccos\left(-\frac{3}{5}\right) = \pi - \arccos\left(\frac{3}{5}\right)$ and begin writing out specific solutions from each family by choosing specific values of k. We leave these details to the reader. $\square$

Example 8.3.2. Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Write a formula for $f(t)$:

1. in the form $E(t) = A\cos(Bt + C) + D$ for $B > 0$
2. in the form $S(t) = A\sin(Bt + C) + D$ for $B > 0$

Solution.

1. Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Write a formula for $f(t)$ in the form $E(t) = A\cos(Bt + C) + D$ for $B > 0$.

   As in Example 8.2.7, we compare the expanded form of $E(t) = A\cos(Bt)\cos(C) - A\sin(Bt)\sin(C) + D$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. We identify $B = 6$ and $D = 0$ and by equating coefficients of $\cos(6t)$ and $\sin(6t)$ get the two equations: $A\cos(C) = 3$ and $A\sin(C) = 4$.

   Using the Pythagorean Identity to eliminate C, we get $A^2 = (A\cos(C))^2 + (A\sin(C))^2 = 3^2 + 4^2 = 25$. We choose $A = 5$ and work to find C.

   Substituting $A = 5$ into our two equations relating A and C, we get $5\cos(C) = 3$, or $\cos(C) = \frac{3}{5}$ and $5\sin(C) = 4$, so $\sin(C) = \frac{4}{5}$. As both $\sin(C)$ and $\cos(C)$ are positive, we know C is a Quadrant I angle. However, because neither the sine nor cosine value of C corresponds to a common angle, we need to express C in terms of either an arcsine or arccosine.

   As the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, we know the angle $C = \arccos\left(\frac{3}{5}\right)$ radians is an acute (Quadrant I) angle which satisfies $\cos(C) = \frac{3}{5}$. Hence, we can take $C = \arccos\left(\frac{3}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arccos\left(\frac{3}{5}\right)\right)$.

   In addition, the real number $t = \arcsin\left(\frac{4}{5}\right)$ satisfies $\sin(t) = \frac{4}{5}$ and $0 < t < \frac{\pi}{2}$. Hence $C = \arcsin\left(\frac{4}{5}\right)$ radians is Quadrant I angle with $\sin(C) = \frac{4}{5}$. This means we could also take $C = \arcsin\left(\frac{4}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arcsin\left(\frac{4}{5}\right)\right)$. (We could also express C in terms of arctangents, if we wanted!)

   We leave it to the reader to verify (both) solutions analytically and graphically.

2. Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Write a formula for $f(t)$ in the form $S(t) = A\sin(Bt + C) + D$ for $B > 0$.

   Once again, we equate the expanded form of $S(t) = A\sin(Bt)\cos(C) + A\cos(Bt)\sin(C) + B$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. Once again, we get $B = 6$ and $D = 0$. Here, our two equations for A and C are $A\cos(C) = -4$ and $A\sin(C) = 3$.

   As before, we get $A^2 = (A\cos(C))^2 + (A\sin(C))^2 = (-4)^2 + 3^2 = 25$, and we choose $A = 5$. Our equations for C become: $\cos(C) = -\frac{4}{5}$ and $\sin(C) = \frac{3}{5}$. Because $\cos(C) < 0$ but $\sin(C) > 0$, we know C is a Quadrant II angle. As before, neither the sine nor cosine value of C corresponds to a common angle, so we need to express C in terms of either an arcsine or arccosine.

   Here, we opt to use the arccosine function, because the range of arccosine, $[0, \pi]$ covers Quadrant II. From $\cos(C) = -\frac{4}{5}$, we get $C = \arccos\left(-\frac{4}{5}\right)$, so $f(t) = 5\sin\left(6t + \arccos\left(-\frac{4}{5}\right)\right)$.

   Had we chosen to work with arcsines, we would need a Quadrant II solution to $\sin(C) = \frac{3}{5}$. Going through the usual machinations, we arrive at $C = \pi - \arcsin\left(\frac{3}{5}\right)$. Hence, an alternative form of our answer is $f(t) = 5\sin\left(6t + \pi - \arcsin\left(\frac{3}{5}\right)\right)$. We leave the checks to the reader. $\square$

8.3.2 Strategies for Solving Equations Involving Circular Functions

In Sections 7.2.2, 7.4 and in the previous subsection, we solved some basic equations involving the trigonometric functions. Below we summarize the techniques we've employed thus far. Note that we use the neutral letter 'u' as the argument of each circular function for generality.

Strategies for Solving Basic Equations Involving the Circular Functions

- To solve $\cos(u) = c$ or $\sin(u) = c$ for $-1 \leq c \leq 1$, first solve for u in the interval $[0, 2\pi)$ and add integer multiples of the period 2π. If $c < -1$ or if $c > 1$, there are no real solutions.
- To solve $\sec(u) = c$ or $\csc(u) = c$ for $c \leq -1$ or $c \geq 1$, convert to cosine or sine, respectively, and solve as above. If $-1 < c < 1$, there are no real solutions.
- To solve $\tan(u) = c$ for any real number c, first solve for u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and add integer multiples of the period π.
- To solve $\cot(u) = c$ for $c \neq 0$, convert to tangent and solve as above. If $c = 0$, the solution to $\cot(u) = 0$ is $u = \frac{\pi}{2} + \pi k$ for integers k.

Using the above guidelines, we can comfortably solve $\sin(x) = \frac{1}{2}$ and find the solution $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. But how do we solve the related equation $\sin(3x) = \frac{1}{2}$?

This equation has the *form* $\sin(u) = \frac{1}{2}$, so we know the solutions take the form $u = \frac{\pi}{6} + 2\pi k$ or $u = \frac{5\pi}{6} + 2\pi k$ for integers k. Because the argument of sine here is $3x$, we have $3x = \frac{\pi}{6} + 2\pi k$ or $3x = \frac{5\pi}{6} + 2\pi k$.

To solve for x, we divide both sides[1] of these equations by 3, and obtain $x = \frac{\pi}{18} + \frac{2\pi}{3}k$ or $x = \frac{5\pi}{18} + \frac{2\pi}{3}k$ for integers k. This is the technique employed in the example below.

Example 8.3.3. Solve the following equations and check your answers analytically. List the solutions which lie in the interval $[0, 2\pi)$ and verify them using a graphing utility.

1. $\cos(2\theta) = -\frac{\sqrt{3}}{2}$
2. $\csc\left(\frac{1}{3}\theta - \pi\right) = \sqrt{2}$
3. $\cot(3t) = 0$
4. $\sec^2(t) = 4$
5. $\tan\left(\frac{x}{2}\right) = -3$
6. $\sin(2x) = 0.87$

Solution.

1. Solve $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ for θ. List the solutions which lie in the interval $[0, 2\pi)$.

   The solutions to $\cos(u) = -\frac{\sqrt{3}}{2}$ are $u = \frac{5\pi}{6} + 2\pi k$ or $u = \frac{7\pi}{6} + 2\pi k$ for integers k.

[1] Don't forget to divide the $2\pi k$ by 3 as well!

As the argument of cosine here is 2θ, this means $2\theta = \frac{5\pi}{6} + 2\pi k$ or $2\theta = \frac{7\pi}{6} + 2\pi k$ for integers k. Solving for θ gives $\theta = \frac{5\pi}{12} + \pi k$ or $\theta = \frac{7\pi}{12} + \pi k$ for integers k.

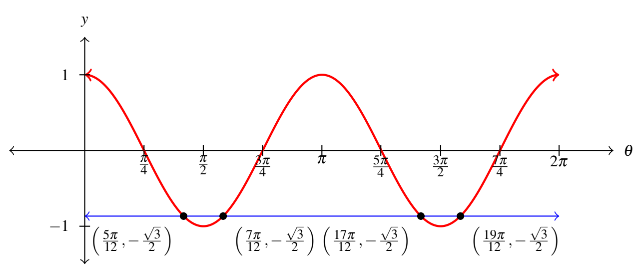
To check these answers analytically, we substitute them into the original equation. For any integer k:

$$\begin{aligned} \cos\left(2\left[\frac{5\pi}{12} + \pi k\right]\right) &= \cos\left(\frac{5\pi}{6} + 2\pi k\right) \\ &= \cos\left(\frac{5\pi}{6}\right) \qquad \text{(the period of cosine is } 2\pi) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

Similarly, we find $\cos\left(2\left[\frac{7\pi}{12} + \pi k\right]\right) = \cos\left(\frac{7\pi}{6} + 2\pi k\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

To determine which of our solutions lie in $[0, 2\pi)$, we substitute integer values for k. The solutions we keep come from the values of $k = 0$ and $k = 1$ and are $\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$ and $\frac{19\pi}{12}$.

Using technology, we graph $y = \cos(2\theta)$ and $y = -\frac{\sqrt{3}}{2}$ over $[0, 2\pi)$ and examine where these two graphs intersect to verify our answers.



$y = \cos(2\theta)$ and $y = -\frac{\sqrt{3}}{2}$

2. Solve $\csc\left(\frac{1}{3}\theta - \pi\right) = \sqrt{2}$ for θ. List the solutions which lie in the interval $[0, 2\pi)$.

   This equation has the form $\csc(u) = \sqrt{2}$, so we rewrite it as $\sin(u) = \frac{\sqrt{2}}{2}$ and find $u = \frac{\pi}{4} + 2\pi k$ or $u = \frac{3\pi}{4} + 2\pi k$ for integers k.

   The argument of cosecant here is $\left(\frac{1}{3}\theta - \pi\right)$, thus $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$ or $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$.

   To solve $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$, we first add π to both sides to get $\frac{1}{3}\theta = \frac{\pi}{4} + 2\pi k + \pi$. A common error is to treat the '$2\pi k$' and 'π' terms as 'like' terms and try to combine them when they are not.

   We can, however, combine the 'π' and '$\frac{\pi}{4}$' terms to get $\frac{1}{3}\theta = \frac{5\pi}{4} + 2\pi k$.

   We now finish by multiplying both sides by 3 to get $\theta = 3\left(\frac{5\pi}{4} + 2\pi k\right) = \frac{15\pi}{4} + 6\pi k$, where k, as always, runs through the integers.

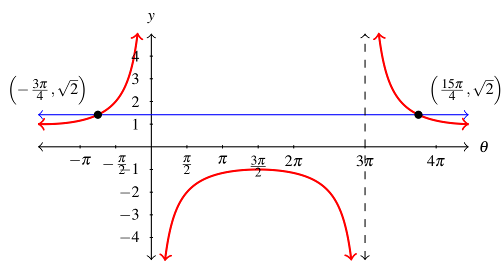
Solving the other equation, $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$ produces $\theta = \frac{21\pi}{4} + 6\pi k$ for integers k. To check the first family of answers, we substitute, combine like terms, and simplify.

$$\begin{array}{rcll} \csc\left(\frac{1}{3}\left[\frac{15\pi}{4} + 6\pi k\right] - \pi\right) & = & \csc\left(\frac{5\pi}{4} + 2\pi k - \pi\right) & \\ & = & \csc\left(\frac{\pi}{4} + 2\pi k\right) & \\ & = & \csc\left(\frac{\pi}{4}\right) & \text{(the period of cosecant is } 2\pi) \\ & = & \sqrt{2} & \end{array}$$

The family $\theta = \frac{21\pi}{4} + 6\pi k$ checks similarly.

Despite having infinitely many solutions, we find that *none* of them lie in $[0, 2\pi)$.

To verify this graphically, we check that $y = \csc\left(\frac{1}{3}\theta - \pi\right)$ and $y = \sqrt{2}$ do not intersect at all over the interval $[0, 2\pi)$.



$y = \csc\left(\frac{1}{3}\theta - \pi\right)$ and $y = \sqrt{2}$

3. Solve $\cot(3t) = 0$ for t. List the solutions which lie in the interval $[0, 2\pi)$.

   Because $\cot(3t) = 0$ has the form $\cot(u) = 0$, we know $u = \frac{\pi}{2} + \pi k$, so, in this case, $3t = \frac{\pi}{2} + \pi k$ for integers k.

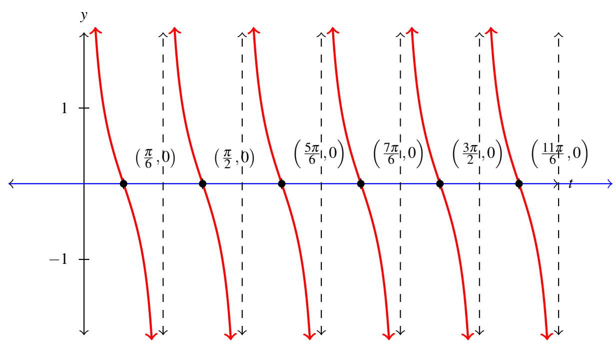
   Solving for t yields $t = \frac{\pi}{6} + \frac{\pi}{3}k$. Checking our answers, we get

$$\begin{array}{rcll} \cot\left(3\left[\frac{\pi}{6} + \frac{\pi}{3}k\right]\right) & = & \cot\left(\frac{\pi}{2} + \pi k\right) & \\ & = & \cot\left(\frac{\pi}{2}\right) & \text{(the period of cotangent is } \pi) \\ & = & 0 & \end{array}$$

   As k runs through the integers, we obtain six answers, corresponding to $k = 0$ through $k = 5$, which lie in $[0, 2\pi)$: $x = \frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$ and $\frac{11\pi}{6}$.

   Graphing $y = \cot(3t)$ and $y = 0$ (the t-axis), we confirm our result.[2]

[2] On many calculators, there is no function button for cotangent. In that case, we would use the quotient identity and graph $y = \frac{\cos(3t)}{\sin(3t)}$ instead. The reader is invited to see what happens if we would graph $y = \frac{1}{\tan(3t)}$ instead.



$y = \cot(3t)$ and $y = 0$

4. Solve $\sec^2(t) = 4$ for t. List the solutions which lie in the interval $[0, 2\pi)$.

   The complication in solving an equation like $\sec^2(t) = 4$ comes not from the argument of secant, which is just t, but rather, the fact the secant is being squared: $\sec^2(t) = (\sec(t))^2 = 4$.

   To get this equation to look like one of the forms listed on page 735, we extract square roots to get $\sec(t) = \pm 2$. Converting to cosines, we have $\cos(t) = \pm\frac{1}{2}$.

   For $\cos(t) = \frac{1}{2}$, we get $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. For $\cos(t) = -\frac{1}{2}$, we get $t = \frac{2\pi}{3} + 2\pi k$ or $t = \frac{4\pi}{3} + 2\pi k$ for integers k.

   If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of $\frac{\pi}{3}$.

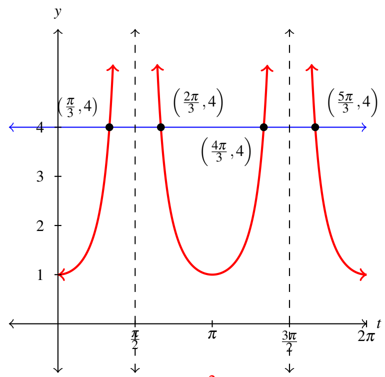
   As a result, these solutions can be combined and we may write our solutions as $t = \frac{\pi}{3} + \pi k$ and $t = \frac{2\pi}{3} + \pi k$ for integers k.

   To check the first family of solutions, we note that, depending on the integer k, $\sec\left(\frac{\pi}{3} + \pi k\right)$ doesn't always equal $\sec\left(\frac{\pi}{3}\right)$. It is true, though, that for all integers k, $\sec\left(\frac{\pi}{3} + \pi k\right) = \pm\sec\left(\frac{\pi}{3}\right) = \pm 2$. (Can you show this?) Hence, checking our first family of solutions gives:

$$\begin{aligned} \sec^2\left(\frac{\pi}{3} + \pi k\right) &= \left(\pm\sec\left(\frac{\pi}{3}\right)\right)^2 \\ &= (\pm 2)^2 \\ &= 4 \end{aligned}$$

   The check for the family of solutions $t = \frac{2\pi}{3} + \pi k$ is similar.

   The solutions which lie in $[0, 2\pi)$ come from the values $k = 0$ and $k = 1$, namely $t = \frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$. Graphing $y = (\sec(t))^2$ and $y = 4$ confirms our results.



$y=(\sec(t))^2$ and $y=4$

5. Solve $\tan\left(\frac{x}{2}\right)=-3$ for x. List the solutions which lie in the interval $[0,2\pi)$.

The equation $\tan\left(\frac{x}{2}\right)=-3$ has the form $\tan(u)=-3$, whose solution is $u=\arctan(-3)+\pi k$.

Hence, $\frac{x}{2}=\arctan(-3)+\pi k$, so $x=2\arctan(-3)+2\pi k$ for integers k. To check, we note

$$\begin{aligned}\tan\left(\frac{2\arctan(-3)+2\pi k}{2}\right) &= \tan\left(\arctan(-3)+\pi k\right) \\ &= \tan\left(\arctan(-3)\right) && \text{(the period of tangent is } \pi) \\ &= -3 && \text{(See Theorem 7.16)}\end{aligned}$$

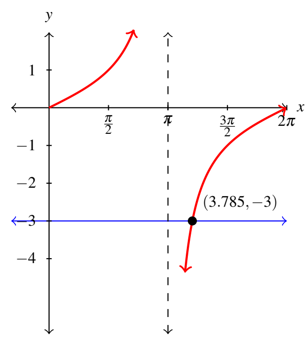
To determine which of our answers lie in the interval $[0,2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator, we proceed analytically, as is our custom.

To get started, we note that because $-3<0$, it $-\frac{\pi}{2}<\arctan(-3)<0$. Hence, $-\pi<2\arctan(-3)<0$. With regard to our solutions, $x=2\arctan(-3)+2\pi k$, we see for $k=0$, we get $x=2\arctan(-3)<0$, so we discard this answer and all answers $x=2\arctan(-3)+2\pi k$ where $k<0$.

Next, we turn our attention to $k=1$ and get $x=2\arctan(-3)+2\pi$. Starting with the inequality $-\pi<2\arctan(-3)<0$, we add through 2π and get $\pi<2\arctan(-3)+2\pi<2\pi$. This means $x=2\arctan(-3)+2\pi$ lies in $[0,2\pi)$.

Advancing k to 2 produces $x=2\arctan(-3)+4\pi$. Once again, we get from $-\pi<2\arctan(-3)<0$ that $3\pi<2\arctan(-3)+4\pi<4\pi$. This is outside the interval of interest, $[0,2\pi)$, so we discard $x=2\arctan(-3)+4\pi$ and all solutions of the form $x=2\arctan(-3)+2\pi k$ for $k>2$.

Graphically, $y=\tan\left(\frac{x}{2}\right)$ and $y=-3$ intersect only once on $[0,2\pi)$ at $x=2\arctan(-3)+2\pi\approx 3.785$.



$y = \tan\left(\frac{x}{2}\right)$ and $y = -3$

6. Solve $\sin(2x) = 0.87$ for x. List the solutions which lie in the interval $[0, 2\pi)$.

   To solve $\sin(2x) = 0.87$, we first note that it has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$ for integers k.

   The argument of sine here is $2x$, thus we get $2x = \arcsin(0.87) + 2\pi k$ or $2x = \pi - \arcsin(0.87) + 2\pi k$ which gives $x = \frac{1}{2}\arcsin(0.87) + \pi k$ or $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. To check,

$$\begin{array}{rcll} \sin\left(2\left[\frac{1}{2}\arcsin(0.87) + \pi k\right]\right) & = & \sin\left(\arcsin(0.87) + 2\pi k\right) & \\ & = & \sin\left(\arcsin(0.87)\right) & \text{(the period of sine is } 2\pi) \\ & = & 0.87 & \text{(See Theorem 7.15)} \end{array}$$

   For the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$, we get

$$\begin{array}{rcll} \sin\left(2\left[\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k\right]\right) & = & \sin\left(\pi - \arcsin(0.87) + 2\pi k\right) & \\ & = & \sin\left(\pi - \arcsin(0.87)\right) & \text{(the period of sine is } 2\pi) \\ & = & \sin\left(\arcsin(0.87)\right) & (\sin(\pi - t) = \sin(t)) \\ & = & 0.87 & \text{(See Theorem 7.15)} \end{array}$$

   To determine which of these solutions lie in $[0, 2\pi)$, we first need to get an idea of the value of $x = \frac{1}{2}\arcsin(0.87)$. Once again, we could use the calculator, but we adopt an analytic route here.

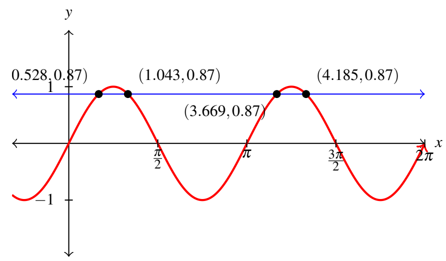
   By definition, $0 < \arcsin(0.87) < \frac{\pi}{2}$ so that multiplying through by $\frac{1}{2}$ gives us $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$.

   Starting with the family of solutions $x = \frac{1}{2}\arcsin(0.87) + \pi k$, we use the same kind of arguments as in our solution to number 5 above and find only the solutions corresponding to $k = 0$ and $k = 1$ lie in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87)$ and $x = \frac{1}{2}\arcsin(0.87) + \pi$.

   Next, we move to the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. Here, we need to get a better estimate of $\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$. From the inequality $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$, we first multiply through by -1 and then add $\frac{\pi}{2}$ to get $\frac{\pi}{2} > \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) > \frac{\pi}{4}$, or $\frac{\pi}{4} < \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) < \frac{\pi}{2}$.

Proceeding with the usual arguments, we find the only solutions which lie in $[0,2\pi)$ correspond to $k=0$ and $k=1$, namely $x=\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)$ and $x=\frac{3\pi}{2}-\frac{1}{2}\arcsin(0.87)$.

All told, we have found four solutions to $\sin(2x)=0.87$ in $[0,2\pi)$: $x=\frac{1}{2}\arcsin(0.87)\approx 0.528$, $x=\frac{1}{2}\arcsin(0.87)+\pi\approx 3.669$, $x=\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)\approx 1.043$ and $x=\frac{3\pi}{2}-\frac{1}{2}\arcsin(0.87)\approx 4.185$. By graphing $y=\sin(2x)$ and $y=0.87$, we confirm our results.



$y=\sin(2x)$ and $y=0.87$

$\square$

If one looks closely at the equations and solutions in Example 8.3.3, an interesting relationship evolves between the frequency of the circular function involved in the equation and how many solutions one can expect in the interval $[0,2\pi)$. This relationship is explored in Exercise 89.

Each of the problems in Example 8.3.3 featured one circular function. If an equation involves two different circular functions or if the equation contains the same circular function but with different arguments, we will need to employ identities and Algebra to reduce the equation to the same form as those given on page 735. We demonstrate these techniques in the following example.

Example 8.3.4. Solve the following equations and list the solutions which lie in the interval $[0,2\pi)$. Verify your solutions on $[0,2\pi)$ graphically.

1. $3\sin^3(\theta)=\sin^2(\theta)$
2. $\sec^2(\theta)=\tan(\theta)+3$
3. $\cos(2t)=3\cos(t)-2$
4. $\cos(3x)=\cos(5x)$
5. $\sin(2x)=\sqrt{3}\cos(x)$
6. $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$
7. $\cos(x)-\sqrt{3}\sin(x)=2$

Solution.

1. Solve $3\sin^3(\theta)=\sin^2(\theta)$ and list the solutions which lie in the interval $[0,2\pi)$.

   One approach to solving $3\sin^3(\theta)=\sin^2(\theta)$ begins with dividing both sides by $\sin^2(\theta)$. Doing so, however, assumes that $\sin^2(\theta)\neq 0$ which means we risk losing solutions.

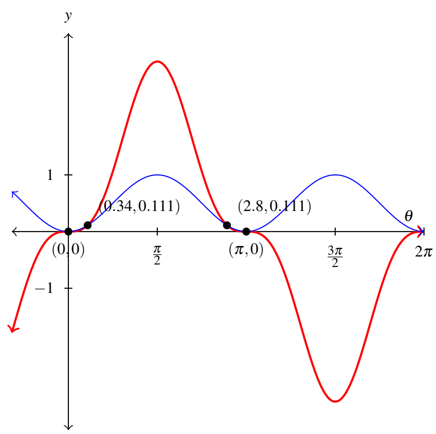
Instead, we take a cue from Chapter 2 (due to the fact that what we have here is a polynomial equation in terms of $\sin(\theta)$) and gather all the nonzero terms on one side and factor:

$$\begin{array}{rcll} 3\sin^3(\theta) & = & \sin^2(\theta) & \\ 3\sin^3(\theta)-\sin^2(\theta) & = & 0 & \\ \sin^2(\theta)(3\sin(\theta)-1) & = & 0 & \text{Factor out } \sin^2(\theta) \text{ from both terms.} \end{array}$$

We get $\sin^2(\theta)=0$ or $3\sin(\theta)-1=0$, so $\sin(\theta)=0$ or $\sin(\theta)=\frac{1}{3}$. The solution to $\sin(\theta)=0$ is $\theta=\pi k$, with $\theta=0$ and $\theta=\pi$ being the two solutions which lie in $[0,2\pi)$.

To solve $\sin(\theta)=\frac{1}{3}$, we use the arcsine function to get $\theta=\arcsin\left(\frac{1}{3}\right)+2\pi k$ or $\theta=\pi-\arcsin\left(\frac{1}{3}\right)+2\pi k$ for integers k. We find the two solutions here which lie in $[0,2\pi)$ to be $\theta=\arcsin\left(\frac{1}{3}\right)\approx 0.34$ and $\theta=\pi-\arcsin\left(\frac{1}{3}\right)\approx 2.80$.

To check graphically, we plot $y=3(\sin(\theta))^3$ and $y=(\sin(\theta))^2$ and identify the θ-coordinates of the intersection points of these two curves.[3] (Some extra zooming may be required near $\theta=0$ and $\theta=\pi$ to verify that these two curves do in fact intersect four times.)



$y=3(\sin(\theta))^3$ and $y=(\sin(\theta))^2$

2. Solve $\sec^2(\theta)=\tan(\theta)+3$ and list the solutions which lie in the interval $[0,2\pi)$.

   We see immediately in the equation $\sec^2(\theta)=\tan(\theta)+3$ that there are two different circular functions present, so we look for an identity to express both sides in terms of the same function.

   We use the Pythagorean Identity $\sec^2(\theta)=1+\tan^2(\theta)$ to exchange $\sec^2(\theta)$ for tangents. What results is a quadratic in disguise:[4]

$$\begin{array}{rcll} \sec^2(\theta) & = & \tan(\theta)+3 & \\ 1+\tan^2(\theta) & = & \tan(\theta)+3 & (\text{Given } \sec^2(\theta)=1+\tan^2(\theta).) \end{array}$$

[3] Note that we do *not* list $\theta=2\pi$ as part of the solution over the interval $[0,2\pi)$ because 2π is not in $[0,2\pi)$.

[4] See Section 0.5.5 for a review of this concept.

$$\begin{array}{rcll}\tan^2(\theta)-\tan(\theta)-2 & = & 0 & \\ u^2-u-2 & = & 0 & \text{Let } u=\tan(\theta). \\ (u+1)(u-2) & = & 0 & \end{array}$$

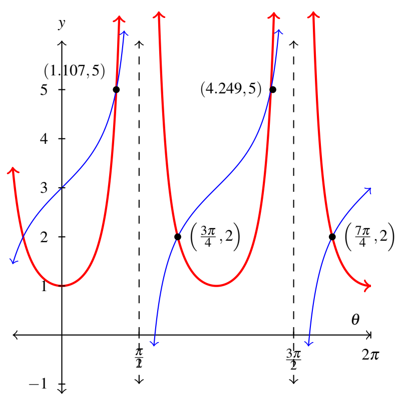
This gives $u=-1$ or $u=2$. As $u=\tan(\theta)$, we have $\tan(\theta)=-1$ or $\tan(\theta)=2$.

From $\tan(\theta)=-1$, we get $\theta=-\frac{\pi}{4}+\pi k$ for integers k. To solve $\tan(\theta)=2$, we employ the arctangent function and get $\theta=\arctan(2)+\pi k$ for integers k.

From the first set of solutions, we get $\theta=\frac{3\pi}{4}$ and $\theta=\frac{7\pi}{4}$ as our answers which lie in $[0,2\pi)$.

Using the same sort of argument we saw in Example 8.3.3, we get $\theta=\arctan(2)\approx 1.107$ and $\theta=\pi+\arctan(2)\approx 4.249$ as answers from our second set of solutions which lie in $[0,2\pi)$.

We verify our solutions below graphically.



$y=(\sec(\theta))^2$ and $y=\tan(\theta)+3$

3. Solve $\cos(2t)=3\cos(t)-2$ and list the solutions which lie in the interval $[0,2\pi)$.

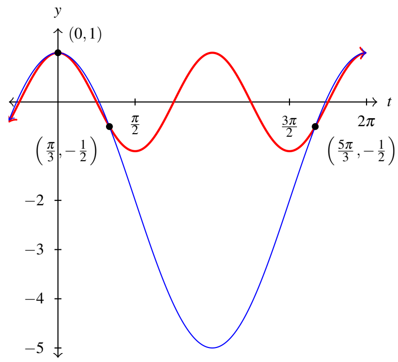
   The good news is that in the equation $\cos(2t)=3\cos(t)-2$, we have the same circular function, cosine, throughout. The bad news is that we have different arguments, $2t$ and t.

   Using the double angle identity $\cos(2t)=2\cos^2(t)-1$ results in another quadratic in disguise:

$$\begin{array}{rcll}\cos(2t) & = & 3\cos(t)-2 & \\ 2\cos^2(t)-1 & = & 3\cos(t)-2 & (\cos(2t)=2\cos^2(t)-1) \\ 2\cos^2(t)-3\cos(t)+1 & = & 0 & \\ 2u^2-3u+1 & = & 0 & \text{Let } u=\cos(t). \\ (2u-1)(u-1) & = & 0 & \end{array}$$

   We get $u=\frac{1}{2}$ or $u=1$, so $\cos(t)=\frac{1}{2}$ or $\cos(t)=1$. Solving $\cos(t)=\frac{1}{2}$, we get $t=\frac{\pi}{3}+2\pi k$ or $t=\frac{5\pi}{3}+2\pi k$ for integers k. From $\cos(t)=1$, we get $t=2\pi k$ for integers k.

The answers which lie in $[0,2\pi)$ are $t=0$, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$. Graphing $y=\cos(2t)$ and $y=3\cos(t)-2$, we find that the curves intersect in three places on $[0,2\pi)$ and confirm our results.



$y=\cos(2t)$ and $y=3\cos(t)-2$

4. Solve $\cos(3x)=\cos(5x)$ and list the solutions which lie in the interval $[0,2\pi)$.

While we could approach solving the equation $\cos(3x)=\cos(5x)$ in the same manner as we did the previous two problems, we choose instead to showcase the utility of the Sum to Product Identities.[5]

From $\cos(3x)=\cos(5x)$, we get $\cos(5x)-\cos(3x)=0$, and it is the presence of 0 on the right hand side that indicates a switch to a product would be a good move.[6]

Using Theorem 8.13, we rewrite $\cos(5x)-\cos(3x)$ as $-2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)=-2\sin(4x)\sin(x)$. Hence, our original equation $\cos(3x)=\cos(5x)$ is equivalent to $-2\sin(4x)\sin(x)=0$.

From $-2\sin(4x)\sin(x)=0$, we get either $\sin(4x)=0$ or $\sin(x)=0$. Solving $\sin(4x)=0$ gives $x=\frac{\pi}{4}k$ for integers k, and the solution to $\sin(x)=0$ is $x=\pi k$ for integers k.

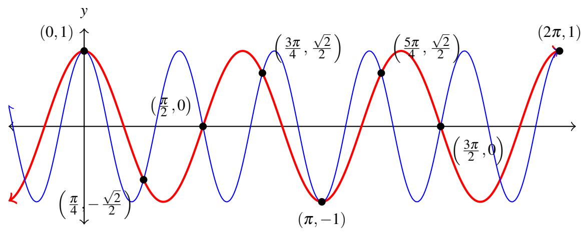
The second set of solutions is contained in the first set of solutions,[7] so our final solution to $\cos(5x)=\cos(3x)$ is $x=\frac{\pi}{4}k$ for integers k.

There are eight of these answers which lie in $[0,2\pi)$: $x=0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{4}$. Our plot of the graphs of $y=\cos(3x)$ and $y=\cos(5x)$ below (after some careful zooming) bears this out.

[5] We invite the reader to try the 'polynomial approach' used in the previous problem to see what difficulties are encountered.

[6] A *product* equalling zero means, necessarily, one or both *factors* is 0. See page 0.1.1.

[7] As always, when in doubt, write it out!



$y=\cos(3x)$ and $y=\cos(5x)$

5. Solve $\sin(2x)=\sqrt{3}\cos(x)$ and list the solutions which lie in the interval $[0,2\pi)$.

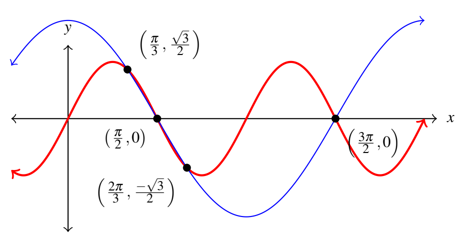
   In the equation $\sin(2x)=\sqrt{3}\cos(x)$, we not only have different circular functions involved, but we also have different arguments to contend with.

   Using the double angle identity $\sin(2x)=2\sin(x)\cos(x)$ makes all of the arguments the same and we proceed to gather all of the nonzero terms on one side of the equation and factor.

$$\begin{array}{rcll}\sin(2x) & = & \sqrt{3}\cos(x) & \\ 2\sin(x)\cos(x) & = & \sqrt{3}\cos(x) & (\sin(2x)=2\sin(x)\cos(x)) \\ 2\sin(x)\cos(x)-\sqrt{3}\cos(x) & = & 0 & \\ \cos(x)(2\sin(x)-\sqrt{3}) & = & 0 & \end{array}$$

   We get $\cos(x)=0$ or $\sin(x)=\frac{\sqrt{3}}{2}$. From $\cos(x)=0$, we obtain $x=\frac{\pi}{2}+\pi k$ for integers k. From $\sin(x)=\frac{\sqrt{3}}{2}$, we get $x=\frac{\pi}{3}+2\pi k$ or $x=\frac{2\pi}{3}+2\pi k$ for integers k.

   The answers which lie in $[0,2\pi)$ are $x=\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, as verified graphically below.



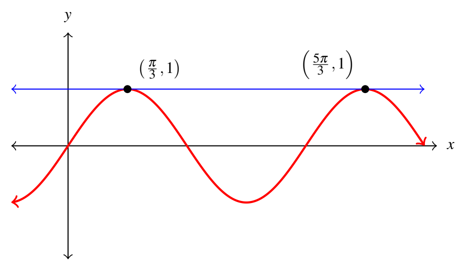
$y=\sin(2x)$ and $y=\sqrt{3}\cos(x)$

6. Solve $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$ and list the solutions which lie in the interval $[0,2\pi)$.

   Unlike the previous problem, there seems to be no quick way to get the circular functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$.

   If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for $\sin\left(x+\frac{x}{2}\right)$. Hence, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right)=1$.

Solving, we find $x = \frac{\pi}{3} + \frac{4\pi}{3}k$ for integers k. Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Graphing $y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$ validates our solutions.



$y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$

7. Solve $\cos(x) - \sqrt{3}\sin(x) = 2$ and list the solutions which lie in the interval $[0, 2\pi)$.

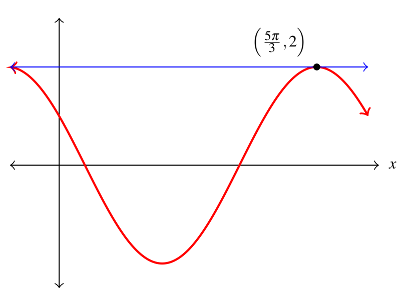
   With the absence of double angles or squares, there doesn't seem to be much we can do with the equation $\cos(x) - \sqrt{3}\sin(x) = 2$.

   However, as the frequencies of the sine and cosine terms are the same, we can rewrite the left hand side of this equation as a sinusoid.

   To fit $f(x) = \cos(x) - \sqrt{3}\sin(x)$ to the form $A\sin(Bt + C) + D$, we use what we learned in Example 8.2.7 and find $A = 2$, $D = 0$, $B = 1$ and $C = \frac{5\pi}{6}$.

   Hence, we can rewrite the equation $\cos(x) - \sqrt{3}\sin(x) = 2$ as $2\sin\left(x + \frac{5\pi}{6}\right) = 2$, or $\sin\left(x + \frac{5\pi}{6}\right) = 1$. Solving, we get $x = -\frac{\pi}{3} + 2\pi k$ for integers k.

   Only one of our solutions, $x = \frac{5\pi}{3}$, which corresponds to $k = 1$, lies in $[0, 2\pi)$. Geometrically, we see that $y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$ intersect just once, supporting our answer.



$y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$

An alternative way to solve this problem is to *introduce* squares in order to exchange sines and cosines using a Pythagorean Identity.

From $\cos(x) - \sqrt{3}\sin(x) = 2$ we get $\sqrt{3}\sin(x) = \cos(x) - 2$ so that $\left(\sqrt{3}\sin(x)\right)^2 = (\cos(x)-2)^2$. Simplifying, we get: $3\sin^2(x) = \cos^2(x) - 4\cos(x) + 4$.

Substituting $\sin^2(x) = 1 - \cos^2(x)$, we get $3(1-\cos^2(x)) = \cos^2(x) - 4\cos(x) + 4$ which results in the quadratic equation: $4\cos^2(x) - 4\cos(x) + 1 = 0$.

Letting $u = \cos(x)$, we get $4u^2 - 4u + 1 = 0$ or $(2u-1)^2 = 0$. We get $u = \cos(x) = \frac{1}{2}$. Solving $\cos(x) = \frac{1}{2}$ gives $x = \frac{\pi}{3} + 2\pi k$ as well as $x = \frac{5\pi}{3} + 2\pi k$ for integers, k.

Of these two families, only solutions of the form $x = \frac{5\pi}{3} + 2\pi k$ checks in our original equation.[8] We leave it the reader to verify this representation of solutions to $\cos(x) - \sqrt{3}\sin(x) = 2$ is equivalent to the one we found previously. $\square$

We repeat here the advice given when solving systems of nonlinear equations in Section 6.2 – when it comes to solving equations involving the circular functions, it helps to just try something.

8.3.3 Harmonic Motion

One of the major applications of the circular functions (sinusoids in particular!) in Science and Engineering is the study of **harmonic motion**, We close this chapter with a brief foray into this topic as it pulls together many important concepts from both Chapters 7 and 8. The equations for harmonic motion can be used to describe a wide range of phenomena, from the motion of an object on a spring, to the response of an electronic circuit. In this subsection, we restrict our attention to modeling a simple spring system. Before we jump into the Mathematics, there are some Physics terms and concepts we need to discuss.

In Physics, 'mass' is defined as a measure of an object's resistance to straight-line motion whereas 'weight' is the amount of force (pull) gravity exerts on an object. An object's mass cannot change,[9] while its weight could change. An object which weighs 6 pounds on the surface of the Earth would weigh 1 pound on the surface of the Moon, but its mass is the same in both places. In the English system of units, 'pounds' (lbs.) is a measure of force (weight), and the corresponding unit of mass is the 'slug'. In the SI system, the unit of force is 'Newtons' (N) and the associated unit of mass is the 'kilogram' (kg).

We convert between mass and weight using the formula[10] $w = mg$. Here, w is the weight of the object, m is the mass and g is the acceleration due to gravity. In the English system, $g = 32\frac{\text{feet}}{\text{second}^2}$, and in the SI system, $g = 9.8\frac{\text{meters}}{\text{second}^2}$. Hence, on Earth a *mass* of 1 slug *weighs* 32 lbs. and a *mass* of 1 kg *weighs* 9.8 N.[11] Suppose

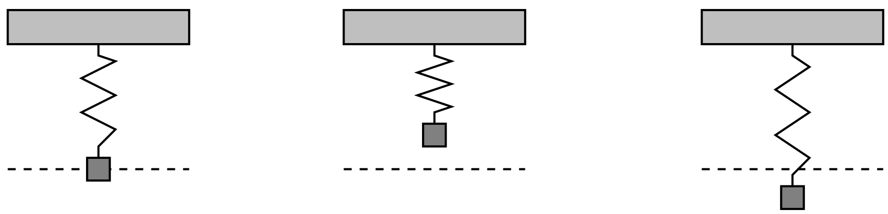
[8] We've seen how squaring both sides can lead to extraneous solutions in Section 0.2 and Chapter 4. Here, squaring both sides admits an entire *family* of extraneous solutions.

[9] Well, assuming the object isn't subjected to relativistic speeds …

[10] This is a consequence of Newton's Second Law of Motion $F = ma$ where F is force, m is mass and a is acceleration. In our present setting, the force involved is weight which is caused by the acceleration due to gravity.

[11] Note that 1 pound $= 1\,\frac{\text{slug foot}}{\text{second}^2}$ and 1 Newton $= 1\,\frac{\text{kg meter}}{\text{second}^2}$.

we attach an object with mass m to a spring as depicted below.



$x(t) = 0$ at the equilibrium position | $x(t) < 0$ above the equilibrium position | $x(t) > 0$ below the equilibrium position

The weight of the object will stretch the spring. The system is said to be in 'equilibrium' when the weight of the object is perfectly balanced with the restorative force of the spring. How far the spring stretches to reach equilibrium depends on the spring's 'spring constant'. Usually denoted by the letter k, the spring constant relates the force F applied to the spring to the amount d the spring stretches in accordance with Hooke's Law $F = kd$.

If the object is released above or below the equilibrium position, or if the object is released with an upward or downward velocity, the object will bounce up and down on the end of the spring until some external force stops it. If we let $x(t)$ denote the object's displacement from the equilibrium position at time t, then $x(t) = 0$ means the object is at the equilibrium position, $x(t) < 0$ means the object is *above* the equilibrium position, and $x(t) > 0$ means the object is *below* the equilibrium position. The function $x(t)$ is called the 'equation of motion' of the object.[12]

If we ignore all other influences on the system except gravity and the spring force, then Physics tells us that gravity and the spring force will battle each other forever and the object will oscillate indefinitely. In this case, we describe the motion as 'free' (meaning there is no external force causing the motion) and 'undamped' (meaning we ignore friction caused by surrounding medium, which in our case is air).

The following theorem, which comes from Differential Equations, gives $x(t)$ as a function of the mass m of the object, the spring constant k, the initial displacement x_0 of the object and initial velocity v_0 of the object.

As with $x(t)$, $x_0 = 0$ means the object is released from the equilibrium position, $x_0 < 0$ means the object is released *above* the equilibrium position and $x_0 > 0$ means the object is released *below* the equilibrium position. As far as the initial velocity v_0 is concerned, $v_0 = 0$ means the object is released 'from rest,' $v_0 < 0$ means the object is heading *upwards* and $v_0 > 0$ means the object is heading *downwards*.[13]

[12] To keep units compatible, if we are using the English system, we use feet (ft.) to measure displacement. If we are in the SI system, we measure displacement in meters (m). Time is always measured in seconds (s).

[13] The sign conventions here are carried over from Physics. If not for the spring, the object would fall towards the ground, which is the 'natural' or 'positive' direction. Because the spring force acts in direct opposition to gravity, any movement upwards is considered 'negative'.

Theorem 8.14. Equation for Free Undamped Harmonic Motion:
Suppose an object of mass m is suspended from a spring with spring constant k. If the initial displacement from the equilibrium position is x_0 and the initial velocity of the object is v_0, then the displacement x from the equilibrium position at time t is given by $x(t) = A\sin(\omega t + \phi)$ where

- $\omega = \sqrt{\dfrac{k}{m}}$ and $A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}$
- $A\sin(\phi) = x_0$ and $A\omega\cos(\phi) = v_0$.

It is a great exercise in 'dimensional analysis' to verify that the formulas given in Theorem 8.14 work out so that ω has units $\frac{1}{sec}$ and A has units ft. or m, depending on which system we choose.

Example 8.3.5. Suppose an object weighing 64 pounds stretches a spring 8 feet.

1. If the object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, $x(t)$. When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?
2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, $x(t)$. What is the longest distance the object travels *above* the equilibrium position? When does this first happen? Confirm your result using a graphing utility.

Solution. In order to use the formulas in Theorem 8.14, we first need to determine the spring constant k and the mass of the object m.

To find k, we use Hooke's Law $F = kd$. We know the object weighs 64 lbs. and stretches the spring 8 ft.. Using $F = 64$ and $d = 8$, we get $64 = k \cdot 8$, or $k = 8\frac{\text{lbs.}}{\text{ft.}}$.

To find m, we use $w = mg$ with $w = 64$ lbs. and $g = 32\frac{\text{ft.}}{s^2}$. We get $m = 2$ slugs. We can now proceed to apply Theorem 8.14.

1. If the object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, $x(t)$. When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?

   With $k = 8$ and $m = 2$, we get $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$. Because the object is released 3 feet *below* the equilibrium position 'from rest,' $x_0 = 3$ and $v_0 = 0$. Therefore, $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + 0^2} = 3$.

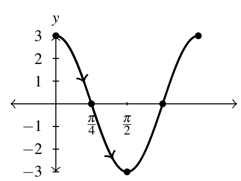
   To determine the phase angle, ϕ, we have $A\sin(\phi) = x_0$, which in this case gives $3\sin(\phi) = 3$ so $\sin(\phi) = 1$. Only $\phi = \frac{\pi}{2}$ and angles coterminal to it satisfy this condition, so we pick[14] $\phi = \frac{\pi}{2}$. Hence,

[14]For confirmation, we note that $A\omega\cos(\phi) = v_0$, which in this case reduces to $6\cos(\phi) = 0$.

the equation of motion is $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$.

To find when the object passes through the equilibrium position we solve $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right) = 0$. Going through the usual analysis we find $t = -\frac{\pi}{4} + \frac{\pi}{2}k$ for integers k. We are interested in the *first* time the object passes through the equilibrium position, so we look for the smallest positive t value which in this case is $t = \frac{\pi}{4} \approx 0.78$ seconds after the start of the motion.

Common sense suggests that if we release the object *below* the equilibrium position, the object should be traveling *upwards* when it first passes through it. To check this answer, we graph one cycle of $x(t)$. Because our applied domain in this situation is $t \geq 0$, and the period of $x(t)$ is $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$, we graph $x(t)$ over the interval $[0, \pi]$. Remembering that $x(t) > 0$ means the object is below the equilibrium position and $x(t) < 0$ means the object is above the equilibrium position, the fact our graph is crossing through the t-axis from positive x to negative x at $t = \frac{\pi}{4}$ confirms our answer.



$x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$

2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, $x(t)$. What is the longest distance the object travels *above* the equilibrium position? When does this first happen? Confirm your result using a graphing utility.

   The only difference between this problem and the previous problem is that we now release the object with an upward velocity of $8\,\frac{\text{ft}}{s}$. We still have $\omega = 2$ and $x_0 = 3$, but now we have $v_0 = -8$, the negative indicating the velocity is directed upwards.

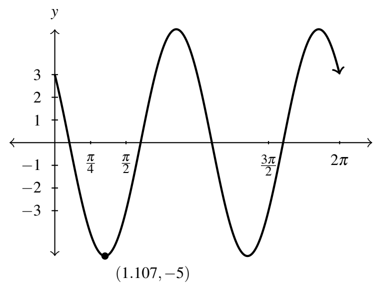
   Here, we get $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + (-4)^2} = 5$. From $A\sin(\phi) = x_0$, we get $5\sin(\phi) = 3$ which gives $\sin(\phi) = \frac{3}{5}$. From $A\omega\cos(\phi) = v_0$, we get $10\cos(\phi) = -8$, or $\cos(\phi) = -\frac{4}{5}$.

   Hence, ϕ is a Quadrant II angle which we can describe in terms of either arcsine or arccosine. The range of arccosine covers Quadrant II, so we choose to express ϕ in terms of the arccosine: $\phi = \arccos\left(-\frac{4}{5}\right)$. Hence, $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$.

   As the amplitude of $x(t)$ is 5, the object will travel at most 5 feet above the equilibrium position. To find when this happens, we solve the equation $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right) = -5$, the negative once again signifying that the object is *above* the equilibrium position.

   Going through the usual machinations, we get $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) - \frac{\pi}{4} + \pi k$ for integers k. The smallest (positive) of these values occurs when $k = 1$, that is, $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) + \frac{3\pi}{4} \approx 1.107$ seconds after the start of the motion.

Graphing $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$, we find the coordinates of the first relative minimum of to be approximately $(1.107, -5)$.



$x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$

□

Though beyond the scope of this course, it is possible to model the effects of friction and other external forces acting on the system.[15]

While we may not have the Physics and Calculus background to *derive* equations of motion for these scenarios, we can certainly analyze them. We examine three cases in the following example.

Example 8.3.6.

1. Write $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ in the form $x(t) = A(t)\sin(Bt + C)$. Graph $x(t)$ using technology.
2. Write $x(t) = (t+3)\sqrt{2}\cos(2t) + (t+3)\sqrt{2}\sin(2t)$ in the form $x(t) = A(t)\sin(Bt + C)$. Graph $x(t)$ using technology.
3. Compute the period of $x(t) = 5\sin(6t) - 5\sin(8t)$. Graph $x(t)$ using technology.

Solution.

1. Write $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ in the form $x(t) = A(t)\sin(Bt + C)$. Graph $x(t)$ using technology.

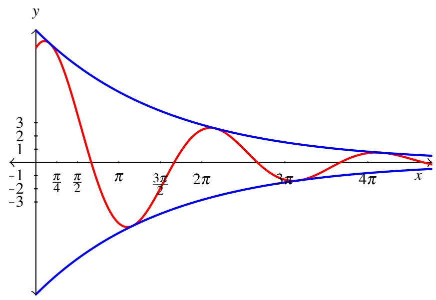
   We start rewriting $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ by factoring out $5e^{-t/5}$ from both terms to get $x(t) = 5e^{-t/5}\left(\cos(t) + \sqrt{3}\sin(t)\right)$. We convert what's left in parentheses to the required form using the technique introduced in Example 8.2.1 from Section 8.2. We find $\left(\cos(t) + \sqrt{3}\sin(t)\right) = 2\sin\left(t + \frac{\pi}{3}\right)$ so that $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$.

   Graphing $x(t)$ reveals some interesting behavior. The sinusoidal nature continues indefinitely, but it is being attenuated. In the sinusoid $A\sin(Bt + C)$, the coefficient A of the sine function is the

[15] Take a good Differential Equations class to see this!

amplitude. In the case of $x(t) = 10e^{-t/5}\sin\left(t+\frac{\pi}{3}\right)$, we can think of the *function* $A(t) = 10e^{-t/5}$ as the amplitude.[16] As $t \to \infty$, $10e^{-t/5} \to 0$ which means the amplitude shrinks towards zero.

Indeed, if we graph $x = \pm 10e^{-t/5}$ along with $x(t) = 10e^{-t/5}\sin\left(t+\frac{\pi}{3}\right)$, we see this attenuation taking place with the exponentials acting as a 'wave envelope.'



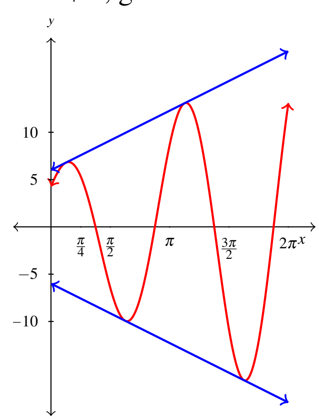
$x(t) = 10e^{-t/5}\sin\left(t+\frac{\pi}{3}\right)$ and $x = \pm 10e^{-t/5}$

In this case, the function $x(t)$ corresponds to the motion of an object on a spring where there is a slight force which acts to 'damp', or slow the motion. An example of this kind of force would be the friction of the object against the air. According to this model, the object oscillates forever, but with increasingly smaller and smaller amplitude.

2. Write $x(t) = (t+3)\sqrt{2}\cos(2t) + (t+3)\sqrt{2}\sin(2t)$ in the form $x(t) = A(t)\sin(Bt+C)$. Graph $x(t)$ using technology.

   Proceeding as in the first example, we factor out $(t+3)\sqrt{2}$ from each term in the function $x(t)$ to get $x(t) = (t+3)\sqrt{2}(\cos(2t)+\sin(2t))$. We find $(\cos(2t)+\sin(2t)) = \sqrt{2}\sin\left(2t+\frac{\pi}{4}\right)$, so an equivalent form of $x(t)$ is $x(t) = 2(t+3)\sin\left(2t+\frac{\pi}{4}\right)$.

   Graphing $x(t)$, we find the sinusoid's amplitude growing. This isn't too surprising because our amplitude function here is $A(t) = 2(t+3) = 2t+6$, grows without bound as $t \to \infty$.



$x(t) = 2(t+3)\sin\left(2t+\frac{\pi}{4}\right)$ and $x = \pm 2(t+3)$

[16]This is the same sort of phenomenon we saw on page 723 in Section 8.2.1.

The phenomenon illustrated here is 'forced' motion. That is, we imagine that the entire apparatus on which the spring is attached is oscillating as well.

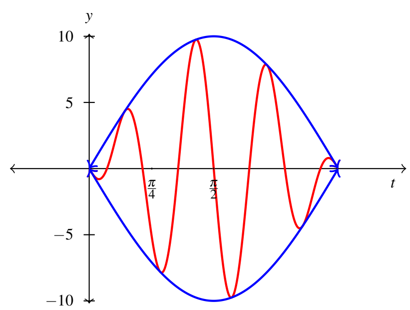
In this particular case, we are witnessing a 'resonance' effect – the frequency of the external oscillation matches the frequency of the motion of the object on the spring. In a mechanical system, this will result in some sort of structural failure.[17]

3. Compute the period of $x(t) = 5\sin(6t) - 5\sin(8t)$. Graph $x(t)$ using technology.

   Last, but not least, we come to $x(t) = 5\sin(6t) - 5\sin(8t)$. To find the period of this function, we need to determine the length of the smallest interval on which both $f(t) = 5\sin(6t)$ and $g(t) = 5\sin(8t)$ complete a whole number of cycles.

   To do this, we take the ratio of their frequencies and reduce to lowest terms: $\frac{6}{8} = \frac{3}{4}$. This tells us that for every 3 cycles f makes, g makes 4. Hence, the period of $x(t)$ is three times the period of $f(t)$ (which is four times the period of $g(t)$), or π. We check our work by graphing $x(t)$ over $[0, \pi]$

   The reader may recognize $x(t)$ an example of the 'beats' phenomenon we first saw on 723 in Section 8.2.1. Indeed, using a sum to product identity, we may rewrite $x(t)$ as $x(t) = -10\sin(t)\cos(7t)$. As we saw on 723 (and Exercises 122 - 125 in Section 8.2), the lower frequency factor, $-10\sin(t)$ determines the 'wave-envelope,' $x = \pm 10\sin(t)$.



$x(t) = 5\sin(6t) - 5\sin(8t)$ and $x = \pm 10\sin(t)$ over $[0, \pi]$

   This equation of motion also results from 'forced' motion, but here the frequency of the external oscillation is different than that of the object on the spring. The sinusoids here have different frequencies, so they are 'out of sync' and do not amplify each other as in the previous example. Instead, through a combination of constructive and destructive interference, the mass continues to oscillate no more than 10 units from its equilibrium position indefinitely. □

8.3.4 Exercises

In Exercises 1 - 20, solve the equation using the techniques discussed in Example 8.3.1 then approximate the solutions which lie in the interval $[0, 2\pi)$ to four decimal places.

[17]The reader is invited to investigate the destructive implications of resonance.

1. $\sin(\theta) = \dfrac{7}{11}$
2. $\cos(\theta) = -\dfrac{2}{9}$
3. $\sin(\theta) = -0.569$
4. $\cos(\theta) = 0.117$
5. $\sin(\theta) = 0.008$
6. $\cos(\theta) = \dfrac{359}{360}$
7. $\tan(t) = 117$
8. $\cot(t) = -12$
9. $\sec(t) = \dfrac{3}{2}$
10. $\csc(t) = -\dfrac{90}{17}$
11. $\tan(t) = -\sqrt{10}$
12. $\sin(t) = \dfrac{3}{8}$
13. $\cos(x) = -\dfrac{7}{16}$
14. $\tan(x) = 0.03$
15. $\sin(x) = 0.3502$
16. $\sin(x) = -0.721$
17. $\cos(x) = 0.9824$
18. $\cos(x) = -0.5637$
19. $\cot(x) = \dfrac{1}{117}$
20. $\tan(x) = -0.6109$

In Exercises 21 - 38, compute all of the exact solutions of the equation and then list those solutions which are in the interval $[0, 2\pi)$.

21. $\sin(5\theta) = 0$
22. $\cos(3t) = \dfrac{1}{2}$
23. $\sin(-2x) = \dfrac{\sqrt{3}}{2}$
24. $\tan(6\theta) = 1$
25. $\csc(4t) = -1$
26. $\sec(3x) = \sqrt{2}$
27. $\cot(2\theta) = -\dfrac{\sqrt{3}}{3}$
28. $\cos(9t) = 9$
29. $\sin\left(\dfrac{x}{3}\right) = \dfrac{\sqrt{2}}{2}$
30. $\cos\left(\theta + \dfrac{5\pi}{6}\right) = 0$
31. $\sin\left(2t - \dfrac{\pi}{3}\right) = -\dfrac{1}{2}$
32. $2\cos\left(x + \dfrac{7\pi}{4}\right) = \sqrt{3}$
33. $\csc(\theta) = 0$
34. $\tan(2t - \pi) = 1$
35. $\tan^2(x) = 3$
36. $\sec^2(\theta) = \dfrac{4}{3}$
37. $\cos^2(t) = \dfrac{1}{2}$
38. $\sin^2(x) = \dfrac{3}{4}$

In Exercises 39 - 62, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$

39. $\sin(\theta) = \cos(\theta)$
40. $\sin(2t) = \sin(t)$
41. $\sin(2x) = \cos(x)$
42. $\cos(2\theta) = \sin(\theta)$
43. $\cos(2t) = \cos(t)$
44. $\cos(2x) = 2 - 5\cos(x)$

45. $3\cos(2\theta)+\cos(\theta)+2=0$

46. $\cos(2t)=5\sin(t)-2$

47. $3\cos(2x)=\sin(x)+2$

48. $2\sec^2(\theta)=3-\tan(\theta)$

49. $\tan^2(t)=1-\sec(t)$

50. $\cot^2(x)=3\csc(x)-3$

51. $\sec(\theta)=2\csc(\theta)$

52. $\cos(t)\csc(t)\cot(t)=6-\cot^2(t)$

53. $\sin(2x)=\tan(x)$

54. $\cot^4(\theta)=4\csc^2(\theta)-7$

55. $\cos(2t)+\csc^2(t)=0$

56. $\tan^3(x)=3\tan(x)$

57. $\tan^2(\theta)=\frac{3}{2}\sec(\theta)$

58. $\cos^3(t)=-\cos(t)$

59. $\tan(2x)-2\cos(x)=0$

60. $\csc^3(\theta)+\csc^2(\theta)=4\csc(\theta)+4$

61. $2\tan(t)=1-\tan^2(t)$

62. $\tan(x)=\sec(x)$

In Exercises 63 - 78, solve the equation, giving the exact solutions which lie in $[0,2\pi)$

63. $\sin(6\theta)\cos(\theta)=-\cos(6\theta)\sin(\theta)$

64. $\sin(3t)\cos(t)=\cos(3t)\sin(t)$

65. $\cos(2x)\cos(x)+\sin(2x)\sin(x)=1$

66. $\cos(5\theta)\cos(3\theta)-\sin(5\theta)\sin(3\theta)=\frac{\sqrt{3}}{2}$

67. $\sin(t)+\cos(t)=1$

68. $\sin(x)+\sqrt{3}\cos(x)=1$

69. $\sqrt{2}\cos(\theta)-\sqrt{2}\sin(\theta)=1$

70. $\sqrt{3}\sin(2t)+\cos(2t)=1$

71. $\cos(2x)-\sqrt{3}\sin(2x)=\sqrt{2}$

72. $3\sqrt{3}\sin(3\theta)-3\cos(3\theta)=3\sqrt{3}$

73. $\cos(3t)=\cos(5t)$

74. $\cos(4x)=\cos(2x)$

75. $\sin(5\theta)=\sin(3\theta)$

76. $\cos(5t)=-\cos(2t)$

77. $\sin(6x)+\sin(x)=0$

78. $\tan(x)=\cos(x)$

In Exercises 79 - 88, solve the equation.

79. $\arccos(2x)=\pi$

80. $\pi-2\arcsin(t)=2\pi$

81. $4\arctan(3x-1)-\pi=0$

82. $6\operatorname{arccot}(2t)-5\pi=0$

83. $4\operatorname{arcsec}\left(\frac{x}{2}\right) = \pi$

84. $12\operatorname{arccsc}\left(\frac{t}{3}\right) = 2\pi$

85. $9\arcsin^2(x) - \pi^2 = 0$

86. $9\arccos^2(t) - \pi^2 = 0$

87. $8\operatorname{arccot}^2(x) + 3\pi^2 = 10\pi\operatorname{arccot}(x)$

88. $6\arctan(t)^2 = \pi\arctan(x) + \pi^2$

89. (a) With the help of your classmates, determine the number of solutions to $\sin(x) = \frac{1}{2}$ in $[0, 2\pi)$. Then find the number of solutions to $\sin(2x) = \frac{1}{2}$, $\sin(3x) = \frac{1}{2}$ and $\sin(4x) = \frac{1}{2}$ in $[0, 2\pi)$. What pattern emerges? Explain how this pattern would help you solve equations like $\sin(11x) = \frac{1}{2}$.
    (b) Repeat the above exercise focusing on $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ and $\sin\left(\frac{5x}{2}\right) = \frac{1}{2}$. What pattern emerges here?
    (c) Replace sine with tangent and $\frac{1}{2}$ with 1 and repeat the whole exploration.

90. Suppose an object weighing 10 pounds is suspended from the ceiling by a spring which stretches 2 feet to its equilibrium position when the object is attached.
    (a) Find the spring constant k in $\frac{\text{lbs.}}{\text{ft.}}$ and the mass of the object in slugs.
    (b) Find the equation of motion of the object if it is released from 1 foot *below* the equilibrium position from rest. When is the first time the object passes through the equilibrium position? In which direction is it heading?
    (c) Find the equation of motion of the object if it is released from 6 inches *above* the equilibrium position with a *downward* velocity of 2 feet per second. Find when the object passes through the equilibrium position heading downwards for the third time.

Section 8.3 Exercise Answers A.1.8

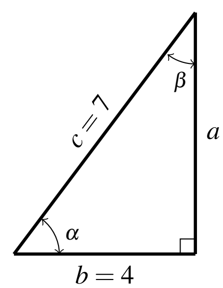
8.4 LAW OF SINES

In this section, we showcase how the the tools we've developed in Chapters 7 and 8 can be applied to Geometry. Our next two sections focus specifically on solving oblique (non-right) Triangles.[1]

Our first example reviews the basics of right triangle trigonometry. The reader is referred to Section 7.2.1 for more details and practice with these concepts.

Example 8.4.1. Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, compute the length of the remaining side and the measures of the remaining angles. Express the angles in decimal degrees, rounded to the nearest hundredth of a degree.

Solution. For definitiveness, we label the triangle below.



To find a, we use the Pythagorean Theorem, Theorem 7.1: $a^2+4^2=7^2$, so $a=\sqrt{33}$ units.

Now that all three sides of the triangle are known, there are several ways we can find α using the inverse trigonometric functions.

To decrease the chances of propagating error, however, we stick to using the data given to us in the problem. In this case, the lengths 4 and 7 were given, so we want to relate these to α.

According to Definition 7.2, $\cos(\alpha)=\frac{4}{7}$. As α is an acute angle, $\alpha=\arccos\left(\frac{4}{7}\right)$ radians $\approx 55.15^\circ$.

Now that we have the measure of angle α, we could find the measure of angle β using the fact that α and β are complements so $\alpha+\beta=90^\circ$.

Once again, in the interests of minimizing propagated error, we opt to use the data given to us in the problem. According to Definition 7.2, $\sin(\beta)=\frac{4}{7}$ so $\beta=\arcsin\left(\frac{4}{7}\right)$ radians $\approx 34.85^\circ$. □

A few remarks about Example 8.4.1 are in order. First, we adhere to the convention that a lower case Greek letter denotes an angle (as well as the measure of said angle) and the corresponding lowercase English letter represents the side (as well as the length of said side) opposite that angle.

[1] Recall that the word 'Trigonometry' literally means 'measuring triangles' so we are returning to our roots here.

More specifically, a is the side opposite α, b is the side opposite β and c is the side opposite γ. Taken together, the pairs (α, a), (β, b) and (γ, c) are called *angle-side opposite pairs*.

Second, as mentioned earlier, we will strive to solve for quantities using the original data given in the problem whenever possible. While this is not always the easiest or fastest way to proceed, it minimizes the chances of propagated error.[2]

Third, as many of the applications which require solving triangles 'in the wild' rely on degree measure, we shall adopt this convention for the time being.

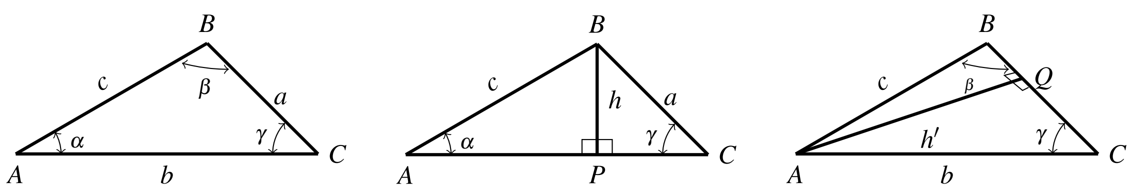
The Pythagorean Theorem along with Definition 7.2 allow us to easily handle any given *right* triangle problem, but what if the triangle isn't a right triangle? In certain cases, we can use the **Law of Sines**.

Theorem 8.15. The Law of Sines:
Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following ratios hold:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \qquad \text{or, equivalently,} \qquad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

The proof of the Law of Sines can be broken into three cases, and, as we'll see, ultimately relies on what we know about right triangles.

For our first case, consider the triangle $\triangle ABC$ below, all of whose angles are acute, with angle-side opposite pairs (α, a), (β, b) and (γ, c).



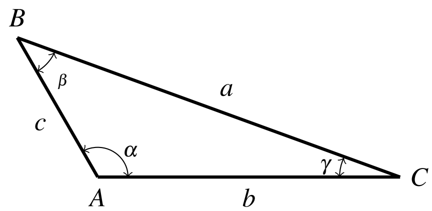
If we drop an altitude from vertex B, we divide the triangle into two right triangles: $\triangle BAP$ and $\triangle BCP$.

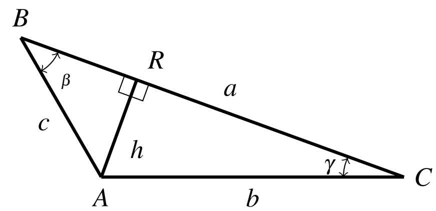
If we call the length of the altitude h (for height), we get from Definition 7.2 that $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$ so that $h = c\sin(\alpha) = a\sin(\gamma)$. Rearranging this last equation, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

Dropping an altitude from vertex A, we can proceed as above using the triangles $\triangle ABQ$ and $\triangle ACQ$. We find that $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$, so we have shown $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$ as required.

For our next case consider the triangle $\triangle ABC$ below with obtuse angle α.

[2]Your Science teachers should thank us for this.

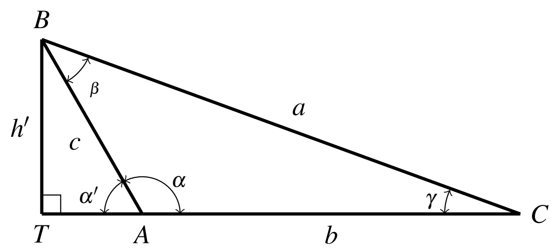




Extending an altitude from vertex A gives two right triangles, as in the previous case: $\triangle ABR$ and $\triangle ACR$.

Proceeding as before, we get $h = b\sin(\gamma)$ and $h = c\sin(\beta)$ so that $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.

Dropping an altitude from vertex B also generates two right triangles, $\triangle ABT$ and $\triangle CBT$.



We see $\sin(\alpha') = \frac{h'}{c}$ so that $h' = c\sin(\alpha')$. Because $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so $h' = c\sin(\alpha)$.

Proceeding to $\triangle BCT$, we get $\sin(\gamma) = \frac{h'}{a}$ so $h' = a\sin(\gamma)$.

As before, we get $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$, so $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$ in this case, too.

The remaining case is when $\triangle ABC$ is a right triangle. In this case, the Law of Sines reduces to the formulas given in Definition 7.2 and is left to the reader.

In order to use the Law of Sines to solve a triangle, we need at least one angle-side opposite pair. The next example showcases some of the power, and the pitfalls, of the Law of Sines.

Example 8.4.2. Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and then sketch the triangle.

1. $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$
2. $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units
3. $\alpha = 30^\circ$, $a = 1$ units, $c = 4$ units
4. $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units
5. $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units
6. $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units

Solution.

1. Solve the triangle with the properties: $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$.

   Knowing an angle-side opposite pair, namely α and a, we may proceed in using the Law of Sines.

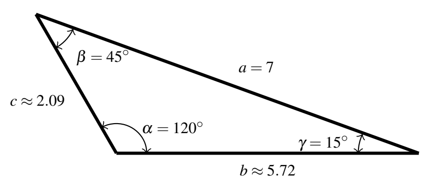
   Given $\beta = 45^\circ$, we use $\frac{b}{\sin(45^\circ)} = \frac{7}{\sin(120^\circ)}$ so $b = \frac{7\sin(45^\circ)}{\sin(120^\circ)} = \frac{7\sqrt{6}}{3} \approx 5.72$ units.

   To find γ, we use the fact that the sum of the measures of the angles in a triangle is 180°. Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$.

   To find c, we have no choice but to used the derived value $\gamma = 15^\circ$, yet we can minimize the propagation of error here by using the given angle-side opposite pair (α, a).

   The Law of Sines gives us $\frac{c}{\sin(15^\circ)} = \frac{7}{\sin(120^\circ)}$ so that $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)} \approx 2.09$ units.[3]

   We sketch this triangle below.



2. Solve the triangle with the properties: $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units.

   In this example, we are not immediately given an angle-side opposite pair, but as we have the measures of α and β, we can solve for γ giving us $\gamma = 180^\circ - 85^\circ - 30^\circ = 65^\circ$.

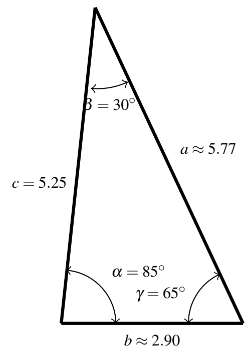
   As in the previous example, we are forced to use a derived value in our computations because the only angle-side pair available is (γ, c).

   The Law of Sines gives $\frac{a}{\sin(85^\circ)} = \frac{5.25}{\sin(65^\circ)}$. Solving, we get $a = \frac{5.25\sin(85^\circ)}{\sin(65^\circ)} \approx 5.77$ units.

   To find b we use the angle-side pair (γ, c): $\frac{b}{\sin(30^\circ)} = \frac{5.25}{\sin(65^\circ)}$. Hence $b = \frac{5.25\sin(30^\circ)}{\sin(65^\circ)} \approx 2.90$ units.

   Next, we sketch this triangle.

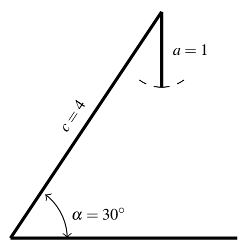
[3] The exact value of $\sin(15^\circ)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus "exact" here means $\frac{7\sin(15^\circ)}{\sin(120^\circ)}$.



3. Solve the triangle with the properties: $\alpha = 30°$, $a = 1$ units, $c = 4$ units.

   We are given (α, a) and c, so we use the Law of Sines to find the measure of γ.

   From $\frac{\sin(\gamma)}{4} = \frac{\sin(30°)}{1}$, we get $\sin(\gamma) = 4\sin(30°) = 2$, which is impossible. (Why?) As seen below, side a is just too short to make a triangle.



The next three examples keep the same values for the measure of α and the length of c while varying the length of a. We will discuss this case in more detail after we see what happens in those examples.

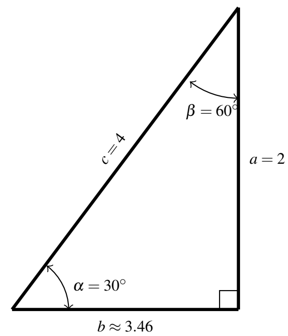
4. Solve the triangle with the properties:$\alpha = 30°$, $a = 2$ units, $c = 4$ units.

   In this case, we have the measure of $\alpha = 30°$, $a = 2$ and $c = 4$. Using the Law of Sines, we get $\frac{\sin(\gamma)}{4} = \frac{\sin(30°)}{2}$ so $\sin(\gamma) = 2\sin(30°) = 1$.

   As γ is an angle in a triangle which also contains $\alpha = 30°$, γ must measure between $0°$ and $150°$ in order to fit inside the triangle with α. The only angle that satisfies this requirement and has $\sin(\gamma) = 1$ is $\gamma = 90°$, so we are working in a right triangle.

   We find the measure of β to be $\beta = 180° - 30° - 90° = 60°$. Using the Law of Sines, we get $b = \frac{2\sin(60°)}{\sin(30°)} = 2\sqrt{3} \approx 3.46$ units.

   As seen below, the side a is just long enough to form a right triangle in this case.



5. Solve the triangle with the properties: $\alpha = 30°$, $a = 3$ units, $c = 4$ units.

   Proceeding as we have in the previous two examples, we use the Law of Sines to find γ.

   In this case, we have $\frac{\sin(\gamma)}{4} = \frac{\sin(30°)}{3}$ or $\sin(\gamma) = \frac{4\sin(30°)}{3} = \frac{2}{3}$. As γ lies in a triangle with $\alpha = 30°$, we must have that $0° < \gamma < 150°$.

   In this case, there are *two* angles that fall in this range: $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81°$ and $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19°$.

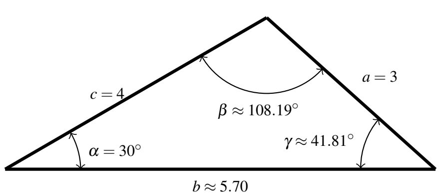
   At this point, we pause to see if it makes sense that we have two cases to consider.

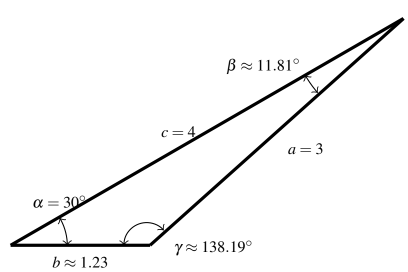
   As $c > a$, it must also be true that γ, which is opposite c, has greater measure than α which is opposite a. In both cases, $\gamma > \alpha$, so both candidates for γ make sense with the given value of c.

   Thus have two triangles on our hands. In the case $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81°$, we find[4] $\beta \approx 180° - 30° - 41.81° = 108.19°$.

   The Law of Sines with the angle-side opposite pair (α, a) and β gives $b \approx \frac{3\sin(108.19°)}{\sin(30°)} \approx 5.70$ units. We sketch this triangle below on the left.

   In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19°$, we repeat the same steps and find $\beta \approx 11.81°$ and $b \approx 1.23$ units.[5] We sketch this triangle below on the right.





[4]To find an exact expression for β, we convert everything back to radians: $\alpha = 30° = \frac{\pi}{6}$ radians, $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians and $180° = \pi$ radians. Hence, $\beta = \pi - \frac{\pi}{6} - \arcsin\left(\frac{2}{3}\right) = \frac{5\pi}{6} - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 108.19°$.

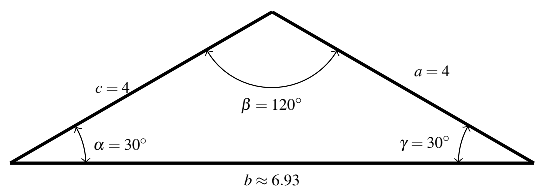
[5]An exact answer for β in this case is $\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{6}$ radians $\approx 11.81°$.

6. Solve the triangle with the properties: $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units.

   For this last problem, we repeat the usual Law of Sines routine to find that $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$ so that $\sin(\gamma) = \frac{1}{2}$. As γ must inhabit a triangle with $\alpha = 30^\circ$, we must have $0^\circ < \gamma < 150^\circ$.

   Because the measure of γ must be *strictly less* than 150°, there is just *one* angle which satisfies both required conditions, namely $\gamma = 30^\circ$.

   Hence, $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$. The Law of Sines gives $b = \frac{4\sin(120^\circ)}{\sin(30^\circ)} = 4\sqrt{3} \approx 6.93$ units. We sketch this triangle below.



$\square$

Some remarks about Example 8.4.2 are in order. First note that if we are given the measures of two of the angles in a triangle, say α and β, the measure of the third angle γ is uniquely determined using the equation $\gamma = 180^\circ - \alpha - \beta$. Knowing the measures of all three angles of a triangle completely determines the triangle's *shape*.

If in addition we are given the length of one of the sides of the triangle, we can then use the Law of Sines to find the lengths of the remaining two sides to determine the *size* of the triangle. Such is the case in numbers 1 and 2 above.

In number 1, the given side is adjacent to just one of the angles – this is called the 'Angle-Angle-Side' (AAS) case.[6] In number 2, the given side is adjacent to both angles which means we are in the so-called 'Angle-Side-Angle' (ASA) case.

If, on the other hand, we are given the measure of just one of the angles in the triangle along with the length of two sides, only one of which is adjacent to the given angle, we are in the 'Side-Side-Angle' (SSA) case. Such was the case in numbers 3, 4, 5, and 6 above.

In number 3, the length of the one given side a was too short to even form a triangle; in number 4, the length of a was just long enough to form a right triangle; in 5, a was long enough, but not too long, so that two triangles were possible; and in number 6, side a was long enough to form a triangle but too long to swing

[6]If this sounds familiar, it should. From Geometry, we know there are four congruence conditions for triangles: Angle-Angle-Side (AAS), Angle-Side-Angle (ASA), Side-Angle-Side (SAS) and Side-Side-Side (SSS). If we are given information about a triangle that meets one of these four criteria, then we are guaranteed that exactly one triangle exists which satisfies said criteria.

back and form two. These four cases exemplify all of the possibilities in the Angle-Side-Side case which are summarized in the following theorem.

Theorem 8.16. Suppose (α, a) and (γ, c) are intended to be angle-side pairs in a triangle where α, a and c are given. Let $h = c\sin(\alpha)$

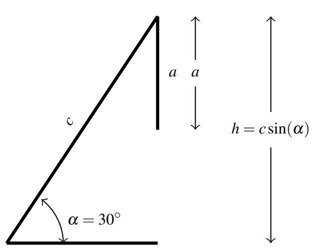
- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists which satisfies the given criteria

Theorem 8.16 is proved on a case-by-case basis. If $a < h$, then $a < c\sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible.

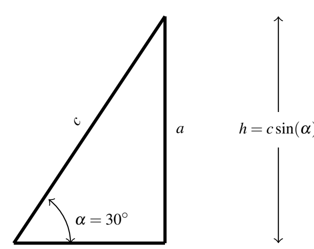
In the figure below on the left, we see geometrically why this is the case. Simply put, if $a < h$, the side a is too short to connect to form a triangle.

This means if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem tell us what kind and how many triangles to expect in each case.

If $a = h$, then $a = c\sin(\alpha)$ and the Law of Sines gives $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} = \frac{a}{a} = 1$. Here, $\gamma = 90^\circ$ as required. This situation is sketched below on the right.



$a < h$, no triangle



$a = h$, $\gamma = 90^\circ$

Moving along, now suppose $h < a < c$. As before, the Law of Sines[7] gives $\sin(\gamma) = \frac{c\sin(\alpha)}{a}$.

As $h < a$, $c\sin(\alpha) < a$ or $\frac{c\sin(\alpha)}{a} < 1$ which means there are two solutions to $\sin(\gamma) = \frac{c\sin(\alpha)}{a}$: an acute angle which we'll call γ_0, and its supplement, $180^\circ - \gamma_0$.

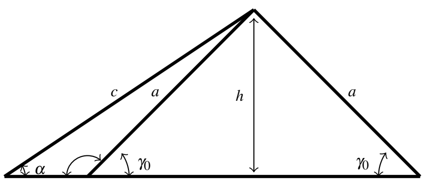
Our job now is to argue that each of these angles 'fit' into a triangle with α. Given (α, a) and (γ_0, c) are angle-side opposite pairs, the assumption $c > a$ in this case gives us $\gamma_0 > \alpha$. Because γ_0 is acute, we must

[7] Remember, we have already argued that a triangle exists in this case!

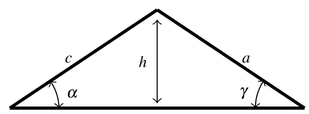
have that α is acute as well. This means one triangle can contain both α and γ_0, giving us one of the triangles promised in the theorem.

If we manipulate the inequality $\gamma_0 > \alpha$ a bit, we have $180^\circ - \gamma_0 < 180^\circ - \alpha$. Adding α to both sides gives $(180^\circ - \gamma_0) + \alpha < 180^\circ$. This proves a triangle can contain both of the angles α and $(180^\circ - \gamma_0)$, giving us the second triangle predicted in the theorem. We sketch the two triangle case below on the left.

To prove the last case in the theorem, we assume $a \geq c$. Then $\alpha \geq \gamma$, which forces γ to be an acute angle. Hence, we get only one triangle in this case, completing the proof.



$h < a < c$, two triangles



$a \geq c$, one triangle

One last comment regarding the Angle-Side-Side case: if you are given an obtuse angle to begin with then it is impossible to have the two triangle case. Think about this before reading further.

In many of the derivations and arguments in this section, we used the height of a given triangle, h, as an intermediate variable to prove equivalences. The height of a triangle can be used to determine the area enclosed by said triangle, thus we can use the methods in this section to reformulate area in terms of side lengths and sines of angles. We state the following theorem and leave its proof as an exercise.

Theorem 8.17. Suppose (α, a), (β, b) and (γ, c) are the angle-side opposite pairs of a triangle. Then the area A enclosed by the triangle is given by

$$A = \frac{1}{2}bc\sin(\alpha) = \frac{1}{2}ac\sin(\beta) = \frac{1}{2}ab\sin(\gamma)$$

That is, the area enclosed by the triangle $A = \frac{1}{2}$ (the product of two sides) sin(of the included angle).

Example 8.4.3. Compute the area of the triangle in Example 8.4.2 number 1. Recall: $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$.

Solution. From our work in Example 8.4.2 number 1, we have all three angles and all three sides to work with. However, to minimize propagated error, we choose $A = \frac{1}{2}ac\sin(\beta)$ from Theorem 8.17 because it uses the most pieces of given information.

We are given $a = 7$ and $\beta = 45^\circ$, and we calculated $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)}$.

Using these values, we find the area $A = \frac{1}{2}(7)\left(\frac{7\sin(15^\circ)}{\sin(120^\circ)}\right)\sin(45^\circ) =\approx 5.18$ square units.

The reader is encouraged to check this answer against the results obtained using the other formulas in Theorem 8.17. □

8.4.1 Bearings

Our last example of the section uses the navigation tool known as **bearings**. Simply put, a bearing is the direction you are heading according to a compass.

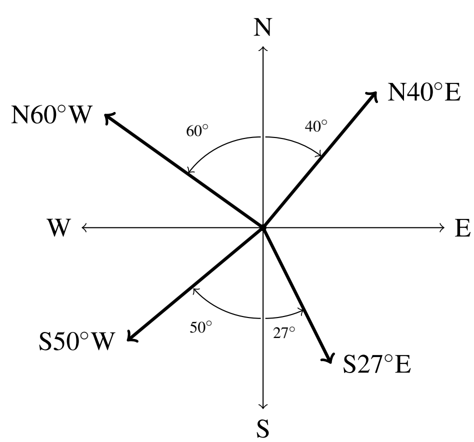
The classic nomenclature for bearings, however, is not given as an angle in standard position, so we must first understand the notation. A bearing is given as an acute angle of rotation (to the east or to the west) away from the north-south (up and down) line of a compass rose.

For example, N40°E (read "40° east of north") is a bearing which is rotated clockwise 40° from due north. If we imagine standing at the origin in the Cartesian Plane, this bearing would have us heading into Quadrant I along the terminal side of $\theta = 50°$.

Similarly, S50°W would point into Quadrant III along the terminal side of $\theta = 220°$ because we started out pointing due south (along $\theta = 270°$) and rotated clockwise 50° back to 220°.

Counter-clockwise rotations would be found in the bearings N60°W (which is on the terminal side of $\theta = 150°$) and S27°E (which lies along the terminal side of $\theta = 297°$).

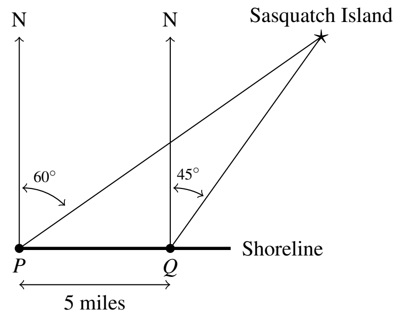
These four bearings are sketched in the plane below.



The cardinal directions north, south, east and west are usually not given as bearings in the fashion described above, but rather, one just refers to them as 'due north', 'due south', 'due east' and 'due west', respectively, and it is assumed that you know which quadrantal angle goes with each cardinal direction.

We make good use of bearings and the Law of Sines in our next example.

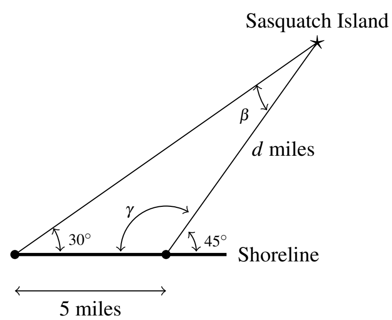
Example 8.4.4. Sasquatch Island lies off the coast of Ippizuti Lake. As illustrated below, from a point P on the shore, the bearing to Sasquatch Island is observed to be N60°E. From a point Q that is 5 miles due East of P, the bearing to the island is observed to be N45°E.



Assuming the coastline continues to run due East, find the distance from the point Q to the island. What point on the shore is closest to the island? How far is the island from this point?

Solution. As illustrated above, the points P, Q, and the location of the island (represented as a point) form a triangle. Using the bearings information given, we get that the angle between the shore and the island at point P is $90^\circ - 60^\circ = 30^\circ$ while the angle between the shore and the island at point Q is $90^\circ - 45^\circ = 45^\circ$.

We pause for a moment to summarize our known (and label our unknown) information below.



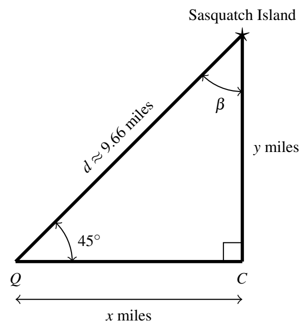
In order to use the Law of Sines to find the distance d from Q to the island, we first need to find the measure of β which is the angle opposite the side of length 5 miles.

As a result of the angles γ and 45° being supplemental, we get $\gamma = 180^\circ - 45^\circ = 135^\circ$. Knowing γ, we now find $\beta = 180^\circ - 30^\circ - \gamma = 180^\circ - 30^\circ - 135^\circ = 15^\circ$.

By the Law of Sines, we have $\frac{d}{\sin(30^\circ)} = \frac{5}{\sin(15^\circ)}$ which gives $d = \frac{5\sin(30^\circ)}{\sin(15^\circ)} \approx 9.66$ miles.

To find the point on the coast closest to the island, which we've labeled as C in the next diagram, we need to find the perpendicular distance from the island to the coast.[8] Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the island.

[8]Do you see why C must lie to the right (East) of Q?



Using Definition 7.2, we get $\sin(45°) = \frac{y}{d}$, so $y = d\sin(45°) \approx 9.66\left(\frac{\sqrt{2}}{2}\right) \approx 6.83$ miles. Hence, the island is approximately 6.83 miles from the coast.

To find the distance from Q to C, we note that $\beta = 180° - 90° - 45° = 45°$ so by symmetry,[9] we get $x = y \approx 6.83$ miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point Q. □

8.4.2 Exercises

In Exercises 1 - 20, solve for the remaining side(s) and angle(s) if possible. As in the text, (α, a), (β, b) and (γ, c) are angle-side opposite pairs.

1. $\alpha = 13°$, $\beta = 17°$, $a = 5$
2. $\alpha = 73.2°$, $\beta = 54.1°$, $a = 117$
3. $\alpha = 95°$, $\beta = 85°$, $a = 33.33$
4. $\alpha = 95°$, $\beta = 62°$, $a = 33.33$
5. $\alpha = 117°$, $a = 35$, $b = 42$
6. $\alpha = 117°$, $a = 45$, $b = 42$
7. $\alpha = 68.7°$, $a = 88$, $b = 92$
8. $\alpha = 42°$, $a = 17$, $b = 23.5$
9. $\alpha = 68.7°$, $a = 70$, $b = 90$
10. $\alpha = 30°$, $a = 7$, $b = 14$
11. $\alpha = 42°$, $a = 39$, $b = 23.5$
12. $\gamma = 53°$, $\alpha = 53°$, $c = 28.01$
13. $\alpha = 6°$, $a = 57$, $b = 100$
14. $\gamma = 74.6°$, $c = 3$, $a = 3.05$
15. $\beta = 102°$, $b = 16.75$, $c = 13$
16. $\beta = 102°$, $b = 16.75$, $c = 18$

[9] Or by Definition 7.2 again ...

17. $\beta = 102^\circ$, $\gamma = 35^\circ$, $b = 16.75$

18. $\beta = 29.13^\circ$, $\gamma = 83.95^\circ$, $b = 314.15$

19. $\gamma = 120^\circ$, $\beta = 61^\circ$, $c = 4$

20. $\alpha = 50^\circ$, $a = 25$, $b = 12.5$

21. Find the area of the triangles given in Exercises 1, 12 and 20 above.

The Grade of a Road: The grade of a road is much like the pitch of a roof (See Example **??**) in that it expresses the ratio of rise/run. In the case of a road, this ratio is always positive because it is measured going uphill and it is usually given as a percentage. For example, a road which rises 7 feet for every 100 feet of (horizontal) forward progress is said to have a 7% grade. However, if we want to apply any Trigonometry to a story problem involving roads going uphill or downhill, we need to view the grade as an angle with respect to the horizontal. In Exercises 22 - 24, we first have you change road grades into angles and then use the Law of Sines in an application.

22. Using a right triangle with a horizontal leg of length 100 and vertical leg with length 7, show that a 7% grade means that the road (hypotenuse) makes about a 4° angle with the horizontal. (It will not be exactly 4°, but it's pretty close.)

23. What grade is given by a 9.65° angle made by the road and the horizontal?[10]

24. Along a long, straight stretch of mountain road with a 7% grade, you see a tall tree standing perfectly plumb alongside the road.[11] From a point 500 feet downhill from the tree, the angle of inclination from the road to the top of the tree is 6°. Use the Law of Sines to find the height of the tree. (Hint: First show that the tree makes a 94° angle with the road.)

Exercises 25 - 31 use the concept of bearings as introduced in Section 8.4.1.

25. Find the angle θ in standard position with $0^\circ \leq \theta < 360^\circ$ which corresponds to each of the bearings given below.

    (a) due west
    (b) S83°E
    (c) N5.5°E
    (d) due south
    (e) N31.25°W
    (f) N45°E
    (g) S45°W

26. The Colonel spots a campfire at a of bearing N42°E from his current position. Sarge, who is positioned 3000 feet due east of the Colonel, reckons the bearing to the fire to be N20°W from his current position. Determine the distance from the campfire to each man, rounded to the nearest foot.

[10] I have friends who live in Pacifica, CA and their road is actually this steep. It's not a nice road to drive.

[11] The word 'plumb' here means that the tree is perpendicular to the horizontal.

27. A hiker starts walking due west from Sasquatch Point and gets to the Chupacabra Trailhead before she realizes that she hasn't reset her pedometer. From the Chupacabra Trailhead she hikes for 5 miles along a bearing of N53°W which brings her to the Muffin Ridge Observatory. From there, she knows a bearing of S65°E will take her straight back to Sasquatch Point. How far will she have to walk to get from the Muffin Ridge Observatory to Sasquach Point? What is the distance between Sasquatch Point and the Chupacabra Trailhead?

28. The captain of the SS Bigfoot sees a signal flare at a bearing of N15°E from her current location. From his position, the captain of the HMS Sasquatch finds the signal flare to be at a bearing of N75°W. If the SS Bigfoot is 5 miles from the HMS Sasquatch and the bearing from the SS Bigfoot to the HMS Sasquatch is N50°E, find the distances from the flare to each vessel, rounded to the nearest tenth of a mile.

29. Carl spies a potential Sasquatch nest at a bearing of N10°E and radios Jeff, who is at a bearing of N50°E from Carl's position. From Jeff's position, the nest is at a bearing of S70°W. If Jeff and Carl are 500 feet apart, how far is Jeff from the Sasquatch nest? Round your answer to the nearest foot.

30. A hiker determines the bearing to a lodge from her current position is S40°W. She proceeds to hike 2 miles at a bearing of S20°E at which point she determines the bearing to the lodge is S75°W. How far is she from the lodge at this point? Round your answer to the nearest hundredth of a mile.

31. A watchtower spots a ship off shore at a bearing of N70°E. A second tower, which is 50 miles from the first at a bearing of S80°E from the first tower, determines the bearing to the ship to be N25°W. How far is the boat from the second tower? Round your answer to the nearest tenth of a mile.

Exercises 32 - 33 use the concepts of 'angle of inclination' and 'angle of depression' introduced in Section 7.2.1 on page 7.2.1 and Exercise 18, respectively.

32. Skippy and Sally decide to hunt UFOs. One night, they position themselves 2 miles apart on an abandoned stretch of desert runway. An hour into their investigation, Skippy spies a UFO hovering over a spot on the runway directly between him and Sally. He records the angle of inclination from the ground to the craft to be 75° and radios Sally immediately to find the angle of inclination from her position to the craft is 50°. How high off the ground is the UFO at this point? Round your answer to the nearest foot. (Recall: 1 mile is 5280 feet.)

33. The angle of depression from an observer in an apartment complex to a gargoyle on the building next door is 55°. From a point five stories below the original observer, the angle of inclination to the gargoyle is 20°. Find the distance from each observer to the gargoyle and the distance from the gargoyle to the apartment complex. Round your answers to the nearest foot. (Use the rule of thumb that one story of a building is 9 feet.)

34. Prove that the Law of Sines holds when $\triangle ABC$ is a right triangle.

35. Discuss with your classmates why knowing only the three angles of a triangle is not enough to determine any of the sides.

36. Discuss with your classmates why the Law of Sines cannot be used to find the angles in the triangle when only the three sides are given. Also discuss what happens if only two sides and the angle between them are given. (Said another way, explain why the Law of Sines cannot be used in the SSS and SAS cases.)

37. Given $\alpha = 30^\circ$ and $b = 10$, choose four different values for a so that

    (a) the information yields no triangle

    (b) the information yields exactly one right triangle

    (c) the information yields two distinct triangles

    (d) the information yields exactly one obtuse triangle

    Explain why you cannot choose a in such a way as to have $\alpha = 30^\circ$, $b = 10$ and your choice of a yield only one triangle where that unique triangle has three acute angles.

38. Use the cases and diagrams in the proof of the Law of Sines (Theorem 8.15) to prove the area formulas given in Theorem 8.17. Why do those formulas yield square units when four quantities are being multiplied together?

SECTION 8.4 EXERCISE ANSWERS A.1.8

8.5 Law of Cosines

In Section 8.4, we developed the Law of Sines (Theorem 8.15) to enable us to solve triangles in the 'Angle-Angle-Side' (AAS), the 'Angle-Side-Angle' (ASA) and the ambiguous 'Side-Side-Angle' (SSA) cases.

In this section, we develop the Law of Cosines which handles solving triangles in the 'Side-Angle-Side' (SAS) and 'Side-Side-Side' (SSS) cases.[1] We state and prove the theorem below.

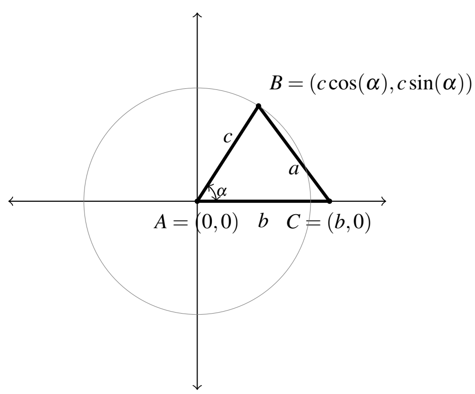
Theorem 8.18. Law of Cosines:
Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following equations hold

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha) \qquad b^2 = a^2 + c^2 - 2ac\cos(\beta) \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

To prove the theorem, we consider a generic triangle with the vertex of angle α at the origin with side b positioned along the positive x-axis as sketched in the diagram below.



From this set-up, we immediately find that the coordinates of A and C are $A(0,0)$ and $C(b,0)$. From Theorem 7.4, we know that because the point $B(x,y)$ lies on a circle of radius c, the coordinates of B are $B(x,y) = B(c\cos(\alpha), c\sin(\alpha))$. (This would be true even if α were an obtuse or right angle so although we have drawn the case when α is acute, the following computations hold for any angle α drawn in standard position where $0 < \alpha < 180°$.)

We note that the distance between the points B and C is none other than the length of side a. Using the distance formula, Equation 1.1, we get

[1] Here, 'Side-Angle-Side' means that we are given two sides and the 'included' angle - that is, the given angle is adjacent to both of the given sides.

$$
\begin{aligned}
a &= \sqrt{(c\cos(\alpha)-b)^2+(c\sin(\alpha)-0)^2} \\
a^2 &= \left(\sqrt{(c\cos(\alpha)-b)^2+c^2\sin^2(\alpha)}\right)^2 \\
a^2 &= (c\cos(\alpha)-b)^2+c^2\sin^2(\alpha) \\
a^2 &= c^2\cos^2(\alpha)-2bc\cos(\alpha)+b^2+c^2\sin^2(\alpha) \\
a^2 &= c^2\left(\cos^2(\alpha)+\sin^2(\alpha)\right)+b^2-2bc\cos(\alpha) \\
a^2 &= c^2(1)+b^2-2bc\cos(\alpha) && \text{As } \cos^2(\alpha)+\sin^2(\alpha)=1 \\
a^2 &= c^2+b^2-2bc\cos(\alpha)
\end{aligned}
$$

The remaining formulas given in Theorem 8.18 can be shown by simply reorienting the triangle to place a different vertex at the origin. We leave these details to the reader.

What's important about a and α in the above proof is that (α, a) is an angle-side opposite pair and b and c are the sides adjacent to α – the same can be said of any other angle-side opposite pair in the triangle.

Notice that the proof of the Law of Cosines relies on the distance formula which has its roots in the Pythagorean Theorem. That being said, the Law of Cosines can be thought of as a *generalization* of the Pythagorean Theorem.

Indeed, in a triangle in which $\gamma = 90^\circ$, (i.e., a right triangle) then $\cos(\gamma) = \cos(90^\circ) = 0$ and we get the familiar relationship $c^2 = a^2 + b^2$. What this means is that in the larger mathematical sense, the Law of Cosines and the Pythagorean Theorem amount to pretty much the same thing.[2]

Example 8.5.1. Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\beta = 50^\circ$, $a = 7$ units, $c = 2$ units
2. $a = 4$ units, $b = 7$ units, $c = 5$ units

Solution.

1. Solve the triangle with the properties: $\beta = 50^\circ$, $a = 7$ units, $c = 2$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

   We are given the lengths of two sides, $a = 7$ and $c = 2$, and the measure of the included angle, $\beta = 50^\circ$. With no angle-side opposite pair to use for the Law of Sines, we apply the Law of Cosines. We get $b^2 = 7^2 + 2^2 - 2(7)(2)\cos(50^\circ)$ which yields $b = \sqrt{53 - 28\cos(50^\circ)} \approx 5.92$ units.

   In order to determine the measures of the remaining angles α and γ, we are forced to used the derived value for b. There are two ways to proceed at this point. We could use the Law of Cosines again, or, now that we have the angle-side opposite pair (β, b) we could use the Law of Sines.

[2] This shouldn't come as too much of a shock. All of the theorems in Trigonometry can ultimately be traced back to the definition of the circular functions along with the distance formula and hence, the Pythagorean Theorem.

The advantage to using the Law of Cosines over the Law of Sines in cases like this is that unlike the sine function, the cosine function distinguishes between acute and obtuse angles. The cosine of an acute is positive, whereas the cosine of an obtuse angle is negative. The sine of both acute and obtuse angles are positive, thus the sine of an angle alone is not enough to determine if the angle in question is acute or obtuse.

As both authors of the textbook prefer the Law of Cosines, we proceed with this method first. When using the Law of Cosines, it's always best to find the measure of the largest unknown angle first, as this will give us the obtuse angle of the triangle, if there is one.

The largest angle is opposite the longest side, so we choose to find α first. To that end, we use the formula $\cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$ and substitute $a = 7$, $b = \sqrt{53-28\cos(50^\circ)}$ and $c = 2$. We get[3]

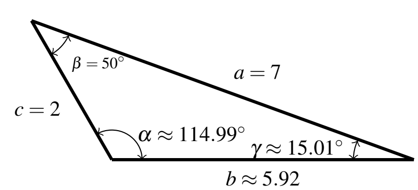
$$\cos(\alpha) = \frac{2-7\cos(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}$$

Because α is an angle in a triangle, we know the radian measure of α must lie between 0 and π radians. This matches the range of the arccosine function, so we have

$$\alpha = \arccos\left(\frac{2-7\cos(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}\right) \text{ radians} \approx 114.99^\circ$$

At this point, we could find γ using $\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 114.99^\circ - 50^\circ = 15.01^\circ$, that is if we trust our approximation for α.

To minimize propagation of error (and obtain an *exact* answer for γ), however, we could use the Law of Cosines again.[4] From $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$ with $a = 7$, $b = \sqrt{53-28\cos(50^\circ)}$ and $c = 2$, we get $\gamma = \arccos\left(\frac{7-2\cos(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}\right)$ radians $\approx 15.01^\circ$. We sketch the triangle below.



As we mentioned earlier, once we've determined b it is possible to use the Law of Sines to find the remaining angles. Here, however, we must proceed with caution as we are in the ambiguous (SSA) case. In the ambiguous case it is advisable to first find the *smallest* of the unknown angles, because we are guaranteed it will be acute.[5]

[3] after simplifying ...
[4] Your instructor will let you know which procedure to use. It all boils down to how much you trust your calculator.
[5] There can only be one *obtuse* angle in the triangle, and if there is one, it must be the largest.

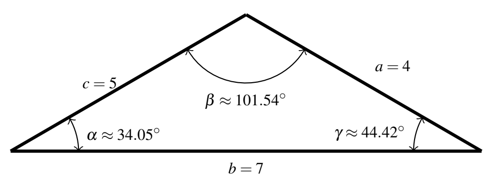
In this case, we would find γ as the side opposite γ is smaller than the side opposite the other unknown angle, α. Using the angle-side opposite pair (β, b), we get $\frac{\sin(\gamma)}{2} = \frac{\sin(50^\circ)}{\sqrt{53-28\cos(50^\circ)}}$. The usual calculations produces $\gamma \approx 15.01^\circ$ and $\alpha = 180^\circ - \beta - \gamma \approx 180^\circ - 50^\circ - 15.01^\circ = 114.99^\circ$.

2. Solve the triangle with the properties: $a = 4$ units, $b = 7$ units, $c = 5$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

All three sides and no angles are given, so we are forced to use the Law of Cosines. Following our discussion in the previous problem, we find β first, as it is opposite the longest side, b. We get $\cos(\beta) = \frac{a^2+c^2-b^2}{2ac} = -\frac{1}{5}$, so $\beta = \arccos\left(-\frac{1}{5}\right)$ radians $\approx 101.54^\circ$.

Now that we have obtained an angle-side opposite pair (β, b), we could proceed using the Law of Sines. The Law of Cosines, however, offers us a rare opportunity to find the remaining angles using *only* the data given to us in the statement of the problem.

Using the Law of Cosines, we get $\gamma = \arccos\left(\frac{5}{7}\right)$ radians $\approx 44.42^\circ$ and $\alpha = \arccos\left(\frac{29}{35}\right)$ radians $\approx 34.05^\circ$. We sketch this triangle below.

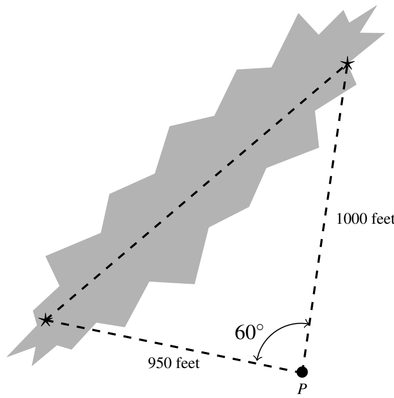


□

We note that, depending on how many decimal places are carried through successive calculations, and depending on which approach is used to solve the problem, the approximate answers you obtain may differ slightly from those the authors obtain in the Examples and the Exercises.

A great example of this is number 2 in Example 8.5.1, where the *approximate* values we record for the measures of the angles sum to 180.01°, which is geometrically impossible.

Example 8.5.2. A researcher wishes to determine the width of a vernal pond as drawn below. From a point P, he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is 60°, find the width of the pond.



Solution. We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle.

Calling this length w (for *width*), we get $w^2 = 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) = 952500$ from which we get $w = \sqrt{952500} \approx 976$ feet. □

In Section 8.4, we used the proof of the Law of Sines to develop Theorem 8.17 as an alternate formula for the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula, the so-called Heron's Formula.[6]

Theorem 8.19. Heron's Formula:

Suppose a, b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a+b+c)$. Then, the area A enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We prove Theorem 8.19 using Theorem 8.17. Using the convention that the angle γ is opposite the side c, we have $A = \frac{1}{2}ab\sin(\gamma)$ from Theorem 8.17.

In order to simplify computations, we start by manipulating the expression for A^2.

$$\begin{aligned} A^2 &= \left(\frac{1}{2}ab\sin(\gamma)\right)^2 \\ &= \frac{1}{4}a^2b^2\sin^2(\gamma) \\ &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \quad \text{as } \sin^2(\gamma) = 1-\cos^2(\gamma). \end{aligned}$$

[6]Or 'Hero's Formula.'

The Law of Cosines tells us $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$, so substituting this into our equation for A^2 gives

$$\begin{aligned}
A^2 &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \\
&= \frac{a^2b^2}{4}\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= \frac{a^2b^2}{4}\left[1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] \\
&= \frac{a^2b^2}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] \\
&= \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16}
\end{aligned}$$

Recognizing $4a^2b^2$ as a perfect square, $4a^2b^2 = (2ab)^2$, we can factor the resulting difference of squares:

$$\begin{aligned}
A^2 &= \frac{(2ab)^2-\left(a^2+b^2-c^2\right)^2}{16} \\
&= \frac{\left(2ab-\left[a^2+b^2-c^2\right]\right)\left(2ab+\left[a^2+b^2-c^2\right]\right)}{16} && \text{difference of squares.} \\
&= \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16}
\end{aligned}$$

Next, we regroup $c^2-a^2+2ab-b^2 = c^2-\left[a^2-2ab+b^2\right]$ and $a^2+2ab+b^2-c^2 = \left[a^2+2ab+b^2\right]-c^2$. Recognizing $a^2-2ab+b^2 = (a-b)^2$ and $a^2+2ab+b^2 = (a+b)^2$, we continue factoring:

$$\begin{aligned}
A^2 &= \frac{\left(c^2-\left[a^2-2ab+b^2\right]\right)\left(\left[a^2+2ab+b^2\right]-c^2\right)}{16} \\
&= \frac{\left(c^2-(a-b)^2\right)\left((a+b)^2-c^2\right)}{16} && \text{perfect square trinomials.} \\
&= \frac{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)}{16} && \text{difference of squares.} \\
&= \frac{(b+c-a)(a+c-b)(a+b-c)(a+b+c)}{16} \\
&= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2}
\end{aligned}$$

At this stage, we recognize the last factor as the semiperimeter, $s = \frac{1}{2}(a+b+c) = \frac{a+b+c}{2}$. To complete the proof, we note that

$$(s-a) = \frac{a+b+c}{2} - a = \frac{a+b+c-2a}{2} = \frac{b+c-a}{2}$$

Similarly, we find $(s-b) = \frac{a+c-b}{2}$ and $(s-c) = \frac{a+b-c}{2}$. Hence, we get

$$\begin{aligned} A^2 &= \frac{(b+c-a)}{2} \cdot \frac{(a+c-b)}{2} \cdot \frac{(a+b-c)}{2} \cdot \frac{(a+b+c)}{2} \\ &= (s-a)(s-b)(s-c)s \end{aligned}$$

so that $A = \sqrt{s(s-a)(s-b)(s-c)}$ as required.

We close with an example of Heron's Formula.

Example 8.5.3. Compute the area enclosed of the triangle in Example 8.5.1 number 2.

Solution. We are given $a = 4$, $b = 7$ and $c = 5$. Using these values, we find $s = \frac{1}{2}(4+7+5) = 8$, $(s-a) = 8-4 = 4$, $(s-b) = 8-7 = 1$ and $(s-c) = 8-5 = 3$.

Per Heron's Formula, $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(8)(4)(1)(3)} = \sqrt{96} = 4\sqrt{6} \approx 9.80$ square units. □

8.5.1 Exercises

In Exercises 1 - 10, use the Law of Cosines to compute the remaining side(s) and angle(s) if possible.

1. $a = 7,\ b = 12,\ \gamma = 59.3^\circ$
2. $\alpha = 104^\circ,\ b = 25,\ c = 37$
3. $a = 153,\ \beta = 8.2^\circ,\ c = 153$
4. $a = 3,\ b = 4,\ \gamma = 90^\circ$
5. $\alpha = 120^\circ,\ b = 3,\ c = 4$
6. $a = 7,\ b = 10,\ c = 13$
7. $a = 1,\ b = 2,\ c = 5$
8. $a = 300,\ b = 302,\ c = 48$
9. $a = 5,\ b = 5,\ c = 5$
10. $a = 5,\ b = 12, ; c = 13$

In Exercises 11 - 16, use any method to solve for the remaining side(s) and angle(s), if possible.

11. $a = 18,\ \alpha = 63^\circ,\ b = 20$
12. $a = 37,\ b = 45,\ c = 26$
13. $a = 16,\ \alpha = 63^\circ,\ b = 20$
14. $a = 22,\ \alpha = 63^\circ,\ b = 20$
15. $\alpha = 42^\circ,\ b = 117,\ c = 88$
16. $\beta = 7^\circ,\ \gamma = 170^\circ,\ c = 98.6$
17. Determine the area of the triangles given in Exercises 6, 8 and 10 above.
18. The hour hand on my antique Seth Thomas schoolhouse clock in 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

19. A geologist wants to measure the diameter of an impact crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

20. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of N8.2°E for 192 miles and it can fly to Bigfoot Falls by following a bearing of S68.5°E for 207 miles.[7] Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

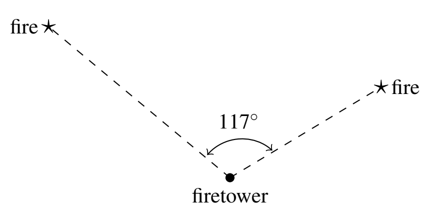
21. Cliffs of Insanity Point and Bigfoot Falls from Exericse 20 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

22. A naturalist sets off on a hike from a lodge on a bearing of S80°W. After 1.5 miles, she changes her bearing to S17°W and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

23. The HMS Sasquatch leaves port on a bearing of N23°E and travels for 5 miles. It then changes course and follows a heading of S41°E for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

24. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of N32°E. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now S70°W, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

25. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression[8] made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot.



[7] Please refer to Section 8.4.1 for an introduction to bearings.
[8] See Exercise 18 in Section 7.2.1 for the definition of this angle.

HINT: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will give you a Side-Angle-Side case in which to apply the Law of Cosines.

26. If you apply the Law of Cosines to the ambiguous Angle-Side-Side (ASS) case, the result is a quadratic equation whose variable is that of the missing side. If the equation has no positive real zeros then the information given does not yield a triangle. If the equation has only one positive real zero then exactly one triangle is formed and if the equation has two distinct positive real zeros then two distinct triangles are formed. Apply the Law of Cosines to Exercises 11, 13 and 14 above in order to demonstrate this result.

SECTION 8.5 EXERCISE ANSWERS A.1.8

CHAPTER 9

VECTORS

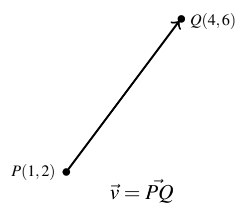
9.1 Vectors

As we have seen numerous times in this book, Mathematics can be used to model and solve real-world problems. For many applications, real numbers suffice; that is, real numbers with the appropriate units attached can be used to answer questions like "How close is the nearest Sasquatch nest?"

There are other times though, when these kinds of quantities do not suffice. Perhaps it is important to know, for instance, how close the nearest Sasquatch nest is as well as the direction in which it lies. To answer questions like these which involve both a quantitative answer, or *magnitude*, along with a *direction*, we use the mathematical objects called **vectors**.[1]

A vector is represented geometrically as a directed line segment where the magnitude of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrow at one endpoint of the segment. When referring to vectors in this text, we shall adopt[2] the 'arrow' notation, so the symbol $\vec{v}$ is read as 'the vector v'. Below is a typical vector $\vec{v}$ with endpoints $P(1,2)$ and $Q(4,6)$.

The point P is called the *initial point* or *tail* of $\vec{v}$ and the point Q is called the *terminal point* or *head* of $\vec{v}$. We can reconstruct $\vec{v}$ completely from P and Q, so we write $\vec{v} = \overrightarrow{PQ}$, where the order of points P (initial point) and Q (terminal point) is important. (Think about this before moving on.)



While it is true that P and Q completely determine $\vec{v}$, it is important to note that because vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\vec{v}$ is considered to be the same vector as $\vec{v}$, regardless of its initial point.

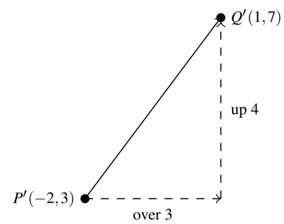
In the case of our vector $\vec{v}$ above, any vector which moves three units to the right and four up[3] from its initial point to arrive at its terminal point is considered the same vector as $\vec{v}$. The notation we use to capture this idea is the *component form* of the vector, $\vec{v} = \langle 3,4 \rangle$, where the first number, 3, is called the *x-component* of $\vec{v}$ and the second number, 4, is called the *y-component* of $\vec{v}$.

For example, if we wanted to reconstruct $\vec{v} = \langle 3,4 \rangle$ with initial point $P'(-2,3)$, then we would find the terminal point of $\vec{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$.

[1] The word 'vector' comes from the Latin *vehere* meaning 'to convey' or 'to carry.'

[2] Other textbook authors use bold vectors such as $\mathbf{v}$. We find that writing in bold font on the chalkboard is inconvenient at best, so we have chosen the 'arrow' notation.

[3] If this idea of 'over' and 'up' seems familiar, it should. The slope of the line segment containing $\vec{v}$ is $\frac{4}{3}$.



$\vec{v} = \langle 3,4\rangle$ with initial point $P'(-2,3)$

$\vec{v} = \langle 3,4\rangle$ with initial point $P'(-2,3)$

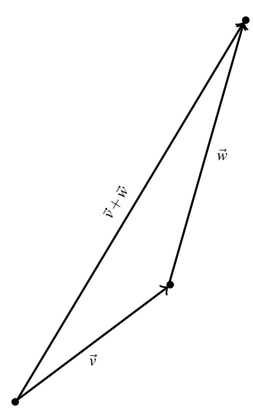
The component form of a vector is what ties these very geometric objects back to Algebra and ultimately Trigonometry. We generalize our example in our definition below.

Definition 9.1. Suppose $\vec{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0\rangle$$

Using the language of components, we have that two vectors are equal if and only if their corresponding components are equal. That is, $\langle v_1, v_2\rangle = \langle v_1', v_2'\rangle$ if and only if $v_1 = v_1'$ and $v_2 = v_2'$. (Again, think about this before reading on.)

We now set about defining operations on vectors. Suppose we are given two vectors $\vec{v}$ and $\vec{w}$. The sum, or *resultant* vector $\vec{v}+\vec{w}$ is obtained as follows. First, plot $\vec{v}$. Next, plot $\vec{w}$ so that its initial point is the terminal point of $\vec{v}$. To plot the vector $\vec{v}+\vec{w}$ we begin at the initial point of $\vec{v}$ and end at the terminal point of $\vec{w}$. It is helpful to think of the vector $\vec{v}+\vec{w}$ as the 'net result' of moving along $\vec{v}$ then moving along $\vec{w}$.



$\vec{v}, \vec{w}$, and $\vec{v}+\vec{w}$

Our next example makes good use of resultant vectors and reviews bearings and the Law of Cosines.[4]

[4]If necessary, review Sections 8.4.1 and 8.5.

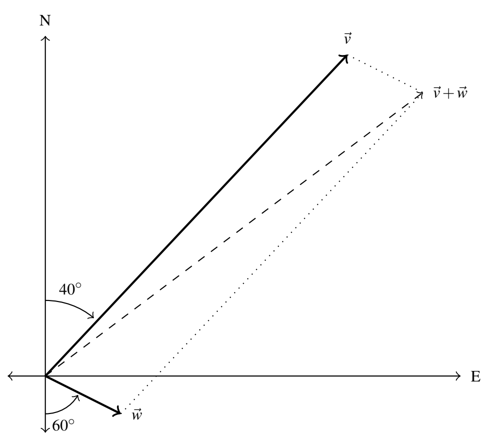
Example 9.1.1. A plane leaves an airport with an airspeed[5] of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

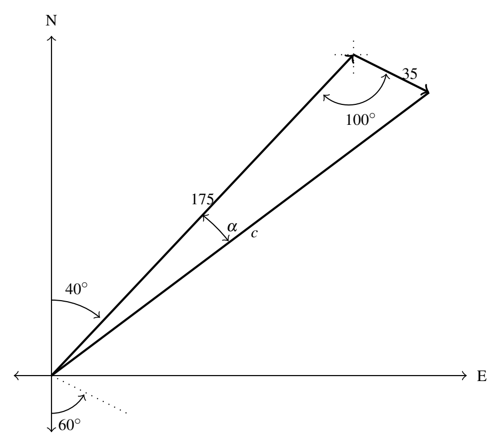
Solution. For both the plane and the wind, we are given their speeds and their directions. Coupling speed (as a magnitude) with direction is the concept of *velocity* which we've seen a few times before.[6]

We let $\vec{v}$ denote the plane's velocity and $\vec{w}$ denote the wind's velocity in the diagram below. The 'true' speed and bearing is found by analyzing the resultant vector, $\vec{v}+\vec{w}$.

From the vector diagram, we get a triangle, the lengths of whose sides are the magnitude of $\vec{v}$, which is 175, the magnitude of $\vec{w}$, which is 35, and the magnitude of $\vec{v}+\vec{w}$, which we'll call c.

From the given bearing information, we go through the usual geometry to determine that the angle between the sides of length 35 and 175 measures 100°.





From the Law of Cosines, we determine $c = \sqrt{31850 - 12250\cos(100^\circ)} \approx 184$, which means the true speed of the plane is (approximately) 184 miles per hour.

To determine the true bearing of the plane, we need to determine the angle α. Using the Law of Cosines once more,[7] we find $\cos(\alpha) = \frac{c^2+29400}{350c}$ so that $\alpha \approx 11^\circ$.

Given the geometry of the situation, we add α to the given 40° and find the true bearing of the plane to be (approximately) N51°E. □

Our next step is to define addition of vectors component-wise to match the geometric action.[8]

[5]That is, the speed of the plane relative to the air around it. If there were no wind, plane's airspeed would be the same as its speed as observed from the ground. How does wind affect this? Keep reading!

[6]See Section 7.1.3, for instance.

[7]Or, as our given angle, 100°, is obtuse, we could use the Law of Sines without any ambiguity here.

[8]In more advanced courses. chief among them Linear Algebra, vectors are actually *defined* as $1 \times n$ or $n \times 1$ matrices, depending on the situation.

Definition 9.2. Suppose $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. The vector $\vec{v} + \vec{w}$ is defined by

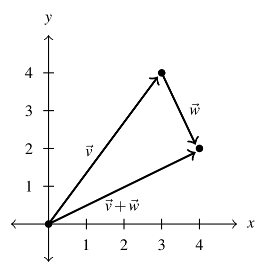
$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

Example 9.1.2. Let $\vec{v} = \langle 3, 4 \rangle$ and suppose $\vec{w} = \overrightarrow{PQ}$ where $P(-3, 7)$ and $Q(-2, 5)$. Compute $\vec{v} + \vec{w}$ and interpret this sum geometrically.

Solution. Before we can add the vectors using Definition 9.2, we need to write $\vec{w}$ in component form. Using Definition 9.1, we get $\vec{w} = \langle -2 - (-3), 5 - 7 \rangle = \langle 1, -2 \rangle$. Thus,

$$\vec{v} + \vec{w} = \langle 3, 4 \rangle + \langle 1, -2 \rangle = \langle 3 + 1, 4 + (-2) \rangle = \langle 4, 2 \rangle .$$

To visualize this sum, we draw $\vec{v}$ with its initial point at $(0, 0)$ (for convenience) so that its terminal point is $(3, 4)$. Next, we graph $\vec{w}$ with its initial point at $(3, 4)$. Moving one to the right and two down, we find the terminal point of $\vec{w}$ to be $(4, 2)$.



We see the vector $\vec{v} + \vec{w}$ has initial point $(0, 0)$ and terminal point $(4, 2)$ so its component form is $\langle 4, 2 \rangle$. □

In order for vector addition to enjoy the same kinds of properties as real number addition, it is necessary to extend our definition of vectors to include a ‘zero vector’, $\vec{0} = \langle 0, 0 \rangle$.

Geometrically, $\vec{0}$ represents a point, which we can (very broadly) think of as a directed line segment with the same initial and terminal points. The reader may well object to the inclusion of $\vec{0}$, because after all, vectors are supposed to have both a magnitude (length) and a direction.

While it seems clear that the magnitude of $\vec{0}$ should be 0, it is not clear what its direction is. As we shall see, the direction of $\vec{0}$ is in fact undefined, but this minor hiccup in the natural flow of things is worth the benefits we reap by including $\vec{0}$ in our discussions. We have the following theorem.

Theorem 9.1. Properties of Vector Addition

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v}+\vec{w}=\vec{w}+\vec{v}$.
- **Associative Property:** For all vectors $\vec{u}, \vec{v}$ and $\vec{w}$, $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$.
- **Identity Property:** For all vectors $\vec{v}$,

$$\vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}.$$

  The vector $\vec{0}$ acts as the additive identity for vector addition.
- **Inverse Property:** For every vector $\vec{v}=\langle v_1, v_2\rangle$, the vector $\vec{w}=\langle -v_1, -v_2\rangle$ satisfies

$$\vec{v}+\vec{w}=\vec{w}+\vec{v}=\vec{0}.$$

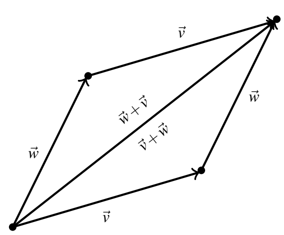
  That is, the additive inverse of a vector is the vector of the additive inverses of its components.

The properties in Theorem 9.1 are easily verified using the definition of vector addition, and are a direct consequence of the definition of vector addition along with properties inherited from real number arithmetic.

For the commutative property, we note that if $\vec{v}=\langle v_1, v_2\rangle$ and $\vec{w}=\langle w_1, w_2\rangle$ then

$$\begin{aligned}\vec{v}+\vec{w} &= \langle v_1, v_2\rangle+\langle w_1, w_2\rangle \\ &= \langle v_1+w_1, v_2+w_2\rangle \\ &= \langle w_1+v_1, w_2+v_2\rangle \\ &= \vec{w}+\vec{v}\end{aligned}$$

Geometrically, we can 'see' the commutative property by realizing that the sums $\vec{v}+\vec{w}$ and $\vec{w}+\vec{v}$ are the same directed diagonal determined by the parallelogram below.



Demonstrating the commutative property of vector addition.

The proofs of the associative and identity properties proceed similarly, and the reader is encouraged to verify them and provide accompanying diagrams.

The additive identity property is likewise verified algebraically using a calculation. If $\vec{v}=\langle v_1, v_2\rangle$, then

$$\vec{v}+\vec{0}=\langle v_1, v_2\rangle+\langle 0,0\rangle=\langle v_1+0, v_2+0\rangle=\langle v_1, v_2\rangle=\vec{v}.$$

From the commutative property of vector addition, we get that $\vec{0}+\vec{v}=\vec{v}$ as well. Again, the reader is encouraged to visualize what this means geometrically.[9]

Regarding additive inverses, we can verify by direct computation that if $\vec{v}=\langle v_1,v_2\rangle$ and $\vec{w}=\langle -v_1,-v_2\rangle$,

$$\vec{v}+\vec{w}=\langle v_1,v_2\rangle+\langle -v_1,-v_2\rangle=\langle v_1+(-v_1),v_2+(-v_2)\rangle=\langle 0,0\rangle=\vec{0}.$$

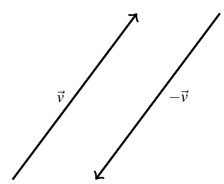
Once again, the commutative property of vector addition assures us that, likewise, $\vec{w}+\vec{v}=\vec{0}$.

Moreover, additive inverses of vectors are *unique*. That is, given a vector $\vec{v}=\langle v_1,v_2\rangle$, there is precisely only *one* vector $\vec{w}$ so that $\vec{v}+\vec{w}=\vec{0}$.

To see this, suppose a vector $\vec{w}=\langle w_1,w_2\rangle$ satisfies $\vec{v}+\vec{w}=\vec{0}$. By the definition of vector addition, we have $\langle v_1+w_1,v_2+w_2\rangle=\langle 0,0\rangle$. Hence, $v_1+w_1=0$ and $v_2+w_2=0$. We get $w_1=-v_1$ and $w_2=-v_2$ so that $\vec{w}=\langle -v_1,-v_2\rangle$ as prescribed in Theorem 9.1.

Hence, every vector $\vec{v}$ has one, and only one, additive inverse. In general, we denote the additive inverse of a vector $\vec{v}$ with the (highly suggestive) notation $-\vec{v}$.

Geometrically, the vectors $\vec{v}=\langle v_1,v_2\rangle$ and $-\vec{v}=\langle -v_1,-v_2\rangle$ have the same length, but opposite directions. As a result, when adding the vectors geometrically, the sum $\vec{v}+(-\vec{v})$ results in starting at the initial point of $\vec{v}$ and ending back at the initial point of $\vec{v}$. That is, the net result of moving $\vec{v}$ then $-\vec{v}$ is not moving at all.



Using the additive inverse of a vector, we can define the difference of two vectors: $\vec{v}-\vec{w}=\vec{v}+(-\vec{w})$. Looking at this at the level of components, we see if $\vec{v}=\langle v_1,v_2\rangle$ and $\vec{w}=\langle w_1,w_2\rangle$ then

$$\begin{aligned}\vec{v}-\vec{w} &= \vec{v}+(-\vec{w})\\ &= \langle v_1,v_2\rangle+\langle -w_1,-w_2\rangle\\ &= \langle v_1+(-w_1),v_2+(-w_2)\rangle\\ &= \langle v_1-w_1,v_2-w_2\rangle\end{aligned}$$

In other words, like vector addition, vector subtraction works component-wise.

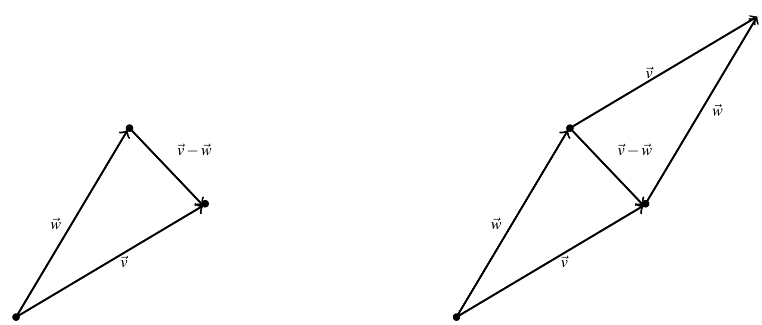
To interpret the vector $\vec{v}-\vec{w}$ geometrically, we note

$$\begin{aligned}\vec{w}+(\vec{v}-\vec{w}) &= \vec{w}+(\vec{v}+(-\vec{w})) && \text{Definition of Vector Subtraction}\\ &= \vec{w}+((-\vec{w})+\vec{v}) && \text{Commutativity of Vector Addition}\\ &= (\vec{w}+(-\vec{w}))+\vec{v} && \text{Associativity of Vector Addition}\\ &= \vec{0}+\vec{v} && \text{Definition of Additive Inverse}\\ &= \vec{v} && \text{Definition of Additive Identity}\end{aligned}$$

[9] Recall, $\vec{0}$ is represented geometrically as a point ...

This means that the 'net result' of moving along $\vec{w}$ then moving along $\vec{v}-\vec{w}$ is just $\vec{v}$ itself.

From the diagram below on the left, we see that $\vec{v}-\vec{w}$ may be interpreted as the vector whose initial point is the terminal point of $\vec{w}$ and whose terminal point is the terminal point of $\vec{v}$.



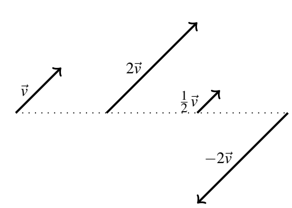
It is also worth mentioning that in the parallelogram determined by the vectors $\vec{v}$ and $\vec{w}$ above on the right, the vector $\vec{v}-\vec{w}$ is one of the diagonals – the other being $\vec{v}+\vec{w}$.

Next, we discuss *scalar* multiplication – that is, taking a real number times a vector.

Definition 9.3. If k is a real number and $\vec{v}=\langle v_1,v_2\rangle$, we define $k\vec{v}$ by

$$k\vec{v}=k\langle v_1,v_2\rangle=\langle kv_1,kv_2\rangle$$

Scalar multiplication by k in vectors can be understood geometrically as scaling the vector (if $k>0$) or scaling the vector and reversing its direction (if $k<0$) as demonstrated below.



Note by definition 9.3, $(-1)\vec{v}=(-1)\langle v_1,v_2\rangle=\langle(-1)v_1,(-1)v_2\rangle=\langle -v_1,-v_2\rangle=-\vec{v}$, which is what we would expect. This and other properties of scalar multiplication are summarized in the next theorem.

Theorem 9.2. Properties of Scalar Multiplication

- **Associative Property:** For every vector $\vec{v}$ and scalars k and r, $(kr)\vec{v} = k(r\vec{v})$.
- **Identity Property:** For all vectors $\vec{v}$, $1\vec{v} = \vec{v}$.
- **Additive Inverse Property:** For all vectors $\vec{v}$, $-\vec{v} = (-1)\vec{v}$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every vector $\vec{v}$ and scalars k and r,
$$(k+r)\vec{v} = k\vec{v} + r\vec{v}$$
- **Distributive Property of Scalar Multiplication over Vector Addition:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k,
$$k(\vec{v}+\vec{w}) = k\vec{v} + k\vec{w}$$
- **Zero Product Property:** If $\vec{v}$ is vector and k is a scalar, then
$$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

The proof of Theorem 9.2, like the proof of Theorem 9.1, ultimately boils down to the definition of scalar multiplication and properties of real numbers.

For example, to prove the associative property, we let $\vec{v} = \langle v_1, v_2 \rangle$. If k and r are scalars then

$$\begin{array}{rcll}
(kr)\vec{v} & = & (kr)\langle v_1, v_2 \rangle & \\
& = & \langle (kr)v_1, (kr)v_2 \rangle & \text{Definition of Scalar Multiplication} \\
& = & \langle k(rv_1), k(rv_2) \rangle & \text{Associative Property of Real Number Multiplication} \\
& = & k\langle rv_1, rv_2 \rangle & \text{Definition of Scalar Multiplication} \\
& = & k\left(r\langle v_1, v_2 \rangle\right) & \text{Definition of Scalar Multiplication} \\
& = & k(r\vec{v}) &
\end{array}$$

The reader is invited to think about what this property means geometrically. The remaining properties are proved similarly and are left as exercises.

Our next example demonstrates how Theorem 9.2 allows us to do the same kind of algebraic manipulations with vectors as we do with variables – multiplication and division of vectors notwithstanding. If the pedantry seems familiar, it should.

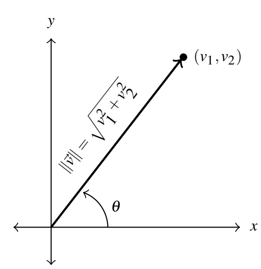
Example 9.1.3. Solve $5\vec{v} - 2\left(\vec{v} + \langle 1, -2 \rangle\right) = \vec{0}$ for $\vec{v}$.

Solution.

$$\begin{aligned}
5\vec{v}-2\left(\vec{v}+\ \langle 1,-2\rangle\right) &= \vec{0} \\
5\vec{v}+(-1)\left[2\left(\vec{v}+\langle 1,-2\rangle\right)\right] &= \vec{0} \\
5\vec{v}+\left[(-1)(2)\right]\left(\vec{v}+\langle 1,-2\rangle\right) &= \vec{0} \\
5\vec{v}+(-2)\left(\vec{v}+\langle 1,-2\rangle\right) &= \vec{0} \\
5\vec{v}+\left[(-2)\vec{v}+(-2)\langle 1,-2\rangle\right] &= \vec{0} \\
5\vec{v}+\left[(-2)\vec{v}+\langle(-2)(1),(-2)(-2)\rangle\right] &= \vec{0} \\
\left[5\vec{v}+(-2)\vec{v}\right]+\langle -2,4\rangle &= \vec{0} \\
(5+(-2))\vec{v}+\langle -2,4\rangle &= \vec{0} \\
3\vec{v}+\langle -2,4\rangle &= \vec{0} \\
(3\vec{v}+\langle -2,4\rangle)+(-\langle -2,4\rangle) &= \vec{0}+(-\langle -2,4\rangle) \\
3\vec{v}+\left[\langle -2,4\rangle+(-\langle -2,4\rangle)\right] &= \vec{0}+(-1)\langle -2,4\rangle \\
3\vec{v}+\vec{0} &= \vec{0}+\langle(-1)(-2),(-1)(4)\rangle \\
3\vec{v} &= \langle 2,-4\rangle \\
\tfrac{1}{3}(3\vec{v}) &= \tfrac{1}{3}(\langle 2,-4\rangle) \\
\left[\left(\tfrac{1}{3}\right)(3)\right]\vec{v} &= \left\langle\left(\tfrac{1}{3}\right)(2),\left(\tfrac{1}{3}\right)(-4)\right\rangle \\
1\vec{v} &= \left\langle\tfrac{2}{3},-\tfrac{4}{3}\right\rangle \\
\vec{v} &= \left\langle\tfrac{2}{3},-\tfrac{4}{3}\right\rangle
\end{aligned}$$

The reader is invited to check our solution in the original equation. □

A vector whose initial point is $(0,0)$ is said to be in **standard position**. If $\vec{v}=\langle v_1,v_2\rangle$ is plotted in standard position, then its terminal point is necessarily (v_1,v_2). (Once more, think about this before reading on.)



$\vec{v}=\langle v_1,v_2\rangle$ in standard position.

Plotting a vector in standard position enables us to more easily quantify the concepts of magnitude and direction of the vector.

Recall the magnitude of vector $\vec{v}$ is the length of the directed line segment representing $\vec{v}$. When plotted in standard position, the length of this line segment is none other than the distance from the origin $(0,0)$ to the point (v_1,v_2). Hence, the magnitude of $\vec{v}$, which we denote $\|\vec{v}\|$, is given by $\|\vec{v}\|=\sqrt{v_1^2+v_2^2}$.

Turning to the notion of direction, we note that the point (v_1,v_2) is on the terminal side of the angle θ depicted in the diagram above. From Theorem 7.4, we have $v_1=\|\vec{v}\|\cos(\theta)$ and $v_2=\|\vec{v}\|\sin(\theta)$. From the definition of scalar multiplication and vector equality, we get

$$\begin{aligned}\vec{v} &= \langle v_1, v_2 \rangle \\ &= \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \sin(\theta) \rangle \\ &= \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle\end{aligned}$$

This motivates the following definition.

Definition 9.4. Suppose $\vec{v}$ is a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$.

- The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$
- The **direction angle** of $\vec{v}$, denoted θ, is given by $\theta = \arctan\left(\dfrac{v_2}{v_1}\right)$

  Note: θ is an angle in standard position whose terminal side contains the point (v_1, v_2).
- If $\vec{v} \neq \vec{0}$, the **directional vector** of $\vec{v}$, denoted $\hat{v}$ is given by $\hat{v} = \langle \cos(\theta), \sin(\theta) \rangle$

Taken together, we get $\vec{v} = \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \sin(\theta) \rangle$.

A few remarks are in order. First, we note that if $\vec{v} \neq 0$ then there are infinitely many angles θ which satisfy Definition 9.4. However, the fact that all of them must contain the same point (v_1, v_2) on their terminal sides means they are all coterminal.

Hence, if θ and θ' both satisfy the conditions of Definition 9.4, then $\cos(\theta) = \cos(\theta')$ and $\sin(\theta) = \sin(\theta')$, and as such, $\langle \cos(\theta), \sin(\theta) \rangle = \langle \cos(\theta'), \sin(\theta') \rangle$ making $\hat{v}$ is well-defined.

For $\vec{0} = \langle 0, 0 \rangle$, note that $\|\vec{0}\| = \sqrt{0^2 + 0^2} = 0$. Hence, $\|\vec{0}\| \langle \cos(\theta), \sin(\theta) \rangle = 0 \langle \cos(\theta), \sin(\theta) \rangle = < 0, 0 >$ for *every* angle θ. In other words, every angle θ satisfies the equation $\vec{v} = \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \sin(\theta) \rangle$ in Definition 9.4, so for this reason, $\hat{0}$ is undefined.

The following theorem summarizes the important facts about the magnitude and direction of a vector.

Theorem 9.3. Properties of Magnitude and Direction:
Suppose $\vec{v}$ is a vector.

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
- For all scalars k, $\|k\,\vec{v}\| = |k|\|\vec{v}\|$.
- If $\vec{v} \neq \vec{0}$ then $\vec{v} = \|\vec{v}\|\hat{v}$, so that $\hat{v} = \left(\dfrac{1}{\|\vec{v}\|}\right)\vec{v}$. [a]

[a] We will see in Definition 9.5 that we also call $\hat{v}$ the **unit vector** in the direction of $\vec{v}$.

The proof of the first property in Theorem 9.3 is a direct consequence of the definition of $\|\vec{v}\|$. Given

$\vec{v} = \langle v_1, v_2 \rangle$, then $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$ which is by definition greater than or equal to 0. Moreover, $\sqrt{v_1^2 + v_2^2} = 0$ if and only of $v_1^2 + v_2^2 = 0$ if and only if $v_1 = v_2 = 0$. Hence, $\|\vec{v}\| = 0$ if and only if $\vec{v} = \langle 0, 0 \rangle = \vec{0}$, as required.

The second property is a result of the definition of magnitude and scalar multiplication along with a property of radicals. If $\vec{v} = \langle v_1, v_2 \rangle$ and k is a scalar then

$$\begin{array}{rcll}
\|k\,\vec{v}\| & = & \|k\,\langle v_1, v_2 \rangle\| & \\
& = & \|\langle kv_1, kv_2 \rangle\| & \text{Definition of scalar multiplication} \\
& = & \sqrt{(kv_1)^2 + (kv_2)^2} & \text{Definition of magnitude} \\
& = & \sqrt{k^2 v_1^2 + k^2 v_2^2} & \\
& = & \sqrt{k^2 (v_1^2 + v_2^2)} & \\
& = & \sqrt{k^2}\sqrt{v_1^2 + v_2^2} & \text{Product Rule for Radicals} \\
& = & |k|\sqrt{v_1^2 + v_2^2} & \sqrt{k^2} = |k| \\
& = & |k|\|\vec{v}\| &
\end{array}$$

The equation $\vec{v} = \|\vec{v}\|\hat{v}$ in Theorem 9.3 is a consequence of the definitions of $\|\vec{v}\|$ and $\hat{v}$ and was worked out in the discussion just prior to Definition 9.4 on page 790. In words, the equation $\vec{v} = \|\vec{v}\|\hat{v}$ says that any given vector is the product of its magnitude and its direction – an important concept to keep in mind when studying and using vectors.

The formula for $\hat{v}$ stated in Theorem 9.3 is a consequence of solving $\vec{v} = \|\vec{v}\|\hat{v}$ for $\hat{v}$ by multiplying[10] both sides of the equation by $\frac{1}{\|\vec{v}\|}$ and using the properties of Theorem 9.2. We leave these details to the reader. We are overdue for an example.

[10] Of course, to go from $\vec{v} = \|\vec{v}\|\hat{v}$ to $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$, we are essentially 'dividing both sides' of the equation by the scalar $\|\vec{v}\|$. The authors encourage the reader, however, to work out the details carefully to gain an appreciation of the properties in play.

Example 9.1.4.

1. Write the component form of the vector $\vec{v}$ with $\|\vec{v}\| = 5$ so that when $\vec{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle[11] with the negative x-axis.

2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, compute $\|\vec{v}\|$ and θ, $0 \leq \theta < 2\pi$ so that $\vec{v} = \|\vec{v}\| \left\langle \cos(\theta), \sin(\theta) \right\rangle$.

3. For the vectors $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, determine the following.

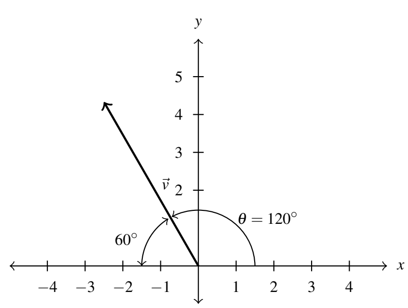
   (a) $\hat{v}$ (b) $\|\vec{v}\| - 2\|\vec{w}\|$ (c) $\|\vec{v} - 2\vec{w}\|$ (d) $\|\hat{w}\|$

Solution.

1. Write the component form of the vector $\vec{v}$ with $\|\vec{v}\| = 5$ so that when $\vec{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle with the negative x-axis.

   We are told that $\|\vec{v}\| = 5$ and are given information about its direction, so we can use the formula $\vec{v} = \|\vec{v}\|\hat{v}$ to get the component form of $\vec{v}$.

   To determine $\hat{v}$, we appeal to Definition 9.4. Because $\vec{v}$ lies in Quadrant II and makes a 60° angle with the negative x-axis, one angle θ satisfying the criteria of Definition 9.4 is $\theta = 120^\circ$.



   Hence, $\hat{v} = \left\langle \cos(120^\circ), \sin(120^\circ) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, so $\vec{v} = \|\vec{v}\|\hat{v} = 5\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$.

2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, compute $\|\vec{v}\|$ and θ, $0 \leq \theta < 2\pi$ so that $\vec{v} = \|\vec{v}\| \left\langle \cos(\theta), \sin(\theta) \right\rangle$.

   For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, we get $\|\vec{v}\| = \sqrt{(3)^2 + (-3\sqrt{3})^2} = 6$. In light of Definition 9.4, we can find the θ we're after by finding a Quadrant IV angle whose terminal side contains the point $(3, -3\sqrt{3})$.

[11] Due to the utility of vectors in 'real-world' applications, we will usually use degree measure for the angle when giving the vector's direction. There are examples and exercises in which radians are used as well.

We compute θ using the definition of the direction angle of $\vec{v}$, $\theta = \arctan\left(\dfrac{v_2}{v_1}\right)$.

$$\begin{aligned} \theta &= \arctan\left(\frac{v_2}{v_1}\right) \\ &= \arctan\left(\frac{-3\sqrt{3}}{3}\right) \\ &= \arctan\left(-\sqrt{3}\right) \\ &= -\tfrac{\pi}{3} + \pi k, \text{ where } k \text{ is any integer} \end{aligned}$$

As the terminal side of $\vec{v}$ is in Quadrant IV, we let $k = 2$ resulting in $\theta = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$.

We may check our answer by verifying $6\left\langle \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right)\right\rangle = \left\langle 3, -3\sqrt{3}\right\rangle = \vec{v}$.

3. (a) For the vectors $\vec{v} = \langle 3, 4\rangle$ and $\vec{w} = \langle 1, -2\rangle$, determine $\hat{v}$.

We are given the component form of $\vec{v}$, sowe'll use the formula $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$. For $\vec{v} = \langle 3, 4\rangle$, we have $\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Hence, $\hat{v} = \frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$.

(b) For the vectors $\vec{v} = \langle 3, 4\rangle$ and $\vec{w} = \langle 1, -2\rangle$, determine $\|\vec{v}\| - 2\|\vec{w}\|$.

We know from our work above that $\|\vec{v}\| = 5$, so to find $\|\vec{v}\| - 2\|\vec{w}\|$, we need only find $\|\vec{w}\|$. Given $\vec{w} = \langle 1, -2\rangle$, we get $\|\vec{w}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Hence, $\|\vec{v}\| - 2\|\vec{w}\| = 5 - 2\sqrt{5}$.

(c) For the vectors $\vec{v} = \langle 3, 4\rangle$ and $\vec{w} = \langle 1, -2\rangle$, determine $\|\vec{v} - 2\vec{w}\|$.

In the expression $\|\vec{v} - 2\vec{w}\|$, notice that the arithmetic on the vectors comes first, then the magnitude. Hence, our first step is to find the component form of the vector $\vec{v} - 2\vec{w}$. We get $\vec{v} - 2\vec{w} = \langle 3, 4\rangle - 2\langle 1, -2\rangle = \langle 1, 8\rangle$. Hence, $\|\vec{v} - 2\vec{w}\| = \|\langle 1, 8\rangle\| = \sqrt{1^2 + 8^2} = \sqrt{65}$.

(d) For the vectors $\vec{v} = \langle 3, 4\rangle$ and $\vec{w} = \langle 1, -2\rangle$, determine $\|\hat{w}\|$.

One approach to find $\|\hat{w}\|$, is to first find $\hat{w}$ and then take the magnitude.

Using the formula $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$ along with $\|\vec{w}\| = \sqrt{5}$, which we found the in the previous problem, we get $\hat{w} = \frac{1}{\sqrt{5}}\langle 1, -2\rangle = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle = \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\right\rangle$.

Hence, $\|\hat{w}\| = \sqrt{\left(\frac{\sqrt{5}}{5}\right)^2 + \left(-\frac{2\sqrt{5}}{5}\right)^2} = \sqrt{\frac{5}{25} + \frac{20}{25}} = \sqrt{1} = 1$.

Alternatively, we can use Theorem 9.3. Given $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$, where $\frac{1}{\|\vec{w}\|} > 0$ is a scalar,

$$\|\hat{w}\| = \left\|\left(\frac{1}{\|\vec{w}\|}\right)\vec{w}\right\| = \frac{1}{\|\vec{w}\|}\|\vec{w}\| = \frac{\|\vec{w}\|}{\|\vec{w}\|} = 1.$$

For a third way to show $\|\hat{w}\| = 1$, we can appeal to Definition 9.4. Because $\hat{w} = \langle \cos(\theta), \sin(\theta)\rangle$ for some angle θ, $\|\hat{w}\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$, where we have used the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. No matter how we approach the problem, $\|\hat{w}\| = 1$. $\square$

Note that the second and third solutions to number 3d in Example 9.1.4 above work for *any* nonzero vector, $\vec{w}$. We will have more to say about this shortly.

The process exemplified by number 1 in Example 9.1.4 above by which we take information about the magnitude and direction of a vector and find the component form of a vector is called **resolving** a vector into its components. As an application of this process, we revisit Example 9.1.1 below.

Example 9.1.5. A plane leaves an airport with an airspeed of 175 miles per hour with bearing N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution. We proceed as we did in Example 9.1.1 and let $\vec{v}$ denote the plane's velocity and $\vec{w}$ denote the wind's velocity, and set about determining $\vec{v}+\vec{w}$.

If we regard the airport as being at the origin, the positive y-axis acting as due north and the positive x-axis acting as due east, we see that the vectors $\vec{v}$ and $\vec{w}$ are in standard position and their directions correspond to the angles 50° and -30°, respectively.

Hence, the component form of $\vec{v} = 175\left\langle \cos(50^\circ), \sin(50^\circ)\right\rangle = \left\langle 175\cos(50^\circ), 175\sin(50^\circ)\right\rangle$ and the component form of $\vec{w} = \left\langle 35\cos(-30^\circ), 35\sin(-30^\circ)\right\rangle$.

We have no convenient way to express the exact values of cosine and sine of 50°, so we leave both vectors in terms of cosines and sines.[12] Adding corresponding components, we find the resultant vector $\vec{v}+\vec{w} = \left\langle 175\cos(50^\circ)+35\cos(-30^\circ), 175\sin(50^\circ)+35\sin(-30^\circ)\right\rangle$. To find the 'true' speed of the plane, we compute the magnitude of this resultant vector

$$\|\vec{v}+\vec{w}\| = \sqrt{(175\cos(50^\circ)+35\cos(-30^\circ))^2+(175\sin(50^\circ)+35\sin(-30^\circ))^2} \approx 184$$

Hence, the 'true' speed of the plane is approximately 184 miles per hour.

To find the true bearing, we need to find the angle θ whose terminal side when graphed in standard position contains $(x,y) = (175\cos(50^\circ)+35\cos(-30^\circ), 175\sin(50^\circ)+35\sin(-30^\circ))$.

As both of these coordinates are positive,[13] we know θ is a Quadrant I angle, as depicted below. Furthermore,

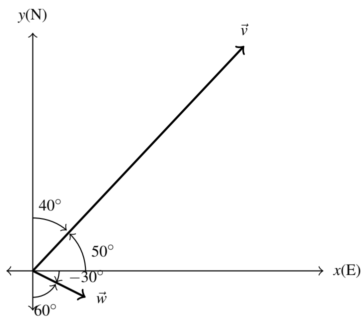
$$\tan(\theta) = \frac{y}{x} = \frac{175\sin(50^\circ)+35\sin(-30^\circ)}{175\cos(50^\circ)+35\cos(-30^\circ)},$$

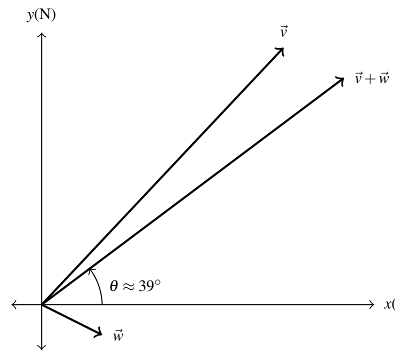
so using the arctangent function,[14] we get $\theta \approx 39^\circ$. Because, for the purposes of bearing, we need the angle between $\vec{v}+\vec{w}$ and the positive y-axis, we take the complement of θ and find the 'true' bearing of the plane to be approximately N51°E.

[12] Keeping things 'calculator' friendly, for once!

[13] Yes, a calculator approximation is the quickest way to see this, but you can also use good old-fashioned inequalities and the fact that $45^\circ \leq 50^\circ \leq 60^\circ$.

[14] We could just have easily used arcsine or arccosine here ...





□

In part 3d of Example 9.1.4, we saw that the length of the direction vector, $\hat{w}$, $\|\hat{w}\| = 1$. Vectors of length 1 play such an important role that they are given a special name.

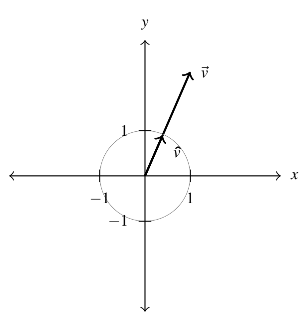
Definition 9.5. Unit Vectors: Let $\vec{v}$ be a vector. If $\|\vec{v}\| = 1$, then we say that $\vec{v}$ is a **unit vector**.

Note that if $\vec{v}$ is a unit vector, then necessarily, $\vec{v} = \|\vec{v}\|\hat{v} = 1 \cdot \hat{v} = \hat{v}$. Conversely, in the solution of part 3d of Example 9.1.4, two different arguments show for any nonzero vector $\vec{v}$, $\|\hat{v}\| = 1$, so $\hat{v}$ is a unit vector.

In other words, unit vectors are direction vectors and vice-versa. Indeed, the vector $\hat{v}$ which we have defined as 'the *directional vector* of $\vec{v}$' is often described as 'the *unit vector in the direction* of $\vec{v}$.'

In practice, if $\vec{v}$ is a unit vector we write it as $\hat{v}$ as opposed to $\vec{v}$ because we have reserved the '$\hat{\ }$' notation for unit vectors. The process of multiplying a nonzero vector by the factor $\frac{1}{\|\vec{v}\|}$ to produce a unit vector is called '**normalizing** the vector.'

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. (You should take the time to show this.) As a result, we visualize normalizing a nonzero vector $\vec{v}$ as shrinking[15] its terminal point, when plotted in standard position, back to the Unit Circle.



Visualizing vector normalization $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$

[15]... if $\|\vec{v}\| > 1$ ...

Of all of the unit vectors, two deserve special mention.

Definition 9.6. The Principal Unit Vectors:

- The vector $\hat{\imath}$ is defined by $\hat{\imath} = \langle 1, 0 \rangle$
- The vector $\hat{\jmath}$ is defined by $\hat{\jmath} = \langle 0, 1 \rangle$

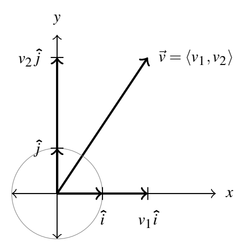
Geometrically, in the xy-plane, the vector $\hat{\imath}$ as represents the positive x-direction, whereas the vector $\hat{\jmath}$ represents the positive y-direction. We have the following 'decomposition' theorem.[16]

Theorem 9.4. Principal Vector Decomposition Theorem:
Let $\vec{v}$ be a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Then $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

The proof of Theorem 9.4 is straightforward. Given $\hat{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \langle 0, 1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$v_1\hat{\imath} + v_2\hat{\jmath} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = \langle v_1, v_2 \rangle = \vec{v}$$

Geometrically, the situation looks like this:



We conclude this section with a classic example which demonstrates how vectors are used in physics to study forces. A 'force' is defined as a 'push' or a 'pull.' The intensity of the push or pull is the magnitude of the force, and is measured in Newtons (N) in the SI system or pounds (lbs.) in the English system.[17]

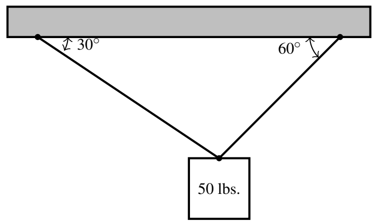
The following example uses all of the concepts in this section, and should be studied in great detail.

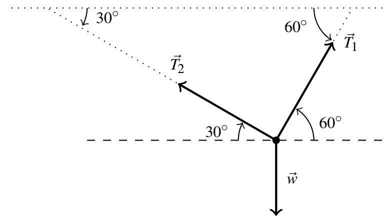
Example 9.1.6. A 50 pound speaker is suspended from the ceiling by two support braces. If one of them makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each of the supports?

Solution. We represent the problem schematically below along with the corresponding vector diagram.

[16] We will see a generalization of Theorem 9.4 in Section 9.2. Stay tuned!

[17] See also Section 8.3.3.





We have three forces acting on the speaker: the weight of the speaker, which we'll call $\vec{w}$, pulling the speaker directly downward, and the forces on the support rods, which we'll call $\vec{T}_1$ and $\vec{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively.

We are looking for the tensions on the support, which are the magnitudes $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$. In order for the speaker to remain stationary,[18] we require $\vec{w}+\vec{T}_1+\vec{T}_2=\vec{0}$.

Viewing the common initial point of these vectors as the origin and the dashed line as the x-axis, we use Theorem 9.3 to get component representations for the three vectors involved. We can model the weight of the speaker as a vector pointing directly downwards with a magnitude of 50 pounds. That is, $\|\vec{w}\|=50$ and $\hat{w}=-\hat{\jmath}=\langle 0,-1\rangle$. Hence, $\vec{w}=50\langle 0,-1\rangle=\langle 0,-50\rangle$. For the force in the first support, we get

$$\begin{aligned}
\vec{T}_1 &= \|\vec{T}_1\|\left\langle \cos(60^\circ), \sin(60^\circ)\right\rangle \\
&= \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2}\right\rangle
\end{aligned}$$

For the second support, we note that the angle 30° is measured from the negative x-axis, so the angle needed to write $\vec{T}_2$ in component form is 150°. Hence

$$\begin{aligned}
\vec{T}_2 &= \|\vec{T}_2\|\left\langle \cos(150^\circ), \sin(150^\circ)\right\rangle \\
&= \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2}\right\rangle
\end{aligned}$$

The requirement $\vec{w}+\vec{T}_1+\vec{T}_2=\vec{0}$ gives us the vector equation:

$$\begin{aligned}
\vec{w}+\vec{T}_1+\vec{T}_2 &= \vec{0} \\
\langle 0,-50\rangle + \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2}\right\rangle + \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2}\right\rangle &= \langle 0,0\rangle \\
\left\langle \frac{\|\vec{T}_1\|}{2} - \frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2} + \frac{\|\vec{T}_2\|}{2} - 50\right\rangle &= \langle 0,0\rangle
\end{aligned}$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$.

[18]This is the criteria for 'static equilbrium'.

$$\begin{cases} (E1) & \dfrac{\|\vec{T}_1\|}{2} - \dfrac{\|\vec{T}_2\|\sqrt{3}}{2} = 0 \\ (E2) & \dfrac{\|\vec{T}_1\|\sqrt{3}}{2} + \dfrac{\|\vec{T}_2\|}{2} - 50 = 0 \end{cases}$$

From $(E1)$, we get $\|\vec{T}_1\| = \|\vec{T}_2\|\sqrt{3}$. Substituting that into $(E2)$ gives $\frac{(\|\vec{T}_2\|\sqrt{3})\sqrt{3}}{2} + \frac{\|\vec{T}_2\|}{2} - 50 = 0$. Solving, we get $2\|\vec{T}_2\| - 50 = 0$, so $\|\vec{T}_2\| = 25$ pounds. Hence, $\|\vec{T}_1\| = \|\vec{T}_2\|\sqrt{3} = 25\sqrt{3}$ pounds. □

Note that the sum of the tensions on the wires in Example 9.1.6 exceed the 50 pounds of the speaker. Explaining why this happens is a good exercise and gets at the heart of the concept of vectors and resolution of forces. Speaking of exercises . . .

9.1.1 Exercises

In Exercises 1 - 10, use the given pair of vectors $\vec{v}$ and $\vec{w}$ to compute the following quantities. State whether the result is a vector or a scalar.

- $\vec{v} + \vec{w}$
- $\vec{w} - 2\vec{v}$
- $\|\vec{v} + \vec{w}\|$
- $\|\vec{v}\| + \|\vec{w}\|$
- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v}$
- $\|\vec{w}\|\hat{v}$

Finally, verify that the vectors satisfy the **Parallelogram Law**

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left[\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2\right]$$

1. $\vec{v} = \langle 12, -5\rangle$, $\vec{w} = \langle 3, 4\rangle$
2. $\vec{v} = \langle -7, 24\rangle$, $\vec{w} = \langle -5, -12\rangle$
3. $\vec{v} = \langle 2, -1\rangle$, $\vec{w} = \langle -2, 4\rangle$
4. $\vec{v} = \langle 10, 4\rangle$, $\vec{w} = \langle -2, 5\rangle$
5. $\vec{v} = \left\langle -\sqrt{3}, 1\right\rangle$, $\vec{w} = \left\langle 2\sqrt{3}, 2\right\rangle$
6. $\vec{v} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$, $\vec{w} = \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle$
7. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$, $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
8. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$, $\vec{w} = \left\langle -1, -\sqrt{3}\right\rangle$
9. $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$, $\vec{w} = -2\hat{\jmath}$
10. $\vec{v} = \frac{1}{2}\left(\hat{\imath} + \hat{\jmath}\right)$, $\vec{w} = \frac{1}{2}\left(\hat{\imath} - \hat{\jmath}\right)$

In Exercises 11 - 25, write the component form of the vector $\vec{v}$ using the information given about its magnitude and direction. Give exact values.

11. $\|\vec{v}\| = 6$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 60° angle with the positive x-axis
12. $\|\vec{v}\| = 3$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 45° angle with the positive x-axis

13. $\|\vec{v}\| = \frac{2}{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes a 60° angle with the positive y-axis

14. $\|\vec{v}\| = 12$; when drawn in standard position $\vec{v}$ lies along the positive y-axis

15. $\|\vec{v}\| = 4$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes a 30° angle with the negative x-axis

16. $\|\vec{v}\| = 2\sqrt{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes a 30° angle with the positive y-axis

17. $\|\vec{v}\| = \frac{7}{2}$; when drawn in standard position $\vec{v}$ lies along the negative x-axis

18. $\|\vec{v}\| = 5\sqrt{6}$; when drawn in standard position $\vec{v}$ lies in Quadrant III and makes a 45° angle with the negative x-axis

19. $\|\vec{v}\| = 6.25$; when drawn in standard position $\vec{v}$ lies along the negative y-axis

20. $\|\vec{v}\| = 4\sqrt{3}$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes a 30° angle with the positive x-axis

21. $\|\vec{v}\| = 5\sqrt{2}$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes a 45° angle with the negative y-axis

22. $\|\vec{v}\| = 2\sqrt{5}$; when drawn in standard position $\vec{v}$ lies in Quadrant I and makes an angle measuring $\arctan(2)$ with the positive x-axis

23. $\|\vec{v}\| = \sqrt{10}$; when drawn in standard position $\vec{v}$ lies in Quadrant II and makes an angle measuring $\arctan(3)$ with the negative x-axis

24. $\|\vec{v}\| = 5$; when drawn in standard position $\vec{v}$ lies in Quadrant III and makes an angle measuring $\arctan\left(\frac{4}{3}\right)$ with the negative x-axis

25. $\|\vec{v}\| = 26$; when drawn in standard position $\vec{v}$ lies in Quadrant IV and makes an angle measuring $\arctan\left(\frac{5}{12}\right)$ with the positive x-axis

In Exercises 26 - 31, approximate the component form of the vector $\vec{v}$ using the information given about its magnitude and direction. Round your approximations to two decimal places.

26. $\|\vec{v}\| = 392$; when drawn in standard position $\vec{v}$ makes a 117° angle with the positive x-axis

27. $\|\vec{v}\| = 63.92$; when drawn in standard position $\vec{v}$ makes a 78.3° angle with the positive x-axis

28. $\|\vec{v}\| = 5280$; when drawn in standard position $\vec{v}$ makes a 12° angle with the positive x-axis

29. $\|\vec{v}\| = 450$; when drawn in standard position $\vec{v}$ makes a 210.75° angle with the positive x-axis

30. $\|\vec{v}\| = 168.7$; when drawn in standard position $\vec{v}$ makes a 252° angle with the positive x-axis

31. $\|\vec{v}\| = 26$; when drawn in standard position $\vec{v}$ makes a 304.5° angle with the positive x-axis

In Exercises 32 - 52, for the given vector $\vec{v}$, compute the magnitude $\|\vec{v}\|$ and the direction angle θ with $0 \leq \theta < 360^\circ$ so that $\vec{v} = \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle$ (See Definition 9.4.) Round approximations to two decimal places.

32. $\vec{v} = \langle 1, \sqrt{3} \rangle$
33. $\vec{v} = \langle 5, 5 \rangle$
34. $\vec{v} = \langle -2\sqrt{3}, 2 \rangle$
35. $\vec{v} = \left\langle -\sqrt{2}, \sqrt{2} \right\rangle$
36. $\vec{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
37. $\vec{v} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
38. $\vec{v} = \langle 6, 0 \rangle$
39. $\vec{v} = \langle -2.5, 0 \rangle$
40. $\vec{v} = \langle 0, \sqrt{7} \rangle$
41. $\vec{v} = -10\hat{\jmath}$
42. $\vec{v} = \langle 3, 4 \rangle$
43. $\vec{v} = \langle 12, 5 \rangle$
44. $\vec{v} = \langle -4, 3 \rangle$
45. $\vec{v} = \langle -7, 24 \rangle$
46. $\vec{v} = \langle -2, -1 \rangle$
47. $\vec{v} = \langle -2, -6 \rangle$
48. $\vec{v} = \hat{\imath} + \hat{\jmath}$
49. $\vec{v} = \hat{\imath} - 4\hat{\jmath}$
50. $\vec{v} = \langle 123.4, -77.05 \rangle$
51. $\vec{v} = \langle 965.15, 831.6 \rangle$
52. $\vec{v} = \langle -114.1, 42.3 \rangle$

53. A small boat leaves the dock at Camp DuNuthin and heads across the Nessie River at 17 miles per hour (that is, with respect to the water) at a bearing of S68°W. The river is flowing due east at 8 miles per hour. What is the boat's true speed and heading? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

54. The HMS Sasquatch leaves port with bearing S20°E maintaining a speed of 42 miles per hour (that is, with respect to the water). If the ocean current is 5 miles per hour with a bearing of N60°E, find the HMS Sasquatch's true speed and bearing. Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

55. If the captain of the HMS Sasquatch in Exercise 54 wishes to reach Chupacabra Cove, an island 100 miles away at a bearing of S20°E from port, in three hours, what speed and heading should she set to take into account the ocean current? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

    HINT: If $\vec{v}$ denotes the velocity of the HMS Sasquatch and $\vec{w}$ denotes the velocity of the current, what does $\vec{v} + \vec{w}$ need to be to reach Chupacabra Cove in three hours?

56. In calm air, a plane flying from the Pedimaxus International Airport can reach Cliffs of Insanity Point in two hours by following a bearing of N8.2°E at 96 miles an hour. (The distance between the airport and the cliffs is 192 miles.) If the wind is blowing from the southeast at 25 miles per hour, what speed and bearing should the pilot take so that she makes the trip in two hours along the original heading? Round the speed to the nearest hundredth of a mile per hour and your angle to the nearest tenth of a degree.

57. The SS Bigfoot leaves Yeti Bay on a course of N37°W at a speed of 50 miles per hour. After traveling half an hour, the captain determines he is 30 miles from the bay and his bearing back to the bay is S40°E. What is the speed and bearing of the ocean current? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

58. A 600 pound Sasquatch statue is suspended by two cables from a gymnasium ceiling. If each cable makes a 60° angle with the ceiling, find the tension on each cable. Round your answer to the nearest pound.

59. Two cables are to support an object hanging from a ceiling. If the cables are each to make a 42° angle with the ceiling, and each cable is rated to withstand a maximum tension of 100 pounds, what is the heaviest object that can be supported? Round your answer down to the nearest pound.

60. A 300 pound metal star is hanging on two cables which are attached to the ceiling. The left hand cable makes a 72° angle with the ceiling while the right hand cable makes a 18° angle with the ceiling. What is the tension on each of the cables? Round your answers to three decimal places.

61. Two drunken college students have filled an empty beer keg with rocks and tied ropes to it in order to drag it down the street in the middle of the night. The stronger of the two students pulls with a force of 100 pounds at a heading of N77°E and the other pulls at a heading of S68°E. What force should the weaker student apply to his rope so that the keg of rocks heads due east? What resultant force is applied to the keg? Round your answer to the nearest pound.

62. Emboldened by the success of their late night keg pull in Exercise 61 above, our intrepid young scholars have decided to pay homage to the chariot race scene from the movie 'Ben-Hur' by tying three ropes to a couch, loading the couch with all but one of their friends and pulling it due west down the street. The first rope points N80°W, the second points due west and the third points S80°W. The force applied to the first rope is 100 pounds, the force applied to the second rope is 40 pounds and the force applied (by the non-riding friend) to the third rope is 160 pounds. They need the resultant force to be at least 300 pounds otherwise the couch won't move. Does it move? If so, is it heading due west?

63. Let $\vec{v} = \langle v_1, v_2 \rangle$ be any non-zero vector. Show that $\dfrac{1}{\|\vec{v}\|}\vec{v}$ has length 1.

64. We say that two non-zero vectors $\vec{v}$ and $\vec{w}$ are **parallel** if they have same or opposite directions. That is, $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$ are parallel if either $\hat{v} = \hat{w}$ or $\hat{v} = -\hat{w}$. Show that this means $\vec{v} = k\vec{w}$ for some non-zero scalar k and that $k > 0$ if the vectors have the same direction and $k < 0$ if they point in opposite directions.

65. The goal of this exercise is to use vectors to describe non-vertical lines in the plane. To that end, consider the line $y = 2x - 4$. Let $\vec{v}_0 = \langle 0, -4 \rangle$ and let $\vec{s} = \langle 1, 2 \rangle$. Let t be any real number. Show that the vector defined by $\vec{v} = \vec{v}_0 + t\vec{s}$, when drawn in standard position, has its terminal point on the line $y = 2x - 4$. (Hint: Show that $\vec{v}_0 + t\vec{s} = \langle t, 2t - 4 \rangle$ for any real number t.) Now consider the non-vertical line $y = mx + b$. Repeat the previous analysis with $\vec{v}_0 = \langle 0, b \rangle$ and let $\vec{s} = \langle 1, m \rangle$. Thus any non-vertical

line can be thought of as a collection of terminal points of the vector sum of $\langle 0, b\rangle$ (the position vector of the y-intercept) and a scalar multiple of the slope vector $\vec{s} = \langle 1, m\rangle$.

66. Prove the associative and identity properties of vector addition in Theorem 9.1.

67. Prove the properties of scalar multiplication in Theorem 9.2.

SECTION 9.1 EXERCISE ANSWERS A.1.9

9.2 Dot Products and Projections

In Section 9.1, we learned how add and subtract vectors and how to multiply vectors by scalars. In this section, we define a product of vectors. We begin with the following definition.

Definition 9.7. Given vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, the **dot product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

For example, if $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$,then $\vec{v} \cdot \vec{w} = \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle = (3)(1) + (4)(-2) = -5$.

Note that the dot product takes two *vectors* and produces a *scalar*. For that reason, the quantity $\vec{v} \cdot \vec{w}$ is often called the **scalar product** of $\vec{v}$ and $\vec{w}$. The dot product enjoys the following properties.

Theorem 9.5. Properties of the Dot Product

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- **Distributive Property:** For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- **Scalar Property:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k, $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.
- **Relation to Magnitude:** For all vectors $\vec{v}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Like most of the theorems involving vectors, the proof of Theorem 9.5 amounts to using the definition of the dot product and properties of real number arithmetic.

For example, to show the commutative property, let $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. Then

$$\begin{array}{rcll} \vec{v} \cdot \vec{w} & = & \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \\ & = & v_1 w_1 + v_2 w_2 & \text{Definition of Dot Product} \\ & = & w_1 v_1 + w_2 v_2 & \text{Commutativity of Real Number Multiplication} \\ & = & \langle w_1, w_2 \rangle \cdot \langle v_1, v_2 \rangle & \text{Definition of Dot Product} \\ & = & \vec{w} \cdot \vec{v} & \end{array}$$

The distributive property is proved similarly and is left as an exercise.

For the scalar property, assume that $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ and k is a scalar. Then

$$\begin{array}{rcll}
(k\vec{v}) \cdot \vec{w} & = & (k\langle v_1, v_2 \rangle) \cdot \langle w_1, w_2 \rangle & \\
& = & \langle kv_1, kv_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Scalar Multiplication} \\
& = & (kv_1)(w_1) + (kv_2)(w_2) & \text{Definition of Dot Product} \\
& = & k(v_1 w_1) + k(v_2 w_2) & \text{Associativity of Real Number Multiplication} \\
& = & k(v_1 w_1 + v_2 w_2) & \text{Distributive Law of Real Numbers} \\
& = & k\langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Dot Product} \\
& = & k(\vec{v} \cdot \vec{w}) &
\end{array}$$

We leave the proof of $k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$ as an exercise.

For the last property, we note that if $\vec{v} = \langle v_1, v_2 \rangle$, then $\vec{v} \cdot \vec{v} = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle = v_1^2 + v_2^2 = \|\vec{v}\|^2$, where the last equality comes courtesy of Definition 9.4.

The following example puts Theorem 9.5 to good use. As in Example 9.1.3, we work out the problem in great detail and encourage the reader to supply the justification for each step.

Example 9.2.1. Prove the identity: $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$.

Solution. We begin by rewriting $\|\vec{v} - \vec{w}\|^2$ in terms of the dot product using Theorem 9.5.

$$\begin{array}{rcl}
\|\vec{v} - \vec{w}\|^2 & = & (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \\
& = & (\vec{v} + [-\vec{w}]) \cdot (\vec{v} + [-\vec{w}]) \\
& = & (\vec{v} + [-\vec{w}]) \cdot \vec{v} + (\vec{v} + [-\vec{w}]) \cdot [-\vec{w}] \\
& = & \vec{v} \cdot (\vec{v} + [-\vec{w}]) + [-\vec{w}] \cdot (\vec{v} + [-\vec{w}]) \\
& = & \vec{v} \cdot \vec{v} + \vec{v} \cdot [-\vec{w}] + [-\vec{w}] \cdot \vec{v} + [-\vec{w}] \cdot [-\vec{w}] \\
& = & \vec{v} \cdot \vec{v} + \vec{v} \cdot [(-1)\vec{w}] + [(-1)\vec{w}] \cdot \vec{v} + [(-1)\vec{w}] \cdot [(-1)\vec{w}] \\
& = & \vec{v} \cdot \vec{v} + (-1)(\vec{v} \cdot \vec{w}) + (-1)(\vec{w} \cdot \vec{v}) + [(-1)(-1)](\vec{w} \cdot \vec{w}) \\
& = & \vec{v} \cdot \vec{v} + (-1)(\vec{v} \cdot \vec{w}) + (-1)(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w} \\
& = & \vec{v} \cdot \vec{v} - 2(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w} \\
& = & \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2
\end{array}$$

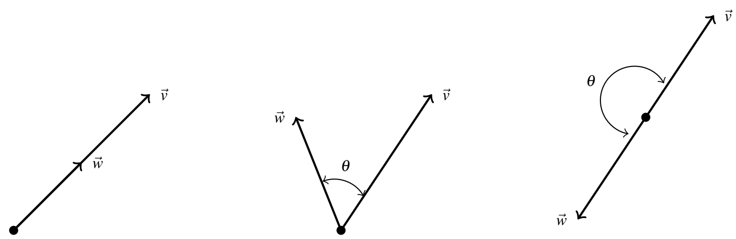
Hence, $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$ as required. □

If we take a step back from the pedantry in Example 9.2.1, we see that the bulk of the work is needed to show that $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - 2(\vec{v} \cdot \vec{w}) + \vec{w} \cdot \vec{w}$. If this looks familiar, it should.

As the dot product enjoys many of the same properties enjoyed by real numbers, the machinations required to expand $(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$ for vectors $\vec{v}$ and $\vec{w}$ match those required to expand $(v-w)(v-w)$ for real numbers v and w, and hence we get similar looking results.

The identity verified in Example 9.2.1 plays a large role in the development of the geometric properties of the dot product, which we now explore.

Suppose $\vec{v}$ and $\vec{w}$ are two nonzero vectors. If we draw $\vec{v}$ and $\vec{w}$ with the same initial point, we define the **angle between** $\vec{v}$ and $\vec{w}$ to be the angle θ determined by the rays containing the vectors $\vec{v}$ and $\vec{w}$, as illustrated below. We require $0 \le \theta \le \pi$. (Think about why this is needed in the definition.)



$\theta = 0$ $\qquad$ $0 < \theta < \pi$ $\qquad$ $\theta = \pi$

The following theorem gives us some insight into the geometric role the dot product plays.

Theorem 9.6. Geometric Interpretation of Dot Product:
If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

We prove Theorem 9.6 in cases. If $\theta = 0$, then $\vec{v}$ and $\vec{w}$ have the same direction. It follows[1] that there is a real number $k > 0$ such that $\vec{w} = k\vec{v}$. Hence, $\vec{v}\cdot\vec{w} = \vec{v}\cdot(k\vec{v}) = k(\vec{v}\cdot\vec{v}) = k\|\vec{v}\|^2$.

Working from the other end of the equation, $\|\vec{v}\|\|\vec{w}\|\cos(\theta) = \|\vec{v}\|\|k\vec{v}\|\cos(0) = \|\vec{v}\|(|k|\|\vec{v}\|)(1) = k\|\vec{v}\|^2$, where $\|k\vec{v}\| = |k|\|\vec{v}\|$ courtesy of Theorem 9.3, and $|k| = k$ because $k > 0$.

Hence, in the case $\theta = 0$, we have shown $\vec{v}\cdot\vec{w} = k\|\vec{v}\|^2$ and $\|\vec{v}\|\|\vec{w}\|\cos(\theta) = k\|\vec{v}\|^2$. Putting these two equations together shows that $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$ holds in this case.

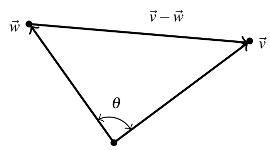
If $\theta = \pi$, $\vec{v}$ and $\vec{w}$ have the exact opposite directions, so there is a real number $k < 0$ with $\vec{w} = k\vec{v}$.

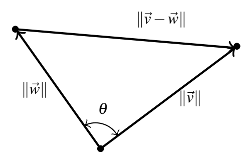
As before, we compute $\vec{v}\cdot\vec{w} = \vec{v}\cdot(k\vec{v}) = k(\vec{v}\cdot\vec{v}) = k\|\vec{v}\|^2$. Because $k < 0$ here, we have $|k| = -k$. Hence, we find $\|\vec{v}\|\|\vec{w}\|\cos(\theta) = \|\vec{v}\|\|k\vec{v}\|\cos(\pi) = \|\vec{v}\|(|k|\|\vec{v}\|)(-1) = \|\vec{v}\|(-k)\|\vec{v}\|(-1) = k\|\vec{v}\|^2$.

Once again, both $\vec{v}\cdot\vec{w} = k\|\vec{v}\|^2$ and $\|\vec{v}\|\|\vec{w}\|\cos(\theta) = k\|\vec{v}\|^2$, so $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$ in this case.

[1] Because $\vec{v} = \|\vec{v}\|\hat{v}$ and $\vec{w} = \|\vec{w}\|\hat{w}$, if $\hat{v} = \hat{w}$ then $\vec{w} = \|\vec{w}\|\hat{v} = \frac{\|\vec{w}\|}{\|\vec{v}\|}(\|\vec{v}\|\hat{v}) = \frac{\|\vec{w}\|}{\|\vec{v}\|}\vec{v}$. In this case, $k = \frac{\|\vec{w}\|}{\|\vec{v}\|} > 0$.

Next, if $0 < \theta < \pi$, the vectors $\vec{v}$, $\vec{w}$ and $\vec{v}-\vec{w}$ determine a triangle with side lengths $\|\vec{v}\|$, $\|\vec{w}\|$ and $\|\vec{v}-\vec{w}\|$, respectively, as seen in the diagram below.





The Law of Cosines yields $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos(\theta)$. From Example 9.2.1, we also have that $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$.

Equating these two expressions for $\|\vec{v}-\vec{w}\|^2$ gives $\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos(\theta) = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$ which reduces to $-2\|\vec{v}\|\|\vec{w}\|\cos(\theta) = -2(\vec{v}\cdot\vec{w})$. Hence, $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$, as required.

An immediate consequence of Theorem 9.6 is the following.

Theorem 9.7. Let $\vec{v}$ and $\vec{w}$ be nonzero vectors and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(\hat{v}\cdot\hat{w})$$

We obtain the formula in Theorem 9.7 by solving the equation given in Theorem 9.6 for θ.

As $\vec{v}$ and $\vec{w}$ are nonzero, so are $\|\vec{v}\|$ and $\|\vec{w}\|$. Hence, we may divide both sides of $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$ by $\|\vec{v}\|\|\vec{w}\|$. Given $0 \leq \theta \leq \pi$ by definition, the values of θ exactly match the range of the arccosine function. Hence,

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} \Rightarrow \theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Using Theorem 9.5, we can rewrite

$$\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \left(\frac{1}{\|\vec{v}\|}\vec{v}\right)\cdot\left(\frac{1}{\|\vec{w}\|}\vec{w}\right) = \hat{v}\cdot\hat{w},$$

giving us the alternative formula listed in Theorem 9.7: $\theta = \arccos(\hat{v}\cdot\hat{w})$. We are overdue for an example.

Example 9.2.2. Compute the angle between the following pairs of vectors. Graph each pair of vectors in standard position to check the reasonableness of your answer.

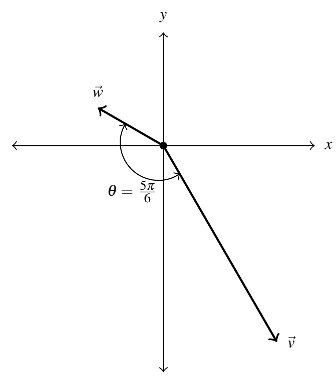
1. $\vec{v} = \left\langle 3, -3\sqrt{3}\right\rangle$, and $\vec{w} = \left\langle -\sqrt{3}, 1\right\rangle$
2. $\vec{v} = \langle 2, 2\rangle$, and $\vec{w} = \langle 5, -5\rangle$

3. $\vec{v} = \langle 3, -4 \rangle$, and $\vec{w} = \langle 2, 1 \rangle$

Solution. We use the formula $\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$ from Theorem 9.7 in each case below.

1. Compute the angle between $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, and $\vec{w} = \left\langle -\sqrt{3}, 1 \right\rangle$.

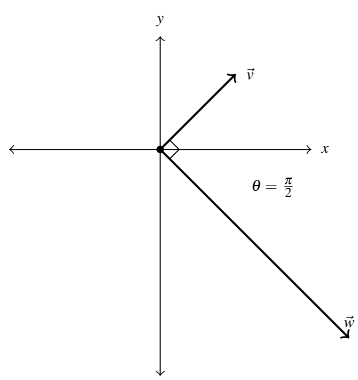
   We have $\vec{v}\cdot\vec{w} = \left\langle 3, -3\sqrt{3} \right\rangle \cdot \left\langle -\sqrt{3}, 1 \right\rangle = -3\sqrt{3} - 3\sqrt{3} = -6\sqrt{3}$. Computing the length of each vector, we find $\|\vec{v}\| = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{36} = 6$ and $\|\vec{w}\| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Hence, we find $\theta = \arccos\left(\frac{-6\sqrt{3}}{12}\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$. We check our answer geometrically by graphing this pair of vectors.



$\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, and $\vec{w} = \left\langle -\sqrt{3}, 1 \right\rangle$

2. Compute the angle between $\vec{v} = \langle 2, 2 \rangle$, and $\vec{w} = \langle 5, -5 \rangle$.

   For $\vec{v} = \langle 2, 2 \rangle$ and $\vec{w} = \langle 5, -5 \rangle$, we find $\vec{v}\cdot\vec{w} = \langle 2, 2 \rangle \cdot \langle 5, -5 \rangle = 10 - 10 = 0$. Hence, it doesn't matter what $\|\vec{v}\|$ and $\|\vec{w}\|$ are, $\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(0) = \frac{\pi}{2}$. We check our answer geometrically by graphing this pair of vectors.

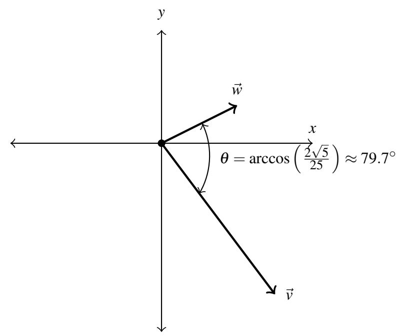


$\vec{v} = \langle 2, 2 \rangle$, and $\vec{w} = \langle 5, -5 \rangle$

3. Compute the angle between $\vec{v} = \langle 3, -4 \rangle$, and $\vec{w} = \langle 2, 1 \rangle$.

   We find $\vec{v}\cdot\vec{w} = \langle 3, -4 \rangle \cdot \langle 2, 1 \rangle = 6 - 4 = 2$. Computing lengths, we find $\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$ and $\vec{w} = \sqrt{2^2 + 1^2} = \sqrt{5}$, so $\theta = \arccos\left(\frac{2}{5\sqrt{5}}\right) = \arccos\left(\frac{2\sqrt{5}}{25}\right)$.

As $\frac{2\sqrt{5}}{25}$ isn't the cosine of one of the 'common angles,' we leave our *exact* answer in terms of the arccosine function. For the purposes of checking our answer, however, we *approximate* $\theta \approx 79.7°$.



$\vec{v} = \langle 3, -4 \rangle$, and $\vec{w} = \langle 2, 1 \rangle$

□

A few remarks about Example 9.2.2 are in order. Note that for nonzero vectors $\vec{v}$ and $\vec{w}$, the lengths $\|\vec{v}\|$ and $\|\vec{w}\|$ are always positive. Theorem 9.6 tells us that $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$, thus we know the sign of $\vec{v} \cdot \vec{w}$ is the same as the sign of $\cos(\theta)$.

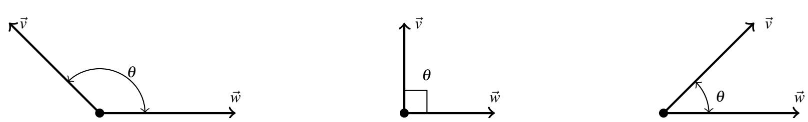
Geometrically, if $\vec{v} \cdot \vec{w} < 0$, then $\cos(\theta) < 0$ so θ is an obtuse angle, demonstrated in number 1 above.

If $\vec{v} \cdot \vec{w} = 0$, then $\cos(\theta) = 0$ so $\theta = \frac{\pi}{2}$ as in number 2. In this case, the vectors $\vec{v}$ and $\vec{w}$ are called **orthogonal**. Geometrically, when orthogonal vectors are sketched with the same initial point, the lines containing the vectors are perpendicular. Hence, if $\vec{v}$ and $\vec{w}$ are orthogonal, we write $\vec{v} \perp \vec{w}$.

Note there is no 'zero product property' for the dot product. As with the vectors in number 2 above, it is quite possible to have $\vec{v} \cdot \vec{w} = 0$ but neither $\vec{v}$ nor $\vec{w}$ be $\vec{0}$.

Finally, if $\vec{v} \cdot \vec{w} > 0$, then $\cos(\theta) > 0$ so θ is an acute angle, as in the case of number 3 above.

We summarize all of our observations in the schematic below.



$\vec{v} \cdot \vec{w} < 0$	$\vec{v} \cdot \vec{w} = 0$	$\vec{v} \cdot \vec{w} > 0$
θ is obtuse	$\theta = \frac{\pi}{2} = 90°$	θ is acute

Of the three cases diagrammed above, the one which has the most mathematical significance moving forward is the orthogonal case. Hence, we state the corresponding theorem below.

Theorem 9.8. For nonzero vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

Basically, Theorem 9.8 tells us that 'the dot product detects orthogonality.' This is a helpful interpretation to keep in mind as you continue your study of vectors in later courses.

We have already argued one direction of Theorem 9.8, namely if $\vec{v} \cdot \vec{w} = 0$ then $\vec{v} \perp \vec{w}$ in the comments following Example 9.2.2.

To show the converse, we note if $\vec{v} \perp \vec{w}$, then the angle between $\vec{v}$ and $\vec{w}$, $\theta = \frac{\pi}{2}$. From Theorem 9.6, we have that $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\left(\frac{\pi}{2}\right) = \|\vec{v}\|\|\vec{w}\| \cdot (0) = 0$, as required.

We can use Theorem 9.8 in the following example to provide a different proof about the relationship between the slopes of perpendicular lines.[2]

Example 9.2.3. Let L_1 be the line $y = m_1 x + b_1$ and let L_2 be the line $y = m_2 x + b_2$. Prove that L_1 is perpendicular to L_2 if and only if $m_1 \cdot m_2 = -1$.

Solution. Our strategy is to find two vectors: $\vec{v}_1$, which has the same direction as L_1, and $\vec{v}_2$, which has the same direction as L_2 and show $\vec{v}_1 \perp \vec{v}_2$ if and only if $m_1 m_2 = -1$.

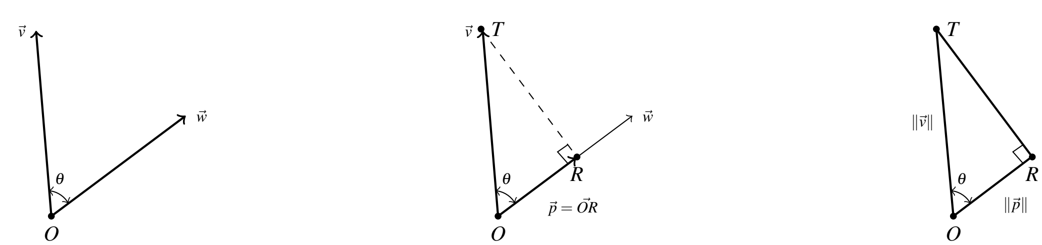
To that end, we substitute $x = 0$ and $x = 1$ into $y = m_1 x + b_1$ to find two points which lie on L_1, namely $P(0, b_1)$ and $Q(1, m_1 + b_1)$. We let $\vec{v}_1 = \overrightarrow{PQ} = \langle 1 - 0, (m_1 + b_1) - b_1 \rangle = \langle 1, m_1 \rangle$. Because $\vec{v}_1$ is determined by two points on L_1, it may be viewed as lying on L_1, so $\vec{v}_1$ has the same direction as L_1.

Similarly, we get the vector $\vec{v}_2 = \langle 1, m_2 \rangle$ which has the same direction as the line L_2. Hence, L_1 and L_2 are perpendicular if and only if $\vec{v}_1 \perp \vec{v}_2$. According to Theorem 9.8, $\vec{v}_1 \perp \vec{v}_2$ if and only if $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Notice that $\vec{v}_1 \cdot \vec{v}_2 = \langle 1, m_1 \rangle \cdot \langle 1, m_2 \rangle = 1 + m_1 m_2$. Hence, $\vec{v}_1 \cdot \vec{v}_2 = 0$ if and only if $1 + m_1 m_2 = 0$, which is true if and only if $m_1 m_2 = -1$, as required. $\square$

9.2.1 Vector Projections

While Theorem 9.8 certainly gives us some insight into what the dot product means geometrically, there is more to the story of the dot product. Consider the two nonzero vectors $\vec{v}$ and $\vec{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\vec{v}$ and $\vec{w}$, θ, is acute.



[2] See Exercise 41 in Section 1.3.1.

We wish to develop a formula for the vector $\vec{p}$, indicated below, which is called the **orthogonal projection of $\vec{v}$ onto $\vec{w}$**. The vector $\vec{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\vec{v}$ to the vector $\vec{w}$ and call the point of intersection R. The vector $\vec{p}$ is then defined as $\vec{p} = \overrightarrow{OR}$.

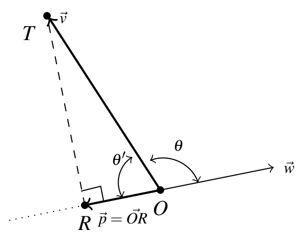
Like any vector, $\vec{p}$ is determined by its magnitude $\|\vec{p}\|$ and its direction $\hat{p}$ according to the formula $\vec{p} = \|\vec{p}\|\hat{p}$. Because we want $\hat{p}$ to have the same direction as $\vec{w}$, we have $\hat{p} = \hat{w}$.

To determine $\|\vec{p}\|$, we apply Definition 7.2 to the right triangle $\triangle ORT$. We find $\cos(\theta) = \frac{\|\vec{p}\|}{\|\vec{v}\|}$, or, equivalently, $\|\vec{p}\| = \|\vec{v}\|\cos(\theta)$. Using Theorems 9.6 and 9.5, we get:

$$\|\vec{p}\| = \|\vec{v}\|\cos(\theta) = \frac{\|\vec{v}\|\|\vec{w}\|\cos(\theta)}{\|\vec{w}\|} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|} = \vec{v}\cdot\left(\frac{1}{\|\vec{w}\|}\vec{w}\right) = \vec{v}\cdot\hat{w}.$$

Hence, $\|\vec{p}\| = \vec{v}\cdot\hat{w}$, and as $\hat{p} = \hat{w}$, we have $\vec{p} = \|\vec{p}\|\hat{p} = (\vec{v}\cdot\hat{w})\hat{w}$.

Now suppose that the angle θ between $\vec{v}$ and $\vec{w}$ is obtuse, and consider the diagram below.



In this case, we see that $\hat{p} = -\hat{w}$ and using the triangle $\triangle ORT$, we find $\|\vec{p}\| = \|\vec{v}\|\cos(\theta')$. Because $\theta + \theta' = \pi$, it follows that $\cos(\theta') = -\cos(\theta)$, which means $\|\vec{p}\| = \|\vec{v}\|\cos(\theta') = -\|\vec{v}\|\cos(\theta)$.

Rewriting this last equation in terms of $\vec{v}$ and $\vec{w}$ as before, we get $\|\vec{p}\| = -(\vec{v}\cdot\hat{w})$. Putting this together with $\hat{p} = -\hat{w}$, we get $\vec{p} = \|\vec{p}\|\hat{p} = -(\vec{v}\cdot\hat{w})(-\hat{w}) = (\vec{v}\cdot\hat{w})\hat{w}$ in this case as well.

If the angle between $\vec{v}$ and $\vec{w}$ is $\frac{\pi}{2}$ then it is easy to show[3] that $\vec{p} = \vec{0}$. Because $\vec{v} \perp \vec{w}$ in this case, $\vec{v}\cdot\vec{w} = 0$. It follows that $\vec{v}\cdot\hat{w} = 0$ and $\vec{p} = \vec{0} = 0\hat{w} = (\vec{v}\cdot\hat{w})\hat{w}$ in this case, too. We have motivated the following.

Definition 9.8. Let $\vec{v}$ and $\vec{w}$ be nonzero vectors.
The **orthogonal projection of $\vec{v}$ onto $\vec{w}$**, denoted $\text{proj}_{\vec{w}}(\vec{v})$ is given by $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w}$.

Definition 9.8 gives us a good idea what the dot product does. The scalar $\vec{v}\cdot\hat{w}$ is a measure of how much of the vector $\vec{v}$ is in the direction of the vector $\vec{w}$ and is thus called the **scalar projection** of $\vec{v}$ onto $\vec{w}$.

While the formula given in Definition 9.8 is theoretically appealing, because of the presence of the normalized unit vector $\hat{w}$, computing the projection using the formula $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w}$ can be messy. We present two other formulas that are often used in practice.

[3]In this case, the point R coincides with the point O, so $\vec{p} = \overrightarrow{OR} = \overrightarrow{OO} = \vec{0}$.

Theorem 9.9. Alternate Formulas for Vector Projections:
If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\right)\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

The proof of Theorem 9.9, which we leave to the reader as an exercise, amounts to using the formula $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$ and properties of the dot product. It is time for an example.

Example 9.2.4. Let $\vec{v} = \langle 1,8\rangle$ and $\vec{w} = \langle -1,2\rangle$. Determine $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$. Check your answer geometrically.

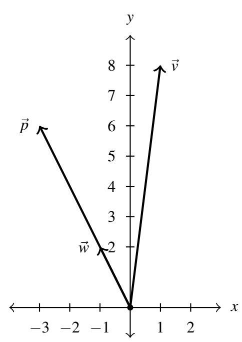
Solution. We find $\vec{v}\cdot\vec{w} = \langle 1,8\rangle\cdot\langle -1,2\rangle = (-1)+16 = 15$ and $\vec{w}\cdot\vec{w} = \langle -1,2\rangle\cdot\langle -1,2\rangle = 1+4 = 5$. Hence,

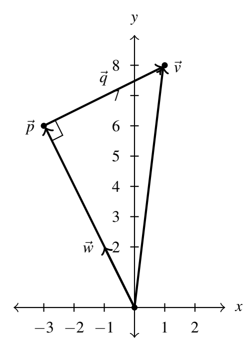
$$\vec{p} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{15}{5}\langle -1,2\rangle = \langle -3,6\rangle.$$

We plot $\vec{v}$, $\vec{w}$ and $\vec{p}$ in standard position below on the left. We see $\vec{p}$ has the same direction as $\vec{w}$, but we need to do more to show $\vec{p}$ in is indeed the *orthogonal* projection of $\vec{v}$ onto $\vec{w}$.

Consider the vector $\vec{q}$ whose initial point is the terminal point of $\vec{p}$ and whose terminal point is the terminal point of $\vec{v}$. From the definition of vector arithmetic, $\vec{p}+\vec{q} = \vec{v}$, so that $\vec{q} = \vec{v}-\vec{p}$.

For $\vec{v} = \langle 1,8\rangle$ and $\vec{p} = \langle -3,6\rangle$, $\vec{q} = \langle 1,8\rangle - \langle -3,6\rangle = \langle 4,2\rangle$. To prove $\vec{q}\perp\vec{w}$, we compute the dot product: $\vec{q}\cdot\vec{w} = \langle 4,2\rangle\cdot\langle -1,2\rangle = (-4)+4 = 0$. Hence, per Theorem 9.8, we know $\vec{q}\perp\vec{w}$ which completes our check.[4]





□

In Example 9.2.4 above, writing $\vec{v} = \vec{p}+\vec{q}$ is an example of what is called a **vector decomposition** of $\vec{v}$. We generalize this result in the following theorem.

[4]Note that, necessarily, $\vec{q}\perp\vec{p}$ as well!

Theorem 9.10. Generalized Decomposition Theorem:
Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. There are unique vectors $\vec{p}$ and $\vec{q}$ such that $\vec{v} = \vec{p} + \vec{q}$ where $\vec{p} = k\vec{w}$ for some scalar k, and $\vec{q} \cdot \vec{w} = 0$.

If the vectors $\vec{p}$ and $\vec{q}$ in Theorem 9.10 are nonzero, then we can say $\vec{p}$ is 'parallel'[5] to $\vec{w}$ and $\vec{q}$ is 'orthogonal' to $\vec{w}$. In this case, the vector $\vec{p}$ is sometimes called the 'vector component of $\vec{v}$ *parallel* to $\vec{w}$' and $\vec{q}$ is called the 'vector component of $\vec{v}$ *orthogonal* to $\vec{w}$.'

To prove Theorem 9.10, we take $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$ and $\vec{q} = \vec{v} - \vec{p}$. Then $\vec{p}$ is, by definition, a scalar multiple of $\vec{w}$. Next, we compute $\vec{q} \cdot \vec{w}$.

$$\begin{array}{rcll}
\vec{q} \cdot \vec{w} & = & (\vec{v} - \vec{p}) \cdot \vec{w} & \text{Definition of } \vec{q}. \\
& = & \vec{v} \cdot \vec{w} - \vec{p} \cdot \vec{w} & \text{Properties of Dot Product} \\
& = & \vec{v} \cdot \vec{w} - \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} \right) \cdot \vec{w} & \vec{p} = \text{proj}_{\vec{w}}(\vec{v}). \\
& = & \vec{v} \cdot \vec{w} - \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) (\vec{w} \cdot \vec{w}) & \text{Properties of Dot Product.} \\
& = & \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} & \\
& = & 0. &
\end{array}$$

Hence, $\vec{q} \cdot \vec{w} = 0$, as required. At this point, we have shown that the vectors $\vec{p}$ and $\vec{q}$ guaranteed by Theorem 9.10 *exist*. Now we need to show that they are *unique* - that is, there is only *one* such way to decompose $\vec{v}$ in the manner described in Theorem 9.10.

Suppose $\vec{v} = \vec{p} + \vec{q} = \vec{p}\,' + \vec{q}\,'$ where the vectors $\vec{p}\,'$ and $\vec{q}\,'$ satisfy the same properties described in Theorem 9.10 as $\vec{p}$ and $\vec{q}$. Then $\vec{p} - \vec{p}\,' = \vec{q}\,' - \vec{q}$, so $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (\vec{q}\,' - \vec{q}) = \vec{w} \cdot \vec{q}\,' - \vec{w} \cdot \vec{q} = 0 - 0 = 0$. The long and short of this computation is that $\vec{w} \cdot (\vec{p} - \vec{p}\,') = 0$.

Now there are scalars k and k' so that $\vec{p} = k\vec{w}$ and $\vec{p}\,' = k'\vec{w}$. This means $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (k\vec{w} - k'\vec{w}) = \vec{w} \cdot ([k - k']\vec{w}) = (k - k')(\vec{w} \cdot \vec{w}) = (k - k')\|\vec{w}\|^2$.

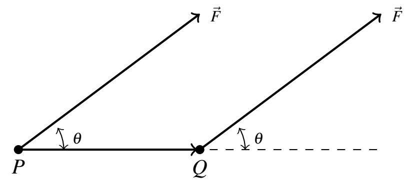
Because $\vec{w} \neq \vec{0}$, $\|\vec{w}\|^2 \neq 0$, which means the only way $\vec{w} \cdot (\vec{p} - \vec{p}\,') = (k - k')\|\vec{w}\|^2 = 0$ is for $k - k' = 0$, or $k = k'$. This means $\vec{p} = k\vec{w} = k'\vec{w} = \vec{p}\,'$. As $\vec{q}\,' - \vec{q} = \vec{p} - \vec{p}\,' = \vec{p} - \vec{p} = \vec{0}$, it must be that $\vec{q}\,' = \vec{q}$ as well.

Hence, we have shown there is only one way to write $\vec{v}$ as a sum of vectors as described in Theorem 9.10, so the decomposition listed there is unique.

We close this section with an application of the dot product. In Physics, if a constant force F is exerted over a distance d, the **work** W done by the force is given by $W = Fd$. Here, the assumption is that the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done.

[5] See Exercise 64 in Section 9.1.

Consider the scenario sketched below in which the constant force $\vec{F}$ is applied to move an object from the point P to the point Q. Here the force is being applied at an angle θ as opposed to being applied directly in the direction of the motion.



To find the work W done in this scenario, we need to find how much of the force $\vec{F}$ is in the direction of the motion $\overrightarrow{PQ}$. This is precisely what the dot product $\vec{F} \cdot \widehat{PQ}$ represents.

The distance the object travels is $\|\overrightarrow{PQ}\|$, so we get $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\|$. As $\overrightarrow{PQ} = \|\overrightarrow{PQ}\|\widehat{PQ}$, we can simplify this formula as follows: $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\| = \vec{F} \cdot (\|\overrightarrow{PQ}\|\widehat{PQ}) = \vec{F} \cdot \overrightarrow{PQ}$.

Using Theorem 9.6, we can rewrite $W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$, where θ is the angle between the applied force $\vec{F}$ and the trajectory of the motion $\overrightarrow{PQ}$. We have proved the following.

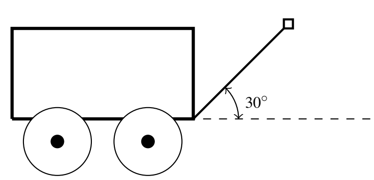
Theorem 9.11. Work as a Dot Product:
Suppose a constant force $\vec{F}$ is applied along the vector $\overrightarrow{PQ}$. The work W done by $\vec{F}$ is given by

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta),$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

We test out our formula for work in the following example.

Example 9.2.5. Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume the force of 10 pounds is exerted at a 30° angle for the duration of the 50 feet.



Solution. There are (at least) two ways to attack this problem. One way is to find the vectors $\vec{F}$ and $\overrightarrow{PQ}$ mentioned in Theorem 9.11 and compute $W = \vec{F} \cdot \overrightarrow{PQ}$.

To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above.

To find the force vector $\vec{F}$, we note the force in this situation is a constant 10 pounds, so $\|\vec{F}\| = 10$. Moreover, the force is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis. Definition 9.4 gives us $\vec{F} = \|\vec{F}\| \langle \cos(\theta), \sin(\theta) \rangle = 10 \langle \cos(30^\circ), \sin(30^\circ) \rangle = \langle 5\sqrt{3}, 5 \rangle$.

The wagon is being pulled along 50 feet in the positive x-direction, so we find the displacement vector is $\overrightarrow{PQ} = 50\hat{\imath} = 50\langle 1,0 \rangle = \langle 50,0 \rangle$.

Per Theorem 9.11, $W = \vec{F} \cdot \overrightarrow{PQ} = \langle 5\sqrt{3}, 5 \rangle \cdot \langle 50,0 \rangle = 250\sqrt{3}$. Force is measured in pounds and distance is measured in feet, giving us $W = 250\sqrt{3}$ foot-pounds.

Alternatively, we can use the formula $W = \|\vec{F}\| \|\overrightarrow{PQ}\| \cos(\theta)$. With $\|\vec{F}\| = 10$ pounds, $\|\overrightarrow{PQ}\| = 50$ feet and $\theta = 30^\circ$, we get $W = (10\,\text{pounds})(50\,\text{feet})\cos(30^\circ) = 250\sqrt{3}$ foot-pounds of work. □

9.2.2 Exercises

In Exercises 1 - 20, use the pair of vectors $\vec{v}$ and $\vec{w}$ to find the following quantities.

- $\vec{v} \cdot \vec{w}$
- The angle θ (in degrees) between $\vec{v}$ and $\vec{w}$
- $\text{proj}_{\vec{w}}(\vec{v})$
- $\vec{q} = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$ (Show that $\vec{q} \cdot \vec{w} = 0$.)

1. $\vec{v} = \langle -2,-7 \rangle$ and $\vec{w} = \langle 5,-9 \rangle$
2. $\vec{v} = \langle -6,-5 \rangle$ and $\vec{w} = \langle 10,-12 \rangle$
3. $\vec{v} = \langle 1,\sqrt{3} \rangle$ and $\vec{w} = \langle 1,-\sqrt{3} \rangle$
4. $\vec{v} = \langle 3,4 \rangle$ and $\vec{w} = \langle -6,-8 \rangle$
5. $\vec{v} = \langle -2,1 \rangle$ and $\vec{w} = \langle 3,6 \rangle$
6. $\vec{v} = \langle -3\sqrt{3},3 \rangle$ and $\vec{w} = \langle -\sqrt{3},-1 \rangle$
7. $\vec{v} = \langle 1,17 \rangle$ and $\vec{w} = \langle -1,0 \rangle$
8. $\vec{v} = \langle 3,4 \rangle$ and $\vec{w} = \langle 5,12 \rangle$
9. $\vec{v} = \langle -4,-2 \rangle$ and $\vec{w} = \langle 1,-5 \rangle$
10. $\vec{v} = \langle -5,6 \rangle$ and $\vec{w} = \langle 4,-7 \rangle$
11. $\vec{v} = \langle -8,3 \rangle$ and $\vec{w} = \langle 2,6 \rangle$
12. $\vec{v} = \langle 34,-91 \rangle$ and $\vec{w} = \langle 0,1 \rangle$
13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
21. A force of 1500 pounds is required to tow a trailer. Find the work done towing the trailer along a flat stretch of road 300 feet. Assume the force is applied in the direction of the motion.

22. Find the work done lifting a 10 pound book 3 feet straight up into the air. Assume the force of gravity is acting straight downwards.

23. Suppose Taylor fills her wagon with rocks and must exert a force of 13 pounds to pull her wagon across the yard. If she maintains a 15° angle between the handle of the wagon and the horizontal, compute how much work Taylor does pulling her wagon 25 feet. Round your answer to two decimal places.

24. In Exercise 61 in Section 9.1, two drunken college students have filled an empty beer keg with rocks which they drag down the street by pulling on two attached ropes. The stronger of the two students pulls with a force of 100 pounds on a rope which makes a 13° angle with the direction of motion. (In this case, the keg was being pulled due east and the student's heading was N77°E.) Find the work done by this student if the keg is dragged 42 feet.

25. Find the work done pushing a 200 pound barrel 10 feet up a 12.5° incline. Ignore all forces acting on the barrel except gravity, which acts downwards. Round your answer to two decimal places.

    HINT: Because you are working to overcome gravity only, the force being applied acts directly upwards. This means that the angle between the applied force in this case and the motion of the object is *not* the 12.5° of the incline!

26. Prove the distributive property of the dot product in Theorem 9.5.

27. Finish the proof of the scalar property of the dot product in Theorem 9.5.

28. Show Theorem 9.10 reduces to Theorem 9.4 in the case $\vec{w} = \hat{\imath}$.

29. Use the identity in Example 9.2.1 to prove the **Parallelogram Law**

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left[\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2\right]$$

30. We know that $|x+y| \leq |x| + |y|$ for all real numbers x and y by the Triangle Inequality established in Exercise 55 in Section 1.4. We can now establish a Triangle Inequality for vectors. In this exercise, we prove that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

    (a) (Step 1) Show that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$.

    (b) (Step 2) Show that $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$. This is the celebrated Cauchy-Schwarz Inequality.[6]

    HINT: Start with $|\vec{u}\cdot\vec{v}| = |\ \|\vec{u}\|\|\vec{v}\|\cos(\theta)\ |$ and use the fact that $|\cos(\theta)| \leq 1$ for all θ.

    (c) (Step 3) Show:

    $$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2.$$

    (d) (Step 4) Use Step 3 to show that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

[6] It is also known by other names. Check out this site for details.

SECTION 9.2 EXERCISE ANSWERS A.1.9

Appendix A

Appendix

A.1 Homework Answers

A.1.0 Chapter 0 Answers

Section 0.1 Answers

1. 6
2. 0
3. $\frac{2}{21}$
4. $\frac{19}{24}$
5. $-\frac{1}{3}$
6. -1
7. $\frac{3}{5}$
8. 18
9. $-\frac{7}{8}$
10. Undefined.
11. 0
12. Undefined.
13. $\frac{23}{9}$
14. $-\frac{4}{99}$
15. $-\frac{24}{7}$
16. 0
17. $\frac{243}{32}$
18. $\frac{13}{48}$
19. $\frac{9}{22}$
20. $\frac{25}{4}$
21. 5
22. $-3\sqrt{3}$
23. $\frac{107}{27}$
24. $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$
25. $\sqrt{10}$
26. $\frac{\sqrt{61}}{2}$
27. $\sqrt{7}$
28. $\frac{-4+\sqrt{2}}{7}$
29. -1
30. $2+\sqrt{5}$
31. $\frac{15}{16}$
32. 13
33. $-\frac{385}{12}$
34. 1.38×10^{10237}

Section 0.2 Answers

1. $3|x|$
2. $2t$
3. $5|y^3|\sqrt{2}$
4. $|2t+1|$
5. $|w-8|$
6. $\sqrt{3x+1}$
7. $\frac{\sqrt{c^2-v^2}}{|c|}$
8. $\frac{2r\sqrt[3]{3\pi r^2}}{L}$
9. $\frac{2\varepsilon^2\sqrt[4]{2\pi}}{|\rho^3|}$
10. $-\frac{1}{\sqrt{x}}$
11. $\frac{3-6t^2}{\sqrt{1-t^2}}$

12. $\dfrac{6-8z}{3(\sqrt[3]{1-z})^2}$

13. $\dfrac{4x-3}{(2x-1)\sqrt[3]{2x-1}}$

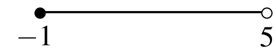
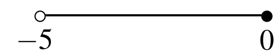
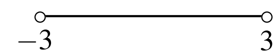
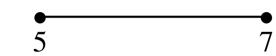
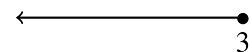
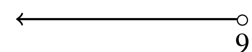
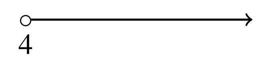
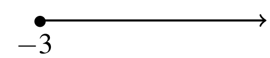
Section 0.3 Answers

1. $2x(1-5x)$
2. $4t^3(3t^2-2)$
3. $4xy(4y-3x)$
4. $-(m+3)^2(4m+7)$
5. $(2x-1)(x-1)$
6. $(t-5)(t^2+1)$
7. $(w-11)(w+11)$
8. $(7-2t)(7+2t)$
9. $(3t-2)(3t+2)(9t^2+4)$
10. $(3z-8y^2)(3z+8y^2)$
11. $-3(y-3)(y+1)$
12. $(x+h)(x+h-1)(x+h+1)$
13. $(y-12)^2$
14. $(5t+1)^2$
15. $3x(2x-3)^2$
16. $(m^2+5)^2$
17. $(3-2x)(9+6x+4x^2)$
18. $t^3(t+1)(t^2-t+1)$
19. $(x-7)(x+2)$
20. $(y-9)(y-3)$
21. $(3t+1)(t+5)$
22. $(2x-5)(3x-4)$
23. $(7-m)(5+m)$
24. $(-2w+1)(w-3)$
25. $3m(m-1)(m+4)$
26. $(x-2)(x+2)(x^2+5)$
27. $(2t-3)(2t+3)(t^2+1)$
28. $(x-3)(x+3)(x-5)$
29. $(t-3)(1-t)(1+t)$
30. $(y^2-y+3)(y^2+y+3)$

Section 0.4 Answers

1. O is the odd natural numbers.
2. $X = \{0,1,4,9,16,\ldots\}$
3. (a) $\dfrac{20}{10} = 2$ and 117
   (b) $\sqrt{3}$ and 5.2020020002
   (c) $\left\{-3, \dfrac{20}{10}, 117\right\}$
   (d) $\left\{-3, -1.02, -\dfrac{3}{5}, 0.57, 1.\overline{23}, \dfrac{20}{10}, 117\right\}$

4.

Subset of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1,5)$	
$\{x \mid 0 \leq x < 3\}$	$[0,3)$	
$\{x \mid 2 < x \leq 7\}$	$(2,7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5,0]$	
$\{x \mid -3 < x < 3\}$	$(-3,3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5,7]$	
$\{x \mid x \leq 3\}$	$(-\infty,3]$	
$\{x \mid x < 9\}$	$(-\infty,9)$	
$\{x \mid x > 4\}$	$(4,\infty)$	
$\{x \mid x \geq -3\}$	$[-3,\infty)$	

5. $(-1,5] \cap [0,8) = [0,5]$

6. $(-1,1) \cup [0,6] = (-1,6]$

7. $(-\infty,4] \cap (0,\infty) = (0,4]$

8. $(-\infty,0) \cap [1,5] = \emptyset$

9. $(-\infty,0) \cup [1,5] = (-\infty,0) \cup [1,5]$

10. $(-\infty,5] \cap [5,8) = \{5\}$

11. $(-\infty,5) \cup (5,\infty)$

12. $(-\infty,-1) \cup (-1,\infty)$

13. $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

14. $(-\infty,0)\cup(0,2)\cup(2,\infty)$

15. $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

16. $(-\infty,-4)\cup(-4,0)\cup(0,4)\cup(4,\infty)$

17. $(-\infty,-1]\cup[1,\infty)$

18. $[2,3)$

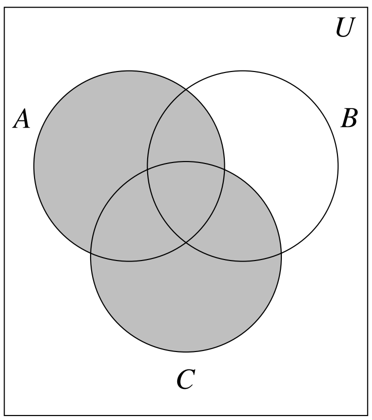
19. $(-\infty,-3]\cup(0,\infty)$

20. $\emptyset$

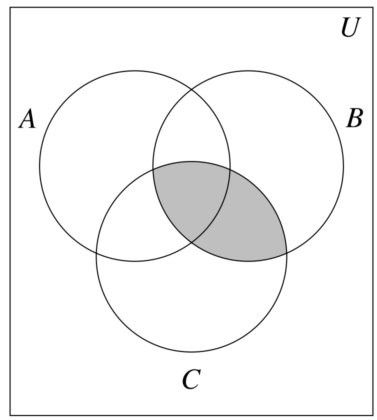
21. $\{-1\}\cup\{1\}\cup(2,\infty)$

22. $(3,4)\cup(4,13)$

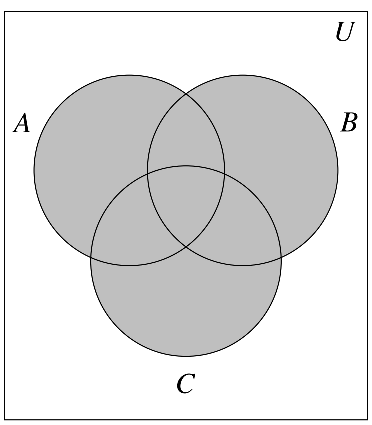
23. $A\cup C$



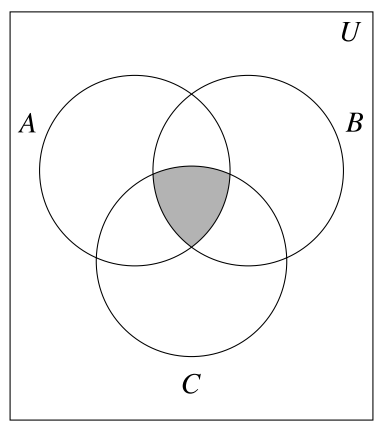
24. $B\cap C$



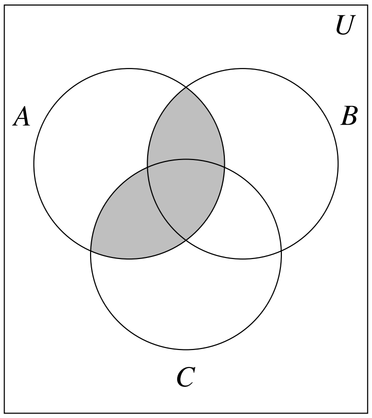
25. $(A\cup B)\cup C$



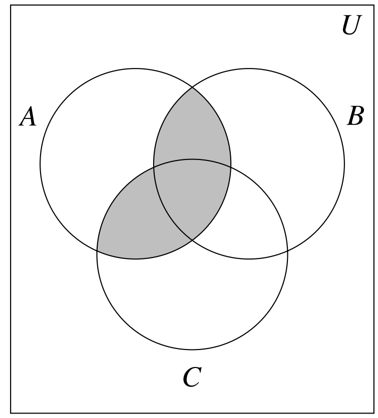
26. $(A\cap B)\cap C$



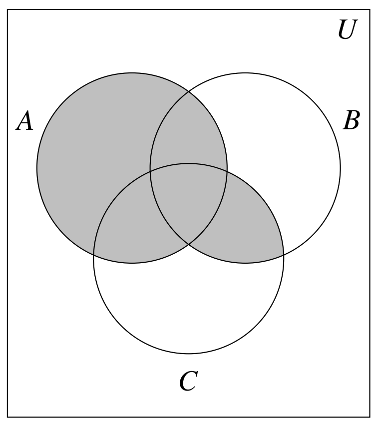
27. $A \cap (B \cup C)$



28. $(A \cap B) \cup (A \cap C)$



29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



Section 0.5 Answers

1. $x = \dfrac{18}{7}$

2. $t = -\dfrac{1}{30}$

3. $w = \dfrac{61}{33}$

4. $y = 50000$

5. All real numbers.

6. No solution.

7. $t = -\dfrac{5}{3\sqrt{7}} = -\dfrac{5\sqrt{7}}{21}$

8. $y = \dfrac{6}{17\sqrt{2}} = \dfrac{3\sqrt{2}}{17}$

9. $x = \dfrac{27}{18+\sqrt{7}}$

10. $y = \dfrac{4-3x}{2}$ or $y = -\dfrac{3}{2}x + 2$

11. $x = \dfrac{4-2y}{3}$ or $x = -\dfrac{2}{3}y + \dfrac{4}{3}$

12. $C = \frac{5}{9}(F-32)$ or $C = \frac{5}{9}F - \frac{160}{9}$

13. $x = \frac{p-15}{-2.5} = \frac{15-p}{2.5}$ or $x = -\frac{2}{5}p+6$.

14. $x = \frac{C-1000}{200}$ or $x = \frac{1}{200}C-5$

15. $y = \frac{x-7}{4}$ or $y = \frac{1}{4}x - \frac{7}{4}$

16. $w = \frac{3v+1}{v}$, provided $v \neq 0$.

17. $v = \frac{1}{w-3}$, provided $w \neq 3$.

18. $y = \frac{3x+1}{x-2}$, provided $x \neq 2$.

19. $\pi = \frac{C}{2r}$, provided $r \neq 0$.

20. $V = \frac{nRT}{P}$, provided $P \neq 0$.

21. $R = \frac{PV}{nT}$, provided $n \neq 0, T \neq 0$.

22. $g = \frac{E}{mh}$, provided $m \neq 0, h \neq 0$.

23. $m = \frac{2E}{v^2}$, provided $v^2 \neq 0$ (so $v \neq 0$).

24. $V_2 = \frac{P_1V_1}{P_2}$, provided $P_2 \neq 0$.

25. $t = \frac{x-x_0}{a}$, provided $a \neq 0$.

26. $x = \frac{y-y_0+mx_0}{m}$ or $x = x_0 + \frac{y-y_0}{m}$, provided $m \neq 0$.

27. $T_1 = \frac{mcT_2-q}{mc}$ or $T_1 = T_2 - \frac{q}{mc}$, provided $m \neq 0, c \neq 0$.

28. $x = -6$ or $x = 6$

29. $t = -3$ or $t = \frac{11}{3}$

30. $w = -3$ or $w = 11$

31. $y = -1$ or $y = 1$

32. $m = -\frac{1}{2}$ or $m = \frac{1}{10}$

33. No solution

34. $x = -3$ or $x = 3$

35. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$

36. $t = \frac{\sqrt{2} \pm 2}{3}$

37. $v = -1$ or $v = 0$

38. No solution

39. $y = \frac{3}{2}$

40. $t = -1$ or $t = 9$

41. $x = -\frac{1}{7}$ or $x = 1$

42. $y = 0$ or $y = \frac{2}{\sqrt{2}-1}$

43. $x = 1$

44. $z = -\frac{3}{10}$

45. $w = \frac{\sqrt{3} \pm 2}{\sqrt{3} \mp 2}$ See footnote[1]

46. $x = -\frac{3}{7}$ or $x = 5$

47. $t = \frac{1}{2}$ or $t = -4$

48. $y = \frac{5}{3}$ or $y = -2$

[1] That is, $w = \frac{\sqrt{3}+2}{\sqrt{3}-2}$ or $w = \frac{\sqrt{3}-2}{\sqrt{3}+2}$

49. $t=0$ or $t=4$

50. $y=-1$ or $y=\frac{3}{2}$

51. $x=\frac{3}{2}$ or $x=\frac{7}{4}$

52. $x=0$ or $x=\pm\frac{3}{4}$

53. $w=-\frac{5}{2}$ or $w=\frac{2}{3}$

54. $w=-5$ or $w=-\frac{1}{2}$

55. $x=3$ or $x=\pm 4$

56. $t=-1, t=-\frac{1}{2}$, or $t=0$

57. $a=\pm 1$

58. $t=-\frac{3}{4}$ or $t=\frac{3}{2}$

59. $x=\frac{2}{3}$

60. $y=\pm 3$

61. $x=-\frac{3}{2}$

62. $y=-1,2$

63. $t=-\frac{\sqrt[3]{3}}{2}$

64. $x=5$

65. $t=\pm 3\sqrt{7}$

66. $x=3$

67. $y=-3$

68. $t=-\frac{1}{3},\frac{2}{3}$

69. $x=\frac{5+\sqrt{57}}{8}$

70. $w=\sqrt{3}$

71. $x=6$

72. $x=4$

73. $h=\sqrt[3]{\frac{12I}{b}}$

74. $a=\frac{2\sqrt[4]{I_0}}{\sqrt[4]{5\sqrt{3}}}$

75. $g=\frac{4\pi^2 L}{T^2}$

76. $v=\frac{c\sqrt{L_0^2-L^2}}{L_0}$

77. $x=\frac{3\pm\sqrt{5}}{6}$

78. $t=-\frac{4}{5},-\frac{2}{5}$

79. $y=\pm 1, \pm\sqrt{5}$

80. $x=\frac{-1\pm\sqrt{5}}{2}$

81. $w=-1,\frac{2}{3}$

82. $y=-2\pm\sqrt{5}$

83. $z=\frac{1\pm\sqrt{65}}{16}$

84. $v=-3,1$

85. No real solution.

86. $t=\frac{-5\pm\sqrt{33}}{4}$

87. $x=0$

88. $y=\frac{2\pm\sqrt{10}}{6}$

89. $w=\pm\sqrt{\frac{\sqrt{13}-3}{2}}$

90. $x=\pm 1$

91. $y=\frac{4\pm\sqrt{6+2\sqrt{13}}}{2}$

92. $x=0,\frac{5\pm\sqrt{17}}{2}$

93. $p=-\frac{1}{3},\pm\sqrt{2}$

94. $v=0,\pm\sqrt{2},\pm\sqrt{5}$

95. $y=\frac{5\sqrt{2}\pm\sqrt{46}}{2}$

96. $x=\frac{\sqrt{2}\pm\sqrt{10}}{2}$

97. $v=-\frac{\sqrt{3}}{2},2\sqrt{3}$

98. $b = \pm\frac{\sqrt{13271}}{50}$

99. $r = \pm\sqrt{\frac{37}{\pi}}$

100. $r = \frac{-4\sqrt{2} \pm \sqrt{54\pi + 32}}{\pi}$

101. $t = \frac{500 \pm 10\sqrt{491}}{49}$

102. $x = \frac{99 \pm 6\sqrt{165}}{13}$

103. $A = \frac{-107 \pm 7\sqrt{70}}{330}$

104. $x = 1, 2, \frac{3 \pm \sqrt{17}}{2}$

105. $x = \pm 1, 2 \pm \sqrt{3}$

106. $x = -\frac{1}{2}, 1, 7$

107. The discriminant is: $D = p^2 - 4p^2 = -3p^2 < 0$. because $D < 0$, there are no real solutions.

108. $t = \frac{v \pm \sqrt{v^2 + 2gh}}{g}$

109. $7i$

110. $3i$

111. -10

112. 10

113. -12

114. 12

115. 3

116. $-3i$

117. $i^5 = i^4 \cdot i = 1 \cdot i = i$

118. $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

119. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

120. $i^8 = i^4 \cdot i^4 = \left(i^4\right)^2 = (1)^2 = 1$

121. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

122. $i^{26} = \left(i^4\right)^6 \cdot i^2 = 1 \cdot (-1) = -1$

123. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

124. $i^{304} = \left(i^4\right)^{76} = 1^{76} = 1$

125. $x = \frac{2 \pm i\sqrt{14}}{3}$

126. $t = 5, \pm\frac{i\sqrt{3}}{3}$

127. $y = \pm 2, \pm i$

128. $w = \frac{1 \pm i\sqrt{7}}{2}$

129. $y = \pm\frac{3i\sqrt{2}}{2}$

130. $x = 0, \frac{1 \pm i\sqrt{2}}{3}$

131. $x = \frac{\sqrt{5} \pm i\sqrt{3}}{2}$

132. $y = \pm i, \pm\frac{i\sqrt{2}}{2}$

133. $z = \pm 2, \pm 2i$

Section 0.6 Answers

1. $\left(-\infty, \frac{3}{4}\right]$

2. $\left(-\infty, \frac{7}{6}\right)$

3. $\left(-\infty, \frac{3}{13}\right]$

4. $\left(-\frac{4}{3}, \infty\right)$

5. No solution.

6. $(-\infty, \infty)$

7. $(4, \infty)$

8. $\left[\dfrac{7}{2 - \sqrt[3]{18}}, \infty\right)$

9. $[0, \infty)$

10. $\left[\dfrac{1}{2}, \dfrac{7}{10}\right]$

11. $\left(-\dfrac{23}{6}, \dfrac{19}{2}\right]$

12. $\left(-\dfrac{13}{10}, -\dfrac{7}{10}\right]$

13. $(-4, 1]$

14. $\{1\} = [1, 1]$

15. $\left[-6, \dfrac{18}{19}\right)$

16. $(-\infty, -1] \cup [0, \infty)$

17. $(-\infty, -7) \cup [4, \infty)$

18. $(-\infty, \infty)$

19. $\left[\dfrac{1}{3}, 3\right]$

20. $\left(-\infty, -\dfrac{12}{7}\right) \cup \left(\dfrac{8}{7}, \infty\right)$

21. $(-3, 2)$

22. $(-\infty, 1] \cup [3, \infty)$

23. No solution

24. $(-\infty, \infty)$

25. $(-\infty, -6 - \sqrt{5}) \cup (6 - \sqrt{5}, \infty)$

26. $\left[-\dfrac{3}{4}, \dfrac{3}{4}\right]$

27. No solution

28. $(-3, 2] \cup [6, 11)$

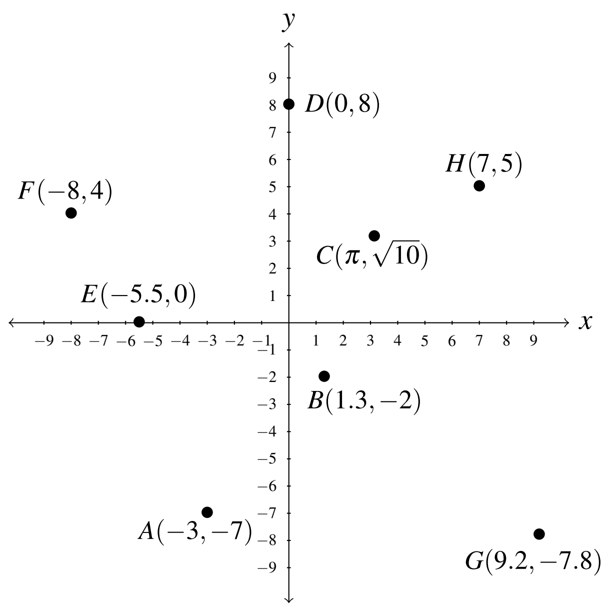
29. $[3, 4) \cup (5, 6]$

30. $\left(\dfrac{2\sqrt{3} - 3}{2}, \dfrac{2\sqrt{3} - 1}{2}\right) \cup \left(\dfrac{2\sqrt{3} + 1}{2}, \dfrac{2\sqrt{3} + 3}{2}\right)$

A.1.1 Chapter 1 Answers

Section 1.1 Answers

1. The required points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$, and $H(7,5)$ are plotted in the Cartesian Coordinate Plane below.



2. (a) The point $A(-3,-7)$ is
    - in Quadrant III
    - symmetric about x-axis with $(-3,7)$
    - symmetric about y-axis with $(3,-7)$
    - symmetric about origin with $(3,7)$

   (b) The point $B(1.3,-2)$ is
    - in Quadrant IV
    - symmetric about x-axis with $(1.3,2)$
    - symmetric about y-axis with $(-1.3,-2)$
    - symmetric about origin with $(-1.3,2)$

   (c) The point $C(\pi,\sqrt{10})$ is
    - in Quadrant I
    - symmetric about x-axis with $(\pi,-\sqrt{10})$
    - symmetric about y-axis with $(-\pi,\sqrt{10})$
    - symmetric about origin with $(-\pi,-\sqrt{10})$

(d) The point $D(0,8)$ is

- on the positive y-axis
- symmetric about x-axis with $(0,-8)$
- symmetric about y-axis with $(0,8)$
- symmetric about origin with $(0,-8)$

(e) The point $E(-5.5,0)$ is

- on the negative x-axis
- symmetric about x-axis with $(-5.5,0)$
- symmetric about y-axis with $(5.5,0)$
- symmetric about origin with $(5.5,0)$

(f) The point $F(-8,4)$ is

- in Quadrant II
- symmetric about x-axis with $(-8,-4)$
- symmetric about y-axis with $(8,4)$
- symmetric about origin with $(8,-4)$

(g) The point $G(9.2,-7.8)$ is

- in Quadrant IV
- symmetric about x-axis with $(9.2,7.8)$
- symmetric about y-axis with $(-9.2,-7.8)$
- symmetric about origin with $(-9.2,7.8)$

(h) The point $H(7,5)$ is

- in Quadrant I
- symmetric about x-axis with $(7,-5)$
- symmetric about y-axis with $(-7,5)$
- symmetric about origin with $(-7,-5)$

3. $d = 5$ units, $M = \left(-1, \frac{7}{2}\right)$
4. $d = 4\sqrt{10}$ units, $M = (1,-4)$
5. $d = \sqrt{26}$ units, $M = \left(1, \frac{3}{2}\right)$
6. $d = \frac{\sqrt{37}}{2}$ units, $M = \left(\frac{5}{6}, \frac{7}{4}\right)$
7. $d = \sqrt{74}$ units, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$
8. $d = 3\sqrt{5}$ units, $M = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$
9. $d = \sqrt{83}$ units, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$
10. $d = 2$ units, $M = (0,0)$
11. $(-3,-4)$, 5 miles, $(4,-4)$

12. The distance from A to B is $|AB| = \sqrt{13}$, the distance from A to C is $|AC| = \sqrt{52}$, and the distance from B to C is $|BC| = \sqrt{65}$. because $\left(\sqrt{13}\right)^2 + \left(\sqrt{52}\right)^2 = \left(\sqrt{65}\right)^2$, we are guaranteed by the converse of the Pythagorean Theorem that the triangle is a right triangle.

Section 1.2 Answers

1. The mapping M is not a function because 'Tennant' is matched with both 'Eleven' and 'Twelve.'
2. The mapping C is a function because each input is matched with only one output. The domain of C is $\{\text{Hartnell}, \text{Cushing}, \text{Hurndall}, \text{Troughton}\}$ and the range is $\{\text{One}, \text{Two}\}$. We can represent C as the following set of ordered pairs: $\{(\text{Hartnell}, \text{One}), (\text{Cushing}, \text{One}), (\text{Hurndall}, \text{One}), (\text{Troughton}, \text{Two})\}$
3. In this case, y is a function of x because each x is matched with only one y.

   The domain is $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is $\{0, 1, 2, 3\}$.

   As ordered pairs, this function is $\{(-3,3), (-2,2), (-1,1), (0,0), (1,1), (2,2), (3,3)\}$
4. In this case, y is not a function of x because there are x values matched with more than one y value. For instance, 1 is matched both to 1 and -1.
5. The mapping is a function because given any word, there is only one answer to 'how many letters are in the word?' For the range, we would need to know what the length of the longest word is and whether or not we could find words of all the lengths between 1 (the length of the word 'a') and it. See here.
6. because Grover Cleveland was both the 22nd and 24th POTUS, neither mapping described in this exercise is a function.
7. The outdoor temperature could never be the same for more than two different times - so, for example, it could always be getting warmer or it could always be getting colder.
8. $f(2) = \frac{7}{4}$, $f(x) = \frac{2x+3}{4}$
9. $f(2) = \frac{5}{2}$, $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$
10. $f(2) = 7$, $f(x) = 2\left(\frac{x}{4}+3\right) = \frac{1}{2}x + 6$
11. $f(2) = \sqrt{7}$, $f(x) = \sqrt{2x+3}$
12. $f(2) = \sqrt{10}$, $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$
13. $f(2) = 2\sqrt{5}$, $f(x) = 2\sqrt{x+3}$
14. For $f(x) = 2x+1$
    - $f(3) = 7$
    - $f(-1) = -1$
    - $f\left(\frac{3}{2}\right) = 4$
    - $f(4x) = 8x+1$
    - $4f(x) = 8x+4$
    - $f(-x) = -2x+1$

- $f(x-4) = 2x-7$
- $f(x)-4 = 2x-3$
- $f\left(x^2\right) = 2x^2+1$

15. For $f(x) = 3-4x$
    - $f(3) = -9$
    - $f(-1) = 7$
    - $f\left(\frac{3}{2}\right) = -3$
    - $f(4x) = 3-16x$
    - $4f(x) = 12-16x$
    - $f(-x) = 4x+3$
    - $f(x-4) = 19-4x$
    - $f(x)-4 = -4x-1$
    - $f\left(x^2\right) = 3-4x^2$
16. For $f(x) = 2-x^2$
    - $f(3) = -7$
    - $f(-1) = 1$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 2-16x^2$
    - $4f(x) = 8-4x^2$
    - $f(-x) = 2-x^2$
    - $f(x-4) = -x^2+8x-14$
    - $f(x)-4 = -x^2-2$
    - $f\left(x^2\right) = 2-x^4$
17. For $f(x) = x^2-3x+2$
    - $f(3) = 2$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 16x^2-12x+2$
    - $4f(x) = 4x^2-12x+8$
    - $f(-x) = x^2+3x+2$
    - $f(x-4) = x^2-11x+30$
    - $f(x)-4 = x^2-3x-2$
    - $f\left(x^2\right) = x^4-3x^2+2$
18. For $f(x) = 6$
    - $f(3) = 6$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = 6$
    - $f(4x) = 6$
    - $4f(x) = 24$
    - $f(-x) = 6$
    - $f(x-4) = 6$
    - $f(x)-4 = 2$
    - $f\left(x^2\right) = 6$
19. For $f(x) = 0$
    - $f(3) = 0$
    - $f(-1) = 0$
    - $f\left(\frac{3}{2}\right) = 0$
    - $f(4x) = 0$
    - $4f(x) = 0$
    - $f(-x) = 0$
    - $f(x-4) = 0$
    - $f(x)-4 = -4$
    - $f\left(x^2\right) = 0$

20. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 5 - 2x$

- $f(2) = 1$
- $f(-2) = 9$
- $f(2a) = 5 - 4a$
- $2f(a) = 10 - 4a$
- $f(a+2) = 1 - 2a$
- $f(a) + f(2) = 6 - 2a$
- $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
- $\frac{f(a)}{2} = \frac{5-2a}{2}$
- $f(a+h) = 5 - 2a - 2h$

22. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

23. For $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

24. For $f(x) = 117$

- $f(2) = 117$
- $f(-2) = 117$
- $f(2a) = 117$
- $2f(a) = 234$
- $f(a+2) = 117$
- $f(a) + f(2) = 234$

- $f\left(\frac{2}{a}\right) = 117$
- $\frac{f(a)}{2} = \frac{117}{2}$
- $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$
    - $f(2) = 1$
    - $f(-2) = -1$
    - $f(2a) = a$
    - $2f(a) = a$
    - $f(a+2) = \frac{a+2}{2}$
    - $f(a)+f(2) = \frac{a}{2}+1$ $= \frac{a+2}{2}$
    - $f\left(\frac{2}{a}\right) = \frac{1}{a}$
    - $\frac{f(a)}{2} = \frac{a}{4}$
    - $f(a+h) = \frac{a+h}{2}$
26. For $f(x) = 2x-1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
27. For $f(x) = 3-\frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$
28. For $f(x) = 2x^2-6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$
29. For $f(x) = x^2-x-12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$
30. Function
31. Function
32. Function
33. Not a function
34. Function
35. Not a function
36. Not a function
37. Function
38. Not a function
39. Function
40. Not a function
41. Function
42. Function
43. Function
44. Not a function
45. Function
    domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
    range = $\{0, 1, 4, 9\}$
46. Not a function
47. Function
    domain = $\{-7, -3, 3, 4, 5, 6\}$
    range = $\{0, 4, 5, 6, 9\}$
48. Function
    domain = $\{1, 4, 9, 16, 25, 36, \ldots\}$
    $= \{x \mid x$ is a perfect square$\}$
    range = $\{2, 4, 6, 8, 10, 12, \ldots\}$
    $= \{y \mid y$ is a positive even integer$\}$
49. Not a function
50. Function
    domain $= \{x \mid x$ is irrational$\}$
    range $= \{1\}$

51. Function
domain $= \{x \mid 1, 2, 4, 8, \ldots\}$
$= \{x \mid x = 2^n$ for some whole number $n\}$
range $= \{0, 1, 2, 3, \ldots\}$
$= \{y \mid y$ is any whole number$\}$

52. Function
domain $= \{x \mid x$ is any integer$\}$
range $= \{y \mid y$ is the square of an integer$\}$

53. Not a function

54. Function
domain $= \{x \mid -2 \leq x < 4\} = [-2, 4)$,
range = $\{3\}$

55. Function
domain $= \{x \mid x$ is a real number$\} = (-\infty, \infty)$
range $= \{y \mid y \geq 0\} = [0, \infty)$

56. Not a function

57. **Horizontal Line Test:** A graph on the xy-plane represents x as a function of y if and only if no **horizontal** line intersects the graph more than once.

58. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

59. Not a function

60. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

61. Not a function

62.
- Number 58 represents x as a function of y.
  domain = $\{-1, 0, 1, 2, 3, 4\}$ and range = $\{-4, -3, -2, -1, 0, 1\}$
- Number 61 represents x as a function of y.
  domain = $(-\infty, \infty)$ and range = $[1, \infty)$

63. Function
domain = $[2, \infty)$
range = $[0, \infty)$

64. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$

65. Not a function

66. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

67. Only number 63 represents v as a function of w; domain = $[0, \infty)$ and range = $[2, \infty)$

68. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

69. Not a function

70. Function
domain = $(-5,4)$
range = $(-4,4)$

71. Function
domain = $[0,3)\cup(3,6]$
range = $(-4,-1]\cup[0,4]$

72. None of numbers 68 - 71 represent t as a function of T.

73. Function
domain = $(-\infty,\infty)$
range = $(-\infty,4]$

74. Function
domain = $(-\infty,\infty)$
range = $(-\infty,4]$

75. Function
domain = $[-2,\infty)$
range = $(-\infty,3]$

76. Function
domain = $(-\infty,\infty)$
range = $(-\infty,\infty)$

77. Only number 75 represents s as a function of H; domain = $(-\infty,3]$ and range = $[-2,\infty)$

78. Function
domain = $(-\infty,0]\cup(1,\infty)$
range = $(-\infty,1]\cup\{2\}$

79. Function
domain = $[-3,3]$
range = $[-2,2]$

80. Not a function

81. Function
domain = $(-\infty,\infty)$
range = $\{2\}$

82. Only number 80 represents t as a function of u; domain = $(-\infty,\infty)$ and range=$\{2\}$.

83. $f(-2)=2$

84. $g(-2)=-5$

85. $f(2)=3$

86. $g(2)=3$

87. $f(0)=-1$

88. $g(0)=0$

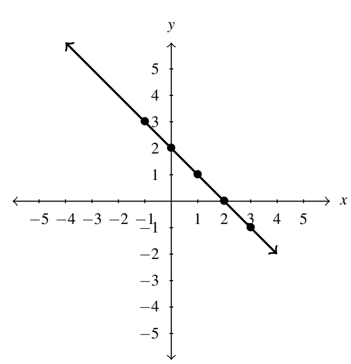
89. $x=-4,-1,1$

90. $t=-4,0,4$

91. Domain: $[-5,3]$, Range: $[-5,4]$.

92. Domain: $[-4,4]$, Range: $[-5,5)$.

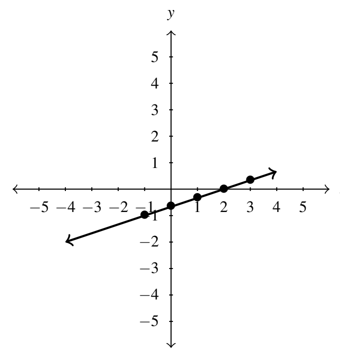
93. $f(x)=2-x$
Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$



94. $g(t) = \dfrac{t-2}{3}$

Domain: $(-\infty, \infty)$

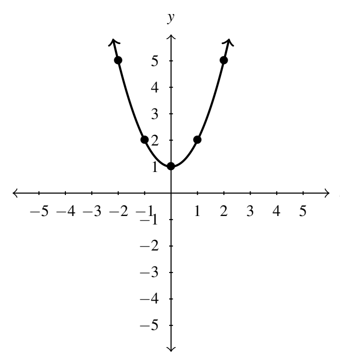
Range: $(-\infty, \infty)$



95. $h(s) = s^2 + 1$

Domain: $(-\infty, \infty)$

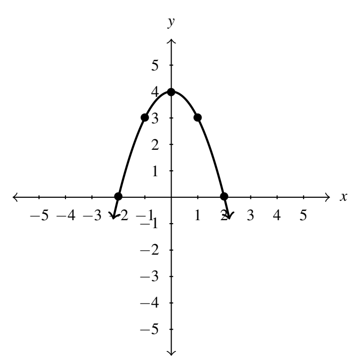
Range: $[1, \infty)$



96. $f(x) = 4 - x^2$

Domain: $(-\infty, \infty)$

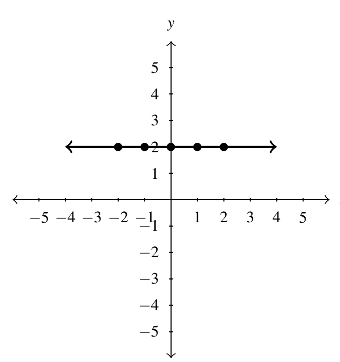
Range: $(-\infty, 4]$



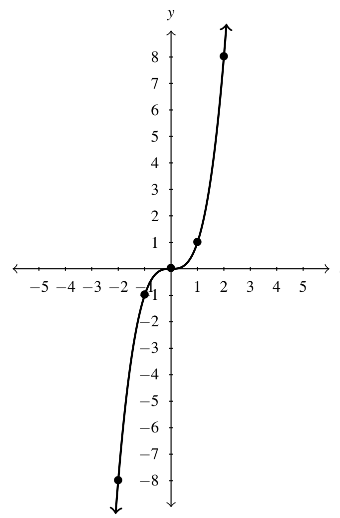
97. $g(t) = 2$

Domain: $(-\infty, \infty)$

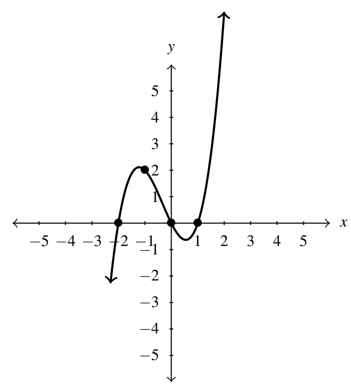
Range: $\{2\}$



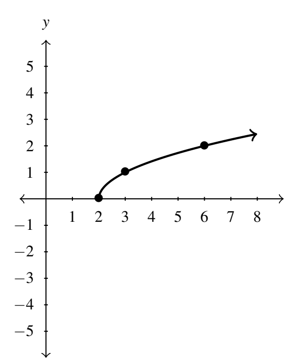
98. $h(s) = s^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



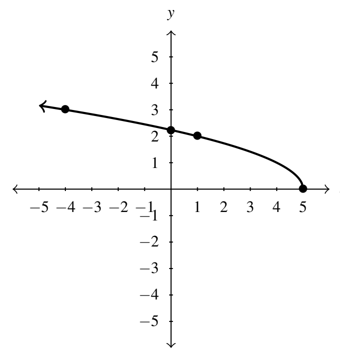
99. $f(x) = x(x-1)(x+2)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



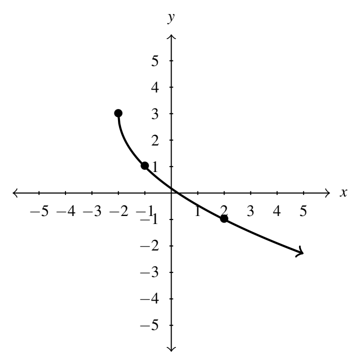
100. $g(t) = \sqrt{t-2}$
Domain: $[2, \infty)$
Domain: $[0, \infty)$



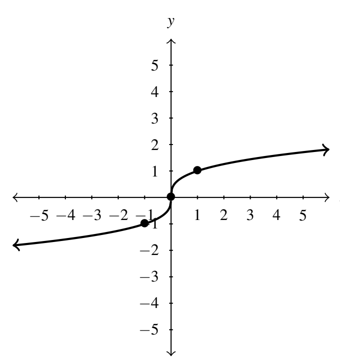
101. $h(s) = \sqrt{5-s}$
Domain: $(-\infty, 5]$
Range: $[0, \infty)$



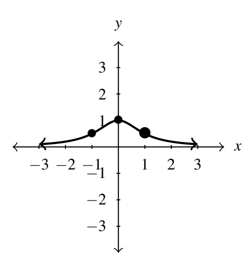
102. $f(x) = 3 - 2\sqrt{x+2}$
Domain: $[-2, \infty)$
Range: $(-\infty, 3]$



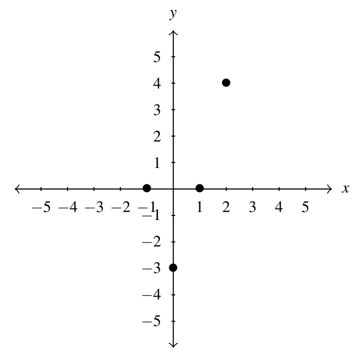
103. $g(t) = \sqrt[3]{t}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



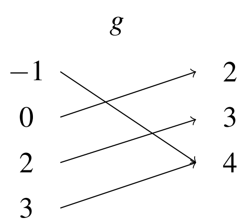
104. $h(s) = \dfrac{1}{s^2+1}$
Domain: $(-\infty, \infty)$
Range: $(0, 1]$



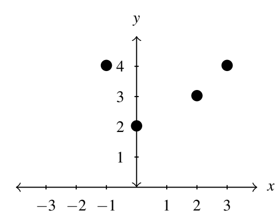
105. (a) domain $= \{-1,0,1,2\}$, range $= \{-3,0,4\}$
(b) $f(0) = -3$, $f(x) = 0$ for $x = -1,1$.
(c) $f = \{(-1,0),(0,-3),(1,0),(2,4)\}$
(d)



106. (a) domain $= \{-1,0,2,3\}$, range $= \{2,3,4\}$
(b)



(c) Find $g(0) = 2$ and $g(x) = 0$ has no solutions.
(d)



107. $F(4) = 4^2 = 16$ (when $t = 4$), the solutions to $F(x) = 4$ are $x = \pm 2$ (when $t = \pm 2$).

108. $G(4) = 7$ (when $t = 2$), the solution to $G(t) = 4$ is $x = -2$ (when $t = -1$)

109. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(\ell) = 36$ are $\ell = \pm 6$. because ℓ is restricted to $\ell > 0$, we only keep $\ell = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. because ℓ represents a length, $\ell > 0$.

110. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. because r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. because r represents a radius (length), $r > 0$.

111. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(s) = 27$ is $s = 3$. This means for the volume enclosed by the cube to

be 27 cubic centimeters, the length of the side needs to 3 centimeters. because x represents a length, $x > 0$.

112. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. because r represents a radius (length), $r > 0$.

113. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. because we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

114. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

115. $C(0) = 27$, so to make 0 pens, it costs[2] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

116. $E(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $E(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $E(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

117. $P(s) = 4s$, $s > 0$.

118. $C(D) = \pi D$, $D > 0$.

119. (a) The amount in the retirement account after 30 years if the monthly payment is \$50.
   (b) The solution to $A(P) = 250000$ is what the monthly payment needs to be in order to have \$250,000 in the retirement account after 30 years.
   (c) $A(P+50)$ is how much is in the retirement account in 30 years if \$50 is added to the monthly payment P. $A(P)+50$ represents the amount of money in the retirement account after 30 years if \$$P$ is invested each month plus an additional \$50. $A(P)+A(50)$ is the sum of money from two retirement accounts after 30 years: one with monthly payment \$$P$ and one with monthly payment \$50.

120. (a) because noon is 4 hours after 8 AM, $P(4)$ gives the chance of precipitation at noon.
   (b) We would need to solve $P(t) \geq 50\%$ or $P(t) \geq 0.5$.

121. The graph in question passes the horizontal line test meaning for each w there is only one v. The domain of g is $[0, \infty)$ (which is the range of f) and the range of g is $[2, \infty)$ which is the domain of f.

[2] This is called the 'fixed' or 'start-up' cost. We'll revisit this concept in Example 1.3.8 in Section 1.3.1.

122. Answers vary.

Section 1.3 Answers

1. $y+1=3(x-3)$
   $y=3x-10$

2. $y-8=-2(x+5)$
   $y=-2x-2$

3. $y+1=-(x+7)$
   $y=-x-8$

4. $y-1=\frac{2}{3}(x+2)$
   $y=\frac{2}{3}x+\frac{7}{3}$

5. $y-4=-\frac{1}{5}(x-10)$
   $y=-\frac{1}{5}x+6$

6. $y-4=\frac{1}{7}(x+1)$
   $y=\frac{1}{7}x+\frac{29}{7}$

7. $y-117=0$
   $y=117$

8. $y+3=-\sqrt{2}(x-0)$
   $y=-\sqrt{2}x-3$

9. $y-2\sqrt{3}=-5(x-\sqrt{3})$
   $y=-5x+7\sqrt{3}$

10. $y+12=678(x+1)$
    $y=678x+666$

11. $y=-\frac{5}{3}x$

12. $y=-2$

13. $y=\frac{8}{5}x-8$

14. $y=\frac{9}{4}x-\frac{47}{4}$

15. $y=5$

16. $y=-8$

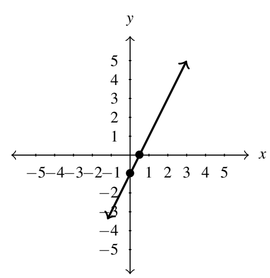
17. $y=-\frac{5}{4}x+\frac{11}{8}$

18. $y=2x+\frac{13}{6}$

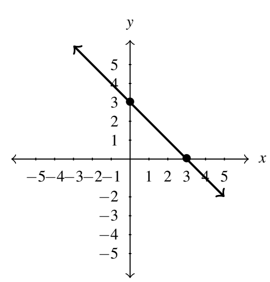
19. $y=-x$

20. $y=\frac{\sqrt{3}}{3}x$

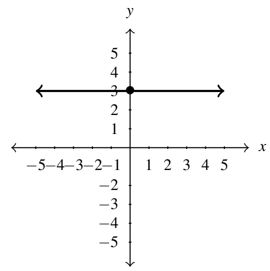
21. $y=2x-1$
    slope: $m=2$
    y-intercept: $(0,-1)$
    x-intercept: $\left(\frac{1}{2},0\right)$



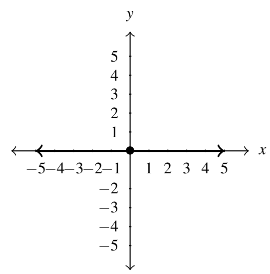
22. $y=3-x$
    slope: $m=-1$
    y-intercept: $(0,3)$
    x-intercept: $(3,0)$



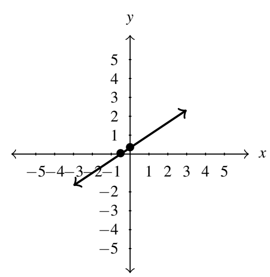
23. $y = 3$
slope: $m = 0$
y-intercept: $(0,3)$
x-intercept: none



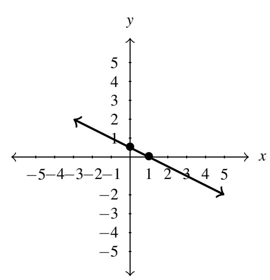
24. $y = 0$
slope: $m = 0$
y-intercept: $(0,0)$
x-intercept: $\{(x,0)\,|\,x \text{ is a real number}\}$



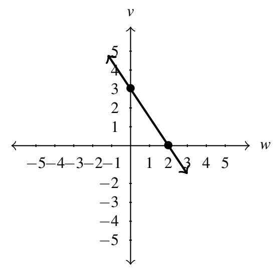
25. $y = \frac{2}{3}x + \frac{1}{3}$
slope: $m = \frac{2}{3}$
y-intercept: $\left(0, \frac{1}{3}\right)$
x-intercept: $\left(-\frac{1}{2}, 0\right)$



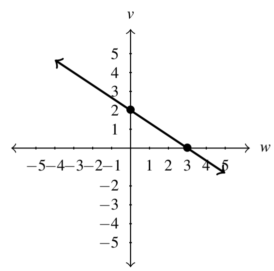
26. $y = \dfrac{1-x}{2}$
slope: $m = -\frac{1}{2}$
y-intercept: $\left(0, \frac{1}{2}\right)$
x-intercept: $(1,0)$



27. $w = -\frac{3}{2}v + 3$
slope: $m = -\frac{3}{2}$
w-intercept: $(0, 3)$
v-intercept: $(2, 0)$



28. $v = -\frac{2}{3}w + 2$
slope: $m = -\frac{2}{3}$
v-intercept: $(0, 2)$
w-intercept: $(3, 0)$



29. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

30. $y = 3x$

31. $y = -6x + 20$

32. $y = \frac{2}{3}x - 4$

33. $y = -\frac{1}{3}x - \frac{2}{3}$

34. $y = -2$

35. $x = -5$

36. $y = -3x$

37. $y = \frac{1}{6}x + \frac{3}{2}$

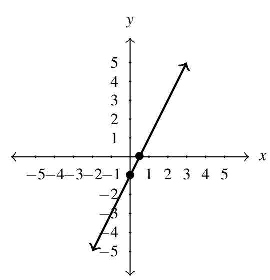
38. $y = -\frac{3}{2}x + 9$

39. $y = 3x - 4$

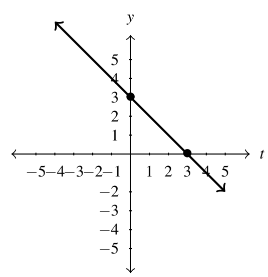
40. $x = 3$

41. $y = 0$

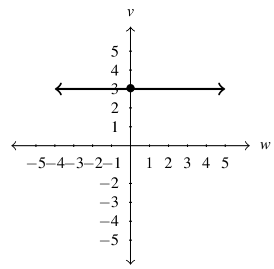
42. $f(x) = 2x - 1$
slope: $m = 2$
y-intercept: $(0, -1)$
x-intercept: $\left(\frac{1}{2}, 0\right)$



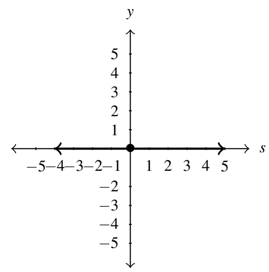
43. $g(t) = 3 - t$
slope: $m = -1$
y-intercept: $(0, 3)$
t-intercept: $(3, 0)$



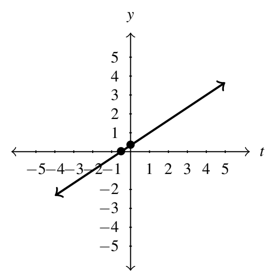
44. $F(w) = 3$
slope: $m = 0$
y-intercept: $(0,3)$
w-intercept: none



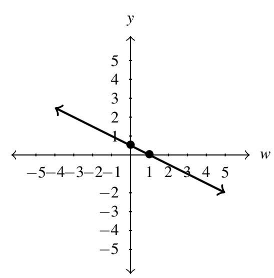
45. $G(s) = 0$
slope: $m = 0$
y-intercept: $(0,0)$
s-intercept: $\{(s,0) \,|\, s \text{ is a real number}\}$



46. $h(t) = \frac{2}{3}x + \frac{1}{3}$
slope: $m = \frac{2}{3}$
y-intercept: $\left(0, \frac{1}{3}\right)$
t-intercept: $\left(-\frac{1}{2}, 0\right)$

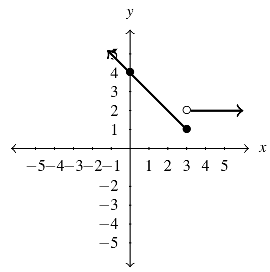


47. $j(w) = \dfrac{1-w}{2}$
slope: $m = -\frac{1}{2}$
y-intercept: $\left(0, \frac{1}{2}\right)$
w-intercept: $(1,0)$



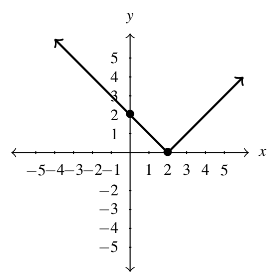
48.

domain: $(-\infty, \infty)$
range: $[1, \infty)$
y-intercept: $(0, 4)$
x-intercept: none



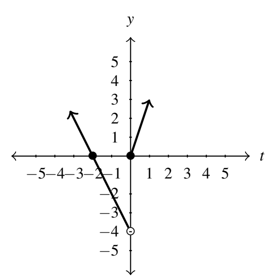
49.

domain: $(-\infty, \infty)$
range: $[0, \infty)$
y-intercept: $(0, 2)$
x-intercept: $(2, 0)$



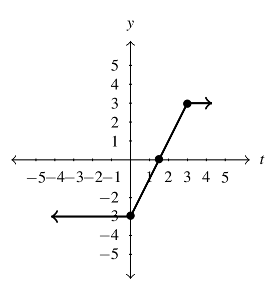
50.

domain: $(-\infty, \infty)$
range: $(-4, \infty)$
y-intercept: $(0, 0)$
t-intercepts: $(-2, 0)$, $(0, 0)$

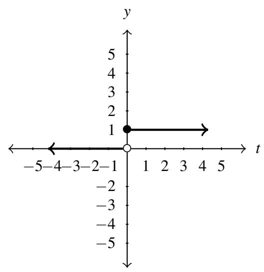


51.

domain: $(-\infty, \infty)$
range: $[-3, 3]$
y-intercept: $(0, -3)$
t-intercept: $\left(\frac{3}{2}, 0\right) = (1.5, 0)$



52. (a)

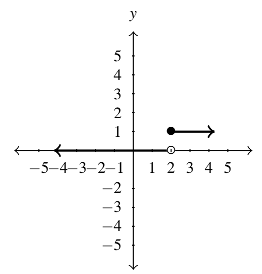


$y = U(t)$

(b) domain: $(-\infty, \infty)$, range: $\{0, 1\}$

(c) U is constant on $(-\infty, 0)$ and $[0, \infty)$.

(d) $U(t-2) = \begin{cases} 0 & \text{if } t < 2, \\ 1 & \text{if } t \geq 2. \end{cases}$



$y = U(t)$

53. $f(x) = -3$

54. $F(t) = \begin{cases} 2 & \text{if } t \leq 1, \\ -3 & \text{if } 1 < t \leq 3, \\ 4 & \text{if } t > 3. \end{cases}$

55. $L(x) = -\frac{3}{5}x + 1$

56. $g(v) = \begin{cases} 3v + 5 & \text{if } v \leq -1, \\ 2 & \text{if } -1 < v \leq 3, \end{cases}$

57. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
    (b) $C(50) = 675$, \$675. $C(51) = 612$, \$612.
    (c) 56 books.

58. (a) $S(10) = 17.5$, \$17.50.
    (b) There is free shipping on orders of 15 or more comic books.

59. (a) $C(750) = 25$, \$25.
    (b) $C(1200) = 45$, \$45.
    (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

60. $d(t) = 3t$, $t \geq 0$.

61. $E(t) = 360t$, $t \geq 0$.

62. $C(x) = 45x + 20$, $x \geq 0$.

63. $C(t) = 80t + 50$, $0 \leq t \leq 8$.

64. $W(x) = 200 + .05x$, $x \geq 0$ She must make \$5500 in weekly sales.

65. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

66. $F(m) = 2.25m + 2.05$ The slope 2.25 means it costs an additional \$2.25 for each mile beyond the first 0.2 miles. $F(0) = 2.05$, so according to the model, it would cost \$2.05 for a trip of 0 miles. Would this ever really happen? Depends on the driver and the passenger, we suppose.

67. (a) $F(T) = \frac{9}{5}T + 32$

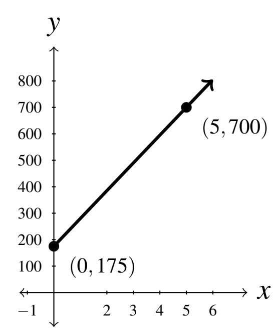
(b) $C(T) = \frac{5}{9}(T - 32) = \frac{5}{9}T - \frac{160}{9}$

(c) $F(-40) = -40 = C(-40)$.

68. $N(T) = -\frac{2}{15}T + \frac{43}{3}$ and $N(20) = \frac{35}{3} \approx 12$ howls per hour.

Having a negative number of howls makes no sense and because $N(107.5) = 0$ we can put an upper bound of $107.5°F$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-42°F$ so that the applied domain is $[-42, 107.5]$.

69. (a) $C(0) = 175$, so our start-up costs are \$175. $C(5) = 700$, so to produce 5 systems, it costs \$700.



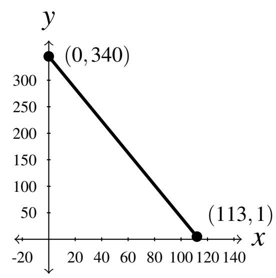
$y = C(x)$

(b) because we can't make a negative number of game systems, $x \geq 0$.

(c) The slope is $m = 105$ so for each additional system produced, it costs an additional \$105.

(d) Solving $C(x) = 15000$ gives $x \approx 141.19$ so 141 can be produced for \$15,000.

70. (a) $p(x) = -3x + 340$, $0 \leq x \leq 113$.



$y = p(x)$

(b) The slope is $m = -3$ so for each \$3 drop in price, we sell one additional game system.

(c) because $x = 150$ is not in the domain of p, $p(150)$ is not defined. (In other words, under these conditions, it is impossible to sell 150 game systems.)

(d) Solving $p(x) = 150$ gives $x \approx 63.33$ so if the price \$150 per system, we would sell 63 systems.

71. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

72. $T(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 9 \\ 12.5n & \text{if} \quad n \geq 10 \end{cases}$

73. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

74. $P(c) = \begin{cases} 0.12c & \text{if} \quad 1 \leq c \leq 100 \\ 12 + 0.1(c - 100) & \text{if} \quad c > 100 \end{cases}$

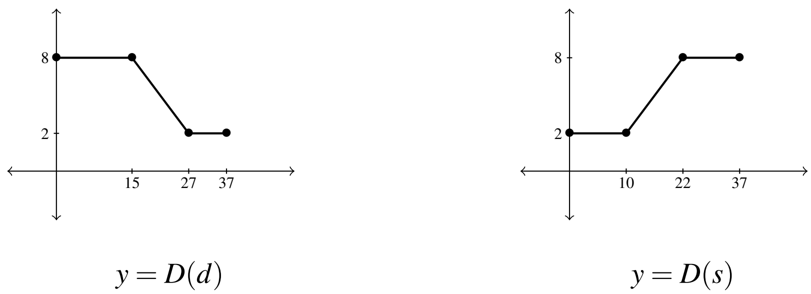
75. (a)

$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$

(b)

$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$

(c)



76. because $I(x) = x$ for all real numbers x, the function I doesn't change the 'identity' of the input at all.

77. If a graph contains more than one y-intercept, it would violate the Vertical Line Test because $x = 0$ would be matched with (at least) two different y-values.

78. Vertical Lines fail the Vertical Line Test.

79. $\left(-\frac{b}{m}, 0\right)$. (Note the importance here of $m \neq 0$.)

80. Plugging in $(c,0)$ for $(x_0, f(x_0))$, we get $f(x) = f(x_0) + m(x - x_0) = 0 + m(x - c)$ or $f(x) = m(x - c)$.

81. because L is linear with slope 3, $L(x) = L(x_0) + m\Delta x = L(100) + (3)(120 - 100) = L(100) + 60$.

82. $\frac{2^3-(-1)^3}{2-(-1)} = 3$

83. $\frac{\frac{1}{5}-\frac{1}{1}}{5-1} = -\frac{1}{5}$

84. $\frac{\sqrt{16}-\sqrt{0}}{16-0} = \frac{1}{4}$

85. $\frac{3^2-(-3)^2}{3-(-3)} = 0$

86. $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7-5} = -\frac{7}{8}$

87. $\frac{(3(2)^2+2(2)-7)-(3(-4)^2+2(-4)-7)}{2-(-4)} = -4$

88. The average rate of change is $\frac{h(2)-h(0)}{2-0} = -32$. During the first two seconds after it is dropped, the object has fallen at an average rate of 32 feet per second.

89. The average rate of change is $\frac{F(28)-F(0)}{28-0} = 0.2372$. From 1980 to 2008, the average fuel economy of passenger cars in the US increased, on average, at a rate of 0.2372 miles per gallon per year.

90. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F. $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F. $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.

    (b) The average rate of change is $\frac{T(8)-T(4)}{8-4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.

    (c) The average rate of change is $\frac{T(12)-T(8)}{12-8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.

    (d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

91. The average rate of change is $\frac{C(5)-C(3)}{5-3} = -2$. As production is increased from 3000 to 5000 pens, the cost decreases at an average rate of \$200 per 1000 pens produced (20¢ per pen.)

92. (a) i. -49.5 so the average velocity of the rocket between 14.9 and 15 seconds after lift off is -49.5 feet per second (49.5 feet per second directed *downwards*.)

    ii. -50.5 so the average velocity of the rocket between 14 and 15.1 seconds after lift off is -50.5 feet per second. (50.5 feet per second directed *downwards*.)

    iii. -49.95 so the average velocity of the rocket between 14.99 and 15 seconds after lift off is -49.95 feet per second. (49.95 feet per second directed *downwards*.)

    iv. -50.05 so the average velocity of the rocket between 15.01 and 15 seconds after lift off is -50.05 feet per second. (50.05 feet per second directed *downwards*.)

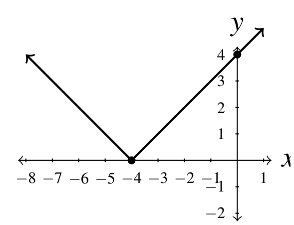
    (b) The average rate of change seem to be approaching -50.

    (c) Line: $y = -50(t-15)+375$ or $y = -50t+1125$. Graphing this line along with the s on a graphing utility we find the two graphs become indistinguishable as we zoom in near $(15, 375)$.

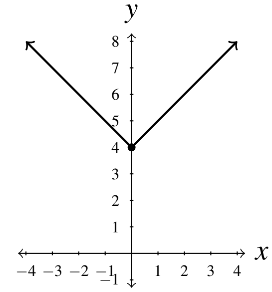
93. (a) i. $L(x) = 3$ ii. $L(x) = -2$ iii. $L(x) = x+1$ iv. $L(x) = -2x+3$

Section 1.4 Answers

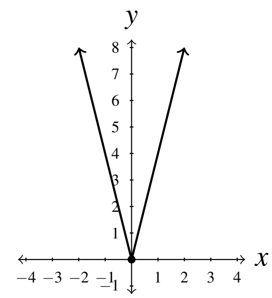
1. $f(x) = |x+4|$
   x-intercept $(-4,0)$
   y-intercept $(0,4)$
   Domain $(-\infty,\infty)$
   Range $[0,\infty)$
   Decreasing on $(-\infty,-4]$
   Increasing on $[-4,\infty)$
   Minimum is 0 at $(-4,0)$
   No maximum



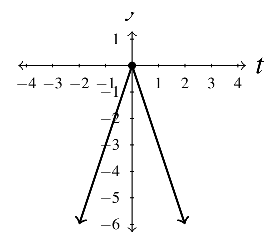
2. $f(x) = |x|+4$
   No x-intercepts
   y-intercept $(0,4)$
   Domain $(-\infty,\infty)$
   Range $[4,\infty)$
   Decreasing on $(-\infty,0]$
   Increasing on $[0,\infty)$
   Minimum is 4 at $(0,4)$
   No maximum



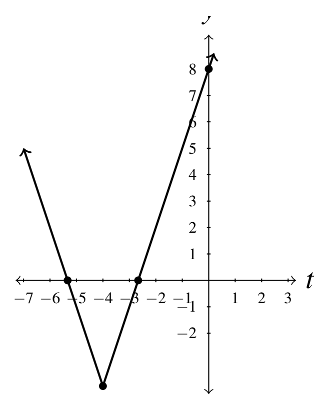
3. $f(x) = |4x|$
   x-intercept $(0,0)$
   y-intercept $(0,0)$
   Domain $(-\infty,\infty)$
   Range $[0,\infty)$
   Decreasing on $(-\infty,0]$
   Increasing on $[0,\infty)$
   Minimum is 0 at $(0,0)$
   No maximum



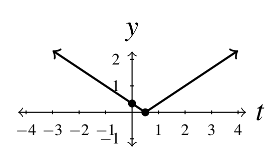
4. $g(t) = -3|t|$
   t-intercept $(0,0)$
   y-intercept $(0,0)$
   Domain $(-\infty,\infty)$
   Range $(-\infty,0]$
   Increasing on $(-\infty,0]$
   Decreasing on $[0,\infty)$
   Maximum is 0 at $(0,0)$
   No minimum



5. $g(t) = 3|t+4|-4$
   t-intercepts $\left(-\frac{16}{3},0\right)$, $\left(-\frac{8}{3},0\right)$
   y-intercept $(0,8)$
   Domain $(-\infty,\infty)$
   Range $[-4,\infty)$
   Decreasing on $(-\infty,-4]$
   Increasing on $[-4,\infty)$
   Minimum is -4 at $(-4,-4)$
   No maximum



6. $g(t) = \frac{1}{3}|2t-1|$
   t-intercepts $\left(\frac{1}{2},0\right)$
   y-intercept $\left(0,\frac{1}{3}\right)$
   Domain $(-\infty,\infty)$
   Range $[0,\infty)$
   Decreasing on $\left(-\infty,\frac{1}{2}\right]$
   Increasing on $\left[\frac{1}{2},\infty\right)$
   Minimum is 0 at $\left(\frac{1}{2},0\right)$
   No maximum



7. $F(x) = 2|x+1| - 3$

8. $F(x) = |x - 1.25| - 2.75$

9. $F(x) = -|x+1| + 2$

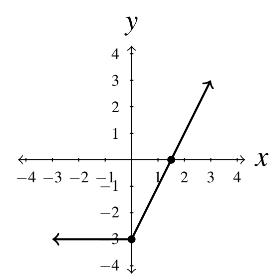
10. $F(x) = -\frac{1}{2}|x+1| + \frac{3}{2}$

11. In each case, the graph of g can be obtained from the graph of f by reflecting the portion of the graph of f which lies below the x-axis about the x-axis. This meshes with Definition 1.12 because what we are doing algebraically is making the negative y-values positive.

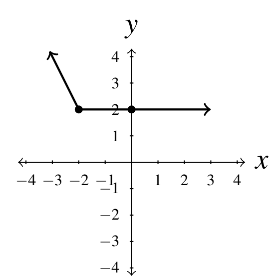
12. If $F(x) = a|x-h| + k$, then for the vertex to be at $(1,-2)$, $h = 1$ and $k = -2$ so $F(x) = a|x-1| - 2$. because $(0,-1)$ is on the graph, $F(0) = -1$ so $-1 = a|0-1| - 2$ which means $a = 1$. This means $F(x) = |x-1| - 2$. However, $(2.6, 0)$ is also on the graph, so it should work out that $F(2.6) = 0$. However, we find $F(2.6) = |2.6-1| - 2 = -0.4 \neq 0$.

$$F(x) = \begin{cases} -x-1 & \text{if } x \leq 1, \\ \frac{5}{4}x - \frac{13}{4} & \text{if } x \geq 1, \end{cases}$$

13. Re-write $f(x) = x + |x| - 3$ as
$f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad x \geq 0 \end{cases}$
x-intercept $\left(\frac{3}{2}, 0\right)$
y-intercept $(0,-3)$
Domain $(-\infty, \infty)$
Range $[-3, \infty)$
Increasing on $[0, \infty)$
Constant on $(-\infty, 0]$
Minimum is -3 at $(x,-3)$ where $x \leq 0$
No maximum



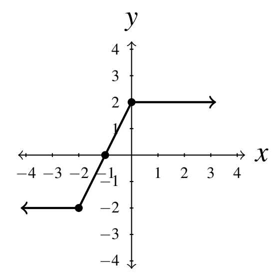
14. Re-write $f(x) = |x+2| - x$ as
$f(x) = \begin{cases} -2x-2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$
No x-intercepts
y-intercept $(0,2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Minimum is 2 at every point $(x,2)$ where $x \geq -2$
No maximum



15. Re-write $f(x) = |x+2| - |x|$ as

$$f(x) = \begin{cases} -2 & \text{if} \quad x < -2 \\ 2x+2 & \text{if} \quad -2 \le x < 0 \\ 2 & \text{if} \quad x \ge 0 \end{cases}$$

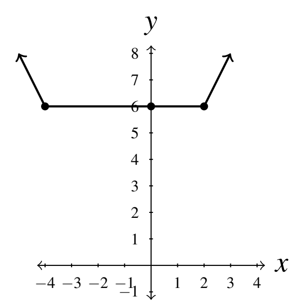
x-intercept $(-1,0)$
y-intercept $(0,2)$
Domain $(-\infty,\infty)$
Range $[-2,2]$
Increasing on $[-2,0]$
Constant on $(-\infty,-2]$
Constant on $[0,\infty)$
Minimum is -2 at $(x,-2)$ where $x \le -2$
Maximum is 2 at $(x,2)$ where $x \ge 0$



16. Re-write $g(t) = |t+4| + |t-2|$ as

$$g(t) = \begin{cases} -2t-2 & \text{if} \quad t < -4 \\ 6 & \text{if} \quad -4 \le t < 2 \\ 2t+2 & \text{if} \quad t \ge 2 \end{cases}$$

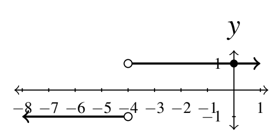
No t-intercept
y-intercept $(0,6)$
Domain $(-\infty,\infty)$
Range $[6,\infty)$
Decreasing on $(-\infty,-4]$
Constant on $[-4,2]$
Increasing on $[2,\infty)$
Minimum is 6 at $(t,6)$ where $-4 \le t \le 2$
No maximum



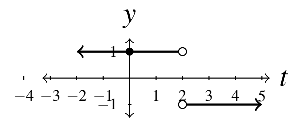
17. Re-write $g(t) = \dfrac{|t+4|}{t+4}$ as

$$g(t) = \begin{cases} -1 & \text{if} \quad t < -4 \\ 1 & \text{if} \quad t > -4 \end{cases}$$

No t-intercept
y-intercept $(0,1)$
Domain $(-\infty,-4) \cup (-4,\infty)$
Range $\{-1,1\}$
Constant on $(-\infty,-4)$
Constant on $(-4,\infty)$
Minimum is -1 at every point $(t,-1)$ where $t < -4$
Maximum is 1 at $(t,1)$ where $t > -4$



18. Re-write $g(t) = \dfrac{|2-t|}{2-t}$ as
$g(t) = \begin{cases} 1 & \text{if} \quad t < 2 \\ -1 & \text{if} \quad t > 2 \end{cases}$
No t-intercept
y-intercept $(0,1)$
Domain $(-\infty,2) \cup (2,\infty)$
Range $\{-1,1\}$
Constant on $(-\infty,2)$
Constant on $(2,\infty)$
Minimum is -1 at $(t,-1)$ where $t > 2$
Maximum is 1 at every point $(t,1)$ where $t < 2$



19. $f(x) = ||x| - 4|$

20. $x = -6$ or $x = 6$
21. $x = -3$ or $x = \frac{11}{3}$
22. $x = -3$ or $x = 11$
23. $t = -1$ or $t = 1$
24. $t = -\frac{1}{2}$ or $t = \frac{1}{10}$
25. no solution
26. $w = -3$ or $w = 3$
27. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
28. $w = -\frac{3}{2}$
29. $x = 0$ or $x = 2$
30. $x = 1$
31. no solution
32. $x = -1$ or $x = 9$
33. $x = -\frac{1}{7}$ or $x = 1$
34. $x = 0$ or $x = 2$
35. $t = 1$
36. $t = -\frac{3}{10}$
37. $t = \frac{1}{5}$ or $t = 5$
38. $\left[\frac{1}{3}, 3\right]$
39. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
40. $(-3,2)$
41. $(-\infty,1] \cup [3,\infty)$
42. No solution
43. $(-\infty,\infty)$
44. $(-3,2] \cup [6,11)$
45. $[3,4) \cup (5,6]$
46. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
47. $(-\infty,-4) \cup \left(\frac{2}{3},\infty\right)$
48. $\left(-\infty, -\frac{4}{3}\right] \cup [6,\infty)$
49. $(-\infty,-5)$
50. No Solution.
51. $\left[-7, \frac{5}{3}\right]$
52. $\left(1, \frac{5}{3}\right)$
53. $(-\infty,\infty)$

Section 1.5 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$

- $(f+g)(2)=9$
- $(f-g)(-1)=-7$
- $(g-f)(1)=-1$
- $(fg)\left(\frac{1}{2}\right)=\frac{35}{4}$
- $\left(\frac{f}{g}\right)(0)=\frac{1}{4}$
- $\left(\frac{g}{f}\right)(-2)=-\frac{6}{5}$

2. For $f(x)=x^2$ and $g(x)=-2x+1$
   - $(f+g)(2)=1$
   - $(f-g)(-1)=-2$
   - $(g-f)(1)=-2$
   - $(fg)\left(\frac{1}{2}\right)=0$
   - $\left(\frac{f}{g}\right)(0)=0$
   - $\left(\frac{g}{f}\right)(-2)=\frac{5}{4}$

3. For $f(x)=x^2-x$ and $g(x)=12-x^2$
   - $(f+g)(2)=10$
   - $(f-g)(-1)=-9$
   - $(g-f)(1)=11$
   - $(fg)\left(\frac{1}{2}\right)=-\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0)=0$
   - $\left(\frac{g}{f}\right)(-2)=\frac{4}{3}$

4. For $f(x)=2x^3$ and $g(x)=-x^2-2x-3$
   - $(f+g)(2)=5$
   - $(f-g)(-1)=0$
   - $(g-f)(1)=-8$
   - $(fg)\left(\frac{1}{2}\right)=-\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0)=0$
   - $\left(\frac{g}{f}\right)(-2)=\frac{3}{16}$

5. For $f(x)=\sqrt{x+3}$ and $g(x)=2x-1$
   - $(f+g)(2)=3+\sqrt{5}$
   - $(f-g)(-1)=3+\sqrt{2}$
   - $(g-f)(1)=-1$
   - $(fg)\left(\frac{1}{2}\right)=0$
   - $\left(\frac{f}{g}\right)(0)=-\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2)=-5$

6. For $f(x)=\sqrt{4-x}$ and $g(x)=\sqrt{x+2}$
   - $(f+g)(2)=2+\sqrt{2}$
   - $(f-g)(-1)=-1+\sqrt{5}$
   - $(g-f)(1)=0$
   - $(fg)\left(\frac{1}{2}\right)=\frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0)=\sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2)=0$

7. For $f(x)=2x$ and $g(x)=\frac{1}{2x+1}$
   - $(f+g)(2)=\frac{21}{5}$
   - $(f-g)(-1)=-1$
   - $(g-f)(1)=-\frac{5}{3}$
   - $(fg)\left(\frac{1}{2}\right)=\frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0)=0$
   - $\left(\frac{g}{f}\right)(-2)=\frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
   - $(f+g)(2) = 7$
   - $(f-g)(-1) = \frac{8}{5}$
   - $(g-f)(1) = -4$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$
9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0)$ is undefined.
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$
10. For $f(x) = x^2+1$ and $g(x) = \frac{1}{x^2+1}$
    - $(f+g)(2) = \frac{26}{5}$
    - $(f-g)(-1) = \frac{3}{2}$
    - $(g-f)(1) = -\frac{3}{2}$
    - $(fg)\left(\frac{1}{2}\right) = 1$
    - $\left(\frac{f}{g}\right)(0) = 1$
    - $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$
11. $(f+g)(-4) = -5$
12. $(f+g)(0) = 5$
13. $(f-g)(4) = -5$
14. $(fg)(-4) = 6$
15. $(fg)(-2) = 0$
16. $(fg)(4) = -6$
17. $\left(\dfrac{f}{g}\right)(0) = \dfrac{3}{2}$
18. $\left(\dfrac{f}{g}\right)(2) = 0$
19. $\left(\dfrac{g}{f}\right)(-1) = 0$
20. The domains of $f+g$, $f-g$ and fg are all $[-4,4]$. The domain of $\frac{f}{g}$ is $[-4,-1)\cup(-1,4]$ and the domain of $\frac{g}{f}$ is $[-4,-2)\cup(-2,2)\cup(2,4]$.
21. $(f+g)(-3) = 2$
22. $(f-g)(2) = 3$
23. $(fg)(-1) = 0$
24. $(g+f)(1) = 0$
25. $(g-f)(3) = 3$
26. $(gf)(-3) = -8$
27. $\left(\frac{f}{g}\right)(-2)$ does not exist
28. $\left(\frac{f}{g}\right)(-1) = 0$
29. $\left(\frac{f}{g}\right)(2) = 4$
30. $\left(\frac{g}{f}\right)(-1)$ does not exist
31. $\left(\frac{g}{f}\right)(3) = -2$
32. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$
33. For $f(x) = 2x+1$ and $g(x) = x-2$
    - $(f+g)(x) = 3x-1$
      Domain: $(-\infty,\infty)$
    - $(f-g)(x) = x+3$
      Domain: $(-\infty,\infty)$

- $(fg)(x) = 2x^2 - 3x - 2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
  Domain: $(-\infty, 2) \cup (2, \infty)$

34. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f+g)(x) = -2x$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = 2 - 6x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
  Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

35. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f+g)(x) = x^2 + 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = x^2 - 3x + 1$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 - x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
  Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

36. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f+g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

37. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f+g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

38. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f+g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$

- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

39. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f+g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f-g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

40. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f+g)(x) = \frac{x^2-2x+2}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(f-g)(x) = \frac{x^2-2x}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

41. For $f(x) = x$ and $g(x) = \sqrt{x+1}$

- $(f+g)(x) = x + \sqrt{x+1}$
  Domain: $[-1, \infty)$
- $(f-g)(x) = x - \sqrt{x+1}$
  Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x+1}$
  Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
  Domain: $(-1, \infty)$

42. For $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

- $(f+g)(x) = 2\sqrt{x-5}$
  Domain: $[5, \infty)$
- $(f-g)(x) = 0$
  Domain: $[5, \infty)$
- $(fg)(x) = x - 5$
  Domain: $[5, \infty)$
- $\left(\frac{f}{g}\right)(x) = 1$
  Domain: $(5, \infty)$

43. One solution is $f(z) = 4z$ and $g(z) = z^3$.
44. One solution is $f(z) = 4z$ and $g(z) = -z^3$.
45. One solution is $f(t) = 3t$ and $h(t) = |2t - 1|$
46. One solution is $f(x) = 3 - x$ and $g(x) = x + 1$.

47. One solution is $f(x) = 3 - x$ and $g(x) = (x+1)^{-1}$.

48. No. The equivalence does not hold when $x = 0$.

49. For $f(x) = x^2$ and $g(t) = 2t + 1$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 16$
- $(g \circ f)(-3) = 19$
- $(f \circ g)\left(\frac{1}{2}\right) = 4$
- $(f \circ f)(-2) = 16$

50. For $f(x) = 4 - x$ and $g(t) = 1 - t^2$,

- $(g \circ f)(0) = -15$
- $(f \circ g)(-1) = 4$
- $(f \circ f)(2) = 2$
- $(g \circ f)(-3) = -48$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4}$
- $(f \circ f)(-2) = -2$

51. For $f(x) = 4 - 3x$ and $g(t) = |t|$,

- $(g \circ f)(0) = 4$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 10$
- $(g \circ f)(-3) = 13$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
- $(f \circ f)(-2) = -26$

52. For $f(x) = |x - 1|$ and $g(t) = t^2 - 5$,

- $(g \circ f)(0) = -4$
- $(f \circ g)(-1) = 5$
- $(f \circ f)(2) = 0$
- $(g \circ f)(-3) = 11$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4}$
- $(f \circ f)(-2) = 2$

53. For $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$,

- $(g \circ f)(0) = \sqrt{5}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = 57$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
- $(f \circ f)(-2) = -7$

54. For $f(x) = \sqrt{3 - x}$ and $g(t) = t^2 + 1$,

- $(g \circ f)(0) = 4$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \sqrt{2}$
- $(g \circ f)(-3) = 7$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2}$
- $(f \circ f)(-2) = \sqrt{3 - \sqrt{5}}$

55. For $f(x) = 6 - x - x^2$ and $g(t) = t\sqrt{t+10}$,

- $(g \circ f)(0) = 24$
- $(f \circ g)(-1) = 0$
- $(f \circ f)(2) = 6$
- $(g \circ f)(-3) = 0$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{27-2\sqrt{42}}{8}$
- $(f \circ f)(-2) = -14$

56. For $f(x) = \sqrt[3]{x+1}$ and $g(t) = 4t^2 - t$,

- $(g \circ f)(0) = 3$
- $(f \circ g)(-1) = \sqrt[3]{6}$
- $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3}+1}$
- $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$
- $(f \circ f)(-2) = 0$

57. For $f(x) = \frac{3}{1-x}$ and $g(t) = \frac{4t}{t^2+1}$,

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)\left(\frac{1}{2}\right) = -5$
- $(f \circ f)(-2)$ is undefined

58. For $f(x) = \frac{x}{x+5}$ and $g(t) = \frac{2}{7-t^2}$,

- $(g \circ f)(0) = \frac{2}{7}$
- $(f \circ g)(-1) = \frac{1}{16}$
- $(f \circ f)(2) = \frac{2}{37}$
- $(g \circ f)(-3) = \frac{8}{19}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$
- $(f \circ f)(-2) = -\frac{2}{13}$

59. For $f(x) = \frac{2x}{5-x^2}$ and $g(t) = \sqrt{4t+1}$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
- $(f \circ f)(-2) = \frac{8}{11}$

60. For $f(x) = \sqrt{2x+5}$ and $g(t) = \frac{10t}{t^2+1}$,

- $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = \sqrt{11}$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$
- $(f \circ f)(-2) = \sqrt{7}$

61. For $f(x) = 2x + 3$ and $g(t) = t^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 2t^2 - 15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

62. For $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$

- $(g \circ f)(x) = 3x^2 - 3x - 2$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = 9t^2 - 33t + 31$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$, domain: $(-\infty, \infty)$

63. For $f(x) = x^2 - 4$ and $g(t) = |t|$

- $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = |t|^2 - 4 = t^2 - 4$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$

64. For $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$

- $(g \circ f)(x) = \sqrt{3x - 5}$, domain: $\left[\frac{5}{3}, \infty\right)$
- $(f \circ g)(t) = 3\sqrt{t} - 5$, domain: $[0, \infty)$
- $(f \circ f)(x) = 9x - 20$, domain: $(-\infty, \infty)$

65. For $f(x) = |x + 1|$ and $g(t) = \sqrt{t}$

- $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
- $(f \circ g)(t) = |\sqrt{t} + 1| = \sqrt{t} + 1$, domain: $[0, \infty)$
- $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$

66. For $f(x) = 3 - x^2$ and $g(t) = \sqrt{t + 1}$

- $(g \circ f)(x) = \sqrt{4 - x^2}$, domain: $[-2, 2]$
- $(f \circ g)(t) = 2 - t$, domain: $[-1, \infty)$
- $(f \circ f)(x) = -x^4 + 6x^2 - 6$, domain: $(-\infty, \infty)$

67. For $f(x) = |x|$ and $g(t) = \sqrt{4 - t}$

- $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
- $(f \circ g)(t) = |\sqrt{4 - t}| = \sqrt{4 - t}$, domain: $(-\infty, 4]$
- $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

68. For $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t-5}$
    - $(g \circ f)(x) = \sqrt{x^2 - x - 6}$, domain: $(-\infty, -2] \cup [3, \infty)$
    - $(f \circ g)(t) = t - 6 - \sqrt{t-5}$, domain: $[5, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$, domain: $(-\infty, \infty)$

69. For $f(x) = 3x - 1$ and $g(t) = \frac{1}{t+3}$
    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(t) = -\frac{t}{t+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

70. For $f(x) = \frac{3x}{x-1}$ and $g(t) = \frac{t}{t-3}$
    - $(g \circ f)(x) = x$, domain: $(-\infty, 1) \cup (1, \infty)$
    - $(f \circ g)(t) = t$, domain: $(-\infty, 3) \cup (3, \infty)$
    - $(f \circ f)(x) = \frac{9x}{2x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$

71. For $f(x) = \frac{x}{2x+1}$ and $g(t) = \frac{2t+1}{t}$
    - $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
    - $(f \circ g)(t) = \frac{2t+1}{5t+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
    - $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

72. For $f(x) = \frac{2x}{x^2-4}$ and $g(t) = \sqrt{1-t}$
    - $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$, domain: $(-\infty, -2) \cup \left[1 - \sqrt{5}, 2\right) \cup \left[1 + \sqrt{5}, \infty\right)$
    - $(f \circ g)(t) = -\frac{2\sqrt{1-t}}{t+3}$, domain: $(-\infty, -3) \cup (-3, 1]$
    - $(f \circ f)(x) = \frac{4x-x^3}{x^4-9x^2+16}$, domain: $\left(-\infty, -\frac{1+\sqrt{17}}{2}\right) \cup \left(-\frac{1+\sqrt{17}}{2}, -2\right) \cup \left(-2, \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right) \cup \left(\frac{-1+\sqrt{17}}{2}, 2\right) \cup \left(2, \frac{1+\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$

73. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$

74. $(h \circ f \circ g)(t) = |-2\sqrt{t}| = 2\sqrt{t}$, domain: $[0, \infty)$

75. $(g \circ f \circ h)(s) = \sqrt{-2|s|}$, domain: $\{0\}$

76. $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$, domain: $(-\infty, \infty)$

77. $(f \circ h \circ g)(t) = -2|\sqrt{t}| = -2\sqrt{t}$, domain: $[0, \infty)$

78. $(f \circ g \circ h)(s) = -2\sqrt{|s|}$, , domain: $(-\infty, \infty)$

79. $(f \circ g)(3) = f(g(3)) = f(2) = 4$

80. $f(g(-1)) = f(-4)$ which is undefined

81. $(f \circ f)(0) = f(f(0)) = f(1) = 3$

82. $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

83. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

84. $g(f(-3)) = g(4)$ which is undefined

85. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

86. $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

87. $g(f(g(0))) = g(f(0)) = g(1) = -3$

88. $f(f(f(-1))) = f(f(0)) = f(1) = 3$

89. $f(f(f(f(f(1))))) = f(f(f(f(3)))) =$
$f(f(f(-1))) = f(f(0)) = f(1) = 3$

90. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

91.
- The domain of $f \circ g$ is $\{-3,-2,0,1,2,3\}$ and the range of $f \circ g$ is $\{1,2,3,4\}$.
- The domain of $g \circ f$ is $\{-2,-1,0,1,3\}$ and the range of $g \circ f$ is $\{-4,-3,0,1,2\}$.

92. $(g \circ f)(1) = 3$

93. $(f \circ g)(3) = 1$

94. $(g \circ f)(2) = 0$

95. $(f \circ g)(0) = 1$

96. $(f \circ f)(4) = 1$

97. $(g \circ g)(1) = 0$

98.
- The domain of $f \circ g$ is $[0,3]$ and the range of $f \circ g$ is $[1,4.5]$.
- The domain of $g \circ f$ is $[0,2] \cup [3,4]$ and the range is $[0,3]$.

99. Let $f(x) = 2x+3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.

100. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.

101. Let $f(t) = 2t - 1$ and $g(t) = \sqrt{t}$, then $h(t) = (g \circ f)(t)$.

102. Let $f(t) = 7 - 3t$ and $g(t) = |t|$, then $H(t) = (g \circ f)(t)$.

103. Let $f(s) = 5s + 1$ and $g(s) = \frac{2}{s}$, then $r(s) = (g \circ f)(s)$.

104. Let $f(s) = s^2 - 1$ and $g(s) = \frac{7}{s}$, then $R(s) = (g \circ f)(s)$.

105. Let $f(z) = |z|$ and $g(z) = \frac{z+1}{z-1}$, then $q(z) = (g \circ f)(z)$.

106. Let $f(z) = z^3$ and $g(z) = \frac{2z+1}{z-1}$, then $Q(z) = (g \circ f)(z)$.

107. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.

108. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.

109. $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \frac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

110. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

111. One solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x-7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x+1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x-7$ and $H(x) = -\frac{1}{2}x+1$.

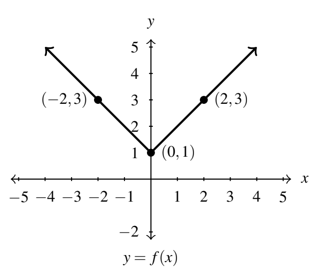
112. $(f \circ g)(x) = \begin{cases} 6x-2 & \text{if } x \leq 3 \\ 13-3x & \text{if } x > 3 \end{cases}$ and $(g \circ f)(x) = \begin{cases} 6x+1 & \text{if } x \leq \dfrac{2}{3} \\ 3-3x & \text{if } x > \dfrac{2}{3} \end{cases}$

113. $V(x) = x^3$ so $V(x(t)) = (t+1)^3$

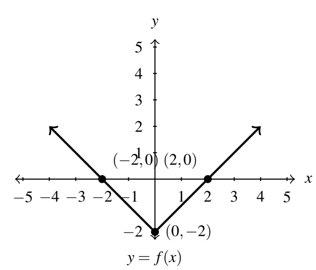
114. (a) $R(x) = 2x$
   (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.
   (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

Section 1.6 Answers

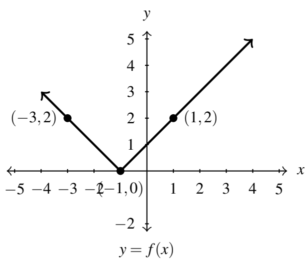
1. $(2,0)$
2. $(-1,-3)$
3. $(2,-4)$
4. $(3,-3)$
5. $(2,-9)$
6. $\left(\frac{2}{3},-3\right)$
7. $(2,3)$
8. $(-2,-3)$
9. $(5,-2)$
10. $(1,-6)$
11. $(2,13)$
12. $y = (1,-10)$
13. $\left(2,-\frac{3}{2}\right)$
14. $\left(\frac{1}{2},-12\right)$
15. $(-1,-7)$
16. $\left(-\frac{1}{2},-3\right)$
17. $\left(\frac{2}{3},-2\right)$
18. $(1,1)$
19. $y = f(x)+1$



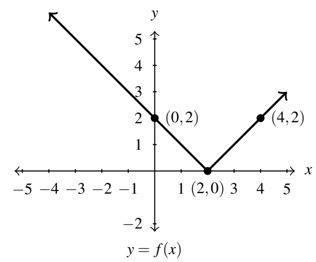
20. $y = f(x)-2$



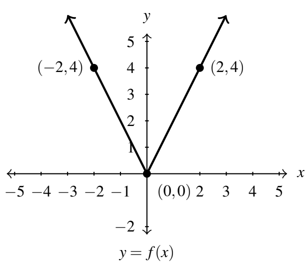
21. $y = f(x+1)$



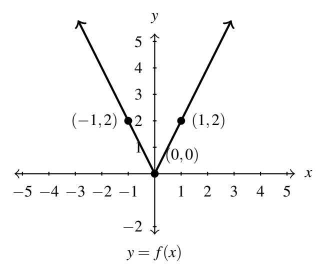
22. $y = f(x-2)$



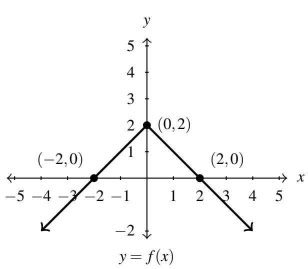
23. $y = 2f(x)$



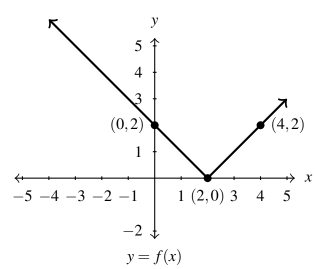
24. $y = f(2x)$



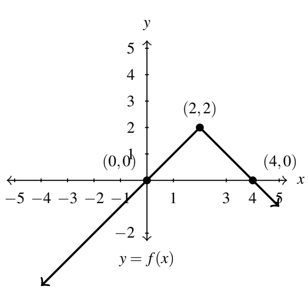
25. $y = 2 - f(x)$



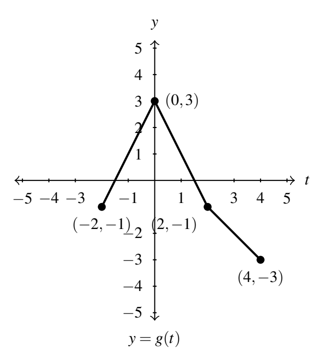
26. $y = f(2-x)$



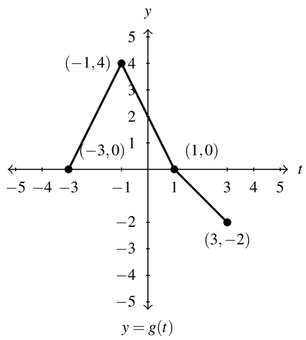
27. $y = 2 - f(2-x)$



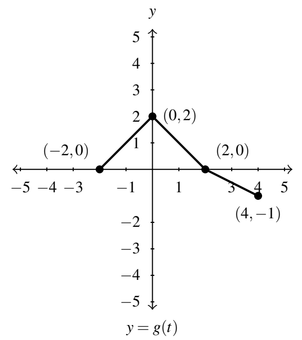
30. $y = g(t) - 1$



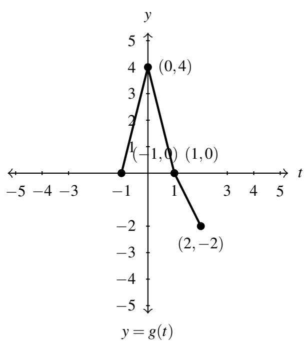
31. $y = g(t+1)$



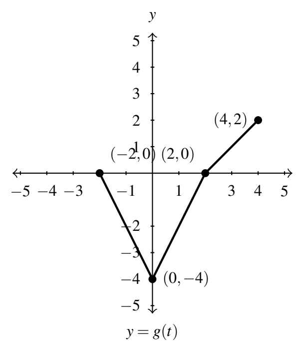
32. $y = \frac{1}{2}g(t)$



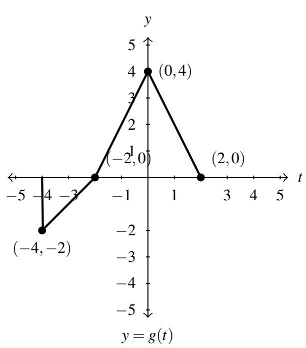
33. $y = g(2t)$



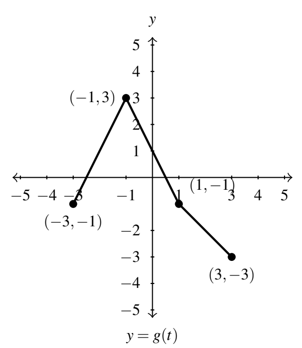
34. $y = -g(t)$



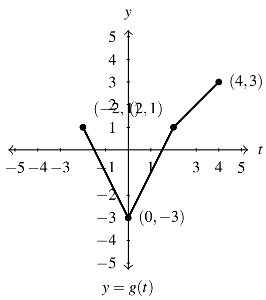
35. $y = g(-t)$



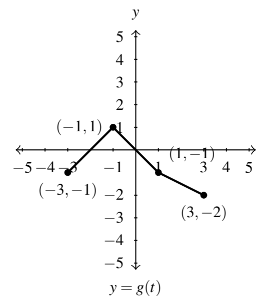
36. $y = g(t+1) - 1$



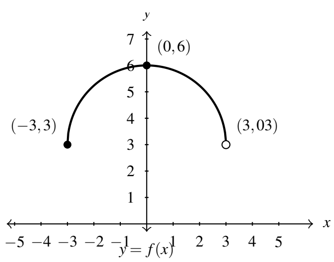
37. $y = 1 - g(t)$



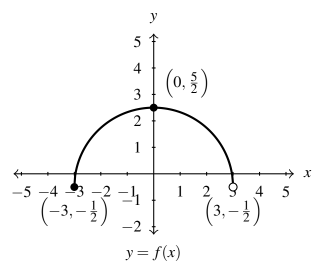
38. $y = \frac{1}{2}g(t+1) - 1$



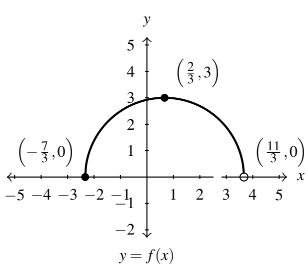
39. $g(x) = f(x) + 3$



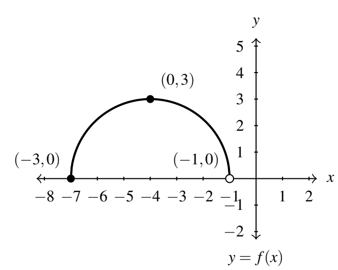
40. $h(x) = f(x) - \frac{1}{2}$



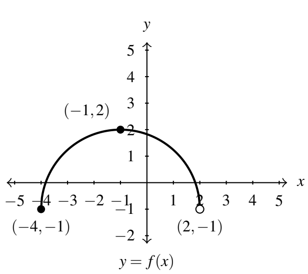
41. $j(x) = f\left(x - \frac{2}{3}\right)$



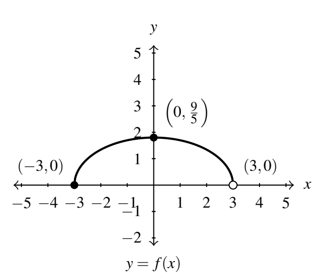
42. $a(x) = f(x+4)$



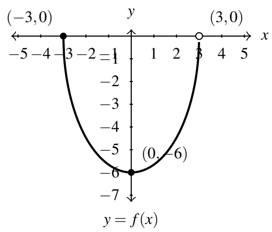
43. $b(x) = f(x+1) - 1$



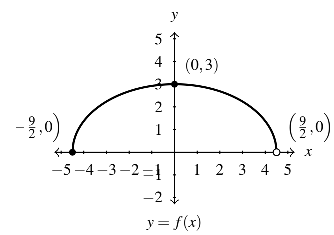
44. $c(x) = \frac{3}{5}f(x)$



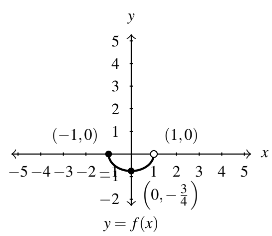
45. $d(x) = -2f(x)$



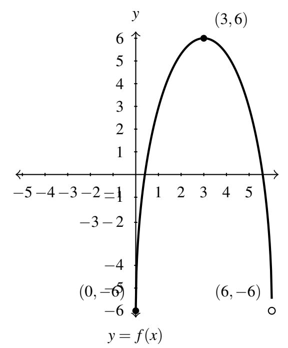
46. $k(x) = f\left(\frac{2}{3}x\right)$



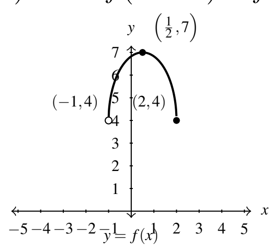
47. $m(x) = -\frac{1}{4}f(3x)$



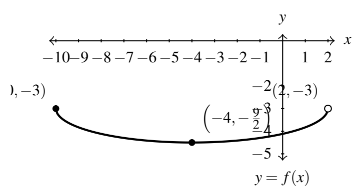
48. $n(x) = 4f(x-3) - 6$



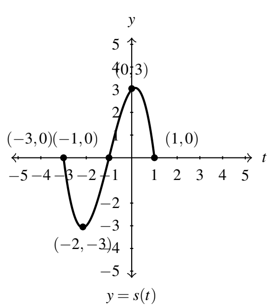
49. $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$



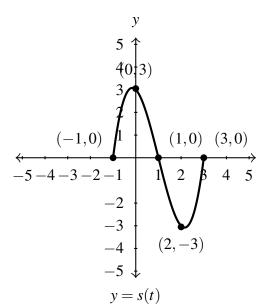
50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x + 2\right) - 3$



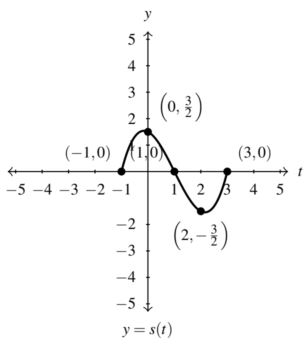
51. $y = S_1(t) = S(t+1)$



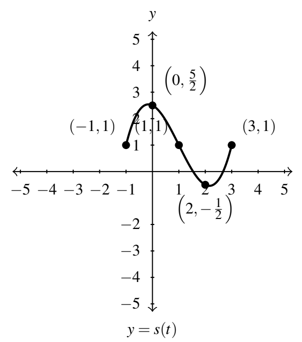
52. $y = S_2(t) = S_1(-t) = S(-t+1)$



53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$



54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$



55. $g(x) = \sqrt{x-2} - 3$

56. $g(x) = \sqrt{x-2} - 3$

57. $g(x) = -\sqrt{x} + 1$

58. $g(x) = -(\sqrt{x} + 1) = -\sqrt{x} - 1$

59. $g(x) = \sqrt{-x+1} + 2$

60. $g(x) = \sqrt{-(x+1)} + 2 = \sqrt{-x-1} + 2$

61. $g(x) = 2\sqrt{x+3} - 4$

62. $g(x) = 2\left(\sqrt{x+3} - 4\right) = 2\sqrt{x+3} - 8$

63. $g(x) = \sqrt{2x-3} + 1$

64. $g(x) = \sqrt{2(x-3)} + 1 = \sqrt{2x-6} + 1$

65. $g(x) = f(x) + 1$

66. $h(x) = f(x-2)$

67. $p(x) = f\left(\frac{x}{2}\right) - 1$

68. $q(x) = -2f(x) = 2f(-x)$

69. $r(x) = 2f(x+1) - 3$

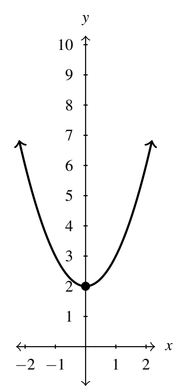
70. $s(x) = 2f(-x+1) - 3 = -2f(x-1) + 3$

71. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

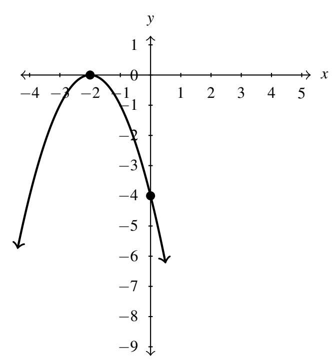
A.1.2 Chapter 2 Answers

Section 2.1 Answers

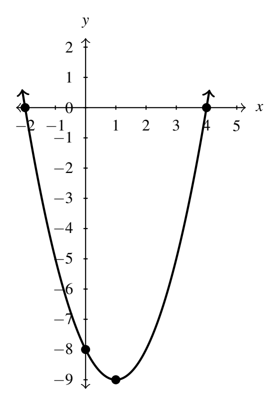
1. $f(x) = x^2 + 2$ (this is both forms!)
   No x-intercepts
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



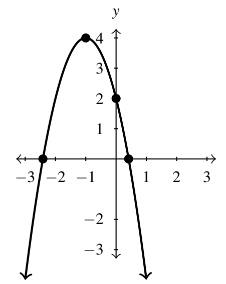
2. $f(x) = -(x+2)^2 = -x^2 - 4x - 4$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$



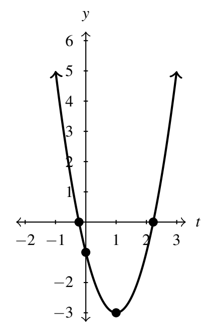
3. $f(x) = x^2 - 2x - 8 = (x-1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$



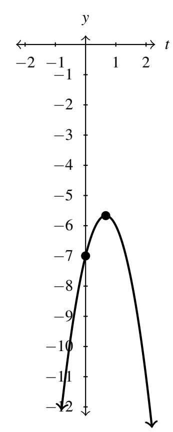
4. $g(t) = -2(t+1)^2 + 4 = -2t^2 - 4t + 2$
   t-intercepts $(-1-\sqrt{2}, 0)$ and $(-1+\sqrt{2}, 0)$
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 4]$
   Increasing on $(-\infty, -1]$
   Decreasing on $[-1, \infty)$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $t = -1$



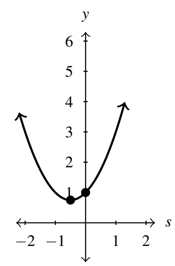
5. $g(t) = 2t^2 - tx - 1 = 2(t-1)^2 - 3$
   t-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
   y-intercept $(0, -1)$
   Domain: $(-\infty, \infty)$
   Range: $[-3, \infty)$
   Increasing on $[1, \infty)$
   Decreasing on $(-\infty, 1]$
   Vertex $(1, -3)$ is a minimum
   Axis of symmetry $t = 1$



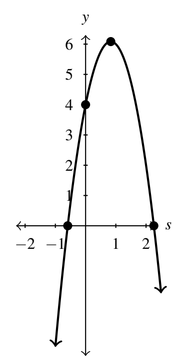
6. $g(t) = -3t^2 + 4t - 7 = -3\left(t - \frac{2}{3}\right)^2 - \frac{17}{3}$
   No t-intercepts
   y-intercept $(0, -7)$
   Domain: $(-\infty, \infty)$
   Range: $\left(-\infty, -\frac{17}{3}\right]$
   Increasing on $\left(-\infty, \frac{2}{3}\right]$
   Decreasing on $\left[\frac{2}{3}, \infty\right)$
   Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
   Axis of symmetry $t = \frac{2}{3}$



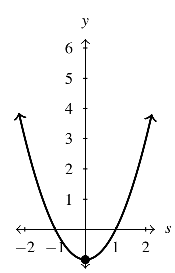
7. $h(s) = s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4}$
No s-intercepts
y-intercept $(0, 1)$
Domain: $(-\infty, \infty)$
Range: $\left[\frac{3}{4}, \infty\right)$
Increasing on $\left[-\frac{1}{2}, \infty\right)$
Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
Axis of symmetry $s = -\frac{1}{2}$



8. $h(s) = -3s^2 + 5s + 4 = -3\left(s - \frac{5}{6}\right)^2 + \frac{73}{12}$
s-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
Axis of symmetry $s = \frac{5}{6}$



9. $h(s) = s^2 - \frac{1}{100}s - 1 = \left(s - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
s-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[3]
Axis of symmetry $s = \frac{1}{200}$



10. $F(x) = (x+2)^2 - 3$

11. $F(x) = \frac{1}{2}(x-2)^2 - 1$

12. $F(x) = -x^2 + 4$

13. $F(x) = -2(x-1.5)^2 + 4.5$

14.
- $P(x) = -2x^2 + 28x - 26$, for $0 \leq x \leq 15$.
- 7 T-shirts should be made and sold to maximize profit.

[3] You'll need to use your calculator to zoom in far enough to see that the vertex is not the y-intercept.

- The maximum profit is \$72.
- The price per T-shirt should be set at \$16 to maximize profit.
- The break even points are $x = 1$ and $x = 13$, so to make a profit, between 1 and 13 T-shirts need to be made and sold.

15.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- because the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

16.
- $P(x) = -3x^2 + 72x - 240$, for $0 \leq x \leq 30$
- 12 cups of lemonade need to be made and sold to maximize profit.
- The maximum profit is 192¢ or \$1.92.
- The price per cup should be set at 54¢ per cup to maximize profit.
- The break even points are $x = 4$ and $x = 20$, so to make a profit, between 4 and 20 cups of lemonade need to be made and sold.

17.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

18.
- $P(x) = -2x^2 + 120x - 1000$, for $0 \leq x \leq 70$
- 30 scooters need to be made and sold to maximize profit.
- The maximum monthly profit is 800 hundred dollars, or \$80,000.
- The price per scooter should be set at 80 hundred dollars, or \$8000 per scooter.
- The break even points are $x = 10$ and $x = 50$, so to make a profit, between 10 and 50 scooters need to be made and sold monthly.

19. 495 cookies

20. The vertex is (approximately) $(29.60, 22.66)$, which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29 – 30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1908.) So, at this point, we are using the model to *predict* the maximum fuel economy.

21. 64° at 2 PM (8 hours after 6 AM.)

22. 5000 pens should be produced for a cost of \$200.

23. 8 feet by 16 feet; maximum area is 128 square feet.

24. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.

25. The largest rectangle has area 12.25 square inches.

26. Make the vertex of the parabola $(0,10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200,100)$ and the point on the top of the right-hand tower is $(200,100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

27. $x = \pm y\sqrt{10}$

28. $x = \pm(y-2)$

29. $x = \dfrac{m \pm \sqrt{m^2+4}}{2}$

30. $y = \dfrac{3 \pm \sqrt{16x+9}}{2}$

31. $y = 2 \pm x$

32. $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

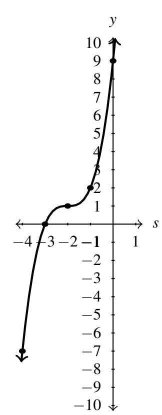
33. (a) i. $L(x) = x^2$ ii. $L(x) = 2x^2 + x$ iii. $L(x) = -x^2 + 5x + 1$

    (c) The three points lie on the same line and we get $L(x) = -x+5$.

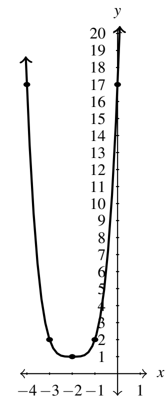
    (d) To obtain a quadratic function, we require that the points are not collinear (i.e., they do not all lie on the same line.)

Section 2.2 Answers

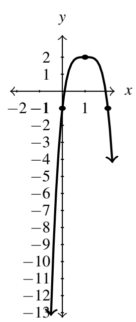
1. $F(x) = (x+2)^3 + 1$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



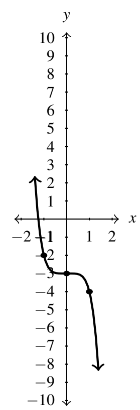
2. $F(x) = (x+2)^4 + 1$
domain: $(-\infty, \infty)$
range: $[1, \infty)$



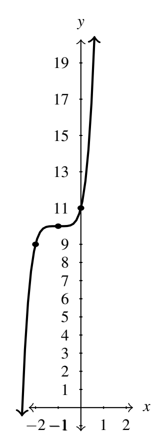
3. $F(x) = 2 - 3(x-1)^4$
domain: $(-\infty, \infty)$
range: $(-\infty, 2]$



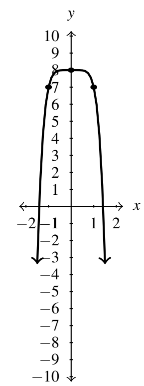
4. $F(x) = -x^5 - 3$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



5. $F(x) = (x+1)^5 + 10$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



6. $F(x) = 8 - x^6$
domain: $(-\infty, \infty)$
range: $(-\infty, 8]$



7. $F(x) = (x-1)^3 - 2$

8. $F(x) = -\frac{1}{2}(x+2)^3 + 3$

9. $F(x) = 2(x+1)^4 - 4$

10. $F(x) = -0.15625x^4 + 2.5$

11. $f(x) = 4 - x - 3x^2$
Degree 2
Leading term $-3x^2$
Leading coefficient -3
Constant term 4
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

12. $g(x) = 3x^5 - 2x^2 + x + 1$
Degree 5
Leading term $3x^5$
Leading coefficient 3
Constant term 1
As $x \to -\infty$, $g(x) \to -\infty$
As $x \to \infty$, $g(x) \to \infty$

13. $q(r) = 1 - 16r^4$
Degree 4
Leading term $-16r^4$
Leading coefficient -16
Constant term 1
As $r \to -\infty$, $q(r) \to -\infty$
As $r \to \infty$, $q(r) \to -\infty$

14. $Z(b) = 42b - b^3$
Degree 3
Leading term $-b^3$
Leading coefficient -1
Constant term 0
As $b \to -\infty$, $Z(b) \to \infty$
As $b \to \infty$, $Z(b) \to -\infty$

15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
Degree 17
Leading term $\sqrt{3}x^{17}$
Leading coefficient $\sqrt{3}$
Constant term $\frac{1}{3}$
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$

16. $s(t) = -4.9t^2 + v_0t + s_0$
Degree 2
Leading term $-4.9t^2$
Leading coefficient -4.9
Constant term s_0
As $t \to -\infty$, $s(t) \to -\infty$
As $t \to \infty$, $s(t) \to -\infty$

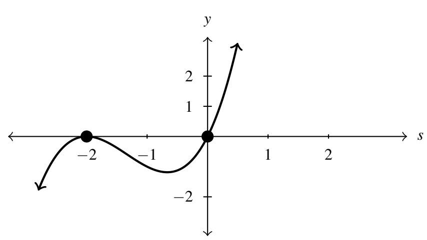
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
Degree 4
Leading term x^4
Leading coefficient 1
Constant term 24
As $x \to -\infty$, $P(x) \to \infty$
As $x \to \infty$, $P(x) \to \infty$

18. $p(t) = -t^2(3-5t)(t^2+t+4)$
Degree 5
Leading term $5t^5$
Leading coefficient 5
Constant term 0
As $t \to -\infty$, $p(t) \to -\infty$
As $t \to \infty$, $p(t) \to \infty$

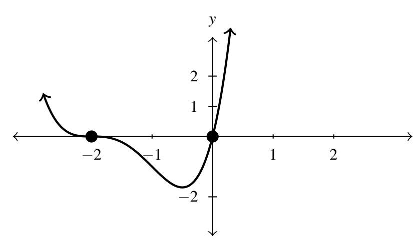
19. $f(x) = -2x^3(x+1)(x+2)^2$
Degree 6
Leading term $-2x^6$
Leading coefficient -2
Constant term 0
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

20. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$
Degree 3
Leading term $4t^3$
Leading coefficient 4
Constant term 8
As $t \to -\infty$, $G(t) \to -\infty$
As $t \to \infty$, $G(t) \to \infty$

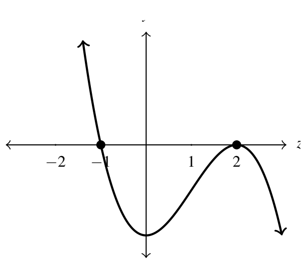
21. $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



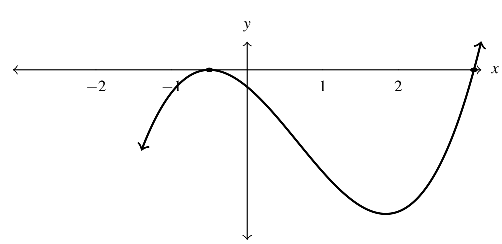
22. $g(t) = t(t+2)^3$
$t = 0$ multiplicity 1
$t = -2$ multiplicity 3



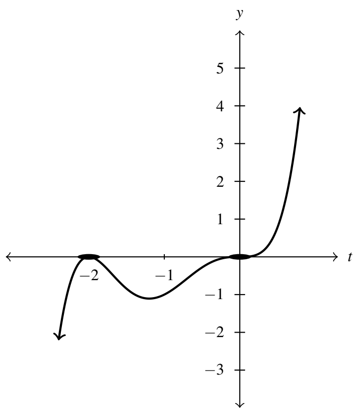
23. $f(z) = -2(z-2)^2(z+1)$
$z = 2$ multiplicity 2
$z = -1$ multiplicity 1



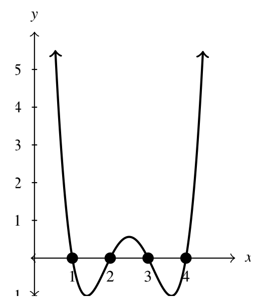
24. $g(x) = (2x+1)^2(x-3)$
$x = -\frac{1}{2}$ multiplicity 2
$x = 3$ multiplicity 1



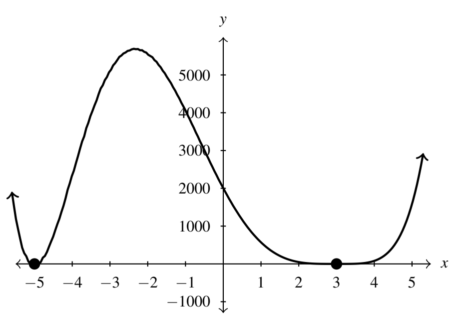
25. $F(t) = t^3(t+2)^2$
$t = 0$ multiplicity 3
$t = -2$ multiplicity 2



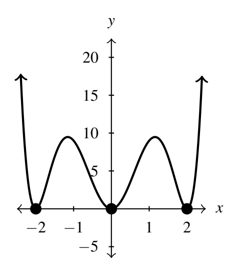
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
$z = 1$ multiplicity 1
$z = 2$ multiplicity 1
$z = 3$ multiplicity 1
$z = 4$ multiplicity 1



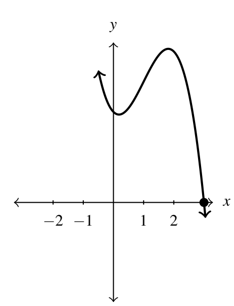
27. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



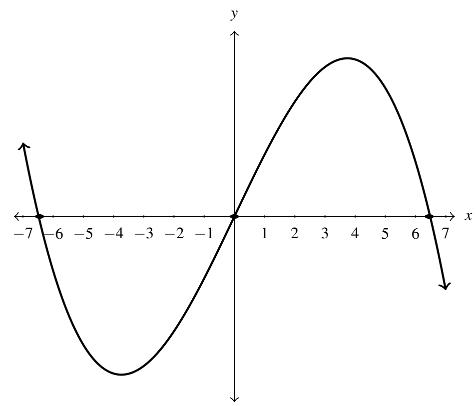
28. $f(t) = t^2(t-2)^2(t+2)^2$
$t = -2$ multiplicity 2
$t = 0$ multiplicity 2
$t = 2$ multiplicity 2



29. $H(z) = (3-z)\left(z^2+1\right)$
$z = 3$ multiplicity 1



30. $Z(x) = x(42 - x^2)$
$x = -\sqrt{42}$ multiplicity 1
$x = 0$ multiplicity 1
$x = \sqrt{42}$ multiplicity 1



31. odd
32. neither
33. even
34. even
35. even
36. neither
37. odd
38. odd
39. even
40. odd
41. neither
42. even
43. odd
44. even
45. even **and** odd

47. (a) even (b) odd (c) neither (d) odd[4]

48. For $f(x) = |x|$, $f(-x) = |-x| = |(-1)x| = |-1||x| = (1)|x| = |x|$. Hence, $f(-x) = f(x)$.

49. $V(x) = x(8.5 - 2x)(11 - 2x) = 4x^3 - 39x^2 + 93.5x$, $0 < x < 4.25$. Volume is maximized when $x \approx 1.58$, so we get the dimensions of the box with maximum volume are: height ≈ 1.58 inches, width ≈ 5.34 inches, and depth ≈ 7.84 inches. The maximum volume is ≈ 66.15 cubic inches.

50. Each of these average rates of change indicate slope of the curve over the given interval. Smaller slopes correspond to 'flatter' curves and higher slopes correspond to 'steeper' curves.

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1	0	0	0	0	0	0
x	1	1	1	1	1	1
x^2	-0.1	0.1	1.9	2.1	3.9	4.1
x^3	0.01	0.01	2.71	3.31	11.41	12.61
x^4	-0.001	0.001	3.439	4.641	29.679	34.481
x^5	0.0001	0.0001	4.0951	6.1051	72.3901	88.4101

[4] You need to first multiply out the expression for $g(x)$ so it is in the form prescribed by Definition 2.7.

51. As we sample points closer to $x = 1$, the slope of the curve approaches the exponent on x.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1	0	0	0	0
x	1	1	1	1
x^2	2	2	2	2
x^3	3.01	3.0001	≈ 3	≈ 3
x^4	4.04	4.0004	≈ 4	≈ 4
x^5	5.1001	≈ 5.001	≈ 5	≈ 5

52. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. because x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. because we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. because y represents the revenue in *thousands* of dollars, the maximum revenue is \$1,115,416.

53. $P(x) = R(x) - C(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$.

54. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. because x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. because we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. because y represents the revenue in *thousands* of dollars, the maximum revenue is \$35,255.

55. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

56. (a) To maximize the volume, we assume we start with the maximum Length + Girth of 130, so the length is $130 - 4x$. The volume of a rectangular box is 'length × width × height' so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.
    (b) Using a graphing utility, we get a (local) maximum of $y = V(x)$ at $(21.67, 20342.59)$. Hence, the maximum volume is 20342.59in.^{3} using a box with dimensions 21.67in. × 21.67in. × 43.32in..
    (c) If we start with Length + Girth = 108 then the length is $108 - 4x$ so $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ shows a (local) maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. × 18.00in. × 36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all numerical results as approximations and list them as such.)

57. (a) as $x \to -\infty$, $p(x) \to -\infty$ and as $x \to \infty$, $p(x) \to -\infty$
    (b) The zeros appear to be: $x = -1.5$, even multiplicity - probably 2 because it doesn't 'look like' the graph is very flat near $x = 2$; $x = 0$, odd multiplicity - probably 1 because the graph seems fairly linear as it passes through the origin; $x = 1$ odd multiplicity - probably 3 or higher because the graph seems fairly 'flat' near $x = 1$.

(c) local minimum: approximately $(-0.773, -2.888)$; local maximums: approximately $(-1.5, 0)$, and $(0.32, 0.532)$

(d) Based on the graph, even degree (at least 6 based on multiplicities) with a negative leading coefficient based on the end behavior.

(e) We only have a *portion* of the graph represented here.

58. We are looking for the largest open interval containing $x = -0.235$ for which the graph of $y = p(x)$ is at or above $y = -1.121$. because each of the gridlines on the x-axis correspond to 0.2 units, we approximate this interval as $(-1.25\,\text{ish}, 1.1\,\text{ish})$.

59. (c) $L(x) = x^2$ (d) $L(x) = x + 1$

Section 2.3 Answers

1. quotient: $5x - 8$, remainder: 9
2. quotient: $3y + 15$, remainder: 38
3. quotient: 3, remainder: 18
4. quotient: $\frac{2}{3}$, remainder: $\frac{11}{3}$
5. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$
6. quotient: $\frac{w^2}{5} + \frac{2w}{5} + \frac{4}{5}$, remainder: 0
7. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$
8. quotient: $2y^2 + y + 1$, remainder: 0
9. quotient: w, remainder: $2w$
10. quotient: $5t$, remainder: $-21t + 1$
11. quotient:[5] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0
12. quotient: $x - 1 - \sqrt{2}$, remainder: 0
13. $\left(3x^2 - 2x + 1\right) = (x-1)(3x+1) + 2$
14. $\left(x^2 - 5\right) = (x-5)(x+5) + 20$
15. $\left(3 - 4t - 2t^2\right) = (t+1)(-2t-2) + 5$
16. $\left(4t^2 - 5t + 3\right) = (t+3)(4t-17) + 54$
17. $\left(z^3 + 8\right) = (z+2)\left(z^2 - 2z + 4\right) + 0$
18. $\left(4z^3 + 2z - 3\right) = (z-3)\left(4z^2 + 12z + 38\right) + 111$
19. $\left(18x^2 - 15x - 25\right) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
20. $\left(4x^2 - 1\right) = \left(x - \frac{1}{2}\right)(4x + 2) + 0$
21. $\left(2t^3 + t^2 + 2t + 1\right) = \left(t + \frac{1}{2}\right)\left(2t^2 + 2\right) + 0$

[5] Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

22. $\left(3t^3-t+4\right)=\left(t-\frac{2}{3}\right)\left(3t^2+2t+\frac{1}{3}\right)+\frac{38}{9}$
23. $\left(2z^3-3z+1\right)=\left(z-\frac{1}{2}\right)\left(2z^2+z-\frac{5}{2}\right)-\frac{1}{4}$
24. $\left(4z^4-12z^3+13z^2-12z+9\right)=\left(z-\frac{3}{2}\right)\left(4z^3-6z^2+4z-6\right)+0$
25. $\left(x^4-6x^2+9\right)=\left(x-\sqrt{3}\right)\left(x^3+\sqrt{3}x^2-3x-3\sqrt{3}\right)+0$
26. $\left(x^6-6x^4+12x^2-8\right)=\left(x+\sqrt{2}\right)\left(x^5-\sqrt{2}x^4-4x^3+4\sqrt{2}x^2+4x-4\sqrt{2}\right)+0$
27. $x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$
28. $x^3-24x^2+192x-512=(x-8)^3$
29. $3t^3+4t^2-t-2=3\left(t-\frac{2}{3}\right)(t+1)^2$
30. $2t^3-3t^2-11t+6=2\left(t-\frac{1}{2}\right)(t+2)(t-3)$
31. $z^3+2z^2-3z-6=(z+2)(z+\sqrt{3})(z-\sqrt{3})$
32. $2z^3-z^2-10z+5=2\left(z-\frac{1}{2}\right)(z+\sqrt{5})(z-\sqrt{5})$
33. $4x^4-28x^3+61x^2-42x+9=4\left(x-\frac{1}{2}\right)^2(x-3)^2$
34. $t^5+2t^4-12t^3-38t^2-37t-12=(t+1)^3(t+3)(t-4)$
35. $125z^5-275z^4-2265z^3-3213z^2-1728z-324=125\left(z+\frac{3}{5}\right)^3(z+2)(z-6)$
36. $x^2-2x-2=(x-(1-\sqrt{3}))(x-(1+\sqrt{3}))$

A.1.3 CHAPTER 3 ANSWERS

Section 3.1 Answers

1. $\dfrac{3(x+3)}{x(x+2)}, x \neq 3$
2. $\dfrac{t}{(3t+4)(t^2+1)}, t \neq 2$
3. $-\dfrac{y(y-3)}{y+4}, y \neq -\dfrac{1}{2}, 3, 4$
4. $\dfrac{2x-1}{3x-1}$
5. $-w-1, w \neq 1$
6. $\dfrac{y}{3}, y \neq 0$
7. $\dfrac{b^2-5b+7}{b-3}$
8. $\dfrac{4x^2+x+4}{(x-4)(2x+1)}$
9. $\dfrac{m+1}{m+2}, m \neq 2$
10. $-\dfrac{2}{x}, x \neq 1$
11. $\dfrac{3}{4-2h}, h \neq 0$
12. $-\dfrac{1}{x(x+h)}, h \neq 0$
13. $\dfrac{8}{3w}$
14. $-\dfrac{2(y^2-7y+9)}{y(y-3)^2}$
15. $-\dfrac{6}{(x-2)^2}$
16. $t^2+t, t \neq 0$
17. $-\dfrac{2(h+6)}{9(h+3)^2}, h \neq 0$
18. $\dfrac{1}{(7-x)(7-x-h)}, h \neq 0$
19. 2, 2.
20. -3, -3.
21. 0, 0
22. $3h+11$, $6x+3h-1$
23. $-h-2$, $-2x-h+2$
24. $4h+16$, $8x+4h$
25. $-h-3$, $-2x-h+1$
26. $h^2+6h+12$, $3x^2+3xh+h^2$
27. m, m
28. $ah+4a+b$, $2ax+ah+b$
29. $\dfrac{2}{\Delta x-1}, \dfrac{-2}{x(x+\Delta x)}$
30. $\dfrac{-3}{2(\Delta x-2)}, \dfrac{3}{(x+\Delta x-1)(x-1)}$
31. $\dfrac{2-\Delta x}{(\Delta x-1)^2}, \dfrac{-(2x+\Delta x)}{x^2(x+\Delta x)^2}$
32. $\dfrac{-1}{2(\Delta x+4)}, \dfrac{-2}{(x+5)(x+\Delta x+5)}$
33. $\dfrac{4}{7(4\Delta x-7)}, \dfrac{-4}{(4x-3)(4x+4\Delta x-3)}$
34. $\dfrac{6}{\Delta x+1}, \dfrac{6}{(x+2)(x+\Delta x+2)}$
35. $\dfrac{9}{10(\Delta x-10)}, \dfrac{-9}{(x-9)(x+\Delta x-9)}$
36. $\dfrac{\Delta x}{2\Delta x-1}, \dfrac{2x^2+2x\Delta x+2x+\Delta x}{(2x+1)(2x+2\Delta x+1)}$
37. $\dfrac{-1}{\sqrt{9-\Delta t}+3}, \dfrac{-1}{\sqrt{9-t-\Delta t}+\sqrt{9-t}}$
38. $\dfrac{2}{\sqrt{2\Delta t+1}+1}, \dfrac{2}{\sqrt{2t+2\Delta t+1}+\sqrt{2t+1}}$

39. $\dfrac{-4}{\sqrt{5-4\Delta t}+\sqrt{5}}, \dfrac{-4}{\sqrt{-4t-4\Delta t+5}+\sqrt{-4t+5}}$

40. $\dfrac{-1}{\sqrt{4-\Delta t}+2}, \dfrac{-1}{\sqrt{4-t-\Delta t}+\sqrt{4-t}}$

41. $\dfrac{a}{\sqrt{a\Delta t+b}+\sqrt{b}}, \dfrac{a}{\sqrt{at+a\Delta t+b}+\sqrt{at+b}}$

42. $(\Delta t)^{\frac{1}{2}}, \dfrac{3t^2+3t\Delta t+(\Delta t)^2}{(t+\Delta t)^{3/2}+t^{3/2}}$

43. $\dfrac{1}{(\Delta t)^{2/3}}, \dfrac{1}{(t+\Delta t)^{2/3}+(t+\Delta t)^{1/3}t^{1/3}+t^{2/3}}$

44. (b) $(f_+ + f_-)(x) = f_+(x)+f_-(x) = \dfrac{f(x)+|f(x)|}{2}+\dfrac{f(x)-|f(x)|}{2} = \dfrac{2f(x)}{2} = f(x).$

(c)

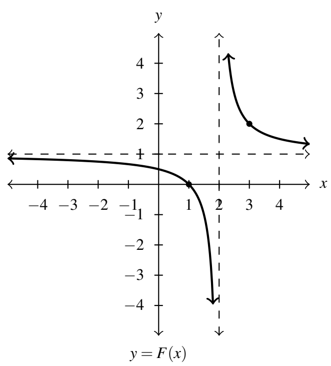
$$f_+(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}, \qquad f_-(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

Section 3.2 Answers

1. $4x^2+3x-1 = (x-3)(4x+15)+44$
2. $2x^3-x+1 = \left(x^2+x+1\right)(2x-2)+(-x+3)$
3. $5x^4-3x^3+2x^2-1 = \left(x^2+4\right)\left(5x^2-3x-18\right)+(12x+71)$
4. $-x^5+7x^3-x = \left(x^3-x^2+1\right)\left(-x^2-x+6\right)+\left(7x^2-6\right)$
5. $9x^3+5 = (2x-3)\left(\frac{9}{2}x^2+\frac{27}{4}x+\frac{81}{8}\right)+\frac{283}{8}$
6. $4x^2-x-23 = \left(x^2-1\right)(4)+(-x-19)$

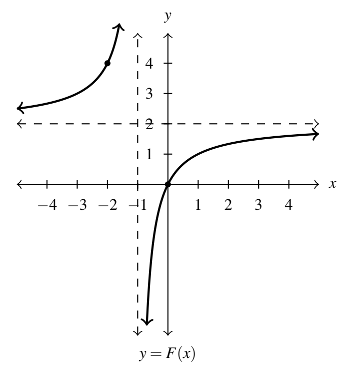
7. $F(x) = \dfrac{1}{x-2} + 1$

Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 1$



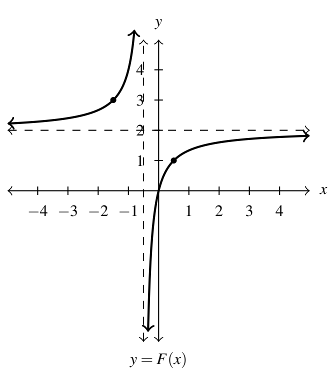
8. $F(x) = \dfrac{2x}{x+1} = \dfrac{-2}{x+1} + 2$

Domain: $(-\infty, -1) \cup (-1, \infty)$
Range: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 2$



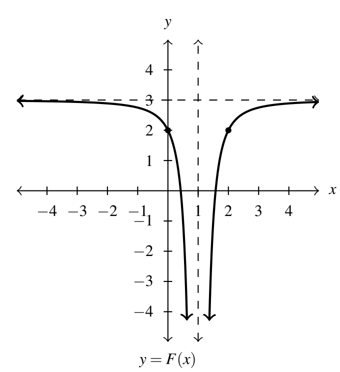
9. $F(x) = 4x(2x+1)^{-1} = \dfrac{4x}{2x+1} = \dfrac{-1}{x+\frac{1}{2}} + 2$

Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
Range: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $y = 2$
Horizontal asymptote: $x = -\frac{1}{2}$



10. $F(x) = -(x-1)^{-2} + 3 = \dfrac{-1}{(x-1)^2} + 3$

Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 3$



11. $F(x) = \dfrac{1}{x+2} - 1$

12. $F(x) = \dfrac{-2}{x-1} + 1$

13. $F(x) = \dfrac{-4}{(x+2)^2} + 4$

14. $F(x) = \dfrac{1}{\left(x-\frac{1}{2}\right)^2} - 4$

15. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty, f(x) \to \frac{1}{3}^-$
As $x \to \infty, f(x) \to \frac{1}{3}^+$

16. $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-, f(x) \to \infty$
As $x \to \frac{5}{2}^+, f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty, f(x) \to -\frac{7}{2}^+$
As $x \to \infty, f(x) \to -\frac{7}{2}^-$

17. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4, x = 3$
As $x \to -4^-, f(x) \to -\infty$
As $x \to -4^+, f(x) \to \infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

18. $g(t) = \dfrac{t}{t^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $t \to -\infty, g(t) \to 0^-$
As $t \to \infty, g(t) \to 0^+$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $t = -3$
As $t \to -3^-, g(t) \to \infty$
As $t \to -3^+, g(t) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[6]As $t \to -\infty, g(t) \to 0^-$
As $t \to \infty, g(t) \to 0^+$

20. $g(t) = \dfrac{t^3+1}{t^2-1} = \dfrac{t^2-t+1}{t-1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $t = 1$
As $t \to 1^-, g(t) \to -\infty$
As $t \to 1^+, g(t) \to \infty$
Hole at $(-1, -\frac{3}{2})$
Slant asymptote: $y = t$
As $t \to -\infty$, the graph is below $y = t$
As $t \to \infty$, the graph is above $y = t$

[6]This is hard to see on the calculator, but trust us, the graph is below the t-axis to the left of $t = -7$.

21. $r(z) = \dfrac{4z}{z^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty, r(z) \to 0^-$
As $z \to \infty, r(z) \to 0^+$

22. $r(z) = \dfrac{4z}{z^2-4} = \dfrac{4z}{(z+2)(z-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
Vertical asymptotes: $z = -2, z = 2$
As $z \to -2^-, r(z) \to -\infty$
As $z \to -2^+, r(z) \to \infty$
As $z \to 2^-, r(z) \to -\infty$
As $z \to 2^+, r(z) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty, r(z) \to 0^-$
As $z \to \infty, r(z) \to 0^+$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6} = \dfrac{z-4}{z-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $z = 2$
As $z \to 2^-, r(z) \to \infty$
As $z \to 2^+, r(z) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $z \to -\infty, r(z) \to 1^+$
As $z \to \infty, r(z) \to 1^-$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to -\infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty, f(x) \to 3^+$
As $x \to \infty, f(x) \to 3^-$

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x+3$
As $x \to -\infty$, the graph is below $y = x+3$
As $x \to \infty$, the graph is above $y = x+3$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = x$
As $x \to -\infty$, the graph is above $y = x$
As $x \to \infty$, the graph is below $y = x$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $t = -\frac{2}{3}$
As $t \to -\frac{2}{3}^-, g(t) \to \infty$
As $t \to -\frac{2}{3}^+, g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to -\infty$, the graph is above $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to \infty$, the graph is below $y = \frac{2}{3}t + \frac{11}{9}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9} = \dfrac{-t^3+4t}{(t-3)(t+3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $t = -3$, $t = 3$
As $t \to -3^-, g(t) \to \infty$
As $t \to -3^+, g(t) \to -\infty$
As $t \to 3^-, g(t) \to \infty$
As $t \to 3^+, g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -t$
As $t \to -\infty$, the graph is above $y = -t$
As $t \to \infty$, the graph is below $y = -t$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1} = \dfrac{-5t^4-3t^3+t^2-10}{(t-1)^3}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $t = 1$
As $t \to 1^-, g(t) \to \infty$
As $t \to 1^+, g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -5t - 18$
As $t \to -\infty$, the graph is above $y = -5t - 18$
As $t \to \infty$, the graph is below $y = -5t - 18$

30. $r(z) = \dfrac{z^3}{1-z}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $z = 1$
As $z \to 1^-, r(z) \to \infty$
As $z \to 1^+, r(z) \to -\infty$
No holes in the graph
No horizontal or slant asymptote
As $z \to -\infty$, $r(z) \to -\infty$
As $z \to \infty$, $r(z) \to -\infty$

31. $r(z) = \dfrac{18-2z^2}{z^2-9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
As $z \to \pm\infty$, $r(z) = -2$

32. $r(z) = \dfrac{z^3-4z^2-4z-5}{z^2+z+1} = z - 5$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = z - 5$
$r(z) = z - 5$ everywhere.

33. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.

(b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.

(c) For \$40000 you could remove about 95.76% of the fish.

34. (a) $\overline{v}(t) = \frac{s(t)-s(5)}{t-5} = \frac{-5t^2+100t-375}{t-5} = -5t+75$, $t \neq 5$. The instantaneous velocity of the rocket when $t_0 = 5$ is $-5(5)+75 = 50$ meaning it is traveling 50 feet per second upwards.

(b) $\overline{v}(t) = \frac{s(t)-s(9)}{t-9} = \frac{-5t^2+100t-495}{t-9} = -5t+55$, $t \neq 9$. The instantaneous velocity of the rocket when $t_0 = 9$ is $-5(9)+55 = 10$, so the rocket has slowed to 10 feet per second (but still heading up.)

(c) $\overline{v}(t) = \frac{s(t)-s(10)}{t-10} = \frac{-5t^2+100t-495}{t-10} = -5t+50$, $t \neq 10$. The instantaneous velocity of the rocket when $t_0 = 10$ is $-5(10)+50 = 0$, so the rocket has momentarily stopped! In Example 1.3.12, we learned the rocket reaches its maximum height when $t = 10$ seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is 0.

(d) $\overline{v}(t) = \frac{s(t)-s(11)}{t-11} = \frac{-5t^2+100t-495}{t-11} = -5t+45$, $t \neq 11$. The instantaneous velocity of the rocket when $t_0 = 11$ is $-5(11)+45 = -10$ meaning the rocket has, indeed, changed direction and is heading downwards at a rate of 10 feet per second. (Note the symmetry here between this answer and our answer when $t = 9$.)

35. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y = 150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

36. (a) $\overline{C}(x) = \frac{100x+2000}{x} = 100 + \frac{2000}{x}$, $x > 0$.

(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.

(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.

(d) As $x \to 0^+$, $\overline{C}(x) \to \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.

(e) As $x \to \infty$, $\overline{C}(x) \to 100^+$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

37. (a) The cost to make 0 items is $C(0) = m(0) + b = b$. Hence, so the fixed costs are b.

(b) $C(x) = mx + b$ is a linear function with slope $m > 0$. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.

(c) $\overline{C}(x) = \frac{C(x)}{x} = \frac{mx+b}{x} = m + \frac{b}{x}$ for $x > 0$.

(d) Since $b > 0$, $\overline{C}(x) = m + \frac{b}{x} > m$ for $x > 0$. As $x \to \infty$, $\frac{b}{x} \to 0$ so $\overline{C}(x) = m + \frac{b}{x} \to m$.

(e) Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\frac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

38. If $p(x) = mx+b$ and $C(x)$ is linear, say $C(x) = rx+s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx+b) - (rx+s)$ which simplifies to $P(x) = mx^2 + (b-r)x - s$. Hence, the average profit $\overline{P}(x) = \frac{P(x)}{x} = \frac{mx^2+(b-r)x-s}{x} = mx + (b-r) - \frac{s}{x}$. We see that as $x \to \infty$, $\frac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b-r)$. Hence, $y = mx + (b-r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

39. (a)

    (b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

    (c) As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

40. $a = -2$ and $c = -18$ so $f(x) = \dfrac{-2x^2+18}{x+3}$.

41. (a) $a = 6$ and $n = 2$ so $f(x) = \dfrac{6x^2-4}{2x^2+1}$ (b) $a = 10$ and $n = 3$ so $f(x) = \dfrac{10x^3-4}{2x^2+1}$.

42. If we define $f(x) = p(x) - p(a)$ then f is a polynomial function with $f(a) = p(a) - p(a) = 0$. The Factor Theorem guarantees $(x-a)$ is a factor of $f(x)$, that is, $f(x) = p(x) - p(a) = (x-a)q(x)$ for some polynomial $q(x)$. Hence, $r(x) = \frac{p(x)-p(a)}{x-a} = \frac{(x-a)q(x)}{x-a} = q(x)$ so the graph of $y = r(x)$ is the same as the graph of the polynomial $y = q(x)$ except for a hole when $x = a$.

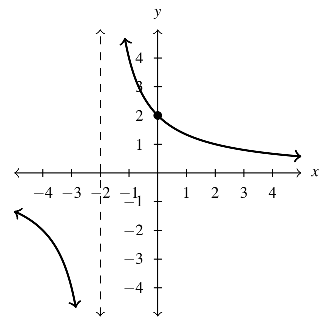
43. The slope of the curves near $x = 1$ matches the exponent on x. This exactly what we saw in Exercise 51 in Section 2.2.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4

Section 3.3 Answers

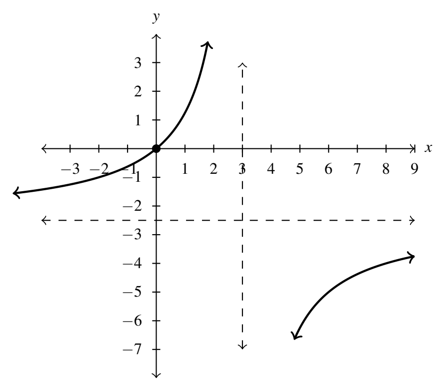
1. $f(x) = \dfrac{4}{x+2}$

   Domain: $(-\infty, -2) \cup (-2, \infty)$
   No x-intercepts
   y-intercept: $(0, 2)$
   Vertical asymptote: $x = -2$
   As $x \to -2^-$, $f(x) \to -\infty$
   As $x \to -2^+$, $f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty$, $f(x) \to 0^-$
   As $x \to \infty$, $f(x) \to 0^+$



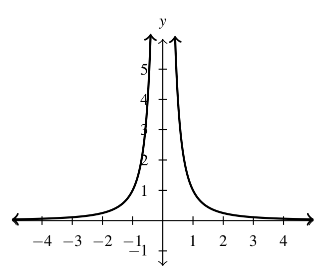
2. $f(x) = 5x(6-2x)^{-1} = \dfrac{5x}{6-2x}$

   Domain: $(-\infty, 3) \cup (3, \infty)$
   x-intercept: $(0, 0)$
   y-intercept: $(0, 0)$
   Vertical asymptote: $x = 3$
   As $x \to 3^-$, $f(x) \to \infty$
   As $x \to 3^+$, $f(x) \to -\infty$
   Horizontal asymptote: $y = -\frac{5}{2}$
   As $x \to -\infty$, $f(x) \to -\frac{5}{2}^+$
   As $x \to \infty$, $f(x) \to -\frac{5}{2}^-$



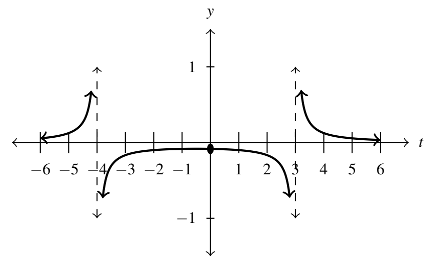
3. $g(t) = t^{-2} = \dfrac{1}{t^2}$

   Domain: $(-\infty, 0) \cup (0, \infty)$
   No t-intercepts
   No y-intercepts
   Vertical asymptote: $t = 0$
   As $t \to 0^-$, $g(t) \to \infty$
   As $t \to 0^+$, $g(t) \to \infty$
   Horizontal asymptote: $y = 0$
   As $t \to -\infty$, $g(t) \to 0^+$
   As $t \to \infty$, $g(t) \to 0^+$



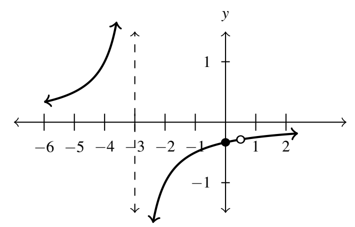
4. $g(t) = \dfrac{1}{t^2+t-12} = \dfrac{1}{(t-3)(t+4)}$

Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
No t-intercepts
y-intercept: $(0,-\frac{1}{12})$
Vertical asymptotes: $t=-4$ and $t=3$
As $t\to -4^-$, $g(t)\to\infty$
As $t\to -4^+$, $g(t)\to -\infty$
As $t\to 3^-$, $g(t)\to -\infty$
As $t\to 3^+$, $g(t)\to \infty$
Horizontal asymptote: $y=0$
As $t\to -\infty$, $g(t)\to 0^+$
As $t\to \infty$, $g(t)\to 0^+$



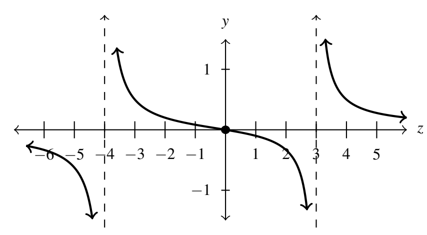
5. $r(z) = \dfrac{2z-1}{-2z^2-5z+3} = -\dfrac{2z-1}{(2z-1)(z+3)}$

Domain: $(-\infty,-3)\cup(-3,\frac{1}{2})\cup(\frac{1}{2},\infty)$
No z-intercepts
y-intercept: $(0,-\frac{1}{3})$
$r(z) = \dfrac{-1}{z+3},\ z\neq\frac{1}{2}$
Hole in the graph at $(\frac{1}{2},-\frac{2}{7})$
Vertical asymptote: $z=-3$
As $z\to -3^-$, $r(z)\to\infty$
As $z\to -3^+$, $r(z)\to -\infty$
Horizontal asymptote: $y=0$
As $z\to -\infty$, $r(z)\to 0^+$
As $z\to \infty$, $r(z)\to 0^-$



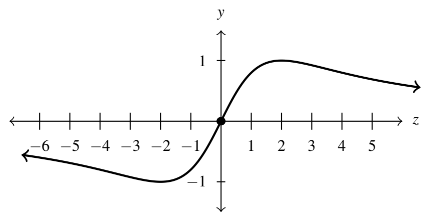
6. $r(z) = \dfrac{z}{z^2+z-12} = \dfrac{z}{(z-3)(z+4)}$

Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
z-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $z=-4$ and $z=3$
As $z\to -4^-$, $r(z)\to -\infty$
As $z\to -4^+$, $r(z)\to \infty$
As $z\to 3^-$, $r(z)\to -\infty$
As $z\to 3^+$, $r(z)\to \infty$
Horizontal asymptote: $y=0$
As $z\to -\infty$, $r(z)\to 0^-$
As $z\to \infty$, $r(z)\to 0^+$



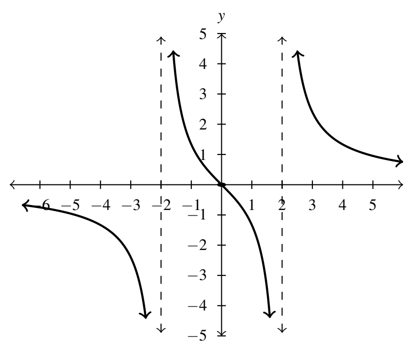
7. $f(x) = 4x(x^2+4)^{-1} = \dfrac{4x}{x^2+4}$

Domain: $(-\infty, \infty)$
x-intercept: $(0,0)$
y-intercept: $(0,0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



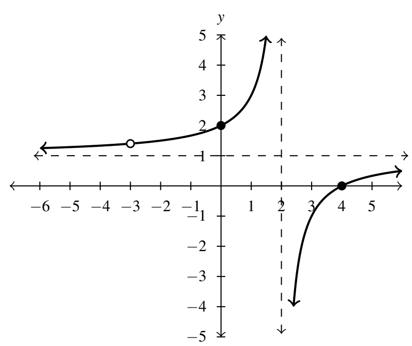
8. $f(x) = 4x(x^2-4)^{-1} = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
x-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $x = -2, x = 2$
As $x \to -2^-, f(x) \to -\infty$
As $x \to -2^+, f(x) \to \infty$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



9. $g(t) = \dfrac{t^2-t-12}{t^2+t-6} = \dfrac{t-4}{t-2}, t \neq -3$

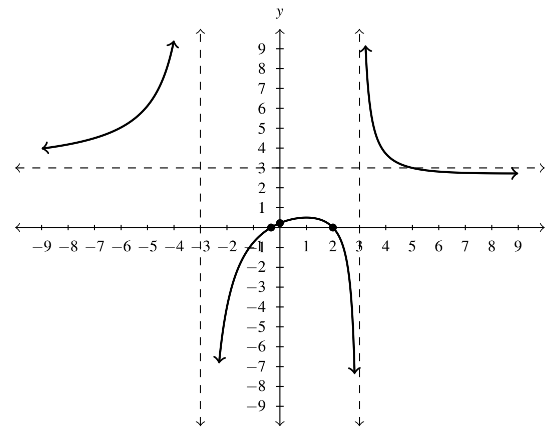
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
t-intercept: $(4,0)$
y-intercept: $(0,2)$
Vertical asymptote: $t = 2$
As $t \to 2^-, g(t) \to \infty$
As $t \to 2^+, g(t) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $t \to -\infty, g(t) \to 1^+$
As $t \to \infty, g(t) \to 1^-$



10. $g(t) = 3 - \dfrac{5t-25}{t^2-9} = \dfrac{3t^2-5t-2}{t^2-9}$

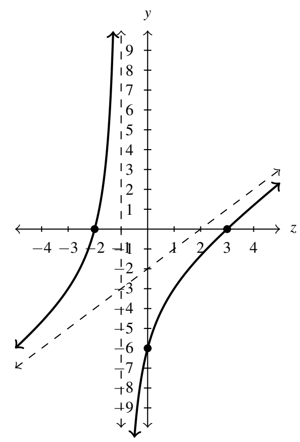
$= \dfrac{(3t+1)(t-2)}{(t+3)(t-3)}$

Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
t-intercepts: $\left(-\frac{1}{3},0\right)$, $(2,0)$
y-intercept: $\left(0,\frac{2}{9}\right)$
Vertical asymptotes: $t=-3, t=3$
As $t\to-3^-, g(t)\to\infty$
As $t\to-3^+, g(t)\to-\infty$
As $t\to3^-, g(t)\to-\infty$
As $t\to3^+, g(t)\to\infty$
Horizontal asymptote: $y=3$
As $t\to-\infty, g(t)\to3^+$
As $t\to\infty, g(t)\to3^-$



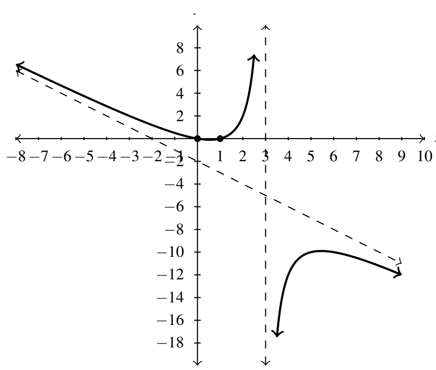
11. $r(z)=\dfrac{z^2-z-6}{z+1}=\dfrac{(z-3)(z+2)}{z+1}$

Domain: $(-\infty,-1)\cup(-1,\infty)$
z-intercepts: $(-2,0)$, $(3,0)$
y-intercept: $(0,-6)$
Vertical asymptote: $z=-1$
As $z\to-1^-, r(z)\to\infty$
As $z\to-1^+, r(z)\to-\infty$
Slant asymptote: $y=z-2$
As $z\to-\infty$, the graph is above $y=z-2$
As $z\to\infty$, the graph is below $y=z-2$



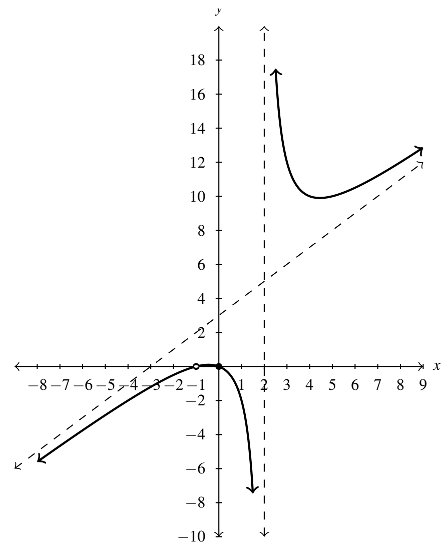
12. $r(z)=-z-2+\dfrac{6}{3-z}=\dfrac{z^2-z}{3-z}$

Domain: $(-\infty,3)\cup(3,\infty)$
z-intercepts: $(1,0)$
y-intercept: None
Vertical asymptote: $z=3$
As $z\to3^-, r(z)\to\infty$
As $z\to3^+, r(z)\to-\infty$
Slant asymptote: $y=-z-2$
As $z\to-\infty$, the graph is above $y=-z-2$
As $z\to\infty$, the graph is below $y=-z-2$



13. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}, x \neq -1$

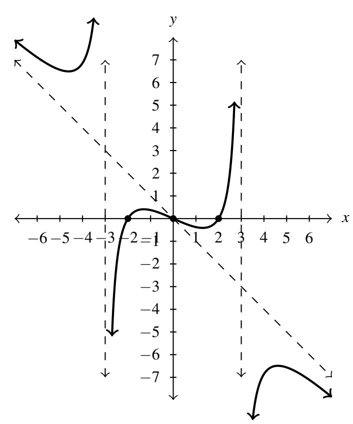
Domain: $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$
x-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptote: $x=2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1,0)$
Slant asymptote: $y = x+3$
As $x \to -\infty$, the graph is below $y = x+3$
As $x \to \infty$, the graph is above $y = x+3$



14. $f(x) = \dfrac{5x}{9-x^2} - x = \dfrac{x^3-4x}{9-x^2}$

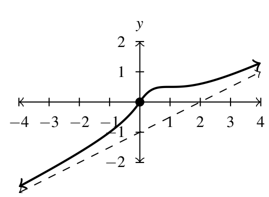
$$= \frac{x(x-2)(x+2)}{-(x-3)(x+3)}$$

Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
x-intercepts: $(-2,0),(0,0),(2,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $x=-3, x=3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
As $x \to 3^-$, $f(x) \to \infty$
As $x \to 3^+$, $f(x) \to -\infty$
Slant asymptote: $y = -x$
As $x \to -\infty$, the graph is above $y=-x$
As $x \to \infty$, the graph is below $y=-x$



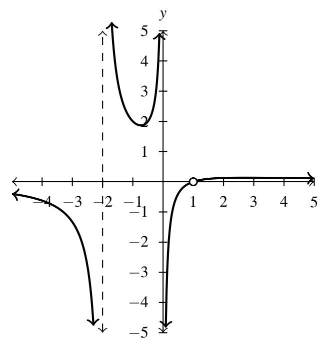
15. $g(t) = \dfrac{1}{2}t - 1 + \dfrac{t+1}{t^2+1} = \dfrac{t(t^2-2t+3)}{2t^2+2}$

Domain: $(-\infty,\infty)$
t-intercept: $(0,0)$
y-intercept: $(0,0)$
Slant asymptote: $y = \frac{1}{2}t - 1$
As $t \to -\infty$, the graph is below $y = \frac{1}{2}t-1$
As $t \to \infty$, the graph is above $y = \frac{1}{2}t-1$



16. $g(t) = \dfrac{t^2-2t+1}{t^3+t^2-2t} = \dfrac{t-1}{t(t+2)}$, $t \neq 1$

Domain: $(-\infty,-2)\cup(-2,0)\cup(0,1)\cup(1,\infty)$
No t-intercepts
No y-intercepts
Vertical asymptotes: $t=-2$ and $t=0$
As $t \to -2^-$, $g(t) \to -\infty$
As $t \to -2^+$, $g(t) \to \infty$
As $t \to 0^-$, $g(t) \to \infty$
As $t \to 0^+$, $g(t) \to -\infty$
Hole in the graph at $(1,0)$
Horizontal asymptote: $y=0$
As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$



17. $f(x) = \dfrac{1}{x-2}$

18. $F(x) = \dfrac{x-3}{(x-2)(x-3)} = \dfrac{x-3}{x^2-5x+6}$

19. $g(t) = \dfrac{t^2-1}{t}$

20. $G(t) = \dfrac{(t^2-1)(t+1)}{t(t+1)} = \dfrac{t^3+t^2-t-1}{t^2+t}$

Section 3.4 Answers

1. $x = -\dfrac{6}{7}$
2. $y = 1, 2$
3. $w = -1$
4. $x = -6, 2$
5. No solution.
6. $y = 0, \pm 2\sqrt{2}$
7. $w = -\sqrt{3}, -1$
8. $x = -\dfrac{3\sqrt{2}}{2}, \sqrt{2}$
9. $x = -\dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{4}$
10. $n = \dfrac{1}{2}$
11. $x = \dfrac{1\pm\sqrt{5}}{2}, \dfrac{-1\pm\sqrt{5}}{2}$
12. $t = -1$
13. $R = \pm\sqrt{\dfrac{0.08}{9.64\pi}}$
14. $x = -\dfrac{231}{400} \approx -0.58, x = \dfrac{77}{200}$
15. $c = \pm\sqrt{\dfrac{4\cdot 6.75\times 10^{16}}{3}} = \pm 3.00\times 10^{8}$
16. $y = \dfrac{1-3x}{x+2}, y \neq -3, x \neq -2$
17. $y = \dfrac{x-1}{x-3}, y \neq 1, x \neq 3$

18. $T_2 = \dfrac{V_2 T_1}{V_1}$, $T_1 \neq 0, T_2 \neq 0, V_1 \neq 0$

19. $t_0 = \dfrac{2}{2t_1 + 1}$, $t_1 \neq -\dfrac{1}{2}$

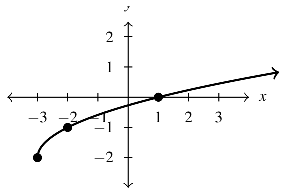
20. $x = \dfrac{1 \pm \sqrt{25v_r^2 + 1}}{5}$, $x \neq \pm v_r$.

21. $R = \dfrac{-(8P-25) \pm \sqrt{(8P-25)^2 - 64P^2}}{2P} = \dfrac{(25-8P) \pm 5\sqrt{25-16P}}{2P}$, $P \neq 0, R \neq -4$

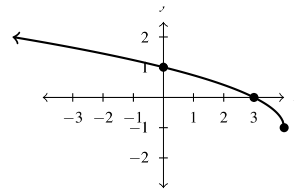
A.1.4 Chapter 4 Answers

Section 4.1 Answers

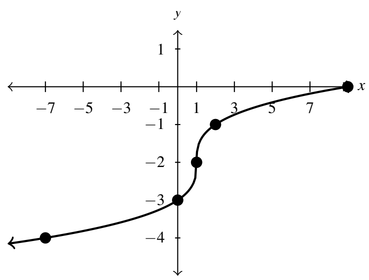
1. $F(x) = \sqrt{x+3} - 2$



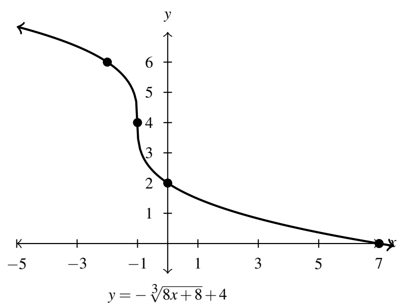
2. $F(x) = \sqrt{4-x} - 1 = \sqrt{-x+4} - 1$



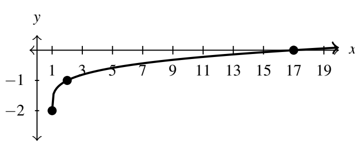
3. $F(x) = \sqrt[3]{x-1} - 2$



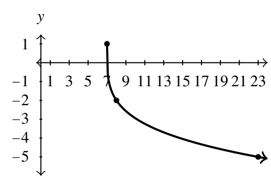
4. $F(x) = -\sqrt[3]{8x+8} + 4$



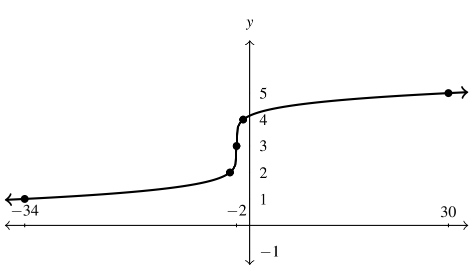
5. $F(x) = \sqrt[4]{x-1} - 2$



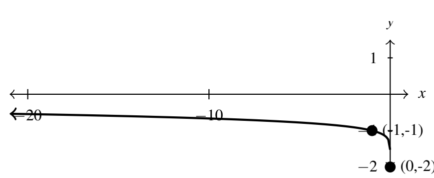
6. $F(x) = -3\sqrt[4]{x-7} + 1$



7. $F(x) = \sqrt[5]{x+2} + 3$



8. $F(x) = \sqrt[8]{-x} - 2$



9. One solution is: $F(x) = -\sqrt{x+4}+2$

10. One solution is: $F(x) = 2\sqrt{-x+1}$

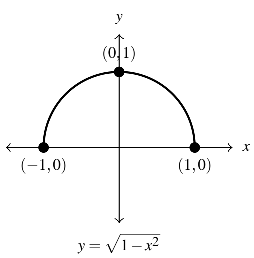
11. One solution is: $F(x) = -\sqrt[3]{2x+1}$

12. One solution is: $F(x) = 2\sqrt[3]{x-1}-2$

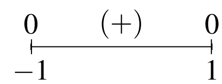
13. (a) $(-\infty, 4]$ (b) $(-2, \infty)$ (c) $\left[\frac{1}{2}, \infty\right)$

    (d) $[-2, 2]$ (e) $(-\infty, -2) \cup (2, \infty)$ (f) $(-\infty, -2] \cup [1, \infty)$

14. $f(x) = \sqrt{1-x^2}$
    Domain: $[-1, 1]$
    Intercepts: $(-1, 0)$, $(1, 0)$
    Graph:

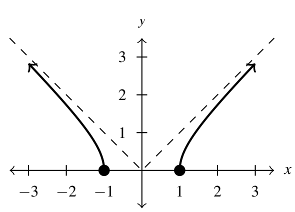


Range: $[0, 1]$
Local maximum: $(0, 1)$
Increasing: $[-1, 0]$, Decreasing: $[0, 1]$
Unusual steepness[7] at $x = -1$ and $x = 1$
Sign Diagram:

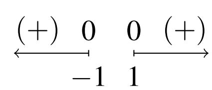


Note: f is even.

15. $f(x) = \sqrt{x^2-1}$
    Domain: $(-\infty, -1] \cup [1, \infty)$
    Intercepts: $(-1, 0)$, $(1, 0)$
    Graph:



As $x \to \pm\infty$, $f(x) \to \infty$
Range: $[0, \infty)$
Increasing: $[1, \infty)$, Decreasing: $(-\infty, -1]$
Unusual steepness[8] at $x = -1$ and $x = 1$
Using Calculus, one can show $y = \pm x$ are slant asymptotes to the graph.
Sign Diagram:

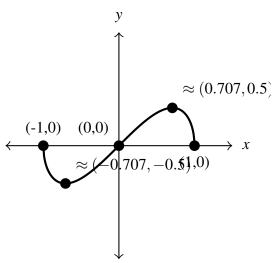


Note: f is even.

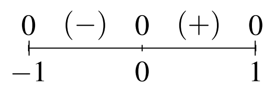
16. $g(t) = t\sqrt{1-t^2}$
    Domain: $[-1, 1]$
    Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$

Graph:

[7] You may need to zoom in to see this.
[8] You may need to zoom in to see this.

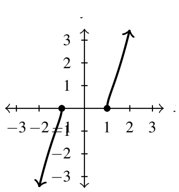


Range: $\approx [-0.5, 0.5]$
Local minimum $\approx (-0.707, -0.5)$
Local maximum: $\approx (0.707, 0.5)$
Increasing: $\approx [-0.707, 0.707]$
Decreasing: $\approx [-1, -0.707]$, $[0.707, 1]$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:

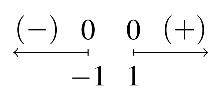


Note: g is odd.

17. $g(t) = t\sqrt{t^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph:

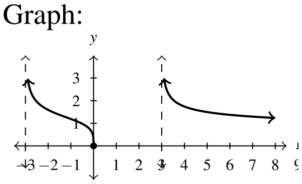


As $t \to -\infty$, $g(t) \to -\infty$
As $t \to \infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Increasing: $(-\infty, -1]$, $[1, \infty)$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:

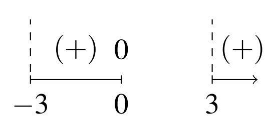


Note: g is odd.

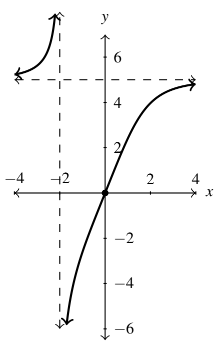
18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
Domain: $(-3, 0] \cup (3, \infty)$
Graph:



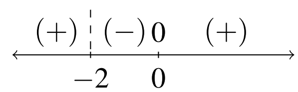
Intercept: $(0, 0)$
As $x \to \infty$, $f(x) \to 0$
Range: $[0, \infty)$
Decreasing: $(-3, 0]$, $(3, \infty)$
Unusual steepness at $x = 0$
Vertical asymptotes: $x = -3$ and $x = 3$
Horizontal asymptote: $y = 0$
Sign Diagram:



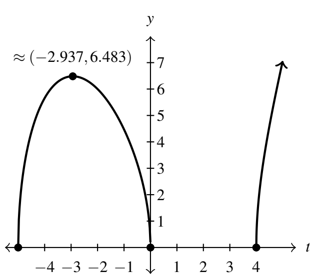
19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3+8}}$
Graph:



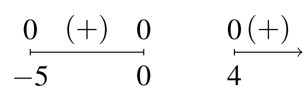
Domain: $(-\infty,-2)\cup(-2,\infty)$
Intercept: $(0,0)$
As $x \to \pm\infty$, $f(x) \to 5$
Range: $(-\infty,5)\cup(5,\infty)$
Increasing: $(-\infty,-2)$, $(-2,\infty)$
Vertical asymptote $x=-2$
Horizontal asymptote $y=5$
Sign Diagram:



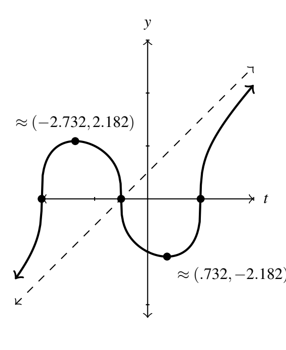
20. $g(t) = \sqrt{t(t+5)(t-4)}$
Domain: $[-5,0]\cup[4,\infty)$
Intercepts $(-5,0)$, $(0,0)$, $(4,0)$
As $t \to \infty$, $g(t) \to \infty$
Graph:



Range: $[0,\infty)$
Local maximum $\approx (-2.937, 6.483)$
Increasing: $\approx [-5,-2.937]$, $[4,\infty)$
Decreasing: $\approx [-2.937,0]$
Unusual steepness at $t=-5, t=0$ and $t=4$
Sign Diagram:



21. $g(t) = \sqrt[3]{t^3+3t^2-6t-8}$
Domain: $(-\infty,\infty)$
Intercepts: $(-4,0)$, $(-1,0)$, $(0,-2)$, $(2,0)$
Graph:



as $t \to -\infty$, $g(t) \to -\infty$
as $t \to \infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Local maximum: $\approx (-2.732, 2.182)$
Local minimum: $\approx (0.732, -2.182)$
Increasing: $\approx (-\infty, -2.732]$, $[0.732, \infty)$
Decreasing: $\approx [-2.732, 0.732]$
Unusual steepness at $t = -4, t = -1$ and $t = 2$

Using Calculus it can be shown that $y = t+1$ is a slant asymptote of this graph.
Sign Diagram:

$$\begin{array}{ccccccc} (-) & 0 & (+) & 0 & (-) & 0 & (+) \\ \hline & -4 & & -1 & & 2 & \end{array}$$

22. $C(x) = 15x + 20\sqrt{100 + (30-x)^2}$, $0 \leq x \leq 30$. The calculator gives the absolute minimum at approximately $(18.66, 582.29)$. This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

23. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.
    (b) $S(r) = \pi r\sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$
    (c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

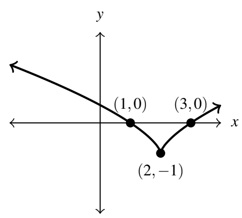
24. $9.8\left(\dfrac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

25. (a) $[0, c)$
    (b)
$$m(.1c) = \frac{m_r}{\sqrt{.99}} \approx 1.005 m_r \quad m(.5c) = \frac{m_r}{\sqrt{.75}} \approx 1.155 m_r$$
$$m(.9c) = \frac{m_r}{\sqrt{.19}} \approx 2.294 m_r \quad m(.999c) = \frac{m_r}{\sqrt{.0.001999}} \approx 22.366 m_r$$
    (c) As $v \to c^-$, $m(x) \to \infty$
    (d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100 m_r$.

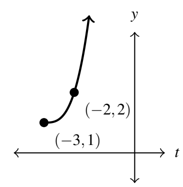
26. $k^{-1}(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

Section 4.2 Answers

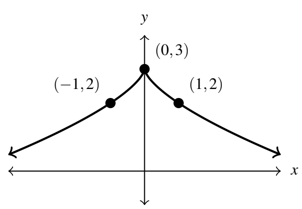
1. $F(x) = (x-2)^{\frac{2}{3}} - 1$



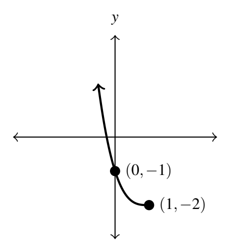
2. $G(t) = (t+3)^{\pi} + 1$



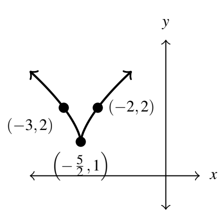
3. $F(x) = 3 - x^{\frac{2}{3}} = (-1)x^{\frac{2}{3}} + 3$



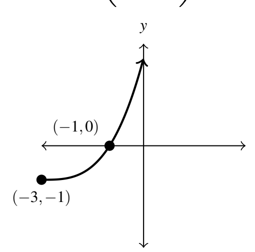
4. $G(t) = (1-t)^{\pi} - 2 = ((-1)t+1)^{\pi} - 2$



5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$



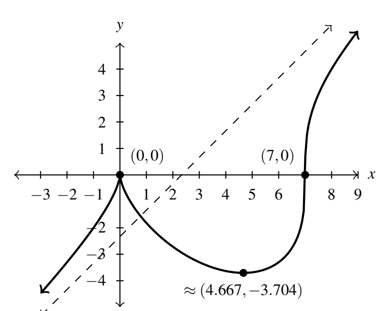
6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1 = \left(\frac{1}{2}t + \frac{3}{2}\right)^{\pi} - 1$



7. One solution is: $F(x) = 2(x-1)^{\frac{2}{3}} - 2$

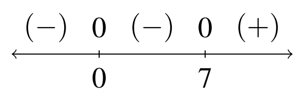
8. One solution is: $F(x) = -(x+1)^{\frac{2}{3}} + 4$

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
   Domain: $(-\infty, \infty)$
   Intercepts: $(0,0)$, $(7,0)$
   Graph:

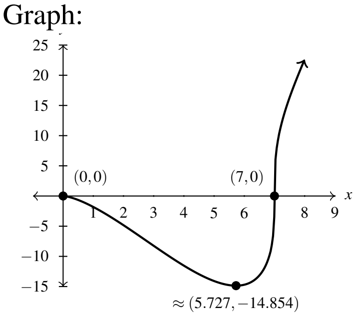


As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (4.667, -3.704)$
Local maximum: $(0,0)$ (this is a cusp)
Increasing: $(-\infty, 0]$, $\approx [4.667, \infty)$
Decreasing: $[0, 4.667]$
Unusual steepness at $x = 7$

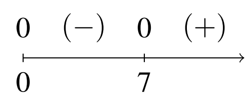
Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.
Sign Diagram:



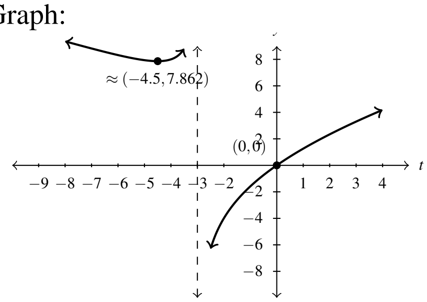
10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Graph:



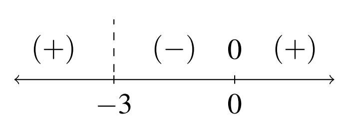
Domain: $[0, \infty)$
Intercepts: $(0,0)$, $(7,0)$
As $x \to \infty$, $f(x) \to \infty$
Range: $\approx [-14.854, \infty)$
Local minimum: $\approx (5.727, -14.854)$
Increasing: $\approx [5.727, \infty)$
Decreasing: $\approx [0, 5.727]$
Unusual steepness at $x = 7$
Sign Diagram:



11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$
Graph:

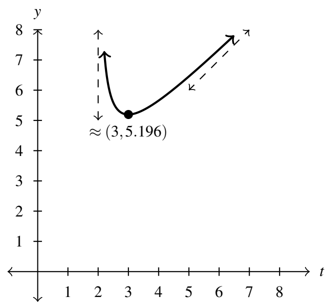


Domain: $(-\infty, -3) \cup (-3, \infty)$
Intercept: $(0,0)$
As $t \to \pm\infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (-4.5, 7.862)$
Increasing: $\approx [-4.5, -3)$, $(-3, \infty)$
Decreasing: $\approx (-\infty, -4.5]$
Vertical Asymptote: $t = -3$
Sign Diagram:

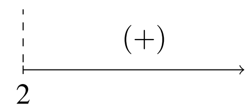


12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
Domain: $(2, \infty)$
As $t \to \infty$, $g(t) \to \infty$

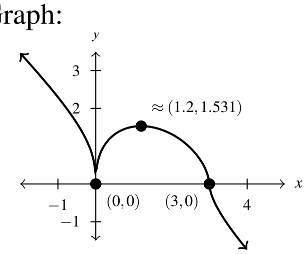
Graph:



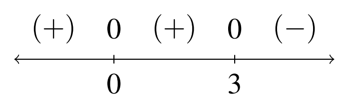
Range: $\approx [5.196, \infty)$
Local minimum: $\approx (3, 5.196)$
Increasing: $\approx [3, \infty)$
Decreasing: $\approx (2, 3]$
Vertical asymptote: $t = 2$
Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.
Sign Diagram:



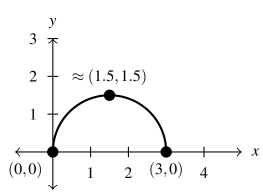
13. $f(x) = x^{0.4}(3-x)^{0.6}$
Domain: $(-\infty, \infty)$
Intercepts: $(0,0)$, $(3,0)$
Graph:



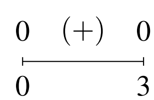
As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$
Range: $(-\infty, \infty)$
Local minimum: $(0,0)$ (this is a cusp)
Local maximum: $\approx (1.2, 1.531)$
Increasing: $\approx [0, 1.2]$
Decreasing: $\approx (-\infty, 0]$, $[1.2, \infty)$
Unusual Steepness: $x = 3$
Sign Diagram:



14. $f(x) = x^{0.5}(3-x)^{0.5}$
Graph:

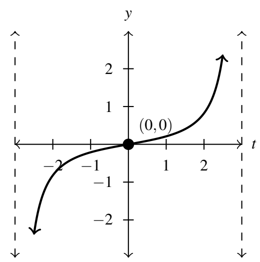


Domain: $[0, 3]$
Intercepts: $(0,0)$, $(3,0)$
Range: $\approx [0, 1.5]$
Increasing: $\approx [0, 1.5]$
Decreasing: $\approx [1.5, 3]$
Unusual Steepness:[9] $x = 0$, $x = 3$
Sign Diagram:

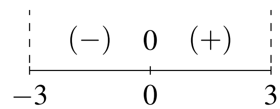


[9]Note you may need to zoom in to see this.

15. $g(t) = 4t(9-t^2)^{-\sqrt{2}}$
    Graph:

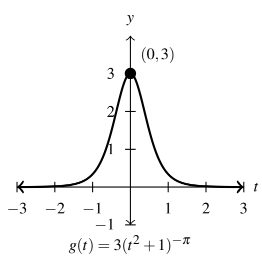


Domain: $(-3, 3)$
Intercepts: $(0, 0)$
Range: $(-\infty, \infty)$
Increasing: $(-3, 3)$
Sign Diagram:

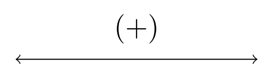


Note: g is odd

16. $g(t) = 3(t^2+1)^{-\pi}$
    Domain: $(-\infty, \infty)$
    Graph:



Intercept: $(0, 3)$
As $t \to \pm\infty$, $g(t) \to 0$
Range: $(0, 3]$
Increasing: $(-\infty, 0]$
Decreasing: $[0, \infty)$
Horizontal asymptote: $y = 0$
Sign Diagram:



Note: g is even

17. As in Exercise 51 in Section 2.2 and Exercise 43 in Section 3.2, the slopes of these curves near $x = 1$ approach the value of the exponent on x.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
$x^{\frac{1}{2}}$	0.5006	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$
$x^{\frac{2}{3}}$	0.6672	0.6667	$\approx \frac{2}{3}$	$\approx \frac{2}{3}$
$x^{-0.23}$	-0.2310	≈ -0.23	≈ -0.23	≈ -0.23
x^{π}	3.1544	3.1417	$\approx \pi$	$\approx \pi$

18. (a) $W \approx 37.55^\circ$F.
    (b) $V \approx 9.84$ miles per hour.

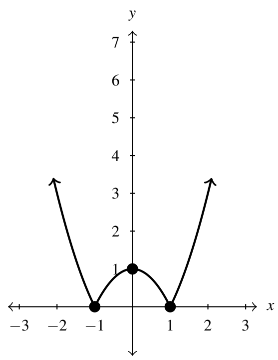
Section 4.3 Answers

1. $x = 3$
2. $x = \frac{1}{4}$
3. $t = -3$
4. $t = -\frac{1}{3},\ \frac{2}{3}$
5. $x = \frac{1}{4}$
6. $x = \frac{\sqrt{3}}{2}$
7. $t = \pm 8$
8. $t = 6$
9. $x = 4$
10. $x = -2, 6$
11. $t = -8, \frac{27}{8}$
12. $t = -1, \frac{1}{27}$
13. $x = 16$
14. $x = \frac{1}{81}, \frac{1}{16}$
15. $K = f(L) = (240)^{\frac{5}{3}} L^{-\frac{2}{3}}$. $f(100) \approx 430.2148$. This means in order for the production level of Sasquatchia to reach 300,000 Bigfoot Bullion with a labor investment of 100,000 hours, the country needs to invest approximately 430 Bigfoot Bullion into capital.

Section 4.4 Answers

15. $(-\infty, -3] \cup [1, \infty)$
16. $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
17. No solution
18. $(-\infty, \infty)$
19. $\{2\}$
20. No solution
21. $\left[-\frac{1}{3}, 4\right]$
22. $(0, 1)$
23. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
24. $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$
25. $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup [-\sqrt{7}, 0) \cup (0, \sqrt{7}] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$
26. $\left[-2-\sqrt{7}, -2+\sqrt{7}\right] \cup [1, 3]$
27. $(-\infty, \infty)$
28. $(-\infty, -1] \cup \{0\} \cup [1, \infty)$
29. $[-6, -3] \cup [-2, \infty)$
30. $(-\infty, 1) \cup \left(2, \frac{3+\sqrt{17}}{2}\right)$
31. 2 seconds.
32. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.
33. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.
34. (a) The applied domain is $[0, \infty)$.
    (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.
    (e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.
    (f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

35. $y = |1 - x^2|$



36. $\left(\frac{3-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right), \left(\frac{3+\sqrt{7}}{2}, \frac{-1-\sqrt{7}}{2}\right)$

37. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$ is minimized when $x = -\frac{2}{5}$. Hence to find the point on $y = 2x+1$ closest to $(0,0)$ we substitute $x = -\frac{2}{5}$ into $y = 2x+1$ to get $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

38. $x = -\frac{6}{7}$

39. $x = 1,\ x = 2$

40. $t = -1$

41. $t = -6,\ x = 2$

42. No solution

43. $z = 0,\ z = \pm 2\sqrt{2}$

44. $(-2, \infty)$

45. $(-2, 3]$

46. $(-\infty, -1) \cup (0, 1)$

47. $[0, \infty)$

48. $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$

49. $\left(-3, -\frac{1}{3}\right) \cup (2, 3)$

50. $(-1, 0] \cup (2, \infty)$

51. $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$

52. $(-\infty, 1] \cup [2, \infty)$

53. $(-\infty, -6) \cup (-1, 2)$

54. $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$

55. No solution

56. $[-4, -1) \cup (-1, 2]$

57. $(-\infty, -6) \cup (-6, -3] \cup [9, \infty)$

58. $[-3, 0) \cup (0, 4) \cup [5, \infty)$

59. $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$

60. $f(x) \geq 0$ on $(-\infty, 0) \cup [3, \infty)$.

61. $f(x) < 1$ on $(0, \infty)$.

62. $g(t) \geq -1$ on $(-\infty, 1] \cup (2, \infty)$.

63. $-1 \leq g(t) < 1$ on $(-\infty, 1] \cup (3, \infty)$.

64. $r(z) \leq 1$ on $(-\infty, -1] \cup (1, \infty)$.

65. $r(z) > 0$ on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

66. The absolute minimum of $y = \overline{C}(x)$ occurs at $\approx (75.73, 59.57)$. Given x represents the number of game systems, we check $\overline{C}(75) \approx 59.58$ and $\overline{C}(76) \approx 59.57$. Hence, to minimize the average cost, 76 systems should be produced at an average cost of $59.57 per system.

67. The width (and depth) should be 10.00 centimeters, the height should be 5.00 centimeters. The minimum surface area is 300.00 square centimeters.

68. The width of the base of the box should be approximately 4.12 inches, the height of the box should be approximately 6.67 inches, and the depth of the base of the box should be approximately 5.09 inches. The minimum surface area is approximately 164.91 square inches.

69. The dimensions are approximately 7 feet by 14 feet. Hence, the minimum amount of fencing required is approximately 28 feet.

70. (a) $V = \pi r^2 h$

(b) $S = 2\pi r^2 + 2\pi rh$

(c) $S(r) = 2\pi r^2 + \frac{67.2}{r}$, Domain $r > 0$

(d) $r \approx 1.749$ in. and $h \approx 3.498$ in.

71. The radius of the drum should be approximately 1.05 feet and the height of the drum should be approximately 2.12 feet. The minimum surface area of the drum is approximately 20.93 cubic feet.

72. $P(t) < 100$ on $(-15, 30)$, and the portion of this which lies in the applied domain is $[0, 30)$. Given $t = 0$ corresponds to the year 1803, from 1803 through the end of 1832, there were fewer than 100 Sasquatch in Portage County.

73. $[-8, 8]$

74. $[-1, 0] \cup [1, \infty)$

75. $\left(-\infty, \frac{1}{3}\right)$

76. $\left[-\frac{\sqrt{5}}{2}, -1\right) \cup \left(1, \frac{\sqrt{5}}{2}\right]$

77. $\left(-\infty, -\frac{3\sqrt{2}}{4}\right] \cup (-1, 1) \cup \left[\frac{3\sqrt{2}}{4}, \infty\right)$

78. $\left[\frac{3}{4}, 1\right) \cup (1, \infty)$

79. $\left(-\infty, \frac{3}{5}\right] \cup (1, \infty)$

80. $(-\infty, 2) \cup (2, 3]$

81. $(2, 6]$

82. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$

83. $(-\infty, -1)$

84. $[4, 7)$

85. $\left(0, \frac{27}{13}\right)$

86. $(-\infty, 0) \cup (0, 3)$

87. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$

A.1.5 Chapter 5 Answers

Section 5.1 Answers

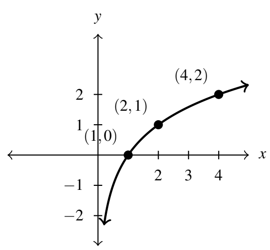
9. $f^{-1}(x) = \dfrac{x+2}{6}$
10. $f^{-1}(x) = 42 - x$
11. $g^{-1}(t) = 3t - 10$
12. $g^{-1}(t) = -\frac{5}{3}t + \frac{1}{3}$
13. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}, x \geq 5$
14. $f^{-1}(x) = (x-2)^2 + 5,\ x \leq 2$
15. $g^{-1}(t) = \frac{1}{9}(t+4)^2 + 1, t \geq -4$
16. $g^{-1}(t) = \frac{1}{8}(t-1)^2 - \frac{5}{2}, t \leq 1$
17. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
18. $f^{-1}(x) = -(x-3)^3 + 2$
19. $g^{-1}(t) = 5 + \sqrt{t+25}$
20. $g^{-1}(t) = -\sqrt{\frac{t+5}{3}} - 4$
21. $f^{-1}(x) = 3 - \sqrt{x+4}$
22. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}, x > 1$
23. $g^{-1}(t) = \dfrac{4t-3}{t}$
24. $g^{-1}(t) = \dfrac{t}{3t+1}$
25. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
26. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
27. $g^{-1}(t) = \dfrac{-3t-2}{t+3}$
28. $g^{-1}(t) = \dfrac{t-2}{2t-1}$
29. (a) None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one. $F^{-1} = \{(0,0),(1,1),(-1,2),(2,3),(-2,4),(3,5),(-3,6)\}$

    (b) Because of the '…' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0,2,4,\ldots$, the ordered pair $\left(n, -\frac{n}{2}\right)$ is in G. For the odd numbers $n = 1,3,5,\ldots$, the ordered pair $\left(n, \frac{n+1}{2}\right)$ is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either $-\frac{n}{2}$ or $\frac{n+1}{2}$, both of which are functions of n. To show G is one to one, we note that if the output from G is 0 or less, then it must be of the form $-\frac{n}{2}$ for an even number n. Moreover, if $-\frac{n}{2} = -\frac{m}{2}$, then $n = m$. In the case we are looking at outputs from G which are greater than 0, then it must be of the form $\frac{n+1}{2}$ for an odd number n. In this, too, if $\frac{n+1}{2} = \frac{m+1}{2}$, then $n = m$. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and $G^{-1} = \{(0,0),(1,1),(-1,2),(2,3),(-2,4),(3,5),(-3,6),\ldots\}$

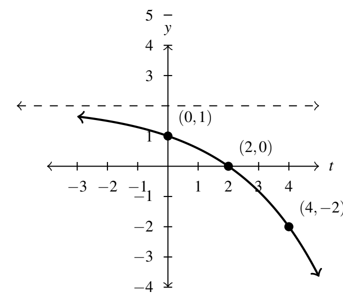
    (c) To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$, then $t = u$. Hence the corresponding outputs, $2t-1$ and $3u-1$, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P, $3t-1 = 3u-1$, then $t = u$. Hence, the corresponding inputs $2t^5$ and $2u^5$ are equal, too. Hence P is one-to-one and $P^{-1} = \{(3t-1, 2t^5) \,|\, t \text{ is a real number.}\}$

(d) To show Q is a function, we note that if we have the same inputs to Q, say $n = m$, then the outputs from Q, namely n^2 and m^2 are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely $n^2 = m^2$, then $n = \pm m$. However because n and m are *natural* numbers, both n and m are positive so $n = m$. Hence Q is one-to-one and $Q^{-1} = \{(n^2, n) \,|\, n \text{ is a } \textit{natural} \text{ number.}\}$.

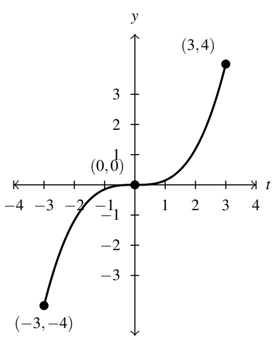
30. $y = f^{-1}(x)$. Asymptote: $x = 0$.



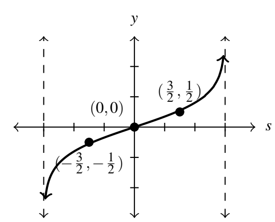
31. $y = g^{-1}(t)$. Asymptote: $y = 2$.



32. $y = S^{-1}(t)$. Domain $[-3, 3]$.



33. $y = R^{-1}(s)$. Asymptotes: $s = \pm 3$.



34. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$

(b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.

(c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$.

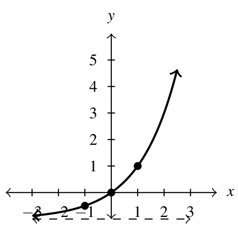
The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. We cannot sell part of a system, so we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

36. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

38. (b) If $b = 0$, then $m = \pm 1$. If $b \neq 0$, then $m = -1$ and b can be any real number.

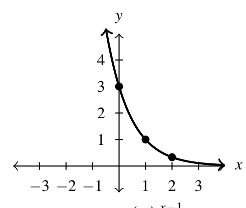
Section 5.2 Answers

1. Domain of g: $(-\infty, \infty)$
Range of g: $(-1, \infty)$
Points: $\left(-1, -\frac{1}{2}\right)$, $(0,0)$, $(1,1)$
Asymptote: $y = -1$



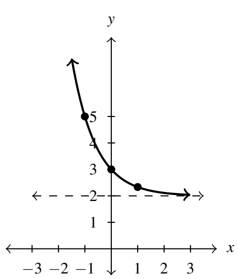
$y = g(x) = 2^x - 1$

2. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$
Points: $(0,3)$, $(1,1)$, $\left(2, \frac{1}{3}\right)$
Asymptote: $y = 0$



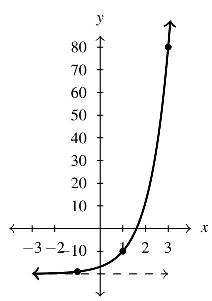
$y = g(x) = \left(\frac{1}{3}\right)^{x-1}$

3. Domain of g: $(-\infty, \infty)$
Range of g: $(2, \infty)$
Points: $\left(1, \frac{7}{3}\right)$, $(0,3)$, $(-1,5)$
Asymptote: $y = 2$



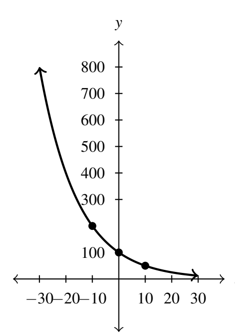
$y = g(x) = 3^{-x} + 2$

4. Domain of g: $(-\infty, \infty)$
Range of g: $(-20, \infty)$
Points: $(-1, -19)$, $(1, -10)$, $(3, 80)$
Asymptote: $y = -20$



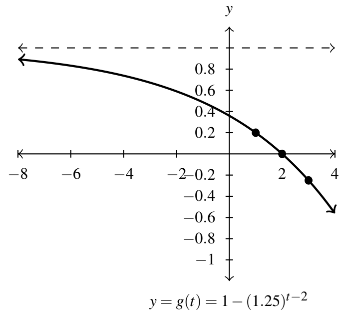
$y = g(x) = 10^{\frac{x+1}{2}} - 20$

5. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$
Points: $(-10, 200)$, $(0, 100)$, $(10, 50)$
Asymptote: $y = 0$

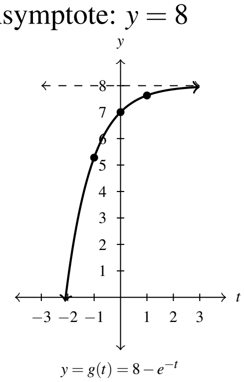


$y = g(t) = 100(0.5)^{0.1t}$

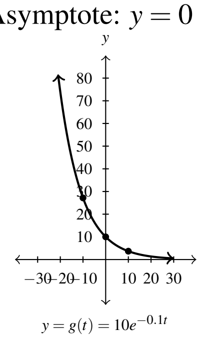
6. Domain of g: $(-\infty, \infty)$
   Range of g: $(-\infty, 1)$
   Points: $(1, 0.2)$, $(2, 0)$, $(3, -0.25)$
   Asymptote: $y = 1$



7. Domain of g: $(-\infty, \infty)$
   Range of g: $(-\infty, 8)$
   Points: $\left(1, 8 - e^{-1}\right) \approx (1, 7.63)$,
   $(0, 7)$, $(-1, 8 - e) \approx (1, 5.28)$
   Asymptote: $y = 8$



8. Domain of g: $(-\infty, \infty)$
   Range of g: $(0, \infty)$
   Points: $\left(10, 10e^{-1}\right) \approx (10, 3.68)$
   $(0, 10)$, $(-10, 10e) \approx (-10, 27.18)$
   Asymptote: $y = 0$



9. $F(x) = 2^{x+1} - 3$
10. $F(x) = -2^{-x} + 3$
11. $F(x) = 2^{2x-6}$
12. $F(x) = 3 \cdot 2^{-2x}$

13. Knowing $2 = 4^{\frac{1}{2}}$, one way to obtain the formulas for $G(x)$ is to use properties of exponents. For example, $F(x) = 2^{x+1} - 3 = \left(4^{\frac{1}{2}}\right)^{x+1} - 3 = 4^{\frac{1}{2}(x+1)} - 3 = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$. In order, the formulas for $G(x)$ are:

   - $G(x) = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$
   - $G(x) = -4^{-\frac{1}{2}x} + 3$
   - $G(x) = 4^{x-3}$
   - $G(x) = 3 \cdot 4^{-x}$

15. One solution is $g(x) = e^{-x}$ and $h(x) = 1$.

16. One solution is $g(x) = e^{2x}$ and $h(x) = x$.

17. One solution is $g(t) = t^2$ and $h(t) = e^{-t}$.

18. One solution is $f(x) = e^x - e^{-x}$ and $g(x) = e^x + e^{-x}$.

19. One solution is $f(x) = -x^2$ and $g(x) = e^x$.
20. One solution is $f(x) = e^{2x} - 1$ and $g(x) = \sqrt{x}$.

Section 5.3 Answers

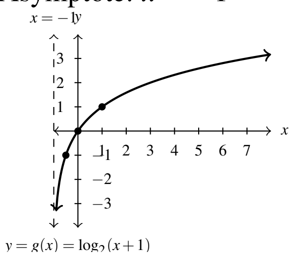
1. $\log_2(8) = 3$
2. $\log_5\left(\frac{1}{125}\right) = -3$
3. $\log_4(32) = \frac{5}{2}$
4. $\log_{\frac{1}{3}}(9) = -2$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
6. $\log(0.001) = -3$
7. $\ln(1) = 0$
8. $5^2 = 25$
9. $(25)^{\frac{1}{2}} = 5$
10. $3^{-4} = \frac{1}{81}$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
12. $10^2 = 100$
13. $10^{-1} = 0.1$
14. $e^1 = e$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
16. $\log_3(27) = 3$
17. $\log_6(216) = 3$
18. $\log_2(32) = 5$
19. $\log_6\left(\frac{1}{36}\right) = -2$
20. $\log_8(4) = \frac{2}{3}$
21. $\log_{36}(216) = \frac{3}{2}$
22. $\log_{\frac{1}{5}}(625) = -4$
23. $\log_{\frac{1}{6}}(216) = -3$
24. $\log_{36}(36) = 1$
25. $\log \frac{1}{1000000} = -6$
26. $\log(0.01) = -2$
27. $\ln\left(e^3\right) = 3$
28. $\log_4(8) = \frac{3}{2}$
29. $\log_6(1) = 0$
30. $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
32. $7^{\log_7(3)} = 3$
33. $36^{\log_{36}(216)} = 216$
34. $\log_{36}\left(36^{216}\right) = 216$
35. $\ln(e^5) = 5$
36. $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
38. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
39. $\log_5\left(3^{\log_3 5}\right) = 1$
40. $\log\left(e^{\ln(100)}\right) = 2$
41. $\log_2\left(3^{-\log_3(2)}\right) = -1$
42. $\ln\left(42^{6\log(1)}\right) = 0$
43. $(-\infty, \infty)$
44. $(-2, \infty)$
45. $(5, \infty)$
46. $(-\infty, -6) \cup (-3, \infty)$
47. $(-2, -1) \cup (1, \infty)$
48. $(-6, -3) \cup (5, \infty)$
49. $(4, 7)$
50. $(5, \infty)$
51. $(-\infty, \infty)$
52. $[1, \infty)$
53. $(-\infty, -7) \cup (1, \infty)$
54. $(13, \infty)$

55. $(0,125)\cup(125,\infty)$

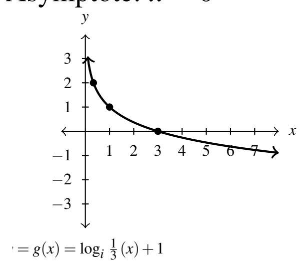
56. No domain

57. $(-\infty,-3)\cup\left(\frac{1}{2},2\right)$

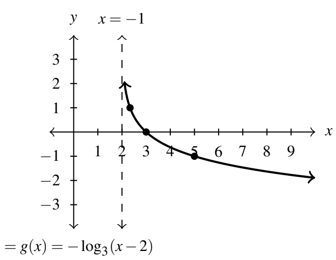
58. Domain of g: $(-1,\infty)$
Range of g: $(-\infty,\infty)$
Points: $\left(-\frac{1}{2},-1\right)$, $(0,0)$, $(1,1)$
Asymptote: $x=-1$



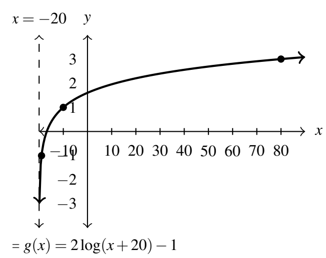
59. Domain of g: $(0,\infty)$
Range of g: $(-\infty,\infty)$
Points: $\left(\frac{1}{3},2\right)$, $(1,1)$, $(3,0)$
Asymptote: $x=0$



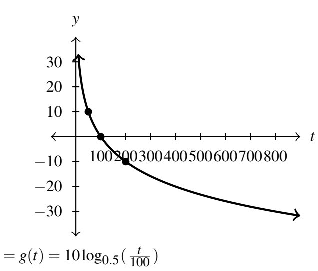
60. Domain of g: $(2,\infty)$
Range of g: $(-\infty,\infty)$
Points: $\left(\frac{7}{3},1\right)$, $(3,0)$, $(5,-1)$
Asymptote: $x=2$



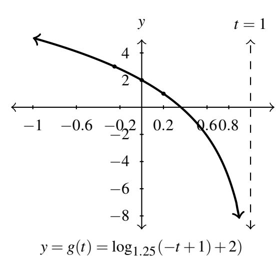
61. Domain of g: $(-20,\infty)$
Range of g: $(-\infty,\infty)$
Points: $(-19,-1)$, $(-10,1)$, $(80,3)$
Asymptote: $x=-20$



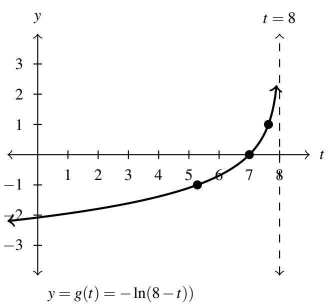
62. Domain of g: $(0,\infty)$
Range of g: $(-\infty,\infty)$
Points: $(50,10)$, $(100,0)$, $(200,-10)$
Asymptote: $t=0$



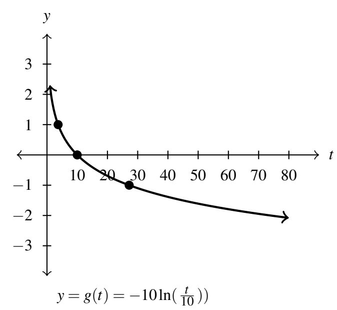
63. Domain of g: $(-\infty,1)$
Range of g: $(-\infty,\infty)$
Points: $(-0.25,3)$, $(0,2)$, $(0.2,1)$
Asymptote: $t=1$



64. Domain of g: $(-\infty, 8)$
Range of g: $(-\infty, \infty)$
Points: $(8-e, -1) \approx (5.28, -1)$, $(7, 0)$, $(8-e^{-1}, 1) \approx (7.63, 1)$
Asymptote: $t = 8$



65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(10e^{-1}, 10) \approx (3.68.10)$ $(10, 0)$, $(10e, -10) \approx (27.18, -10)$
Asymptote: $t = 0$



66. $F(x) = \log_2(x+3) - 1$

67. $F(x) = -\log_2(-x+3)$

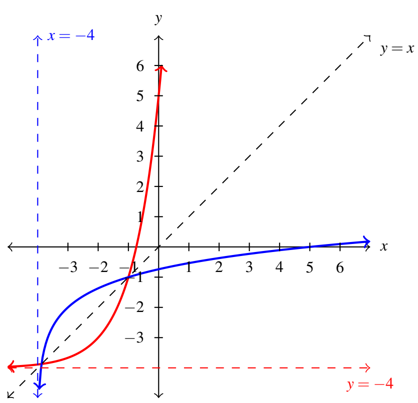
68. $F(x) = \frac{1}{2}\log_2(x) + 3$

69. $F(x) = -\frac{1}{2}\log_2\left(\frac{x}{3}\right)$

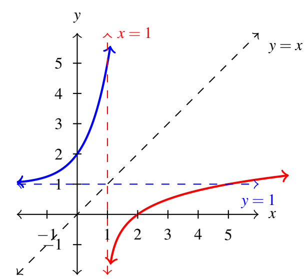
70. In order, the formulas for $G(x)$ are:

- $G(x) = 2\log_4(x+3) - 1$
- $G(x) = -2\log_4(-x+3)$
- $G(x) = \log_4(x) + 3$
- $G(x) = -\log_4\left(\frac{x}{3}\right)$

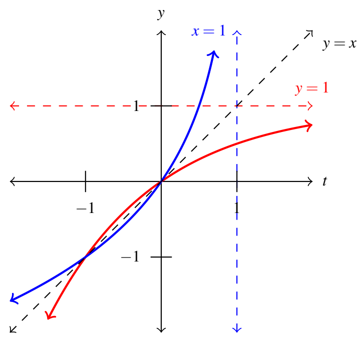
71. $y = f(x) = 3^{x+2} - 4$
$y = f^{-1}(x) = \log_3(x+4) - 2$



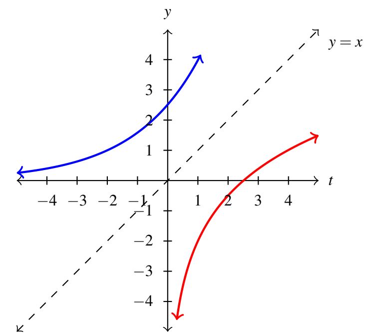
72. $y = f(x) = \log_4(x-1)$
$y = f^{-1}(x) = 4^x + 1$



73. $y = g(t) = -2^{-t} + 1$
$y = g^{-1}(t) = -\log_2(-t+1)$



74. $y = g(t) = 5\log(t) - 2$
$y = g^{-1}(t) = 10^{\frac{t+2}{5}}$



75. One solution is $g(x) = \log_2(x+3)$ and $h(x) = 4$.

76. One solution is $g(x) = \log(2x)$ and $h(x) = e^{-x}$.

77. One solution is $g(t) = 3t$ and $h(t) = \log(t)$.

78. One solution is $f(x) = \ln(x)$ and $g(x) = x$.

79. One solution is $f(t) = t^2 + 1$ and $g(t) = \ln(t)$.

80. One solution is $f(z) = \ln(z)$ and $g(z) = z^2$.

81. (a) $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0$.
    (b) $M(80,000) = \log\left(\frac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9$.

82. (a) $L(10^{-6}) = 60$ decibels.
    (b) $I = 10^{-.5} \approx 0.316$ watts per square meter.
    (c) Because $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

83. (a) The pH of pure water is 7.
    (b) If $[\mathrm{H}^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.
    (c) $[\mathrm{H}^+] = 10^{-0.7} \approx .1995$ moles per liter.

Section 5.4 Answers

1. $3\ln(x)+2\ln(y)$
2. $7-\log_2(x^2+4)$
3. $3\log_5(z)-6$
4. $\log(1.23)+37$
5. $\frac{1}{2}\ln(z)-\ln(x)-\ln(y)$
6. $\log_5(x-5)+\log_5(x+5)$
7. $3\log_{\sqrt{2}}(x)+4$
8. $-2+\log_{\frac{1}{3}}(x)+\log_{\frac{1}{3}}(y-2)+\log_{\frac{1}{3}}(y^2+2y+4)$
9. $3+3\log(x)+5\log(y)$
10. $2\log_3(x)-4-4\log_3(y)$
11. $\frac{1}{4}\ln(x)+\frac{1}{4}\ln(y)-\frac{1}{4}-\frac{1}{4}\ln(z)$
12. $12-12\log_6(x)-4\log_6(y)$
13. $\frac{5}{3}+\log(x)+\frac{1}{2}\log(y)$
14. $-2+\frac{2}{3}\log_{\frac{1}{2}}(x)-\log_{\frac{1}{2}}(y)-\frac{1}{2}\log_{\frac{1}{2}}(z)$
15. $\frac{1}{3}\ln(x)-\ln(10)-\frac{1}{2}\ln(y)-\frac{1}{2}\ln(z)$
16. $\ln(x^4y^2)$
17. $\log_2\left(\frac{xy}{z}\right)$
18. $\log_3\left(\frac{x}{y^2}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2z}\right)$
20. $\ln\left(\frac{x^2}{y^3z^4}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
22. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
24. $\log\left(\frac{1000}{x}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
26. $\ln\left(x\sqrt{e}\right)$
27. $\log_2\left(x^{3/2}\right)$
28. $\log_2\left(x\sqrt{x-1}\right)$
29. $\log_2\left(\frac{x}{x-1}\right)$
30. $7^{x-1}=e^{(x-1)\ln(7)}$
31. $\log_3(x+2)=\frac{\log(x+2)}{\log(3)}$
32. $\left(\frac{2}{3}\right)^x=e^{x\ln\left(\frac{2}{3}\right)}$
33. $\log(x^2+1)=\frac{\ln(x^2+1)}{\ln(10)}$
34. $\log_3(12)\approx 2.26186$
35. $\log_5(80)\approx 2.72271$
36. $\log_6(72)\approx 2.38685$
37. $\log_4\left(\frac{1}{10}\right)\approx -1.66096$
38. $\log_{\frac{3}{5}}(1000)\approx -13.52273$
39. $\log_{\frac{2}{3}}(50)\approx -9.64824$

Section 5.5 Answers

1. $x=\frac{3}{4}$
2. $x=4$
3. $x=2$
4. $t=-\frac{1}{4}$
5. $t=-\frac{7}{3}$
6. $t=-1,0,1$
7. $x=\frac{16}{15}$
8. $x=-\frac{2}{11}$
9. $x=\dfrac{\ln(5)}{2\ln(3)}$

10. $t = -\dfrac{\ln(2)}{\ln(5)}$

11. No solution.

12. $t = \dfrac{\ln(29)+\ln(3)}{\ln(3)}$

13. $x = \dfrac{\ln(3)}{12\ln(1.005)}$

14. $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730} = \dfrac{\ln(2)}{5730}$

15. $t = \dfrac{\ln(2)}{0.1} = 10\ln(2)$

16. $t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$

17. $t = \dfrac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$

18. $t = -10\ln\left(\frac{5}{3}\right) = 10\ln\left(\frac{3}{5}\right)$

19. $x = \ln(2)$

20. $t = \frac{1}{3}\ln(2)$

21. $t = \dfrac{\ln\left(\frac{1}{29}\right)}{-0.8} = \dfrac{5}{4}\ln(29)$

22. $x = \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)} = \dfrac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}$

23. $x = \ln(2)$

24. $t = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$

25. $x = \dfrac{\ln(3)}{\ln(3)-\ln(2)}$

26. $x = \dfrac{\ln(3)+5\ln\left(\frac{1}{2}\right)}{\ln(3)-\ln\left(\frac{1}{2}\right)} = \dfrac{\ln(3)-5\ln(2)}{\ln(3)+\ln(2)}$

27. $x = \dfrac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$

28. $t = \ln(5)$

29. $t = \ln(3)$

30. $t = \dfrac{\ln(3)}{\ln(2)}$

31. $x = \ln(3)$

32. $x = \ln(3), \ln(5)$

33. $x = \dfrac{\ln(5)}{\ln(3)}$

34. $(-\infty,\infty)$

35. $(-\infty,0)\cup(0,\infty)$

36. $\left(\frac{1}{2}\ln(3),\infty\right)$

37. $(-\infty,\infty)$

38. $(-\infty,\ln(3))$

39. $(\ln(2),\infty)$

40. $f^{-1} = \ln\left(x+\sqrt{x^2+1}\right)$. Both f and f^{-1} have domain $(-\infty,\infty)$ and range $(-\infty,\infty)$.

Section 5.6 Answers

1. $x = \frac{5}{4}$

2. $x = 1$

3. $t = -2$

4. $t = -3, 4$

5. $x = -1$

6. $x = \frac{9}{2}$

7. $t = \pm 10$

8. $t = -2, 5$

9. $x = -\frac{17}{7}$

10. $x = 10^{1.7}$

11. $x = 10^{-5.4}$

12. $x = 10^{3}$

13. $t = \frac{25}{2}$

14. $t = e^{3/4}$

15. $t = 5$

16. $t = \frac{1}{2}$

17. $x = 2$

18. $x = \frac{1}{e^3 - 1}$

19. $t = 6$

20. $t = 4$

21. $x = 81$

22. $x = e^{e^3}$

23. $t = 10^{-3}, 10^5$

24. $t = 1, x = e^2$

25. $(-\infty, e) \cup (e, \infty)$

26. $(0, e) \cup (e, \infty)$

27. $(0, 100]$

28. $(0, \infty)$

29. $(1, \infty)$

30. $(1, e)$

31. $y = \dfrac{3}{5e^{2x} + 1}$

33. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

To see why we rewrite this in this form, see Exercise **??** in Section **??**.

The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

Section 5.7 Answers

1.
 - $A(t) = 500\left(1 + \frac{0.0075}{12}\right)^{12t}$
 - $A(5) \approx \$519.10$, $A(10) \approx \$538.93$, $A(30) \approx \$626.12$, $A(35) \approx \$650.03$
 - It will take approximately 92 years for the investment to double.
 - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of \$4.85 per year at this point.

2.
 - $A(t) = 500e^{0.0075t}$
 - $A(5) \approx \$519.11$, $A(10) \approx \$538.94$, $A(30) \approx \$626.16$, $A(35) \approx \$650.09$
 - It will take approximately 92 years for the investment to double.
 - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.86. This means that the investment is growing at an average rate of \$4.86 per year at this point.

3.
 - $A(t) = 1000\left(1 + \frac{0.0125}{12}\right)^{12t}$
 - $A(5) \approx \$1064.46$, $A(10) \approx \$1133.07$, $A(30) \approx \$1454.71$, $A(35) \approx \$1548.48$

- It will take approximately 55 years for the investment to double.
- The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of \$19.23 per year at this point.

4. - $A(t) = 1000e^{0.0125t}$
   - $A(5) \approx \$1064.49$, $A(10) \approx \$1133.15$, $A(30) \approx \$1454.99$, $A(35) \approx \$1548.83$
   - It will take approximately 55 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.24. This means that the investment is growing at an average rate of \$19.24 per year at this point.

5. - $A(t) = 5000\left(1+\frac{0.02125}{12}\right)^{12t}$
   - $A(5) \approx \$5559.98$, $A(10) \approx \$6182.67$, $A(30) \approx \$9453.40$, $A(35) \approx \$10512.13$
   - It will take approximately 33 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of \$116.80 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of \$220.83 per year at this point.

6. - $A(t) = 5000e^{0.02125t}$
   - $A(5) \approx \$5560.50$, $A(10) \approx \$6183.83$, $A(30) \approx \$9458.73$, $A(35) \approx \$10519.05$
   - It will take approximately 33 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.91. This means that the investment is growing at an average rate of \$116.91 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 221.17. This means that the investment is growing at an average rate of \$221.17 per year at this point.

8. $P = \frac{2000}{e^{0.0025\cdot 3}} \approx \$1985.06$

9. $P = \frac{5000}{\left(1+\frac{0.0225}{12}\right)^{12\cdot 10}} \approx \$3993.42$

10. (a) $A(8) = 2000\left(1+\frac{0.0025}{12}\right)^{12\cdot 8} \approx \$2040.40$

    (b) $t = \dfrac{\ln(2)}{12\ln\left(1+\frac{0.0025}{12}\right)} \approx 277.29$ years

(c) $P = \dfrac{2000}{\left(1+\frac{0.0025}{12}\right)^{36}} \approx \$1985.06$

11. (a) $A(8) = 2000\left(1+\frac{0.0225}{12}\right)^{12\cdot 8} \approx \$2394.03$

(b) $t = \dfrac{\ln(2)}{12\ln\left(1+\frac{0.0225}{12}\right)} \approx 30.83$ years

(c) $P = \dfrac{2000}{\left(1+\frac{0.0225}{12}\right)^{36}} \approx \$1869.57$

(d) $\left(1+\frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

12. $A(3) = 5000e^{0.299\cdot 3} \approx \$12,226.18$, $A(6) = 5000e^{0.299\cdot 6} \approx \$30,067.29$

14.
- $k = \frac{\ln(1/2)}{5.27} \approx -0.1315$
- $A(t) = 50e^{-0.1315t}$
- $t = \frac{\ln(0.1)}{-0.1315} \approx 17.51$ years.

15.
- $k = \frac{\ln(1/2)}{14} \approx -0.0495$
- $A(t) = 2e^{-0.0495t}$
- $t = \frac{\ln(0.1)}{-0.0495} \approx 46.52$ days.

16.
- $k = \frac{\ln(1/2)}{27.7} \approx -0.0250$
- $A(t) = 75e^{-0.0250t}$
- $t = \frac{\ln(0.1)}{-0.025} \approx 92.10$ days.

17.
- $k = \frac{\ln(1/2)}{432.7} \approx -0.0016$
- $A(t) = 0.29e^{-0.0016t}$
- $t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11$ years.

18.
- $k = \frac{\ln(1/2)}{704} \approx -0.0010$
- $A(t) = e^{-0.0010t}$
- $t = \frac{\ln(0.1)}{-0.0010} \approx 2302.58$ million years, or 2.30 billion years.

19. $t = \frac{\ln(0.1)}{k} = -\frac{\ln(10)}{k}$

20. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$

21. (a) $G(0) = 9743.77$ This means that the GDP of the US in 2000 was \$9743.77 billion dollars.

(b) $G(7) = 13963.24$ and $G(10) = 16291.25$, so the model predicted a GDP of \$13,963.24 billion in 2007 and \$16,291.25 billion in 2010.

22. (a) $D(0) = 15$, so the tumor was 15 millimeters in diameter when it was first detected.

(b) $t = \frac{\ln(2)}{0.0277} \approx 25$ days.

23. (a) $k = \frac{\ln(2)}{20} \approx 0.0346$

(b) $N(t) = 1000e^{0.0346t}$

(c) $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes

24. (a) $k = \frac{1}{2}\frac{\ln(6)}{2.5} \approx 0.4377$

(b) $N(t) = 2.5e^{0.4377t}$

(c) $t = \frac{\ln(2)}{0.4377} \approx 1.58$ hours

25. $N_0 = 52$, $k = \frac{1}{3}\ln\left(\frac{118}{52}\right) \approx 0.2731$, $N(t) = 52e^{0.2731t}$. $N(6) \approx 268$.

26. $N_0 = 2649$, $k = \frac{1}{60}\ln\left(\frac{7272}{2649}\right) \approx 0.0168$, $N(t) = 2649e^{0.0168t}$. $N(150) \approx 32923$, so the population of Painesville in 2010 based on this model would have been 32,923.

27. (a) $P(0) = \frac{120}{4.167} \approx 29$. There are 29 Sasquatch in Bigfoot County in 2010.
    (b) $P(3) = \frac{120}{1+3.167e^{-0.05(3)}} \approx 32$ Sasquatch.
    (c) $t = 20\ln(3.167) \approx 23$ years.
    (d) As $t \to \infty$, $P(t) \to 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.

28. (b) The average rates of change are listed in order below. They suggest slope at $(1,5)$ is 2.5.

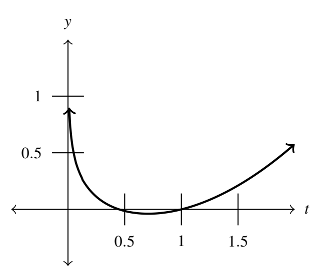
    - ≈ 2.487
    - ≈ 2.498
    - ≈ 2.500
    - ≈ 2.500
    - ≈ 2.498
    - ≈ 2.487

29. (a) $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$
    (b) $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains
    (c) $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old

30. $A(t) = 2.3e^{-0.0138629t}$

32. (a) $T(t) = 75 + 105e^{-0.005005t}$
    (b) The roast would have cooled to 140°F in about 95 minutes.

33. From the graph, it appears that as $x \to 0^+$, $y \to \infty$. This is due to the presence of the $\ln(x)$ term in the function. This means that Fritzy will never catch Chewbacca, which makes sense as Chewbacca has a head start and Fritzy only runs as fast as he does.



$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

34. The steady state current is 2 amps.

36. 630 feet.

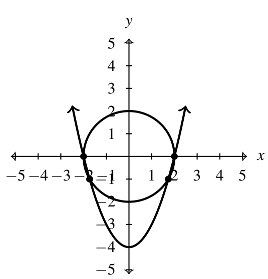
A.1.6 Chapter 6 Answers

Section 6.1 Answers

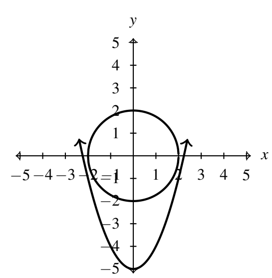
1. Consistent independent
   Solution $\left(6, -\frac{1}{2}\right)$
2. Consistent independent
   Solution $\left(-\frac{7}{3}, -3\right)$
3. Consistent independent
   Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$
4. Consistent independent
   Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$
5. Consistent dependent
   Solution $\left(t, \frac{3}{2}t+3\right)$
   for all real numbers t
6. Consistent dependent
   Solution $(6-4t, t)$
   for all real numbers t
7. Inconsistent
   No solution
8. Inconsistent
   No solution
9. 13 chose the basic buffet and 14 chose the deluxe buffet.
10. Mavis needs 20 pounds of $3 per pound coffee and 30 pounds of $8 per pound coffee.
11. Skippy needs to invest $6000 in the 3% account and $4000 in the 8% account.
12. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.

Section 6.2 Answers

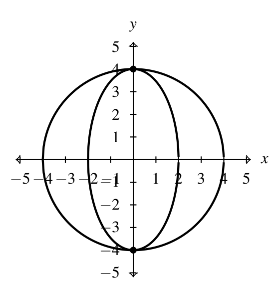
1. $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$



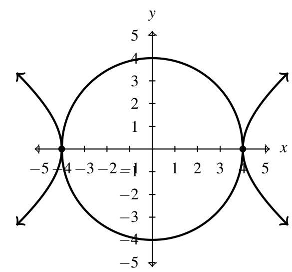
2. No solution



3. $(0, \pm 4)$

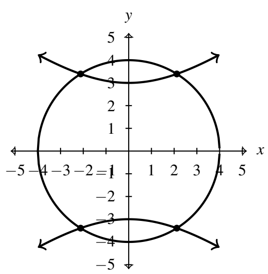


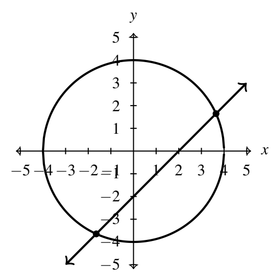
4. $(\pm 4, 0)$



5. $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$

6. $\left(1+\sqrt{7}, -1+\sqrt{7}\right)$, $\left(1-\sqrt{7}, -1-\sqrt{7}\right)$





7. $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$

8. $(0, 1)$

9. $(0, \pm 1)$, $(2, 0)$

10. $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{3}\right)$

11. $(3, 4)$, $(-4, -3)$

12. $(\pm 3, 4)$

13. $(-4, -56)$, $(1, 9)$, $(2, 16)$

14. $(-2, 2)$, $(2, -2)$

15. Initially, there are $\frac{250000}{49} \approx 5102$ bacteria. It will take $\frac{5\ln(49/5)}{\ln(7/5)} \approx 33.92$ minutes for the colony to grow to 50,000 bacteria.

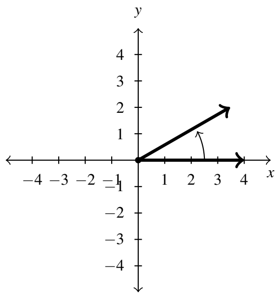
16. $\left(-\sqrt[3]{5}, 49\right)$

17. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

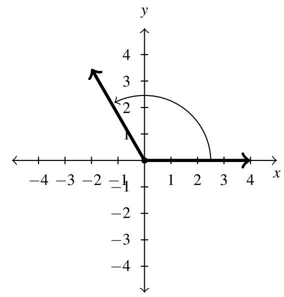
A.1.7 Chapter 7 Answers

Section 7.1 Answers

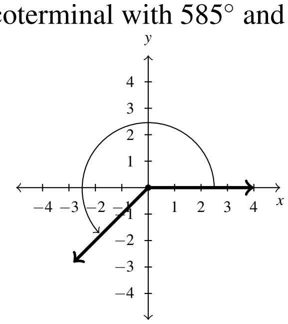
1. 30° is a Quadrant I angle coterminal with 390° and -330°



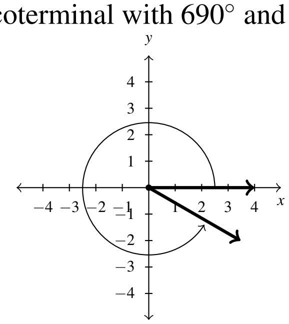
2. 120° is a Quadrant II angle coterminal with 480° and -240°



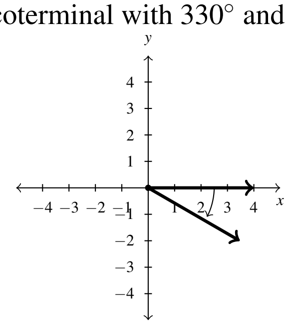
3. 225° is a Quadrant III angle coterminal with 585° and -135°



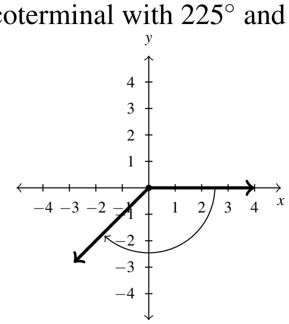
4. 330° is a Quadrant IV angle coterminal with 690° and -30°



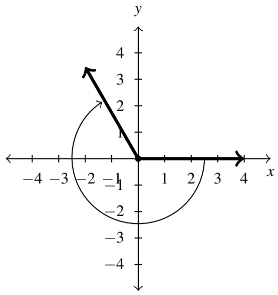
5. -30° is a Quadrant IV angle coterminal with 330° and -390°



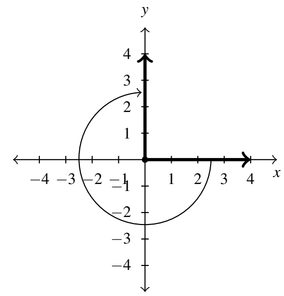
6. -135° is a Quadrant III angle coterminal with 225° and -495°



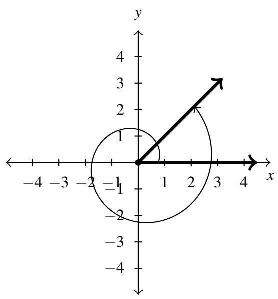
7. -240° is a Quadrant II angle coterminal with 120° and -600°



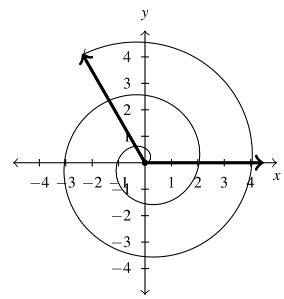
8. -270° is a quadrantal angle coterminal with 90° and -630°



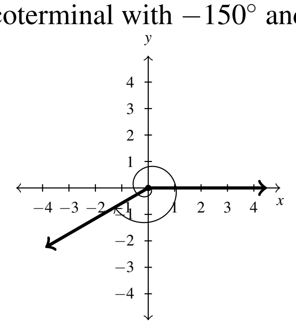
9. 405° is a Quadrant I angle coterminal with 45° and -315°



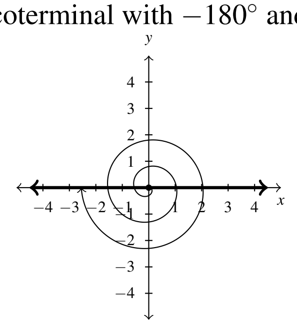
10. 840° is a Quadrant II angle coterminal with 120° and -240°



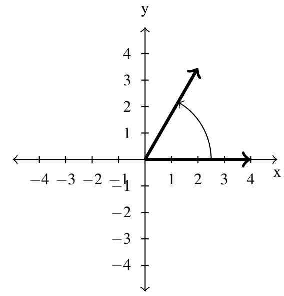
11. -510° is a Quadrant III angle coterminal with -150° and 210°



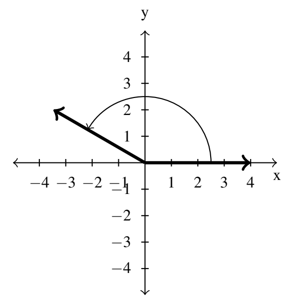
12. -900° is a quadrantal angle coterminal with -180° and 180°



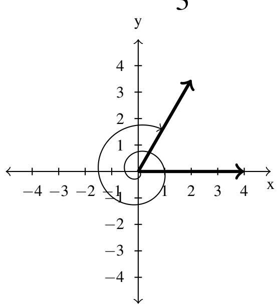
13. $\dfrac{\pi}{3}$ is a Quadrant I angle coterminal with $\dfrac{7\pi}{3}$ and $-\dfrac{5\pi}{3}$



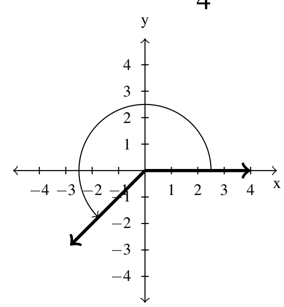
14. $\frac{5\pi}{6}$ is a Quadrant II angle
coterminal with $\frac{17\pi}{6}$ and $-\frac{7\pi}{6}$



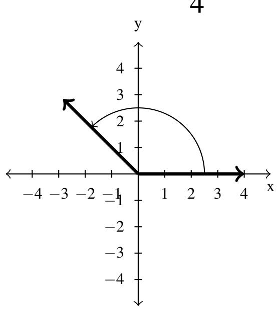
15. $-\frac{11\pi}{3}$ is a Quadrant I angle
coterminal with $\frac{\pi}{3}$ and $-\frac{5\pi}{3}$



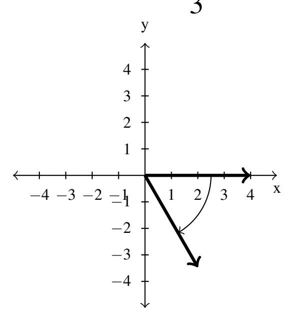
16. $\frac{5\pi}{4}$ is a Quadrant III angle
coterminal with $\frac{13\pi}{4}$ and $-\frac{3\pi}{4}$



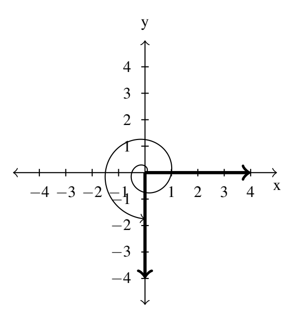
17. $\frac{3\pi}{4}$ is a Quadrant II angle
coterminal with $\frac{11\pi}{4}$ and $-\frac{5\pi}{4}$



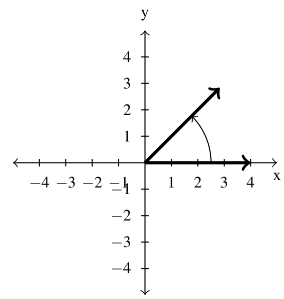
18. $-\frac{\pi}{3}$ is a Quadrant IV angle
coterminal with $\frac{5\pi}{3}$ and $-\frac{7\pi}{3}$



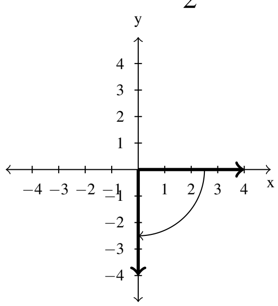
19. $\frac{7\pi}{2}$ lies on the negative y-axis
coterminal with $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$



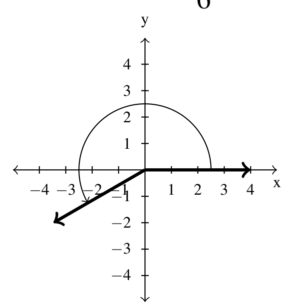
20. $\frac{\pi}{4}$ is a Quadrant I angle
coterminal with $\frac{9\pi}{4}$ and $-\frac{7\pi}{4}$



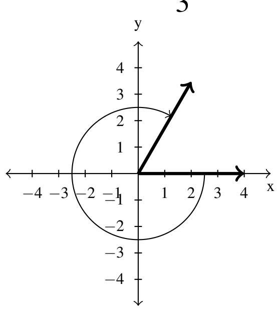
21. $-\frac{\pi}{2}$ lies on the negative y-axis
coterminal with $\frac{3\pi}{2}$ and $-\frac{5\pi}{2}$



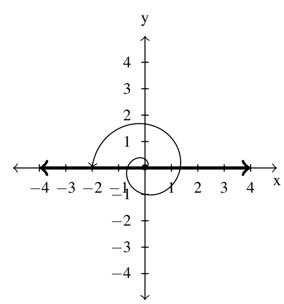
22. $\frac{7\pi}{6}$ is a Quadrant III angle
coterminal with $\frac{19\pi}{6}$ and $-\frac{5\pi}{6}$



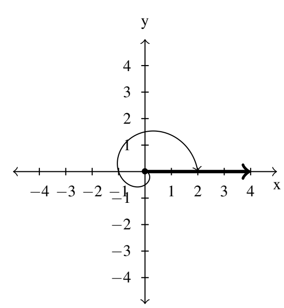
23. $-\frac{5\pi}{3}$ is a Quadrant I angle
coterminal with $\frac{\pi}{3}$ and $-\frac{11\pi}{3}$



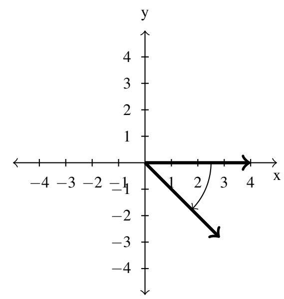
24. 3π lies on the negative x-axis
coterminal with π and $-\pi$



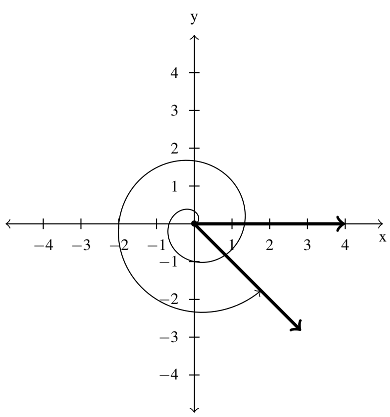
25. -2π lies on the positive x-axis
coterminal with 2π and -4π



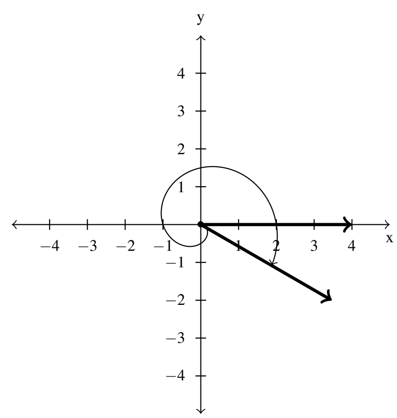
26. $-\dfrac{\pi}{4}$ is a Quadrant IV angle coterminal with $\dfrac{7\pi}{4}$ and $-\dfrac{9\pi}{4}$



27. $\dfrac{15\pi}{4}$ is a Quadrant IV angle coterminal with $\dfrac{7\pi}{4}$ and $-\dfrac{\pi}{4}$



28. $-\dfrac{13\pi}{6}$ is a Quadrant IV angle coterminal with $\dfrac{11\pi}{6}$ and $-\dfrac{\pi}{6}$



29. 0

30. $\dfrac{4\pi}{3}$

31. $\dfrac{3\pi}{4}$

32. $-\dfrac{3\pi}{2}$

33. $-\dfrac{7\pi}{4}$

34. $\dfrac{5\pi}{6}$

35. $\dfrac{\pi}{4}$

36. $-\dfrac{5\pi}{4}$

37. 180°

38. -120°

39. 210°

40. 330°

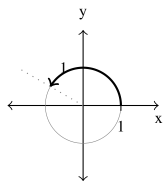
41. 60°

42. 300°

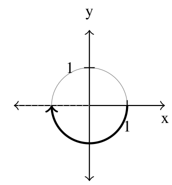
43. -30°

44. 90°

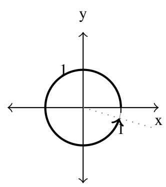
45. $t = \frac{5\pi}{6}$



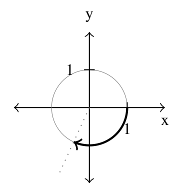
46. $t = -\pi$



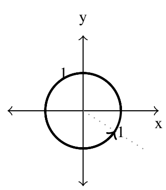
47. $t = 6$



48. $t = -2$



49. $t = 12$ (between 1 and 2 revolutions)



50. $\frac{3375\pi}{352}$ miles per hour

51. $\frac{19712}{\pi}$ revolutions per minute

52. $\frac{35\pi}{33}$ miles per hour

53. $\frac{375\pi}{22}$ miles per hour

54. 70 miles per hour

55. $\frac{1920\pi}{1397}$ miles per hour

57. 12π square units

58. 6250π square units

59. $79.2825\pi \approx 249.07$ square units

60. $\frac{\pi}{2}$ square units

61. $\frac{50\pi}{3}$ square units

62. $38.025\pi \approx 119.46$ square units

Section 7.2 Answers

1. $\theta = 30°$, $a = 3\sqrt{3}$, $c = \sqrt{108} = 6\sqrt{3}$

2. $\alpha = 56^\circ$, $b = 12\tan(34^\circ) = 8.094$, $c = 12\sec(34^\circ) = \dfrac{12}{\cos(34^\circ)} \approx 14.475$

3. $\theta = 43^\circ$, $a = 6\cot(47^\circ) = \dfrac{6}{\tan(47^\circ)} \approx 5.595$, $c = 6\csc(47^\circ) = \dfrac{6}{\sin(47^\circ)} \approx 8.204$

4. $\beta = 40^\circ$, $b = 2.5\tan(50^\circ) \approx 2.979$, $c = 2.5\sec(50^\circ) = \dfrac{2.5}{\cos(50^\circ)} \approx 3.889$

5. The side opposite θ has length $10\sin(15^\circ) \approx 2.588$

6. The hypoteneuse has length $14\csc(38.2^\circ) = \dfrac{14}{\sin(38.2^\circ)} \approx 22.639$

7. The side adjacent to θ has length $3.98\cos(2.05^\circ) \approx 3.977$

8. $\cos(0) = 1,\ \sin(0) = 0$

9. $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2},\ \sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

10. $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2},\ \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

11. $\cos\left(\dfrac{\pi}{2}\right) = 0,\ \sin\left(\dfrac{\pi}{2}\right) = 1$

12. $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2},\ \sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

13. $\cos\left(\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2},\ \sin\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

14. $\cos(\pi) = -1,\ \sin(\pi) = 0$

15. $\cos\left(\dfrac{7\pi}{6}\right) = -\dfrac{\sqrt{3}}{2},\ \sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$

16. $\cos\left(\dfrac{5\pi}{4}\right) = -\dfrac{\sqrt{2}}{2},\ \sin\left(\dfrac{5\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

17. $\cos\left(\dfrac{4\pi}{3}\right) = -\dfrac{1}{2},\ \sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

18. $\cos\left(\dfrac{3\pi}{2}\right) = 0,\ \sin\left(\dfrac{3\pi}{2}\right) = -1$

19. $\cos\left(\dfrac{5\pi}{3}\right) = \dfrac{1}{2},\ \sin\left(\dfrac{5\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

20. $\cos\left(\dfrac{7\pi}{4}\right) = \dfrac{\sqrt{2}}{2},\ \sin\left(\dfrac{7\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

21. $\cos\left(\dfrac{23\pi}{6}\right) = \dfrac{\sqrt{3}}{2},\ \sin\left(\dfrac{23\pi}{6}\right) = -\dfrac{1}{2}$

22. $\cos\left(-\dfrac{13\pi}{2}\right) = 0,\ \sin\left(-\dfrac{13\pi}{2}\right) = -1$

23. $\cos\left(-\dfrac{43\pi}{6}\right) = -\dfrac{\sqrt{3}}{2},\ \sin\left(-\dfrac{43\pi}{6}\right) = \dfrac{1}{2}$

24. $\cos\left(-\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2},\ \sin\left(-\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

25. $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2},\ \sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$

26. $\cos\left(\dfrac{10\pi}{3}\right) = -\dfrac{1}{2},\ \sin\left(\dfrac{10\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

27. $\cos(117\pi) = -1,\ \sin(117\pi) = 0$

28. $\sin(\theta) = \dfrac{1}{2}$ when $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{5\pi}{6} + 2\pi k$ for any integer k.

29. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ when $\theta = \frac{5\pi}{6} + 2\pi k$ or $\theta = \frac{7\pi}{6} + 2\pi k$ for any integer k.

30. $\sin(\theta) = 0$ when $\theta = \pi k$ for any integer k.

31. $\cos(\theta) = \frac{\sqrt{2}}{2}$ when $\theta = \frac{\pi}{4} + 2\pi k$ or $\theta = \frac{7\pi}{4} + 2\pi k$ for any integer k.

32. $\sin(\theta) = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ for any integer k.

33. $\cos(\theta) = -1$ when $\theta = (2k+1)\pi$ for any integer k.

34. $\sin(\theta) = -1$ when $\theta = \frac{3\pi}{2} + 2\pi k$ for any integer k.

35. $\cos(\theta) = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{11\pi}{6} + 2\pi k$ for any integer k.

36. $\cos(\theta) = -1.001$ never happens

37. $\cos(t) = 0$ when $t = \frac{\pi}{2} + \pi k$ for any integer k.

38. $\sin(t) = -\frac{\sqrt{2}}{2}$ when $t = \frac{5\pi}{4} + 2\pi k$ or $t = \frac{7\pi}{4} + 2\pi k$ for any integer k.

39. $\cos(t) = 3$ never happens.

40. $\sin(t) = -\frac{1}{2}$ when $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for any integer k.

41. $\cos(t) = \frac{1}{2}$ when $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for any integer k.

42. $\sin(t) = -2$ never happens

43. $\cos(t) = 1$ when $t = 2\pi k$ for any integer k.

44. $\sin(t) = 1$ when $t = \frac{\pi}{2} + 2\pi k$ for any integer k.

45. $\cos(t) = -\frac{\sqrt{2}}{2}$ when $t = \frac{3\pi}{4} + 2\pi k$ or $t = \frac{5\pi}{4} + 2\pi k$ for any integer k.

46. $\cos(\theta) = -\frac{7}{25}$, $\sin(\theta) = \frac{24}{25}$

47. $\cos(\theta) = \frac{3}{5}$, $\sin(\theta) = \frac{4}{5}$

48. $\cos(\theta) = \frac{5\sqrt{106}}{106}$, $\sin(\theta) = -\frac{9\sqrt{106}}{106}$

49. $\cos(\theta) = -\frac{2\sqrt{5}}{25}$, $\sin(\theta) = -\frac{11\sqrt{5}}{25}$

50. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, then $\cos(\theta) = \frac{24}{25}$.

51. If $\cos(\theta) = \frac{4}{9}$ with θ in Quadrant I, then $\sin(\theta) = \frac{\sqrt{65}}{9}$.

52. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, then $\cos(\theta) = -\frac{12}{13}$.

53. If $\cos(\theta) = -\frac{2}{11}$ with θ in Quadrant III, then $\sin(\theta) = -\frac{\sqrt{117}}{11}$.

54. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, then $\cos(\theta) = -\frac{\sqrt{5}}{3}$.

55. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, then $\sin(\theta) = -\frac{45}{53}$.

56. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\cos(\theta) = -\frac{\sqrt{5}}{5}$.

57. If $\cos(\theta) = \frac{\sqrt{10}}{10}$ and $2\pi < \theta < \frac{5\pi}{2}$, then $\sin(\theta) = \frac{3\sqrt{10}}{10}$.

58. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, then $\cos(\theta) = -\sqrt{0.8236} \approx -0.9075$.

59. If $\cos(\theta) = -0.98$ and $\frac{\pi}{2} < \theta < \pi$, then $\sin(\theta) = \sqrt{0.0396} \approx 0.1990$.

60. One solution is $g(t) = 3t$ and $h(t) = \sin(2t)$.

61. One solution is $g(\theta) = 3\cos(\theta)$ and $h(\theta) = \sin(4\theta)$.

62. One solution is $g(t) = e^{-0.1t}$ and $h(t) = \sin(3t)$.

63. One solution is $f(t) = \sin(t)$ and $g(t) = t$.

64. One solution is $f(\theta) = 3\cos(\theta)$ and $g(\theta) = \sqrt{\theta}$.

65. As we zoom in towards 0, the average rate of change of $\sin(kt)$ approaches k.

$S(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\sin(t)$	≈ 0.9983	≈ 1	≈ 1
$\sin(2t)$	≈ 1.9867	≈ 1.9999	≈ 2
$\sin(3t)$	≈ 2.9552	≈ 2.9995	≈ 3
$\sin(4t)$	≈ 3.8942	≈ 3.9989	≈ 4

66. $r = 1.125$ inches, $\omega = 9000\pi \frac{\text{radians}}{\text{minute}}$, $x = 1.125\cos(9000\pi t)$, $y = 1.125\sin(9000\pi t)$. Here x and y are measured in inches and t is measured in minutes.

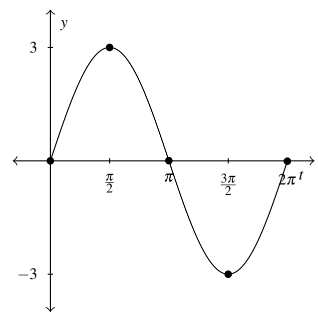
67. $r = 28$ inches, $\omega = \frac{2\pi}{3} \frac{\text{radians}}{\text{second}}$, $x = 28\cos\left(\frac{2\pi}{3}t\right)$, $y = 28\sin\left(\frac{2\pi}{3}t\right)$. Here x and y are measured in inches and t is measured in seconds.

68. $r = 1.25$ inches, $\omega = 14400\pi \frac{\text{radians}}{\text{minute}}$, $x = 1.25\cos(14400\pi t)$, $y = 1.25\sin(14400\pi t)$. Here x and y are measured in inches and t is measured in minutes.

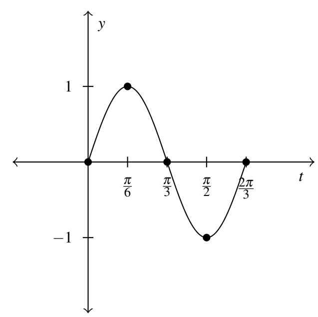
69. $r = 64$ feet, $\omega = \frac{4\pi}{127} \frac{\text{radians}}{\text{second}}$, $x = 64\cos\left(\frac{4\pi}{127}t\right)$, $y = 64\sin\left(\frac{4\pi}{127}t\right)$. Here x and y are measured in feet and t is measured in seconds

Section 7.3 Answers

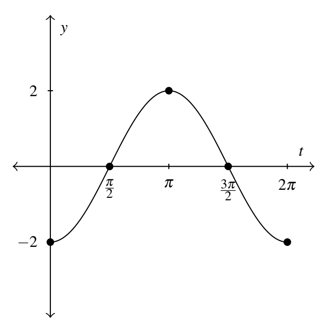
1. $f(t) = 3\sin(t)$
Period: 2π
Amplitude: 3
Phase Shift: 0
Vertical Shift: 0



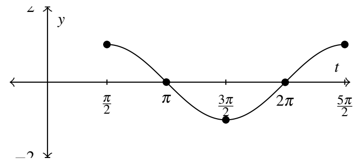
2. $g(t) = \sin(3t)$
Period: $\frac{2\pi}{3}$
Amplitude: 1
Phase Shift: 0
Vertical Shift: 0



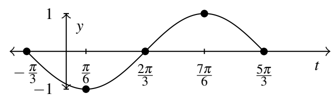
3. $h(t) = -2\cos(t)$
Period: 2π
Amplitude: 2
Phase Shift: 0
Vertical Shift: 0



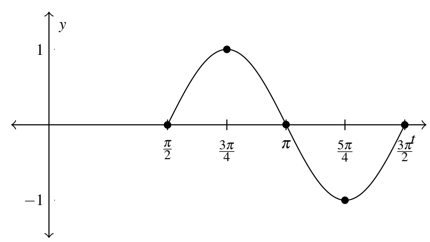
4. $f(t) = \cos\left(t - \frac{\pi}{2}\right)$
Period: 2π
Amplitude: 1
Phase Shift: $\frac{\pi}{2}$
Vertical Shift: 0



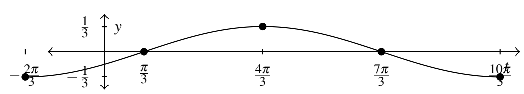
5. $g(t) = -\sin\left(t + \frac{\pi}{3}\right)$
Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{3}$
Vertical Shift: 0



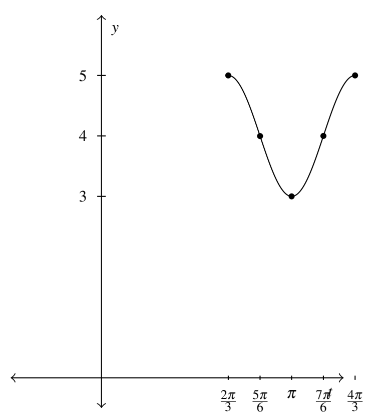
6. $h(t) = \sin(2t - \pi)$
Period: π
Amplitude: 1
Phase Shift: $\frac{\pi}{2}$
Vertical Shift: 0



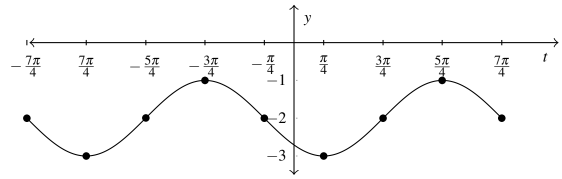
7. $f(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
Period: 4π
Amplitude: $\frac{1}{3}$
Phase Shift: $-\frac{2\pi}{3}$
Vertical Shift: 0



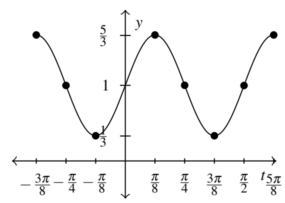
8. $g(t) = \cos(3t - 2\pi) + 4$
Period: $\frac{2\pi}{3}$
Amplitude: 1
Phase Shift: $\frac{2\pi}{3}$
Vertical Shift: 4



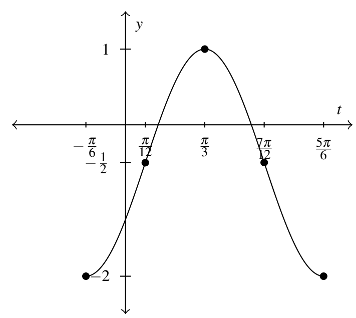
9. $h(t) = \sin\left(-t - \frac{\pi}{4}\right) - 2$
   Period: 2π
   Amplitude: 1
   Phase Shift: $-\frac{\pi}{4}$ (You need to use $y = -\sin\left(t + \frac{\pi}{4}\right) - 2$ to find this.)[10]
   Vertical Shift: -2



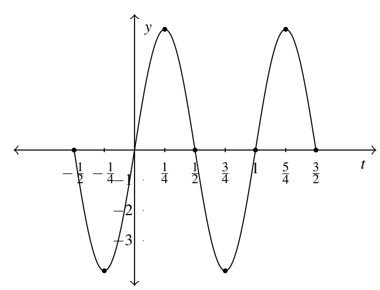
10. $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4t\right) + 1$
    Period: $\frac{\pi}{2}$
    Amplitude: $\frac{2}{3}$
    Phase Shift: $\frac{\pi}{8}$ (You need to use $y = \frac{2}{3}\cos\left(4t - \frac{\pi}{2}\right) + 1$ to find this.)[11]
    Vertical Shift: 1



11. $g(t) = -\frac{3}{2}\cos\left(2t + \frac{\pi}{3}\right) - \frac{1}{2}$
    Period: π
    Amplitude: $\frac{3}{2}$
    Phase Shift: $-\frac{\pi}{6}$
    Vertical Shift: $-\frac{1}{2}$



12. $h(t) = 4\sin(-2\pi t + \pi)$
    Period: 1
    Amplitude: 4
    Phase Shift: $\frac{1}{2}$ (You need to use $h(t) = -4\sin(2\pi t - \pi)$ to find this.)[12]
    Vertical Shift: 0



[10] Two cycles of the graph are shown to illustrate the discrepancy discussed on page 625.
[11] Again, we graph two cycles to illustrate the discrepancy discussed on page 625.
[12] This will be the last time we graph two cycles to illustrate the discrepancy discussed on page 625.

13. $S(t) = 4\sin\left(t + \frac{\pi}{4}\right), C(t) = 4\cos\left(t - \frac{\pi}{4}\right)$

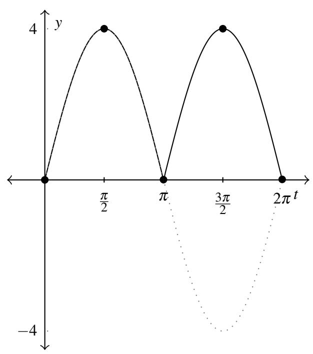
14. $S(t) = -3\sin(t) + 3, C(t) = -3\cos\left(t - \frac{\pi}{2}\right) + 3$

15. $S(t) = 3\sin\left(2t - \frac{\pi}{3}\right), C(t) = 3\cos\left(2t - \frac{5\pi}{6}\right)$

16. $S(t) = \frac{7}{2}\sin(\pi t) + \frac{1}{2}, C(t) = \frac{7}{2}\cos\left(\pi t \frac{\pi}{2}\right) + \frac{1}{2}$

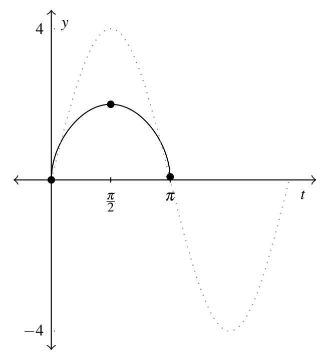
17. (a) $y = |4\sin(t)|$. Period: π.

Two cycles are graphed below.

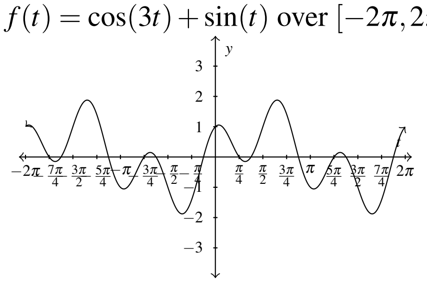


(b) $y = \sqrt{4\sin(t)}$. Period: 2π.

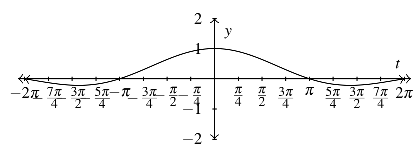
One cycle is graphed below.



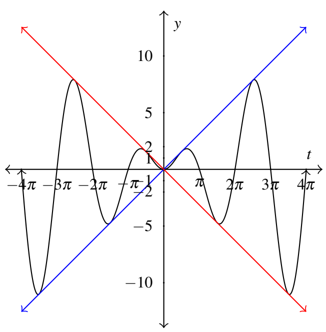
18. $f(t) = \cos(3t) + \sin(t)$ over $[-2\pi, 2\pi]$



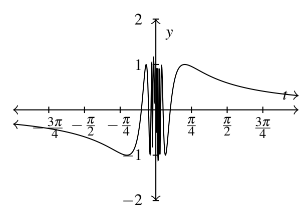
19. $f(t) = \frac{\sin(t)}{t}$ over $[-2\pi, 2\pi]$



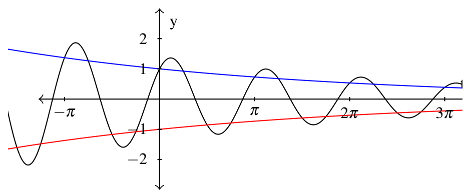
20. $f(t) = t\sin(t)$ over $[-4\pi, 4\pi]$



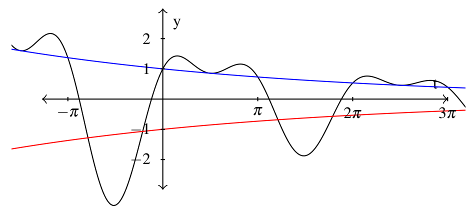
21. $f(t) = \sin\left(\frac{1}{t}|\right)$ over $[-\pi, \pi]$



22. $f(t) = e^{-0.1t}(\cos(2t) + \sin(2t)))$ over $[-\pi, 3\pi]$



23. $f(t) = e^{-0.1t}(\cos(2t) + 2\sin(t)))$ over $[-\pi, 3\pi]$



25. $S(t) = \sin(880\pi t)$

26. $V(t) = 220\sqrt{2}\sin(120\pi t)$

27. $h(t) = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 67.5$

28. $x(t) = 67.5\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) = 67.5\sin\left(\frac{\pi}{15}t\right)$

29. $h(t) = 28\sin\left(\frac{2\pi}{3}t - \frac{\pi}{2}\right) + 30$

30. (a) $\theta(t) = \theta_0\sin\left(\sqrt{\frac{g}{l}}\,t + \frac{\pi}{2}\right)$ (b) $\theta(t) = \frac{\pi}{12}\sin\left(4\pi t + \frac{\pi}{2}\right)$

Section 7.4 Answers

1. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = \frac{4}{5}, \tan(\theta) = \frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = \frac{5}{4}, \cot(\theta) = \frac{4}{3}$
2. $\sin(\theta) = \frac{12}{13}, \cos(\theta) = \frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = \frac{13}{12}, \sec(\theta) = \frac{13}{5}, \cot(\theta) = \frac{5}{12}$
3. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$
4. $\sin(\theta) = \frac{4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = 4\sqrt{3}, \csc(\theta) = \frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = \frac{\sqrt{3}}{12}$
5. $\sin(\theta) = \frac{\sqrt{91}}{10}, \cos(\theta) = \frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = \frac{10\sqrt{91}}{91}, \sec(\theta) = \frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$
6. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = \frac{23\sqrt{530}}{530}, \tan(\theta) = \frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = \frac{\sqrt{530}}{23}, \cot(\theta) = 23$
7. $\sin(\theta) = \frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = 2, \csc(\theta) = \frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = \frac{1}{2}$
8. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = \frac{1}{4}, \tan(\theta) = \sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = 4, \cot(\theta) = \frac{\sqrt{15}}{15}$
9. $\sin(\theta) = \frac{\sqrt{6}}{6}, \cos(\theta) = \frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = \sqrt{6}, \sec(\theta) = \frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$
10. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$
11. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$
12. $\sin(\theta) = \frac{1}{5}, \cos(\theta) = \frac{2\sqrt{6}}{5}, \tan(\theta) = \frac{\sqrt{6}}{12}, \csc(\theta) = 5, \sec(\theta) = \frac{5\sqrt{6}}{12}, \cot(\theta) = 2\sqrt{6}$
13. $\sin(\theta) = \frac{\sqrt{110}}{11}, \cos(\theta) = \frac{\sqrt{11}}{11}, \tan(\theta) = \sqrt{10}, \csc(\theta) = \frac{\sqrt{110}}{10}, \sec(\theta) = \sqrt{11}, \cot(\theta) = \frac{\sqrt{10}}{10}$

14. $\sin(\theta) = \frac{\sqrt{95}}{10}, \cos(\theta) = \frac{\sqrt{5}}{10}, \tan(\theta) = \sqrt{19}, \csc(\theta) = \frac{2\sqrt{95}}{19}, \sec(\theta) = 2\sqrt{5}, \cot(\theta) = \frac{\sqrt{19}}{19}$

15. $\sin(\theta) = \frac{\sqrt{21}}{5}, \cos(\theta) = \frac{2}{5}, \tan(\theta) = \frac{\sqrt{21}}{2}, \csc(\theta) = \frac{5\sqrt{21}}{21}, \sec(\theta) = \frac{5}{2}, \cot(\theta) = \frac{2\sqrt{21}}{21}$

16. The tree is about 47 feet tall.

17. The lights are about 75 feet apart.

18. (b) The fire is about 4581 feet from the base of the tower.
    (c) The Sasquatch ran $200\cot(6^\circ) - 200\cot(6.5^\circ) \approx 147$ feet in those 10 seconds. This translates to ≈ 10 miles per hour. At the scene of the second sighting, the Sasquatch was ≈ 1755 feet from the tower, which means, if it keeps up this pace, it will reach the tower in about 2 minutes.

19. The tree is about 41 feet tall.

20. The boat has traveled about 244 feet.

21. The tower is about 682 feet tall. The guy wire hits the ground about 731 feet away from the base of the tower.

22. $\tan\left(\frac{\pi}{4}\right) = 1$

23. $\sec\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$

24. $\csc\left(\frac{5\pi}{6}\right) = 2$

25. $\cot\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}$

26. $\tan\left(-\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{3}$

27. $\sec\left(-\frac{3\pi}{2}\right)$ is undefined

28. $\csc\left(-\frac{\pi}{3}\right) = -\frac{2\sqrt{3}}{3}$

29. $\cot\left(\frac{13\pi}{2}\right) = 0$

30. $\tan(117\pi) = 0$

31. $\sec\left(-\frac{5\pi}{3}\right) = 2$

32. $\csc(3\pi)$ is undefined

33. $\cot(-5\pi)$ is undefined

34. $\tan\left(\frac{31\pi}{2}\right)$ is undefined

35. $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

36. $\csc\left(-\frac{7\pi}{4}\right) = \sqrt{2}$

37. $\cot\left(\frac{7\pi}{6}\right) = \sqrt{3}$

38. $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

39. $\sec(-7\pi) = -1$

40. $\csc\left(\frac{\pi}{2}\right) = 1$

41. $\cot\left(\frac{3\pi}{4}\right) = -1$

42. Quadrant II.

43. Quadrant III.

44. Quadrant I.

45. Quadrant IV.

46. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$

47. $\sin(\theta) = -\frac{12}{13}, \cos(\theta) = -\frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = -\frac{13}{12}, \sec(\theta) = -\frac{13}{5}, \cot(\theta) = \frac{5}{12}$

48. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$

49. $\sin(\theta) = \frac{-4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = -4\sqrt{3}, \csc(\theta) = -\frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = -\frac{\sqrt{3}}{12}$

50. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$

51. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = -\frac{23\sqrt{530}}{530}, \tan(\theta) = -\frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = -\frac{\sqrt{530}}{23}, \cot(\theta) = -23$

52. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$

53. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = -\frac{1}{4}, \tan(\theta) = -\sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = -4, \cot(\theta) = -\frac{\sqrt{15}}{15}$

54. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$

55. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$

56. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$

57. $\sin(\theta) = \frac{1}{5}, \cos(\theta) = -\frac{2\sqrt{6}}{5}, \tan(\theta) = -\frac{\sqrt{6}}{12}, \csc(\theta) = 5, \sec(\theta) = -\frac{5\sqrt{6}}{12}, \cot(\theta) = -2\sqrt{6}$

58. $\sin(\theta) = -\frac{\sqrt{110}}{11}, \cos(\theta) = -\frac{\sqrt{11}}{11}, \tan(\theta) = \sqrt{10}, \csc(\theta) = -\frac{\sqrt{110}}{10}, \sec(\theta) = -\sqrt{11}, \cot(\theta) = \frac{\sqrt{10}}{10}$

59. $\sin(\theta) = -\frac{\sqrt{95}}{10}, \cos(\theta) = \frac{\sqrt{5}}{10}, \tan(\theta) = -\sqrt{19}, \csc(\theta) = -\frac{2\sqrt{95}}{19}, \sec(\theta) = 2\sqrt{5}, \cot(\theta) = -\frac{\sqrt{19}}{19}$

60. $\csc(78.95^\circ) \approx 1.019$

61. $\tan(-2.01) \approx 2.129$

62. $\cot(392.994) \approx 3.292$

63. $\sec(207^\circ) \approx -1.122$

64. $\csc(5.902) \approx -2.688$

65. $\tan(39.672^\circ) \approx 0.829$

66. $\cot(3^\circ) \approx 19.081$

67. $\sec(0.45) \approx 1.111$

68. $\tan(\theta) = \sqrt{3}$ when $\theta = \dfrac{\pi}{3} + \pi k$ for any integer k

69. $\sec(\theta) = 2$ when $\theta = \dfrac{\pi}{3} + 2\pi k$ or $\theta = \dfrac{5\pi}{3} + 2\pi k$ for any integer k

70. $\csc(\theta) = -1$ when $\theta = \dfrac{3\pi}{2} + 2\pi k$ for any integer k.

71. $\cot(\theta) = \dfrac{\sqrt{3}}{3}$ when $\theta = \dfrac{\pi}{3} + \pi k$ for any integer k

72. $\tan(\theta) = 0$ when $\theta = \pi k$ for any integer k

73. $\sec(\theta) = 1$ when $\theta = 2\pi k$ for any integer k

74. $\csc(\theta) = 2$ when $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{5\pi}{6} + 2\pi k$ for any integer k.

75. $\cot(\theta) = 0$ when $\theta = \dfrac{\pi}{2} + \pi k$ for any integer k

76. $\tan(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

77. $\sec(\theta) = 0$ never happens

78. $\csc(\theta) = -\dfrac{1}{2}$ never happens

79. $\sec(\theta) = -1$ when $\theta = \pi + 2\pi k = (2k+1)\pi$ for any integer k

80. $\tan(\theta) = -\sqrt{3}$ when $\theta = \dfrac{2\pi}{3} + \pi k$ for any integer k

81. $\csc(\theta) = -2$ when $\theta = \dfrac{7\pi}{6} + 2\pi k$ or $\theta = \dfrac{11\pi}{6} + 2\pi k$ for any integer k

82. $\cot(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

83. $\cot(t) = 1$ when $t = \dfrac{\pi}{4} + \pi k$ for any integer k

84. $\tan(t) = \dfrac{\sqrt{3}}{3}$ when $t = \dfrac{\pi}{6} + \pi k$ for any integer k

85. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{5\pi}{6} + 2\pi k$ or $t = \dfrac{7\pi}{6} + 2\pi k$ for any integer k

86. $\csc(t) = 0$ never happens

87. $\cot(t) = -\sqrt{3}$ when $t = \dfrac{5\pi}{6} + \pi k$ for any integer k

88. $\tan(t) = -\dfrac{\sqrt{3}}{3}$ when $t = \dfrac{5\pi}{6} + \pi k$ for any integer k

89. $\sec(t) = \dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{\pi}{6} + 2\pi k$ or $t = \dfrac{11\pi}{6} + 2\pi k$ for any integer k

90. $\csc(t) = \dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{\pi}{3} + 2\pi k$ or $t = \dfrac{2\pi}{3} + 2\pi k$ for any integer k

91. One solution is $g(t) = 3t^2$ and $h(t) = 2\tan(3t)$.

92. One solution is $g(\theta) = \sec(\theta)$ and $h(\theta) = \tan(\theta)$.

93. One solution is $g(t) = -\csc(t)$ and $h(t) = \cot(t)$.

94. One solution is $f(t) = \tan(3t)$ and $g(t) = t$.

95. One solution is $f(\theta) = 4\theta$ and $g(\theta) = \tan(\theta)$.

96. As $\sec^2(\theta) = (\sec(\theta))^2$, one solution is $f(\theta) = \sec(\theta)$ and $g(\theta) = \theta^2$.

97. One solution is $f(x) = \sin(x)$ and $g(x) = \ln(x)$.

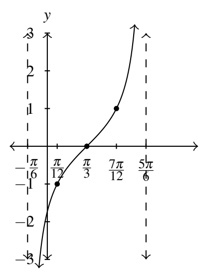
98. One solution is $f(\theta) = \sec(\theta)$, $g(\theta) = \tan(\theta)$, and $h(\theta) = \ln|\theta|$.

100. As we zoom in towards 0, the average rate of change of $\tan(kt)$ approaches k. This is the same trend we observed for $\sin(kt)$ in Section 7.2.2 number 65.

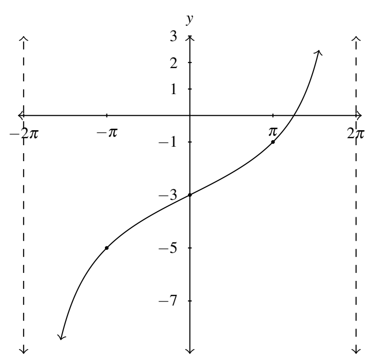
$T(t)$	$[-0.1, 0.1]$	$[-0.01, 0.01]$	$[-0.001, 0.001]$
$\tan(t)$	≈ 1.0033	≈ 1	≈ 1
$\tan(2t)$	≈ 2.0271	≈ 2.0003	≈ 2
$\tan(3t)$	≈ 3.0933	≈ 3.0009	≈ 3
$\tan(4t)$	≈ 4.2279	≈ 4.0021	≈ 4

Section 7.5 Answers

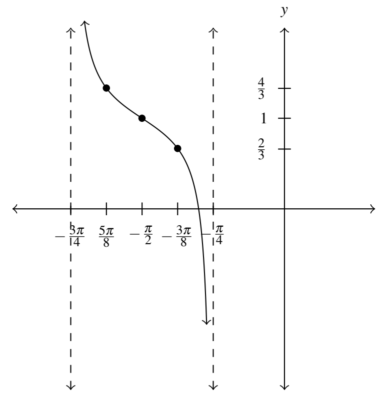
1. $y = \tan\left(t - \dfrac{\pi}{3}\right)$
   Period: π



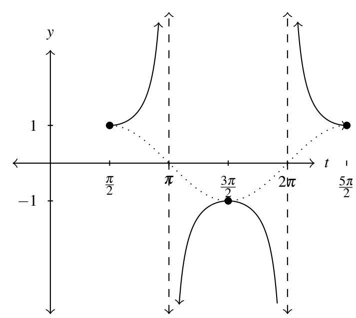
2. $y = 2\tan\left(\dfrac{1}{4}t\right) - 3$
   Period: 4π



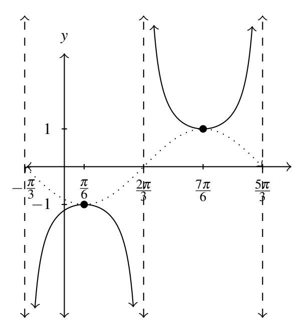
3. $y = \frac{1}{3}\tan(-2t - \pi) + 1$
is equivalent to
$y = -\frac{1}{3}\tan(2t + \pi) + 1$
via the Even / Odd identity for tangent.
Period: $\frac{\pi}{2}$



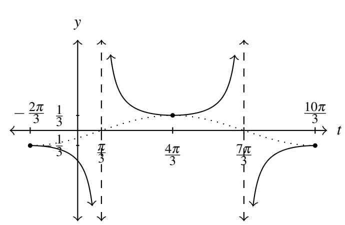
4. $y = \sec\left(t - \frac{\pi}{2}\right)$
Start with $y = \cos\left(t - \frac{\pi}{2}\right)$
Period: 2π



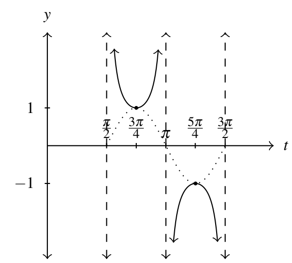
5. $y = -\csc\left(t + \frac{\pi}{3}\right)$
Start with $y = -\sin\left(t + \frac{\pi}{3}\right)$
Period: 2π



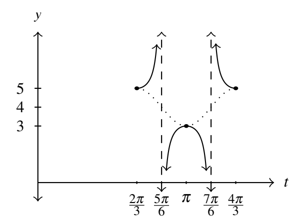
6. $y = -\frac{1}{3}\sec\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
Start with $y = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$
Period: 4π



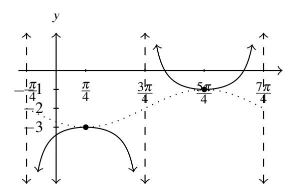
7. $y = \csc(2t - \pi)$
   Start with $y = \sin(2t - \pi)$
   Period: π



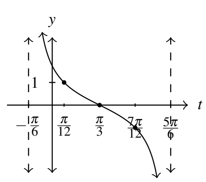
8. $y = \sec(3t - 2\pi) + 4$
   Start with $y = \cos(3t - 2\pi) + 4$
   Period: $\dfrac{2\pi}{3}$



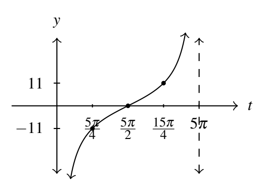
9. $y = \csc\left(-t - \dfrac{\pi}{4}\right) - 2$
   Start with $y = \sin\left(-t - \dfrac{\pi}{4}\right) - 2$
   Period: 2π



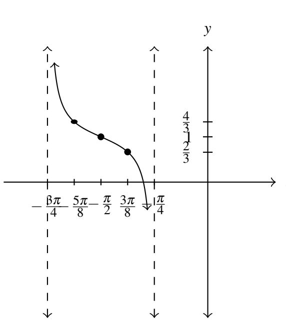
10. $y = \cot\left(t + \dfrac{\pi}{6}\right)$
    Period: π



11. $y = -11\cot\left(\dfrac{1}{5}t\right)$
    Period: 5π



12. $y = \dfrac{1}{3}\cot\left(2t + \dfrac{3\pi}{2}\right) + 1$
    Period: $\dfrac{\pi}{2}$



13. $F(t) = 2\sec(t-\pi)$, $G(t) = 2\csc\left(t-\frac{\pi}{2}\right)$

14. $F(t) = \sec\left(\frac{\pi}{2}t\right)+1$, $G(t) = \csc\left(\frac{\pi}{2}t+\frac{\pi}{2}\right)+1$

15. $J(t) = -\tan\left(t+\frac{\pi}{4}\right)$, $K(t) = \cot\left(t-\frac{\pi}{4}\right)$

16. $J(t) = \tan\left(\frac{\pi}{4}t\right)+1$, $K(t) = -\cot\left(\frac{\pi}{4}t+\frac{\pi}{2}\right)+1$

17. (a) $\csc\left(t+\frac{\pi}{2}\right) = \sec(t)$ and $\sec\left(t-\frac{\pi}{2}\right) = \csc(t)$.

(b) $f(t) = \sec\left(2t-\frac{7\pi}{6}\right)-1 = \csc\left(\left[2t-\frac{7\pi}{6}\right]+\frac{\pi}{2}\right)-1 = \csc\left(2t-\frac{2\pi}{3}\right)-1$, in terms of cosecants.

18. $f(t) = -\sec\left(2t-\frac{\pi}{6}\right)-1$ and $f(t) = -\csc\left(2t+\frac{\pi}{3}\right)-1$ are two answers

Section 7.6 Answers

1. $\arcsin(-1) = -\dfrac{\pi}{2}$

2. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\pi}{3}$

3. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$

4. $\arcsin\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

5. $\arcsin(0) = 0$

6. $\arcsin\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

7. $\arcsin\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$

8. $\arcsin\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

9. $\arcsin(1) = \dfrac{\pi}{2}$

10. $\arccos(-1) = \pi$

11. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

12. $\arccos\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\pi}{4}$

13. $\arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

14. $\arccos(0) = \dfrac{\pi}{2}$

15. $\arccos\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

16. $\arccos\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$

17. $\arccos\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$

18. $\arccos(1) = 0$

19. $\arctan\left(-\sqrt{3}\right) = -\dfrac{\pi}{3}$

20. $\arctan(-1) = -\dfrac{\pi}{4}$

21. $\arctan\left(-\dfrac{\sqrt{3}}{3}\right) = -\dfrac{\pi}{6}$

22. $\arctan(0) = 0$

23. $\arctan\left(\dfrac{\sqrt{3}}{3}\right) = \dfrac{\pi}{6}$

24. $\arctan(1) = \dfrac{\pi}{4}$

25. $\arctan\left(\sqrt{3}\right) = \dfrac{\pi}{3}$

26. $\operatorname{arccot}\left(-\sqrt{3}\right) = \dfrac{5\pi}{6}$

27. $\operatorname{arccot}(-1) = \dfrac{3\pi}{4}$

28. $\operatorname{arccot}\left(-\dfrac{\sqrt{3}}{3}\right) = \dfrac{2\pi}{3}$

29. $\operatorname{arccot}(0) = \dfrac{\pi}{2}$

30. $\operatorname{arccot}\left(\dfrac{\sqrt{3}}{3}\right) = \dfrac{\pi}{3}$

31. $\operatorname{arccot}(1) = \dfrac{\pi}{4}$

32. $\operatorname{arccot}\left(\sqrt{3}\right) = \dfrac{\pi}{6}$

33. $\operatorname{arcsec}(2) = \dfrac{\pi}{3}$

34. $\operatorname{arccsc}(2) = \frac{\pi}{6}$

35. $\operatorname{arcsec}\left(\sqrt{2}\right) = \frac{\pi}{4}$

36. $\operatorname{arccsc}\left(\sqrt{2}\right) = \frac{\pi}{4}$

37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$

38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{3}$

39. $\operatorname{arcsec}(1) = 0$

40. $\operatorname{arccsc}(1) = \frac{\pi}{2}$

41. $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$

42. $\operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$

43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$

44. $\operatorname{arcsec}(-1) = \pi$

45. $\operatorname{arccsc}(-2) = -\frac{\pi}{6}$

46. $\operatorname{arccsc}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$

48. $\operatorname{arccsc}(-1) = -\frac{\pi}{2}$

49. $\operatorname{arcsec}(-2) = \frac{4\pi}{3}$

50. $\operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

51. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{7\pi}{6}$

52. $\operatorname{arcsec}(-1) = \pi$

53. $\operatorname{arccsc}(-2) = \frac{7\pi}{6}$

54. $\operatorname{arccsc}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

55. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{4\pi}{3}$

56. $\operatorname{arccsc}(-1) = \frac{3\pi}{2}$

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right) = -\frac{\sqrt{2}}{2}$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

60. $\sin(\arcsin(-0.42)) = -0.42$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$ is undefined.

62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\sqrt{2}}{2}$

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right) = \frac{5}{13}$

65. $\cos(\arccos(-0.998)) = -0.998$

66. $\cos(\arccos(\pi))$ is undefined.

67. $\tan(\arctan(-1)) = -1$

68. $\tan\left(\arctan\left(\sqrt{3}\right)\right) = \sqrt{3}$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{12}$

70. $\tan(\arctan(0.965)) = 0.965$

71. $\tan(\arctan(3\pi)) = 3\pi$

72. $\cot(\operatorname{arccot}(1)) = 1$

73. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right) = -\sqrt{3}$

74. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right) = -\frac{7}{24}$

75. $\cot(\operatorname{arccot}(-0.001)) = -0.001$

76. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right) = \frac{17\pi}{4}$

77. $\sec(\operatorname{arcsec}(2)) = 2$

78. $\sec(\operatorname{arcsec}(-1)) = -1$

79. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$ is undefined.

80. $\sec(\operatorname{arcsec}(0.75))$ is undefined.

81. $\sec(\operatorname{arcsec}(117\pi)) = 117\pi$

82. $\csc\left(\operatorname{arccsc}\left(\sqrt{2}\right)\right) = \sqrt{2}$

83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right) = -\frac{2\sqrt{3}}{3}$

84. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$ is undefined.

85. $\csc(\operatorname{arccsc}(1.0001)) = 1.0001$

86. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$ is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

99. $\arctan(\tan(\pi)) = 0$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$ is undefined

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right) = -\frac{\pi}{3}$

102. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right) = \frac{3\pi}{4}$

104. $\operatorname{arccot}(\cot(\pi))$ is undefined

105. $\text{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right)=\frac{\pi}{2}$

106. $\text{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)=\frac{2\pi}{3}$

107. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$

108. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)=\frac{2\pi}{3}$

109. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)=\frac{5\pi}{6}$

110. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

111. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)=\frac{\pi}{3}$

112. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$

113. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)=-\frac{\pi}{4}$

114. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{3}$

115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)=-\frac{\pi}{2}$

116. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)=-\frac{\pi}{6}$

117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)=\frac{11\pi}{12}$

118. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)=-\frac{\pi}{8}$

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$

120. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)=\frac{4\pi}{3}$

121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)=\frac{7\pi}{6}$

122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)=\frac{\pi}{3}$

124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$

125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)=\frac{5\pi}{4}$

126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{3}$

127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)=\frac{3\pi}{2}$

128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)=\frac{7\pi}{6}$

129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)=\frac{13\pi}{12}$

130. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)=\frac{9\pi}{8}$

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)=\frac{\sqrt{3}}{2}$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)=\frac{4}{5}$

133. $\sin(\arctan(-2))=-\frac{2\sqrt{5}}{5}$

134. $\sin\left(\text{arccot}\left(\sqrt{5}\right)\right)=\frac{\sqrt{6}}{6}$

135. $\sin(\text{arccsc}(-3))=-\frac{1}{3}$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)=\frac{12}{13}$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right) = \frac{\sqrt{2}}{4}$

138. $\cos(\operatorname{arccot}(3)) = \frac{3\sqrt{10}}{10}$

139. $\cos(\operatorname{arcsec}(5)) = \frac{1}{5}$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right) = -2$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$

142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right) = \frac{4}{3}$

143. $\tan(\operatorname{arccot}(12)) = \frac{1}{12}$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right) = \frac{5}{12}$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}$

146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right) = 2$

147. $\cot(\arctan(0.25)) = 4$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{2\sqrt{3}}{3}$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right) = \frac{13}{5}$

150. $\sec(\arctan(10)) = \sqrt{101}$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right) = -\sqrt{11}$

152. $\csc(\operatorname{arccot}(9)) = \sqrt{82}$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{5}{3}$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{13}}{2}$

155. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right) = \frac{17\sqrt{2}}{26}$

156. $\cos(\operatorname{arcsec}(3) + \arctan(2)) = \frac{\sqrt{5} - 4\sqrt{10}}{15}$

157. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right) = \frac{1}{3}$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right) = -\frac{24}{25}$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right) = \frac{120}{169}$

160. $\sin(2\arctan(2)) = \frac{4}{5}$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right) = \frac{7}{25}$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right) = -\frac{527}{625}$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right) = \frac{2}{3}$

164. $\sin\left(\frac{\arctan(2)}{2}\right) = \sqrt{\frac{5-\sqrt{5}}{10}}$

165. $f(x) = \sin(\arccos(x)) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$

166. $f(x) = \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$ for all x

167. $f(x) = \tan(\arcsin(x)) = \dfrac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

168. $f(x) = \sec(\arctan(x)) = \sqrt{1+x^2}$ for all x

169. $f(x) = \csc(\arccos(x)) = \dfrac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

170. $f(x) = \sin(2\arctan(x)) = \dfrac{2x}{x^2+1}$ for all x

171. $f(x) = \sin(2\arccos(x)) = 2x\sqrt{1-x^2}$ for $-1 \leq x \leq 1$

172. $f(x) = \cos(2\arctan(x)) = \dfrac{1-x^2}{1+x^2}$ for all x

173. $f(x) = \sin(\arccos(2x)) = \sqrt{1-4x^2}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

174. $f(x) = \sin\left(\arccos\left(\dfrac{x}{5}\right)\right) = \dfrac{\sqrt{25-x^2}}{5}$ for $-5 \leq x \leq 5$

175. $f(x) = \cos\left(\arcsin\left(\dfrac{x}{2}\right)\right) = \dfrac{\sqrt{4-x^2}}{2}$ for $-2 \leq x \leq 2$

176. $f(x) = \cos(\arctan(3x)) = \dfrac{1}{\sqrt{1+9x^2}}$ for all x

177. $f(x) = \sin(2\arcsin(7x)) = 14x\sqrt{1-49x^2}$ for $-\dfrac{1}{7} \leq x \leq \dfrac{1}{7}$

178. $f(x) = \sin\left(2\arcsin\left(\dfrac{x\sqrt{3}}{3}\right)\right) = \dfrac{2x\sqrt{3-x^2}}{3}$ for $-\sqrt{3} \leq x \leq \sqrt{3}$

179. $f(x) = \cos(2\arcsin(4x)) = 1-32x^2$ for $-\dfrac{1}{4} \leq x \leq \dfrac{1}{4}$

180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x)) = 2x\sqrt{1+4x^2}$ for all x

181. $f(x) = \sin(\arcsin(x) + \arccos(x)) = 1$ for $-1 \leq x \leq 1$

182. $f(x) = \cos(\arcsin(x) + \arctan(x)) = \dfrac{\sqrt{1-x^2}-x^2}{\sqrt{1+x^2}}$ for $-1 \leq x \leq 1$

183.[13] $f(x) = \tan(2\arcsin(x)) = \dfrac{2x\sqrt{1-x^2}}{1-2x^2}$ for x in $\left(-1, -\dfrac{\sqrt{2}}{2}\right) \cup \left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right) \cup \left(\dfrac{\sqrt{2}}{2}, 1\right)$

[13] The equivalence for $x = \pm 1$ can be verified independently of the derivation of the formula, but Calculus is required to fully understand what is happening at those x values. You'll see what we mean when you work through the details of the identity for $\tan(2t)$. For now, we exclude $x = \pm 1$ from our answer.

184. $f(x) = \sin\left(\frac{1}{2}\arctan(x)\right) = \begin{cases} \sqrt{\dfrac{\sqrt{x^2+1}-1}{2\sqrt{x^2+1}}} & \text{for } x \geq 0 \\ -\sqrt{\dfrac{\sqrt{x^2+1}-1}{2\sqrt{x^2+1}}} & \text{for } x < 0 \end{cases}$

185. $\theta + \sin(2\theta) = \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2}$

186. $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta) = \frac{1}{2}\arctan\left(\frac{x}{7}\right) - \frac{7x}{x^2+49}$

187. $4\tan(\theta) - 4\theta = \sqrt{x^2-16} - 4\operatorname{arcsec}\left(\frac{x}{4}\right)$

188. $\left[-\frac{1}{5}, \frac{1}{5}\right]$

189. $\left[-\frac{1}{3}, 1\right]$

190. $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

191. $(-\infty, -\sqrt{5}] \cup [-\sqrt{3}, \sqrt{3}] \cup [\sqrt{5}, \infty)$

192. $(-\infty, \infty)$

193. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

194. $\left(\frac{1}{2}, \infty\right)$

195. $\left[\frac{1}{2}, \infty\right)$

196. $\left(-\infty, -\frac{1}{12}\right] \cup \left[\frac{1}{12}, \infty\right)$

197. $(-\infty, -6] \cup [-4, \infty)$

198. $(-\infty, -2] \cup [2, \infty)$

199. $[0, \infty)$

A.1.8 Chapter 8 Answers

Section 8.1 Answers

1. $\csc(\theta) = \sqrt{5}$.
2. $\cos(\theta) = -\frac{1}{4}$.
3. $\cot(t) = \frac{1}{3}$.
4. $\sin(\theta) = -\frac{12}{13}$.
5. $\sec(\theta) = -\sqrt{5}$.
6. $\csc(t) = \sqrt{5}$.
7. $\tan(\theta) = -2\sqrt{2}$.
8. $\cos(\theta) = -\frac{\sqrt{5}}{3}$.
9. $\cos(t) \approx 0.9075$.
10. $\tan(\theta) \approx -0.6074$.
11. $\csc(t) \approx -4.079$.
12. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$
13. $\sin(\theta) = -\frac{12}{13}, \cos(\theta) = -\frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = -\frac{13}{12}, \sec(\theta) = -\frac{13}{5}, \cot(\theta) = \frac{5}{12}$
14. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$
15. $\sin(\theta) = \frac{-4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = -4\sqrt{3}, \csc(\theta) = -\frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = -\frac{\sqrt{3}}{12}$
16. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$
17. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = -\frac{23\sqrt{530}}{530}, \tan(\theta) = -\frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = -\frac{\sqrt{530}}{23}, \cot(\theta) = -23$
18. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$
19. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = -\frac{1}{4}, \tan(\theta) = -\sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = -4, \cot(\theta) = -\frac{\sqrt{15}}{15}$
20. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$
21. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$
22. $\sin(t) = \frac{\sqrt{5}}{5}, \cos(t) = \frac{2\sqrt{5}}{5}, \tan(t) = \frac{1}{2}, \csc(t) = \sqrt{5}, \sec(t) = \frac{\sqrt{5}}{2}, \cot(t) = 2$
23. $\sin(t) = \frac{1}{5}, \cos(t) = -\frac{2\sqrt{6}}{5}, \tan(t) = -\frac{\sqrt{6}}{12}, \csc(t) = 5, \sec(t) = -\frac{5\sqrt{6}}{12}, \cot(t) = -2\sqrt{6}$
24. $\sin(t) = -\frac{\sqrt{110}}{11}, \cos(t) = -\frac{\sqrt{11}}{11}, \tan(t) = \sqrt{10}, \csc(t) = -\frac{\sqrt{110}}{10}, \sec(t) = -\sqrt{11}, \cot(t) = \frac{\sqrt{10}}{10}$
25. $\sin(t) = -\frac{\sqrt{95}}{10}, \cos(t) = \frac{\sqrt{5}}{10}, \tan(t) = -\sqrt{19}, \csc(t) = -\frac{2\sqrt{95}}{19}, \sec(t) = 2\sqrt{5}, \cot(t) = -\frac{\sqrt{19}}{19}$
26. No, Skippy is not correct. In order to be an identity, an equation must hold for *all* applicable angles. For example, $\cos(\theta) + \sin(\theta) = 1$ does not hold when $\theta = \pi$.

Section 8.2 Answers

7. $\cos(75^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$
8. $\sec(165^\circ) = -\frac{4}{\sqrt{2}+\sqrt{6}} = \sqrt{2} - \sqrt{6}$

9. $\sin(105^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$
10. $\csc(195^\circ) = \frac{4}{\sqrt{2}-\sqrt{6}} = -(\sqrt{2}+\sqrt{6})$
11. $\cot(255^\circ) = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$
12. $\tan(375^\circ) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$
13. $\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{6}+\sqrt{2}}{4}$
14. $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$
15. $\tan\left(\frac{13\pi}{12}\right) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$
16. $\cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$
17. $\tan\left(\frac{17\pi}{12}\right) = 2+\sqrt{3}$
18. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$
19. $\cot\left(\frac{11\pi}{12}\right) = -(2+\sqrt{3})$
20. $\csc\left(\frac{5\pi}{12}\right) = \sqrt{6}-\sqrt{2}$
21. $\sec\left(-\frac{\pi}{12}\right) = \sqrt{6}-\sqrt{2}$
22. (a) $\cos(\alpha+\beta) = -\frac{\sqrt{2}}{10}$ (b) $\sin(\alpha+\beta) = \frac{7\sqrt{2}}{10}$
    (c) $\tan(\alpha+\beta) = -7$ (d) $\cos(\alpha-\beta) = -\frac{\sqrt{2}}{2}$
    (e) $\sin(\alpha-\beta) = \frac{\sqrt{2}}{2}$ (f) $\tan(\alpha-\beta) = -1$
23. (a) $\cos(\alpha+\beta) = -\frac{4+7\sqrt{2}}{30}$ (b) $\sin(\alpha+\beta) = \frac{28-\sqrt{2}}{30}$
    (c) $\tan(\alpha+\beta) = \frac{-28+\sqrt{2}}{4+7\sqrt{2}} = \frac{63-100\sqrt{2}}{41}$ (d) $\cos(\alpha-\beta) = \frac{-4+7\sqrt{2}}{30}$
    (e) $\sin(\alpha-\beta) = -\frac{28+\sqrt{2}}{30}$ (f) $\tan(\alpha-\beta) = \frac{28+\sqrt{2}}{4-7\sqrt{2}} = -\frac{63+100\sqrt{2}}{41}$
24. (a) $\sin(\alpha+\beta) = \frac{16}{65}$ (b) $\cos(\alpha-\beta) = \frac{33}{65}$ (c) $\tan(\alpha-\beta) = \frac{56}{33}$
25. (a) $\csc(\alpha-\beta) = -\frac{5}{4}$ (b) $\sec(\alpha+\beta) = \frac{125}{117}$ (c) $\cot(\alpha+\beta) = \frac{117}{44}$
26. $f(t) = \sqrt{2}\sin(t)+\sqrt{2}\cos(t)+1 = 2\sin\left(t+\frac{\pi}{4}\right)+1 = 2\cos\left(t+\frac{7\pi}{4}\right)+1$
27. $f(t) = 3\sqrt{3}\sin(3t)-3\cos(3t) = 6\sin\left(3t+\frac{11\pi}{6}\right) = 6\cos\left(3t+\frac{4\pi}{3}\right)$
28. $f(t) = -\sin(t)+\cos(t)-2 = \sqrt{2}\sin\left(t+\frac{3\pi}{4}\right)-2 = \sqrt{2}\cos\left(t+\frac{\pi}{4}\right)-2$
29. $f(t) = -\frac{1}{2}\sin(2t)-\frac{\sqrt{3}}{2}\cos(2t) = \sin\left(2t+\frac{4\pi}{3}\right) = \cos\left(2t+\frac{5\pi}{6}\right)$
30. $f(t) = 2\sqrt{3}\cos(t)-2\sin(t) = 4\sin\left(t+\frac{2\pi}{3}\right) = 4\cos\left(t+\frac{\pi}{6}\right)$
31. $f(t) = \frac{3}{2}\cos(2t)-\frac{3\sqrt{3}}{2}\sin(2t)+6 = 3\sin\left(2t+\frac{5\pi}{6}\right)+6 = 3\cos\left(2t+\frac{\pi}{3}\right)+6$
32. $f(t) = -\frac{1}{2}\cos(5t)-\frac{\sqrt{3}}{2}\sin(5t) = \sin\left(5t+\frac{7\pi}{6}\right) = \cos\left(5t+\frac{2\pi}{3}\right)$
33. $f(t) = -6\sqrt{3}\cos(3t)-6\sin(3t)-3 = 12\sin\left(3t+\frac{4\pi}{3}\right)-3 = 12\cos\left(3t+\frac{5\pi}{6}\right)-3$
34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t)-\frac{5\sqrt{2}}{2}\cos(t) = 5\sin\left(t+\frac{7\pi}{4}\right) = 5\cos\left(t+\frac{5\pi}{4}\right)$
35. $f(t) = 3\sin\left(\frac{t}{6}\right)-3\sqrt{3}\cos\left(\frac{t}{6}\right) = 6\sin\left(\frac{t}{6}+\frac{5\pi}{3}\right) = 6\cos\left(\frac{t}{6}+\frac{7\pi}{6}\right)$

54. $\cos(75^\circ) = \frac{\sqrt{2-\sqrt{3}}}{2}$

55. $\sin(105^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2}$

56. $\cos(67.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

57. $\sin(157.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

58. $\tan(112.5^\circ) = -\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = -1-\sqrt{2}$

59. $\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{2-\sqrt{3}}}{2}$

60. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$

61. $\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

62. $\sin\left(\frac{5\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

63. $\tan\left(\frac{7\pi}{8}\right) = -\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = 1-\sqrt{2}$

64.
- $\sin(2\theta) = -\frac{336}{625}$
- $\cos(2\theta) = \frac{527}{625}$
- $\tan(2\theta) = -\frac{336}{527}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{7\sqrt{2}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{7}$

65.
- $\sin(2\theta) = \frac{2520}{2809}$
- $\cos(2\theta) = -\frac{1241}{2809}$
- $\tan(2\theta) = -\frac{2520}{1241}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{106}}{106}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{9\sqrt{106}}{106}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{5}{9}$

66.
- $\sin(2\theta) = \frac{120}{169}$
- $\cos(2\theta) = -\frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{3\sqrt{13}}{13}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{2\sqrt{13}}{13}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{3}{2}$

67.
- $\sin(2\theta) = -\frac{\sqrt{15}}{8}$
- $\cos(2\theta) = \frac{7}{8}$
- $\tan(2\theta) = -\frac{\sqrt{15}}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{8+2\sqrt{15}}}{4}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{8-2\sqrt{15}}}{4}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}}$
  $\tan\left(\frac{\theta}{2}\right) = 4+\sqrt{15}$

68.
- $\sin(2\theta) = \frac{24}{25}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{5}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$

69.
- $\sin(2\theta) = \frac{24}{25}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{5}}{5}$
- $\tan\left(\frac{\theta}{2}\right) = -2$

70.
- $\sin(2\theta) = -\frac{120}{169}$
- $\cos(2\theta) = \frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{5\sqrt{26}}{26}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{5}$

71.
- $\sin(2\theta) = -\frac{120}{169}$
- $\cos(2\theta) = \frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{26}}{26}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\tan\left(\frac{\theta}{2}\right) = 5$

72.
- $\sin(2\theta) = -\frac{4}{5}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\tan(2\theta) = \frac{4}{3}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5-\sqrt{5}}{5+\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5-5\sqrt{5}}{10}$

73.
- $\sin(2\theta) = -\frac{4}{5}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\tan(2\theta) = \frac{4}{3}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5+5\sqrt{5}}{10}$

93. $\tan(t) = \frac{x}{\sqrt{1-x^2}}$

94. $\sec(\theta) = \sqrt{1+x^2}$

95. $\tan(\theta) = -\sqrt{x^2-1}$

96. $\cos(2t) = 1 - \dfrac{x^2}{2}$

97. $\sin(2\theta) = \dfrac{14x}{x^2+49}$

98. $\ln|\sec(t)+\tan(t)| = \ln|x+\sqrt{x^2+16}| - \ln(4)$

110. $\dfrac{\cos(2\theta)+\cos(8\theta)}{2}$

111. $\dfrac{\cos(5t)-\cos(9t)}{2}$

112. $\dfrac{\sin(8x)+\sin(10x)}{2}$

113. $\dfrac{\cos(4\theta)+\cos(8\theta)}{2}$

114. $\dfrac{\cos(t)-\cos(5t)}{2}$

115. $\dfrac{\sin(2x)+\sin(4x)}{2}$

116. $2\cos(4\theta)\cos(\theta)$

117. $-2\cos\left(\frac{9}{2}t\right)\sin\left(\frac{5}{2}t\right)$

118. $2\sin\left(\frac{11}{2}x\right)\sin\left(\frac{1}{2}x\right)$

119. $2\cos(4\theta)\sin(5\theta)$

120. $\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)$

121. $-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

122. $f(t) = [2\cos(t)]\cos(4t)$, $A(t) = 2\cos(t)$, wave-envelope: $y = \pm 2\cos(t)$.

123. $f(t) = \left[6\sin\left(\frac{1}{2}t\right)\right]\sin\left(\frac{11}{2}t\right)$, $A(t) = 6\sin\left(\frac{1}{2}t\right)$, wave-envelope: $y = \pm 6\sin\left(\frac{1}{2}t\right)$.

124. $f(t) = [\cos(4t)]\sin(5t)$, $A(t) = \cos(4t)$, wave-envelope: $y = \pm\cos(4t)$.

125. $f(t) = \left[-\frac{4}{3}\sin\left(\frac{5}{2}t\right)\right]\cos\left(\frac{9}{2}t\right)$, $A(t) = -\frac{4}{3}\sin\left(\frac{5}{2}t\right)$, wave-envelope: $y = \pm\frac{4}{3}\sin\left(\frac{5}{2}t\right)$.

Section 8.3 Answers

1. $\theta = \arcsin\left(\dfrac{7}{11}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\dfrac{7}{11}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.6898, 2.4518$
2. $\theta = \arccos\left(-\dfrac{2}{9}\right) + 2\pi k$ or $\theta = -\arccos\left(-\dfrac{2}{9}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.7949, 4.4883$
3. $\theta = \pi + \arcsin(0.569) + 2\pi k$ or $\theta = 2\pi - \arcsin(0.569) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 3.7469, 5.6779$
4. $\theta = \arccos(0.117) + 2\pi k$ or $\theta = 2\pi - \arccos(0.117) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.4535, 4.8297$
5. $\theta = \arcsin(0.008) + 2\pi k$ or $\theta = \pi - \arcsin(0.008) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0080, 3.1336$
6. $\theta = \arccos\left(\dfrac{359}{360}\right) + 2\pi k$ or $\theta = 2\pi - \arccos\left(\dfrac{359}{360}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0746, 6.2086$
7. $t = \arctan(117) + \pi k$, in $[0, 2\pi)$, $t \approx 1.56225, 4.70384$

8. $t = \arctan\left(-\frac{1}{12}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 3.0585, 6.2000$

9. $t = \arccos\left(\frac{2}{3}\right) + 2\pi k$ or $t = 2\pi - \arccos\left(\frac{2}{3}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.8411, 5.4422$

10. $t = \pi + \arcsin\left(\frac{17}{90}\right) + 2\pi k$ or $t = 2\pi - \arcsin\left(\frac{17}{90}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 3.3316, 6.0932$

11. $t = \arctan\left(-\sqrt{10}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 1.8771, 5.0187$

12. $t = \arcsin\left(\frac{3}{8}\right) + 2\pi k$ or $t = \pi - \arcsin\left(\frac{3}{8}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.3844, 2.7572$

13. $x = \arccos\left(-\frac{7}{16}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{7}{16}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.0236, 4.2596$

14. $x = \arctan(0.03) + \pi k$, in $[0, 2\pi)$, $x \approx 0.0300, 3.1716$

15. $x = \arcsin(0.3502) + 2\pi k$ or $x = \pi - \arcsin(0.3502) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.3578, 2.784$

16. $x = \pi + \arcsin(0.721) + 2\pi k$ or $x = 2\pi - \arcsin(0.721) + 2\pi k$, in $[0, 2\pi)$, $x \approx 3.9468, 5.4780$

17. $x = \arccos(0.9824) + 2\pi k$ or $x = 2\pi - \arccos(0.9824) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.1879, 6.0953$

18. $x = \arccos(-0.5637) + 2\pi k$ or $x = -\arccos(-0.5637) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.1697, 4.1135$

19. $x = \arctan(117) + \pi k$, in $[0, 2\pi)$, $x \approx 1.5622, 4.7038$

20. $x = \arctan(-0.6109) + \pi k$, in $[0, 2\pi)$, $x \approx 2.5932, 5.7348$

21. $\theta = \frac{\pi k}{5}$; $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}$

22. $t = \frac{\pi}{9} + \frac{2\pi k}{3}$ or $t = \frac{5\pi}{9} + \frac{2\pi k}{3}$; $t = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

23. $x = \frac{2\pi}{3} + \pi k$ or $x = \frac{5\pi}{6} + \pi k$; $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

24. $\theta = \frac{\pi}{24} + \frac{\pi k}{6}$; $\theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{41\pi}{24}, \frac{15\pi}{8}$

25. $t = \frac{3\pi}{8} + \frac{\pi k}{2}$; $t = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

26. $x = \frac{\pi}{12} + \frac{2\pi k}{3}$ or $x = \frac{7\pi}{12} + \frac{2\pi k}{3}$; $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$

27. $\theta = \frac{\pi}{3} + \frac{\pi k}{2}$; $\theta = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

28. No solution
29. $x = \frac{3\pi}{4} + 6\pi k$ or $x = \frac{9\pi}{4} + 6\pi k$; $x = \frac{3\pi}{4}$
30. $\theta = -\frac{\pi}{3} + \pi k$; $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$
31. $t = \frac{3\pi}{4} + \pi k$ or $t = \frac{13\pi}{12} + \pi k$; $t = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$
32. $x = -\frac{19\pi}{12} + 2\pi k$ or $x = \frac{\pi}{12} + 2\pi k$; $x = \frac{\pi}{12}, \frac{5\pi}{12}$
33. No solution
34. $t = \frac{5\pi}{8} + \frac{\pi k}{2}$; $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
35. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
36. $\theta = \frac{\pi}{6} + \pi k$ or $\theta = \frac{5\pi}{6} + \pi k$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
37. $t = \frac{\pi}{4} + \frac{\pi k}{2}$; $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
38. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
39. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$
40. $t = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$
41. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$
42. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
43. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$
44. $x = \frac{\pi}{3}, \frac{5\pi}{3}$
45. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \arccos\left(\frac{1}{3}\right), 2\pi - \arccos\left(\frac{1}{3}\right)$
46. $t = \frac{\pi}{6}, \frac{5\pi}{6}$
47. $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)$
48. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \arctan\left(\frac{1}{2}\right), \pi + \arctan\left(\frac{1}{2}\right)$
49. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$
50. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$
51. $\theta = \arctan(2), \pi + \arctan(2)$
52. $t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

53. $x = 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

54. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$

55. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

56. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

57. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

58. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

59. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

60. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

61. $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

62. No solution

63. $\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{8\pi}{7}, \frac{9\pi}{7}, \frac{10\pi}{7}, \frac{11\pi}{7}, \frac{12\pi}{7}, \frac{13\pi}{7}$

64. $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

65. $x = 0$

66. $\theta = \frac{\pi}{48}, \frac{11\pi}{48}, \frac{13\pi}{48}, \frac{23\pi}{48}, \frac{25\pi}{48}, \frac{35\pi}{48}, \frac{37\pi}{48}, \frac{47\pi}{48}, \frac{49\pi}{48}, \frac{59\pi}{48}, \frac{61\pi}{48}, \frac{71\pi}{48}, \frac{73\pi}{48}, \frac{83\pi}{48}, \frac{85\pi}{48}, \frac{95\pi}{48}$

67. $t = 0, \frac{\pi}{2}$

68. $x = \frac{\pi}{2}, \frac{11\pi}{6}$

69. $\theta = \frac{\pi}{12}, \frac{17\pi}{12}$

70. $t = 0, \pi, \frac{\pi}{3}, \frac{4\pi}{3}$

71. $x = \frac{17\pi}{24}, \frac{41\pi}{24}, \frac{23\pi}{24}, \frac{47\pi}{24}$

72. $\theta = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{5\pi}{6}, \frac{17\pi}{18}, \frac{3\pi}{2}, \frac{29\pi}{18}$

73. $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

74. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

75. $\theta = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

76. $t = \frac{\pi}{7}, \frac{\pi}{3}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{5\pi}{3}, \frac{13\pi}{7}$

77. $x = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

78. $x = \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 0.6662, \pi - \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 2.4754$

79. $x = -\frac{1}{2}$

80. $t = -1$

81. $x = \frac{2}{3}$

82. $t = -\frac{\sqrt{3}}{2}$

83. $x = 2\sqrt{2}$

84. $t = 6$

85. $x = \pm\frac{\sqrt{3}}{2}$

86. $t = \frac{1}{2}$

87. $x = -1, 0$

88. $t = -\sqrt{3}$

90. (a) $k = 5\,\frac{\text{lbs.}}{\text{ft.}}$ and $m = \frac{5}{16}$ slugs
   (b) $x(t) = \sin\left(4t + \frac{\pi}{2}\right)$. The object first passes through the equilibrium point when $t = \frac{\pi}{8} \approx 0.39$ seconds after the motion starts. At this time, the object is heading upwards.
   (c) $x(t) = \frac{\sqrt{2}}{2}\sin\left(4t + \frac{7\pi}{4}\right)$. The object passes through the equilibrium point heading downwards for the third time when $t = \frac{17\pi}{16} \approx 3.34$ seconds.

Section 8.4 Answers

1. $\alpha = 13^\circ$ $\beta = 17^\circ$ $\gamma = 150^\circ$
   $a = 5$ $b \approx 6.50$ $c \approx 11.11$

2. $\alpha = 73.2^\circ$ $\beta = 54.1^\circ$ $\gamma = 52.7^\circ$
   $a = 117$ $b \approx 99.00$ $c \approx 97.22$

3. Information does not produce a triangle

4. $\alpha = 95^\circ$ $\beta = 62^\circ$ $\gamma = 23^\circ$
   $a = 33.33$ $b \approx 29.54$ $c \approx 13.07$

5. Information does not produce a triangle

6. $\alpha = 117^\circ$ $\beta \approx 56.3^\circ$ $\gamma \approx 6.7^\circ$
   $a = 45$ $b = 42$ $c \approx 5.89$

7. $\alpha = 68.7^\circ$ $\beta \approx 76.9^\circ$ $\gamma \approx 34.4^\circ$
   $a = 88$ $b = 92$ $c \approx 53.36$

   $\alpha = 68.7^\circ$ $\beta \approx 103.1^\circ$ $\gamma \approx 8.2^\circ$
   $a = 88$ $b = 92$ $c \approx 13.47$

8. $\alpha = 42^\circ$ $\beta \approx 67.66^\circ$ $\gamma \approx 70.34^\circ$
   $a = 17$ $b = 23.5$ $c \approx 23.93$

   $\alpha = 42^\circ$ $\beta \approx 112.34^\circ$ $\gamma \approx 25.66^\circ$
   $a = 17$ $b = 23.5$ $c \approx 11.00$

9. Information does not produce a triangle

10. $\alpha = 30^\circ$ $\beta = 90^\circ$ $\gamma = 60^\circ$
    $a = 7$ $b = 14$ $c = 7\sqrt{3}$

11. $\alpha = 42^\circ$ $\beta \approx 23.78^\circ$ $\gamma \approx 114.22^\circ$
    $a = 39$ $b = 23.5$ $c \approx 53.15$

12. $\alpha = 53^\circ$ $\beta = 74^\circ$ $\gamma = 53^\circ$
    $a = 28.01$ $b \approx 33.71$ $c = 28.01$

13. $\alpha = 6^\circ$ $\beta \approx 169.43^\circ$ $\gamma \approx 4.57^\circ$
    $a = 57$ $b = 100$ $c \approx 43.45$

    $\alpha = 6^\circ$ $\beta \approx 10.57^\circ$ $\gamma \approx 163.43^\circ$
    $a = 57$ $b = 100$ $c \approx 155.51$

14. $\alpha \approx 78.59^\circ$ $\beta \approx 26.81^\circ$ $\gamma = 74.6^\circ$
    $a = 3.05$ $b \approx 1.40$ $c = 3$

    $\alpha \approx 101.41^\circ$ $\beta \approx 3.99^\circ$ $\gamma = 74.6^\circ$
    $a = 3.05$ $b \approx 0.217$ $c = 3$

15. $\alpha \approx 28.61^\circ$ $\beta = 102^\circ$ $\gamma \approx 49.39^\circ$
    $a \approx 8.20$ $b = 16.75$ $c = 13$

16. Information does not produce a triangle

17. $\alpha = 43^\circ$ $\beta = 102^\circ$ $\gamma = 35^\circ$
$a \approx 11.68$ $b = 16.75$ $c \approx 9.82$

18. $\alpha = 66.92^\circ$ $\beta = 29.13^\circ$ $\gamma = 83.95^\circ$
$a \approx 593.69$ $b = 314.15$ $c \approx 641.75$

19. Information does not produce a triangle

20. $\alpha = 50^\circ$ $\beta \approx 22.52^\circ$ $\gamma \approx 107.48^\circ$
$a = 25$ $b = 12.5$ $c \approx 31.13$

21. The area of the triangle from Exercise 1 is about 8.1 square units.
The area of the triangle from Exercise 12 is about 377.1 square units.
The area of the triangle from Exercise 20 is about 149 square units.

22. $\arctan\left(\frac{7}{100}\right) \approx 0.699$ radians, which is equivalent to 4.004°

23. About 17%

24. About 53 feet

25. (a) $\theta = 180^\circ$ (b) $\theta = 353^\circ$ (c) $\theta = 84.5^\circ$ (d) $\theta = 270^\circ$
(e) $\theta = 121.25^\circ$ (f) $\theta = 45^\circ$ (g) $\theta = 225^\circ$

26. The Colonel is about 3193 feet from the campfire.
Sarge is about 2525 feet to the campfire.

27. The distance from the Muffin Ridge Observatory to Sasquach Point is about 7.12 miles.
The distance from Sasquatch Point to the Chupacabra Trailhead is about 2.46 miles.

28. The SS Bigfoot is about 4.1 miles from the flare.
The HMS Sasquatch is about 2.9 miles from the flare.

29. Jeff is about 371 feet from the nest.

30. She is about 3.02 miles from the lodge

31. The boat is about 25.1 miles from the second tower.

32. The UFO is hovering about 9539 feet above the ground.

33. The gargoyle is about 44 feet from the observer on the upper floor.
The gargoyle is about 27 feet from the observer on the lower floor.
The gargoyle is about 25 feet from the other building.

Section 8.5 Answers

1. $\alpha \approx 35.54°$ $\beta \approx 85.16°$ $\gamma = 59.3°$
$a = 7$ $b = 12$ $c \approx 10.36$

2. $\alpha = 104°$ $\beta \approx 29.40°$ $\gamma \approx 46.60°$
$a \approx 49.41$ $b = 25$ $c = 37$

3. $\alpha \approx 85.90°$ $\beta = 8.2°$ $\gamma \approx 85.90°$
$a = 153$ $b \approx 21.88$ $c = 153$

4. $\alpha \approx 36.87°$ $\beta \approx 53.13°$ $\gamma = 90°$
$a = 3$ $b = 4$ $c = 5$

5. $\alpha = 120°$ $\beta \approx 25.28°$ $\gamma \approx 34.72°$
$a = \sqrt{37}$ $b = 3$ $c = 4$

6. $\alpha \approx 32.31°$ $\beta \approx 49.58°$ $\gamma \approx 98.21°$
$a = 7$ $b = 10$ $c = 13$

7. Information does not produce a triangle

8. $\alpha \approx 83.05°$ $\beta \approx 87.81°$ $\gamma \approx 9.14°$
$a = 300$ $b = 302$ $c = 48$

9. $\alpha = 60°$ $\beta = 60°$ $\gamma = 60°$
$a = 5$ $b = 5$ $c = 5$

10. $\alpha \approx 22.62°$ $\beta \approx 67.38°$ $\gamma = 90°$
$a = 5$ $b = 12$ $c = 13$

11. $\alpha = 63°$ $\beta \approx 98.11°$ $\gamma \approx 18.89°$
$a = 18$ $b = 20$ $c \approx 6.54$

$\alpha = 63°$ $\beta \approx 81.89°$ $\gamma \approx 35.11°$
$a = 18$ $b = 20$ $c \approx 11.62$

12. $\alpha \approx 55.30°$ $\beta \approx 89.40°$ $\gamma \approx 35.30°$
$a = 37$ $b = 45$ $c = 26$

13. Information does not produce a triangle

14. $\alpha = 63°$ $\beta \approx 54.1°$ $\gamma \approx 62.9°$
$a = 22$ $b = 20$ $c \approx 21.98$

15. $\alpha = 42°$ $\beta \approx 89.23°$ $\gamma \approx 48.77°$
$a \approx 78.30$ $b = 117$ $c = 88$

16. $\alpha \approx 3°$ $\beta = 7°$ $\gamma = 170°$
$a \approx 29.72$ $b \approx 69.2$ $c = 98.6$

17. The area of the triangle given in Exercise 6 is $\sqrt{1200} = 20\sqrt{3} \approx 34.64$ square units.
The area of the triangle given in Exercise 8 is $\sqrt{51764375} \approx 7194.75$ square units.
The area of the triangle given in Exercise 10 is exactly 30 square units.

18. The distance between the ends of the hands at four o'clock is about 8.26 inches.

19. The diameter of the crater is about 5.22 miles.

20. About 313 miles

21. N31.8°W

22. She is about 3.92 miles from the lodge and her bearing to the lodge is N37°E.

23. It is about 4.50 miles from port and its heading to port is S47°W.

24. It is about 229.61 miles from the island and the captain should set a course of N16.4°E to reach the island.

25. The fires are about 17456 feet apart. (Try to avoid rounding errors.)

A.1.9 Chapter 9 Answers

Section 9.1 Answers

1.
   - $\vec{v}+\vec{w}=\langle 15,-1\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=\sqrt{226}$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\langle -21,77\rangle$, vector
   - $\vec{w}-2\vec{v}=\langle -21,14\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=18$, scalar
   - $\|w\|\hat{v}=\left\langle \frac{60}{13},-\frac{25}{13}\right\rangle$, vector
2.
   - $\vec{v}+\vec{w}=\langle -12,12\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=12\sqrt{2}$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\langle -34,-612\rangle$, vector
   - $\vec{w}-2\vec{v}=\langle 9,-60\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=38$, scalar
   - $\|w\|\hat{v}=\left\langle -\frac{91}{25},\frac{312}{25}\right\rangle$, vector
3.
   - $\vec{v}+\vec{w}=\langle 0,3\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=3$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -6\sqrt{5},6\sqrt{5}\right\rangle$, vector
   - $\vec{w}-2\vec{v}=\langle -6,6\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=3\sqrt{5}$, scalar
   - $\|w\|\hat{v}=\langle 4,-2\rangle$, vector
4.
   - $\vec{v}+\vec{w}=\langle 8,9\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=\sqrt{145}$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -14\sqrt{29},6\sqrt{29}\right\rangle$, vector
   - $\vec{w}-2\vec{v}=\langle -22,-3\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=3\sqrt{29}$, scalar
   - $\|w\|\hat{v}=\langle 5,2\rangle$, vector
5.
   - $\vec{v}+\vec{w}=\left\langle \sqrt{3},3\right\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=2\sqrt{3}$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle 8\sqrt{3},0\right\rangle$, vector
   - $\vec{w}-2\vec{v}=\left\langle 4\sqrt{3},0\right\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=6$, scalar
   - $\|w\|\hat{v}=\left\langle -2\sqrt{3},2\right\rangle$, vector
6.
   - $\vec{v}+\vec{w}=\left\langle -\frac{1}{5},\frac{7}{5}\right\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=\sqrt{2}$, scalar
   - $\|\vec{v}\|\vec{w}-\|\vec{w}\|\vec{v}=\left\langle -\frac{7}{5},-\frac{1}{5}\right\rangle$, vector
   - $\vec{w}-2\vec{v}=\langle -2,-1\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=2$, scalar
   - $\|w\|\hat{v}=\left\langle \frac{3}{5},\frac{4}{5}\right\rangle$, vector
7.
   - $\vec{v}+\vec{w}=\langle 0,0\rangle$, vector
   - $\|\vec{v}+\vec{w}\|=0$, scalar
   - $\vec{w}-2\vec{v}=\left\langle -\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right\rangle$, vector
   - $\|\vec{v}\|+\|\vec{w}\|=2$, scalar

- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v} = \left\langle -\sqrt{2}, \sqrt{2} \right\rangle$, vector
- $\|w\|\hat{v} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$, vector

8.
- $\vec{v} + \vec{w} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$, vector
- $\vec{w} - 2\vec{v} = \left\langle -2, -2\sqrt{3} \right\rangle$, vector
- $\|\vec{v} + \vec{w}\| = 1$, scalar
- $\|\vec{v}\| + \|\vec{w}\| = 3$, scalar
- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v} = \left\langle -2, -2\sqrt{3} \right\rangle$, vector
- $\|w\|\hat{v} = \left\langle 1, \sqrt{3} \right\rangle$, vector

9.
- $\vec{v} + \vec{w} = \langle 3, 2 \rangle$, vector
- $\vec{w} - 2\vec{v} = \langle -6, -10 \rangle$, vector
- $\|\vec{v} + \vec{w}\| = \sqrt{13}$, scalar
- $\|\vec{v}\| + \|\vec{w}\| = 7$, scalar
- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v} = \langle -6, -18 \rangle$, vector
- $\|w\|\hat{v} = \left\langle \frac{6}{5}, \frac{8}{5} \right\rangle$, vector

10.
- $\vec{v} + \vec{w} = \langle 1, 0 \rangle$, vector
- $\vec{w} - 2\vec{v} = \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle$, vector
- $\|\vec{v} + \vec{w}\| = 1$, scalar
- $\|\vec{v}\| + \|\vec{w}\| = \sqrt{2}$, scalar
- $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v} = \left\langle 0, -\frac{\sqrt{2}}{2} \right\rangle$, vector
- $\|w\|\hat{v} = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$, vector

11. $\vec{v} = \left\langle 3, 3\sqrt{3} \right\rangle$
12. $\vec{v} = \left\langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right\rangle$
13. $\vec{v} = \left\langle \frac{\sqrt{3}}{3}, \frac{1}{3} \right\rangle$
14. $\vec{v} = \langle 0, 12 \rangle$
15. $\vec{v} = \left\langle -2\sqrt{3}, 2 \right\rangle$
16. $\vec{v} = \left\langle -\sqrt{3}, 3 \right\rangle$
17. $\vec{v} = \left\langle -\frac{7}{2}, 0 \right\rangle$
18. $\vec{v} = \left\langle -5\sqrt{3}, -5\sqrt{3} \right\rangle$
19. $\vec{v} = \langle 0, -6.25 \rangle$
20. $\vec{v} = \left\langle 6, -2\sqrt{3} \right\rangle$
21. $\vec{v} = \langle 5, -5 \rangle$
22. $\vec{v} = \langle 2, 4 \rangle$
23. $\vec{v} = \langle -1, 3 \rangle$
24. $\vec{v} = \langle -3, -4 \rangle$
25. $\vec{v} = \langle 24, -10 \rangle$
26. $\vec{v} \approx \langle -177.96, 349.27 \rangle$
27. $\vec{v} \approx \langle 12.96, 62.59 \rangle$
28. $\vec{v} \approx \langle 5164.62, 1097.77 \rangle$
29. $\vec{v} \approx \langle -386.73, -230.08 \rangle$
30. $\vec{v} \approx \langle -52.13, -160.44 \rangle$
31. $\vec{v} \approx \langle 14.73, -21.43 \rangle$
32. $\|\vec{v}\| = 2$, $\theta = 60^\circ$
33. $\|\vec{v}\| = 5\sqrt{2}$, $\theta = 45^\circ$
34. $\|\vec{v}\| = 4$, $\theta = 150^\circ$
35. $\|\vec{v}\| = 2$, $\theta = 135^\circ$
36. $\|\vec{v}\| = 1$, $\theta = 225^\circ$
37. $\|\vec{v}\| = 1$, $\theta = 240^\circ$
38. $\|\vec{v}\| = 6$, $\theta = 0^\circ$
39. $\|\vec{v}\| = 2.5$, $\theta = 180^\circ$
40. $\|\vec{v}\| = \sqrt{7}$, $\theta = 90^\circ$
41. $\|\vec{v}\| = 10$, $\theta = 270^\circ$
42. $\|\vec{v}\| = 5$,
$\theta = \arctan\left(\frac{4}{3}\right) \approx 53.13^\circ$
43. $\|\vec{v}\| = 13$,
$\theta = \arctan\left(\frac{5}{12}\right) \approx 22.62^\circ$

44. $\|\vec{v}\| = 5$,
$\theta = \arctan\left(-\frac{3}{4}\right) \approx 143.13^\circ$

45. $\|\vec{v}\| = 25$,
$\theta = \arctan\left(-\frac{24}{7}\right) \approx 106.26^\circ$

46. $\|\vec{v}\| = \sqrt{5}$,
$\theta = \arctan\left(\frac{1}{2}\right) + \pi \approx 206.57^\circ$

47. $\|\vec{v}\| = 2\sqrt{10}$,
$\theta = \arctan(3) + \pi \approx 251.57^\circ$

48. $\|\vec{v}\| = \sqrt{2}$,
$\theta = 45^\circ$

49. $\|\vec{v}\| = \sqrt{17}$,
$\theta = \arctan(-4) + \pi \approx 284.04^\circ$

50. $\|\vec{v}\| \approx 145.48$,
$\theta = \arctan\left(-\frac{77.05}{123.4}\right) + \pi \approx 328.02^\circ$

51. $\|\vec{v}\| \approx 1274.00$,
$\theta = \arctan\left(\frac{831.6}{965.15}\right) \approx 40.75^\circ$

52. $\|\vec{v}\| \approx 121.69$,
$\theta = \arctan\left(-\frac{42.3}{114.1}\right) \approx 159.66^\circ$

53. The boat's true speed is about 10 miles per hour at a heading of S50.6°W.

54. The HMS Sasquatch's true speed is about 41 miles per hour at a heading of S26.8°E.

55. She should maintain a speed of about 35 miles per hour at a heading of S11.8°E.

56. She should fly at 83.46 miles per hour with a heading of N22.1°E

57. The current is moving at about 10 miles per hour bearing N54.6°W.

58. The tension on each of the cables is about 346 pounds.

59. The maximum weight that can be held by the cables in that configuration is about 133 pounds.

60. The tension on the left hand cable is 285.317 lbs. and on the right hand cable is 92.705 lbs.

61. The weaker student should pull about 60 pounds. The net force on the keg is about 153 pounds.

62. The resultant force is only about 296 pounds so the couch doesn't budge. Even if it did move, the stronger force on the third rope would have made the couch drift slightly to the south as it traveled down the street.

Section 9.2 Answers

1. $\vec{v} = \langle -2, -7 \rangle$ and $\vec{w} = \langle 5, -9 \rangle$
$\vec{v} \cdot \vec{w} = 53$
$\theta = 45^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{5}{2}, -\frac{9}{2} \right\rangle$
$\vec{q} = \left\langle -\frac{9}{2}, -\frac{5}{2} \right\rangle$

2. $\vec{v} = \langle -6, -5 \rangle$ and $\vec{w} = \langle 10, -12 \rangle$
$\vec{v} \cdot \vec{w} = 0$
$\theta = 90^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
$\vec{q} = \langle -6, -5 \rangle$

3. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$ and $\vec{w} = \left\langle 1, -\sqrt{3} \right\rangle$
   $\vec{v} \cdot \vec{w} = -2$
   $\theta = 120^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$
   $\vec{q} = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle$

4. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle -6, -8 \rangle$
   $\vec{v} \cdot \vec{w} = -50$
   $\theta = 180^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 3, 4 \rangle$
   $\vec{q} = \langle 0, 0 \rangle$

5. $\vec{v} = \langle -2, 1 \rangle$ and $\vec{w} = \langle 3, 6 \rangle$
   $\vec{v} \cdot \vec{w} = 0$
   $\theta = 90^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
   $\vec{q} = \langle -2, 1 \rangle$

6. $\vec{v} = \left\langle -3\sqrt{3}, 3 \right\rangle$ and $\vec{w} = \left\langle -\sqrt{3}, -1 \right\rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta = 60^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{3\sqrt{3}}{2}, -\frac{3}{2} \right\rangle$
   $\vec{q} = \left\langle -\frac{3\sqrt{3}}{2}, \frac{9}{2} \right\rangle$

7. $\vec{v} = \langle 1, 17 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$
   $\vec{v} \cdot \vec{w} = -1$
   $\theta \approx 93.37^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 1, 0 \rangle$
   $\vec{q} = \langle 0, 17 \rangle$

8. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 5, 12 \rangle$
   $\vec{v} \cdot \vec{w} = 63$
   $\theta \approx 14.25^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle$
   $\vec{q} = \left\langle \frac{192}{169}, -\frac{80}{169} \right\rangle$

9. $\vec{v} = \langle -4, -2 \rangle$ and $\vec{w} = \langle 1, -5 \rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta \approx 74.74^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{3}{13}, -\frac{15}{13} \right\rangle$
   $\vec{q} = \left\langle -\frac{55}{13}, -\frac{11}{13} \right\rangle$

10. $\vec{v} = \langle -5, 6 \rangle$ and $\vec{w} = \langle 4, -7 \rangle$
    $\vec{v} \cdot \vec{w} = -62$
    $\theta \approx 169.94^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{248}{65}, \frac{434}{65} \right\rangle$
    $\vec{q} = \left\langle -\frac{77}{65}, -\frac{44}{65} \right\rangle$

11. $\vec{v} = \langle -8, 3 \rangle$ and $\vec{w} = \langle 2, 6 \rangle$
    $\vec{v} \cdot \vec{w} = 2$
    $\theta \approx 87.88^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1}{10}, \frac{3}{10} \right\rangle$
    $\vec{q} = \left\langle -\frac{81}{10}, \frac{27}{10} \right\rangle$

12. $\vec{v} = \langle 34, -91 \rangle$ and $\vec{w} = \langle 0, 1 \rangle$
    $\vec{v} \cdot \vec{w} = -91$
    $\theta \approx 159.51^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -91 \rangle$
    $\vec{q} = \langle 34, 0 \rangle$

13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
    $\vec{v} \cdot \vec{w} = -4$
    $\theta \approx 108.43^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -1 \rangle$
    $\vec{q} = \langle 3, 0 \rangle$

14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
$\vec{v} \cdot \vec{w} = -48$
$\theta \approx 163.74^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle -24, 0 \rangle$
$\vec{q} = \langle 0, 7 \rangle$

15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
$\vec{v} \cdot \vec{w} = 0$
$\theta = 90^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
$\vec{q} = \left\langle \frac{3}{2}, \frac{3}{2} \right\rangle$

16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
$\vec{v} \cdot \vec{w} = 33$
$\theta \approx 59.49^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{99}{25}, \frac{132}{25} \right\rangle$
$\vec{q} = \left\langle \frac{224}{25}, \frac{168}{25} \right\rangle$

17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{6}-\sqrt{2}}{4}$
$\theta = 75^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1-\sqrt{3}}{4}, \frac{\sqrt{3}-1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{1+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right\rangle$

18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{2}-\sqrt{6}}{4}$
$\theta = 105^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{2}-\sqrt{6}}{8}, \frac{3\sqrt{2}-\sqrt{6}}{8} \right\rangle$
$\vec{q} = \left\langle \frac{3\sqrt{2}+\sqrt{6}}{8}, \frac{\sqrt{2}+\sqrt{6}}{8} \right\rangle$

19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = -\frac{\sqrt{6}+\sqrt{2}}{4}$
$\theta = 165^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, \frac{\sqrt{3}+1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{\sqrt{3}-1}{4}, \frac{1-\sqrt{3}}{4} \right\rangle$

20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
$\vec{v} \cdot \vec{w} = \frac{\sqrt{6}+\sqrt{2}}{4}$
$\theta = 15^\circ$
$\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, -\frac{\sqrt{3}+1}{4} \right\rangle$
$\vec{q} = \left\langle \frac{1-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4} \right\rangle$

21. $(1500\,\text{pounds})(300\,\text{feet})\cos(0^\circ) = 450,000$ foot-pounds

22. $(10\,\text{pounds})(3\,\text{feet})\cos(0^\circ) = 30$ foot-pounds

23. $(13\,\text{pounds})(25\,\text{feet})\cos(15^\circ) \approx 313.92$ foot-pounds

24. $(100\,\text{pounds})(42\,\text{feet})\cos(13^\circ) \approx 4092.35$ foot-pounds

25. $(200\,\text{pounds})(10\,\text{feet})\cos(77.5^\circ) \approx 432.88$ foot-pounds