Restricted Increases in Risk Aversion and Their Application

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Louis Eeckhoudt, Liqun Liu, and Jack Meyer*

Abstract: This paper proposes two restricted forms of an increase in risk aversion. Using examples from portfolio choice, self-protection and insurance demand, it is shown that these stronger notions of increased risk aversion facilitate clear-cut comparative statics analysis in environments where traditional concepts of increased risk aversion are insufficient.

Key Words: comparative risk aversion, portfolio choice, self-protection, insurance demand

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Eeckhoudt: IESEG School of Management, LEM, Lille, France, and CORE Universite Catholique de Louvain, Louvain-la-Neuve, Belgium, eeckhoudt@fucam.ac.be

Liu: Private Enterprise Research Center, Texas A&M University, College Station, TX 77843, lliu@tamu.edu

Meyer: Department of Economics, Michigan State University, East Lansing, MI 48824, jmeyer@msu.edu

Corresponding author:

Liqun Liu
Private Enterprise Research Center
Texas A&M University
4231 TAMU
College Station, TX 77843
lliu@tamu.edu
979-845-7723 (phone)
979-845-6636(fax)

1. Introduction

Increased risk aversion as defined by Arrow (1971) and Pratt (1964) is the most common method used to compare the risk attitudes of two decision makers. Arrow and Pratt show that one individual is always willing to pay more than another to eliminate risk in its entirety if and only if the former is Arrow-Pratt more risk averse than the latter. In many situations it has been shown that an Arrow-Pratt more risk averse decision maker would undertake more of a risk-reducing activity than one who is less risk averse. For example, in the standard portfolio choice problem where investors allocate their wealth between a riskless asset, and a risky asset whose expected return is higher than the risk-free return, an Arrow-Pratt more risk averse investor will invest less in the risky asset and more in the riskless asset.[1] As another example, in pure insurance demand models it is well known that when the insurance is unfairly priced, risk averse individuals will choose partial coverage, and an Arrow-Pratt more risk averse individual will choose more insurance coverage than a less risk averse one.[2]

As Ross (1981) points out, however, there are situations in which the assumption of Arrow-Pratt more risk averse is not sufficiently strong to ensure unambiguous comparative static results. Specifically, Ross shows that comparisons of the willingness to pay for partial rather than total risk reduction can be made when one decision maker is Ross more risk averse than another. The same is not true for Arrow-Pratt more risk averse, a ranking that is weaker than or implied by Ross more risk averse. Ross also shows that in a more general portfolio choice problem in which initial wealth is allocated between two risky assets with one having a "larger

[1] A direct application of this result is that under decreasing absolute risk aversion or DARA, the amount invested in the risky asset increases as an investor's initial wealth increases (Pratt 1964).

[2] When the insurance is fairly priced, all risk averse individuals will choose full coverage (Mossin 1968).

and riskier" return than the other,[3] being Ross more risk averse implies, whereas being Arrow-Pratt more risk averse does not imply, choosing to invest a smaller share of wealth in the asset with the "larger and riskier" return.

In spite of these additional findings, there are still other decision models in which even Ross more risk averse is not a strong enough assumption to guarantee definite comparative static results. In addition to more general portfolio decisions, two other prominent examples discussed later in this paper are the model of self-protection analyzed by Dionne and Eeckhoudt (1985), and the insurance demand model with positive probability of insurer default, presented by Schlesinger and Schulenburg (1987).[4] Each of these papers uses analysis and specific examples to show that the assumption of Arrow-Pratt more risk averse is not sufficient to demonstrate the intuitively appealing result that more risk aversion implies a higher degree of self-protection or more insurance coverage, respectively. Their examples can also be used to show that the stronger Ross more risk averse assumption is insufficient as well. This is discussed further in Subsections 2.3 and 3.3, respectively, where a more detailed analysis of the two models is presented.

This paper proposes two restricted versions of the Ross more risk averse partial order and illustrates their application. Specifically, the linearly-restricted and the quadratically-restricted versions of the Ross more risk averse order are defined by requiring the utility functions being compared to differ by either a linear or a quadratic polynomial, respectively. Properties of these restricted increases in risk aversion are presented, and their usefulness is demonstrated in

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[3] According to Ross, $\tilde{y}$ is "larger and riskier" than $\tilde{x}$ if there is $\tilde{z}$ such that $\tilde{z}$ dominates $\tilde{x}$ in FSD and is a mean-preserving contraction from $\tilde{y}$.

[4] See also Doherty and Schlesinger (1990) and Schlesinger (2000).

situations where neither the Arrow-Pratt nor the Ross more risk averse assumption can generate clear-cut comparative static results.

While these restricted definitions of increased risk aversion involve strong assumptions, they also lead to interesting and powerful findings. In a sense, the approach taken here parallels in a logical way that carried out earlier in the "increases in risk" literature, where a variety of strong restrictions are placed on the Rothschild and Stiglitz (1970) definition of an increase in risk so that additional comparative static findings can be derived. Very early in that literature, Sandmo (1971) examines the effect of a risk increase caused by a linear transformation of a random variable. This narrowly defined risk increase was later generalized using less restrictive assumptions. Adjectives such as simple, strong, relatively strong, relatively weak and others have been used to describe restricted subsets of the more general set of Rothschild and Stiglitz risk increases.[5]

It is also the case that the linear and quadratic restrictions employed here lead to one and two parameter families of utility functions. This property allows changes in risk aversion to be modeled as a parameter change. This parametric property is a useful simplification and partially explains why constant absolute or relative risk aversion is often assumed.[6]

The traditional Arrow-Pratt or Ross definitions of increased risk aversion often fail to yield a definitive prediction concerning the risk-reducing effort made by the more risk averse individual because attitudes toward higher degrees of risk also affect this decision, and neither Ross nor Arrow-Pratt more risk averse restricts the way higher degrees of risk aversion vary

[5] Meyer and Ormiston (1985), Black and Bulkley (1989), Dionne, Eeckhoudt and Gollier (1993), Gollier (1995) and many others define and use these restricted forms of Rothschild and Stiglitz increases in risk.

[6] The CRRA family is often used in macroeconomic and financial models, and the two parameter "linex" family is also commonly used (Bell and Fishburn 2001 and Denuit et al. 2013).

across decision makers. For instance, in the self-protection model, or the insurance demand with insurer default, more risk-reducing effort also causes an increase in downside risk. This can lead to an Arrow-Pratt or Ross more risk averse individual, who is also more averse to downside risk, to choose less rather than more of a risk-reducing effort in order to avoid an increase in downside risk. Separating the effects of simultaneous changes in risk of different degrees by restricting the definition of more risk averse is one of the goals of the work discussed here.

The linearly- and quadratically-restricted increases in risk aversion definitions are specifically chosen to remedy this particular inadequacy of the traditional notions of increased risk aversion. These restricted increases in risk aversion can be viewed as a way to impose a strong *ceteris paribus* assumption, a common feature of economic analysis and often key to scientific inquiry. For the two restricted forms of increased risk aversion defined here, changes in attitudes towards increases in risk of higher-degree are kept constant to the extent that is possible or even well defined.

A consensus is lacking as to how to measure higher degrees of risk aversion. Several alternatives have been suggested and defended. Linearly- and quadratically-restricted increases in risk aversion keep some of these suggested measures fixed, and allow other measures to vary. The linearly-restricted assumption, for instance, keeps $-u'''(x)/u''(x)$ fixed, but allows $u'''(x)/u'(x)$ to change. Each of these two ratios of derivatives of utility has been suggested as a way to measure downside risk aversion. Linearly- and quadratically-restricted increases in risk aversion are formulated so as to minimize or eliminate the impact of higher degrees of risk aversion on the comparison of the risk-reducing effort chosen by two decision makers. Using these definitions allows additional unambiguous comparative static findings to be derived.

2. Linearly-Restricted More Risk Aversion and Its Application

Throughout the paper utility functions u(x) are assumed to be differentiable at least three times with u'(x) > 0 and u''(x) < 0. The support of all random variables is assumed to be in a closed finite interval denoted [A, B]. When necessary, we additionally require the utility functions to display third-degree risk aversion or even "mixed risk aversion". Recall that a utility function u(x) is mixed risk averse if it is *n*th-degree risk averse or $(-1)^{n+1}u^{(n)}(x) > 0$, for all $n \geq 1$ on some bounded domain [A, B].[7]

2.1. Linearly-Restricted More Risk Aversion

Because the restrictions imposed here further restrict the Ross more risk averse definition, we first give that definition in its extended form. *n*th-degree Ross more risk averse is defined below for any $n \geq 2$.

Definition 1. *Suppose that both u(x) and v(x) are nth-degree risk averse. u(x) is nth-degree Ross more risk averse than v(x) on [A, B] if*

$$\frac{(-1)^{n+1}u^{(n)}(x)}{u'(y)} \geq \frac{(-1)^{n+1}v^{(n)}(x)}{v'(y)} \qquad \textit{for all } x, y \in [A, B],$$

or equivalently, if there exists $\lambda > 0$, *such that*

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)} \qquad \textit{for all } x, y \in [A, B].$$

[7] Caballe and Pomansky (1996) study the properties of the utility functions that display mixed risk aversion. Almost all commonly used utility functions are mixed risk averse. Ekern (1980) characterizes $(-1)^{n+1}u^{(n)}(x) > 0$ as aversion to "*n*th-degree risk increases", which include as special cases the first-degree stochastically dominated change (*n* = 1), the Rothschild and Stiglitz (1970) mean-preserving spread (*n* = 2), and the Menezes et al. (1980) downside risk increase (*n* = 3). Eeckhoudt and Schlesinger (2006) further establish that mixed risk aversion is characterized by preference for combining "good" with "bad".

Definition 1, due to Jindapon and Neilson (2007), includes as special cases (for n = 2 and n = 3 respectively) the more well-known definition of Ross more risk averse (Ross 1981) and Ross more downside risk averse (Modica and Scarsini 2005).[8]

It has been shown that u(x) is nth-degree Ross more risk averse than v(x) if and only if u(x) is always willing to pay a weakly larger risk premium than v(x) to avoid an nth-degree risk increase. It is also important to recall that u(x) is nth-degree Ross more risk averse than v(x) if and only if there exist $k > 0$ and a function $\phi(x)$ such that $u(x) \equiv kv(x) + \phi(x)$, where $\phi'(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ (Li 2009 and Denuit and Eeckhoudt 2010).[9]

The first restriction placed on the definition of Ross more risk averse involves requiring this $\phi(x)$ function to be linear with a nonpositive first derivative.

<u>Definition 2</u>. u(x) is linearly-restricted more risk averse than v(x) if there are k > 0 and some linear form $a + bx$*, where b ≤ 0, such that* $u(x) \equiv kv(x) + a + bx$.

As an example, consider the family of $u_l(x) = lx + w(x)$, where $l > 0$ and $w(x)$ is any function that is increasing and concave on [A, B]. It is easy to see that, whenever $l_2 < l_1$, $u_{l_2}(x)$ is linearly-restricted more risk averse than $u_{l_1}(x)$ on [A, B]. If we let $w(x) = -e^{-cx}$, this family of utility functions is the linex (linear plus exponential) family studied in Bell and Fishburn (2001) and Denuit et al. (2013).

Several properties of the linearly-restricted more risk averse definition follow directly because the ϕ(x) function satisfies the requirement for nth-degree Ross more risk averse. This is formally stated as the first property, and from it the other two properties follow. Properties of

[8] The notion of nth-degree Ross more risk aversion in Definition 1 is further generalized by Liu and Meyer (2013).

[9] Crainich and Eeckhoudt (2008) and Li and Liu (2014) further establish the close relationship between the monetary utility premium for (higher-degree) risk increases and (higher-degree) Ross more risk aversion.

the linearly-restricted more risk averse order include: i) if u(x) is linearly-restricted more risk averse than v(x), then u(x) is *n*th-degree Ross more risk averse than v(x) for all $n \geq 2$; ii) linearly-restricted more risk averse is transitive; iii) linearly-restricted more risk averse is preserved under independent background risk; that is if u(x) is linearly-restricted more risk averse than v(x), then $U(x) \equiv Eu(x+\tilde{z})$ is linearly-restricted more risk averse than $V(x) \equiv Ev(x+\tilde{z})$, where $\tilde{z}$ is any random variable.

The following proposition provides the main tool for using linearly-restricted more risk aversion in comparative static analysis.

<u>Proposition1.</u> *If* $E\tilde{x} \leq E\tilde{y}$ *and* $Ev(\tilde{x}) > Ev(\tilde{y})$, *then* $Eu(\tilde{x}) > Eu(\tilde{y})$ *for all u(x) that are linearly-restricted more risk averse than v(x).*

<u>Proof</u>: Given $E\tilde{x} \leq E\tilde{y}$ and $Ev(\tilde{x}) > Ev(\tilde{y})$, u(x) being linearly-restricted more risk averse than v(x) implies

$$\begin{aligned} Eu(\tilde{x}) &= kEv(\tilde{x}) + a + bE\tilde{x} \\ &> kEv(\tilde{y}) + a + bE\tilde{y} = Eu(\tilde{y}). \end{aligned}$$

Q.E.D.

The intuition behind Proposition 1 is the following. Random variables $\tilde{x}$ and $\tilde{y}$ have many properties, including their mean value and all second- and higher-degree risk properties, and each can be important to decision makers when choosing between $\tilde{x}$ and $\tilde{y}$. When it is observed that $\tilde{x}$ is preferred to $\tilde{y}$ by v(x), then it must be the case that $\tilde{x}$ has some properties that are preferred to those of $\tilde{y}$ by this decision maker. Of course, there can also be properties that are not preferred, but in total, v(x) accepts $\tilde{x}$ over $\tilde{y}$. Now one property of a random variable becomes the focus of the analysis, and this is the mean value. It is assumed that $E\tilde{x} \le E\tilde{y}$; that is, the preferred random variable has a lower mean value. The question of interest is which other decision makers will also choose $\tilde{x}$ over $\tilde{y}$? The answer to this is given using the linearly-restricted more risk averse ranking. This is because all decision makers with utility functions u(x) which are linearly-restricted more risk averse than v(x) choose between $\tilde{x}$ and $\tilde{y}$ based on properties other than the mean in a way that is similar to that of v(x), and these decision makers with utility functions u(x) are even more willing than v(x) to accept a lower mean. As a consequence, these decision makers also rank $\tilde{x}$ over $\tilde{y}$.

To examine this explanation in more detail, observe that those u(x) which are linearly-restricted more risk averse than v(x) have marginal utilities satisfying $u'(x) = k \cdot v'(x) + b$, where $b \le 0$ is assumed. All other derivatives of these u(x) functions satisfy $u^{(n)} = k \cdot v^{(n)}$ for $n \ge 2$. Thus, the Arrow-Pratt and Ross measures of risk aversion are increased when a decision maker is linearly-restricted more risk averse, and the measures of higher-degrees of risk aversion may or may not be altered depending on the measure chosen for comparison. For downside risk aversion or prudence, for instance, the measure $u'''(x)/u'(x)$ is increased, but the measure $-u'''(x)/u''(x)$ stays the same. Similar statements can be made for all higher-degrees of risk aversion.

Finally, the discussion of how a decision maker chooses between random variables often uses terminology indicating that a decision maker “pays” for one change by accepting another. For those who are linearly-restricted more risk averse than v(x), their willingness to pay by accepting a reduced mean is larger than the similar willingness to pay by v(x). Since

$(-1)^{n+1}u^{(n)}(x)/u'(x) \geq (-1)^{n+1}v^{(n)}(x)/v'(x)$ for all $n \geq 2$, u(x) can be said to be willing to pay more in terms of a reduced mean value than v(x) for these second- and/or higher-degree risk changes.

2.2. Portfolio Choice and Linearly-Restricted More Risk Aversion

A very general two asset portfolio choice problem involves the decision maker choosing $\lambda \in [0,1]$ to maximize $Eu[\lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2]$. No restrictions are placed on $\tilde{x}_1$ or $\tilde{x}_2$. Assume that $Eu[\lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2]$ is strictly concave in λ so that the maximum is unique for every utility function u(x) considered here.[10] The effect of increased risk aversion on portfolio choice has been addressed in two quite restricted portfolio models using the Arrow-Pratt and the Ross more risk averse assumptions. The following proposition states the effect of the linearly-restricted more risk averse assumption on portfolio choice in this considerably more general model.

Proposition 2: *Without loss of generality, assume that* $E\tilde{x}_1 \leq E\tilde{x}_2$. $\lambda_u \geq \lambda_v$ *when u(x) is linearly-restricted more risk averse than v(x), where* $\lambda_u = \arg\max_{\lambda} Eu[\lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2]$ *and* $\lambda_v = \arg\max_{\lambda} Ev[\lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2]$.

Proof: We use proof by contradiction. Assume that u(x) is linearly-restricted more risk averse than v(x), but that $\lambda_u < \lambda_v$.

Consider the following two portfolios: $\tilde{x} = \lambda_v \tilde{x}_1 + (1-\lambda_v)\tilde{x}_2$ and $\tilde{y} = \lambda_u \tilde{x}_1 + (1-\lambda_u)\tilde{x}_2$. Obviously, $E\tilde{x} \leq E\tilde{y}$ and $Ev(\tilde{x}) > Ev(\tilde{y})$. Then according to Proposition 1, $Eu(\tilde{x}) > Eu(\tilde{y})$, which contradicts that λ_u is the optimal choice for u(x). Therefore, it must be $\lambda_u \geq \lambda_v$. Q.E.D.

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[10] This is equivalent to requiring $E\left\{u''[\lambda \tilde{x}_1 + (1-\lambda)\tilde{x}_2](\tilde{x}_1 - \tilde{x}_2)^2\right\} < 0$ for all $\lambda \in [0,1]$. Note that the unique maximum does not have to be an interior solution.

Note that in Proposition 2, other than $E\tilde{x}_1 \le E\tilde{x}_2$ which is without loss of generality, no conditions on the other moments of the random returns, or even on the dependence between these random returns is assumed. This portfolio model is as general as possible in a two asset portfolio model. The strong assumption of linearly-restricted more risk averse leads to a very strong portfolio result. More linearly-restricted risk averse investors choose portfolios with smaller mean returns. As noted, when u(x) is linearly-restricted more risk averse than v(x), u(x) is also *n*th-degree Ross more risk averse than v(x) for all $n \ge 2$. This implies that u(x) is willing to pay more, that is, accept a larger reduction in the mean, than v(x) to achieve any given change in the 2nd- or higher-degrees (Denuit and Eeckhoudt 2010). By choosing λ_v and λ_u, respectively, v(x) and u(x) reveal their willingness to pay. A larger λ represents a larger willingness to pay in terms of a reduced portfolio mean.

Proposition 2 completes the strengthening of the Arrow-Pratt portfolio theorem that was begun by Ross. Arrow and Pratt assume that the portfolio was composed of one risky and one riskless asset and show that being Arrow-Pratt more risk averse implies choosing less of the risky asset. Ross improves on this portfolio finding by allowing the two assets to be risky with one having a return that is larger and riskier than the other's. Ross then shows that Ross more risk averse implies choosing less of the larger and riskier asset.[11] The finding in Proposition 2 continues this movement toward a more and more general portfolio model at the expense of a stronger and stronger definition of more risk averse. The generality of the portfolio model is significantly improved since the returns on the two assets are unrestricted. The assumption of

[11] For the precise meaning of "larger and riskier," see Footnote 3.

linearly-restricted more risk averse, however, is a strong one. In this setting, when any investor forms a portfolio of two risky assets, Proposition 2 indicates that a linearly-restricted more risk averse investor will always form a portfolio with less of the asset with the higher mean return.

2.3. Self-Protection and Linearly-Restricted More Risk Aversion

In the standard two-state world of self-protection, loss L occurs with probability $p(I)$, where I is the investment on self-protection, $p'(I) < 0$ and $p''(I) > 0$. Letting w_0 be the initial wealth, the random final wealth distribution under self-protection investment I is

$$\tilde{w}(I) = \begin{cases} w_0 - L - I & \text{with probability } p(I) \\ w_0 - I & \text{with probability 1- } p(I) \end{cases}$$

Therefore, individual with utility function u(x) solves the following problem of expected utility maximization,

$$\max_I \, p(I)u(w_0 - L - I) + \left[1 - p(I)\right]u(w_0 - I).$$

Let I_u, which may be either an interior or a corner solution, denote the optimal self-protection investment for u(x).

It is well known that the relationship between risk aversion and self-protection investment is a complicated one. In particular, Dionne and Eeckhoudt (1985) find that, as a major difference between self-insurance and self-protection, the assumption of Arrow-Pratt more risk averse does not necessarily imply more self-protection. One of the examples used by Dionne and Eeckhoudt to illustrate the possibility that an Arrow-Pratt more risk averse individual may choose less self-protection compares the choices made by two decision makers with quadratic utility functions. For these quadratic utility functions, the Arrow-Pratt more risk averse comparison is the same as the Ross more risk averse comparison. Thus, that particular

Dionne and Eeckhoudt example also shows that the assumption of Ross more risk averse is not sufficient to imply more self-protection.

The question posed here is whether or not the assumption of linearly-restricted more risk averse can lead to more clear-cut implications for the self-protection decision. The following proposition indicates that whether added self-protection increases or reduces the mean of final wealth is the determining factor.

Proposition 3. *Suppose that u(x) is linearly-restricted more risk averse than v(x). When additional self-protection always reduces (or increases) the mean of final wealth distribution, i.e.* $p'(I)L+1\geq 0$ *for all I (or* $p'(I)L+1\leq 0$ *for all I), then* $I_u \geq I_v$ *(or* $I_u \leq I_v$*).*

Proof: We only demonstrate the case where additional self-protection always (weakly) reduces the mean of final wealth distribution. The reverse case can be proved similarly.

Assume that u(x) is linearly-restricted more risk averse than v(x), but that $I_u < I_v$. Let $\tilde{x} = \tilde{w}(I_v)$ and $\tilde{y} = \tilde{w}(I_u)$. Then $E\tilde{x} \leq E\tilde{y}$ because additional self-protection always (weakly) reduces the mean of final wealth distribution, and, by definition, $\tilde{x}$ is preferred to $\tilde{y}$ by v(x). According to Proposition 1, $\tilde{x}$ is also preferred to $\tilde{y}$ by u(x) because u(x) is linearly-restricted more risk averse than v(x). This contradicts the fact that I_u is optimal for u(x). Q.E.D.

These two findings, one for portfolios with two assets, and one for the decision to self-protect, illustrate a general finding that is true for all decision models. Whenever it is the case that the increased level of the decision variable either always increases or always decreases the mean of the outcome variable, then the assumption of linearly-restricted more risk averse is sufficient to determine whether the more risk averse decision maker chooses a larger or smaller value for that decision variable.

3. Quadratically-Restricted More Risk Aversion and Its Application

The usefulness of the linearly-restricted more risk aversion definition was demonstrated in the last section. A drawback of this simple restriction is that it can only be used to rank utility functions that differ by a linear function. In this section, this restriction is weakened to allow the more general case of two utility functions that differ by a quadratic function.

3.1 Quadratically-Restricted More Risk Aversion

Definition 3. *u(x) is quadratically-restricted more risk averse than v(x) if there are k > 0 and some quadratic form* $a+bx+cx^2$*, where both the first and second derivatives of the quadratic form are nonpositive (corresponding to* $b+2cx\le 0$ *and* $c\le 0$*, respectively), such that* $u(x)\equiv kv(x)+a+bx+cx^2$.

Obviously, linearly-restricted more risk averse is a special case of quadratically-restricted more risk averse where $c = 0$. Again it is the case that the ϕ(x) function specified in this definition satisfies the restriction for *n*th-degree Ross more risk averse. As a result, transitivity and preservation under background risk are maintained for this restriction as well.

The following proposition provides the main tool for using quadratically-restricted more risk averse in comparative static analysis.

Proposition 4. *Suppose that* $E\tilde{x}\le E\tilde{y}$ *and* $Var(\tilde{x})\le Var(\tilde{y})$*. If* $Ev(\tilde{x})>Ev(\tilde{y})$*, then* $Eu(\tilde{x})>Eu(\tilde{y})$ *for all u(x) that are quadratically-restricted more risk averse than v(x).*

Proof: Given $E\tilde{x} \leq E\tilde{y}$, $Var(\tilde{x}) \leq Var(\tilde{y})$ and $Ev(\tilde{x}) > Ev(\tilde{y})$, u(x) being quadratically-restricted more risk averse than v(x) implies

$$\begin{aligned}
&Eu(\tilde{x})\\
&= kEv(\tilde{x}) + a + bE\tilde{x} + c\,E\tilde{x}^2 = kEv(\tilde{x}) + a + bE\tilde{x} + cVar(\tilde{x}) + c\left(E\tilde{x}\right)^2\\
&> kEv(\tilde{y}) + a + bE\tilde{x} + cVar(\tilde{x}) + c\left(E\tilde{x}\right)^2\\
&\geq kEv(\tilde{y}) + a + bE\tilde{y} + cVar(\tilde{y}) + c\left(E\tilde{y}\right)^2 = kEv(\tilde{y}) + a + bE\tilde{y} + cE\tilde{y}^2\\
&= Eu(\tilde{y}),
\end{aligned}$$

where the strict inequality is due to $Ev(\tilde{x}) > Ev(\tilde{y})$, and the weak inequality is due to $E\tilde{x} \leq E\tilde{y}$ and $Var(\tilde{x}) \leq Var(\tilde{y})$ as well as $c \leq 0$ and that $a + bx + cx^2$ is weakly decreasing in *x*. Q.E.D.

The intuition behind Proposition 4 is similar to that provided for Proposition 1. Again random variables $\tilde{x}$ and $\tilde{y}$ have many properties, including the mean, the variance, and all higher-degree risk properties. The decision maker considers each of these properties when choosing between $\tilde{x}$ and $\tilde{y}$. Now two properties rather than one are the focus of the analysis, and these are the mean and the variance. It is assumed that $E\tilde{x} \leq E\tilde{y}$ and $Var(\tilde{x}) \leq Var(\tilde{y})$; that is, the preferred random variable has both a smaller mean and a smaller variance. The question of interest is again which other decision makers will also choose $\tilde{x}$ over $\tilde{y}$, and the answer this time is given using the quadratically-restricted more risk averse ranking of decision makers. All decision makers with utility functions u(x) which are quadratically-restricted more risk averse than v(x) choose between $\tilde{x}$ and $\tilde{y}$ based on properties other than the mean and variance in the same way as v(x), and these decision makers are even more willing than v(x) to accept a lower mean along with a reduced variance. As a consequence, these decision makers also rank $\tilde{x}$ over $\tilde{y}$.

Now it is the case that u'(x) = k·v'(x) + b + 2cx and u"(x) = k·v"(x) + 2c. All other derivatives of these u(x) satisfy $u^{(n)} = k\cdot v^{(n)}$ for n ≥ 3. Arrow-Pratt and Ross measures of risk aversion are increased, and for downside risk aversion or prudence, the measure u'"(x)/u'(x) is increased and the measure -u'"(x)/u"(x) is decreased.

3.2 Portfolio Choice and Quadratically-Restricted More Risk Aversion

Consider again the very general two asset portfolio choice problem where $\lambda \in [0,1]$ is chosen to maximize $Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$. Because the restriction imposed by quadratically-restricted more risk averse is weaker than that imposed by linearly-restricted more risk averse, more structure must be imposed on the portfolio model in order to make determinate comparative static statements. In this section two such comparative static propositions concerning portfolio choice are presented. The first proposition is similar to, and fits the pattern of, the Arrow-Pratt, the Ross, and the linearly-restricted portfolio results. The second proposition extends a different literature concerning diversification.

Proposition 5: *Assume that $E\tilde{x}_1 \geq E\tilde{x}_2$ and $Var(\tilde{x}_1) \geq Var(\tilde{x}_2)$ and that $\lambda_v \geq 1/2$. Then $\lambda_u \leq \lambda_v$ when u(x) is quadratically-restricted more risk averse than v(x), where $\lambda_u = \arg\max_\lambda Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$ and $\lambda_v = \arg\max_\lambda Ev[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$.*

Proof: First, it is obvious that an increase in λ weakly increases the mean of the portfolio. In addition, it can be shown that an increase in λ weakly increases the variance of the portfolio for $\lambda \geq 1/2$.

To see this, let $\mu_1 = E\tilde{x}_1, \mu_2 = E\tilde{x}_2, \sigma_1^2 = Var(\tilde{x}_1)$ and $\sigma_2^2 = Var(\tilde{x}_2)$. Then the portfolio variance is $\sigma_w^2 = \lambda^2\sigma_1^2 + (1 - \lambda)^2 \cdot \sigma_2^2 + 2\lambda(1 - \lambda)\rho_{12} \cdot \sigma_1 \cdot \sigma_2$. Differentiating gives $d\sigma_w^2/d\lambda = 2\lambda \cdot \sigma_1^2 + 2\lambda \cdot \sigma_2^2 - 2\sigma_2^2 + 2\rho_{12} \cdot \sigma_1 \cdot \sigma_2 - 4 \cdot \lambda \cdot \rho_{12} \cdot \sigma_1 \cdot \sigma_2$. This expression can be written as $d\sigma_w^2/d\lambda = 2\lambda(\sigma_1^2 + \sigma_2^2 - 2\rho_{12} \cdot \sigma_1 \cdot \sigma_2) - 2(\sigma_2^2 - \rho_{12} \cdot \sigma_1 \cdot \sigma_2)$. Now it is the case that this expression is linear in ρ_{12}, and is decreasing when $\lambda \geq 1/2$. Therefore the expression is at its minimum with respect to ρ_{12} when $\rho_{12} = 1$. At $\rho_{12} = 1$, $d\sigma_w^2/d\lambda = 2\lambda(\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \cdot \sigma_2) - 2(\sigma_2^2 - \sigma_1 \cdot \sigma_2)$ which for $\lambda \geq 1/2$ is greater than $(\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \cdot \sigma_2) - 2(\sigma_2^2 - \sigma_1 \cdot \sigma_2) = \sigma_1^2 - \sigma_2^2 \geq 0$. Thus, $d\sigma_w^2/d\lambda \geq 0$ for $\lambda \geq 1/2$.

Now we use proof by contradiction. Assume that $\lambda_v \geq ½$, u(x) is quadratically-restricted more risk averse than v(x), but $\lambda_u > \lambda_v$. Let $\tilde{x} = \lambda_v \tilde{x}_1 + (1 - \lambda_v)\tilde{x}_2$ and $\tilde{y} = \lambda_u \tilde{x}_1 + (1 - \lambda_u)\tilde{x}_2$. Because $\lambda_u > \lambda_v \geq 1/2$, we have $E\tilde{x} \leq E\tilde{y}$ and $Var(\tilde{x}) \leq Var(\tilde{y})$ based on what is demonstrated above. By definition, $Ev(\tilde{x}) > Ev(\tilde{y})$. Then, according to Proposition 4, $Eu(\tilde{x}) > Eu(\tilde{y})$, which contradicts that λ_u is the optimal choice for u(x). Therefore, it must be $\lambda_u \leq \lambda_v$.

Q.E.D.

Proposition 5 says that if we observe a portfolios of two assets where one asset has a smaller mean and a smaller variance than the other, and any investor chooses to invest more in the asset with larger mean and variance than the other, then all those investors who are more quadratically-restricted more risk averse would reduce their holding of the larger mean and larger variance asset in favor of increasing their holding of the lower mean and lower variance asset. It is important to recognize that this theorem is quite strong since it does not require independence or any particular dependence structure for the returns on the two assets.

A second portfolio proposition extends a finding of Rothschild and Stiglitz (1971) who show that if $\tilde{x}_1$ and $\tilde{x}_2$ are independently and identically distributed (i.i.d.), then any risk averse individual will divide his wealth equally between the two assets. Indeed, this is true because any alternative feasible portfolio is a mean-preserving spread from the perfectly diversified portfolio.[12]

Now assume that $\tilde{x}_1$ and $\tilde{x}_2$ are independently distributed returns with equal mean and variance, but are not necessarily i.i.d. In this more general case, although the perfectly diversified portfolio yields the smallest variance among all feasible portfolios, it does not necessarily have the least risk in the sense of Rothschild and Stiglitz. The riskiness of portfolios is not likely to be comparable using the Rothschild and Stiglitz definition. As a result, there is no guarantee that all risk averse decision makers will choose λ = 1/2. The question then arises as to whether a more risk averse individual would choose an allocation that is "closer" to the perfectly diversified portfolio than the less risk averse individual. The assumption of quadratically-restricted more risk averse allows this question to be answered.

Formally, suppose that u(x) chooses $\lambda \in [0,1]$ to maximize $Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$, where $\tilde{x}_1$ and $\tilde{x}_2$ are independent, with $E\tilde{x}_1 = E\tilde{x}_2 = \mu$ and $Var(\tilde{x}_1) = Var(\tilde{x}_2) = \sigma^2$. The following proposition says that a quadratically-restricted more risk averse individual will choose an allocation that is "closer" to the 50 – 50 perfect diversification than the less risk averse individual.

[12] Gollier (2001, pages 45-46) generalizes this result to *n* assets with returns that are i.i.d.

<u>Proposition 6.</u> *Given the portfolio choice problem with the stated conditions, if u(x) is quadratically-restricted more risk averse than v(x), then* $\left|\lambda_u - \frac{1}{2}\right| \le \left|\lambda_v - \frac{1}{2}\right|$*, where* $\lambda_u = \arg\max_{\lambda} Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$ *and* $\lambda_v = \arg\max_{\lambda} Ev[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$.

<u>Proof:</u> Obviously, $E[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2] = \mu$ and $Var[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2] = 2\sigma^2\left[\left(\lambda - \frac{1}{2}\right)^2 + \frac{1}{4}\right]$. In particular, the mean of the portfolio $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$ is invariant with respect to λ, and the variance of the portfolio strictly increases in $\left|\lambda - \frac{1}{2}\right|$ with the smallest variance occurring at $\lambda = \frac{1}{2}$ (perfect diversification).

We now use proof by contradiction. Assume that u(x) is quadraticly-restricted more risk averse than v(x), but $\left|\lambda_u - \frac{1}{2}\right| > \left|\lambda_v - \frac{1}{2}\right|$. Consider the following two portfolios: $\tilde{x} = \lambda_v\tilde{x}_1 + (1-\lambda_v)\tilde{x}_2$ and $\tilde{y} = \lambda_u\tilde{x}_1 + (1-\lambda_u)\tilde{x}_2$. Obviously, $E\tilde{x} = E\tilde{y}$, $Var(\tilde{x}) < Var(\tilde{y})$ and, by definition, $\tilde{x}$ is preferred to $\tilde{y}$ by v(x). Then, according to Proposition 4, $\tilde{x}$ must also be preferred to $\tilde{y}$ by u(x), which contradicts the assumption that λ_u is the optimal choice for u(x). Therefore, it must be that $\left|\lambda_u - \frac{1}{2}\right| \le \left|\lambda_v - \frac{1}{2}\right|$.

Q.E.D.

To conclude this subsection, we note that Ross more risk averse, and hence Arrow-Pratt more risk averse, does not necessarily imply that a more risk averse investor chooses an optimal portfolio that is (weakly) closer to the perfectly diversified portfolio. An example confirming this is included in the appendix.

3.3 Insurance with Default Risk and Quadratically-Restricted More Risk Aversion

In simple insurance demand models, it is well-known that a risk averse individual would choose full coverage under fair insurance pricing and less-than-full coverage under unfair pricing. Moreover, under unfair pricing, an Arrow-Pratt more risk averse individual will choose more insurance coverage than a less risk averse one.[13] Schlesinger and Schulenburg (1987) and Schlesinger (2000) point out that these same results do not hold when there exists a positive probability of insurer default. The question addressed in this subsection asks if these results would follow even with a positive probability of insurer default if the more risk averse order is strengthened by assuming quadratically-restricted more risk averse.

Following Schlesinger (2000), suppose that a loss of size L occurs with probability p, $0 < p < 1$. Let $\alpha \geq 0$ be the share (coverage) of the loss paid by the insurer when the loss occurs AND when the insurer does not default. Assume that conditional on the loss occurring, default happens with probability q, $0 < q < 1$. Therefore, a decision maker with utility function u(x) chooses α to maximize $Eu\left[\tilde{w}(\alpha)\right]$, where the random final wealth $\tilde{w}(\alpha)$ when the chosen coverage is α is given by:

$$\tilde{w}(\alpha) = \begin{cases} w_0 - (1+\lambda)\alpha pqL & \text{with probability } 1-p \\ w_0 - (1+\lambda)\alpha pqL - L + \alpha L & \text{with probability } pq \\ w_0 - (1+\lambda)\alpha pqL - L & \text{with probability } p(1-q) \end{cases} \tag{1}$$

In (1), w_0 is the initial wealth, $(1+\lambda)\alpha pqL$ is the insurance premium for coverage α, and $\lambda \geq 0$ is the loading factor with $\lambda = 0$ corresponding to actuarially fair insurance pricing.

Schlesinger (2000) shows that, even under fair pricing ($\lambda = 0$), a risk averse individual will choose partial coverage ($\alpha < 1$) and an Arrow-Pratt more risk averse individual does not necessarily choose more coverage than a less risk averse one. Using an argument very similar to

[13] See, for example, Mossin (1968) and Schlesinger (2000).

that of Schlesinger (2000), we show in the appendix that a Ross more risk averse individual does not necessarily choose more insurance coverage either. In contrast, the analysis which follows shows that a quadratically-restricted more risk averse decision maker does choose more insurance coverage.

To show this, we begin by establishing a preliminary result that is used in the proof of the main result, Proposition 7. The following lemma, which assumes that decision makers are prudent, strengthens the Schlesinger result that the optimally chosen insurance coverage under default risk is less than full.

<u>Lemma.</u> *If a risk averse individual is also prudent (i.e., $u'''(x) > 0$), then the optimally chosen coverage satisfies* $\alpha^* < \dfrac{1-p}{1-pq} < 1$.

<u>Proof:</u> From (1),

$$\frac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} = pqL\left\{\begin{aligned}&-(1+\lambda)(1-p)u'\left(w_0-(1+\lambda)\alpha pqL\right)+\left[1-(1+\lambda)pq\right]u'\left(w_0-(1+\lambda)\alpha pqL+(\alpha-1)L\right)\\&-(1+\lambda)p(1-q)u'\left(w_0-(1+\lambda)\alpha pqL-L\right)\end{aligned}\right\} \tag{2}$$

Then it is obvious that, for a risk averse u(x), $\dfrac{d^2Eu\left(\tilde{w}(\alpha)\right)}{d\alpha^2} < 0$. Therefore, if $\dfrac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} < 0$ at some $\bar{\alpha} > 0$, we would know that $\alpha^* < \bar{\alpha}$.[14]

Now we complete the proof by showing that $\dfrac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} < 0$ at $\bar{\alpha} = \dfrac{1-p}{1-pq} < 1$. Indeed, let

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[14] One can easily show from (2) that $\dfrac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} < 0$ at $\alpha = 1$ based on u(x) being risk averse alone. So $\alpha^* < 1$, a result obtained in Schlesinger (2000).

$$w_3 = w_0 - (1+\lambda)\bar{\alpha}\, pqL$$
$$w_2 = w_0 - (1+\lambda)\bar{\alpha}\, pqL + (\bar{\alpha}-1)L\,.$$
$$w_1 = w_0 - (1+\lambda)\bar{\alpha}\, pqL - L$$

Because $w_3 > w_2 > w_1$ and $u'''(x) > 0$, we have from the intermediate value theorem that

$$\frac{u'(w_3) - u'(w_2)}{w_3 - w_2} > \frac{u'(w_2) - u'(w_1)}{w_2 - w_1},$$

which implies that

$$\begin{aligned} &\bar{\alpha} u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL\right) - u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL + (\bar{\alpha}-1)L\right) \\ &+ (1-\bar{\alpha})u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL - L\right) > 0 \end{aligned}$$

or, replacing some $\bar{\alpha}$ in the above inequality with $\dfrac{1-p}{1-pq}$ and multiplying through by

$-(1+\lambda)(1-pq)$,

(3) $$\begin{aligned} &-(1+\lambda)(1-p)u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL\right) + (1+\lambda)(1-pq)u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL + (\bar{\alpha}-1)L\right) \\ &-(1+\lambda)\, p(1-q)u'\left(w_0 - (1+\lambda)\bar{\alpha}\, pqL - L\right) < 0. \end{aligned} \quad .$$

From (2), and using (3) and noting $\lambda \geq 0$, we have $\dfrac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} < 0$ at $\bar{\alpha} = \dfrac{1-p}{1-pq} < 1$.

Therefore, $\alpha^* < \dfrac{1-p}{1-pq} < 1$. Q.E.D.

<u>Proposition 7.</u> *Assuming prudence in the stated model of insurance demand with default risk, if u(x) is quadratically-restricted more risk averse than v(x), then* $\alpha_u \geq \alpha_v$, *where*

$\alpha_u = \arg\max_{\alpha} Eu\left[\tilde{w}(\alpha)\right]$ *and* $\alpha_v = \arg\max_{\alpha} Ev\left[\tilde{w}(\alpha)\right]$.

<u>Proof:</u> We use proof by contradiction. Assume that u(x) is quadratically-restricted more risk averse than v(x), but that $\alpha_u < \alpha_v$. Then from the lemma, we know $\alpha_u < \alpha_v < \frac{1-p}{1-pq}$.

It is obvious that $E\tilde{w}(\alpha) = w_0 - pL - \lambda\alpha pqL$, which says in particular that $E\tilde{w}(\alpha)$ is non-increasing in α (strictly decreasing in α if $\lambda > 0$). Further, straightforward calculations yield

$$\frac{dVar(\tilde{w}(\alpha))}{d\alpha} = 2pqL^2(1-pq)\left[\alpha - \frac{1-p}{1-pq}\right],$$

which says that $Var(\tilde{w}(\alpha))$ decreases in α for $\alpha \in [0, \frac{1-p}{1-pq})$.

Now let $\tilde{x} = \tilde{w}(\alpha_v)$ and $\tilde{y} = \tilde{w}(\alpha_u)$. Under the assumption that $\alpha_u < \alpha_v < \frac{1-p}{1-pq}$, we have $E\tilde{x} \le E\tilde{y}$ and $Var(\tilde{x}) < Var(\tilde{y})$. By definition, $\tilde{x}$ is preferred to $\tilde{y}$ by v(x). Then, according to Proposition 4, $\tilde{x}$ must also be preferred to $\tilde{y}$ by u(x), which contradicts that α_u is the optimal choice for u(x). Therefore, it must be that $\alpha_u \ge \alpha_v$. Q.E.D.

3.4 Self-Protection and Quadratically-Restricted More Risk Aversion

In Subsection 2.3, we have shown that when additional self-protection always reduces the mean of the final wealth distribution, $\tilde{w}(I)$, a linearly-restricted more risk averse individual will choose more self-protection. Because the linearly-restricted more risk averse assumption is a strong one, we now investigate whether the less demanding quadratically-restricted more risk averse assumption is sufficient to imply more self-protection without imposing significant additional restrictions on the self-protection model.[15]

Suppose that $E\tilde{w}(I)$ is weakly decreasing in *I*. To use Proposition 4, we want the variance of the final wealth, $Var\left(\tilde{w}(I)\right)$, to also decrease in *I*.

[15] Note that the condition that additional self-protection always reduces the mean of the final wealth, or $p'(I)L + 1 \ge 0$ for all *I*, is satisfied as long as $p'(0)L + 1 \ge 0$, given the standard assumption that $p''(I) > 0$.

$$\frac{dVar\left(\tilde{w}(I)\right)}{dI} = L^2 \cdot \frac{d\left[p(I)(1-p(I))\right]}{dI} = L^2 p'(I)\left[1-2p(I)\right].$$

Thus, the necessary and sufficient condition for $Var\left(\tilde{w}(I)\right)$ to also decrease in *I* is $p(0) \le \frac{1}{2}$.

The following proposition is similar to Proposition 3, replacing the linearly-restricted more risk averse assumption with that of quadratically-restricted more risk averse, but adding the condition that p(0) ≤ 1/2.

<u>Proposition 8</u>. *Suppose that u(x) is quadratically-restricted more risk averse than v(x). If* $p(0) \le \frac{1}{2}$ *and additional self-protection always (weakly) reduces the mean of final wealth, i.e.* $p'(I)L+1 \ge 0$ *for all I, then* $I_u \ge I_v$

<u>Proof</u>: Assume that u(x) is quadratically-restricted more risk averse than v(x), but that $I_u < I_v$. Let $\tilde{x} = \tilde{w}(I_v)$ and $\tilde{y} = \tilde{w}(I_u)$. Then $E\tilde{x} \le E\tilde{y}$ and $Var(\tilde{x}) < Var(\tilde{y})$ because additional self-protection always (weakly) reduces the mean of final wealth, and, under the condition $p(0) \le \frac{1}{2}$, $Var\left(\tilde{w}(I)\right)$ decrease in *I* as well. Further, by definition, $\tilde{x}$ is preferred to $\tilde{y}$ by v(x). Then, according to Proposition 4, $\tilde{x}$ is also preferred to $\tilde{y}$ by u(x). This contradicts the fact that I_u is optimal for u(x).

Q.E.D.

Relaxing the assumption of linearly-restricted more risk averse and replacing it with quadratically-restricted more risk averse, the only additional structure on the self-protection model that is required is that $p(0) \le \frac{1}{2}$. This requirement states that the bad outcome where a loss occurs is not as likely as no loss occurring at all even without an investment in self-protection. It implies that additional self-protection always reduces the variance of final wealth. This is a feature of many losses which decision makers protect against.

Proposition 8 complements a result in Dachraoui et al. (2004). Dachraoui et al. propose a notion of increased risk aversion, referred to as more risk aversion MRA (mixed risk aversion). Utility function u(x) is defined to be more risk averse MRA than v(x), if u(x) is more risk averse than v(x) in the sense of Arrow-Pratt, more prudent than v(x) in the sense of Kimball (1990),….[16] They find that, for u(x) and v(x) such that u(x) is more risk averse MRA than v(x), if $I_u > I_v$, then it must be the case that $p(I_v) < \frac{1}{2}$. In other words, the loss probability being sufficiently low is a necessary condition for maintaining an expected relationship between more risk aversion MRA and the self-protection decision. Related to this, Proposition 8 indicates that a sufficiently low loss probability is a sufficient condition for showing that quadratically-restricted more risk averse decision makers choose more self-protection.

4. Conclusion

The two primary definitions of increased risk aversion currently in use, Arrow-Pratt or Ross more risk averse, are important tools when doing comparative static analysis in decision models with uncertainty. While these definitions of increased risk aversion are sufficient in some settings, they are not sufficient in others to ensure that more risk averse individuals undertake more intensive risk-reducing activities. The portfolio choice, self-protection and insurance demand with insurer default models illustrate this, and open up the possibility of using a stronger definition of more risk averse. One of the reasons why a stronger definition of more risk averse is needed is that neither the Arrow-Pratt nor the Ross definition seeks to hold attitudes toward risk changes of a higher degree fixed. Without a *ceteris paribus* assumption that

[16] All commonly used utility functions display mixed risk aversion as defined by Caballe and Pomansky (1996).

does this, too many factors partially determine a decision makers' choice of risk-reducing effort, and determining the effect of changing one of them is difficult if not impossible.

The analysis here proposes two definitions of a restricted increase in risk aversion, linearly-restricted more risk aversion and quadratically-restricted more risk aversion. These restrictions are formulated to hold constant the effects of risk increases of higher degrees, and are shown to be sufficient to generate clear-cut and intuitive comparative statics results in settings where the Arrow-Pratt and Ross definitions are not adequate.

Of course, this line of research can be extended. Other polynomials and additional simple one parameter functions can replace the linear and quadratic functions used here. What is assumed to be held fixed when utility functions differ by such a function is an interesting topic, and left for future research.

Appendix

Ross More Risk Aversion Is not Sufficient for Proposition 6

Let v(x) be a quadratic utility function that is increasing and concave in the relevant interval, and let $u(x) \equiv kv(x) + \phi(x)$, where k > 0 and $\phi(x)$ is a chosen *cubic* function with $\phi'(x) \le 0$, $\phi''(x) \le 0$ and $\phi'''(x) > 0$ in the relevant interval. Further assume that v(x), k and $\phi(x)$ are chosen in such a way that u(x) is still increasing in the relevant interval (the concavity of u(x) is guaranteed due to $\phi''(x) \le 0$). From Ross (1981), it is the case that this u(x) is Ross more risk averse than v(x).

Let $\tilde{x}_1$ and $\tilde{x}_2$ be independent, with $E\tilde{x}_1 = E\tilde{x}_2 = \mu$, $Var(\tilde{x}_1) = Var(\tilde{x}_2) = \sigma^2$ and $E(\tilde{x}_1^3) > E(\tilde{x}_2^3)$. Because v(x) is quadratic and risk averse, $Ev[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]$ is maximized at $\lambda_v = \frac{1}{2}$.

Therefore, although u(x) is Ross more risk averse than v(x), the optimally chosen portfolio for u(x), represented by λ_u, cannot be closer to perfect diversification than that chosen by v(x). Indeed, we can show that $\lambda_u > \frac{1}{2}$. To see this, note that given the requirements for the construction of u(x) we can express $u(x) = ax^3 + bx^2 + cx + d$ with $a > 0$. We prove $\lambda_u > \frac{1}{2}$ by showing that $d\left\{Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]\right\}/d\lambda > 0$ at $\lambda = \frac{1}{2}$. Indeed,

$$\frac{d\left\{Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]\right\}}{d\lambda} = 3aE\left[\left(\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2\right)(\tilde{x}_1 - \tilde{x}_2)\right].$$
$$+2bE\left[\left(\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2\right)(\tilde{x}_1 - \tilde{x}_2)\right] + cE(\tilde{x}_1 - \tilde{x}_2)$$

Therefore,

$$\frac{d\left\{Eu[\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2]\right\}}{d\lambda}\bigg|_{\lambda=\frac{1}{2}} = \frac{3a}{4}E\left[(\tilde{x}_1 + \tilde{x}_2)^2(\tilde{x}_1 - \tilde{x}_2)\right] = \frac{3a}{4}E(\tilde{x}_1^3 - \tilde{x}_2^3) > 0.$$

<u>Ross More Risk Aversion and Insurance with Default Risk</u>

We follow Schlesinger (2000) to assume fair pricing ($\lambda = 0$) and an interior solution. From (2), the optimal coverage for u(x), denoted by α_u, satisfies the equation

$$\frac{dEu\left(\tilde{w}(\alpha)\right)}{d\alpha} = pqL\left\{-(1-p)u'\left(Y_1\right)+(1-pq)u'\left(Y_2\right)-p(1-q)u'\left(Y_3\right)\right\}=0,$$

where

$$\begin{aligned} Y_1 &= w_0 - \alpha pqL \\ Y_2 &= w_0 - \alpha pqL + (\alpha - 1)L \\ Y_3 &= w_0 - \alpha pqL - L \end{aligned}$$

Now choose an *arbitrary* $\phi(x)$ such that $\phi'(x) \le 0$ and $\phi''(x) \le 0$, and let $v(x) \equiv u(x) + \phi(x)$. We have that v(x) is Ross more risk averse than u(x) (Ross 1981). Then consider

$$\frac{dEv\left(\tilde{w}(\alpha)\right)}{d\alpha}\Big|_{\alpha_u} = pqL\left\{-(1-p)\phi'\left(Y_1\right)+(1-pq)\phi'\left(Y_2\right)-p(1-q)\phi'\left(Y_3\right)\right\}.$$

Although we know that $\phi'\left(Y_1\right)$, $\phi'\left(Y_2\right)$, and $\phi'\left(Y_3\right)$ are all nonpositive and that $\phi'\left(Y_1\right) \le \phi'\left(Y_2\right) \le \phi'\left(Y_3\right)$, these conditions are not sufficient for signing $\frac{dEv\left(\tilde{w}(\alpha)\right)}{d\alpha}\Big|_{\alpha_u}$. Indeed, $\frac{dEv\left(\tilde{w}(\alpha)\right)}{d\alpha}\Big|_{\alpha_u}$ can be of any sign due to the arbitrariness of $\phi(x)$.

Therefore, the Ross more risk aversion does not necessarily imply more coverage in the insurance model with default risk.

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