THURSTON POLYTOPES

A Thesis

by

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ABSTRACT

We explore a 3-manifold and link invariant called the Thurston norm which provides a deep understanding of the submanifold structure of a 3-manifold. Recently, methods to compute the Thurston norm ball (a symmetric rational polytope) have been developed, providing a doorway through which we can hope to understand more about this invariant. In particular we use these techniques in order to find patterns in these Thurston norm balls which give rise to new conjectures. We also showcase some existing literature in the field to highlight relationships between $\|\cdot\|_T$ and other properties/invariants of manifolds and links. This work embarks on a journey through low dimensional topology making stops in fields as diverse as combinatorial group theory, Floer homology, and hyperbolic geometry. In this way, we hope to convince the reader that this invariant is well worth study.

DEDICATION

I dedicate this thesis to Thurston Waffles.

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All images of knots and links in this document are either taken from the Knot Atlas [1] or are produced using SnapPy [2].

All other work conducted for the thesis was completed by the student independently.

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1. INTRODUCTION

In 1986 William Thurston published a paper titled ‘A Norm for the Homology of 3-Manifolds’ [3]. Thurston wrote the paper about a decade prior and its unpublished results had already circulated prompting other mathematians, notably David Gabai who was Thurston’s student at the time, to utilize the paper’s results. The keystone result of the paper was the construction of a seminorm on the second real homology of a compact connected orientable manifold which studies the topology of embedded surfaces in a 3-manifold. Since then, this norm has found wide application across low dimensional topology. The most notable property of the Thurston norm is that the shape of its norm ball is a symmetric polytope with rational vertices. A longstanding question asks which symmetric rational polytopes are Thurston polytopes for some 3-manifold or for some link.

This question is the prime motivation for this work. Recent advances in algorithmic low-dimensional topology have made computing Thurston polytopes very feasible. We seek to utilize these advances to compute Thurston polytopes of link complements in the hopes of finding some patterns and formulating some conjectures about the structure of these polytopes. The most interesting conjecture which we formulate is the following:

Conjecture 7.2. *The Newton polytope of the Alexander polynomial of an alternating 3-component link L has slope $s \in [1/2, 1]$.*

We also demonstrate some elementary, but nonetheless interesting results.

Theorem 7.6. *There exists an infinite class of links with the same Thurston norm ball. In particular, there are infinitely many links whose norm ball is the unit diamond in $\mathbb{R}^2$.*

If $\epsilon : L \to L$ is a change of orientation, we show the following result.

Theorem 7.9. *Let $\vec{L}$ be a link. If $\vec{L} = \epsilon(\vec{L})$ for any arbitrary ϵ, then $(\mathbb{Z}_2)^\ell \leq \mathrm{Sym}(\mathrm{B}_{\tilde{\mathrm{L}}})$.*

In chapter 2 we provide background on manifolds and links in S^3 which are the basic objects of study in this work. After this in chapter 3 we define the Thurston norm and explain some of its basic properties. In chapter 4 we connect the Thurston norm to the study of knots and links via the Alexander polynomial. In chapter 5 we look at some applications of the Thurston norm with the goal of highlighting diverse applications across various fields in topology, geometry, and algebra. In chapter 6 we explain methods to compute Thurston norm balls for link complements.In 7 we explore Thurston polytopes of links and formulate some conjectures as well as demonstrate some results.

2. MANIFOLDS, KNOTS, AND HYPERBOLIC GEOMETRY

2.1 Manifolds

Here we review some facts about 3-dimensional manifolds, giving a general overview of the major structural theorems in the field, and providing background for what is needed later. As is tradition in 3-manifold topology, Y will denote a 3-manifold (for obvious reasons). The guiding philosophy of studying a 3-manifold is to examine the types of surfaces living in the manifold. To avoid pathologies, all of our embeddings are assumed to be tame embeddings. Additionally when $\partial Y \neq \emptyset$, then we also want our embeddings to be proper. An embedding $f : \Sigma \to Y$ of a surface is proper if $\partial f(\Sigma) \subset \partial Y$.

Theorem 2.1. *(Alexander-Schönflies Theorem) Let $\iota : S^2 \hookrightarrow S^3$ be an embedding of the sphere, then $S^3 - \iota(S^2) = B^3 \sqcup B^3$.*

Theorem 2.2. *(Sphere Theorem) Let Y be an orientable 3-manifold with nontrivial $\pi_2(Y)$, then there exists a nontrivial $\alpha \in \pi_2(M)$ admitting a representative that is an embedding $S^2 \hookrightarrow M$.*

Definition 2.1. *Let $\Sigma \hookrightarrow Y$ be an embedded surface. A compression disk $D \hookrightarrow Y$ is an embedded disk such that $D \cap \Sigma = \partial D$, however D is not the boundary of a disk in Σ. We say that Σ is incompressible if no such compression disk exists.*

If Σ is compressible, then we may perform a simplifying operation called compression. Find a compression disk D. In a local region about that disk we can find a copy of $D \times I$ such that $\Sigma \cap (D \times I) = \partial D \times I = S^1 \times I$. We then cut Σ at $S^1 \times \{0\}$ and $S^1 \times \{1\}$ and remove the cylinder in the middle, and then cap off both ends with disks.

Lemma 2.1. *Compressing Σ increases the Euler characteristic χ by 2*

Proof. On a cellular level, a compression operation cuts the surface which involves deleting a 0-cell and a 1-cell which leaves χ unchanged, and then one glues two 2-cells, thus χ increases by 2. □

An embedded surface induces a map on fundamental groups by inclusion $\iota_* : \pi_1(\Sigma) \to \pi_1(Y)$. When this map is injective, we say that Σ is π_1-injective. A nontrivial result in geometric 3-manifold topology is the following theorem.

Theorem 2.3. *A surface $\Sigma \hookrightarrow Y$ is incompressible if and only if it is π_1-injective.*

Note that spheres are trivially incompressible. Topologists often alter the definition of incompressible to exclude spheres, unless the sphere does not bound an embedded B^3. In this later case we say that the sphere is *essential*. Moving forward, we take the convention that spheres are not incompressible, and spheres that do not bound a B^3 are essential.

Definition 2.2. *A 3-manifold Y is said to be irreducible if it contains no essential spheres.*

3-manifolds can decompose into simpler pieces. There is a notion of 'prime' which resembles primeness in rings. The operation on the set of manifolds in question is a connect sum. Since manifolds are locally homeomorphic to $\mathbb{R}^n$ we can find n-balls that are homotopically trivial. For two 3-manifolds Y_1 and Y_2, find two topologically trivial B^3s and delete their interiors. Since there is only one orientation preserving homeomorphism of the sphere up to homotopy, it follows that there is a unique way of glueing together $Y_1 - \mathrm{int(B^3)}$ and $Y_2 - \mathrm{int(B^3)}$ via identifying the boundary spheres. We call this operation the connect sum and it is independent over the choice of 3-balls so long as they are homotopically trivial. We denote this operation $Y_1 \# Y_2$. Now consider an open 3-ball removed from the 3-sphere S^3. Theorem 2.1 (Alexander-Schoenflies) tells us that this resulting space is in fact homeomorphic to B^3. Now if we remove a trivial 3-ball B from a 3-manifold Y and take the connect sum with S^3, we see that we are really just gluing this 3-ball back into Y. This motivates the following notion of prime.

Definition 2.3. *A compact orientable 3 manifold is said to be prime if $Y = M \# N$ implies that either M or N is S^3.*

Notice that irreducible manifolds are prime since if we express an irreducible manifold Y as a connect sum $M \# N$, then gluing together M and N along S^2 implies that either M or N with a B^3

removed is homeomorphic to a B^3, thus M or N is S^3 and therefore that Y is prime. Most prime manifolds are irreducible with the notable exception of $S^2 \times S^1$.

One of the first 'deep' results in 3-manifold topology is the prime decomposition theorem whose existence was proven by Kneser followed by Milnor proving uniqueness.

Theorem 2.4. *(Prime Decomposition) If Y is a compact connected orientable 3-manifold, then Y can be written as the connect sum of finitely many prime manifolds:*

$$Y = P_1 \# P_2 \# ... \# P_n$$

Moreover this decomposition is unique up to connect sums of S^3 which act as units.

Note that since connect sum is an abelian operation, the set of compact connected orientable manifolds under connect sum form an abelian monoid with S^3 representing the unit element.

Definition 2.4. *A compact orientable 3-manifold is said to be Haken if it is prime and contains a properly embedded connected incompressible surface*

Another way to break down a 3-manifold into simpler pieces is via a Heegaard decomposition.

Definition 2.5. *Let Y be a closed connected orientable 3-manifold. A Heegaard decomposition of Y consists of two g-genus handle bodies H_0, H_1 along with a homeomorphism $\varphi : \Sigma_g \to \Sigma_g$ such that*

$$Y \cong H_0 \cup_\varphi H_1$$

where ∂H_0 is glued to ∂H_1 via φ. The surface Σ_g is called the Heegaard surface.

To prove the existence of Heegaard diagrams, one uses the fact that all closed orientable compact 3-manifolds are triangulable, then the 1-skeleton and its dual 1-skeleton for some triangulation form the cores of genus g handle bodies which are then identified at their boundary. Another standard proof involves finding a self-indexing Morse function and showing that $f^{-1}(-\infty, 3/2)$ and $f^{-1}(3/2, \infty)$ form handlebodies of the same genus.

Example 1. *Recall from the Alexander-Schönflies theorem that a 2-sphere Σ in S^3 bounds two disjoint 3-balls; $S^3 = B_0^3 \cup_\Sigma B_1^3$. This is precisely a genus 0 Heegaard decomposition of S^3. Moreover, S^3 is the only manifold with a Genus 0 Heegaard decomposition.*

Topology without any additional structure is often an unwieldy field. To offset this, topologists impose additional or auxiliary structures and explore their restrictions on the topology. Since every smooth manifold admits a Riemannian metric, topologists will study manifolds with some geometric characteristics in order to learn something about the underlying topology. The geometrization theorem is one such result. The goal of the geometrization program is to demonstrate a relationship betwen the topology of a manifold and its geometry; namely to show that there is a unique decomposition such that each piece of the decomposition admits a unique model geometry.

Definition 2.6. *Let X be a space, and let G be a Lie group which acts transitively on X where the point stabilizers are compact subgroups of G. The pair (X, G) is said to be a model geometry if there are subgroups $\Gamma \leq G$ such that X/Γ form a manifold, and such that G is maximal with respect to these conditions.*

In dimension 2, there are three such model geometries: spherical, Euclidean, and hyperbolic. Their topological type is encapsulated by the famous Gauß-Bonnet theorem.

Theorem 2.5. *Let Σ be a closed Riemannian 2-manifold*

$$2\pi\chi = \int_\Sigma K dA$$

where $K : \Sigma \to \mathbb{R}$ is the sectional curvature (Gaussian curvature), and χ denotes the Euler characteristic of Σ.

As a consequence, χ determines whether Σ admits a constant Riemannian metric of some geometric type.

- If $\chi < 0$: hyperbolic

- If $\chi = 0$: Euclidean
- If $\chi > 0$: spherical

Thurston's goal was to find a geometrization theorem for 3-manifolds. He proposed that given a 3-manifold, that one could decompose it into pieces which admit one of 8 model geometries. Thurston himself had proven the conjecture for Haken manifolds, but the full geometrization conjecture was finally proven in 2003 by Grigori Perelman.

Theorem 2.6. *(Perelman) Every oriented closed prime 3-manifold Y can be cut along tori such that the interior of each piece admits a finite volume geometric structure of one of eight types:* $\mathbb{H}^3$, $\mathbb{E}^3$, S^3, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, Nil, *or* Sol.

Unlike the connect sum, these decompositions are defined by cutting along tori. A JSJ decomposition[1] is another natural way to define a sum operation is by cutting via tori.

Definition 2.7. *A JSJ decomposition of a closed orientable irreducible 3-manifold is a decomposition of Y into pieces which are either atoroidal or Seifert fibered.*

Definition 2.8. *A Seifert fibered manifold is a circle bundle such that each fiber has a neighborhood that is a standard fibered torus.*

Definition 2.9. *A manifold is said to be aspherical if its universal cover is contractible.*

This is equivalent to saying that all higher homotopy groups are trivial.

2.2 Knots and Links

Definition 2.10. *An n-component oriented link $\vec{L} \subset S^3$ is a smooth embedding $L : \sqcup_{i=1}^n S^1 \to S^3$ where each S^1 is endowed with an orientation. We often identify the map with its image, referring to L as both the image of the smooth embedding and the embedding itself. Moreover, a one-component link is called a knot.*

[1] The name JSJ stems from William Jaco, Peter Shalen, and Klaus Johannson.

Topologically a link is determined only by the number of link components, but knot theory requires a stricter notion of link equivalence in order to capture the intuitive essence of a link as a collection of closed strings which cannot be cut or allowed to pass through itself.

Definition 2.11. *(Ambient isotopy) An ambient isotopy of embeddings $p, q : N \to M$ is a (PL or smooth) homotopy $H : M \times I \to M$ such that $H(-, t) = h_t : M \to M$ is a (PL or smooth) homeomorphism $\forall t$ where $h_0 = Id$ and $q = h_1 \circ p$.*

In other words, it is an isotopy of the ambient manifold M which deforms one submanifold to another.

Definition 2.12. *(Link Equivalence) 2 links $\vec{L}_1, \vec{L}_2 \subset S^3$ are said to be equivalent if there exists an ambient isotopy H such that $H(\vec{L}_1, 1) = \vec{L}_2$.*

Although ambient isotopies give us the desired framework in which to describe knots, it can be hopelessly challenging to explicitely write one down or to show that none even exist. Instead of working explicitly with 3-space, we pass to the study of knot diagrams. Knot diagrams are projections of knots onto $\mathbb{S}^2$ *(which we think of $\mathbb{R}^2$ compactified at infinity)*. The projection takes the appearance of a 4-valent graph where the vertices are called crossings. Each crossing contains an additional piece of data regarding which strand passes over the other.

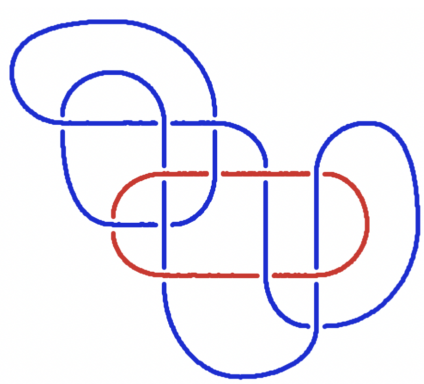


Figure 2.1: Link Diagram (L13a2915)

Equivalence as we have defined it for knots via isotopy can be expressed in diagrams via the *Reidemeister moves*.

$$\{\text{knots / isotopy }\} \cong \{\text{ diagrams / Reidemeister moves }\}$$

We have thus replaced a hard problem with another hard problem which is much more approachable. The goal of knot/link theory is the classification of knots and links. This classification is generally intractable so instead knot theorists develop invariants that can distinguish various classes of knots. Utilizing diagrams and similar combinatorial representations of links yield methods for finding invariants computable from the diagrams such as the Goeritz matrix, linking numbers, Arf invariant, Jones polynomial, Alexander polynomial, etc. Moreover, Knot/link invariants fall into two categories: combinatorial ones as described earlier, or topological which we shall describe next. The more topological approach to finding link invariants is to study the exterior of a link.

Definition 2.13. *Let $L \subset S^3$ be a link. We define the link exterior to be*

$$X_L = \overline{S^3 - \nu(L)}$$

where $\nu(L)$ is a tubular neighborhood of L.

The exterior[2] is a connected compact orientable manifold with toroidal boundary, with the number of boundary components equal to the number of link components. It is clear that the homeomorphism type of X_L is an invariant of the knot since any isotopy of L induces a homeomorphism on X_L.

[2]Here we break slightly from the convention that 3-manifolds are denoted by Y. The notation X_L is appropriate as we are studying the link EXterior

Proposition 2.1. *Let $L = \sqcup_{i=1}^{n} L_i$ be a n-component link in S^3,*

$$\widetilde{H}_k(X_L) = \begin{cases} \mathbb{Z}^n & k = 1 \\ \mathbb{Z}^{n-1} & k = 2 \\ 0 & \textit{else} \end{cases}$$

moreover the generators of $H_1(X_L)$ are represented by meridians μ_i of each L_i, moreover if L is an oriented link, then the orientations $\vec{\mu_i}$ are taken to be compatible with the "right hand rule".

Definition 2.14. *A Seifert surface of K is a connected compact orientable surface $F \subset S^3$ such that $\partial F = K$.*

Proposition 2.2. *Let X_K be the exterior of a knot $K \hookrightarrow S^3$, then $H_2(X_k, \partial X_k) \cong \mathbb{Z}$ and moreover, generators of $H_2(X_K, \partial X_K)$ correspond to Seifert surfaces of K*

Proof. By Theorem 2.1 it follows that

$$H_n(X_K) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

and that ∂X_k is homeomorphic to a torus, thus

$$H_n(\partial X_K) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

Now consider the following long exact sequence

$$\begin{array}{ccccccccccc} \longrightarrow & H_2(X_k) & \longrightarrow & H_2(X_k, \partial X_k) & \xrightarrow{\partial} & H_1(\partial X_k) & \xrightarrow{\iota_*} & H_1(X_k) & \xrightarrow{j_*} & H_1(X_k, \partial X_k) & \longrightarrow \\ & \wr\| & & & & \wr\| & & \wr\| & & & \\ & 0 & & & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & & \end{array}$$

Notice that ∂ is injective. Since ι_* is the induced map by inclusion, we know that $\mathrm{im}\iota_*$ is the homology class generated by meridians of K, thus $\mathrm{im}\iota_* = [\mu] \cong \mathbb{Z}$, thus it follows that $H_2(X_k, \partial X_k) \cong \mathbb{Z}$ and moreover is generated by surfaces whose boundary corresponds to longitudes of K, in other words, generated by Seifert surfaces F of K. □

The following proposition follows via a similar argument.

Proposition 2.3. *Let X_L be the complement of an n-component link $L \hookrightarrow S^3$, then $H_2(X_L, \partial X_L) \cong \mathbb{Z}^n$ and generators of $H_2(X_L, \partial X_L)$ correspond to surfaces which cobound components of L*

So the homology type of X_L is determined by the number of components of the link. Other obvious invariants would be the homotopy groups of X_L, in particular the fundamental group $\pi_1(X_L)$ which is called the knot (link) group. We discuss methods to compute the link group in Appendix B, but for the moment what can we say about its properties? We know that $\pi_1^{ab}(X_L) = H_1(X_L) = \mathbb{Z}^n$. Additionally, $\pi_1(X_L)$ is finitely presented. Higher homotopy groups turn out to be trivial.

Theorem 2.7. *Let L be a non-split link in S^3, then $X_L = S^3 - L$ is a $K(\pi_1(X_L), 1)$ space.*

Proof. If π_2 is non-trival, then by the Theorem 2.2 (Sphere theorem), there exists an element $[f : S^2 \to X_L] \in \pi_2(X_L)$ that admits an embedding Σ. Since this embedding misses L we can also think of it as an embedding into S^3. By the Alexander-Schönflies theorem it follows that this embedding bounds two B^3's: $S^3 = B_1^3 \cup_\Sigma B_2^3$. Since L is non-split, we may assume WLOG $L \subset B_1^3$, but then $\Sigma = \partial B_2^3$ in $S^3 - L$, thus Σ is nullhomotopic in X_L, and thus $\pi_2(X_L) = 0$. Since $\overline{S^3 - L} \simeq S^3 - L$ is open, it follows from Proposition 3.29 in Hatcher[4] that $H_3(X_L) = 0$. Since $H_i(X_L) = 0$ for all $i \geq 2$ it follows from the Hurewicz theorem that $\pi_i(X_L) = 0$ for all $i \geq 2$. □

Proposition 2.4. *Non-split link exteriors are Haken, and in particular knot exteriors are Haken*

Proof. Seifert surfaces are compact properly embedded surfaces. If F is a genus minimizing Seifert surface, then F is incompressible. It remains to show that the exteriors of non-split links are prime. This follows from the fact that non-split link exteriors are irreducible since any embedded sphere $S^2 \hookrightarrow S^3 - L$ is the boundary of a B^3 after applying the Alexander-Schönflies theorem. Since irreducible manifolds are prime, the result follows. $\square$

How strong is the homeomorphism type of X_L? If L is a knot, then it is in fact a complete invariant! A famous theorem of Gordon and Luecke states that $X_{K_1} \cong X_{K_2}$ if and only if $K_1 \sim K_2$.

Theorem 2.8. *(Gordon-Luecke [5]) Let K_1 and K_2 be knots in S^3, then $K_1 \sim K_2$ if and only if $X_{K_1} \cong X_{K_2}$*

Unfortunately, the same is not true for links with more than one component. Nevertheless, manifolds obtained as link exteriors are still interesting in and of themselves, especially in the case where their interiors admit hyperbolic structures.

2.3 Hyperbolic Geometry

One can always endow a smooth manifold with a Riemannian metric. A smooth manifold M is said to be hyperbolic if it admits a Riemannian metric with constant negative sectional curvature: $K(p, \sigma) = -1$ for all $p \in M$ and $\sigma \in \mathrm{Gr}_2(\mathrm{T_pM})$. Hyperbolic manifolds are locally modelled on hyperbolic space $\mathbb{H}^n$. This means that for any $p \in M$, there is a neighborhood isometric to $\mathbb{H}^n$. There are several isometric descriptions for models of hyperbolic space, the most common being the following.

Upper half space: $\mathbb{H}^n = \{(x_1, ..., x_n) | x_n > 0\}$ where

$$ds^2 = \frac{dx_1^2 + ...dx_n^2}{x_n^2}$$

The Poincaré disk (or ball model) is given by $\mathbb{H}^n = \{(x_1, ..., x_n) |\ |x_n| < 1\}$:

$$ds^2 = \frac{dx_1^2 + ... + dx_n^2}{(1 - x_1^2 - ... - x_n^2)^2}$$

By the Cartan-Hadamard theorem, every complete, connected, simply connected hyperbolic manifold is isometric to $\mathbb{H}^n$. Moreover, this implies that $\mathbb{H}^n$ is the universal cover for every complete connected hyperbolic manifold. Since $\mathbb{H}^n$ is contractible this also implies that hyperbolic manifolds are aspherical. If M is a smooth manifold with $\dim M = n \geq 3$ that admits a hyperbolic structure with finite volume, then that hyperbolic structure is unique. This is known as Mostow rigidity, and it tells us that geometric invariants are also topological invariants.

Compact orientable hyperbolic manifolds admit a finite volume metric if and only if ∂M is empty or toroidal and $M \neq S^1 \times D^2$, $M \neq T^2 \times I$. Suppose M is hyperbolic with finite volume with n (toroidal) boundary components, then if we remove the boundary we obtain a *cusped 3-manifold*. The neighborhoods around the "boundary" are called cusps. The region near a boundary looks like $T^2 \times [1, \infty)$ (infinity going towards the end of the manifold). These regions have a local geometric structure isometric to the subset of the upper half plane model with $z \geq 1$. In this region, each cross section $t \times T^2$ is an embedded torus in $M - \partial M$; however, when $0 < t < 1$ there exists a point t_{min} where the torus fails to be embedded and becomes immersed. We call $\{(x, y, t) | t \geq t_{min}\}$ a maximal cusp.

A more algebraic viewpoint of studying hyperbolic manifolds is to study the group of isometries of $\mathbb{H}^3$. Namely this group is $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{I}\} = \mathrm{Isom}^+(\mathbb{H}^3)$. This group acts on $\mathbb{H}^3$ via fractional linear transformations. Let $(z, t) \in (\mathbb{C}, \mathbb{R}_{>0})$ be a point in the upper half space model. Let $A \in \mathrm{PSL}_2(\mathbb{C})$, then the action is

$$A(z, t) = (Az, At) = \left(\frac{az + b}{cz + d}, \frac{at + b}{ct + d} \right).$$

Notice that $\mathrm{PSL}_2(\mathbb{C})$ acts on the boundary sphere $\partial \mathbb{H}^3$ by fractional linear transformations.

Definition 2.15. *A triangle is the convex hull of 3 distinct non-colinear points in* $\mathbb{H}^3$. *A triangle is said to be* ideal *if its vertices lie on* $\partial \mathbb{H}^3$.

Theorem 2.9. *Given* $a, b, c \in \partial \mathbb{H}^3$*, there is a unique* $A \in \mathrm{Isom}^+(\mathbb{H}^3)$ *sending the hyperbolic ideal triangle* $(a, b, c) \mapsto (0, 1, \infty)$ *and this isometry is defined by*

$$A(p) = \frac{(p - a)(c - b)}{(b - a)(c - p)}$$

Definition 2.16. *A topological group G is said to be discrete if it contains no limit points. A subgroup H of a topological group is said to be discrete if it is a discrete group endowed with the subspace topology.*

Example 2. $\mathbb{Z} < \mathbb{R}$ *is a discrete subgroup of* $\mathbb{R}$ *with the usual topology with the group operation given by addition.*

Definition 2.17. *A* Kleinian Group Γ *is a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$

Kleinian groups act on $\mathbb{H}^3$ and may contain limit points on the boundary sphere. $p \in \partial\mathbb{H}^3 = S^2_\infty$ is a limit point if for a sequence of elements $g_n \in \Gamma$ and for some $\xi \in \mathbb{H}^3$

$$\lim_{n\to\infty} g_n(\xi) = p.$$

The limit set $\Lambda(\Gamma)$ is the set of limit points under the action of Γ. The complement $\Omega(\Gamma) = S^2_\infty - \Lambda(\Gamma)$ is called the regular set.

Discreteness implies that Kleinian groups act discontinuously on $\mathbb{H}^3$, thus we can construct fundamental domains.

Definition:

A fundamental domain corresponding to a Kleinian group G is a closed subset $D \subset \mathbb{H}^3$ such that the following 3 conditions are met:

- The orbit under G is the entire space $\mathbb{H}^3$: $\bigcup_{g\in G} gD = \mathbb{H}^3$
- $D^o \cap gD^o = \emptyset$ for all nonidentity $g \in G$
- ∂D is measure 0

These exist and are easily constructed; choose $p \in \mathbb{H}^3$ such that $g(p) \neq p$ for all $g \in G$, then

$$D_p(G) = \{q \in \mathbb{H}^3 | d(q,p) \leq d(g(q),p), \forall g \in G\}.$$

Such constructions $D_p(G)$ are called Dirichlet domains. If a Kleinian group G has a fundamental domain D of finite hyperbolic volume , then G is said to be of finite covolume.

Example 3. *Bianchi Groups: Let* d *be a square free positive integer. Let* $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{-d})$ *(this ring depends on whether* $\mathrm{d} \equiv 1 \bmod 4$ *or* $3 \bmod 4$*) denote the ring of integers of the field* $\mathbb{Q}(\sqrt{-d})$*. We define the Bianchi groups to be* $\mathrm{PSL}_2(\mathcal{O}_\mathrm{d})$ *which forms a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *and thus are examples of Kleinian groups. As an example consider* $\mathrm{d} = 1$*, then* $\mathcal{O}_1 = \mathbb{Z}[i]$*, the ring of Gaussian integers which form the ring of integers for* $\mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1})$*. Furthermore* $\mathrm{PSL}_2(\mathcal{O}_1)$ *consists of the set of* 2×2 *matrices of determinant equal to 1 or* i *mod its center.*

Kleinian groups play a fundamental role in the study of hyperbolic 3-manifolds. When $Y = \mathbb{H}^3/\Gamma$ is a manifold, one sees that Γ acts via deck transformations on $\mathbb{H}^3$ thus $\Gamma = \pi_1(Y)$. Conversely, if M is hyperbolic with fundamental group Γ, then Γ acts as the group of deck transformations on its universal cover $\mathbb{H}^3$ (by Cartan-Hadamard) so it can be identified as a Kleinian group. We see that hyperbolic 3-manifolds are determined by their fundamental group. We also have sufficient conditions for when $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ determines a hyperbolic manifold.

Theorem 2.10. *Any complete hyperbolic 3-manifold* M *is equivalent to* $\mathbb{H}^3/\Gamma$ *where* Γ *is a torsion free Kleinian group.*

One interesting consequence of what we just described is that if Y is the complement of a hyperbolic knot in S^3, then the knot is determined by the knot group by the Gordon Luecke Theorem. In practical applications, it is difficult to discern when two groups are isomorphic. Fortunately, the word problem on fundamental groups of Haken manifolds (which include hyperbolic manifolds) is solvable [6].

Example 4. *(Bianchi Groups) The spaces* $M_G = \mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_\mathrm{d})$ *obtained from Bianchi groups are* *generally not manifolds but orbifolds which are noncompact and with finite volume*

$$\mathrm{Vol(M)} = \frac{|\mathrm{D}|^{3/2}}{4\pi}\zeta_{\mathbb{Q}(\sqrt{-\mathrm{d}})}(2)$$

where D is the discriminant of $\mathbb{Q}(-d)$. Additionally, since

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an element of order 2 in $\Gamma = \mathrm{PSL}_2(\mathcal{O}_1)$, it follows that M_Γ is not a manifold.

3. THURSTON NORM: DEFINITION AND BASIC PROPERTIES

The goal of this section is to present the construction of the Thurston norm, and explore its basic properties. In addition to Thurston's original paper this presentation takes inspiration from Calegari's Foliations and Geometry of 3-Manifolds [7].

Lemma 3.1. *Let Y be a compact oriented 3-manifold. Every element in $H_2(Y, \partial Y; \mathbb{Z})$ is represented by the fundamental class of an embedded oriented surface Σ. If α is divisible by $k \in \mathbb{N}$, then Σ is a union of k subsurfaces, each representing α/k.*

Proof. Suppose $\alpha \in H_2(Y, \partial Y; \mathbb{Z})$ and let $D_Y(\alpha) \in H^1(Y)$ be its Poincaré dual. Since S^1 is a $K(\mathbb{Z}, 1)$ space, it follows that $H^1(Y) \cong [Y, S^1]$, so there exists a unique homotopy class of maps $f : Y \to S^1$ such that $f_\alpha^*(u) = D_Y(\alpha)$, where $u \in H^1(S^1)$. We may assume that f_α is smooth. If $x \in S^1$ is a regular value of f_α (that is a value for which $df_\alpha \neq 0$) then $N = f_\alpha^{-1}(x)$ is a codimension 1 submanifold of Y whose fundamental class in Y yields α.

Now let $\alpha \in k\beta \in H_2(Y, \partial Y; \mathbb{Z})$ with corresponding maps $f_\alpha, f_\beta : M \to S^1$. Let $p : S^1 \to S^1$ be a k-fold covering map, then

$$f_\alpha^*(u) = D_Y(\alpha) = kD_Y(\beta) = f_\beta^*(ku) = (p \circ f_\beta)^*(u)$$

f_α and $p \circ f_\beta$ are homotopic so by the lifting property we may homotope f_β such that $f_\alpha = kf_\beta$, then $p^{-1}(x) = x_1, ..., x_k$ are regular values of f_β, thus $f_\alpha^{-1}(x) = f_\beta^{-1}(x_1) \cup ... \cup f_\beta^{-1}(x_k)$ which is disjoint union of surfaces representing β. □

Another route of proving the previous lemma is the following. Take a triangulation of Y, so every class in $H_2(Y, \partial Y; \mathbb{Z})$ is a sum of 2-simplices. For a given 2-simplex there may be n in the sum. Push each off slightly. At each edge in the triangulation we need to pair up simplices. Since our class is in $Z_2(Y; \mathbb{Z})$ there is a way to pair all the triangles meeting edges. The neighborhood around an edge looks like $I \times D^2$ and for a fixed disc $p \times D^2$ the triangles meet this in segments extending from p, thus meeting $\partial D^2 = S^1$ in a fixed number of points. Keeping in mind the orientations we can then draw line segments connecting the paired points and thus resolve the interior of the edges. We are left now with a surface with singularities at each vertex. These can be resolved by taking a ball around each vertex and noting that $S \cap \partial B^3$ will be a collection of S^1s (This is called the link of the singularity, e.g. Figure 3.1).

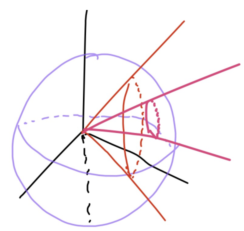


Figure 3.1: Link of a Singularity

We can take an innermost S^1 and cap it off by gluing a disk then pushing it off. Repeating this process of taking innermost circles and capping them resolves the singularity. This does not change the homology class since we are simply changing the class via the boundary of a higher dimensional cell. Moreover what we are left with after this pair will be copies of an embedded compact surface collectively representing $\alpha \in H_2(Y, \partial Y; \mathbb{Z})$.

Given two oriented surfaces $\Sigma_1, \Sigma_2 \subset Y$ with Y closed which intersect transversely, we can perform an operation respecting orientation, χ, and $H_2(Y, \partial Y; \mathbb{Z})$. By transversality $\Sigma_1 \cap \Sigma_2 = \gamma_1 \sqcup ... \sqcup \gamma_n$, where γ_i are simple closed curves. At each γ remove an annular neighborhood of γ in Σ_1 and then attach two annuli such that the resultant is an oriented surface.

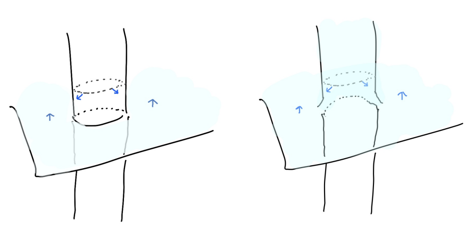


Figure 3.2: Oriented Sum

Lemma 3.2. *The oriented sum of two transverse surfaces Σ_1 and Σ_2 preserves χ and $H_2(Y;\mathbb{Z})$*

Proof. Oriented sum is defined by deleting an annulus $\mathbb{A}$ and gluing back two annuli such that the resultant is an embedded oriented surface. Since $\chi(\mathbb{A}) = 0$, it follows that this procedure leaves χ unchanged. Now the sum of two oriented surfaces forms a cycle in Y, and the oriented sum defined by cutting and pasting preserves the cycle in C_2 and then perturbing it so that the two are pushed off *(Figure 3.3 for a 2-d example)*, thus the resultant is homologous to $\Sigma_1 + \Sigma_2$.

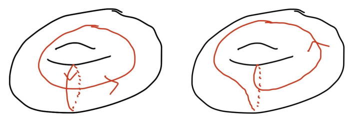


Figure 3.3: Oriented Sum of Curves

□

A compact surface Σ is classified by its genus and its number of boundary components. Notice that increasing the genus by one decreases the Euler characteristic χ by 2. Similarly, puncturing the surface decreases the Euler characteristic by 1. Intuitively more complex surfaces are those with smaller χ. This leads to the following definition.

Definition 3.1. *Let $\Sigma = \sqcup_{i=1}^{n}\Sigma_i$ be a possibly disconnected surface. The complexity χ_- of Σ is defined as*

$$\chi_-(\Sigma) = \sum_{i=1}^{n} \max\{0, -\chi(\Sigma_i)\}$$

where χ is the Euler characteristic.

Definition 3.2. *Let Y be a compact connected oriented 3-manifold. The integral Thurston norm $\|\cdot\|_T : H_2(Y, \partial Y; \mathbb{Z}) \to \mathbb{Z}_+$ is a function defined by*

$$\|\alpha\|_T = \inf\{\chi_-(\Sigma) | \Sigma \hookrightarrow S^3 \quad [\Sigma, \partial\Sigma] = \alpha\}$$

By Poincaré duality we can equivalently define $\|\cdot\|_T$ on $H^1(Y;\mathbb{Z})$. Next we will show that $\|\cdot\|_T$ defines a seminorm on $H_2(Y,\partial Y;\mathbb{Z})$ which then uniquely extends to a seminorm on $H_2(Y;\partial Y;\mathbb{R})$

Theorem 3.1. *Let $k \in \mathbb{Z}$ and $\alpha, \beta \in H_2(Y,\partial Y;\mathbb{Z})$, then $|k|\|\alpha\|_T = \|k\alpha\|_T$, $\|\alpha+\beta\|_T \leq \|\alpha\|_T + \|\beta\|_T$.*

Proof. Let $\alpha \in H_2(Y,\partial Y;\mathbb{Z})$ be represented by norm minimizing surface S, $||-\alpha|| = ||\alpha||$ as $-\alpha$ is represented by S endowed with the opposite orientation. Lemma 3.1 tells us that $\|k\alpha\|$ can be represented by $|k|$ disjoint surfaces S_α which represent α. Take $S_\alpha = S$ to be norm minimizing and we see that $|k|\|\alpha\| = \|k\alpha\|$.

Let $\alpha, \beta \in H_2(Y,\partial Y;\mathbb{Z})$ and let $\Sigma_\alpha, \Sigma_\beta$ be norm minimizing surfaces for α and β respectively. WLOG α and β intersect transversally. $\Sigma_\alpha \cup \Sigma_\beta$ is a representative of class $\alpha+\beta$. The goal is to use the fact that the oriented sum preserves both Euler characteristic and homology class; however, before doing so we need to remove possible surfaces of positive Euler characteristic *(since for example $\chi_-(\Sigma + S^2) = \chi_-(\Sigma)$ but $\chi(\Sigma + S^2) \neq \chi(\Sigma)$).*

By transversality, $\Sigma_\alpha \cap \Sigma_\beta$ consists of finitely many arcs and simple closed curves. Suppose $C \in \Sigma_\alpha \cap \Sigma_\beta$ is an innermost closed curve which bounds a disk in Σ_α (wlog Σ_β), then we may perform a surgery on Σ_β by removing a tubular neighborhood C and gluing in two disks on either side. This operation does not affect the homology class β. It clearly does not increase the norm and cannot decrease it by assumption that Σ_β was norm minimizing. Repeat this step to remove all components of $\Sigma_\alpha \cap \Sigma_\beta$ which bound disks on either Σ_α or Σ_β. Similarly we can remove components which bound disks relative to the boundary. For every arc in $\Sigma_\alpha \cap \Sigma_\beta$ homotopic (relative to the endpoints) to boundary of Σ_α (wlog Σ_β), this arc spans a disk D on Σ_α and assuming the arc is innermost we may perform a surgery by removing a neighborhood around I and attaching disks D_1 and D_2 that run parallel to the disk D illustrated in figure 3.4.

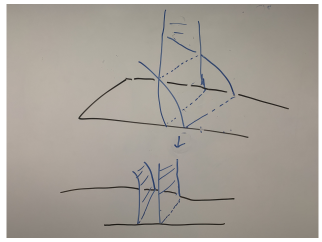


Figure 3.4: Resolving Nonessential Arcs

This surgery neither changes homology class, nor increase the complexity.

After performing these operations we obtain modified surfaces Σ'_α and Σ'_β. Having eliminated "waste" in the sense that we have removed components of positive Euler characteristic (spheres and disks), we can perform the oriented sum of S'_a and S'_b. By lemma 3.2, the oriented sum preserves Euler characteristic as well as homology class in $H_2(Y, \partial Y)$, so we learn that $\|a+b\|_T \leq \|a\|_T + \|b\|_T$. $\square$

The first property tells allows us to linearly extend $\|\cdot\|$ to $H_2(Y, \partial Y; \mathbb{Q})$ and then the second property allows us uniquely extend to $H_2(Y, \partial Y; \mathbb{R})$. To see how we use the following lemma

Lemma 3.3. *Any integral (semi-)norm (positive, absolutely homogeneous, and subadditive)* $\|\cdot\|_\mathbb{Z} : \mathbb{Z}^n \to \mathbb{Z}$ *extends to a (semi-)norm* $\|\cdot\|_\mathbb{R} \to \mathbb{R}$.

Proof. First we extend to a rational norm $\|\cdot\|_\mathbb{Q} : \mathbb{Q}^n \to \mathbb{Q}$. For any nonzero $v \in \mathbb{Q}^n$, there exists $m \in \mathbb{Z}$ such that $mv \in \mathbb{Z}^n$, then we define the rational extension by

$$\|\alpha\|_\mathbb{Q} = \frac{1}{m}\|m\alpha\|_\mathbb{Z}$$

which is well defined since $\|\cdot\|_\mathbb{Z}$ is absolutely homogenous and linear on rays. Since $\|\cdot\|_\mathbb{Q}$ is Lipschitz it has a continuous extension to $\mathbb{R}^n \to \mathbb{R}$. $\square$

Definition: Thurston Norm

The Thurston Norm $||\cdot||_T$ is the unique extension of $||\cdot||$ to $H_2(Y, \partial Y; \mathbb{R})$ as described above.

Let K_T denote the set of classes α in $H_2(Y, \partial Y; \mathbb{R})$ such that $||\alpha||_T = 0$

Lemma:

K_T is precisely the subspace spanned by lattice points p such that $||p||_T = 0$.

Proof. Indeed suppose that $a \in K_T$, then by linearity of $||\cdot||_T$ it follows that $||\cdot||_T$ vanishes on all points along the ray. There exists some $s \in \mathbb{R}$, and lattice point $z \in \mathbb{Z}^d$ such that $||sa - z||_{euc} <<< 1$. It follows then that since $||\cdot||_T$ takes integer values on lattice points, that $||z||_T = 0$. It follows again by linearity on the rays that the entire ray along z has zero Thurston norm, moreover this holds for any lattice point arbitrarily close to the ray defined by a. It then follows that K_T is indeed contained in the closure of the linear span of lattice points p such that $||p||_T = 0$. $\square$

Thus K_T is spanned by classes for which a homology class may be represented by surfaces of non-negative Euler characteristic. Note that since hyperbolic manifolds admit no essential spheres, tori, nor annuli, it follows that $\|\cdot\|_T$ is non-degenerate for hyperbolic manifolds.

Recall that a norm is determined entirely by the shape of its “norm ball”.

Definition 3.3. *The Thurston norm ball of Y is the following set:*

$$B_Y = \{\phi \in H^1(Y; \mathbb{R}); ||\phi||_T \leq 1\}.$$

When $\|\cdot\|_T$ is non-degenerate, its norm ball turns out to be a symmetric polytope with finitely many rational vertices. This is a consequence of a more general theorem:

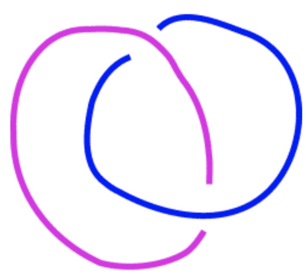
Theorem 3.2. *Let $||\cdot||$ be a norm a finite dimensional $\mathbb{R}$-vector space containing an embedded $\mathbb{Z}$-lattice such that $||\cdot||$ takes integral values on the lattice. Then the unit ball of $||\cdot||$ is a finite sided polytope with rational vertices.*

Example 5. *Let $K \hookrightarrow S^3$ be a knot. Let $X_k = S^3 \backslash N(K)$ be the knot exterior. Then $H_2(X_K, \partial X_k; \mathbb{R}) = \mathbb{R}$, where by proposition 2.2 the generator is represented by Seifert surfaces F of K. Moreover,*

$$||[F]||_T = \begin{cases} 0 & K = 0_1 \\ 2g-1 & else \end{cases}$$

So for knots, the Thurston norm is given by the knot genus. The norm ball is simply the line segment $[-1/(2g-1), 1/(2g-1)] \subset \mathbb{R}$.

Example 6. *Let L be the Hopf link, that is the following two component link:*



Each link component λ_1 and λ_2 bounds a properly embedded annulus. To see this take a disk bounded by λ_i which intersects the other component transversely once, thus in the link complement, we remove a neighborhood about λ_2 intersecting the disk, yielding an annulus. Let $\mu_{1,2}$ be generators of $H_2(M, \partial M)$, and recall by proposition 2.3 surfaces representing $\mu_{1,2}$ correspond to surfaces whose boundary is $\lambda_{1,2}$. Since $\chi(\mathbb{A}) = 0$, it follows that $||\mu_1||_T = ||\mu_2||_T = 0$, thus $||\cdot||_T$ is degenerate on X_L, and the unit ball is simply the entire space $\mathbb{R}^2$.

Proposition 3.1. *If Y is irreducible and Σ is norm minimizing for $[\Sigma] \in H_2(Y, \partial Y; \mathbb{Z})$ then Σ is incompressible.*

Proof. Suppose Σ is norm minimizing and compressible, then recall by lemma 2.1 that compression increases the Euler characteristic by 2.

Suppose compressing Σ along a curve $\gamma \subset \Sigma$ is nonseperating. Since Y is incompressible, the surface Σ' obtained by compression on Σ is not a sphere, thus $\chi_-(\Sigma') < \chi_-(\Sigma)$ and since compression preserves homology class it follows that Σ is not norm minimizing, hence a contradiction.

If compression along γ is seperating then the result is two surfaces Σ_1, Σ_2 such that their Euler characteristics sum to $\Sigma + 2$ and by irreducibility of Y, neither Σ_1 nor Σ_2 is a sphere and thus $\chi_-([)\Sigma') < \chi_-(\Sigma)$ and again we have a contradiction.

□

Example 7. *Suppose $Y = S^3/\Gamma$ is a spherical 3-manifold, then we know that $\pi_1(Y)$ is finite (and isomorphic to Γ). By the Hurewicz theorem, we know that $H_1(Y)$ must also be finite, and so the universal coefficient theorem tells us that $H^1(Y)$ has rank 0, thus the Thurston norm is zero. Another viewpoint is that any incompressible surface Σ is π_1 injective and thus has finite fundamental group and thus is either a sphere or non-orientable.*

An important property of $\|\cdot\|_T$ is that K_T may have nonzero dimension. What can we learn about our manifold from K_T? Well clearly if $\dim K_T > 0$ then Y is not hyperbolic; However, is $\dim K_T = 0$ sufficient to tell that Y is hyperbolic?

It turns out that not only do we need to know $\dim K_T(Y)$ but $\dim K_T(\widetilde{Y})$ for all finite coverings of Y. Let $b_1(Y) = \dim(H_1(Y; \mathbb{R}))$,

$$r(Y) = \begin{cases} 0 & b_1(Y) = 0 \\ \frac{K_T(Y)}{b_1(Y)} & b_1(Y) > 0 \end{cases}$$

and if $C(Y)$ is the collection of finite covers of Y, define $\hat{r}(Y) = \sup_{\widetilde{Y} \in C(Y)} r(\widetilde{Y})$. If Y is hyperbolic, recall that $\pi_1(Y) \leq \mathrm{PSL}_2(\mathbb{C})$ is Kleinian and moreover by the Galois correspondence any finite cover $\widetilde{Y}$ corresponds to a finite index subgroup H of $\pi_1(Y)$. Then $H \leq \pi_1(Y) \leq \mathrm{PSL}_2(\mathbb{C})$ is also discrete and thus Kleinian and thus $\widetilde{Y}$ is hyperbolic. Thus we know that $K_T = 0$ for all finite regular covers of Y when Y is hyperbolic ($\hat{r}(Y) = 0$). When Y is also aspherical, then the converse also holds as seen in the following theorem.

Theorem 3.3. *[8] Let Y be an aspherical 3-manifold with empty or toroidal boundary, then Y is hyperbolic if and only if $\hat{r}(Y) = 0$*

A finite cover $\hat{Y}$ is said to be subregular if it correspond to a subnormal subgroup of $\pi_1(Y)$. Let $\rho(Y)$ be the infimum of $r(\hat{Y})$ for all subregular covers of Y and moreover let $\hat{\rho}(Y) = \sup_{\widetilde{Y} \in C(Y)} \rho(\widetilde{Y})$. A graph manifold is a manifold admitting no hyperbolic components in its JSJ decomposition.

Theorem 3.4. *[8] Let Y be an aspherical 3-manifold with empty or toroidal boundary. If Y is a graph manifold, then $\hat{\rho}(Y) = 1$. If Y is not a graph manifold, then $\hat{\rho}(Y) = 0$.*

3.1 Fibered Faces

If $\alpha \in H^1(Y;\mathbb{Z})$ can be represented by a nonsingular closed 1-form ω with integral periods, then $f = \int_\gamma \omega : M \to S^1$ is a smooth map to S^1 where every $p \in S^1$ is a regular value. By the rank theorem, $f^{-1}(p)$ is a codimension 1 submanifold of Y for all p, and thus defines a fibration:

$$\begin{tikzcd} \Sigma \arrow[r, hook] & Y \arrow[d, "f"] \\ & S^1 \end{tikzcd}$$

where Σ is a surface representing α. What Thurston realized is that for any class α that defines a fibration over S^1, any other class in the cone of the top dimensional face including α also realizes such a fibration. If α defines a fibration over S^1 and $\Sigma \hookrightarrow Y$ is a norm realizing fiber, then

$$Y \cong \frac{\Sigma \times [0,1]}{\sim}$$

where $\Sigma \times \{0\}$ is identified with $\Sigma \times \{1\}$ by a surface automorphism $\psi : \Sigma \to \Sigma$. If Y is hyperbolic, then by the Thurston-Nielson classification of surface automorphisms and Thurston's geometrization conjecture, ψ is pseudo-Anosov. Later, Thurston and Cannon realized that if Y is closed hyperbolic and fibers over the circle; that Σ admits a hyperbolic structure and the lift of the surface to an action on universal covers $\mathbb{H}^2$ and $\mathbb{H}^3$ induces a continuous surjective map from $\partial\mathbb{H}^2 \to \partial\mathbb{H}^3$, that is a space filling curve [9].

What the Thurston norm tells us about these fibered manifolds is that they do not fiber uniquely. Given a closed hyperbolic manifold Y which fibers over the circle with $b_2(Y) \geq 2$, there are infinitely many ways in which Y fibers. This rests upon the following result.

Theorem 3.5. *[3] If Y is connected compact oriented 3-manifold and $\alpha \in H^1(Y;\mathbb{Z})$ determines a fibration over S^1 then the ray determined by α lives in the interior of a top dimensional face of $B_{\|\cdot\|_T}$ and any class in this face determines a fibration over S^1. Such faces are called fibered faces.*

Thus any primitive class in an open cone of a fibered face determines a unique fibration. Moreover, $\mathrm{D_Y}(\alpha)$ is the fundamental class for a fiber of this fibration. Moreover since α must live in the interior of a face, classes which are in the rays of the vertices of $B_{\|\cdot\|_T}$ are realized by incompressible surfaces which do not represent a fiber.

Lemma 3.4. *[10] Let $\Sigma \hookrightarrow Y \xrightarrow{f} S^1$ be a fibration and let $\alpha = [f^*(d\theta)]$ where $d\theta$ generates $H^1(S^1)$. Then $\|\alpha\|_T = -\chi(\Sigma)$*

Proof. Suppose α is primitive (that is the image of $\pi_1(Y)$ via the corresponding homotopy class induced by α surjects onto $\mathbb{Z} = \pi_1(S^1)$) and thus Σ is connected, then $\Sigma \times \mathbb{R}$ is the infinite cyclic cover of Y defined by α. If $\chi(\Sigma) \geq 0$ then the result follows, but now assume that $\chi(\Sigma) < 0$. Let F be any surface representing α then for any component $F_i \subset F$, the inclusion of its fundamental group is a subgroup of $\pi_1(\Sigma)$. Assume $\chi_-(F) = \|\alpha\|$ then we know F is an incompressible surface (if not then we could compress F without changing the homology class) and it follows from Theorem 2.3 that $\pi_1(F_i) \hookrightarrow \pi_1(\Sigma \times \mathbb{R}) = \pi_1(\Sigma)$. We conclude that F_i is a finite cover of Σ and thus $\chi_-(F) \geq \chi_-(F_i) \geq \chi_-(\Sigma)$ and the result follows. $\square$

3.2 Some 4 Dimensional Analogs

In this section X denotes a closed orientable 4 manifold (for obvious reasons). As in dimensional 3, we can analogously show that every class $\alpha \in H_2(X, \mathbb{Z})$ may be represented by a smoothly embedded surface.

Proof. Poincaré duality tells us that $H^2(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$ and there again exists a bijective corresponondance between $H^2(X)$ and $[X, K(\mathbb{Z},2)] = [X, \mathbb{CP}^\infty]$. By the cellular approximation theorem, every class in $[X, \mathbb{CP}^\infty]$ can be represented by a cellular map, and since $\dim X = 4$, it follows that they map into the 4 skeleton of $\mathbb{CP}^\infty$ which is simply $\mathbb{CP}^2$, thus $H^2(X) \cong [X, \mathbb{CP}^2]$. Recall that $H^2(\mathbb{CP}^2) = \mathbb{Z}$ is freely generated by the Poincaré dual of the fundamental class of the complex projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ sitting inside. Thus for $\alpha \in H^2(X;\mathbb{Z})$,

$$\alpha = f_\alpha^*(D^{-1}([\mathbb{CP}^1]))$$

We can further assume that f is smooth and transverse to $\mathbb{CP}^1$ and thus $f^{-1}(\mathbb{CP}^1)$ is a smoothly embedded surface in X. □

This proof is more or less identical to our first proof of lemma 3.1. Smoothness is a necessary assumption since the PL, topologically flat, and smooth categories are vastly different for 4-manifolds. See the literature on knot concordance for a taste of this theory [11]. Generally computations for the complexity of such homology classes are significantly more difficult than in dimension 3. Prior to the advent of Seiberg-Witten theory topologists were struggling to show whether homology class could be represented by a sphere. After Seiberg-Witten theory a lot more became known, in particular the so called adjunction inequality paved the way for many results.

Theorem 3.6. *Let X be a simply connected connected 4 manifold with $b_2^+(X) \geq 2$, and Σ any homologically non trivial embedded connected surface such that*

$$\Sigma \cdot \Sigma \geq 0$$

then for every basic class κ of X,

$$\chi(\Sigma) + \Sigma \cdot \Sigma \leq \kappa \cdot \Sigma$$

(where b_2^+ refers to the number of positive eigenvalues of the intersection form).

In essense the adjunction inequality yields a lower bound on the complexity of a surface representing a homology class. We will later see an analog of the adjunction inequality in the section on foliations. This basic class, defined by Kronheimer and Mrowka [12], is less nebulous when X is symplectic.

Theorem 3.7. *If X is a symplectic closed 4-manifold with $b_2^+ \geq 2$, Σ a homologically nontrivial smoothly embedded oriented surface with $\Sigma \cdot \Sigma \geq 0$ then*

$$\chi(\Sigma) + \Sigma \cdot \Sigma \leq \langle c_1(X), \Sigma \rangle$$

where $c_1(X)$ is the first Chern class.

Some cases for the minimal genus problem are known, including the following which was an outstanding conjecture until the 1990s.

Theorem 3.8. *(Thom Conjecture) In $\mathbb{CP}^2$, the minimum genus of a smooth surface representing $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ is*

$$g = \frac{1}{2}(d-1)(d-2)$$

which was proven by Kronheimer and Mrowka using Seiberg-Witten invariants [13]. Unfortunately little has been achieved in obtaining a seminorm structure on $H_2(X; \mathbb{Z})$.

Lemma 3.5. *[14] Let Σ be a connected embedded surface in a 4-manifold X. Assume either $\Sigma \cdot \Sigma > 0$ or $\Sigma \cdot \Sigma = 0$ with Σ not a sphere, then for every integer n, the class $n[\Sigma]$ may be represented by a connected embedded surface Σ_n such that*

$$\chi(\Sigma_n) + \Sigma_n \cdot \Sigma_n = n(\chi(\Sigma) + \Sigma \cdot \Sigma).$$

4. ALEXANDER MODULES AND POLYNOMIALS

Let X be a (path connected) topological space with $G = \pi_1(X)$ its fundamental group and G' its derived subgroup which is the kernel of the Hurewicz homomorphism $G \to H_1(X)$. As $G' \lhd G$ there exists a normal covering space $\widetilde{X}$ such that $G/G' \cong H_1(X)$ forms the group of deck transformations of $\widetilde{X}$. Assume momentarily that $G/G' = \langle t \rangle$. If X is a CW complex, then the action on any cell by t induces an action on the cell complex endowing $C_k(\widetilde{X})$ the enhanced structure of a $\mathbb{Z}[t^{\pm 1}]$ module (The cell complex itself is a free abelian group generated by the cells and thus a $\mathbb{Z}$-module). This then descends to $H_1(\widetilde{X})$; thus identifying $H_1(\widetilde{X})$ as a $\mathbb{Z}[t^{\pm 1}]$-module. Now suppose $X = X_L$ is a link complement of an n-component link, then $H_1(X_L) \cong \mathbb{Z}^n$ and so $H_1(\widetilde{X}_L) \cong \mathbb{Z}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$-module. This is the Alexander module of a link. Let

$$R^r \xrightarrow{A} R^m \longrightarrow H_1(\widetilde{X}_L) \longrightarrow 0$$

be a finite presentation with $R = \mathbb{Z}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$. The determinants of the $(m-1) \times (m-1)$ minors of A generate an ideal in R called the first elementary ideal also called the Alexander ideal. The GCD of the Alexander ideal is the (multivariate) Alexander polynomial.

A long standing fact is that the degree of the Alexander polynomial of a knot forms a lower bound on the knot genus. In 1998 McMullen generalized this result to links where he showed that the Alexander norm forms lower bound for the Thurston norm.

Let X_L be a link complement for an l-component link. Let $A(L)$ be its Alexander module identified as a $\mathbb{Z}[t_1^{\pm 1}, ..., t_l^{\pm 1}]$ module. For $\sum_i a_i t^{\alpha_i} \in A(L)$, the set $\alpha = (\alpha_1, ..., \alpha_n)$ is an element in $H_1(X_L; \mathbb{Z})$ after identifying $t_1, ..., t_l$ as a basis. The polytope formed as the convex hull of the exponent vectors of the Alexander polynomial is a subset of $H_1(X_L)$ is the Alexander dual norm ball with its polar dual polytope[1] being the true Alexander norm ball. Let $\phi \in (\phi_1, ..., \phi_b) \in H^1(M; \mathbb{Z})$ defined by $\phi(\alpha) = \sum_k \phi_i \alpha_i$ where α_i are the components of vectors corresponding to

[1]See Appendix A for details

elements in the Newton polytope of the Alexander polynomial.

$$\|\phi\|_A = \text{len}(\phi(\text{Newt}(\Delta_L)))$$

As $\text{Newt}(\Delta_L)$ is symmetric and convex, the dual norm ball of $|| \cdot ||_A$ is the dual polytope of $\text{Newt}(\Delta_L)$. McMullen then proves the following result:

Theorem 4.1. *[15] Let M be a cmpt. con. or. 3-manifold s.t. $\partial M = \sqcup_i T^2$, then*

$$||\phi||_A \leq ||\phi||_T + \begin{cases} 0 & b_1(M) \geq 2 \\ 1 + b_3(M) & b_1(M) = 1, H^1(M; \mathbb{Z}) = \mathbb{Z}\phi \end{cases}$$

with equality when $\phi : \pi_1(M) \to \mathbb{Z}$ is represented by fibration $M \to S^1$ where the fibers are surfaces of non-negative Euler characteristic

A result of Ozsváth and Szabó in 2007 is that the Newton polytope forms the dual Thurston norm ball for alternating links.

Corollary 4.1. *[16] Let L be a link with ℓ components which admits a connected, alternating projection. Consider the convex hull of all points in ℓ-dimensional space which correspond to non-zero terms in the multi-variable Alexander polynomial of L (i.e. the Newton polytope of the multi-variable Alexander polynomial). This polytope scaled by a factor of two, is the dual Thurston polytope of X_L*

Ideally one would like a set of conditions which would describe which symmetric integer coefficient Laurent polynomials are realized as Alexander polynomials for links. This result is known for knots as any Alexander polynomial of a knot can be computed: $\Delta_K(t) = \det(tA - A^T)$, where

A is a Seifert matrix [2]. However, for links this problem is more difficult. Such a characterisation would yield a characterisation for Thurston norm balls of certain large classes of links, notably alternating links.

4.1 Properties of Alexander Polynomials

So which polynomials are in fact Alexander polynomials. In the case of knots, we can characterize Alexander polynomials. We know that $\Delta_K(t)$ can be directly computed from a Seifert matrix A as $\det(tA - A^T)$ which are characterised by the following theorem.

Theorem 4.2. *Let A be a square matrix with integer coefficients. A is a Seifert matrix of some knot K if and only if* $\det(A - A^T) = \pm 1$.

Sketch of Proof. $\Rightarrow$ The forward direction is a corollary to Lickorish Theorem 6.10 (ii) [17]. The essence of the argument is that entries in $A - A^T$ yield precisely the intersection form on F, which must have determinant plus or minus one.

$\Leftarrow$ Notice that $A - A^T$ is anti-symmetric, so in order for $\det(A - A^T)$ to be nonzero, A must be even-dimensional. Now we can build a Seifert surface of genus g by glueing (g) pairs of “bands" (as in figure 6.1 of [17]) and twisting and linking them properly to obtain the coefficients in A. $\square$

Unfortunately, no characterization exists for links with more than one component. Nevertheless Torres was able to provide some necessary conditions about Alexander polynomials.

[2] Seifert matrices are integral square matrices satisfying $\det(A - A^T) = \pm 1$

Torres Conditions [18]

If a polynomial $\Delta(t_1, ..., t_\mu) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_\mu^{\pm 1}]$ is the Alexander polynomial of a link L with $\mu > 1$ components, then

1) $\Delta(t_1, ..., t_\mu) = (-1)^\mu t_1^{n_1} ... t_\mu^{n_\mu} \Delta(t_1^{-1}, ..., , t_\mu^{-1})$ for some integers $n_1, ..., n_\mu$

2a) If $\mu(L) = 2$

$$\Delta(t_1, 1) = \frac{t_1^\ell - 1}{t_1 - 1} \Delta(t_1)$$

where ℓ is the linking number of L and $\Delta(t_1)$ is the Alexander polynomial of the first component L_1

2b) If $\mu(L) > 2$, then

$$\Delta(t_1, ..., t_{\mu-1}, 1) = (t_1^{\ell_1} .... t_{\mu-1}^{\ell_{\mu-1}} - 1) \Delta(t_1, ..., t_{\mu-1})$$

where ℓ_i is the linking number of the ith and μth components and $\Delta(t_1, ..., t_{\mu-1})$ is the Alexander polynomial of $L - L_\mu$.

The Torres' conditions have since been shown to be insufficient [19], but they do yield insight into the structure of Alexander polynomials. It is an active area of research to discern relationships between properties of links and the Alexander polynomial.

Recall that the Thurston polytope for the complement of a knot in S^3 is determined entirely by the knot genus. Since $b_1(X_k) = 1$ there is one way in which it could fiber over S^1. If X_K fibers over S_1 then we say that K is a fibered knot. Examples of fibered knots include the trefoil and figure eight knot. Fibered knots can be characterized by the following theorem:

Theorem 4.3. *[20] (Stallings 61') A knot $K \hookrightarrow S^3$ is fibered if and only if $[\pi_1(X_k), \pi_1(X_k)]$ is finitely generated.*

Theorem 4.4. *If K is fibered, then $\Delta_K(t)$ is monic.*

So Alexander polynomials provide a simple obstruction to whether K fibers. As an example, one can easily compute Alexander polynomials of twist knots T_n

$$\Delta_{T_n}(t) = n(t^2 - 2t + 1) + t.$$

Since they are not monic when $n > 1$, twist knots do not fiber.

4.2 Link Floer Homology

A theme in modern low dimensional topology is the pursuit of categorifications of classical invariants. The Euler characteristic χ of a CW-complex is a classical invariant, but it is well known that

$$\chi = \sum_i (-1)^i \mathrm{rk}(\mathrm{H_i}(\mathrm{X}; \mathbb{Z})).$$

Thus the cellular homology of a space yields a stronger invariant which captures the Euler characteristic. More recently mathematicians study categorifications of polynomial invariants after the discovery of various homology theories including the Khovanov homology which categorifies the Jones polynomial. In this section we briefly study the link Floer homology which is a categorification of the Alexander polynomial. Link Floer homology is defined as the Heegaard-Floer homology of a Heegaard decomposition of S^3 with additional basepoints on the Heegaard Surface making the diagram compatible with a link $L \subset S^3$.

Example 8. *The following example illustrates a Heegaard decomposition of S^3 compatible with* 3_1.

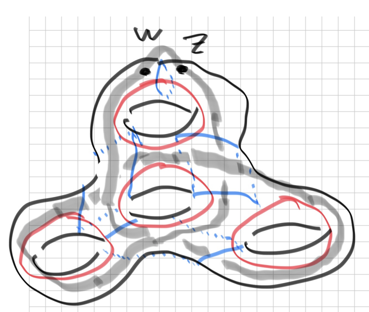


Figure 4.1: Heegaard Diagram of the Trefoil

The link Floer homology is a bigraded $\mathbb{F}_2$ vector space.

$$\widehat{\mathrm{HFL}}(L) = \bigoplus_{s \in \mathbb{H}, d \in \mathbb{Z}} \widehat{\mathrm{HFL}}_d(L, s)$$

where $\mathbb{H} = H_1(S^3 - L; \mathbb{Z})$. Let $\mathbb{Z}[\mathbb{H}]$ denote the integral group ring of $\mathbb{H}$ with elements written in the form

$$\sum_{h \in H} a_h \cdot e^h \in \mathbb{Z}[\mathbb{H}]$$

This categorifies the Alexander polynomial as seen in the following expression.

$$\sum_{h \in \mathbb{H}} \chi(\widehat{\mathrm{HFL}}_*(L, h)) \cdot e^h = \begin{cases} \left(\prod_{i=1}^{\ell}(T_i^{1/2} - T_i^{-1/2})\right) \cdot \Delta_L, & \ell > 1 \\ \Delta_L \quad \ell = 1 \end{cases}$$

where Δ_L is the Alexander polynomial.

Theorem 4.5. *(Ozsváth and Szabó [16]) Let L be an ℓ component link in S^3. Let $\alpha \in H^1(X_L; \mathbb{Z})$, then*

$$\|\alpha\|_T + \sum_{i=1}^{\ell} |\langle h, \mu_i \rangle| = 2 \max_{\{s \in H_1(X_L) | \widehat{\mathrm{HFL}}(L,s) \neq 0\}} \langle s, h \rangle$$

There is another beautiful interpretation of Ozsváth and Szabó's result via Juhasz's sutured Floer homology which combines David Gabai's sutured manifolds with Heegaard Floer homology and link Floer homology.

Definition 4.1. *[21] A sutured manifold (Y, γ) is a compact oriented 3 manifold with boundary together with a set $\gamma \subset \partial Y$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The interior of each component in $A(\gamma)$ contains a homologically nontrivial simple closed curve called a suture. The collection of sutures is denoted as $s(\gamma)$. Let $R(\gamma) = \partial Y - \mathrm{int}(\gamma)$*

Every component in $R(\gamma)$ inherits an orientation where $R_+(\gamma)$ (resp. $R_-(\gamma)$) denotes compo-

nents whose normal vector points out of (into) Y.

If $\sigma \in \partial R$ *is given the boundary orientation, then* σ *represents the same homology class as some suture.*

Sutured manifolds are powerful constructions which enabled David Gabai to prove a plethora of results, including several related to $\|\cdot\|_T$. In 2006 András Juhász created a novel variant of Heegaard Floer homology called Sutured Floer homology [22].

5. APPLICATIONS OF THE THURSTON NORM AND RELATIONSHIP TO OTHER NORMS

In this chapter we study some applications of the Thurston norm. The purpose is to highlight diverse applications across various fields in topology, geometry, and algebra. First we examine some foliation theory, followed by studying the Gromov norm. The core message is that in the 3-manifold world, the structure of immersed and embedded surfaces are in many respects identical. After this we show that the Thurston norm yields estimates on the hyperbolic L^2 norm which has interesting consequences for hyperbolic and in particular arithmetic hyperbolic 3-manifolds. We also see a beautiful connection between $\|\cdot\|_T$ and the study of minimal surfaces. Lastly we examine an application of $\|\cdot\|_T$ in cryptography.

5.1 Foliations

In the early days of $\|\cdot\|_T$, David Gabai made extensive use of Thurston's norm to improve the theory of 3-manifold foliations [21]. A codimension r foliation of a manifold M is a decomposition of M into $(n-r)$-dimensional submanifolds called leaves which locally stack on top of each other. In topology, foliations served an important role, starting with Alexander's proof of the irreducibility of $\mathbb{R}^3$ [23]. Exposition on foliation theory of 3-manifolds may be found in Appendix C. We discuss the role of Thurston norm in foliation theory including a 3-dimensional analog of the adjunction inequality.

Our first goal is to obtain a 3-dimensional analog of the adjunction inequality. Let $E \to M$ denote a smooth oriented vector bundle over an oriented compact manifold M. M can be canonically identified as the zero section of E. Any smooth section $\sigma : M \to E$ can be taken to intersect the zero section transversely, thus $\mathrm{im}(\sigma) \cap \mathrm{M}$ is a smoothly embedded submanifold of M. We can now define the Euler class of E.

Definition 5.1. *Let $E \to M$ be a smooth oriented vector bundle over an oriented compact manifold M. The Euler class $e(E) \in H^*(M)$ is the Poincaré dual of the fundamental class of the zero locus* *of a generic section in E.*

$e(E)$ is independent of the choice of generic section. Some obvious conclusions follow. If E admits a non-zero section, then $e(E) = 0$ since the intersection with the zero section is trivial.

Lemma 5.1. *(Poincaré-Hopf Index Theorem) Suppose M is an oriented compact smooth manifold and $V \in \Gamma(M)$ is a vector field with isolated singularities, and if $\partial M \neq \emptyset$ then V points in the outward normal direction along the boundary, then*

$$\sum_i \mathrm{Ind(x_i)} = \chi(\mathrm{M})$$

where $x_i \in M$ are the isolated singularities of V.

Proposition 5.1. *If $T\Sigma$ denotes the tangent bundle of a compact oriented surface Σ, then $e(TS) \cdot [\Sigma, \partial\Sigma] = \chi(\Sigma)$*

Sketch of Proof. First $e(T\Sigma)$ will be a multiple of $[\Sigma, \partial\Sigma]$. The result follows once one considers a generic section as a vector field on the tangent bundle and evaluates it using the Poincaré-Hopf index theorem. □

The following is known as the Thurston-Roussarie theorem.

Theorem 5.1. *Suppose $\mathcal{F}$ is taut, and Σ an immmersed incompressible surface, then then either Σ is homotopic to a leaf or Σ may be homotoped to intersect $\mathcal{F}$ in saddle tangencies.*

The idea is to use the fact that we can find a Riemannian metric on Y such that each leaf is a minimal surface (Theorem C.5). Then by a theorem of Schoen and Yau, Σ is homotopic to an immersed minimal representative which may only have saddle singularities with with $\mathcal{F}$. Now let Σ be a minimal immersed surface which intersects $\mathcal{F}$ in saddle tangences. At each saddle tangency, the orientations of Σ and $\mathcal{F}$ may agree or not, so define $I_p(\Sigma)$ ($I_n(\Sigma)$) to be the number of positive (negative) saddle intersections.

Lemma 5.2. *If $e(T\mathcal{F})$ denotes the Euler class of the tangent bundle of the foliation (the defining distribution) then*

$$e(T\mathcal{F}) \cap [S] = I_n - I_p$$

Proof. Recall that any foliation $\mathcal{F}$ is defined by a rank 2 distribution $D \subset TY$ which is a subbundle of TY. Let $\iota : S \to Y$ be the inclusion of S. We can pull back the distribution by ι to a bundle

$$\begin{tikzcd} \iota^*(D) \arrow[d] & D \subset TY \arrow[l] \arrow[d] \\ \Sigma \arrow[r, "\iota"] & Y \end{tikzcd}$$

The Euler class of $D = T\mathcal{F}$ is the orientation class of the zero locus of a generic section. The pullback of a generic vector field on $D|_{\iota(\Sigma)}$ will be nonsingular on non-saddle points and will wind around each singularity either in a positive spiral or negative spiral depending on whether the orientations agree or not, and of course the vector fields vanish at the singularities, and so we see that $e(T \cap F) \cap [S] = I_n - I_p$

□

Notice then by Proposition 5.1 that since saddle tangencies are index 1 critical points, $I_p + I_n = \chi(S)$. Now we can prove the 3-dimensional analog of the adjunction inequality,

Theorem 5.2. *[3] Let $\mathcal{F}$ be a taut oriented foliation on Y, and Σ an immersed oriented surface, then*

$$|e(T\mathcal{F}) \cap [\Sigma]| \leq \|\Sigma\|$$

where $\|\Sigma\| = \chi_-(\Sigma)$.

Proof. If Σ is compressible, then we may first compress it until it is incompressible, and since compression preserves $[\Sigma]$ we may work with the new incompressible surface Σ'. Now by the Thurston-Roussarie theorem we may homotope Σ' to either a leaf of the foliation or such that it meets $\mathcal{F}$ in saddle tangencies. If it is homotopic to a leaf then use Proposition 5.1. If it has saddle tangencies, then by the previous theorem and our observation,

$$e(T\mathcal{F}) \cap [\Sigma] = e(T\mathcal{F}) \cap [\Sigma'] = I_n - I_p \leq I_n + I_p = -\chi(\Sigma') \leq -\chi(\Sigma)$$

so

$$|e(T\mathcal{F}) \cap [\Sigma]| = |I_n - I_p| \leq I_n + I_p = -\chi(\Sigma)$$

□

Moreover if Σ is incompressible, equality holds when all the saddles are of the same sign or when Σ is homotopic to a leaf. The theorem statement and its proof can be appropriately modified for when $\partial Y \neq \emptyset$. The following theorem follows as a corollary.

Theorem 5.3. *(Thurston [3]) Let Y be a compact orientable 3-manifold and $\mathcal{F}$ a codimension 1 transversely oriented Reebless foliation transverse to ∂Y. If $\lambda \in \mathcal{F}$ is a compact leaf, then λ is Thurston norm minimizing*

Proof. Suppose λ is not a sphere or disk, in which case the theorem would be trivial. From previous results we can find an incompresssible surface representative λ' in the same homology class with $\chi_-(\lambda') \leq \chi_-(\lambda)$. We know from the previous results,

$$\begin{aligned} \|\lambda\| &= -\chi(\lambda) \\ &= |e(T\mathcal{F} \cap [\lambda]| \\ &= |e(T\mathcal{F} \cap ([\lambda'])| \leq \|[\lambda']\| \end{aligned}$$

but since $\|\lambda\| \leq \|[\lambda']\| \leq \|\lambda\|$ the result follows. □

Theorem 5.4. *(Gabai [21]) Let Y be a compact orientable irreducible 3 manifold with toroidal boundary. If Σ is a Thurston norm minimizing surface in Y, then there exists a taut finite depth* *foliation $\mathcal{F}$ on Y such that $\mathcal{F}$ is transverse to ∂Y, where Σ is a leaf of $\mathcal{F}$, and $\mathcal{F}|_{\partial Y}$ is a suspension of homeomorphisms of S^1.*

An easy corollary is the following:

Corollary 5.1. *[21] Let Y be a compact connected orientable irreducible 3 manifold with $H_2(Y, \partial Y; \mathbb{Z}) \neq 0$, then Y has a taut foliation.*

Proof. If $H_2(Y, \partial Y; \mathbb{Z})$ is nonzero then we can always find a norm-minimizing surface that is properly embedded in Y, then by Theorem 5.4 the result follows. □

Next is the Property R theorem which was also proven by David Gabai.

Theorem 5.5. *(Property R) If the 0-surgery of a knot $K \subset S^3$ is homeomorphic to $S^1 \times S^2$, then K is the unknot.*

The hard part of Gabai's proof is to show that the Thurston norm remains unchanged under a 0-framed Dehn filling. This requires a heavy amount of foliation theory. The remainder of the proof is simple. Suppose that under a 0-framed Dehn filling, one obtained $S^1 \times S^2$, then we know that $\|\cdot\|_T$ is trivial since $\{p\} \times S^2$ represents a generator for $H_2(S^1 \times S^2; \mathbb{Z})$, but this implies that $S^3 - K$ has trivial Thurston norm which can only happen if K admits a genus zero Seifert surface which can only happen if $K = 0_1$.

5.2 Gromov Norm

The Gromov norm was introduced in order to define a 'volume' for a manifold without specifying a metric [24]. The Gromov ℓ^1-norm is a semi-norm defined on $H_n(X; \mathbb{R})$ by asking what the least number of simplices one needs to represent $\alpha \in H_n(X; \mathbb{R})$. We define this norm at the chain level; we have $B_n \leq Z_n \leq C_n$, and for $\alpha \in H_n(X; \mathbb{R})$, $\alpha = [c]$ for $c = \sum_{i=1}^k \lambda_i c_i \in Z_n$. We define $\|c\|_{\ell^1} = \sum_{i=1}^k |\lambda_i|$.

Definition 5.2. *Let $\alpha \in H_n(X;\mathbb{R})$, the Gromov simplicial norm (or ℓ^1-norm) is defined as*

$$\|\alpha\|_{\ell^1} = \inf_{c \in Z_n} \{\|c\|_{\ell^1} \ : \ [c] = \alpha\}.$$

In essence this measures the simplicial volume for α. Also, this norm is functorial.

Theorem 5.6. *If $f : X \to Y$ is a continuous map, then*

$$\|f_*(\alpha)\|_{\ell^1} \leq \|\alpha\|_{\ell^1}.$$

Proof. Let $\alpha \in H_n(X;\mathbb{R})$ and let $[c] = \alpha$ with $c = \sum_{i=1}^k \lambda_i c_i$, then

$$\|f_*(c)\|_{\ell^1} = \left\| f_*\left(\sum_{i=1}^k \lambda_i c_i\right)\right\|_{\ell^1} = \left\|\left(\sum_{i=1}^k \lambda_i f_*(c_i)\right)\right\|_{\ell^1} \leq \sum_{i=1}^k |\lambda_i| = \|c\|_{\ell^1}$$

taking $\inf$ over all c on both sides we see that $\|f_*(\alpha)\|_{\ell^1} \leq \|\alpha\|_{\ell^1}$ □

It's not hard to see from this that if $f : X \to Y$ is a homotopy, then it induces norm preserving isomorphisms on $H_n(X;\mathbb{R})$. One application of the Gromov norm is to measure the simplicial volume of a manifold.

Definition 5.3. *Let M be an orientable n-dimensional manifold possibly with boundary ∂M, and let $[M] = [M, \partial M] \in H_n(M, \partial M;\mathbb{R})$ be its fundamental class, then the simplicial volume of M is*

$$\|M\|_1 = \|[M]\|_{\ell^1}.$$

If M is not orientable, then let N be the double cover of M, then the simplicial volume of M is $\|M\|_1 = \frac{1}{2}\|[N]\|_{\ell^1}$

Theorem 5.7. *Let $f : M \to N$ be a map between closed connected orientable manifolds with degree $\deg(f)$, then*

$$\|M\|_1 \geq |\deg(f)|\|N\|_1$$

Proof. This follows immediately from Theorem 5.6,

$$\begin{aligned} \|M\|_1 \geq \|f_*([M])\|_1 &= \| \deg(f) \cdot [N]\|_1 \\ &= | \deg(f)|\|N\|_1 \end{aligned}$$

□

Corollary 5.2. *If $f : M \to M$ has $\deg(f) > 1$, then $\|M\|_1 = 0$.*

Corollary 5.3. *$\|S^n\|_1 = 0$ for $n \geq 1$, and $\|T^n\|_1 = 0$.*

Theorem 5.8. *If $p : \widetilde{M} \to M$ is a finite covering of degree d, then $d\|M\|_1 = \|\widetilde{M}\|_1$*

Proof. Let α be the fundamental class of M, and let $\sum \lambda_i c_i$ be a representative of α. Each $c_i \in C_i(M)$ lifts to d simplices in $\widetilde{M}$. From here we see that

$$\|p_*(\alpha)\|_1 \leq \left\| p_* \left(\sum_i \lambda_i c_i \right) \right\|_1 = \left\| \sum_i \sum_{j=1}^{d} \lambda_i \widetilde{c}_{ij} \right\|_1 = d \left\| \sum_i \lambda_i c_i \right\|_1$$

Since the choice of representative was arbitrary, it follows that $\|p_*(\alpha)\|_1 \leq d\|\alpha\|$ and by Theorem 5.7 the reverse inequality holds. □

Our goal is to relate the Gromov norm and the Thurston Norm, but to do this we first need to understand the simplicial volume of surfaces.

Theorem 5.9. *If Σ is a closed connected orientable surface of genus ≥ 2, then $\|\Sigma\|_1 = 2\chi_-(\Sigma)$*

Proof. Any such surface can be given a triangulation with 1 vertex. For example in figure 5.1.

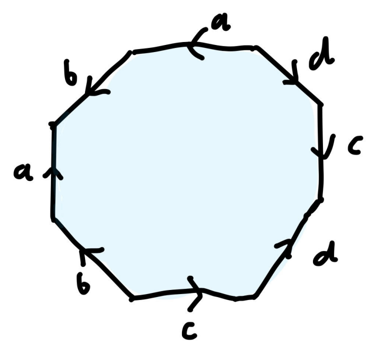


Figure 5.1: Genus 2 Surface

We can subdivide the face into triangles with the vertices on the exterior of the face (which are all identified). It follows that $\chi(\Sigma) = v - e + f = 1 - e + f$. Moreover since each triangle in a triangulation has 3 edges, and each edge shares 2 triangles, it follows that $2e = 3f$, thus $\chi(\Sigma) = 1 - \frac{3}{2}f + f = 1 - \frac{f}{2}$, so $2\chi_-(\Sigma) + 2 = f \geq \|S\|_1$.

Recall that for any $\deg d$ cover of Σ, both χ and $\|\cdot\|_1$ are multiplicative. If Σ^d is a degree d cover, then

$$\|\Sigma\|_1 = \|\Sigma^d\|/d \leq \frac{2 + 2\chi_-(\Sigma^d)}{d} = \frac{2 + 2d\chi_-(\Sigma)}{d} = \frac{2}{d} + 2\chi_-(\Sigma)$$

Since such covers exist for any $d \geq 0$, we can take $\lim_{d\to\infty}$, thus

$$\|\Sigma\|_1 \leq 2\chi_-(\Sigma)$$

The reverse inequality follows from the following result of Gromov [24]; if Σ is an oriented closed connected hyperbolic surface, then $\|\Sigma\|_1 \geq \mathrm{Area}(\Sigma)/\pi$. Then we see that

$$\|\Sigma\|_1 \geq \frac{\mathrm{Area}(\Sigma)}{\pi} = -2\chi(\Sigma) = 2\chi_-(\Sigma)$$

$\square$

The above theorem can be appropriately modified when $\partial\Sigma \neq \emptyset$. Theorem 5.9 serves as an example of Gromov proportionality, where given an oriented closed connected hyperbolic manifold M, $\|M\|_1 = \mathrm{Vol(M)/v_n}$ where v_n is the largest volume of an ideal n-simplex.

Let us now study the Gromov norm on $H_2(Y, \partial Y; \mathbb{R})$ for a compact orientable irreducible 3-manifold. $\|\cdot\|_1$ essentially measures the complexity of immersed surfaces in Y. It is not obvious that the immersed complexity should be equal to the embedded complexity. This is certainly false for 4-manifolds. The following theorem of Gabai tells us that the immersed and embedded complexities are actually equivalent, and that the two norms $\|\cdot\|_T$ and $\|\cdot\|_1$ are in fact the same.

Theorem 5.10. *[21] Let Y be a connected compact oriented 3-manifold, then $\|\cdot\|_T = \frac{1}{2}\|\cdot\|_1$ on $H_2(Y, \partial Y; \mathbb{R})$.*

Corollary 5.4. *Let Y be a compact connected orientable 3-manifold with an n-fold cover $Y^n \xrightarrow{p} Y$. Then for any $\alpha \in H^1(Y)$, $\|p^*(\alpha)\|_T = n\|\alpha\|_T$*

Proof. This immediately follows from Theorem 5.10 and Theorem 5.8 □

To prove Theorem 5.10 we need an intermediate norm called the Thurston singular norm.

Definition 5.4. *The singular Thurston norm $\|\cdot\|_S : H^1(Y) :\to \mathbb{R}_{\geq 0}$*

$$\|\varphi\|_s = \left\{\frac{1}{k}\chi_-(\Sigma) : f : (\Sigma, \partial\Sigma) \to (M, \partial M) \textit{ is proper map s.t. } f_*([\Sigma, \partial\Sigma]) \textit{ is dual to } k\varphi\right\}$$

Lemma 5.3. *[21] The singular Thurston norm and Thurston norm are equivalent: $\|\cdot\|_s = \|\cdot\|_T$.*

Lemma 5.4. *The Gromov norm is half the singular Thurston norm: $\|\cdot\|_1 = 2\|\cdot\|_s$.*

Proof. Recall Theorem 5.9 that $\|[\Sigma, \partial\Sigma]\|_1 = 2\chi_-(\Sigma)$. When Σ admits a non-negative χ, all norms in consideration vanish so we will assume moving forward that $\chi(\Sigma) < 0$.

Let $kz \in H_2(Y, \partial Y)$ be represented by a singular surface Σ, then

$$\|z\|_1 \leq 2\frac{\chi(\Sigma)}{k}$$

and since our choice of singular surface was arbitrary, it follows that $\|z\|_1 \leq 2\|z\|_s$. On the other hand if $z \in H_2(Y, \partial Y)$ and $z = [\Sigma_i \lambda_i c_i]$ is represented by some cycle with $\lambda_i \in \mathbb{Z}$, then we can paste together the singular simplices in the cycle to form a surface and a proper map $f : \Sigma \to Y$ such that $f_*([\Sigma, \partial\Sigma]) = z$ thus $\|z\|_s \leq \chi_-(\Sigma) \leq \sum_i |\lambda_i|$. Now for any cycle $\sum_i \lambda_i c_i$, there is a rational cycle $\sum_i \lambda_i' c_i$, $\lambda_i' \in \mathbb{Q}$ where $\sum_i |\lambda_i'| \leq \sum_i |\lambda_i| + \varepsilon$ with $\varepsilon << 1$. By continuity of $\|\cdot\|_1$ and $\|\cdot\|_s$ the result follows. $\square$

5.3 Harmonic norm

Here we present a relationship between $\|\cdot\|_T$ and some more familiar norms on $H^1(M)$ coming from differential geometry, in particular De Rham cohomology. The work we present is that of Nathan Dunfield, Jeffrey, and Xiaolung Hans Han.

Theorem 5.11. *(Dunfield and Brock) Let M be a closed orientable hyperbolic 3-manifold, then*

$$\frac{\pi}{\sqrt{Vol(M)}}\|\cdot\|_T \leq \|\cdot\|_{L^2} \leq \frac{10\pi}{\sqrt{inj(M)}}\|\cdot\|_T$$

on $H^1(M;\mathbb{R})$.

Background on minimal surface theory can be found in Appendix D. In the following, M will be a hyperbolic closed orientable 3 manifold. By Mostow rigidity, the hyperbolic structure on M is unique.

Definitions: Let $\varphi \in H^1(M;\mathbb{R})$ and let $\alpha \in \Omega^1(M)$ be a representative of φ, then $|\alpha_p|$ is the operator norm of $\alpha_p : T_pM \to \mathbb{R}$ and $|\alpha| : M \to \mathbb{R}$ is a smooth function.

$$\|\alpha\|_{L^1} = \int_M |\alpha| d\text{Vol}$$

$$\|\alpha\|_{L^2} = \sqrt{\int_M |\alpha|^2 d\text{Vol}}$$

$$\|\alpha\|_{L^\infty} = \max_{p\in M} |\alpha_p|$$

which then extend to norms on $H^1(M,\mathbb{R})$ by

$$\|\varphi\|_{L^1} = \inf\{\|\alpha\|_{L^1} | \alpha \in [\varphi]\}$$

$$\|\varphi\|_{L^2} = \inf\{\|\alpha\|_{L^2} | \alpha \in [\varphi]\} = \|\beta\|_{L^2},\ \beta \text{ is harmonic}$$

$$\|\varphi\|_{L^\infty} = \inf\{\|\alpha\|_{L^\infty} | \alpha \in [\varphi]\}$$

Let $\Sigma \hookrightarrow M$ be an smooth embedded surface dual to $\varphi \in H^1(M;\mathbb{Z})$

$$\|\varphi\|_{LA} = \inf\{\text{Area}(\Sigma) |\ \Sigma \text{ dual to } \varphi\}$$

We then continuously extend $\|\cdot\|_{LA}$ to $H^1(M;\mathbb{R})$

As we saw, standard results from geometric measure theory guarantee existence of minimal surfaces in each class φ and further results by Uhlenbeck guarantee existence of stable minimal surfaces in each class.

Lemma 5.5. $\|\varphi\|_{LA} = \|\varphi\|_{L^1}$

Sketch of Proof. We only prove $\|\varphi\|_{LA} \leq \|\varphi\|_{L^1}$: Let $\varphi \in H^1(M;\mathbb{Z})$ and let $\alpha \in [\varphi]$ then since α is a representative from an integral class, there exists a map $f : M \to S^1$ such that $\alpha = f^*(dt)$ where $dt = [\mathbb{R}/\mathbb{Z}] = [S^1]$. Since f is smooth, $f^{-1}(t) = S_t$ is a surface for almost every $t \in [0,1]$

thus the coarea formula [25] tells us

$$\|\alpha\|_{L^1} = \int_M |\alpha| d\text{Vol} = \int_0^1 \text{Area}(S_t) dt$$

i.e. the average area of surfaces representing α. Thus there exists many t such that $\|\alpha\|_{L^1} \geq \text{Area}(S_t) \geq \|\varphi\|_{LA}$ □

Lemma 5.6. *For any stable minimal surface S in a hyperbolic 3-manifold Y*

$$\pi\chi_-(S) \leq \mathit{Area}(S) \leq 2\pi\chi_-(S)$$

Sketch of Proof. The proof of the first inequality is found in [26] *(it uses the formula for second variation of area and then Gauß-Bonnet to obtain the inequality)*. Now let S be a minimal representative of φ. Since S is a smooth surface embedded in a hyperbolic 3-manifold, it has intrinsic curvature $K : S \to \mathbb{R}$ bounded above by -1, thus by the Gauß-Bonnet theorem

$$2\pi\chi_-(S) = -2\pi\chi(S) = -\int K dA \geq \int dA = \text{Area}(S)$$

□

Proposition 5.2. *[27]* $\pi\|\cdot\|_T \leq \|\cdot\|_{LA} \leq 2\pi\|\cdot\|_T$

Proof. Let S be dual to φ such that $\chi_-(S) = \|\varphi\|_T$. There is a least area stable minimal surface in its isotopy class [28]. By Lemma 5.6: $\|\varphi\|_{LA} \leq \text{Area}(S) \leq 2\pi\chi_-(S) = 2\pi\|\varphi\|_T$. Now let S be a least area representative of φ. S must be stable so $\pi\|\varphi\|_T \leq \pi\chi_-(S) \leq \text{Area}(S) = \|\varphi\|_{LA}$ □

Lemma 5.7. *[27]* $\|\cdot\|_{L^\infty} \leq \frac{5}{\sqrt{\mathit{inj}(M)}}\|\cdot\|_{L^2}$ *on* $H^1(M;\mathbb{R})$

Lemma 5.7 is proven in section 4 of Brock and Dunfield, and uses some interesting techniques from harmonic analysis. Now we prove the main theorem.

Proof. By the previous work $\pi\|\varphi\|_T \leq \|\varphi\|_{LA} = \|\varphi\|_{L^1}$. Let $\alpha \in [\varphi]$ be harmonic, then

$$\begin{aligned}
\pi\|\varphi\|_T \leq \|\varphi\|_{L^1} \leq & \|\alpha\|_{L^1} \\
= & \int_M |\alpha| d\text{Vol} \\
= & \int_M |\alpha \cdot 1| d\text{Vol} \\
\leq & \sqrt{\int_M |\alpha|^2 d\text{Vol}} \sqrt{\int_M |1|^2 d\text{Vol}} \quad \text{Cauchy-Schwarz} \\
= & \|\varphi\|_{L^2} \text{Vol}(M) \quad \text{since } \alpha \text{ is harmonic}
\end{aligned}$$

Now let $\varphi \in H^1(M;\mathbb{Z})$ and let S be dual to φ such that the area is at most $2\pi\|\varphi\|_T$. For every closed 2-form $\beta \in \Omega^2(M)$ $\int_M \beta \wedge \alpha = \int_S \beta$. If α is harmonic (representing φ) then $0 = d^*\alpha = - * d * \alpha$, thus $*\alpha$ is closed. Thus

$$\begin{aligned}
\|\alpha\|_{L^2}^2 = \int_M \alpha \wedge *\alpha = & \int_M *\alpha \wedge \alpha \\
= & \int_S *\alpha \\
\leq & \int_S |\alpha| dA \\
\leq & \int_S \|\alpha\|_{L^\infty} dA \\
\leq & \|\alpha\|_{L^\infty} \text{Area}(S) \\
\leq & 2\pi \|\alpha\|_{L^\infty} \|\varphi\|_T \\
\leq & \frac{10\pi}{\sqrt{\text{inj}(M)}} \|\alpha\|_{L^2} \|\varphi\|_T \quad \text{by Lemma 5.7}
\end{aligned}$$

thus since α is harmonic $\|\varphi\|_{L^2} = \|\alpha\|_{L^2} \leq \frac{10\pi}{\sqrt{\text{inj}(M)}} \|\varphi\|_T$ □

Dunfield and Brock worked with closed hyperbolic 3-manifolds, but many important hyperbolic manifolds are hyperbolic link complements, which are not closed and not even compact. Link exteriors are examples of cusped hyperbolic 3-manifolds. One can ask whether the results of

Brock and Dunfield extend to cusped hyperbolic manifolds, however, there are a number of difficulties in doing so. The first is that in the noncompact case the injectivity radius can be zero which makes the RHS inequality meaningless. The other subtlety relies on the fact that stable minimal surfaces might not be well defined in the noncompact case. Hans Han developed the corresponding theory for the non-compact case by utilizing a less naive compactly supported cohomology theory.

Theorem 5.12. *(Hans Han [29]) For any cusped orientable hyperbolic 3-manifold Y one has the following:*

$$\frac{\pi}{\mathrm{Vol(Y)}}\|\cdot\|_T \leq \|\cdot\|_{L^2} \leq \max\left\{\frac{10\pi}{\sqrt{\mathrm{sys(Y)}}}, 4.86\pi\sqrt{1+\frac{d^2}{2}}\right\}\|\cdot\|_T$$

on $\mathcal{H}^1$

5.4 Cryptographic Application

One refreshing application of the Thurston norm lives in the field of cryptography. Flores, Kahrobaei, and Koberda developed cryptographic schemes using subgroup distortion methods invoking a hyperbolic 3-manifold with finite volume [30]. We begin by introducing subgroup distortion followed by an explanation of the Thurston norm based cryptographic scheme.

Let G be a finitely presented group with X a finite generating set. Endow G with the word metric with respect to X, and let $d_X(e,w) = \ell_X(w)$ be the word length. Let $H \leq G$ be a subgroup with finite generating set Y, then

$$\ell_{X\cup Y}(h) \leq \ell_Y(h)$$

Definition 5.5. *Let G be a finitely generated group with S a generating set, and H a finitely generated subgroup with generating set T, then the distortion of H in G is the function*

$$DIST_H^G : \mathbb{N} \to \mathbb{N}$$

given by $n \mapsto \max\{\ell_T(h) | \ell_S(h) \leq n\}$.

In essence it measures how much a given word length is distorted by restricting to a subgroup presentation. Notice the dependency on the choice of presentations for G and H.

Example 9. *Consider the Baumslag-Solitar group* $BS(1,2) = \langle a, b | aba^{-1} = b^2 \rangle$ *Consider the subgroup* $H = \langle b \rangle$*, then*

$$a^n b a^{-n} = b^{2^n}$$

so $\ell_H(b^{2^n}) = 2^n$ *and* $\ell_{BS(2,1)}(b^{2^n}) = \ell_{BS(2,1)}(a^n b a^{-n}) = 2n + 1$*. We then see that* H *is exponentially distorted.*

Group theoretic cryptographic systems are typically based on some type of decision problem (problems with a yes or no answer), where given a property P and object O, does O have property P? We will be interested in the following decision problem.

Definition 5.6. *Membership Problem: Suppose we have a finitely presented group* G *given by presentation* S *with a finitely presented subgroup* H *presented by* T*. Given an element* $g \in G$*, does* g *belong to* H*.*

Theorem 5.13. *[30] Let* Y *be a hyperbolic manifold that fibers over* S^1 *with fiber* F*. Then* $\pi_1(F) \leq \pi_1(Y)$ *is exponentially distorted and the membership problem is solvable in finite time.*

Proof. Since Y is hyperbolic and fibers over S^1 it follows that Y is the mapping torus of a pseudo-Anosov mapping class of F. First apply the homotopy exact sequence for fibrations

$$0 \longrightarrow \pi_1(F) \longrightarrow \pi_1(Y) \xrightarrow{\varphi} \pi_1(S^1) \longrightarrow 0$$

Note that $\pi_1(Y)$ is the semidirect product of $\mathbb{Z}$ with $\pi_1(F)$ where the generator of $\mathbb{Z}$ acts on the fiber by the pseudo-Anosov mapping class lifted to an automorphism of $\pi_1(F)$. Associated to every pseudo-Anosov map ψ is a real number $\lambda_\psi > 1$ called the stretch factor. It is well known that the word length for $\psi^n(\gamma)$ grows as λ_ψ^n. Since ψ acts via conjugation on $\pi_1(\Sigma)$, thus $\psi^n(\gamma) = t^{-n}\gamma t^n$ so the word length is linear in n. It follows that $\pi_1(F) < \pi_1(Y)$ is exponentially distorted.

Now we address the membership problem. Let $g \in \pi_1(Y)$. Since $g \in \pi_1(\Sigma)$ if $g \in \ker \varphi$ where $\varphi : \pi_1(Y) \to \mathbb{Z}$. Thus if we express g in terms of generators $g = g_1 \dots g_k$, then we simply compute $\varphi(g) = \varphi(g_1) \dots \varphi(g_k) = \sum_{i=1}^{k} n_i$ and check whether this sum is 0 in which case $g \in \pi_1(S)$. Since this computation is linear in k, the result follows. $\square$

A symmetric key encryption scheme between two parties Alice and Bob starts by the two agreeing on a private key beforehand. These parties then communicate over some insecure public channel *(such as the internet, radio frequencies, etc.)*. Alice encrypts the data using the private key and sends the encrypted data over the public channel to Bob who then uses the private key to decrypt the data. An asymmetric (public) key encryption scheme relies on the premise that Alice and Bob are not able to communicate a private key beforehand, and must do all their communication via a public channel. These schemes rely on a *one-way function* for which the inverse problem is significantly more difficult. A public key is made available for Bob who then encrypts his data and sends it to Alice who then is able to use her solution to the inverse problem to decrypt the data.

5.4.1 Symmetric Key Scheme

The following symmetric-key encryption system was proposed by the authors. The public information consists of a hyperbolic 3-manifold Y along with a presentation of $\pi_1(Y)$. Alice and Bob agree beforehand on a primitive fibered cohomology class $\alpha \in H^1(Y;\mathbb{Z})$. Alice and Bob then have a private subgroup corresponding to the fiber subgroup and agree on a presentation of this group with some set of generators $\{h_1, \dots h_m\}$, and they have a surface automorphism $\psi_\alpha : \Sigma_\alpha \to \Sigma_\alpha$. The scheme proceeds as follows.

Alice relays an integer $n \in \mathbb{Z}$ to Bob over the public channel. Both compute $\psi^n(g_i)$ for all generators in the private presentation of $\pi_1(\Sigma_\alpha)$. The shared key then is $\ell_{\max} = \max_i \|\psi^n(g_i)\|$.

The cryptographic system relies on the fact that a fibered 3-manifold admits infinitely many non-unique ways to fiber, and choosing a primitive class over which Y fibers produces a membership problem that is linear only for those who know the choice of fibered class as the fundamental groups of fibers of hyperbolic mapping tori are exponentially distorted. Another strength of this scheme is that there are myriads of examples of fibered hyperbolic 3-manifolds, allowing there to be plenty of examples that may be used. Of course, since hyperbolic 3-manifolds are determined by their fundamental group, there is little secrecy in the manifold choice.

The public key scheme that the authors develop relies on an AAG scheme to communicate the choice of fibered cohomology class, and so does not add anything from the perspective of the Thurston norm. For more information regarding subgroup distortion based cryptography we encourage the reader to check out [31].

6. COMPUTATIONAL METHODS

Given a 3-manifold Y, determining its Thurston polytope directly appears difficult. Each homology class admits an uncountable number of surface representatives, and determining the minimum complexity requires some machinery. A reasonable question to ask is given a compact orientable 3-manifold Y, does an algorithm exist that can compute the Thurston polytope in finite time?

- 1998: McMullen shows that Alexander polynomial determines the Thurston Polytope for alternating links [15]. Methods to compute the Alexander polynomial are well known [32].
- 2008: Oszváth and Szabó show that link Floer homology determines $B_{\|\cdot\|_T}$ for all links [16]. The work of Manolescu, Sarkar, and others yields algorithms to compute Heegaard-Floer homology and subsequently link Floer homology [33].
- 2009: Cooper and Tillmann use normal surface theory to compute the Thurston norm on closed hyperbolic orientable 3-manifolds of finite volume [34]
- 2022: Cooper, Tillmann, and Worden generalize this to hyperbolic compact orientable 3-manifolds of finite volume [35]. Worden publishes a program to compute $\|\cdot\|_T$ [36].

So one has a (small) selection of methods to choose from to compute $\|\cdot\|_T$.

6.1 Normal Surface Approach

The idea of a normal surface was first introduced by Hellmuth Kneser, and Wolfgang Haken later developed normal surface theory [37]. Haken utilized it to find algorithms in low-dimensional topology. One of Haken's most famous results was showing that the unknotting problem was solvable, that is there exists an algorithm that can determine in a finite number of steps whether a given knot diagram is equivalent to the unknot. Later Thurston developed the theory of *spun-normal surfaces* in order to develop similar algorithms for manifolds with boundary.

It is a nontrivial fact that every 3-manifold admits a triangulation. These combinatorial structures lend themselves well to computation, and normal surface theory begins by studies surfaces with respect to a triangulation. Let Δ be a triangle. A normal arc α is an embedded simple arc in Δ such that the boundary $\partial\alpha = \{v_0, v_1\}$ meets two different edges of Δ. Two such arcs are equivalent if we can find a homeomorphism from Δ to itself preserving each face of the triangle and taking one arc to the other. Now let T be a tetrahedron. A normal disk D is an embedded disk in T such that ∂D is a collection of normal arcs where each normal arc meets a different face of T. Similarly, two such disks are equivalent if there exists a homeomorphism of T to itself preserving its faces taking one to the other.

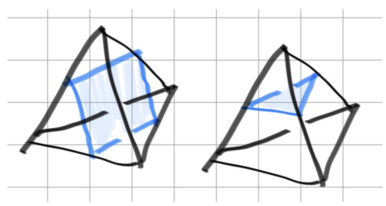


Figure 6.1: Normal Disks

We can equip each normal arc or disk with a transverse orientation. In a triangle there are 6 choices for transversely oriented normal arcs, and in a tetrahedron there are 14 choices for transversely oriented normal disks. If $\mathscr{T}$ is a triangulation with d tetrahedron, then we define the transversly oriented disk space $\mathrm{ND}(\mathscr{T})$ to be the real vector space generated on a basis of transversely oriented normal disks in each tetrahedron in $\mathscr{T}$, thus it is a $(14d)$-dimensional vector space. If $\Sigma \subset Y$ is an oriented surface that meets any 3-simplex in $\mathscr{T}$ in transversely oriented normal disks, then we say that Σ is a normal surface. Moreover, any normal surface can be specified by a unique point in $\mathrm{ND}(\mathscr{T})$.

$\mathrm{ND}(\mathscr{T})$ is a very large space and most of the time we are interested in normal surfaces with desirable properties (e.g. closed surfaces). So we do not need the entire normal surface space when developing algorithms. So we can do dimension reduction using a set of *matching equations*.

The set of matching equations will vary depending on the problem, but they are typically linear equations that match certain faces together.

6.2 Computing Link Floer Homology

Ozsváth and Szabó showed $\widehat{HFL}$ determined the Thurston Polytope for a link complement. Unfortunately, link Floer homology is generally very difficult to compute, but there is hope since various combinatorial reformulations for Link Floer homology exist. The first prominent computational Floer theory for knots and links was grid homology [38]. Grid homology however is in many respects too computationally inefficient for most practical implementations. The modern approach is based on *bordered algebras* [39].

6.3 Our Approach

To compute Thurston polytopes of links, we first separate links into alternating and non-alternating classes. This is due to the significant computational advantage of computing the Thurston polytope directly from the Alexander polynomial. In the non-alternating case, we use Worden's program *Tnorm* based on spun-normal surfaces.

A computer cluster is a collection of machines arranged to work together to achieve a task. It is structured into a collection of nodes that are delegated individual tasks which may work independently or in parallel. In order to improve computational efficiency, we are using TAMU Math's Whistler cluster. At the time of writing, Whistler is a 33-node cluster consisting of 68 16-Core intel Xeon Gold processors, 8.25 TB total system memory, and 24.2 TB SSD usable storage.

All of our code is written in Sage and Python and utilizes SnapPy [2], Regina [40], and Tnorm [36]. Many examples that we compute are from the Hoste-Thistlethwaite link table which is displayed on the Knot-Atlas [1].

7. REALIZABLE THURSTON POLYTOPES

A Thurston polytope is a convex symmetric polytope with finitely many rational vertices. One question we can ask is given a polytope P satisfying those conditions; is there a manifold that realizes P as its Thurston polytope? In general, this question is unanswered; however several classes of polytopes have been realized as Thurston polytopes. In his original paper, Thurston proved the following theorem.

Theorem 7.1. *(Corollary [3]) Every norm on $\mathbb{R}^2$ which takes even integer values on lattice points is the Thurston norm on $H^1(Y, \partial Y)$ for some Y*

It is not known if this parity property shown by Thurston generalizes to Thurston polytopes in higher dimensions. Sane gave a partial extension to even dimensional Thurston polytopes [41].

Theorem 7.2. *[41] If P is a homologically nontrivial polytope[1], then it is the dual Thurston polytope of some 3-manifold Y.*

It might also be feasible to classify Thurston polytopes for more restricted classes of manifolds. For example, Joan Licata [42] classified Thurston polygons for Pretzel links of the form

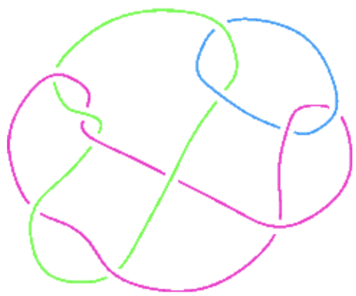
$$P(-2r_1 - 1, 2q_1 - 2q_2, 2r_2 + 1)$$

using techniques from link Floer homology. With this in mind, one could try to find relationships between invariants of links to properties of the norm ball in hopes of finding potential restrictions. We explored some of these relationships in Chapter 4. In particular, any restrictions on polynomials admissible as Alexander polynomials of alternating links yield restrictions on polytopes admissible as Thurston polytopes.

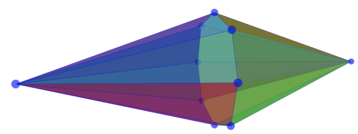
[1] See [41] for exposition on homologically nontrivial polytopes.

7.1 Links and their Polytopes

In this section, we explore Thurston polytopes of links with a focus on producing explicit examples via the methods discussed in chapter 6.



(a) L11a444, Reprinted from [1]



(b)

The Thurston norm of a link is a generalization of the genus of a knot. It tells us the minimum complexity of an oriented surface cobounding some collection of link components. Our first impulse might be to study Seifert surfaces both of the individual link components and of the entire link itself. Let us begin by studying Seifert surfaces in links.

Theorem 7.3. *Let $L = L_1 \sqcup L_2$ be a two component link, then* $\mathrm{lk}(\mathrm{L}_1, \mathrm{L}_2) = \xi(\mathrm{L}_1, \mathrm{F}_{\mathrm{L}_2}) = \xi(\mathrm{F}_{\mathrm{L}_1}, \mathrm{L}_2)$ *where F_{L_i} is a Seifert surface of L_i and $\xi(\cdot,\cdot)$ denotes the algebraic intersection form.*

Sketch of Proof. First consider a diagram for L. Without loss of generality perform Seifert's algorithm on L_1 and call this Seifert surface F. If necessary isotope L_0 such that it meets L_1 far from any crossings and record the intersection number and linking number. Now after performing the first part of Seifert's algorithm, L_1 consists of Seifert circles. Notice that neither the linking number nor algebraic intersection number changed as a result of this process. Now notice that locally on each Seifert circle, the linking number and the algebraic intersection number are the same, then after summing up over all Seifert circles it follows that $\mathrm{lk}(\mathrm{L}_0, \mathrm{L}_1) = \xi(\mathrm{L}_0, \mathrm{F})$.

Now let G be an arbitrary Seifert surfaces for L_1. It follows that if we reverse the orientation of G and glue the boundaries of F and G, then we obtain an oriented closed surface, and that the intersection number of a knot L_0 with the new surface formed from F and G must be zero, thus $\mathrm{lk}(\mathrm{L_0}, \mathrm{L_1}) = \xi(\mathrm{L_0}, \mathrm{F}) = \xi(\mathrm{L_1}, \mathrm{G})$ for all Seifert surfaces of L_1. □

Now let us investigate some properties of the norm ball by studying the reduced Alexander polynomials $\widehat{\Delta}_L(t)$ for an oriented link L.

Theorem 7.4. *[17] If L is an ℓ-component link in S^3 and $\widehat{\Delta}_L(t)$ denotes the reduced Alexander polynomial of L, and $\deg \widehat{\Delta}(t)$ is the breadth of polynomial, then*

$$2g + \ell - 1 \geq \deg \widehat{\Delta}(t) \tag{7.1}$$

If L is alternating, then equation 7.1 is an equality.

Thus if L is a 2-component link, then $2g + 1 = \deg \hat{\Delta}(t)$, and notice that $\deg \hat{\Delta}(t) - 1 = (2g + 1) - 1 = \chi_-(F)$ for a Seifert surface of L, thus F represents $\alpha = (1, 1) \in H_2(X_L, \partial X_L)$. Notice that in order for $(1, 1)$ to be a vertex of $B_{\|\cdot\|_T}(L)$, $\chi_-(F) = 1 = 2g$ which has no integral solutions. Furthermore, when $\ell \geq 2$, then $2g + \ell > 1$ thus $(1, ..., 1)$ can not be in our norm ball, and by reversing orientations neither could any point $(\pm 1, \ldots, \pm 1)$.

In general

$$\ell - 2 \leq \deg \widehat{\Delta}_L(t) = \chi_-(F_{min}) + 1$$

. This does tell us though where to find the point on the face of the norm balls. If $\deg \hat{\Delta}_L(t) = d$, then $d - 1 = \chi_-(F_{min})$, so the point in question on the norm ball is likely at $(\frac{1}{d-1}, \ldots, \frac{1}{d-1})$.

This tells us the following. Given a non-oriented link L, we can compute the points on the faces of Thurston norm ball along the rays going towards $(\pm 1, \ldots, \pm 1)$ by placing appropriate orientations and computing the degree of the reduced Alexander polynomial.

There is one thing we did not address, namely if a representative of these classes might be represented with a disconnected surface. We have the following theorem from Lickorish:

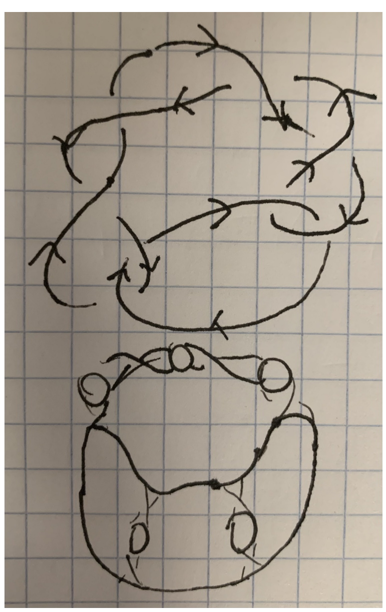
Theorem 7.5. *[17] Suppose an oriented link L bounds a disconnected oriented surface in S^3, then $\widehat{\Delta}_L(t) = 0$.*

So if L is alternating, and $\widehat{\Delta}_L(t)$ is non-zero for all orientations, then we know that these connected surfaces are norm-minimizing. This allows us to prove some more interesting results.

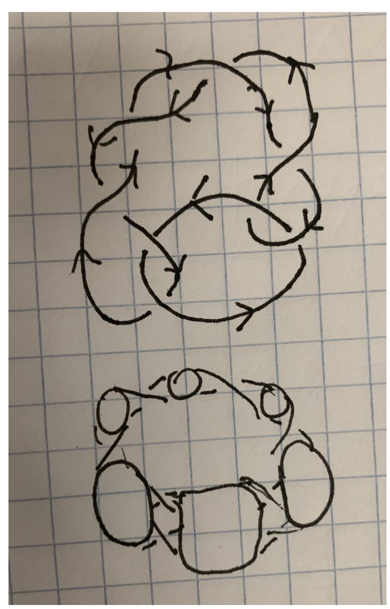
Theorem 7.6. *There exists an infinite class of links with the same Thurston norm ball. In particular, there are infinitely many links whose norm ball is the unit diamond in $\mathbb{R}^2$.*

The proof follows from first studying the following example.

Example 10. *Consider $L8a6$. Notice that each link component is the boundary for a twice-punctured annulus. One can readily verify that these annuli are norm-minimizing. Now fix orientations on each component and use Seifert's algorithm to obtain a Seifert surface. One sees that this surface has genus 1, and thus $\chi_-(F) = 2$. Now switch the orientation of the second link component and we see that again we obtain a Seifert surface of genus 1.*



(c)



(d)

Computing the reduced Alexander polynomials we see that they are non-zero. Both the link on the left and the link on the right have the following Seifert matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 4 \end{pmatrix}$$

, thus $\widehat{\Delta}_L(t) = 4(t-1)^3 - t^2 + t$*. It follows then that* $(\pm 1/2, \pm 1/2)$ *are on the boundary of the Thurston norm ball, and since we now have 3 colinear points in each quadrant, it follows that the Thurston norm ball is the diamond formed by the convex hull of* $(\pm 1, 0), (0, \pm 1)$.

Proof. Consider the previous example, and now notice that we can increase the number of twists on the twisted component. This constitutes an infinite class of links that we may reasonably call twist links T_n. Increasing the number of twists does not change the fact that twice-punctured annuli are still norm-minimizing for the generators of $H^1(X_{T_n}; \mathbb{Z})$. When $n = 2k$ is even, performing Seifert's algorithm on these links yields the same surfaces.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & n \end{pmatrix}, \qquad \widehat{\Delta}_L = n(t-1)^3 - t^2 + t$$

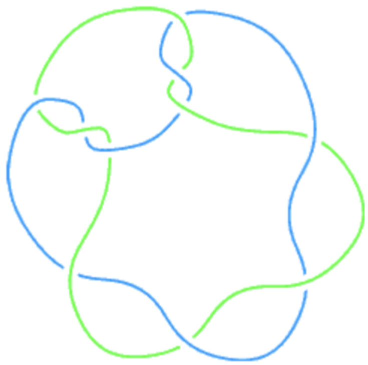
Thus $B_{T_{2k}}$ is a diamond with vertices at $(\pm 1, 0), (0, \pm 1)$ for all even n at least 4. $\square$

7.1.1 Regular Polytopes

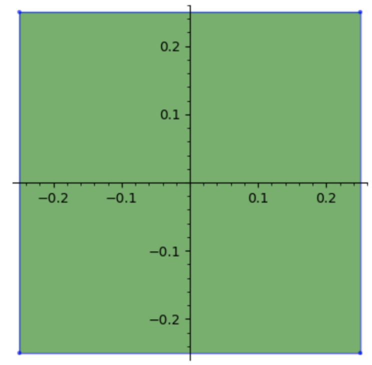
Perhaps the most famous convex polytopes are regular polytopes. In dimension 2, these are the regular polygons.

Definition 7.1. *A polytope is said to be regular if its symmetry group acts transitively.*

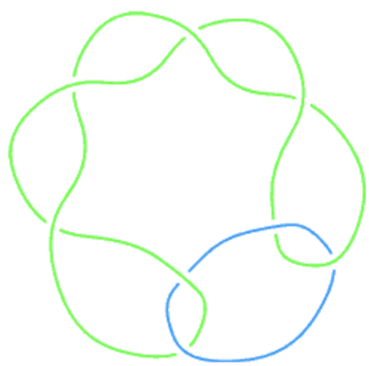
Example 11. *The convex hull of the* n*th roots of unity form a regular* n*-polygon.*



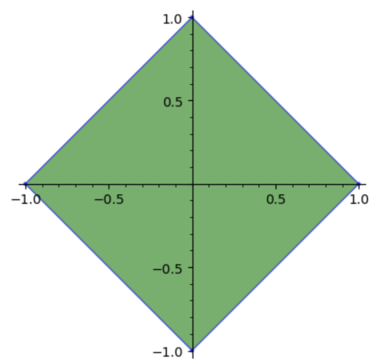
(e) L10a115, Reprinted from [1]



(f)



(g) L8a6, Reprinted from [1]



(h)

We saw examples of squares (L10a115) and diamonds (L8a6) which are Thurston polygons for some links. Which other regular polygons are admissible? The answer is quite sad but interesting nonetheless.

Theorem 7.7. *A regular n-polygon admits some irrational vertices for* $n > 4$.

Proof. Suppose that a regular polygon admitted rational vertices, then we could scale it by a rational number to obtain a regular polygon with integral vertices. The area of a regular polygon with side length ℓ is

$$\frac{n\ell^2}{4}\cot\frac{\pi}{n}$$

but by Pick's theorem the area of such a polygon with integral vertices is given by

$$i+\frac{b}{2}-1$$

where i, b are vertices. Now $\cot(\pi/n)$ is irrational for all $n \in \mathbb{N}_{\geq 3}$ with $n \neq 4$. Now what we need now is to rule out the existence of a sufficient side length ℓ.

Now we know that every triangle formed by the coordinates of successive vertices of the n-gon has two integer side lengths. And moreover it then follows from the Pythagorean theorem that ℓ^2 is an integer. Thus no choice of ℓ can make the area of our integral regular polygon rational. [2]

$\square$

A requirement of any Thurston polytope is that it must have vertices at rational coordinates, and thus Theorem 7.7 bars the majority of these polygons from being Thurston polygons. Can we do better when $\mathrm{rk}(\mathrm{H}^1(\mathrm{Y})) = 3$? The regular convex 3-polytopes are the platonic solids of which there are five. We can produce plenty of examples of 3 component links whose Thurston polytope is a regular octahedron. A non-extensive list includes L7a7, L8a15, L9a45, L10a26.

So which other Platonic solids are Thurston polyhedra? The answer is not too promising. The tetrahedron fails to be symmetric about each octant, thus there is no tetrahedral Thurston polytope. The next ones to consider are the dodecahedron and icosehedron which both have pentagonal faces. However neither of these can be embedded into $\mathbb{R}^3$ with rational vertices as coordinates. A rigorous proof of this fact can be found in [43]. A less rigorous but intuitive argument follows by

[2] An alternative method of proof would be to notice that the rational lattice in $\mathbb{R}^2$ is equivalent as a set to the field $\mathbb{Q}(i)$, now if a regular polygon could be embedded in $\mathbb{Q}(i)$ then its center is also in $\mathbb{Q}(i)$ since its simply the sum of the vertices divided by the number of vertices, then we could assume the center is at the origin and then scale by some vertex which would send a vertex to 1, and then notice that the vertices of the polygon are simply the roots of $x^n - 1$ which does not split over $\mathbb{Q}(i)$.

considering the following coordinates yielding a regular icosahedron.

$$(\pm 1, \pm\varphi, 0)$$
$$(\pm\varphi, 0, \pm 1)$$
$$(0, \pm 1, \pm\varphi)$$

Similarly, coordinates for the dodecahedron are

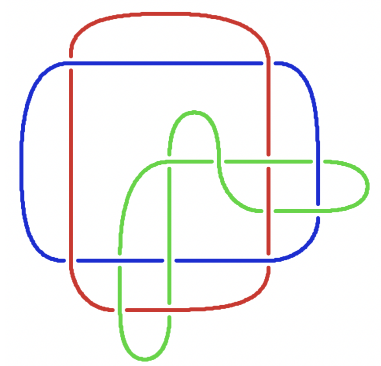
$$(\pm 1, \pm 1, \pm 1)$$
$$(0, \pm\varphi, \pm\frac{1}{\varphi})$$
$$(\pm\frac{1}{\varphi}, 0, \pm\varphi)$$
$$(\pm\varphi, \pm\frac{1}{\varphi}, 0)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio. A simple proof of the irrationality of φ is to notice that $\varphi^2 = 1 + \varphi$ and then rearrange to see that φ is a root of the polynomial

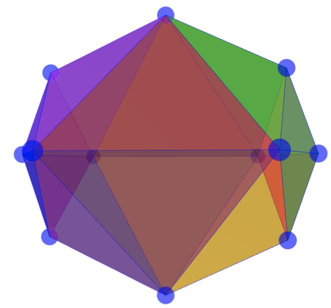
$$x^2 - x - 1$$

which by the rational root test is clearly irreducible over $\mathbb{Q}$. Thus neither the dodecahedron nor icosahedron can be Thurston polytopes for any 3-manifold Y. The closest we might hope for is a polytope with 20 triangular faces.

Example 12. *The following polyhedron is such a 'non-regular icosahedron.' This Thurston norm ball has 20 triangular faces.*



(i) L14a29515



(j)

The only remaining regular convex polyhedron is the cube, the polar dual of the octahedron.

Conjecture 7.1. *There exists a link $L \subset S^3$ such that its Thurston polytope is a cube.*

7.1.2 Slopes

Combinatorially a convex 3-polytope is simply a collection of vertices, edges, and faces embedded in $\mathbb{R}^3$, so one can define

$$\psi : \mathcal{P}_3 \to \mathcal{F}_3 \subset \mathbb{Z}^3$$

where $(v, e, f) \in \mathbb{Z}^3$ counts the number of vertices, edges, and faces. Elements in $\mathcal{F}_3$ are called f-vectors [3] and it generalizes in an obvious way to polytopes of arbitrary fixed finite dimension. Every convex 3-polytope P is homeomorphic to S^2 and thus $\chi(P) = v - e + f = 2$. This yields a strong restriction on f-vectors admissible as representing some 3-polytope. Furthermore, these combinatorial 3-polytopes were completely characterized in 1906 by Ernst Steinitz.

Lemma 7.1. *(Steinitz' Lemma) The set of all f-vectors of 3-polytopes is given by*

$$\mathcal{F}_3 = \{(v, e, f) \in \mathbb{Z}^3 : v - e + f = 2, f \leq 2v - 4, v \leq 2f - 4\}$$

[3] The reader may notice that they are not vectors as $\mathbb{Z}^3$ is not a vector space. The term "vector" likely is more closely interpreted in the computer science tradition of refering to a tuple of numbers

The $\chi = 2$ criterion tells us that the space of combinatorial convex polytopes is a 2-dimensional linear subspace with the two inequalities defining a cone in which combinatorial convex polytopes exist. The apex of this cone is the element $(4, 6, 4)$ which corresponds to a simplex, which can be realized by a tetrahedron.

Definition 7.2. *The slope φ for a polytope P with f-vector $(f, e, v) \in \mathcal{F}_3$ is defined by*

$$\varphi(P) = \frac{f-4}{v-4}.$$

By the Stenitz Lemma, barring the apex of the cone of $\mathcal{F}_3$, the values for φ lie between $1/2$ and 2. A cube, with f-vector $(8, 12, 6)$, has slope $1/2$. Steinitz later proved a stronger result.

Theorem 7.8. *(Steinitz' Theorem) There is a bijective correspondence between the set of 3-connected planar graphs and combinatorial polytopes.*

One method of proof for Steinitz' theorem is that any combinatorial polytope admits an embedding with rational (and thus integral) vertices [44].

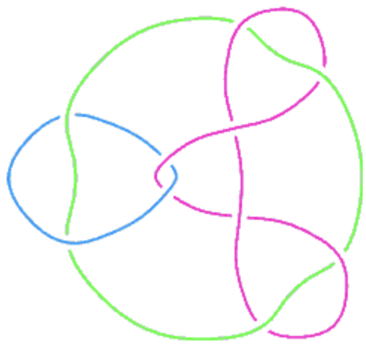
Definition 7.3. *A d-dimensional polytope is simple if each vertex is contained in d faces.*

There are many examples of simple polytopes including the truncated tetrahedron, truncated octahedron, and the chamfered dodecahedron or as it is more commonly called, a Fußball. The duals of simple polytopes are *simplicial polytopes* where each face has precisely d vertices.

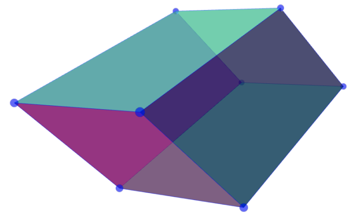
Proposition 7.1. *Let P be a 3-polytope with slope $\varphi(P)$. Then the polar dual P^* has slope $1/\varphi(P)$.*

Proof. This simply follows from the definition and the fact that polarity replaces faces with vertices and vertices with faces. $\square$

Example 13. *The following link L10n71 admits a slope 1/2 polytope.*

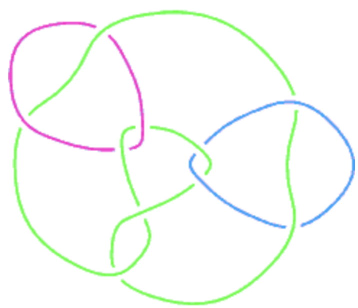


(**k**) L10n71, Reprinted from [1]

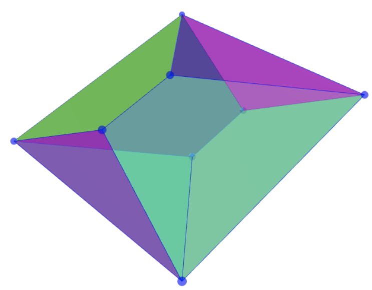


(l)

Example 14. *The following link L10a154 admits a slope 1 polytope.*

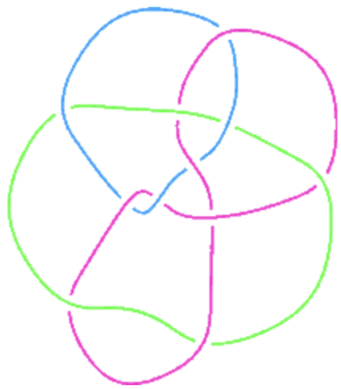


(**m**) L10a154, Reprinted from [1]

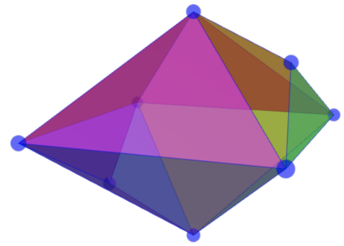


(n)

Example 15. *The following link L11a506 admits a slope 2 polytope.*

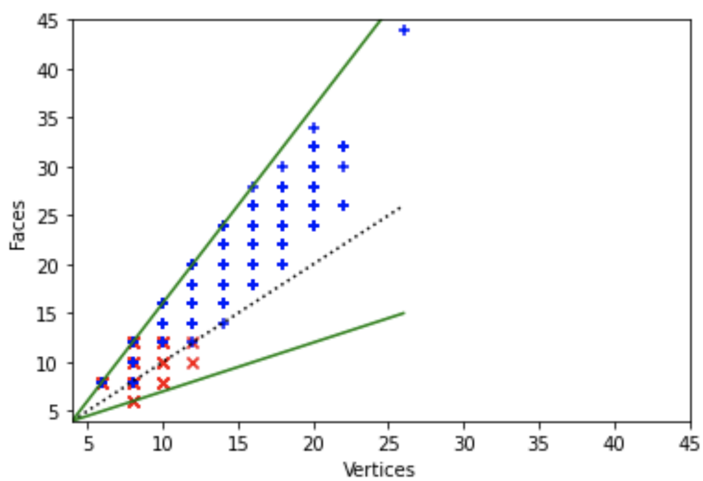


(o) L11a506, Reprinted from [1]



(p)

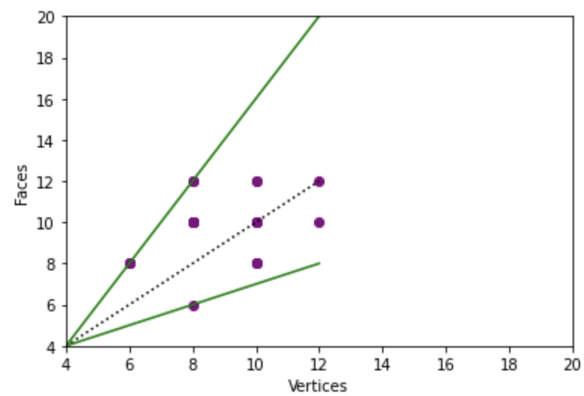
Now we present some novel findings which lead to some interesting conjectures.



The previous figure plots the vertex-face data for Thurston polytopes of alternating (blue +) and non-alternating (red x) 3-component links[4]. What one notices is that non-alteranting links appear to span the entire cone of admissible combinatorial polytopes, whereas the alternating links only occupy a subcone taking on slope values in the range $[1, 2]$. One might first expect that since

[4]Due to computational limitations, less Thurston polytopes for non-alternating links were computed than alternating. This does not take away from the observations

the Alexander polytope is equivalent to the Thurston polytope for alternating links and not for non-alternating links that perhaps this result somehow follows from a restriction on all Alexander polynomials or 3-component links such as the Torres' conditions. However this is not the case. The following plot shows the vertex-face data for *Alexander polytopes* of non-alternating links.



Even for this dataset, the non-alternating links span the entire cone. This leads to the following conjecture.

Conjecture 7.2. *The Newton polytope of the Alexander polynomial of an alternating 3-component link L has slope $s \in [1/2, 1]$*

At present the author is not aware of this being observed prior and thus constitutes a potentially new discovery. No sufficient conditions are known for 3-variable Alexander polynomials, and these findings suggest that studying their Newton polytopes constitutes a viable path.

7.1.3 Symmetry Groups?

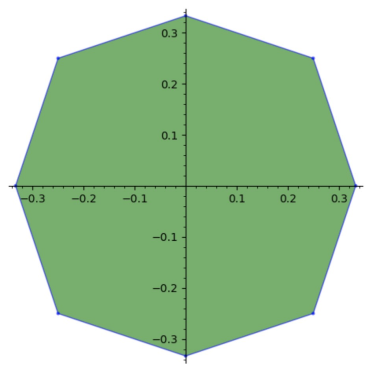
Finite group theory has been applied to the theory of polytopes by studying their symmetries. Let $\mathrm{Sym(P)}$ denote the symmetry group of a polytope P.

Proposition 7.2. *Let L be a link in S^3, and P_L denote its Thurston polytope, then $\mathbb{Z}_2 \leq \mathrm{Sym(P_L)}$*

Proof. The Thurston polytope is symmetric about $P \mapsto -P$. $\square$

Let us explore some examples.

Example 16. *Consider the link $L9a34$. Its Thurston norm polygon is the following:*

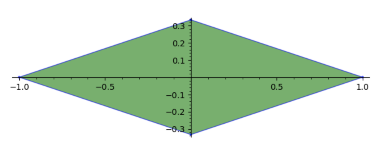


which has vertices at

$$(\pm 1/3, 0), (\pm 1/4, \pm 1/4), (0, \pm 1/3)$$

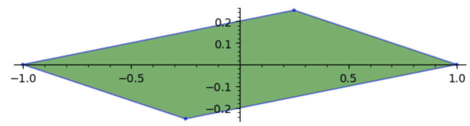
Now inside of this polygon sits a square with vertices at $(\pm 1/3, 0)$ and $(0, \pm 1/3)$. If we perform an isometry of the square formed by these vertices then notice that we are also sending the remaining 4 vertices to each other, thus the symmetry group of this polygon is D_8.

Example 17. *Consider the link L9a19. Its Thurston polygon is the following.*



Notice the two obvious reflectional symmetries about each axis and the lack of rotational symmetry. Thus the symmetry group for this is the Klein four group $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Example 18. *Consider the link L11a97. Its Thurston polygon is the following*



which has one symmetry, rotating about 180°. Thus its symmetry group is $\mathbb{Z}_2$.

Conjecture 7.3. *There exists a two-component link L whose Thurston polytope has $\mathbb{Z}_4$ symmetry.*

How do we interpret D_8 symmetry? Recall that all of our links are *oriented* and that choice of orientation matters when computing the infinite cyclic cover of the link complement, and hence when we compute the Alexander polynomial and Alexander polytope. When the Thurston polygon admits D_8 symmetry, it first tells that orientation choice of the components did not matter. Let the rank of $H^1(Y) = \ell$. There are maps

$$\mathscr{O} : \bigsqcup_{i=1}^{\ell} S^1 \longrightarrow \{\pm 1\}^{\ell}$$

giving each component of the link an orientation. As there is no canonical choice of orientation, fix an orientation on L, and define this to be $L \mapsto (1, \dots, 1)$. Now suppose $\epsilon : \{+1\}^{\ell} \to \{\pm 1\}^{\ell}$ is an arbitrary change of orientation. Call the new link $\epsilon(L)$.

Theorem 7.9. *Let $\vec{L}$ be a link. If $\vec{L} = \epsilon(\vec{L})$ for any arbitrary ϵ, then $(\mathbb{Z}_2)^{\ell} \leq \mathrm{Sym}(\mathrm{B}_{\vec{\mathrm{L}}})$.*

Proof. Fix an arbitrary orientation on L and let $\{E_i\}_{i=1}^{\ell}$ be a basis for $H_1(X_{\vec{L}}; \mathbb{R})$ determined by this orientation. Now swap the orientation of L_i, this acts on the Thurston polytope by a reflection. The reflected norm ball is also the Thurston norm ball of $\epsilon(\vec{L})$ which by assumption is equivalent to $\vec{L}$. Thus $B_{\vec{L}} = B_{\epsilon\vec{L}}$. Since we could choose any $i \in \{1, \dots, \ell\}$ the result follows. □

We see that the symmetries of the Thurston polytope form an obstruction to the link being invariant under orientations.

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APPENDIX A

POLYTOPES

Definition A.1. *A polytope in $\mathbb{R}^n$ is the convex hull of a finite non-empty subset of $\mathbb{R}^n$*

Definition A.2. *Let S be a subset of $\mathbb{R}^d$, then the polar set $S^* \subset \mathbb{R}^d$ is the following set defined by*

$$S^* = \{y \in \mathbb{R}^d | \forall x \in S : \langle x, y\rangle \leq 1\}$$

When $S = P$ is a polytope then we say that P^* is the polar dual polytope of P.

If P is an n-polytope, then polarity also replaces the number of k-faces with $(n-k)-$ faces.

Example 19. *Let P be the unit cube, then $P^* = \mathrm{Conv}(\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\})$ is the octahedron.*

Polytopes may admit certain symmetries. Thurston polytopes admit a reflectional symmetry about the origin. Other shapes like the octahedron have more symmetries.

Definition A.3. *The symmmetry group of a polytope is the group of transformations (isometries) of that polytope which preserve the polytope.*

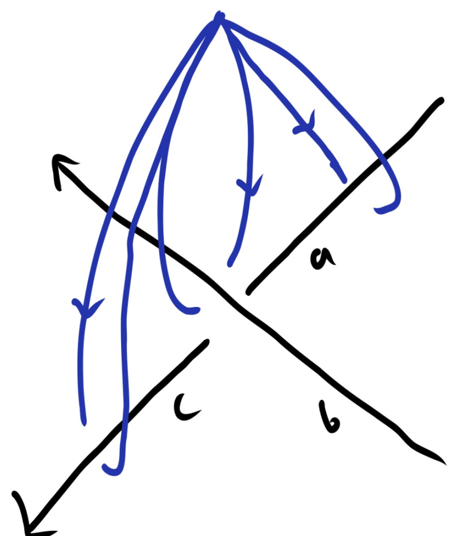
Example 20. *Consider the tetrahedron. This object has 4 order 3 rotational symmetries where one fixes a vertex and rotates the other 3. It also has reflectional symmetries where one fixes a plane passing through one edge and the middle of the opposite edge. If we consider the vertices as a set of order 4, then the rotations correspond to order 3 permutations, and the reflections correspond to transpositions thus the group of symmetries is S_4.*

Polar duals admit the same symmetries. The dual of an octahedron is a cube which admits reflectional and rotation symmetries, thus its symmetry group is S_4. Consequently S_4 is also the symmetry group of the octahedron.

APPENDIX B

COMPUTING ALEXANDER POLYNOMIALS

In this section, we will present a method for computing the Alexander polynomial of a link. Recall the link group $\pi_1(X_L)$ is the fundamental group of its exterior. An easy way to compute this group is by using the Wirtinger presentation. Consider a link diagram and label each strand. Choose a basepoint x_0 away from the link and for each strand choose a loop based at x_0 passing under that strand in the direction compatible with the right hand rule. These loops inherit the labels of their associated strands and they form the generators of $\pi_1(X_L)$. Relations come from each crossing as follows



$$abc^{-1} = b$$

One the one hand, the Wirtinger presentation is easy to compute, but not a very efficient presentation. Nevertheless we can use it in computing the Alexander polynomial. We need one more notion; namely the Fox derivatives. Let $F_n = \langle x_1, ..., x_n | - \rangle$ be the free group on n generators. We define the Fox free derivative $\frac{\partial}{\partial x_i} : \langle x_1, ..., x_n | - \rangle \to \mathbb{Z}[F_n]$ to be the map defined by

$$\frac{\partial}{\partial x_i} e = 0$$
$$\frac{\partial}{\partial x_i} x_j = \delta_{ij}$$
$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}$$

From these rules it is straightforward to show that $\frac{\partial(x^{-1})}{\partial x} = -x^{-1}$. Fox was able to show that given a finite group presentation of $\pi_1(X_L)$, the image of the "Jacobian" matrix of the relations under the abelianization homomorphism, call it A, may be used to yield the Alexander ideal.

Theorem (Fox '60) [32]

Let X_L be a link complement and $\pi_1(X_L)$ given a finite presentation

$$\pi_1(X_L) = \langle x_1, ..., x_n | r_1, ..., r_m \rangle$$

and let A be the image of the "Jacobian" matrix reduced to words in $\mathbb{Z}[H_1(X_L)]$

$$A = \psi \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{pmatrix}$$

where ψ is the abelianization homomorphism. Then the $(n-1) \times (n-1)$ minors of A generate the Alexander ideal and

$$\Delta_{X_L} = \gcd\{(n-1) \times (n-1) \text{ minors of } A\}$$

To understand how this machinery works, we recommend reading chapter 9 of [45].

Another method for computing the Alexander polynomial follows a more geometric approach. Let K be a knot in S^3 and F a Seifert surface of K. A Seifert form is a bilinear form which takes pairs of classes in $H_1(F)$ and returns the linking number of a loop in $[f]$ and the positive pushoff, $\iota_+[f]$. For details see [17]. Such a form can be represented by a $(2g \times 2g)$ matrix, where g is the genus of F. Seifert forms for a knot K admit different matrix representations up to S-equivalence (see [17]). We can now construct the maximal Abelian cover. Take a Seifert surface F of K, and cut the knot complement X_K at F. This new space Y has two boundary components F_+ and F_-. There is a homeomorphism $\varphi : F_+ \to F_-$ which can recover the original space X. Take infinitely many copies Y_i and glue F_+ of Y_i to F_- of Y_{i+1}. One can clearly see that there is a group of deck transformations which take Y_i to Y_{i+1}, and let us call this action t. Denote this total space by $X_\infty = \cup_{\varphi_i} Y_i$ which has an infinite cyclic group of deck transformations. This action then descends to $H_1(X_\infty; \mathbb{Z})$ turning it into a $\mathbb{Z}[t, t^{-1}]$-module. We can now compute the Alexander module.

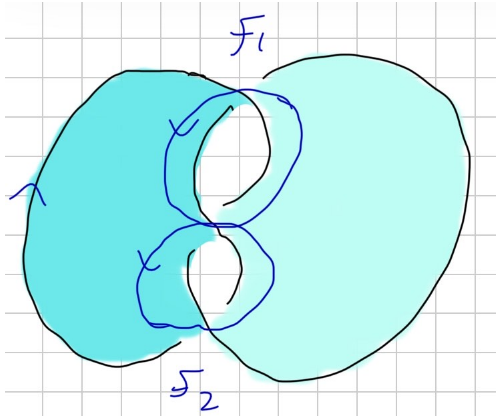
Theorem B.1. *[17] Let F be a Seifert surface for an oriented knot K and let A be the corresponding Seifert matrix w.r.t. any basis of $H_1(F; \mathbb{Z})$, then $tA - A^T$ is a matrix which presents $H_1(X_\infty; \mathbb{Z})$ that is the Alexander module. Moreover,* $\det(tA - A^T) = \Delta_K(t)$.

A proof can be found in Lickorish [17], and in 1958 Fujitsugu Hosokawa gave an extension of this method to compute the reduced Alexander polynomial for links [46].

Corollary B.1. *Let K be a knot with minimum genus g_{min}, then* $\deg \Delta_K(t) \leq 2g_{min}$.

This follows from the fact that since $\Delta_K(t)$ is the determinant of a matrix of size $2g$ for a Seifert surface, the Alexander polynomial can have a degree at most $2g_{min}$.

Example 21. *Consider the following Seifert surface of the Trefoil* 3_1.



We can now find the entries of the Seifert matrix.

$$lk(f_1, f_1^+) = 1$$
$$lk(f_1, f_2^+) = -1$$
$$lk(f_2, f_1^+) = 0$$
$$lk(f_2, f_2^+) = 1$$

So A,

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

We can compute the Alexander Polynomial easily as $\Delta_{3_1}(t) = t^2 - t + 1$*, which can be normalized by* t^{-1} *as* $\Delta_{3_1}(t) = t + t^{-1} - 1$*. The signature can also be easily computed:* $\sigma(3_1) = -2$*. As the trefoil is alternating, it follows that* $g_{min}(3_1) = (1/2)\deg(\Delta_{3_1}(t)) = 1$*.*

APPENDIX C

BACKGROUND ON FOLIATIONS

Definition C.1. *A codimension 1 foliation $\mathcal{F}$ of a 3-manifold Y is a decomposition of Y into connected surfaces called leaves such that Y is covered by charts $\{U_\alpha\}$ called foliation charts of the form $\mathbb{R}^2 \times \mathbb{R}$ such that each leaf passes through U_α in slices of the form $\mathbb{R}^2 \times \{p\}$.*

Definition C.1 generalizes in an obvious way to codimenion k foliations of manifolds of dimension $n > k$.

Example 22. *If $\Sigma \hookrightarrow Y \to S^1$ is a fibration over S^1 with connected fibers, then the fibers form the leaves of a foliation of Y.*

A rank 2 distribution $\mathscr{D}$ on a 3-manifold Y is a choice of a 2-dimensional subspace $\mathscr{D}(p)$ of T_pY for each $p \in Y$ such that there exists a neighborhood U of p and smooth vector fields X_1, X_2 on U such that $\mathscr{D}(q) = \mathrm{Span(X_1(q), X_2(q))}$ for all $q \in U$ (such a distribution is also called a hyperplane field). A vector field X is tangent to a distribution $\mathscr{D}$ if $X(p) \in \mathscr{D}(p)$ for all $p \in Y$.

Definition C.2. *A distribution $\mathscr{D}$ is called involutive if for any pair of vector fields X and Y tangent to $\mathscr{D}$ their Lie bracket $[X, Y]$ is also tangent to $\mathscr{D}$.*

From a smooth foliation $\mathcal{F}$ one obtains a rank 2 distribution on Y.

Definition C.3. *A rank 2 distribution $\mathscr{D}$ which defines a foliation is said to be integrable. That is if for every $p \in Y$ there is a unique germ of an embedded surface Σ such that $T\Sigma = \mathscr{D}$ in a neighborhood around p.*

An m-dimensional subspace W of a vector space V can be described via a choice of a basis for this subspace. A dual way to describe W is as the intersection of kernels of $(n - m)$ linear forms (elements of V^*). With this in mind, one can define a rank 2 distribution as the kernel of a locally defined one form ω. Furthermore the theorem of Frobenius relates all these concepts.

Theorem C.1. *(Frobenius) Let $\mathscr{D}$ be a rank 2 distribution on a 3 manifold Y. The following statements are equivalent:*

- *$\mathscr{D}$ is integrable*
- *If ω is a 1-form such that locally $\ker \omega = \mathscr{D}$ then $\omega \wedge d\omega = 0$*
- *$\mathscr{D}$ is involutive.*

Each leaf of a foliation $\mathcal{F}$ is defined by an immersion $\iota : \Sigma \to Y$. We can take the normal bundle of the leaf $\iota(\Sigma)$, which is defined as the quotient bundle $TN = TY|_{\iota(Y)}/T\Sigma$, and paste the normal bundles of each leaf in $\mathcal{F}$ together. If each leaf is equipped with a Riemannian metric, then we can view the normal bundle as the set of normal vectors to the leaves, i.e. a normal vector field.

Definition C.4. *A foliation $\mathcal{F}$ is said to be transversely oriented if it is integrable to a distribution $\mathscr{D}$ whose normal bundle is orientable.*

Definition C.5. *A foliation $\mathcal{F}$ is said to be taut if there exists a properly embedded circle transverse to $\mathcal{F}$ which intersects every leaf of $\mathcal{F}$.*

Definition C.6. *A dead end component of a transversely oriented foliation $\mathcal{F}$ of Y is an embedded 3-manifold $N \subset Y$ such that for some transverse orientation on N the vectors point into N along ∂N.*

Dead end components obstruct a foliation from being taut. In fact a theorem of Sue Goodman says that a transversely oriented foliation is taut if and only if it admits no dead end components [47]. An important example of a dead end component is a Reeb component which was discovered by Georges Reeb in the 1950s where Reeb described a foliation of S^3.

Example 23. *(Reeb Foliation [48]) Consider a decomposition of S^3 via two solid tori glued together via their boundary. We will define a foliation on each solid torus $T_0 \cup_T T_1$.*

Consider the upper half space $H = \mathbb{R}^2 \times [0, \infty)$ with the product foliation of planes $\mathbb{R}^2 \times \{z\}$. Remove the origin and consider the $\mathbb{Z}$-action generated by $p \mapsto 2p$. The quotient $(H\backslash\{0\})/\mathbb{Z}$ is a *solid torus and the induced foliation is known as a Reeb component. Notice that each leaf in the Reeb component is homeomorphic to $\mathbb{R}^2$ save the boundary torus which is the only compact leaf. The Reeb foliation on S^3 is the foliation defined by gluing together two Reeb components.*

Theorem C.2. *(Lickorish [49]) Every closed orientable 3-manifold M admits a codimension 1 foliation.*

Sketch of Proof. The idea of the proof is first to use the fact that if Y is closed connected and orientable, then Y contains a fibered link $L \subset Y$. Foliate the complement $Y - \nu(L)$ by the fibers. $\nu(L)$ consists of a finite disjoint union of solid tori. Endow each solid torus with a Reeb foliation and then fill in $\nu(L)$. $\square$

Theorem C.3. *(Reeb Stability Theorem [48]) Let $\mathcal{F}$ be a cooriented foliation of Y such that some leaf λ is a sphere. Then $Y = S^2 \times S^1$, and $\mathcal{F}$ is simply the product foliation by spheres $S^2 \times \{p\}$.*

The following theorem is due to Novikov and was later improved by Rosenberg.

Theorem C.4. *(Novikov [50])(Rosenberg [51]) Suppose $\mathcal{F}$ is a taut foliataion of Y and that Y is not finitely covered by $S^1 \times S^2$, then the following hold:*

- *Y is irreducible*
- *Leaves of $\mathcal{F}$ are incompressible.*
- *Every transverse loop to $\mathcal{F}$ is esential in $\pi_1(Y)$.*

Definition C.7. *Let Y be a compact oriented 3-manifold with a codimension 1 foliation $\mathcal{F}$. A leaf λ is depth 0 if it is compact. If $\bar{\lambda} - \lambda$ consists of leaves of depth at most n, then λ is at depth at most n+1. If every leaf of $\mathcal{F}$ is depth at most n then the foliation $\mathcal{F}$ is said to be depth n, and if n is finite then we say $\mathcal{F}$ has finite depth.*

We see then that the Reeb foliation has depth 1. The following theorem is due to Sullivan

Theorem C.5. *[52] Let $\mathcal{F}$ be a taut foliation on Y. Then there is a smooth metric g on Y for which leaves $\lambda \in \mathcal{F}$ are area minimizing.*

APPENDIX D

MINIMAL SURFACE THEORY

There is a vast literature on minimal submanifolds dating back to Euler and Lagrange. Let M be a Riemannian manifold. Let Σ be an immersed submanifold of M given the induced metric. At points in Σ a natural way to decompose TM is via the orthogonal decomposition:

$$TM = T\Sigma \oplus (T\Sigma)^{\perp}.$$

Now let $X, Y \in \Gamma(T\Sigma)$ and let $\widetilde{X}, \widetilde{Y}$ be arbitrary extensions of X, Y to M, then we may define the second fundamental form II : $\Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma((T\Sigma)^{\perp})$ by

$$\mathrm{II}(X, Y) = (\nabla_{\widetilde{X}}\widetilde{Y})^{\perp}$$

It is easy to see that II is symmetric and independent of the extensions of X, Y. The mean curvature H is then defined as

$$H = \frac{1}{n}\mathrm{tr}(\mathrm{II})$$

where $n = \dim M$. Generalizing the notion of how geodesics, curves which locally minimize arclength, have vanishing Gaussian curvature, minimal submanifolds are submanifolds with vanishing H. If we consider all isometric immersions of a submanifold $\Sigma \to M$, minimal surfaces will be the critical points of the area functional.

Theorem D.1. *Let Σ be a compact Riemannnian manifold with $f : \Sigma \to M$ an isometric immersion of an n dimensional submanifold with mean curvature H, and let $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a smooth family of immersions fixing the boundary with $F(\cdot, 0) = f$. Let $v = \frac{\partial}{\partial s}F(t, s)$ be the vector* *field of the variation. The first variational formula is*

$$\frac{d}{ds}A(f_s)|_{s=0} = -\int_{\Sigma} \langle H, nv \rangle \, dA$$

A minimal surface Σ is said to be stable if its second variation formula is non-negative for all variations fixing the boundary of Σ.

$$\frac{d^2}{ds^2}A(f_s)|_{s=0} = -\int_{\Sigma} \langle L, nv \rangle \, dA$$

where L is the Jacobi operator.

Fix $\alpha \in H_2(M, \partial M)$ and let $\mathcal{F} = \{f : \Sigma \to M : F$ orientable. , f smooth, $f_*([\Sigma, \partial\Sigma]) = \alpha\}$ be the set of smooth immersions of surfaces representing α. Standard results in geometric measure theory guarantee that minimal surface representatives exist in each class α. If $\partial M = \emptyset$ then Joel Hass argues in *lemma 2.1* [53] that minimal surface representatives do in fact exist for elements in $H_2(M; \mathbb{Z})$ as another direct application of the compactness theorem of geometric measure theory ,see [25](*page 371*). Moreover, it is easy to see that this argument holds for classes in $H_2(M, \partial M; \mathbb{Z})$. Thus there exists a surface $f : \Sigma \to M$ such that $Area(f) = \inf\{Area(\tilde{f})|\tilde{f} \in \mathcal{F}\}$. Moreover f is also a critical point of the area functional and is a minimal surface. Further results by Uhlenbeck guarantee the existence of stable minimal surfaces in each class [54].