INFORMATION DESIGN IN LINEAR-QUADRATIC-GAUSSIAN GAMES

A Dissertation

by

FURKAN SEZER

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Chair of Committee,	Ceyhun Eksin
Committee Members,	Alfredo Garcia
	Joseph Geunes
	Rodrigo Velez
Head of Department,	Lewis Ntaimo

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ABSTRACT

Information design in an incomplete information game involves a designer that influences players' actions through signals generated from a designed probability distribution to optimize its objective function. For quadratic objective functions, if players have quadratic payoffs that depend on players' actions and an unknown payoff-relevant state, and signals on the state that follow a Gaussian distribution conditional on the state realization (LQG game), the information design problem is a semi-definite program (SDP). The doctoral research is pursued in three thrusts: analytical and numerical characterization of optimal information design to maximize social welfare and the agreement among players' action in LQG games, analysis of individual information preferences of agents in LQG network games, and robust optimal information design in LQG games under perturbed utilities.

Firstly, it is shown that full information disclosure maximizes social welfare under common payoff state, under homogeneous payoff dependencies, or under public signals. When the objective is to maximize agreement among players' actions, no information disclosure is optimal. Under joint objective, full information optimality condition on weight of agreement is determined for public information structures and common payoffs. In the second thrust, conditions for which rational agents are expected to benefit from full information are characterized in general network games. In star networks, the central agent benefits more than a peripheral agent from full information unless the competition is strong and the number of peripheral agents is small enough. Numerical simulations of *ex-post* preferences showed that a risk-averse central agent may prefer no information under strong competition. In the third thrust, we consider the setting where the designer has partial information about players' payoffs. We develop robust convex programs considering the worst-case realization of players' payoffs under general perturbations. We obtain optimality conditions of no and full information disclosure based on uncertainty set radius and perturbation shifts under ellipsoid uncertainty. Numerical experiments show that the designer is inclined to reveal less informative signals as its uncertainty about the game increases.

DEDICATION

To my family

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NOMENCLATURE

BNE	Bayesian Nash Equilibrium
BCE	Bayesian Correlated Equilibrium
LQG	Linear-Quadratic-Gaussian
SDP	Semidefinite Program
α-BNE	α-approximate Bayesian Nash Equilibrium
α-BCE	α-approximate Bayesian Correlated Equilibrium
G	Incomplete information game
$BNE(G)$	Set of BNE strategies in an incomplete information game G
γ	Payoff state vector
γ_i	Payoff state of agent i
ψ	Prior distribution on payoff state
n	Number of agents
a	Action profile
a_i	Action of agent i
s	Strategy profile
s_i	Strategy of agent i
ω_i	Signal for agent i
u_i	Utility function of agent i
ζ	Information structure
ϕ	Action distribution
λ	Weight on agreement objective
$[\lambda]_i$	i^{th} eigenvalue of a matrix

β	Common off-diagonal element of H in homogeneous LQG games
ρ	Uncertainty set radius
ϵ	Matrix of shifts over payoff coefficents matrix H
η	Matrix of linear coefficients over payoff coefficents matrix H in objective coefficients matrix F
ν_i	i^{th} perturbation vector
ξ	Weight of consensus term in the Beauty contest game
δ_i	Risk reduction coefficient of agent i
r_i	Risk of infection for agent i
U_ε	Space of utility functions parameterized by constant $\varepsilon > 0$
$\mathcal{G}_\varepsilon$	Set of games with utilities $u \in U_\varepsilon$
$\mathcal{Z}$	Feasible set over information structures
$C(\mathcal{Z})$	Set of equilibrium action distributions
$\mathcal{N}$	Set of agents
Γ	Set of payoff state vectors
$\mathcal{A}$	Set of action profiles
$\mathcal{A}_i$	Set of actions of agent i
Ω_i	Set of signals for agent i
$\mathcal{V}_i$	i^{th} uncertainty set
$\mathcal{Y}_i$	i^{th} set of indices over H corresponding to i^{th} BCE constraint
H	Utility coefficients matrix
F	Objective coefficients matrix
F^C	Agreement objective coefficient matrix
F^{SW}	Social welfare objective coefficient matrix
X	Covariance matrix of actions and payoff state

X^*	Optimal covariance matrix of actions and payoff state
R_i	Coefficient matrix for i^{th} BCE constraint
$\bullet$	Frobenius product
$\odot$	Hadamard product
P^m	Set of $m \times m$ symmetric matrices
P_+^m	Set of $m \times m$ symmetric positive semi-definite matrices
$tr(\cdot)$	Trace of a matrix
$\mathbf{1}$	Column vector of all ones
O	Zero matrix
$A_{i,j}$	ith row and jth column of matrix A
$[A]_{i,j}$	i,jth submatrix of A
$\|\|\cdot\|\|_F$	Frobenius matrix norm

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1. INTRODUCTION AND LITERATURE REVIEW

The chapter starts with the introduction to general information design problem in non-cooperative incomplete information games. It continues with a section on linear-quadratic-Gaussian (LQG) games which are the underlying game structure in the research performed. Following these definitions, we present the SDP formulation of the information design problem in LQG games. We conclude the section with an overview of our results in the thesis, and contributions to the literature[1].

1.1 The Information Design Problem

Information design originated from Bayesian persuasion framework which involves a sender, i.e. information designer and a single receiver, i.e. agent. The goal is to maximize sender's objective by persuading a rational agent regarding a payoff state affecting receiver's utility via revealing credible information [1]. In multi-agent systems, the information designer optimizes a system level objective, such as social welfare, by selecting an information structure, and the corresponding induced rational behavior of multiple agents to overcome undesirable or inefficient outcomes [2, 3]. In general, the designer's problem is computationally intractable as it involves identifying the "best" distribution over the space of distributions that induces desirable rational behavior [4]. Thus, the current approaches make structural assumptions about the state/action space, the system designer's objective, and the game payoffs in order to attain analytical results [5].

To further illustrate the information design problem, we consider an example from pandemic control (see Fig. 1.1). Consider a public health institute, e.g., Centers for Disease Control and Prevention (CDC) or a local health department, that determines the fidelity of information revealed about the unknown risks of an emerging infectious disease, in order to eliminate or reduce the risks of an outbreak. The agents in the population can be in susceptible, infected, and recovered

[1]Part of this chapter is reprinted with permission from F. Sezer, H. Khazaei, and C. Eksin, "Maximizing social welfare and agreement via information design in linear-quadratic-gaussian games," IEEE Transactions on Automatic Control, pp. 1–8, 2023, ©2023 IEEE and F. Sezer and C. Eksin, "Information preferences of individual agents in linear-quadratic-gaussian network games," IEEE Control Systems Letters, vol. 6, pp. 3235–3240, 2022, ©2022 IEEE.

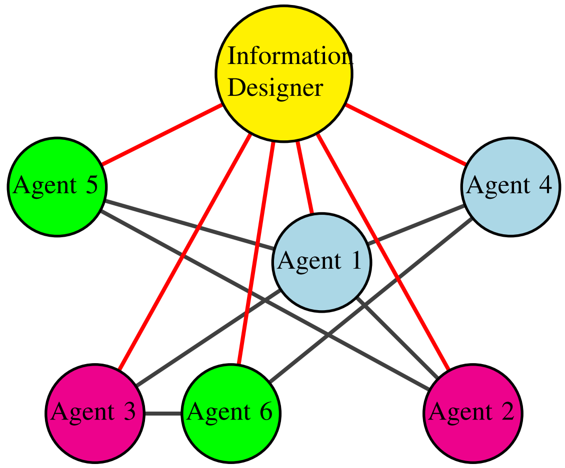


Figure 1.1: Information designer sends optimally designed signals on the risks of infection from an emerging infectious disease to the population with individuals who are susceptible (blue), infected (green) or recovered (magenta), so that they follow the recommended health measures, e.g. social distancing or masking that reduce the risk of an outbreak. An individual's infection or disease transmission risk is determined by its contacts (shown by black edges). For instance agent 1 (susceptible) has one infected neighbor (agent 5) that it can contract the disease from. See also Example 3.

states. A susceptible agent determines its level of adherence to recommended preventative health measures based on its assessment of risks of an infection, and costs of the measures. For instance, both susceptible agents (agent 1 and 4) have a single infected contact (agent 5 and 6) that they can contract the disease from (shown by black lines in Fig. 1.1). An infectious agent pits the risks associated with transmitting the disease to other agents against the costs of taking the isolation measures. For instance, both infected agents (agent 5 and 6) have one susceptible and one recovered contact. The higher is the number of susceptible contacts, the stronger is the risk for disease transmission. A local health department (designer) has more information about the potential risks associated with an emerging infectious disease. Its goal is to induce actions that will reduce the disease transmission in the population by sending signals on infection risk or campaigning for certain actions, e.g., social distancing, or masking indoors. In this setting, the health department has to reveal credible information based on the actual (realized) severity of the infectious disease.

Information design assumes at least a single designer under commitment assumption and multiple agents who play a game with each other. Commitment assumption refers to fixing information structure before realization of payoff state. There are also simpler models where a sender tries to persuade a single receiver. Based on the commitment assumption, there are two types of sender-receiver models. The older model called "Cheap Talk" [6, 7] does not assume commitment whereas the newer model called Bayesian persuasion [1] assumes commitment. Information design is closer in the sense of commitment to Bayesian persuasion. Commitment assumption brings truthful communication of realized signal from the sender to the receiver. This removes the strategic environment which exists in Cheap Talk i.e the sender and the receiver do not play a game with each other in Bayesian persuasion. Commitment assumption causes sender's utility to be weakly higher under Bayesian persuasion compared to under any other strategic communication games where commitment does not exist [1].

Information design in operations gained importance recently, see [5] for a detailed review. On further development of information design methodology, [8] presents equilibrium existence results when there are competing information designers, and [9] develops approximation algorithms for persuasion under limited communication. In revenue maximization, [10] studies optimal signalling mechanism in second price auctions. Value of personalized information provisioning compared to public information is studied by [11]. [12] presents results on optimal information policy in time-locked sales. In queuing systems, [13] considers a designer who gives information on state of the queue to customers to maximize expected revenue. [14] considers information design in queues where customers make a payment to learn the signal regarding the state of the queue. In public health management, informing the public about approaching health emergencies such as epidemics [15] and workplace occupancy control, i.e., in person vs remote via informing about infection risk parameter [16] is studied. In platform management, nudging suppliers to use platform is studied by [17]. [18] studies an e-commerce platform as a designer who studies joint promotion and information policy in e-commerce platforms. In network settings, [19] considers a social network where it recommends to engage or not to engage with a post to a user with the objective of minimizing misinformation. This work focuses on public information structures and identifies which networks are more amenable to persuasion.

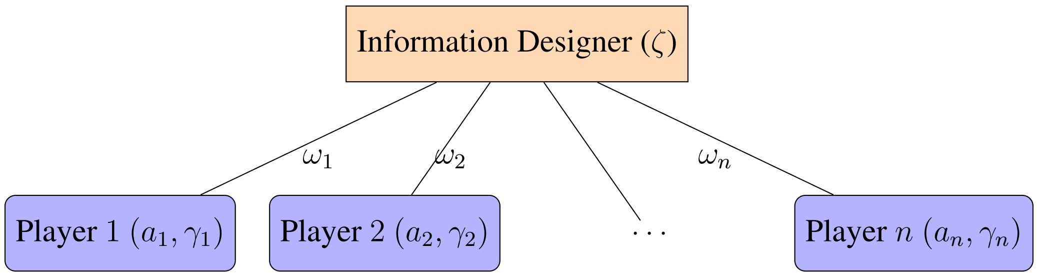


Figure 1.2: An information designer sends a signal ω_i drawn from information structure $\zeta(\omega|\gamma)$ to each player i who takes action a_i in a game with other players under payoff state γ_i.

In this thesis, we study information design linear-quadratic-Gaussian (LQG) games. In an LQG game, players have quadratic payoff functions, and the state and the signals come from a Gaussian distribution. Under certain assumptions, the optimal strategy in LQG games defined by the Bayesian Nash equilibrium (BNE) is unique and linear in the signals received [20]. The linearity of BNE strategies allow the information design problem to be a semi-definite program (SDP) when the information designer's objective is a quadratic function of the players' actions and the payoff-relevant states [21].

Next, we provide the mathematical notation and preliminaries that our results build on.

1.1.1 Mathematical Framework for the Information Design Problem

A non-cooperative incomplete information game involves a set of n players belonging to the set N, each of which selects actions $a_i \in \mathcal{A}_i$ to maximize the expectation of its payoff function $u_i(a, \gamma)$ where $a \equiv (a_i)_{i \in \mathcal{N}} \in \mathcal{A}$ and $\gamma \equiv (\gamma_i)_{i \in \mathcal{N}} \in \Gamma$ correspond to an action profile and an unknown payoff state, respectively. Players form expectations about their payoffs based on their signals/types ω_i about the state given a common prior ψ. We represent the incomplete information game by the tuple $G := \{\mathcal{N}, \mathcal{A}, \{u_i\}_{i \in \mathcal{N}}, \{\omega_i\}_{i \in \mathcal{N}}\}$.

A strategy of player i maps each possible value of the private signal $\omega_i \in \Omega_i$ to an action $s_i(\omega_i) \in \mathcal{A}_i$, i.e., $s_i : \Omega_i \to \mathcal{A}_i$. A strategy profile $s = (s_i)_{i \in N}$ is a BNE with information structure ζ, if it satisfies

$$E_\zeta[u_i(s_i(\omega_i), s_{-i}, \gamma)|\omega_i] \geq E_\zeta[u_i(a_i', s_{-i}, \gamma)|\omega_i], \tag{1.1}$$

for all $a'_i \in \mathcal{A}_i, \omega_i \in \Omega_i, i \in \mathcal{N}$ where $s_{-i} = (s_j(\omega_j))_{j \neq i}$ is the equilibrium strategy of all the players except i, and E_ζ is the expectation operator with respect to the signal distribution ζ and the prior on the payoff state ψ. The above definition ensures that no player has a unilateral profitable deviation from a BNE strategy to another action at any signal realization given the information structure ζ.

An information designer optimizes the expected value of a design objective $f(a, \gamma)$, e.g., social welfare, by deciding on an information structure ζ from a set of signal generating distributions $\mathcal{Z}$, i.e.,

$$\max_{\zeta \in \mathcal{Z}} E_\zeta[f(s, \gamma)] \tag{1.2}$$

where s is a BNE strategy profile for the game G under the information structure of the game ζ. The information structure of the game $\zeta(\omega|\gamma)$ is the conditional distribution of $\omega \equiv (\omega_i)_{i \in N}$ given γ. That is, an information structure comprises signal transmission rules and the probability distribution from which signals are generated. Signals transmitted to players convey information about payoff relevant states.

Information design follows the given timeline (Fig. 1.2):

1. Designer selects $\zeta \in Z$ and notifies all players.
2. Payoff state γ is realized.
3. Players observe signals $\{\omega_i\}_{i \in N}$ drawn from $\zeta(\omega|\gamma)$.
4. Players act according to BNE under ζ.

The information designer's problem in (1.2) is intractable for general incomplete information games with continuous actions because it is a linear program with an infinite number of variables [21]. We focus on LQG games that admit a tractable SDP formulation for (1.2) when $f(\cdot)$ is quadratic and signals come from a Gaussian distribution.

1.2 Linear-Quadratic-Gaussian Games

In an LQG game, player i's payoff function is quadratic,

$$u_i(a,\gamma) = -H_{i,i}a_i^2 - 2\sum_{j\neq i} H_{i,j}a_i a_j + 2\gamma_i a_i + d_i(a_{-i},\gamma), \tag{1.3}$$

where $H_{i,j}$ for $i \in N$, $j \in N$ are real-valued coefficients with $H_{i,i} > 0$, $d_i(a_{-i},\gamma)$ is an arbitrary function of the opponents' actions $a_{-i} \equiv \{a_j\}_{j\neq i}$ and state γ, and we have $a \in A \equiv \mathbb{R}^n$, and $\gamma \in \Gamma \equiv \mathbb{R}^n$. We collect the payoff function coefficients in a matrix $H = [H_{i,j}] \in \mathbb{R}^{n\times n}$. We note that the function is quadratic in player i's action but it need not be quadratic in others' actions and payoff state as per the term $d_i(a_{-i},\gamma)$. Indeed, this term cannot be controlled by player i, i.e., it does not affect its strategy. Here, we focus on scalar actions, i.e., $a_i \in \mathbb{R}$.

Remark 1. *The results in the dissertation can be extended to cover the case where $a_i \in \mathbb{R}^{m_i}$ for $m_i \in \mathbb{N}$, as long as $u_i(a,\gamma)$ remains quadratic in actions.*

Payoff state γ follows a normal distribution $\psi(\mu,\Sigma)$ with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ. Player i receives a private signal $\omega_i \in \mathbb{R}$. We assume the joint distribution over the random variables (ω,γ) is normal; thus, ζ is assumed to be a normal distribution. Next, we provide three examples of games with quadratic payoffs.

Example 1 (Cournot competition)**.** *Firms determine the production quantities for their goods (a_i) facing a marginal cost of production (γ_i) [22]. The price is a function of the production quantities, $p_i(a) = \vartheta - \varpi a_i - \varrho\sum_{j\neq i} a_j$ with positive constants ϑ, ϖ and ϱ. The payoff function of the firm i is its profit given by its revenue $a_i p_i(a)$ minus the cost of production $\gamma_i a_i$,*

$$u_i(a,\gamma) = a_i p_i(a) - \gamma_i a_i. \tag{1.4}$$

Example 2 (Beauty Contest Game)**.** *Payoff function of player i is given by*

$$u_i(a,\gamma) = -(1-\xi)(a_i-\gamma)^2 - \xi(a_i - \bar{a}_{-i})^2, \tag{1.5}$$

where $\xi \in [0,1]$ *and* $\bar{a}_{-i} = \sum_{j\neq i} a_j/(n-1)$ *represents the average action of other players. The first term in* (1.5) *denotes the players' urge for taking actions close to the payoff state* γ, *i.e estimation . The second term accounts for players' tendency towards taking actions in compliance with the rest of the population i.e consensus. The constant* ξ *gauges the importance between the two terms. The payoff captures settings where the valuation of a good, e.g., stock, depends not just on the performance of the company but also on what other players think about its value [23].*

Example 3 (Social Distancing Game)**.** *Action of agent* i, $a_i \in \mathbb{R}^+ \cup \{0\}$ *is the amount of social distancing the agent exerts to avoid the risk of contracting an infectious disease. The risk of infection depends on unknown disease specific parameters, e.g., severity, infection rate, and the social distancing actions individuals in contact with agent* i. *We define the payoff function of player* i *as follows,*

$$u_i(a,\gamma) = -H_{i,i}a_i^2 - (1-\delta_i a_i) r_i(a,\gamma) \tag{1.6}$$

where the risk of infection is

$$r_i = \gamma - 2\sum_{i\neq j} H_{i,j} a_j, \tag{1.7}$$

$0 < \delta_i < 1$ *is the risk reduction coefficient. In the definition of risk* r_i (1.7)*,* γ *denotes the risk rate of the disease such as infection rate or severity, and* $H_{i,j}$ *determines the contacts of agent* i *and the intensity of the contacts. First term in* (1.6) *represents cost of social distancing to agent* i. *Second term in* (1.6) *denotes the overall risk of infection that scales with agent* i*'s social distancing efforts.*

We continue with the description of BCE under LQG games. The following result by [20] states a sufficient condition for having an unique BNE strategy, and provides a set of linear equations to determine the coefficients of the linear BNE strategy in LQG games.

Proposition 1 (Theorem 5, [20])**.** *Suppose that* $H + H^T$ *and the variance of the private signals* var(ω_i) *are positive definite for each* $i \in \mathcal{N}$. *Then LQG game has a unique Bayesian Nash equilibrium given by*

$$s_i(\omega_i) = \bar{a}_i + b_i^T(\omega_i - E_\zeta[\omega_i]) \text{ for } i \in N, \tag{1.8}$$

where $b_1, ...b_n$ are determined by the following systems of linear equations:

$$\sum_{j \in N} H_{i,j} cov(\omega_i, \omega_j) b_j = cov(\omega_i, \gamma_i) \textit{ for } i \in N, \tag{1.9}$$

and $cov(\cdot, \cdot)$ represents the covariance between two random variables.

Assumptions of Proposition 1 guarantee the existence and uniqueness of a linear strategy (1.8) that satisfy (1.1) with coefficients obtained by solving (1.9). We assume the sufficient conditions above, i.e., the existence of an unique BNE, throughout the dissertation.

An action distribution represents the probability of observing an action profile $a \in \mathcal{A}$ when agents follow a strategy profile s under ζ. The action distribution ϕ is defined as a sum over information structures ζ:

$$\phi(a|\gamma) = \sum_{\omega : s(\omega) = a} \zeta(\omega|\gamma) \tag{1.10}$$

A Bayesian correlated equilibrium (BCE) is an action distribution in which no individual would profit by unilaterally deviating from selecting actions according to the given action distribution. The formal definition follows.

Definition 1. *An action distribution ϕ under ζ is a BCE if and only if it satisfies*

$$E_\phi[u_i((a_i, a_{-i}), \gamma)|a_i] \geq E_\phi[u_i((a'_i, a_{-i}), \gamma)|a_i] \tag{1.11}$$

for all $a_i, a'_i \in \mathcal{A}_i$ and $i \in \mathcal{N}$ where $E_\phi[\cdot|a_i]$ is the conditional expectation with respect to the action distribution ϕ and information structure ζ given action $a_i \in \mathcal{A}_i$.

An equilibrium action distribution ϕ, corresponding to a BNE strategy profile s under ζ, i.e., $\phi(a|\gamma) = \sum_{\omega:s(\omega)=a} \zeta(\omega|\gamma)$, satisfies (1.11) as stated in the following result.

Proposition 2 (Corollary 2, [2])**.** *An equilibrium action distribution is a BCE under any information structure. If a BCE corresponds to an equilibrium action distribution, a corresponding information structure exists.*

Using Propositions 1 and 2, we can derive a necessary and sufficient condition for an action distribution comprised of jointly normally distributed action profile and payoff state.

Proposition 3 (Proposition 3, [21])**.** *An action distribution ϕ comprised of jointly normally distributed action profile and a payoff state is a BCE if and only if the following conditions hold*

$$E_\phi[a] = \overline{a} \tag{1.12}$$

$$\sum_{j\in N} H_{i,j} cov(a_i, a_j) = cov(a_i, \gamma_i). \tag{1.13}$$

Solution of (1.8) and (1.9) by $b_i = 1$ and $\overline{a}_i = E_\zeta[\omega_i]$ constitute a necessary and sufficient condition for a BCE by Proposition 1. Conditions (1.12) and (1.13) correspond to this solution; thus, Proposition 3 is established.

1.3 A SDP Formulation of Information Design Problem given Quadratic Design Objectives

In this section, we provide preliminary results on the information design problem in LQG games. The first result represents the problem in (1.2) as a SDP with the decision variable

$$X := \begin{bmatrix} var(a) & cov(a,\gamma) \\ cov(\gamma, a) & var(\gamma) \end{bmatrix} \tag{1.14}$$

and objective coefficients matrix

$$F := \begin{bmatrix} [F]_{1,1} & [F]_{1,2} \\ [F]_{1,2} & [F]_{2,2} \end{bmatrix} \tag{1.15}$$

where $[F]_{i,j}$ indicates the $n \times n$ block matrix for $i, j \in \{1, 2\}$.

Proposition 4 (Section 3.2, [21])**.** *If the objective function $f(a, \gamma)$ is quadratic in its arguments, and the payoff matrix H is such that $H + H^T$ is positive definite, then the information design* *problem in* (1.2) *can be restated as the following SDP,*

$$\max_{X \in P_+^{2n}} F \bullet X \tag{1.16}$$

$$\textit{s.t.} \;\; R_k \bullet X = 0 \quad \forall\, k \in \{1,..,n\}, \tag{1.17}$$

$$M_{k,l} \bullet X = \operatorname{cov}(\gamma_k, \gamma_l), \quad \forall\, k, l \in N \textit{ with } k \leq l \tag{1.18}$$

where $R_k \in P^{2n}$ *and* $M_{k,l} \in P^{2n}$ *are defined as*

$$[R_k]_{i,j} = \begin{cases} H_{k,k} & if \quad i = j = k, \\ H_{k,j}/2 & if \quad i = k, 1 \leq j \leq n, j \neq k, \\ -1/2 & if \quad i = k, j = n + k, \\ H_{k,i}/2 & if \quad j = k, 1 \leq i \leq n, i \neq k \\ -1/2 & if \quad j = k, i = n + k, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$[M_{k,l}]_{i,j} = \begin{cases} 1/2 & \textit{if } k < l, i = n + k, j = n + l, \\ 1/2 & \textit{if } k < l, i = n + l, j = n + k, \\ 1 & \textit{if } k = l, i = n + k, j = n + l, \\ 0 & \textit{otherwise}, \end{cases}$$

This result, due to [21], represents the original information design problem (1.2) as the maximization of a linear function of a positive semi-definite matrix X subject to linear constraints. The result leverages the fact there is a unique BNE that is a linear function of the signals whose coefficients can be obtained by solving a set of linear equations in an LQG game with payoff matrix H where $H + H^T \in P_+^n$ [20]. The linear strategies allow a mapping from strategies to signals, which then means selecting the best distribution over the signals is equivalent to selecting the best distribution over the actions subject to the BNE constraints. Accordingly, the selection of the information structure in (1.2) reduces to determining the covariance between the realized actions and payoff states in (1.16). Note that we can assume $[F]_{2,2}$ is a zero matrix $O_{n\times n}$, because $var(\gamma)$ is given by nature, and cannot be altered by choosing an information structure. Again by leveraging the linear mapping of strategies from signal space to action space, one can express the BNE equations with the set of linear constraints in (1.17). The set of constraints in (1.18) assigns the given covariance matrix of the payoff states to the corresponding sub-matrix in X, i.e., it is equivalent to $[X]_{2,2} = \text{var}(\gamma)$. We note that we assume the conditions in Proposition 4 hold throughout all chapters.

Next, we consider an important special case.

Definition 2 (Public Information Structure)**.** *A public information structure has $\omega_1 = ... = \omega_n$ with probability one. The set of public information structures is a subset of the general information structures.*

In the public information design problem, all players receive the same signal, and it is common knowledge that they will receive the same signal. We define two important feasible solutions to (1.16) - (1.18) (no and full information disclosure), [21].

Definition 3 (No information disclosure)**.** *No information disclosure refers to the case when there is no informative signal sent to the players. In this case, the equilibrium action profile is given by $a = H^{-1}\mu$. The induced decision variable and the objective value is respectively given by*

$$X = \begin{bmatrix} O & O \\ O & \text{var}(\gamma) \end{bmatrix} \text{ and } F \bullet X = 0. \tag{1.19}$$

Definition 4 (Full information disclosure)**.** *The signals sent to the players reveal all elements of payoff state γ under full information disclosure. Equilibrium action profile is given by $a = H^{-1}\gamma$.*

The induced decision variable

$$X = \begin{bmatrix} H^{-1}\text{var}(\gamma)(H^{-1})^T & H^{-1}\text{var}(\gamma) \\ \text{var}(\gamma)(H^{-1})^T & \text{var}(\gamma) \end{bmatrix} \tag{1.20}$$

and the objective value is $F \bullet X = F_H \bullet \text{var}(\gamma)$ *where* $F_H = (H^{-1})^T([F]_{1,1} + [F]_{1,2}H + H^T[F]_{2,1})H^{-1}$

The information structure is *public* when all players receive a common signal. Otherwise, when the players receive individual signals, the signal structure is *private*. Another distinction is based on the fidelity of information carried by the signals. A signal can carry *no*, *partial*, or *full information*. No information disclosure does not improve the prior information of the players about the payoff relevant state, while signals reveal the payoff relevant state under full information disclosure. A partial information disclosure is when the signals carry some information, but do not fully reveal the payoff relevant state to the players.

Next result states the conditions for the optimality of full information disclosure solution when we consider the set of public information structures.

Proposition 5 (Proposition 6-7, [21])**.** *Let* $\text{var}(\gamma) = DD^T$ *such that* D *is an* $n \times k$ *matrix of rank* k *where* k *is the rank of* $\text{var}(\gamma)$ *and* F_H *is as given in* (4.73).

- *Assume* $D^T F_H D \neq O$ *is negative semi-definite. Then, no information disclosure is optimal in the set of public information structures.*

- *Assume* $D^T F_H D \neq O$ *is positive semi-definite. Then, full information disclosure is optimal in the set of public information structures.*

Remark 2. *The SDP formulation of the information design problem in* (1.2) *poses the problem as the determination of a distribution over actions not signals. A natural question is: how can the designer use the solution* X *and* ϕ *instead of the distribution over signals* ζ*? As per the information design timeline, when* X *is decided and* γ *is realized, the designer can draw the* *suggested actions from $\phi(a|\gamma)$ which has a Gaussian distribution. These suggested actions can be used as coordinating signals instead of the private signals ω_i.*

In next sections, we continue with a motivation for the models and results in the upcoming chapters, and an overview of related literature and the results included in this thesis.

1.4 Summary of Chapter 2: Welfare and Agreement Maximization in LQG Games

In this chapter, we focus on the maximization of welfare and agreement. Welfare refers to sum of all agents' utilities. Maximization of welfare bring alignment between the designer and agents, in the sense that information disclosure is optimal. The level of information disclosure depends on the symmetry between agents. Total symmetry among payoff states, signals or payoff coefficients of agents make full information optimal disclosure policy. As the symmetry reduces, partial information becomes optimal solution.

Agreement objective is the sum of squared differences between agent's actions and mean actions. This objective is misaligned with agent's utilities because of the nature of the objective and utilities as functions. In this case, no information disclosure is optimal in general. The results appeared in [24].

1.4.1 Contributions

Building on the SDP formulation of the information design problem, we analyze optimal information structures when the system level objective is to maximize social welfare (Section 2.3), maximize agreement among players' actions (Section 2.4), or a weighted combination of these two objectives (Section 2.5).

In this thrust, we provide analytical and computational insights about the value of information and optimal information structures by focusing on particular objectives for the designer (social welfare and agreement). Our contributions are fourfold:

1) Given the social welfare design objective, we show that full information disclosure is optimal if there is a common payoff state (Proposition 7), when the dependency of payoffs on others' actions is homogeneous (Theorem 1), or if we only consider the set of public signals (Proposition

8). These results follow the intuition that the designer would like to reveal as much information as possible when the payoffs of players are aligned with the system-level objective [3].

2) When the objective is to maximize the agreement between players' actions, we show that no information disclosure is optimal for any LQG game (Proposition 9). That is, by hiding information, players' actions are closer to each other.

3) If the information designer maximizes social welfare and agreement, we identify a critical weight on the agreement term of the objective based on game payoffs below which full information disclosure is preferred to no information disclosure (Propositions 10 and 11). That is, the benefit of revealing information outweighs the increase in disagreement.

4) Numerical solutions to the SDP formulation reveal optimal private signal distributions that outperform both full and no information disclosure schemes. These contributions mentioned above build on the SDP formulation of the information design problem that considers generic quadratic design objectives in LQG games [21], but are distinct in that they provide specific insights about the practically-relevant social welfare and agreement design objectives.

1.4.2 Related literature

Other intervention mechanisms, besides information design, include providing financial incentives in the form of taxes and rewards [25], system utility design [26], and nudging or player control during learning dynamics [27–30]. In contrast to these approaches, the information design framework manages the uncertainties of players so that their expected payoffs align with the objective of a system designer. That is, the system designer does not control agents directly, rather it determines the information revealed to the players, so that players' evaluation of their payoffs lead to better outcomes from the system designer's perspective. In this sense, there is a limit to the system designer's capability to achieve its goal. This limit determines the value of information.

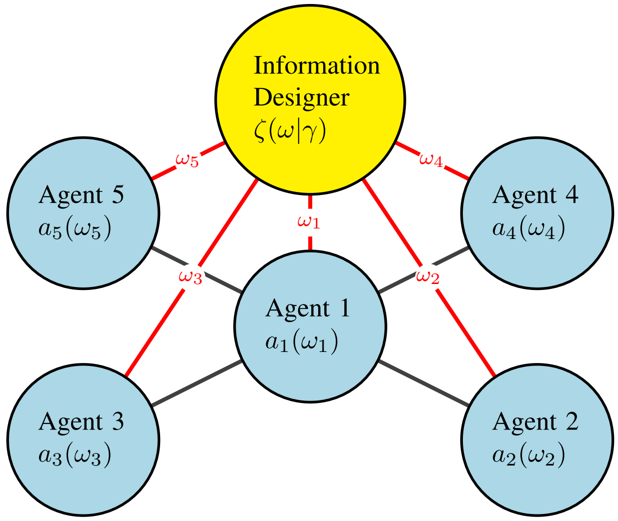


Figure 1.3: Agents play a network game with individual payoffs that depend on their neighbors' actions and an unknown payoff state γ. An information designer sends a signal w_i drawn from information structure $\zeta(\omega|\gamma)$ to each agent i. Agent i takes an equilibrium action a_i based on the received signal ω_i to maximize its expected utility (©2022 IEEE).

1.5 Summary of Chapter 3: Information Preferences of Individual Agents in LQG Network Games

In Chapter 2, we show that full information disclosure is the optimal solution to social welfare maximization under public information structures and/or common payoff states (see Propositions 7-8). While full information disclosure may be optimal from the system designer's perspective, its effect on individual player's payoff is not clear. In Chapter 3, we analyze the effect of such information disclosure policy on the payoffs of individual agents and its dependence on the centrality of the agents in network games. The results appeared in [31].

1.5.1 Contributions

We identify sufficient conditions for the individual preference of informative signals based on the payoff coefficients prior to realization of the state (ex-ante) in Theorem 2. We leverage this result, and identify that both central and peripheral agents in a star network structure prefer information disclosure ex-ante for homogeneous LQG games (Proposition 12). In computing the benefit of information disclosure to individual agents, we find that a peripheral agent can benefit more than the central agent under full information disclosure if competition is strong and number of agents is small (Proposition 13). In sum, the incentives of the agents and the system designer are in congruence ex-ante given the conditions considered.

We find that joint incentives of individual agents and the system designer can cease to exist ex-post, i.e., after the realization of the payoff state. In contrast to Proposition 12, the central agent prefers no information disclosure ex-post if realization of the payoff state is lower than expected. In the context of Bertrand competition among firms in networked markets, these results imply that central firms may not benefit from information disclosure when the competition among firms is strong. Ex-post analysis is not useful because agents do not observe the realized payoff state when taking actions, but they observe signals generated by the information designer based on the realized state. Still, the ex-post analysis of incentives imply that a risk averse central agent can prefer uninformative signals ex-ante. These results extend prior knowledge on the information design problem [3, 21] by providing a characterization of the benefit of informative signals on players' payoffs and its dependence on centrality of the players in network games with incomplete information.

1.5.2 Related Literature

Prior studies in network games with quadratic payoffs focus on computation and characterization of equilibria, and analyze the changes to the equilibria or social welfare when network topology is modified via adding/removing links or nodes [32–36]. In contrast, Chapter 3 considers the effects of information design on individual payoffs when the design objective is to maximize social welfare.

Among the papers that consider network games, a standard thread of research is on developing dynamics that reach a Nash equilibrium in complete information network games with quadratic or other types of utilities—see [37] for a recent extensive review on this topic. Another thread focuses on characterizing the Nash equilibrium, showing its relation to network centrality metrics in order to identify key players in the game [32,35]. These research threads are also extended to incomplete information network games [33, 34, 38]. There are two sources for uncertainty considered in these works: network and utility. Network uncertainty refers to players not knowing their neighbors, while the payoff uncertainty considers an unknown payoff relevant state, same as the setting in this chapter. Again the equilibrium computation or characterization is the main focus of these studies.

In terms of design/intervention in network games, studies either focus on seeding, i.e., controlling a given number players, [39, 40], on taxation/subsidy schemes [25], or on changing the network structure to maximize a system level objective. Orthogonal to these works, the intervention mechanism considered in this thrust is information design [3,5,24]. That is, a designer chooses the level of information revealed to the players in an incomplete information network game so that social welfare is maximized. In addition, this chapter analyzes the potential gains of information design to the players based on their centrality. This thrust tries to answer the question "which players benefit the most and how much from optimal information design in incomplete information network games?" We also note that the result in Theorem 2 provides a condition for an agent i to benefit from full information disclosure for general networks. The ensuing results (Proposition 12 and 13) exemplify some of the consequences of this general result for star networks.

1.6 Summary of Chapter 4: Robust Information Design in LQG Games

In Chapter 4, we propose a robust optimization approach to the information design problem considering the fact that the designer cannot exactly know the game players are in. Indeed, while the designer may be knowledgeable about the payoff relevant random state, it may have uncertainty about the payoff coefficients of the players. For instance, in the pandemic control example above while the public health department may have near-certain information about the potential risks of a disease or intervention, it may not know how the society weights the risks and benefits in their decision-making. Here, we assume the designer has partial knowledge about players' utilities, and wants to perform information design over the payoff relevant states. The results appeared in [41].

1.6.1 Contributions

When the payoffs of the players are unknown, the designer cannot be sure of the rational behavior under a chosen information structure. We formulate this problem as a robust optimization problem where the designer chooses the "best" optimal information structure for the worst possible realization of the payoffs. That is, we do not make any assumptions on the distribution of the players' payoff coefficients.

Specifically, we assume the players have linear-quadratic payoffs with coefficients unknown by the designer. We further assume that the payoff relevant states and signals generated by the designer come from a Gaussian distribution. In this setting, we show that the robust information design problem can be formulated as a tractable SDP given ellipsoid (Theorem 4), interval (Theorem 5) and general cone perturbations (Theorem 6) on the payoff coefficients–see Section 4.4. Using the tractable SDP formulation under ellipsoid perturbations, we establish the optimality of canonical information disclosure settings, namely no and full information disclosure, given the set of general (Theorem 7) and public (Theorem 8 and 9) information structures.

1.6.2 Related Literature

It is worth noting that there are existing works in the framework of Bayesian persuasion and information design that consider robustness in a different sense than our framework. Both [42] and [43] consider the Bayesian persuasion setting with a sender and a receiver agents, where the nature may send additional information to the receiver, and aim to find the sender's optimal information structures. These works assume the receiver's payoff is known by the sender, but consider robustness against the side information provided by the nature to the receiver.

In another Bayesian persuasion setting, [44] considers the situation where the sender does not know the prior distribution on payoff state and the sender learns it over repeated interactions. The goal of the sender is to minimize regret with respect to the setting where the sender knows the prior distribution on payoff state. Their main result proves that for any persuasion setting satisfying certain regularity conditions, their proposed algorithm achieves $O(\sqrt{T \log T})$ regret with high probability under the horizon length T. In contrast, here we consider a one-shot (static) problem, and we assume the prior distribution on the payoff state is known by the designer.

In [45], an adversarial approach is taken for the setting where the sender does not know the utility of the receiver. That is, the robustness is against the unknown receiver utilities similar to ours. They show that the sender can achieve a low regret under the assumption that it knows the receiver's ordinal preferences over the states of nature upon adoption. However, they formulate the problem as regret minimizing, whereas we formulate the problem using robust optimization. The difference is that we assume the designer knows the perturbed payoff functions, but it does not know the amount of perturbation. Lastly, robustness against the receiver's error prone (non-Bayesian probabilistic inference over payoff state) calculations is considered in [46]. In our case, we assume the receiving agents are capable of updating their expected payoffs without error.

Besides the mentioned distinctions between our framework and [42–46], we focus on the multi-player setting, i.e., information design, and assume an incomplete information game among the players.

2. MAXIMIZING SOCIAL WELFARE AND AGREEMENT VIA INFORMATION DESIGN IN LINEAR-QUADRATIC-GAUSSIAN GAMES

2.1 Introduction

We focus on identification of optimal solutions for welfare maximization and agreement objectives. We analytically derive optimal information structure to welfare maximization under common payoff states, homogeneous games and public information structure settings. We numerically study sensitivity of optimal information structure to asymmetry in payoff states and in interaction terms.

We continue with derivation of analytical optimal solution to information design problem under agreement objective. We conclude with analysis of joint welfare maximization and agreement objective under public information structures and common payoff states settings. The results in this chapter appear in [24] [1].

2.2 Objectives

We start with formal definitions of social welfare and agreement design objectives.

Definition 5 (Social Welfare)**.** *Social welfare is the sum of individual utility functions,*

$$f(a, \gamma) = \sum_{i=1}^{n} u_i(a, \gamma). \tag{2.1}$$

Social welfare is a common design objective used in congestion [25, 47], global [23] or public goods games [15].

Definition 6 (Agreement)**.** *The information designer would like players to agree by minimizing the deviation of players' actions from the mean action, i.e., by maximizing*

$$f(a, \gamma) = -\sum_{i=1}^{n} (a_i - \bar{a})^2, \text{ where } \bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i, \tag{2.2}$$

[1] Part of this chapter is reprinted with permission from F. Sezer, H. Khazaei, and C. Eksin, "Maximizing social welfare and agreement via information design in linear-quadratic-gaussian games," IEEE Transactions on Automatic Control, pp. 1–8, 2023, ©2023 IEEE.

where we assume $a_i \in A_i \equiv \mathbb{R}$. The objective is suitable in settings where consensus is desirable but not exactly attainable. For instance, this objective can be used in reducing consumption variability in demand response [48], or coordinated autonomous movement [38].

2.2.1 Design objectives in Matrix Form

We focus on two specific quadratic design objectives: social welfare (2.1) and agreement (2.2). According to Proposition 1, we can express the information design problem in (1.2) for these objectives as in (1.16). The following are the objective coefficients matrices

$$F^W = \begin{bmatrix} -H & I \\ I & O \end{bmatrix}, \text{ and } F^C = \begin{bmatrix} \frac{1}{n}\mathbf{1}\mathbf{1}^T - I & O \\ O & O \end{bmatrix}, \tag{2.3}$$

corresponding to (2.1) and (2.2), respectively. We obtain F^W by substituting the quadratic payoffs (1.3) in (2.1), and taking the expectation. See Lemma 2 in the appendix for the derivation of F^C.

2.3 Results on Social Welfare Maximization

Our first result shows that full information disclosure will be preferred to no information disclosure in social welfare maximization.

Proposition 6. *Assume H is symmetric. Then, full information disclosure never performs worse than no information disclosure for maximizing social welfare objective.*

Proof. No information disclosure has the objective value $F \bullet X = 0$ regardless of F as per Definition 3. Objective value of full information disclosure is $F^W \bullet X = F^W_H \bullet \text{var}(\gamma)$ as per Definition 4. Given (2.3), $F^W_H = H^{-1}$. We have $F^W_H = H^{-1} \succ 0$ because eigenvalues of H^{-1} is equal to reciprocals of eigenvalues of H which are positive because H is positive definite by the assumption that $H + H^T \succ 0$ and H is symmetric. The result follows from the fact that $F^W_H \bullet \text{var}(\gamma) \geq 0$ given $\text{var}(\gamma) \succeq 0$. □

The result implies that no information disclosure cannot be an optimal information structure for social welfare maximization given symmetric payoff coefficients, since it cannot do better than full information disclosure. Next, we show that full information disclosure maximizes social welfare for some important special cases.

2.3.1 Common Payoff State

We consider a scenario in which the payoff states are identical for all the players.

Proposition 7. *Assume H is symmetric and $\gamma_i = \gamma_j$, $\forall\, i, j \in N$. Then, full information disclosure (X in* (1.20)*) is optimal for social welfare maximization.*

Proof. The objective function $f(\cdot)$ with coefficients matrix F^W in (2.3) is such that $F^W_{i,n+j} = 0$ $\forall\, i, j \in N$ with $i \neq j$. Moreover, we have $F^W_{n+i,n+j} = 0$, $\forall\, i, j \in N$. Therefore,

$$F^W \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} F^W_{i,j} \text{cov}(a_i, a_j) + 2 \sum_{i=1}^{n} F^W_{i,n+i} \text{cov}(a_i, \gamma_i). \tag{2.4}$$

Using the BNE condition in (1.17), which is equivalent to

$$\sum_{j \in N} H_{i,j} \text{cov}(a_i, a_j) = \text{cov}(a_i, \gamma_i),\ \forall\, i, j \in N, \tag{2.5}$$

for the corresponding terms in (2.4), we obtain

$$F^W \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} (F^W_{i,j} + 2F^W_{i,n+i} H_{i,j}) \text{cov}(a_i, a_j). \tag{2.6}$$

We can express $F^W \bullet X = E \bullet \text{var}(a)$ where we define $E := [F^W]_{1,1} + [F^W]_{2,1} \circ H + [F^W]_{1,2} \circ H^T$ using (2.6).

Substituting F^W (2.3) in E, we get $E = H^T$. Since H is symmetric, we have $E = H$.

We have that if $E = \kappa H$ for some constant $\kappa > 0$, then full information disclosure is optimal under common payoff states (Proposition 9 in [21]). In our setting, the condition holds with $\kappa = 1$. □

Proposition 7 establishes that full information disclosure is the optimal information structure among all possible information structures if the payoff state is common and H is symmetric.

In the following example, we analyze the discrepancy between the optimal objective value obtained by solving the SDP in (1.16)-(1.18) and full information disclosure, as we gradually relax the assumptions of Proposition 7. In particular, we allow partially correlated payoff states, and asymmetric game coefficients H.

Example (Asymmetric payoffs and correlated payoff states):

Figure 2.1 shows that full information disclosure becomes increasingly suboptimal as asymmetry grows and correlation between payoff states decreases.

Note that when $Corr(\gamma_i, \gamma_j) = 1$, there is a common payoff state and full information disclosure is optimal for symmetric H. If we consider the beauty contest game with symmetric H and a single stock, full information disclosure on the stock price, i.e, payoff state, is optimal for maximizing social welfare by Proposition 7. If we deviate from common payoff state assumption, this means that stock price is not the same for players when they buy the stock. If we deviate from the symmetry assumption, it means the effect of a player i's action on j's payoff is different than the effect of player j's action on i's payoff. In these scenarios, full information disclosure is no longer optimal.

2.3.2 Homogeneous LQG games

We consider the following payoff matrix H:

$$H_{i,j} = \begin{cases} 1 & \text{if} \quad i = j;\ \forall\, i, j \in N \\ h & \text{if} \quad i \neq j;\ \forall\, i, j \in N \end{cases} \tag{2.7}$$

in which the off-diagonal terms are identical. For the Cournot competition (1.4), we have a homogeneous payoff matrix with $h = \frac{\varrho}{2\varpi}$ when cost is common, i.e., when $\gamma_i = \gamma_j$ for all $i, j \in N$. For the beauty contest game (1.5), we have an homogeneous payoff matrix with $h = -\frac{\xi}{n-1}$.

Theorem 1. *Assume H is given in* (2.7)*, and* $\text{tr}(\text{var}(\gamma)) \geq 2\sum_{i=1}^{n}\sum_{j \in N \setminus \{i\}} cov(\gamma_i, \gamma_j)$*. If* $-\frac{1}{n-1} < h < 1$*, then full information disclosure is optimal for the social welfare maximization objective under general information structures.*

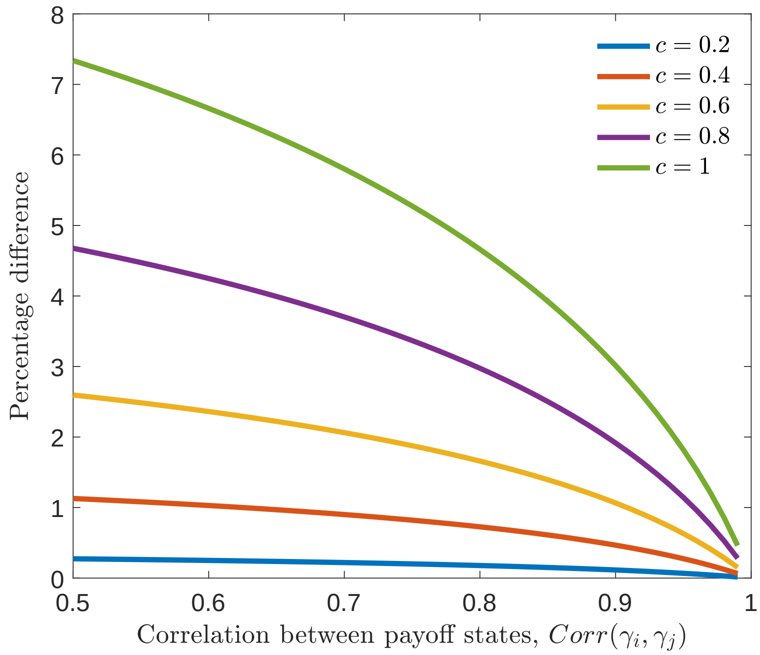


Figure 2.1: Percentage difference between optimal objective value (1.16) and objective value of full information disclosure versus correlation between payoff states. We consider a game with asymmetric payoffs given by $H_{i,i} = 4$ for $i \in N$, and $H_{i,j} = 1 + cU_{i,j}$ for $i \in N$, and $j \in N \setminus \{i\}$ where $U_{i,j} \in [-1, 1]$ is a uniformly distributed random variable‘ for $i, j \in N$, and $c \in [0, 1]$ is a constant determining the magnitude of the asymmetry. The suboptimality of full information disclosure increases with growing asymmetry and decreasing correlation (©2023 IEEE).

Proof. See Appendix A.2 for the proof. □

Theorem 1 shows that full information disclosure is optimal when the effects of others' actions on payoffs are homogeneous and belong to the given region. We note that the LQG game is submodular if $h > 0$, and it is supermodular if $h < 0$. In a submodular game, an increase in a player's action reduces the incentive for other players to increase their actions. In a supermodular game, the effect is reversed, i.e., increasing a player's action increases the incentive for other players to increase their actions—see [49] for formal definitions. Accordingly, social welfare maximization objective is aligned with the incentives of players, when the game is submodular. In contrast, when we have a supermodular game, the optimality of full information disclosure is optimal as long as the effect of another players' actions on a player's action is small, i.e., $h > \frac{-1}{n-1}$.

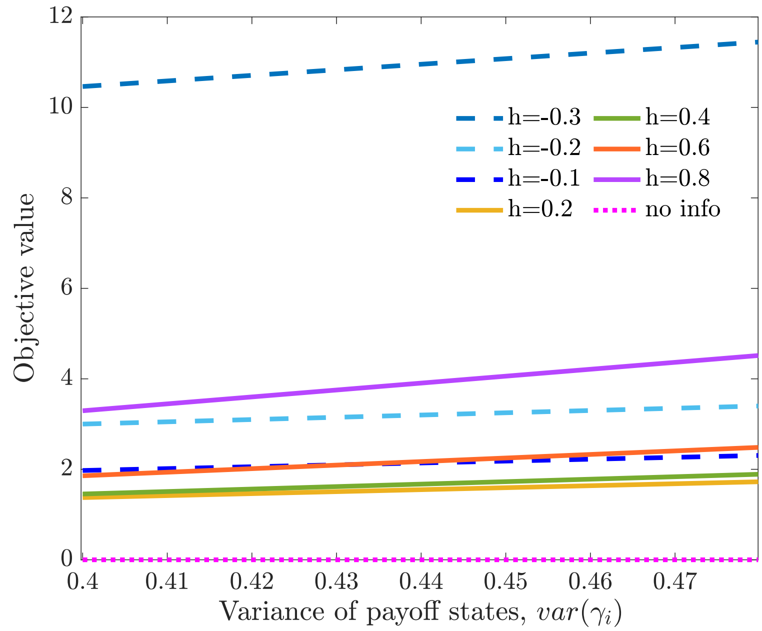


Figure 2.2: Comparison of the social welfare values under full information and no information disclosure. We consider homogeneous games with $\frac{-1}{n-1} < h < 1$. We let $\text{var}(\gamma)_{i,j} = 0.2$ for $i \in N$ and $j \in N \setminus \{i\}$ as we vary $\text{var}(\gamma_i)$ for all $i \in N$. As $\text{var}(\gamma_i)$ increases, the value of full information disclosure increases compared to no information disclosure (©2023 IEEE).

We note that the condition $h > \frac{-1}{n-1}$ stems from the requirement that H needs to be positive definite. Considering the beauty contest game in stock markets with $h = \frac{-\beta}{n-1} < 0$, the information disclosure is always optimal because $\beta < 1$. In Cournot competition, full information disclosure is optimal as long as $\frac{\varrho}{2\varpi} < 1$ according to Theorem 1.

A sufficient condition for optimality of full information disclosure in Theorem 1 is the diagonal dominance of the covariance matrix of the payoff state. In the following numerical example, we identify that the full information disclosure remains optimal even when the diagonal dominance assumption does not hold in homogeneous LQG games.

Example (Relaxing the diagonal dominance of $\text{var}(\gamma)$):

We consider a submodular game among $n = 4$ players with homogeneous payoff coefficients with h ranging from -0.3 to 0.8 (see Fig. 2.2).

When we compare the social welfare value under full information disclosure solution (1.20) with the optimal solution to the information design problem in (1.16)-(1.18), we find that they are identical for all values of var(γ_i) $\in [0.4, 0.48]$. In this interval of var(γ_i), the diagonal dominance assumption is not satisfied. This suggests that full information disclosure remains optimal even when the diagonal assumption is not satisfied. Fig. 2.2 also shows that as the dependence of the payoffs on other players' actions, i.e., $|h|$, increases, objective value for full information disclosure increases. This means the value of revealing information increases as competition increases.

2.3.3 Public information structures

Next, we show that full information disclosure maximizes social welfare under public signals.

Proposition 8. *Assume H is symmetric and consider the set of public information structures as the feasible set. Then, full information disclosure maximizes social welfare* (2.1).

Proof. From Definition 4 and F^W in (2.3), we have

$$F_H^W = (H^{-1})^T(-H + IH + H^T I)H^{-1} = H^{-1}.$$

H^{-1} is positive definite because eigenvalues of H^{-1} are equal to reciprocals of eigenvalues of H which are positive. Therefore, $K^T F_H^W K \neq 0$ is positive definite for any matrix K. The result follows from Proposition 5. □

Together with the previous results in this section, Proposition 8 implies that for full information disclosure to be suboptimal in welfare maximization, the payoff has to include individual payoff states or asymmetric payoff matrix, and the designer has to consider private signals.

2.4 Maximizing Agreement

We show that no information disclosure is an optimal information structure that maximizes agreement objective (2.2).

Proposition 9. *No information disclosure is a maximizer of the objective function in* (2.2) *under general information structures.*

Proof. The objective coefficients matrix F^C has $n-1$ eigenvalues with value -1 and $n+1$ eigenvalues with value of 0. Thus, F^C is negative semi-definite. The decision matrix X is positive semi-definite. We deduce that $F^C \bullet X \leq 0$. Objective value of no information disclosure is 0 by (1.19); thus, no information disclosure is optimal. □

Proposition 9 implies that the information designer achieves the maximum similarity between players' actions by revealing uninformative signals to the players. Broadly, hiding information from players is optimal when there is a conflict between the utility functions of the players and the information designer's objective. We compare this with the objective value attained by full information disclosure to provide further intuition for this result. Given F^C, we have that $F^C \bullet X = F^C_H \bullet \text{var}(\gamma)$ where $F^C_H = (H^{-1})^T [F^C_{1,1}] H^{-1}$. We know that var$(\gamma)$ is positive definite, and F^C_H is negative semi-definite because $[F^C]_{1,1}$ is negative semi-definite as per the proof of the Proposition. Thus, we have that full information disclosure can never be better than no information disclosure for the agreement objective.

In the context of Cournot competition, we can envision a market regulator that wants to reduce the variability in quantities produced by each firm. The result above states that the designer can achieve minimum variability by not revealing information about the marginal cost of production.

Remark 3. *Agreement objective is an example of misaligned objectives between the designer and agents. In this case, obfuscating information is preferred by the designer [3]. Therefore, optimality of no information disclosure complies with this insight.*

In Cheap Talk [7], obfuscating information corresponds to a meaningless case because the point is lying and changing receiver's perception on the private information about the bias the sender has. When no information is given to the receiver about its type, Cheap Talk does not even occur. In contrast, information design focuses on selecting the level of information given to agents so that the designer objective is maximized. In this sense, no information carries a meaning in information design.

Another insight on optimality of no information disclosure for agreement objective comes from concavity of the objective function. In Remark 1 and Section 3 of [1], it is indicated that the *sender does not benefit from persuasion under concave sender utility i.e no disclosure is optimal for concave sender objectives.*

2.5 Maximizing Welfare vs. Agreement

We consider an information design problem in which the designer aims to maximize social welfare and agreement at the same time by considering a weighted combination of (2.1) and (2.2). The objective coefficients matrix is given by

$$F := ((1-\lambda)F^W + \lambda F^C) = \begin{bmatrix} \lambda[F^C]_{1,1} - (1-\lambda)H & (1-\lambda)I \\ (1-\lambda)I & \boldsymbol{O} \end{bmatrix}, \tag{2.8}$$

for weight $\lambda \in [0, 1]$.

The constant λ quantifies the importance of agreement. On one hand full information disclosure is optimal when the design objective is social welfare under common payoff state, homogeneous games, or public signals. On the other hand, no information disclosure is optimal when the objective is to maximize agreement. In the following results we show that full information disclosure remains preferred under public information structures and common payoff states given homogeneous games, if social welfare term gets a large enough weight relative to the agreement term.

Proposition 10. *Assume H has the form in* (2.7) *with $h \in (0, 1)$, and common payoff states $\gamma_i = \gamma_j$, $\forall\ i, j \in N$. If $\lambda < \frac{1-h}{2-h}$ for $\lambda \in (0, 1)$, full information never performs worse than no information for information design problem with objective coefficients in* (2.8).

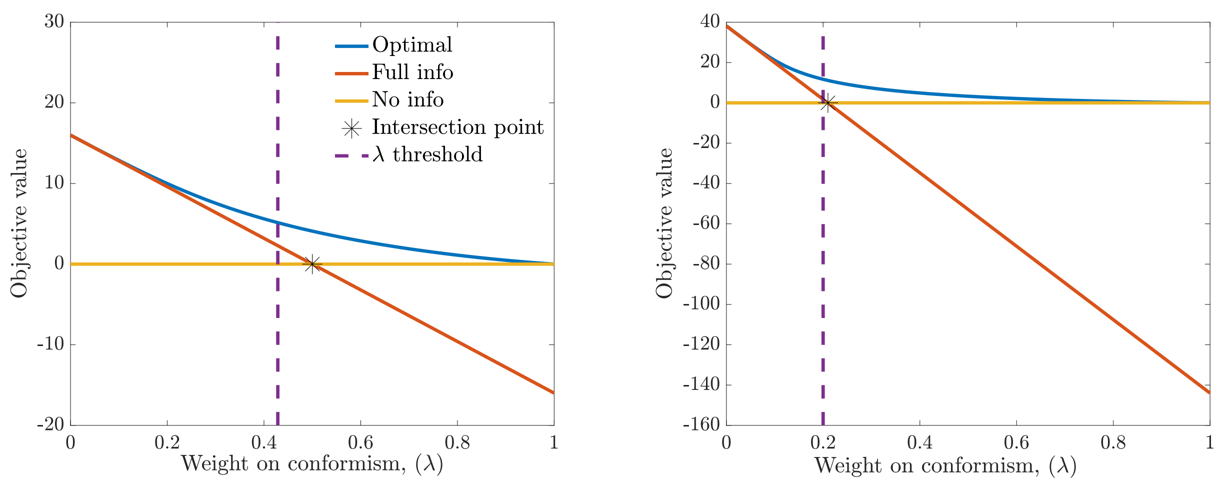
Proof. Following identical steps to Proposition 7, we obtain the matrix $E = [F]_{1,1} + [F]_{2,1} \circ H + [F]_{1,2} \circ H^T$ that provides $F \bullet X = E \bullet \text{var}(a)$. Substituting in the coefficients from (2.8), we simplify $E = \lambda[F^C]_{1,1} + (1-\lambda)H$.

First eigenvalue of E is equal to $[(n-1)h + 1](1-\lambda)$. The rest of the eigenvalues of E are equal to $-\lambda + (1-\lambda)(1-h)$. E is positive definite because both eigenvalues are greater than zero when $\lambda < \frac{1-h}{2-h}$.

If E is positive definite, then the objective value $E \bullet X_{11} = E \bullet \text{var}(a) \geq 0$. Thus, full information performs better or the same compared to no information disclosure. $\square$

Proposition 11. *Assume H has the form in (2.7) with $h \in (0,1)$. If $\lambda < \frac{1-h}{2-h}$ for $\lambda \in (0,1)$, then full information disclosure is optimal for the information design problem with objective coefficients given in (2.8) under the feasible set of public information structures.*

Proof. We know $F \bullet X = F_H \bullet \text{var}(\gamma)$ where F_H is given in Definition 4. Substituting in for the sub-matrices in (2.8), we have $F_H = (H^{-1})^T E H^{-1}$, where $E = \lambda[F^C]_{1,1} + (1-\lambda)H$ is as in Proposition 10. We know from the proof of Proposition 10 that E is positive definite for $\lambda < \frac{1-h}{2-h}$. Thus, full information disclosure is optimal for public information structures by the fact that F_H is positive semi-definite and by Proposition 5. $\square$



(a) $h = 0.25$ (b) $h = 0.75$

Figure 2.3: Objective values for optimal, full, no information disclosure under varying weights $\lambda \in [0,1]$. Optimal information disclosure is obtained by solving (1.16)-(1.18) under general information structures. The game payoff coefficients H is as in (2.7) with $h \in \{0.25, 0.75\}$. Let $\text{var}(\gamma)$ be such that $\text{var}(\gamma)_{i,i} = 4$ for $i \in N$ and $\text{var}(\gamma)_{i,j} = 1$ for $i \in N$ and $j \in N \setminus \{i\}$. Full information disclosure is preferred over no information disclosure for larger weight values λ than the λ threshold given in Proposition 11 (dashed line) (©2023 IEEE).

Propositions 10 and 11 specify the threshold for λ below which social welfare dominates the agreement term so that no information disclosure can no longer be optimal. It is worth noting that the λ threshold for the superiority of full information disclosure are identical in both results because we can reduce the objective function evaluation in Proposition 10 and the optimality condition in Proposition 11 to positive definiteness evaluation of E. According to the threshold $\lambda < \frac{1-h}{2-h}$, the region in which no information disclosure is not optimal increases to $\lambda \in (0, 0.5)$ as $h \to 0^+$. The region in which no information disclosure is not optimal shrinks to $\lambda = 0$ as $h \to 1$.

That is, as the dependence of players' payoffs on others' actions increases, no information disclosure can no longer be ruled out as sub-optimal, unless social welfare maximization is the objective of the designer, i.e., $\lambda = 0$.

Next, we assess the tightness of the threshold for λ, and the optimality of no and full information disclosures for the class of general information structures in a numerical example.

Numerical example: Fig. 2.3 shows that the region for the weight λ where full information information disclosure is preferable by the information designer over no information disclosure under public information structures is larger than the region given by the condition $\lambda < \frac{1-h}{2-h}$. The gap between the analytical threshold (dashed line) and the numerical threshold (shown by $*$) for λ decreases as h increases. Fig. 2.3 also shows the optimal value achieved by solving the information design problem under general information structures. We observe that general information structures that send partial signals to players perform better than no and full information disclosure for $\lambda \in (0, 1)$.

As mentioned in Section 2.3.2, Cournot competition is submodular. Fig. 2.3 indicates that as $h = \frac{\varrho}{2\varpi}$ decreases, i.e., the value of information disclosure increases. In other words, in settings where competition is fierce, hiding information is preferred when agreement is a design factor.

3. INFORMATION PREFERENCES OF INDIVIDUAL AGENTS IN LINEAR-QUADRATIC-GAUSSIAN NETWORK GAMES

3.1 Introduction

The information design problem ensures that optimal solutions satisfy equilibrium constraints so that agents will be in obedience to optimal information structures in expectation. Because rationality is defined as maximization of expected utility, agents will comply to the proscribed equilibrium actions to them in any information structure which satisfies equilibrium constraints.

However, agents could obtain different utility values under different information structures. This raises the question that what affects agents' preferences towards information structure. In this chapter, this question is answered in the context of LQG network games under the consideration of agents' position in the network for analytical and numerical studies and of their level of risk aversion in numerical studies. The results in this chapter appear in [31] [1].

3.2 Information Design in LQG Network Games

We consider LQG network games where the nodes are the players $\mathcal{N}$, and edges $\mathcal{E}$ determine the payoff dependencies, i.e., if $(i, j) \notin \mathcal{E}$ then $H_{ij} = 0$, otherwise $H_{ij} \in \mathbb{R}$ for $(i, j) \in \mathcal{E}$. Next, we provide an example.

Example 4 (Bertrand Competition in Networked Markets)**.** *Firms determine the price for their goods (a_i) facing a marginal cost of production (γ_i). Firms compete over markets that are connected [50]. The demand is a function of the price of all the firms, $q_i = \vartheta - \varpi a_i + \varrho \sum_{j \neq i} a_j$ with positive constants ϑ, ϖ and ϱ.*

Firm i's profit is its revenue $q_i a_i$ minus the cost of production $\gamma_i q_i$,

$$u_i(a, \gamma) = q_i a_i - \gamma_i q_i. \tag{3.1}$$

[1] Part of this chapter is reprinted with permission from F. Sezer and C. Eksin, "Information preferences of individual agents in linear-quadratic-gaussian network games," IEEE Control Systems Letters, vol. 6, pp. 3235–3240, 2022, ©2022 IEEE.

Nodes of networks correspond to a firm in Bertrand competition. If two nodes share an edge, they compete over the same market. For a star network, the central node can be a multinational firm competing with local competitors (peripheral nodes).

3.3 Social Welfare Maximization via Information Design

Social welfare is the sum of agents' (quadratic) utilities:

$$f(a, \boldsymbol{\gamma}) = \sum_{i=1}^{n}(-H_{ii}a_i^2 - 2\sum_{j\neq i} H_{ij}a_i a_j + 2\gamma_i a_i + d_i(a_{-i}, \boldsymbol{\gamma})). \tag{3.2}$$

Given quadratic utilities and Gaussian information structure, the information design problem (1.2) is transformed to the maximization of a linear function of a positive semi-definite covariance matrix ($X = \text{cov}(a, \boldsymbol{\gamma})$) subject to linear constraints stemming from the BNE condition in (1.1). That is, the information design problem is a semi-definite program (SDP) given in (1.16)-(1.18).

Using this SDP formulation, it is shown in Chapter 2 that *full information disclosure*, i.e., signals that reveal the payoff state, is an optimal strategy for the information designer under common payoff states (Proposition 7 in chapter 2) and public information structures (Proposition 8 in chapter 2).

We interpret the results in Propositions 7-8 for network games where $H_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. According to Proposition 7, full information disclosure is optimal given a common payoff state and symmetric H. A common payoff state corresponds to a common marginal cost for firms in Bertrand competition. This result implies that each firm should receive a fully informed signal on the marginal cost to maximize the social welfare.

Proposition 8 implies that if H is diagonally dominant, then full information disclosure is optimal for public signal structures. In the context of Bertrand competition, this result implies that if firms receive the same signal on the cost of their production, it is preferable to reveal the realized cost of production.

Next, we analyze the ex-ante information structure preferences of individual agents based on their position in the network.

3.4 Ex-ante Information Structure Preferences of Agents based on Network Structure

When there is a common payoff state γ, i.e., $\gamma_i = \gamma$, for $i \in \mathcal{N}$ and public signals $\omega_i = \bar{\omega}$ for $i \in \mathcal{N}$, individual equilibrium actions under full and no information disclosure are given by $a_i = \gamma[H^{-1}\mathbf{1}]_i$ and $a_i = \mu[H^{-1}\mathbf{1}]_i$, respectively for $i \in \mathcal{N}$ where $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones and $[\cdot]_i$ represents the ith element of a vector—see Appendix B.1 for the derivation. In this section, we treat the actions as random variables where we assume $\gamma \sim \psi(\mu, \sigma^2)$ and $\mu \sim \psi(\mu_0, \sigma_0^2)$.

Theorem 2. *Consider an LQG network game with common payoff state γ and public information structures. Define*

$$V_i(H) := [H^{-1}\mathbf{1}]_i \bigg(2 - H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j \neq i} H_{ij}[H^{-1}\mathbf{1}]_j\bigg). \tag{3.3}$$

If $V_i(H) > 0$, then full information disclosure is preferable by agent $i \in \mathcal{N}$ over no information disclosure.

Proof. If agent i's expected utility given full information disclosure is larger than its expected utility at no information disclosure, then full information disclosure is preferable. We start with computing agent i's expected utility under full information disclosure by plugging in the equilibrium action profile $a = \gamma H^{-1}\mathbf{1}$ (see Lemma 3 in appendix B.1) into (1.3):

$$E[u_i(a, \gamma)] = E[\gamma^2][H^{-1}\mathbf{1}]_i \bigg(2 - H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j \neq i} H_{ij}[H^{-1}\mathbf{1}]_j\bigg) + E[d_i(a_{-i}, \gamma)] \tag{3.4}$$

Next, we plug in the equilibrium action profile for no information disclosure $a = \mu H^{-1}\mathbf{1}$ (see Lemma 3) into (1.3):

$$E[u_i(a, \gamma)] = [H^{-1}\mathbf{1}]_i \bigg(E[\mu^2](-H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j \neq i} H_{ij}[H^{-1}\mathbf{1}]_j) + 2E[\gamma\mu]\bigg) + E[d_i(a_{-i}, \gamma)] \tag{3.5}$$

We subtract (3.5) from (3.4) to obtain the difference between expected utilities under full information and no information:

$$E[\Delta u_i(a,\gamma)] = [H^{-1}\mathbf{1}]_i\bigg(E[\gamma^2-\mu^2](-H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j\neq i} H_{ij}[H^{-1}\mathbf{1}]_j) + 2E[\gamma^2-\gamma\mu]\bigg) \tag{3.6}$$

$$= \sigma^2[H^{-1}\mathbf{1}]_i\bigg(2 - H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j\neq i} H_{ij}[H^{-1}\mathbf{1}]_j\bigg). \tag{3.7}$$

To get the second equality, we use $E[\mu^2]-\mu_0^2=\sigma_0^2$, $E[\gamma^2]=\sigma^2+\sigma_0^2+\mu_0^2$ and $E[\gamma\mu]=\sigma_0^2+\mu_0^2$ given that $\gamma\sim\psi(\mu,\sigma^2)$ and $\mu\sim\psi(\mu_0,\sigma_0^2)$. If $E[\Delta u_i(a,\gamma)]>0$, full information is preferred. The condition $E[\Delta u_i(a,\gamma)]>0$ is equivalent to $V_i(H)>0$ by the fact that $\sigma^2>0$. □

3.4.1 Information Structure Preferences under Star Network

We showcase Theorem 2 by applying to LQG games over star networks. A star network is comprised of a central agent ($i=1$) and $n-1$ peripheral agents ($j\in\mathcal{N}\backslash\{1\}$). We derive conditions for information structure preferences of both the central and peripheral agents in homogeneous games.

Definition 7 (Homogeneous LQG games)**.** *An LQG network game with a payoff coefficients matrix where $H_{ii}=1$ and $H_{ij}=\beta$, for $(i,j)\in\mathcal{E}$, and $\beta\in\mathbb{R}$ is homogeneous.*

Proposition 12. *If the LQG game is homogeneous and $(n-1)|\beta|<1$, then full information disclosure is preferred over no information disclosure by both the central and peripheral agents in the star network.*

Proof. We can compute $[H^{-1}\mathbf{1}]_i$ in close form for star networks,

$$[H^{-1}\mathbf{1}]_i = \frac{|\mathcal{N}_i|\beta-1}{(n-1)\beta^2-1} \text{ for } i\in\mathcal{N}, \tag{3.8}$$

where $\mathcal{N}_i:\{j:(i,j)\in\mathcal{E}\}$ denotes the neighbors of agent i, and $|\mathcal{N}_i|$ denotes its cardinality. We check the condition $V_i(H)>0$ for the central agent, say $i=1$, by substituting in (3.8),

$|\mathcal{N}_1| = n-1$ and $|\mathcal{N}_j| = 1$ for $j \in \mathcal{N} \setminus \{1\}$,

$$\frac{(n-1)\beta - 1}{(n-1)\beta^2 - 1}\left(2 - \frac{(n-1)\beta - 1 + 2(n-1)\beta(\beta-1)}{(n-1)\beta^2 - 1}\right) > 0. \tag{3.9}$$

We simplify (3.9) to get $((n-1)\beta-1)((n-1)\beta-3) > 0$. Given that $(n-1)\beta < 1$, the inequality is always true. Thus, full information disclosure is always preferable to no information disclosure by the central agent.

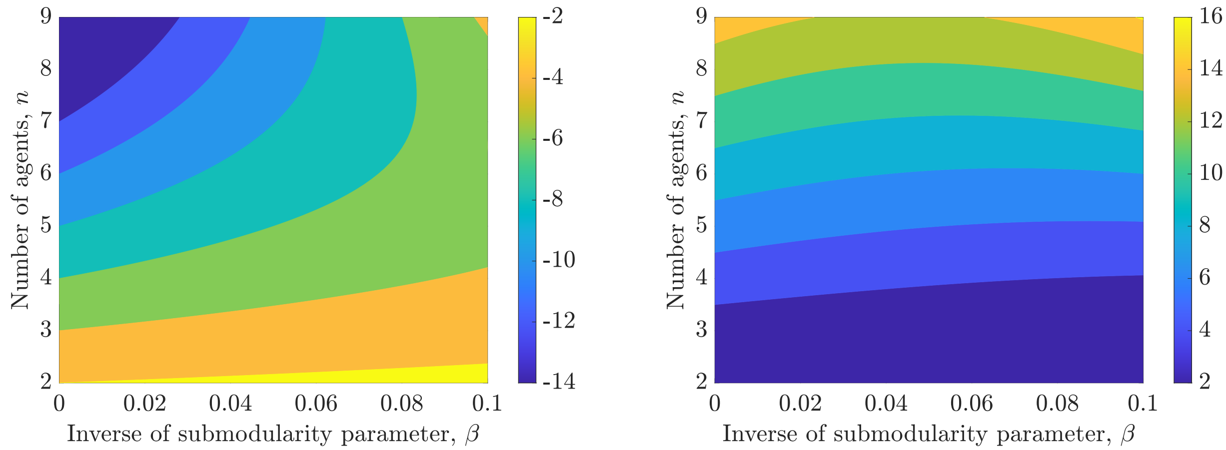
Now we consider peripheral agents $j \in \mathcal{N} \setminus \{1\}$. We check the condition $V_i(H) > 0$ for a peripheral agent by substituting in (3.8), $|\mathcal{N}_1| = n-1$, and $|\mathcal{N}_j| = 1$ for $j \in \mathcal{N} \setminus \{1\}$:

$$\frac{\beta - 1}{(n-1)\beta^2 - 1}\left(2 - \frac{\beta(\beta-1) + 2((n-1)\beta - 1)}{(n-1)\beta^2 - 1}\right) > 0. \tag{3.10}$$

(3.10) simplifies to $(\beta-1)^2 > 0$ which is always satisfied. This means $E[\Delta u_i(a,\gamma)]$ is always positive. Therefore, full information disclosure is always preferable over no information disclosure by the peripheral agents. □

This result shows that all agents regardless of their position in the star network are expected to benefit from information disclosure. We analyze the change in the value of information as a function of competition and number of players in homogeneous games in Fig. 3.1. We note that for homogeneous games $V_i(H) = V_i(\beta, n)$, and $V_i(\beta, n)$ is given by (3.9) and (3.10), respectively for central and peripheral agents. We observe that $V_i(\beta, n)$ is a decreasing function for the central agent while it is an increasing function for a peripheral agent with respect to β. Also, $\frac{\partial V_i(\beta,n)}{\partial \beta}$ decreases further as β decreases or the number of agents increases for the central agent. In contrast, $\frac{\partial V_i(\beta,n)}{\partial \beta}$ is not affected much by a change in the value of β for a peripheral agent.

Next, we identify the region for β where the expected benefit of the information disclosure to a peripheral agent is higher than that of the central agent.



(a) Central agent

(b) Peripheral agent

Figure 3.1: Contour plot of $\frac{\partial V_i(\beta,n)}{\partial \beta}$ for central (a) and peripheral (b) agents under homogeneous payoff matrix H where $H_{ii} = 1$ and $H_{ij} = \beta$, for $(i, j) \in \mathcal{E}$, and $\beta \in \mathbb{R}$. $\frac{\partial V_i(\beta,n)}{\partial \beta} < 0$ for the central agent and $\frac{\partial V_i(\beta,n)}{\partial \beta} > 0$ for a peripheral agent (©2022 IEEE).

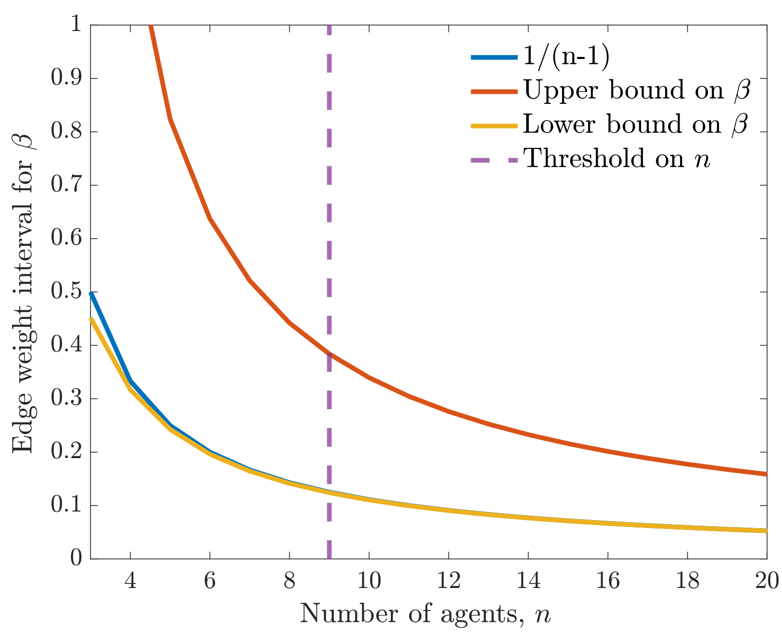


Figure 3.2: We plot (3.11) for number of agents from 3 to 20. We also plot positive definiteness condition we impose on β, i.e., $(n-1)\beta < 1$. Indeed, the positive definiteness line $(1/(n-1))$ crosses below the lower bound in (3.11) at $n > 9$, indicating that the central agent benefits more than a peripheral agent from information disclosure (©2022 IEEE).

Proposition 13. *If the LQG game is homogeneous and*

$$\frac{2(n-1)-\sqrt{\nu(n)}}{n(n-2)} < \beta < \frac{2(n-1)+\sqrt{\nu(n)}}{n(n-2)} \tag{3.11}$$

where $\nu(n) = n^2 - 2n + 4$*, then the gain of a peripheral agent from information disclosure is higher than the gain of the central agent. For* β *values outside interval* (3.11)*, the gain of the central agent is higher than that of a peripheral agent.*

Proof. We consider the difference between $E[\Delta u_1(a,\gamma)]$ in (3.9), i.e., central agent's benefit from information disclosure, and $E[\Delta u_j(a,\gamma)]$ in (3.10) for $j \in \mathcal{N} \setminus \{1\}$, i.e., a peripheral agent's benefit, to get

$$E[\Delta u_1(a,\gamma)] - E[\Delta u_j(a,\gamma)] = \frac{((n-1)\beta-1)((n-1)\beta-3)-(\beta-1)^2}{((n-1)\beta^2-1)^2} > 0. \tag{3.12}$$

We remove the positive valued denominator, and simplify the numerator to get

$$n(n-2)\beta^2 - 4(n-1)\beta + 3 > 0. \tag{3.13}$$

Solving quadratic inequality (3.13) indicates that when β is in the range given in (3.11), $E[\Delta u_1(a,\gamma)] - E[\Delta u_j(a,\gamma)] < 0$. Thus, a peripheral agent benefits more than the central agent from full information disclosure. The second part of the result follows from the fact that we have $E[\Delta u_1(a,\gamma)] - E[\Delta u_j(a,\gamma)] > 0$ for β values outside the interval (3.11). □

In Fig. 3.2, we plot the upper and lower bound values in (3.11) as a function of n. We observe the bounds get closer as n increases. When we contrast these bounds with the bound for positive-definiteness, i.e., $\beta < 1/(n-1)$, we observe that the upper bound is not realized for any β value. For $n > 9$, the positive definiteness condition implies that the lower bound cannot be exceeded. Thus, the central agent always benefits more than a peripheral agent for $n > 9$.

3.5 Ex-post Information Structure Preferences

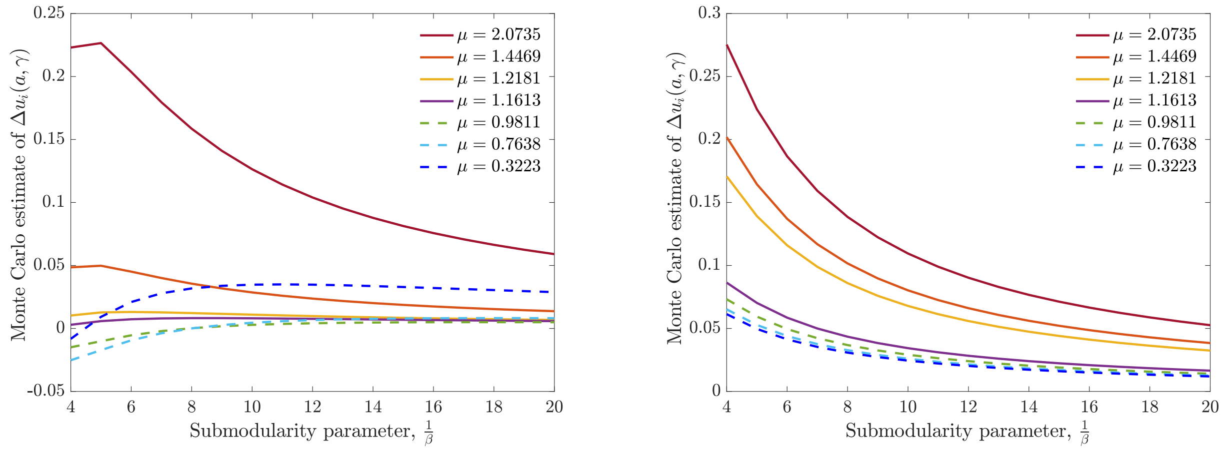
Depending upon the realizations of μ and γ, agents may prefer no information disclosure ex-post. We can say an agent prefers full information disclosure over no information disclosure if its change in the utility function from information disclosure $\Delta u_i(a,\gamma) > 0$, upon realization of μ and γ. We express $\Delta u_i(a,\gamma)$ as following by removing the expectation operator in (3.6),

$$\Delta u_i(a,\gamma) = (\gamma-\mu)[H^{-1}\mathbf{1}]_i\bigg((\gamma+\mu)(-H_{ii}[H^{-1}\mathbf{1}]_i - 2\sum_{j\neq i} H_{ij}[H^{-1}\mathbf{1}]_j) + 2\gamma\bigg). \quad (3.14)$$

We estimate (3.14) numerically via Monte Carlo simulation for homogeneous submodular ($\beta < 0$) and supermodular ($\beta > 0$) games. In submodular games, agents' actions are strategic substitutes, i.e., when agent j increases its action agent i's incentive to increase its action decreases ($\frac{\partial^2 u_i}{\partial a_i \partial a_j} < 0$). In supermodular games, agents' actions complement each other, i.e., when agent j increases its action, agent i's incentive to increase its action increases ($\frac{\partial^2 u_i}{\partial a_i \partial a_j} > 0$)—see [35, Section 3]. The Bertrand competition with payoffs in (3.1) is an example of a supermodular game.

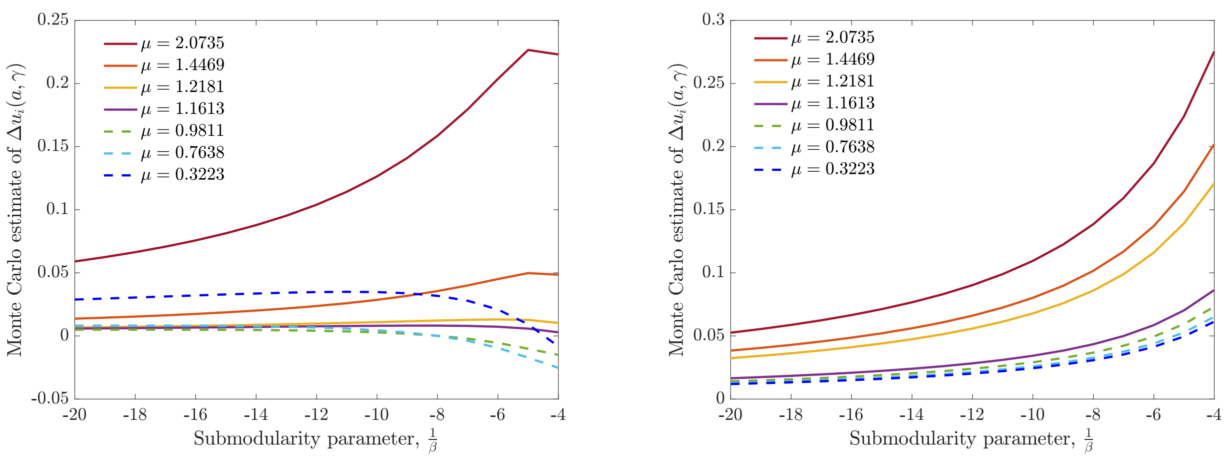
We compute $\Delta u_i(a,\gamma)$ for submodular and supermodular games in Figs. 3.3 and 3.4, respectively. In particular, we generate μ values from $\psi(\mu_0 = 1, 0.3^2)$, and γ values from $\psi(\mu, 0.1^2)$ where ψ denotes the normal distribution. We estimate $\Delta u_i(a,\gamma)$ for every combination of β and μ value by averaging over realizations of γ.

In both types of games, the average change in utility function over realizations of μ is positive indicating that information disclosure is preferable and confirming Proposition 12. The value of information decreases on average for both central and peripheral agents in both types of games as submodularity parameter $\frac{1}{|\beta|}$ increases. This is reasonable because the dependence of the payoffs on others' actions reduces as $|\beta|$ decreases. In both of the games, central agent prefers no information disclosure ex-post when realized μ is less than μ_0 and the absolute value of submodularity parameter is low (Figs. 3.3(a) and 3.4(a)). Otherwise, the central agent prefers full information disclosure ex-post. This indicates a risk-averse central agent may prefer no information disclosure ex-ante. For instance, a multinational company in a Bertrand competition with local firms may



(a) Central agent (b) Peripheral agent

Figure 3.3: Ex-post information preference estimates of central and peripheral agents in submodular games on a star network with $n = 4$. Lines show seven realized μ values generated from $\mu \sim \psi(\mu_0 = 1, 0.3^2)$. Dashed lines indicate $\mu < \mu_0$. Solid lines indicate $\mu > \mu_0$. For each μ and β value, 1000 γ values are generated from $\psi(\mu, 0.1^2)$. We estimate $\Delta u_i(a, \gamma)$ by averaging the values over γ realizations. For large β values, full information disclosure may not be preferred by the central agent when $\mu < \mu_0$ (©2022 IEEE).



(a) Central agent (b) Peripheral agent

Figure 3.4: Ex-post information preference estimates of central and peripheral agents in supermodular games on a star network with $n = 4$. Lines show seven realized μ values generated from $\mu \sim \psi(\mu_0 = 1, 0.3^2)$. Dashed lines indicate $\mu < \mu_0$. Solid lines indicate $\mu > \mu_0$. For each μ and β value, 1000 γ values are generated from $\psi(\mu, 0.1^2)$. We estimate $\Delta u_i(a, \gamma)$ by averaging the values over γ realizations. For large $|\beta|$ values, full information disclosure is not preferred by the central agent when $\mu < \mu_0$ (©2022 IEEE).

prefer that information remains hidden when the production costs are high and competition is stiff. In contrast, a peripheral agent always prefers full information disclosure regardless of the realized μ values (Figs. 3.3(b) and 3.4(b)).

4. ROBUST OPTIMIZATION APPROACH TO INFORMATION DESIGN IN LINEAR-QUADRATIC-GAUSSIAN GAMES

4.1 Introduction

Information design rests on the strong assumption that the designer knows agents' utilities in full. This assumption ignores the privacy of and possible errors in utilities. To overcome this issue in various settings, this chapter develops robust optimization models to perform information design when there are uncertain utilities. Robust optimization models are built for utilities under ellipsoid, interval and conic perturbations.

Robust optimality conditions for full and no information disclosures are derived for LQG games under ellipsoid perturbations. Numerical studies analyze the relation between optimal information structure and the level of uncertainty for LQG games under ellipsoid perturbations with welfare design objective. The results in this chapter appear in [41].

4.2 Generic Robust Information Design Problem

An incomplete information game involves a set of n players belonging to the set $\mathcal{N} := \{1, \ldots, n\}$, each of which selects actions $a_i \in \mathcal{A}_i$ to maximize the expectation of its individual payoff function $u_i^\theta(a, \gamma)$ where $a \equiv (a_i)_{i\in\mathcal{N}} \in \mathcal{A}$ is the action profile, $\gamma \equiv (\gamma_i)_{i\in\mathcal{N}} \in \Gamma$ is the payoff state vector, and $\theta \in \Theta$ is a payoff parameter. Players know the payoff parameter θ, but they do not know the payoff state γ. Player i forms expectation about the payoff state γ based on the prior on the state ψ and its signal/type $\omega_i \in \Omega_i$.

The information designer does not know the payoff parameter θ, but is more informed about the payoff state γ than the players. Specifically, given θ an information designer aims to maximize a system level objective function $f^\theta : \mathcal{A} \times \Gamma \to \mathbb{R}$, e.g., social welfare, that depends on the actions of the players (a), and the state realization (γ) by deciding on an information structure ζ belonging to the set of probability distributions over the signals $\mathcal{Z}$. That is, ζ is a conditional probability on the signals $\{\omega_i\}_{i\in\mathcal{N}}$ given the payoff state vector γ, i.e., $(P(\omega \mid \gamma))$ belonging to the space of all such conditional probability distributions $\mathcal{Z}$. The information structure determines the fidelity of signals $\{\omega_i\}_{i\in\mathcal{N}}$ that will be revealed to the players given a realization of the payoff state vector γ.

We represent the incomplete information game given $\theta \in \Theta$ and a prior ψ on the state γ by the tuple $G_\theta := \{\mathcal{N}, \mathcal{A}, \Gamma, \{u_i^\theta\}_{i\in\mathcal{N}}, \{\omega_i\}_{i\in\mathcal{N}}, \zeta, \psi\}$. We use $\mathcal{G}_\Theta := \{G_\theta : \theta \in \Theta\}$ to refer to the set of possible games.

A strategy of player i maps each possible value of the private signal $\omega_i \in \Omega_i$ to an action $s_i(\omega_i) \in \mathcal{A}_i$, i.e., $s_i : \Omega_i \to \mathcal{A}_i$. A strategy profile $s = (s_i)_{i\in\mathcal{N}}$ is a Bayesian Nash equilibrium (BNE) with information structure ζ of the game G_θ, if it satisfies the following inequality

$$E_\zeta[u_i^\theta(s_i(\omega_i), s_{-i}, \gamma)|\omega_i] \geq E_\zeta[u_i^\theta(a_i', s_{-i}, \gamma)|\omega_i], \tag{4.1}$$

for all $a_i' \in \mathcal{A}_i, \omega_i \in \Omega_i, i \in \mathcal{N}$, and $s_{-i} = (s_j(\omega_j))_{j\neq i}$ is the equilibrium strategy of all the players except player i, and E_ζ is the expectation operator with respect to the distribution ζ and the prior ψ. We denote the set of BNE strategies in a game G_θ with $BNE(G_\theta)$.

In this paper, the designer does not make any distributional assumptions on the payoff parameter θ, and aims to select the best signal distribution for the worst case scenario, i.e.,

$$\min_{\theta\in\Theta} \max_{\zeta\in\mathcal{Z}} E_\zeta[f^\theta(s, \gamma)] \quad \text{s.t.} \;\; s \in BNE(G_\theta). \tag{4.2}$$

The outer optimization problem in (4.2) evaluates to the designer's objective under the worst possible payoff parameter realization, and BNE actions given a signal distribution ζ. The designer wants to do the best it can to maximize the system objective assuming the realization of the worst-case scenario.

We denote the optimal solution to (4.2) by ζ^*. Given the robust optimal information structure ζ^*, the information design timeline is given in the following:

1. Designer notifies players about ζ^*

2. Realization of the payoff parameter θ and payoff state γ, and with subsequent draw of signals w_i, $\forall i \in \mathcal{N}$ from $\zeta^*(\omega, \gamma)$

3. Players take action according to BNE strategies under information structure ζ^* in game G_θ.

The generic robust information design problem in (4.2) is not tractable in general. We also note that the information design problem is not a Stackelberg (leader-follower) game, since the players are not strategic against the designer's strategy and objective [1].

In the following we make assumptions on the payoff structure and the signal distribution to attain a tractable formulation.

4.2.1 Linear-Quadratic-Gaussian (LQG) Games

An LQG game corresponds to an incomplete information game with quadratic payoff functions and Gaussian information structures. Specifically, each player $i \in \mathcal{N}$ decides on his action $a_i \in \mathcal{A}_i \equiv \mathbb{R}$ according to a payoff function

$$u_i^\theta(a, \gamma) = -H_{i,i}a_i^2 - 2\sum_{j \neq i} H_{i,j}a_i a_j + 2\gamma_i a_i + d_i(a_{-i}, \gamma) \tag{4.3}$$

where $\mathcal{A} \equiv \mathbb{R}^n$ and $\Gamma \equiv \mathbb{R}^n$ that is a quadratic function of player i's action, and is bilinear with respect to a_i and a_j, and a_i and γ. The term $d_i(a_{-i}, \gamma)$ is an arbitrary function of the opponents' actions $a_{-i} \equiv (a_j)_{j \neq i}$ and payoff state γ. We collect the coefficients of the quadratic payoff function in a matrix $H = [H_{i,j}]_{n \times n}$. The payoff parameter θ, unknown to the designer in (4.3), is the coefficients matrix H, i.e. $\theta \equiv H$.

Payoff state γ follows a Gaussian distribution, i.e., $\gamma \sim \psi(\mu, \Sigma)$ where ψ is a multivariate normal probability distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ. Each player $i \in \mathcal{N}$ receives a private signal $\omega_i \in \Omega_i \equiv \mathbb{R}$. We define the information structure of the game $\zeta(\omega|\gamma)$ as the conditional distribution of $\omega \equiv (\omega_i)_{i \in N}$ given γ. We assume the joint distribution over the random variables (ω, γ) is Gaussian; thus, ζ is a Gaussian distribution.

Next we state the main structural assumption on the unknown payoff parameter H of the LQG game.

Assumption 1. *We assume the following affine perturbation structure on the payoff matrix H,*

$$H_{i,j} = [H_0]_{i,j} + v_{i,j}\epsilon_{i,j}, \quad \forall i, j \in \mathcal{N} \tag{4.4}$$

where H_0 is the nominal payoff matrix, $v_{i,j} \in \mathbb{R}$, is an element of the unknown perturbation matrix $v \in \mathbb{R}^{n \times n}$ which covers a given closed and convex perturbation set $\mathcal{V}$ such that $0 \in \mathcal{V}$ and $\epsilon_{i,j}$ is the constant shift.

We note that while the actual payoff parameters H are unknown to the designer, they are known by the players. The designer only knows the nominal payoff matrix H_0, potentially obtained from past data.

4.2.2 From signal to action distributions

We will reformulate the problem in (4.2) in order to obtain a tractable formulation. The reformulation will first entail changing the design variables from signals to actions. We define the distribution of actions induced by the information structure under a given strategy profile as follows.

Definition 8 (Action distribution)**.** *An action distribution is the probability of observing an action profile $a \in \mathcal{A}$ when players follow a strategy profile s under ζ, which can be computed as*

$$\phi(a|\gamma) = \sum_{\omega: s(\omega)=a} \zeta(\omega|\gamma). \tag{4.5}$$

According to the definition, the probability of observing the action profile a is the sum of the conditional probabilities of all signal profiles ω under ζ that induce action profile a given the strategy profile s.

Definition 9 (*Equilibrium* action distribution set)**.** *The set of equilibrium action distributions induced by BNE strategies under an information structure $\zeta \in \mathcal{Z}$ for game G_θ is*

$$C_\theta(\zeta) = \{\phi : \phi \textit{ satisfies } (4.5) \textit{ for } s \in BNE(G_\theta) \textit{ given } \zeta \in \mathcal{Z}\}. \tag{4.6}$$

We begin by stating the BNE condition in (4.1) by a set of linear constraints for LQG games given the payoff matrix H.

Lemma 1. *Define the covariance matrix $X \in P_+^{2n}$ as follows:*

$$X := \begin{bmatrix} var(a) & cov(a,\gamma) \\ cov(\gamma,a) & var(\gamma) \end{bmatrix}. \tag{4.7}$$

For a given payoff matrix H such that $H + H^T$ is positive definite, the BNE condition in (4.1) *can be written as the following set of equality constraints,*

$$\sum_{j\in\mathcal{N}} H_{i,j}X_{i,j} - X_{i,n+i} = 0, \quad i \in \mathcal{N} \tag{4.8}$$

where $X_{i,j} = cov(a_i, a_j)$ for $i \le n$, and $j \le n$, and $X_{i,n+i} = cov(a_i, \gamma_i)$.

Proof. See Appendix C.1. □

The condition in (4.8) ensures that X is a Bayesian correlated equilibrium (BCE)—see [3] for a definition. When θ is known, we can state the designer's maximization problem in (4.2) as the determination of an action distribution subject to the constraint that actions belong to $C_\theta(\zeta)$, i.e., $\max_{\phi\in C_\theta(\zeta)} E_\phi[f(a,\gamma)]$. Indeed, we can state the design problem as a SDP using X in (4.7) as the decision variable, subject to the BCE constraints in (4.8)—see [21]. In such a case, the players would not benefit from deviating from the recommended actions because they would satisfy the obedience condition as per the revelation principle, see [3, Proposition 1]. However, this principle does not apply in the setting where θ is chosen adversarially. Next, we address this issue in the finite scenario and ellipsoid, interval and conic perturbations settings.

4.3 Robust Information Design under Finite Scenarios

In the following, we express the robust information design problem under a finite set of scenarios as a mixed integer SDP.

Theorem 3 (Finite-case)**.** *Suppose Assumption 1 holds, and assume the design objective coefficients do not depend on H. Let the design objective $f^\theta(a,\gamma)$ be quadratic in its arguments with the coefficients stored in matrix $F \in \mathbb{R}^{2n\times 2n}$, i.e., $f^\theta(a,\gamma) = [a\ \gamma]^T F [a\ \gamma]$. Consider a finite perturbation vector with C scenarios, and let $v_c \in \mathbb{R}^{n\times n}$ refer to perturbation vectors corresponding to one of the scenarios $c \in \mathcal{C} = \{1,\ldots,C\}$. We relax the robust information design problem in* (4.2) *as the following mixed-integer SDP:*

$$\min_{y_c\in\{0,1\},c\in\mathcal{C}}\ \max_{X\in P_+^{2n}}\ F\bullet X \tag{4.9}$$

$$\textit{s.t.}\quad \sum_{c=1}^{C} y_c = 1, \tag{4.10}$$

$$y_c(R_{0,l}\bullet X + \sum_{(i,j)\in\mathcal{Y}_l}[v_c]_{i,j}\epsilon_{i,j}X_{i,j}) = 0, \forall l\in\mathcal{N}, c\in\mathcal{C} \tag{4.11}$$

$$M_{k,l}\bullet X = cov(\gamma_k,\gamma_l),\quad \forall k,l\in\mathcal{N}\ \textit{with}\ k\le l, \tag{4.12}$$

where X is defined in (4.7)*, $R_{0,l}\in P^{2n}, l\in\mathcal{N}$ is given as:*

$$[R_{0,l}]_{i,j} = \begin{cases} [H_0]_{l,l} & if \quad i=j=l, \\ [H_0]_{l,j}/2 & if \quad i=l, 1\le j\le n, j\ne l, \\ -1/2 & if \quad i=l, j=n+l, \\ [H_0]_{i,l}/2 & if \quad j=l, 1\le i\le n, i\ne l \\ -1/2 & if \quad j=l, i=n+l, \\ 0 & \textit{otherwise}, \end{cases} \tag{4.13}$$

and $M_{k,l} \in P^{2n}$ is given as:

$$[M_{k,l}]_{i,j} = \begin{cases} 1/2 & \textit{if } k < l, i = n+k, j = n+l, \\ 1/2 & \textit{if } k < l, i = n+l, j = n+k, \\ 1 & \textit{if } k = l, i = n+k, j = n+l, \\ 0 & \textit{otherwise,} \end{cases} \tag{4.14}$$

and $\mathcal{Y}_l$ refer to the elements of the perturbation vector with

$$\mathcal{Y}_l := \{\{i,j\} : i = j = l \vee i = l, 1 \leq j \leq n, j \neq l \vee j = l, 1 \leq i \leq n, i \neq l\}. \tag{4.15}$$

Proof. We can express the expected objective using the Frobenius product as follows,

$$E_\phi[f(a,\gamma)] = E_\phi\Big[\begin{bmatrix} a^T, & \gamma^T \end{bmatrix} F \begin{bmatrix} a \\ \gamma \end{bmatrix}\Big] = F \bullet X \tag{4.16}$$

where $F = \begin{bmatrix} [F]_{1,1} & [F]_{1,2} \\ [F]_{1,2} & [F]_{2,2} \end{bmatrix} \in P^{2n}$, and note that $[F]_{i,j}$ denotes the i,jth $n \times n$ submatrix.

Let c^* be the worst-case scenario from the perspective of the designer. The designer chooses X^* that maximizes its objective $F \bullet X$ subject to rational behavior of players in the worst case scenario. As per Lemma 1, we have

$$\sum_{j \in \mathcal{N}} H_{i,j} X^*_{i,j} - X^*_{i,n+i} = 0, \quad \forall i \in \mathcal{N} \tag{4.17}$$

$$\sum_{j \in \mathcal{N}} ([H_0]_{i,j} + [v_{c^*}]_{i,j}\epsilon_{i,j}) X^*_{i,j} - X^*_{i,n+i} = 0, \forall i \in \mathcal{N}. \tag{4.18}$$

We rewrite (4.18) in terms of matrices $R_{0,l}, \forall l \in \mathcal{N}$ as in (4.13) and X as in (4.7) to obtain (4.11). Minimization over $y_c, \{1, 2, .., C\}$ enforces the constraint c^* among the set of constraints in (4.11) to be selected. Constraint (4.12) corresponds to the assignment of $var(\gamma)$ to $[X]_{2,2}$. Constraint

(4.12) is not affected by perturbations to H. □

According to the formulation in (4.9)-(4.12), the solution entails finding the covariance matrix X that maximizes $F \bullet X$ for the worst possible scenario. That is, the solution X does not necessarily satisfy the BCE constraints for every scenario. We note that an alternative equivalent formulation can entail C covariance matrices, i.e., $X_1, \ldots, X_C$, and leave out the integer variables $\{y_c\}_{c=1,\ldots,C}$.

We use the scenario-based formulation (4.9)-(4.12) to motivate the tractable robust design formulations under ellipsoid uncertainty set. For illustration purposes, consider $C = 2$ scenarios. Assume $c = 1$ is the worst case scenario, i.e., $y_1 = 1$ and $y_2 = 0$. In such a case, X^* will satisfy the BNE condition (4.18) for $c = 1$ exactly, while the BCE condition will be approximately satisfied for $c = 2$. Specifically, we have

$$\sum_{j\in\mathcal{N}}([H_0]_{i,j} + [v_2]_{i,j}\epsilon_{i,j})X^*_{i,j} - X^*_{i,n+i} =$$

$$\sum_{j\in\mathcal{N}}([H_0]_{i,j} + [v_2]_{i,j}\epsilon_{i,j} + [v_1]_{i,j}\epsilon_{i,j} - [v_1]_{i,j}\epsilon_{i,j})X^*_{i,j} - X^*_{i,n+i} \tag{4.19}$$

$$= \sum_{j\in\mathcal{N}}([v_2]_{i,j}\epsilon_{i,j} - [v_1]_{i,j}\epsilon_{i,j})X^*_{i,j} > 0, \quad \forall i \in \mathcal{N}. \tag{4.20}$$

We can interpret this relation as the optimal solution X^* being an approximate BNE for the good scenario $c = 2$.

Remark 4. *The standard robust optimization problem in* (4.2) *requires that X^* is feasible for every $\theta \in \Theta$. The formulation for this problem would entail getting rid of the integer variables from the formulation in* (4.9)-(4.12)*, i.e., $y_c = 1$ for each* (4.11) *and removing* (4.10)*. This formulation may restrict the feasibility region drastically, as is often the issue with robust optimization problems with equality constraints [51].*

4.4 Robust Welfare Maximizing Information Design under Ellipsoid Uncertainty

We assume the following ellipsoidal structural form for the perturbation vectors in (4.4) that affect the BCE constraints, for $l \in \mathcal{N}$,

$$\mathcal{V}_l = \text{Ball}_\rho = \{v : ||v_l||_2 \leq \rho, v_l = \{[v_l]_{i,j}\}_{\{i,j\} \in \mathcal{Y}_l}\}, \tag{4.21}$$

where $\mathcal{Y}_l$ is given in (4.15). Under convex continuous uncertainty sets as the one above, the number of scenarios C is infinite. Thus, the formulation in Theorem 3 where we enforce BCE constraints in (4.8) exactly for the worst-case scenario, and annul the other cases using integer variables may not be viable. Moreover, enforcing the BCE constraints in (4.8) for all perturbations $v \in \mathcal{V}_l$ may limit the solution space drastically [51]. Instead, here we relax the BCE constraint in (4.8) as follows

$$|\sum_{j \in \mathcal{N}} H_{i,j} X_{i,j} - X_{i,n+i}| \leq \alpha, \quad i \in \mathcal{N} \tag{4.22}$$

where $\alpha \geq 0$ is a finite constant. This relaxation guarantees an approximate tractable solution to the information design problem in which the designer aims to maximize social welfare under ellipsoidal perturbations.

Theorem 4. *Consider the social welfare in* (2.1) *as the designer's objective* $f^\theta(a, \gamma)$*. Assume* H *is given by* (4.4) *and perturbation vectors* $v_l, \forall l \in \mathcal{N}$ *exhibit ellipsoid uncertainty* (4.21)*. Then the robust information design problem in* (1.2) *can be approximated by the following convex problem* *for* $\alpha \geq 0$*:*

$$\max_{X \in P_+^{2n}, t} \quad t \tag{4.23}$$

$$\text{s.t. } F_0 \bullet X - \frac{n^2\rho}{2n-1}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_{i,j}X_{i,j})^2} \geq t, \tag{4.24}$$

$$R_{0,l} \bullet X + \rho\sqrt{\sum_{(i,j)\in\mathcal{Y}_l}(\epsilon_{i,j}X_{i,j})^2} \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.25}$$

$$-R_{0,l} \bullet X + \rho\sqrt{\sum_{(i,j)\in\mathcal{Y}_l}(\epsilon_{i,j}X_{i,j})^2} \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.26}$$

$$M_{k,l} \bullet X = cov(\gamma_k, \gamma_l), \quad \forall k, l \in \mathcal{N} \text{ with } k \leq l, \tag{4.27}$$

where $F_0 = \begin{bmatrix} -H_0 & I \\ I & O \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$*, and the matrices* $R_{0,l}$ *and* $M_{k,l}$ *are as defined in* (4.13) *and* (4.14)*, respectively. Moreover, the optimal objective value for* (1.2) *is equal to* (4.23)-(4.27) *with* $\alpha = 0$.

Proof. We can express the social welfare objective in (5) in the form $F \bullet X$ with $F = \begin{bmatrix} -H & I \\ I & O \end{bmatrix}$– see [24]. We start by writing the social welfare objective as a constraint $F \bullet X \geq t$ under ellipsoid uncertainty:

$$F \bullet X = F_0 \bullet X + \sum_{i=1}^{n}\sum_{j=1}^{n} v_{i,j}\epsilon_{i,j}X_{i,j} \geq t \tag{4.28}$$

where t represents the designer's objective value. In the above summation, all elements of the perturbation matrix v are involved. Given the assumption of ellipsoid perturbations in (4.21), it is guaranteed that v is within a ball of radius $\frac{n^2\rho}{2n-1}$, i.e. $v \in \text{Ball}_{\frac{n^2\rho}{2n-1}}$. We can write (4.28) as a minimization problem that aims to find the worst case scenario:

$$\min_{||v|| \leq \frac{n^2\rho}{2n-1}} \sum_{i=1}^{n}\sum_{j=1}^{n} v_{i,j}\epsilon_{i,j}X_{i,j} \leq F_0 \bullet X - t \tag{4.29}$$

Solution to (4.29) is the tractable robust constraint given in (4.24) [51, Section 1.3]. Next, we substitute H in (4.4) into (4.22) to get,

$$|\sum_{j\in\mathcal{N}}([H_0]_{ij}+v_{ij}\epsilon_{ij})X_{i,j}-X_{i,n+i}|\leq\alpha\quad\forall i\in\mathcal{N},v\in\mathcal{V}_l \tag{4.30}$$

We can rewrite (4.30) in terms of matrices $R_{0,l},\forall l\in\mathcal{N}$ and X as in (4.7):

$$|R_{0,l}\bullet X+\sum_{(i,j)\in\mathcal{Y}_l}v_{i,j}\epsilon_{i,j}X_{i,j}|\leq\alpha,\quad\forall l\in\mathcal{N}. \tag{4.31}$$

We split the absolute value into two linear constraints (positive and negative sides). When we write the maximization problem over the uncertain constraint (4.31) for the positive side, we have

$$\max_{||v_l||\leq\rho}\sum_{(i,j)\in\mathcal{Y}_l}v_{i,j}\epsilon_{i,j}X_{i,j}\leq\alpha-R_{0,l}\bullet X,\quad\forall l\in\mathcal{N} \tag{4.32}$$

where $\mathcal{Y}_l$ is given by (4.15). Solution to (4.32) give us the tractable constraint (4.25) [51, Section 1.3]. Repeating the same steps for the negative side yields (4.26).

Constraint (4.27) enforces assignment of known covariance matrix of payoff states, $cov(\gamma)$ to the respective place in X. □

When $\alpha=0$, the formulation in (4.23)-(4.27) is equivalent to (4.2). It is easy to check that no information disclosure $X_{no}=\begin{bmatrix}O & O\\ O & \text{var}(\gamma)\end{bmatrix}$ is a feasible solution even when $\alpha=0$—see [21] for the derivation of X_{no} derivation. As noted, this formulation may be too restrictive. When $\alpha>0$, the incentive compatibility of the solution X^* is compromised, but the set of feasible solutions increases.

Given an optimal solution X^*, the designer can draw actions from a Gaussian distribution with mean 0 and covariance matrix X^*, and send these values to the players as signals.

4.5 Robust Information Design under Interval and Conic Uncertanties

We consider robust information design problem (4.2) under interval and conic uncertainties. We develop robust convex programs for the information design problem (4.2). In this section, we assume that F does not depend upon H. The extension to that case is similar to welfare maximization objective in ellipsoid uncertainty (see Theorem 4).

4.5.1 Robust Model under Interval Uncertainty

Interval uncertainty is defined by the box perturbation sets $\mathcal{V}_l, \forall l \in \mathcal{N}$:

$$\mathcal{V}_l = \text{Box}_1 \equiv \{v_l \in \mathbb{R}^{2n-1} : ||v_l||_\infty \leq 1\}, \quad \forall l \in \mathcal{N}. \tag{4.33}$$

Theorem 5. *Assume H is given by* (4.4) *and perturbation vectors $v_l, \forall l \in \mathcal{N}$ exhibit interval uncertainty* (4.33) *over constraint* (4.22) *and F does not depend on H. Then robust information design model is given as the following convex program:*

$$\max_{X \in P_+^{2n}} F \bullet X \tag{4.34}$$

$$\text{s.t.} \quad R_{0,l} \bullet X + \sum_{(i,j) \in \mathcal{Y}_l} \overline{u}_{i,j} \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.35}$$

$$- R_{0,l} \bullet X + \sum_{(i,j) \in \mathcal{Y}_l} \overline{u}_{ij} \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.36}$$

$$- \overline{u}_{i,j} \leq \epsilon_{i,j} X_{i,j} \leq \overline{u}_{i,j}, \quad \forall i, j \in \mathcal{N} \tag{4.37}$$

$$M_{k,l} \bullet X = cov(\gamma_k, \gamma_l), \quad \forall k, l \in \mathcal{N} \text{ with } k \leq l, \tag{4.38}$$

where $\overline{u}_{i,j}, \forall i, j \in \mathcal{N}$ denotes bounds on $\epsilon_{i,j} X_{i,j}, \forall i, j \in \mathcal{N}$.

Proof. The proof is the same as the proof of Theorem 4 up to (4.31). Now we write the perturbation maximization problem over the uncertain constraint (4.31) under interval uncertainty (see Example

1.3.2 in [51]):

$$\max_{-1 \le v_{ij} \le 1,\, \forall (i,j) \in \mathcal{Y}_l} \sum_{(i,j) \in \mathcal{Y}_l} v_{i,j} \epsilon_{i,j} X_{i,j} \le \alpha - R_{0,l} \bullet X, \quad \forall l \in \mathcal{N} \tag{4.39}$$

$$\max_{-1 \le v_{ij} \le 1,\, \forall (i,j) \in \mathcal{Y}_l} \sum_{(i,j) \in \mathcal{Y}_l} v_{ij} \epsilon_{ij} X_{ij} \le \alpha + R_{0,l} \bullet X, \quad \forall l \in \mathcal{N} \tag{4.40}$$

Solutions to (4.39)-(4.40) give us the robust constraints which include absolute value terms:

$$R_{0,l} \bullet X + \sum_{(i,j) \in \mathcal{Y}_l} |\epsilon_{i,j} X_{i,j}| \le \alpha \tag{4.41}$$

$$-R_{0,l} \bullet X + \sum_{(i,j) \in \mathcal{Y}_l} |\epsilon_{ij} X_{ij}| \le \alpha \tag{4.42}$$

We linearize absolute value terms in (4.41)-(4.42) and obtain tractable robust constraints (4.35)-(4.37). □

4.5.2 Robust Model under Conic Uncertainty

Consider the perturbation sets $\mathcal{V}_l, \forall l \in \mathcal{N}$ given by a *conic representation*:

$$\mathcal{V}_l = \{v_l \in \mathbb{R}^{2n-1} : \exists \pi_l \in \mathbb{R}^K : P_l v_l + Q_l \pi_l + p_l \in \boldsymbol{K}_l\}, \quad \forall l \in \mathcal{N} \tag{4.43}$$

where $\boldsymbol{K}_l$ is a closed convex pointed cone in $\mathbb{R}^N$ with a nonempty interior, P_l, Q_l are given matrices and p_l is a given vector. We assume that this representation is strictly feasible if $\boldsymbol{K}_l$ is not a polyhedral cone:

$$\exists (\overline{v}_l, \overline{\pi}_l) : P_l \overline{v}_l + Q_l \overline{\pi}_l + p_l \in \text{int}\boldsymbol{K}_l. \tag{4.44}$$

Theorem 6. *Assume H is given by* (4.4) *and perturbation vectors $v_l, \forall l \in \mathcal{N}$ exhibit conic uncertainty* (4.43) *over constraint* (4.22) *and F does not depend on H. Then robust information design* *model is given as the following convex program:*

$$\max_{X \in P_+^{2n}, y_l \in \mathbb{R}^N} \; F \bullet X \tag{4.45}$$

$$\textit{s.t.} \;\; p_l^T y_l + R_{0,l} \bullet X \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.46}$$

$$p_l^T y_l - R_{0,l} \bullet X \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.47}$$

$$Q_l^T y_l = 0, \quad \forall l \in \mathcal{N} \tag{4.48}$$

$$(P_l^T y_l)_{i,j} + \epsilon_{i,j} X_{i,j} = 0, \quad \forall (i,j) \in \mathcal{Y}_l, \forall l \in \mathcal{N} \tag{4.49}$$

$$y_l \in \boldsymbol{K}_{l*}, \quad \forall l \in \mathcal{N} \tag{4.50}$$

$$M_{k,l} \bullet X = cov(\gamma_k, \gamma_l), \quad \forall k, l \in \mathcal{N} \textit{ with } k \leq l, \tag{4.51}$$

where $\boldsymbol{K}_{l*} = \{y_l : y_l^T z \in \boldsymbol{K}_l\}$ *is the cone dual to* $\boldsymbol{K}_l$*.*

Proof. The proof is the same as the proof of Theorem 4 up to (4.31). We will show the equivalency of (4.31) to (4.46)-(4.50) in robust sense i.e under worst case perturbations (see Theorem 1.3.4 in [51]).

We start with the claim that X is feasible to (4.31). This is equivalent to following:

$$\sup_{v_l \in \mathcal{V}_l} \{R_{0,l} \bullet X - \alpha + \sum_{(ij) \in \mathcal{Y}_l} v_{i,j} \epsilon_{i,j} X_{i,j}\} \leq 0, \quad \forall l \in \mathcal{N} \tag{4.52}$$

$$\sup_{v_l \in \mathcal{V}_l} \{-R_{0,l} \bullet X - \alpha + \sum_{(ij) \in \mathcal{Y}_l} v_{ij} \epsilon_{ij} X_{ij}\} \leq 0, \quad \forall l \in \mathcal{N} \tag{4.53}$$

We take unaffected terms in (4.52)-(4.53) out of supremum and put to right hand side:

$$\sup_{v_l \in \mathcal{V}_l} \{\sum_{(ij) \in \mathcal{Y}_l} v_{i,j} \epsilon_{i,j} X_{i,j}\} \leq -R_{0,l} \bullet X + \alpha, \quad \forall l \in \mathcal{N} \tag{4.54}$$

$$\sup_{v_l \in \mathcal{V}_l} \{\sum_{(ij) \in \mathcal{Y}_l} v_{ij} \epsilon_{ij} X_{ij}\} \leq R_{0,l} \bullet X + \alpha, \quad \forall l \in \mathcal{N} \tag{4.55}$$

We substitute conic uncertainty set definition (4.43) into (4.54)-(4.55) for $\mathcal{V}_l, \forall l \in \mathcal{N}$:

$$\max_{v_l,\kappa_l}\{\sum_{(i,j)\in\mathcal{Y}_l} v_{i,j}\epsilon_{i,j}X_{i,j} : P_l v_l + Q_l\kappa_l + p_l \in \boldsymbol{K}_l\} \leq -R_{0,l} \bullet X + \alpha, \quad \forall l \in \mathcal{N} \tag{4.56}$$

$$\max_{v_l,\theta_l}\{\sum_{(ij)\in\mathcal{Y}_l} v_{ij}\epsilon_{ij}X_{ij} : P_l v_l + Q_l\theta_l + p_l \in \boldsymbol{K}_l\} \leq R_{0,l} \bullet X + \alpha, \quad \forall l \in \mathcal{N} \tag{4.57}$$

Conditions (4.56)-(4.57) indicate that X is feasible for (4.31) if and only if the optimal values in the following conic perturbation maximization problems

$$\max_{v_l,\kappa_l}\{\sum_{(i,j)\in\mathcal{Y}_l} v_{i,j}\epsilon_{i,j}X_{i,j} : P_l v_l + Q_l\kappa_l + p_l \in \boldsymbol{K}_l\}, \quad \forall l \in \mathcal{N} \tag{4.58}$$

$$\max_{v_l,\theta_l}\{\sum_{(ij)\in\mathcal{Y}_l} v_{ij}\epsilon_{ij}X_{ij} : P_l v_l + Q_l\theta_l + p_l \in \boldsymbol{K}_l\}, \quad \forall l \in \mathcal{N} \tag{4.59}$$

are less than or equal to $-R_{0,l} \bullet X + \alpha$ for (4.58) and $R_{0,l} \bullet X + \alpha$ for (4.59). We need to consider two cases: $\boldsymbol{K}$ is not a polyhedral cone (4.44) and $\boldsymbol{K}$ is a polyhedral cone.

When $\boldsymbol{K}$ is not a polyhedral cone, conic programs (4.58)-(4.59) are strictly feasible. Strong duality property of conic duality theorem states that if either primal or dual problems is strictly feasible and bounded, then the other problem is solvable and optimal value of primal objective is equal to optimal value of dual objective (Theorem A.2.1, [51]).

Strict feasibility of primal problem is satisfied because $\boldsymbol{K}$ is not a polyhedral cone. Bound property of primal problems is evident in (4.56)-(4.57) . Therefore, via strong duality property of conic duality, the optimal value in (4.58) is $\leq -R_{0,l} \bullet X + \alpha$ and the optimal value in (4.59) is $\leq R_{0,l} \bullet X + \alpha$ if and only if the optimal value in the conic dual program to the (4.58)-(4.59)

$$\min_{y_l}\{p_l^T y_l : Q_l^T y_l = 0, (P_l^T y_l)_{i,j} + \epsilon_{i,j}X_{i,j} = 0, y_l \in \boldsymbol{K}_{l*}, \forall(i,j) \in \mathcal{Y}_l, \forall l \in \mathcal{N}\} \tag{4.60}$$

is obtained and the optimal value is less than or equal to $\min\{R_{0,l} \bullet X + \alpha, -R_{0,l} \bullet X + \alpha\}$.

When $\boldsymbol{K}$ is a polyhedral cone, LP duality theorem leads us to the same conclusion: the optimal value in (4.58) is $\leq -R_{0,l} \bullet X + \alpha$ and the optimal value in (4.59) is $\leq R_{0,l} \bullet X + \alpha$ if and only if the optimal value in (4.60) is attained and is less than or equal to $\min\{R_{0,l}\bullet X+\alpha, -R_{0,l}\bullet X+\alpha\}$. □

We continue with an example for conic perturbation sets.

Example 5 (Budgeted Uncertainty)**.** *We consider the case where $\mathcal{V}_l$ is the intersection of $||\cdot||_\infty-$ and $||\cdot||_1-$ balls:*

$$\mathcal{V}_l = \{v_l \in \mathbb{R}^{2n-1} : ||v||_\infty \le 1, ||v||_1 \le \overline{\rho}\}, \quad \forall l \in \mathcal{N} \tag{4.61}$$

where $\overline{\rho}, 1 \le \overline{\rho} \le 2n-1$, is a given uncertainty budget.

We now write (4.61) *in terms of* (4.43)*:*

$$\mathcal{V}_l = \{v_l \in \mathbb{R}^{2n-1} : P_{l,1}v_l + p_{l,1} \in \boldsymbol{K}_{l,1}, P_{l,2}v_l + p_{l,2} \in \boldsymbol{K}_{l,2}\}, \quad \forall l \in \mathcal{N}. \tag{4.62}$$

where

- $P_{l,1}v_l \equiv [v_l, 0]$, $p_{l,1} = [0_{2n-1\times 1}, 1]$ *and* $\boldsymbol{K}_{l,1} = \{[\nu_l; t_l] \in \mathbb{R}^{2n-1} \times \mathbb{R} : t_l \ge ||\nu_l||_\infty\}$*, from which* $\boldsymbol{K}_{l,1,*} = \{\nu_l; t_l] \in \mathbb{R}^{2n-1} \times \mathbb{R} : t_l \ge ||\nu_l||_1\}$*,* $\forall l \in \mathcal{N}$

- $P_{l,2}v_l \equiv [v_l, 0]$, $p_{l,2} = [0_{2n-1\times 1}, \overline{\rho}]$ *and* $\boldsymbol{K}_{l,2} = \boldsymbol{K}_{l,1,*} = \{\nu_l; t_l] \in \mathbb{R}^{2n-1} \times \mathbb{R} : t_l \ge ||\nu_l||_1\}$ *from which* $\boldsymbol{K}_{l,2,*} = \boldsymbol{K}_{l,1}, \forall l \in \mathcal{N}$*.*

Setting $y_l^1 = [\nu_l, \tau_{1,l}], y_l^2 = [w_l, \tau_{2,l}]$ *with one-dimensional* τ_l *and* $2n-1$*-dimensional* ν_l, w_l*, system* (4.46)-(4.50) *transforms into the following system of constraints in variables* $\tau_1, \tau_2, \nu, w, X$:

$$\tau_{1,l} + \overline{\rho}\tau_{2,l} + R_{0,l} \bullet X \le \alpha, \quad \forall l \in \mathcal{N} \tag{4.63}$$

$$\tau_{1,l} + \overline{\rho}\tau_{2,l} - R_{0,l} \bullet X \le \alpha, \quad \forall l \in \mathcal{N} \tag{4.64}$$

$$(\nu_l + w_l)_{i,j} = -\epsilon_{i,j}X_{i,j}, \quad \forall i,j \in \mathcal{N}, \forall l \in \mathcal{N} \tag{4.65}$$

$$||\nu_l||_1 \le \tau_{1,l}, \quad \forall l \in \mathcal{N} \tag{4.66}$$

$$||w_l||_\infty \le \tau_{2,l}, \quad \forall l \in \mathcal{N} \tag{4.67}$$

We can eliminate τ variables to obtain a simpler model in variables X, w, ν :

$$\sum_{(i,j)\in Y_l} |[\nu_l]_{i,j}| + \overline{\rho} \max_{(i,j)\in Y_l} |[w_l]_{ij}| + R_{0,l} \bullet X \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.68}$$

$$\sum_{(i,j)\in Y_l} |[\nu_l]_{ij}| + \overline{\rho} \max_{(ij)\in Y_l} |[w_l]_{ij}| - R_{0,l} \bullet X \leq \alpha, \quad \forall l \in \mathcal{N} \tag{4.69}$$

$$[\nu_l]_{i,j} + [w_l]_{i,j} = -\epsilon_{i,j} X_{i,j}, \quad \forall i, j \in \mathcal{N}. \tag{4.70}$$

4.6 Optimality conditions for no and full information disclosures

We aim to obtain optimality conditions for canonical information structures under ellipsoid perturbations (4.21). We consider two canonical solutions to the information design problem, namely no and full information disclosure (see [21]). The equilibrium action profile under no information disclosure is $a = H^{-1}E[\gamma]$. The corresponding solution matrix X and the associated design objective value is given as follows,

$$X = \begin{bmatrix} O & O \\ O & \text{var}(\gamma) \end{bmatrix} \text{ and } F \bullet X = 0. \tag{4.71}$$

Solution matrix X for no information disclosure given in (4.71) is positive semi-definite and it satisfies the BCE condition in (4.25)-(4.26) at all times because when we substitute X in (4.71) into (4.25)-(4.26), we obtain $0 \leq \alpha$ which is always true. We have (4.27) satisfied, because $[X]_{2,2} = \text{var}(\gamma)$. Thus, no information disclosure is a feasible solution.

The equilibrium action profile under full information disclosure is $a = H^{-1}\gamma$. The corresponding solution matrix X and the design objective value is given as follows,

$$X = \begin{bmatrix} H^{-1}\text{var}(\gamma)(H^{-1})^T & H^{-1}\text{var}(\gamma) \\ \text{var}(\gamma)(H^{-1})^T & \text{var}(\gamma) \end{bmatrix} \text{ and } F \bullet X = F_H \bullet \text{var}(\gamma) \tag{4.72}$$

where

$$F_H = (H^{-1})^T([F]_{1,1} + [F]_{1,2}H + H^T[F]_{2,1})H^{-1}, \tag{4.73}$$

and $[F]_{i,j}$ denotes the i, jth $n \times n$ submatrix. We also define the perturbed version of F_H as following:

$$[F_0]_H = (H_0^{-1})^T([F_0]_{1,1} + [F_0]_{1,2}H_0 + H_0^T[F_0]_{2,1})H_0^{-1}. \tag{4.74}$$

X in (4.72) is positive semi-definite. We also have that for large enough α, X in (4.72) will satisfy (4.25)-(4.26). In the following analysis, we assume that we have a large enough α to ensure feasibility of full information disclosure.

4.6.1 Optimality of no information disclosure under general information structures

We consider general information structures, in which each player can receive private signals generated from different signal fidelities.

Theorem 7. *Let F_0 and F be symmetric matrices with dimensions $2n \times 2n$ for which we have*

$$[F_0]_{1,1} = \eta \odot H_0 \quad \textit{and} \quad [F]_{1,1} = \eta \odot H \tag{4.75}$$

where $\odot$ is the Hadamard product, η is a $n \times n$ constant coefficient matrix and H is as given in (4.4) *under ellipsoid perturbations* (4.21). *Let $[\lambda_0]_j$ and $[\lambda]_j$ denote the j^{th} largest eigenvalues of F_0 and F, respectively. If F_0 is negative definite with the largest eigenvalue*

$$|[\lambda_0]_{2n}| \geq \max(\overline{\eta_D \epsilon_D}, \overline{\eta_O \epsilon_O})\frac{n^2\rho}{2n-1}, \tag{4.76}$$

where we define $\overline{\eta_D} := \max_{i=j}(|\eta_{i,j}|)$, $\overline{\eta_O} := \max_{i\neq j}(|\eta_{i,j}|)$, $\overline{\epsilon_D} := \max_{i=j}(\epsilon_{i,j})$, and $\overline{\epsilon_O} := \max_{i\neq j}(\epsilon_{i,j})$, then no information disclosure is an optimal solution to the SDP defined by (4.75) *and* (4.25)-(4.27) *jointly.*

Proof. Consider Frobenius matrix norm $||\cdot||_F$ of the difference between F_0 and F,

$$||F_0 - F||_F \leq \left(\sum_i (\overline{\eta_D\epsilon_D} v_{i,i})^2 + 2\sum_i\sum_j (\overline{\eta_O\epsilon_O} v_{i,j})^2\right)^{\frac{1}{2}} \tag{4.77}$$

$$\leq \left(\max((\overline{\eta_D\epsilon_D})^2, (\overline{\eta_O\epsilon_O})^2)(\sum_i v_{i,i}^2 + 2\sum_i\sum_j v_{i,j}^2)\right)^{\frac{1}{2}} \tag{4.78}$$

$$\leq \max(\overline{\eta_D\epsilon_D}, \overline{\eta_O\epsilon_O})\frac{n^2\rho}{2n-1}. \tag{4.79}$$

We obtain (4.78) by taking the largest multiplier among $\overline{\eta_D\epsilon_D}$ and $\overline{\eta_O\epsilon_O}$. We attain (4.79) by using the ellipsoid perturbations in (4.21). By Lemma 4 in Appendix C and (4.79), we have that

$$\max_j |[\lambda]_j - [\lambda_0]_j| \leq \max(\overline{\eta_D\epsilon_D}, \overline{\eta_O\epsilon_O})\frac{n^2\rho}{2n-1}. \tag{4.80}$$

Given that all the eigenvalues of F_0 are negative, (4.76) and (4.80) together imply that $[\lambda]_i \leq 0,\ i = 1,..,2n$. This means F is negative definite. Thus, no information disclosure is an optimal solution that achieves the objective value zero. □

In the above result, we assume that the payoff matrix and perturbations only affect F_{11}, as is the case in social welfare maximization (2.3). Given this assumption, we show that no information disclosure is the optimal solution to the program given by (4.75) and (4.25)-(4.27) jointly, if the perturbed objective coefficients matrix F_0 is symmetric negative definite with the magnitude of the largest eigenvalue ($[\lambda_0]_{2n} < 0$) greater than a constant ($\max(\overline{\eta_D\epsilon_D}, \overline{\eta_P\epsilon_O})\frac{n^2\rho}{2n-1}$), that depends on the perturbation set radius ρ and the maximum of the product between the payoff coefficients and off-diagonal or diagonal shifts. That is, if the perturbed coefficient matrix is negative definite with sufficiently negative eigenvalue, then no information disclosure is guaranteed to be optimal. We note that there can be cases where no information disclosure is optimal even when the conditions of Theorem 7 are not satisfied. Moreover, if the perturbation radius of H is large enough, F_0 can be indefinite or positive definite even if F is negative definite. In these cases, one needs to solve the SDP given by (4.75) and (4.25)-(4.27) jointly to determine the optimal information structure.

4.6.2 Public information structures

We restrict our attention to public information structures where each player receives the same signal. In this context, Proposition 5 provides conditions for the optimality of no and full information disclosures based on the known perturbed matrix $[F_0]_H$. In the following results, we use Proposition 5 to find sufficient conditions for the optimality of no and full information disclosures based on the perturbed matrix F_0 and the perturbation set structure.

Theorem 8. *Let F_0 and F be symmetric matrices with dimensions $2n \times 2n$ for which* (4.75) *is valid. Let $[\lambda_0]_j$ and $[\lambda]_j$ denote the j^{th} largest eigenvalues of $[F_0]_{1,1} + 2\varsigma H_0, \varsigma \in [0,1]$ and $[F]_{1,1} + 2\varsigma H, \varsigma \in [0,1]$, respectively. Assume $[F_0]_{1,2} = [F_0]_{2,1} = [F]_{1,2} = [F]_{2,1} = \varsigma I$ for $\varsigma \in [0,1]$. If $[F_0]_{1,1}$ is negative definite and the largest eigenvalue of $[F_0]_{1,1} + 2\varsigma H_0$ is bounded as follows*

$$|[\lambda_0]_{2n}| \geq \max((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D}, (\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O})\frac{n^2\rho}{2n-1}, \quad \lambda \in [0,1], \tag{4.81}$$

where $\overline{\epsilon_O}, \overline{\epsilon_D}$ and $\overline{\eta_O}, \overline{\eta_D}$ are as defined in Theorem 7, then no information disclosure is the optimal solution to the SDP defined by (4.75) *and* (4.25)-(4.27) *jointly under public information structures.*

Proof. We start with (4.73) to calculate $[F]_H$ and $[F_0]_H$. We plug $\varsigma I, \varsigma \in [0,1]$ into (4.73) for $[F_0]_{1,2}, [F_0]_{2,1}, [F]_{1,2}$ and $[F]_{2,1}$. Then,

$$[F]_H = (H^{-1})^T([F]_{1,1} + 2\varsigma H)H^{-1} \text{ and } [F_0]_H = (H_0^{-1})^T([F_0]_{1,1} + 2\varsigma H_0)H_0^{-1}. \tag{4.82}$$

It is enough to show that $[F]_{1,1} + 2\varsigma H$ is negative definite. By (4.75) and (4.4),

$$||[F_0]_{1,1} + 2\varsigma H_0 - [F]_{1,1} - 2\varsigma H||_F = ||[F_0]_{1,1} - [F]_{1,1} + 2\varsigma v\epsilon||_F \tag{4.83}$$

$$\leq \left(\sum_i ((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D} v_{i,i})^2 + 2 \sum_i \sum_j ((\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O} v_{i,j})^2 \right)^{\frac{1}{2}} \tag{4.84}$$

$$\leq \left(\max((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D})^2, ((\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O})^2)(\sum_i v_{i,i}^2 + 2 \sum_i \sum_j v_{i,j}^2) \right)^{\frac{1}{2}} \tag{4.85}$$

$$\leq \max((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D}, (\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O}) \frac{n^2 \rho}{2n-1}. \tag{4.86}$$

We obtain (4.85) by taking the largest multiplier among $\overline{\eta_D \epsilon_D}$ and $\overline{\eta_O \epsilon_O}$. We attain (4.86) by using the ellipsoid perturbations in (4.21). By Lemma 4 in Appendix C and (4.86), we have that

$$\max_j |[\lambda]_j - [\lambda_0]_j| \leq \max((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D}, (\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O}) \frac{n^2 \rho}{2n-1}. \tag{4.87}$$

Given that all eigenvalues of $[F_0]_{1,1} + 2\varsigma H_0$ are negative, (4.81) and (4.87) together imply that $[\lambda]_i \leq 0, i = 1, .., 2n$. This means $[F]_{1,1} + 2\varsigma H$ is negative definite, and no information disclosure is optimal via Proposition 5. □

We continue with a result identifying when full information disclosure is optimal under public information structures.

Theorem 9. *Let F_0 and F be $2n \times 2n$ symmetric matrices for which* (4.75) *is valid. Let $[\lambda_0]_j$ and $[\lambda]_j$ denote the j^{th} largest eigenvalues of $[F_0]_{1,1} + 2\varsigma H_0$ and $[F]_{1,1} + 2\varsigma H$, respectively for some $\varsigma \in [0, 1]$. Assume $[F_0]_{1,2} = [F_0]_{2,1} = [F]_{1,2} = [F]_{2,1} = \varsigma I$ for $\varsigma \in [0, 1]$. If $[F_0]_{11}$ is positive definite, and the smallest eigenvalue of $[F_0]_{1,1} + 2\varsigma H_0$ is such that*

$$[\lambda_0]_1 \geq \max((\overline{\eta_D} + 2\varsigma)\overline{\epsilon_D}, (\overline{\eta_O} + 2\varsigma)\overline{\epsilon_O}) \frac{n^2 \rho}{2n-1}, \quad \varsigma \in [0, 1] \tag{4.88}$$

where $\overline{\epsilon_O}, \overline{\epsilon_D}$ and $\overline{\eta_O}, \overline{\eta_D}$ are as defined in Theorem 7, then full information disclosure is the optimal solution to the SDP defined by (4.75) *and* (4.25)-(4.27) *jointly under public information structures.*

Proof. The proof is similar to Theorem 8's proof, and thus is omitted. □

When there is no uncertainty on H, the perturbation set radius ρ is zero. In such a case, Theorem 8 and 9 recover the result in Proposition 5. However, if there is uncertainty regarding H, then (4.81) and (4.88) are sufficient conditions to claim no and full information disclosure, respectively, are optimal solutions in conjunction with definiteness conditions on F_0. In general, adding additional constraints on a feasible set makes an optimal solution worse or does not improve it. This implies that we cannot prove the optimality of no or full information disclosure for some objective functions for which they can be deemed optimal if there was no uncertainty.

We note that the coefficients of the social welfare objective in (2.3) satisfy the assumed structure in Theorems 8 and 9 with $\varsigma = 1$, and $[F_0]_{1,1} = -H_0$. So, we can check the eigenvalues of $[F_0]_{1,1} + 2\varsigma H_0 = H_0$ to determine the optimality of no or full information disclosure in welfare maximization. In the following, we obtain the conditions that the payoff coefficients in Cournot competition and the beauty contest games need to satisfy.

Example 6 (Welfare maximization during Cournot competition)**.** *We seek the condition that guarantees full information disclosure is the robust optimal public information structure under welfare maximization for the Cournot competition with payoffs in* (1.4)*. For n players, with $H_{i,i} = 1$ for all i, and $H_{i,j} = \frac{\varrho}{2\varpi}$ for all $i \neq j$, we have the following eigenvalue repeated $n-1$ times $[\lambda]_j = 1 - \frac{\varrho}{2\varpi}$ for $j = 1, \ldots, n$, and $[\lambda]_n = (n-1)\frac{\varrho}{2\varpi} + 1$. The minimum eigenvalue of H is equal to $1 - \frac{\varrho}{2\varpi}$. We also know $\overline{\eta_D} = \overline{\eta_O} = 1$ for the welfare maximization objective. Then the condition* (4.88) *reduces to the following condition on the payoff constants in* (1.4)*,*

$$\frac{\varrho}{2\varpi} \leq 1 - \frac{3n^2\rho}{2n-1} \max(\overline{\epsilon_D}, \overline{\epsilon_O}). \tag{4.89}$$

Since the payoff constants ϱ and ϖ are positive, the optimality condition in (4.89) *will only be true for small enough n and ρ given ϱ and ϖ.*

Example 7 (Welfare maximization during the Beauty contest)**.** *We look for the condition that guarantees full information disclosure is the robust optimal public information structure under* *welfare maximization for the beauty contest with payoffs in* (1.5)*. For n players, with $H_{i,i} = 1$ for all i and $H_{i,j} = \frac{-\xi}{n-1}$ for all $i \neq j$, we have the following smallest eigenvalue $[\lambda]_1 = 1 - \xi$, and the largest eigenvalue repeated $n-1$ times $[\lambda]_j = 1 + \frac{\xi}{n-1}$ for $j = 2, \ldots, n$. We also know $\overline{\eta_D} = \overline{\eta_O} = 1$ for the welfare maximization objective. Then the condition* (4.88) *reduces to the following condition on the payoff constants in* (1.5)*,*

$$\xi \leq 1 - \frac{3n^2\rho}{2n-1}\max(\overline{\epsilon_D}, \overline{\epsilon_O}). \tag{4.90}$$

Cournot competition is a submodular game $(H_{i,j} = \frac{\varrho}{2\varpi} \geq 0)$ while the Beauty contest is a supermodular game $(H_{i,j} = \frac{-\xi}{n-1} \leq 0)$. The direction of the competition, i.e., $H_{i,j}$ renders different optimality conditions for full information disclosure given by (4.89) and (4.90). We can consider (4.89) as $1 - H_{i,j} \geq \frac{3n^2\rho}{2n-1}\max(\overline{\epsilon_D}, \overline{\epsilon_O})$ whereas (4.90) has the form $1 + (n-1)H_{i,j} \geq \frac{3n^2\rho}{2n-1}\max(\overline{\epsilon_D}, \overline{\epsilon_O})$. Therefore when $|H_{i,j}| = |\frac{\varrho}{2\varpi}| = |\frac{-\xi}{n-1}|$, optimality condition for full information disclosure is more restrictive under the beauty contest than under the Cournot competition.

4.7 Numerical Experiments

We consider a designer that wants to maximize the social welfare of $n = 5$ players. The designer knows the perturbed payoff coefficients as follows

$$[H_0]_{i,j} = \begin{cases} 5 & \text{if} \quad i = j;\ i,j \in \{1,2,..,5\} \\ -1 & \text{if} \quad i \neq j;\ i,j \in \{1,2,..,5\}. \end{cases} \tag{4.91}$$

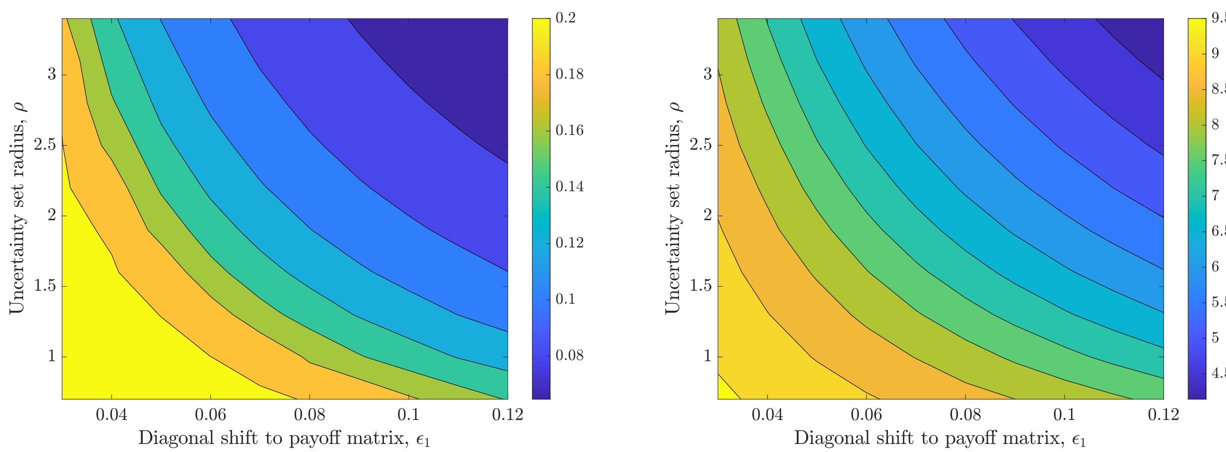
The variance of the unknown payoff state γ is given as follows

$$var(\gamma)_{i,j} = \begin{cases} 5, & \text{if } i = j;\ i,j \in \{1,2,.,5\} \\ 0.5, & \text{if } i \neq j;\ i,j \in \{1,2,.,5\}. \end{cases} \tag{4.92}$$

We consider ellipsoid perturbations with $\rho \in \{0.7, 1, 1.3, .., 3.4\}$ and let $\alpha = 0.1$. Given the setup, we solve the robust convex program (4.23)-(4.27) in order to obtain the robust optimal information design $X_{optimal}$.

We analyze the effects of constant shifts $\epsilon_{i,j}$ by assuming the diagonal elements and off-diagonal elements of shift matrix are homogeneous, i.e., $\epsilon_{i,i} = \epsilon_1$ and $\epsilon_{i,j} = \epsilon_2$ for all $i, j = 1, \ldots, n$ for constants ϵ_1 and ϵ_2.

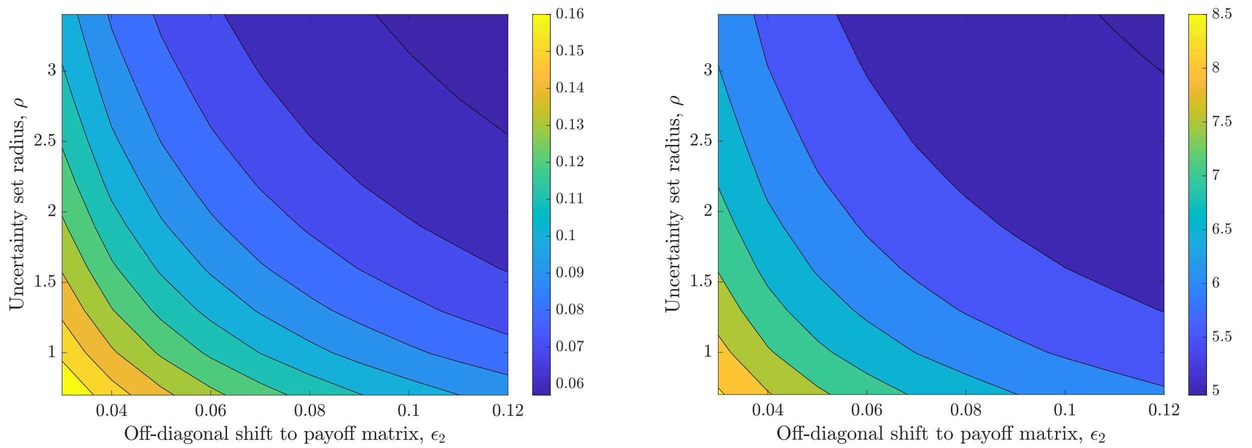
In order to systematically analyze the effects of the shifts, we fix the off-diagonal shifts to a small value $\epsilon_2 = 0.001$, and vary the diagonal shift $\epsilon_1 \in \{0.03, 0.04, 0.05, .., 0.12\}$. Fig. 4.1 (a) shows that as the uncertainty ball radius ρ and diagonal shift ϵ_1 increases, the optimal information structure remains a partial information disclosure but gets closer to the no information disclosure. Fig. 4.1 (b) confirms the same result by also showing a decrease in optimal social welfare under increasing uncertainty.



(a) $||X_{optimal} - X_{no}||_F$ (b) Optimal objective value

Figure 4.1: Contour plots of (a) normalized Frobenius matrix norm distance $||X_{optimal} - X_{no}||_F$ between optimal covariance matrix and no information disclosure covariance matrix and (b) optimal objective value with respect to uncertainty ball radius ρ and diagonal shift ϵ_1 to coefficient matrix H under a symmetric supermodular game with social welfare objective. Optimal solution, that is partial information disclosure, approaches to no information disclosure as ρ and ϵ_1 increase.

Next, we consider the same setup but with a small diagonal shift $\epsilon_1 = 0.001$ and larger off-diagonal shifts $\epsilon_2 \in \{0.03, 0.04, 0.05, .., 0.12\}$. As expected, we see a similar trend in fig. 4.2(a) to fig. 4.1 (a) where the optimal information structure approaches no information disclosure as the uncertainty in the system increases. This trend toward no information disclosure is faster when ϵ_2 increases than when ϵ_1 increases. This is expected as the off-diagonal shifts appear in more of the terms in (4.23)-(4.27).



(a) $||X_{optimal} - X_{no}||_F$ (b) Optimal objective value

Figure 4.2: Contour plots of (a) normalized Frobenius matrix norm distance $||X_{optimal} - X_{no}||_F$ between optimal covariance matrix and no information disclosure covariance matrix and (b) optimal objective value with respect to uncertainty ball radius ρ and off-diagonal shift ϵ_2 to coefficient matrix H under a symmetric supermodular game with social welfare objective. Optimal solution, that is partial information disclosure, approaches to no information disclosure as ρ and ϵ_2 increase.

We can discuss figures 4.1 and 4.2 in terms of the Beauty contest which is a supermodular game. If we consider the common goods in the Beauty contest game as a stock, we see that a social welfare maximizing information designer i.e the company whose stock is traded releases less information about stock price γ when uncertainty about payoff coefficient matrix H increases.

5. SUMMARY AND CONCLUSIONS

5.1 Summary

The overarching theme in the dissertation was the determination of optimal information structures for a given objective under equilibrium constraints. The focus was on linear-quadratic-Gaussian games due to tractability purposes.

Chapter 2 laid out analytical and numerical study of the welfare and agreement maximization in the various contexts of homogeneous games, public information structures and common payoff states. Second chapter is concluded with a study of joint maximization problem of the welfare and agreement.

In Chapter 3, reverse perspective, that is agents' point of view, is analyzed in comparison to designer perspective which is welfare maximization. The relation between agents' preference towards an information structure and their positions in the network is studied. Numerical studies are conducted to understand distributional robustness of agents' preferences.

In Chapter 4, the main topic was perturbation aware information design. Private nature of agents' information is encoded via partially known utilities under uncertainty sets over utility parameters. The designer's problem under uncertain utilities is addressed with the development of tractable robust optimization models from intractable semi-infinite programs under ellipsoid, interval and general cone uncertainties. Numerical studies are carried out to understand the effects of uncertainty level on optimal information structure and optimal welfare.

5.2 Conclusions

The context of research was information design in linear-quadratic-Gaussian games. Analysis of optimal information structures for the objectives of maximizing social welfare and agreement formed the first part of research. It was found out that full information disclosure optimizes social welfare under three configurations: common payoffs, homogeneous games, and public information structures. The study showed that agreement maximizing information structure is no information disclosure. In homogeneous games, a bound on the strategic interaction coefficient is determined which signifies where full information disclosure becomes sub-optimal in the contexts of public information structures and common payoff states.

Moreover, agents' preferences toward given information structures was analyzed in the context of LQG network games. A full information preference condition based on network information is provided. Using this general result, it is found that all agents in a star network prefer full information over no information. A peripheral agent benefits more than the central agent from full information disclosure if competition is strong and the number of agents is small. The value of information for the central agent decreases if strategic interaction coefficient increases. In line with this result, ex-post benefit estimates shows that a risk averse central agent could prefer no information disclosure ex-ante. In contrast, full information disclosure is distributionally robust against uncertain payoff mean for peripheral agents.

Furthermore, robust information design problem under perturbed utilities is considered. For finite scenarios case, a mixed integer SDP is developed. In the harder case of continuous perturbation sets, semi-infinite nature of problem is overcome through the development of convex programs in the cases of ellipsoid, interval and general cone perturbation sets. Incorporation of perturbed objectives into convex programs as tractable constraints is demonstrated on social welfare objective under ellipsoid perturbations. Perturbation aware full and no information structure optimality conditions are developed and discussed through the beauty contest and Cournot competition games. Numerical studies showed that increasing uncertainty moves optimal information structure towards no information disclosure and reduces optimal welfare.

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APPENDIX A

APPENDIX TO MAXIMIZING SOCIAL WELFARE AND AGREEMENT VIA INFORMATION DESIGN IN LINEAR-QUADRATIC-GAUSSIAN GAMES

A.1 Coefficients matrix of the agreement objective

Lemma 2 (Agreement Objective)**.** *The expected value of* $f(a, \gamma)$ *for* (1.16) *can be written as* $F^C \bullet X$ *where* F^C *is given in* (2.3).

Proof. By expanding and regrouping the terms in (2.2),

$$E[-\sum_{i=1}^{n}(a_i - \bar{a})^2] = \sum_{i=1}^{n} \frac{1-n}{n} E[a_i^2] + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E[a_i a_j]. \tag{A.1}$$

Because $E[a_i]$ is constant for all $i \in N$, we can write (A.1) as $F^C \bullet X$ using the definition of var(a). □

A.2 Proof of Theorem 1

We verify that the full information disclosure solution satisfies the KKT conditions. We denote the dual variables associated with constraints (1.17), (1.18) and $X \in P_+^{2n}$ by $\overline{\nu} \in \mathbb{R}^n$, $\overline{\mu} \in \mathbb{R}^{n(n+1)/2}$ and $\overline{\Gamma}$, respectively. Primal feasibility conditions in (1.17)-(1.18) are satisfied by full information disclosure. Next we respectively state the rest of the KKT conditions, i.e., dual feasibility, first order optimality and complementary slackness condition,

$$\overline{\Gamma} \in P_+^{2n}, \tag{A.2}$$

$$F^W + \sum_{k=1}^{n} \overline{\nu}_k R_k + \sum_{k=1}^{n} \sum_{l=1}^{k} \overline{\mu}_{(n-1)k+l} M_{k,l} + \overline{\Gamma} = 0, \tag{A.3}$$

$$X \bullet \overline{\Gamma} = 0. \tag{A.4}$$

Let $\overline{X} \in P_{+}^{2n}$ denote the full information disclosure solution as given in (1.20) to the social welfare maximization problem (2.1) with coefficients F^W. We check whether the above KKT conditions are satisfied by $\overline{X}$. We look for a uniform dual variable $\overline{\nu}$, i.e., $\overline{\nu}_k = \nu, \forall\ k \in N$ where $\nu \in \mathbb{R}$, that satisfies (A.3). We define $\Xi = -\sum_{k=1}^{n}\sum_{l=1}^{k}\overline{\mu}_{(n-1)k+l}M_{k,l}$ in matrix form and assume $\Xi = \mu I, \mu > 0$ i.e uniformity over $\bar{\mu}_{(n-1)k+1}$. We can express the dual variable $\overline{\Gamma}$ using (A.3) and substituting in (2.3) for F^W,

$$\overline{\Gamma} = \begin{bmatrix} (1-\nu)H & (\frac{\nu}{2}-1)I \\ (\frac{\nu}{2}-1)I & \Xi. \end{bmatrix} \tag{A.5}$$

We use Schur complement to analyze the positive definiteness of $\overline{\Gamma}$ in (A.5). A strict version of dual feasibility condition $\overline{\Gamma} \succ 0$ is satisfied if and only if Ξ is positive definite and Schur complement

$$\overline{\Gamma}/\Xi = (1-\lambda)H - \frac{(\frac{\nu}{2}-1)^2 I}{\mu}. \tag{A.6}$$

of block matrix Ξ of matrix $\overline{\Gamma}$ is positive definite. Sum of each row of $\overline{\Gamma}/\Xi$ is equal to $(1-\nu) - (\frac{\nu}{2}-1)^2/\mu + (n-1)(1-\nu)h$. This is the first eigenvalue of $\overline{\Gamma}/\Xi$. Rest of the eigenvalues of $\overline{\Gamma}/\Xi$ are equal to $(1-\nu)(1+h) - \frac{(\nu/2-1)^2}{\mu}$. We have all of the eigenvalues of $\overline{\Gamma}/\Xi$ positive and $\Xi \succ 0$, when

$$\mu > \max\{\frac{(\frac{\nu}{2}-1)^2}{(1-\nu)(1+h)}, 0\}. \tag{A.7}$$

Hence, if μ satisfies (A.7), then $\overline{\Gamma}$ is positive definite.

Next, we show that there exists $\nu \in \mathbb{R}$ and μ as in (A.7) satisfying (A.4). We can express the inverse of H in (2.7) as follows for $n \geq 3$

$$H_{i,j}^{-1} = \begin{cases} \frac{(n-2)h+1}{-(n-1)h^2+(n-2)h+1} & \text{if} \quad i = j;\ i, j \in N \\ \frac{-h}{-(n-1)h^2+(n-2)h+1} & \text{if} \quad i \neq j;\ i, j \in N \end{cases} \tag{A.8}$$

When $X = \overline{X}$ is given by (1.20) and $\overline{\Gamma}$ is as in (A.5), we obtain the following equation by computing the Frobenius product terms within (A.4) corresponding to each of the four sub-matrices,

$$\overline{X} \bullet \overline{\Gamma} = n^2(1-\nu)^2 * (\tau + h\phi) + 2\frac{(\nu-2)[((2-n)h-1)\tau + h\phi]}{(n-1)h^2-(n-2)h-1} + \mu\tau = 0, \tag{A.9}$$

where we let $\tau = \text{tr}(\text{var}(\gamma))$ and $\phi = 2\sum_{i=1}^{n}\sum_{j\in N\setminus\{i\}} \text{cov}(\gamma_i, \gamma_j)$ to simplify the exposition.

Next we show that there exists at least one real root of (A.9) $\nu \in \mathbb{R}$ and μ as in (A.7). If there is a real root, there exists a $\nu \in \mathbb{R}$ satisfying the KKT conditions.

First, we consider the case $\mu = \frac{(\frac{\nu}{2}-1)^2}{(1-\nu)(1+h)} + \epsilon, \epsilon > 0$. In this case, (A.9) becomes

$$\overline{X} \bullet \overline{\Gamma} = n^2(1-\nu)^2(\tau + h\phi) + (\frac{(\frac{\nu}{2}-1)^2}{(1-\nu)(1+h)} + \epsilon)\tau + \frac{2(\nu-2)[((2-n)h-1)\tau + h\phi]}{(n-1)h^2-(n-2)h-1} = 0. \tag{A.10}$$

When we equalize the denominators, (A.10) becomes a cubic equation in ν. The cubic equation with real coefficients always has at least one real root.

Secondly, we consider the case $\mu = \epsilon, \epsilon > 0$. In this case, (A.9) is a quadratic function of ν

$$a\nu^2 + b\nu + c = 0, \tag{A.11}$$

where we define the constants a , b and c as

$$a = n^2(\tau + h\phi) \tag{A.12}$$

$$b = -2n^2(\tau + h\phi) + 2\frac{((2-n)h-1)\tau + h\phi}{(n-1)h^2+(2-n)h-1} \tag{A.13}$$

$$c = n^2(\tau + h\phi) + \frac{-4((2-n)h-1)\tau - 4h\phi}{(n-1)h^2+(2-n)h-1} + \epsilon\tau \tag{A.14}$$

We want to show $b^2 - 4ac > 0$, so that there exists a real root. Note that $(n-1)h^2-(n-2)h-1 < 0$ for $\frac{-1}{n-1} < h < 1$.

Also, by our assumption $\tau \geq h\phi$. We can deduce that the discriminant $(b^2 - 4ac)$ is positive, i.e.,

$$b^2 - 4ac = \frac{8n^2(\tau + h\phi)[((2-n)h-1)\tau + h\phi]}{(n-1)h^2 + (2-n)h - 1} + 4\left(\frac{((2-n)h-1)\tau + h\phi}{(n-1)h^2 + (2-n)h - 1}\right)^2$$
$$+ n^2(\tau + h\phi)\epsilon\tau > 0. \tag{A.15}$$

Therefore the roots of (A.11) are real. We also need to show at least one of the roots of (A.11) (ν_r) is such that $\nu_r > 1$ so that $\mu = \epsilon$ as per (A.7). We consider the larger root,

$$\nu_r = 1 - \frac{((2-n)h-1)\tau + h\phi}{n^2(\tau + h\phi)[(n-1)h^2 + (2-n)h - 1]} + \frac{\sqrt{b^2 - 4ac}}{2a} > 1. \tag{A.16}$$

We know $a > 0$. Also, it can be deduced that the third term in (A.16) is greater than the absolute value of the second term in (A.16). Thus, $\nu_r > 1$.

APPENDIX B

APPENDIX TO INFORMATION PREFERENCES OF INDIVIDUAL AGENTS IN LINEAR-QUADRATIC-GAUSSIAN NETWORK GAMES

B.1 BNE under Public Information and Common Value Payoff States

The expectation of the payoff state γ given two Gaussian signals (prior μ and public signal $\bar{\omega}$) as follows

$$E[\gamma|\omega_i = \bar{\omega}] = (1 - \xi_i)\mu + \xi_i \bar{\omega} \tag{B.1}$$

where $\xi_i(\nu) = \frac{var(\gamma)}{var(\gamma)+var(\bar{\omega})}$, and ν is the covariance matrix of the distribution $\zeta(\omega|\gamma)$.

Lemma 3. *Bayesian Nash equilibrium of LQG network game given public signals $\bar{\omega}$ and common payoff state γ can be represented by the following function*

$$a_i^*(\bar{\omega}) = E[\gamma|\bar{\omega}][H^{-1}\mathbf{1}]_i \qquad \forall i \in \mathcal{N}, \tag{B.2}$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones, and $[\cdot]_i$ indicates the ith element of a vector.

Proof. First order condition of the expectation of the utility function in (1.3) with respect to a_i yields

$$\frac{\partial E[u_i|\{\omega_i = \bar{\omega}\}]}{\partial a_i} = -H_{ii}a_i^*(\bar{\omega}) - \sum_{i \neq j} H_{ij} E[a_j^*|\{\omega_i = \bar{\omega}\}] + E[\gamma|\{\omega_i = \bar{\omega}\}] = 0, \forall i \in \mathcal{N} \tag{B.3}$$

We incorporate (B.1) into (B.3):

$$H_{ii}a_i^*(\bar{\omega}) = -\sum_{i \neq j} H_{ij} E[a_j^*|\bar{\omega}] + (1 - \xi_i)\mu + \xi_i \bar{\omega} = 0, \forall i \in \mathcal{N} \tag{B.4}$$

We assume agent $i \in \mathcal{N}$'s strategy is linear in its information $a_i^*(\bar{\omega}) = \alpha_{i1}\bar{\omega} + \alpha_{i2}\mu$ with coefficients

α_{i1} and α_{i2}. We substitute linear actions in (B.4) to get

$$H_{ii}(\alpha_{i1}\bar{\omega} + \alpha_{i2}\mu) = -\sum_{i \neq j} H_{ij}(\alpha_{j1}\bar{\omega} + \alpha_{j2}\mu) + (1 - \xi_i)\mu + \xi_i\bar{\omega} = 0, \forall i \in \mathcal{N} \tag{B.5}$$

We solve for the action coefficients $\boldsymbol{\alpha}_1 = [\alpha_{11}, \ldots, \alpha_{n1}] \in \mathbb{R}^n$ and $\boldsymbol{\alpha}_2 = [\alpha_{12}, \ldots, \alpha_{n2}] \in \mathbb{R}^n$: $\boldsymbol{\alpha}_1 = \mathbf{1} - \boldsymbol{\alpha}_2 = H^{-1}\xi$ where $\xi = [\xi_1, \ldots, \xi_n]$ and ξ_i is as in (B.1). Thus, $a^*(\bar{\omega}) = H^{-1}\xi\mathbf{1}\bar{\omega} + (I - H^{-1})\xi\mathbf{1}\mu$ where I is the identity matrix. (B.2) follows from rearranging terms in a^* and using (B.1). $\square$

APPENDIX C

APPENDIX TO ROBUST OPTIMIZATION APPROACH TO INFORMATION DESIGN IN LINEAR-QUADRATIC-GAUSSIAN GAMES

C.1 Proof of Lemma 1

We start with writing the first order condition equivalent to (4.1) for a given $\theta \equiv H$:

$$E_\zeta\left[\frac{\partial}{\partial a_i}u_i^\theta(s(\omega),\gamma)|\omega_i\right] = -2H_{i,i}s_i(\omega_i) - 2\sum_{i\neq j} H_{i,j}E_\zeta[s_j|\omega_i] + 2E_\zeta[\gamma_i|\omega_i] = 0 \tag{C.1}$$

We solve (C.1) for the best response $s_i(\omega_i), \forall i \in \mathcal{N}$:

$$H_{i,i}s_i(\omega_i) = -\sum_{i\neq j} H_{i,j}E_\zeta[s_j|\omega_i] + E_\zeta[\gamma_i|\omega_i], \quad i \in \mathcal{N} \tag{C.2}$$

We look for an equilibrium strategy of the form given below:

$$s_i(\omega_i) = \bar{a}_i + b_i^T(\omega_i - E_\zeta[\omega_i]), \quad \forall i \in \mathcal{N}, \tag{C.3}$$

where $\bar{a}_i$ and $b_i^T, \forall i \in \mathcal{N}$ are constants and constant vectors, respectively. We plug (C.3) into the first order condition (C.2):

$$\sum_{j\in\mathcal{N}} H_{i,j}E[\bar{a}_j + b_j^T(\omega_j - E_\zeta[\omega_j])|\omega_i = \bar{\omega}_i] = E[\gamma_i|\omega_i = \bar{\omega}_i], \forall \bar{\omega}_i \in \mathbb{R}, i \in \mathcal{N}.$$

Via conditional expectation rule over multivariate normal distribution, we obtain following:

$$\begin{aligned}&\sum_{j\in\mathcal{N}} H_{i,j}(b_j^T cov(\omega_j,\omega_i)var(\omega_i)^{-1}(\bar{\omega}_i - E_\zeta[\omega_i]) + \bar{a}_j)\\ &= E[\gamma_i] + cov(\omega_i,\gamma_i)^T var(\omega_i)^{-1}(\bar{\omega}_i - E_\zeta[\omega_i]), \forall \bar{\omega}_i \in \mathbb{R}, i \in \mathcal{N}.\end{aligned} \tag{C.4}$$

Vectors $b_i, i \in \mathcal{N}$ and constants $\bar{a}_i, i \in \mathcal{N}$ are determined by following set equations when we separate (C.4) into respective parts: For $i \in \mathcal{N}$

$$\sum_{j \in \mathcal{N}} H_{i,j} b_j^T cov(\omega_j, \omega_i) var(\omega_i)^{-1} = cov(\omega_i, \gamma_i)^T var(\omega_i)^{-1}, \tag{C.5}$$

$$\sum_{j \in \mathcal{N}} H_{i,j} \bar{a}_j = E[\gamma_i], \quad \forall i \in \mathcal{N}. \tag{C.6}$$

We divide both sides of (C.5) by $var(\omega_i)^{-1}$ and obtain the following set of equations:

$$\sum_{j \in \mathcal{N}} H_{i,j} b_j^T cov(\omega_j, \omega_i) = cov(\omega_i, \gamma_i), \quad \forall i \in \mathcal{N} \tag{C.7}$$

For scalar signals $\omega_i \in \mathbb{R}$, if we let $b_i = 1$ and $\bar{a}_i = E_\zeta[\omega_i]$ for $i \in \mathcal{N}$, then we have $a_i = \omega_i$ by (C.3). Moreover, the set of equations in (C.7) is equivalent to (4.8).

C.2 Eigenvalue Bounds for Symmetric Matrices

Lemma 4 (Theorem 8.1, [52]). *Let $G = [g_{i,j}]$ and $\hat{G} = [\hat{g}_{i,j}]$ be two symmetric matrices with eigenvalues $[\lambda]_1 \leq [\lambda]_2 \leq \cdots [\lambda]_n$ and $[\hat{\lambda}]_1 \leq [\hat{\lambda}]_2 \leq \cdots [\hat{\lambda}]_n$, respectively. Then,*

$$\max_j \left| [\lambda_j] - [\hat{\lambda}]_j \right| \leq \left| \left| G - \hat{G} \right| \right|_F \tag{C.8}$$

where $||\cdot||_F$ denotes Frobenius matrix norm.