AN EXTENSION OF DESCARTES' RULE OF SIGNS TO MULTIPLE VARIABLES, WITH APPLICATIONS TO MODELS OF BIOLOGICAL SYSTEMS

A Thesis

by

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ABSTRACT

Many biological systems are modeled mathematically using systems of differential equations. Descartes' rule of signs is a common tool for analyzing these systems. In this thesis we explore a new perspective on Descartes' rule of signs which allows us to begin extending the rule to polynomials in multiple variables, as introduced in two papers by Maté Telek and Elisenda Feliu. We explain the relationship between Descartes' rule of signs and this new perspective in detail, with several examples. Then we present a result for the situation when a polynomial in multiple variables has only one negative coefficient, with proof. Finally, we present some additional thoughts and potential areas for future study.

DEDICATION

To Grandpa Jay, who earned his masters degree studying polynomials 50 years ago.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis committee consisting of Professors Anne Shiu [advisor], Eric Rowell, and Frank Sottile of the Department of Mathematics and Professor Irina Gaynanova of the Department of Statistics.

All work conducted for the thesis was completed by the student independently.

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1. INTRODUCTION

The inspiration and main reference for this thesis is the paper *On generalizing Descartes' rule of signs to hypersurfaces* by Elisenda Feliu and Máté L. Telek [1]. We will also reference the follow-up paper by the same authors, *Topological descriptors of the parameter region of multistationarity: Deciding upon connectivity* [2].

1.1 Chemical Reaction Networks

One area of mathematical biology involves the study of chemical reaction networks, a tool for modeling the interactions between reactions in a biochemical system. More precisely, a *reaction network* is a set of reactions $R_1, ..., R_m$ between species $S_1, ..., S_n$, where each reaction connects two $\mathbb{Z}_{\geq 0}$-linear combinations of species:

$$R_j : a_{1_j} S_1 + \cdots + a_{n_j} S_n \to b_{1_j} S_1 + \cdots + b_{n_j} S_n.$$

Given a chemical reaction network, we use the assumption of mass-action kinetics to build a system of ordinary differential equations representing the rate of change of the quantity of each species, with parameters $\kappa_1, ..., \kappa_m$ representing the rate at which each reaction occurs.

Example 1.1.1. *Consider the reaction network given by*

$$R_1 : 2S + T \to 2T$$
$$R_2 : S \to 2S$$
$$R_3 : T \to 0$$

where κ_1, κ_2, and κ_3 represent the rates at which R_1, R_2, and R_3 occur, respectively. Then the system of ordinary differential equations which describes the rate of change of quantities of S and

T is given below.

$$\frac{dS}{dt} = -2\kappa_1 S^2 T + \kappa_2 S$$
$$\frac{dT}{dt} = \kappa_1 S^2 T - \kappa_3 T$$

Observe that each reaction contributes one term to each ordinary differential equation (may be zero). The coefficient of the term is the reaction rate constant κ multiplied by the net change in the quantity of the respective species. The rest of the term is the product of each species required for the reaction to occur (given in the left hand side of the reaction) raised to the power of the coefficient of that species in the left hand side of the reaction.

One goal in the study of reaction networks is to determine the long-term behavior of the biological system, which is accomplished by analysing the steady states of the corresponding system of ordinary differential equations. In this thesis we focus specifically on the task of determining which values of $\kappa_1, \ldots, \kappa_n$ allow for more than one (positive) steady state. When this occurs we say the system exhibits *multistationarity*. The goal of [1] and [2] is to develop methods for determining whether multistationarity can occur in a given system and understanding the properties of the set of parameters which result in this phenomenon. Descartes' rule of signs is a common tool for accomplishing this task.

1.2 Descartes' Rule of Signs

Descartes was a French mathematician who lived during the 17th century. In 1637 he published *La Géométrie*, a book on the connections between algebra and geometry. One of the results included in this book was his rule of signs [7, page 96].

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients. The *sign sequence* of $p(x)$ is the order of the signs of the nonzero coefficients a_i. A change in the sign sequence occurs any time the sign switches from $+$ to $-$, or from $-$ to $+$.

Theorem 1.2.1 (Descartes' Rule of Signs)**.** *The number of changes in the sign sequence of a poly-*

nomial provides a strict upper bound on the number of positive roots of the polynomial, counted with multiplicity. Additionally, the parity of the number of sign changes matches the parity of the number of positive roots, counted with multiplicity.

The number of corollaries and extensions of this single result over the span of almost 400 years hints at the broad range of situations in which it is useful. The second half of the rule, which relates to the parity of the number of positive roots, was added by Gauss in 1828. Later, in 1918, Curtiss found a new proof which showed the rule works for any polynomial with real exponents [1, page 1]. Others have continued to search for applications and extensions to this rule, leading to such results as the Budan-Fourier Theorem [6, page 14]. In addition to the perspective presented in this thesis, other efforts have also been made to generalize Descartes' rule to polynomials in more than one variable (for example, see [4] and [5]).

Consider the following example of Descartes' rule of signs.

Example 1.2.2. *The polynomial* $p(x) = 3x^7 - 4x^6 - x^4 + 2x - 1$ *has sign sequence* $(+ - - + -)$. *The number of sign changes is three: from positive to negative, back to positive, then back to negative. Thus, Descartes' rule of signs tells us this polynomial has at most three positive roots, and the number is odd. So there are either one or three positive roots (counting multiplicity). In fact there is one, as illustrated in Figure 1.1.*

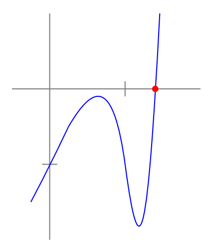

Figure 1.1: The graph of $p(x) = 3x^7 - 4x^6 - x^4 + 2x - 1$, with its one positive root emphasized.

1.3 Definitions

We conclude the introduction by presenting some necessary definitions from [1] and [3], followed by the statement of our main theorems.

For a point $\mu = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$, we use the notation $x^\mu = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

Definition 1.3.1. *A* signomial *is a function $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ such that $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, with $c_\mu \in \mathbb{R} \backslash \{0\}$ and $\sigma(f) \subseteq \mathbb{R}^n$ a finite set.*

The set $\sigma(f)$ is often referred to as the *support* of the signomial f.

Below are some examples of signomials:

$$f(x_1, x_2) = x_1 x_2^3 + 5x_2^{-1}$$
$$g(x_1, x_2, x_3) = -1.3x_1 x_2 x_3 + x_2^{1/2} x_3^2 - x_3$$

In this thesis we only consider *hypersurfaces* of the following form:

$$\{(x_1, \cdots, x_n, y) \in \mathbb{R}^n_{>0} \times \mathbb{R} \mid f(x_1, \cdots, x_n) - y = 0\},$$

where f is a signomial.

Example 1.3.2. *A parabola (see Figure 1.2) is an example of a hypersurface in $\mathbb{R}^2$, $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - y = 0\}$.*

For a signomial $f : \mathbb{R}^n_{>0} \to \mathbb{R}$, we denote the *positive variety of f*

$$V_{>0}(f) = \{x \in \mathbb{R}^n_{>0} \mid f(x) = 0\},$$

and we denote its complement

$$V^c_{>0}(f) = \mathbb{R}^n_{>0} \setminus V_{>0}(f) = \{x \in \mathbb{R}^n_{>0} \mid f(x) \neq 0\}.$$

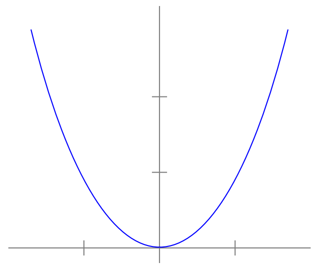

Figure 1.2: An example of a hypersurface in $\mathbb{R}^2$.

The following ideas from topology will also be relevant.

Definition 1.3.3. *We say a region $B \subseteq \mathbb{R}^n$ is* convex *if for any two points $x, y \in B$ the line $\beta(t) = xt + y(1-t)$ has $\beta([0,1]) \subseteq B$.*

Definition 1.3.4. *We say a region $A \subseteq \mathbb{R}^n$ is* path-connected *if for any two points $x, y \in A$ there is a continuous function (called a* path*) $\alpha : [a,b] \to A$ such that $\alpha(a) = x$ and $\alpha(b) = y$.*

Remark 1.3.5. *Later we will want to prove that a set is* connected*. This property is implied by path-connectedness, so we will use the above definition. See chapters 8 and 9 in [3] for more on the relationship between connectedness and path-connectedness.*

Definition 1.3.6. *Let $S \subseteq \mathbb{R}^n$. We call a set $A \subseteq S$ a* connected component *of S if it is the largest set containing A which is connected; in other words, given $B \subset S$ such that $A \subset B$, either $A = B$ or B is not connected.*

Given a connected component A of $V^c_{>0}(f)$ where f is a signomial, we classify A as a positive connected component *if $f(A) \subseteq \mathbb{R}_{>0}$ or a* negative connected component *if $f(A) \subseteq \mathbb{R}_{<0}$.*

The following example illustrates the above definitions.

Example 1.3.7. *Consider the signomial $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3 = \frac{1}{2}(x-1)(x-2)(x-3)$. The graph of this signomial is given in Figure 1.3, and a number line showing its positive and negative connected components is given in Figure 1.4.*

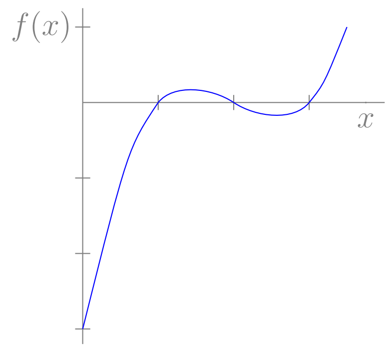

Figure 1.3: A graph of the signomial $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3 = \frac{1}{2}(x-1)(x-2)(x-3)$.

Figure 1.4: A number line representing the positive and negative connected components of the signomial $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3 = \frac{1}{2}(x-1)(x-2)(x-3)$. The positive connected components, $(1, 2)$ and $(3, \infty)$, are blue, while the negative connected components, $(0, 1)$ and $(2, 3)$, are red and bold.

Below are the results we will focus on. The first is not explicitly stated in [1], but is straightforward to prove. The second result is **Theorem 3.4** in [1], and will be proven in Section 3.

Theorem 1.3.8. *If a signomial f has no negative coefficients, then $V^c_{>0}(f)$ has no negative connected components.*

Theorem 1.3.9. *If a signomial f has exactly one negative coefficient, then $V^c_{>0}(f)$ has at most one negative connected component.*

Proof of Theorem 1.3.8. Let $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$ be a signomial with no negative coefficients, so $c_\mu > 0$ for all μ. Then take any $x \in \mathbb{R}^n_{>0}$. It is straightforward to observe that $c_\mu x^\mu > 0$ for each μ, so the sum $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu > 0$ as well. Thus $f(x)$ is positive for every $x \in \mathbb{R}^n_{>0}$, so $V^c_{>0}$ has one connected component, and it is positive. $\square$

Example 1.3.10. *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a signomial $\mathbb{R}_{>0} \to \mathbb{R}$ with all positive coefficients, $a_n > 0$ and $a_i \geq 0$ for $0 \leq i < n$. Then the sign sequence is $(+ \cdots +)$, which has no* *sign changes, so Descartes' rule of signs tells us there are no positive roots, $\rho = 0$. By considering the end behavior of this polynomial, we see it must take positive values everywhere. An example of this can be seen in Figure 1.5.*

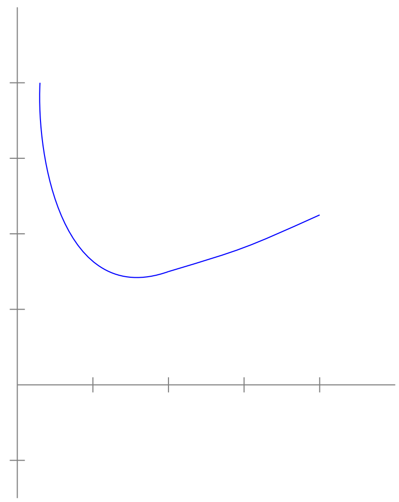

Figure 1.5: The graph of a signomial with only positive coefficients, $f(x) = x^{-1} + \frac{1}{2}x$.

2. A NEW PERSPECTIVE ON DESCARTES' RULE OF SIGNS

As we begin our attempt to extend Descartes' rule of signs to signomials in more than one variable, two clear difficulties arise. First, as soon as more than one variable is introduced, the task of ordering terms based on their exponents becomes ambiguous. What is the correct way, for example, to order the terms of the signomial $x_1^2 - x_1x_2 + x_1^2x_2 - x_2 + x_2^2$? Additionally, signomials in more than one variable typically have an uncountable number of solutions.

Because of these difficulties, our first step in generalizing Descartes' rule of signs is establishing a new perspective that can be generalized to higher dimensions. In [1], the authors accomplish this by developing a "dual" perspective on Descartes' rule of signs, focusing on the number of connected components of $V^c_{>0}(f)$. We call this perspective "dual" because it focuses on regions which are *not* solutions to the signomial, rather than regions which are. How does this new perspective relate to the old? Feliu and Telek wrote:

> If we write $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with $a_n \neq 0$, and let ρ be the Descartes' bound on the number of positive roots, then there are at most $\rho + 1$ connected components. If ρ is odd, the upper bounds for the number of components where f is positive or negative agree, while if ρ is even, then there are at most $\frac{\rho}{2} + 1$ connected components where f attains the sign of a_n [1, page 1].

To make the connection between the two perspectives more explicit, we present the following examples (in one variable).

First, suppose $f(x)$ has exactly one negative coefficient. There are two possibilities:

(i) The negative coefficient is first $(- + \cdots +)$ or last $(+ \cdots + -)$. Then there is one sign change, which by Descartes' Rule of Signs means there must be exactly one positive root, $\rho = 1$. Thus, there are at most $\rho + 1 = 2$ connected components, one negative and one positive. In fact, there will always be one of each. See Figure 2.1 for an example with the negative coefficient on the term with highest degree, and Figure 2.2 for an example with the negative coefficient on the term with lowest degree.

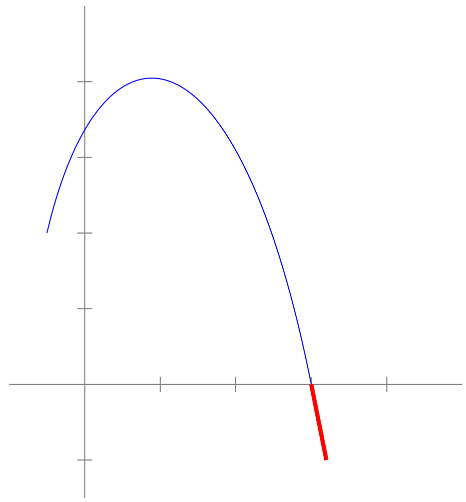

Figure 2.1: An example of a signomial with one negative coefficient, $f(x) = -x^2 + 2x + 3$. The negative connected component is highlighted in red/bold.

(ii) The negative coefficient occurs in the middle of the polynomial, as $(+ \cdots + - + \cdots +)$. In this case, Descartes' Rule of Signs tells us there are possibilities for either zero or two positive roots (remembering to count multiplicity). Taking the maximum, we get $\rho = 2$, so at most we get $\rho + 1 = 3$ connected components, two positive and one negative, because the leading coefficient is positive. This creates three distinct possibilities, which can be seen in Figures 2.3, 2.4, and 2.5, respectively.

For an example with more than one negative coefficient, recall Figure 1.3, the graph of the signomial $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3$. Here there are three sign changes, so Descartes' rule of signs gives $\rho = 3$. Then there are at most $\rho + 1 = 4$ connected components; two positive and two negative. This is an example where the maximum number of connected components is achieved.

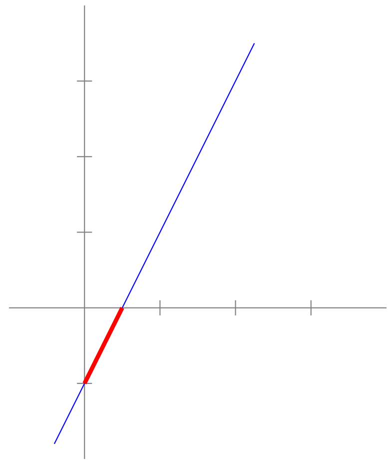

Figure 2.2: An example of a signomial with one negative coefficient, $f(x) = 2x - 1$. The negative connected component is highlighted in red/bold.

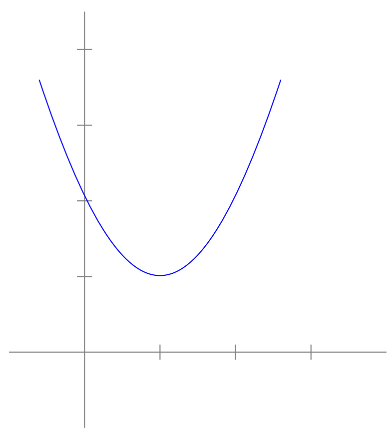

Figure 2.3: An example of a signomial with one negative coefficient, one positive connected component, and no negative connected components, $f(x) = x^2 - 2x + 2$.

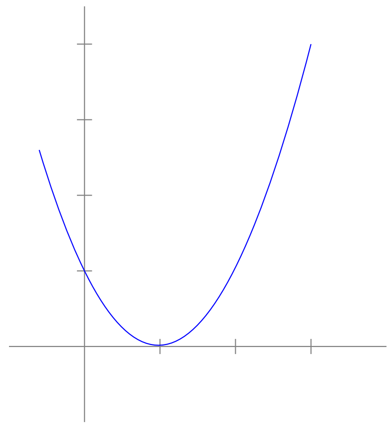

Figure 2.4: An example of a signomial with one negative coefficient, two positive connected components, and no negative connected components, $f(x) = x^2 - 2x + 1$.

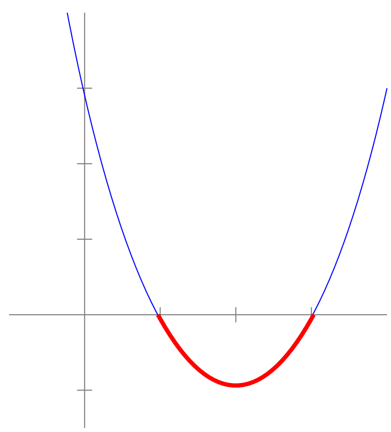

Figure 2.5: An example of a signomial with one negative coefficient, two positive connected components, and one negative connected component, $f(x) = x^2 - 4x + 3$. The negative connected component is highlighted in red/bold.

3. PROOF OF MAIN RESULT

We now present a few more definitions and lemmas from [1] which we will use to prove the main result.

3.1 Definitions and Lemmas

Recall the notation $x^\mu = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, and consider a signomial $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$. Then we create a partition of the support $\sigma(f)$ based on the signs of the coefficients c_μ, as follows:

$$\sigma_+(f) = \{\mu \in \sigma(f) \mid c_\mu > 0\} \text{ and } \sigma_-(f) = \{\mu \in \sigma(f) \mid c_\mu < 0\}.$$

Definition 3.1.1. *Given $v \in \mathbb{R}^n \backslash \{0\}$ and $a \in \mathbb{R}$, we define the* positive (negative) half-spaces *of* $\mathbb{R}^n$*:*

$$\mathcal{H}_{v,a}^+ = \{\mu \in \mathbb{R}^n \mid v \cdot \mu \geq a\} \textit{ and } \mathcal{H}_{v,a}^- = \{\mu \in \mathbb{R}^n \mid v \cdot \mu \leq a\}.$$

Similarly, we define the interiors *of these half-spaces as:*

$$\mathcal{H}_{v,a}^{+,\circ} = \{\mu \in \mathbb{R}^n \mid v \cdot \mu > a\} \textit{ and } \mathcal{H}_{v,a}^{-,\circ} = \{\mu \in \mathbb{R}^n \mid v \cdot \mu < a\}.$$

Remark 3.1.2. *In [1, and personal communication], the authors allow for $v = 0$. They made this decision because it allows for a nice structure on the set of* separating vectors *[1, Definition 3.2(i)]. However, we have chosen to exclude $v = 0$ in this thesis because it does not result in half-spaces, but depending on the choice of a defines the empty set or the entirety of $\mathbb{R}^n$. Allowing $v = 0$ does not invalidate any of the following results, but often results in vacuous statements.*

Let us consider some examples of half-spaces:

Example 3.1.3. *Given $v = (1, 2)$ and $a = 0$, the corresponding half-spaces are illustrated in Figure 3.1.*

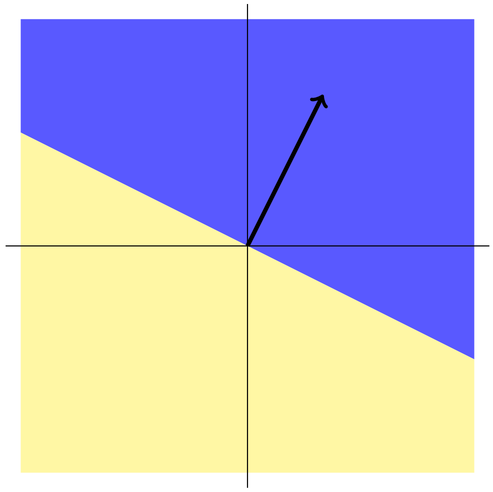

Figure 3.1: The half-spaces associated with $v = (1, 2)$ and $a = 0$, with $\mathcal{H}^+_{v,a}$ in blue (darker) and $\mathcal{H}^-_{v,a}$ in yellow (lighter)

Example 3.1.4. *For another example, consider $v = (1, 0)$ and $a = -1$, with positive and negative half-spaces illustrated in Figure 3.2*

Definition 3.1.5. *Let $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$. A vector $v \in \mathbb{R}^n$ is called an* enclosing vector *of f if there are $a, b \in \mathbb{R}$ with $a \leq b$ and $\sigma_-(f) \subseteq \mathcal{H}^+_{v,a} \cap \mathcal{H}^-_{v,b}$ and $\sigma_+(f) \subseteq \mathbb{R}^n \setminus \left(\mathcal{H}^{+,\circ}_{v,a} \cap \mathcal{H}^{-,\circ}_{v,b}\right)$.*

To better understood the above definition, consider the following example:

Example 3.1.6. *Consider the signomial*

$$f(x_1, x_2) = x_1^2 x_2 + 5x_1^{-2} x_2^{1.5} + 2x_2^{-2} - 2x_1^{.75} x_2^2 - 4x_2^{.3}$$

where the support has partition $\sigma_+ = \{(2, 1), (-2, 1.5), (0, -2)\}$ and $\sigma_- = \{(.75, 2), (0, .3)\}$. Then $v = (-2, 1)$ is an enclosing vector of f with $a = 0$ and $b = 1$, as demonstrated in Figure 3.3.

Given a signomial $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, our next step is to induce a signomial in one variable so we can apply Descartes' rule of signs. For any $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n_{>0}$, consider the *induced*

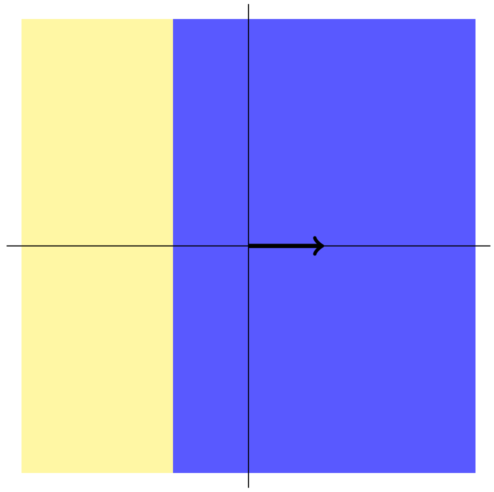

Figure 3.2: The half-spaces associated with $v = (1, 0)$ and $a = -1$, with $\mathcal{H}^+_{v,a}$ in blue (darker) and $\mathcal{H}^-_{v,a}$ in yellow (lighter)

signomial function in one variable [1, Equation (8)]:

$$f_{v,x} : \mathbb{R}_{>0} \to \mathbb{R} \qquad t \mapsto \sum_{\mu \in \sigma(f)} c_\mu x^\mu t^{v \cdot \mu} \tag{3.1}$$

Remark 3.1.7. *Note that $f_{v,x}(1) = f(x)$. In words, the value of the induced signomial function in one variable at $t = 1$ equals the value of the original signomial at x.*

We present the following lemmas which build up to our proof of the main theorem.

Lemma 3.1.8. *[1, Lemma 3.1] Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be a signomial in one variable such that $g(1) < 0$. If the sign sequence of the coefficients of g has at most two sign changes, and the leading coefficient is positive, then there is a unique $\rho \in (1, \infty)$ such that:*

(a) $g(\rho) = 0$,

(b) $g(t) < 0$ *for all* $t \in [1, \rho)$*, and*

(c) $g(t) > 0$ *for all* $t \in (\rho, \infty)$.

If the sign sequence of the coefficients of g has at most one sign change, and the leading coefficient is negative, then $g(t) < 0$ for all $t \in [1, \infty)$.

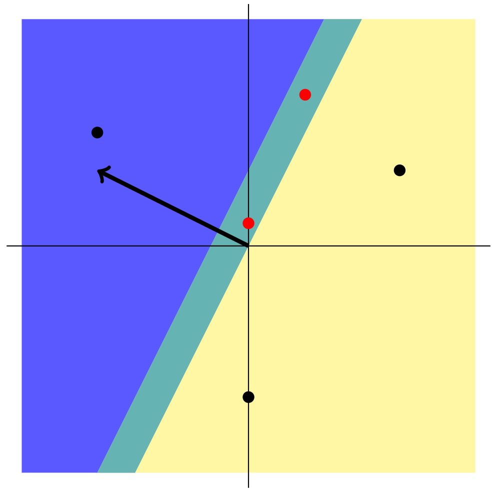

Figure 3.3: The enclosing vector $v = (-2, 1)$, with $a = 0$ and $b = 1$. The half-plane $\mathcal{H}^+_{v,a}$ is shaded blue (darker) and $\mathcal{H}^-_{v,b}$ is shaded yellow (lighter). Their intersection is shaded green. Points in σ_+ are black, while points in σ_- are red.

Proof. This proof is a direct application of Descartes' rule of signs.

Suppose the sign sequence of the coefficients of g has at most two sign changes and the leading coefficient is positive. Then Descartes' rule of signs tells us there are at most two positive roots. Additionally, because the leading coefficient is positive, $\lim_{t\to\infty} g(t) = \infty$. By assumption $g(1) < 0$, so the intermediate value theorem says there is some $\rho \in (1, \infty)$ where $g(\rho) = 0$. Then because there are at most two roots in the interval $(0, \infty)$, we conclude $g(t) < 0$ for all $t \in [1, \rho)$ and $g(t) > 0$ for all $t \in (\rho, \infty)$ as desired.

Now suppose the sign sequence of the coefficients of g has at most one sign change, and the leading coefficient is negative. Then Descartes' rule of signs tells us there is exactly one positive root. Additionally, because the leading coefficient is negative, $\lim_{t\to\infty} g(t) = -\infty$. By assumption $g(1) < 0$, so g cannot have any roots in $(1, \infty)$ (recall that the number from Descartes' rule of signs includes multiplicity of roots). Thus, $g(t) < 0$ for all $t \in [1, \infty)$. $\square$

Lemma 3.1.9. *[1, Lemma 3.3 (i')] Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. Suppose we have $x \in \mathbb{R}^n_{>0}$ such that $f(x) < 0$. If v is an enclosing vector of f, then there is a unique $\rho \in (1, \infty]$ such that $f_{v,x}(t) < 0$ for all $t \in [1, \rho)$ and $f_{v,x}(t) > 0$ for all $t > \rho$ (note that ρ might be ∞).*

Proof. Let $x \in \mathbb{R}^n_{>0}$, and suppose $f(x) < 0$. Additionally, suppose we have an enclosing vector v of f. Then the exponents $v \cdot \mu$ of the induced signomial in one variable, $f_{v,x}(t)$, are ordered such that the sign sequence is $(+ \cdots + - \cdots - + \cdots +)$, where either (or both) of the blocks of repeating $+$ signs might be absent. We now consider two cases. Recall from Remark 3.1.7 that $f_{v,x}(1) = f(x) < 0$.

(a) If the leading coefficient of $f_{v,x}$ is positive, then Lemma 3.1.8 (i), with $g(t) = f_{v,x}(t)$, tells us there exists a unique $\rho \in (1, \infty)$ such that $f_{v,x}(t) < 0$ for all $t \in [1, \rho)$ and $f_{v,x}(t) > 0$ for all $t > \rho$.

(b) If the leading coefficient of $f_{v,x}$ is negative, the sign sequence must be $(- \cdots - + \cdots +)$, with the positive signs potentially absent. Then $f_{v,x}$ has at most one sign change in its coefficients. Then Lemma 3.1.8 (ii), with $g(t) = f_{v,x}(t)$, tells us $f_{v,x}(t) < 0$ for all $t \in [1, \infty)$.

□

Definition 3.1.10. *[1, Equation (6)] Let* $\log(x)$ *represent the natural log of* x. *Then we define the* coordinate-wise natural logarithm map, *denoted* $\text{Log} : \mathbb{R}^n_{>0} \to \mathbb{R}^n$, *by*

$$\text{Log}(x_1, x_2, ..., x_n) = (\log(x_1), \log(x_2), ..., \log(x_n)).$$

3.2 Main Theorem

With the above definitions and lemmas, we are now ready to prove the main theorem, a restatement of Theorem 1.3.9.

Main Theorem. *[1, Theorem 3.4] Let* $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ *be a signomial. If* f *has exactly one negative coefficient, then the preimage of the negative real line*

$$f^{-1}(\mathbb{R}_{<0}) = \{x \in \mathbb{R}^n_{>0} \mid f(x) < 0\}$$

has at most one negative connected component.

Remark 3.2.1. *In [1], the authors prove the stronger result that the negative connected component is log-convex, meaning there is a line connecting* $\mathrm{Log}(x)$ *and* $\mathrm{Log}(y)$ *in the logarithmic space* $\mathrm{Log}(f^{-1}(\mathbb{R}_{<0}))$.

Proof. Suppose f has exactly one negative coefficient, $c_{\mu^\star}$. Here we use $\mu^\star$ to denote the exponent vector of the term with negative coefficient. If $f^{-1}(\mathbb{R}_{<0})$ is empty, then we are done. (For instance, see Examples 2.3 and 2.4.)

Now suppose $f^{-1}(\mathbb{R}_{<0})$ is nonempty; let $x, y \in f^{-1}(\mathbb{R}_{<0})$. To show connectedness, we need to find a path in $f^{-1}(\mathbb{R}_{<0})$ which connects x and y. Define $v = \mathrm{Log}(y) - \mathrm{Log}(x)$. We will show that the induced signomial in one variable, $f_{v,x}(t)$, gives our desired path on the interval $[1, e]$.

In Remark 3.1.7 we observed that $f_{v,x}(1) = f(x)$. The following calculation shows that $f_{v,x}(e) = f(y)$:

$$\begin{aligned} f_{v,x}(e) &= \sum_{\mu \in \sigma(f)} c_\mu x^\mu e^{v \cdot \mu} \\ &= \sum_{\mu \in \sigma(f)} c_\mu x^\mu e^{(\mathrm{Log}(y) - \mathrm{Log}(x)) \cdot \mu} \\ &= \sum_{\mu \in \sigma(f)} c_\mu x^\mu e^{\mathrm{Log}(y) \cdot \mu - \mathrm{Log}(x) \cdot \mu} \\ &= \sum_{\mu \in \sigma(f)} c_\mu x^\mu y^\mu x^{-\mu} \\ &= \sum_{\mu \in \sigma(f)} c_\mu y^\mu \\ &= f(y). \end{aligned}$$

Thus $f_{v,x}(t)$ is a path from x to y on the interval $[1, e]$, but it remains to show that this path stays in $f^{-1}(\mathbb{R}_{<0})$; in other words, we need to show that $f_{v,x}(t) < 0$ for all $t \in [1, e]$.

It is straightforward to check that v is an enclosing vector of f, with $a = b = \mu^\star \cdot v$. Then by Lemma 3.1.9, we know $f_{v,x}(t) < 0$ for all $t \in [1, \rho)$ for some $\rho \in (1, \infty]$, and if $\rho < \infty$, then $f_{v,x}(t) > 0$ for all $t > \rho$.

Observe $f_{v,x}(e) = f(y) < 0$ by assumption that $y \in f^{-1}(\mathbb{R}_{<0})$; thus $e < \rho$. So $f_{v,x}(t) < 0$ on the interval $[1, e] \subset [1, \rho)$, and $f_{v,x}(t)$ defines a path connecting x and y in $f^{-1}(\mathbb{R}_{<0})$. Thus, $f^{-1}(\mathbb{R}_{<0})$ is connected. $\square$

4. CONCLUSION AND OBSERVATIONS ABOUT FURTHER STUDY

4.1 Conclusion

We have proven one result about a new perspective used for extending Descartes' rule of signs to signomials, or generalized polynomials in multiple variables. In [2], Feliu and Telek use this and other results from [1] to develop an algorithm for determining whether the parameter region of a reaction network which allows for multistationarity is connected. This is just one example of a situation in which this new perspective may be useful.

For the remainder of this thesis, we consider some questions that arise while studying the negative connected components of signomials, with examples. These questions may provide interesting directions for future research.

4.2 Boundedness

The first question we consider is whether the negative connected components are bounded. In [1], every example of a negative connected component is unbounded. Is this always the case? Or is it possible for a negative connected component to be bounded?

Definition 4.2.1. *We say a set $S \subset \mathbb{R}^n$ is* bounded*, or* bounded away from infinity*, if we can find $M \in \mathbb{R}$ such that $|s| < M$ for all $s \in S$.*

For $n = 1$, the answer depends only on the sign of the leading coefficient. If the leading coefficient is positive, any negative connected components will be bounded. However, if the leading coefficient is negative, all but one of the negative connected components will be bounded, and the last will be unbounded. This can be realized by considering the end behavior of the polynomial.

For $n > 1$, the regions are not as easy to classify. However, even restricting ourselves to signomials with one negative coefficient we can find examples where the negative connected component is bounded and other examples where the negative connected component is unbounded.

Example 4.2.2. *Consider the signomial $f(x_1, x_2) = x_1^2 - 4x_1 + x_2$. The set of points $(x_1, x_2) \in \mathbb{R}^2$*

for which $f(x_1, x_2) < 0$ is connected. We calculate this region as follows:

$$x_1^2 - 4x_1 + x_2 < 0$$
$$-x_1^2 + 4x_1 < x_2$$
$$-x_1(x_1 - 4) < x_2$$

This is the region between the downward-facing parabola with roots at $x_1 = 0, 4$ (see Figure 4.1) and the x-axis, which is bounded.

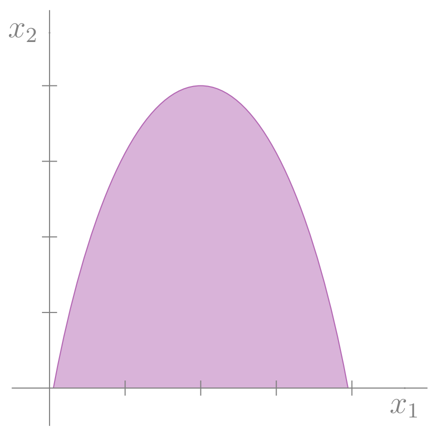

Figure 4.1: The violet shaded region is the bounded negative connected component of the support of the signomial $f(x_1, x_2) = x_1^2 - 4x_1 + x_2$.

Example 4.2.3. *Consider the signomial $f(x_1, x_2) = x_1 - 3x_2^{-1}$. The support of this signomial has one negative connected component, the region where $x_1 - 3x_2^{-1} < 0$. We calculate this region as* *follows:*

$$\begin{aligned} x_1 - 3x_2^{-1} &< 0 \\ x_1 &< 3x_2^{-1} \\ x_1^{-1} &> 3x_2 \\ (3x_1)^{-1} &> x_2 \end{aligned}$$

This is the region above the curve $x_2 = \frac{1}{3x_1}$ *(see Figure 4.2), which is unbounded.*

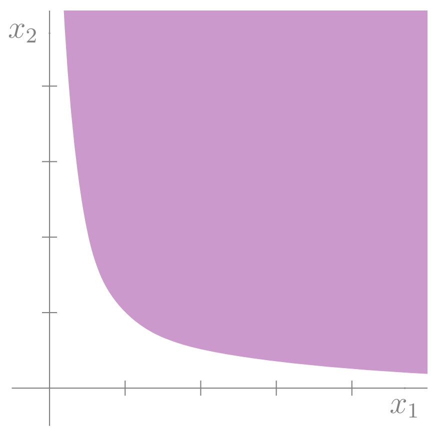

Figure 4.2: The violet shaded region is the unbounded negative connected component of the support of the signomial $f(x_1, x_2) = x_1 - x_2^{-1}$.

Because the domain we consider for signomials is $\mathbb{R}^n_{>0}$, it is also beneficial to consider the following stronger definition:

Definition 4.2.4. *We say a set* $S \subset \mathbb{R}^n_{>0}$ *is* bounded *if we can bound each component away from the axes as well as away from infinity. More precisely, a set is bounded away from the axes if there is* $\varepsilon > 0$ *such that for every* $x = (x_1, \ldots, x_n) \in S$*,* $x_i > \varepsilon$ *for all* $1 \leq i \leq n$*.*

There are many examples of signomials which satisfy this stronger property, including the following:

Example 4.2.5. *Consider the signomial* $f(x_1, x_2) = 1 + x_1^2 x_2 + x_1 x_2^2 - 4x_1 x_2$. *Then* $V^c_{>0}(f)$ *has one negative connected component, which is bounded away from infinity* and *away from all axes (see Figure 4.3).*

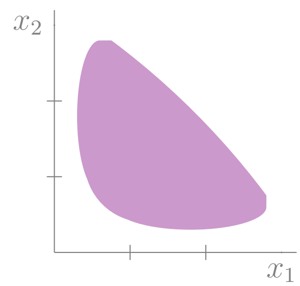

Figure 4.3: The violet shaded region is the negative connected component of the support of the signomial $f(x_1, x_2) = 1 + x_1^2 x_2 + x_1 x_2^2 - 4x_1 x_2$, which is bounded away from infinity and all axes.

Future research might be conducted to determine what is possible when we allow for more than one negative coefficient: for example, can we construct families of examples with any given combination of bounded and unbounded connected components?

4.3 Existence of enclosing vectors

In the proof of Theorem 3.2, one step uses that the existence of an enclosing vector ensures an induced signomial function in one variable with at most two sign changes. This leads us to wonder how essential the existence of an enclosing vector is to determining the number of negative connected components. However, when the number of negative coefficients of a signomial is greater than one, we can find examples where the existence of an enclosing vector is neither necessary nor sufficient to ensure at most one negative connected component.

In [1, Theorem 3.6] they prove that, under the stronger hypothesis of the existence of a *strict separating vector* [1, Definition 3.2(ii)], we are guaranteed at most one negative connected component. They also prove that the existence of a *strict enclosing vector* [1, Definition 3.2(i)] is sufficient to guarantee at most two negative connected components [1, Theorem 3.8].

Example 4.3.1. *Consider the signomial* $f(x_1, x_2) = x_1x_2 - x_1 - x_2 + 2$*. Although an enclosing vector exists for* $\sigma_+ = \{(1,1),(0,0)\}$ *and* $\sigma_- = \{(1,0),(0,1)\}$*, the support of this signomial has two negative connected components.*

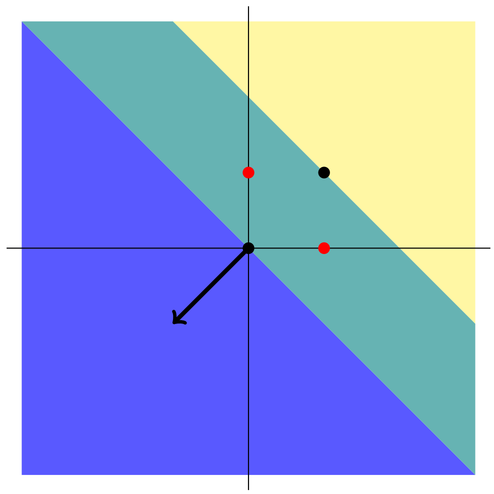

Figure 4.4: An example of an enclosing vector, $\langle 0.5, 0.5\rangle$, with $a = -2$ and $b = 0$, for the sets $\sigma_+ = \{(1,1),(0,0)\}$ and $\sigma_- = \{(1,0),(0,1)\}$.

It is also possible to find examples of signomials with one negative connected component but no enclosing vector, starting in one dimension. Consider the following example.

Example 4.3.2. *Recall* $p(x) = 3x^7 - 4x^6 - x^4 + 2x - 1$ *from Example 1.2.2. There is no enclosing vector for this signomial, as can be seen by considering the graph of the support (Figure 4.6 below). This signomial has exactly one negative connected component, as can be seen by referring to the graph in Figure 1.1.*

Descartes' rule of signs only provides an upper bound on the number of positive roots. This leads us to consider a final question: will any bounds we find on the number of negative connected components of a signomial be upper bounds only?

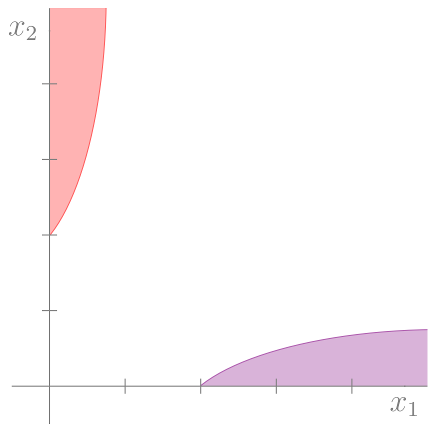

Figure 4.5: The red and violet shaded regions are the two negative connected components of the support of the signomial $f(x_1, x_2) = x_1x_2 - x_1 - x_2 + 2$.

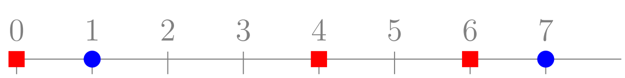

Figure 4.6: The support of $p(x) = 3x^7 - 4x^6 - x^4 + 2x - 1$ with positive elements marked with blue circles and negative elements marked with red squares.

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