Comparative Risk Apportionment*

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Abstract: A decision maker who would rather apportion an independent risk in a state with a good lottery than in a state with a bad lottery is said to have a preference for risk apportionment (Eeckhoudt & Schlesinger, 2006). In this paper, we propose a measure for the strength of nth-degree risk apportionment preference based on Pratt's probability premium (Pratt, 1964). Under expected utility theory, we analyze the relationship between a greater preference for risk apportionment and both the Ross and Arrow-Pratt versions of comparative risk aversion.

Keywords: risk apportionment, risk aversion, downside risk aversion, prudence

JEL Classification: D81

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1 Introduction

Using 50-50 compound lotteries, Eeckhoudt and Schlesinger (2006) define decision makers with a preference for risk apportionment as those who would rather apportion an independent risk in the state with a good lottery than in the state with a bad lottery. They show that, under expected utility theory, the preference for risk apportionment is implied by risk aversion of various degrees.[1] The risk apportionment framework of Eeckhoudt and Schlesinger (2006) helps deepen the understanding of higher-degree risk increases. Despite the time that has passed since Ekern's (1980) definition of an nth-degree risk increase, little is known beyond the fact that the nth-degree risk increase is a special case of the nth-degree stochastic dominance in which the first $n-1$ moments of the two random variables are kept the same. The risk apportionment framework makes it very easy to decompose a higher-degree risk increase into lower-degree risk increases, thereby providing an intuitive interpretation of nth-degree risk increases in terms of the well-understood 1st-degree risk increases (leftward shifts in the probability mass) and/or 2nd-degree risk increases (mean-preserving spreads). The risk apportionment framework also facilitates a general treatment in a large category of models regarding the effects of exogenous changes in a risky environment (Menegatti and Peter, 2020; Nocetti, 2016). In addition, this framework has been extended to characterize preferences for disaggregating two multiplicative risks (Chiu et al., 2012; Wang & Li, 2010), to study multiattribute risk preferences (Denuit & Rey 2013; Gollier 2019; Jokung, 2011; Tsetlin & Winkler, 2009), to better understand the relationship between stochastic dominance and the corresponding preferences (Courbage et al., 2018; Ebert et al., 2018; Huang et al., 2020; Tsetlin & Winkler, 2018), and to shed light on nth-degree risk aversion in non-EU models (Eeckhoudt et al., 2020).

[1]As the opposite of risk apportionment, Crainich et al. (2013) study preferences for combining two independent risk increases of various degrees in a single state, as opposed to putting them in separate states, and show that these preferences can be characterized by risk loving at all even degrees and risk aversion at all odd degrees. See also Ebert (2013).

To the best of our knowledge, however, there exists almost no analysis on how to measure the strength of the preference for risk apportionment.[2] While the notion of risk aversion was mathematically formalized by Daniel Bernoulli in the 1700s (Bernoulli, 1954), our understanding of decision making under risk has been greatly enhanced by the more recent research on the measurement of risk aversion by Arrow (1971) and Pratt (1964). The present paper sets out to quantify the strength of the preference for risk apportionment, extending the literature on risk apportionment to include an analysis of comparative risk apportionment. Thus, our framework can be useful in applied work as well as in experimental settings when researchers try to compare the strength of risk apportionment preferences between different individuals. For $n \geq 3$, we define a measure of preference for nth-degree risk apportionment, called the nth-degree probability premium and denoted by p_n. This is a generalization of Pratt's (1964) probability premium, which is defined as p such that a decision maker is indifferent when choosing between 0 and a lottery that yields $k > 0$ with probability $1/2 + p$ and $-k$ with probability $1/2 - p$. Pratt proves that $p > 0$ for a risk averter and p is larger for a decision maker who is more risk averse. We find that, under an expected utility representation of preferences, if a decision maker is risk averse and nth-degree risk averse (Ekern, 1980), the corresponding nth-degree probability premium, p_n, will be positive. Moreover, an nth-degree generalization of Ross more risk aversion (Ross, 1980) is a sufficient condition for the interpersonal comparison of the nth-degree probability premiums, whereas the corresponding nth-degree Arrow-Pratt more risk aversion is a necessary condition.

We also adopt the generalized framework of nth-degree risk apportionment developed

[2] In his Geneva Risk Economics Lecture in 2012, Louis Eeckhoudt distinguished between research addressing the direction of risk attitudes and research addressing their intensity (Eeckhoudt, 2012). Much progress has been made on understanding risk apportionment as a directional preference, but less is known about how to measure its intensity. Jindapon (2010) and Watt (2011) theoretically investigate the intensity of 3rd-degree risk apportionment, while Ebert and Wiesen (2014) experimentally elicit dollar measures of the intensity of 3rd- and 4th-degree risk apportionment.

by Eeckhoudt et al. (2009), based on mutual aggravation of mth- and $(n-m)$th-degree risks, where $n > m \geq 1$, and define another measure of preference for nth-degree risk apportionment, called (n/m)th-degree probability premium, or $p_{n/m}$. We find that, under an expected utility representation of preferences, if decision makers are nth- and mth-degree risk averse, the corresponding (n/m)th-degree probability premium will be positive. Moreover, the (n/m)th-degree Ross more risk aversion, as defined by Liu and Meyer (2013), is a sufficient condition for the interpersonal comparison of the (n/m)th-degree probability premiums, whereas the corresponding (n/m)th-degree Arrow-Pratt more risk aversion is a necessary condition.

Since the original concept of nth-degree risk apportionment in Eeckhoudt and Schlesinger (2006) is a special case of Eeckhoudt et al. (2009) with $m = 2$, this generalization allows us to compare the strength of preference for nth-degree risk apportionment in a broader set of situations. For example, we can compare the strength of the nth-degree risk apportionment preference between two decision makers without the assumption that both are risk averse. As an alternative, we propose a variation of the 50-50 lottery pairs with $m = 1$, so we can use the concept of the $(n/1)$th-degree probability premium, or $p_{n/1}$, as a measure of the nth-degree risk apportionment of an individual who may be risk averse, risk loving, or neither. For example, given $n = 3$, we have two possible measures of 3rd-degree risk apportionment preference: $p_{3/2}$ and $p_{3/1}$. To compare the strength of the 3rd-degree risk apportionment preferences of two individuals, the former measure, $p_{3/2}$, requires both individuals to be risk averse, while the latter measure, $p_{3/1}$, requires only monotonicity.

The paper is organized as follows. Section 2 provides definitions and preliminary results related to nth-degree risk aversion and nth-degree risk apportionment. In Section 3, we introduce the concept of the nth-degree probability premium, based on Eeckhoudt and Schlesinger (2006), i.e., p_n, and show how it can be characterized by the nth-degree Ross and Arrow-Pratt measures of higher-order risk aversion. In Section 4, we define a more general nth-degree probability premium, i.e., $p_{n/m}$, using the risk-apportionment framework by Eeckhoudt et al. (2009), and derive analogous results regarding comparative risk apportionment. We focus on the case of prudence (i.e., $n = 3$) in Section 5 and conclude in Section 6.

2 Risk Apportionment

Throughout the paper, we let $[L_1, p_1; L_2, p_2]$ denote a binary compound lottery that yields lottery L_i with probability p_i for $i = 1, 2$. In this section, we first review Eeckhoudt and Schlesinger's (2006) definition of nth-degree risk apportionment. For $n = 3$, we consider two lotteries, $A_3 = [-k + \tilde{\epsilon}_3, 1/2; 0, 1/2]$ and $B_3 = [-k, 1/2; \tilde{\epsilon}_3, 1/2]$, where $k > 0$ and $\tilde{\epsilon}_3$ is a nondegenerate zero-mean risk. A decision maker who prefers more to less and dislikes risk would regard both $-k$ and $\tilde{\epsilon}$ as "bads." A decision maker who displays a preference for risk apportionment—a preference for putting two independent bads in separate states, as opposed to combining them in a single state—prefers lottery B_3 to lottery A_3, for any k and $\tilde{\epsilon}_3$. To put it differently, the only difference between lotteries A_3 and B_3 is that the zero-mean risk $\tilde{\epsilon}$ occurs in the high-wealth state of B_3 and in the low-wealth state of A_3. According to Menezes et al. (1980), A_3 has more downside risk than B_3. Therefore, a preference for risk apportionment in this example is the same as an aversion to downside risk increases. See Figure 1.

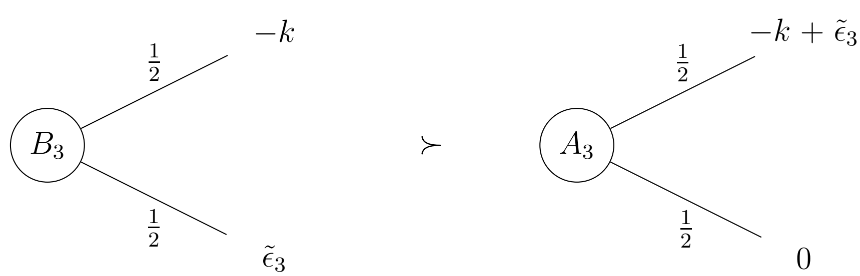


Figure 1: 3rd-degree risk apportionment

This notion of a preference for 3rd-degree risk apportionment does not require the existence of an expected-utility representation of a decision maker's preferences. If a decision maker has an initial wealth of w and his preferences are represented by utility function u, however, both Menezes et al. (1980) and Eeckhoudt and Schlesinger (2006) demonstrate that $Eu(w + B_3) > Eu(w + A_3)$ for all w, k, and $\tilde{\epsilon}$ if and only if $u'''(x) > 0$ for all x in the wealth domain. For a larger value of n, Eeckhoudt and Schlesinger (2006) define lotteries A_n and B_n to represent nth-degree risk apportionment as follows.

Definition 1. *(Eeckhoudt & Schlesinger, 2006) Let k be a strictly positive real number, $\tilde{\epsilon}_n$, for $n \geq 2$, be a zero-mean nondegenerate random variable, and all $\tilde{\epsilon}_n$ be mutually independent. Define lotteries $B_1 = B_2 = 0$, $A_1 = -k$, and $A_2 = \tilde{\epsilon}_2$. For $n \geq 3$,*

$$A_n = [A_{n-2} + \tilde{\epsilon}_n, 1/2; B_{n-2}, 1/2]$$

$$B_n = [A_{n-2}, 1/2; B_{n-2} + \tilde{\epsilon}_n, 1/2]$$

A decision maker prefers nth-degree risk apportionment if

(i) For $n = 1$, $B_1 \succ A_1$ for all $k > 0$.

(ii) For $n = 2$, $B_2 \succ A_2$ for all $\tilde{\epsilon}_n$.

(iii) For $n \geq 3$, $B_n \succ A_n$ for all A_{n-2}, B_{n-2}, and $\tilde{\epsilon}_n$.

Eeckhoudt and Schlesinger's (2006) definition of a preference for nth-degree risk apportionment is illustrated in Figure 2. Let $F(x)$ and $G(x)$ represent the cumulative distribution functions (CDFs) of two random variables whose supports are contained in a finite interval denoted $[a, b]$ with no probability mass at point a. This implies that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Let $F^{[1]}(x) = F(x)$ and $F^{[j]}(x) = \int_a^x F^{[j-1]}(y)dy$ for any integer $j \geq 2$. Similar notation applies to $G(x)$. Ekern (1980) gives the following definition.

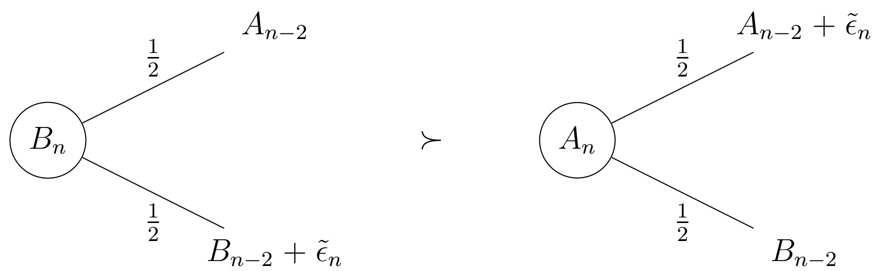


Figure 2: nth-degree risk apportionment

Definition 2. *(Ekern, 1980)*

1. *For any integer $n \geq 1$, $G(x)$ has more nth-degree risk than $F(x)$ if*

   (i) $G^{[j]}(b) = F^{[j]}(b)$ for $j = 1, 2, ..., n$, and

   (ii) $G^{[n]}(x) \geq F^{[n]}(x)$ for all $x \in [a, b]$ with "$>$" holding for some $x \in (a, b)$.

2. *A decision maker exhibits nth-degree risk aversion if $F(x) \succ G(x)$ for all $F(x)$ and $G(x)$ such that $G(x)$ has more nth-degree risk than $F(x)$.*

Condition (i) guarantees that the first $n - 1$ moments of $F(x)$ and $G(x)$ are held the same across the two distributions, and conditions (i) and (ii) together imply that $F(x)$ dominates $G(x)$ by nth-degree stochastic dominance. Thus, the nth-degree risk increase is a special case of nth-degree stochastic dominance in which the first $n - 1$ moments are kept the same. Also note that an increase in 1st-degree risk is equivalent to a first-degree stochastically dominated shift, that an increase in 2nd-degree risk is equivalent to a sequence of mean-preserving spreads, as in Rothschild and Stiglitz (1970), and that an increase in 3rd-degree risk is equivalent to a downside risk increase, as in Menezes et al. (1980). A decision maker who always prefers any distribution with less nth-degree risk is said to be nth-degree risk averse.

Under expected utility theory, Ekern (1980) proves that a decision maker is nth-degree risk averse if and only if $(-1)^{n-1}u^{(n)}(x) > 0$ for all $x \in [a, b]$. Eeckhoudt and Schlesinger

(2006) find that, under expected utility theory, a decision maker has a preference for nth-degree risk apportionment if and only if $(-1)^{n-1}u^{(n)}(x) > 0$ for all $x \in [a, b]$, and hence a preference for nth-degree risk apportionment is equivalent to nth-degree risk aversion.

3 Comparative Risk Apportionment

In this section, we define the nth-degree probability premium, based on Eeckhoudt and Schlesinger (2006), to compare the intensity of preference for nth-degree risk apportionment between two decision makers. Like Definition 1, the following definition of the probability premium does not rely on the existence of an expected utility representation of the preferences.[3]

Definition 3. *Given A_n and B_n in Definition 1, a decision maker's nth-degree probability premium is p_n such that $A'_n \sim B'_n$ where*

$$A'_n = [A_{n-2} + \tilde{\epsilon}_n, 1/2 - p_n; B_{n-2}, 1/2 + p_n]$$

$$B'_n = [A_{n-2}, 1/2 - p_n; B_{n-2} + \tilde{\epsilon}_n, 1/2 + p_n]$$

for $n \geq 3$.

The definition of nth-degree probability premium, denoted by p_n, is illustrated in Figure 3. The intuition for using p_n as a measure of the strength of nth-degree risk apportionment is the following. According to Figure 2, nth-degree risk apportionment is characterized by $B_n \succ A_n$. Note that for a risk-averse individual, the lower (upper) random wealth in lottery A_n is better (worse) than the lower (upper) random wealth in lottery B_n. Therefore, the attractiveness of lottery A_n relative to lottery B_n can be improved by moving some

[3]Note that the preference relation over lotteries, in general, must satisfy continuity and monotonicity to guarantee the existence of a unique probability premium defined in Definition 3. Note also that the utility functions considered in this paper satisfy the required continuity and monotonicity.

probability mass from the upper state to the lower state in both lotteries. The required probability mass that makes the two sides equally attractive, as in Figure 3, indicates how difficult it is to offset the individual's preference for nth-degree risk apportionment by such a movement of probability mass and, therefore, serves as a measure of the strength of nth-degree risk apportionment.

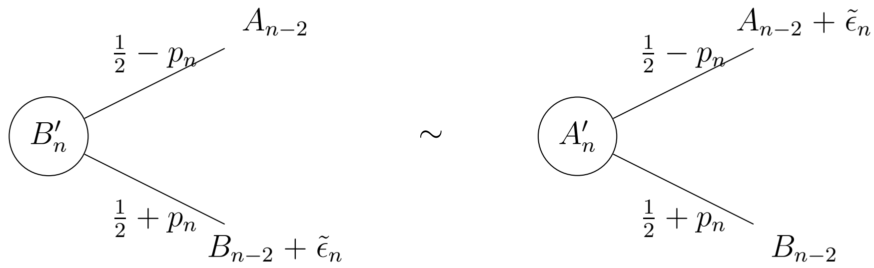


Figure 3: nth-degree probability premium

Under expected utility theory, we can derive u's probability premium, denoted by p_n^u, according to Definition 3 as

$$\begin{aligned}\left(\frac{1}{2} - p_n^u\right) Eu(w + A_{n-2} + \tilde{\epsilon}_n) + \left(\frac{1}{2} + p_n^u\right) Eu(w + B_{n-2}) \\ = \left(\frac{1}{2} - p_n^u\right) Eu(w + A_{n-2}) + \left(\frac{1}{2} + p_n^u\right) Eu(w + B_{n-2} + \tilde{\epsilon}_n)\end{aligned} \tag{1}$$

where w is initial wealth. We define the utility premium of $\tilde{\epsilon}_n$, given random wealth $\tilde{w}$, as

$$\Delta^u_{\tilde{\epsilon}_n}(\tilde{w}) = Eu(\tilde{w}) - Eu(\tilde{w} + \tilde{\epsilon}_n). \tag{2}$$

Using (1) and (2), we can write p_n^u as

$$p_n^u = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{\epsilon}_n}(w + A_{n-2}) - \Delta^u_{\tilde{\epsilon}_n}(w + B_{n-2})}{\Delta^u_{\tilde{\epsilon}_n}(w + A_{n-2}) + \Delta^u_{\tilde{\epsilon}_n}(w + B_{n-2})}\right] \tag{3}$$

and find that it is positive for any decision maker who is risk averse and nth-degree risk averse.

Theorem 1. *If u is risk-averse and nth-degree risk averse for $n \geq 3$, then $0 < p_n^u < \frac{1}{2}$.*

Proof. See Appendix A.

Now we study the relationship between the interpersonal comparison of our proposed strength measure of nth-degree risk apportionment, namely the nth-degree probability premium, and two related concepts of comparative nth-degree risk aversion under expected utility theory.

Theorem 2. *Suppose that u and v are risk averse decision makers who are also nth-degree risk averse for $n \geq 3$. Then, statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) $\frac{(-1)^n u^{(n)}(x)}{u''(y)} \geq \frac{(-1)^n v^{(n)}(x)}{v''(y)}$ *for all* $x, y \in [a, b]$.

(ii) $p_n^u \geq p_n^v$ *for all* w, $\tilde{\epsilon}_n$, A_{n-2}, *and* B_{n-2} *defined in Definition 1.*

(iii) $\frac{(-1)^n u^{(n)}(x)}{u''(x)} \geq \frac{(-1)^n v^{(n)}(x)}{v''(x)}$ *for all* $x \in [a, b]$.

Proof. See Appendix B.

In Theorem 2, we derive a sufficiency condition and a necessary condition for u to have a larger nth-degree probability premium than v. The sufficiency condition (i) is a generalization of Ross more risk aversion (Ross 1981) while the necessary condition (iii) is a generalization of Arrow-Pratt more risk aversion (Arrow 1971 and Pratt 1964) to the nth degree ($n \geq 3$). While it is obvious that condition (i) is stronger than condition (iii), we find that it is possible for conditions (ii) and (iii) to be equivalent. See Remark 1 and the discussion at the end of Section 5.

4 Generalization

In the previous section, we analyze the nth-degree probability premium based on Eeckhoudt and Schlesinger's (2006) definition of nth-degree risk apportionment. Eeckhoudt and Schlesinger's (2006) framework has been generalized by Eeckhoudt et al. (2009) who adopt a more general class of 50-50 lottery pairs that differ by nth-degree riskiness. In this section, we extend our results to this class of lotteries and show how to generalize the proposed intensity measure of an individual's preference for nth-degree risk apportionment.

4.1 (n/m)th-degree risk apportionment

Based on Eeckhoudt et al. (2009), we define lotteries $A_{n,m}$ and $B_{n,m}$ as follows.

Definition 4. *For $n > m \geq 1$,*

$$A_{n,m} = [\tilde{y}_{n-m} + \tilde{y}_m, 1/2; \tilde{x}_{n-m} + \tilde{x}_m, 1/2]$$

$$B_{n,m} = [\tilde{y}_{n-m} + \tilde{x}_m, 1/2; \tilde{x}_{n-m} + \tilde{y}_m, 1/2]$$

are compound binary lotteries such that

(i) $\tilde{x}_{n-m}$ and $\tilde{y}_{n-m}$ are independent of $\tilde{x}_m$ and $\tilde{y}_m$, and

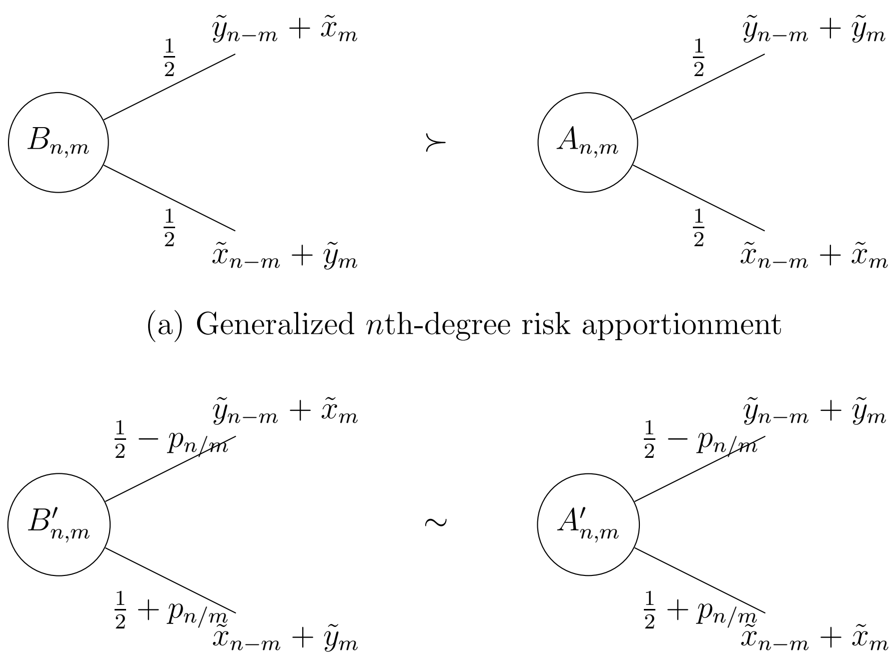
(ii) $\tilde{y}_i$ has more ith-degree risk than $\tilde{x}_i$ for $i = m, n-m$.

If the preference ordering satisfies both mth- and $(n-m)$th-degree risk aversion, we have

$$\tilde{x}_{n-m} + \tilde{x}_m \succ \left\{ \begin{array}{c} \tilde{y}_{n-m} + \tilde{x}_m \\ \\ \tilde{x}_{n-m} + \tilde{y}_m \end{array} \right\} \succ \tilde{y}_{n-m} + \tilde{y}_m.$$

Note that the two inner risks in the above rankings (i.e., $\tilde{y}_{n-m} + \tilde{x}_m$ and $\tilde{x}_{n-m} + \tilde{y}_m$) cannot be ranked without further information. Eeckhoudt et al. (2009) point out that for a decision maker to prefer B to A, he must prefer the 50-50 lottery of two “inner risks” to the 50-50 lottery of two “outer risks.”[4] They further prove that $A_{n,m}$ has more nth-degree risk than $B_{n,m}$.[5] Using the lotteries $A_{n,m}$ and $B_{n,m}$ defined in Definition 4, a more general definition of nth-degree risk apportionment than Definition 1 can be given below.

Definition 5. *For any integer $n \geq 3$, preferences are said to satisfy nth-degree risk apportionment if $B_{n,m} \succ A_{n,m}$ for all $A_{n,m}$ and $B_{n,m}$ given in Definition 4 and for every positive integer $m < n$.*



(a) Generalized nth-degree risk apportionment

(b) (n/m)th-degree probability premium $p_{n/m}$

Figure 4: Generalized probability premium

The preference relation defining the nth-degree risk apportionment is depicted in Figure 4 (a). Note that the lottery pair used to show nth-degree risk apportionment in Eeckhoudt

[4]The terminology is from Menezes and Wang (2005).

[5]Eeckhoudt et al. (2009) present theorems both for the case where the relatively bad is an nth-degree risk increase from the relatively good, and for the case where the relatively bad is nth-degree stochastically dominated by the relatively good. For the purpose of the present paper, we only need to consider the case of risk increases.

and Schlesinger (2006) in Definition 1 is a special case of $A_{n,m}$ and $B_{n,m}$ in Definition 4 with $m = 2$. Even though this preference relation does not hinge on the existence of an expected utility representation, when the decision maker's preferences satisfy the expected utility axioms and can be represented by utility function $u(x)$, the general notion of nth-degree risk apportionment is also characterized by $(-1)^{n-1}u^{(n)}(x) > 0$ for all $x \in [a, b]$. Moreover, the preference relation $B \succ A$ is equivalent to the following inequality:

$$Eu(w+\tilde{y}_{n-m}+\tilde{x}_m)-Eu(w+\tilde{y}_{n-m}+\tilde{y}_m) > Eu(w+\tilde{x}_{n-m}+\tilde{x}_m)-Eu(w+\tilde{x}_{n-m}+\tilde{y}_m). \quad (4)$$

Both sides in the above inequality represent the utility loss from an mth-degree risk increase (i.e., from $\tilde{x}_m$ to $\tilde{y}_m$), with the left-hand side being associated with $\tilde{y}_{n-m}$ and the right-hand side with $\tilde{x}_{n-m}$. Therefore, the above inequality states that the pain from an mth-degree risk increase in one asset component increases as the other asset component undergoes an $(n-m)$th-degree risk increase (i.e., from $\tilde{x}_{n-m}$ to $\tilde{y}_{n-m}$). See Denuit and Rey (2010) and Ebert et al. (2018).

4.2 (n/m)th-degree probability premiums

Using the concept of nth-degree risk apportionment given in Definition 5, we are now ready to define the (n/m)th-degree probability premium to measure the intensity of preference for nth-degree risk apportionment illustrated in Figure 4 (a). Like Definition 5, the following definition of $p_{n/m}$, which is illustrated in Figure 4 (b), does not rely on the existence of an expected utility representation of the preferences.[6]

[6] The remarks in Footnote 3 also apply here.

Definition 6. *Given*

$$A'_{n,m} = [\tilde{y}_{n-m} + \tilde{y}_m, 1/2 - p_{n/m}; \tilde{x}_{n-m} + \tilde{x}_m, 1/2 + p_{n/m}]$$
$$B'_{n,m} = [\tilde{y}_{n-m} + \tilde{x}_m, 1/2 - p_{n/m}; \tilde{x}_{n-m} + \tilde{y}_m, 1/2 + p_{n/m}],$$

a decision maker's (n/m)*th-degree probability premium is* $p_{n/m}$ *such that* $A' \sim B'$.

The intuition for using $p_{n/m}$ defined above as a measure of the strength of nth-degree risk apportionment is the following. According to Definition 5, nth-degree risk apportionment is characterized by $B_{n,m} \succ A_{n,m}$. Note that for an mth-degree risk averse individual, the lower (upper) random wealth in lottery A is better (worse) than the lower (upper) random wealth in lottery B. Therefore, the attractiveness of lottery A relative to lottery B can be improved by moving some probability mass from the upper state to the lower state in both lotteries. The required probability mass that makes the two sides equally attractive, denoted by $p_{n/m}$, indicates how difficult it is to offset the individual's preference for nth-degree risk apportionment by such a movement of probability mass, and can serve as a measure of the strength of nth-degree risk apportionment. Note that the nth-degree probability premium p_n defined in the previous section is a special case of $p_{n/m}$ given $m = 2$.

Under expected utility theory, we can derive u's (n/m)th-degree probability premium, denoted by $p^u_{n/m}$, given an initial wealth w as follows.

$$\begin{aligned}\left(\frac{1}{2} - p^u_{n/m}\right) & Eu(w + \tilde{y}_{n-m} + \tilde{y}_m) + \left(\frac{1}{2} + p^u_{n/m}\right) Eu(w + \tilde{x}_{n-m} + \tilde{x}_m) \\ &= \left(\frac{1}{2} - p^u_{n/m}\right) Eu(w + \tilde{y}_{n-m} + \tilde{x}_m) + \left(\frac{1}{2} + p^u_{n/m}\right) Eu(w + \tilde{x}_{n-m} + \tilde{y}_m) \quad (5)\end{aligned}$$

Let $\tilde{z}_m = \tilde{y}_m - \tilde{x}_m$. Using (5) and (2), we can write $p^u_{n/m}$ as

$$p^u_{n/m} = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{z}_m}(w + \tilde{y}_{n-m} + \tilde{x}_m) - \Delta^u_{\tilde{z}_m}(w + \tilde{x}_{n-m} + \tilde{x}_m)}{\Delta^u_{\tilde{z}_m}(w + \tilde{y}_{n-m} + \tilde{x}_m) + \Delta^u_{\tilde{z}_m}(w + \tilde{x}_{n-m} + \tilde{x}_m)}\right] \quad (6)$$

and find that it is positive for any decision maker who is nth- and mth-degree risk averse.

Theorem 3. *If u is nth- and mth-degree risk averse, then* $0 < p^u_{n/m} < 1/2$.

Proof. See Appendix C.

Consider the mixed risk averters defined by Cabellé and Pomansky (1996) as our special case. If the decision maker is mixed risk averse, then he is both mth- and nth-degree risk averse, and hence $0 < p^u_{n/m} < 1/2$ for every pair of A and B satisfying Definition 4, for all n and m such that $n > m \geq 1$. It is also possible for the mixed risk lovers defined by Crainich et al. (2013) to have a positive $p^u_{n/m}$ since they are nth-degree risk averse for an odd n. So if both m and n are odd, a mixed risk lover will be both mth- and nth-degree risk averse, and his corresponding $p^u_{n/m}$ will be positive.

As we argued after Theorem 1 in Section 2, although both nth-degree risk apportionment and nth-degree risk aversion are characterized by $(-1)^{n-1}u^{(n)} > 0$ when the preferences of a decision maker have an expected utility representation, this does not necessarily imply that the intensity measure for nth-degree risk apportionment must be the same as the intensity measure for nth-degree risk aversion. Indeed, it is clear upon examination that our general measure of nth-degree risk apportionment, $p^u_{n/m}$, is different from all the measures of nth-degree risk aversion proposed in Liu and Meyer (2013) and Liu and Neilson (2019). Nevertheless, it is interesting to compare $p^u_{n/m}$ with the "rate of substitution" measure of nth-degree risk aversion, denoted by T_u, in Liu and Meyer (2013). First, note that (6) can be rewritten as

$$p^u_{n/m} = \frac{1}{2}\left[\frac{Eu(w + B_{n,m}) - Eu(w + A_{n,m})}{Eu(w + D_{n,m}) - Eu(w + C_{n,m})}\right] \tag{7}$$

where

$$C_{n,m} = [\tilde{y}_{n-m} + \tilde{y}_m, 1/2; \tilde{x}_{n-m} + \tilde{y}_m, 1/2]$$
$$D_{n,m} = [\tilde{y}_{n-m} + \tilde{x}_m, 1/2; \tilde{x}_{n-m} + \tilde{x}_m, 1/2]$$

and lotteries $A_{n,m}$ and $B_{n,m}$ are given in Definition 4. Since $A_{n,m}$ has more nth-degree risk than $B_{n,m}$, and $C_{n,m}$ has more mth-degree risk than $D_{n,m}$, $p_{n/m}$ can be interpreted as a half of the ratio of the utility premium of an nth-degree risk reduction to the utility premium of an mth-degree risk reduction. In contrast,

$$T_u = \frac{Eu(w+F) - Eu(w+G)}{Eu(w+F) - Eu(w+H)} \tag{8}$$

where G has more nth-degree risk than F and H has more mth- degree risk than F. While both $p^u_{n/m}$ and T_u are ratios of an nth-degree utility premium to an mth-degree utility premium, there are two subtle differences between them. First, the nth- and the mth-degree risk increases happen to the same F in T_u, which is not the case for $p^u_{n/m}$. Second, F, G, and H in (8) can be any random variables as long as G has more nth-degree risk than F and H has more mth-degree risk than F, but the risk comparisons among $A_{n,m}$, $B_{n,m}$, $C_{n,m}$, and $D_{n,m}$ in (7) are based on the risk apportionment framework.[7]

In a related study, Liu and Neilson (2019) define an mth-degree probability premium for an nth-degree risk increase, which, in the framework of expected utility, can be written

[7] It is impossible to find lotteries $A_{n,m}$, $B_{n,m}$, $C_{n,m}$, and $D_{n,m}$ so that $p^u_{n/m}$ in (7) coincides with 1/2 of T_u in (8). If we assume that $B_{n,m}$ and $D_{n,m}$ in (7) are identical so that both lotteries can be represented by F in (8), then we impose that $\tilde{x}_m = \tilde{y}_m$ which in turn contradicts $A_{n,m}$ and $B_{n,m}$ defined in Definition 4.

as

$$
\begin{aligned}
p_u &= \frac{Eu(w+F) - Eu(w+G)}{Eu(w+Z) - Eu(w+G)} \\
&= \frac{Eu(w+F) - Eu(w+G)}{[Eu(w+Z) - Eu(w+F)] + [Eu(w+F) - Eu(w+G)]}.
\end{aligned}
\tag{9}
$$

where G has more nth-degree risk than F, and Z has less mth-degree risk than F. Besides the fact that 1/2 is present in (7) but not in (9), the two versions of probability premium have two main differences. First, although the numerators in (7) and (9) are both nth-degree utility premiums, the denominator in (7) is an mth-degree utility premium whereas the denominator in (9) is the sum of an mth-degree utility premium and an nth-degree utility premium. Second, and more importantly, F, G and Z in (9) can be any random variables as long as G has more nth-degree risk than F and F has more mth-degree risk than Z, but the four random variables in (7) are more restricted 50-50 compound lotteries.

4.3 Comparative risk apportionment

In this section, we study the relationship between the interpersonal comparison of our proposed strength measure of nth-degree risk apportionment, namely the (n/m)th-degree probability premium, and two related concepts of comparative nth-degree risk aversion under expected utility theory. First, we introduce the two generalized concepts of comparative nth-degree risk aversion defined by Liu and Meyer (2013).

Definition 7. *(Liu & Meyer 2013) For $n > m \geq 1$, u is (n/m)th-degree Arrow-Pratt more risk averse than v on $[a,b]$ if*

$$
\frac{(-1)^{(n-1)}u^{(n)}(x)}{(-1)^{(m-1)}u^{(m)}(x)} \geq \frac{(-1)^{(n-1)}v^{(n)}(x)}{(-1)^{(m-1)}v^{(m)}(x)}
\tag{10}
$$

for all $x \in [a,b]$.

Definition 8. *(Liu & Meyer 2013) For $n > m \geq 1$, u is (n/m)th-degree Ross more risk averse than v on $[a, b]$ if*

$$\frac{(-1)^{(n-1)}u^{(n)}(x)}{(-1)^{(m-1)}u^{(m)}(y)} \geq \frac{(-1)^{(n-1)}v^{(n)}(x)}{(-1)^{(m-1)}v^{(m)}(y)} \tag{11}$$

for all $x, y \in [a, b]$.

By choosing $y = x$ in Definition 8, it follows immediately that (n/m)th-degree Ross more risk averse is a stronger condition than (n/m)th-degree Arrow-Pratt more risk averse.[8] The theorem below states how the interpersonal comparison of our proposed measure of the strength of nth-degree risk apportionment–the (n/m)th-degree probability premium–is related to the above two notions of (n/m)th-degree more risk averse.

Theorem 4. *Let $p^u_{n/m}$ and $p^v_{n/m}$ be (n/m)th-degree probability premiums for decision makers u and v respectively. If both are mth- and nth-degree risk averse, then statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) u is (n/m)th-degree Ross more risk averse than v.

(ii) $p^u_{n/m} \geq p^v_{n/m}$ for all w, $\tilde{x}_m$, $\tilde{y}_m$, $\tilde{x}_{n-m}$, $\tilde{y}_{n-m}$.

(iii) u is (n/m)th-degree Arrow-Pratt more risk averse than v.

Proof. See Appendix D.

According to Theorem 4, $p^u_{n/m} \geq p^v_{n/m}$ is necessary but not sufficient for u being (n/m)th-degree Ross more risk averse than v. This result is not as strong as Liu and

[8] The notion of (n/m)th-degree Arrow-Pratt more risk averse given by Definition 7 includes many lower-degree versions as special cases: Arrow (1971) and Pratt (1964) for $n = 2$ and $m = 1$, Kimball (1990) and Chiu (2005) for $n = 3$ and $m = 2$, Crainich and Eeckhoudt (2008) for $n = 3$ and $m = 1$, and Crainich and Eeckhoudt (2011) for $n = 4$ and $m = 1, 2, 3$. Similarly, the notion of (n/m)th-degree Ross more risk averse given by Definition 8 also includes many lower-degree versions as special cases: Ross (1981) and Machina and Neilson (1987) for $n = 2$ and $m = 1$, Modica and Scarsini (2005) for $n = 3$ and $m = 1$, and Jindapon and Neilson (2007), Li (2009), and Denuit and Eeckhoudt (2010) for $n \geq 2$ and $m = 1$. Crainich et al. (2020) further extend Definition 8 to bivariate situations.

Neilson's (2019) finding regarding the mth-degree probability premium for an nth-degree risk increase, which is proved to be larger for u than for v *if and only if* u is (n/m)th-degree Ross more risk averse than v. The reason for this difference is that the random variables determining $p^u_{n/m}$ in (7) are 50-50 lotteries which are more restricted than the random variables determining p_u in (9).

5 Prudence

The focus of this section is a preference for 3rd-degree risk apportionment, which is also known as prudence and downside risk aversion. A prudent decision maker prefers $B_3 = [-k, 1/2; \tilde{\epsilon}, 1/2]$ to $A_3 = [-k + \tilde{\epsilon}, 1/2; 0, 1/2]$, as illustrated in Figure 1. Using Definition 6 with $n = 3$, we can define two prudence probability premiums, $p_{3/1}$ and $p_{3/2}$, which are illustrated in Figures 5 (a) and 5 (b). Given a utiliy function u, the two probability premiums can be expressed under expected utility theory as

$$p^u_{3/1} = \frac{1}{2}\left[\frac{\Delta^u_{-k}(w+\tilde{\epsilon}) - \Delta^u_{-k}(w)}{\Delta^u_{-k}(w+\tilde{\epsilon}) + \Delta^u_{-k}(w)}\right] \tag{12}$$

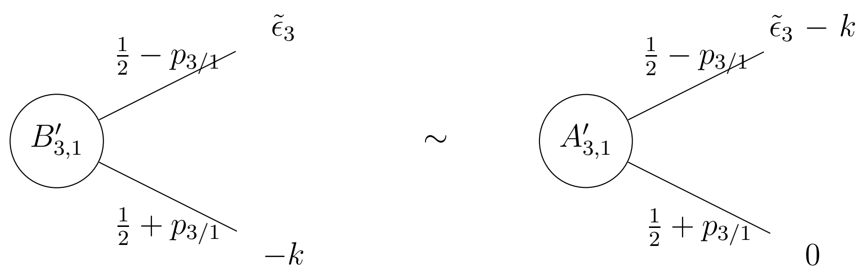
and

$$p^u_{3/2} = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{\epsilon}}(w-k) - \Delta^u_{\tilde{\epsilon}}(w)}{\Delta^u_{\tilde{\epsilon}}(w-k) + \Delta^u_{\tilde{\epsilon}}(w)}\right]. \tag{13}$$

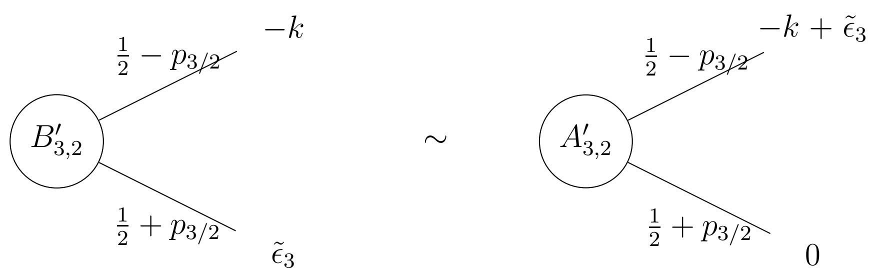
In the literature, two studies have previously used a probability premium to measure the strength of prudence or downside risk aversion; Watt (2011) proposed a probability premium concept similar to $p_{3/2}$, while Jindapon's (2010) probability premium is defined slightly different. Specifically, Jindapon's prudence probability premium is q such that $B_3 \sim A^*_3$ where

$$A^*_3 = [-k + \tilde{\epsilon}, 1/2 - q; 0, 1/2 + q].$$

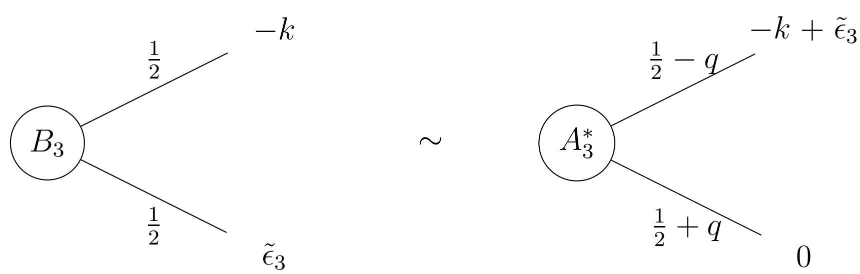
See Figure 5 (c) for an illustration of q. The key difference between q and the first two



(a) Prudence probability premium $p_{3/1}$



(b) Prudence probability premium $p_{3/2}$



(c) Prudence probability premium q

Figure 5: Various concepts of prudence probability premium

probability premiums is that the probability of each state in B_3 that we use to derive q is unchanged. To compare prudence probability premiums between two expected-utility maximizers, Jindapon (2010) identifies a sufficient condition for $q^u > q^v$. However, its application is quite limited, because his sufficient condition depends not only on the utility functions, but also on $\tilde{\epsilon}$. Watt's (2011) sufficient condition for $p^u_{3/2} > p^v_{3/2}$ is considered incomplete for the same reason. Based on our results from the previous section, we can provide a sufficient condition for comparing each probability premium concept without a restriction on $\tilde{\epsilon}$.

Under expected utility theory, we have

$$q^u = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{\epsilon}}(w-k) - \Delta^u_{\tilde{\epsilon}}(w)}{u(w) - Eu(w-k+\tilde{\epsilon})}\right] = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{\epsilon}}(w-k) - \Delta^u_{\tilde{\epsilon}}(w)}{\Delta^u_{-k}(w) + \Delta^u_{\tilde{\epsilon}}(w-k)}\right]. \tag{14}$$

Consider the ratio inside the last brackets. Each of these conditions, $u'(x) > 0$, $u''(x) < 0$, and $u'''(x) > 0$, implies $\Delta_{-k}(w) > 0$, $\Delta_{\tilde{\epsilon}}(w-k) > 0$, and $\Delta_{\tilde{\epsilon}}(w-k) - \Delta_{\tilde{\epsilon}}(w) > 0$, respectively. Thus, q^u is positive for any prudent risk averter. Following the proof of Theorem 4, we can derive a sufficient condition for $q^u \geq q^v$ given any $\tilde{\epsilon}$. We summarize sufficient conditions for comparing prudence probability premiums between two decision makers as follows.

Corollary 1. *Suppose that u and v are prudent.*

(i) If $\frac{u'''(x)}{u'(y)} \geq \frac{v'''(x)}{v'(y)}$ for all $x, y \in [a,b]$, then $p^u_{3/1} \geq p^v_{3/1}$ for all w, k, and $\tilde{\epsilon}$.

(ii) If both u and v are risk averse and $-\frac{u'''(x)}{u''(y)} \geq -\frac{v'''(x)}{v''(y)}$ for all $x, y \in [a,b]$, then $p^u_{3/2} \geq p^v_{3/2}$ for all w, k, and $\tilde{\epsilon}$.

(iii) If both u and v are risk averse, $\frac{u'''(x)}{u'(y)} \geq \frac{v'''(x)}{v'(y)}$ and $-\frac{u'''(x)}{u''(y)} \geq -\frac{v'''(x)}{v''(y)}$ for all $x, y \in [a,b]$, then $q^u \geq q^v$ for all w, k, and $\tilde{\epsilon}$.

Note that the comparison between $p^u_{3/1}$ and $p^v_{3/1}$ does not need both agents to be risk averse. The sufficient condition in Corollary 1 (*i*), i.e., more (3/1)th-degree Ross more risk averse, is actually equivalent to being more strongly downside risk averse, as defined by Modica and Scarsini (2005). Parts (*ii*) and (*iii*) provide sufficient conditions for comparing Watt's and Jindapon's probability premiums, respectively. Based on the derivation of each probability premium concept in (12), (13), and (14), we can see how we obtain such sufficient conditions. Specifically, $p_{3/1}$ is half of the ratio of the utility premium of a third-degree risk increase to the utility premium of a first-degree risk increase; $p_{3/2}$ is half of the ratio of the utility premium of a third-degree risk increase to the utility premium of a second-degree risk increase; and q is half of the ratio of the utility premium of a third-degree risk increase to the utility premium of a second-degree stochastically dominated change (of which both the first-degree risk increase and the second-degree risk increase are special cases).[9]

Finally, we focus on the case of constant absolute risk aversion (CARA), i.e., $u(x) = -e^{-\alpha x}$. We find that, between two CARA decision makers, the one with a larger risk preference parameter α has a larger 3rd-degree probability premium.

Remark 1. *Suppose that $u(x) = -e^{-\alpha x}$ and $v(x) = -e^{-\beta x}$. The following three statements are equivalent:*

(i) $\alpha > \beta > 0$.

(ii) $p^u_{3/1} \geq p^v_{3/1}$ *for any* $w > k > 0$, *and any zero-mean risk* $\tilde{\epsilon}$.

(iii) $p^u_{3/2} \geq p^v_{3/2}$ *for any* $w > k > 0$, *and any zero-mean risk* $\tilde{\epsilon}$.

Proof. See Appendix E.

[9]In general, the utility premium refers to the reduction in expected utility caused by a change in the random wealth variable. While it has long been recognized that the utility premium is not interpersonally comparable, the ratio of two utility premiums is. See, for example, Crainich and Eeckhoudt (2008), Eeckhoudt and Schlesinger (2009), Denuit and Rey (2010), Menegatti (2011), and Li and Liu (2014).

Note that, given a CARA utility function, $p^u_{3/1}$ and $p^u_{3/2}$ do not depend on w or $\tilde{\epsilon}$. Therefore, Given $n = 3$ and the CARA functional form, (ii) $\Leftrightarrow$ (iii) in Theorem (4). This unambiguous positive effect of the preference parameter in the CARA utility function on both versions of the prudence probability premium is in fact not obvious. While the CARA parameter has a positive effect on the measure of absolute prudence (Kimball, 1990) and the measure of downside risk aversion (Crainich and Eeckhoudt, 2008; Modica and Scarsini, 2005), it has a negative effect on the Schwarzian derivative of Keenan and Snow (2002, 2012) and no effect on Liu and Meyer's (2012) measure of decreasing absolute risk aversion (DARA). Peter (2020) discusses these four measures in the context of comparative downside risk aversion and optimal prevention. Using CARA utility as an example, Peter (2020) finds that the change in optimal prevention depends not only on the direction of a change in the CARA parameter, but also on the magnitude of the change.

6 Conclusion

In this paper, we propose a concept of probability premium that can be used to compare the strength of preference for nth-degree risk apportionment between two individuals. Specifically, based on the general framework of Eeckhoudt et al. (2009), we define the (n/m)th-degree probability premium, where $n > m \geq 1$, denoted by $p_{n/m}$, and prove that, under expected utility theory, the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013) is a sufficient condition for comparative nth-degree risk apportionment, whereas the corresponding (n/m)th-degree Arrow-Pratt more risk aversion is a necessary condition.

While there are $n-1$ ways to measure the strength of nth-degree risk apportionment by using probability premiums $p_{n/m}$ where $m = 1, 2, ..., n-1$, the special case of $m = 2$ can be dealt with in the original risk apportionment framework of Eeckhoudt and Schlesinger (2006), where $p_{n/2}$ or simply p_n emerges as a measure of preference for nth-degree risk apportionment. For example, we can use $p_{3/2}$ as a measure of prudence and $p_{4/2}$ as a measure of temperance. As Eeckhoudt and Schlesinger (2006) point out, the preference relation between two 50-50 lotteries—which involve two independent risks—that they use to define temperance can also be used to define proper risk aversion (Pratt & Zeckhauser, 1987), risk vulnerability (Gollier & Pratt, 1996), and standard risk aversion (Kimball, 1993), as long as the risks in the 50-50 lotteries are given appropriate reinterpretations. Therefore, the way we define the $p_{4/2}$ measure for temperance can be used to formulate measures of proper risk aversion, risk vulnerability, and standard risk aversion. Such extensions/applications of our comparative risk apportionment approach are left for future research.

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Appendix

A. Proof of Theorem 1

The result is a special case of Theorem 3 and can be obtained by letting $\tilde{x}_m = 0$ and $\tilde{y}_m = \tilde{\epsilon}_n$ so that $m = 2$.

B. Proof of Theorem 2

The result is a special case of Theorem 4 and can be obtained by letting $\tilde{x}_m = 0$ and $\tilde{y}_m = \tilde{\epsilon}_n$ so that $m = 2$.

C. Proof of Theorem 3

Since u is mth-degree risk averse and $\tilde{x}$ has more mth-degree risk than $\tilde{y}$, then,

$$\Delta^u_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) = Eu(w + \tilde{x}_m + \tilde{x}_{n-m}) - Eu(w + \tilde{y}_m + \tilde{x}_{n-m}) > 0 \tag{15}$$

and

$$\Delta^u_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) = Eu(w + \tilde{x}_m + \tilde{y}_{n-m}) - Eu(w + \tilde{y}_m + \tilde{y}_{n-m}) > 0. \tag{16}$$

Since u is nth-degree risk averse and A has more nth-degree risk than B, then $B \succ A$. Under expected utility theory, we can write

$$\frac{1}{2}[Eu(w+\tilde{x}_m+\tilde{y}_{n-m})+Eu(w+\tilde{y}_m+\tilde{x}_{n-m})] > \frac{1}{2}[Eu(w+\tilde{y}_m+\tilde{y}_{n-m})+Eu(w+\tilde{x}_m+\tilde{x}_{n-m})] \tag{17}$$

which is equivalent to

$$\Delta^u_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^u_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) > 0. \tag{18}$$

Given $p_{n/m}$ in (6), we find that (15), (16), and (18) jointly imply $0 < p^u_{n/m} < 1/2$.

D. Proof of Theorem 4

Part 1. (i) ⇒ (ii)

Given $p_{n/m}$ in (6), we can write

$$p^v_{n/m} = \frac{s}{2t} \tag{19}$$

where

$$s = \Delta^v_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^v_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) \tag{20}$$

and

$$t = \Delta^v_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^v_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}). \tag{21}$$

Since v is nth- and mth-degree risk averse, both s and t are positive (see the proof of Theorem 3). Given that u is Ross more risk averse than v, we can write

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)} \tag{22}$$

for all $x, y \in [a, b]$ and some $\lambda > 0$. Liu and Meyer (2013) show that this condition is equivalent to

$$u(x) = \lambda v(x) + \phi(x) \tag{23}$$

where $(-1)^{m-1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n-1}\phi^{(n)}(x) \geq 0$ for all $x \in [a, b]$. By substituting (23) into (6), we can write

$$p^u_{n/m} = \frac{1}{2}\left[\frac{\lambda s + \Delta^\phi_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^\phi_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})}{\lambda t + \Delta^\phi_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^\phi_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})}\right]. \tag{24}$$

It follows from (19) and (24) that $p^u_{n/m} \geq p^v_{n/m}$ if and only if

$$t[\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})] \geq$$

$$s[\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})]. \tag{25}$$

Since $(-1)^{m-1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n-1}\phi^{(n)}(x) \geq 0$ for all $x \in [a, b]$, then

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) \leq 0, \tag{26}$$

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) \leq 0, \tag{27}$$

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) \geq 0. \tag{28}$$

As a result, the inequality in (25) always holds and, therefore, $p^u_{n/m} \geq p^v_{n/m}$.

Part 2. (ii) $\Rightarrow$ (iii)

Suppose that (iii) is false, i.e., $u(x)$ is not (n/m)th-degree Arrow-Pratt more risk averse than $v(x)$. Then, there exists $x \in [a, b]$ such that

$$\frac{(-1)^{n-1}u^n(x)}{(-1)^{m-1}u^m(x)} < \frac{(-1)^{n-1}v^n(x)}{(-1)^{m-1}v^m(x)}. \tag{29}$$

Since both u and v are mth- and nth-degree risk averse, the above inequality implies

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} < \frac{u^{(m)}(x)}{v^{(m)}(x)}. \tag{30}$$

Due to continuity, there exists $\mu > 0$ and $[c, d] \subset [a, b]$ such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)} \tag{31}$$

for all $y, z \in [c, d]$. Define $\psi(x) = u(x) - \mu v(x)$. Differentiating yields

$$(-1)^{m-1}\psi^m(x) = (-1)^{m-1}u^m(x) - \mu(-1)^{m-1}v^m(x) > 0 \tag{32}$$

and

$$(-1)^{n-1}\psi^n(x) = (-1)^{n-1}u^n(x) - \mu(-1)^{n-1}v^n(x) < 0 \tag{33}$$

for all $x \in [c, d]$. If we pick w, k, and $\tilde{\epsilon}$ so that the support of all possible levels of final wealth is a subset of $[c, d]$, then we have

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) > 0, \tag{34}$$

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) > 0, \tag{35}$$

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) < 0. \tag{36}$$

It follows that the inequality in 25 is reversed so that $p^u_{n/m} < p^v_{n/m}$. Therefore, (ii) is false.

E. Proof of Remark 1

Given $\Delta^u_{-k}(w) = (e^{\alpha k} - 1)e^{-\alpha w}$ and $\Delta^u_{-k}(w + \tilde{\epsilon}) = (e^{\alpha k} - 1)E[e^{-\alpha\tilde{\epsilon}}]$, we can write $p^u_{3/1}$ in (6) as

$$p^u_{3/1} = \frac{1}{2}\left[\frac{\Delta^u_{-k}(w + \tilde{\epsilon}) - \Delta^u_{-k}(w)}{\Delta^u_{-k}(w + \tilde{\epsilon}) + \Delta^u_{-k}(w)}\right] = \frac{1}{2}\left(\frac{E[e^{-\alpha\tilde{\epsilon}}] - 1}{E[e^{-\alpha\tilde{\epsilon}}] + 1}\right). \tag{37}$$

Since $E[\tilde{\epsilon}] = 0$ and $\alpha > 0$, then $dE[e^{-\alpha\tilde{\epsilon}}]/d\alpha = -E[\alpha e^{-\alpha\tilde{\epsilon}}] > 0$. Given $\alpha > \beta > 0$, then

$$p^u_{3/1} = \frac{1}{2}\left(\frac{E[e^{-\alpha\tilde{\epsilon}}] - 1}{E[e^{-\alpha\tilde{\epsilon}}] + 1}\right) > \frac{1}{2}\left(\frac{E[e^{-\beta\tilde{\epsilon}}] - 1}{E[e^{-\beta\tilde{\epsilon}}] + 1}\right) = p^v_{3/1}. \tag{38}$$

Given $\Delta^u_{\tilde{\epsilon}}(w) = e^{-\alpha w}(1 - E[e^{-\alpha\tilde{\epsilon}}])$ and $\Delta^u_{\tilde{\epsilon}}(w - k) = e^{-\alpha(w-k)}(1 - E[e^{-\alpha\tilde{\epsilon}}])$, we can write

$p^u_{3/2}$ in (6) as

$$p^u_{3/2} = \frac{1}{2}\left[\frac{\Delta^u_{\tilde{\epsilon}}(w-k) - \Delta^u_{\tilde{\epsilon}}(w)}{\Delta^u_{\tilde{\epsilon}}(w-k) + \Delta^u_{\tilde{\epsilon}}(w)}\right] = \frac{1}{2}\left(\frac{e^{\alpha k}-1}{e^{\alpha k}+1}\right). \tag{39}$$

Since $\alpha > \beta > 0$, then

$$p^u_{3/2} = \frac{1}{2}\left(\frac{e^{\alpha k}-1}{e^{\alpha k}+1}\right) > \frac{1}{2}\left(\frac{e^{\beta k}-1}{e^{\beta k}+1}\right) = p^v_{3/2}. \tag{40}$$