SUBCONVEXITY FOR TWISTED L-FUNCTIONS ON $GL(3) \times GL(2)$ AND $GL(3)$

A Dissertation

by

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ABSTRACT

Let ϕ be the symmetric-square lift of an $SL_2(\mathbb{Z})$ Hecke-Maass form. Let q be an odd cube-free positive integer, and let χ be a primitive Dirichlet character modulo q such that χ is not quadratic. Let f be an even Hecke-normalized Hecke-Maass newform of level dividing q, central character $\overline{\chi}^2$, and spectral parameter t_f. In this thesis, we show the following subconvexity bounds for twisted L-functions on $GL(3) \times GL(2)$ and $GL(3)$:

$$\begin{aligned} L\left(\frac{1}{2}, \phi \times f \times \chi\right) &\ll_{\phi, t_f, \epsilon} q^{\frac{5}{4}+\epsilon}, \\ L\left(\frac{1}{2}+it, \phi \times \chi\right) &\ll_{\phi, t, \epsilon} q^{\frac{5}{8}+\epsilon}, \end{aligned} \tag{1}$$

for every $\epsilon > 0$, where the dependence of the implied constants on t_f, t are polynomial.

DEDICATION

Dedicated to my dear parents and to my sweet sister.

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All other work conducted for the dissertation was completed by the student under the guidance of Professor Matthew Young.

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TABLE OF CONTENTS

Page

ABSTRACT ii
DEDICATION iii
ACKNOWLEDGMENTS iv
CONTRIBUTORS AND FUNDING SOURCES v
TABLE OF CONTENTS vi
1. INTRODUCTION 1
2. STATEMENT OF RESULTS 6
3. TECHNIQUE 8
4. L-FUNCTION DATA 11
5. STANDARD FORMULAE AND DEFINITIONS 13
6. SETUP 17
- 6.1 Asymptotic analysis of G_β 23

7. ARCHIMEDEAN ASPECTS 28
- 7.1 Oscillatory case 28
  - 7.1.1 Asymptotic analysis of $J_{\sigma,\mathcal{I}_1}$ 28
  - 7.1.2 Asymptotic analysis of $\mathcal{J}_{\sigma,\mathcal{I}_1}$ 30
  - 7.1.3 Asymptotic analysis of $\mathcal{K}_{\beta,\sigma,\mathcal{I}}$ 35
- 7.2 Non-oscillatory case 39
  - 7.2.1 Asymptotic analysis of $\mathcal{J}_{\sigma,\mathcal{I}}$ 39
  - 7.2.2 Asymptotic analysis of $\mathcal{K}_{\beta,\sigma,\mathcal{I}}$ 41

8. ARITHMETIC ASPECTS 44
- 8.1 Summary of character sum computation 49
- 8.2 Simplifying C' 53
- 8.3 Simplifying C'_2 54
- 8.4 Simplifying C'_1 55

8.5 Simplifying N_1' 56
8.6 Simplifying N_3' 56
8.7 Simplifying B' 57
8.8 Simplifying A' 57
8.9 Simplifying C_0 60
8.10 Collecting the Conrey-Iwaniec phase term 63
8.11 Putting everything together 64

9. THE Z-FUNCTION 66

9.1 Factoring $Z_{\text{fin},1}$ 70
9.2 Bounds for $Z_{\text{fin},1}$ 71
9.3 Factoring $Z_{\text{fin},2}$ 74
9.4 Bounds for $Z_{\text{fin},2}$ 77
9.5 Large sieve inequalities 80
9.6 Bounds for Z 82
9.7 Bounding Z_0 83
9.8 Bounding Z_1 84

10. COMPLETING THE PROOF 91

10.1 Oscillatory case 91
10.2 Non-oscillatory case 93

REFERENCES 96

1. INTRODUCTION

Let us begin with some motivation. Consider

$$R_3(n) := \{(x, y, z) \in \mathbb{Z}^3 \mid x^2 + y^2 + z^2 = n\}. \tag{1.1}$$

One may then ask: are the points in $\frac{R_3(n)}{\sqrt{n}}$ equidistributed on the unit sphere in $\mathbb{R}^3$? Questions like this can be answered by finding good bounds on the Fourier coefficients of certain modular forms or Maass forms, and often such bounds come from deep connections with subconvexity bounds for L-functions. Duke and Schulze-Pillot [1] found that the answer to the above equidistribution question is in the affirmative. More generally, subconvexity estimates of L-functions and a local-to-global principle were the key ingredients of Cogdell, Piatetski-Shapiro, and Sarnak's preprint [2], which essentially resolved the final open case of Hilbert's 11$^{\text{th}}$ problem, which lets us answer interesting questions such as the following: which integers in $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ can we write as sums of 3 squares? Replace $\mathbb{Q}(\sqrt{5})$ with any fixed totally real number field and "write as sums of 3 squares" with "integrally represent by any given positive definite integral ternary quadratic form" for the general strength of their result; please see [3] for an exposition of their ideas.

The *correspondence principle* in physics roughly states that the quantum mechanical behavior of systems approaches classical mechanical behavior in high-energy limits. Unlike in classical mechanics, energy values in quantum mechanics forms a discrete set (they are "quantized"), and these values are related to the eigenvalues of Laplace eigenfunctions. The Quantum ergodicity theorem (QE) of Shnirelman [4], Zelditch [5], and Colin de Verdiere [6] states the following: let the geodesic flow on a compact smooth Riemannian manifold X without boundary be ergodic (sufficiently chaotic) with respect to the normalized Liouville measure, and let $\{\phi_j\}_{j\geq 0}$ be an orthonormal basis of $L^2(X)$ composed of Laplace-Beltrami eigenfunctions such that the sequence of corresponding eigenvalues $\{\lambda_j\}_{j\geq 0}$ satisfies $\lambda_j \geq 0$

and $\lambda_j \to \infty$; then there exists a density 1 subsequence of $\{\phi_j\}_{j\geq 0}$ that equidistributes in the cotangent bundle T^*X (phase space). Based on evidence from a favorite toy model, Rudnick and Sarnak [7] conjectured that if additionally X has negative sectional curvature, then the entire sequence $\{\phi_j\}_{j\geq 0}$ equidistributes in phase space; this is known as the Quantum Unique Ergodicity conjecture (QUE). *Arithmetic* QUE asks if this is true specifically for surfaces of arithmetic nature (such as modular curves). In several cases, Arithmetic QUE follows from certain triple product identities and subconvexity bounds for certain L-functions of high degree. Please refer to Rudnick [8], Zelditch [9], and Sarnak [10], [11] for more details on QE and QUE.

Michel [12] and Iwaniec-Sarnak [13] provide us with several other applications of subconvexity bounds for L-functions, including Duke's theorem on equidistribution of Heegner points in the hyperbolic plane.

For automorphic L-functions, consider the bound $L(s,\pi) \ll_\epsilon Q(s,\pi)^{\delta+\epsilon}$ on the $\Re(s) = \frac{1}{2}$ line for all $\epsilon > 0$ with $Q(\cdot,\pi)$ being the analytic conductor of $L(\cdot,\pi)$ and $\delta \geq 0$ a fixed number. The Lindelöf hypothesis conjectures that we can take $\delta = 0$, but that is not yet known in any case. We can take $\delta = \frac{1}{4}$ in all cases; this is known as the *convexity* bound, and it follows from the Phragmen-Lindelöf principle from complex analysis combined with the functional equation of the L-function. Bounds with $0 \leq \delta < \frac{1}{4}$ are therefore aptly called *sub*convexity bounds. Subconvexity bounds are not yet known in all cases; establishing and improving subconvexity bounds is an active area of research not only because of their applicability (such as to equidistribution problems or to QUE), but also since they are interesting and challenging problems in their own right. Our results in this thesis are subconvex in the q-aspect.

The first subconvexity bound was due to Hardy and Littlewood based on the work of Weyl on a shifting method for finding nontrivial bounds for certain exponential sums: $\zeta\left(\frac{1}{2}+it\right) \ll t^{\frac{1}{6}+\epsilon}$; the proof of the same bound for Dirichlet L-functions is similar. The best bound known result for ζ today is $\delta = \frac{13}{84}$ due to Bourgain [14]. A subconvexity bound with $\delta = \frac{1}{6}$ is known as a *Weyl bound*.

The first subconvexity bound in the q-aspect was proved by Burgess [15] using cancellations in short character sums and Weil's Riemann hypothesis for curves over finite fields: $L\left(\frac{1}{2}+it,\chi\right) \ll_\epsilon q^{\frac{3}{16}+\epsilon}$ for fixed t and any $\epsilon > 0$. A subconvexity bound with $\delta = \frac{3}{16}$ is known as a *Burgess bound*. Heath-Brown [16] proved the hybrid Burgess bound in t and q aspect for Dirichlet L-functions.

After nearly four decades, the q-aspect Burgess bound for Dirichlet L-functions was improved to a Weyl bound for primitive quadratic Dirichlet characters of odd conductor by Conrey and Iwaniec [17]. They employed cubic moments of central values of L-functions, spectral theory of $GL(2)$ automorphic forms, Waldspurger's result on nonnegativity of central values of automorphic L-functions, and Deligne's solution of the Weil conjectures for varieties over finite fields. This celebrated paper has inspired several subsequent results, including Young [18], [19], Petrow [20], [21], Petrow and Young [22], [23], [24]; this series of papers culminated in the hybrid Weyl bound in q and t-aspects for all Dirichlet L-functions. Specifically, in [23], Petrow and Young proved the Weyl bound for any Dirichlet L-function of cube-free conductor, and in [24], they dropped the cube-free requirement by performing meticulous study of fourth moments of Dirichlet L-functions along cosets of certain groups of Dirichlet characters. Djordje Milićević [25] obtained sub-Weyl subconvexity for Dirichlet L-functions to prime-power moduli using a p-adic method of exponent pairs of van der Corput, Phillips, and Rankin.

In the $GL(2)$ realm, the first subconvexity result was a Weyl bound due to Good [26]: $L(\frac{1}{2}+it,f) \ll_\epsilon (1+|t|)^{\frac{1}{3}+\epsilon}$ for f a holomorphic Hecke cusp form of level 1. The widely used *amplification* method was developed by Iwaniec [27] to study the spectral aspect for Hecke L-functions. An influential series of papers by Duke, Friedlander, and Iwaniec [28], [29], [30], [31], [32], [33], [34], [35] played a major role in establishing subconvexity as an attractive and rich area of research. In a very general treatment, Michel and Venkatesh [36] showed subconvexity in the $GL(1)$ and $GL(2)$ settings uniformly in all aspects.

Xiaoqing Li [37] proved the first subconvexity bound for $GL(3)$: for ϕ the symmetric-square lift of a fixed $SL_2(\mathbb{Z})$ Hecke-Maass form and u_j an orthonormal basis of even Hecke-Maass forms for $SL_2(\mathbb{Z})$ with spectral parameter $t_j \geq 0$, she showed $L\left(\frac{1}{2}, \phi \times u_j\right) \ll_{\epsilon,\phi} (1+|t_j|)^{\frac{11}{8}+\epsilon}$ and $L\left(\frac{1}{2}+it, \phi\right) \ll_{\epsilon,\phi} (1+|t|)^{\frac{11}{16}+\epsilon}$. This result depended on Lapid's theorem [38] on the nonnegativity of $L\left(\frac{1}{2}, \phi \times u_j\right)$. Xiaoqing Li's results were subsequently improved by McKee, Sun, Ye [39] and Nunes [40].

Blomer [41] followed the Conrey-Iwaniec approach and Xiaoqing Li [37] to prove impressive q-aspect subconvexity results: $L\left(\frac{1}{2}, \phi \times f \times \chi\right) \ll_{\phi,f,\epsilon} q^{\frac{5}{4}+\epsilon}$ and $L\left(\frac{1}{2}+it, \phi \times \chi\right) \ll_{\phi,t,\epsilon} q^{\frac{5}{8}+\epsilon}$ with ϕ the symmetric-square lift of a fixed $SL_2(\mathbb{Z})$ Hecke-Maass form and χ a primitive, quadratic Dirichlet character modulo q for q an odd prime. Under the same assumptions on ϕ, χ, Huang [42] followed the approach of Young [18] to prove hybrid subconvexity results $L\left(\frac{1}{2}, \phi \times u_j \times \chi\right) \ll_{\epsilon,\phi} (q(1+|t_j|))^{\frac{3}{2}-\theta+\epsilon}$ and $L\left(\frac{1}{2}+it, \phi \times \chi\right) \ll_{\epsilon,\phi} (q(1+|t|))^{\frac{3}{4}-\frac{\theta}{2}+\epsilon}$, where $\theta = \frac{1}{23}$. Qi [43] proved Blomer's bounds for ϕ a self-dual Hecke automorphic cusp form for $SL_3(\mathbb{Z}[i])$ and $q \in \mathbb{Z}[i]$ a Gaussian prime.

Munshi [44], [45], [46] partially complemented Blomer's results by showing subconvexity for $L\left(\frac{1}{2}, \phi \times \chi\right)$ with ϕ being the symmetric-square lift of a fixed $SL_2(\mathbb{Z})$ Hecke-Maass form and χ a primitive Dirichlet character (not necessarily quadratic) of conductor q^l for q prime; he looked at two different aspects: either keep q fixed and let $l \to \infty$ or keep l fixed and let $q \to \infty$. In his breakthrough series on subconvexity via the circle method, Munshi [47], [48], [49], [50], [51] used his $GL(2)$ δ-symbol method that detects equality of integers using the Petersson trace formula. One benefit of this approach was that it allowed him to bypass any nonnegativity requirement on central values of L-functions, which is an important aspect of the moment method used in Conrey and Iwaniec [17], Xiaoqing Li [37], Blomer [41], Petrow and Young [23] among others. As a result, Munshi was able to drop the self-duality requirement (symmetric-square lift requirement) on the $SL_3(\mathbb{Z})$ Hecke-Maass cusp form. In particular, in [51], Munshi showed that $L\left(\frac{1}{2}, \pi \times \chi\right) \ll_{\pi,\epsilon} q^{\frac{3}{4}-\frac{1}{308}+\epsilon}$, where π is an $SL_3(\mathbb{Z})$ Hecke-Maass cusp form, and χ is a primitive Dirichlet character modulo q, with q being prime. Holowinsky and Nelson [52] simplified Munshi's proof by replacing the $GL(2)$ δ-symbol method with a formula obtained using Poisson summation that expresses χ of prime conductor q in terms of additive characters and twisted Kloosterman sums; they also improved the exponent: $L\left(\frac{1}{2}, \pi \times \chi\right) \ll_{\pi,\epsilon} q^{\frac{3}{4}-\frac{1}{36}+\epsilon}$. By a variant of the Munshi and Holowinsky-Nelson methods, Lin [53] showed hybrid subconvexity in q (prime) and t aspects: $L\left(\frac{1}{2}+it, \pi \times \chi\right) \ll_{\pi,\epsilon} (q(1+|t|))^{\frac{3}{4}-\frac{1}{36}+\epsilon}$. Using the δ-method, Sharma [54] obtained an improvement in the exponent when q is prime: $L\left(\frac{1}{2}, \pi \times \chi\right) \ll_{\pi,\epsilon} q^{\frac{3}{4}-\frac{1}{32}+\epsilon}$.

In a major breakthrough, Nelson [55] recently settled the subconvexity problem for $GL(n)$ standard L-functions for all $n \geq 1$ in the t-aspect. He also addressed the spectral aspect in case of uniform parameter growth.

In the current thesis, we broaden the result of Blomer [41] in two ways using the Petrow and Young [23] approach while maintaining the strength of the exponents: we remove the quadratic requirement for χ, and we allow q to be any cube-free odd positive integer.

2. STATEMENT OF RESULTS

Let ϕ be the symmetric-square lift of an $SL_2(\mathbb{Z})$ Hecke-Maass form. Let q be an odd cube-free positive integer, and let χ be a primitive Dirichlet character modulo q such that χ is not quadratic. For ψ a Dirichlet character modulo q, let $\mathcal{H}_{it}(k,\psi)$ denote the (possibly empty) set of Hecke-normalized Hecke-Maass newforms of level $k|q$, central character ψ, and spectral parameter t. Then we show the following:

Theorem 2.0.1. *For $T \geq 1$, we have*

$$\sum_{k|q} \sum_{|t_j| \leq T} \sum_{f \in \mathcal{H}_{it_j}(k,\overline{\chi}^2)} L\left(\frac{1}{2}, \phi \times f \times \chi\right) + \int_{-T}^{T} \left| L\left(\frac{1}{2} + it, \phi \times \chi\right)\right|^2 dt \ll_{\phi,T,\epsilon} q^{\frac{5}{4}+\epsilon}, \tag{2.1}$$

where the dependence of the implied constant on T is polynomial.

The following corollaries of theorem 2.0.1 extend results of Blomer [41] that assume that χ is quadratic and q is prime. Corollary 2.0.1.2 has some advantages compared to the results of Munshi [51] and Holowinsky-Nelson [52]: it holds on the entire critical line $\Re(s) = \frac{1}{2}$, it lacks the primality assumption on q, and the bound has a better exponent; however, their results are more flexible in the sense that they hold for general $SL_3(\mathbb{Z})$ Hecke-Maass cusp forms ϕ (not necessarily symmetric-square lifts).

Corollary 2.0.1.1. *Let f be an even Hecke-normalized Hecke-Maass newform of level dividing q, central character $\overline{\chi}^2$, and spectral parameter t_f. We have*

$$L\left(\frac{1}{2}, \phi \times f \times \chi\right) \ll_{\phi,t_f,\epsilon} q^{\frac{5}{4}+\epsilon}, \tag{2.2}$$

where the dependence of the implied constant on t_f is polynomial.

Corollary 2.0.1.2.

$$L\left(\frac{1}{2} + it, \phi \times \chi\right) \ll_{\phi,t,\epsilon} q^{\frac{5}{8}+\epsilon}, \tag{2.3}$$

where the dependence of the implied constant on t is polynomial.

Corollaries 2.0.1.1 and 2.0.1.2 provide us with subconvexity in the q-aspect; the corresponding convexity bounds are $q^{\frac{3}{2}+\epsilon}$ and $q^{\frac{3}{4}+\epsilon}$ respectively.

3. TECHNIQUE

Here we sketch the proof of our results for the convenience of the reader.

Denote the set of newform Eisenstein series by $\mathcal{H}_{it,\mathrm{Eis}}(k,\psi) = \{E_{\chi_1,\chi_2}\left(z,\frac{1}{2}+it\right) \mid q_1q_2 = k \text{ and } \chi_1\overline{\chi_2} \simeq \psi\}$, where $\eta \simeq \psi$ means that η and ψ have equal underlying primitive characters. Let $h_0(t) = e^{-t^2}\left(t^2+\frac{1}{4}\right)$. Consider the following first moment of degree 6 L-functions:

$$\begin{aligned}\mathcal{M} = &\sum_{j=1}^{\infty} h_0(t_j) \sum_{lk=q} \sum^{+}_{f\in\mathcal{H}_{it_j}(k,\overline{\chi}^2)} w_{f,l} L\left(\frac{1}{2}, \phi\times f\times\chi\right) + \\ &\frac{1}{4\pi}\int_{-\infty}^{\infty} h_0(t) \sum_{lk=q} \sum^{+}_{E\in\mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)} w_{E,l} L\left(\frac{1}{2}, \phi\times E\times\chi\right)\, dt.\end{aligned} \tag{3.1}$$

Here $w_{f,l} \gg q^{-1}(q(1+|t_j|))^{-\epsilon}$, $w_{E,l} \gg q^{-1}(q(1+|t|))^{-\epsilon}$, and $\sum^{+}$ denotes summation over *even* Maass forms or Eisenstein series. We apply an approximate functional equation (theorem 5.3, [56]) at the cost of a small error to prepare for applying the $GL(2)$ Bruggeman-Kuznetsov trace formula (proposition 2.1, [23]). Bruggeman-Kuznetsov replaces the $GL(2)$ spectral aspects of our moment with twisted Kloosterman sums and some standard integral transforms. Applying a Hecke relation to the $GL(3)$ Fourier coefficients (Fourier coefficients of ϕ) and opening the twisted Kloosterman sum allows us to extract a sum involving additive twists of $GL(3)$ Fourier coefficients; such a sum is primed for an application of the $GL(3)$ Voronoi formula (lemma 3, [41]), which leads to a reduction in the length of the sum in our case. We complete the setup of the problem by applying dyadic partitions of unity to localize the variables. We fix some of the variables to have their most typical values to reduce sources of distraction in this summary:

$$\mathcal{M} \approx \sum_{\sigma,\beta\in\{\pm1\}} \sum_{\substack{N,C,N_2\\ \text{dyadic}}} \frac{1}{N^{\frac{1}{2}}C^4} \sum_{q|c} \sum_{n_2=1}^{\infty} A_\phi(n_2,1)\mathcal{T}_{\beta,\sigma}(c,n_2,\chi)\mathcal{K}_{\beta,\sigma,\mathcal{I}}(c,n_2), \tag{3.2}$$

where $\mathcal{I} = (q,N,C,N_2)$ and $c \asymp C$, $n \asymp N$, $n_2 \asymp N_2$ with $1 \ll N \ll q^{3+\epsilon}$ (small $\epsilon > 0$),

$q \ll C \ll q^{100}$, $1 \ll N_2 \ll q^{10^4}$. Here, the values of $A_\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ are Fourier coefficients of ϕ, $\mathcal{T} := \mathcal{T}_{\beta,\sigma}(c, n_2, \chi)$ is a character sum similar to the one in [41], and $\mathcal{K}_{\beta,\sigma,\mathcal{I}}(c, n_2)$ is an integral transform from Voronoi summation. To find asymptotic expressions for $\mathcal{K}_{\beta,\sigma,\mathcal{I}}(c, n_2)$, we apply integration by parts and stationary phase (see [57]) on several layers of oscillatory integrals. Then we carefully craft a Petrow-Young-style Z-function (see [23]) involving $\mathcal{T}$ such that after careful simplification involving numerous integer variables, new L-functions on the dual side are revealed. After setting some variables to their most typical values to highlight the essence of the message, it is essentially the following:

$$Z(s_1, s_2) \approx \frac{1}{\varphi(q)} \sum_{\psi(q)} L(s_1, \phi \times \psi) L(s_2, \overline{\psi}) Z_{\text{fin}}(s_1, s_2), \tag{3.3}$$

where $Z_{\text{fin}}(s_1, s_2)$ is analogous to the one of Petrow and Young [23]; the philosophy for bounding $Z_{\text{fin}}(s_1, s_2)$ is same as that of Petrow and Young: factor over primes and perform local computations until it boils down to bounding

$$g(\chi, \psi) = \sum_{t,u \,(\text{mod}\, q)} \chi(t)\overline{\chi}(t+1)\overline{\chi}(u)\chi(u+1)\psi(ut-1). \tag{3.4}$$

Petrow and Young showed that $g(\chi, \psi) \ll_\epsilon q^{1+\epsilon}$ using a combination of classical methods and Deligne's Riemann hypothesis for varieties. Bounding $Z(s_1, s_2)$ is completed by using Cauchy's inequality followed by classical large sieve inequalities to bound second moments of $L(\cdot, \phi \times \psi)$ and $L(\cdot, \overline{\psi})$. Finally, by an argument of Petrow and Young [23], there exists an $E \in \mathcal{H}_{it,\text{Eis}}(k, \psi)$ such that $L\left(\frac{1}{2}, \phi \times E \times \chi\right) = \left|L\left(\frac{1}{2} + it, \phi \times \chi\right)\right|^2$, which completes the proof of theorem 2.0.1.

We deduce corollary 2.0.1.1 from theorem 2.0.1 by invoking a result of Lapid [38] on the nonnegativity of $L\left(\frac{1}{2}, \phi \times f \times \chi\right)$ for self-dual (symmetric-square lift) ϕ. Corollary 2.0.1.2 follows from theorem 2.0.1 after dropping the complete cuspidal spectrum followed by a standard method of extracting an individual bound of L-functions from an integral bound.

We conclude this chapter with a few comments.

- Lapid's theorem only works for self-dual (symmetric-square lift) ϕ and only at the central point $\frac{1}{2}$. Other methods need to be investigated in order to remove the self-dual assumption on ϕ or to prove corollary 2.0.1.1 at $\frac{1}{2}+it$ for nonzero t.

- Like Blomer's results in [41], theorem 2.0.1 is unfortunately not Lindelöf on average; therefore corollaries 2.0.1.1 and 2.0.1.2 fall short of the Weyl bound even though the same strategy resulted in the Weyl bound for Dirichlet L-functions in Conrey-Iwaniec [17] and Petrow-Young [23]. The large sieve estimates for the second moments of the $L(\cdot, \phi \times \psi)$ and $L(\cdot, \overline{\psi})$ on the dual side (see definition of $Z(s_1, s_2)$) are $\ll q^{\frac{3}{2}+\epsilon}$ and $\ll q^{1+\epsilon}$ respectively. Therefore, by Cauchy-Schwarz, we get the bound $(q^{\frac{3}{2}+\epsilon})^{\frac{1}{2}}(q^{1+\epsilon})^{\frac{1}{2}} = q^{\frac{5}{4}+\epsilon}$ in theorem 2.0.1. The $\ll q^{\frac{3}{2}+\epsilon}$ for $L(\cdot, \phi \times \psi)$ above is worse than Lindelöf on average ($\ll q^{1+\epsilon}$); it is an unfortunate combination of high conductor of the L-function (q^3) leading to a length of $q^{\frac{3}{2}+\epsilon}$ after truncation in the approximate functional equation and the nature of the large sieve inequality.

- This project combines the approaches of Blomer [41] and Petrow-Young [23] and tremendously benefited from these projects. However, we faced new difficulties compared to both. The already complicated character sum $\mathcal{T}$ handled by Blomer had a more convoluted incarnation here in the sense that now χ was not longer quadratic and q was not necessarily prime. The Z-function tackled in this project is same in spirit as Petrow and Young's, but significant amount of unpacking was needed in our version of Z to realize that semblance; this is partly due to the presence $GL(3)$ Fourier coefficients and $\mathcal{T}$ in our version.

4. L-FUNCTION DATA

Let ϕ be the symmetric-square lift of an $SL_2(\mathbb{Z})$ Hecke-Maass form having spectral parameter t. Let the Whittaker-Fourier coefficients of ϕ be denoted by (the values of) $A_\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$. Let χ be a primitive Dirichlet character modulo $q \in \mathbb{N}$. For $\Re(s) > 1$, consider the following three absolutely convergent series.

(1) The *Godement-Jacquet L-function* or *standard L-function* given by

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{A_\phi(1,n)}{n^s} = \prod_{p \text{ prime}} (1 - A_\phi(1,p)p^{-s} + A_\phi(p,1)p^{-2s} - p^{-3s})^{-1}. \tag{4.1}$$

(2) The twisted L-function

$$L(\phi \times \chi, s) = \sum_{n=1}^{\infty} \frac{A_\phi(n,1)\chi(n)}{n^s}. \tag{4.2}$$

(3) For $f \in \mathcal{H}_{it}(k, \overline{\chi}^2) \cup \mathcal{H}_{it,\text{Eis}}(k, \overline{\chi}^2)$ and f even, the Rankin-Selberg convolution of ϕ and $f \times \chi$ given by

$$L(\phi \times f \times \chi, s) = \sum_{\substack{m,n \geq 1 \\ (m,q)=1}} \frac{A_\phi(n,m)\lambda_f(n)\chi(n)}{(m^2 n)^s}, \tag{4.3}$$

where the Fourier coefficients of f are denoted by $\lambda_f(n)$.

$L(\phi, \cdot)$, $L(\phi \times \chi, \cdot)$, and $L(\phi \times f \times \chi, \cdot)$ can be analytically continued to entire functions that are L-functions in the sense of [56] chapter 5 having conductors 1, q^3, q^6 respectively.

For all $s \in \mathbb{C}$, the corresponding completed L-functions are given by

$$\Lambda(\phi, s) = \pi^{-\frac{3s}{2}} \prod_{j=1}^{3} \Gamma\left(\frac{s+\alpha_j}{2}\right) L(\phi, s) = \Lambda(\phi, 1-s),$$
$$\Lambda(\phi \times \chi, s) = \left(\frac{q}{\pi}\right)^{\frac{3s}{2}} \prod_{j=1}^{3} \Gamma\left(\frac{s+\theta_0+\alpha_j}{2}\right) L(\phi \times \chi, s) = i^{-\theta_0}\frac{\tau(\chi)^2}{\tau(\overline{\chi})\sqrt{q}}\Lambda(\phi \times \overline{\chi}, 1-s),$$
$$\Lambda(\phi \times f \times \chi, s) = \left(\frac{q}{\pi}\right)^{3s} \prod_{\pm} \prod_{j=1}^{3} \Gamma\left(\frac{s+\theta_0 \pm it-\alpha_j}{2}\right) L(\phi \times f \times \chi, s) = \Lambda(\phi \times f \times \chi, 1-s), \tag{4.4}$$

where $\alpha_1 = 2it, \alpha_2 = 0, \alpha_3 = -2it$, and

$$\theta_0 = \begin{cases} 0 & \text{if } \chi(-1) = +1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \tag{4.5}$$

$\Lambda(\phi, \cdot)$, $\Lambda(\phi \times \chi, \cdot)$, $\Lambda(\phi \times f \times \chi, \cdot)$ are all entire functions. The root number of $L(\phi \times f \times \chi, \cdot)$ is $(\varepsilon(f \times \chi))^3$, and $\varepsilon(f \times \chi)$ equals the parity of f; see section 2.3 of [23].

5. STANDARD FORMULAE AND DEFINITIONS

Throughout the rest of this document, we will use the following notation: $e(z) := \exp(2\pi i z)$.

The following is similar to Proposition 2.1 of [23].

Lemma 5.0.0.1 (Bruggeman-Kuznetsov trace formula)**.** *Let h be a function such that there exists $\delta > 0$ such that*

- *h is even, i.e. $h(-z) = h(z)$,*
- *h is holomorphic in the strip $|\Im(z)| \leq \frac{1}{2} + \delta$,*
- *$|h(z)| \ll (1 + |z|)^{-2-\delta}$ for z in the above strip.*

Suppose χ is primitive of conductor q and not quadratic. There exist positive weights $w_{f,l} \gg q^{-1}(q(1+|t_j|))^{-\epsilon}$ and $w_{E,l} \gg q^{-1}(q(1+|t|))^{-\epsilon}$ such that for any $(n_1 n_2, q) = 1$ and $\operatorname{sgn}(n_1 n_2) = \sigma \in \{1, -1\}$, we have

$$\sum_{j=1}^{\infty} h(t_j) \sum_{lk=q} \sum_{f \in \mathcal{H}_{it_j}(k, \overline{\chi}^2)} w_{f,l} \lambda_f(n_1) \overline{\lambda_f(n_2)} + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \sum_{lk=q} \sum_{E \in \mathcal{H}_{it,Eis}(k, \overline{\chi}^2)} w_{E,l} \lambda_E(n_1) \overline{\lambda_E(n_2)} \, dt$$
$$= \delta_{n_1=n_2} g_0 + \sum_{q|c} \frac{S_{\overline{\chi}^2}(n_1, n_2; c)}{c} g_\sigma\left(\frac{4\pi\sqrt{|n_1 n_2|}}{c}\right), \tag{5.1}$$

where

$$g_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \tanh(\pi t) t \, h(t) \, dt,$$
$$g_\sigma(x) = \kappa_\sigma \int_{-\infty}^{\infty} K_\sigma(x, t) t \, h(t) \, dt, \qquad \sigma \in \{1, -1\}, \tag{5.2}$$

with

$$K_\sigma(x,t) = \begin{cases} \frac{J_{2it}(x)}{\cosh(\pi t)} & \sigma = +1 \\ K_{2it}(x)\sinh(\pi t) & \sigma = -1 \end{cases} \tag{5.3}$$

and

$$\kappa_\sigma = \begin{cases} 2i & \sigma = +1 \\ \frac{4}{\pi} & \sigma = -1 \end{cases} \tag{5.4}$$

□

Next, we have the Hecke relation, which follows from Möbius inversion and theorem 6.4.11 of [58].

Lemma 5.0.0.2 (Hecke relation)**.**

$$A_\phi(n,m) = \sum_{d|(n,m)} \mu(d) A_\phi\left(\frac{n}{d},1\right) A_\phi\left(1,\frac{m}{d}\right). \tag{5.5}$$

□

Let w be a smooth compactly supported function, and let $\widetilde{w}$ be its Mellin transform. For $\sigma_0 > \frac{7}{32}$, $\beta \in \{1,-1\}$, let

$$\mathcal{W}_\beta(x) := \frac{x}{2\pi i}\int_{(\sigma_0)} (\pi^3 x)^{-s} \left(\prod_{j=1}^{3} \frac{\Gamma\left(\frac{s+\alpha_j}{2}\right)}{\Gamma\left(\frac{1-s-\alpha_j}{2}\right)} - i\beta \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1+s+\alpha_j}{2}\right)}{\Gamma\left(\frac{2-s-\alpha_j}{2}\right)}\right) \widetilde{w}(1-s)\, ds, \tag{5.6}$$

with $\alpha_1 = 2it, \alpha_2 = 0, \alpha_3 = -2it$ being the local parameters at infinity of ϕ. The following is [41] lemma 3.

Lemma 5.0.0.3 ($GL(3)$ Voronoi summation)**.** *Let c,d be integers with $c \neq 0$ and $(c,d)=1$. Then*

$$\sum_{n=1}^{\infty} A_\phi(m,n) e\left(\frac{n\overline{d}}{c}\right) w(n) = \frac{\pi^{\frac{3}{2}} c}{2} \sum_{\beta\in\{\pm 1\}} \sum_{n_1|cm} \sum_{n_2=1}^{\infty} \frac{A_\phi(n_2,n_1)}{n_1 n_2} S\left(md, \beta n_2, \frac{mc}{n_1}\right) \mathcal{W}_\beta\left(\frac{n_2 n_1^2}{c^3 m}\right). \tag{5.7}$$

□

Now, consider the following renormalization of $\mathcal{W}_\beta$

$$\mathcal{K}_\beta(x) := \frac{\pi^{\frac{3}{2}}\mathcal{W}_\beta(x)}{2x} = \frac{1}{2\pi i}\int_{(\sigma_0)} (8\pi^3 x)^{-s} G_\beta(s)\widetilde{w}(1-s)\,ds, \tag{5.8}$$

where

$$G_\beta(s) = \left(e^{i\beta\frac{3\pi s}{2}} + \varsigma e^{-i\beta\frac{\pi s}{2}}\right)\prod_{j=1}^{3}\Gamma(s+\alpha_j), \tag{5.9}$$

where $\varsigma = \sum_{j=1}^{3} e^{i\beta\pi\alpha_j}$. We used Legendre duplication and reflection for Γ and some elementary trigonometric identities to get the simplified formula for G_β. Note that since $\alpha_1+\alpha_2+\alpha_3 = 0$, we have $\varsigma = 1 + 2\cos(\pi\alpha_1)$.

The following is a corollary of 5.0.0.3.

Corollary 5.0.0.1. *Let $c, q \in \mathbb{N}$, $u, v \in \mathbb{Z}$ such that $q|c$. Let χ be a Dirichlet character modulo q. Then,*

$$\sum_{n=1}^{\infty} A_\phi(1,n)\chi(n)S_{\overline{\chi}^2}(un, v; c)w(n) = \frac{1}{c}\sum_{\beta\in\{\pm1\}}\sum_{c_1|c}\sum_{n_1 n_3 = c_1}\sum_{n_2=1}^{\infty}\frac{A_\phi(n_2,n_1)n_1}{c_1^2}\mathcal{T}\mathcal{K}_\beta\left(\frac{n_2 n_1^2}{c_1^3}\right), \tag{5.10}$$

where

$$\mathcal{T} = \mathcal{T}_{\beta,u,v}(c, c_1, n_3, n_2, \chi) = \sideset{}{^*}\sum_{b(c_1)}\ \sideset{}{^*}\sum_{d(c)}\sum_{a(c)}\ \sideset{}{^*}\sum_{f(n_3)}\chi^2(d)\chi(a)e\left(\frac{v\overline{d}+uad}{c}\right)e\left(-\frac{\overline{b}a}{c_1}\right)\times e\left(\frac{bf+\beta n_2\overline{f}}{n_3}\right). \tag{5.11}$$

Proof: Call the left hand side S. Opening the twisted Kloosterman sum and splitting the n-sum into residue classes $\pmod c$, we get

$$\begin{aligned} S &= \sum_{a(c)}\sum_{n\equiv a(c)} A_\phi(1,n)\chi(n)w(n)\sideset{}{^*}\sum_{d(c)}\chi^2(d)e\left(\frac{v\overline{d}+und}{c}\right) \\ &= \sideset{}{^*}\sum_{d(c)}\sum_{a(c)}\chi^2(d)\chi(a)e\left(\frac{v\overline{d}+uad}{c}\right)\sum_{n\equiv a(c)} A_\phi(1,n)w(n). \end{aligned} \tag{5.12}$$

Next, we detect $n \equiv a \pmod{c}$ using primitive additive characters modulo $c_1|c$.

$$\begin{aligned} S &= \sum_{d(c)}^{*} \sum_{a(c)} \chi^2(d)\chi(a) e\left(\frac{v\bar{d}+uad}{c}\right) \sum_{n=1}^{\infty} A_\phi(1,n) w(n) \frac{1}{c} \sum_{c_1|c} \sum_{b(c_1)}^{*} e\left(\frac{\bar{b}(n-a)}{c_1}\right) \\ &= \frac{1}{c} \sum_{c_1|c} \sum_{b(c_1)}^{*} \sum_{d(c)}^{*} \sum_{a(c)} \chi^2(d)\chi(a) e\left(\frac{v\bar{d}+uad}{c}\right) e\left(-\frac{\bar{b}a}{c_1}\right) \sum_{n=1}^{\infty} A_\phi(1,n) e\left(\frac{\bar{b}n}{c_1}\right) w(n). \end{aligned} \tag{5.13}$$

Applying lemma 5.0.0.3 gives

$$\begin{gathered} S = \frac{1}{c} \sum_{c_1|c} \sum_{b(c_1)}^{*} \sum_{d(c)}^{*} \sum_{a(c)} \chi^2(d)\chi(a) e\left(\frac{v\bar{d}+uad}{c}\right) e\left(-\frac{\bar{b}a}{c_1}\right) \times \\ \sum_{\beta\in\{\pm1\}} \sum_{n_1|c_1} \sum_{n_2=1}^{\infty} \frac{A_\phi(n_2,n_1)n_1}{c_1^2} S\left(b, \beta n_2, \frac{c_1}{n_1}\right) \mathcal{K}_\beta\left(\frac{n_2 n_1^2}{c_1^3}\right). \end{gathered} \tag{5.14}$$

Opening the Kloosterman sum completes the proof. □

The following definition is from [57].

Definition 5.0.0.1 (Inert functions)**.** *Let $\mathcal{F}$ be a set and $X : \mathcal{F} \to \mathbb{R}_{\geq 1}$ be a function whose value at $T \in \mathcal{F}$ is denoted by X_T. A family $\{w_T\}_{T\in\mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}^d_{>0}$ is called X-inert if for each $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d_{\geq 0}$ we have*

$$C(j_1,\ldots,j_d) := \sup_{T\in\mathcal{F}} \sup_{(x_1,\ldots,x_d)\in\mathbb{R}^d_{>0}} X_T^{-j_1-\cdots-j_d} \left| x_1^{j_1}\cdots x_d^{j_d} w_T^{(j_1,\ldots,j_d)}(x_1,\ldots,x_d)\right| < \infty. \tag{5.15}$$

6. SETUP

Let us set up our moment problem now. For $T \geq 1$, let

$$h_0(t) = \exp\left(-\left(\frac{t}{T}\right)^2\right)\frac{t^2+\frac{1}{4}}{T^2}. \tag{6.1}$$

Consider the following 1st moment of degree 6 L-functions.

$$\begin{aligned}\mathcal{M} = \mathcal{M}(q,\chi) = \sum_{j=1}^{\infty} h_0(t_j) \sum_{lk=q} \sideset{}{^+}\sum_{f\in\mathcal{H}_{it_j}(k,\overline{\chi}^2)} w_{f,l} L\left(\frac{1}{2}, \phi\times f\times\chi\right) + \\ \frac{1}{4\pi}\int_{-\infty}^{\infty} h_0(t) \sum_{lk=q} \sideset{}{^+}\sum_{E\in\mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)} w_{E,l} L\left(\frac{1}{2}, \phi\times E\times\chi\right) dt,\end{aligned} \tag{6.2}$$

where $\sum^+$ denotes summation over *even* Maass forms or Eisenstein series. By theorem 5.3 of [56], we have

$$\begin{aligned} L\left(\frac{1}{2}, \phi\times f\times\chi\right) &= 2\sum_{\substack{n,d\geq 1\\(d,q)=1}} \frac{A_\phi(n,d)\lambda_f(n)\chi(n)}{(nd^2)^{\frac{1}{2}}} V\left(\frac{nd^2}{q^3}, t_j\right), \\ L\left(\frac{1}{2}, \phi\times E\times\chi\right) &= 2\sum_{\substack{n,d\geq 1\\(d,q)=1}} \frac{A_\phi(n,d)\lambda_E(n)\chi(n)}{(nd^2)^{\frac{1}{2}}} V\left(\frac{nd^2}{q^3}, t\right),\end{aligned} \tag{6.3}$$

where

$$V(x,t) = \frac{1}{2\pi i}\int_{(2)} (\pi^3 x)^{-u} \frac{\prod_{\pm}\prod_{j=1}^{3}\Gamma\left(\frac{\frac{1}{2}+u+\theta_0\pm it-\alpha_j}{2}\right)}{\prod_{\pm}\prod_{j=1}^{3}\Gamma\left(\frac{\frac{1}{2}+\theta_0\pm it-\alpha_j}{2}\right)} e^{u^2}\frac{du}{u}, \tag{6.4}$$

where $\theta_0 = \begin{cases} 0 & \text{if } \chi(-1) = +1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}.$

Therefore, we have

$$\frac{\mathcal{M}}{2} = \sum_{j=1}^{\infty} h_0(t_j) \sum_{lk=q} \sum_{f\in\mathcal{H}_{it_j}(k,\overline{\chi}^2)}^{+} w_{f,l} \sum_{\substack{n,d\geq 1\\(d,q)=1}} \frac{A_\phi(n,d)\lambda_f(n)\chi(n)}{(nd^2)^{\frac{1}{2}}} V\left(\frac{nd^2}{q^3}, t_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_0(t) \sum_{lk=q} \sum_{E\in\mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)}^{+} w_{E,l} \sum_{\substack{n,d\geq 1\\(d,q)=1}} \frac{A_\phi(n,d)\lambda_E(n)\chi(n)}{(nd^2)^{\frac{1}{2}}} V\left(\frac{nd^2}{q^3}, t\right)\, dt. \tag{6.5}$$

The absolute convergence of these sums follows from the rapid decay of h_0 and since

$$V(x,t) \ll_A \left(1 + \frac{x}{1+|t|^3}\right)^{-A}, \tag{6.6}$$

for any $A > 0$ (analogous to lemma 10.1 of [23]). Interchanging the order of summation, we have

$$\frac{\mathcal{M}}{2} = \sum_{\substack{n,d\geq 1\\(d,q)=1}} \frac{A_\phi(n,d)\chi(n)}{(nd^2)^{\frac{1}{2}}} \mathcal{M}_0(n,d), \tag{6.7}$$

where

$$\mathcal{M}_0(n,d) = \sum_{j=1}^{\infty} h\left(t_j, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{f\in\mathcal{H}_{it_j}(k,\overline{\chi}^2)}^{+} w_{f,l}\lambda_f(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(t, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{E\in\mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)}^{+} w_{E,l}\lambda_E(n)\, dt, \tag{6.8}$$

with

$$h(t,y) = h_0(t)V(y,t). \tag{6.9}$$

Applying lemma 5.0.0.2 (Hecke relation) gives

$$\frac{\mathcal{M}}{2} = \sum_{\substack{n,d,\delta\geq 1\\(\delta d,q)=1}} \frac{\mu(\delta)A_\phi(n,1)A_\phi(1,d)\chi(\delta n)}{(nd^2\delta^3)^{\frac{1}{2}}} \mathcal{M}_0(\delta n, \delta d). \tag{6.10}$$

Next, we apply dyadic partitions of unity to n, δ to get

$$\mathcal{M} = \sum_{N,\Delta} w_{N,\Delta}(n,\delta) \sum_{\substack{n,d,\delta\geq 1 \\ (\delta d,q)=1}} \frac{\mu(\delta)A_\phi(n,1)A_\phi(1,d)\chi(\delta n)}{(nd^2\delta^3)^{\frac{1}{2}}} \mathcal{M}_0(\delta n, \delta d), \tag{6.11}$$

where $w_{N,\Delta}$ is a family of 1-inert functions with support on $[N,2N] \times [\Delta, 2\Delta]$.

Observe the following:

- For $\epsilon' > 0$, $h_0(t)$ is small for $|t| > T^{1+\epsilon'}q^{\epsilon'}$, and
- by (6.6), $V(x,t)$ is small for $|t| \leq T^{1+\epsilon'}q^{\epsilon'}$, $x \geq T^{3+\epsilon}q^{\epsilon}$ for $\epsilon > 0$ depending upon ϵ'.

Due to the above, the $h\left(t, \frac{nd^2\delta^3}{q^3}\right)$ terms in $\mathcal{M}_0(\delta n, \delta d)$ are small when $nd^2\delta^3 > (qT)^{3+\epsilon}$. This allows us to truncate the sums above at the cost of small errors so that $d^2 \leq (qT)^{3+\epsilon}$ and $N\Delta^3 \leq \frac{(qT)^{3+\epsilon}}{d^2}$. Further, since n, δ are positive integers, we have $\frac{1}{2} \leq N, \Delta$. In other words, we have

$$\mathcal{M} = \sum_{\substack{d^2\leq (qT)^{3+\epsilon} \\ (d,q)=1}} \frac{A_\phi(1,d)}{d} \sum_{N,\Delta} w_{N,\Delta}(n,\delta) \sum_{\substack{n,\delta\geq 1 \\ (\delta,q)=1}} \frac{\mu(\delta)A_\phi(n,1)\chi(\delta n)}{(n\delta^3)^{\frac{1}{2}}} \mathcal{M}_0(\delta n, \delta d) + O_\epsilon((qT)^{-100}), \tag{6.12}$$

where we have omitted the bounds on N, Δ for brevity of notation. We detect the evenness of the Maass forms and Eisenstein series in $\mathcal{M}_0(n,d)$ by inserting indicator functions $(1 + \lambda_f(-1))$ and $(1 + \lambda_E(-1))$ as follows

$$\begin{aligned}\mathcal{M}_0(n,d) = \sum_{j=1}^{\infty} h\left(t_j, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{f\in\mathcal{H}_{it_j}(k,\overline{\chi}^2)} w_{f,l}(1+\lambda_f(-1))\lambda_f(n) + \\ \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(t, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{E\in\mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)} w_{E,l}(1+\lambda_E(-1))\lambda_E(n)\,dt.\end{aligned} \tag{6.13}$$

We rewrite this as

$$\mathcal{M}_0(n,d) = \mathcal{M}_1(n,d) + \mathcal{M}_{-1}(n,d), \tag{6.14}$$

where, for $\sigma \in \{1, -1\}$, we have,

$$\begin{aligned}\mathcal{M}_\sigma(n,d) = \sum_{j=1}^{\infty} h\left(t_j, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{f \in \mathcal{H}_{it_j}(k,\overline{\chi}^2)} w_{f,l}\lambda_f(\sigma)\lambda_f(n) + \\ \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(t, \frac{nd^2}{q^3}\right) \sum_{lk=q} \sum_{E \in \mathcal{H}_{it,\mathrm{Eis}}(k,\overline{\chi}^2)} w_{E,l}\lambda_E(\sigma)\lambda_E(n)\, dt.\end{aligned} \tag{6.15}$$

By the Bruggeman-Kuznetsov trace formula (5.0.0.1), we have

$$\mathcal{M}_\sigma(n,d) = \mathbb{1}_{n=\sigma} g_0\left(\frac{nd^2}{q^3}\right) + \sum_{q|c} \frac{S_{\overline{\chi}^2}(n,\sigma;c)}{c} g_\sigma\left(\frac{4\pi\sqrt{n}}{c}, \frac{nd^2}{q^3}\right), \tag{6.16}$$

where

$$\begin{aligned} g_0(y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \tanh(\pi t) t\, h(t,y)\, dt, \\ g_\sigma(x,y) &= \kappa_\sigma \int_{-\infty}^{\infty} K_\sigma(x,t) t\, h(t,y)\, dt, \quad \sigma \in \{1,-1\}. \end{aligned} \tag{6.17}$$

Thus

$$\mathcal{M} = \mathcal{D} + \mathcal{S}_{+1} + \mathcal{S}_{-1} + O_\epsilon((qT)^{-100}), \tag{6.18}$$

where the diagonal term from Bruggeman-Kuznetsov gives

$$\mathcal{D} = \sum_{N,\Delta} w_{N,\Delta}(1,1) \sum_{\substack{d^2 \le (qT)^{3+\epsilon} \\ (d,q)=1}} \frac{A_\phi(1,d)}{d} g_0\left(\frac{d^2}{q^3}\right), \tag{6.19}$$

and for $\sigma \in \{1, -1\}$, we have

$$\begin{aligned}\mathcal{S}_\sigma = \sum_{\substack{d^2 \le (qT)^{3+\epsilon} \\ (d,q)=1}} \frac{A_\phi(1,d)}{d} \sum_{N,\Delta} w_{N,\Delta}(n,\delta) \sum_{\substack{n,\delta \ge 1 \\ (\delta,q)=1}} \frac{\mu(\delta)A_\phi(n,1)\chi(\delta n)}{(n\delta^3)^{\frac{1}{2}}} \sum_{q|c} \frac{S_{\overline{\chi}^2}(\delta n,\sigma;c)}{c} \times \\ g_\sigma\left(\frac{4\pi\sqrt{\delta n}}{c}, \frac{nd^2\delta^3}{q^3}\right).\end{aligned} \tag{6.20}$$

Absolute convergence of the sum over c is the consequence of the following: by the Weil bound, we have $|S_{\overline{\chi}^2}(n,\sigma;c)| \ll_\epsilon c^{\frac{1}{2}+\epsilon} q^{\frac{1}{2}}$, and analogous to [23] lemmas 10.2 and 10.4, we have $g_{+1}(x,y) \ll xT$, $g_{-1}(x,y) \ll_\epsilon x^{1-\epsilon}T^{1+\epsilon}$ for sufficiently small $\epsilon > 0$.

Now, by Rankin-Selberg theory, we have

$$\sum_{n \le x} |A_\phi(1,n)|^2 \ll x. \tag{6.21}$$

By a trivial bound on g_0 followed by partial summation, Cauchy-Schwarz, and (6.21), we get

$$\mathcal{D} \ll \sum_{d^2 \le (qT)^{3+\epsilon}} \frac{|A_\phi(1,d)|}{d} g_0\left(\frac{d^2}{q^3}\right) \ll_\epsilon T^{2+\epsilon} q^\epsilon. \tag{6.22}$$

Next, we apply a dyadic partition of unity to c in $\mathcal{S}_\sigma$ to get

$$\mathcal{S}_\sigma = \sum_{\substack{d^2 \le (qT)^{3+\epsilon} \\ (d,q)=1}} \frac{A_\phi(1,d)}{d} \sum_{N,\Delta,C} \mathcal{S}_{N,\Delta,C,\sigma}, \tag{6.23}$$

where

$$\mathcal{S}_{N,\Delta,C,\sigma} = \frac{1}{C\sqrt{N}} \sum_{\substack{n,\delta \ge 1 \\ (\delta,q)=1}} \frac{\mu(\delta) A_\phi(n,1) \chi(\delta n)}{\delta^{\frac{3}{2}}} \sum_{q|c} S_{\overline{\chi}^2}(\delta n, \sigma; c) J_{\sigma,\mathcal{I}_0}\left(\frac{4\pi\sqrt{\delta n}}{c}, n, \delta, c\right), \tag{6.24}$$

where

$$J_{\sigma,\mathcal{I}_0}(x,n,\delta,c) = w_{\mathcal{I}_0}(n,\delta,c) \int_{-\infty}^{\infty} K_\sigma(x,t) t\, h\left(t, \frac{nd^2\delta^3}{q^3}\right) dt, \qquad \sigma \in \{1,-1\}, \tag{6.25}$$

with $\mathcal{I}_0 = (q,T,d,\Delta,N,C)$ and $w_{\mathcal{I}_0}$ being a family of 1-inert functions with support on $[N,2N] \times [\Delta, 2\Delta] \times [C,2C]$.

Since c is a positive integer satisfying $q|c$, we have $\frac{q}{2} \le C$. Now, note the following:

- By the Weil bound, we have $|S_{\overline{\chi}^2}(\delta n, \sigma; c)| \ll_\epsilon C^{\frac{1}{2}+\epsilon} q^{\frac{1}{2}}$.
- Analogous to [23] lemmas 10.2 and 10.4, we have $J_{+1,\mathcal{I}_0}(x,n,\delta,c) \ll xT$, $J_{-1,\mathcal{I}_0}(x,n,\delta,c) \ll_\epsilon x^{1-\epsilon}T^{1+\epsilon}$.

- $A_\phi(n,1) \ll_{\phi,\epsilon} n^{\frac{1}{2}+\epsilon} \ll_\epsilon N^{\frac{1}{2}+\epsilon}$ and $A_\phi(1,d) \ll_{\phi,\epsilon} d^{\frac{1}{2}+\epsilon}$ for sufficiently small $\epsilon > 0$.

Using the above, one can conclude the crude bound $\mathcal{S}_\sigma \ll_\epsilon (qT)^{20(1+\epsilon)} \sum_C C^{-\frac{1}{2}+20\epsilon}$ for sufficiently small $\epsilon > 0$; the contribution to this from $C > (qT)^{100}$ is absorbed into the error term in the expression for $\mathcal{M}$ in (6.51). We can therefore assume that $C \le (qT)^{100}$.

To prepare for an application of $GL(3)$ Voronoi summation, we interchange sums to write

$$\mathcal{S}_{N,\Delta,C,\sigma} = \frac{1}{C\sqrt{N}} \sum_{\substack{\delta \ge 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{\frac{3}{2}}} \sum_{q|c} \sum_{n=1}^{\infty} A_\phi(n,1)\chi(n) S_{\overline{\chi}^2}(\delta n, \sigma; c) J_{\sigma,\mathcal{I}_0}\left(\frac{4\pi\sqrt{\delta n}}{c}, n, \delta, c\right). \tag{6.26}$$

By $GL(3)$ Voronoi summation formula (5.0.0.1) followed by application of dyadic partitions of unity to new variables c_2, n_1, n_2 resulting from the Voronoi summation, we get

$$\mathcal{S}_{N,\Delta,C,\sigma} = \sum_{\beta \in \{\pm 1\}} \sum_{C_2,N_1,N_2} \mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}, \tag{6.27}$$

where

$$\begin{aligned}\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta} = \frac{N_1 C_2^2}{C^2\sqrt{N}} \sum_{\substack{\delta \ge 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{\frac{3}{2}}} \sum_{q|c} \sum_{c_1 c_2 = c} \sum_{n_1 n_3 = c_1} \sum_{n_2=1}^{\infty} \frac{1}{c^2} A_\phi(n_2,n_1)\mathcal{T} \times \\ \mathcal{K}_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2 n_1^2}{c_1^3}, \delta, c, c_2, n_1, n_2\right),\end{aligned} \tag{6.28}$$

where

$$\mathcal{T} = \mathcal{T}_{\beta,\delta,\sigma}(c, c_1, n_3, n_2, \chi) = \sideset{}{^*}\sum_{b(c_1)} \sideset{}{^*}\sum_{g(c)} \sum_{a(c)} \sideset{}{^*}\sum_{f(n_3)} \chi^2(g)\chi(a) e\left(\frac{\sigma\overline{g} + \delta a g}{c}\right) e\left(-\frac{\overline{b}a}{c_1}\right) e\left(\frac{bf + \beta n_2 \overline{f}}{n_3}\right), \tag{6.29}$$

and

$$\mathcal{K}_{\beta,\sigma,\mathcal{I}}(y, \delta, c, c_2, n_1, n_2) = \frac{1}{2\pi i} \int_{(\sigma_0)} (8\pi^3 y)^{-s} G_\beta(s) \mathcal{J}_{\sigma,\mathcal{I}}(s, \delta, c, c_2, n_1, n_2)\, ds, \tag{6.30}$$

for $\sigma_0 > \frac{7}{32}$, where

$$\mathcal{J}_{\sigma,\mathcal{I}}(s,\delta,c,c_2,n_1,n_2) = \int_0^\infty J_{\sigma,\mathcal{I}}\left(\frac{4\pi\sqrt{\delta x}}{c}, x, \delta, c, c_2, n_1, n_2\right) x^{-s}\, dx, \tag{6.31}$$

where

$$J_{\sigma,\mathcal{I}}(x,n,\delta,c,c_2,n_1,n_2) = w_{\mathcal{I}}(n,\delta,c,c_2,n_1,n_2)\int_{-\infty}^{\infty} K_\sigma(x,t)t\, h\left(t, \frac{nd^2\delta^3}{q^3}\right) dt, \tag{6.32}$$

with $\mathcal{I} = (q, T, d, N, \Delta, C, C_2, N_1, N_2)$ and $w_{\mathcal{I}}$ being a family of 1-inert functions with support on $[N, 2N] \times [\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2]$. We have $\frac{1}{2} \leq C_2, N_1, N_2$ since c_2, n_1, n_2 are positive integers.

Next, we will truncate the sum in (6.27). For that, we need some control on G_β. We use this opportunity to establish bounds for G_β which will be useful on multiple occasions. We begin by writing

$$\begin{aligned}\mathcal{K}_{\beta,\sigma,\mathcal{I}}(y,\delta,c,c_2,n_1,n_2) = \frac{1}{2\pi}\sum_{\theta\in\{\pm1\}}\int_0^\infty (8\pi^3 y)^{-(\sigma_0+i\theta t)} G_\beta(\sigma_0+i\theta t)\times \\ \mathcal{J}_{\sigma,\mathcal{I}}(\sigma_0+i\theta t,\delta,c,c_2,n_1,n_2)\, dt.\end{aligned} \tag{6.33}$$

6.1 Asymptotic analysis of G_β

Recall that $\alpha_1 = 2it, \alpha_2 = 0, \alpha_3 = -2it$. Let us fix $\sigma_0 > 0$ and vary $t \geq 0$.

Lemma 6.1.0.1. *Let $a > 0$.*

(1) For $t > a$, we have

$$\begin{aligned} G_\beta(\sigma_0 + i\beta t) &\ll_{t,\sigma_0,a} e^{-\frac{\pi t}{2}}, \\ G_\beta(\sigma_0 - i\beta t) &\ll_{t,\sigma_0,a} t^{3(\sigma_0-\frac{1}{2})}. \end{aligned} \tag{6.34}$$

(2) For $t \leq a$, we have

$$G_\beta(\sigma_0 \pm it) \ll_{t,\sigma_0,a} 1. \tag{6.35}$$

(3) For $t > a$, $B > \frac{3}{2}$,

$$G_\beta(\sigma_0 - i\beta t) = t^{3(\sigma_0 - \frac{1}{2})}\left(\frac{t}{e}\right)^{-3i\beta t} W_{\beta,t,\sigma_0,B}(t) + O_{t,\sigma_0,B,a}(t^{-B}), \tag{6.36}$$

where

$$t^k \frac{\partial^k}{\partial t^k} W_{\beta,t,\sigma_0,B}(t) \ll_{t,\sigma_0,B,a,k} 1, \tag{6.37}$$

for $t > a$.

Proof: (2) follows from continuity of G_β on the vertical line $\Re(s) = \sigma_0$.

Let $s = \sigma_0 + i\theta t$ with $\theta \in \{1, -1\}$ as in (6.33). By Stirling's approximation, there exists $t_{t,\sigma_0} > 0$ such that for $t > t_{t,\sigma_0}$, we have

$$\prod_{j=1}^{3} \frac{\Gamma(s+\alpha_j)}{\Gamma(s)} = \prod_{j=1}^{3} s^{\alpha_j}(1 + O_{\alpha_j,\sigma_0}(t^{-1})) = 1 + O_{t,\sigma_0}(t^{-1}), \tag{6.38}$$

and

$$\begin{aligned} \prod_{j=1}^{3} \Gamma(s+\alpha_j) &= (\Gamma(s))^3 \prod_{j=1}^{3} \frac{\Gamma(s+\alpha_j)}{\Gamma(s)} \\ &= k_{\theta,\sigma_0} t^{3(\sigma_0-\frac{1}{2})} e^{-\frac{3\pi t}{2}} \left(\frac{t}{e}\right)^{3i\theta t} (1 + O_{t,\sigma_0}(t^{-1})), \end{aligned} \tag{6.39}$$

where $k_{\theta,\sigma_0} = (2\pi)^{\frac{3}{2}} \exp\left(3i\theta\frac{\pi}{2}\left(\sigma_0 - \frac{1}{2}\right)\right)$. This immediately gives

$$G_\beta(s) = k_{\theta,\sigma_0}\left(e^{i\beta\frac{3\pi\sigma_0}{2}} e^{-(\beta\theta+1)\frac{3\pi t}{2}} + \varsigma e^{-i\beta\frac{\pi\sigma_0}{2}} e^{(\beta\theta-3)\frac{\pi t}{2}}\right) t^{3(\sigma_0-\frac{1}{2})}\left(\frac{t}{e}\right)^{3i\theta t}(1 + O_{t,\sigma_0}(t^{-1})). \tag{6.40}$$

(6.40) implies that for $t > t_{t,\sigma_0}$, we have

$$\begin{aligned} G_\beta(\sigma_0 + i\beta t) &\ll_{t,\sigma_0} e^{-\frac{\pi t}{2}}, \\ G_\beta(\sigma_0 - i\beta t) &\ll_{t,\sigma_0} t^{3(\sigma_0-\frac{1}{2})}. \end{aligned} \tag{6.41}$$

If $t_{t,\sigma_0} \le a$, then (1) is proved. Otherwise, for $a < t \le t_{t,\sigma_0}$, we use (2) to see that $G_\beta(\sigma_0 \pm it) \ll_{t,\sigma_0} 1$, which in turn implies (1).

Upon using more terms from Stirling's approximation to refine the $(1 + O_{t,\sigma_0}(t^{-1}))$ term in (6.40), we get the following asymptotic expansion for $t > t_{t,\sigma_0}$, $N \geq 1$,

$$G_\beta(\sigma_0 - i\beta t) = t^{3(\sigma_0 - \frac{1}{2})} \left(\frac{t}{e}\right)^{-3i\beta t} \left(\sum_{j=0}^{N-1} \frac{c_{t,\sigma_0,\beta,j}}{t^j} + O_{t,\sigma_0,N}(t^{-N})\right) + O_{t,\sigma_0}(e^{-\pi t}). \tag{6.42}$$

Here $c_{t,\sigma_0,\beta,0} = (2\pi)^{\frac{3}{2}} \exp\left(\frac{3i\beta\pi}{4}\right)$, and the above is an asymptotic expansion, i.e. the sequence $\{c_{t,\sigma_0,\beta,j}\}_{j=0}^{\infty}$ does not depend upon N. Now, let $N = \lceil 3\left(\sigma_0 - \frac{1}{2}\right) + B \rceil$ for some $B > \frac{3}{2}$, and for $t > 0$, let

$$W_{\beta,t,\sigma_0,B}(t) := \sum_{j=0}^{N-1} \frac{c_{t,\sigma_0,j}}{t^j}. \tag{6.43}$$

Then we get

$$G_\beta(\sigma_0 - i\beta t) = t^{3(\sigma_0 - \frac{1}{2})} \left(\frac{t}{e}\right)^{-3i\beta t} W_{\beta,t,\sigma_0,B}(t) + O_{t,\sigma_0,B}(t^{-B}), \tag{6.44}$$

for $t > t_{t,\sigma_0}$. Again, if $t_{t,\sigma_0} \leq a$, then (3) is proved. Otherwise, for $a < t \leq t_{t,\sigma_0}$, we use (2) to see that $G_\beta(\sigma_0 - i\beta t) \ll_{t,\sigma_0} 1$, which in turn implies (3). □

Now, analogous to [23] lemmas 10.2 and 10.4, we have

$$\begin{aligned} n^{\lambda_2} \frac{\partial^{\lambda_1+\lambda_2}}{\partial x^{\lambda_1} \partial n^{\lambda_2}} J_{+1,\mathcal{I}}(x, n, \delta, c, c_2, n_1, n_2) &\ll_{\lambda_1,\lambda_2} x(x^{-\lambda_1} + x^{\lambda_1}) T^{1+\lambda_1}, \\ n^{\lambda_2} \frac{\partial^{\lambda_1+\lambda_2}}{\partial x^{\lambda_1} \partial n^{\lambda_2}} J_{-1,\mathcal{I}}(x, n, \delta, c, c_2, n_1, n_2) &\ll_{\lambda_1,\lambda_2,\epsilon} x^{1-\epsilon}(x^{-\lambda_1} + x^{\lambda_1}) T^{1+\lambda_1+\epsilon}, \end{aligned} \tag{6.45}$$

for sufficiently small $\epsilon > 0$, and the implied constants do not depend upon $\mathcal{I}$. Integrating by parts k times followed by applying these derivative bounds gives

$$\begin{aligned}
&\mathcal{J}_{\sigma,\mathcal{I}}(\sigma_0+i\theta t,\delta,c,c_2,n_1,n_2)\\
&\ll_k |\sigma_0+i\theta t|^{-k}\max_{\substack{0\le\lambda_1+\lambda_2\le k\\ \lambda_1,\lambda_2\ge 0}}\int_N^{2N}\left|J_{\sigma,\mathcal{I}}^{(\lambda_1,\lambda_2,0,\ldots,0)}\left(\frac{4\pi\sqrt{\delta x}}{c},x,\cdot\right)\right|\left(\frac{4\pi\sqrt{\delta x}}{c}\right)^{\lambda_1}x^{\lambda_2-\sigma_0}\,dx \quad (6.46)\\
&\ll_{k,\sigma_0}\frac{(\max(1,P))^{2k+1}T^{k+2}N}{t^k},
\end{aligned}$$

for $t>0$, where $P=\frac{4\pi\sqrt{\Delta N}}{C}$; here the implied constant does not depend upon $\mathcal{I}$. Combining this with lemma 6.1.0.1, we have

$$\begin{aligned}
&\mathcal{K}_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2n_1^2}{c_1^3},\delta,c,c_2,n_1,n_2\right)\\
&\ll_{\sigma_0}\left(\frac{n_2n_1^2}{c_1^3}\right)^{-\sigma_0}\sum_{\theta\in\{\pm1\}}\int_0^\infty|G_\beta(\sigma_0+i\theta t)\mathcal{J}_{\sigma,\mathcal{I}}(\sigma_0+i\theta t,\delta,c,c_2,n_1,n_2)|\,dt\\
&\ll_{t,\sigma_0,k}\left(\frac{N_2N_1^2C_2^3}{C^3}\right)^{-\sigma_0}\max(1,P)T^2N\left[1+(\max(1,P))^{2k}T^k\int_1^\infty\frac{e^{-\frac{\pi t}{2}}+t^{3(\sigma_0-\frac{1}{2})}}{t^k}\,dt\right]. \quad (6.47)
\end{aligned}$$

Let $k=\left(2+\lceil 3(\sigma_0-\frac{1}{2})\rceil\right)$ to get

$$\begin{aligned}
&\mathcal{K}_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2n_1^2}{c_1^3},\delta,c,c_2,n_1,n_2\right)\\
&\ll_{t,\sigma_0}\left(\frac{N_2N_1^2C_2^3}{C^3}\right)^{-\sigma_0}(\max(1,P))^{2k+1}T^{k+2}N \quad (6.48)\\
&=\left(\frac{N_2N_1^2C_2^3}{C^3}\right)^{-\sigma_0}(\max(1,P))^{2\lceil 3(\sigma_0-\frac{1}{2})\rceil+5}T^{\lceil 3(\sigma_0-\frac{1}{2})\rceil+4}N,
\end{aligned}$$

where the implied constant does not depend upon $\mathcal{I}$. Next, we note the following crude bounds

$$\frac{1}{c^2}|\mathcal{T}|\le\frac{c^2}{n_1c_2^2}\ll\frac{C^2}{N_1C_2^2}, \quad (6.49)$$

and

$$A_\phi(n_2,n_1)\ll_\epsilon(n_2n_1^2)^{\frac{1}{2}+\epsilon}\ll_\epsilon(N_2N_1^2)^{\frac{1}{2}+\epsilon}\quad\text{for }\epsilon>0. \quad (6.50)$$

After choosing large enough σ_0, say $\sigma_0 = 103$, combining the bounds above implies the crude bound $\mathcal{S}_\sigma \ll_{t,\epsilon} (qT)^{10^5(1+\epsilon)} \sum_{C_2,N_1,N_2} (N_2 N_1 C_2)^{-100}$ for sufficiently small $\epsilon > 0$; contributions to this from all three pieces $C_2 > (qT)^{10^4}$, $N_1 > (qT)^{10^4}$, and $N_2 > (qT)^{10^4}$ are absorbed into the error term in the expression for $\mathcal{M}$ in (6.51). We can therefore assume that $C_2, N_1, N_2 \leq (qT)^{10^4}$. We summarize this chapter in the following proposition.

Proposition 6.1.0.1.

$$\mathcal{M} = \sum_{\substack{\sigma\in\{\pm 1\} \\ \beta\in\{\pm 1\}}} \sum_{\substack{d^2 \leq (qT)^{3+\epsilon} \\ (d,q)=1}} \frac{A_\phi(1,d)}{d} \sum_{N,\Delta,C,C_2,N_1,N_2} \mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta} + O_\epsilon(T^{2+\epsilon} q^\epsilon), \tag{6.51}$$

for $0 < \epsilon < 10^{-10}$*, where* $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ *is defined in* (6.28)*. The dyadic support variables* N, C, C_2, N_1, N_2 *satisfy the following:*

$$\frac{1}{2} \leq N, \Delta, \qquad N\Delta^3 \leq \frac{(qT)^{3+\epsilon}}{d^2}, \qquad \frac{q}{2} \leq C \leq (qT)^{100}, \qquad \frac{1}{2} \leq C_2, N_1, N_2 \leq (qT)^{10^4}. \tag{6.52}$$

To prove theorem 2.0.1, it is sufficient to show the following:

Proposition 6.1.0.2. *$\mathcal{M} \ll_\epsilon T^B q^{\frac{1}{4}+\epsilon}$ for some absolute constant $B > 0$.*

By proposition 6.1.0.1, to prove proposition 6.1.0.2, it is sufficient to show the following:

Proposition 6.1.0.3. *All $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta} \ll_\epsilon T^B q^{\frac{1}{4}+\epsilon}$ for some absolute constant $B > 0$.*

The rest of this document is dedicated to proving proposition 6.1.0.3.

7. ARCHIMEDEAN ASPECTS

Let $P := \frac{4\pi\sqrt{\Delta N}}{C}$ and $\mathcal{K}_{\beta,\sigma,\mathcal{I}} = \mathcal{K}_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2 n_1^2}{c_1^3}, \delta, c, c_2, n_1, n_2\right)$.

7.1 Oscillatory case

Let us apply a dyadic partition of unity to t.

$$\mathcal{K}_{\beta,\sigma,\mathcal{I}} = \sum_{\theta \in \{\pm 1\}} \sum_{j=-\infty}^{\infty} \int_0^\infty \left(\frac{8\pi^3 n_2 n_1^2}{c_1^3}\right)^{-(\sigma_0 + i\theta t)} G_\beta(\sigma_0 + i\theta t)\mathcal{J}_{\sigma,\mathcal{I}_1}(\sigma_0 + i\theta t, \delta, c, c_2, n_1, n_2, t)\, dt, \tag{7.1}$$

where

$$\mathcal{J}_{\sigma,\mathcal{I}_1}(s, \delta, c, c_2, n_1, n_2, t) = \int_0^\infty J_{\sigma,\mathcal{I}_1}\left(\frac{4\pi\sqrt{\delta x}}{c}, x, \delta, c, c_2, n_1, n_2, t\right) x^{-s}\, dx, \tag{7.2}$$

where

$$J_{\sigma,\mathcal{I}_1}(x, n, \delta, c, c_2, n_1, n_2, t) = w_{\mathcal{I}_1}(n, \delta, c, c_2, n_1, n_2, t) \int_{-\infty}^{\infty} K_\sigma(x, r) r\, h\left(r, \frac{nd^2\delta^3}{q^3}\right) dr, \tag{7.3}$$

with $\mathcal{I}_1 = (q, T, d, N, \Delta, C, C_2, N_1, N_2, j)$ and $w_{\mathcal{I}_1}$ being a family of 1-inert functions with support on

$$[N, 2N] \times [\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2] \times [2^{\frac{j}{2}} P, 2^{1+\frac{j}{2}} P]. \tag{7.4}$$

7.1.1 Asymptotic analysis of $J_{\sigma,\mathcal{I}_1}$

The following are analogs of [23] lemmas 10.2, 10.3, 10.4, and 10.5. The derivative bounds hold as expected for all mixed partial derivatives.

Lemma 7.1.0.1.

$$\frac{\partial^k}{\partial x^k} J_{+1,\mathcal{I}_1}(x, n, \delta, c, c_2, n_1, n_2, t) \ll_k x(x^{-k} + x^k)T^{k+1}, \tag{7.5}$$

where the implied constant does not depend upon $\mathcal{I}_1$.

$J_{+1,\mathcal{I}_1}$ *is a family of* 1*-inert functions with respect to the variables* $n, \delta, c, c_2, n_1, n_2, t$ *(while varying over all* $\mathcal{I}_1$*); these variables are supported on* (7.4).

Lemma 7.1.0.2. *Suppose for some* $\epsilon > 0$ *that* $1 \leq T^{2+\epsilon} \ll x$. *Then, for any* $A > 0$,

$$J_{+1,\mathcal{I}_1}(x, n, \delta, c, c_2, n_1, n_2, t) = \sum_{\lambda \in \{\pm 1\}} T^2 x^{-\frac{1}{2}} e^{\lambda i x} W_{\epsilon,A,\mathcal{I}_2}(x, n, \delta, c, c_2, n_1, n_2, t) + O_{A,\epsilon}(x^{-A}), \tag{7.6}$$

where the implied constant does not depend upon $\mathcal{I}_1$. *Here*

$$\mathcal{I}_2 = (q, T, d, N, \Delta, C, C_2, N_1, N_2, j, \lambda). \tag{7.7}$$

We have

$$x^k \frac{\partial^k}{\partial x^k} W_{\epsilon,A,\mathcal{I}_2}(x, n, \delta, c, c_2, n_1, n_2, t) \ll_{k,A,\epsilon} 1, \tag{7.8}$$

where the implied constant does not depend upon $\mathcal{I}_2$.

$W_{\epsilon,A,\mathcal{I}_2}$ *is a family of* 1*-inert functions with respect to the variables* $n, \delta, c, c_2, n_1, n_2, t$ *(while varying over all* $\mathcal{I}_2$*); these variables are supported on* (7.4).

Lemma 7.1.0.3.

$$\frac{\partial^k}{\partial x^k} J_{-1,\mathcal{I}_1}(x, n, \delta, c, c_2, n_1, n_2, t) \ll_{k,\epsilon} x^{1-\epsilon}(x^{-k} + x^k)T^{1+k+\epsilon}, \tag{7.9}$$

for all $\epsilon > 0$, *and the implied constant does not depend upon* $\mathcal{I}_1$.

$J_{-1,\mathcal{I}_1}$ *is a family of* 1*-inert functions with respect to the variables* $n, \delta, c, c_2, n_1, n_2, t$ *(while varying over all* $\mathcal{I}_1$*); these variables are supported on* (7.4).

Lemma 7.1.0.4. *Suppose for some $\epsilon > 0$ that $1 \leq T^{1+\epsilon} \ll x$. Then*

$$J_{-1,\mathcal{I}_1}(x, n, \delta, c, c_2, n_1, n_2, t) \ll_{A,\epsilon} x^{-A}, \tag{7.10}$$

where the implied constant does not depend upon $\mathcal{I}_1$.

7.1.2 Asymptotic analysis of $\mathcal{J}_{\sigma,\mathcal{I}_1}$

Let $\mathcal{J}_{\sigma,\mathcal{I}_1} = \mathcal{J}_{\sigma,\mathcal{I}_1}(\sigma_0 + i\theta t, \delta, c, c_2, n_1, n_2, t)$ as in (7.2).

Lemma 7.1.0.5. ***Oscillatory Case*** *Fix $\vartheta > 0$. Let $P \geq T^3 q^{\vartheta}$. Note that in this case $P \geq 1$.*

(1)

$$\mathcal{J}_{\sigma,\mathcal{I}_1} \ll_{k,\sigma_0} \frac{q^4 T^{k+6} P^{2k+1}}{t^k}, \tag{7.11}$$

for $t > 0$.

(2)

$$\mathcal{J}_{-1,\mathcal{I}_1} \ll_{B,\sigma_0} P^{-B} N, \tag{7.12}$$

for $B > 0$.

(3) For $\mathcal{I}_1$ varying over $\{\mathcal{I}_1 \mid j > 0 \text{ or } j < -6\}$, we have

$$\mathcal{J}_{+1,\mathcal{I}_1} \ll_{B,\sigma_0} P^{-B} N, \tag{7.13}$$

for $B > 0$.

(4) For $\mathcal{I}_1$ varying over $\{\mathcal{I}_1 \mid -6 \leq j \leq 0\}$, we have,

$$\mathcal{J}_{+1,\mathcal{I}_1} = T^2 P^{-1} N^{1-\sigma_0} \left(\frac{t^2 c^2}{4 e^2 \pi^2 \delta} \right)^{-i\theta t} \varrho_{B,\theta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) + O_{B,\sigma_0,\theta}(P^{-B} N), \tag{7.14}$$

where $\varrho_{B,\theta,\mathcal{I}_1}$ is a 1-inert family of functions (varying $\mathcal{I}_1$ over $\{\mathcal{I}_1 \mid -6 \leq j \leq 0\}$) with support on (7.37).

Proof:

(1) Similar to (6.46), we have, by lemma 7.1.0.1 or lemma 7.1.0.3 depending upon $\sigma \in \{\pm 1\}$, that

$$\mathcal{J}_{\sigma,\mathcal{I}_1} \ll_{k,\sigma_0} \frac{T^{k+2}P^{2k+1}N}{t^k}, \tag{7.15}$$

for $t > 0$. The bound $N \ll (qT)^{3+\epsilon}$ from (6.52) then gives

$$\mathcal{J}_{\sigma,\mathcal{I}_1} \ll_{k,\sigma_0} \frac{q^4T^{k+6}P^{2k+1}}{t^k}, \tag{7.16}$$

for $t > 0$. The implied constant does not depend upon $\mathcal{I}_1$.

(2) By lemma 7.1.0.4,

$$\mathcal{J}_{-1,\mathcal{I}_1} \ll_{B,\sigma_0} P^{-B}N, \tag{7.17}$$

for $B > 0$; here the implied constant does not depend upon $\mathcal{I}_1$.

(3) By lemma 7.1.0.2,

$$\begin{aligned} \mathcal{J}_{+1,\mathcal{I}_1} = T^2P^{-\frac{1}{2}}N^{-\sigma_0} \sum_{\lambda\in\{\pm1\}} \int_0^\infty & z_{B,\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t)\times \\ & \exp\left(i\Phi_{\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t)\right)\, dx+ \\ & O_{B,\sigma_0}(P^{-B}N), \end{aligned} \tag{7.18}$$

for $B > 0$; here the implied constant does not depend upon $\mathcal{I}_1$.

$$z_{B,\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t) = \left(\frac{\Delta}{\delta}\right)^{\frac{1}{4}} \left(\frac{N}{x}\right)^{\sigma_0+\frac{1}{4}} \left(\frac{c}{C}\right)^{\frac{1}{2}} W_{B,\mathcal{I}_2}\left(\frac{4\pi\sqrt{\delta x}}{c}, x,\delta,c,c_2,n_1,n_2,t\right), \tag{7.19}$$

is a 1-inert family of functions (while varying over all $\mathcal{I}_2$) supported on (7.4).

$$\Phi_{\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t) = \lambda P\left(\frac{\delta x}{\Delta N}\right)^{\frac{1}{2}}\left(\frac{C}{c}\right) - \theta t\log(x). \tag{7.20}$$

Now, on the support of $z_{B,\mathcal{I}_2}$, for integer $a \geq 1$, we have

$$\frac{\partial^a}{\partial x^a}\Phi_{\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t) \ll_a \frac{Y}{N^a}, \tag{7.21}$$

where $Y = \max(1, 2^{\frac{j}{2}})P$, and the implied constant does not depend upon $\mathcal{I}_2$. Now, for $\mathcal{I}_2$ varying over $\mathcal{F}_1 = \{\mathcal{I}_2 \mid (\lambda = -\theta) \text{ or } (\lambda = \theta \text{ and } (j > 0 \text{ or } j < -6))\}$, we have, on the support of $z_{B,\mathcal{I}_2}$, that

$$\frac{\partial}{\partial x}\Phi_{\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t) \gg \frac{Y}{N}, \tag{7.22}$$

where the implied constant does not depend upon $\mathcal{I}_2 \in \mathcal{F}_1$. Therefore, by [59] lemma 4.2, for $\mathcal{I}_2 \in \mathcal{F}_1$, we have

$$\int_0^\infty z_{B,\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t)\exp\left(i\Phi_{\mathcal{I}_2}(x,\delta,c,c_2,n_1,n_2,t)\right)\,dx \ll_B Y^{-(B+1)}N, \tag{7.23}$$

where the implied constant does not depend upon $\mathcal{I}_2 \in \mathcal{F}_1$. Therefore, for $\mathcal{I}_1$ varying over $\mathcal{F}_2 = \{\mathcal{I}_1 \mid j > 0 \text{ or } j < -6\}$, we have

$$\mathcal{J}_{+1,\mathcal{I}_1} \ll_{B,\sigma_0} P^{-B}N, \tag{7.24}$$

and for $\mathcal{I}_1$ varying over $\mathcal{F}_3 = \{\mathcal{I}_1 \mid -6 \leq j \leq 0\}$, we have,

$$\mathcal{J}_{+1,\mathcal{I}_1} = T^2P^{-\frac{1}{2}}N^{-\sigma_0}\int_0^\infty \mu_{B,\theta,\mathcal{I}_1}(x,\delta,c,c_2,n_1,n_2,t)\exp\left(i\Psi_{\mathcal{I}_1}(x,\delta,c,c_2,n_1,n_2,t)\right)\,dx + O_{B,\sigma_0}(P^{-B}N), \tag{7.25}$$

where the $\mu_{B,\theta,\mathcal{I}_1} = z_{B,\mathcal{I}_2}$ and $\Psi_{\mathcal{I}_1} = \Phi_{\mathcal{I}_2}$ with $\lambda = \theta$ in $\mathcal{I}_2$. $\mu_{B,\theta,\mathcal{I}_1}$ is a 1-inert family of functions (while varying over all $\mathcal{I}_1 \in \mathcal{F}_3$) with support on (7.4). We write down $\Psi_{\mathcal{I}_1}$

explicitly below:

$$\Psi_{\mathcal{I}_1}(x,\delta,c,c_2,n_1,n_2,t) = \theta\left(P\left(\frac{\delta x}{\Delta N}\right)^{\frac{1}{2}}\left(\frac{C}{c}\right) - t\log(x)\right). \tag{7.26}$$

(4) Now we wish to analyze the integral in (7.25); we start by assuming that $\mathcal{I}_1 \in \mathcal{F}_3$; in particular, we have $-6 \leq j \leq 0$. Write $y = \frac{x\pi^2\delta}{t^2c^2}$. On the support of $\mu_{B,\theta,\mathcal{I}_1}$, we have

$$2^{-8} \leq 2^{-j-8} = \frac{N\pi^2\Delta}{(2^{1+\frac{j}{2}}P)^2(2C)^2} \leq y \leq \frac{(2N)\pi^2(2\Delta)}{(2^{\frac{j}{2}}P)^2C^2} = 2^{-j-2} \leq 2^4. \tag{7.27}$$

Performing the substitution $x = \frac{yt^2c^2}{\pi^2\delta}$ in the integral followed by a dyadic partition of unity to y gives

$$\begin{aligned}\mathcal{J}_{+1,\mathcal{I}_1} = T^2P^{-\frac{1}{2}}N^{1-\sigma_0}\left(\frac{t^2c^2}{\pi^2\delta}\right)^{-i\theta t}\sum_{k=-18}^{8}\int_0^\infty \xi_{B,\theta,\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\times \\ \exp\left(i\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\right)\,dy+ \\ O_{B,\sigma_0}(P^{-B}N).\end{aligned} \tag{7.28}$$

Here $\mathcal{I}_3 = (q,T,d,N,\Delta,C,C_2,N_1,N_2,j,k)$ is varying over

$$\mathcal{F}_4 = \{\mathcal{I}_3 \mid -6 \leq j \leq 0,\ -18 \leq k \leq 8\}. \tag{7.29}$$

$\xi_{B,\theta,\mathcal{I}_3}$ is a 1-inert family of functions (while varying over all $\mathcal{I}_3 \in \mathcal{F}_4$) with support on $([2^{\frac{k}{2}},2^{1+\frac{k}{2}}]\cap[2^{-8},2^4])\times[\Delta,2\Delta]\times[C,2C]\times[C_2,2C_2]\times[N_1,2N_1]\times[N_2,2N_2]\times[2^{\frac{j}{2}}P,2^{1+\frac{j}{2}}P]$, and

$$\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t) = \theta t\left(4\sqrt{y} - \log y\right). \tag{7.30}$$

We have truncated $\sum_{k=-\infty}^{\infty}$ to $\sum_{k=-18}^{8}$ since $[2^{\frac{k}{2}},2^{1+\frac{k}{2}}] \cap [2^{-8},2^4] = \emptyset$ for $k < -18$, $k > 8$.

Now, on the support of $\xi_{B,\theta,\mathcal{I}_3}$, for non-negative integers $a_1, \ldots, a_6$, we have

$$\frac{\partial^{a_1+\cdots+a_7}\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)}{\partial y^{a_1}\partial\delta^{a_2}\partial c^{a_3}\partial c_2^{a_4}\partial n_1^{a_5}\partial n_2^{a_6}\partial t^{a_7}} \ll_{a_1,\ldots,a_7} \frac{P}{(2^{\frac{k}{2}})^{a_1}\Delta^{a_2}C^{a_3}C_2^{a_4}N_1^{a_5}N_2^{a_6}(2^{\frac{j}{2}}P)^{a_7}}, \quad (7.31)$$

where the implied constant does not depend upon $\mathcal{I}_3$.

For $\mathcal{I}_3$ varying over $\mathcal{F}_5 = \{\mathcal{I}_3 \mid -6 \leq j \leq 0 \text{ and } (-18 \leq k \leq -7 \text{ or } -3 \leq k \leq 8)\}$, we have, on the support of $\xi_{B,\theta,\mathcal{I}_3}$, that

$$\frac{\partial}{\partial y}\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t) \gg \frac{P}{2^{\frac{k}{2}}}, \quad (7.32)$$

where the implied constant does not depend upon $\mathcal{I}_3 \in \mathcal{F}_5$. Therefore, by [59] lemma 4.2, for $\mathcal{I}_3 \in \mathcal{F}_5$, we have

$$\int_0^\infty \xi_{B,\theta,\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\exp\left(i\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\right)\,dy \ll_{B,\theta} P^{-(B+1)}, \quad (7.33)$$

where the implied constant does not depend upon $\mathcal{I}_3 \in \mathcal{F}_5$. Therefore,

$$\begin{aligned}\mathcal{J}_{+1,\mathcal{I}_1} = T^2P^{-\frac{1}{2}}N^{1-\sigma_0}\left(\frac{t^2c^2}{\pi^2\delta}\right)^{-i\theta t}\sum_{k=-6}^{-4}\int_0^\infty \xi_{B,\theta,\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\times \\ \exp\left(i\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\right)\,dy+ \\ O_{B,\sigma_0,\theta}(P^{-B}N).\end{aligned} \quad (7.34)$$

Now, for $\mathcal{I}_3$ varying over $\mathcal{F}_6 = \{\mathcal{I}_3 \mid -6 \leq j \leq 0,\ -6 \leq k \leq -4\}$, we have, on the support of $\xi_{B,\theta,\mathcal{I}_3}$, that

$$\frac{\partial^2}{\partial y^2}\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t) \gg \frac{P}{(2^{\frac{k}{2}})^2}, \quad (7.35)$$

where the implied constant does not depend upon $\mathcal{I}_3 \in \mathcal{F}_6$. Therefore, by [59] lemma

4.3, for $\mathcal{I}_3 \in \mathcal{F}_6$, we have

$$\begin{aligned} \int_0^\infty \xi_{B,\theta,\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t) \exp\left(i\Theta_{\mathcal{I}_3}(y,\delta,c,c_2,n_1,n_2,t)\right) \, dy \\ = \frac{2^{\frac{k}{2}}}{\sqrt{P}}(2e)^{2i\theta t}\Xi_{B,\theta,\mathcal{I}_3}(\delta,c,c_2,n_1,n_2,t) + O_{B,\theta}(P^{-(B+1)}), \end{aligned} \tag{7.36}$$

where the implied constant does not depend upon $\mathcal{I}_3 \in \mathcal{F}_6$. Here $\Xi_{B,\theta,\mathcal{I}_3}$ is a 1-inert family of functions (while varying over $\mathcal{I}_3 \in \mathcal{F}_6$) with support on

$$[\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2] \times [2^{\frac{j}{2}}P, 2^{1+\frac{j}{2}}P]. \tag{7.37}$$

Therefore,

$$\begin{aligned} \mathcal{J}_{+1,\mathcal{I}_1} = T^2P^{-1}N^{1-\sigma_0}\left(\frac{t^2c^2}{4e^2\pi^2\delta}\right)^{-i\theta t} \varrho_{B,\theta,\mathcal{I}_1}(\delta,c,c_2,n_1,n_2,t) + \\ O_{B,\sigma_0,\theta}(P^{-B}N), \end{aligned} \tag{7.38}$$

where

$$\varrho_{B,\theta,\mathcal{I}_1}(\delta,c,c_2,n_1,n_2,t) = \sum_{k=-6}^{-4} 2^{\frac{k}{2}}\Xi_{B,\theta,\mathcal{I}_3}(\delta,c,c_2,n_1,n_2,t). \tag{7.39}$$

$\varrho_{B,\theta,\mathcal{I}_1}$ is a 1-inert family of functions (varying $\mathcal{I}_1 \in \mathcal{F}_3$) with support on (7.37).

□

7.1.3 Asymptotic analysis of $\mathcal{K}_{\beta,\sigma,\mathcal{I}}$

Lemma 7.1.0.6. ***Oscillatory Case*** *Fix $\vartheta > 0$. Let $P \geq T^3q^\vartheta$. Note that in this case $P \geq 1$. Then*

(1) $\mathcal{K}_{\beta,-1,\mathcal{I}} \ll_{\vartheta,\sigma_0,t} (qT)^{-100}$.

(2) $\mathcal{K}_{\beta,+1,\mathcal{I}} \ll_{\vartheta,\sigma_0,t} (qT)^{-100}$ for $\mathcal{I}$ varying over $\{\mathcal{I} \mid N_2N_1^2C_2^3 \not\asymp CP\Delta\}$.

(3) For $\mathcal{I}$ varying over $\{\mathcal{I} \mid N_2N_1^2C_2^3 \asymp CP\Delta\}$, we have

$$\mathcal{K}_{\beta,+1,\mathcal{I}} = \mathcal{P}^{-1}T^2P^{-2}NL_{\vartheta,\sigma_0,t,\beta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2) + O_{\vartheta,\sigma_0,t,\epsilon}((qT)^{-100}), \tag{7.40}$$

with

$$L_{\vartheta,\sigma_0,t,\beta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2) = \int_{|\mathbf{u}|\ll(qT)^{\epsilon}} F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\mathbf{u}) \left(\frac{N_2}{n_2}\right)^{u_1} \left(\frac{C}{c}\right)^{u_2} \left(\frac{N_1}{n_1}\right)^{u_3} \left(\frac{C_2}{c_2}\right)^{u_4} \left(\frac{\Delta}{\delta}\right)^{u_5} d\mathbf{u}. \tag{7.41}$$

Here $\mathcal{P} = e\left(-\frac{\beta n_2 n_1^2 c_2^3}{c\delta}\right)$ is the Conrey-Iwaniec phase term, and the integral is over 5 vertical lines in the complex plane such that $\Re(u_k) = \sigma_k$ for $1 \leq k \leq 5$. Here $F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}$ is entire and $F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\mathbf{u}) \ll_{\vartheta,\sigma_0,t,\boldsymbol{\sigma},A} (1+|\mathbf{u}|)^{-A}$ for $A>0$, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5)$.

Proof: By lemma 7.1.0.5 (1) and lemma 6.1.0.1 (1), the contribution to the right hand side of (7.1) from all $j>0$ such that $2^{\frac{j}{2}} > P^A$ for some large $A>0$ depending upon ϑ, σ_0, t is $O_{\vartheta,\sigma_0,t}((qT)^{-100})$.

By lemma 7.1.0.5 (2) and lemma 6.1.0.1, taking $B>0$ sufficiently large depending upon A we get that for $\sigma=-1$, the contribution to the right hand side of (7.1) from all integers j such that $2^{\frac{j}{2}} \leq P^A$ is $O_{\vartheta,\sigma_0,t}((qT)^{-100})$. Therefore, we have

$$\mathcal{K}_{\beta,-1,\mathcal{I}} \ll_{\vartheta,\sigma_0,t} (qT)^{-100}. \tag{7.42}$$

By lemma 7.1.0.5 (3) and lemma 6.1.0.1, taking $B>0$ sufficiently large depending upon A we get that for $\sigma=+1$, the contribution to the right hand side of (7.1) from all integers $j \notin [-6,0]$ such that $2^{\frac{j}{2}} \leq P^A$ is $O_{\vartheta,\sigma_0,t}((qT)^{-100})$. Therefore, we have

$$\mathcal{K}_{\beta,+1,\mathcal{I}} = \sum_{\theta\in\{\pm1\}} \sum_{j=-6}^{0} \int_0^{\infty} \left(\frac{8\pi^3 n_2 n_1^2}{c_1^3}\right)^{-(\sigma_0+i\theta t)} G_\beta(\sigma_0+i\theta t)\mathcal{J}_{+1,\mathcal{I}_1}(\sigma_0+i\theta t,\delta,c,c_2,n_1,n_2,t)\,dt + O_{\vartheta,\sigma_0,t}((qT)^{-100}). \tag{7.43}$$

Next, by lemma 7.1.0.5 (1) and lemma 6.1.0.1, specifically the exponential decay of $G_\beta(\sigma_0 + i\beta t)$, we have

$$\mathcal{K}_{\beta,+1,\mathcal{I}} = \sum_{j=-6}^{0} \int_0^\infty \left(\frac{8\pi^3 n_2 n_1^2}{c_1^3}\right)^{-(\sigma_0 - i\beta t)} G_\beta(\sigma_0 - i\beta t)\mathcal{J}_{+1,\mathcal{I}_1}(\sigma_0 - i\beta t, \delta, c, c_2, n_1, n_2, t)\, dt + O_{\vartheta,\sigma_0,t}((qT)^{-100}). \tag{7.44}$$

$j \geq -6$ implies that we can truncate the integral above so that $t > 2^{\frac{j}{2}} P \geq 2^{-3} P \geq 2^{-3}$. This allows us to apply lemma 6.1.0.1 (3) to $G_\beta(\sigma_0 - i\beta t)$. By lemma 7.1.0.5 (4), we have

$$\begin{aligned}\mathcal{K}_{\beta,+1,\mathcal{I}} = \left(\frac{8\pi^3 n_2 n_1^2}{c_1^3}\right)^{-\sigma_0} T^2 P^{3\sigma_0 - \frac{5}{2}} N^{1-\sigma_0} \sum_{j=-6}^{0} \int_0^\infty \zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) \times \\ \exp(i\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t))\, dt + \\ O_{\vartheta,\sigma_0,t}((qT)^{-100}),\end{aligned} \tag{7.45}$$

where

$$\zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) = \left(\frac{t}{P}\right)^{3(\sigma_0 - \frac{1}{2})} W_{\beta,t,\sigma_0,B}(t)\varrho_{B,-\beta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t), \tag{7.46}$$

for some large $B > 0$ depending upon ϑ, σ_0, t, and

$$\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) = \beta t \log\left(\frac{2\pi e n_2 n_1^2 c_2^3}{ct\delta}\right). \tag{7.47}$$

$\zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}$ is a 1-inert family of functions (varying $\mathcal{I}_1 \in \mathcal{F}_3$) with support on (7.37).

Now, for $\mathcal{I}_1$ varying over $\mathcal{F}_7 = \left\{\mathcal{I}_1 \mid \left|\log\left(\frac{2\pi N_2 N_1^2 C_2^3}{CP\Delta}\right)\right| > 100,\ -6 \leq j \leq 0\right\}$, on the support of $\zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}$, we have $\frac{\partial}{\partial t}\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) \gg \frac{Y}{Z}$ and $\frac{\partial^a}{\partial t^a}\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) \ll_a \frac{Y}{Z^a}$ for $a \geq 2$ with $Y = P$, $Z = 2^{\frac{j}{2}} P$; therefore, by [59] lemma 4.2, we have

$$\int_0^\infty \zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t)\exp(i\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t))\, dt \ll_{\vartheta,\sigma_0,t,B} P^{-B}, \tag{7.48}$$

for $B > 0$ arbitrarily large, where the implied constant does not depend upon $\mathcal{I}_1 \in \mathcal{F}_7$. Thus, for $\mathcal{I}$ varying over $\mathcal{F}_8 = \left\{\mathcal{I} \mid \left|\log\left(\frac{2\pi N_2 N_1^2 C_2^3}{CP\Delta}\right)\right| > 100\right\}$, we have

$$\mathcal{K}_{\beta,+1,\mathcal{I}} \ll_{\vartheta,\sigma_0,t} (qT)^{-100}, \tag{7.49}$$

where the implied constant does not depend upon $\mathcal{I}$.

Now, for $\mathcal{I}_1$ varying over $\mathcal{F}_9 = \left\{\mathcal{I}_1 \mid \left|\log\left(\frac{2\pi N_2 N_1^2 C_2^3}{CP\Delta}\right)\right| \leq 100,\ -6 \leq j \leq 0\right\}$, on the support of $\zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}$, we have $\frac{\partial^2}{\partial t^2}\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) \gg \frac{Y}{Z^2}$ and

$$\frac{\partial^{a_1+\cdots+a_6}}{\partial t^{a_1}\partial\delta^{a_2}\partial c^{a_3}\partial c_2^{a_4}\partial n_1^{a_5}\partial n_2^{a_6}}\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t) \ll_{a_1,\ldots,a_6} \frac{Y}{Z^{a_1}\Delta^{a_2}C^{a_3}C_2^{a_4}N_1^{a_5}N_2^{a_6}}, \tag{7.50}$$

for $a_1 \geq 1$, $a_2, \ldots, a_6 \geq 0$ with $Y = P$, $Z = 2^{\frac{j}{2}}P$; therefore, by [59] lemma 4.3, we have

$$\begin{aligned}
&\int_0^\infty \zeta_{\vartheta,\sigma_0,t,\beta,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t)\exp(i\Omega_{\mathcal{I}_1}(\delta, c, c_2, n_1, n_2, t))\,dt \\
&= \frac{2^{\frac{j}{2}}P}{\sqrt{P}}\exp\left(2\pi i\frac{\beta n_2 n_1^2 c_2^3}{c\delta}\right)E_{\vartheta,\sigma_0,t,\beta,B,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2) + O_{\vartheta,\sigma_0,t,B}(P^{-B}),
\end{aligned} \tag{7.51}$$

for $B > 0$ arbitrarily large, where the implied constant does not depend upon $\mathcal{I}_1 \in \mathcal{F}_9$. Here $E_{\vartheta,\sigma_0,t,\beta,B,\mathcal{I}_1}$ is a 1-inert family of functions (while varying over $\mathcal{I}_1 \in \mathcal{F}_9$) with support on $[\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2]$. Thus, for $\mathcal{I}$ varying over $\mathcal{F}_{10} = \left\{\mathcal{I} \mid \left|\log\left(\frac{2\pi N_2 N_1^2 C_2^3}{CP\Delta}\right)\right| \leq 100\right\}$, we have

$$\begin{aligned}
\mathcal{K}_{\beta,+1,\mathcal{I}} = \left(\frac{N_2 N_1^2 C_2^3}{C^3}\right)^{-\sigma_0} T^2 P^{3\sigma_0-2} N^{1-\sigma_0} e\left(\frac{\beta n_2 n_1^2 c_2^3}{c\delta}\right) L_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\delta, c, c_2, n_1, n_2) + \\
O_{\vartheta,\sigma_0,t}((qT)^{-100}),
\end{aligned} \tag{7.52}$$

where the implied constant does not depend upon $\mathcal{I}$. Here

$$\begin{aligned}
&L_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\delta, c, c_2, n_1, n_2) := \\
&\left(\frac{N_2 N_1^2 C_2^3}{C^3}\right)^{\sigma_0}\left(\frac{8\pi^3 n_2 n_1^2 c_2^3}{c^3}\right)^{-\sigma_0}\sum_{j=-6}^{0} 2^{\frac{j}{2}} E_{\vartheta,\sigma_0,t,\beta,B,\mathcal{I}_1}(\delta, c, c_2, n_1, n_2),
\end{aligned} \tag{7.53}$$

for a large enough $B > 0$ that depends upon upon $\vartheta, \sigma_0, t, \beta$. We have that $L_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}$ is a 1-inert family of functions (while varying over $\mathcal{I} \in \mathcal{F}_{10}$) with support on $[\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2]$. By Mellin inversion, we have

$$\begin{aligned} &L_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\delta, c, c_2, n_1, n_2) = \\ &\int_{\prod_{k=1}^{5}(\sigma_k)} f_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\mathbf{u}) \left(\frac{N_2}{n_2}\right)^{u_1} \left(\frac{C}{c}\right)^{u_2} \left(\frac{N_1}{n_1}\right)^{u_3} \left(\frac{C_2}{c_2}\right)^{u_4} \left(\frac{\Delta}{\delta}\right)^{u_5} d\mathbf{u}, \end{aligned} \tag{7.54}$$

where rapid decay of $f_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}$ allows us to truncate the quadruple integral such that $|\mathbf{u}| \ll (qT)^{\epsilon}$. Now, $\left|\log\left(\frac{2\pi N_2 N_1^2 C_2^3}{CP\Delta}\right)\right| \leq 100$ implies

$$\frac{NN_2N_1^2C_2^3}{C^3P^3} \asymp \frac{NCP\Delta}{C^3P^3} = \frac{N\Delta}{C^2P^2} = \frac{1}{16\pi^2}, \tag{7.55}$$

which implies

$$\left(\frac{N_2N_1^2C_2^3}{C^3}\right)^{-\sigma_0} P^{3\sigma_0} N^{-\sigma_0} = \left(\frac{NN_2N_1^2C_2^3}{C^3P^3}\right)^{-\sigma_0} \ll_{\sigma_0} 1. \tag{7.56}$$

Let $F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}} = \left(\frac{NN_2N_1^2C_2^3}{C^3P^3}\right)^{-\sigma_0} f_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}$ to complete the proof. □

7.2 Non-oscillatory case

7.2.1 Asymptotic analysis of $\mathcal{J}_{\sigma,\mathcal{I}}$

Let $\mathcal{J}_{\sigma,\mathcal{I}} = \mathcal{J}_{\sigma,\mathcal{I}}(\sigma_0 + i\theta t, \delta, c, c_2, n_1, n_2)$.

Lemma 7.2.0.1.

$$\begin{aligned} \frac{\partial^\lambda}{\partial x^\lambda} J_{+1,\mathcal{I}}(x, n, \delta, c, c_2, n_1, n_2) &\ll_\lambda x^2(x^{-\lambda} + x^\lambda)T^\lambda, \\ \frac{\partial^\lambda}{\partial x^\lambda} J_{-1,\mathcal{I}}(x, n, \delta, c, c_2, n_1, n_2) &\ll_{\lambda,\epsilon} x^{2-\epsilon}(x^{-\lambda} + x^\lambda)T^{\lambda+\epsilon}, \end{aligned} \tag{7.57}$$

for sufficiently small $\epsilon > 0$, and the implied constants do not depend upon $\mathcal{I}$.

Further, $J_{+1,\mathcal{I}}$ and $J_{-1,\mathcal{I}}$ are families of 1-inert functions with respect to the variables $n, \delta, c, c_2, n_1, n_2$ (while varying over all $\mathcal{I}$) with these variables being supported on $[N, 2N] \times [\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2]$.

Proof: Recall that

$$J_{\sigma,\mathcal{I}}(x,n,\delta,c,c_2,n_1,n_2) = w_{\mathcal{I}}(n,\delta,c,c_2,n_1,n_2)\int_{-\infty}^{\infty} K_\sigma(x,t)t\,h\left(t,\frac{nd^2\delta^3}{q^3}\right)\,dt, \qquad \sigma\in\{1,-1\}. \tag{7.58}$$

To prove the bound for $J_{+1,\mathcal{I}}$, we mimic the proof of [23] lemma 10.2, except that we move the line of integration to $\Im(t) = -1$ instead of $\Im(t) = -\frac{1}{2}$. To prove the bound for $J_{-1,\mathcal{I}}$, we mimic the proof of [23] lemma 10.4, except that we apply [60] 8.486.10 twice instead of just once. To prove the final statement on inertness, we follow the proofs of [23] lemmas 10.2 and 10.4. □

Lemma 7.2.0.2. ***Non-oscillatory Case*** *Fix $\vartheta > 0$. Let $P \leq T^3 q^\vartheta$. For $0 < \epsilon < 1$, we can write*

$$\mathcal{J}_{\sigma,\mathcal{I}} = T^\epsilon P^{2-\epsilon} N^{1-\sigma_0} H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2,t), \tag{7.59}$$

where

$$\frac{\partial^k}{\partial t^k} H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2,t) \ll_{\vartheta,\sigma_0,B,\epsilon,k} \left(\frac{X_{\mathcal{I}}}{t+1}\right)^B, \tag{7.60}$$

with $X_{\mathcal{I}} = T\max(1,P)^2$ for $B > 0$. Further, $H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}$ is an $X_{\mathcal{I}}$-inert family with respect to δ, c, c_2, n_1, n_2 (varying over all $\mathcal{I}$); these variables are supported on $[\Delta, 2\Delta]\times[C,2C]\times[C_2,2C_2]\times[N_1,2N_1]\times[N_2,2N_2]$. All mixed partial derivative bounds for $H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}$ behave as expected.

Proof: Write

$$\eta_{\vartheta,\sigma,\sigma_0,\epsilon,\mathcal{I}}(x,\delta,c,c_2,n_1,n_2) := \left(\frac{N}{x}\right)^{\sigma_0} T^{-\epsilon}P^{-2+\epsilon} J_{\sigma,\mathcal{I}}\left(\frac{4\pi\sqrt{\delta x}}{c},x,\delta,c,c_2,n_1,n_2\right). \tag{7.61}$$

By lemma 7.2.0.1, we get that $\eta_{\vartheta,\sigma,\sigma_0,\epsilon,\mathcal{I}}$ is an $X_{\mathcal{I}}$-inert family of functions (while varying over all $\mathcal{I}$) with $X_{\mathcal{I}} = T\max(1,P)^2$; the functions in this family have support on $[N,2N]\times$

$[\Delta, 2\Delta] \times [C, 2C] \times [C_2, 2C_2] \times [N_1, 2N_1] \times [N_2, 2N_2]$. Finally, let

$$H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}(\delta, c, c_2, n_1, n_2, t) := N^{-1} \int_0^\infty \eta_{\vartheta,\sigma,\sigma_0,\epsilon,\mathcal{I}}(x, \delta, c, c_2, n_1, n_2) x^{-i\theta t}\, dx. \tag{7.62}$$

(7.60) follows by repeated integration by parts. □

7.2.2 Asymptotic analysis of $\mathcal{K}_{\beta,\sigma,\mathcal{I}}$

Lemma 7.2.0.3. ***Non-oscillatory Case*** *Fix $\vartheta > 0$. Let $P \le T^3 q^\vartheta$. For $0 < \epsilon < 1$, we can write*

$$\begin{aligned}\mathcal{K}_{\beta,\sigma,\mathcal{I}} = \mathcal{P}^{-1} \left(\frac{P^2 P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0} T^\epsilon P^{2-\epsilon} N X_{\mathcal{I}}^{\frac{1}{2}} L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\delta, c, c_2, n_1, n_2) + \\ O_{\vartheta,\sigma_0,\epsilon}((qT)^{-100}),\end{aligned} \tag{7.63}$$

where

$$\begin{aligned}L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\delta, c, c_2, n_1, n_2) = \int_{|\mathbf{u}| \ll X_{\mathcal{I}}(qT)^\epsilon} F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\mathbf{u}) \int_{|t| \ll (qT)^\epsilon + P'} \left(\frac{n_2 n_1^2 c_2^3}{c\delta}\right)^{-it} f_{\beta,\sigma,\mathcal{I}}(t) \times \\ \left(\frac{N_2}{n_2}\right)^{u_1} \left(\frac{C}{c}\right)^{u_2} \left(\frac{N_1}{n_1}\right)^{u_3} \left(\frac{C_2}{c_2}\right)^{u_4} \left(\frac{\Delta}{\delta}\right)^{u_5} dt d\mathbf{u}.\end{aligned} \tag{7.64}$$

Here $X_{\mathcal{I}} = T \max(1, P)^2$, $P' = \frac{N_2 N_1^2 C_2^3}{C\Delta}$, and $\mathcal{P} = e\left(-\frac{\beta n_2 n_1^2 c_2^3}{c\delta}\right)$ is the Conrey-Iwaniec phase term. We have $f_{\beta,\sigma,\mathcal{I}}(t) \ll (1 + |t|)^{-\frac{1}{2}}$. The $\mathbf{u}$-integral is over 5 vertical lines in the complex plane such that $\Re(u_k) = \sigma_k$ for $1 \le k \le 5$. Here $F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ is entire and $F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\mathbf{u}) \ll_{\vartheta,\sigma_0,\epsilon,\boldsymbol{\sigma},A} \prod_{k=1}^5 \left(1 + \frac{|u_k|}{X_{\mathcal{I}}}\right)^{-A}$ for $A > 0$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$.

In particular, when $\frac{P^2 P'}{X_{\mathcal{I}}^3} \gg_\epsilon (qT)^\epsilon$, taking large σ_0 depending upon ϵ gives $\mathcal{K}_{\beta,\sigma,\mathcal{I}} \ll_{\vartheta,\epsilon} (qT)^{-100}$.

Proof: By lemma 7.2.0.2, we can write

$$\mathcal{K}_{\beta,\sigma,\mathcal{I}} = T^\epsilon P^{2-\epsilon} N^{1-\sigma_0} \left(\frac{N_2 N_1^2 C_2^3}{C^3}\right)^{-\sigma_0} X_{\mathcal{I}}^{3(\sigma_0 - \frac{1}{2}) + 2} L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\delta, c, c_2, n_1, n_2), \tag{7.65}$$

where

$$L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2) = \left(\frac{N_2N_1^2C_2^3}{C^3}\right)^{\sigma_0}\frac{X_{\mathcal{I}}^{-3(\sigma_0-\frac{1}{2})-2}}{2\pi}\sum_{\theta\in\{\pm 1\}}\int_0^\infty\left(\frac{8\pi^3n_2n_1^2c_2^3}{c^3}\right)^{-(\sigma_0+i\theta t)}\times$$
$$G_\beta(\sigma_0+i\theta t)H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}(\delta,c,c_2,n_1,n_2,t)\,dt. \tag{7.66}$$

By differentiation under the integral sign, lemma 6.1.0.1 (1) and (2), and rapid decay with respect to t of mixed partial derivatives of $H_{\vartheta,\sigma,\sigma_0,\theta,\epsilon,\mathcal{I}}$ (see lemma 7.2.0.2), we have that $L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ is an $X_{\mathcal{I}}$-inert family of functions (varying over all $\mathcal{I}$) with support on $[\Delta,2\Delta]\times[C,2C]\times[C_2,2C_2]\times[N_1,2N_1]\times[N_2,2N_2]$, where $X_{\mathcal{I}} = T\max(1,P)^2$.

We wish to incorporate the Conrey-Iwaniec phase term $\mathcal{P}$ in our expression for $\mathcal{K}_{\beta,\sigma,\mathcal{I}}$. For that, consider a smooth function w on $(0,\infty)$ that is compactly supported and is identically 1 on $[\frac{1}{4},64]$. Let $P' := \frac{N_2N_1^2C_2^3}{C\Delta}$ and $g_{\beta,\sigma,\mathcal{I}}(x) := e(-\beta\sigma x)w\left(\frac{x}{P'}\right)$. Note that on the support of $L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$, $\frac{1}{4}\le\frac{n_2n_1^2c_2^3}{c\delta P'}\le 64$. By Mellin inversion, we have

$$g_{\beta,\sigma,\mathcal{I}}(x) = \int_{-\infty}^{\infty}x^{-it}f_{\beta,\sigma,\mathcal{I}}(t)\,dt, \tag{7.67}$$

where

$$f_{\beta,\sigma,\mathcal{I}}(t) = \frac{1}{2\pi}\int_0^\infty g_{\beta,\sigma,\mathcal{I}}(x)x^{it}\,\frac{dx}{x}. \tag{7.68}$$

We wish to analyze $\mathcal{P}w\left(\frac{n_2n_1^2c_2^3}{c\delta P'}\right) = g_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2n_1^2c_2^3}{c\delta}\right)$. There are 2 cases.

- Let $P'\le(qT)^\epsilon$ (non-oscillatory subcase). $g_{\beta,\sigma,\mathcal{I}}$ has support $\asymp P'$ and satisfies the derivative bounds satisfied by an $(qT)^\epsilon$-inert family of functions (varying over all $\mathcal{I}$). By repeated integration by parts, we have that $f_{\beta,\sigma,\mathcal{I}}(t)\ll_A(qT)^{A\epsilon}(1+|t|)^{-A}$; this allows us to truncate the t-integral to get

$$\mathcal{P}w\left(\frac{n_2n_1^2c_2^3}{c\delta P'}\right) = \int_{|t|\ll(qT)^{2\epsilon}}\left(\frac{n_2n_1^2c_2^3}{c\delta}\right)^{-it}f_{\beta,\sigma,\mathcal{I}}(t)\,dt + O_{\epsilon,B}((qT)^{-B}), \tag{7.69}$$

  for $B>0$.

- Let $P' > (qT)^\epsilon$ (oscillatory subcase). In the x-integral, the phase is $-2\pi\beta\sigma x + t\log(x)$, and the derivative of that with respect to x is $-2\pi\beta\sigma + \frac{t}{x}$. Therefore, we perform repeated integration by parts to show that $f_{\beta,\sigma,\mathcal{I}}(t)$ is small when $|t| \not\asymp P'$. When $|t| \asymp P'$, we apply stationary phase to get $f_{\beta,\sigma,\mathcal{I}}(t) \ll |t|^{-\frac{1}{2}}$. To be precise, we get

$$\mathcal{P}w\left(\frac{n_2 n_1^2 c_2^3}{c\delta P'}\right) = \int_{|t|\asymp P'} \left(\frac{n_2 n_1^2 c_2^3}{c\delta}\right)^{-it} f_{\beta,\sigma,\mathcal{I}}(t)\, dt + O_{\epsilon,B}((qT)^{-B}), \tag{7.70}$$

  for $B > 0$.

Next, similar to lemma 7.1.0.6, we apply Mellin inversion to $L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ and use the decay properties of its Mellin transform to truncate the quadruple integral at $|\mathbf{u}| \ll X_{\mathcal{I}}(qT)^\epsilon$ (redefine $L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ to be this truncated integral).

Also, notice that

$$\frac{NN_2N_1^2C_2^3}{C^3} = \frac{N\Delta}{C^2}\frac{N_2N_1^2C_2^3}{C\Delta} = \frac{P^2P'}{16\pi^2}. \tag{7.71}$$

Finish the proof by redefining $L_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ again to absorb $(16\pi^2)^{\sigma_0}$. □

Ramanujan sums will be denoted by

$$R_q(n) := \sum_{a(q)}^{*} e\left(\frac{na}{q}\right) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(\frac{B}{(q,n)}\right)}. \tag{8.1}$$

We will also heavily use weak reciprocity, which says that if $a, b \in \mathbb{N}$ such that $(a,b) = 1$, then for any $c \in \mathbb{Z}$, we have

$$e\left(\frac{c}{ab}\right) = e\left(\frac{\bar{b}c}{a}\right) e\left(\frac{\bar{a}c}{b}\right). \tag{8.2}$$

Lemma 8.0.0.1. *Let $q_1, q_2 \in \mathbb{N}$ and $m, n \in \mathbb{Z}$ such that $(m, q_1) = 1$ and $n \equiv 0\,(mod\,(q_1, q_2))$. Then*

$$R_{q_1 q_2}(m+n) = \begin{cases} R_{q_1}(m+n)R_{q_2}(m+n) & \text{if } (q_1,q_2) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{8.3}$$

Proof: The $(q_1, q_2) = 1$ case follows from weak reciprocity and change of variables. Assume that $(q_1, q_2) > 1$; we have

$$R_{q_1 q_2}(m+n) = \mu\left(\frac{q_1 q_2}{(q_1 q_2, m+n)}\right)\frac{\varphi(q_1 q_2)}{\varphi\left(\frac{q_1 q_2}{(q_1 q_2, m+n)}\right)}. \tag{8.4}$$

If p is a prime such that $p|(q_1, q_2)$, then $p^2|q_1 q_2$ whereas $p \not| (q_1 q_2, m+n)$ making the μ factor above vanish. $\square$

For $n \in \mathbb{N}$ and $m_1, m_2 \in \mathbb{Z}$, let

$$\text{free}(n) := \prod_{\substack{p \| n \\ p \text{ prime}}} p, \tag{8.5}$$

and

$$\mathcal{U}(m_1,m_2,n) := \sum_{a,b(n)}^{*} e\left(\frac{ab+m_2a+m_1b}{n}\right),$$

$$\begin{aligned}\mathcal{V}(m_1,m_2,n) &:= e\left(\frac{m_1m_2}{n}\right)\mathcal{U}(m_1,m_2,n)\\ &= \sum_{a,b(n)}^{*} e\left(\frac{(a+m_1)(b+m_2)}{n}\right)\\ &= \sum_{\substack{x,y(n)\\ ((x-m_1)(y-m_2),n)=1}} e\left(\frac{xy}{n}\right).\end{aligned} \tag{8.6}$$

$\mathcal{U}(m_1,m_2,n)$ and $\mathcal{V}(m_1,m_2,n)$ are symmetric in m_1,m_2. If $(m_2,n)=1$, then $\mathcal{U}(m_1,m_2,n) = \mathcal{U}(m_1m_2,1,n)$ and $\mathcal{V}(m_1,m_2,n) = \mathcal{V}(m_1m_2,1,n)$. If $(m,n)=1$, then

$$\begin{aligned}\mathcal{V}(m,1,n) &= \sum_{a,b(n)}^{*} e\left(\frac{m(a+1)(b+1)}{n}\right)\\ &= \sum_{\substack{x,y(n)\\ ((x-1)(y-1),n)=1}} e\left(\frac{mxy}{n}\right).\end{aligned} \tag{8.7}$$

Lemma 8.0.0.2. *If $n|m_2^\infty$, then*

$$\mathcal{U}(m_1,m_2,n) = \mu(n)R_n(m_1). \tag{8.8}$$

Consequently, if n is not square-free, then $\mathcal{U}(m_1,m_2,n)$ vanishes in this case. If n is square-free, then $n|m_2$.

Further, if $(m_1,n)=1$, then evaluating the above Ramanujan sum gives

$$\mathcal{U}(m_1,m_2,n) = (\mu(n))^2. \tag{8.9}$$

Proof: In

$$\mathcal{U}(m_1,m_2,n) = \sum_{a,b(n)}^{*} e\left(\frac{ab+m_2a+m_1b}{n}\right), \tag{8.10}$$

we evaluate the Ramanujan sum over a to get

$$\mathcal{U}(m_1, m_2, n) = \sum_{b(n)}^{*} e\left(\frac{m_1 b}{n}\right) \mu\left(\frac{n}{(n, b+m_2)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,b+m_2)}\right)}. \tag{8.11}$$

Now, suppose that for some $c \in \mathbb{Z}$, $(n, c+m_2) > 1$; let p be a prime such that $p|(n, c+m_2)$. Then $p|n|(m_2)^\infty \implies p|m_2 \implies p|c$; that is, $p|(n, c)$. Therefore, if $(n, b) = 1$, then $(n, b+m_2) = 1$, from which the result follows. □

Lemma 8.0.0.3. *If* $(m, n) = 1$*, then*

$$\begin{aligned} \mathcal{V}(m, 1, n) &= n \sum_{d|\text{free}(n)} \frac{1}{d} e\left(\frac{m\overline{(n/d)}}{d}\right) \\ &= \sum_{\substack{d|\text{free}(n) \\ k=\frac{n}{d}}} k\, e\left(\frac{m\overline{k}}{d}\right). \end{aligned} \tag{8.12}$$

Proof: In

$$\mathcal{V}(m, 1, n) = \sum_{a,b(n)}^{*} e\left(\frac{(a+m)(b+1)}{n}\right), \tag{8.13}$$

the sum over a is a Ramanujan sum, which we evaluate to get

$$\begin{aligned} \mathcal{V}(m, 1, n) &= \varphi(n) \sum_{b(n)}^{*} \frac{\mu\left(\frac{n}{(n,b+1)}\right)}{\varphi\left(\frac{n}{(n,b+1)}\right)} e\left(\frac{m(b+1)}{n}\right) \\ &= \varphi(n) \sum_{j|n} \frac{\mu\left(\frac{n}{j}\right)}{\varphi\left(\frac{n}{j}\right)} \sum_{\substack{y(n) \\ (y-1,n)=1 \\ (n,y)=j}} e\left(\frac{my}{n}\right) \\ &= \varphi(n) \sum_{lj=n} \frac{\mu(l)}{\varphi(l)} \sum_{\substack{x(l) \\ (xj-1,n)=1}}^{*} e\left(\frac{mx}{l}\right), \end{aligned} \tag{8.14}$$

where $x = y/j$. Observing that $(xj-1, n) = (xj-1, l)$, consider the inner sum

$$L := \sideset{}{^*}\sum_{\substack{x(l) \\ (xj-1,l)=1}} e\left(\frac{mx}{l}\right), \tag{8.15}$$

where l is square-free due to the $\mu(l)$. We detect the coprimality condition by using Möbius function to get

$$L = \sum_{d_1|l} \mu(d_1) \sideset{}{^*}\sum_{\substack{x(l) \\ xj\equiv 1(d_1)}} e\left(\frac{mx}{l}\right). \tag{8.16}$$

Since l is square-free, if $l = d_1 d_2$, then $(d_1, d_2) = 1$. Write $x = d_1 u x_2 + d_2 v x_1$ where $u, v \in \mathbb{Z}$ such that $d_1 u + d_2 v = 1$, $x_1 \pmod{d_1}$, $x_2 \pmod{d_2}$. Then

$$\begin{aligned} L &= \sum_{d_1 d_2 = l} \mu(d_1) \sideset{}{^*}\sum_{\substack{x_1(d_1) \\ x_1 j\equiv 1(d_1)}} e\left(\frac{\overline{d_2} m x_1}{d_1}\right) \sideset{}{^*}\sum_{x_2(d_2)} e\left(\frac{\overline{d_1} m x_2}{d_2}\right) \\ &= \sum_{d_1 d_2 = l} \mu(d_1)\mu(d_2) \sideset{}{^*}\sum_{\substack{x_1(d_1) \\ x_1 j\equiv 1(d_1)}} e\left(\frac{\overline{d_2} m x_1}{d_1}\right) \\ &= \mu(l) \sum_{d_1 d_2 = l} \sideset{}{^*}\sum_{\substack{x_1(d_1) \\ x_1 j\equiv 1(d_1)}} e\left(\frac{\overline{d_2} m x_1}{d_1}\right), \end{aligned} \tag{8.17}$$

where we used the fact that $(m, n) = 1$ to show that the Ramanujan sum over $x_2 \pmod{d_2}$ is $\mu(d_2)$. The conditions $(x_1, d_1) = 1$ and $x_1 j \equiv 1 \pmod{d_1}$ force $(d_1, j) = 1$ and $x_1 \equiv \overline{j} \pmod{d_1}$, giving

$$L = \mu(l) \sum_{\substack{d_1 d_2 = l \\ (d_1, j)=1}} e\left(\frac{m\overline{(d_2 j)}}{d_1}\right). \tag{8.18}$$

We use this in (8.14) to get

$$\begin{aligned}\mathcal{V}(m,1,n) &= \varphi(n)\sum_{lj=n}\frac{(\mu(l))^2}{\varphi(l)}\sum_{\substack{d_1d_2=l\\(d_1,j)=1}} e\left(\frac{m\overline{(d_2j)}}{d_1}\right)\\ &= \varphi(n)\sum_{d_1|\,\mathrm{free}(n)} e\left(\frac{m\overline{(n/d_1)}}{d_1}\right)\sum_{d_1d_2|n}\frac{\mu(d_1d_2)^2}{\varphi(d_1d_2)}.\end{aligned} \tag{8.19}$$

Let $n_* = \prod_{\substack{p|n\\ p \text{ prime}}} p$, which is square-free. Then the inner sum

$$\begin{aligned}\sum_{d_1d_2|n}\frac{\mu(d_1d_2)^2}{\varphi(d_1d_2)} &= \frac{1}{\varphi(d_1)}\sum_{d_2|\frac{n_*}{d_1}}\frac{1}{\varphi(d_2)}\\ &= \frac{1}{\varphi(d_1)}\frac{\frac{n_*}{d_1}}{\varphi\left(\frac{n_*}{d_1}\right)}\\ &= \frac{n_*}{d_1\varphi(n_*)}\\ &= \frac{n}{d_1\varphi(n)}.\end{aligned} \tag{8.20}$$

Therefore

$$\mathcal{V}(m,1,n) = n\sum_{d_1|\,\mathrm{free}(n)}\frac{1}{d_1}e\left(\frac{m\overline{(n/d_1)}}{d_1}\right), \tag{8.21}$$

as claimed. □

8.1 Summary of character sum computation

For the ease of the reader, we make a list of variables that will be used in the process below.

$$
\begin{aligned}
&c = c'c_0 = qr = c_1c_2\\
&r = r'r_0\\
&c_1 = c_1'c_{1,0} = n_1n_3\\
&c_2 = c_2'c_{2,0}\\
&n_1 = n_1'n_{1,0}\\
&n_2 = n_2'n_{2,0}\\
&n_3 = n_3'n_{3,0}\\
&m_1 = m_1'm_{1,0} = n_2n_1c_2\\
&m_2 = m_2'm_{2,0} = n_1c_2\\
&m_3 = m_3'm_{3,0} = c_2\\
&m = m'm_0 = m_1m_2m_3 = n_2n_1^2c_2^3
\end{aligned}
\tag{8.22}
$$

where all the variables with a $'$ superscript are coprime to q; that is,

$$(r'c'c_1'n_1'n_2'n_3'c_2'm_1'm_2'm_3'm', q) = 1, \tag{8.23}$$

and all the variables with 0 subscript divide q^∞; that is,

$$r_0c_0c_{1,0}n_{1,0}n_{2,0}n_{3,0}c_{2,0}m_{1,0}m_{2,0}m_{3,0}m_0|q^\infty. \tag{8.24}$$

Thus

$$
\begin{aligned}
c' &= r' \\
c' &= c'_1 c'_2 \\
c'_1 &= n'_1 n'_3 \\
m'_1 &= n'_2 n'_1 c'_2 \\
m'_2 &= n'_1 c'_2 \\
m'_3 &= c'_2 \\
m' &= m'_1 m'_2 m'_3 = n'_2 n'^2_1 c'^3_2
\end{aligned} \tag{8.25}
$$

and

$$
\begin{aligned}
c_0 &= q r_0 \\
c_0 &= c_{1,0} c_{2,0} \\
c_{1,0} &= n_{1,0} n_{3,0} \\
m_{1,0} &= n_{2,0} n_{1,0} c_{2,0} \\
m_{2,0} &= n_{1,0} c_{2,0} \\
m_{3,0} &= c_{2,0} \\
m_0 &= m_{1,0} m_{2,0} m_{3,0} = n_{2,0} n^2_{1,0} c^3_{2,0}
\end{aligned} \tag{8.26}
$$

Additionally, let

$$B = (n'_3, n'_2), \ \ A = \frac{n'_3}{B}. \tag{8.27}$$

Let

$$F = \mathrm{free}(A) = \prod_{\substack{p \| A \\ p \text{ prime}}} p. \tag{8.28}$$

Finally, recall that δ is square-free and $(\delta, q) = 1$ due to the $\mu(\delta)\chi(\delta)$ in (6.51). Let

$$\begin{aligned} \delta_1 &= (\delta, c') = (\delta, c) = (\delta, r) \\ \delta_2 &= \frac{\delta}{\delta_1} \\ \delta_3 &= (n_2', \delta_2) \\ \delta_4 &= \frac{\delta_2}{\delta_3} \end{aligned} \tag{8.29}$$

We also note down the following definition from section 5.1 of [23]:

$$\begin{aligned} &H_\chi(j_1, j_2, j_3, \boldsymbol{r}) := \\ &\sum_{u,t(q)} \chi(t)\overline{\chi}(u)\overline{\chi}(-j_2 + \boldsymbol{r}t)\chi(-j_1 + \boldsymbol{r}u) e\left(\frac{j_3(-j_1 + \boldsymbol{r}u)(-j_2 + \boldsymbol{r}t) - j_1 j_2 j_3}{c}\right). \end{aligned} \tag{8.30}$$

Proposition 8.1.0.1. *$\mathcal{T}$ is 0 if any of the following conditions is not satisfied.*

$$\begin{aligned} &\delta_1 = c_2' \ \textit{(also } m_3' = c_2' \textit{ by definition)} \\ &(c', \delta_2) = 1 \\ &(c_1', \delta) = 1 \\ &(n_1', n_3') = 1 \\ &(A, n_2') = 1 \\ &(\mu(B))^2 = 1 \\ &(n_2', \delta_4) = 1 \\ &(m_{1,0}, r_0) = (n_{2,0} n_{1,0} c_{2,0}, r_0) = 1 \ \textit{(and consequently } n_{1,0}c_{2,0} | q) \end{aligned} \tag{8.31}$$

If all of the above conditions hold, then

$$\mathcal{T} = \mathcal{P}\mathcal{T}_0\mathcal{T}', \tag{8.32}$$

where $\mathcal{P} = e\left(-\frac{\beta\sigma m}{c\delta}\right) = e\left(-\frac{\beta\sigma n_2 n_1^2 c_2^3}{c\delta}\right)$ *is the Conrey-Iwaniec phase term,*

$$\begin{aligned}\mathcal{T}_0 &= \frac{\varphi(c_{1,0})\varphi(n_{3,0})}{(\varphi(c_0))^2}\frac{\chi(-\sigma)\overline{\chi}(\delta)qr_0^2}{\varphi(q)}\sum_{\psi(q)}\widehat{H}(\psi)\psi(-\beta\sigma m')\overline{\psi}(c'\delta),\\ \mathcal{T}' &= c'\mu(m_2')\sum_{\substack{D_1|F\\ D_2=\frac{A}{D_1}}}\frac{D_2}{\varphi(D_1\delta_4)}\sum_{\lambda(D_1\delta_4)}\tau(\overline{\lambda})\lambda(\beta\sigma m_0 m_1')\overline{\lambda}(\delta_3 c_0 B D_2),\end{aligned} \tag{8.33}$$

where

$$\widehat{H}(\psi) = \sum_{v(q)} H_{\overline{\chi}}(m_{1,0}, m_{2,0}, m_{3,0}v, r_0)\overline{\psi}(v). \tag{8.34}$$

□

The proof will involve repeated applications of weak reciprocity and lemma 8.0.0.1 to collect the conditions in (8.31). At first, let us write $\mathcal{T}$ as a sum modulo c.

$$\mathcal{T} = \frac{\varphi(c_1)\varphi(n_3)}{(\varphi(c))^2}C, \tag{8.35}$$

where

$$\begin{aligned}C &= \sum_{\substack{a,b,d,f(c)\\(bdf,c)=1}}\chi^2(d)\chi(a)e\left(\frac{\sigma\overline{d} - \overline{b}ac_2 + \delta da + bfn_1c_2 + \beta n_2 n_1 c_2\overline{f}}{c}\right)\\ &= \sum_{\substack{a,b,d,f(c)\\(bdf,c)=1}}\chi^2(d)\chi(a)e\left(\frac{\sigma\overline{d} - \overline{b}am_3 + \delta da + bfm_2 + \beta m_1\overline{f}}{c}\right).\end{aligned} \tag{8.36}$$

By weak reciprocity, we can write $C = C'C_0$, where

$$\begin{aligned}C' &= \sum_{\substack{a,b,d,f(c')\\(bdf,c')=1}} e\left(\frac{\overline{c_0}(\sigma\overline{d} - \overline{b}am_3 + \delta da + bfm_2 + \beta m_1\overline{f})}{c'}\right),\\ C_0 &= \sum_{\substack{a,b,d,f(c_0)\\(bdf,c_0)=1}}\chi^2(d)\chi(a)e\left(\frac{\overline{c'}(\sigma\overline{d} - \overline{b}am_3 + \delta da + bfm_2 + \beta m_1\overline{f})}{c_0}\right).\end{aligned} \tag{8.37}$$

8.2 Simplifying C'

$$C' = \sum_{\substack{a,b,d,f(c') \\ (bdf,c')=1}} e\left(\frac{\overline{c_0}(\sigma\overline{d} - \overline{b}am_3 + \delta da + bfm_2 + \beta m_1\overline{f})}{c'}\right). \tag{8.38}$$

The sum over a is 0 unless $\delta d \equiv \overline{b}m_3 \pmod{c'}$ which implies $(\delta, c') = (m_3, c') = (c_2, c')$, giving

$$\delta_1 = m_3' = c_2'. \tag{8.39}$$

From this point onward, we will use δ_1, c_2', m_3' interchangeably. By (8.39) and $(c', \delta) = \delta_1$, we get

$$(c_1', \delta_2) = \left(\frac{c'}{c_2'}, \frac{\delta}{\delta_1}\right) = 1. \tag{8.40}$$

Since δ is square-free, we also have

$$(m_3', \delta_2) = (c_2', \delta_2) = (\delta_1, \delta_2) = 1. \tag{8.41}$$

Combining the above, we get that

$$(c', \delta_2) = 1. \tag{8.42}$$

After $d \mapsto \sigma\overline{c_0}d$, $a \mapsto \sigma c_0^2\overline{\delta_2}a$, $b \mapsto \sigma c_0\overline{\delta_2}b$, $f \mapsto \sigma\delta_2 f$, we get

$$C' = \sum_{\substack{a,b,d,f(c') \\ (bdf,c')=1}} e\left(\frac{\overline{d} - \overline{b}am_3 + \delta_1 da + bfm_2 + \beta\sigma m_1\overline{c_0\delta_2}\,\overline{f}}{c'}\right). \tag{8.43}$$

After $b \mapsto m_{3,0}b$, $f \mapsto \overline{m_{2,0}m_{3,0}}f$

$$C' = \sum_{\substack{a,b,d,f(c') \\ (bdf,c')=1}} e\left(\frac{\overline{d} - \overline{b}am_3' + \delta_1 da + bfm_2' + \beta\sigma m_0 m_1'\overline{c_0\delta_2}\,\overline{f}}{c'}\right). \tag{8.44}$$

After $d \mapsto \overline{d}$ followed by $a \mapsto da$, we get

$$\begin{aligned} C' &= \sum_{\substack{a,b,d,f(c') \\ (bdf,c')=1}} e\left(\frac{d - \overline{b}dam_3' + \delta_1 a + bfm_2' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c'}\right) \\ &= \sum_{\substack{a,b,f(c') \\ (bf,c')=1}} e\left(\frac{\delta_1 a + bfm_2' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c'}\right) R_{c_1' m_3'}(1 - \overline{b}am_3'). \end{aligned} \tag{8.45}$$

Therefore, we can assume that

$$(c_1', \delta_1) = (c_1', c_2') = (c_1', m_3') = 1, \tag{8.46}$$

since otherwise, by lemma 8.0.0.1, all the Ramanujan sums above will vanish. (8.40) and (8.46) together imply

$$(c_1', \delta) = 1. \tag{8.47}$$

We use (8.46) in (8.44) to write $C' = C_1' C_2'$ where

$$\begin{aligned} C_1' &= \sum_{\substack{a,b,d,f(c_1') \\ (bdf,c_1')=1}} e\left(\frac{\overline{c_2'}(\overline{d} - \overline{b}am_3' + \delta_1 da + bfm_2' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{c_1'}\right), \\ C_2' &= \sum_{\substack{a,b,d,f(c_2') \\ (bdf,c_2')=1}} e\left(\frac{\overline{c_1'}(\overline{d} - \overline{b}am_3' + \delta_1 da + bfm_2' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{c_2'}\right). \end{aligned} \tag{8.48}$$

8.3 Simplifying C_2'

The last four terms in the numerator can be removed since $m_3' \equiv \delta_1 \equiv m_1' \equiv 0 \pmod{c_2'}$; thus

$$\begin{aligned} C_2' &= \delta_1(\varphi(\delta_1))^2 \sum_{d(c_2')}^{*} e\left(\frac{\overline{c_1'}d}{c_2'}\right) \\ &= \delta_1(\varphi(\delta_1))^2 \mu(\delta_1), \end{aligned} \tag{8.49}$$

where the last sum was a Ramanujan sum.

8.4 Simplifying C_1'

$$C_1' = \sum_{\substack{a,b,d,f(c_1') \\ (bdf,c_1')=1}} e\left(\frac{\overline{c_2'}(\overline{d} - \overline{b}am_3' + \delta_1 da + bfm_2' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{c_1'}\right). \tag{8.50}$$

After $d \mapsto \overline{\delta_1} d$, $a \mapsto \delta_1 a$, $b \mapsto \delta_1 b$, and $f \mapsto \overline{\delta_1} f$, we get

$$C_1' = \sum_{\substack{a,b,d,f(c_1') \\ (bdf,c_1')=1}} e\left(\frac{\overline{d} - \overline{b}a + da + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c_1'}\right). \tag{8.51}$$

Next, let us evaluate the sum over a followed by that over b.

$$\begin{aligned} C_1' &= c_1' \sum_{\substack{b,d,f(c_1') \\ (bdf,c_1')=1 \\ d\equiv\overline{b}(c_1')}} e\left(\frac{\overline{d} + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c_1'}\right) \\ &= c_1' \sum_{\substack{b,f(c_1') \\ (bf,c_1')=1}} e\left(\frac{b + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c_1'}\right) \\ &= c_1' \sideset{}{^*}\sum_{f(c_1')} e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{c_1'}\right) R_{n_1'n_3'}(1 + fn_1'). \end{aligned} \tag{8.52}$$

Therefore, we can assume that

$$(n_1', n_3') = 1, \tag{8.53}$$

since otherwise, by lemma 8.0.0.1, all the Ramanujan sums above will vanish. This condition enables us to write $C_1' = c_1' N_1' N_3'$, where

$$\begin{aligned} N_1' &= \sum_{\substack{b,f(n_1') \\ (bf,n_1')=1}} e\left(\frac{\overline{n_3'}(b + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{n_1'}\right), \\ N_3' &= \sum_{\substack{b,f(n_3') \\ (bf,n_3')=1}} e\left(\frac{\overline{n_1'}(b + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{n_3'}\right). \end{aligned} \tag{8.54}$$

8.5 Simplifying N_1'

The last two terms in the numerator can be removed since $m_1' \equiv 0 \pmod{n_1'}$.

$$\begin{aligned} N_1' &= \varphi(n_1') \sum_{b(n_1')}^{*} e\left(\frac{\overline{n_3'}b}{n_1'}\right) \\ &= \varphi(n_1')\mu(n_1'), \end{aligned} \tag{8.55}$$

where the last sum was a Ramanujan sum.

8.6 Simplifying N_3'

$$N_3' = \sum_{\substack{b,f(n_3') \\ (bf,n_3')=1}} e\left(\frac{\overline{n_1'}(b + bfn_1' + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f})}{n_3'}\right). \tag{8.56}$$

After $b \mapsto n_1'b$ and $f \mapsto \overline{n_1'}f$, we get

$$N_3' = \sum_{\substack{b,f(n_3') \\ (bf,n_3')=1}} e\left(\frac{b + bf + \beta\sigma m_0 m_1' \overline{c_0\delta_2}\,\overline{f}}{n_3'}\right). \tag{8.57}$$

After $b \mapsto \overline{f}b$ followed by $f \mapsto \overline{f}$, we get

$$\begin{aligned} N_3' &= \sum_{\substack{b,f(n_3') \\ (bf,n_3')=1}} e\left(\frac{bf + b + \beta\sigma m_0 m_1' \overline{c_0\delta_2} f}{n_3'}\right) \\ &= \sum_{b(n_3')}^{*} e\left(\frac{b}{n_3'}\right) R_{AB}(b + \beta\sigma m_0 m_1' \overline{c_0\delta_2}). \end{aligned} \tag{8.58}$$

Note that since $n_2' \equiv 0 \pmod{B}$, we have $m_1' = n_2' n_1' c_2' \equiv 0 \pmod{(A, B)}$. Therefore, we can assume that

$$(A, B) = 1, \tag{8.59}$$

since otherwise, by lemma 8.0.0.1, all the Ramanujan sums above will vanish. This enables us to write $N_3' = A'B'$ where

$$A' = \sum_{\substack{b,f(A)\\(bf,A)=1}} e\left(\frac{\overline{B}(bf + b + \beta\sigma m_0 m_1' \overline{c_0\delta_2} f)}{A}\right),$$
$$B' = \sum_{\substack{b,f(B)\\(bf,B)=1}} e\left(\frac{\overline{A}(bf + b + \beta\sigma m_0 m_1' \overline{c_0\delta_2} f)}{B}\right). \tag{8.60}$$

8.7 Simplifying B'

Again, since $m_1' \equiv 0 \pmod{B}$, we can remove the last term in the numerator and get

$$B' = \sum_{\substack{b,f(B)\\(bf,B)=1}} e\left(\frac{\overline{A}(bf + b)}{B}\right). \tag{8.61}$$

Evaluating the Ramanujan sum over f followed by that over b, we obtain

$$B' = (\mu(B))^2, \tag{8.62}$$

which lets us assume that B is square-free.

8.8 Simplifying A'

$$A' = \sum_{\substack{b,f(A)\\(bf,A)=1}} e\left(\frac{\overline{B}(bf + b + \beta\sigma m_0 m_1' \overline{c_0\delta_2} f)}{A}\right). \tag{8.63}$$

After $b \mapsto Bb$, we get

$$A' = \sum_{\substack{b,f(A)\\(bf,A)=1}} e\left(\frac{bf + b + \beta\sigma m_0 m_1' \overline{c_0\delta_2 B} f}{A}\right) = \mathcal{U}(1, \beta\sigma m_0 m_1' \overline{c_0\delta_2 B}, A). \tag{8.64}$$

Now, $(n_3', n_2') = B \implies \left(A, \frac{n_2'}{B}\right) = \left(\frac{n_3'}{B}, \frac{n_2'}{B}\right) = 1$; this combined with $(A, B) = 1$ gives $(A, n_2') = 1$. Consequently $(A, \beta\sigma m_0 m_1' \overline{c_0 \delta_2 B}) = 1$. Therefore, by lemma 8.0.0.3, we get

$$A' = e\left(-\frac{\beta\sigma m_0 m_1' \overline{c_0 \delta_2 B}}{A}\right) \sum_{\substack{D_1 | F \\ D_2 = \frac{A}{D_1}}} D_2 e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 \delta_2 B}\, \overline{D_2}}{D_1}\right). \tag{8.65}$$

Here $F = \text{free}(A)$ as defined earlier.

We request the reader to keep in mind that c_1' and hence all its divisors are coprime to δ; see (8.47). Now we prepare for extracting the Conrey-Iwaniec phase term; $D_1 D_2 = A$ implies

$$\begin{aligned} A' = e\left(-\frac{\overline{\delta_2}\beta\sigma m_0 m_1' \overline{c_0 B}}{A}\right) e\left(-\frac{\overline{A}\beta\sigma m_0 m_1' \overline{c_0 B}}{\delta_2}\right) \sum_{\substack{D_1 | F \\ D_2 = \frac{A}{D_1}}} D_2 e\left(\frac{\overline{\delta_2}\beta\sigma m_0 m_1' \overline{c_0 B}\, \overline{D_2}}{D_1}\right) \times \\ e\left(\frac{\overline{D_1}\beta\sigma m_0 m_1' \overline{c_0 B}\, \overline{D_2}}{\delta_2}\right) \\ = \varkappa \sum_{\substack{D_1 | F \\ D_2 = \frac{A}{D_1}}} D_2 e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 B}\, \overline{D_2}}{D_1 \delta_2}\right), \end{aligned} \tag{8.66}$$

where

$$\varkappa = e\left(-\frac{\beta\sigma m_0 m_1' \overline{c_0 B}}{A\delta_2}\right). \tag{8.67}$$

Now we wish to find the Fourier expansion of the term $e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 B}\, \overline{D_2}}{D_1 \delta_2}\right)$ with respect to Dirichlet characters. To simplify our work, we first ensure that the base (denominator) of this complex exponential is coprime to the numerator; currently δ_2 might share a common factor with n_2'. Recall that $\delta_3 = (n_2', \delta_2)$ and $\delta_4 = \frac{\delta_2}{\delta_3}$. Since $(D_1, \delta) = 1$ and since δ is square-free, we have that $(D_1\delta_4, \delta_3) = 1$. Therefore,

$$\begin{aligned} e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{D_1\delta_2}\right) &= e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{D_1\delta_4\delta_3}\right) \\ &= e\left(\frac{\overline{\delta_3}\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{D_1\delta_4}\right) e\left(\frac{\overline{D_1\delta_4}\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{\delta_3}\right) \\ &= e\left(\frac{\overline{\delta_3}\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{D_1\delta_4}\right), \end{aligned} \tag{8.68}$$

where the last equality follows from $m_1' = n_2' n_1' c_2' \equiv 0 \pmod{\delta_3}$, which itself is a result of $n_2' \equiv 0 \pmod{\delta_3}$. Now

$$\begin{aligned} &(n_2', \delta_2) = \delta_3 \\ \Longrightarrow &\left(\frac{n_2'}{\delta_3}, \frac{\delta_2}{\delta_3}\right) = 1 \\ \Longrightarrow &\left(\frac{n_2'}{\delta_3}, \delta_4\right) = 1. \end{aligned} \tag{8.69}$$

Also, we have already recorded that $(\delta_3, \delta_4) = 1$. Combining these, we have

$$(n_2', \delta_4) = 1. \tag{8.70}$$

This implies that $(\overline{\delta_3}\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}, D_1\delta_4) = 1$. Therefore,

$$e\left(\frac{\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}}{D_1\delta_2}\right) = \frac{1}{\varphi(D_1\delta_4)} \sum_{\lambda(D_1\delta_4)} \tau(\overline{\lambda})\lambda(\overline{\delta_3}\beta\sigma m_0 m_1' \overline{c_0 B}\,\overline{D_2}). \tag{8.71}$$

In other words,

$$A' = \varkappa \sum_{\substack{D_1|F \\ D_2=\frac{A}{D_1}}} \frac{D_2}{\varphi(D_1\delta_4)} \sum_{\lambda(D_1\delta_4)} \tau(\overline{\lambda})\lambda(\beta\sigma m_0 m_1')\overline{\lambda}(\delta_3 c_0 B D_2). \tag{8.72}$$

8.9 Simplifying C_0

$$C_0 = \sum_{\substack{a,b,d,f(c_0)\\(bdf,c_0)=1}} \chi^2(d)\chi(a)e\left(\frac{\overline{c'}(\sigma\overline{d} - m_3\overline{b}a + \delta da + m_2bf + \beta m_1\overline{f})}{c_0}\right). \tag{8.73}$$

Because of the $\chi(a)$ and since $c_0|q^\infty$, we can take $(a, c_0) = 1$. After $d \mapsto \sigma\overline{c'}d$, $a \mapsto \sigma c'^2\overline{\delta}a$, $b \mapsto \sigma c' m_3'\overline{\delta}b$, $f \mapsto \sigma\overline{m_2'm_3'}\delta f$, we get

$$C_0 = \chi(\sigma)\overline{\chi}(\delta)\sum_{a,b,d,f(c_0)}^{*} \chi^2(d)\chi(a)e\left(\frac{\overline{d} - m_{3,0}\overline{b}a + da + m_{2,0}bf - \omega_0 m_{1,0}\overline{f}}{c_0}\right), \tag{8.74}$$

where

$$\begin{aligned}&\omega_0 \in \mathbb{Z} \text{ such that } \omega_0 \equiv -\beta\sigma m'\overline{c'\delta} \pmod{c_0}\\ &\text{we have } (\omega_0, c_0) = (\omega_0, q) = 1.\end{aligned} \tag{8.75}$$

The sum over a is 0 unless $d \equiv \overline{b}m_{3,0} \pmod{r_0}$, which implies that

$$(m_{3,0}, r_0) = (c_{2,0}, r_0) = 1. \tag{8.76}$$

Let

$$x_1 = \overline{f}, x_2 = bf, x_3 = \overline{b}a, x_4 = da. \tag{8.77}$$

Then

$$a = x_1x_2x_3, b = x_1x_2, d = \overline{x_1x_2x_3}x_4, f = \overline{x_1}, \tag{8.78}$$

and

$$C_0 = \chi(\sigma)\overline{\chi}(\delta) \sideset{}{^*}\sum_{x_1,x_2,x_3,x_4(c_0)} \chi(\overline{x_1x_2x_3}x_4^2) e\left(\frac{x_1x_2x_3\overline{x_4} - m_{3,0}x_3 + x_4 + m_{2,0}x_2 - \omega_0 m_{1,0}x_1}{c_0}\right)$$
$$= \chi(\sigma)\overline{\chi}(\delta) \sideset{}{^*}\sum_{x_2,x_3,x_4(c_0)} \chi(x_4)\overline{\chi}(x_2x_3\overline{x_4}) e\left(\frac{-m_{3,0}x_3 + x_4 + m_{2,0}x_2}{c_0}\right) \times$$
$$\sideset{}{^*}\sum_{x_1(c_0)} \overline{\chi}(x_1) e\left(\frac{(x_2x_3\overline{x_4} - \omega_0 m_{1,0})x_1}{c_0}\right). \tag{8.79}$$

We assume $x_2x_3\overline{x_4} \equiv \omega_0 m_{1,0} \pmod{r_0}$ since otherwise the sum over x_1 is 0. This condition implies $(\omega_0 m_{1,0}, r_0) = 1$; in particular,

$$(m_{1,0}, r_0) = (n_{2,0}n_{1,0}c_{2,0}, r_0) = 1. \tag{8.80}$$

Note that (8.80) makes (8.76) redundant. Let

$$x_5 = \frac{x_2x_3\overline{x_4} - \omega_0 m_{1,0}}{r_0}. \tag{8.81}$$

Then

$$x_2x_3\overline{x_4} = r_0x_5 + \omega_0 m_{1,0}. \tag{8.82}$$

Evaluating the x_1 sum and eliminating x_2 gives

$$C_0 = \chi(\sigma)\overline{\chi}(\delta) r_0 \tau(\overline{\chi}) \sideset{}{^*}\sum_{x_3,x_4(c_0)} \sum_{x_5(q)} \chi(x_4)\overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5) \times$$
$$e\left(\frac{-m_{3,0}x_3 + x_4 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})\overline{x_3}x_4}{c_0}\right)$$
$$= \chi(\sigma)\overline{\chi}(\delta) r_0 \tau(\overline{\chi}) \sideset{}{^*}\sum_{x_3(c_0)} \sum_{x_5(q)} \overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5) e\left(\frac{-m_{3,0}x_3}{c_0}\right) \times$$
$$\sideset{}{^*}\sum_{x_4(c_0)} \chi(x_4) e\left(\frac{(1 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})\overline{x_3})x_4}{c_0}\right). \tag{8.83}$$

We assume $1 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})\overline{x_3} \equiv 0 \pmod{r_0}$ since otherwise the sum over x_4 is 0.

This condition implies $(r_0, m_{2,0}(r_0x_5 + \omega_0 m_{1,0})) = 1$ which is redundant because of (8.80). Let

$$x_6 = \frac{x_3 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})}{r_0}. \tag{8.84}$$

Then

$$x_3 = x_6 r_0 - m_{2,0}(r_0x_5 + \omega_0 m_{1,0}). \tag{8.85}$$

Since χ is primitive modulo q,

$$\alpha := \chi(\sigma)\overline{\chi}(\delta) r_0^2 \tau(\overline{\chi})\tau(\chi) = \chi(-\sigma)\overline{\chi}(\delta) q r_0^2. \tag{8.86}$$

Let

$$\Omega := e\left(\frac{\omega_0 m_0}{c_0}\right) = e\left(-\frac{\beta\sigma m \overline{c'\delta}}{c_0}\right). \tag{8.87}$$

Evaluating the x_4 sum gives

$$\begin{aligned} C_0 &= \chi(\sigma)\overline{\chi}(\delta) r_0^2 \tau(\overline{\chi})\tau(\chi) \sideset{}{^*}\sum_{x_3(c_0)} \sum_{x_5(q)} \overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5)\times \\ &\qquad\qquad e\left(\frac{-m_{3,0}x_3}{c_0}\right)\overline{\chi}\left(\frac{1 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})\overline{x_3}}{r_0}\right) \\ &= \alpha \sideset{}{^*}\sum_{x_3(c_0)} \sum_{x_5(q)} \overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5)\chi(x_3)\overline{\chi}\left(\frac{x_3 + m_{2,0}(r_0x_5 + \omega_0 m_{1,0})}{r_0}\right) e\left(\frac{-m_{3,0}x_3}{c_0}\right) \\ &= \alpha \sum_{x_5, x_6(q)} \overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5)\chi(x_6 r_0 - m_{2,0}(r_0x_5 + \omega_0 m_{1,0}))\overline{\chi}(x_6)\times \\ &\qquad\qquad e\left(\frac{-m_{3,0}(x_6 r_0 - m_{2,0}(r_0x_5 + \omega_0 m_{1,0}))}{c_0}\right) \\ &= \alpha\Omega \sum_{x_5, x_6(q)} \overline{\chi}(r_0x_5 + \omega_0 m_{1,0})\chi(x_5)\chi(x_6 r_0 - m_{2,0}(r_0x_5 + \omega_0 m_{1,0}))\overline{\chi}(x_6) \\ &\qquad\qquad e\left(\frac{-m_{3,0}(x_6 - m_{2,0}x_5)}{q}\right). \end{aligned} \tag{8.88}$$

After $x_5 \mapsto -\omega_0 x_5$ and $x_6 \mapsto -\omega_0 x_6$, we get

$$
\begin{aligned}
C_0 = \alpha\Omega \sum_{x_5, x_6(q)} & \overline{\chi}(r_0 x_5 - m_{1,0})\chi(x_5)\chi(x_6 r_0 - m_{2,0}(r_0 x_5 - m_{1,0}))\overline{\chi}(x_6)\times \\
& \qquad\qquad e\left(\frac{\omega_0 m_{3,0}(x_6 - m_{2,0}x_5)}{q}\right) \\
= \alpha\Omega H_{\overline{\chi}}&(m_{1,0}, m_{2,0}, m_{3,0}\omega_0, r_0).
\end{aligned} \tag{8.89}
$$

We perform Fourier expansion to get

$$
\begin{aligned}
C_0 &= \frac{\alpha\Omega}{\varphi(q)} \sum_{\psi(q)} \widehat{H}(\psi)\psi(\omega_0) \\
&= \frac{\alpha\Omega}{\varphi(q)} \sum_{\psi(q)} \widehat{H}(\psi)\psi(-\beta\sigma m')\overline{\psi}(c'\delta),
\end{aligned} \tag{8.90}
$$

where

$$
\widehat{H}(\psi) = \widehat{H} = \widehat{H}(\psi, \overline{\chi}, m_{1,0}, m_{2,0}, m_{3,0}, r_0) = \sum_{v(q)} H_{\overline{\chi}}(m_{1,0}, m_{2,0}, m_{3,0}v, r_0)\overline{\psi}(v). \tag{8.91}
$$

8.10 Collecting the Conrey-Iwaniec phase term

Recall that

$$
\varkappa = e\left(-\frac{\beta\sigma m_0 m_1' \overline{c_0 B}}{A\delta_2}\right), \tag{8.92}
$$

and

$$
\Omega = e\left(-\frac{\beta\sigma m \overline{c'\delta}}{c_0}\right). \tag{8.93}
$$

We have

$$
\begin{aligned}
\varkappa &= e\left(-\frac{\beta\sigma m\, \overline{c_0 B m_2' m_3'}}{A\delta_2}\right) \\
&= e\left(-\frac{\beta\sigma m\, \overline{c_0 B n_1' c_2' \delta_1}}{A\delta_2}\right),
\end{aligned} \tag{8.94}
$$

since $m_3' = c_2' = \delta_1$ and $m_2' = n_1'c_2'$. Since $B|n_2'$ and $c_2' = \delta_1$, we have $m = m_0m' = m_0n_2'n_1'^2c_2'^3 \equiv 0 \pmod{Bn_1'c_2'\delta_1}$. Also, $Bn_1'c_2'\delta_1A\delta_2 = c'\delta$. By weak reciprocity, we have

$$\begin{aligned}\varkappa &= e\left(-\frac{\beta\sigma m\,\overline{c_0}}{Bn_1'c_2'\delta_1A\delta_2}\right)e\left(\frac{\beta\sigma m\,\overline{c_0A\delta_2}}{Bn_1'c_2'\delta_1}\right)\\ &= e\left(-\frac{\beta\sigma m\,\overline{c_0}}{c'\delta}\right).\end{aligned} \tag{8.95}$$

Finally, we have

$$\mathcal{P} := \varkappa\Omega = e\left(-\frac{\beta\sigma m\,\overline{c_0}}{c'\delta}\right)e\left(-\frac{\beta\sigma m\overline{c'\delta}}{c_0}\right) = e\left(-\frac{\beta\sigma m}{c_0c'\delta}\right) = e\left(-\frac{\beta\sigma n_2n_1^2c_2^3}{c\delta}\right), \tag{8.96}$$

which is the Conrey-Iwaniec phase term.

8.11 Putting everything together

When all of the conditions in (8.31) are satisfied, we have

$$\mathcal{T} = \frac{\varphi(c_1)\varphi(n_3)}{(\varphi(c))^2}C, \tag{8.97}$$

where

$$C = C_0C' = C_0C_2'C_1' = C_0C_2'c_1'N_1'N_3' = c_1'C_0C_2'N_1'B'A', \tag{8.98}$$

with

$$\begin{aligned}C_0 &= \frac{\alpha\Omega}{\varphi(q)}\sum_{\psi(q)}\widehat{H}(\psi)\psi(-\beta\sigma m')\overline{\psi}(c'\delta)\\ C_2' &= \delta_1(\varphi(\delta_1))^2\mu(\delta_1)\\ N_1' &= \varphi(n_1')\mu(n_1')\\ B' &= (\mu(B))^2\\ A' &= \varkappa\sum_{\substack{D_1|F\\ D_2=\frac{A}{D_1}}}\frac{D_2}{\varphi(D_1\delta_4)}\sum_{\lambda(D_1\delta_4)}\tau(\overline{\lambda})\lambda(\beta\sigma m_0m_1')\overline{\lambda}(\delta_3c_0BD_2),\end{aligned} \tag{8.99}$$

with

$$\alpha = \chi(-\sigma)\overline{\chi}(\delta)qr_0^2. \tag{8.100}$$

Note that since we have added $(\mu(B))^2 = 1$ to (8.31), we can remove $B' = (\mu(B))^2$ from our final expression for $\mathcal{T}$. The result is obtained by multiplying the above expressions, simplifying a bit, and using the fact that $\mathcal{P} = \varkappa\Omega$.

Define

$$Z = Z(s_1, s_2, s_3, s_4, s_5) = \sum_{\substack{\delta \geq 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{r\geq 1 \\ n_2 \geq 1 \\ c=qr}} \sum_{c_1c_2=c} \sum_{n_1n_3=c_1} \frac{A_\phi(n_2,n_1)}{n_2^{s_1} r^{s_2} n_1^{s_3} c_2^{s_4}} \frac{1}{qr^2} \mathcal{T}\mathcal{P}^{-1}. \tag{9.1}$$

This Z-function will serve as our analog of the Z-function from section 5 of [23].

Let us perform some simplifications. By proposition 8.1.0.1, we get

$$Z = \sum_{\substack{\delta \geq 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{n_2',n_1',A\geq 1 \\ c_2'\delta_3\delta_4=\delta \\ B\delta_3|n_2' \\ F=\text{free}(A) \\ (\mu(B))^2=1 \\ (n_2',A\delta_4 q)=1 \\ (n_1',AB\delta q)=1 \\ (B,A\delta q)=1 \\ (A,\delta q)=1}} \sum_{\substack{D_1|F \\ D_2=\frac{A}{D_1}}} \frac{1}{\varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q)}} A_\phi(n_2',n_1') \times$$
$$\frac{\mu(n_1'c_2')\tau(\overline{\lambda})\lambda(n_2'n_1'c_2')\overline{\lambda}(\delta_3 BD_2)\psi(n_2'n_1'c_2'^2)\overline{\psi}(AB\delta)}{n_2'^{s_1} D_2^{s_2} D_1^{s_2+1} n_1'^{s_2+s_3+1} c_2'^{s_2+s_4+1} B^{s_2+1}} Z_{\text{fin},1}, \tag{9.2}$$

where

$$\begin{aligned} Z_{\text{fin},1} &= Z_{\text{fin},1}(\lambda,\psi) \\ &= \omega_1 \sum_{\substack{r_0 n_{2,0}|q^\infty \\ n_{1,0}c_{2,0}|q \\ (n_{2,0}n_{1,0}c_{2,0},r_0)=1}} \frac{A_\phi(n_{2,0},n_{1,0})}{n_{2,0}^{s_1} r_0^{s_2} n_{1,0}^{s_3} c_{2,0}^{s_4}} \lambda(n_{2,0}n_{1,0}^2 c_{2,0}^3)\overline{\lambda}(r_0) \frac{\varphi\left(\frac{qr_0}{c_{2,0}}\right)\varphi\left(\frac{qr_0}{n_{1,0}c_{2,0}}\right)}{(\varphi(qr_0))^2} \widehat{H}(\psi), \end{aligned} \tag{9.3}$$

with

$$\widehat{H}(\psi) = \widehat{H} = \widehat{H}(\psi, \overline{\chi}, n_{2,0}n_{1,0}c_{2,0}, n_{1,0}c_{2,0}, c_{2,0}, r_0), \tag{9.4}$$

and

$$\omega_1 = \psi(-1)(\lambda\psi)(\beta\sigma)\overline{\lambda}(q)\chi(-\sigma)\overline{\chi}(\delta). \tag{9.5}$$

Changing orders of summing, we get

$$Z = \sum_{\substack{\delta \geq 1 \\ (\delta, q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{\delta_4|\delta \\ D_1 \geq 1 \\ (\mu(D_1))^2=1 \\ (D_1, \delta q)=1}} \frac{1}{D_1^{s_2+1}\varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q)}} \overline{\psi}(D_1\delta_4)\tau(\overline{\lambda})Z'Z_{\text{fin},1}, \tag{9.6}$$

where

$$Z' = \sum_{\substack{n_2', n_1', D_2 \geq 1 \\ c_2'\delta_3\delta_4 = \delta \\ B\delta_3 | n_2' \\ (\mu(B))^2=1 \\ (n_2', D_1D_2\delta_4 q)=1 \\ (n_1', D_1D_2B\delta q)=1 \\ (B, D_1D_2\delta q)=1 \\ (D_2, D_1\delta q)=1}} \frac{A_\phi(n_2', n_1')\mu(n_1'c_2')(\lambda\psi)(n_2'n_1'c_2')\overline{\lambda\psi}(D_2B\delta_3)}{n_2'^{s_1}D_2^{s_2}n_1'^{s_2+s_3+1}c_2'^{s_2+s_4+1}B^{s_2+1}}, \tag{9.7}$$

since

$$(\mu(D_1))^2 = 1 \text{ and } D_1 | F = \text{free}(A) = \text{free}(D_1D_2) \iff (D_1, D_2) = 1. \tag{9.8}$$

Then

$$Z' = \sum_{\substack{n_1', B \geq 1 \\ c_2'\delta_3\delta_4 = \delta \\ (\mu(B))^2=1 \\ (n_1', D_1B\delta q)=1 \\ (B, D_1\delta q)=1}} \frac{\mu(n_1'c_2')(\lambda\psi)(n_1'c_2')\overline{\lambda\psi}(B\delta_3)}{n_1'^{s_2+s_3+1}c_2'^{s_2+s_4+1}B^{s_2+1}} Z'', \tag{9.9}$$

where

$$Z'' = \sum_{\substack{n_2', D_2 \geq 1 \\ B\delta_3 | n_2' \\ (n_2', D_2)=1 \\ (n_2', D_1\delta_4 q)=1 \\ (D_2, n_1'BD_1\delta q)=1}} \frac{A_\phi(n_2', n_1')(\lambda\psi)(n_2')\overline{\lambda\psi}(D_2)}{n_2'^{s_1}D_2^{s_2}}. \tag{9.10}$$

Next, we detect the condition $(n_2', D_2) = 1$ using the Möbius function:

$$Z'' = \sum_{\substack{n_2', D_2 \geq 1 \\ B\delta_3 | n_2' \\ (n_2', D_1\delta_4 q)=1 \\ (D_2, n_1'BD_1\delta q)=1}} \sum_{\rho | (n_2', D_2)} \frac{\mu(\rho)A_\phi(n_2', n_1')(\lambda\psi)(n_2')\overline{\lambda\psi}(D_2)}{n_2'^{s_1}D_2^{s_2}}. \tag{9.11}$$

Now, $(D_2, B\delta) = 1 \implies (D_2, B\delta_3) = 1 \implies (\rho, B\delta_3) = 1$ since $\rho | D_2$. As a result, $\rho B\delta_3 | n_2'$.

Let

$$n_2' = n_{2,1}n_{2,2},$$
$$D_2 = \rho D_3, \tag{9.12}$$

where $\rho B\delta_3|n_{2,1}|(\rho B\delta_3 n_1')^\infty$ and $(n_{2,2}, \rho B\delta_3 n_1') = 1$. Switching order of summing, we get

$$Z'' = \sum_{\substack{\rho\geq 1 \\ (\rho, n_1' BD_1\delta q)=1}} \frac{\mu(\rho)\overline{\lambda\psi}(\rho)}{\rho^{s_2}} \sum_{\rho B\delta_3|n_{2,1}|(\rho B\delta_3 n_1')^\infty} \frac{A_\phi(n_{2,1}, n_1')(\lambda\psi)(n_{2,1})}{n_{2,1}^{s_1}} Z''', \tag{9.13}$$

where

$$Z''' = \sum_{\substack{n_{2,2}, D_3\geq 1 \\ (n_{2,2}, \rho n_1' BD_1\delta_3\delta_4 q)=1 \\ (D_3, n_1' BD_1\delta q)=1}} \frac{A_\phi(n_{2,2}, 1)(\lambda\psi)(n_{2,2})\overline{\lambda\psi}(D_3)}{n_{2,2}^{s_1} D_3^{s_2}}. \tag{9.14}$$

We have omitted $(n_{2,1}, D_1\delta_4 q) = 1$ since that follows from $n_{2,1}|(\rho B\delta_3 n_1')^\infty$. We can now write

$$Z''' = L(s_1, \phi\times(\lambda\psi))L(s_2, \overline{\lambda\psi})Z'''', \tag{9.15}$$

where

$$\begin{aligned} Z'''' &= \left[\prod_{p|n_1' BD_1\delta q}\left(\sum_{k=0}^\infty \frac{\overline{\lambda\psi}(p^k)}{p^{ks_2}}\right)^{-1}\right]\left[\prod_{p|\rho n_1' BD_1\delta_3\delta_4 q}\left(\sum_{k=0}^\infty \frac{A_\phi(p^k, 1)(\lambda\psi)(p^k)}{p^{ks_1}}\right)^{-1}\right] \\ &= \left[\prod_{p|n_1' BD_1\delta q} I(p, s_2)\right]\left[\prod_{p|\rho n_1' BD_1\delta_3\delta_4 q} J(p, s_1)\right], \end{aligned} \tag{9.16}$$

where

$$\begin{aligned} I(p,s) &= \left(1 - \frac{\overline{\lambda\psi}(p)}{p^s}\right), \\ J(p,s) &= \prod_{j=1}^3\left(1 - \frac{\alpha_j(p)(\lambda\psi)(p)}{p^s}\right), \end{aligned} \tag{9.17}$$

with $\alpha_j(p), j \in \{1,2,3\}$ being the local parameters. Therefore,

$$Z = \sum_{\substack{\delta \geq 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{\delta_4 | \delta \\ D_1 \geq 1 \\ (\mu(D_1))^2=1 \\ (D_1,\delta q)=1}} \frac{1}{D_1^{s_2+1}\varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q)}} \overline{\psi}(D_1\delta_4)\tau(\overline{\lambda})L(s_1, \phi \times (\lambda\psi))L(s_2, \overline{\lambda\psi})Z_{\text{fin}}, \tag{9.18}$$

where

$$Z_{\text{fin}} = Z_{\text{fin}}(\lambda,\psi) = Z_{\text{fin},1}(\lambda,\psi)Z_{\text{fin},2}(\lambda,\psi), \tag{9.19}$$

with

$$\begin{aligned} Z_{\text{fin},2} = Z_{\text{fin},2}(\lambda,\psi) = & \sum_{\substack{n_1',B \geq 1 \\ c_2'\delta_3\delta_4=\delta \\ (\mu(B))^2=1 \\ (n_1',Bc_2'\delta_3)=1 \\ (B,c_2'\delta_3)=1}} \frac{\mu(n_1'c_2')(\lambda\psi)(n_1'c_2')\overline{\lambda\psi}(B\delta_3)}{n_1'^{s_2+s_3+1}c_2'^{s_2+s_4+1}B^{s_2+1}} \times \\ & \sum_{\substack{\rho \geq 1 \\ (\rho,n_1'Bc_2'\delta_3)=1}} \frac{\mu(\rho)\overline{\lambda\psi}(\rho)}{\rho^{s_2}} \sum_{\rho B\delta_3 | n_{2,1} | (\rho B\delta_3 n_1')^\infty} \frac{A_\phi(n_{2,1},n_1')(\lambda\psi)(n_{2,1})}{n_{2,1}^{s_1}} \times \\ & \left[\prod_{p | n_1'Bc_2'\delta_3} I(p,s_2)\right]\left[\prod_{p|\rho n_1'B\delta_3} J(p,s_1)\right]. \end{aligned} \tag{9.20}$$

Note that we have simplified the coprimality conditions since some of them are detected by $\lambda\psi, \overline{\lambda\psi} \pmod{D_1\delta_4 q}$. Now we present the reader with bounds for Z_{fin}, which are proved in the following sections.

Proposition 9.0.0.1. *Let*

- $\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2}$; *then*

$$Z_{fin} \ll_{\epsilon,\gamma} \delta^\epsilon q^{\frac{3}{2}+\epsilon}, \tag{9.21}$$

- $\sigma_1 \geq \gamma_1 > 1,\ \sigma_2 \geq \gamma_2 > 1,\ \sigma_3 \geq \gamma_3 > 1,\ \sigma_4 \geq \gamma_4 > 1$ *and* ψ *is the trivial Dirichlet* *character modulo q (conductor = 1); then*

$$Z_{fin} \ll_{\epsilon,\gamma} \delta^{\epsilon} q^{1+\epsilon}, \tag{9.22}$$

where in both cases, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. □

9.1 Factoring $Z_{\mathbf{fin},1}$

At first, let us factor $Z_{\text{fin},1}$ over prime powers to simplify our work. Notice that $r_0 | q^\infty \implies \varphi(qr_0) = r_0\varphi(q)$; thus,

$$Z_{\text{fin},1} = \frac{\omega_1}{(\varphi(q))^2} \sum_{\substack{r_0 n_{2,0} | q^\infty \\ n_{1,0} c_{2,0} | q \\ (n_{2,0} n_{1,0} c_{2,0}, r_0) = 1}} \frac{A_\phi(n_{2,0}, n_{1,0})}{n_{2,0}^{s_1} r_0^{s_2+2} n_{1,0}^{s_3} c_{2,0}^{s_4}} \lambda(n_{2,0} n_{1,0}^2 c_{2,0}^3) \overline{\lambda}(r_0) \varphi\left(\frac{qr_0}{c_{2,0}}\right) \varphi\left(\frac{qr_0}{n_{1,0} c_{2,0}}\right) \widehat{H}(\psi). \tag{9.23}$$

Now write

$$Z_{\text{fin},1} = \frac{\omega_1 \omega_2}{(\varphi(q))^2} \prod_{\substack{p^j \| q \\ p \text{ prime}}} Z_{\text{fin},1,p}, \tag{9.24}$$

where

$$Z_{\text{fin},1,p} = \sum_{\substack{ad | p^\infty \\ bc | p^j \\ (abc, d) = 1}} \frac{A_\phi(a,b)}{a^{s_1} d^{s_2+2} b^{s_3} c^{s_4}} (\lambda \eta_p)(ab^2c^3) \overline{\lambda \eta_p}(d) \varphi\left(\frac{p^j d}{c}\right) \varphi\left(\frac{p^j d}{bc}\right) \widehat{H}(\psi_p, \overline{\chi_p}, abc, bc, c, d), \tag{9.25}$$

where ω_2 is some complex number of absolute value 1 depending on ψ, η_p is some Dirichlet character depending on ψ and p, and ψ_p, χ_p are the p-parts of ψ, χ respectively. Here we will assume that q is cube-free; therefore $p^j \| q \implies j \in \{1, 2\}$.

9.2 Bounds for $Z_{\mathbf{fin},1}$

χ_p is a primitive Dirichlet character modulo p^j (conductor $= p^j$). ψ_p is a Dirichlet character modulo p^j. We handle this in 3 cases; for the first two cases, we assume

$$\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2}. \tag{9.26}$$

Case 1 ψ_p is primitive modulo p^j (conductor $= p^j$). Then $a = b = c = d = 1$ is forced. We have

$$Z_{\text{fin},1,p} = (\varphi(p^j))^2 \widehat{H}(\psi_p, \overline{\chi_p}, 1, 1, 1, 1) = (\varphi(p^j))^2 \tau(\overline{\psi_p}) g(\overline{\chi_p}, \psi_p), \tag{9.27}$$

from lemma 6.4 in [23]. Thus, by theorem 6.9 in [23], we have

$$|Z_{\text{fin},1,p}| \ll (\varphi(p^j))^2 p^{\frac{j}{2}} p^j = (\varphi(p^j))^2 p^{\frac{3j}{2}}. \tag{9.28}$$

Case 2 ψ_p is modulo p^2 $(j = 2)$ with conductor p. By lemma 6.8 of [23], the only 2 terms that survive correspond to $a = b = c = 1, d = p$ and $a = b = d = 1, c = p$. Thus

$$Z_{\text{fin},1,p} = \frac{(\varphi(p))^2 (\lambda\eta_p)(p^3)}{p^{s_4}} \widehat{H}(\psi_p, \overline{\chi_p}, p, p, p, 1) + \frac{(\varphi(p^3))^2 \overline{\lambda\eta_p}(p)}{p^{s_2+2}} \widehat{H}(\psi_p, \overline{\chi_p}, 1, 1, 1, p). \tag{9.29}$$

By lemma 6.8 of [23], we have

$$\begin{aligned}
|Z_{\text{fin},1,p}| &\ll \frac{(\varphi(p))^2}{p^{\sigma_4}} p^{6-\frac{3}{2}} + \frac{(\varphi(p^3))^2}{p^{\sigma_2+2}} p^{4-\frac{1}{2}} \\
&= (\varphi(p^2))^2 (p^{\frac{5}{2}-\sigma_4} + p^{\frac{7}{2}-\sigma_2}) \\
&\leq (\varphi(p^2))^2 (p^{\frac{5}{2}+\frac{1}{2}} + p^{\frac{7}{2}-\frac{1}{2}}) \\
&= 2(\varphi(p^2))^2 p^3 \\
&= 2(\varphi(p^j))^2 p^{\frac{3j}{2}} \\
&\ll (\varphi(p^j))^2 p^{\frac{3j}{2}}.
\end{aligned} \tag{9.30}$$

Case 3 ψ_p is trivial modulo p^j (conductor = 1). By lemma 6.5 of [23], we have

$$\widehat{H}(\psi_p, \overline{\chi_p}, abc, bc, c, d) = \chi_0(d)R_{p^j}(abc)R_{p^j}(bc)R_{p^j}(c) + p^j R_{p^j}(d)\chi(-1)\chi_0(ab^2c^3). \tag{9.31}$$

Writing $a = p^{a_1}, b = p^{b_1}, c = p^{c_1}, d = p^{d_1}$, we have

$$\begin{aligned} Z_{\text{fin},1,p} &= p^j\chi(-1)\sum_{d_1=0}^{\infty}\frac{(\varphi(p^{j+d_1}))^2\overline{\lambda\eta_p}(p^{d_1})R_{p^j}(p^{d_1})}{p^{d_1(s_2+2)}} + \\ &\sum_{\substack{a_1,b_1,c_1\geq 0\\ b_1+c_1\leq j}}\frac{A_\phi(p^{a_1},p^{b_1})\varphi(p^{j-c_1})\varphi(p^{j-b_1-c_1})(\lambda\eta_p)(p^{a_1+2b_1+3c_1})}{p^{a_1s_1+b_1s_3+c_1s_4}}\times \\ &\qquad R_{p^j}(p^{a_1+b_1+c_1})R_{p^j}(p^{b_1+c_1})R_{p^j}(p^{c_1}). \end{aligned} \tag{9.32}$$

By using the fact that $\varphi(p^{j+d_1}) = \varphi(p^j)p^{d_1}$ and by evaluating $R_{p^j}(p^{b_1+c_1})R_{p^j}(p^{c_1})$, we have

$$\begin{aligned} \frac{Z_{\text{fin},1,p}}{(\varphi(p^j))^2} &= p^j\chi(-1)\sum_{d_1=0}^{\infty}\frac{\overline{\lambda\eta_p}(p^{d_1})R_{p^j}(p^{d_1})}{p^{d_1s_2}} + \\ &\sum_{\substack{a_1,b_1,c_1\geq 0\\ b_1+c_1\leq j}}\frac{A_\phi(p^{a_1},p^{b_1})\mu(p^{j-c_1})\mu(p^{j-b_1-c_1})(\lambda\eta_p)(p^{a_1+2b_1+3c_1})R_{p^j}(p^{a_1+b_1+c_1})}{p^{a_1s_1+b_1s_3+c_1s_4}}. \end{aligned} \tag{9.33}$$

Therefore

$$|Z_{\text{fin},1,p}| \leq (\varphi(p^j))^2(p^jS_1 + S_2), \tag{9.34}$$

where

$$S_1 = \sum_{d_1=0}^{\infty}\frac{(p^j,p^{d_1})}{p^{d_1\sigma_2}} = \sum_{d_1=0}^{j-1}p^{d_1(1-\sigma_2)} + p^j\sum_{d_1=j}^{\infty}\frac{1}{p^{d_1\sigma_2}}, \tag{9.35}$$

and

$$
\begin{aligned}
S_2 &= \sum_{\substack{a_1,b_1,c_1\geq 0\\ b_1+c_1\leq j}} \frac{|A_\phi(p^{a_1},p^{b_1})|(p^j,p^{a_1+b_1+c_1})}{p^{a_1\sigma_1+b_1\sigma_3+c_1\sigma_4}} \\
&\leq p^j \sum_{\substack{a_1,b_1,c_1\geq 0\\ b_1+c_1\leq j}} \frac{|A_\phi(p^{a_1},p^{b_1})|}{p^{a_1\sigma_1+b_1\sigma_3+c_1\sigma_4}} \\
&\ll_\epsilon p^{j+\epsilon} \sum_{a_1\geq 0} p^{(\frac{1}{2}-\sigma_1)a_1} \sum_{\substack{b_1,c_1\geq 0\\ b_1+c_1\leq j}} p^{(\frac{1}{2}-\sigma_3)b_1-\sigma_4 c_1},
\end{aligned}
\tag{9.36}
$$

for $\epsilon > 0$ since

$$
|A_\phi(p^a,p^b)| \ll_\epsilon p^{\frac{a+b}{2}+\epsilon} \quad \text{for } \epsilon > 0. \tag{9.37}
$$

Subcase 1 $\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2}$

$$
\begin{aligned}
S_1 &\leq \sum_{d_1=0}^{j-1} p^{d_1(1-\frac{1}{2})} + p^j \sum_{d_1=j}^{\infty} \frac{1}{p^{\frac{d_1}{2}}} = \sum_{d_1=0}^{j-1} p^{\frac{d_1}{2}} + \frac{p^{\frac{j}{2}}}{1-\frac{1}{p^{\frac{1}{2}}}} \leq jp^{\frac{j}{2}} + \frac{p^{\frac{j}{2}}}{1-\frac{1}{2^{\frac{1}{2}}}} \ll p^{\frac{j}{2}}. \\
S_2 &\ll_\epsilon p^{j+\epsilon} \sum_{a_1\geq 0} p^{(\frac{1}{2}-\gamma_1)a_1} \sum_{\substack{b_1,c_1\geq 0\\ b_1+c_1\leq j}} p^{\frac{b_1+c_1}{2}} \leq \frac{p^{\frac{3j}{2}+\epsilon}}{1-p^{\frac{1}{2}-\gamma_1}} \sum_{\substack{b_1,c_1\geq 0\\ b_1+c_1\leq j}} 1 \\
&\leq \frac{p^{\frac{3j}{2}+\epsilon}(j+1)^2}{1-2^{\frac{1}{2}-\gamma_1}} \ll_{\gamma_1} p^{\frac{3j}{2}+\epsilon}.
\end{aligned}
$$

Therefore $|Z_{\mathrm{fin},1,p}| \ll_{\epsilon,\gamma_1} (\varphi(p^j))^2 p^{\frac{3j}{2}+\epsilon}$.

(9.38)

Subcase 2 $\sigma_1 \geq \gamma_1 > 1,\ \sigma_2 \geq \gamma_2 > 1,\ \sigma_3 \geq \gamma_3 > 1,\ \sigma_4 \geq \gamma_4 > 1$

$$
\begin{aligned}
S_1 &\leq \sum_{d_1=0}^{j-1} 1 + p^j \sum_{d_1=j}^{\infty} \frac{1}{p^{d_1}} = j + \frac{1}{1-\frac{1}{p}} \leq j + \frac{1}{1-\frac{1}{2}} \ll 1. \\
S_2 &\ll_\epsilon p^{j+\epsilon} \sum_{a_1\geq 0} p^{-\frac{1}{2}a_1} \sum_{b_1\geq 0} p^{-\frac{1}{2}b_1} \sum_{c_1\geq 0} p^{-c_1} = \frac{p^{j+\epsilon}}{\left(1-p^{-\frac{1}{2}}\right)^2 (1-p^{-1})} \\
&\leq \frac{p^{j+\epsilon}}{\left(1-2^{-\frac{1}{2}}\right)^2 (1-2^{-1})} \ll p^{j+\epsilon}.
\end{aligned}
\tag{9.39}
$$

Therefore $|Z_{\mathrm{fin},1,p}| \ll_\epsilon (\varphi(p^j))^2 p^{j+\epsilon}$.

Now, let us combine the above information to obtain bounds for $Z_{\text{fin},1}$; we perform this in 2 cases.

- $\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2}$

  Combine (9.24), (9.28), (9.30), (9.38) to obtain

$$|Z_{\text{fin},1}| = \frac{1}{(\varphi(q))^2} \prod_{p^j \| q} |Z_{\text{fin},1,p}| \ll_{\epsilon,\gamma_1} \frac{q^{\epsilon}}{(\varphi(q))^2} \prod_{p^j \| q} (\varphi(p^j))^2 p^{\frac{3j}{2}} = q^{\frac{3}{2}+\epsilon}. \tag{9.40}$$

- $\sigma_1 \geq \gamma_1 > 1,\ \sigma_2 \geq \gamma_2 > 1,\ \sigma_3 \geq \gamma_3 > 1,\ \sigma_4 \geq \gamma_4 > 1$ and ψ is the trivial Dirichlet character modulo q (conductor $= 1$).

  Combine (9.24) and (9.39) to get

$$|Z_{\text{fin},1}| = \frac{1}{(\varphi(q))^2} \prod_{p^j \| q} |Z_{\text{fin},1,p}| \ll_{\epsilon} \frac{q^{\epsilon}}{(\varphi(q))^2} \prod_{p^j \| q} (\varphi(p^j))^2 p^{j} = q^{1+\epsilon}. \tag{9.41}$$

9.3 Factoring $Z_{\text{fin},2}$

At first, we factor $Z_{\text{fin},2}$ over primes to make our work simpler. Consider the prime factorization $\frac{\delta}{\delta_4} = \prod_{p \text{ prime}} p^{b_p}$. Note that since δ is square-free, we have $b_p \in \{0,1\}$. Then

$$Z_{\text{fin},2} = \prod_{p \text{ prime}} Z_{\text{fin},2,p}, \tag{9.42}$$

with

$$Z_{\text{fin},2,p} = \sum_{\substack{n_1', c_2', \delta_3, B, \rho \in \{0,1\} \\ c_2'+\delta_3 = b_p \\ n_1'+b_p+B+\rho \in \{0,1\}}} \frac{\mu(p^{n_1'+c_2'+\rho})(\lambda\psi)(p^{n_1'+c_2'})\overline{\lambda\psi}(p^{B+\delta_3+\rho})}{p^{n_1'(s_2+s_3+1)+c_2'(s_2+s_4+1)+B(s_2+1)+\rho s_2}} \times$$
$$\sum_{\rho+B+\delta_3 \leq n_{2,1} \leq (\rho+B+\delta_3+n_1')\infty} \frac{A_\phi(p^{n_{2,1}}, p^{n_1'})(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \times \tag{9.43}$$
$$\left[\prod_{\substack{P | p^{n_1'+B+b_p} \\ P \text{ prime}}} I(P,s_2)\right] \left[\prod_{\substack{P | p^{\rho+n_1'+B+\delta_3} \\ P \text{ prime}}} J(P,s_1)\right],$$

where we have retained the variable names for respective exponents, $n_{2,1} \le 0\infty$ is taken to mean $n_{2,1} \le 0$, and $n_{2,1} \le 1\infty$ is interpreted as $n_{2,1} < \infty$. We handle this in 2 cases.

$(b_p = 1)$ We have $n'_1 = B = \rho = 0$. We will break the sum into 2 parts depending on whether $c'_2 = 1, \delta_3 = 0$ or $\delta_3 = 1, c'_2 = 0$.

$$
\begin{aligned}
Z_{\mathrm{fin},2,p} = & \sum_{\substack{c'_2,\delta_3\in\{0,1\} \\ c'_2+\delta_3=1}} \frac{\mu(p^{c'_2})(\lambda\psi)(p^{c'_2})\overline{\lambda\psi}(p^{\delta_3})}{p^{c'_2(s_2+s_4+1)}} \times \\
& \sum_{\delta_3\le n_{2,1}\le \delta_3\infty} \frac{A_\phi(p^{n_{2,1}},1)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} I(p,s_2) \prod_{\substack{P\mid p^{\delta_3} \\ P \text{ prime}}} J(P,s_1) \\
= & -\frac{(\lambda\psi)(p)}{p^{s_2+s_4+1}} I(p,s_2) + \overline{\lambda\psi}(p) I(p,s_2) J(p,s_1) \sum_{1\le n_{2,1}<\infty} \frac{A_\phi(p^{n_{2,1}},1)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \\
= & -\frac{(\lambda\psi)(p)}{p^{s_2+s_4+1}} I(p,s_2) + \overline{\lambda\psi}(p) I(p,s_2)(1 - J(p,s_1)),
\end{aligned}
\tag{9.44}
$$

since

$$
\sum_{1\le n_{2,1}<\infty} \frac{A_\phi(p^{n_{2,1}},1)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} = (J(p,s_1))^{-1} - 1. \tag{9.45}
$$

$(b_p = 0)$ We have $c'_2 = \delta_3 = 0$. Also, the contribution from $n'_1 = B = \rho = 0$ is 1. Therefore

$$
\begin{aligned}
Z_{\mathrm{fin},2,p} = 1 + & \sum_{\substack{n'_1,B,\rho\in\{0,1\} \\ n'_1+B+\rho=1}} \frac{\mu(p^{n'_1+\rho})(\lambda\psi)(p^{n'_1})\overline{\lambda\psi}(p^{B+\rho})}{p^{n'_1(s_2+s_3+1)+B(s_2+1)+\rho s_2}} \times \\
& \sum_{\rho+B\le n_{2,1}\le(\rho+B+n'_1)\infty} \frac{A_\phi(p^{n_{2,1}},p^{n'_1})(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \times \\
& \left[\prod_{\substack{P\mid p^{n'_1+B} \\ P \text{ prime}}} I(P,s_2)\right] \left[\prod_{\substack{P\mid p^{\rho+n'_1+B} \\ P \text{ prime}}} J(P,s_1)\right].
\end{aligned}
\tag{9.46}
$$

Now we will break the sum into 3 parts depending on which one of n_1', B, ρ is 1.

$$\begin{aligned} Z_{\mathrm{fin},2,p} &= 1 \\ &- \frac{(\lambda\psi)(p)}{p^{s_2+s_3+1}} I(p,s_2) J(p,s_1) \sum_{0 \le n_{2,1} < \infty} \frac{A_\phi(p^{n_{2,1}}, p)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \\ &+ \frac{\overline{\lambda\psi}(p)}{p^{s_2+1}} I(p,s_2) J(p,s_1) \sum_{1 \le n_{2,1} < \infty} \frac{A_\phi(p^{n_{2,1}}, 1)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \\ &- \frac{\overline{\lambda\psi}(p)}{p^{s_2}} J(p,s_1) \sum_{1 \le n_{2,1} < \infty} \frac{A_\phi(p^{n_{2,1}}, 1)(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}}. \end{aligned} \tag{9.47}$$

By Hecke relation, for $n_{2,1} \ge 1$, we have

$$A_\phi(p^{n_{2,1}}, p) = A_\phi(p^{n_{2,1}}, 1) A_\phi(1, p) - A_\phi(p^{n_{2,1}-1}, 1). \tag{9.48}$$

Therefore

$$\begin{aligned} Z_{\mathrm{fin},2,p} = 1 &- \frac{(\lambda\psi)(p)}{p^{s_2+s_3+1}} I(p,s_2) J(p,s_1) A_\phi(1,p) \\ &- \frac{(\lambda\psi)(p)}{p^{s_2+s_3+1}} I(p,s_2) J(p,s_1) \sum_{1 \le n_{2,1} < \infty} \frac{(A_\phi(p^{n_{2,1}}, 1) A_\phi(1,p) - A_\phi(p^{n_{2,1}-1}, 1))(\lambda\psi)(p^{n_{2,1}})}{p^{n_{2,1}s_1}} \\ &+ \frac{\overline{\lambda\psi}(p)}{p^{s_2+1}} I(p,s_2)(1 - J(p,s_1)) \\ &- \frac{\overline{\lambda\psi}(p)}{p^{s_2}} (1 - J(p,s_1)). \end{aligned} \tag{9.49}$$

Simplifying, we get

$$\begin{aligned} Z_{\mathrm{fin},2,p} = 1 &- \frac{(\lambda\psi)(p)}{p^{s_2+s_3+1}} I(p,s_2) A_\phi(1,p) + \frac{(\lambda\psi)(p^2)}{p^{s_1+s_2+s_3+1}} I(p,s_2) + \\ &\frac{\overline{\lambda\psi}(p)}{p^{s_2+1}} I(p,s_2)(1 - J(p,s_1)) - \frac{\overline{\lambda\psi}(p)}{p^{s_2}} (1 - J(p,s_1)). \end{aligned} \tag{9.50}$$

9.4 Bounds for $Z_{\mathbf{fin},2}$

For $\sigma_2 \geq 0$, $|I(p, s_2)| \leq \left(1 + \frac{1}{p^{\sigma_2}}\right) \leq 2$. We know

$$1 - J(p, s_1) = \frac{A_\phi(1,p)(\lambda\psi)(p)}{p^{s_1}} - \frac{A_\phi(p,1)(\lambda\psi)(p^2)}{p^{2s_1}} + \frac{(\lambda\psi)(p^3)}{p^{3s_1}}. \tag{9.51}$$

Thus, for $\sigma_1 \geq 0$, $|1 - J(p, s_1)| \leq \frac{2|A_\phi(1,p)|}{p^{\sigma_1}} + \frac{1}{p^{3\sigma_1}}$. We are interested in the following two situations

- $\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2}$
- $\sigma_1 \geq \gamma_1 > 1,\ \sigma_2 \geq \gamma_2 > 1,\ \sigma_3 \geq \gamma_3 > 1,\ \sigma_4 \geq \gamma_4 > 1$

In both these situations, we can perform the following estimations, which are handled in two cases.

($b_p = 1$) Since the Rankin-Selberg L-function associated with $\phi \times \overline{\phi}$ exists, we have

$$|A_\phi(1,p)| < 3p^{\frac{1}{2}} \quad \forall p \text{ prime}. \tag{9.52}$$

Therefore,

$$\frac{|A_\phi(1,p)|}{p^{\sigma_1}} < 3, \tag{9.53}$$

and thus

$$|Z_{\text{fin},2,p}| \leq \frac{2}{p^{\sigma_2+\sigma_4+1}} + 2\left(\frac{2|A_\phi(1,p)|}{p^{\sigma_1}} + \frac{1}{p^{3\sigma_1}}\right) \ll 1. \tag{9.54}$$

The divisor bound implies

$$\prod_{\substack{p \text{ prime} \\ b_p=1}} |Z_{\text{fin},2,p}| \ll_\epsilon \left(\frac{\delta}{\delta_4}\right)^\epsilon \leq \delta^\epsilon. \tag{9.55}$$

$(b_p = 0)$

$$
\begin{aligned}
|Z_{\mathrm{fin},2,p}| \leq 1 &+ \frac{2|A_\phi(1,p)|}{p^{\sigma_2+\sigma_3+1}} + \frac{2}{p^{\sigma_1+\sigma_2+\sigma_3+1}} + \\
&\frac{2}{p^{\sigma_2+1}}\left(\frac{2|A_\phi(1,p)|}{p^{\sigma_1}} + \frac{1}{p^{3\sigma_1}}\right) + \frac{1}{p^{\sigma_2}}\left(\frac{2|A_\phi(1,p)|}{p^{\sigma_1}} + \frac{1}{p^{3\sigma_1}}\right) \\
&\leq 1 + \frac{2|A_\phi(1,p)|}{p^{\sigma_2+\sigma_3+1}} + \frac{2}{p^{\sigma_1+\sigma_2+\sigma_3+1}} + \frac{3}{p^{\sigma_2}}\left(\frac{2|A_\phi(1,p)|}{p^{\sigma_1}} + \frac{1}{p^{3\sigma_1}}\right) \\
&\leq \left(1 + \frac{2|A_\phi(1,p)|}{p^{\sigma_2+\sigma_3+1}}\right)\left(1 + \frac{2}{p^{\sigma_1+\sigma_2+\sigma_3+1}}\right)\left(1 + \frac{6|A_\phi(1,p)|}{p^{\sigma_1+\sigma_2}}\right)\left(1 + \frac{3}{p^{3\sigma_1+\sigma_2}}\right) \\
&\leq \left(1 + \frac{|A_\phi(1,p)|}{p^{\sigma_2+\sigma_3+1}}\right)^2\left(1 + \frac{1}{p^{\sigma_1+\sigma_2+\sigma_3+1}}\right)^2\left(1 + \frac{|A_\phi(1,p)|}{p^{\sigma_1+\sigma_2}}\right)^6\left(1 + \frac{1}{p^{3\sigma_1+\sigma_2}}\right)^3 .
\end{aligned}
\tag{9.56}
$$

By Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|Z_{\mathrm{fin},2,p}| \leq &\left(1 + \frac{1}{p^{\sigma_2+\sigma_3+1}}\right)\left(1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_2+\sigma_3+1}}\right)\left(1 + \frac{1}{p^{\sigma_1+\sigma_2+\sigma_3+1}}\right)^2 \times \\
&\left(1 + \frac{1}{p^{\sigma_1+\sigma_2}}\right)^3\left(1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_1+\sigma_2}}\right)^3\left(1 + \frac{1}{p^{3\sigma_1+\sigma_2}}\right)^3 \\
&\leq \left(1 + \frac{1}{p^{\sigma_2+\sigma_3+1}}\right)^3\left(1 + \frac{1}{p^{\sigma_1+\sigma_2}}\right)^6\left(1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_2+\sigma_3+1}}\right)\left(1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_1+\sigma_2}}\right)^3 .
\end{aligned}
\tag{9.57}
$$

Let us focus on the terms involving $A_\phi(1,p)$ for a moment. We have

$$
\begin{aligned}
1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_2+\sigma_3+1}} &\leq \left(\sum_{k_1,k_2\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\sigma_2+\sigma_3+1)(2k_2+k_1)}}\right) \\
&= \left(1 - \frac{1}{p^{3(\sigma_2+\sigma_3+1)}}\right)\left(\sum_{k_1,k_2,k_3\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\sigma_2+\sigma_3+1)(3k_3+2k_2+k_1)}}\right) \\
&\leq \left(\sum_{k_1,k_2,k_3\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\sigma_2+\sigma_3+1)(3k_3+2k_2+k_1)}}\right).
\end{aligned}
\tag{9.58}
$$

Similarly,

$$
1 + \frac{|A_\phi(1,p)|^2}{p^{\sigma_1+\sigma_2}} \leq \sum_{k_1,k_2,k_3\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\sigma_1+\sigma_2)(3k_3+2k_2+k_1)}}. \tag{9.59}
$$

Therefore

$$\begin{aligned}|Z_{\text{fin},2,p}| \leq \left(1+\frac{1}{p^{\gamma_2+\gamma_3+1}}\right)^3 \left(1+\frac{1}{p^{\gamma_1+\gamma_2}}\right)^6 \times \\ \left(\sum_{k_1,k_2,k_3\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\gamma_2+\gamma_3+1)(3k_3+2k_2+k_1)}}\right)\left(\sum_{k_1,k_2,k_3\geq 0} \frac{|A_\phi(p^{k_2},p^{k_1})|^2}{p^{(\gamma_1+\gamma_2)(3k_3+2k_2+k_1)}}\right)^3.\end{aligned} \tag{9.60}$$

Since every factor on the right hand side exceeds 1, after applying the above inequality, we can extend from $\prod_{\substack{p \text{ prime} \\ b_p=0}}$ to $\prod_{p \text{ prime}}$ to get

$$\begin{aligned}\prod_{\substack{p \text{ prime} \\ b_p=0}} |Z_{\text{fin},2,p}| \leq \left(\frac{\zeta(\gamma_2+\gamma_3+1)}{\zeta(2(\gamma_2+\gamma_3+1))}\right)^3 \left(\frac{\zeta(\gamma_1+\gamma_2)}{\zeta(2(\gamma_1+\gamma_2))}\right)^6 L(\phi\times\overline{\phi},\gamma_2+\gamma_3+1)\times \\ (L(\phi\times\overline{\phi},\gamma_1+\gamma_2))^3 \\ \ll_{\gamma_1,\gamma_2,\gamma_3} 1,\end{aligned} \tag{9.61}$$

where the last two are Rankin-Selberg L-functions.

Combining the bounds from both of the above cases, we have

$$Z_{\text{fin},2} \ll_{\gamma,\epsilon} \delta^\epsilon, \tag{9.62}$$

where $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$. The following is a useful result which was not used above but is worth recording for the future.

Lemma 9.4.0.1. *For $\epsilon > 0$, we have*

$$\begin{aligned}\prod_{p \text{ prime}} \left(1+\frac{|\alpha_1(p)|+|\alpha_2(p)|+|\alpha_3(p)|}{p^s}\right) \ll_\epsilon 1 \text{ for } \Re(s) \geq 1+\epsilon \\ \prod_{\substack{p|N \\ p \text{ prime}}} \left(1+\frac{|\alpha_1(p)|+|\alpha_2(p)|+|\alpha_3(p)|}{p^s}\right) \ll_\epsilon N^\epsilon \text{ for } \Re(s) \geq \frac{1}{2}+\epsilon.\end{aligned} \tag{9.63}$$

Proof: Since $\alpha_1(p)\alpha_2(p)\alpha_3(p) = 1$, the next lemma implies that for $1 \le k \le 3$,

$$|\alpha_k(p)| \le |A_\phi(1,p)| + |A_\phi(p,1)| + 1 = 2|A_\phi(1,p)| + 1, \tag{9.64}$$

where the last equality is due to $A_\phi(1,p) = \overline{A_\phi(p,1)}$. Therefore

$$|\alpha_1(p)| + |\alpha_2(p)| + |\alpha_3(p)| \le 6|A_\phi(1,p)| + 3, \tag{9.65}$$

from which the lemma follows.

Lemma 9.4.0.2. *Suppose $z_1, \ldots, z_n \in \mathbb{C}$, and for $1 \le j \le n$, let $p_j(z_1, \ldots, z_n)$ be the j^{th} elementary symmetric polynomial in $z_1, \ldots, z_n$. Then, for any $1 \le k \le n$,*

$$|z_k| \le \max\left(1, \sum_{j=1}^{n} |p_j(z_1, \ldots, z_n)|\right). \tag{9.66}$$

Proof:

$$z_k^n = \sum_{j=1}^{n} (-1)^{j-1} p_j(z_1, \ldots, z_n) z_k^{n-j} \implies |z_k|^n \le \sum_{j=1}^{n} |p_j(z_1, \ldots, z_n)||z_k|^{n-j}. \tag{9.67}$$

If $|z_k| > 1$, then dividing by $|z_k|^{n-1}$ proves the claim.

9.5 Large sieve inequalities

The following is a hybrid large sieve inequality that combines theorems 2 and 3 of [61].

Lemma 9.5.0.1. *Let $q \in \mathbb{N}, x \in \mathbb{R}_{\ge 1}$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |c_n|$ is convergent. For $T \ge \theta > 0$, we have*

$$\sum_{\substack{D \le x \\ (q,D)=1}} \frac{qD}{\varphi(qD)} \sideset{}{^*}\sum_{\eta(D)} \sum_{\chi(q)} \int_{-T}^{T} \left| \sum_{n=1}^{\infty} c_n \chi(n) \eta(n) n^{it} \right|^2 dt \ll_\theta \sum_{n=1}^{\infty} (Tqx^2 + n)|c_n|^2. \tag{9.68}$$

For the next lemma, let $L(f, \cdot)$ be an L-function as in chapter 5 of [56]. Specifically, we have the following:

- degree $= d$,
- conductor $= q = q(f) = q(\overline{f})$,
- Dirichlet series $\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$ absolutely convergent for $\Re(s) > 1$,
- local parameters at infinity κ_j for $1 \leq j \leq d$,
- as mentioned in the proof of [62] lemma 3.4, the local parameters at infinity of $L(\overline{f}, \cdot)$ are $\overline{\kappa_j}$ for $1 \leq j \leq d$,
- the completed L-function is $\Lambda(f, \cdot)$,
- $$\mathfrak{q}_\infty(f, s) = \prod_{j=1}^{d}(|s + \kappa_j| + 3) \text{ and } \mathfrak{q}_\infty(\overline{f}, s) = \prod_{j=1}^{d}(|s + \overline{\kappa_j}| + 3), \tag{9.69}$$
- $$\mathfrak{q}(f, s) = q\mathfrak{q}_\infty(f, s) \text{ and } \mathfrak{q}(\overline{f}, s) = q\mathfrak{q}_\infty(\overline{f}, s). \tag{9.70}$$

The following is essentially [62] lemma 3.4.

Lemma 9.5.0.2. *For $L(f, \cdot)$, let $\Lambda(f, \cdot)$ be entire, and let*

$$0 < A \leq \frac{1}{2} + \Re(\kappa_j) \leq B \quad \text{for } 1 \leq j \leq d. \tag{9.71}$$

Let $\psi = \{\psi_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $|\lambda_f(n)| \leq \psi_n$ for $n \geq 1$ and such that $\sum_{n=1}^{\infty} \frac{\psi_n}{n^k}$ converges for $k > 1$.

For $t \in \mathbb{R}$ and $Q \geq \mathfrak{q}\left(f, \frac{1}{2} + it\right)$, we have

$$\left|L\left(f, \frac{1}{2} + it\right)\right|^2 \ll_{\epsilon,d} Q^{\epsilon} \int_{-\log Q}^{\log Q} \left|\sum_{n=1}^{N} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\epsilon+it+iv}}\right|^2 dv + O_{A,B,\psi}(Q^{-200}), \tag{9.72}$$

where $N = \lfloor Q^{\frac{1}{2}+\epsilon} \rfloor$, $\epsilon > 0$. The implied constants do not depend upon particular t, Q.

The following two lemmas are a consequence of lemmas 9.5.0.1, 9.5.0.2, and the Phragmen-Lindelöf principle for vertical strips as in [56] p.150.

Lemma 9.5.0.3. *Let* $q \in \mathbb{N}$, $x, U \geq 1$. *For* $\sigma \geq \frac{1}{2}$, $T \in \mathbb{R}$, *we have*

$$\begin{aligned} \sum_{\substack{D \leq x \\ (D,q)=1}} \frac{qD}{\varphi(qD)} \sum_{\substack{\rho(D) \\ \eta(q) \\ \rho\eta \neq \rho_0\eta_0}} |L(\sigma + iT, \rho\eta)|^2 &\ll_\epsilon x^{2+\epsilon} q^{1+\epsilon} (1 + |T|)^{\frac{1}{2}+\epsilon}, \\ \sum_{\substack{D \leq x \\ (D,q)=1}} \frac{qD}{\varphi(qD)} \sum_{\substack{\rho(D) \\ \eta(q) \\ \rho\eta \neq \rho_0\eta_0}} \int_{-U}^{U} |L(\sigma + it, \rho\eta)|^2 \, dt &\ll_\epsilon x^{2+\epsilon} (qU)^{1+\epsilon}, \end{aligned} \tag{9.73}$$

for all $\epsilon > 0$. *Here* ρ_0, η_0 *denote the principal Dirichlet characters modulo* D, q *respectively.*

Lemma 9.5.0.4. *Let* $q \in \mathbb{N}$, $x, U \geq 1$. *For* $\sigma \geq \frac{1}{2}$, $T \in \mathbb{R}$, *we have*

$$\begin{aligned} \sum_{\substack{D \leq x \\ (D,q)=1}} \frac{qD}{\varphi(qD)} \sum_{\substack{\rho(D) \\ \eta(q)}} |L(\phi \times \rho\eta, \sigma + iT)|^2 &\ll_{\phi,\epsilon} x^{2+\epsilon} (q(1 + |T|))^{\frac{3}{2}+\epsilon}, \\ \sum_{\substack{D \leq x \\ (D,q)=1}} \frac{qD}{\varphi(qD)} \sum_{\substack{\rho(D) \\ \eta(q)}} \int_{-U}^{U} |L(\phi \times \rho\eta, \sigma + it)|^2 \, dt &\ll_{\phi,\epsilon} x^{2+\epsilon} (qU)^{\frac{3}{2}+\epsilon}, \end{aligned} \tag{9.74}$$

for all $\epsilon > 0$.

9.6 Bounds for Z

Recall that

$$Z = \sum_{\substack{\delta \geq 1 \\ (\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{\delta_4 | \delta \\ D_1 \geq 1 \\ (\mu(D_1))^2=1 \\ (D_1,\delta q)=1}} \frac{1}{D_1^{s_2+1} \varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q)}} \overline{\psi}(D_1\delta_4) \tau(\overline{\lambda}) L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi}) Z_{\text{fin}}. \tag{9.75}$$

Let us write Z as a sum of two parts.

$$Z = Z_0 + Z_1, \tag{9.76}$$

where

$$\begin{aligned}Z_0 &= Z_0(s_1,s_2,s_3,s_4,s_5)\\ &= \sum_{\substack{\delta\geq 1\\(\delta,q)=1}}\frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}}\sum_{\substack{\delta_4\mid\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}}\frac{\overline{\psi_0}(D_1\delta_4)\tau(\overline{\lambda_0})}{D_1^{s_2+1}\varphi(D_1\delta_4 q)}L(s_1,\phi\times(\lambda_0\psi_0))L(s_2,\overline{\lambda_0\psi_0})Z_{\mathrm{fin}}(\lambda_0,\psi_0),\end{aligned} \tag{9.77}$$

and

$$\begin{aligned}&Z_1 = Z_1(s_1,s_2,s_3,s_4,s_5)\\ &= \sum_{\substack{\delta\geq 1\\(\delta,q)=1}}\frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}}\sum_{\substack{\delta_4\mid\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}}\frac{1}{D_1^{s_2+1}\varphi(D_1\delta_4 q)}\sum_{\substack{\lambda(D_1\delta_4)\\\psi(q)\\\lambda\psi\neq\lambda_0\psi_0}}\overline{\psi}(D_1\delta_4)\tau(\overline{\lambda})L(s_1,\phi\times(\lambda\psi))L(s_2,\overline{\lambda\psi})Z_{\mathrm{fin}},\end{aligned} \tag{9.78}$$

with λ_0,ψ_0 being the principal Dirichlet characters modulo $D_1\delta_4$, q respectively.

9.7 Bounding Z_0

We will work under the following assumption:

$$\sigma_1\geq\gamma_1>1,\ \sigma_2\geq\gamma_2>1,\ \sigma_3\geq\gamma_3>1,\ \sigma_4\geq\gamma_4>1,\ \sigma_5\geq\gamma_5>0. \tag{9.79}$$

Write

$$\begin{aligned}Z_0 = L(s_1,\phi)\zeta(s_2)\sum_{\substack{\delta\geq 1\\(\delta,q)=1}}\frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}}\sum_{\substack{\delta_4\mid\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}}\frac{\overline{\psi_0}(D_1\delta_4)\tau(\overline{\lambda_0})}{D_1^{s_2+1}\varphi(D_1\delta_4 q)}Z_{\mathrm{fin}}(\lambda_0,\psi_0)\times\\ \prod_{\substack{p\mid D_1\delta_4 q\\p\text{ prime}}}\left((1-p^{-s_2})\prod_{j=1}^{3}(1-\alpha_j(p)p^{-s_1})\right).\end{aligned} \tag{9.80}$$

Now, $L(s_1,\phi)\zeta(s_2) \ll_{\gamma_1,\gamma_2} 1$ by absolute convergence. We have, by proposition 9.0.0.1, that for every $\epsilon > 0$,

$$\begin{aligned} |Z_0| &\ll_{\epsilon,\gamma} q^{1+\epsilon} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{1}{\delta^{\frac{3}{2}-\epsilon}} \sum_{\substack{\delta_4|\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}} \frac{|\tau(\overline{\lambda_0})|}{D_1^2\varphi(D_1\delta_4 q)} \prod_{\substack{p|D_1\delta_4 q\\p \text{ prime}}} \left((1+p^{-1})\prod_{j=1}^{3}(1+|\alpha_j(p)|p^{-1})\right) \\ &\ll_{\epsilon,\gamma} q^{1+\epsilon} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{1}{\delta^{\frac{3}{2}-\epsilon}} \sum_{\substack{\delta_4|\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}} \frac{|\tau(\overline{\lambda_0})|(D_1\delta_4 q)^{\epsilon}}{D_1^2\varphi(D_1\delta_4 q)}, \end{aligned} \tag{9.81}$$

since $|\alpha_i(p)| < \sqrt{p}$ and where $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5)$. Using the following basic facts

$$\begin{aligned} &\frac{1}{\varphi(D_1\delta_4 q)} \ll_{\epsilon} \frac{1}{(D_1\delta_4 q)^{1-\epsilon}} \quad \forall \epsilon > 0, \\ &|\tau(\overline{\lambda})| \leq \sqrt{D_1\delta_4}, \end{aligned} \tag{9.82}$$

we get that for every $\epsilon > 0$,

$$|Z_0| \ll_{\epsilon,\gamma} q^{\epsilon} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{1}{\delta^{\frac{3}{2}-\epsilon}} \sum_{\substack{\delta_4|\delta\\D_1\geq 1\\(\mu(D_1))^2=1\\(D_1,\delta q)=1}} \frac{1}{D_1^{\frac{5}{2}-\epsilon}\delta_4^{\frac{1}{2}-\epsilon}} \ll_{\epsilon} q^{\epsilon}. \tag{9.83}$$

9.8 Bounding Z_1

The following is a well known application of Abel's partial summation to Dirichlet series; we will state it without proof.

Lemma 9.8.0.1. *Let $\sigma_0 \geq 0$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Let $f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$.*

For $x \geq 1$, let $A(x) := \sum_{n\leq x} a_n$. We assume the following: given $\epsilon > 0$ there exists $k_\epsilon > 0$ independent of x such that $A(x) \leq k_\epsilon x^{\sigma_0+\epsilon}$ $\forall x \geq 1$. Then

- *f converges and is an analytic function in the half-plane $\Re(s) > \sigma_0$ allowing term-by-term differentiation.*

- *f converges absolutely in the half-plane $\Re(s) > \sigma_0 + 1$.*

- $$f(s) = s\int_1^{\infty} \frac{A(x)}{x^{s+1}}\,dx \qquad \textit{for } \ \Re(s) > \sigma_0. \tag{9.84}$$

- $$|f(s)| \leq \frac{|s|k_\epsilon}{\Re(s) - (\sigma_0 + \epsilon)} \qquad \textit{for } \ \Re(s) > \sigma_0 \ \ \textit{and} \ \ 0 < \epsilon < \Re(s) - \sigma_0. \tag{9.85}$$

- *If there is a sequence of positive real numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0$ and $k_{\epsilon_n} \to k$, then*
$$|f(s)| \leq \frac{|s|k}{\Re(s) - \sigma_0}. \tag{9.86}$$

Note that the particular case where $A(x) \leq cx^{\sigma_0}$ $\forall x \geq 1$ with c independent of x is a special case of the above with $k_\epsilon = c$ $\forall \epsilon > 0$. □

We will work under the following assumption.

$$\sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2},\ \sigma_5 \geq \gamma_5 > 0. \tag{9.87}$$

We bound Z_{fin} by proposition 9.0.0.1 and we bound the Gauss sum by (9.82); for every $\epsilon > 0$, we get

$$\begin{aligned}
|Z_1| &\leq \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{1}{\delta^{\gamma_5+\frac{3}{2}}} \sum_{\substack{\delta_4|\delta\\ D_1\geq 1\\ (\mu(D_1))^2=1\\ (D_1,\delta q)=1}} \frac{1}{D_1^{\gamma_2+1}\varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} |\tau(\overline{\lambda})||L(s_1,\phi\times(\lambda\psi))L(s_2,\overline{\lambda\psi})||Z_{\text{fin}}| \\
&\ll_{\epsilon,\gamma} q^{\frac{3}{2}+\epsilon} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{1}{\delta^{\gamma_5+\frac{3}{2}-\epsilon}} \sum_{\substack{\delta_4|\delta\\ D_1\geq 1\\ (D_1,q)=1}} \frac{\delta_4^{\frac{1}{2}}}{D_1^{\gamma_2+\frac{1}{2}}\varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} |L(s_1,\phi\times(\lambda\psi))L(s_2,\overline{\lambda\psi})|,
\end{aligned} \tag{9.88}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, and by nonnegativity, we have dropped $(\mu(D_1))^2 = 1$ and $(D_1, \delta) = 1$. Interchanging sums over δ_4 and δ, we get

$$\begin{aligned}
Z_1 &\ll_{\epsilon,\gamma} q^{\frac{3}{2}+\epsilon} \sum_{\substack{\delta_4 \geq 1 \\ (\delta_4, q)=1}} \sum_{\substack{\delta_4 | \delta \\ (\delta, q)=1}} \frac{1}{\delta^{\gamma_5+\frac{3}{2}-\epsilon}} \sum_{\substack{D_1 \geq 1 \\ (D_1, q)=1}} \frac{\delta_4^{\frac{1}{2}}}{D_1^{\gamma_2+\frac{1}{2}} \varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})| \\
&= q^{\frac{3}{2}+\epsilon} \sum_{\substack{\delta_4 \geq 1 \\ (\delta_4, q)=1}} \sum_{\substack{\delta_5 \geq 1 \\ (\delta_5, q)=1}} \frac{1}{(\delta_4\delta_5)^{\gamma_5+\frac{3}{2}-\epsilon}} \sum_{\substack{D_1 \geq 1 \\ (D_1, q)=1}} \frac{\delta_4^{\frac{1}{2}}}{D_1^{\gamma_2+\frac{1}{2}} \varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})| \\
&\ll_{\epsilon,\gamma} q^{\frac{3}{2}+\epsilon} \sum_{\substack{\delta_4 \geq 1 \\ (\delta_4, q)=1}} \sum_{\substack{D_1 \geq 1 \\ (D_1, q)=1}} \frac{1}{D_1^{\gamma_2+\frac{1}{2}} \delta_4^{\gamma_5+1-\epsilon} \varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})| \\
&\leq q^{\frac{3}{2}+\epsilon} \sum_{\substack{\delta_4 \geq 1 \\ (\delta_4, q)=1}} \sum_{\substack{D_1 \geq 1 \\ (D_1, q)=1}} \frac{1}{(D_1\delta_4)^{1+\epsilon} \varphi(D_1\delta_4 q)} \sum_{\substack{\lambda(D_1\delta_4) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})|,
\end{aligned} \tag{9.89}$$

where $\delta_5 = \frac{\delta}{\delta_4}$ and $0 < \epsilon \leq \min\left(\gamma_2 - \frac{1}{2}, \frac{\gamma_5}{2}\right)$. Let $D = D_1\delta_4$ to get

$$\begin{aligned}
Z_1 &\ll_{\epsilon,\gamma} q^{\frac{3}{2}+\epsilon} \sum_{\substack{D \geq 1 \\ (D, q)=1}} \frac{d(D)}{D^{1+\epsilon} \varphi(Dq)} \sum_{\substack{\lambda(D) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})| \\
&\ll_{\epsilon} q^{\frac{3}{2}+\epsilon} \sum_{\substack{D \geq 1 \\ (D, q)=1}} \frac{1}{D^{1+\frac{\epsilon}{2}} \varphi(Dq)} \sum_{\substack{\lambda(D) \\ \psi(q) \\ \lambda\psi \neq \lambda_0\psi_0}} |L(s_1, \phi \times (\lambda\psi)) L(s_2, \overline{\lambda\psi})|,
\end{aligned} \tag{9.90}$$

where $d(\cdot)$ is the divisor function.

Now, let

$$Q := Z_1(\sigma_1 + it_1, \sigma_2 + it_2, \sigma_3 + it_3, \sigma_4 + it_4, \sigma_5 + it_5), \tag{9.91}$$

where $t_j \in \mathbb{R}$ and $|t_j| \leq U$ for $1 \leq j \leq 5$. Then

$$Q \ll_{\epsilon,\gamma} q^{\frac{1}{2}+\epsilon} \sum_{D=1}^{\infty} \frac{\Upsilon_D}{D^{2+\frac{\epsilon}{2}}}, \tag{9.92}$$

where

$$\Upsilon_D = \mathbf{1}_{(D,q)=1}\left(\frac{Dq}{\varphi(Dq)}\sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}}|L(\sigma_1+it_1,\phi\times(\lambda\psi))L(\sigma_2+it_2,\overline{\lambda\psi})|\right), \tag{9.93}$$

where $\mathbf{1}$ denotes indicator function. Now, we have

$$\sum_{D\leq x}\Upsilon_D = \sum_{\substack{D\leq x\\ (D,q)=1}}\frac{Dq}{\varphi(Dq)}\sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}}|L(\sigma_1+it_1,\phi\times(\lambda\psi))L(\sigma_2+it_2,\overline{\lambda\psi})|. \tag{9.94}$$

By Cauchy-Schwarz,

$$\begin{aligned}\sum_{D\leq x}\Upsilon_D \leq &\left(\sum_{\substack{D\leq x\\ (D,q)=1}}\frac{Dq}{\varphi(Dq)}\sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}}|L(\sigma_1+it_1,\phi\times(\lambda\psi))|^2\right)^{\frac{1}{2}}\times\\ &\left(\sum_{\substack{D\leq x\\ (D,q)=1}}\frac{Dq}{\varphi(Dq)}\sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}}|L(\sigma_2+it_2,\overline{\lambda\psi})|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{9.95}$$

By lemmas 9.5.0.4, 9.5.0.3 we get

$$\sum_{D\leq x}\Upsilon_D \ll_{\phi,\epsilon} x^{2+\epsilon}q^{\frac{5}{4}+\epsilon}U^{1+\epsilon}, \tag{9.96}$$

for all $\epsilon>0$. Lemma 9.8.0.1 implies

$$Q \ll_{\phi,\gamma,\epsilon} q^{\frac{1}{2}+\epsilon}\left(q^{\frac{5}{4}+\epsilon}U^{1+\epsilon}\right). \tag{9.97}$$

Next, let $f : \mathbb{R} \to \mathbb{C}$ be a function satisfying $f(t) \ll (1+|t|)^{-\frac{1}{2}}$. Let

$$I := \int_{-T_0}^{T_0} |f(t)||Z_1(\sigma_1+it_1+it, \sigma_2+it_2-it, \sigma_3+it_3+2it, \sigma_4+it_4+3it, \sigma_5+it_5-it)|\,dt, \tag{9.98}$$

where the t_j are as before. Then

$$I \ll_{\epsilon,\gamma} q^{\frac{1}{2}+\epsilon} \sum_{D=1}^{\infty} \frac{\beta_D}{D^{2+\frac{\epsilon}{2}}}, \tag{9.99}$$

where

$$\beta_D = \mathbf{1}_{(D,q)=1} \left(\frac{Dq}{\varphi(Dq)} \sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} \int_{-T_0}^{T_0} |f(t)||L(\sigma_1+it_1+it, \phi\times(\lambda\psi))L(\sigma_2+it_2-it, \overline{\lambda\psi})|\,dt \right). \tag{9.100}$$

Now, we have

$$\sum_{D\leq x} \beta_D = \sum_{\substack{D\leq x\\ (D,q)=1}} \frac{Dq}{\varphi(Dq)} \sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} \int_{-T_0}^{T_0} |f(t)||L(\sigma_1+it_1+it, \phi\times(\lambda\psi))L(\sigma_2+it_2-it, \overline{\lambda\psi})|\,dt. \tag{9.101}$$

By Cauchy-Schwarz,

$$\sum_{D\leq x} \beta_D \leq \left(\sum_{\substack{D\leq x\\ (D,q)=1}} \frac{Dq}{\varphi(Dq)} \sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} \int_{-T_0}^{T_0} |f(t)||L(\sigma_1+it_1+it, \phi\times(\lambda\psi))|^2\,dt \right)^{\frac{1}{2}} \times \left(\sum_{\substack{D\leq x\\ (D,q)=1}} \frac{Dq}{\varphi(Dq)} \sum_{\substack{\lambda(D)\\ \psi(q)\\ \lambda\psi\neq\lambda_0\psi_0}} \int_{-T_0}^{T_0} |f(t)||L(\sigma_2+it_2-it, \overline{\lambda\psi})|^2\,dt \right)^{\frac{1}{2}}. \tag{9.102}$$

In order to handle the $|f(t)|$ in the integral associated with the twisted $GL(3)$ L-function above, we apply the variable substitution $t \mapsto t - t_1$ followed by a dyadic partition of unity to the variable t. To each piece of the partitioned integral, we apply lemma 9.5.0.4 after dropping the condition $\lambda\psi \neq \lambda_0\psi_0$ by positivity.

Similarly, to handle the $|f(t)|$ in the integral associated with the Dirichlet L-function above, we apply the variable substitution $t \mapsto t + t_2$ followed by a dyadic partition of unity to the variable t. To each piece of the partitioned integral, we apply lemma 9.5.0.3. This gives us the following:

$$\sum_{D \leq x} \beta_D \ll_{\phi,\epsilon} x^{2+\epsilon} (qU)^{\frac{5}{4}+\epsilon} T_0^{\frac{3}{4}+\epsilon}, \tag{9.103}$$

for all $\epsilon > 0$. Lemma 9.8.0.1 implies

$$I \ll_{\phi,\gamma,\epsilon} q^{\frac{1}{2}+\epsilon} \left((qU)^{\frac{5}{4}+\epsilon} T_0^{\frac{3}{4}+\epsilon} \right). \tag{9.104}$$

We summarize our results from this chapter below.

Proposition 9.8.0.1. *Let q be cube-free. Consider the regions*

- $R_1 : \sigma_1 \geq \gamma_1 > 1,\ \sigma_2 \geq \gamma_2 > 1,\ \sigma_3 \geq \gamma_3 > 1,\ \sigma_4 \geq \gamma_4 > 1,\ \sigma_5 \geq \gamma_5 > 0,$
- $R_2 : \sigma_1 \geq \gamma_1 > \frac{1}{2},\ \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2},\ \sigma_5 \geq \gamma_5 > 0.$

Then we can write $Z = Z_0 + Z_1$ where

- *Z_0 is analytic in R_1. It is meromorphic in R_2 with a simple pole at $s_2 = 1$ and no other poles.*
- *Z_1 is analytic in R_2.*

In R_1, we have

$$Z_0(s_1, s_2, s_3, s_4, s_5) \ll_{\gamma,\epsilon} q^{\epsilon}. \tag{9.105}$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a function satisfying $f(t) \ll (1 + |t|)^{-\frac{1}{2}}$. Let $t_j \in \mathbb{R}$ and $|t_j| \leq U$ for

$1 \leq j \leq 5$. *In* R_2, *we have*

$$Z_1(\sigma_1+it_1,\sigma_2+it_2,\sigma_3+it_3,\sigma_4+it_4,\sigma_5+it_5) \ll_{\phi,\gamma,\epsilon} q^{\frac{7}{4}+\epsilon}U^{1+\epsilon},$$
$$\int_{-T_0}^{T_0} |f(t)||Z_1(\sigma_1+it_1+it,\sigma_2+it_2-it,\sigma_3+it_3+2it,\sigma_4+it_4+3it,\sigma_5+it_5-it)|\,dt$$
$$\ll_{\phi,\gamma,\epsilon} q^{\frac{7}{4}+\epsilon}U^{\frac{5}{4}+\epsilon}T_0^{\frac{3}{4}+\epsilon}, \tag{9.106}$$

for all $\epsilon > 0$. *The exact statement of* (9.106) *holds with* Z_1 *replaced with* Z_0 *in the region*

$$\sigma_1 \geq \gamma_1 > \frac{1}{2},\ 0.99 \geq \sigma_2 \geq \gamma_2 > \frac{1}{2},\ \sigma_3 \geq \gamma_3 > 0,\ \sigma_4 \geq \gamma_4 > -\frac{1}{2},\ \sigma_5 \geq \gamma_5 > 0. \tag{9.107}$$

□

10. COMPLETING THE PROOF

We will complete the proof of proposition 6.1.0.3 in this section. Recall that

$$\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta} = \frac{N_1 C_2^2}{C^2\sqrt{N}} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{\frac{3}{2}}} \sum_{q|c} \sum_{c_1c_2=c} \sum_{n_1n_3=c_1} \sum_{n_2=1}^{\infty} \frac{1}{c^2} A_\phi(n_2,n_1)\mathcal{T}\times \\ \mathcal{K}_{\beta,\sigma,\mathcal{I}}\left(\frac{n_2n_1^2}{c_1^3}, \delta, c, c_2, n_1, n_2\right), \tag{10.1}$$

and

$$Z = \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{s_5+\frac{3}{2}}} \sum_{\substack{r\geq 1\\n_2\geq 1\\c=qr}} \sum_{c_1c_2=c} \sum_{n_1n_3=c_1} \frac{A_\phi(n_2,n_1)}{n_2^{s_1} r^{s_2} n_1^{s_3} c_2^{s_4}} \frac{1}{qr^2} \mathcal{T}\mathcal{P}^{-1}. \tag{10.2}$$

10.1 Oscillatory case

Fix $\vartheta > 0$. Let $P \geq T^3 q^\vartheta$. By lemma 7.1.0.6, we may assume that $\sigma = +1$ and

$$N_2 N_1^2 C_2^3 \asymp CP\Delta, \tag{10.3}$$

in which case, up to an $O_{\vartheta,\sigma_0,t,\epsilon}((qT)^{-100})$ error, $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ is

$$\begin{aligned} &= \frac{N_1 C_2^2}{C^2\sqrt{N}} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{\frac{3}{2}}} \sum_{q|c} \sum_{c_1c_2=c} \sum_{n_1n_3=c_1} \sum_{n_2=1}^{\infty} \frac{1}{c^2} A_\phi(n_2,n_1)\mathcal{T}\mathcal{P}^{-1}T^2P^{-2}N\times \\ &\int_{|\mathbf{u}|\ll(qT)^\epsilon} F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\mathbf{u}) \left(\frac{N_2}{n_2}\right)^{u_1} \left(\frac{C}{c}\right)^{u_2} \left(\frac{N_1}{n_1}\right)^{u_3} \left(\frac{C_2}{c_2}\right)^{u_4} \left(\frac{\Delta}{\delta}\right)^{u_5} d\mathbf{u} \\ &= N_2^{u_1} C^{u_2-2} N_1^{u_3+1} C_2^{u_4+2} \Delta^{u_5} T^2 P^{-2} N^{\frac{1}{2}} \int_{|\mathbf{u}|\ll(qT)^\epsilon} F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}(\mathbf{u}) \frac{Z(u_1,u_2,u_3,u_4,u_5)}{q^{u_2+1}}\, d\mathbf{u}, \end{aligned} \tag{10.4}$$

for $\epsilon > 0$. We will use proposition 9.8.0.1. In the last integral, we first take the lines of integration with

$$L_1 : \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1+\epsilon,\ \sigma_5 = \epsilon, \tag{10.5}$$

and we write $Z = Z_0 + Z_1$ following proposition 9.8.0.1. For Z_0, we maintain the lines at L_1, while for Z_1, we move to

$$L_2 : \sigma_1 = \sigma_2 = \frac{1}{2} + \epsilon,\ \sigma_3 = \epsilon,\ \sigma_4 = -\frac{1}{2} + \epsilon,\ \sigma_5 = \epsilon. \tag{10.6}$$

By the decay properties of $F_{\vartheta,\sigma_0,t,\beta,\mathcal{I}}$, the integrals along horizontal segments arising from these contour shifts are absorbed into the error term $O_{\vartheta,\sigma_0,t,\epsilon}((qT)^{-100})$. Therefore, the contribution to $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ from Z_0 is

$$\begin{aligned}
&\ll q^{\epsilon-2}T^{2+\epsilon}N_2^{1+\epsilon}C^{-1+\epsilon}N_1^{2+\epsilon}C_2^{3+\epsilon}\Delta^{\epsilon}P^{-2}N^{\frac{1}{2}} \\
&\ll q^{\epsilon-2}T^{2+\epsilon}P^{-1}\Delta N^{\frac{1}{2}} \quad \text{(by (6.52) and (10.3))} \\
&\ll q^{\epsilon-2}T^{-1+\epsilon}\Delta N^{\frac{1}{2}} \quad (\text{since } P \geq T^3 q^{\vartheta} \geq T^3) \\
&\ll q^{\epsilon-2}T^{-1+\epsilon}(qT)^{\frac{3}{2}} \quad \text{(by (6.52))} \\
&\ll q^{\epsilon-\frac{1}{2}}T^{\frac{1}{2}+\epsilon} \quad \text{(by (6.52))},
\end{aligned} \tag{10.7}$$

and the contribution from Z_1 is

$$\begin{aligned}
&\ll q^{\frac{1}{4}+\epsilon}T^{2+\epsilon}N_2^{\frac{1}{2}+\epsilon}C^{-\frac{3}{2}+\epsilon}N_1^{\epsilon+1}C_2^{\frac{3}{2}+\epsilon}\Delta^{\epsilon}P^{-2}N^{\frac{1}{2}} \\
&\ll q^{\frac{1}{4}+\epsilon}T^{2+\epsilon}(CP\Delta)^{\frac{1}{2}}C^{-\frac{3}{2}}P^{-2}N^{\frac{1}{2}} \quad \text{(by (6.52) and (10.3))} \\
&= q^{\frac{1}{4}+\epsilon}T^{2+\epsilon}P^{-\frac{3}{2}}C^{-1}(\Delta N)^{\frac{1}{2}} \\
&\ll q^{\frac{1}{4}+\epsilon}T^{2+\epsilon}P^{-\frac{1}{2}} \quad \left(\text{since } P = \frac{4\pi\sqrt{\Delta N}}{C}\right) \\
&\ll q^{\frac{1}{4}+\epsilon}T^{\frac{1}{2}+\epsilon} \quad (\text{since } P \geq T^3),
\end{aligned} \tag{10.8}$$

where the implied constants depend upon $\vartheta, \sigma_0, t, \epsilon$.

Combining the contributions from Z_0 and Z_1 we get

$$\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta} \ll_{\vartheta,\sigma_0,t,\epsilon} q^{\frac{1}{4}+\epsilon}T^{\frac{1}{2}+\epsilon}. \tag{10.9}$$

10.2 Non-oscillatory case

Fix $\vartheta > 0$. Let $P \leq T^3 q^\vartheta$. By lemma 7.2.0.3, we may assume

$$\frac{P^2 P'}{X_{\mathcal{I}}^3} \ll_\epsilon (qT)^\epsilon, \tag{10.10}$$

in which case, up to an $O_{\vartheta,\sigma_0,\epsilon}((qT)^{-100})$ error, $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ is

$$\begin{aligned}
&= \frac{N_1 C_2^2}{C^2\sqrt{N}} \sum_{\substack{\delta\geq 1\\(\delta,q)=1}} \frac{\mu(\delta)\chi(\delta)}{\delta^{\frac{3}{2}}} \sum_{q|c} \sum_{c_1c_2=c} \sum_{n_1n_3=c_1} \sum_{n_2=1}^{\infty} \frac{1}{c^2} A_\phi(n_2,n_1)\mathcal{T}\mathcal{P}^{-1}\left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0} T^\epsilon P^{2-\epsilon} N X_{\mathcal{I}}^{\frac{1}{2}} \times \\
&\int_{|\mathbf{u}|\ll X_{\mathcal{I}}(qT)^\epsilon} F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\mathbf{u}) \int_{|t|\ll (qT)^\epsilon + P'} \left(\frac{n_2 n_1^2 c_2^3}{c\delta}\right)^{-it} f_{\beta,\sigma,\mathcal{I}}(t) \times \\
&\qquad \left(\frac{N_2}{n_2}\right)^{u_1} \left(\frac{C}{c}\right)^{u_2} \left(\frac{N_1}{n_1}\right)^{u_3} \left(\frac{C_2}{c_2}\right)^{u_4} \left(\frac{\Delta}{\delta}\right)^{u_5} dt d\mathbf{u} \\
&= N_2^{u_1} C^{u_2-2} N_1^{u_3+1} C_2^{u_4+2} \Delta^{u_5} \left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0} T^\epsilon P^{2-\epsilon} X_{\mathcal{I}}^{\frac{1}{2}} N^{\frac{1}{2}} \times \\
&\int_{|\mathbf{u}|\ll X_{\mathcal{I}}(qT)^\epsilon} F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}(\mathbf{u}) \int_{|t|\ll (qT)^\epsilon + P'} f_{\beta,\sigma,\mathcal{I}}(t) \frac{Z(u_1+it, u_2-it, u_3+2it, u_4+3it, u_5-it)}{q^{1+u_2-it}} dt d\mathbf{u},
\end{aligned} \tag{10.11}$$

for $0 < \epsilon < 1$.

We will use proposition 9.8.0.1 again. With the lines at L_2, the contribution from Z_1 to $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ is

$$\begin{aligned}
&\ll (qT)^\epsilon N_2^{\frac{1}{2}+\epsilon} C^{-\frac{3}{2}+\epsilon} N_1^{\epsilon+1} C_2^{\frac{3}{2}+\epsilon} \Delta^\epsilon \left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0} P^{2-\epsilon} X_{\mathcal{I}}^{\frac{1}{2}} N^{\frac{1}{2}} \cdot X_{\mathcal{I}}^5 \cdot \frac{q^{\frac{7}{4}} X_{\mathcal{I}}^{\frac{5}{4}+\epsilon}((qT)^\epsilon + P'^{\frac{3}{4}+\epsilon})}{q^{\frac{3}{2}}} \\
&\ll q^{\frac{1}{4}+\epsilon} T^\epsilon N_2^{\frac{1}{2}} C^{-\frac{3}{2}} N_1 C_2^{\frac{3}{2}} \left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0} P^2 N^{\frac{1}{2}} X_{\mathcal{I}}^{\frac{27}{4}+\epsilon}((qT)^\epsilon + P'^{\frac{3}{4}+\epsilon}) \\
&\ll q^{\frac{1}{4}+\epsilon} T^\epsilon \left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{\frac{1}{2}-\sigma_0} P^2 X_{\mathcal{I}}^{\frac{33}{4}+\epsilon}((qT)^\epsilon + P'^{\frac{3}{4}+\epsilon}) \quad \left(\text{since } P = \frac{4\pi\sqrt{\Delta N}}{C} \text{ and } P' = \frac{N_2 N_1^2 C_2^3}{C\Delta}\right).
\end{aligned} \tag{10.12}$$

For $0 < \sigma_0 < \frac{1}{2}$, this is

$$\begin{aligned}
&\ll q^{\frac{1}{4}+\epsilon} T^{\epsilon} P^2 X_{\mathcal{I}}^{\frac{33}{4}+\epsilon}((qT)^{\epsilon} + P'^{\frac{3}{4}+\epsilon}) \\
&\ll q^{\frac{1}{4}+\epsilon} T^{\epsilon} P^2 X_{\mathcal{I}}^{\frac{21}{2}+\epsilon}(1 + P^{-\frac{3}{2}}) \quad (\text{since } X_{\mathcal{I}} \geq 1) \\
&= q^{\frac{1}{4}+\epsilon} T^{\epsilon} X_{\mathcal{I}}^{\frac{21}{2}+\epsilon} P^{\frac{1}{2}}(P^{\frac{3}{2}} + 1) \\
&\ll q^{\frac{1}{4}+\epsilon} T^{\epsilon} (T(T^3 q^{\vartheta})^2)^{\frac{21}{2}+\epsilon} (T^3 q^{\vartheta})^2 \quad (\text{since } X_{\mathcal{I}} = T \max(1, P)^2) \\
&\ll q^{\frac{1}{4}+100\vartheta+\epsilon} T^{100}.
\end{aligned} \tag{10.13}$$

Here the implied constants depend upon $\vartheta, \sigma_0, \epsilon$. We used (6.52) and (10.10) repeatedly above and readjusted the ϵ a number of times.

Next, we focus on the contribution from Z_0. We handle this in the 2 following cases.

- If $P' \gg T^{7+\epsilon} q^{2\vartheta+\epsilon} \geq (qT)^{\epsilon}$, for some $\epsilon > 0$, then we may assume that $f_{\beta,\sigma,\mathcal{I}}$ is supported on $|t| \asymp P'$. We move contours to

$$L_3 : \sigma_1 = \frac{1}{2} + \epsilon,\ 0.99 \geq \sigma_2 = \frac{1}{2} + \epsilon,\ \sigma_3 = \epsilon,\ \sigma_4 = -\frac{1}{2} + \epsilon,\ \sigma_5 = \epsilon. \tag{10.14}$$

  The poles of Z_0 occur at $u_2 - it = 1$, which requires $t_2 = t$. However, $t \gg T^{7+\epsilon} q^{2\vartheta+\epsilon}$ whereas t_2 lies in the complementary range since $t_2 \ll X_{\mathcal{I}}(qT)^{\epsilon} = (qT)^{\epsilon} T \max(1, P)^2 \ll (qT)^{\epsilon} T (T^3 q^{\vartheta})^2 = T^{7+\epsilon} q^{2\vartheta+\epsilon}$. Therefore, no poles are encountered, and the horizontal integrals arising from this contour shift are negligible since $F_{\vartheta,\sigma,\sigma_0,\beta,\epsilon,\mathcal{I}}$ is small at this height. By the final sentence of proposition 9.8.0.1, the contribution from Z_0 in this case is no worse than the one in (10.13).

- Finally, consider the case when $P' \ll T^{7+\epsilon} q^{2\vartheta+\epsilon}$. In this case, we maintain the lines at

L_1. The contribution to $\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}$ from Z_0 in this case is

$$\begin{aligned}
&\ll q^{\epsilon-2}T^{\epsilon}N_2^{1+\epsilon}C^{\epsilon-1}N_1^{\epsilon+2}C_2^{\epsilon+3}\Delta^{\epsilon}\left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0}P^{2-\epsilon}X_{\mathcal{I}}^{\frac{1}{2}}N^{\frac{1}{2}}\cdot X_{\mathcal{I}}^5\int_{|t|\ll(qT)^{\epsilon}+P'}|f_{\beta,\sigma,\mathcal{I}}(t)|\,dt\\
&\ll q^{\epsilon-2}T^{\epsilon}N_2C^{-1}N_1^2C_2^3\left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{-\sigma_0}P^2X_{\mathcal{I}}^{\frac{11}{2}}N^{\frac{1}{2}}((qT)^{\epsilon}+P'^{\frac{1}{2}})\\
&\ll q^{\epsilon-2}T^{\epsilon}\left(\frac{P^2P'}{X_{\mathcal{I}}^3}\right)^{1-\sigma_0}X_{\mathcal{I}}^{\frac{17}{2}}\Delta N^{\frac{1}{2}}((qT)^{\epsilon}+P'^{\frac{1}{2}})\quad\left(\text{since } P'=\frac{N_2N_1^2C_2^3}{C\Delta}\right)\\
&\ll q^{\epsilon-2}T^{\epsilon}X_{\mathcal{I}}^{\frac{17}{2}}\Delta N^{\frac{1}{2}}((qT)^{\epsilon}+P'^{\frac{1}{2}})\quad(\text{for } 0<\sigma_0<1)\\
&\ll q^{\epsilon+\vartheta-2}T^{\frac{7}{2}+\epsilon}X_{\mathcal{I}}^{\frac{17}{2}}\Delta N^{\frac{1}{2}}\\
&\ll q^{\epsilon+\vartheta-2}T^{\frac{7}{2}+\epsilon}X_{\mathcal{I}}^{\frac{17}{2}}(qT)^{\frac{3}{2}}\\
&\ll q^{\epsilon+\vartheta-2}T^{\frac{7}{2}+\epsilon}(T(T^3q^{\vartheta})^2)^{\frac{17}{2}}(qT)^{\frac{3}{2}}\quad(\text{since } X_{\mathcal{I}}=T\max(1,P)^2)\\
&\ll q^{\epsilon+100\vartheta-\frac{1}{2}}T^{100},
\end{aligned}\tag{10.15}$$

where we used the fact that $f_{\beta,\sigma,\mathcal{I}}(t)\ll(1+|t|)^{-\frac{1}{2}}$, we used (6.52) and (10.10), and we adjusted the ϵ a few times. The implied constants depend upon $\vartheta,\sigma_0,\epsilon$.

Combining the contributions from Z_0 and Z_1 we get

$$\mathcal{S}_{N,\Delta,C,C_2,N_1,N_2,\sigma,\beta}\ll_{\vartheta,\sigma_0,\epsilon} q^{\frac{1}{4}+100\vartheta+\epsilon}T^{100}. \tag{10.16}$$

We take ϑ to be arbitrarily small to complete the proof.

REFERENCES

[1] W. Duke and R. Schulze-Pillot, "Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids," *Inventiones mathematicae*, vol. 99, no. 1, pp. 49–57, 1990.

[2] J. W. Cogdell, I. I. Piatetski-Shapiro, and P. Sarnak, "Estimates on the critical line for Hilbert modular L-functions and applications," *preprint*, vol. 2, no. 3, p. 35, 2001.

[3] J. W. Cogdell, "On sums of three squares," *Journal de théorie des nombres de Bordeaux*, vol. 15, no. 1, pp. 33–44, 2003.

[4] A. Shnirelman, "Uspenski math," *Nauk*, vol. 29, no. 6, pp. 79–88, 1974.

[5] S. Zelditch, "Uniform distribution of eigenfunctions on compact hyperbolic surfaces," *Duke mathematical journal*, vol. 55, no. 4, pp. 919–941, 1987.

[6] Y. Colin de Verdière, "Ergodicité et fonctions propres du laplacien," *Communications in Mathematical Physics*, vol. 102, no. 3, pp. 497–502, 1985.

[7] Z. Rudnick and P. Sarnak, "The behaviour of eigenstates of arithmetic hyperbolic manifolds," *Communications in Mathematical Physics*, vol. 161, no. 1, pp. 195–213, 1994.

[8] Z. Rudnick, "Quantum chaos?," *Notices of the AMS*, vol. 55, no. 1, pp. 32–34, 2008.

[9] S. Zelditch, "Mathematics of quantum chaos in 2019," *Notices of the American Mathematical Society*, vol. 66, no. 9, pp. 1412–1421, 2019.

[10] P. Sarnak, "Arithmetic quantum chaos.," *Blyth Lectures. Toronto*, 1993.

[11] P. Sarnak, "Spectra of hyperbolic surfaces," *Bulletin of the American Mathematical Society*, vol. 40, no. 4, pp. 441–478, 2003.

[12] P. Michel, "Analytic number theory and families of automorphic L-functions," tech. rep., American Mathematical Society, 2007.

[13] H. Iwaniec and P. Sarnak, "Perspectives on the analytic theory of L-functions," in *Visions in Mathematics*, pp. 705–741, Springer, 2000.

[14] J. Bourgain, "Decoupling, exponential sums and the Riemann zeta function," *Journal of the American Mathematical Society*, vol. 30, no. 1, pp. 205–224, 2017.

[15] D. A. Burgess, "On character sums and L-series. II," *Proceedings of the London Mathematical Society*, vol. 3, no. 1, pp. 524–536, 1963.

[16] D. Heath-Brown, "Hybrid bounds for Dirichlet L-functions," *Inventiones mathematicae*, vol. 47, no. 2, pp. 149–170, 1978.

[17] J. B. Conrey and H. Iwaniec, "The cubic moment of central values of automorphic L-functions," *Annals of mathematics*, vol. 151, no. 3, pp. 1175–1216, 2000.

[18] M. P. Young, "Weyl-type hybrid subconvexity bounds for twisted L-functions and Heegner points on shrinking sets," *Journal of the European Mathematical Society*, vol. 19, no. 5, pp. 1545–1576, 2017.

[19] M. P. Young, "Explicit calculations with Eisenstein series," *Journal of Number Theory*, vol. 199, pp. 1–48, 2019.

[20] I. N. Petrow, "A twisted Motohashi formula and Weyl-subconvexity for L-functions of weight two cusp forms," *Mathematische Annalen*, vol. 363, no. 1, pp. 175–216, 2015.

[21] I. Petrow, "Bounds for traces of Hecke operators and applications to modular and elliptic curves over a finite field," *Algebra & Number Theory*, vol. 12, no. 10, pp. 2471–2498, 2019.

[22] I. Petrow and M. P. Young, "A generalized cubic moment and the Petersson formula for newforms," *Mathematische Annalen*, vol. 373, no. 1, pp. 287–353, 2019.

[23] I. Petrow and M. P. Young, "The Weyl bound for Dirichlet L-functions of cube-free conductor," *Annals of Mathematics*, vol. 192, no. 2, pp. 437–486, 2020.

[24] I. Petrow and M. P. Young, "The fourth moment of Dirichlet L-functions along a coset and the Weyl bound," *arXiv preprint arXiv:1908.10346*, 2019.

[25] D. Milićević, "Sub-Weyl subconvexity for Dirichlet L-functions to prime power moduli," *Compositio Mathematica*, vol. 152, no. 4, pp. 825–875, 2016.

[26] A. Good, "The square mean of Dirichlet series associated with cusp forms," *Mathematika*, vol. 29, no. 2, pp. 278–295, 1982.

[27] H. Iwaniec, "The spectral growth of automorphic L-functions.," 1992.

[28] W. Duke, J. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions," *Inventiones mathematicae*, vol. 112, no. 1, pp. 1–8, 1993.

[29] W. Duke, J. Friedlander, and H. Iwaniec, "A quadratic divisor problem," *Inventiones mathematicae*, vol. 115, no. 1, pp. 209–217, 1994.

[30] W. Duke, J. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions. II," *Inventiones mathematicae*, vol. 115, no. 1, pp. 219–239, 1994.

[31] W. Duke, J. Friedlander, and H. Iwaniec, "Class group L-functions," *Duke Mathematical Journal*, vol. 79, no. 1, pp. 1–56, 1995.

[32] W. Duke, J. Friedlander, and H. Iwaniec, "Bilinear forms with Kloosterman fractions," *Inventiones mathematicae*, vol. 128, no. 1, pp. 23–43, 1997.

[33] W. Duke, J. Friedlander, and H. Iwaniec, "Representations by the determinant and mean values of L-functions," *London Mathematical Society Lecture Note Series*, vol. 1, no. 237, pp. 109–116, 1996.

[34] W. Duke, J. B. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions. III," *Inventiones mathematicae*, vol. 143, no. 2, pp. 221–248, 2001.

[35] W. Duke, J. B. Friedlander, and H. Iwaniec, "The subconvexity problem for Artin L-functions," *Inventiones mathematicae*, vol. 149, no. 3, pp. 489–577, 2002.

[36] P. Michel and A. Venkatesh, "The subconvexity problem for GL_2," *Publications Mathématiques de l'IHÉS*, vol. 111, pp. 171–271, 2010.

[37] X. Li, "Bounds for $GL(3) \times GL(2)$ L-functions and $GL(3)$ L-functions," *Annals of mathematics*, pp. 301–336, 2011.

[38] E. M. Lapid, "On the nonnegativity of Rankin-Selberg L-functions at the center of symmetry," *International Mathematics Research Notices*, vol. 2003, no. 2, pp. 65–75, 2003.

[39] M. McKee, H. Sun, and Y. Ye, "Improved subconvexity bounds for $GL(2) \times GL(3)$ and $GL(3)$ L-functions by weighted stationary phase," *Transactions of the American Mathematical Society*, vol. 370, no. 5, pp. 3745–3769, 2018.

[40] R. M. Nunes, "Subconvexity for $GL(3)$ L-functions," *arXiv preprint arXiv:1703.04424*, 2017.

[41] V. Blomer, "Subconvexity for twisted L-functions on $GL(3)$," *American Journal of Mathematics*, vol. 134, no. 5, pp. 1385–1421, 2012.

[42] B. Huang, "Hybrid subconvexity bounds for twisted L-functions on $GL(3)$," *Science China Mathematics*, vol. 64, no. 3, pp. 443–478, 2021.

[43] Z. Qi, "Subconvexity for twisted L-functions on GL_3 over the Gaussian number field," *Transactions of the American Mathematical Society*, vol. 372, no. 12, pp. 8897–8932, 2019.

[44] R. Munshi, "Bounds for twisted symmetric square L-functions," *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 2013, no. 682, pp. 65–88, 2013.

[45] R. Munshi, "Bounds for twisted symmetric square L-functions - II,"

[46] R. Munshi, "Bounds for twisted symmetric square L-functions - III," *Advances in Mathematics*, vol. 235, pp. 74–91, 2013.

[47] R. Munshi, "The circle method and bounds for L-functions - I," *Mathematatische Annalen*, vol. 358, no. 1, pp. 389–401, 2014.

[48] R. Munshi, "The circle method and bounds for L-functions - II: Subconvexity for twists of $GL(3)$ L-functions," *American Journal of Mathematics*, vol. 137, no. 3, pp. 791–812, 2015.

[49] R. Munshi, "The circle method and bounds for L-functions - III: t-aspect subconvexity for $GL(3)$ L-functions," *Journal of the American Mathematical Society*, vol. 28, no. 4, pp. 913–938, 2015.

[50] R. Munshi, "The circle method and bounds for L-functions - IV: Subconvexity for twists of $GL(3)$ L-functions," *Annals of Mathematics*, pp. 617–672, 2015.

[51] R. Munshi, "Twists of $GL(3)$ L-functions," in *Relative Trace Formulas*, pp. 351–378, Springer, 2021.

[52] R. Holowinsky and P. D. Nelson, "Subconvex bounds on $GL(3)$ via degeneration to frequency zero," *Mathematische Annalen*, vol. 372, no. 1, pp. 299–319, 2018.

[53] Y. Lin, "Bounds for twists of $GL(3)$ L-functions," *Journal of the European Mathematical Society*, vol. 23, no. 6, pp. 1899–1924, 2021.

[54] P. Sharma and W. Sawin, "Subconvexity for $GL(3) \times GL(2)$ twists," *Advances in Mathematics*, vol. 404, p. 108420, 2022.

[55] P. D. Nelson, "Bounds for standard L-functions," *arXiv preprint arXiv:2109.15230*, 2021.

[56] H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53. American Mathematical Soc., 2021.

[57] E. M. Kıral, I. Petrow, and M. P. Young, "Oscillatory integrals with uniformity in parameters," *Journal de Théorie des Nombres de Bordeaux*, vol. 31, no. 1, pp. 145–159, 2019.

[58] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$*, vol. 99. Cambridge University Press, 2006.

[59] R. Khan and M. P. Young, "Moments and hybrid subconvexity for symmetric-square L-functions," *Journal of the Institute of Mathematics of Jussieu*, pp. 1–45, 2021.

[60] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products.* Academic press, 2014.

[61] P. X. Gallagher, "A large sieve density estimate near $\sigma = 1$," *Inventiones mathematicae*, vol. 11, no. 4, pp. 329–339, 1970.

[62] X. Li and M. P. Young, "The L^2 restriction norm of a GL_3 maass form," *Compositio Mathematica*, vol. 148, no. 3, pp. 675–717, 2012.