ASYMPTOTICS OF THE RELATIVE RESHETIKHIN-TURAEV INVARIANTS

A Dissertation

by

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ABSTRACT

In this dissertation, we study the asymptotic expansion conjecture of the relative Reshetikhin-Turaev invariants proposed by T. Yang and the author in [65] for all pairs (M, L) satisfying the property that $M \setminus L$ is homeomorphic to some fundamental shadow link complement. The hyperbolic cone structure of such (M, L) can be described by using the logarithmic holonomies of the meridians of the fundamental shadow link. We show that when the logarithmic holonomies are sufficiently small and all cone angles are less than π, the asymptotic expansion conjecture of (M, L) is true. Especially, we verify the asymptotic expansion conjecture of the relative Reshetikhin-Turaev invariants for all pairs (M, L) satisfying the property that $M \setminus L$ is homeomorphic to some fundamental shadow link complement, with cone angles sufficiently small. Furthermore, we show that if M is obtained by doing rational surgery on a fundamental shadow link complement with sufficiently large surgery coefficients, then the cone angles can be pushed to any value less than π.

DEDICATION

To my mother, father, brother and best friends :)

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1. INTRODUCTION

Quantum topology and hyperbolic geometry provide two promising approaches to understand 3-dimensional topology. In quantum topology, by using the representation theory of quantum groups [47, 48, 53], one can define quantum invariants of links and 3-manifolds, including the famous Jones polynomial and its generalizations. In hyperbolic geometry, by putting a hyperbolic structure on a manifold [52, 45], one can define geometric invariants, including the hyperbolic volume, the Chern-Simons invariant and the adjoint twisted Reidemeister torsion among many others. These two very different approaches of studying 3-dimensional topology turn out to be related by *Volume Conjectures*, which are a set of conjectures that relate the asymptotics of quantum invariants to hyperbolic geometry.

1.1 Overview of volume conjectures

1.1.1 Kashaev-Murakami-Murakami volume conjecture

For a hyperbolic link L in $\mathbb{S}^3$ (i.e. the link complement $\mathbb{S}^3 \setminus L$ admits a complete hyperbolic structure), the Kashaev-Murakami-Murakami volume conjecture [25, 26, 34] and its generalization [35] of the colored Jones polynomials suggests that the exponential growth rate of the N-th (normalized) colored Jones polynomials of L evaluated at the root of unity $t = e^{\frac{2\pi i}{N}}$ captures the hyperbolic volume of $\mathbb{S}^3 \setminus L$.

Conjecture 1.1. *([25, 26, 34, 35]) Let L be a hyperbolic link in $\mathbb{S}^3$. For $N \in \mathbb{N}$, let $\mathrm{J}_N(L,t)$ be the N-th (normalized) colored Jones polynomial of L. Then*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left|\mathrm{J}_N\big(L, t = e^{\frac{2\pi i}{N}}\big)\right| = \mathrm{Vol}(\mathbb{S}^3 \setminus L),$$

where $\mathrm{Vol}(M)$ is the hyperbolic volume of $\mathbb{S}^3 \setminus L$.

The conjecture is proved for several knots and links [1, 20, 27, 40, 42, 41, 57, 68] and it is also generalized to the fundamental shadow links in connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$ [10] and knotted graphs in $\mathbb{S}^3$ [12, 58], providing surprising connection between quantum invariants and the geometry of hyperbolic polyhedra.

1.1.2 Chen-Yang volume conjecture

Besides, for a closed, oriented hyperbolic 3-manifold M with finite volume, the Chen-Yang volume conjecture [8] of the Reshetikhin-Turaev invariant of at $q = e^{\frac{2\pi i}{r}}$, where $r \geq 3$ odd, suggests that the exponential growth rate of the invariant also captures the hyperbolic volume and the Chern-Simons invariants of M.

Conjecture 1.2. *([8]) Let M be a closed oriented hyperbolic 3-manifold. For an odd integer $r \geq 3$, let $\mathrm{RT}_r(M)$ be the r-th relative Reshetikhin-Turaev invariant of M evaluated at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$. Then as r varies over all positive odd integers,*

$$\lim_{r\to\infty} \frac{4\pi}{r} \log \mathrm{RT}_r(M) = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M),$$

where $\mathrm{Vol}(M)$ and $\mathrm{CS}(M)$ are the hyperbolic volume and the Chern-Simons invariant of M respectively.

Conjecture 1.2 is proved for every closed, oriented hyperbolic 3-manifold obtained by doing an integral Dehn surgery on the figure eight knot complement [43]. More recently, in [61], Conjecture 1.2 is proved for every closed, oriented hyperbolic 3-manifold obtained by doing a rational Dehn surgery on the figure eight knot complement.

1.1.3 Volume conjecture of the relative Reshetikhin-Turaev invariants

In [62], joint with T. Yang, we proposed the volume conjecture for the relative Reshetikhin-Turaev invariants of a pair (M, L), where M is a closed oriented 3-manifold and L is a framed link inside M. The conjecture suggests that the asymptotics of the invariants capture the hyperbolic volume and the Chern-Simons invariant of the cone manifold M with the singular locus L and cone angles θ determined by the sequence of colorings of the framed link.

Conjecture 1.3. *([62, Conjecture 1.1]) Let M be a closed oriented 3-manifold and let L be a framed hyperbolic link in M with n components. For an odd integer,* $r \geq 3$*, let* $\mathbf{m} = (\mathrm{m}_1, \ldots, \mathrm{m}_n)$ *and let* $\mathrm{RT}_r(M, L, \mathbf{m})$ *be the* r*-th relative Reshetikhin-Turaev invariant of M with L colored by* $\mathbf{m}$ *and evaluated at the root of unity* $q = e^{\frac{2\pi\sqrt{-1}}{r}}$*. For a sequence* $\mathbf{m}^{(r)} = (\mathrm{m}_1^{(r)}, \ldots, \mathrm{m}_n^{(r)})$*, let*

$$\theta_k = \left| 2\pi - \lim_{r\to\infty} \frac{4\pi m_k^{(r)}}{r} \right|$$

and let $\boldsymbol{\theta} = (\theta_1, .., \theta_n)$*. If* $M_{L_{\boldsymbol{\theta}}}$ *is a hyperbolic cone manifold consisting of M and a hyperbolic cone metric on M with singular locus L and cone angles* $\boldsymbol{\theta}$*, then*

$$\lim_{r\to\infty} \frac{4\pi}{r} \log \mathrm{RT}_r(M, L, \mathbf{m}^{(r)}) = \mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) + \sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}})$$

where r varies over all positive odd integers.

Conjecture 1.3 is related to Conjectures 1.1 and 1.2 as follows. For $r = 2N + 1$, when $M = \mathbb{S}^3$, L is a framed link inside M and $\mathbf{m}^{(r)} = \mathbf{N} = (N, \ldots, N)$, the relative Reshetikhin-Turaev invariant of the pair $(\mathbb{S}^3, L)$ is, up to some factor, equal to the $\mathbf{N}$-th colored Jones polynomial of L evaluated at the root of unity $t = e^{\frac{2\pi i}{N+1/2}}$. Moreover, the cone angles θ_k's are all equal to zero and this corresponds to the complete hyperbolic structure of $M \setminus L$. Besides, when $\mathbf{m}^{(r)} = (0, \ldots, 0)$, the relative Reshetikhin-Turaev invariant recovers the Reshetikhin-Turaev invariant of the ambient closed oriented 3-manifold and the cones angles θ_k's are all equal to 2π. In particular, L is no longer a singularity and the manifold M admits a complete hyperbolic structure. Thus, the relative Reshetikhin-Turaev invariant can be regarded as a generalization of the colored Jones polynomials of links at roots of unity and the Reshetikhin-Turaev invariants of 3-manifolds. In this sense, Conjecture 1.3 can be understood as an interpolation between the Kashaev-Murakami-Murakami volume conjecture and the Chen-Yang volume conjecture. In particular, this provides a new approach of studying the Chen-Yang volume conjecture by deforming the cone angles from 0 to 2π.

In [62], joint with T. Yang, we study Conjecture 1.3 for all pairs (M, L) obtained by doing a change-of-pair operation from the pair (M_c, L_{FSL}), where $M_c = \#^{c+1}(\mathbb{S}^2 \times \mathbb{S}^1)$ for some $c \in \mathbb{N}$ and L_{FSL} is a fundamental shadow link inside M_c. Here we recall from [62, Proposition 1.3 and 1.4] that the change-of-pair operation is a topological move that changes a pair (M, L) to another pair (M^*, L^*) without changing the complement, i.e. $M \setminus L \simeq M^* \setminus L^*$. In particular, M^* can be obtained by doing *integral* Dehn fillings on the boundary components of $M \setminus L$. Moreover, if two pairs (M, L) and (M^*, L^*) share the same complement, i.e. $M \setminus L \simeq M^* \setminus L^*$, then they are related by a sequence of change-of-pair operations. In this case, M^* can be obtained by doing *rational* Dehn fillings on the boundary components of $M \setminus L$. In [62], Conjecture 1.3 has been proved for all pairs (M, L) obtained by doing a change-of-pair operation from the pair (M_c, L_{FSL}), where $M_c = \#^{c+1}(\mathbb{S}^2 \times \mathbb{S}^1)$ for some $c \in \mathbb{N}$ and L_{FSL} is a fundamental shadow link inside M_c, with sufficiently small cone angles. Especially, since every closed, oriented 3-manifolds can be obtained by doing integral Dehn-fillings on the boundary of some fundamental shadow link complement [13], if the cone angle can be pushed from sufficiently close to 0 all the way to 2π, then one can prove the Chen-Yang volume conjecture. Besides, to show that it is possible to push the cone angle, in [63], joint with T. Yang, we proved Conjecture 1.3 for all pairs (M, L) with $M \setminus L$ homeomorphic to the figure eight knot complement in $\mathbb{S}^3$, for all cone angle from 0 to 2π, except finitely many cases corresponding to the exceptional surgery of the figure eight knot.

1.1.4 Asymptotic expansion conjecture of the relative Reshetikhin-Turaev invariants

Furthermore, in [65], joint with T. Yang, we refined Conjectue 1.3 by studying the asymptotic expansion formula of the relative Reshetikhin-Turaev invariants. Let M be a closed oriented 3-manifold and let L be a framed hyperbolic link in M with n components. Let $\{\mathbf{m}^{(r)}\} = \{(\mathrm{m}_1^{(r)}, \dots, \mathrm{m}_n^{(r)})\}$ be a sequence of colorings of the components of L by the elements of $\{0, \dots, r-2\}$ such that for each $k \in \{1, \dots, n\}$, either $\mathrm{m}_k^{(r)} > \frac{r}{2}$ for all r sufficiently large or $\mathrm{m}_k^{(r)} < \frac{r}{2}$ for all r sufficiently large. In the former case we let $\mu_k = 1$ and in the latter case we let $\mu_k = -1$, and we let

$$\theta_k^{(r)} = \mu_k \left(\frac{4\pi \mathrm{m}_k^{(r)}}{r} - 2\pi \right).$$

Let $\boldsymbol{\theta}^{(r)} = (\theta_1^{(r)}, \dots, \theta_n^{(r)})$. Suppose for all r sufficiently large, a hyperbolic cone metric on M with singular locus L and cone angles $\boldsymbol{\theta}^{(r)}$ exists. We denote M with such a hyperbolic cone metric by $M^{(r)}$, let $\mathrm{Vol}(M^{(r)})$ and $\mathrm{CS}(M^{(r)})$ respectively be the volume and the Chern-Simons invariant of $M^{(r)}$, and let $\mathrm{H}^{(r)}(\gamma_1), \dots, \mathrm{H}^{(r)}(\gamma_n)$ be the logarithmic holonomies in $M^{(r)}$ of the parallel copies $(\gamma_1, \dots, \gamma_n)$ of the core curves of L given by the framing. Let $\rho_{M^{(r)}} : \pi_1(M \smallsetminus L) \to \mathrm{PSL}(2;\mathbb{C})$ be the holonomy representation of the restriction of $M^{(r)}$ to $M \smallsetminus L$, and let $\mathbb{T}_{(M\smallsetminus L, \boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])$ be the Reidemeister torsion of $M \smallsetminus L$ twisted by the adjoint action of $\rho_{M^{(r)}}$ with respect to the system of meridians $\boldsymbol{\Upsilon}$ of a tubular neighborhood of the core curves of L (see Section 2.5 for more details).

Conjecture 1.4. *([65, Conjecture 1.1]) Suppose $\{\theta^{(r)}\}$ converges as r tends to infinity. Then as r varies over all positive odd integers and at $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, the relative Reshetikhin-Turaev invariants*

$$\mathrm{RT}_r(M, L, \mathbf{m}^{(r)}) = C \frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm \mathbb{T}_{(M\smallsetminus L, \boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)}) + \sqrt{-1}\mathrm{CS}(M^{(r)})\right)} \left(1 + O\Big(\frac{1}{r}\Big)\right),$$

where C is a quantity of norm 1 independent of the geometric structure on M.

In [65], Conjecture 1.4 has been proved for all pairs (M, L) obtained by doing a change-of-pair operation from the pair (M_c, L_{FSL}) with sufficiently small cone angles. Similar to the relationship between Conjecture 1.3 and the Chen-Yang volume conjecture, Conjecture 1.4 also provides a new approach to understand the asymptotic expansion of the Reshetikhin-Turaev invariants for closed, oriented 3-manifolds discussed in [18] and [44].

1.2 Methodology

In this dissertation, we combine the ideas in [61], [62], [63] and [65] to study the asymptotic expansion conjecture for the relative Reshetikhin-Turaev invariants of any pair (M, L) with $M \smallsetminus L$ homeomorphic to some fundamental shadow link complement $M_c \smallsetminus L_{\mathrm{FSL}}$. Roughly speaking, our method involves the following 5 steps.

1. Write the invariants as a (multi-)sum of a holomorphic function evaluated at integral points.

2. Apply the Poisson Summation Formula to write the invariants as a sum of the Fourier coef-

ficients together with some error terms, where each Fourier coefficient is of the form

$$\iint_D g(z_1,\dots,z_n)e^{rf(z_1,\dots,z_n)}dz_1\dots dz_n,$$

for some $n\in\mathbb{N}$, $D\subset\mathbb{C}^n$ and holomorphic functions $f,g:\mathbb{C}^n\to\mathbb{C}$. The function f is called the *potential function* of the Fourier coefficient.

3. Obtain the asymptotics of the leading Fourier coefficient by applying the saddle point approximation, which says that under certain technical assumptions, the asymptotics of the integral is determined by certain critical value of the function $f(z_1,\dots,z_n)$ as follows.

$$\iint_D g(z_1,\dots,z_n)e^{rf(z_1,\dots,z_n)}dz_1\dots dz_n=\Big(\frac{2\pi}{r}\Big)^{\frac{n}{2}}\frac{g(\mathbf{z})}{\sqrt{-\det\mathrm{Hess}(f(\mathbf{z}))}}e^{rf(\mathbf{z})}\Big(1+O\Big(\frac{1}{r}\Big)\Big),$$

   where $\mathbf{z}$ is certain critical point and $\mathrm{Hess}(f(\mathbf{z}))$ is the Hessian matrix of f evaluated at $\mathbf{z}$.

4. Relate the critical value and the determinant of the Hessian matrix of f with geometric quantities, including the hyperbolic volume, the Chern-Simons invariant and the adjoint twisted Reidemeister torsion of the related manifold.

5. Show that the other Fourier coefficients and error terms are negligible.

The idea of combining the Poisson Summation formula and the saddle point approximation to prove volume conjectures was used by T. Ohtsuki and his collaborators in [40, 41, 42, 43] and they have obtained promising results for relatively simple knots and manifolds. However, for general cases, the technical arguments in analysis and the highly non-trivial connection with hyperbolic geometry remain the main obstacles of applying the above strategy to study the asymptotics of quantum invariants.

The main goal of this dissertation is to study the asymptotic expansion conjecture for the relative Reshetikhin-Turaev invariants of any pair (M,L) with $M\setminus L$ homeomorphic to some $M_c\setminus L_{\mathrm{FSL}}$. The main contribution is to overcome the problems mentioned above by revealing the geometry of the potential function and connecting the technical argument in analysis with the hyperbolic geometry of the related 3-manifolds.

1.3 Main results

Let $\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)$ be the logarithmic holonomies of the meridians of $L_{\text{FSL}} \subset M_c$. For any pair (M, L) with $M \setminus L$ homeomorphic to $M_c \setminus L_{\text{FSL}}$, near the complete structure, the hyperbolic cone structure of (M, L) can be described by using the parameters $\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)$.

Theorem 1.5. *Given a fundamental shadow link $L_{FSL} \subset M_c$ with n components. There exists $\delta > 0$ (depending on L_{FSL}) such that if (M, L) is a pair with $M \setminus L$ homeomorphic to $M_c \setminus L_{FSL}$ and with a hyperbolic cone structure satisfying $|\mathrm{H}(u_k)| < \delta$ and $\theta_k \in [0, \pi)$ for all $k = 1, \dots, n$, then Conjecture 1.4 is true for (M, L).*

As a special case of Theorem 1.5,

Theorem 1.6. *Given a fundamental shadow link $L_{FSL} \subset M_c$, if (M, L) is a pair with $M \setminus L$ homeomorphic to $M_c \setminus L_{FSL}$, then there exists $\epsilon > 0$ (depending on L_{FSL}) such that Conjecture 1.4 is true for (M, L) for any cone angles $\boldsymbol{\theta} \in [0, \epsilon)^n$.*

Note that M in Theorem 1.6 covers all 3-manifolds M obtained by doing surgery on some fundamental shadow link complement. It is expected that in Theorem 1.6. when M is a closed, oriented hyperbolic 3-manifold, the cone angles can be pushed to 2π so that the Chen-Yang volume conjecture for the Reshetikhin-Turaev invariants of M is true. In this paper, we restrict our attention to the case where M is a hyperbolic 3-manifolds obtained by doing rational surgery on some fundamental shadow link complement with sufficiently large surgery coefficients and all the cone angles are less than π.

Theorem 1.7. *Given a fundamental shadow link $L_{FSL} \subset M_c$ with n components, there exists a constant $C > 0$ (depending on L_{FSL}) such that if*

- *$M \setminus L$ is homeomorphic to $M_c \setminus L_{FSL}$; and*

- *M is obtained by doing a $\{(p_k, q_k)\}_{k=1}^n$ surgery on the boundaries of $M_c \setminus L_{FSL}$ with*

$$|p_k| + |q_k| > C$$

for all $k = 1, \ldots, n$,

then Conjecture 1.4 is true for (M, L) for any cone angles $\boldsymbol{\theta} \in [0, \pi)^n$.

The following result follows immediately from Theorem 1.6 and 1.7.

Theorem 1.8. *Conjecture 1.3 is true for all the pair (M, L) with $M \setminus L$ homeomorphic to some fundamental shadow link complement, with small cone angles.*

Theorem 1.9. *Conjecture 1.3 is true for all the pair (M, L) described in Theorem 1.7, with all cone angles less than π.*

Plan of this paper

In Section 2, we give a brief review for the preliminary knowledge required for the proof of Theorem 1.6. The materials in this section can be found in [61, 62, 64, 65]. In Section 3, we compute the relative Reshetikhin-Turaev invariants of (M, L) and express it as a sum of the evaluation of certain holomorphic function at some integral points (Proposition 3.4). An important step is to use a Gauss sum formula (Lemma 3.3) to simplify the relative Reshetikhin-Turaev invariants. In Section 4, we apply the Poisson summation formula to write the invariants as the sum of the Fourier coefficients together with some error terms (Proposition 4.1 and 4.2). We also gives a simplified expression for the leading Fourier coefficients (Proposition 4.3). In Section 5, we apply the saddle point approximation (Proposition 5.1) to study the asymptotic expansions of those Fourier coefficients. To do that, in Proposition 5.11, we show that certain critical values of the function in this Fourier coefficients give the complex volume of the cone manifold. The key observation is that the critical point equations of the function involved coincide with the cone angle equations of the cone manifold M with singular locus L. Moreover, in Proposition 5.12, we verify that under certain technical assumptions, all the conditions required for applying the saddle point method are satisfied. In Proposition 5.13, we show that the twisted Reideimester torsion appears in the asymptotics of the leading Fourier coefficient. The main idea is to apply the relationship between the torsion and the Gram matrix function studied in [64] and [65]. In Proposition 5.18, we obtain the asymptotic expansions for the leading Fourier coefficients, which capture the complex volume and the twisted Reidemeister torsion of the manifold with the cone structure determined by the sequence of colorings. Finally, in Proposition 5.19, 5.22, 5.23 and 5.24, we show under certain technical assumption, the sum of all the other Fourier coefficients and the error term in Proposition 4.2 are negligible. In Lemma 5.25, 5.26 and 5.27, we show respectively that in the contexts of Theorem 1.5, 1.6 and 1.7, all the technical assumptions mentioned above are satisfied. This completes the proof of the main theorems.

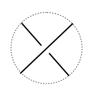
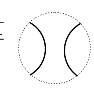
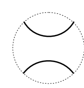
2. PRELIMINARIES*

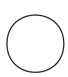
The materials in this section are from [61, 62, 64, 65]. We include the materials here for the reader's convenience.

2.1 Relative Reshetikhin-Turaev invariants

In this article we will follow the skein theoretical approach of the relative Reshetikhin-Turaev invariants [6, 30] and focus on the $SO(3)$-theory and the values at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ for odd integers $r \geqslant 3$.

A framed link in an oriented 3-manifold M is a smooth embedding L of a disjoint union of finitely many thickened circles $\mathrm{S}^1 \times [0, \epsilon]$, for some $\epsilon > 0$, into M. The Kauffman bracket skein module $\mathrm{K}_r(M)$ of M is the $\mathbb{C}$-module generated by the isotopic classes of framed links in M modulo the follow two relations:

(1) *Kauffman Bracket Skein Relation:*  $= e^{\frac{\pi\sqrt{-1}}{r}}$  $+\ e^{-\frac{\pi\sqrt{-1}}{r}}$ .

(2) *Framing Relation:* $L \cup$  $= (-e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}})\ L.$

There is a canonical isomorphism

$$\langle\ \rangle : \mathrm{K}_r(\mathrm{S}^3) \to \mathbb{C}$$

defined by sending the empty link to 1. The image $\langle L\rangle$ of the framed link L is called the Kauffman bracket of L.

Let $\mathrm{K}_r(A \times [0, 1])$ be the Kauffman bracket skein module of the product of an annulus A with a closed interval. For any link diagram D in $\mathbb{R}^2$ with k ordered components and $b_1, \dots, b_k \in \mathrm{K}_r(A \times [0, 1])$, let

$$\langle b_1, \dots, b_k\rangle_D$$

*Subsections 2.1, 2.2, 2.3, 2.4 and 2.6 are reproduced from the published paper "Relative Reshetikhin–Turaev Invariants, Hyperbolic Cone Metrics and Discrete Fourier Transforms I" by Ka Ho Wong and Tian Yang in Communications in Mathematical Physics (2022) with permission from Springer Nature.

be the complex number obtained by cabling $b_1, \dots, b_k$ along the components of D considered as a element of $K_r(\mathrm{S}^3)$ then taking the Kauffman bracket $\langle \ \rangle$.

On $\mathrm{K}_r(A \times [0,1])$ there is a commutative multiplication induced by the juxtaposition of annuli, making it a $\mathbb{C}$-algebra; and as a $\mathbb{C}$-algebra $\mathrm{K}_r(A \times [0,1]) \cong \mathbb{C}[z]$, where z is the core curve of A. For an integer $n \geqslant 0$, let $e_n(z)$ be the n-th Chebyshev polynomial defined recursively by $e_0(z) = 1$, $e_1(z) = z$ and $e_n(z) = ze_{n-1}(z) - e_{n-2}(z)$. Let $\mathrm{I}_r = \{0, 2, \dots, r-3\}$ be the set of even integers in between 0 and $r-2$. Then the Kirby coloring $\Omega_r \in \mathrm{K}_r(A \times [0,1])$ is defined by

$$\Omega_r = \mu_r \sum_{n \in \mathrm{I}_r} [n+1] e_n,$$

where

$$\mu_r = \frac{2 \sin \frac{2\pi}{r}}{\sqrt{r}}$$

and $[n]$ is the quantum integer defined by

$$[n] = \frac{e^{\frac{2n\pi\sqrt{-1}}{r}} - e^{-\frac{2n\pi\sqrt{-1}}{r}}}{e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}}.$$

Let M be a closed oriented 3-manifold and let L be a framed link in M with n components. Suppose M is obtained from S^3 by doing a surgery along a framed link L', $D(L')$ is a standard diagram of L' (ie, the blackboard framing of $D(L')$ coincides with the framing of L'). Then L adds extra components to $D(L')$ forming a linking diagram $D(L \cup L')$ with $D(L)$ and $D(L')$ linking in possibly a complicated way. Let U_+ be the diagram of the unknot with framing 1, $\sigma(L')$ be the signature of the linking matrix of L' and $\mathbf{m} = (m_1, \dots, m_n)$ be a multi-elements of I_r. Then the r-th *relative Reshetikhin-Turaev invariant of M with L colored by* $\mathbf{m}$ is defined as

$$\mathrm{RT}_r(M, L, \mathbf{m}) = \mu_r \langle e_{m_1}, \dots, e_{m_n}, \Omega_r, \dots, \Omega_r \rangle_{D(L \cup L')} \langle \Omega_r \rangle_{U_+}^{-\sigma(L')}. \tag{2.1}$$

Note that if $L = \emptyset$ or $m_1 = \cdots = m_n = 0$, then $\mathrm{RT}_r(M, L, \mathbf{m}) = \mathrm{RT}_r(M)$, the r-th Reshetikhin-Turaev invariant of M; and if $M = S^3$, then $\mathrm{RT}_r(M, L, \mathbf{m}) = \mu_r \mathrm{J}_{\mathbf{m},L}(q^2)$, the value of the $\mathbf{m}$-th unnormalized colored Jones polynomial of L at $t = q^2$.

2.2 Hyperbolic cone manifolds

According to [9], a 3-dimensional *hyperbolic cone-manifold* is a 3-manifold M, which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to a standard sphere and M is equipped with a complete path metric such that the restriction of the metric to each simplex is isometric to a hyperbolic geodesic simplex. The *singular locus* L of a cone-manifold M consists of the points with no neighborhood isometric to a ball in a Riemannian manifold. It follows that

(1) L is a link in M such that each component is a closed geodesic.

(2) At each point of L there is a *cone angle* θ which is the sum of dihedral angles of 3-simplices containing the point.

(3) The restriction of the metric on $M \smallsetminus L$ is a smooth hyperbolic metric, but is incomplete if $L \neq \emptyset$.

Hodgson-Kerckhoff [23] proved that hyperbolic cone metrics on M with singular locus L are locally parametrized by the cone angles provided all the cone angles are less than or equal to 2π, and Kojima [28] proved that hyperbolic cone manifolds (M, L) are globally rigid provided all the cone angles are less than or equal to π. It is expected to be globally rigid if all the cone angles are less than or equal to 2π.

Given a 3-manifold N with boundary a union of tori $T_1, \dots, T_n$, a choice of generators (u_i, v_i) for each $\pi_1(T_i)$ and pairs of relatively prime integers (p_i, q_i), one can do the $(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$-Dehn filling on N by attaching a solid torus to each T_i so that $p_i u_i + q_i v_i$ bounds a disk. If $\mathrm{H}(u_i)$ and $\mathrm{H}(v_i)$ are respectively the logarithmic holonomy for u_i and v_i, then a solution to

$$p_i \mathrm{H}(u_i) + q_i \mathrm{H}(v_i) = \sqrt{-1}\theta_i \tag{2.2}$$

near the complete structure gives a cone-manifold structure on the resulting manifold M with the cone angle θ_i along the core curve L_i of the solid torus attached to T_i; it is a smooth structure if $\theta_1 = \cdots = \theta_n = 2\pi$.

In this setting, the Chern-Simons invariant for a hyperbolic cone manifold (M, L) can be defined by using the Neumann-Zagier potential function [39]. To do this, we need a framing on each component, namely, a choice of a curve γ_i on T_i that is isotopic to the core curve L_i of the solid torus attached to T_i. We choose the orientation of γ_i so that $(p_i u_i + q_i v_i) \cdot \gamma_i = 1$. Then we consider the following function

$$\frac{\Phi(\mathrm{H}(u_1), \ldots, \mathrm{H}(u_n))}{\sqrt{-1}} - \sum_{i=1}^{n} \frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4\sqrt{-1}} + \sum_{i=1}^{n} \frac{\theta_i \mathrm{H}(\gamma_i)}{4},$$

where Φ is the Neumann-Zagier potential function (see [39]) defined on the deformation space of hyperbolic structures on $M \setminus L$ parametrized by the holonomy of the meridians $\{\mathrm{H}(u_i)\}$, characterized by

$$\begin{cases} \frac{\partial \Phi(\mathrm{H}(u_1), \ldots, \mathrm{H}(u_n))}{\partial \mathrm{H}(u_i)} = \frac{\mathrm{H}(v_i)}{2}, \\ \\ \Phi(0, \ldots, 0) = \sqrt{-1}\Big(\mathrm{Vol}(M \setminus L) + \sqrt{-1}\mathrm{CS}(M \setminus L)\Big) \qquad \mod \ \pi^2\mathbb{Z}, \end{cases} \tag{2.3}$$

where $M \setminus L$ is with the complete hyperbolic metric. Another important feature of Φ is that it is even in each of its variables $\mathrm{H}(u_i)$.

Following the argument in [39, Sections 4 & 5], one can prove that if the cone angles of components of L are $\theta_1, \ldots, \theta_n$, then

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) = \mathrm{Re}\bigg(\frac{\Phi(\mathrm{H}(u_1), \ldots, \mathrm{H}(u_n))}{\sqrt{-1}} - \sum_{i=1}^{n} \frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4\sqrt{-1}} + \sum_{i=1}^{n} \frac{\theta_i \mathrm{H}(\gamma_i)}{4}\bigg). \tag{2.4}$$

Indeed, in this case, one can replace the 2π in Equations (33) (34) and (35) of [39] by θ_i, and as a consequence can replace the $\frac{\pi}{2}$ in Equations (45), (46) and (48) by $\frac{\theta_i}{4}$, proving the result.

In [66], Yoshida proved that when $\theta_1 = \cdots = \theta_n = 2\pi$,

$$\mathrm{Vol}(M)+\sqrt{-1}\mathrm{CS}(M) = \frac{\Phi(\mathrm{H}(u_1),\dots,\mathrm{H}(u_n))}{\sqrt{-1}} - \sum_{i=1}^n \frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4\sqrt{-1}} + \sum_{i=1}^n \frac{\theta_i \mathrm{H}(\gamma_i)}{4} \mod \sqrt{-1}\pi^2\mathbb{Z}.$$

Therefore, we can make the following

Definition 2.1. *The* Chern-Simons invariant *of a hyperbolic cone manifold $M_{L_{\boldsymbol\theta}}$ with a choice of the framing $(\gamma_1,\dots,\gamma_n)$ is defined as*

$$\mathrm{CS}(M_{L_{\boldsymbol\theta}}) = \mathrm{Im}\bigg(\frac{\Phi(\mathrm{H}(u_1),\dots,\mathrm{H}(u_n))}{\sqrt{-1}} - \sum_{i=1}^n \frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4\sqrt{-1}} + \sum_{i=1}^n \frac{\theta_i \mathrm{H}(\gamma_i)}{4}\bigg) \mod \pi^2\mathbb{Z}.$$

Then together with (2.4), we have

$$\mathrm{Vol}(M_{L_{\boldsymbol\theta}})+\sqrt{-1}\mathrm{CS}(M_{L_{\boldsymbol\theta}}) = \frac{\Phi(\mathrm{H}(u_1),\dots,\mathrm{H}(u_n))}{\sqrt{-1}} - \sum_{i=1}^n \frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4\sqrt{-1}} + \sum_{i=1}^n \frac{\theta_i \mathrm{H}(\gamma_i)}{4} \mod \sqrt{-1}\pi^2\mathbb{Z}. \tag{2.5}$$

2.3 Quantum $6j$-symbols

A triple (m_1,m_2,m_3) of even integers in $\{0,2,\dots,r-3\}$ is r*-admissible* if

(1) $m_i+m_j-m_k \geqslant 0$ for $\{i,j,k\}=\{1,2,3\}$,

(2) $m_1+m_2+m_3 \leqslant 2(r-2)$.

Recall that for $n \in \mathbb{Z}_{\geq 0}$, the quantum factorial $[n]!$ is defined by $[0]! = 1$ and

$$[n]! = \prod_{k=1}^n [k]$$

for $n \geq 0$. For an r-admissible triple (m_1,m_2,m_3), define

$$\Delta(m_1,m_2,m_3) = \sqrt{\frac{[\frac{m_1+m_2-m_3}{2}]![\frac{m_2+m_3-m_1}{2}]![\frac{m_3+m_1-m_2}{2}]!}{[\frac{m_1+m_2+m_3}{2}+1]!}}$$

with the convention that $\sqrt{x} = \sqrt{|x|}\sqrt{-1}$ when the real number x is negative.

A 6-tuple $(m_1, \dots, m_6)$ is r-*admissible* if the triples (m_1, m_2, m_3), (m_1, m_5, m_6), (m_2, m_4, m_6) and (m_3, m_4, m_5) are r-admissible

Definition 2.2. *The* quantum $6j$-symbol *of an r-admissible 6-tuple $(m_1, \dots, m_6)$ is*

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \end{vmatrix} = \sqrt{-1}^{-\sum_{i=1}^{6} m_i} \Delta(m_1, m_2, m_3)\Delta(m_1, m_5, m_6)\Delta(m_2, m_4, m_6)\Delta(m_3, m_4, m_5)$$
$$\sum_{k=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^k [k+1]!}{[k-T_1]![k-T_2]![k-T_3]![k-T_4]![Q_1-k]![Q_2-k]![Q_3-k]!},$$

where $T_1 = \frac{m_1+m_2+m_3}{2}$, $T_2 = \frac{m_1+m_5+m_6}{2}$, $T_3 = \frac{m_2+m_4+m_6}{2}$ *and* $T_4 = \frac{m_3+m_4+m_5}{2}$, $Q_1 = \frac{m_1+m_2+m_4+m_5}{2}$, $Q_2 = \frac{m_1+m_3+m_4+m_6}{2}$ *and* $Q_3 = \frac{m_2+m_3+m_5+m_6}{2}$.

Definition 2.3. *An r-admissible* 6-*tuple* $(m_1, \dots, m_6)$ *is of the* hyperideal type *if for* $\{i, j, k\} = \{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}$ *and* $\{3, 4, 5\}$,

(1) $0 \leqslant m_i + m_j - m_k < r - 2$, *and*

(2) $r - 2 \leqslant m_i + m_j + m_k \leqslant 2(r - 2)$.

Here we recall a classical result of Costantino [10] which was originally stated at the root of unity $q = e^{\frac{\pi\sqrt{-1}}{r}}$. At the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, see [5, Appendix] for a detailed proof.

Theorem 2.4 ([10])**.** *Let* $\{(m_1^{(r)}, \dots, m_6^{(r)})\}$ *be a sequence of r-admissible* 6-*tuples, and let*

$$\theta_i = \left| \pi - \lim_{r\to\infty} \frac{2\pi m_i^{(r)}}{r} \right|.$$

If $\theta_1, \dots, \theta_6$ are the dihedral angles of a truncated hyperideal tetrahedron Δ, then as r varies over all the odd integers

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{matrix} m_1^{(r)} & m_2^{(r)} & m_3^{(r)} \\ m_4^{(r)} & m_5^{(r)} & m_6^{(r)} \end{matrix} \right|_{q=e^{\frac{2\pi\sqrt{-1}}{r}}} = \mathrm{Vol}(\Delta).$$

Closely related, a triple $(\alpha_1, \alpha_2, \alpha_3) \in [0, 2\pi]^3$ is *admissible* if

(1) $\alpha_i + \alpha_j - \alpha_k \geqslant 0$ for $\{i, j, k\} = \{1, 2, 3\}$,

(2) $\alpha_i + \alpha_j + \alpha_k \leqslant 4\pi$.

A 6-tuple $(\alpha_1, \dots, \alpha_6) \in [0, 2\pi]^6$ is *admissible* if the triples $\{1, 2, 3\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$ and $\{3, 4, 5\}$ are admissible.

Definition 2.5. *A* 6*-tuple* $(\alpha_1, \dots, \alpha_6) \in [0, 2\pi]^6$ *is of the* hyperideal type *if for* $\{i, j, k\} = \{1, 2, 3\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$ *and* $\{3, 4, 5\}$,

(1) $0 \leqslant \alpha_i + \alpha_j - \alpha_k \leqslant 2\pi$, *and*

(2) $2\pi \leqslant \alpha_i + \alpha_j + \alpha_k \leqslant 4\pi$.

2.4 Fundamental shadow links

In this section we recall the construction and basic properties of the fundamental shadow links. The building blocks for the fundamental shadow links are truncated tetrahedra as in the left of Figure 2.1. If we take c building blocks $\Delta_1, \dots, \Delta_c$ and glue them together along the triangles of truncation, we obtain a (possibly non-orientable) handlebody of genus $c + 1$ with a link in its boundary consisting of the edges of the building blocks, such as in the right of Figure 2.1. By taking the orientable double (the orientable double covering with the boundary quotient out by the deck involution) of this handlebody, we obtain a link L_{FSL} inside $M_c = \#^{c+1}(S^2 \times S^1)$. We call a link obtained this way a *fundamental shadow link*, and its complement in M_c a *fundamental shadow link complement*.

The fundamental importance of the family of the fundamental shadow links is the following.

Theorem 2.6 ([13]). *Any compact oriented* 3*-manifold with toroidal or empty boundary can be obtained from a suitable fundamental shadow link complement by doing an integral Dehn-filling to some of the boundary components.*

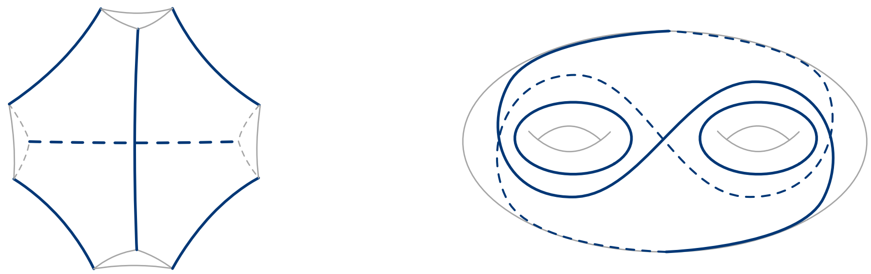


Figure 2.1: The handlebody on the right is obtained from the truncated tetrahedron on the left by identifying the triangles on the top and the bottom by a horizontal reflection and the triangles on the left and the right by a vertical reflection.

A hyperbolic cone metric on M_c with singular locus L_{FSL} and with sufficiently small cone angles $\theta_1, \dots, \theta_n$ can be constructed as follows. For each $s \in \{1, \dots, c\}$, let $e_{s_1}, \dots, e_{s_6}$ be the edges of the building block Δ_s, and θ_{s_j} be the cone angle of the component of L containing e_{s_j}. If θ_i's are sufficiently small, then $\big\{\frac{\theta_{s_1}}{2}, \dots, \frac{\theta_{s_6}}{2}\big\}$ form the set of dihedral angles of a truncated hyperideal tetrahedron, by abuse of notation still denoted by Δ_s. Then the hyperbolic cone manifold M_c with singular locus L_{FSL} and cone angles $\theta_1, \dots, \theta_n$ is obtained by glueing Δ_s's together along isometries of the triangles of truncation, and taking the double. In this metric, the logarithmic holonomy of the meridian u_i of the tubular neighborhood $N(L_i)$ of L_i satisfies

$$\mathrm{H}(u_i) = \sqrt{-1}\theta_i. \tag{2.6}$$

A preferred longitude v_i on the boundary of $N(L_i)$ can be chosen as follows. Recall that a fundamental shadow link is obtained from the double of a set of truncated tetrahedra (along the hexagonal faces) glued together by orientation preserving homeomorphisms between the trice-punctured spheres coming from the double of the triangles of truncation, and recall also that the mapping class group of trice-punctured sphere is generated by mutations, which could be represented by the four 3-braids in Figure 2.2. For each mutation, we assign an integer ± 1 to each component of the braid as in Figure 2.2; and for a composition of a sequence of mutations, we assign the sum of the ± 1 assigned by the mutations to each component of the 3-braid.

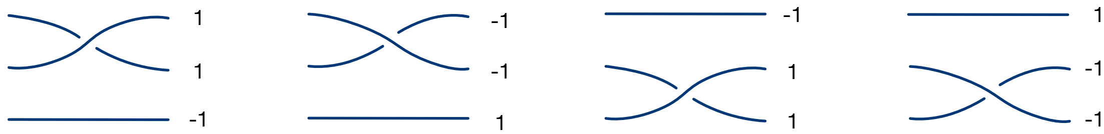


Figure 2.2: Assigning integers to 3-braids

In this way, each orientation preserving homeomorphisms between the trice-punctured spheres assigns three integers to three of the components of L_{FSL}, one for each. For each $i \in \{1, \dots, n\}$, let ι_i be the sum of all the integers on L_i assigned by the homeomorphisms between the trice-punctured spheres. Then we can choose a preferred longitude v_i such that $u_i \cdot v_i = 1$ and the logarithmic holonomy satisfies

$$\mathrm{H}(v_i) = -l_i + \frac{\iota_i\sqrt{-1}\theta_i}{2}, \tag{2.7}$$

where l_i is the length of the closed geodesic L_i. In this way, a framing on L_i gives an integer p_i in the way that the parallel copy of L_i on $N(L_i)$ is isotopic to the curve representing $p_i u_i + v_i$.

Proposition 2.7 ([10, 11])**.** *If $L_{FSL} = L_1 \cup \cdots \cup L_n \subset M_c$ is a framed fundamental shadow link with framing p_i on L_i, and $\mathbf{m} = (m_1, \dots, m_n)$ is a coloring of its components with even integers in $\{0, 2, \dots, r-3\}$, then*

$$\mathrm{RT_r}(M_c, L_{FSL}, \mathbf{m}) = \left(\frac{2\sin\frac{2\pi}{r}}{\sqrt{r}}\right)^{-c} \prod_{i=1}^{n} (-1)^{\frac{\iota_i m_i}{2}} q^{(p_i + \frac{\iota_i}{2})\frac{m_i(m_i+2)}{2}} \prod_{s=1}^{c} \begin{vmatrix} m_{s_1} & m_{s_2} & m_{s_3} \\ m_{s_4} & m_{s_5} & m_{s_6} \end{vmatrix},$$

where $m_{s_1}, \dots, m_{s_6}$ are the colors of the edges of the building block Δ_s inherited from the color $\mathbf{m}$ on L_{FSL}.

Next, we talk about the volume and the Chern-Simons invariant of $M_c \setminus L_{\mathrm{FSL}}$ at the complete hyperbolic structure. In the complete hyperbolic metric, since $M_c \setminus L_{\mathrm{FSL}}$ is the union of $2c$ regular ideal octahedra, we have

$$\mathrm{Vol}(M_c \setminus L_{\mathrm{FSL}}) = 2cv_8. \tag{2.8}$$

For the Chern-Simons invariant, in the case that the truncated tetrahedra $\Delta_1, \dots, \Delta_c$ are glued together along the triangles of truncation via orientation reversing maps, $M_c \smallsetminus L_{\mathrm{FSL}}$ is the ordinary double of the orientable handlebody, which admits an orientation reversing self-homeomorphism. Hence by [37, Corollary 2.5],

$$\mathrm{CS}(M_c \smallsetminus L_{\mathrm{FSL}}) = 0 \qquad \mod \pi^2\mathbb{Z}$$

at the complete hyperbolic structure. In the general case, a fundamental shadow link complement $M_c \smallsetminus L_{\mathrm{FSL}}$ can be obtained from one from the previous case by doing a sequence of mutations along the thrice-punctured spheres coming from the double of the triangles of truncation. Therefore, by [36, Theorem 2.4] that a mutation along an incompressible trice-punctured sphere in a hyperbolic three manifold changes the Chern-Simons invariant by $\frac{\pi^2}{2}$, we have

$$\mathrm{CS}(M_c \smallsetminus L_{\mathrm{FSL}}) = \Big(\sum_{i=1}^{n} \frac{\iota_i}{2}\Big)\pi^2 \qquad \mod \pi^2\mathbb{Z}. \tag{2.9}$$

Together with Theorem 2.4 and the construction of the hyperbolic cone structure, we see that Conjecture 1.3 is true for (M_c, L_{FSL}). This was first proved by Costantino in [10] at the root of unity $q = e^{\frac{\pi\sqrt{-1}}{r}}$.

2.5 Twisted Reidemeister torsion

Let C_* be a finite chain complex

$$0 \to \mathrm{C}_d \xrightarrow{\partial} \mathrm{C}_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{C}_1 \xrightarrow{\partial} \mathrm{C}_0 \to 0$$

of $\mathbb{C}$-vector spaces, and for each C_k choose a basis $\mathbf{c}_k$. Let H_* be the homology of C_*, and for each H_k choose a basis $\mathbf{h}_k$ and a lift $\widetilde{\mathbf{h}}_k \subset \mathrm{C}_k$ of $\mathbf{h}_k$. We also choose a basis $\mathbf{b}_k$ for each image $\partial(\mathrm{C}_{k+1})$ and a lift $\widetilde{\mathbf{b}}_k \subset \mathrm{C}_{k+1}$ of $\mathbf{b}_k$. Then $\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k$ form a basis of C_k. Let $[\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k; \mathbf{c}_k]$ be the determinant of the transition matrix from the standard basis $\mathbf{c}_k$ to the new basis $\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k$. Then the Reidemeister torsion of the chain complex C_* with the chosen bases $\mathbf{c}_*$ and $\mathbf{h}_*$ is defined by

$$\mathrm{Tor}(\mathrm{C}_*, \{\mathbf{c}_k\}, \{\mathbf{h}_k\}) = \pm \prod_{k=0}^{d} [\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k; \mathrm{c}_k]^{(-1)^{k+1}}.$$

It is easy to check that $\mathrm{Tor}(\mathrm{C}_*, \{\mathbf{c}_k\}, \{\mathbf{h}_k\})$ depends only on the choice of $\{\mathbf{c}_k\}$ and $\{\mathbf{h}_k\}$, and does not depend on the choices of $\{\mathbf{b}_k\}$ and the lifts $\{\widetilde{\mathbf{b}}_k\}$ and $\{\widetilde{\mathbf{h}}_k\}$.

We recall the twisted Reidemeister torsion of a CW-complex following the conventions in [46]. Let K be a finite CW-complex and let $\rho : \pi_1(M) \to \mathrm{SL}(N; \mathbb{C})$ be a representation of its fundamental group. Consider the twisted chain complex

$$\mathrm{C}_*(K; \rho) = \mathbb{C}^N \otimes_\rho \mathrm{C}_*(\widetilde{K}; \mathbb{Z})$$

where $\mathrm{C}_*(\widetilde{K}; \mathbb{Z})$ is the simplicial complex of the universal covering of K and $\otimes_\rho$ means the tensor product over $\mathbb{Z}$ modulo the relation

$$\mathbf{v} \otimes (\gamma \cdot \mathbf{c}) = \Big(\rho(\gamma)^T \cdot \mathbf{v}\Big) \otimes \mathbf{c},$$

where T is the transpose, $\mathbf{v} \in \mathbb{C}^N$, $\gamma \in \pi_1(K)$ and $\mathbf{c} \in \mathrm{C}_*(\widetilde{K}; \mathbb{Z})$. The boundary operator on $\mathrm{C}_*(K; \rho)$ is defined by

$$\partial(\mathbf{v} \otimes \mathbf{c}) = \mathbf{v} \otimes \partial(\mathbf{c})$$

for $\mathbf{v} \in \mathbb{C}^N$ and $\mathbf{c} \in \mathrm{C}_*(\widetilde{K}; \mathbb{Z})$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ be the standard basis of $\mathbb{C}^N$, and let $\{c_1^k, \dots, c_{d^k}^k\}$ denote the set of k-cells of K. Then we call

$$\mathbf{c}_k = \Big\{\mathbf{e}_i \otimes c_j^k \;\Big|\; i \in \{1, \dots, N\}, j \in \{1, \dots, d^k\}\Big\}$$

the standard basis of $\mathrm{C}_k(K; \rho)$. Let $\mathrm{H}_*(K; \rho)$ be the homology of the chain complex $\mathrm{C}_*(K; \rho)$ and let $\mathbf{h}_k$ be a basis of $\mathrm{H}_k(K; \rho)$. Then the Reidemeister torsion of K twisted by ρ with basis $\{\mathbf{h}_k\}$ is

$$\mathrm{Tor}(K, \{\mathbf{h}_k\}; \rho) = \mathrm{Tor}(\mathrm{C}_*(K; \rho), \{\mathbf{c}_k\}, \{\mathbf{h}_k\}).$$

By [45], $\mathrm{Tor}(K, \{\mathbf{h}_k\}; \rho)$ depends only on the conjugacy class of ρ. By for e.g. [53], the Reidemeister torsion is invariant under elementary expansions and elementary collapses of CW-complexes, and by [33] it is invariant under subdivisions, hence defines an invariant of PL-manifolds and of topological manifolds of dimension less than or equal to 3.

We list some results by Porti [45] for the Reidemeister torsions of hyperbolic 3-manifolds twisted by the adjoint representation $\mathrm{Ad}_\rho = \mathrm{Ad} \circ \rho$ of the holonomy ρ of the hyperbolic structure. Here Ad is the adjoint acton of $\mathrm{PSL}(2; \mathbb{C})$ on its Lie algebra $\mathbf{sl}(2; \mathbb{C}) \cong \mathbb{C}^3$.

For a closed oriented hyperbolic 3-manifold M with the holonomy representation ρ, by the Weil local rigidity theorem and the Mostow rigidity theorem, $\mathrm{H}_k(M; \mathrm{Ad}_\rho) = 0$ for all k. Then the twisted Reidemeister torsion

$$\mathrm{Tor}(M; \mathrm{Ad}_\rho) \in \mathbb{C}^*/\{\pm 1\}$$

is defined without making any additional choice.

For a compact, orientable 3-manifold M with boundary consisting of n disjoint tori $T_1 \dots, T_n$ whose interior admits a complete hyperbolic structure with finite volume, let $\mathrm{X}(M)$ be the $\mathrm{SL}(2; \mathbb{C})$-character variety of M, let $\mathrm{X}_0(M) \subset \mathrm{X}(M)$ be the distinguished component containing the character of a chosen lifting of the holonomy representation of the complete hyperbolic structure of M, and let $\mathrm{X}^{\mathrm{irr}}(M) \subset \mathrm{X}(M)$ be consisting of the irreducible characters.

Theorem 2.8. *([45, Section 3.3.3]) For a generic character $[\rho] \in \mathrm{X}_0(M) \cap \mathrm{X}^{irr}(M)$ we have:*

(1) For $k \neq 1, 2$, $\mathrm{H}_k(M; \mathrm{Ad}\rho) = 0$.

(2) For $i \in \{1, \dots, n\}$, let $\mathbf{I}_i \in \mathbb{C}^3$ be up to scalar the unique invariant vector of $\mathrm{Ad}_\rho(\pi_1(T_i))$. Then

$$\mathrm{H}_1(M; \mathrm{Ad}\rho) \cong \bigoplus_{i=1}^{n} \mathrm{H}_1(T_i; \mathrm{Ad}\rho) \cong \mathbb{C}^n,$$

and for each $\alpha = ([\alpha_1], \dots, [\alpha_n]) \in \mathrm{H}_1(\partial M; \mathbb{Z}) \cong \bigoplus_{i=1}^n \mathrm{H}_1(T_i; \mathbb{Z})$ has a basis

$$\mathbf{h}^1_{(M,\alpha)} = \{\mathbf{I}_1 \otimes [\alpha_1], \dots, \mathbf{I}_n \otimes [\alpha_n]\}.$$

(3) Let $([T_1], \ldots, [T_n]) \in \bigoplus_{i=1}^n \mathrm{H}_2(T_i;\mathbb{Z})$ *be the fundamental classes of* $T_1, \ldots, T_n$. *Then*

$$\mathrm{H}_2(M;\mathrm{Ad}\rho) \cong \bigoplus_{i=1}^n \mathrm{H}_2(T_i;\mathrm{Ad}\rho) \cong \mathbb{C}^n,$$

and has a basis

$$\mathbf{h}_M^2 = \{\mathbf{I}_1 \otimes [T_1], \ldots, \mathbf{I}_n \otimes [T_n]\}.$$

Remark 2.9 ([45]). Important examples of the generic characters in Theorem 2.8 include the characters of the lifting in $\mathrm{SL}(2;\mathbb{C})$ of the holonomy of the complete hyperbolic structure on the interior of M, the restriction of the holonomy of the closed 3-manifold M_μ obtained from M by doing the hyperbolic Dehn surgery along the system of simple closed curves μ on ∂M, and by [23] the holonomy of a hyperbolic structure on the interior of M whose completion is a conical manifold with cone angles less than 2π.

For $\alpha \in \mathrm{H}_1(M;\mathbb{Z})$, define $\mathbb{T}_{(M,\alpha)}$ on $\mathrm{X}_0(M)$ by

$$\mathbb{T}_{(M,\alpha)}([\rho]) = \mathrm{Tor}(M, \{\mathbf{h}^1_{(M,\alpha)}, \mathbf{h}^2_M\}; \mathrm{Ad}_\rho)$$

for the generic $[\rho] \in \mathrm{X}_0(M) \cap \mathrm{X}^{\mathrm{irr}}(M)$ in Theorem 2.8, and equals 0 otherwise.

Theorem 2.10. *([45, Theorem 4.1]) Let M be a compact, orientable 3-manifold with boundary consisting of n disjoint tori $T_1 \ldots, T_n$ whose interior admits a complete hyperbolic structure with finite volume. Let $\mathbb{C}(\mathrm{X}_0(M))$ be the ring of rational functions over $\mathrm{X}_0(M)$. Then there is up to sign a unique function*

$$\begin{array}{ccc} \mathrm{H}_1(\partial M;\mathbb{Z}) & \to & \mathbb{C}(\mathrm{X}_0(M)) \\ \alpha & \mapsto & \mathbb{T}_{(M,\alpha)} \end{array}$$

which is a $\mathbb{Z}$-multilinear homomorphism with respect to the direct sum $\mathrm{H}_1(\partial M;\mathbb{Z}) \cong \bigoplus_{i=1}^n \mathrm{H}_1(T_i;\mathbb{Z})$ satisfying the following properties:

(i) *For all* $\alpha \in \mathrm{H}_1(\partial M;\mathbb{Z})$, *the domain of definition of* $\mathbb{T}_{(M,\alpha)}$ *contains an open subset* $\mathrm{X}_0(M) \cap \mathrm{X}^{irr}(M)$.

(ii) (Change of curves formula). *Let* $\mu = \{\mu_1, \dots, \mu_n\}$ *and* $\gamma = \{\gamma_1, \dots, \gamma_n\}$ *be two systems of simple closed curves on* ∂M. *If* $\mathrm{H}(\mu_1), \dots, \mathrm{H}(\mu_n)$ *and* $\mathrm{H}(\gamma_1), \dots, \mathrm{H}(\gamma_n)$ *are respectively the logarithmic holonomies of the curves in* μ *and* γ, *then we have the equality of rational functions*

$$\mathbb{T}_{(M,\mu)} = \pm \det\left(\frac{\partial \mathrm{H}(\mu_i)}{\partial \mathrm{H}(\gamma_j)}\right)_{ij} \mathbb{T}_{(M,\gamma)}.$$

(iii) (Surgery formula). *Let* $[\rho_\mu] \in \mathrm{X}_0(M)$ *be the character induced by the holonomy of the closed* 3*-manifold* M_μ *obtained from* M *by doing the hyperbolic Dehn surgery along the system of simple closed curves* μ *on* ∂M. *If* $\mathrm{H}(\gamma_1), \dots, \mathrm{H}(\gamma_n)$ *are the logarithmic holonomies of the core curves* $\gamma_1, \dots, \gamma_n$ *of the solid tori added. Then*

$$\mathrm{Tor}(M_\mu; \mathrm{Ad}_{\rho_\mu}) = \pm \mathbb{T}_{(M,\mu)}([\rho_\mu]) \prod_{i=1}^{n} \frac{1}{4\sinh^2 \frac{\mathrm{H}(\gamma_i)}{2}}.$$

Next, we list some results for the computation of twisted Reidemeister torsions from [64]. We first recall that if $\mathrm{M}_{4\times 4}(\mathbb{C})$ is the space of 4×4 matrices with complex entries, then the *Gram matrix function*

$$\mathbb{G} : \mathbb{C}^6 \to \mathrm{M}_{4\times 4}(\mathbb{C})$$

is defined by

$$\mathbb{G}(\mathbf{z}) = \begin{bmatrix} 1 & -\cosh z_1 & -\cosh z_2 & -\cosh z_6 \\ -\cosh z_1 & 1 & -\cosh z_3 & -\cosh z_5 \\ -\cosh z_2 & -\cosh z_3 & 1 & -\cosh z_4 \\ -\cosh z_6 & -\cosh z_5 & -\cosh z_4 & 1 \end{bmatrix} \tag{2.10}$$

for $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6$. The value of $\mathbb{G}$ at different $\mathbf{u}$ recover the Gram matrices of deeply truncated tetrahedra of all the types. See [4, Section 2.1] for more details.

Theorem 2.11. *([64, Theorem 1.1]) Let $M = \#^{c+1}(S^2 \times S^1) \smallsetminus L_{FSL}$ be the complement of a fundamental shadow link L_{FSL} with n components $L_1, \dots, L_n$, which is the orientable double of the union of truncated tetrahedra $\Delta_1, \dots, \Delta_c$ along pairs of the triangles of truncation, and let $\mathrm{X}_0(M)$ be the distinguished component of the $\mathrm{SL}(2;\mathbb{C})$ character variety of M containing a lifting of the holonomy representation of the complete hyperbolic structure.*

(1) Let $\mathbf{u} = (u_1, \dots, u_n)$ be the system of the meridians of a tubular neighborhood of the components of L_{FSL}. For a generic irreducible character $[\rho]$ in $\mathrm{X}_0(M)$, let $\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)$ be the logarithmic holonomies of $\mathbf{u}$. For each $s \in \{1, \dots, c\}$, let $L_{s_1}, \dots, L_{s_6}$ be the components of L_{FSL} intersecting Δ_s, and let $\mathbb{G}_s$ be the value of the Gram matrix function at $\left(\frac{\mathrm{H}(u_{s_1})}{2}, \dots, \frac{\mathrm{H}(u_{s_6})}{2}\right)$. Then

$$\mathbb{T}_{(M,\mathbf{u})}([\rho]) = \pm 2^{3c} \prod_{s=1}^{c} \sqrt{\det \mathbb{G}_s}.$$

(2) In addition to the assumptions and notations of (1), let $\boldsymbol{\Upsilon} = (\Upsilon_1, \dots, \Upsilon_n)$ be a system of simple closed curves on ∂M, and let $(\mathrm{H}(\Upsilon_1), \dots, \mathrm{H}(\Upsilon_n))$ be their logarithmic holonomies which are functions of $(\mathrm{H}(u_1), \dots, \mathrm{H}(u_n))$. Then

$$\mathbb{T}_{(M,\boldsymbol{\Upsilon})}[(\rho)] = \pm 2^{3c} \det \left(\left. \frac{\partial \mathrm{H}(\Upsilon_i)}{\partial \mathrm{H}(u_j)} \right|_{[\rho]} \right)_{ij} \prod_{s=1}^{c} \sqrt{\det \mathbb{G}_s}.$$

(3) Suppose $M_{\boldsymbol{\Upsilon}}$ is the closed 3-manifold obtained from M by doing the hyperbolic Dehn surgery along a system of simple closed curves $\boldsymbol{\Upsilon} = (\Upsilon_1, \dots, \Upsilon_n)$ on ∂M and $\rho_{\boldsymbol{\Upsilon}}$ is the restriction of the holonomy representation of $M_{\boldsymbol{\Upsilon}}$ to M. Let $(\mathrm{H}(\Upsilon_1), \dots, \mathrm{H}(\Upsilon_n))$ be the logarithmic holonomies of $\boldsymbol{\Upsilon}$ which are functions of the logarithmic holonomies of the meridians $\mathbf{u}$. Let $(\gamma_1, \dots, \gamma_n)$ be a system of simple closed curves on ∂M that are isotopic to the core curves of the solid tori filled in and let $\mathrm{H}(\gamma_1), \dots, \mathrm{H}(\gamma_n)$ be their logarithmic holonomies in $[\rho_\mu]$. Let $\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)$ be the logarithmic holonomies of the meridians $\mathbf{u}$ in $[\rho_\mu]$ and for each $s \in \{1, \dots, c\}$, let $L_{s_1}, \dots, L_{s_6}$ be the components of L_{FSL} intersection Δ_s and let $\mathbb{G}_s$ be the

value of the Gram matrix function at $\left(\frac{\mathrm{H}(u_{s_1})}{2}, \dots, \frac{\mathrm{H}(u_{s_6})}{2}\right)$*. Then*

$$\mathrm{Tor}(M_{\Upsilon}; \mathrm{Ad}_{\rho_{\Upsilon}}) = \pm 2^{3c-2n} \det\left(\frac{\partial \mathrm{H}(\Upsilon_i)}{\partial \mathrm{H}(u_j)}\bigg|_{[\rho_\mu]}\right)_{ij} \prod_{s=1}^{c} \sqrt{\det \mathbb{G}_s} \prod_{i=1}^{n} \frac{1}{\sinh^2 \frac{\mathrm{H}(\gamma_i)}{2}}.$$

2.6 Dilogarithm and quantum dilogarithm functions

Let $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the standard logarithm function defined by

$$\log z = \log |z| + \sqrt{-1} \arg z$$

with $-\pi < \arg z < \pi$.

The dilogarithm function $\mathrm{Li}_2 : \mathbb{C} \setminus (1, \infty) \to \mathbb{C}$ is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du$$

where the integral is along any path in $\mathbb{C} \setminus (1, \infty)$ connecting 0 and z, which is holomorphic in $\mathbb{C} \setminus [1, \infty)$ and continuous in $\mathbb{C} \setminus (1, \infty)$.

The dilogarithm function satisfies the follow properties (see eg. Zagier [67]).

(1)

$$\mathrm{Li}_2\Big(\frac{1}{z}\Big) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\big(\log(-z)\big)^2. \tag{2.11}$$

(2) In the unit disk $\big\{z \in \mathbb{C} \,\big|\, |z| < 1\big\}$,

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \tag{2.12}$$

(3) On the unit circle $\big\{z = e^{2\sqrt{-1}\theta} \,\big|\, 0 \leqslant \theta \leqslant \pi\big\}$,

$$\mathrm{Li}_2\Big(e^{2\sqrt{-1}\theta}\Big) = \frac{\pi^2}{6} + \theta(\theta - \pi) + 2\sqrt{-1}\Lambda(\theta). \tag{2.13}$$

Here $\Lambda : \mathbb{R} \to \mathbb{R}$ is the Lobachevsky function defined by

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t| dt, \tag{2.14}$$

which is an odd function of period π. See eg. Thurston's notes [52, Chapter 7].

The following variant of Faddeev's quantum dilogarithm functions [16, 17] will play a key role in the proof of the main result. Let $r \geqslant 3$ be an odd integer. Then the following contour integral

$$\varphi_r(z) = \frac{4\pi\sqrt{-1}}{r}\int_\Omega \frac{e^{(2z-\pi)x}}{4x\sinh(\pi x)\sinh(\frac{2\pi x}{r})}\, dx \tag{2.15}$$

defines a holomorphic function on the domain

$$\Big\{z \in \mathbb{C} \,\Big|\, -\frac{\pi}{r} < \mathrm{Re} z < \pi + \frac{\pi}{r}\Big\},$$

where the contour is

$$\Omega = \big(-\infty, -\epsilon\big] \cup \big\{z \in \mathbb{C} \,\big|\, |z| = \epsilon, \mathrm{Im} z > 0\big\} \cup \big[\epsilon, \infty\big),$$

for some $\epsilon \in (0, 1)$. Note that the integrand has poles at $n\sqrt{-1}$, $n \in \mathbb{Z}$, and the choice of Ω is to avoid the pole at 0.

The function $\varphi_r(z)$ satisfies the following fundamental properties, whose proof can be found in [61, Section 2.3].

Lemma 2.12. *(1) For $z \in \mathbb{C}$ with $0 < \mathrm{Re} z < \pi$,*

$$1 - e^{2\sqrt{-1}z} = e^{\frac{r}{4\pi\sqrt{-1}}\Big(\varphi_r\big(z-\frac{\pi}{r}\big) - \varphi_r\big(z+\frac{\pi}{r}\big)\Big)}. \tag{2.16}$$

(2) For $z \in \mathbb{C}$ with $-\frac{\pi}{r} < \mathrm{Re} z < \frac{\pi}{r}$,

$$1 + e^{r\sqrt{-1}z} = e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r(z) - \varphi_r\left(z+\pi\right)\right)}. \tag{2.17}$$

Using (2.16) and (2.17), for $z \in \mathbb{C}$ with $\pi + \frac{2(n-1)\pi}{r} < \mathrm{Re} z < \pi + \frac{2n\pi}{r}$, we can define $\varphi_r(z)$ inductively by the relation

$$\prod_{k=1}^{n}\left(1 - e^{2\sqrt{-1}\left(z - \frac{(2k-1)\pi}{r}\right)}\right) = e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(z - \frac{2n\pi}{r}\right) - \varphi_r(z)\right)}, \tag{2.18}$$

extending $\varphi_r(z)$ to a meromorphic function on $\mathbb{C}$. The poles of $\varphi_r(z)$ have the form $(a+1)\pi + \frac{b\pi}{r}$ or $-a\pi - \frac{b\pi}{r}$ for all nonnegative integer a and positive odd integer b.

Let $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, and let

$$(q)_n = \prod_{k=1}^{n}(1 - q^{2k}).$$

Lemma 2.13. *(1) For $0 \leqslant n \leqslant r-2$,*

$$(q)_n = e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{\pi}{r}\right) - \varphi_r\left(\frac{2\pi n}{r} + \frac{\pi}{r}\right)\right)}. \tag{2.19}$$

(2) For $\frac{r-1}{2} \leqslant n \leqslant r-2$,

$$(q)_n = 2e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{\pi}{r}\right) - \varphi_r\left(\frac{2\pi n}{r} + \frac{\pi}{r} - \pi\right)\right)}. \tag{2.20}$$

We consider (2.20) because there are poles in $(\pi, 2\pi)$, and to avoid the poles we move the variables to $(0, \pi)$ by subtracting π.

For $n \in \mathbb{Z}_{\geq 0}$, let $\{0\} = 1$, $\{n\} = q^n - q^{-n}$, $\{0\}! = 1$ and

$$\{n\}! = \prod_{k=1}^{n}\{k\}.$$

Since

$$\{n\}! = (-1)^n q^{-\frac{n(n+1)}{2}}(q)_n,$$

as a consequence of Lemma 2.13, we have

Lemma 2.14. *(1) For* $0 \leqslant n \leqslant r-2$,

$$\{n\}! = e^{\frac{r}{4\pi\sqrt{-1}}\left(-2\pi\left(\frac{2\pi n}{r}\right)+\left(\frac{2\pi}{r}\right)^2(n^2+n)+\varphi_r\left(\frac{\pi}{r}\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}\right)\right)}. \tag{2.21}$$

(2) For $\frac{r-1}{2} \leqslant n \leqslant r-2$,

$$\{n\}! = 2e^{\frac{r}{4\pi\sqrt{-1}}\left(-2\pi\left(\frac{2\pi n}{r}\right)+\left(\frac{2\pi}{r}\right)^2(n^2+n)+\varphi_r\left(\frac{\pi}{r}\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)\right)}. \tag{2.22}$$

The function $\varphi_r(z)$ and the dilogarithm function are closely related as follows.

Lemma 2.15. *(1) For every* z *with* $0 < \mathrm{Re} z < \pi$,

$$\varphi_r(z) = \mathrm{Li}_2(e^{2\sqrt{-1}z}) + \frac{2\pi^2 e^{2\sqrt{-1}z}}{3(1-e^{2\sqrt{-1}z})}\frac{1}{r^2} + O\Big(\frac{1}{r^4}\Big). \tag{2.23}$$

(2) For every z *with* $0 < \mathrm{Re} z < \pi$,

$$\varphi_r'(z) = -2\sqrt{-1}\log(1-e^{2\sqrt{-1}z}) + O\Big(\frac{1}{r^2}\Big). \tag{2.24}$$

(3) [43, Formula (8)(9)]

$$\varphi_r\Big(\frac{\pi}{r}\Big) = \mathrm{Li}_2(1) + \frac{2\pi\sqrt{-1}}{r}\log\Big(\frac{r}{2}\Big) - \frac{\pi^2}{r} + O\Big(\frac{1}{r^2}\Big).$$

2.7 Continued fractions

We recall some notations related to the continued fraction of rational numbers, which will be used in the computation of the Reshetikhin-Turaev invariants (Proposition 3.4). For a pair of relatively prime integers (p, q), let

$$\frac{p}{q} = a_k - \frac{1}{a_{k-1} - \frac{1}{\cdots - \frac{1}{a_1}}}$$

be a continued fraction. For each $l \in \{1, \ldots, k\}$, consider the matrix

$$\begin{bmatrix} A_l & B_l \\ C_l & D_l \end{bmatrix} = T^{a_l} S \cdots T^{a_1} S, \tag{2.25}$$

where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and as a convention let

$$\begin{bmatrix} A_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.26}$$

Lemma 2.16. *[24, Proposition 2.5]*

(1) For $l \in \{1, \ldots, k\}$, $A_l = a_l A_{l-1} - C_{l-1}$ *and* $C_l = A_{l-1}$.

(2) For $l \in \{1, \ldots, k\}$, $B_l = a_l B_{l-1} - D_{l-1}$ *and* $D_l = B_{l-1}$.

(3) We have

$$\frac{A_k}{C_k} = \frac{p}{q}.$$

(4) For $l \in \{1, \ldots, k\}$,

$$\frac{B_l}{A_l} = -\Big(\frac{1}{A_1} + \frac{1}{A_2 A_1} + \cdots + \frac{1}{A_l A_{l-1}}\Big).$$

We observe that A_k and C_k are relatively prime because $A_k D_k - B_k C_k = \det(T^{a_k} S \cdots T^{a_1} S) = 1$. By Lemma 2.16 (3), $\begin{bmatrix} A_k \\ C_k \end{bmatrix} = \pm \begin{bmatrix} p \\ q \end{bmatrix}$. Since a (p, q) Dehn-surgery and a $(-p, -q)$ Dehn-surgery provide the same 3-manifold M, we may without loss of generality assume that

$$\begin{bmatrix} A_k \\ C_k \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{2.27}$$

As a consequence, by Lemma 2.16 (1), we have

$$\begin{bmatrix} A_{k-1} \\ C_{k-1} \end{bmatrix} = \begin{bmatrix} q \\ -p + a_k q \end{bmatrix}. \tag{2.28}$$

We also let

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} D_k \\ -B_k \end{bmatrix} \tag{2.29}$$

so that $pp' + qq' = 1$. In particular, by Lemma 2.16 (1), (2) and (4) we have

$$\frac{1}{A_1} + \frac{1}{A_2 A_1} + \cdots + \frac{1}{A_{k-1} A_{k-2}} = -\frac{B_{k-1}}{A_{k-1}} = -\frac{D_k}{C_k} = -\frac{p'}{q}. \tag{2.30}$$

For $l \in \{1, \ldots, k\}$, we also consider the quantity

$$K_l = \frac{(-1)^{l+1} \sum_{j=1}^{l} a_j C_j}{C_l}. \tag{2.31}$$

The following Lemma 2.17 and 2.18 from [61] are crucial in the computation of the relative Reshetikhin-Turaev invariants and the study of their asymptotics.

Lemma 2.17. *[61, Lemma 3.2] $C_{k-1} K_{k-1} + C_{k-1} q$ is an even integer.*

Lemma 2.18. *[61, Lemma 3.3]*

(1) Let

$$I : \{0, \ldots, |q| - 1\} \to \{0, \ldots, 2|q| - 1\}$$

be the map defined by

$$I(s) = -C_{k-1}(2s+1+K_{k-1}) \quad (\textit{mod}\ 2|q|).$$

Then I is injective with image the set of integers in $\{0, \ldots, 2|q|-1\}$ with parity that of $1-q$. In particular, there exist a unique $s^+ \in \{0, \ldots, |q|-1\}$ and a unique integer m^+ such that

$$I(s^+) = 1 - q + 2m^+q,$$

and a unique $s^- \in \{0, \ldots, |q|-1\}$ and a unique integer m^- such that

$$I(s^-) = -1 - q + 2m^-q.$$

Moreover,

$$s^+ - s^- \equiv p' \quad (\textit{mod}\ q). \tag{2.32}$$

(2) Let

$$J : \{0, \ldots, |q|-1\} \to \mathbb{Q}$$

be the map defined by

$$J(s) = \frac{2s+1}{q} + (-1)^k \sum_{i=1}^{k-1} \frac{(-1)^{i+1}K_i}{C_{i+1}}.$$

Then for the s^+ and s^- in (1),

$$J(s^+) \equiv \frac{p'}{q} \quad (\textit{mod}\ \mathbb{Z})$$

and

$$J(s^-) \equiv -\frac{p'}{q} \quad (\textit{mod}\ \mathbb{Z}).$$

Moverover,

$$J(s^+) \equiv -J(s^-) \quad (\textit{mod } 2\mathbb{Z}).$$

(3) Let

$$K : \{0, \dots, |q|-1\} \to \mathbb{Q}$$

be the map defined by

$$K(s) = \frac{C_{k-1}(2s+1+K_{k-1})^2}{q} + \sum_{i=1}^{k-2} \frac{C_i K_i^2}{C_{i+1}}.$$

Then for the s^+ *and* s^- *in (1),*

$$K(s^+) \equiv -\frac{p'}{q} \quad (\textit{mod } \mathbb{Z})$$

and

$$K(s^-) \equiv -\frac{p'}{q} \quad (\textit{mod } \mathbb{Z}).$$

2.8 Rational Dehn surgery

Given a link $K = K_1 \cup \cdots \cup K_n \subset \mathbb{S}^3$ with n component, let $I \subset \{1, 2, \dots, n\}$, $J = \{1, 2, \dots, n\} \setminus I$ and

$$\frac{p_i}{q_i} = a_{i,\zeta_i} - \frac{1}{a_{i,\zeta_i - 1} - \frac{1}{\dots - \frac{1}{a_{i,1}}}},$$

where $a_{i,1}, \dots, a_{i,\zeta_i}$ are integers for all i. For each $i \in I$, we choose a pair of meridian and longitude $\{(u_i, v_i)\}_{i \in I}$ of the fundamental group of the boundary of the tubular neighborhood of K_i. Recall from [50, p.273] that doing (p_i, q_i) surgery on the K_i is the same as doing $(a_{i,\zeta_i}, a_{i,\zeta_i - 1}, \dots, a_{i,1})$ surgery on the framed link $\tilde{K}_i$ obtained by adding a chain of framed simple loops around K_i as shown in Figure 2.3. Let L_i be a simple loop with framing $a_{i,0}$ as shown in Figure 2.4.

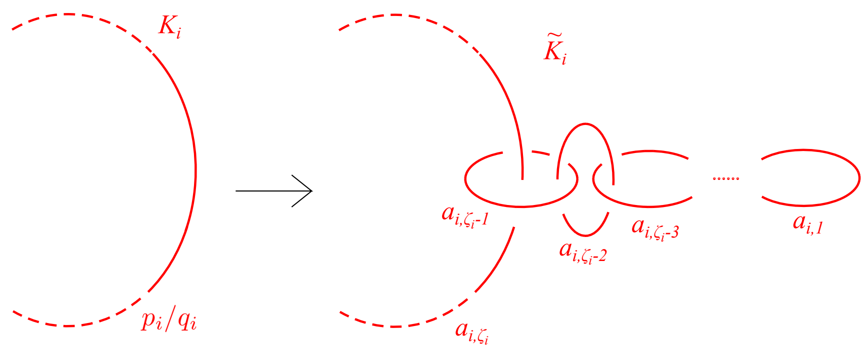


Figure 2.3: doing (p_i, q_i) surgery on K_i is equivalent to doing $(a_{i,\zeta_i}, a_{i,\zeta_i-1}, \ldots, a_{i,1})$ on $\tilde{K}_i$

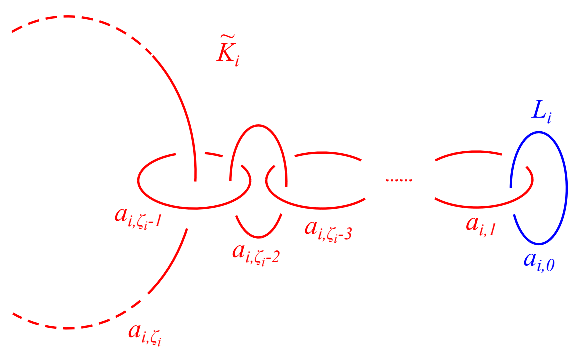


Figure 2.4: Changing each K_i to $\tilde{K}_i$

Consider the continued fraction

$$a_{i,\zeta_i} - \cfrac{1}{a_{i,\zeta_i-1} - \cfrac{1}{\cdots - \cfrac{1}{a_{i,1} - \cfrac{1}{a_{i,0}}}}}.$$

Note that by (2.25), (2.29) and Lemma 2.16 (3), since

$$\begin{bmatrix} p_i & -q_i' \\ q_i & p_i' \end{bmatrix} = T^{a_{i,\zeta_i}} S \cdots T^{a_{i,1}} S,$$

we have

$$T^{a_{i,\zeta_i}}S\cdots T^{a_{i,1}}ST^{a_{i,0}}S = \begin{bmatrix} p_i & -q_i' \\ q_i & p_i' \end{bmatrix}\begin{bmatrix} a_{i,0} & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{i,0}p_i - q_i' & -p_i \\ a_{i,0}q_i + p_i' & q_i \end{bmatrix}. \tag{2.33}$$

This implies that the parallel copy γ_i of L_i given by the framing $a_{i,0}$ is isotopic to curve $-q_i'u_i + p_i'v_i + a_{i,0}(p_iu_i + q_iv_i)$ on the boundary of the tubular neighborhood of K_i. Consider the hyperbolic cone structure on the closed oriented 3-manifold obtained by doing (p_i, q_i) surgery on $\{K_i\}_{i\in I}$ and $(1,0)$ surgery on $\{K_j\}_{j\in J}$ with singular locus $\{L_i\}_{i\in I} \cup \{K_j\}_{j\in J}$ and cone angles $(\theta_1, \dots, \theta_n)$. Since

$$p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i) = \theta_i\sqrt{-1}$$

for all $i \in I$, we have

$$\begin{aligned}\mathrm{H}(\gamma_i) &= -q_i'\mathrm{H}(u_i) + p_i'\mathrm{H}(v_i) + a_{i,0}(p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i)) \\ &= -q_i'\mathrm{H}(u_i) + p_i'\mathrm{H}(v_i) + a_{i,0}\theta_i\sqrt{-1}.\end{aligned} \tag{2.34}$$

3. COMPUTATION OF THE RELATIVE RESHETIKHIN-TURAEV INVARIANTS

Let $L_{\text{FSL}} = L_{\text{FSL},1} \cup \cdots \cup L_{\text{FSL},n}$ be a fundamental shadow link in $M_c = \#^{c+1}(S^2 \times S^1)$ for some $c \in \mathbb{N}$, and let $L' \subset S^3$ be the disjoint union of $c+1$ unknots with the 0-framings by doing surgery along which we get M_c. Let M be a closed oriented 3-manifold and $L = L_1 \cup \cdots \cup L_n$ be a framed link inside M with n components. Suppose $M \setminus L$ is homeomorphic to $M_c \setminus L_{\text{FSL}}$. Then, up to reordering if necessary, there exist a partition $\{I, J\}$ of $\{1, 2, .., n\}$ together with $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{Z} \setminus \{0\}$ for each $i \in I$ such that

1. M is obtained by doing (p_i/q_i) surgery along $L_{\text{FSL},i}$ and $(1,0)$ surgery along $L_{\text{FSL},j}$ in M_c,
2. the i-th component of L in $M_c \setminus L_{\text{FSL}}$ is isotopic to a curve on the boundary of the tubular neighborhood of $L_{\text{FSL},i}$ that intersects the (p_i, q_i)-curve of the boundary at exactly one point, and
3. L_j and $L_{\text{FSL},j}$ are isotopic in M_c for all $j \in J$.

For each $i \in I$, consider a continued fraction expansion

$$\frac{p_i}{q_i} = a_{i,\zeta_i} - \frac{1}{a_{i,\zeta_i - 1} - \frac{1}{\cdots - \frac{1}{a_{i,1}}}},$$

where $\zeta_i \in \mathbb{N}$. We replace $L_{\text{FSL},i}$ by another framed link $\tilde{L}_{\text{FSL},i}$ of ζ_i many components with framings $a_{i,1}, \ldots, a_{i,\zeta_i}$ according to Figure 3.1. Let $\tilde{L}_{\text{FSL},I} = \bigcup_{i \in I} \tilde{L}_{\text{FSL},i}$. By eg [50, p.273], M can also be obtained by doing surgery along the framed link $\tilde{L}_{\text{FSL},I} \cup L' \subset \mathbb{S}^3$.

We let $\mathbf{n}_I = (n_i)_{i \in I} \in I_r^{|I|}$ and $\mathbf{m}_J = (m_j)_{j \in J} \in I_r^{|J|}$ be colors on the I and J components of L respectively. Denote the framings of the I and J components by $a_{i,0}$ and $a_{j,0}$ respectively, where $i \in I$ and $j \in J$. First of all, we compute the $(\mathbf{n}_I, \mathbf{m}_J)$-th relative Reshetikhin-Turaev invariants of the pair (M, L).

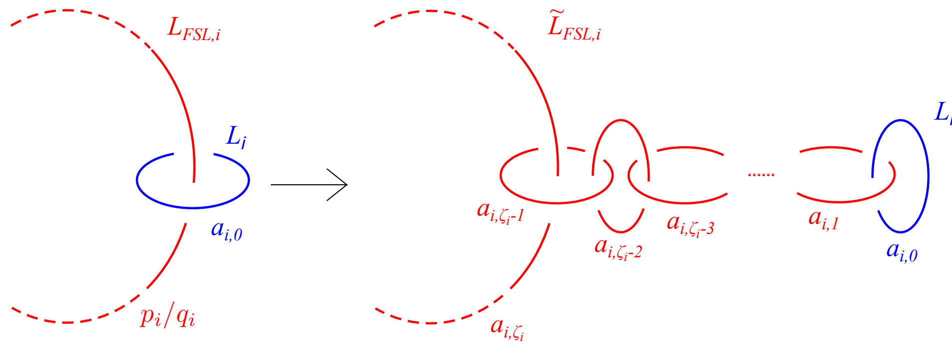


Figure 3.1: Changing each $L_{\mathrm{FSL},i}$ to $\tilde{L}_{\mathrm{FSL},i}$

Proposition 3.1.

$$\begin{aligned}
&\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J)) \\
&= \frac{\mu_r^{\sum_{i\in I}\zeta_i - c}}{\{1\}^{\sum_{i\in I}\zeta_i}} e^{-\sigma(\tilde{L}_{FSL,I}\cup L')(-\frac{3}{r}-\frac{r+1}{4})\sqrt{-1}\pi} \prod_{i\in I} q^{\frac{a_{i,0}n_i(n_i+2)}{2}} \prod_{j\in J}(-1)^{\frac{\iota_j m_j}{2}} q^{\left(a_{j,0}+\frac{\iota_j}{2}\right)\frac{m_j(m_j+2)}{2}} \\
&\quad \times \sum_{\mathbf{m}_I, \mathbf{m}_{\zeta_I}} \Bigg[\left(\prod_{i\in I} \{(n_i+1)(m_{i,1}+1)\}\{(m_{i,1}+1)(m_{i,2}+1)\}\dots\{(m_{i,\zeta_i-1}+1)(m_{i,\zeta_i}+1)\} \right) \\
&\qquad \left(\prod_{i\in I} (-1)^{\frac{\iota_i m_{i,\zeta_i}}{2}} q^{\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}m_{i,l}(m_{i,l}+2)}{2} + \left(a_{i,\zeta_i}+\frac{\iota_i}{2}\right)\frac{m_{i,\zeta_i}(m_{i,\zeta_i}+2)}{2}} \right) \prod_{s=1}^{c} \begin{vmatrix} m_{s_1} & m_{s_2} & m_{s_3} \\ m_{s_4} & m_{s_5} & m_{s_6} \end{vmatrix} \Bigg],
\end{aligned} \tag{3.1}$$

where the sum is over multi-even integers $\mathbf{m}_I = (\mathbf{m}_i)_{i\in I} \in \{0, 2, \dots, r-3\}^{\sum_{i\in I}\zeta_i - |I|}$ *with each* $(\mathbf{m}_i) = (m_{i,1}, \dots, m_{i,\zeta_i - 1}) \in \{0, 2, \dots, r-3\}^{\zeta_i - 1}$ *and multi-even integers* $\mathbf{m}_{\zeta_I} = (m_{i,\zeta_i})_{i\in I} \in \{0, 2, \dots, r-3\}^{|I|}$, *and* $m_{s_1}, \dots, m_{s_6}$ *are the colors of the edges of the building block* Δ_s *inherited from the colors on* L_{FSL}.

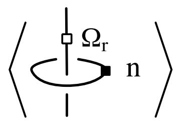
Proof. The terms

$$\prod_{j\in J} q^{\frac{a_{j,0}m_j(m_j+2)}{2}}, \quad \prod_{i\in I} q^{\frac{a_{i,0}n_i(n_i+2)}{2}} \quad \text{and} \quad q^{\frac{\sum_{l=1}^{\zeta_i} a_{i,l}m_{i,l}(m_{i,l}+2)}{2}}$$

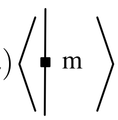
come from changing the framings of all the link components to zero. Besides, the term

$$\left(\prod_{i\in I}\{(n_i+1)(m_{i,1}+1)\}\{(m_{i,1}+1)(m_{i,2}+1)\}\dots\{(m_{i,\zeta_i-1}+1)(m_{i,\zeta_i}+1)\}\right)$$

comes from the skein computation



$$= \mu_r \sum_{m\in \mathrm{I}_r} \mathrm{H}(m,n)$$



where

$$\mathrm{H}(m,n) = (-1)^{m+n}\frac{\{(m+1)(n+1)\}}{\{1\}} = \frac{\{(m+1)(n+1)\}}{\{1\}}$$

for even integers m, n. Together with Proposition 2.7, the result follows. □

Recall from [62] that

Proposition 3.2. *([62, Proposition 3.1]) The quantum $6j$-symbol at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ can be computed as*

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \end{vmatrix} = \frac{\{1\}}{2}\sum_{k=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3,r-2\}} e^{\frac{r}{4\pi\sqrt{-1}}U_r\left(\frac{2\pi m_1}{r},\dots,\frac{2\pi m_6}{r},\frac{2\pi k}{r}\right)},$$

where U_r is defined as follows. If $(m_1,\dots,m_6)$ is of hyperideal type, then

$$\begin{aligned} U_r(\alpha_1,\dots,\alpha_6,\xi) =& \pi^2 - \left(\frac{2\pi}{r}\right)^2 + \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^3(\eta_j-\tau_i)^2 - \frac{1}{2}\sum_{i=1}^4\left(\tau_i+\frac{2\pi}{r}-\pi\right)^2 \\ &+ \left(\xi+\frac{2\pi}{r}-\pi\right)^2 - \sum_{i=1}^4(\xi-\tau_i)^2 - \sum_{j=1}^3(\eta_j-\xi)^2 \\ &- 2\varphi_r\left(\frac{\pi}{r}\right) - \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^3\varphi_r\left(\eta_j-\tau_i+\frac{\pi}{r}\right) + \frac{1}{2}\sum_{i=1}^4\varphi_r\left(\tau_i-\pi+\frac{3\pi}{r}\right) \\ &- \varphi_r\left(\xi-\pi+\frac{3\pi}{r}\right) + \sum_{i=1}^4\varphi_r\left(\xi-\tau_i+\frac{\pi}{r}\right) + \sum_{j=1}^3\varphi_r\left(\eta_j-\xi+\frac{\pi}{r}\right), \end{aligned} \tag{3.2}$$

where $\tau_1 = \frac{\alpha_1+\alpha_2+\alpha_3}{2}$, $\tau_2 = \frac{\alpha_1+\alpha_5+\alpha_6}{2}$, $\tau_3 = \frac{\alpha_2+\alpha_4+\alpha_6}{2}$ *and* $\tau_4 = \frac{\alpha_3+\alpha_4+\alpha_5}{2}$, $\eta_1 = \frac{\alpha_1+\alpha_2+\alpha_4+\alpha_5}{2}$, $\eta_2 = \frac{\alpha_1+\alpha_3+\alpha_4+\alpha_6}{2}$ *and* $\eta_3 = \frac{\alpha_2+\alpha_3+\alpha_5+\alpha_6}{2}$. *If* $(m_1, \dots, m_6)$ *is not of the hyperideal type, then* U_r *will be changed according to Lemma 2.14.*

We are going to apply the Gauss sum formula (Lemma 3.3 below) to write the relative Reshetikhin-Turaev invariants as a sum of the evaluation of certain holomorphic function (Proposition 3.4). Recall that for each $i \in I$, we have

$$\frac{p_i}{q_i} = a_{i,\zeta_i} - \frac{1}{a_{i,\zeta_i-1} - \frac{1}{\dots - \frac{1}{a_{i,1}}}}.$$

With respect to the continued fraction $[a_{i,1}, \dots, a_{i,\zeta_i}]$, for each $i \in I$, let

- $A_{i,l}, B_{i,l}, C_{i,l}, D_{i,l}$ be the integers defined in (2.25) for each $l = 1, \dots, \zeta_i$,
- p'_i, q'_i and $K_{i,l}$ be the quantities defined in (2.29) and (2.31) for each $l = 1, \dots, \zeta_i$,
- $I_i(s_i)$, $J_i(s_i)$ and $K_i(s_i)$ be the functions defined in Lemma 2.18, where $s_i \in \{0, 1, \dots, |q_i| - 1\}$.

For any $n_i, m_{i,\zeta_i} \in \mathbb{N}$, consider the sum

$$\begin{aligned} &S_i(m_{i,\zeta_i}, n_i) \\ &= \sum_{m_{i,1},\dots,m_{i,\zeta_i-1}=0}^{r-1} (-1)^{\sum_{l=1}^{\zeta_i-1} a_{i,l} m_{i,l}} q^{\sum_{l=1}^{\zeta_i-1} \frac{a_{i,l} m_{i,l}^2}{2}} (q^{n_i m_{i,1}} - q^{-n_i m_{i,1}}) q^{\sum_{l=1}^{\zeta_i-1} m_{i,l} m_{i,l+1}}, \end{aligned} \tag{3.3}$$

where $q = e^{\frac{2\pi\sqrt{-1}}{r}}$. This sum will appear later in Proposition 3.4.

Lemma 3.3. *At* $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, *we have*

$$S_i(m_{i,\zeta_i}, n_i) = \frac{(-1)^{\zeta_i+1}(\sqrt{-1}r)^{\frac{\zeta_i-1}{2}}}{\sqrt{q_i}} \sum_{s_i=0}^{|q_i|-1} \left(e^{\frac{r}{4\pi\sqrt{-1}} Z_i^+\left(s_i, \frac{2\pi m_{i,\zeta_i}}{r}, \frac{2\pi n_i}{r}\right)} - e^{\frac{r}{4\pi\sqrt{-1}} Z_i^-\left(s_i, \frac{2\pi m_{i,\zeta_i}}{r}, \frac{2\pi n_i}{r}\right)} \right),$$

where

$$Z_i^{\pm}(s,\alpha,\beta) = \frac{C_{i,\zeta_i-1}}{q_i}(\alpha-\pi)^2 \mp \frac{2(\beta-\pi)(\alpha-\pi)}{q_i} - \frac{2\pi(I_i(s)\pm 1)}{q_i}(\alpha-\pi) + K_i(s)\pi^2 - \frac{p_i'}{q_i}\beta^2 \mp 2\beta\pi J_i(s).$$

Proof. First of all, we consider a closely related sum

$$\tilde{S}_i(m_{i,\zeta_i},n_i) = \sum_{m_{i,1},\dots,m_{i,\zeta_i-1}=0}^{r-1} (-1)^{\sum_{l=1}^{\zeta_i-1} a_{i,l}m_{i,l}} q^{\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}m_{i,l}^2}{2}} (q^{n_i m_{i,1}} - q^{-n_i m_{i,1}}) \times \prod_{l=1}^{\zeta_i-1}\Big(q^{m_{i,l}m_{i,l+1}} - q^{-m_{i,l}m_{i,l+1}}\Big).$$

By considering the transformation $m_{i,l} \mapsto r - m_{i,l}$, a direct computation shows that

$$\tilde{S}_i(m_{i,\zeta_i},n_i) = 2^{\zeta_i-1} S_i(m_{i,\zeta_i},n_i).$$

As a result, by Lemma 3.6 in [61] (note that our variable q is the variable $t^{\frac{1}{2}}$ in [61]),

$$\begin{aligned} S_i(m_{i,\zeta_i},n_i) =& \frac{1}{2^{\zeta_i-1}}\tilde{S}_i(m_{i,\zeta_i},n_i) \\ =& \tau_i^+ \sum_{s_i=0}^{2|q_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_i r+\frac{rK_{i,\zeta_i-1}}{2}-\frac{(-1)^{\zeta_i}n_i}{C_{i,\zeta_i-1}}\right)^2} \\ & - \tau_i^- \sum_{s_i=0}^{2|q_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_i r+\frac{rK_{i,\zeta_i-1}}{2}+\frac{(-1)^{\zeta_i}n_i}{C_{i,\zeta_i-1}}\right)^2}, \end{aligned}$$

where

$$\tau_i^{\pm} = \frac{(\sqrt{-1}r)^{\frac{\zeta_i-1}{2}}}{2\sqrt{q_i}} \times e^{-\frac{\pi\sqrt{-1}}{r}n_i^2\left(\sum_{l=1}^{\zeta_i-2}\frac{1}{C_{i,l}C_{i,l+1}}\right) - \frac{\pi\sqrt{-1}r}{4}\sum_{l=1}^{\zeta_i-2}\frac{C_{i,l}K_{i,l}^2}{C_{i,l+1}} \mp \pi\sqrt{-1}n_i\left(\sum_{l=1}^{\zeta_i-2}(-1)^{l+1}\frac{K_{i,l}}{C_{i,l+1}}\right)}.$$

Moreover, since all of C_{i,ζ_i}, m_{i,ζ_i}, s_i, q_i, n_i and r are integers, a direct computation shows that

$$\frac{e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+(s_i+q_i)r+\frac{rK_{i,\zeta_i-1}}{2}\pm\frac{(-1)^{\zeta_i}n_i}{C_{i,\zeta_i-1}}\right)^2}}{e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_ir+\frac{rK_{i,\zeta_i-1}}{2}\pm\frac{(-1)^{\zeta_i}n_i}{C_{i,\zeta_i-1}}\right)^2}}=e^{-\pi\sqrt{-1}r(K_{i,\zeta_i-1}C_{i,\zeta_i-1}+C_{i,\zeta_i-1}q_i)}=1,$$

where the last equality comes from Lemma 2.17 that $K_{i,\zeta_i-1}C_{i,\zeta_i-1}+C_{i,\zeta_i-1}q_i$ is an even integer. Thus, for each $s_i\in\{0,\dots,|q_i|-1\}$,

$$e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+(s_i+q_i)r+\frac{rK_{i,\zeta_i-1}}{2}\pm\frac{(-1)^{\zeta_i}n_i}{C_{i,\zeta_i-1}}\right)^2}=e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_ir+\frac{rK_{i,\zeta_i-1}}{2}\pm\frac{(-1)^{\zeta_i}n_i}{C_{i,\xi_i-1}}\right)^2}.$$

In particular, we can write

$$S_i(m_{i,\zeta_i},n_i)=\tau_i\sum_{s_i=0}^{|q_i|-1}e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_ir+\frac{rK_{i,\zeta_i-1}}{2}\right)^2}$$
$$\times\left(e^{-\pi\sqrt{-1}n_i\left(\frac{(-1)^{\zeta}}{rq_i}(2m_{i,\zeta_i}+2s_ir)+\sum_{l=1}^{\zeta_i-1}\frac{(-1)^{l+1}K_{i,l}}{C_{i,l+1}}\right)}-e^{\pi\sqrt{-1}n_i\left(\frac{(-1)^{\zeta}}{rq_i}(2m_{i,\zeta_i}+2s_ir)+\sum_{l=1}^{\zeta_i-1}\frac{(-1)^{l+1}K_{i,l}}{C_{i,l+1}}\right)}\right),$$

where
$$\tau_i=\frac{(\sqrt{-1}r)^{\frac{\zeta_i-1}{2}}}{\sqrt{q_i}}e^{-\frac{\pi\sqrt{-1}}{r}n_i^2\left(\sum_{l=1}^{\zeta_i-1}\frac{1}{C_{i,l}C_{i,l+1}}\right)-\frac{\pi\sqrt{-1}r}{4}\sum_{l=1}^{\zeta_i-2}\frac{C_{i,l}K_{i,l}^2}{C_{i,l+1}}}.$$

By a direct computation,

$$e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i,\zeta_i-1}}{q_i}\left(m_{i,\zeta_i}+s_ir+\frac{rK_{i,\zeta_i-1}}{2}\right)^2-\frac{\pi\sqrt{-1}r}{4}\sum_{l=1}^{\zeta_i-2}\frac{C_{i,l}K_{i,l}^2}{C_{i,l+1}}}$$
$$=e^{\frac{r}{4\pi\sqrt{-1}}\left(\frac{C_{i,\zeta_i-1}}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2-\frac{2\pi I_i(s_i)}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)+K_i(s_i)\pi^2\right)}.$$

Besides, by Lemma 2.16 and (2.30),

$$e^{-\frac{\pi\sqrt{-1}}{r}n_i^2\left(\sum_{l=1}^{\zeta_i-1}\frac{1}{C_{i,l}C_{i,l+1}}\right)}=e^{\frac{r}{4\pi\sqrt{-1}}\left(-\frac{p_i'}{q_i}\right)\left(\frac{2\pi n_i}{r}\right)^2}.$$

Moreover,

$$
\begin{aligned}
& e^{\mp\pi\sqrt{-1}n_i\left(\frac{(-1)^{\zeta_i}}{rq_i}(2m_{i,\zeta_i}+2s_ir)+\sum_{l=1}^{\zeta_i-1}\frac{(-1)^{l+1}K_{i,l}}{C_{i,l+1}}\right)} \\
&= e^{\pm(-1)^{\zeta_i}\frac{r}{4\pi\sqrt{-1}}\left(2\left(\frac{2\pi n_i}{r}\left(\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)\frac{1}{q_i}+\pi J_i(s_i)\right)\right)\right)} \\
&= e^{\pm(-1)^{\zeta_i}\frac{r}{4\pi\sqrt{-1}}\left(\frac{2}{q_i}\left(\frac{2\pi n_i}{r}-\pi\right)\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)+\frac{2\pi}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)+2\pi\left(\frac{2\pi n_i}{r}\right)J_i(s_i)\right)}.
\end{aligned}
$$

The result follows from the above computation. □

Proposition 3.4.

$$
\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J)) = Z_r \sum_{\mathbf{s}_I, \mathbf{m}_{\zeta_I}, \mathbf{k}, \mathbf{E}_I} g_r^{\mathbf{E}_I}(\mathbf{s}_I, \mathbf{m}_{\zeta_I}, \mathbf{k}),
$$

where

1. *Z_r is given by*

$$
Z_r = \frac{(-1)^{\sum_{i\in I}(\zeta_i+1+\sum_{l=1}^{\zeta_i}a_{i,l})}(\sqrt{-1}r)^{\sum_{i\in I}\frac{\zeta_i-1}{2}}\mu_r^{\sum_{i\in I}\zeta_i-c}}{2^c\{1\}^{\sum_{i\in I}\zeta_i-c}\sqrt{\prod_{i\in I}q_i}} e^{\frac{\pi\sqrt{-1}}{r}\sum_{i\in I}\sum_{l=1}^{\zeta_i-1}a_{i,l}-\frac{r\pi\sqrt{-1}}{4}(\sum_{i\in I}(a_{i,0}+a_{i,\zeta_i})+\sum_{j\in J}a_{j,0})+\sigma(\tilde{L}_{FSL,I}\cup L')(\frac{3}{r}+\frac{r+1}{4})\sqrt{-1}\pi},
$$

2. *$\mathbf{s}_I = (s_i)_{i\in I}$ where each s_i runs over all integer in $\{0, \ldots, |q_i| - 1\}$,*

3. *$\mathbf{m}_{\zeta_I} = (\mathbf{m}_{i,\zeta_i})_{i\in I}$ runs over all multi-even integers in $\{0, 2, \ldots, r-3\}$ so that for each $s \in \{1, \ldots, c\}$, the triples $(m_{s_1}, m_{s_2}, m_{s_3})$, $(m_{s_1}, m_{s_5}, m_{s_6})$, $(m_{s_2}, m_{s_4}, m_{s_6})$ and $(m_{s_3}, m_{s_4}, m_{s_5})$ are r-admissible,*

4. *$\mathbf{k} = (k_1, \ldots, k_c)$ runs over all multi-integers with each k_s lying in between $\max\{T_{s_i}\}$ and $\min\{Q_{s_j}, r-2\}$,*

5. *$\mathbf{E}_I = (E_i)_{i\in I} \in \{-1, 1\}^{|I|}$ runs over all multi-sign,*

6. *the function* $g_r^{\mathbf{E}_I}(\mathbf{s}_I, \mathbf{m}_{\zeta_I}, \mathbf{k})$ *is defined by*

$$g_r^{\mathbf{E}_I}(\mathbf{s}_I, \mathbf{m}_{\zeta_I}, \mathbf{k}) = \left(\prod_{i\in I} E_i\right) e^{\sqrt{-1}P_r^{\mathbf{E}_I}\left(\mathbf{s}_I, \frac{2\pi\mathbf{m}_{\zeta_I}}{r}\right) + \frac{r}{4\pi\sqrt{-1}}W_r\left(\mathbf{s}_I, \frac{2\pi\mathbf{m}_{\zeta_I}}{r}, \frac{2\pi\mathbf{k}}{r}\right)},$$

where $\frac{2\pi\mathbf{k}}{r} = (\frac{2\pi k_1}{r}, \dots, \frac{2\pi k_c}{r})$, $\frac{2\pi\mathbf{m}_{\zeta_I}}{r} = \left(\frac{2\pi m_{i,\zeta_i}}{r}\right)_{i\in I}$, $\mathbf{s}_I = (s_i)_{i\in I}$,

$$\begin{aligned} P_r^{\mathbf{E}_I}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}) = &\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i - \pi) + \frac{p_i}{q_i}(\alpha_{i,\zeta_i} - \pi) + \frac{E_i(\alpha_{i,\zeta_i} + \beta_i - 2\pi)}{q_i}\right) \\ &+ \pi\sum_{i\in I}\left(\frac{I_i(s_i) + E_i}{q_i} + \frac{p_i'}{q_i} + E_i J_i(s_i)\right) \\ &+ \sum_{i\in I} a_{i,0}\beta_i + \sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i} + \sum_{j\in J}\left(a_{j,0} + \frac{\iota_j}{2}\right)\alpha_j \end{aligned}$$

and

$$\begin{aligned} W_r(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = &-\sum_{i\in I}\left(a_{i,0} + \frac{p_i'}{q_i}\right)(\beta_i - \pi)^2 - \sum_{j\in J}\left(a_{j,0} + \frac{\iota_j}{2}\right)(\alpha_j - \pi)^2 \\ &- \sum_{i=1}^{n}\left(\frac{p_i}{q_i} + \frac{\iota_i}{2}\right)(\alpha_{i,\zeta_i} - \pi)^2 - \sum_{i\in I}\frac{2\pi(I_i(s_i) - E_i)}{q_i}(\alpha_{i,\zeta_i} - \pi) \\ &- \sum_{i\in I}\frac{2E_i(\alpha_{i,\zeta_i} - \pi)(\beta_i - \pi)}{q_i} + \sum_{i\in I} 2\pi\beta_i\Big(-E_i J_i(s_i) - \frac{p_i}{q_i}\Big) \\ &+ \sum_{s=1}^{c} U_r(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s) + \sum_{i\in I}\pi^2\left(K_i(s_i) + \frac{p_i'}{q_i}\right) + \Big(\sum_{i=1}^{n}\frac{\iota_i}{2}\Big)\pi^2 \\ &+ \frac{4\pi^2}{r^2}h_I \end{aligned}$$

with $\beta_i = \frac{2\pi n_i}{r}$ *for* $i \in I$, $\alpha_j = \frac{2\pi m_j}{r}$ *for* $j \in J$ *and* $h_I = \sum_{i\in I}\frac{C_{i,\zeta_i - 1} + 2E_i - p_i'}{q_i}$.

Proof. By Proposition 3.1, we have

$$\begin{aligned}
&\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J))\\
=&\frac{\mu_r^{\sum_{i\in I}\zeta_i - c}}{\{1\}^{\sum_{i\in I}\zeta_i}} e^{-\sigma(\tilde{L}_{\mathrm{FSL},I}\cup L')(-\frac{3}{r}-\frac{r+1}{4})\sqrt{-1}\pi} \prod_{i\in I} q^{\frac{a_{i,0}n_i(n_i+2)}{2}} \prod_{j\in J}(-1)^{\frac{\iota_j m_j}{2}} q^{\left(a_{j,0}+\frac{\iota_j}{2}\right)\frac{m_j(m_j+2)}{2}}\\
&\quad\times \sum_{\mathbf{m}_I}\sum_{\boldsymbol{\epsilon}_I}\sum_{\mathbf{m}_{\zeta_{\mathbf{i}}}} \left(\prod_{i\in I} S_i^{(\epsilon_{i,1},\dots,\epsilon_{i,\zeta_i-1})}(m_{i,1},\dots,m_{i,\zeta_i})\right)\left(\prod_{i\in I}(-1)^{\frac{\iota_i m_{i,\zeta_i}}{2}} q^{\left(a_{i,\zeta_i}+\frac{\iota_i}{2}\right)\frac{m_{i,\zeta_i}(m_{i,\zeta_i}+2)}{2}}\right)\\
&\quad\times \prod_{s=1}^{c}\begin{vmatrix} m_{s_1} & m_{s_2} & m_{s_3}\\ m_{s_4} & m_{s_5} & m_{s_6}\end{vmatrix},
\end{aligned}$$

where

- $\mathbf{m}_I = (\mathbf{m}_i)_{i\in I}$ with $\mathbf{m}_i = (m_{i,1},\dots,m_{i,\zeta_i-1})$ runs over all multi-even integers in $\{0, 2, \dots, r-3\}$,
- $\mathbf{m}_{\zeta_I} = (m_{i,\zeta_i})_{i\in I}$ runs over all multi-even integers in $\{0, 2, \dots, r-3\}$ so that for each $s \in \{1, \dots, c\}$, the triples $(m_{s_1}, m_{s_2}, m_{s_3})$, $(m_{s_1}, m_{s_5}, m_{s_6})$, $(m_{s_2}, m_{s_4}, m_{s_6})$ and $(m_{s_3}, m_{s_4}, m_{s_5})$ are r-admissible,
- $\boldsymbol{\epsilon}_I = (\boldsymbol{\epsilon}_i)_{i\in I}$ with each $\boldsymbol{\epsilon}_i = (\epsilon_{i,1},\dots,\epsilon_{i,\zeta_i-1}) \in \{0,1\}^{\zeta_i-1}$ and
- $S_i^{(\epsilon_{i,1},\dots,\epsilon_{i,\zeta_i-1})}(m_{i,1},\dots,m_{i,\zeta_i-1})$ is given by

$$\begin{aligned}
&S_i^{(\epsilon_{i,1},\dots,\epsilon_{i,\zeta_i-1})}(m_{i,1},\dots,m_{i,\zeta_i-1})\\
=&\left(q^{(n_i+1)(m_{i,1}+1)} - q^{-(n_i+1)(m_{i,1}+1)}\right)(-1)^{\sum_{l=1}^{\zeta_i-1}\epsilon_{i,l}}(-1)^{\sum_{l=1}^{\zeta_i-1}a_{i,l}m_{i,l}}\\
&\times q^{\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}m_{i,l}(m_{i,l}+2)}{2}+\sum_{l=1}^{\zeta_i-1}(-1)^{\epsilon_{i,l}}(m_{i,l}+1)(m_{i,l+1}+1)}.
\end{aligned}$$

Note that in the formula of $S_i^{(\epsilon_l^i,\dots,\epsilon_{\zeta_i-1}^i)}$, the term $(-1)^{\sum_{l=1}^{\zeta_i}a_{i,l}m_{i,l}}$ is equal to 1 when $m_{i,l}$ is even. Nevertheless, we need this term for the next computation.

A direct computation shows that for each $i \in I$, we have

$$S_i^{(1,\epsilon_{i,2},\ldots,\epsilon_{i,\zeta_i-1})}(m_{i,1}, m_{i,2}, \ldots, m_{i,\zeta_i-1}) = S_i^{(0,\epsilon_{i,2},\ldots,\epsilon_{i,\zeta_i-1})}(r-2-m_{i,1}, m_{i,2}, \ldots, m_{i,\zeta_i-1}).$$

Note that since r is odd, $r-2-m_{i,1}$ runs through all odd integers from 0 to $r-2$. More generally, we have

$$\begin{aligned}&S_i^{(0,\ldots,0,1,\epsilon_{i,l+1},\ldots,\epsilon_{i,\zeta_i-1})}(m_{i,1}, \ldots, m_{i,l-1}, m_{i,l}, m_{i,l+1}, \ldots, m_{i,\zeta_i-1})\\ &= S_i^{(0,\ldots,0,0,\epsilon_{i,l+1},\ldots,\epsilon_{i,\zeta_i-1})}(r-2-m_{i,1}, \ldots, r-2-m_{i,l-1}, r-2-m_{i,l}, m_{i,l+1}, \ldots, m_{i,\zeta_i-1}).\end{aligned} \tag{3.4}$$

Originally, $\mathbf{m}_i = (m_{i,1}, \ldots, m_{i,\zeta_i-1})$ run through all multi-even integers in $\{0, 2, \ldots, r-3\}^{\zeta_i-1}$. By (3.4), we can change the sum $\mathbf{m}_I$ to be over all integers in $\{0, 1, \ldots, r-2\}^{\sum_{i\in I}\zeta_i-|I|}$ and write

$$\begin{aligned}&\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J))\\ =&\frac{\mu_r^{\sum_{i\in I}\zeta_i-c}}{\{1\}^{\sum_{i\in I}\zeta_i}} e^{-\sigma(\tilde{L}_{\mathrm{FSL},I}\cup L')(-\frac{3}{r}-\frac{r+1}{4})\sqrt{-1}\pi} \prod_{i\in I} q^{\frac{a_{i,0}n_i(n_i+2)}{2}} \prod_{j\in J}(-1)^{\frac{\iota_j m_j}{2}} q^{\left(a_{j,0}+\frac{\iota_j}{2}\right)\frac{m_j(m_j+2)}{2}}\\ &\quad\sum_{\mathbf{m}_{\zeta_\mathbf{i}}}\left(\prod_{i\in I}(-1)^{\frac{\iota_i m_{i,\zeta_i}}{2}} q^{\left(a_{i,\zeta_i}+\frac{\iota_i}{2}\right)\frac{m_{i,\zeta_i}(m_{i,\zeta_i}+2)}{2}} S_i^{(0,0,\ldots,0)}(m_{i,1}, m_{i,2}, \ldots, m_{i,\zeta_i-1})\right)\\ &\quad\times\prod_{s=1}^{c}\begin{vmatrix} m_{s_1} & m_{s_2} & m_{s_3}\\ m_{s_4} & m_{s_5} & m_{s_6}\end{vmatrix},\end{aligned}$$

where

$$\begin{aligned}&S_i^{(0,0,\ldots,0)}(m_{i,1}, m_{i,2}, \ldots, m_{i,\zeta_i-1})\\ &= \sum_{\mathbf{m}_I}\left(q^{(n_i+1)(m_{i,1}+1)} - q^{-(n_i+1)(m_{i,1}+1)}\right)(-1)^{\sum_{l=1}^{\zeta_i-1} a_{i,l}m_{i,l}} q^{\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}m_{i,l}(m_{i,l}+2)}{2}+\sum_{l=1}^{\zeta_i-1}(m_{i,l}+1)(m_{i,l+1}+1)}.\end{aligned}$$

Note that

$$S_i^{(0,0,\dots,0)}(m_{i,1},m_{i,2},\dots,m_{i,\zeta_i-1})=(-1)^{\sum_{l=1}^{\zeta_i-1}a_{i,l}}q^{-\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}}{2}}S_i(m_{i,\zeta_i}+1,n_i+1),$$

where $S_i(n_{\zeta_i},m_{i,\zeta_i})$ is the sum introduced in (3.3). By Lemma 3.3, we have

$$\begin{aligned}S_i^{(0,0,\dots,0)}(m_{i,1},m_{i,2},\dots,m_{i,\zeta_i-1})=(-1)^{\sum_{l=1}^{\zeta_i-1}a_{i,l}}q^{-\sum_{l=1}^{\zeta_i-1}\frac{a_{i,l}}{2}}\frac{(-1)^{\zeta_i+1}(\sqrt{-1}r)^{\frac{\zeta_i-1}{2}}}{\sqrt{q_i}}\\ \sum_{s_i=0}^{|q_i|-1}\left(e^{\frac{r}{4\pi\sqrt{-1}}Z_i^{+}\left(s_i,\frac{2\pi m_{i,\zeta_i}}{r}+\frac{2\pi}{r},\frac{2\pi n_i}{r}+\frac{2\pi}{r}\right)}-e^{\frac{r}{4\pi\sqrt{-1}}Z_i^{-}\left(s_i,\frac{2\pi m_{i,\zeta_i}}{r}+\frac{2\pi}{r},\frac{2\pi n_i}{r}+\frac{2\pi}{r}\right)}\right),\end{aligned}$$

where

$$\begin{aligned}Z_i^{\pm}(s,\alpha,\beta)=\frac{C_{i,\zeta_i-1}}{q_i}(\alpha-\pi)^2\mp\frac{2(\beta-\pi)(\alpha-\pi)}{q_i}-\frac{2\pi(I_i(s)\pm1)}{q_i}(\alpha-\pi)\\ +K_i(s)\pi^2-\frac{p_i'}{q_i}(\beta-\pi)^2+2\pi\beta\Big(-\frac{p_i'}{q_i}\mp J_i(s)\Big)+\frac{p_i'}{q_i}\pi^2.\end{aligned}\tag{3.5}$$

By a direct computation,

$$\begin{aligned}&Z_i^{\pm}\Big(s,\alpha+\frac{2\pi}{r},\beta+\frac{2\pi}{r}\Big)\\ =&Z_i^{\pm}(s,\alpha,\beta)+\frac{4\pi C_{i,\zeta_i-1}}{rq_i}(\alpha-\pi)+\frac{4\pi^2C_{i,\zeta_i-1}}{q_ir^2}\mp\frac{4\pi}{rq_i}(\alpha+\beta-2\pi)\mp\frac{8\pi^2}{r^2q_i}\\ &-\frac{4\pi^2(I_i(s)\pm1)}{rq_i}-\frac{p_i'}{q_i}\Big(\frac{4\pi}{r}(\beta-\pi)+\frac{4\pi^2}{r^2}\Big)+\frac{4\pi^2}{r}\Big(-\frac{p_i'}{q_i}\mp J_i(s)\Big),\end{aligned}$$

which implies that

$$\begin{aligned}&e^{\frac{r}{4\pi\sqrt{-1}}Z_i^{\pm}\left(s_i,\frac{2\pi m_{i,\zeta_i}}{r}+\frac{2\pi}{r},\frac{2\pi n_i}{r}+\frac{2\pi}{r}\right)}\\ =&e^{\sqrt{-1}\left(-\frac{C_{i,\zeta_i-1}}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)+\frac{pi'}{q_i}(\frac{2\pi n_i}{r}-\pi)\pm\frac{\left(\frac{2\pi m_{i,\zeta_i}}{r}+\frac{2\pi n_i}{r}-2\pi\right)}{q_i}+\pi\left(\frac{I_i(s_i)\pm1}{q_i}+\frac{p_i'}{q_i}\pm J_i(s_i)\right)\right)}\\ &\times e^{\frac{r}{4\pi\sqrt{-1}}\left(Z_i^{\pm}(s_i,\frac{2\pi m_{i,\zeta_i}}{r},\frac{2\pi n_i}{r})+\frac{4\pi^2}{r^2}\left(\frac{C_{i,\zeta_i-1}\mp2-p_i'}{q_i}\right)\right)}.\end{aligned}\tag{3.6}$$

Next, by a direct computation, for any even integer $n \in \mathbb{N}$, we have

$$q^{\frac{n(n+2)}{2}} = \left(e^{\frac{\pi\sqrt{-1}}{4}}\right)^{-r} q^{\frac{1}{2}\left(n-\frac{r}{2}\right)^2+n} = \left(e^{\frac{\pi\sqrt{-1}}{4}}\right)^{-r} e^{\sqrt{-1}\left(\frac{2\pi n}{r}\right)} e^{\frac{r}{4\pi\sqrt{-1}}\left(-\left(\frac{2\pi n}{r}-\pi\right)^2\right)} \tag{3.7}$$

By using Equation (3.7), we can write

$$q^{\frac{a_{j,0}m_j(m_j+2)}{2}} = \left(e^{-\frac{r\pi\sqrt{-1}}{4}a_{j,0}}\right)\left(e^{a_{j,0}\sqrt{-1}\left(\frac{2\pi m_j}{r}\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-a_{j,0}\left(\frac{2\pi m_j}{r}-\pi\right)^2\right)}\right), \tag{3.8}$$

$$q^{\frac{a_{i,0}n_i(n_i+2)}{2}} = \left(e^{-\frac{r\pi\sqrt{-1}}{4}a_{i,0}}\right)\left(e^{a_{i,0}\sqrt{-1}\left(\frac{2\pi n_i}{r}\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-a_{i,0}\left(\frac{2\pi n_i}{r}-\pi\right)^2\right)}\right) \tag{3.9}$$

and

$$\begin{aligned} &q^{\frac{a_{i,\zeta_i}m_{i,\zeta_i}(m_{i,\zeta_i}+2)}{2}} \\ &= \left(e^{-\frac{r\pi\sqrt{-1}}{4}a_{i,\zeta_i}}\right)\left(e^{a_{i,\zeta_i}\sqrt{-1}\left(\frac{2\pi m_{i,\zeta_i}}{r}\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-a_{i,\zeta_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2\right)}\right) \\ &= (-1)^{a_{i,\zeta_i}}\left(e^{-\frac{r\pi\sqrt{-1}}{4}a_{i,\zeta_i}}\right)\left(e^{a_{i,\zeta_i}\sqrt{-1}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-a_{i,\zeta_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2\right)}\right). \end{aligned} \tag{3.10}$$

In particular, by Lemma 2.16, the first term in (3.5) and the last term of (3.10) can be grouped together to give

$$\frac{C_{i,\zeta_i-1}}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2 - a_{i,\zeta_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2 = -\frac{p_i}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r}-\pi\right)^2.$$

Besides, for each $i \in I$, (3.6) and the third term in (3.10) can be grouped together to give

$$\begin{aligned}
& e^{\frac{r}{4\pi\sqrt{-1}} Z_i^{\pm}\left(s_i, \frac{2\pi m_{i,\zeta_i}}{r} + \frac{2\pi}{r}, \frac{2\pi n_i}{r} + \frac{2\pi}{r}\right)} e^{a_{i,\zeta_i}\sqrt{-1}\left(\frac{2\pi m_{i,\zeta_i}}{r} - \pi\right)} \\
=& e^{\sqrt{-1}\left(\frac{p_i'}{q_i}\left(\frac{2\pi m_{i,\zeta_i}}{r} - \pi\right) + \frac{pi'}{q_i}(\frac{2\pi n_i}{r} - \pi) \pm \frac{\left(\frac{2\pi m_{i,\zeta_i}}{r} + \frac{2\pi n_i}{r} - 2\pi\right)}{q_i} + \pi\left(\frac{I_i(s_i)\pm 1}{q_i} + \frac{p_i'}{q_i} \pm J_i(s_i)\right)\right)} \\
& \times e^{\frac{r}{4\pi\sqrt{-1}}\left(Z_i^{\pm}(s_i, \frac{2\pi m_{i,\zeta_i}}{r}, \frac{2\pi n_i}{r}) + \frac{4\pi^2}{r^2}\left(\frac{C_{i,\zeta_i - 1} \mp 2 - p_i'}{q_i}\right)\right)}.
\end{aligned}$$

For any even integer $n \in \mathbb{N}$, we have

$$(-1)^{\frac{n}{2}} q^{\frac{n(n+2)}{4}} = \left(e^{\frac{\pi\sqrt{-1}}{8}}\right)^{-r} q^{\frac{1}{4}\left(n - \frac{r}{2}\right)^2 + \frac{n}{2}} = e^{\frac{r}{4\pi\sqrt{-1}}\left(\frac{\pi^2}{2}\right)} e^{\frac{\sqrt{-1}}{2}\left(\frac{2\pi n}{r}\right)} e^{\frac{r}{4\pi\sqrt{-1}}\left(-\frac{1}{2}\left(\frac{2\pi n}{r} - \pi\right)^2\right)}. \tag{3.11}$$

By using Equation (3.11), we can write

$$(-1)^{\frac{\iota_j m_j}{2}} q^{\frac{\iota_j m_j (m_j + 2)}{4}} = \left(e^{\frac{r}{4\pi\sqrt{-1}}\left(\frac{\iota_j \pi^2}{2}\right)}\right)\left(e^{\frac{\iota_j\sqrt{-1}}{2}\left(\frac{2\pi m_j}{r}\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-\frac{\iota_j}{2}\left(\frac{2\pi m_j}{r} - \pi\right)^2\right)}\right) \tag{3.12}$$

and

$$(-1)^{\frac{\iota_i m_{i,\zeta_i}}{2}} q^{\frac{\iota_i m_{i,\zeta_i}(m_{i,\zeta_i} + 2)}{4}} = \left(e^{\frac{r}{4\pi\sqrt{-1}}\left(\frac{\iota_i \pi^2}{2}\right)}\right)\left(e^{\frac{\iota_i\sqrt{-1}}{2}\left(\frac{2\pi m_{i,\zeta_i}}{r}\right)}\right)\left(e^{\frac{r}{4\pi\sqrt{-1}}\left(-\frac{\iota_i}{2}\left(\frac{2\pi m_{i,\zeta_i}}{r} - \pi\right)^2\right)}\right). \tag{3.13}$$

By Proposition 3.2, we have

$$\prod_{s=1}^{c} \begin{vmatrix} m_{s_1} & m_{s_2} & m_{s_3} \\ m_{s_4} & m_{s_5} & m_{s_6} \end{vmatrix} = \frac{\{1\}^c}{2^c} \sum_{\mathbf{k}} e^{\frac{r}{4\pi\sqrt{-1}} \sum_{s=1}^{c} U_r\left(\frac{2\pi m_{s_1}}{r}, \dots, \frac{2\pi m_{s_6}}{r}, \frac{2\pi k_s}{r}\right)}, \tag{3.14}$$

where $\mathbf{k} = (k_1, \dots, k_c)$ runs over all multi-integers with each k_s lying in between $\max\{T_{s_i}\}$ and $\min\{Q_{s_j}, r-2\}$. The result then follows from above computations.

$\square$

4. POISSON SUMMATION FORMULA

To apply the Poisson Summation Formula to the summation in Proposition 3.4, we consider the following regions and a bump function over them.

For a fixed $(\boldsymbol{\alpha}_j)_{j\in J} \in [0,2\pi]^{|J|}$, we let $\boldsymbol{\alpha}_{\zeta_I} = (\alpha_{i,\zeta_i})_{i\in I} \in [0,2\pi]^{|I|}$, $\boldsymbol{\xi} = (\xi_1,\dots,\xi_c) \in \mathbb{R}^c$,

$$D_A = \{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) \in \mathbb{R}^{|I|+c} \mid (\alpha_{s_1},\dots,\alpha_{s_6}) \text{ is admissible, } \max\{\tau_{s_i}\} \leq \xi_s \leq \min\{\eta_{s_j},2\pi\}\},$$

and

$$D_H = \{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) \in D_A \mid (\alpha_{s_1},\dots,\alpha_{s_6}) \text{ is of hyperideal type}\}.$$

For sufficiently small $\delta > 0$, we let

$$D_H^\delta = \{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) \in D_H \mid d((\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}),\partial D_H) > \delta\},$$

where d is the Euclidean distance on $\mathbb{R}^{|I|+c}$. Moreover, we let $\psi : \mathbb{R}^{|I|+c} \to [0,1]$ be a C^∞-smooth bump function satisfying

$$\begin{aligned} \psi(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) &= 1 \quad \text{for} \quad (\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) \in \overline{D_H^\delta}, \\ \psi(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) &= 0 \quad \text{for} \quad (\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) \notin D_H \end{aligned}$$

and $\psi \in (0,1)$ elsewhere.

Moreover, we let

$$f_r^{\mathbf{E}_I}(\mathbf{s}_I,\mathbf{m}_{\zeta_I},\mathbf{k}) = \psi\left(\frac{2\pi\mathbf{m}_{\zeta_I}}{r},\frac{2\pi\mathbf{k}}{r}\right) g_r^{\mathbf{E}_I}(\mathbf{s}_I,\mathbf{m}_{\zeta_I},\mathbf{k}).$$

To apply the Poisson summation formula, for each $i \in I$, we let $m_{i,\zeta_i} = 2\widehat{m}_{i,\zeta_i}$ and $\widehat{\mathbf{m}}_{\zeta_I} =$

$(\widehat{\mathbf{m}}_{i,\zeta_i})_{i\in I}$. Then from Proposition 3.4, we have

$$\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J)) = Z_r \sum_{(\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) \in \mathbb{Z}^{|I|+c}} \left(\sum_{\mathbf{E}_I, \mathbf{s}_I} f_r^{\boldsymbol{E}_I}(\mathbf{s}_I, 2\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) \right) + \text{error term.} \tag{4.1}$$

Let

$$f_r(2\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) = \sum_{\mathbf{E}_I, \mathbf{s}_I} f_r^{\mathbf{E}_I}(\mathbf{s}_I, 2\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}).$$

Since f_r is in the Schwartz space on $\mathbb{R}^{|I|+c}$, by the Poisson summation formula (see e.g. [51, Theorem 3.1]),

$$\sum_{(\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) \in \mathbb{Z}^{|I|+c}} f_r(2\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) = \sum_{(\mathbf{A}_{\zeta_I}, \mathbf{B}) \in \mathbb{Z}^{|I|+c}} \widehat{f}_r(\mathbf{A}_{\zeta_I}, \mathbf{B}),$$

where $\mathbf{A}_{\zeta_I} = (A_{i,\zeta_i})_{i\in I} \in \mathbb{Z}^{|I|}$, $\mathbf{B} = (B_1, \dots, B_c) \in \mathbb{Z}^c$ and $\widehat{f}_r(\mathbf{A}_{\zeta_I}, \mathbf{B})$ is the $(\mathbf{A}_{\zeta_I}, \mathbf{B})$-th Fourier coefficient of f_r defined by

$$\widehat{f}_r(\mathbf{A}_{\zeta_I}, \mathbf{B}) = \int_{\mathbb{R}^{|I|+c}} f_r(2\widehat{\mathbf{m}}_{\zeta_I}, \mathbf{k}) e^{\sum_{i\in I} 2\pi\sqrt{-1} A_{i,\zeta_i} \widehat{m}_{i,\zeta_i} + \sum_{s=1}^{c} 2\pi\sqrt{-1} B_l k_l} d\widehat{\mathbf{m}}_{\zeta_I} d\mathbf{k}$$

where $d\widehat{\mathbf{m}}_{\zeta_I} d\mathbf{k} = \prod_{i\in I} d\widehat{m}_{i,\zeta_i} \prod_{s=1}^{c} dk_l$.

By change of variables $\widehat{m}_{i,\zeta_i} = \frac{r}{4\pi}\alpha_{i,\zeta_i}$ and $k_l = \frac{r}{2\pi}\xi_s$, the Fourier coefficients can be computed as follows.

Proposition 4.1.

$$\widehat{f}_r(\mathbf{A}_{\zeta_I}, \mathbf{B}) = \sum_{\mathbf{E}_I, \mathbf{s}_I} \widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I, \mathbf{A}_{\zeta_I}, \boldsymbol{B})$$

with

$$\begin{aligned} \widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I, \mathbf{A}_{\zeta_I}, \boldsymbol{B}) &= \frac{r^{|I|+c}\left(\prod_{i\in I} E_i\right)}{2^{2|I|+c}\pi^{|I|+c}} \\ &\times \int_{D_H} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) - \sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i} - \sum_{s=1}^{c} 4\pi B_s \xi_s\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}, \end{aligned}$$

where $d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} = \prod_{i\in I} d\alpha_{i,\zeta_i} \prod_{s=1}^{c} d\xi_s$*,*

$$\phi_r(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \psi(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) e^{\sqrt{-1} P_r^{\mathbf{E}_I}(\mathbf{s}_I, \boldsymbol{\alpha}_{\boldsymbol{\xi_I}})},$$

$$\begin{aligned} P_r^{\mathbf{E}_I}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}) =& \sum_{i\in I} \Big(\frac{p_i'}{q_i}(\beta_i - \pi) + \frac{p_i}{q_i}(\alpha_{i,\zeta_i} - \pi) + \frac{E_i(\alpha_{i,\zeta_i} + \beta_i - 2\pi)}{q_i} \Big) \\ &+ \pi \sum_{i\in I} \Big(\frac{I_i(s_i) + E_i}{q_i} + \frac{p_i'}{q_i} + E_i J_i(s_i) \Big) \\ &+ \sum_{i\in I} a_{i,0}\beta_i + \sum_{i\in I} \Big(\frac{\iota_i}{2}\Big)\alpha_{i,\zeta_i} + \sum_{j\in J} \Big(a_{j,0} + \frac{\iota_j}{2} \Big)\alpha_j \end{aligned}$$

and

$$\begin{aligned} W_r^{\mathbf{E}_I}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) =& -\sum_{i\in I} \Big(a_{i,0} + \frac{p_i'}{q_i} \Big)(\beta_i - \pi)^2 - \sum_{j\in J} \Big(a_{j,0} + \frac{\iota_j}{2} \Big)(\alpha_j - \pi)^2 \\ &- \sum_{i\in I} \Big(\frac{p_i}{q_i} + \frac{\iota_i}{2} \Big)(\alpha_{i,\zeta_i} - \pi)^2 - \sum_{i\in I} \frac{2\pi(I_i(s_i) + E_i)}{q_i}(\alpha_{\xi_i} - \pi) \\ &- \sum_{i\in I} \frac{2E_i(\alpha_{i,\zeta_i} - \pi)(\beta_i - \pi)}{q_i} + \sum_{i\in I} 2\pi\beta_i \Big(-E_i J_i(s_i) - \frac{p_i}{q_i} \Big) \\ &+ \sum_{s=1}^{c} U_r(\alpha_{s_1}, \ldots, \alpha_{s_6}, \xi_s) + \sum_{i\in I} \pi^2 \Bigg(K_i(s_i) + \frac{p_i'}{q_i} \Bigg) + \Big(\sum_{i=1}^{n} \frac{\iota_i}{2} \Big)\pi^2 \\ &+ \frac{4\pi^2}{r^2} h_I \end{aligned}$$

with $\beta_i = \frac{2\pi n_i}{r}$ *for* $i \in I$*,* $\alpha_j = \frac{2\pi m_j}{r}$ *for* $j \in J$ *and* $h_I = \sum_{i\in I} \frac{C_{i,\xi_i - 1} - 2E_i - p_i'}{q_i}$*.*

Together with Equation (4.1), we have

Proposition 4.2.

$$\mathrm{RT}_r(M, L, (\boldsymbol{n}_I, \boldsymbol{m}_J)) = Z_r \sum_{(\mathbf{A}_{\zeta_I}, \mathbf{B}) \in \mathbb{Z}^{|I|+c}} \widehat{f}_r(\mathbf{A}_{\zeta_I}, \boldsymbol{B}) + \textit{error term.}$$

The error term in Proposition 4.2 will be estimated in Proposition 5.24 of Section 5.

For each $i \in I$, with respect to the continued fraction

$$\frac{p_i}{q_i} = [a_{i,1}, \dots, a_{i,\zeta_i}] = a_{i,\zeta_i} - \frac{1}{a_{i,\zeta_i-1} - \frac{1}{\dots - \frac{1}{a_{i,1}}}},$$

let $s_i^{\pm}$ and $m_i^{\pm}$ be the integers $s^{\pm}$ and $m^{\pm}$ defined in Lemma 2.18 (1). For each multi-sign $\mathbf{E}_I = (E_i)_{i\in I} \in \{-1,1\}^{|I|}$, define $\mathbf{s}^{\mathbf{E}_I} = (s_i^{E_I})_{i\in I} \in \mathbb{Z}^{|I|}$ and $\mathbf{m}^{\mathbf{E}_I} = (m_i^{E_I})_{i\in I} \in \mathbb{Z}^{|I|}$ by

$$s_i^{E_I} = \begin{cases} s_i^+ & \text{if } E_i = -1 \\ s_i^- & \text{if } E_i = 1 \end{cases}$$

and

$$m_i^{E_I} = \begin{cases} m_i^+ & \text{if } E_i = -1 \\ m_i^- & \text{if } E_i = 1 \end{cases}.$$

In particular, by Lemma 2.18 (1), (2) and definitions of $s_i^{E_I}$ and $m_i^{E_I}$, we have

$$I_i(s_i^{E_I}) = -E_i - q_i + 2m_i^{E_I} q_i \quad \text{and} \quad E_i J_i(s_i^{E_I}) \equiv -\frac{p_i}{q_i} \pmod{\mathbb{Z}}. \tag{4.2}$$

Let $\mathbf{1} - \mathbf{2m}^{\mathbf{E_I}} = (1 - 2m_i^{E_i})_{i\in I} \in \mathbb{Z}^{|I|}$. In Section 5, we will show that $\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1} - \mathbf{2m}^{\mathbf{E_I}}, \mathbf{0})$ are the leading Fourier coefficients. The following proposition gives a simplified expression for the Fourier coefficients that will be used later.

Proposition 4.3. *We have*

$$\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1} - \mathbf{2m}^{\mathbf{E_I}}, \mathbf{0}) = \frac{Y(\mathbf{E}_I) r^{|I|+c}}{2^{|I|+c}\pi^{|I|+c}} \int_{D_H} \phi_r\left(\mathbf{s}^{\mathbf{E}_I}, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi},$$

where

$$Y(\mathbf{E}_I) = -(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}+E_iJ_i(s_i^{E_I})\right)+|I|}\left(\prod_{i\in I}E_i\right)e^{\frac{r\pi}{4\sqrt{-1}}\sum_{i\in I}\left(4m_i^{E_I}-2+K_i(s_i^{E_I})+\frac{p_i'}{q_i}\right)}, \tag{4.3}$$

$$\begin{aligned}&\phi_r(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = \psi(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\\ &\times e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I}a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)}\end{aligned}$$

and

$$\begin{aligned}G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = &\sum_{i\in I}\left[-\left(a_{i,0}+\frac{p_i'}{q_i}\right)(\beta_i-\pi)^2-\frac{p_i(\alpha_{i,\zeta_i}-\pi)^2+2E_i(\beta_i-\pi)(\alpha_{i,\zeta_i}-\pi)}{q_i}\right]\\ &-\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)(\alpha_j-\pi)^2-\sum_{i\in I}\frac{\iota_i}{2}(\alpha_{i,\zeta_i}-\pi)^2\\ &+\sum_{s=1}^{c}U_r(\alpha_{s_1},\dots,\alpha_{s_6},\xi_s)+\Big(\sum_{i=1}^{n}\frac{\iota_i}{2}\Big)\pi^2+\frac{4\pi^2}{r^2}h_I.\end{aligned} \tag{4.4}$$

Proof. By definition of the Fourier coefficient and the bump function ϕ, we can write

$$\begin{aligned}&\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}^{\mathbf{E}_I},\mathbf{1}-\mathbf{2m}^{\mathbf{E_I}},\mathbf{0}) = \frac{r^{|I|+c}}{2^{|I|+c}\pi^{|I|+c}}\\ &\times\int_{D_H}(-1)^{\sum_{i\in I}(1-2m_i^{E_i})}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I}2\pi(1-2m_i^{E_I})\alpha_{i,\zeta_i}\right)}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}.\end{aligned}$$

By (4.2),

$$e^{\sqrt{-1}\pi\sum_{i\in I}\left(\frac{I_i(s_i)+E_i}{q_i}+\frac{p_i'}{q_i}+E_iJ_i(s_i)\right)} = -(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}+E_iJ_i(s_i)\right)}.$$

Besides, by (4.2), we can write

$$\begin{aligned}
&W_r^{\mathbf{E}_I}\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I}2\pi(1-2m_i^{E_I})\alpha_{i,\zeta_i}\\
=&G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})-2\pi^2\sum_{i\in I}(1-2m_i^{E_i})+\sum_{i\in I}2\pi\beta_i\Big(-E_iJ_i(s_i^{E_I})-\frac{p_i}{q_i}\Big)\\
&+\sum_{i\in I}\pi^2\left(K_i(s_i^{E_I})+\frac{p_i'}{q_i}\right)+\frac{4\pi^2}{r^2}h_I.
\end{aligned}$$

Furthermore, since $\beta_i=\frac{2\pi n_i}{r}$ and n_i is even for all $i\in I$, by (4.2),

$$\frac{r}{4\pi\sqrt{-1}}\Big(2\pi\beta_i\Big(-E_iJ_i(s_i^{E_I})-\frac{p_i}{q_i}\Big)\Big)\in 2\pi\sqrt{-1}\mathbb{Z}\quad\text{and}\quad e^{\frac{r}{4\pi\sqrt{-1}}\sum_{i\in I}2\pi\beta_i\left(-E_iJ_i(s_i^{E_I})-\frac{p_i}{q_i}\right)}=1.$$

The result follows from a direct computation. $\square$

Define

$$Y=-(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}-J_i(s_i^+)\right)}e^{\frac{r\pi}{4\sqrt{-1}}\sum_{i\in I}\left(4m_i^+-2+K_i(s_i^+)+\frac{p_i'}{q_i}\right)}. \tag{4.5}$$

The following lemma ensures that the leading Fourier coefficients in Proposition 4.3 do not cancel out with each other.

Lemma 4.4. *For any $\mathbf{E}_I\in\{1,-1\}^{|I|}$, we have $Y(\mathbf{E}_I)=Y$.*

Proof. Note that Y is equal to $Y(\mathbf{E}_I)$ with $E_i=-1$ for all $i\in I$. We claim that

$$(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}-J_i(s_i^+)\right)}e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^+-2+K_i(s^+)+\frac{p_i'}{q_i}\right)}=-(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}+J_i(s_i^-)\right)}e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^--2+K_i(s^-)+\frac{p_i'}{q_i}\right)}$$

for any $i\in I$. This shows that $Y(\mathbf{E}_I)$ is invariant when we change E_i to $-E_i$ for any $i\in I$. By changing all E_i to -1, we get the desired result.

To prove the claim, first, from Lemma 2.18 (2), since

$$J_i(s_i^+) \equiv -J_i(s_i^-) \pmod{2\mathbb{Z}},$$

we have

$$(-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}-J_i(s_i^+)\right)} = (-1)^{\sum_{i\in I}\left(\frac{p_i'}{q_i}+J_i(s_i^-)\right)}.$$

Moreover, from the definition of K in Lemma 2.18 (3), we get

$$K_i(s_i^+) - K_i(s_i^-) + 4(m_i^+ - m_i^-) = \frac{4C_{i,\zeta_i-1}}{q_i}(s_i^+ + s_i^- + 1 + K_{i,\zeta_i-1})(s_i^+ - s_i^-) + 4(m_i^+ - m_i^-). \tag{4.6}$$

Besides, from the definition of I and Lemma 2.18 (1),

$$I_i(s_i^+) + I_i(s_i^-) = -2C_{i,\zeta_i-1}(s_i^+ + s_i^- + 1 + K_{i,\zeta_i-1}) = 2q_i(m_i^+ + m_i^- - 1). \tag{4.7}$$

From (4.6) and (4.7), we have

$$K_i(s_i^+) - K_i(s_i^-) + 4(m_i^+ - m_i^-) = 4((1 - m_i^+ - m_i^-)(s_i^+ - s_i^-) + m_i^+ + m_i^-).$$

In particular,

$$\begin{aligned}
\frac{e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^+-2+K_i(s^+)+\frac{p_i'}{q_i}\right)}}{-e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^--2+K_i(s^-)+\frac{p_i'}{q_i}\right)}} &= -e^{\frac{r\pi}{4\sqrt{-1}}(K_i(s_i^+)-K_i(s_i^-)+4(m_i^+-m_i^-))} \\
&= -e^{-\pi\sqrt{-1}((1-m_i^+-m_i^-)(s_i^++s_i^-)+m_i^++m_i^-)} \\
&= -(-1)^{(m_i^+-m_i^-)(s_i^++s_i^-+1)+(s_i^++s_i^-)}.
\end{aligned}$$

We claim that the integer $(m_i^+ - m_i^-)(s_i^+ + s_i^- + 1) + (s_i^+ + s_i^-)$ is always odd. Note that by Lemma

2.18 (1) and the definition of I, we have

$$-2C_{i,\zeta_i-1}(s_i^+ - s_i^-) = I(s^+) - I(s^-) = 2(m^+ - m^-)q_i + 2,$$

which implies that

$$(m^+ - m^-)q_i + C_{i,\zeta_i-1}(s_i^+ - s_i^-) = -1.$$

In particular, at least one of $(m^+ - m^-)$ and $(s_i^+ - s_i^-)$ must be odd. Note that if $(s_i^+ - s_i^-)$ is even, then $(m^+ - m^-)$ is odd. In particular, $(m_i^+ - m_i^-)(s_i^+ + s_i^- + 1) + (s_i^+ + s_i^-)$ is odd. If $(s_i^+ - s_i^-)$ is odd, then $(s_i^+ + s_i^-)$ is odd and $(s_i^+ + s_i^- + 1)$ is even. In particular, $(m_i^+ - m_i^-)(s_i^+ + s_i^- + 1) + (s_i^+ + s_i^-)$ is odd.

Altogether,

$$\frac{e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^+ - 2 + K_i(s^+) + \frac{p_i'}{q_i}\right)}}{-e^{\frac{r\pi}{4\sqrt{-1}}\left(4m_i^- - 2 + K_i(s^-) + \frac{p_i'}{q_i}\right)}} = -(-1)^{(m_i^+ - m_i^-)(s_i^+ + s_i^- + 1) + (s_i^+ + s_i^-)} = 1.$$

This completes the proof. $\square$

For $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $\mathrm{Re}(\mathbf{z}) = (\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_n))$, where $\mathrm{Re}\, z_i$ is the real part of z_i for $i = 1, \dots, n$. Let

$$D_{H,\mathbb{C}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \mathbb{C}^{|I|+c} \mid (\mathrm{Re}(\boldsymbol{\alpha}_{\zeta_I}), \mathrm{Re}(\boldsymbol{\xi})) \in D_H\}.$$

To end this section, we consider a closely related function $G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) : D_{H,\mathbb{C}} \to \mathbb{C}$ given by

$$\begin{aligned} G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = &\sum_{i\in I} \left[-\left(\frac{p_i'}{q_i} + a_{i,0}\right)(\beta_i - \pi)^2 - \frac{p_i(\alpha_{i,\zeta_i} - \pi)^2 + 2E_i(\beta_i - \pi)(\alpha_{i,\zeta_i} - \pi)}{q_i}\right] \\ &- \sum_{j\in J} a_{j,0}(\alpha_j - \pi)^2 - \sum_{i=1}^{n} \frac{\iota_i}{2}(\alpha_i - \pi)^2 + \sum_{s=1}^{c} U(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s) + \Big(\sum_{i=1}^{n} \frac{\iota_i}{2}\Big)\pi^2, \end{aligned} \tag{4.8}$$

where U is defined by

$$\begin{aligned}U(\alpha_1,\dots,\alpha_6,\xi)=&\pi^2+\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{3}(\eta_j-\tau_i)^2-\frac{1}{2}\sum_{i=1}^{4}(\tau_i-\pi)^2\\&+(\xi-\pi)^2-\sum_{i=1}^{4}(\xi-\tau_i)^2-\sum_{j=1}^{3}(\eta_j-\xi)^2\\&-2\operatorname{Li}_2(1)-\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{3}\operatorname{Li}_2\left(e^{2i(\eta_j-\tau_i)}\right)+\frac{1}{2}\sum_{i=1}^{4}\operatorname{Li}_2\left(e^{2i(\tau_i-\pi)}\right)\\&-\operatorname{Li}_2\left(e^{2i(\xi-\pi)}\right)+\sum_{i=1}^{4}\operatorname{Li}_2\left(e^{2i(\xi-\tau_i)}\right)+\sum_{j=1}^{3}\operatorname{Li}_2\left(e^{2i(\eta_j-\xi)}\right).\end{aligned}\tag{4.9}$$

Note that when both $G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ and $G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ are defined, they are related by

$$\lim_{r\to\infty}G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})=G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}).$$

More preciesly, by Lemma 2.15, the differenece between $G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ and $G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is given by the following lemma.

Lemma 4.5. *On any compact subset of $D_{H,\mathbb{C}}$, we have*

$$G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})=G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})-\frac{4c\pi\sqrt{-1}}{r}\log\left(\frac{r}{2}\right)+\frac{4\pi\sqrt{-1}\kappa(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})}{r}+\frac{\upsilon_r(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})}{r^2},\tag{4.10}$$

where

$$\begin{aligned}&\kappa(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\\=&\sum_{s=1}^{c}\Big(\frac{1}{2}\sum_{i=1}^{4}\sqrt{-1}\tau_{s_i}-\sqrt{-1}\xi_s-\sqrt{-1}\pi-\frac{\sqrt{-1}\pi}{2}\\&+\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{3}\log\left(1-e^{2\sqrt{-1}(\eta_{s_j}-\tau_{s_i})}\right)-\frac{3}{4}\sum_{i=1}^{4}\log\left(1-e^{2\sqrt{-1}(\tau_{s_i}-\pi)}\right)\\&+\frac{3}{2}\log\left(1-e^{2\sqrt{-1}(\xi_s-\pi)}\right)-\frac{1}{2}\sum_{i=1}^{4}\log\left(1-e^{2\sqrt{-1}(\xi_s-\tau_{s_i})}\right)-\frac{1}{2}\sum_{j=1}^{3}\log\left(1-e^{2\sqrt{-1}(\eta_{s_j}-\xi_s)}\right)\Big)\end{aligned}$$

and $|\nu_r(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})|$ *is bounded from above by a constant independent of* r.

Proof. Note that

$$G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \sum_{s=1}^{c}(U_r(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s) - U(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s)) + O\left(\frac{1}{r^2}\right).$$

Thus, it suffices to study the difference between $U_r(\alpha_1, \dots, \alpha_6, \xi)$ and $U(\alpha_1, \dots, \alpha_6, \xi)$.

By Lemma 2.15(3),

$$\varphi_r\Big(\frac{\pi}{r}\Big) = \mathrm{Li}_2(1) + \frac{2\pi\sqrt{-1}}{r}\log\left(\frac{r}{2}\right) - \frac{\pi^2}{r} + O\Big(\frac{1}{r^2}\Big).$$

Besides, by using Lemma 2.15(1), we have

$$\varphi_r\Big(\eta_j - \tau_i + \frac{\pi}{r}\Big) = \mathrm{Li}_2\left(e^{2\sqrt{-1}(\eta_j - \tau_i + \frac{\pi}{r})}\right) + \frac{2\pi^2 e^{2\sqrt{-1}(\eta_j - \tau_i + \frac{\pi}{r})}}{3\left(1 - e^{2\sqrt{-1}(\eta_j - \tau_i + \frac{\pi}{r})}\right)}\frac{1}{r^2} + O\left(\frac{1}{r^4}\right).$$

In particular, on a given compact subset of $D_{H,\mathbb{C}}$, by continuity, we have

$$\varphi_r\Big(\eta_j - \tau_i + \frac{\pi}{r}\Big) = \mathrm{Li}_2\left(e^{2\sqrt{-1}(\eta_j - \tau_i + \frac{\pi}{r})}\right) + O\left(\frac{1}{r^2}\right).$$

Next, by considering the Talyor series expansion of $\mathrm{Li}_2\left(e^{2\sqrt{-1}(\eta_j - \tau_i + w)}\right)$ at $w = 0$, we have

$$\varphi_r\Big(\eta_j - \tau_i + \frac{\pi}{r}\Big) = \mathrm{Li}_2\left(e^{2\sqrt{-1}(\eta_j - \tau_i)}\right) - 2\sqrt{-1}\log(1 - e^{2\sqrt{-1}(\eta_j - \tau_i)})\left(\frac{\pi}{r}\right) + O\left(\frac{1}{r^2}\right).$$

Similar computations show that

$$\begin{aligned}
\varphi_r\Big(\tau_i - \pi + \frac{3\pi}{r}\Big) &= \mathrm{Li}_2\left(e^{2\sqrt{-1}(\tau_i-\pi)}\right) - 2\sqrt{-1}\log(1-e^{2\sqrt{-1}(\tau_i-\pi)})\left(\frac{3\pi}{r}\right) + O\left(\frac{1}{r^2}\right),\\
\varphi_r\Big(\xi - \pi + \frac{3\pi}{r}\Big) &= \mathrm{Li}_2\left(e^{2\sqrt{-1}(\xi-\pi)}\right) - 2\sqrt{-1}\log(1-e^{2\sqrt{-1}(\xi-\pi)})\left(\frac{3\pi}{r}\right) + O\left(\frac{1}{r^2}\right),\\
\varphi_r\Big(\xi - \tau_i + \frac{\pi}{r}\Big) &= \mathrm{Li}_2\left(e^{2\sqrt{-1}(\xi-\tau_i)}\right) - 2\sqrt{-1}\log(1-e^{2\sqrt{-1}(\xi-\tau_i)})\left(\frac{\pi}{r}\right) + O\left(\frac{1}{r^2}\right),\\
\varphi_r\Big(\eta_j - \xi + \frac{\pi}{r}\Big) &= \mathrm{Li}_2\left(e^{2\sqrt{-1}(\eta_j-\xi)}\right) - 2\sqrt{-1}\log(1-e^{2\sqrt{-1}(\eta_j-\xi)})\left(\frac{\pi}{r}\right) + O\left(\frac{1}{r^2}\right).
\end{aligned}$$

Equation (4.10) then follows from a direct computation. □

5. ASYMPTOTICS OF THE INVARIANTS

In this section, we will find the asymptotics of the leading Fourier coefficients and estimate the other.

5.1 Preliminary

5.1.1 Saddle point approximation

First, to obtain the asymptotic of the invariants, we recall the following proposition from [62].

Proposition 5.1. *[62] Let $D_{\mathbf{z}}$ be a region in $\mathbb{C}^n$ and let $D_{\mathbf{a}}$ be a region in $\mathbb{R}^k$. Let $f(\mathbf{z},\mathbf{a})$ and $g(\mathbf{z},\mathbf{a})$ be complex valued functions on $D_{\mathbf{z}}\times D_{\mathbf{a}}$ which are holomorphic in $\mathbf{z}$ and smooth in $\mathbf{a}$. For each positive integer r, let $f_r(\mathbf{z},\mathbf{a})$ be a complex valued function on $D_{\mathbf{z}}\times D_{\mathbf{a}}$ holomorphic in $\mathbf{z}$ and smooth in $\mathbf{a}$. For a fixed $\mathbf{a}\in D_{\mathbf{a}}$, let $f^{\mathbf{a}}$, $g^{\mathbf{a}}$ and $f^{\mathbf{a}}_r$ be the holomorphic functions on $D_{\mathbf{z}}$ defined by $f^{\mathbf{a}}(\mathbf{z})=f(\mathbf{z},\mathbf{a})$, $g^{\mathbf{a}}(\mathbf{z})=g(\mathbf{z},\mathbf{a})$ and $f^{\mathbf{a}}_r(\mathbf{z})=f_r(\mathbf{z},\mathbf{a})$. Suppose $\{\mathbf{a}_r\}$ is a convergent sequence in $D_{\mathbf{a}}$ with $\lim_r\mathbf{a}_r=\mathbf{a}_0$, $f^{\mathbf{a}_r}_r$ is of the form*

$$f^{\mathbf{a}_r}_r(\mathbf{z})=f^{\mathbf{a}_r}(\mathbf{z})+\frac{\upsilon_r(\mathbf{z},\mathbf{a}_r)}{r^2},$$

$\{S_r\}$ is a sequence of embedded real n-dimensional closed disks in $D_{\mathbf{z}}$ sharing the same boundary and converging to an embedded n-dimensional disk S_0, and $\mathbf{c}_r$ is a point on S_r such that $\{\mathbf{c}_r\}$ is convergent in $D_{\mathbf{z}}$ with $\lim_r\mathbf{c}_r=\mathbf{c}_0$. If for each r

(1) $\mathbf{c}_r$ is a critical point of $f^{\mathbf{a}_r}$ in $D_{\mathbf{z}}$,

(2) $\mathrm{Re}f^{\mathbf{a}_r}(\mathbf{c}_r)>\mathrm{Re}f^{\mathbf{a}_r}(\mathbf{z})$ for all $\mathbf{z}\in S_r\smallsetminus\{\mathbf{c}_r\}$,

(3) the domain $\{\mathbf{z}\in D_{\mathbf{z}}\mid \mathrm{Re}f^{\mathbf{a}_r}(\mathbf{z})<\mathrm{Re}f^{\mathbf{a}_r}(\mathbf{c}_r)\}$ deformation retracts to $S_r\smallsetminus\{\mathbf{c}_r\}$,

(4) $|g^{\mathbf{a}_r}(\mathbf{c}_r)|$ is bounded from below by a positive constant independent of r,

(5) $|\upsilon_r(\mathbf{z},\mathbf{a}_r)|$ is bounded from above by a constant independent of r on $D_{\mathbf{z}}$, and

(6) the Hessian matrix $\mathrm{Hess}(f^{\mathbf{a}_0})$ *of* $f^{\mathbf{a}_0}$ *at* $\mathbf{c}_0$ *is non-singular,*

then

$$\int_{S_r} g^{\mathbf{a}_r}(\mathbf{z}) e^{r f_r^{\mathbf{a}_r}(\mathbf{z})} d\mathbf{z} = \Big(\frac{2\pi}{r}\Big)^{\frac{n}{2}} \frac{g^{\mathbf{a}_r}(\mathbf{c}_r)}{\sqrt{-\det \mathrm{Hess}(f^{\mathbf{a}_r})(\mathbf{c}_r)}} e^{r f^{\mathbf{a}_r}(\mathbf{c}_r)} \Big(1 + O\Big(\frac{1}{r}\Big)\Big).$$

In Section 6.4, We will apply Proposition 5.1 to obtain the asymptotic expansion formula for the leading Fourier coefficient (see Proposition 5.12 for more details).

5.1.2 Convexity and preliminary estimate

Next, to show that conditions in Proposition 5.1 are satisfied, we need the following result about the function U defined in (4.9). Recall that the function $U(\alpha_1, \dots, \alpha_6, \xi)$ in (4.9) is given by

$$\begin{aligned} U(\alpha_1, \dots, \alpha_6, \xi) =& \pi^2 + \frac{1}{2}\sum_{i=1}^4 \sum_{j=1}^3 (\eta_j - \tau_i)^2 - \frac{1}{2}\sum_{i=1}^4 (\tau_i - \pi)^2 \\ &+ (\xi - \pi)^2 - \sum_{i=1}^4 (\xi - \tau_i)^2 - \sum_{j=1}^3 (\eta_j - \xi)^2 \\ &- 2\,\mathrm{Li}_2(1) - \frac{1}{2}\sum_{i=1}^4 \sum_{j=1}^3 \mathrm{Li}_2\left(e^{2i(\eta_j - \tau_i)}\right) + \frac{1}{2}\sum_{i=1}^4 \mathrm{Li}_2\left(e^{2i(\tau_i - \pi)}\right) \\ &- \mathrm{Li}_2\left(e^{2i(\xi - \pi)}\right) + \sum_{i=1}^4 \mathrm{Li}_2\left(e^{2i(\xi - \tau_i)}\right) + \sum_{j=1}^3 \mathrm{Li}_2\left(e^{2i(\eta_j - \xi)}\right). \end{aligned}$$

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_6)$ and $\mathrm{Re}(\boldsymbol{\alpha}) = (\mathrm{Re}(\alpha_1), \dots, \mathrm{Re}(\alpha_6))$, where $\mathrm{Re}(\alpha_i)$ is the real part of α_i for $i = 1, \dots, 6$. Let

$$B_{H,\mathbb{C}} = \left\{ (\boldsymbol{\alpha}, \xi) \in \mathbb{C}^7 \,\middle|\, \begin{array}{l} \mathrm{Re}(\boldsymbol{\alpha}) \text{ is of the hyperideal type,} \\ \max\{\mathrm{Re}(\tau_i)\} \leqslant \mathrm{Re}(\xi) \leqslant \min\{\mathrm{Re}(\eta_j), 2\pi\} \end{array} \right\}$$

and

$$B_H = B_{H,\mathbb{C}} \cap \mathbb{R}^7.$$

By (2.14), on B_H, we have

$$U(\boldsymbol{\alpha}, \xi) = 2\pi^2 + 2\sqrt{-1}V(\boldsymbol{\alpha}, \xi), \tag{5.1}$$

where $V : B_H \to \mathbb{R}$ is defined by

$$\begin{aligned} V(\boldsymbol{\alpha}, \xi) &= \delta(\alpha_1, \alpha_2, \alpha_3) + \delta(\alpha_1, \alpha_5, \alpha_6) + \delta(\alpha_2, \alpha_4, \alpha_6) + \delta(\alpha_3, \alpha_4, \alpha_5) \\ &\quad - \Lambda(\xi) + \sum_{i=1}^{4} \Lambda(\xi - \tau_i) + \sum_{j=1}^{3} \Lambda(\eta_j - \xi) \end{aligned} \tag{5.2}$$

with

$$\delta(x, y, z) = -\frac{1}{2}\Lambda\left(\frac{x+y-z}{2}\right) - \frac{1}{2}\Lambda\left(\frac{y+z-x}{2}\right) - \frac{1}{2}\Lambda\left(\frac{z+x-y}{2}\right) + \frac{1}{2}\Lambda\left(\frac{x+y+z}{2}\right).$$

The function V has been studied by Costantino in [10]. In particular, in the proof of [10, Theorem 3.9], he proved that for each α of the hyperideal type,

1. $V(\boldsymbol{\alpha}, \xi)$ is strictly concave down in ξ,

2. there exists a unique $\xi(\boldsymbol{\alpha})$ so that

$$(\boldsymbol{\alpha}, \xi(\boldsymbol{\alpha})) \in B_H \quad \text{and} \quad \left.\frac{\partial V(\boldsymbol{\alpha}, \xi)}{\partial \xi}\right|_{\xi = \xi(\boldsymbol{\alpha})} = 0, \text{ and}$$

3. $V(\boldsymbol{\alpha}, \xi)$ attains its maximum at $\xi(\boldsymbol{\alpha})$ with the critical value $V(\boldsymbol{\alpha}, \xi(\boldsymbol{\alpha}))$ given by

$$V(\boldsymbol{\alpha}, \xi(\boldsymbol{\alpha})) = \mathrm{Vol}(\Delta_{|\alpha - \pi|}),$$

   where $\mathrm{Vol}(\Delta_{|\alpha-\pi|})$ is the volume of the ideal or the truncated hyperideal tetrahedron with dihedral angles $|\alpha_1 - \pi|, \ldots, |\alpha_6 - \pi|$.

As a special case, when $\alpha_1 = \cdots = \alpha_6 = \pi$, by direct computation we have

$$\xi(\pi, \ldots, \pi) = \frac{7\pi}{4}. \tag{5.3}$$

Furthermore, for $i, j \in \{1, \ldots, 6\}$ with $i \neq j$, at $\left(\pi, \ldots, \pi, \frac{7\pi}{4}\right)$ we have

$$\frac{\partial^2 V}{\partial \alpha_i^2} = -2, \quad \frac{\partial^2 V}{\partial \alpha_i \alpha_j} = -1, \quad \frac{\partial^2 V}{\partial \alpha_i \partial \xi} = 2 \quad \text{and} \quad \frac{\partial^2 V}{\partial \xi^2} = -8.$$

From this, we have the following lemma, which will be used later to prove the convexity result in Proposition 5.8.

Lemma 5.2. *The Hessian matrix of $V(\boldsymbol{\alpha}, \xi)$ is negative definite at $\left(\pi, \ldots, \pi, \frac{7\pi}{4}\right)$.*

We also need to following estimation of V from [5].

Lemma 5.3. *For each $(\alpha_1, \ldots, \alpha_6, \xi) \in B_H$, we have $V(\alpha_1, \ldots, \alpha_6, \xi) \leq v_8$, where v_8 is the volume of the regular ideal octahedron. Moreover, the equality holds if and only if $(\alpha_1, \ldots, \alpha_6, \xi) = \left(\pi, \ldots, \pi, \frac{7\pi}{4}\right)$.*

Proof. This result is proved in [5, Lemma 3.5]. To be precise, the authors of [5] studied the maximum of V on boundary points of B_H, the non-smooth points and the critical points of the interior smooth points. From this, they proved that V attains its maximum at the unique maximum point $(\alpha_1, \ldots, \alpha_6, \xi) = \left(\pi, \ldots, \pi, \frac{7\pi}{4}\right)$ with value v_8. See [5] for more details. $\square$

For $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{C}^n$, let d_∞ be the real maximum norm on $\mathbb{C}^n$ defined by

$$d_\infty((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{i \in \{1, \ldots, n\}} \{|\operatorname{Re}(x_i) - \operatorname{Re}(y_i)|, |\operatorname{Im}(x_i) - \operatorname{Im}(y_i)|\}.$$

Lemma 5.4. *There exists $\delta_1 > 0$ such that if $d_\infty\left((\alpha_1, \ldots, \alpha_6, \xi), (\pi, \ldots, \pi, \frac{7\pi}{4})\right) < \delta_1$, then*

$$\left|\frac{\partial \operatorname{Im} U}{\partial \operatorname{Im} \xi}\right| < 2\pi.$$

Proof. The result follows from the facts that $\operatorname{Im} U$ is smooth and $\frac{\partial U}{\partial \xi}(\pi, \ldots, \pi, \frac{7\pi}{4}) = 0$. $\square$

5.1.3 Geometry of 6j-symbol

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_6) \in \mathbb{C}^6$ such that $(\mathrm{Re}(\alpha_1), \dots, \mathrm{Re}(\alpha_6))$ is of the hyperideal type, let $U_{\boldsymbol{\alpha}}(\xi) = U(\boldsymbol{\alpha}, \xi)$ and let $\xi(\boldsymbol{\alpha})$ be such that

$$\frac{dU_{\boldsymbol{\alpha}}(\xi)}{d\xi}\Big|_{\xi=\xi(\boldsymbol{\alpha})} = \frac{\partial U(\boldsymbol{\alpha}, \xi)}{\partial \xi}\Big|_{\xi=\xi(\boldsymbol{\alpha})} = 0. \tag{5.4}$$

It is proved in [4] that such $\xi(\boldsymbol{\alpha})$ exists. In particular, for $\boldsymbol{\alpha} \in \mathbb{C}^6$ so that $(\boldsymbol{\alpha}, \xi(\boldsymbol{\alpha})) \in B_{H,\mathbb{C}}$, we define

$$W(\boldsymbol{\alpha}) = U(\boldsymbol{\alpha}, \xi(\boldsymbol{\alpha})). \tag{5.5}$$

Theorem 5.5. *([4, Theorem 3.5]) For a partition (I, J) of $\{1, \dots, 6\}$ and a deeply truncated tetrahedron Δ of type (I, J), we let $\{l_i\}_{i\in I}$ and $\{\theta_i\}_{i\in I}$ respectively be the lengths of and dihedral angles at the edges of deep truncation, and let $\{\theta_j\}_{j\in J}$ and $\{l_j\}_{j\in J}$ respectively be the dihedral angles at and the lengths of the regular edges. Then*

$$W\big((\pi \pm \sqrt{-1}l_i)_{i\in I}, (\pi \pm \theta_j)_{j\in J}\big) = 2\pi^2 + 2\sqrt{-1}\mathrm{Cov}\big((l_i)_{i\in I}, (\theta_j)_{j\in J}\big)$$

where Cov *is the co-volume function defined by*

$$\mathrm{Cov}\big((l_i)_{i\in I}, (\theta_j)_{j\in J}\big) = \mathrm{Vol}(\Delta) + \frac{1}{2}\sum_{i\in I}\theta_i l_i,$$

which for $i \in I$ satisfies

$$\frac{\partial \mathrm{Cov}}{\partial l_i} = \frac{\theta_i}{2}$$

and for $j \in J$ satisfies

$$\frac{\partial \mathrm{Cov}}{\partial \theta_j} = -\frac{l_j}{2}.$$

5.1.4 Neumann-Zagier potential functions of fundamental shadow link complements

Finally, to understand the geometry of the critical points of the function $G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ defined in (5.7), we need the following result from [62].

For $s \in \{1, \dots, c\}$, let $\boldsymbol{\alpha}_s = (\alpha_{s_1}, \dots, \alpha_{s_6})$. Consider the following function

$$\mathcal{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\alpha}_J) = -\sum_{i=1}^{n} \frac{\iota_i}{2}(\alpha_i - \pi)^2 + \sum_{s=1}^{c} U(\boldsymbol{\alpha}_s, \xi(\boldsymbol{\alpha}_s)) + \Big(\sum_{i=1}^{n} \frac{\iota_i}{2}\Big)\pi^2.$$

for all $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\alpha}_J)$ such that $(\boldsymbol{\alpha}_s, \xi(\boldsymbol{\alpha}_s)) \in B_{H,\mathbb{C}}$ for all $s \in \{1, \dots, c\}$. Then we have

Proposition 5.6. *([62, Proposition 4.1]) For each component T_i of the boundary of $M_c \setminus L_{FSL}$, choose the basis (u_i, v_i) of $\pi_1(T_i)$ as in (2.6) and (2.7), and let Φ be the Neumann-Zagier potential function characterized by*

$$\begin{cases} \frac{\partial \Phi(\mathrm{H}(u_1), \dots, \mathrm{H}(u_n))}{\partial \mathrm{H}(u_i)} = \frac{\mathrm{H}(v_i)}{2}, \\ \\ \Phi(0, \dots, 0) = \sqrt{-1}\Big(\mathrm{Vol}(M_c \setminus L_{FSL}) + \sqrt{-1}\mathrm{CS}(M_c \setminus L_{FSL})\Big) \qquad \mod \pi^2\mathbb{Z}, \end{cases} \tag{5.6}$$

where $M_c \setminus L_{FSL}$ is with the complete hyperbolic metric. If $\mathrm{H}(u_i) = \pm 2\sqrt{-1}(\alpha_{i,\zeta_i} - \pi)$ for each $i \in I$ and $\mathrm{H}(u_j) = \pm 2\sqrt{-1}(\alpha_j - \pi)$ for each $j \in J$, then

$$\mathcal{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\alpha}_J) = 2c\pi^2 + \Phi(\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)).$$

5.2 Convexity

In this section we study the convexity of the function $G^{\mathbf{E}_I}$. Recall from (5.7) that

$$\begin{aligned} G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \sum_{i \in I} \Bigg[-\left(\frac{p_i'}{q_i} + a_{i,0}\right)(\beta_i - \pi)^2 - \frac{p_i(\alpha_{i,\zeta_i} - \pi)^2 + 2E_i(\beta_i - \pi)(\alpha_{i,\zeta_i} - \pi)}{q_i} \Bigg] \\ - \sum_{j \in J} a_{j,0}(\alpha_j - \pi)^2 - \sum_{i=1}^{n} \frac{\iota_i}{2}(\alpha_i - \pi)^2 + \sum_{s=1}^{c} U(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s) + \Big(\sum_{i=1}^{n} \frac{\iota_i}{2}\Big)\pi^2. \end{aligned} \tag{5.7}$$

For $\delta > 0$, we denote by $D_{\delta,\mathbb{C}}$ the δ-neighborhood of $(\pi,\dots,\pi,\frac{7\pi}{4},\dots,\frac{7\pi}{4})$ in $\mathbb{C}^{|I|+c}$ with respect to the maximum norm, that is

$$D_{\delta,\mathbb{C}} = \left\{ (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \mathbb{C}^{|I|+c} \middle| d_\infty\left((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), \left(\pi,\dots,\pi,\frac{7\pi}{4},\dots,\frac{7\pi}{4}\right) \right) < \delta \right\},$$

where d_∞ is the real maximum norm on $\mathbb{C}^n$ defined by

$$d_\infty((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max_{i\in\{1,\dots,n\}} \{|\operatorname{Re}(x_i) - \operatorname{Re}(y_i)|, |\operatorname{Im}(x_i) - \operatorname{Im}(y_i)|\}.$$

We will also consider the region

$$D_\delta = D_{\delta,\mathbb{C}} \cap \mathbb{R}^{|I|+c}.$$

Let

$$\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = -\sum_{j\in J} a_{j,0}(\alpha_j - \pi)^2 - \sum_{i=1}^n \frac{\iota_i}{2}(\alpha_i - \pi)^2 + \sum_{s=1}^c U(\alpha_{s_1},\dots,\alpha_{s_6},\xi_s) + \Big(\sum_{i=1}^n \frac{\iota_i}{2}\Big)\pi^2. \tag{5.8}$$

Let $\delta_1 > 0$ be the constant in Lemma 5.4.

Proposition 5.7. *There exists a $\delta_0 \in (0,\delta_1)$ such that if all $\{\alpha_j\}_{j\in J}$ are in $(\pi-\delta_0, \pi+\delta_0)$, then $\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave down in $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Re}(\xi_s)\}_{s=1}^c$ and is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$ on $D_{\delta_0,\mathbb{C}}$.*

Proof. Note that when all $\{\alpha_{i,\zeta_i}\}_{i\in I}$ and $\{\xi_s\}_{s=1}^c$ are real, by (5.1) we have

$$\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \sum_{s=1}^c 2V(\alpha_{s_1},\dots,\alpha_{s_6},\xi_s).$$

Therefore, when $\alpha_{i,\zeta_i} = \pi$ for all $i \in I$ and $\xi_s = \frac{7\pi}{4}$ for $s = 1,\dots,c$, by Lemma 5.2, the Hessian matrix of $\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is negative definite in $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Re}(\xi_s)\}_{s=1}^c$.

By continuity, we can find a sufficiently small $\delta_0 \in (0,\delta_1)$ such that for all $\{\alpha_j\}_{j\in J}$ in $(\pi -$

$\delta_0, \pi + \delta_0)$ and $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in D_{\delta_0,\mathbb{C}}$, the Hessian matrix of $\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is negative definite in $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Re}(\xi_s)\}_{s=1}^c$. As a result, $\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave down in $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Re}(\xi_s)\}_{s=1}^c$. Finally, by the holomorphicity of the function $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$, $\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$. □

Proposition 5.8 and 5.9 are analogue of Proposition 5.3 and 5.4 in [62].

Proposition 5.8. *For any* $\mathbf{E}_I \in \{1,-1\}^{|I|}$, $\operatorname{Im} G^{\boldsymbol{E_I}}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ *is strictly concave down in* $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ *and* $\{\operatorname{Re}(\xi_s)\}_{s=1}^c$ *and is strictly concave up in* $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ *and* $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$ *on* $D_{\delta_0,\mathbb{C}}$.

Proof. Note that

$$\begin{aligned}&G^{\boldsymbol{E_I}}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})\\ =&\sum_{i\in I}\left[-\left(a_{i,0}+\frac{p_i'}{q_i}\right)(\beta_i-\pi)^2-\frac{p_i(\alpha_{i,\zeta_i}-\pi)^2+2E_i(\beta_i-\pi)(\alpha_{i,\zeta_i}-\pi)}{q_i}\right]+\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}).\end{aligned}$$

In particular, the $\operatorname{Im}(G^{\boldsymbol{E_I}}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}))$ is a linear function in $\{\operatorname{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$. Since the convexity of a function does not change under addition of linear functions, the result follows from Proposition 5.7. □

Proposition 5.9. *If all* $\{\alpha_j\}_{j\in J}$ *are in* $(\pi-\delta_0, \pi+\delta_0)$*, then the Hessian matrix* $\operatorname{Hess}(G^{\boldsymbol{E_I}})$ *with respect to* $\{\alpha_{i,\zeta_i}\}_{i\in I}$ *and* $\{\xi_s\}_{s=1}^c$ *is non singular on* $D_{\delta_0,\mathbb{C}}$.

Proof. From Proposition 5.8, we see that the real part of $\operatorname{Hess}(G^{\mathbf{E}_I})$ is negative definite. By [[31], Lemma], the matrix $\operatorname{Hess}(G^{\mathbf{E}_I})$ is non-singular. □

Remark 5.10. The constant $\delta_0 > 0$ in Proposition 5.8 and 5.9 depends only on the fundamental shadow link but not on (p_i, q_i), $\mathbf{E}_I$ and β_i.

5.3 Critical Points and critical values

In Proposition 5.11 we will prove that certain critical value of the function $G^{\boldsymbol{E_I}}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ gives the hyperbolic volume and the Chern-Simons invariant of the cone manifold $M_{L_{\boldsymbol{\theta}}}$.

For $i \in I$, let $\theta_i = 2|\beta_i - \pi|$ and let $\mu_i = 1$ if $\beta_i - \pi \geq 0$, $\mu_i = -1$ if $\beta_i - \pi \leq 0$. By definition, we have $\theta_i = 2\mu_i(\beta_i - \pi)$. Consider the (p_i, q_i) Dehn-filling equation with cone angle θ_i

$$p_i \mathrm{H}(u_i) + q_i \mathrm{H}(v_i) = \sqrt{-1}\theta_i, \tag{5.9}$$

where $\mathrm{H}(u_i)$ and $\mathrm{H}(v_i)$ are the logarithmic holonomies of the meridian and the longitude respectively.

Then we have the following analogue of Proposition 5.2 in [62].

Proposition 5.11. *For each $i \in I$, let $\mathrm{H}(u_i)$ be the logarithmic holonomy of u_i of the hyperbolic cone manifold $M_{L_{\boldsymbol{\theta}}}$ and let*

$$\alpha_i^* = \pi + \frac{E_i \mu_i \sqrt{-1}}{2} \mathrm{H}(u_i). \tag{5.10}$$

For $s \in \{1, 2, \dots, c\}$, let $\xi^* = \xi(\alpha_{s_1}^*, \dots, \alpha_{s_6}^*)$ be as defined in (5.4). Assume that

$$\mathbf{z}^{\mathbf{E}_I} = ((\alpha_i^*)_{i \in I}, (\xi_s^*)_{s=1}^c) \in D_{\delta_0, \mathbb{C}}$$

for δ_0 defined in Proposition 5.8. Then $G^{\boldsymbol{E_I}}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ has a critical point

$$\mathbf{z}^{\mathbf{E}_I} = ((\alpha_i^*)_{i \in I}, (\xi_s^*)_{s=1}^c)$$

in $D_{\delta_0, \mathbb{C}}$ with critical value

$$2c\pi^2 + \sqrt{-1}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) + \sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}}))$$

Proof. For $s \in \{1, \dots, c\}$, we let $\boldsymbol{\alpha}_s = (\alpha_{s_1}, \dots, \alpha_{s_6})$ and $\boldsymbol{\alpha}_s^* = (\alpha_{s_1}^*, \dots, \alpha_{s_6}^*)$. For each $s \in \{1, \dots, c\}$, by Equation (5.4),

$$\left.\frac{\partial G^{\boldsymbol{E}_I}}{\partial \xi_s}\right|_{\mathbf{z}^{\mathbf{E}_I}} = \left.\frac{\partial U(\boldsymbol{\alpha}_s, \xi_s)}{\partial \xi_s}\right|_{\xi_s^*} = 0 \tag{5.11}$$

Besides, by the chain rule, for each $s \in \{1, \dots, c\}$ and $i \in I$,

$$\frac{\partial U(\boldsymbol{\alpha}_s, \xi(\boldsymbol{\alpha}_s))}{\partial \alpha_{i,\zeta_i}}\Big|_{\boldsymbol{\alpha}_s^*} = \frac{\partial U(\boldsymbol{\alpha}_s, \xi_s)}{\partial \alpha_{i,\zeta_i}}\Big|_{(\boldsymbol{\alpha}_s^*, \xi_s^*)} + \frac{\partial U(\boldsymbol{\alpha}_s, \xi_s)}{\partial \xi_s}\Big|_{(\boldsymbol{\alpha}_s^*, \xi_s^*)} \cdot \frac{\partial \xi(\boldsymbol{\alpha}_s)}{\partial \alpha_{i,\zeta_i}}\Big|_{\alpha_s} = \frac{\partial U(\boldsymbol{\alpha}_s, \xi_s)}{\partial \alpha_{i,\zeta_i}}\Big|_{(\boldsymbol{\alpha}_s^*, \xi^*)}.$$

Hence, by (5.6),

$$\frac{\sum_{s=1}^{c} \partial U(\boldsymbol{\alpha}_s, \xi(\boldsymbol{\alpha}_s))}{\partial \alpha_{i,\zeta_i}}\Big|_{\mathbf{z}^{\mathbf{E}_I}} = \frac{\partial \mathcal{U}}{\partial \alpha_{i,\zeta_i}}\Big|_{(\alpha_i^*)_{i\in I}} = -E_i\mu_i\sqrt{-1}\mathrm{H}(v_i). \tag{5.12}$$

As a result,

$$\begin{aligned}
\frac{\partial G^{\boldsymbol{E}_I}}{\partial \alpha_{i,\zeta_i}}\Big|_{\mathbf{z}^{\mathbf{E}_I}} &= \frac{-2p_i(\alpha_i^* - \pi) - 2E_i(\beta_i - \pi)}{q_i} + \frac{\partial \mathcal{U}}{\partial \alpha_{i,\zeta_i}}\Big|_{(\alpha_i^*)_{i\in I}} \\
&= \frac{-2p_i(\alpha_i^* - \pi) - 2E_i(\beta_i - \pi)}{q_i} - E_i\mu_i\sqrt{-1}\mathrm{H}(v_i) \\
&= -\frac{E_i\mu_i\sqrt{-1}}{q_i}(p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i) - \sqrt{-1}\theta_i) \\
&= 0,
\end{aligned} \tag{5.13}$$

where the last equality comes from the (p_i, q_i) Dehn-filling equation with cone angle θ_i. Thus, from Equations (5.11) and (5.13), we see that $\mathbf{z}^{\mathbf{E}_I}$ is a critical point of $G^{\boldsymbol{E}_I}$.

To compute the critical value, by Proposition 5.6, we have

$$\mathcal{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\alpha}_J) = 2c\pi^2 + \Phi(\mathrm{H}(u_1), \dots, \mathrm{H}(u_n)). \tag{5.14}$$

For each $i \in I$, let $\gamma_i = (-q_i' u_i + p_i' v_i) + a_{i,0}(p_i u_i + q_i v_i)$ so that it is the curve on the boundary of a tubular neighborhood of L_i that is isotopic to L_i given by the framing $a_{i,0}$ of L_i and with the orientation so that $(p_i u_i + q_i v_i).\gamma_i = 1$. By definition, we have $\theta_i = 2\mu_i(\beta_i - \pi)$ and $\mathrm{H}(u_i) = -2\sqrt{-1}E_i\mu_i(\alpha_i^* - \pi)$. Besides, by the (p_i, q_i) Dehn-filling equation $p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i) = \sqrt{-1}\theta_i$, we have

$$\mathrm{H}(v_i) = \frac{\sqrt{-1}\theta_i - p_i\mathrm{H}(u_i)}{q_i} = \frac{2\mu_i\sqrt{-1}}{q_i}[(\beta_i - \pi) + p_i E_i(\alpha_i^* - \pi)]. \tag{5.15}$$

As a result,

$$-\frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4} = -\frac{E_i(\alpha_i^* - \pi)(\beta - \pi)}{q_i} - \frac{p_i(\alpha_i^* - \pi)^2}{q_i}. \tag{5.16}$$

Besides, by (2.34), we have

$$\begin{aligned}
\mathrm{H}(\gamma_i) &= -q_i'\mathrm{H}(u_i) + p_i'\mathrm{H}(v_i) + a_{i,0}\theta_i\sqrt{-1} \\
&= -\left(q_i' + \frac{p_i p_i'}{q_i}\right)\mathrm{H}(u_i) + \left(\frac{p_i'}{q_i} + a_{i,0}\right)\theta_i\sqrt{-1} \\
&= -\frac{\mathrm{H}(u_i)}{q_i} + \left(\frac{p_i'}{q_i} + a_{i,0}\right)\theta_i\sqrt{-1}.
\end{aligned} \tag{5.17}$$

This implies that

$$\frac{\sqrt{-1}\theta_i\mathrm{H}(\gamma_i)}{4} = -\frac{E_i(\alpha_i^* - \pi)(\beta_i - \pi)}{q_i} - \left(\frac{p_i'}{q_i} + a_{i,0}\right)(\beta_i - \pi)^2. \tag{5.18}$$

By Equations (5.16) and (5.18), we have

$$\begin{aligned}
&-\sum_{i\in I}\frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4} + \sum_{i\in I}\frac{\sqrt{-1}\theta_i\mathrm{H}(\gamma_i)}{4} \\
&= -\sum_{i\in I}\frac{2E_i(\alpha_i^* - \pi)(\beta_i - \pi)}{q_i} - \sum_{i\in I}\frac{p_i}{q_i}(\alpha_i^* - \pi)^2 - \sum_{i\in I}\left(\frac{p_i'}{q_i} + a_{i,0}\right)(\beta_i - \pi)^2.
\end{aligned} \tag{5.19}$$

For each $j \in J$, let $\gamma_j = a_{j,0}u_j + v_j$ so that the curve on the boundary of a tubular neighborhood of L_j that is isotopic to L_j given by the framing $a_{j,0}$ of L_j and with the orientation such that $u_j.\gamma_j = 1$.

Then we have $\theta_j = 2|\alpha_j - \pi| = 2\mu_j(\alpha_j - \pi)$ for some $\mu_j \in \{-1, 1\}$, $\mathrm{H}(u_j) = 2\sqrt{-1}|\alpha_j - \pi|$ and

$\mathrm{H}(\gamma_j) = a_{j,0}\mathrm{H}(u_j) + \mathrm{H}(v_j)$. As a consequence, we have

$$-\sum_{j\in J}\frac{\mathrm{H}(u_j)\mathrm{H}(v_j)}{4} + \sum_{j\in J}\frac{\sqrt{-1}\theta_j\mathrm{H}(\gamma_j)}{4} = -\sum_{j\in J}\frac{\mathrm{H}(u_j)\mathrm{H}(v_j)}{4} + \sum_{j\in J}\frac{\mathrm{H}(u_j)(a_{j,0}\mathrm{H}(u_j)+\mathrm{H}(v_j))}{4}$$
$$= -\sum_{j\in J} a_{j,0}(\alpha_j - \pi)^2. \tag{5.20}$$

From (5.14), (5.19), (5.20) and (2.5), we have

$$\begin{aligned}
G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}) &= \sum_{i\in I}\left[-\left(\frac{p_i'}{q_i} + a_{i,0}\right)(\beta_i - \pi)^2 - \frac{p_i(\alpha_i^* - \pi)^2 + 2E_i(\beta_i - \pi)(\alpha_i^* - \pi)}{q_i}\right] \\
&\quad - \sum_{j\in J} a_{j,0}(\alpha_j - \pi)^2 + \mathcal{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\alpha}_J) \\
&= 2c\pi^2 + \Phi(\mathrm{H}(u_1),\dots,\mathrm{H}(u_n)) - \sum_{i=1}^{n}\frac{\mathrm{H}(u_i)\mathrm{H}(v_i)}{4} + \sum_{i=1}^{n}\frac{\sqrt{-1}\theta_i\mathrm{H}(\gamma_i)}{4} \\
&= 2c\pi^2 + \sqrt{-1}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) + \sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}}))
\end{aligned}$$

□

5.4 Asymptotics of the leading Fourier coefficients

Proposition 5.12. *Let $\boldsymbol{E}_I \in \{1,-1\}^{|I|}$ and let $\mathbf{z}^{\boldsymbol{E}_I}$ be the critical point described in Proposition 5.11. Assume that*

1. $\mathbf{z}^{\boldsymbol{E}_I} \in D_{\delta_0,\mathbb{C}}$ *and*
2. $\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max_{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\overline{D_H\smallsetminus D_{\delta_0}}} \mathrm{Im}\,\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$*, where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H\smallsetminus D_{\delta_0}}$ is the closure of $D_H\smallsetminus D_{\delta_0}$.*

Then the asymptotics of the integral on the right hand side of Proposition 4.3

$$\int_{D_H}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\xi_I},\boldsymbol{\zeta})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}$$
$$=\left(\frac{2}{r}\right)^c\left(\frac{2\pi}{r}\right)^{\frac{|I|+c}{2}}(4\pi\sqrt{-1})^{\frac{|I|+c}{2}}\frac{(-1)^{-\frac{rc}{2}}C^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I})}{\sqrt{-\det\mathrm{Hess}(G^{\mathbf{E}_I})(\mathbf{z}^{\mathbf{E_I}})}}e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})+\sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}}))}\Big(1+O\Big(\frac{1}{r}\Big)\Big),$$

where each $C^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I})$ depends continuously on $\{\beta_i\}_{i\in I}$ and $\{\alpha_j\}_{j\in J}$ and is given by

$$\begin{aligned}
&C^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I})\\
&=e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_i^*-\pi)+\frac{E_i(\alpha_i^*+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I}a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_i^*+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)+\kappa(\mathbf{z}^{\mathbf{E}_I})},
\end{aligned} \tag{5.21}$$

where κ is defined in Lemma 4.5.

Proof. Let $\delta_0>0$ be as in Proposition 5.8. We write

$$\begin{aligned}
&\int_{D_H}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I},\boldsymbol{\zeta})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}\\
=&\int_{D_{\delta_0}}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I},\boldsymbol{\zeta})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}+\int_{D_H\smallsetminus D_{\delta_0}}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I},\boldsymbol{\zeta})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}.
\end{aligned}$$

Step 1: Estimation of the integral over $D_H\smallsetminus D_{\delta_0}$.

From (5.7), on $D_H\smallsetminus D_{\delta_0}$ we have

$$\operatorname{Im}G^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})=\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}),$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is defined in (5.8). By assumption (2), we can find $\epsilon>0$ such that

$$\left|\int_{D_H\smallsetminus D_{\delta_0}}\phi_r\left(\mathbf{s}^{\mathbf{E}_I},\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I},\boldsymbol{\zeta})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}\right|=O\left(e^{\frac{r}{4\pi}\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon}\right).$$

Step 2: Deforming the integral over D_{δ_0}.

Consider the surface $S^{\mathbf{E}_I}=S^{\mathbf{E}_I}_{\text{top}}\cup S^{\mathbf{E}_I}_{\text{bottom}}$ defined by

$$S^{\mathbf{E}_I}_{\text{top}}=\{(\boldsymbol{\alpha}_{\zeta_I},\xi)\in D_{\delta_0,\mathbb{C}}\mid\operatorname{Im}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})=\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I})\}$$

and

$$S^{\mathbf{E}_I}_{\text{side}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + t\sqrt{-1}\,\mathrm{Im}(\mathbf{z}^{\mathbf{E}_I}) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_\delta, t \in [0,1])\}.$$

By the definition of the bump function ψ, on D_{δ_0} we have

$$\begin{aligned}
&\int_{D_{\delta_0}} \phi_r\left(\mathbf{s}^{\mathbf{E}_I}, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \\
&= \int_{D_{\delta_0}} e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I} a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)} \\
&\qquad \times e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \\
&= \int_{S^{\mathbf{E}_I}} e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I} a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)} \\
&\qquad \times e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi},
\end{aligned} \tag{5.22}$$

where the last equality follows from the analyticity of the integrand and $\partial D_{\delta_0} = \partial S^{\mathbf{E}_I}$.

By Lemma 4.5, we have

$$\begin{aligned}
&\int_{S^{\mathbf{E}_I}} e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I} a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)} \\
&\qquad \times e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \\
&= \left(\frac{r}{2}\right)^{-c} \int_{S^{\mathbf{E}_I}} e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p_i'}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I} a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)} \\
&\qquad\qquad \times e^{\kappa(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \frac{r}{4\pi\sqrt{-1}}\left(G^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta}) + \frac{\upsilon_r(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})}{r^2}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}.
\end{aligned} \tag{5.23}$$

Step 3: Verification of the conditions in Proposition 5.1.

Now, we apply Proposition 5.1 to the integral in (5.23). We check the conditions (1)-(6) below:

1. By the definition of $S^{\mathbf{E}_I}_{\text{top}}$ and Proposition 5.11, we have $\mathbf{z}^{\mathbf{E}_I} \in S^{\mathbf{E}_I}_{\text{top}}$.

2. On $S^{\mathbf{E}_I}_{\text{top}}$, by Proposition 5.8, since $\mathrm{Im}\, G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave down in $\{\mathrm{Re}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\mathrm{Re}(\xi_s)\}_{s=1}^{c}$, $\mathrm{Im}\, G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ attains its unique maximum at $\mathbf{z}^{\mathbf{E}_I}$.

On $S^{\mathbf{E}_I}_{\text{sides}}$, by Proposition 5.8, since $\operatorname{Im} G^{\mathbf{E}_I}$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^{c}$, for each $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_{\delta_0}$ and $t \in [0,1]$ we have

$$\operatorname{Im} G^{\mathbf{E}_I}((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})+t\sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I})) < \max\{\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), \operatorname{Im} G^{\mathbf{E}_I}((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})+\sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I}))\}.$$

For $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_{\delta_0}$, by assumption (2) we have

$$\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) < \operatorname{Im} G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}).$$

For $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I}) \in S^{\mathbf{E}_I}_{\text{top}}$, since on $S^{\mathbf{E}_I}_{\text{top}}$ the function $\operatorname{Im} G^{\mathbf{E}_I}$ attains its maximum at $\mathbf{z}^{\mathbf{E}_I}$, we have

$$\operatorname{Im} G^{\mathbf{E}_I}((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I})) < \operatorname{Im} G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}).$$

Altogether, on $S^{\mathbf{E}_I}$, $\operatorname{Im} G^{\mathbf{E}_I}$ has a unique maximum at $\mathbf{z}^{\mathbf{E}_I}$.

3. For any $k \in \mathbb{N}$ and any k-tuple of complex number $(z_1, \dots, z_k) \in \mathbb{C}^k$, we let

$$\operatorname{Re}(z_1, \dots, z_k) = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_k) \in \mathbb{R}^k,$$

where $\operatorname{Re} z_i$ is the real part of z_i for $i = 1, \dots, k$. For any $(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\xi}) \in D_{\delta_0}$, we consider the set

$$P_{(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\xi})} = \left\{ (\tilde{\boldsymbol{\alpha}}_{\boldsymbol{\xi_I}}, \tilde{\boldsymbol{\xi}}) \in D_{\delta_0,\mathbb{C}} \;\middle|\; \begin{array}{l} \operatorname{Re}(\tilde{\boldsymbol{\alpha}}_{\xi_I}, \tilde{\boldsymbol{\xi}}) = \operatorname{Re}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\xi}), \\ \operatorname{Im} G^{\mathbf{E}_I}(\tilde{\boldsymbol{\alpha}}_{\xi_I}, \tilde{\boldsymbol{\xi}}) < \operatorname{Im} G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}) \end{array} \right\}.$$

Note that for $(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\xi}) = \mathbf{z}^{\mathbf{E}_I}$, since it is a critical point of $G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$, by the Cauchy-Riemann equation we know that

$$\frac{\partial}{\partial \operatorname{Im} \alpha_{i,\zeta}} \operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \frac{\partial}{\partial \operatorname{Im} \xi_k} \operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = 0$$

for $i \in I$ and $k = 1, \dots, c$. By Proposition 5.8, since $\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$ on $D_{\delta_0,\mathbb{C}}$, we know that $P_{\mathbf{z}^{\mathbf{E}_I}}$ is an empty set.

Next, for any $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I}_{\text{top}}$, by Proposition 5.8, we know that $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in P_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})}$. Moreover, by Proposition 5.8, since $\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$ on $D_{\delta_0,\mathbb{C}}$, $P_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})}$ is a convex set. This implies that each $P_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})}$ with $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I}_{\text{top}}$ is a topological $(|I|+c)$-dimensional disk which admits a deformation retract to the point $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$. This verifies condition (3).

4. By continuity and compactness of $S^{\mathbf{E}_I}$,

$$\left| e^{\sqrt{-1}\left(\sum_{i\in I} a_{i,0}\beta_i + \sum_{i\in I}\left(a_{i,\zeta_i} + \frac{\iota_i}{2}\right)\alpha_{i,\zeta_i} + \sum_{j\in J}\left(a_{j,0} + \frac{\iota_j}{2}\right)\alpha_j \sum_{i\in I}\left(\left(\frac{p_i'}{q_i}\right)(\beta_i - \pi) + \frac{E_i(\alpha_{i,\zeta_i} + \beta_i - 2\pi)}{q_i}\right)\right) + \kappa(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})} \right|$$

   is non-zero and bounded below by a positive constant independent of r.

5. By Lemma 4.5, $|\upsilon_r(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})|$ is bounded from above by a constant independent of r on any compact subset of $D_{H,\mathbb{C}}$.

6. Note that

$$\lim_{r\to\infty} \operatorname{Hess} G_r^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}) = \operatorname{Hess} G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}).$$

   By Proposition 5.9, the Hessian matrix $\operatorname{Hess} G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I})$ is non-singular. By continuity, the Hessian matrix $\operatorname{Hess} G_r^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I})$ is non-singular

The result then follows from Proposition 5.1. □

5.5 Reidemeister torsion

The goal of this section is to prove Proposition 5.13 and 5.18, which relates the asymptotics of the leading Fourier coefficients obtained in Proposition 5.12 with the adjoint twisted Reideimester torsion of the cone manifold $M \smallsetminus L$.

Proposition 5.13. *Consider the system of meridians $\boldsymbol{\Upsilon} = (\Upsilon_1, \dots, \Upsilon_n)$ with $\Upsilon_i = p_i u_i + q_i v_i$ and $\Upsilon_j = u_j$. Let $\mathbb{T}_{(M\smallsetminus L, \boldsymbol{\Upsilon})}([\rho_{M_{L_{\boldsymbol{\theta}}}}])$ be the Reidemeister torsion of $M \smallsetminus L$ twisted by the adjoint action* *of $\rho_{M^{(r)}}$ with respect to the system of meridians $\boldsymbol{\Upsilon}$. Then we have*

$$\frac{C^{\mathbf{E}_I}(\mathbf{z}^{\boldsymbol{E}_I})}{\sqrt{-\left(\prod_{i\in I} q_i\right)\det\mathrm{Hess}(G^{\mathbf{E}_I})(\mathbf{z}^{\mathbf{E_I}})}} = \frac{e^{\left(\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\right)\sqrt{-1}\pi+\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}(\gamma_k)}}{2^{\frac{|I|+c}{2}}\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M_{L_{\boldsymbol{\theta}}}}])}}.$$

For each $s\in\{1,\dots,c\}$, we let $I_s=\{s_1,\dots,s_6\}\cap I$ and $J_s=\{s_1,\dots,s_6\}\cap J$. We also let $\alpha^*_{I_s}=(\alpha^*_{s_i})_{s_i\in I_s}$, $\alpha_{J_s}=(\alpha_{s_j})_{j\in J}$, $\xi^*_s=\xi(\alpha^*_{I_s},\alpha_{J_s})$ and $z^*_s=(\alpha^*_{I_s},\alpha_{J_s},\xi^*_s)$.

To prove Proposition 5.13, we need Lemmas 5.14, 5.15, 5.16 and 5.17.

Lemma 5.14. *For each $i\in I$, consider the system of meridian $\Upsilon_i=p_iu_i+q_iv_i$. Then*

$$-\left(\prod_{i\in I} q_i\right)\det\mathrm{Hess}\, G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}) = -(-2)^{|I|}\det\left(\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial\mathrm{H}(u_{i_2})}\right)_{i_1,i_2\in I}\prod_{s=1}^c \frac{\partial^2 U}{\partial\xi_s^2}\bigg|_{z^*_s}.$$

Proof. The proof is similar to the proof of Lemma 3.3 in [65]. For $s\in\{1,\dots,c\}$ and $i\in I$, we denote by $s\sim i$ if the tetrahedron Δ_s intersects the component $L_{\mathrm{FSL},i}$ of L_{FSL}, and for $\{i_1,i_2\}\subset I$ we denote by $s\sim i_1,i_2$ if Δ_s intersects both L_{FSL,i_1} and L_{FSL,i_2}. For $s\in\{1,\dots,c\}$, let $\alpha_s=(\alpha_{s_1},\dots,\alpha_{s_6})$ and let $\alpha^*_s=(\alpha^*_{I_s},\alpha_{J_s})$. The following claims (1)-(3) are from [65, Lemma 3.3] and we include the proof below for reader's convenience. Claims (4)-(5) can be proved by suitably modifying the proof of (4)-(5) in [65, Lemma 3.3].

(1) For $s\in\{1,\dots,c\}$,

$$\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\xi_s^2}\bigg|_{\mathbf{z}^{\mathbf{E}_I}} = \frac{\partial^2 U}{\partial\xi_s^2}\bigg|_{z^*_s}.$$

(2) For $\{s_1,s_2\}\subset\{1,\dots,c\}$,

$$\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\xi_{s_1}\partial\xi_{s_2}}\bigg|_{\mathbf{z}^{\mathbf{E}_I}} = 0.$$

(3) For $i\in I$ and $s\in\{1,\dots,c\}$,

$$\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\alpha_i\partial\xi_s}\bigg|_{\mathbf{z}^{\mathbf{E}_I}} = -\frac{\partial^2 U}{\partial\xi_s^2}\bigg|_{z^*_s}\frac{\xi_s(\alpha_s)}{\partial\alpha_i}\bigg|_{\alpha^*_s}.$$

(4) For $i \in I$,

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial \alpha_i^2}\right|_{\mathbf{z}^{\mathbf{E}_I}} = -\frac{2}{q_i}\frac{\partial \mathrm{H}(\Upsilon_i)}{\partial \mathrm{H}(u_i)} + \sum_{s\sim i}\left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{z_s^*}\left(\left.\frac{\xi_s(\alpha_s)}{\partial \alpha_i}\right|_{\alpha_s^*}\right)^2.$$

(5) For $\{i_1, i_2\} \subset I$,

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial \alpha_{i_1}\partial \alpha_{i_2}}\right|_{\mathbf{z}^{\mathbf{E}_I}} = -\frac{2}{q_{i_1}}\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial \mathrm{H}(u_{i_2})} + \sum_{s\sim i_1, i_2}\left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{z_s^*}\left.\frac{\xi_s(\alpha_s)}{\partial \alpha_{i_1}}\right|_{\alpha_s^*}\left.\frac{\xi_s(\alpha_s)}{\partial \alpha_{i_2}}\right|_{\alpha_s^*}.$$

Assuming these claims, then

$$\mathrm{Hess}G^{\mathbf{E}_I}(\mathbf{z}^{\mathbf{E}_I}) = A\cdot D\cdot A^T, \tag{5.24}$$

with D and A defined as follows. The matrix D is a block matrix with the left-top block the $|I|\times|I|$ matrix

$$\left(-\frac{2}{q_{i_1}}\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial \mathrm{H}(u_{i_2})}\right)_{i_1,i_2\in I},$$

the right-top and the left-bottom blocks consisting of 0's, and the right-bottom block the $c\times c$ diagonal matrix with the diagonal entries $\left.\frac{\partial^2 U}{\partial \xi_1^2}\right|_{z_1^*}, \dots, \left.\frac{\partial^2 U}{\partial \xi_c^2}\right|_{z_c^*}$. Then

$$\begin{aligned}\det D =& \frac{(-2)^{|I|}}{\prod_{i\in I} q_i}\det\left(\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial \mathrm{H}(u_{i_2})}\right)_{i_1,i_2\in I}\prod_{s=1}^{c}\left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{z_s^*}\\ =& \frac{(-2)^{|I|}}{\prod_{i\in I} q_i}\det\left(\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial \mathrm{H}(u_{i_2})}\right)_{i_1,i_2\in I}\prod_{s=1}^{c}\left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{z_s^*}.\end{aligned} \tag{5.25}$$

The matrix A is a block matrix with the left-top and the right-bottom blocks respectively the $|I|\times|I|$ and $c\times c$ identity matrices, the left-bottom block consisting of 0's and the right-top block the $|I|\times c$ matrix with entries a_{is}, $i\in I$ and $s\in\{1,\dots,c\}$, given by

$$a_{is} = -\left.\frac{\xi_s(\alpha_s)}{\partial \alpha_i}\right|_{\alpha_s^*}$$

if $s\sim i$ and $a_{is}=0$ if otherwise. Since A is upper triangular with all diagonal entries equal to 1,

$$\det A = 1. \tag{5.26}$$

The result then follows from (5.24), (5.25) and (5.26), and we are left to prove the claims (1) – (5).

Claims (1) and (2) are straightforward from the definition of $G^{\mathbf{E}_I}$. For (3), we have

$$\left.\frac{\partial G^{\mathbf{E}_I}}{\partial \xi_s}\right|_{\left((\alpha_i)_{i\in I},\xi_1,\ldots,\xi_c\right)} = \left.\frac{\partial U}{\partial \xi_s}\right|_{(\alpha_s,\xi_s)}. \tag{5.27}$$

Let

$$f(\alpha_s,\xi_s) \doteq \left.\frac{\partial U}{\partial \xi_s}\right|_{(\alpha_s,\xi_s)}$$

and

$$g(\alpha_s) \doteq f(\alpha_s,\xi_s(\alpha_s)).$$

Then

$$g(\alpha_s) = \left.\frac{\partial U}{\partial \xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))} = \left.\frac{dU_{\alpha_s}}{d\xi_s}\right|_{\xi_s(\alpha_s)} \equiv 0,$$

and hence

$$\left.\frac{\partial g}{\partial \alpha_{s_i}}\right|_{\alpha_s} = 0. \tag{5.28}$$

On the other hand, we have

$$\begin{aligned}\left.\frac{\partial g}{\partial \alpha_{s_i}}\right|_{\alpha_s} &= \left.\frac{\partial f}{\partial \alpha_{s_i}}\right|_{(\alpha_s,\xi_s(\alpha_s))} + \left.\frac{\partial f}{\partial \xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))} \left.\frac{\partial \xi_s(\alpha_s)}{\partial \alpha_{s_i}}\right|_{\alpha_s} \\ &= \left.\frac{\partial^2 U}{\partial \alpha_{s_i}\partial \xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))} + \left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{(\alpha_s,\xi_s(\alpha_s))} \left.\frac{\partial \xi_s(\alpha_s)}{\partial \alpha_{s_i}}\right|_{\alpha_s}.\end{aligned} \tag{5.29}$$

Putting (5.28) and (5.29) together, we have

$$\left.\frac{\partial^2 U}{\partial \alpha_{s_i}\partial \xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))} = -\left.\frac{\partial^2 U}{\partial \xi_s^2}\right|_{(\alpha_s,\xi_s(\alpha_s))} \left.\frac{\partial \xi_s(\alpha_s)}{\partial \alpha_{s_i}}\right|_{\alpha_s}, \tag{5.30}$$

and (3) follows from (5.27) and (5.30).

For (4) and (5), we have

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial \alpha_i^2}\right|_{\mathbf{z}^{\mathbf{E}_I}} = -\frac{2p_i}{q_i} - \iota_i + \sum_{s\sim i} \left.\frac{\partial^2 U}{\partial \alpha_i^2}\right|_{z_s^*}, \tag{5.31}$$

and

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\alpha_{i_1}\partial\alpha_{i_2}}\right|_{\mathbf{z}^{\mathbf{E}_I}} = \sum_{s\sim i_1,i_2} \left.\frac{\partial^2 U}{\partial\alpha_{i_1}\partial\alpha_{i_2}}\right|_{z_s^*}. \tag{5.32}$$

Let W be the function defined in (5.5). By the Chain Rule and (5.4), we have

$$\left.\frac{\partial U}{\partial\xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))} = \left.\frac{dU_{\alpha_s}}{d\xi_s}\right|_{\xi_s(\alpha_s)} = 0,$$

and hence for $j\in\{1,\dots,6\}$,

$$\left.\frac{\partial W}{\partial\alpha_{s_j}}\right|_{\alpha_s} = \left.\frac{\partial U}{\partial\alpha_{s_j}}\right|_{\alpha_s} + \left.\frac{\partial U}{\partial\xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))}\left.\frac{\partial\xi_s(\alpha_s)}{\partial\alpha_{s_j}}\right|_{\alpha_s} = \left.\frac{\partial U}{\partial\alpha_{s_j}}\right|_{\alpha_s}.$$

Then using the Chain Rule again, for $j,k\in\{1,\dots,6\}$ we have

$$\left.\frac{\partial^2 W}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{\alpha_s} = \left.\frac{\partial^2 U}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{(\alpha_s,\xi_s(\alpha_s))} + \left.\frac{\partial^2 U}{\partial\alpha_{s_k}\partial\xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))}\left.\frac{\partial\xi_s(\alpha_s)}{\partial\alpha_{s_j}}\right|_{\alpha_s}.$$

Together with (5.30), for $j,k\in\{1,\dots,6\}$ we have

$$\begin{aligned}\left.\frac{\partial^2 U}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{(\alpha_s,\xi_s(\alpha_s))} &= \left.\frac{\partial^2 W}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{\alpha_s} - \left.\frac{\partial^2 U}{\partial\alpha_{s_k}\partial\xi_s}\right|_{(\alpha_s,\xi_s(\alpha_s))}\left.\frac{\partial\xi_s(\alpha_s)}{\partial\alpha_{s_j}}\right|_{\alpha_s} \\ &= \left.\frac{\partial^2 W}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{\alpha_s} + \left.\frac{\partial^2 U}{\partial\xi_s^2}\right|_{(\alpha_s,\xi_s(\alpha_s))}\left.\frac{\partial\xi_s(\alpha_s)}{\partial\alpha_{s_j}}\right|_{\alpha_s}\left.\frac{\partial\xi_s(\alpha_s)}{\partial\alpha_{s_k}}\right|_{\alpha_s}.\end{aligned} \tag{5.33}$$

By (5.31), (5.32) and (5.33) and we have

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\alpha_i^2}\right|_{\mathbf{z}^{\mathbf{E}_I}} = -\frac{2p_i}{q_i} - \iota_i + \sum_{s\sim i}\sum_{s_k}\left.\frac{\partial^2 W}{\partial\alpha_{s_k}^2}\right|_{\alpha_s^*} + \sum_{s\sim i}\left.\frac{\partial^2 U}{\partial\xi_s^2}\right|_{z_s^*}\left(\left.\frac{\xi_s(\alpha_s)}{\partial\alpha_i}\right|_{\alpha_s^*}\right)^2, \tag{5.34}$$

where the second sum in the third term of the right hand side is over s_k such that the edge e_{s_k} in Δ_s intersects the component $L_{\mathrm{FSL},i}$; and

$$\left.\frac{\partial^2 G^{\mathbf{E}_I}}{\partial\alpha_{i_1}\partial\alpha_{i_2}}\right|_{\mathbf{z}^{\mathbf{E}_I}} = \sum_{s\sim i_1,i_2}\sum_{s_j,s_k}\left.\frac{\partial^2 W}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\right|_{\alpha_s^*} + \sum_{s\sim i_1,i_2}\left.\frac{\partial^2 U}{\partial\xi_s^2}\right|_{z_s^*}\left.\frac{\xi_s(\alpha_s)}{\partial\alpha_{i_1}}\right|_{\alpha_s^*}\left.\frac{\xi_s(\alpha_s)}{\partial\alpha_{i_2}}\right|_{\alpha_s^*}, \tag{5.35}$$

where the second sum in the first term of the right hand side is over s_j, s_k such that the edge e_{s_j} in Δ_s intersects the component L_{FSL,i_1} and the edge e_{s_k} in Δ_s intersects the component L_{FSL,i_2}.

At a hyperbolic cone metric on M_c with singular locus L_{FSL}, by Theorem 5.5, for $i, j \in \{1, \dots, 6\}$ we have

$$\frac{\partial^2 W}{\partial \alpha_{s_i} \partial \alpha_{s_j}}\bigg|_{\alpha_s^*} = -\sqrt{-1}\frac{E_{s_i}\mu_{s_i}}{E_{s_j}\mu_{s_j}}\frac{\partial l_{s_i}}{\partial \theta_{s_j}}, \tag{5.36}$$

where l_{s_k} is the length of e_{s_k} of Δ_s, and if e_{s_k} intersects $L_{\mathrm{FSL},i}$ then $E_{s_k} = E_i$, $\mu_{s_k} = \mu_i$ and $\theta_{s_k} = \frac{\theta_i}{2}$ is the half of the cone angle at $L_{\mathrm{FSL},i}$. We also observe that that

$$l_i = \sum_{s\sim i}\sum_{s_k} l_{s_k}, \tag{5.37}$$

where the second sum is over s_k such that the edge e_{s_k} in Δ_s intersects the component $L_{\mathrm{FSL},i}$.

Then by (5.36), (5.37) (2.6) and (2.7) we have

$$\begin{aligned}
-\frac{2p_i}{q_i} - \iota_i + \sum_{s\sim i}\sum_{s_k}\frac{\partial^2 W}{\partial \alpha_{s_k}^2}\bigg|_{\alpha_s^*} &= -\frac{2p_i}{q_i} - \iota_i - \sqrt{-1}\sum_{s\sim i}\sum_{s_k}\frac{\partial l_{s_k}}{\partial \theta_{s_k}}\\
&= -\frac{2p_i}{q_i} - \iota_i - 2\sqrt{-1}\frac{\partial l_i}{\partial \theta_i}\\
&= -\frac{2}{q_i}\frac{\partial\big(p_i\sqrt{-1}\theta_i - q_i l_i + \frac{q_i\iota_i}{2}\sqrt{-1}\theta_i\big)}{\partial(\sqrt{-1}\theta_i)}\\
&= -\frac{2}{q_i}\frac{\partial\big(p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i)\big)}{\partial(\sqrt{-1}\theta_i)} = -\frac{2}{q_i}\frac{\partial \mathrm{H}(\Upsilon_i)}{\partial \mathrm{H}(u_i)}.
\end{aligned} \tag{5.38}$$

From (5.34) and (5.38), (4) holds at hyperbolic cone metrics on M_c with singular locus L_{FSL}. By the analyticity of the involved functions (see for e.g. [63, Lemma 4.2]), (5.38) still holds in a neighborhood of the complete hyperbolic structure on $M_c \setminus L_{\mathrm{FSL}}$, from which (4) follows.

By (5.36), (5.37), (2.6) and (2.7) we have

$$\begin{aligned}\sum_{s\sim i_1,i_2}\sum_{s_j,s_k}\frac{\partial^2 W}{\partial\alpha_{s_j}\partial\alpha_{s_k}}\bigg|_{\alpha_s^*} &= -2\sqrt{-1}\sum_{s\sim i_1}\sum_{s_j}\frac{E_{s_j}\mu_{s_j}}{E_{i_2}\mu_{i_2}}\frac{\partial l_{s_j}}{\partial\theta_{i_2}}\\ &= -2\sqrt{-1}\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial l_{i_1}}{\partial\theta_{i_2}}\\ &= -2\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial\big(\frac{p_{i_1}}{q_{i_1}}\sqrt{-1}\theta_{i_1}-l_{i_1}+\frac{l_{i_1}}{2}\sqrt{-1}\theta_{i_1}\big)}{\partial(\sqrt{-1}\theta_{i_2})}\\ &= -\frac{2}{q_{i_1}}\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial\big(p_{i_1}\mathrm{H}(u_{i_1})+q_{i_1}\mathrm{H}(v_{i_1})\big)}{\partial(\sqrt{-1}\theta_{i_2})}\\ &= -\frac{2}{q_{i_1}}\frac{E_{i_1}\mu_{i_1}}{E_{i_2}\mu_{i_2}}\frac{\partial\mathrm{H}(\Upsilon_{i_1})}{\partial\mathrm{H}(u_{i_2})},\end{aligned}\tag{5.39}$$

where the second sum on the left hand side is over s_j, s_k such that the edge e_{s_j} in Δ_s intersects the component L_{i_1} and the edge e_{s_k} in Δ_s intersects the component L_{i_2}, the second sum on the right hand side of the first equation is over s_j such that the edge e_{s_j} in Δ_s intersects the component L_{i_1}, and the third equality comes from the fact that

$$\frac{\partial(\sqrt{-1}\theta_{i_1})}{\partial(\sqrt{-1}\theta_{i_2})}=\frac{\partial\mathrm{H}(u_{i_1})}{\partial\mathrm{H}(u_{i_2})}=0.$$

From (5.35) and (5.39), (5) holds at hyperbolic cone metrics on $M_c\smallsetminus L_{\mathrm{FSL}}$. By the analyticity of the involved functions, (5.39) still holds in a neighborhood of the complete hyperbolic structure on $M_c\smallsetminus L_{\mathrm{FSL}}$, from which (5) follows. $\square$

The following lemma is from [65].

Lemma 5.15. *([65, Lemma 3.4])*

$$\begin{aligned}\kappa(\mathbf{z}^{\mathbf{E}_I}) =& -\frac{\sqrt{-1}}{2}\sum_{k=1}^{6}\frac{\partial U}{\partial\alpha_{s_k}}\bigg|_{z_s^*}\\ &-\frac{\sqrt{-1}}{2}\sum_{s_i\in I_s}\alpha^*_{s_i}-\frac{\sqrt{-1}}{2}\sum_{s_j\in J_s}\alpha_{s_j}+2\sqrt{-1}\xi_s^*-\frac{1}{2}\sum_{i=1}^{4}\log\big(1-e^{2\sqrt{-1}(\xi_s^*-\tau^*_{s_i})}\big),\end{aligned}$$

where with the notation that $\alpha^*_{s_j}=\alpha_{s_j}$ *for* $s_j\in J_s$, $\tau^*_{s_1}=\frac{\alpha^*_{s_1}+\alpha^*_{s_2}+\alpha^*_{s_3}}{2}$, $\tau^*_{s_2}=\frac{\alpha^*_{s_1}+\alpha^*_{s_5}+\alpha^*_{s_6}}{2}$,

$\tau^*_{s_3} = \frac{\alpha^*_{s_2}+\alpha^*_{s_4}+\alpha^*_{s_6}}{2}$ *and* $\tau^*_{s_4} = \frac{\alpha^*_{s_3}+\alpha^*_{s_4}+\alpha^*_{s_5}}{2}$.

The following lemma is an analogue of Lemma 3.5 in [65].

Lemma 5.16. *For $i \in I$ recall that $\gamma_i = (-q'_i u_i + p'_i v_i) + a_{i,0}(p_i u_i + q_i v_i)$ is the parallel of copy of L_i given by the framing $a_{i,0}$, and for each $j \in J$ recall that $\gamma_j = a_{j,0}u_j + v_j$ is the parallel copy of L_j given by the framing $a_{j,0}$. Then*

$$\begin{aligned}
&\sqrt{-1}\Bigg[\sum_{i\in I} a_{i,0}\beta_i + \sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha^*_i + \sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j \\
&\qquad + \sum_{i\in I}\left(\left(\frac{p'_i}{q_i}\right)(\beta_i-\pi) + \frac{p_i}{q_i}(\alpha^*_i-\pi) + \frac{E_i(\alpha^*_i+\beta_i-2\pi)}{q_i}\right)\Bigg] \\
&\qquad\qquad - \frac{\sqrt{-1}}{2}\sum_{s=1}^{c}\left(\sum_{k=1}^{6}\frac{\partial U}{\partial \alpha_{s_k}}\bigg|_{z^*_s}\right) \\
=&\left(\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right) + \sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\right)\sqrt{-1}\pi + \frac{1}{2}\sum_{k=1}^{n}\mu_k \mathrm{H}(\gamma_k).
\end{aligned}$$

Proof. We first prove the result for the case that M_c is with a hyperbolic cone metric with singular locus L_{FSL},

$$-\frac{\sqrt{-1}}{2}\sum_{s=1}^{c}\left(\sum_{k=1}^{6}\frac{\partial U}{\partial \alpha_{s_k}}\bigg|_{z^*_s}\right) = -\frac{1}{2}\sum_{i\in I} E_i\mu_i\Big(\mathrm{H}(v_i) - \frac{\iota_i}{2}\mathrm{H}(u_i)\Big) - \frac{1}{2}\sum_{j\in J}\mu_j l_j. \tag{5.40}$$

In this case, the hyperbolic cone manifold $M_c \smallsetminus L_{\mathrm{FSL}}$ is obtained by gluing hyperideal tetrahedra $\Delta_1, \dots, \Delta_s$ together along the hexagonal faces then taking the orientable double. For each $s \in \{1, \dots, c\}$ let $e_{s_1}, \dots, e_{s_6}$ be the edges of Δ_s and for each $k \in \{1, \dots, 6\}$ let l_{s_k} and θ_{s_k} respectively be the length of and the dihedral angle at e_{s_k}. If e_{s_k} intersects the component $L_{\mathrm{FSL},i}$ of L_{FSL} for some $i \in I$, then $\mathrm{H}(u_i) = \sqrt{-1}\theta_i = 2\sqrt{-1}\theta_{s_k}$ and let $\alpha_{s_k} = \alpha^*_i = \pi + \frac{E_i\mu_i\sqrt{-1}\mathrm{H}(u_i)}{2} = \pi - E_{s_k}\mu_{s_k}\theta_{s_k}$, where $E_{s_k} = E_i$ and $\mu_{s_k} = \mu_i$; and if e_{s_k} intersects the component $L_{\mathrm{FSL},j}$ of L_{FSL} for some $j \in J$, then $\theta_j = 2\theta_{s_k}$ and let $\alpha_{s_k} = \alpha_j = \pi + \frac{\mu_j\theta_j}{2} = \pi + \mu_{s_k}\theta_{s_k}$, where $\mu_{s_k} = \mu_j$. We claim that for $s_k \in I_s$

$$\frac{\partial U}{\partial \alpha_{s_k}}\bigg|_{z^*_s} = \sqrt{-1}E_{s_k}\mu_{s_k}l_{s_k},$$

and for $s_k \in J_s$.

$$\left.\frac{\partial U}{\partial \alpha_{s_k}}\right|_{z_s^*} = -\sqrt{-1}\mu_{s_k} l_{s_k}.$$

Indeed, let W again be the function defined in (5.5). Then by Theorem 5.5, we have for $s_k \in I_s$

$$\left.\frac{\partial W}{\partial \alpha_{s_k}}\right|_{\left(a^*_{I_s}, \alpha_{J_s}\right)} = \sqrt{-1} E_{s_k} \mu_{s_k} l_{s_k} \tag{5.41}$$

and for $s_k \in J_s$

$$\left.\frac{\partial W}{\partial \alpha_{s_k}}\right|_{\left(a^*_{I_s}, \alpha_{J_s}\right)} = -\sqrt{-1}\mu_{s_k} l_{s_k}. \tag{5.42}$$

On the other hand, by the Chain Rule and (5.4), we have for $k \in \{1, \dots, 6\}$,

$$\left.\frac{\partial W}{\partial \alpha_{s_k}}\right|_{\left(a^*_{I_s}, \alpha_{J_s}\right)} = \left.\frac{\partial U}{\partial \alpha_{s_k}}\right|_{z_s^*} + \left.\frac{\partial U}{\partial \xi_s}\right|_{z_s^*} \left.\frac{\partial \xi_s(\alpha_s)}{\partial \alpha_{s_k}}\right|_{\left(a^*_{I_s}, \alpha_{J_s}\right)} = \left.\frac{\partial U}{\partial \alpha_{s_k}}\right|_{z_s^*}. \tag{5.43}$$

Putting (5.41), (5.42) and (5.43) together, we have

$$\begin{aligned}
\sum_{s=1}^{c}\left(\sum_{k=1}^{6}\left.\frac{\partial U}{\partial \alpha_{s_k}}\right|_{z_s^*}\right) =& \sqrt{-1}\sum_{i\in I}\sum_{s_k\sim i} E_{s_k}\mu_{s_k} l_{s_k} - \sqrt{-1}\sum_{j\in J}\sum_{s_k\sim j}\mu_{s_k} l_{s_k} \\
=& \sqrt{-1}\sum_{i\in I} E_i\mu_i\Big(\sum_{s_k\sim i} l_{s_k}\Big) - \sqrt{-1}\sum_{j\in J}\mu_j\Big(\sum_{s_k\sim j} l_{s_k}\Big) \\
=& \sqrt{-1}\sum_{i\in I} E_i\mu_i l_i - \sqrt{-1}\sum_{j\in J}\mu_j l_j \\
=& -\sqrt{-1}\sum_{i\in I} E_i\mu_i\Big(\mathrm{H}(v_i) - \frac{l_i}{2}\mathrm{H}(u_i)\Big) - \sqrt{-1}\sum_{j\in J}\mu_j l_j,
\end{aligned} \tag{5.44}$$

where $s_k \sim i$ if e_{s_k} intersects $L_{\mathrm{FSL},i}$ for $i \in I$ and $s_k \sim j$ if e_{s_k} intersects $L_{\mathrm{FSL},j}$ for $j \in J$, and the last equality come from that $\mathrm{H}(u_i) = \sqrt{-1}\theta_i$ and $\mathrm{H}(v_i) = -l_i + \frac{l_i}{2}\sqrt{-1}\theta_i$.

Next, recall that for each $i \in I$, $\alpha_i^* = \pi + \frac{E_i\mu_i\sqrt{-1}\mathrm{H}(u_i)}{2}$ and $\beta_i = \pi + \frac{\mu_i\theta_i}{2}$, and for each $j \in J$,

$\alpha_j = \pi + \frac{\mu_j\theta_j}{2}$. For $i \in I$, we have

$$\begin{aligned}
&\sqrt{-1}\Big(\Big(\frac{\iota_i}{2}\Big)\alpha_i^* + \frac{E_i}{q_i}(\beta_i - \pi) + \frac{p_i}{q_i}(\alpha_i^* - \pi)\Big) - \frac{E_i\mu_i}{2}\Big(\mathrm{H}(v_i) - \frac{\iota_i}{2}\mathrm{H}(u_i)\Big)\\
=&\sqrt{-1}\Big(\Big(\frac{\iota_i}{2}\Big)\Big(\pi + \frac{E_i\mu_i\sqrt{-1}\mathrm{H}(u_i)}{2}\Big) + \frac{E_i}{q_i}\Big(\frac{\mu_i\theta_i}{2}\Big) + \frac{p_i}{q_i}\Big(\frac{E_i\mu_i\sqrt{-1}\mathrm{H}(u_i)}{2}\Big)\Big)\\
&-\frac{E_i\mu_i}{2}\Big(\mathrm{H}(v_i) - \frac{\iota_i}{2}\mathrm{H}(u_i)\Big)\\
=&\Big(\frac{\iota_i}{2}\Big)\sqrt{-1}\pi + \frac{E_i\mu_i}{2}\Big(-\frac{\iota_i}{2}\mathrm{H}(u_i) + \frac{\sqrt{-1}\theta_i}{q_i} - \frac{p_i}{q_i}\mathrm{H}(u_i) - \mathrm{H}(v_i) + \frac{\iota_i}{2}\mathrm{H}(u_i)\Big)\\
=&\Big(\frac{\iota_i}{2}\Big)\sqrt{-1}\pi,
\end{aligned} \tag{5.45}$$

where the last equality comes from $p_i\mathrm{H}(u_i) + q_i\mathrm{H}(v_i) = \sqrt{-1}\theta_i$. For $i \in I$, we also have

$$\begin{aligned}
&\sqrt{-1}\Big(a_{i,0}\beta_i + \frac{E_i}{q_i}(\alpha_i^* - \pi) + \frac{p_i'}{q_i}(\beta_i - \pi)\Big)\\
=&\sqrt{-1}\Big(a_{i,0}\Big(\pi + \frac{\mu_i\theta_i}{2}\Big) + \frac{E_i}{q_i}\Big(\frac{E_i\mu_i\sqrt{-1}\mathrm{H}(u_i)}{2}\Big) + \frac{p_i'}{q_i}\Big(\frac{\mu_i\theta_i}{2}\Big)\Big)\\
=&a_{i,0}\sqrt{-1}\pi + \frac{\mu_i}{2}\Big(a_{i,0}\sqrt{-1}\theta_i - \frac{\mathrm{H}(u_i)}{q_i} + \frac{pi'}{q_i}\sqrt{-1}\theta_i\Big)\\
=&a_{i,0}\sqrt{-1}\pi + \frac{\mu_i}{2}\mathrm{H}(\gamma_i),
\end{aligned} \tag{5.46}$$

where the last equality comes from Equation (5.17).

For each $j \in J$, we have

$$\begin{aligned}
\sqrt{-1}\Big(a_{j,0} + \frac{\iota_j}{2}\Big)\alpha_j - \frac{\mu_j}{2}l_j =&\sqrt{-1}\Big(a_{j,0} + \frac{\iota_j}{2}\Big)\Big(\pi + \frac{\mu_j\theta_j}{2}\Big) - \frac{\mu_j}{2}l_j\\
=&\Big(a_{j,0} + \frac{\iota_j}{2}\Big)\sqrt{-1}\pi + \frac{\mu_j}{2}\Big(a_{j,0}\sqrt{-1}\theta_j + \frac{\iota_j}{2}\sqrt{-1}\theta_j - l_j\Big)\\
=&\Big(a_{j,0} + \frac{\iota_j}{2}\Big)\sqrt{-1}\pi + \frac{\mu_j}{2}\Big(a_{j,0}\mathrm{H}(u_j) + \mathrm{H}(v_j)\Big)\\
=&\Big(a_{j,0} + \frac{\iota_j}{2}\Big)\sqrt{-1}\pi + \frac{\mu_j}{2}\mathrm{H}(\gamma_j).
\end{aligned} \tag{5.47}$$

Then the result follows from (5.45), (5.46), (5.47) and Lemma 5.40. For the general case, the result follows the analyticity of the involved functions. □

Finally, we need the following lemma from [65].

Lemma 5.17. *([65, Lemma 3.6]) For $s \in \{1, \dots, c\}$, let $u_{s_1}, \dots, u_{s_6}$ be the meridians of a tubular neighborhood of the components of L_{FSL} intersecting the six edges of Δ_s. Then*

$$\frac{e^{-\sqrt{-1}\sum_{s_i \in I_s} \alpha^*_{s_i} - \sqrt{-1}\sum_{s_j \in J_s} \alpha_{s_j} + 4\sqrt{-1}\xi^*_s - \sum_{i=1}^4 \log\left(1 - e^{2\sqrt{-1}(\xi^*_s - \tau^*_{s_i})}\right)}}{\frac{\partial^2 U}{\partial \xi_s^2}\Big|_{z^*_s}} = \frac{-1}{16\sqrt{\det \mathbb{G}\left(\frac{\mathrm{H}(u_{s_1})}{2}, \dots, \frac{\mathrm{H}(u_{s_6})}{2}\right)}}. \tag{5.48}$$

Proof of Proposition 5.13. From (5.21), Lemmas 5.14, 5.15, 5.40, 5.16 and 5.17, we have

$$\begin{aligned}
&\frac{C^{\mathbf{E}_I}(\mathbf{z}^{\boldsymbol{E}_I})}{\sqrt{-\det \mathrm{Hess}(G^{\mathbf{E}_I})(\mathbf{z}^{\mathbf{E_I}})}} \\
=&\frac{e^{\left(\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\right)\sqrt{-1}\pi + \frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}(\gamma_k)}}{\sqrt{-(-16)^c(-2)^{|I|}\det\left(\frac{\partial \mathrm{H}(\Upsilon_{i_1})}{\partial \mathrm{H}(u_{i_2})}\right)_{i_1,i_2\in I}\prod_{s=1}^c\sqrt{\det \mathbb{G}\left(\frac{\mathrm{H}(u_{s_1})}{2}, \dots, \frac{\mathrm{H}(u_{s_6})}{2}\right)}}} \\
=&\frac{e^{\left(\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\right)\sqrt{-1}\pi + \frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}(\gamma_k)}}{2^{\frac{|I|+c}{2}}\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M_{L_{\boldsymbol{\theta}}}}])}},
\end{aligned}$$

where the last equality follows from Theorem 2.11 (2). ☐

Proposition 5.18. *Under the assumptions in Proposition 5.12, we have*

$$\sum_{\mathbf{E}_I}\widehat{f}_r(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1}-\mathbf{2m}^{\mathbf{E_I}}, \mathbf{0}) = C_1\frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)})+\sqrt{-1}\mathrm{CS}(M^{(r)})\right)}\left(1+O\Big(\frac{1}{r}\Big)\right),$$

where $C_1 = Y 2^c r^{\sum_{i\in I}\frac{\zeta_i}{2}-\frac{c}{2}}(-1)^{-\frac{rc}{2}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}$ and Y is defined in (4.5).

Proof. By Proposition 4.3, Lemma 4.4, Proposition 5.12 and 5.13,

$$\begin{aligned}
&\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1}-2\mathbf{m}^{\mathbf{E_I}}, \mathbf{0})\\
=&\frac{Yr^{|I|+c}}{2^{|I|+c}\pi^{|I|+c}}\left(\frac{2}{r}\right)^{c}\left(\frac{2\pi}{r}\right)^{\frac{|I|+c}{2}}(4\pi\sqrt{-1})^{\frac{|I|+c}{2}}\\
&\qquad\frac{(-1)^{-\frac{rc}{2}}C^{\mathbf{E}_I}(\mathbf{z}^{\boldsymbol{E}_I})}{\sqrt{-\left(\prod_{i\in I}q_i\right)\det\operatorname{Hess}(G^{\mathbf{E}_I})(\mathbf{z}^{\mathbf{E_I}})}}e^{\frac{r}{4\pi}(\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})+\sqrt{-1}\operatorname{CS}(M_{L_{\boldsymbol{\theta}}}))}\Big(1+O\Big(\frac{1}{r}\Big)\Big)\\
=&C_0\frac{e^{\frac{1}{2}\sum_{k=1}^{n}\mu_k\mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}}e^{\frac{r}{4\pi}\left(\operatorname{Vol}(M^{(r)})+\sqrt{-1}\operatorname{CS}(M^{(r)})\right)}\left(1+O\Big(\frac{1}{r}\Big)\right),
\end{aligned}$$

where $C_0 = Y2^{-|I|+c}r^{\frac{|I|-c}{2}}(-1)^{-\frac{rc}{2}+\frac{|I|+c}{4}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}$. Thus, we have

$$\begin{aligned}
&\sum_{\mathbf{E}_I}\widehat{f}_r(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1}-2\mathbf{m}^{\mathbf{E_I}}, \mathbf{0})\\
=&C_1\frac{e^{\frac{1}{2}\sum_{k=1}^{n}\mu_k\mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}}e^{\frac{r}{4\pi}\left(\operatorname{Vol}(M^{(r)})+\sqrt{-1}\operatorname{CS}(M^{(r)})\right)}\left(1+O\Big(\frac{1}{r}\Big)\right),
\end{aligned}$$

where $C_1 = Y2^{c}r^{\sum_{i\in I}\frac{|I|-c}{2}}(-1)^{-\frac{rc}{2}+\frac{|I|+c}{4}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}$. $\square$

5.6 Estimate of other Fourier coefficients

Proposition 5.19. *Assume that*

$$\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{\max_{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\overline{D_H\smallsetminus D_{\delta_0}}}\operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}), 2cv_8-4\pi\delta_0\right\},$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H\smallsetminus D_{\delta_0}}$ is the closure of $D_H\smallsetminus D_{\delta_0}$. Then for any $\mathbf{E}^I$ and $\mathbf{s}_I$, there exists $\epsilon'>0$ such that if $B_{k_0}\neq 0$ for some $k_0\in\{1,2,\dots,c\}$, then

$$\left|\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I, \mathbf{A}_{\zeta_I}, \mathbf{B})\right| < O\Big(e^{\frac{r}{4\pi}(\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon')}\Big).$$

Proof. Let

$$G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = W_r^{\mathbf{E}_I}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) - 2\pi\sum_{i\in I} A_{i,\zeta_i}\alpha_{i,\zeta_i} - 4\pi\sum_{s=1}^{c} B_s\xi_s.$$

Recall from Proposition 4.1 that

$$\begin{aligned}\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I,\mathbf{A}_{\zeta_I},\boldsymbol{B}) = &\frac{r^{|I|+c}\left(\prod_{i\in I}E_i\right)}{2^{|I|+c}\pi^{|I|+c}}\\ &\times\int_{D_H}(-1)^{\sum_{i\in I}A_{i,\zeta_i}}\phi_r\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}.\end{aligned}$$

When $\alpha_{i,\zeta_i}\in\mathbb{R}$ for all $i\in I$, by Lemma 4.10, on any compact subset of $D_{H,\mathbb{C}}$, $\operatorname{Im} G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ converges uniformly to

$$\operatorname{Im} G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = \operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}).$$

We first estimate the integral on $D_H\smallsetminus D_{\delta_0}$. By assumption, we have

$$\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max_{D_H\smallsetminus D_{\delta_1}}\operatorname{Im}\left(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\right)+\epsilon'. \tag{5.49}$$

for some $\epsilon'>0$. Thus, we have

$$\begin{aligned}&\left|\int_{D_H\smallsetminus D_{\delta_0}}(-1)^{\sum_{i\in I}A_{i,\zeta_i}}\phi_r\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)e^{\frac{r}{4\pi\sqrt{-1}}G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})}d\boldsymbol{\alpha}_{\zeta_I}d\boldsymbol{\xi}\right|\\ =&o\left(e^{\frac{r}{4\pi}(\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon')}\right).\end{aligned} \tag{5.50}$$

Next, we estimate the integral on D_{δ_0}. For simplicity, we assume that $B_c\neq 0$. The following arguments also work for other possibilities.

First, we consider the case where $B_c>0$. Consider the surface $S^+=S^+_{\text{top}}\cup S^+_{\text{sides}}$ in the closure of $D_{\delta_0,\mathbb{C}}$, where

$$S^+_{\text{top}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in D_{\delta_0,\mathbb{C}}\}$$

and

$$S^+_{\text{sides}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, t\sqrt{-1}\delta_0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_{\delta_0}, t \in [0,1]\}.$$

On S^+_{top}, by the Mean Value Theorem,

$$\begin{aligned}
&\operatorname{Im} G^{\mathbf{E}_I}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0)\big) - \operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \\
=&\frac{\partial \operatorname{Im} G^{\mathbf{E}_I}}{\partial \operatorname{Im} \xi_c}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0')\big) \cdot \delta_0
\end{aligned} \tag{5.51}$$

for some $\delta_0' \in (0, \delta_0)$. Note that

$$\frac{\partial \operatorname{Im} G^{\mathbf{E}_I}}{\partial \operatorname{Im} \xi_c}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0')\big) = \frac{\partial \operatorname{Im} U}{\partial \operatorname{Im} \xi}\bigg|_{(\boldsymbol{\alpha}_s, \xi_s + \sqrt{-1}\delta_0')} - 4\pi B_c < 2\pi - 4\pi = -2\pi,$$

where the last inequality follows from Lemma 5.4 and $B_c \geq 1$. This implies that

$$\begin{aligned}
&\operatorname{Im} G^{\mathbf{E}_I}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0)\big) \\
<&\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi\delta_0 < 2cv_8 - 2\pi\delta_0 < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}),
\end{aligned}$$

where the second last inequality follows from Lemma 5.3 and the last inequality follows from the assumption (1). By making $\epsilon' > 0$ smaller if necessary, we have

$$\operatorname{Im} G^{\mathbf{E}_I}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0, \dots, 0, \sqrt{-1}\delta_0)\big) < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon'. \tag{5.52}$$

Next, on S^{+}_{sides}, by Proposition 5.8, for $t \in [0,1]$ we have

$$\begin{aligned}
&\operatorname{Im} G^{\mathbf{E}_I}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + t(0,\dots,0,\sqrt{-1}\delta_0)\big) \\
\le &\max\{\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), \operatorname{Im} G^{\mathbf{E}_I}\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + (0,\dots,0,\sqrt{-1}\delta_0)\big)\} < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon', \qquad (5.53)
\end{aligned}$$

where the last inequality follows from (5.52) and the assumption. From (5.52) and (5.53), we have

$$\begin{aligned}
&\left| \int_{D_{\delta_0}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \right| \\
=&\left| \int_{S^+} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \right| \\
=&o\left(e^{\frac{r}{4\pi}(\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon')}\right).
\end{aligned}$$

If $B_c < 0$, then we consider the surface $S^- = S^-_{\text{top}} \cup S^-_{\text{sides}}$ in the closure of $D_{\delta_0,\mathbb{C}}$, where

$$S^-_{\text{top}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - (0,0,\dots,0,\sqrt{-1}\delta_0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in D_{\delta_0,\mathbb{C}}\}$$

and

$$S^-_{\text{sides}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - (0,\dots,0,t\sqrt{-1}\delta_0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_{\delta_0}, t \in [0,1]\}.$$

Using the same arguments as in the previous case, on S^- we have

$$\operatorname{Im} G^{(\boldsymbol{A}_{0,I}, \boldsymbol{A}_{\zeta_I}, \boldsymbol{B}_I)}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon' \qquad (5.54)$$

This completes the proof. $\square$

From Proposition 5.19, it remains to consider the Fourier coefficients with $\mathbf{B} = \mathbf{0} = (0,\dots,0)$. To do this, for $i \in I$, consider the functions $k_i^{\pm} : \{0,1,\dots,|q_i|-1\} \times \mathbb{Z} \to \mathbb{R}$ defined by

$$k_i^{\pm}(s_i, A_{\zeta_i}) = \frac{I_i(s_i) \mp 1}{q_i} + A_{\zeta_i}.$$

Lemma 5.20. *When $|q_i|$ is odd,*

- $k_i^+(s_i, A_{\zeta_i}) = 0$ *if and only if* $(s_i, A_{\zeta_i}) = (s_i^+, 1 - 2m_i^+)$*;*
- $k_i^-(s_i, A_{\zeta_i}) = 0$ *if and only if* $(s_i, A_{\zeta_i}) = (s_i^-, 1 - 2m_i^-)$.

Moreover,

- *if* $(s_i, A_{\zeta_i}) \neq (s_i^+, 1 - 2m_i^+)$*, then* $\left|k^+(s_i, A_{\zeta_i})\right| \geq \frac{1}{|q_i|}$;
- *if* $(s_i, A_{\zeta_i}) \neq (s_i^-, 1 - 2m_i^-)$*, then* $\left|k^-(s_i, A_{\zeta_i})\right| \geq \frac{1}{|q_i|}$.

Proof. Suppose $k_i^{\pm}(s_i, A_{\zeta_i}) = 0$ for some $(s_i, A_{\zeta_i}) \in \{0, 1, \ldots, |q_i| - 1\} \times \mathbb{Z}$. Then we have

$$I_i(s_i, A_{\zeta_i}) = \pm 1 - q_i A_{\zeta_i} = \pm 1 - q_i - q_i(A_{\zeta_i} - 1).$$

Suppose A_{ζ_i} is even. Then $I_i(s_i, A_{\zeta_i}) = \pm 1 \pmod{2|q_i|}$ is an odd number. However, by Lemma 2.18, the image of I_i^+ has the same parity of $1 - q_i$, which is even when $|q_i|$ is odd. This leads to a contradiction. Thus, A_{ζ_i} is odd. Then we have $I_i(s_i, A_{\zeta_i}) = \pm 1 - q_i - q_i(A_{\zeta_i} - 1) \equiv \pm 1 - q_i \pmod{2|q_i|}$. Since $I_i : \{0, 1, \ldots, |q_i| - 1\} \to \{0, 1, \ldots, 2|q_i| - 1\}$ is injective, we have $s_i = s_i^{\pm}$. This proves the first claim.

For the second claim, if $(s_i, A_{\zeta_i}) \neq (s_i^+, 1 - 2m_i^+)$, then

$$|q_i k_i^+(s_i, A_{\zeta_i})| = |I_i(s_i) - 1 - q A_{\zeta_i}|$$

is a non-zero integer. The other part can be proved similarly. □

Lemma 5.21. *When $|q_i|$ is even, there exist $\tilde{s}_i^+, \tilde{s}_i^- \in \{0, 1, \ldots, |q_i| - 1\}$ and $\tilde{m}_i^+, \tilde{m}_i^- \in \mathbb{Z}$ such that*

- $k_i^+(s_i, A_{\zeta_i}) = 0$ *if and only if* $(s_i, A_{\zeta_i}) = (s_i^+, 1 - 2m_i^+)$ *or* $(\tilde{s}_i^+, -2\tilde{m}_i^+)$*; and*
- $k_i^-(s_i, A_{\zeta_i}) = 0$ *if and only if* $(s_i, A_{\zeta_i}) = (s_i^-, 1 - 2m_i^-)$ *or* $(\tilde{s}_i^-, -2\tilde{m}_i^-)$.

Furthermore, $\tilde{s}_i^{\pm} = s_i^{\pm} + \frac{q_i}{2} \pmod{|q_i|}$. *Moreover,*

- *if* $(s_i, A_{\zeta_i}) \notin \{(s_i^+, 1-2m_i^+), (\tilde{s}_i^+, -2\tilde{m}_i^+)\}$, *then* $|k^+(s_i, A_{\zeta_i})| \geq \frac{1}{|q_i|}$;
- *if* $(s_i, A_{\zeta_i}) \notin \{(s_i^-, 1-2m_i^-), (\tilde{s}_i^-, -2\tilde{m}_i^-)\}$, *then* $|k^-(s_i, A_{\zeta_i})| \geq \frac{1}{|q_i|}$.

Proof. Note that when $k^{\pm}(s_i, A_{\zeta_i}) = 0$, we have $\frac{I_i(s_i)\mp 1}{q_i} \in \mathbb{Z}$. By Lemma 2.18, we have $I(s_i) = \pm 1$ or $\pm 1 + |q_i|$. Recall that $I_i(s_i^{\pm}) = \pm 1 - q_i + 2m_i^{\pm} q_i$. In particular, we have $k^{\pm}(s_i^{\pm}, 1-2m_i^{\pm}) = 0$.

Besides, let $\tilde{s}_i^{\pm} \in \{0, 1, \ldots, |q_i|-1\}$ such that

$$\tilde{s}_i^{\pm} \equiv s_i^{\pm} + \frac{|q_i|}{2} \pmod{|q_i|}.$$

By the definition of I_i, we have

$$I_i(\tilde{s}_i^{\pm}) \equiv I(s_i^{\pm}) - C_{k-1}|q_i| \pmod{2|q_i|}.$$

Since $q_i = A_{i,\zeta_i - 1}$ is even and $(A_{i,\zeta_i-1}, C_{i,\zeta_i-1})$ is a pair of coprime integers, C_{i,ζ_i-1} must be odd. Thus,

$$I(\tilde{s}_i^{\pm}) \equiv I(s_i^{\pm}) + q_i \pmod{2|q_i|} \equiv \pm 1 \pmod{2|q_i|}.$$

Define $\tilde{m}_i^{\pm} \in \mathbb{Z}$ such that

$$I(\tilde{s}_i^{\pm}) = \pm 1 + 2\tilde{m}_i^{\pm} q_i. \tag{5.55}$$

Then $k(\tilde{s}_i^{\pm}, -2\tilde{m}_i^{\pm}) = 2\tilde{m}_i^{\pm} - 2\tilde{m}_i^{\pm} = 0$. Since I is injective, $s_i^{\pm}$ is the unique integer in $\{0, \ldots, |q_i|-1\}$ such that $I(\tilde{s}_i^{\pm}) \equiv 1 \pmod{2|q_i|}$.

Finally, if $(s_i, A_{\zeta_i}) \notin \{(s_i^+, 1-2m_i^+), (\tilde{s}_i^+, -2\tilde{m}_i^+)\}$,

$$|q_i k_i^+(s_i, A_{\zeta_i})| = |I_i(s_i) - 1 - q_i A_{\zeta_i}|$$

is a non-zero integer. The other part can be proved similarly. $\square$

For $i \in I$, let

$$S_i = \begin{cases} \left\{(s_i^+, 1-2m_i^+), (s_i^-, 1-2m_i^-)\right\} & \text{if } |q_i| \text{ is odd,} \\ \left\{(s_i^+, 1-2m_i^+), (s_i^-, 1-2m_i^-), (\tilde{s}_i^+, -2\tilde{m}_i^+), (\tilde{s}_i^-, -2\tilde{m}_i^-)\right\} & \text{if } |q_i| \text{ is even.} \end{cases} \tag{5.56}$$

Proposition 5.22. *Assume that* $\theta_i = 2|\beta_i - \pi| < \pi$ *for all* $i \in I$ *and*

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \setminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}).$$

Then there exists $\epsilon' > 0$ *such that if* $(s_{i_0}, A_{\zeta_{i_0}}) \notin S_{i_0}$ *for some* $i_0 \in I$*, then*

$$\left|\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I, \mathbf{A}_{\zeta_I}, \mathbf{0})\right| < O\left(e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon')}\right).$$

Proof. Recall that

$$G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{0}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = W_r^{\mathbf{E}_I}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i \in I} A_{i,\zeta_i} \alpha_{i,\zeta_i}.$$

By Proposition 4.1,

$$\begin{aligned} \widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}_I, \mathbf{A}_{\zeta_I}, \mathbf{0}) &= \frac{r^{|I|+c}\left(\prod_{i \in I} E_i\right)}{2^{|I|+c}\pi^{|I|+c}} \\ &\times \int_{D_H} (-1)^{\sum_{i \in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{0}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}. \end{aligned}$$

Let $I_0 = \{i_0 \in I \mid (s_{i_0}, A_{\zeta_{i_0}}) \notin S_{i_0}\}$. By a direct computation, we have

$$G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = G_r^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i \in I} k_i^{E_i}(s_i, A_{\zeta_i})(\boldsymbol{\alpha}_{i,\zeta_i} - \pi) + C^{\mathbf{E}_I}(\mathbf{s}_I)$$

where

$$k_i^{E_i}(s_i, A_{\zeta_i}) = \begin{cases} k_i^+(s_i, A_{\zeta_i}) & \text{if } E_i = -1, \\ k_i^-(s_i, A_{\zeta_i}) & \text{if } E_i = 1 \end{cases}$$

and $C^{\mathbf{E}_I}(\mathbf{s}_I)$ is a real numbers independent of $\boldsymbol{\alpha}_{\zeta_I}$ and $\boldsymbol{\xi}$. By Lemma 5.20 and 5.21,

$$G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = G_r^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i \in I_0} k_i^{E_i}(s_i, A_{\zeta_i})(\boldsymbol{\alpha}_{i,\zeta_i} - \pi) + C^{\mathbf{E}_I}(\mathbf{s}_I).$$

Let $i_0 \in I_0$, $\mathbf{E}_I \in \{-1,1\}^{|I|}$ and let $\mathbf{E}'_I \in \{-1,1\}^{|I|}$ be obtained by changing E_{i_0} in $\mathbf{E}_I$ into $-E_{i_0}$. Since

$$\frac{-2E_{i_0}(\alpha_{i_0,\zeta_{i_0}} - \pi)(\beta_{i_0} - \pi)}{q_{i_0}} = \frac{-2(-E_{i_0})(\alpha_{i_0,\zeta_{i_0}} - \pi)(\beta_{i_0} - \pi)}{q_{i_0}} - \frac{4E_{i_0}(\alpha_{i_0,\zeta_{i_0}} - \pi)(\beta_{i_0} - \pi)}{q_{i_0}},$$

by a direct computation, we have

$$\begin{aligned} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) =& G_r^{\mathbf{E}'_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i \in I_0 \smallsetminus \{i_0\}} k_i^{E_i}(s_i, A_{\zeta_i})(\boldsymbol{\alpha}_{i,\zeta_i} - \pi) \\ & - 2\pi\Big(k_{i_0}^{E_{i_0}}(s_{i_0}, A_{\zeta_{i_0}}) + \frac{2E_{i_0}(\beta_{i_0} - \pi)}{\pi q_{i_0}}\Big)(\boldsymbol{\alpha}_{i_0,\zeta_{i_0}} - \pi) + C^{\mathbf{E}_I}(\mathbf{s}_I), \end{aligned}$$

For all $i \in I$, under the assumption that $\theta_i = 2|\beta_i - \pi| < \pi$, we have

$$\left|\frac{2E_i(\beta_i - \pi)}{\pi q_i}\right| < \frac{1}{q_i}.$$

By Lemma 5.20 and 5.21, since $|k_{i_0}^{E_{i_0}}(s_{i_0}, A_{\zeta_{i_0}})| \geq \frac{1}{q_{i_0}}$, $k_{i_0}^{E_{i_0}}(s_{i_0}, A_{\zeta_{i_0}})$ and $k_{i_0}^{E_{i_0}}(s_{i_0}, A_{\zeta_{i_0}}) + \frac{2E_{i_0}(\beta_{i_0}-\pi)}{\pi q_{i_0}}$ are either both positive or both negative. Besides, by Proposition 5.11, we know that the α_i component of $\mathbf{z}^{\mathbf{E}_I}$ and that of $\mathbf{z}^{\mathbf{E}'_I}$ have opposite sign. Altogether, for each $\mathbf{E}_I \in \{-1,1\}^{|I|}$, by changing some E_i in $\mathbf{E}_I$ into $-E_i$ if necessary, we can always find $\mathbf{E}_I'' \in \{-1,1\}^{|I|}$ such that

$$G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = G_r^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) - 2\pi\sum_{i\in I_0} k_i(\boldsymbol{\alpha}_{i,\zeta_i}-\pi) + C^{\mathbf{E}_I}(\mathbf{s}_I), \tag{5.57}$$

where $k_i \in \mathbb{R}\setminus\{0\}$ is some nonzero constant such that the product of k_i and the imaginary part of the α_i component of $\mathbf{z}^{\mathbf{E}_I''}$ is less than or equal to 0 for all $i \in I_0$.

By Lemma 4.10, on any compact subset of $D_{H,\mathbb{C}}$, $G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ converges uniformly to

$$G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) - 2\pi\sum_{i\in I_0} k_i(\boldsymbol{\alpha}_{i,\zeta_i}-\pi) + C^{\mathbf{E}_I}(\mathbf{s}_I),$$

where $G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is defined in (5.7).

In particular,

$$\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big) = \operatorname{Im}\Big(G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) - 2\pi\sum_{i\in I_0} k_i(\boldsymbol{\alpha}_{i,\zeta_i}-\pi)\Big). \tag{5.58}$$

on $D_{H,\mathbb{C}}$ and

$$\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big) = \operatorname{Im}\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}). \tag{5.59}$$

on D_H.

We first estimate the integral on $D_H\setminus D_{\delta_0}$. By assumption, we can find $\epsilon' > 0$ such that

$$\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max_{D_H\setminus D_{\delta_1}} \operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big) + \epsilon'. \tag{5.60}$$

Thus, we have

$$\begin{aligned}&\left|\int_{D_H\setminus D_{\delta_0}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{0}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\ &= o\left(e^{\frac{r}{4\pi}(\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon')}\right).\end{aligned} \tag{5.61}$$

Next, we estimate the integral on D_{δ_0}. Let $i_0 \in I_0$ such that $(s_{i_0}, A_{\zeta_{i_0}}) \notin S_{i_0}$. Consider the surface $S^{\mathbf{E}''_I} = S^{\mathbf{E}''_I}_{\text{top}} \cup S^{\mathbf{E}''_I}_{\text{bottom}}$ defined by

$$S^{\mathbf{E}''_I}_{\text{top}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \xi) \in D_{\delta_0,\mathbb{C}} \mid \mathrm{Im}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \mathrm{Im}(\mathbf{z}^{\mathbf{E}''_I})\}$$

and

$$S^{\mathbf{E}''_I}_{\text{side}} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + t\sqrt{-1}\,\mathrm{Im}(\mathbf{z}^{\mathbf{E}''_I}) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial D_\delta, t \in [0,1])\}.$$

On $S^{\mathbf{E}''_I}_{\text{top}}$, in the proof of Proposition 5.12, we showed that $\mathrm{Im}\, G^{\mathbf{E}''_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ attains its unique maximum at $\mathbf{z}^{\mathbf{E}''_I} = ((\alpha_i^*)_{i\in I}, (\xi_s^*)_{s=1}^c)$. By (5.58), for $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}''_I}_{\text{top}}$,

$$\begin{aligned}
\mathrm{Im}\left(G^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})\right) &= \mathrm{Im}\left(G^{\mathbf{E}''_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i\in I_0} k_i(\boldsymbol{\alpha}_{i,\zeta_i} - \pi)\right) \\
&= \mathrm{Im}\, G^{\mathbf{E}''_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi \sum_{i\in I_0} k_i \mathrm{Im}(\alpha_i^*) \\
&\leq \mathrm{Im}\, G^{\mathbf{E}''_I}(\mathbf{z}^{\mathbf{E}''_I}) - 2\pi \sum_{i\in I_0} k_i \mathrm{Im}(\alpha_i^*). \qquad (5.62)
\end{aligned}$$

From Proposition 5.11, we know that $\mathrm{Im}\, G^{\mathbf{E}''_I}(\mathbf{z}^{\mathbf{E}''_I}) = \mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})$. Thus,

$$\mathrm{Im}\left(G^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})\right) \leq \mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) - 2\pi \sum_{i\in I_0} k_i \mathrm{Im}(\alpha_i^*). \qquad (5.63)$$

We have the following two cases:

Case 1: $\mathrm{Im}(\alpha_{i_0}^*) \neq 0$ **for some** $i_0 \in I_0$

By Lemma 5.20 and 5.21, since $|k_i^{E_i}(s_i, A_{\zeta_i})| \geq \frac{1}{q_i}$ for all $i \in I$, we have

$$|k_i| \geq \frac{1}{q_i} - \frac{2|\beta_i - \pi|}{\pi q_i} > 0. \qquad (5.64)$$

Besides, recall that we choose $\mathbf{E}''_I$ in such a way that $k_{i_0}\,\mathrm{Im}(\alpha_{i_0}^*) \leq 0$. As a result, from (5.63), by making $\epsilon' > 0$ smaller if necessary,

$$\begin{aligned}
&\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big)\\
\leq&\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - 2\pi\min\left\{\left|\Big(\frac{1}{q_i}-\frac{2|\beta_i-\pi|}{\pi q_i}\Big)\operatorname{Im}(\alpha_i^*)\right| \quad\Big|\quad i\in I, \operatorname{Im}(\alpha_i^*)\neq 0\right\}\\
\leq&\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon'.
\end{aligned} \tag{5.65}$$

Next, on $S^{\mathbf{E}_I''}_{\text{sides}}$, by Proposition 5.8, $\operatorname{Im} G^{\mathbf{E}_I''}$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$. Besides,

$$\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) - G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big) = \operatorname{Im}\Bigg(-2\pi\sum_{i\in I_0}k_i(\boldsymbol{\alpha}_{i,\zeta_i}-\pi)\Bigg)$$

is a linear function in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$. As a result, $\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big)$ is also strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^c$. By convexity, for each $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\partial D_{\delta_0}$ and $t\in[0,1]$ we have

$$\begin{aligned}
&\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}\Big(\mathbf{s}_I,(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+t\sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I})\Big)\Big)\\
<&\max\Big\{\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big), \operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}\Big(\mathbf{s}_I,(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I})\Big)\Big)\Big\}.
\end{aligned}$$

For $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\partial D_{\delta_1}$, by (5.60) we have

$$\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon'.$$

For $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}\operatorname{Im}(\mathbf{z}^{\mathbf{E}_I''})\in S^{\mathbf{E}_I''}_{\text{top}}$, by (5.65), we have

$$\operatorname{Im} G^{\mathbf{E}_I}((\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}\operatorname{Im}(z^{\mathbf{E}_I})) < \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon'.$$

Thus, we have

$$\begin{aligned}
&\left|\int_{D_{\delta_0}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
=&\left|\int_{S^{\mathbf{E}_I''}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
=&o\left(e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon')}\right).
\end{aligned} \tag{5.66}$$

The result follows from (5.60) and (5.66).

Case 2: $\mathrm{Im}(\alpha_i^*) = 0$ **for all** $i \in I_0$ From (5.62) and (5.63), on $S^{\mathbf{E}_I''}$ we have

$$\mathrm{Im}\left(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})\right) \leq \mathrm{Im}\, G^{\mathbf{E}_I''}(\mathbf{z}^{\mathbf{E}_I''}) - 2\pi \sum_{i\in I_0} k_i \mathrm{Im}(\alpha_i^*) = \mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) \tag{5.67}$$

and equality holds if and only if $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = \mathbf{z}^{\mathbf{E}_I''}$. Since $\mathbf{z}^{\mathbf{E}_I''}$ is a critical point of $G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$, there exists $\delta \in (0, \delta_0)$ depending of $\mathbf{E}_I''$ such that for any $i \in I$,

$$\left|\frac{\partial\, \mathrm{Im}\, G^{\mathbf{E}_I''}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})}{\partial\, \mathrm{Im}\, \alpha_i}\right| < \pi \min_{i\in I}\left\{\frac{1}{q_i} - \frac{2|\beta_i - \pi|}{\pi q_i}\right\} \tag{5.68}$$

whenever $d_\infty\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), \mathbf{z}^{\mathbf{E}_I''}\big) < \delta$. Let

$$S^{\mathbf{E}_I'',\delta} = \{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I''}_{\text{top}} \mid d_\infty\big((\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), \mathbf{z}^{\mathbf{E}_I''}\big) \leq \delta\}$$

By the compactness of the closure of $S^{\mathbf{E}_I''} \smallsetminus S^{\mathbf{E}_I'',\delta}$, we can find $\epsilon' > 0$ such that

$$\mathrm{Im}\left(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})\right) < \mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon' \tag{5.69}$$

for any $(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I''} \smallsetminus S^{\mathbf{E}_I'',\delta}$. Thus, we have

$$\begin{aligned}
&\left| \int_{S^{\mathbf{E}_I''} \smallsetminus S^{\mathbf{E}_I'',\delta}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) - \sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi} \right| \\
=&o\left(e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon')}\right). \qquad (5.70)
\end{aligned}$$

Assume that $k_i > 0$ for some $i \in I_0$. For simplicity we assume that $i = 1$. Consider the surface $S^{\mathbf{E}_I'',\delta,+} = S^{\mathbf{E}_I'',\delta,+}_{\text{top}} \cup S^{\mathbf{E}_I'',\delta,+}_{\text{bottom}}$ defined by

$$S^{\mathbf{E}_I'',\delta,+}_{\text{top}} = \left\{ (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}(\delta, 0, \dots, 0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I'',\delta} \right\}$$

and

$$S^{\mathbf{E}_I'',\delta,+}_{\text{side}} = \left\{ (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + t\sqrt{-1}(\delta, 0, \dots, 0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial S^{\mathbf{E}_I'',\delta} \right\}.$$

Note that on $S^{\mathbf{E}_I'',\delta,+}_{\text{top}}$, by the Mean Value Theorem,

$$\begin{aligned}
&\operatorname{Im} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}\big(\mathbf{s}_I, (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}(\delta, 0, \dots, 0)\big) - \operatorname{Im} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \\
=&\frac{\partial \operatorname{Im} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}}{\partial \operatorname{Im} \alpha_1}\big(\mathbf{s}_I, (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}(\delta', 0, \dots, 0)\big) \cdot \delta' \qquad (5.71)
\end{aligned}$$

for some $\delta' \in (0, \delta)$. Note that from (5.57),

$$\begin{aligned}
&\frac{\partial \operatorname{Im} G_r^{\mathbf{E}_I, \mathbf{A}_{\zeta_I}, \mathbf{B}}}{\partial \operatorname{Im} \alpha_1}\big(\mathbf{s}_I, (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) + \sqrt{-1}(\delta', 0 \dots, 0)\big) \\
=&\left.\frac{\partial \operatorname{Im} G_r^{\mathbf{E}_I''}}{\partial \operatorname{Im} \alpha_1}\right|_{(\boldsymbol{\alpha}_s, \xi_s + \sqrt{-1}\delta_0')} - 2\pi k_1 < -\pi \min_{i\in I}\left\{ \frac{1}{q_i} - \frac{2|\beta_i - \pi|}{\pi q_i} \right\},
\end{aligned}$$

where the last inequality follows from (5.64) and (5.68). This implies that

$$\begin{aligned}
&\operatorname{Im} G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}\big(\mathbf{s}_I,(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}(\delta,0,\dots,0)\big)\\
&<\operatorname{Im} G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})-\pi\min_{i\in I}\Big\{\frac{1}{q_i}-\frac{2|\beta_i-\pi|}{\pi q_i}\Big\}\\
&\leq \operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\pi\min_{i\in I}\Big\{\frac{1}{q_i}-\frac{2|\beta_i-\pi|}{\pi q_i}\Big\},
\end{aligned}\tag{5.72}$$

where the last equality follows from (5.67).

Next, on $S_{\text{sides}}^{\mathbf{E}_I'',\delta,+}$, since $\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big)$ is strictly concave up in $\{\operatorname{Im}(\alpha_{i,\zeta_i})\}_{i\in I}$ and $\{\operatorname{Im}(\xi_s)\}_{s=1}^{c}$, for each $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\partial S^{\mathbf{E}_I'',\delta}$ and $t\in[0,1]$ we have

$$\begin{aligned}
&\operatorname{Im} G_r^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}\big(\mathbf{s}_I,(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+t\sqrt{-1}(\delta,0,\dots,0)\big)\\
&<\max\Big\{\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\Big),\operatorname{Im}\Big(G^{\mathbf{E}_I,\mathbf{A}_{\zeta_I},\mathbf{B}}\Big(\mathbf{s}_I,(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}(\delta,0,\dots,0)\Big)\Big)\Big\}.
\end{aligned}$$

For $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\partial S^{\mathbf{E}_I'',\delta}$, by (5.69) we have

$$\operatorname{Im} G^{\mathbf{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})<\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon'.$$

For $(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}(\delta,0,\dots,0)\in S_{\text{top}}^{\mathbf{E}_I'',\delta,+}$, by (5.72), by making ϵ' smaller if necessary, we have

$$\operatorname{Im} G^{\mathbf{E}_I}((\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})+\sqrt{-1}\operatorname{Im}(z^{\mathbf{E}_I}))<\operatorname{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon'.$$

Thus, we have

$$
\begin{aligned}
&\left|\int_{D_{\delta_0}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
=&\left|\int_{S^{\mathbf{E}_I''}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
\leq&\left|\int_{S^{\mathbf{E}_I''}\smallsetminus S^{\mathbf{E}_I'',\delta}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
&+\left|\int_{S^{\mathbf{E}_I'',\delta,+}} (-1)^{\sum_{i\in I} A_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}\left(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}\right)-\sum_{i\in I} 2\pi A_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}\right| \\
=&o\left(e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})-\epsilon')}\right).
\end{aligned}
$$

This finishes the proof under the assumption that $k_1 > 0$. For $k_1 < 0$, we consider the surface $S^{\mathbf{E}_I'',\delta,-} = S_{\text{top}}^{\mathbf{E}_I'',\delta,-} \cup S_{\text{bottom}}^{\mathbf{E}_I'',\delta,-}$ defined by

$$
S_{\text{top}}^{\mathbf{E}_I'',\delta,-} = \left\{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - \sqrt{-1}(\delta, 0, \dots, 0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in S^{\mathbf{E}_I'',\delta}\right\}
$$

and

$$
S_{\text{side}}^{\mathbf{E}_I'',\delta,-} = \left\{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - t\sqrt{-1}(\delta, 0, \dots, 0) \mid (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial S^{\mathbf{E}_I'',\delta}\right\}.
$$

The result follows from a similar argument as the previous case.

□

According to Proposition 5.19 and 5.22, it remains to study the asymptotics of the $\widehat{f}_r(\mathbf{s}, \mathbf{A}_{\zeta_\mathbf{I}}, \mathbf{0})$ with $(s_i, A_{\zeta_i}) \in S_i$ for all $i \in I$. where S_i is defined in (5.56). If $|q_i|$'s are odd for all $i \in I$, the asymptotics of $\widehat{f}_r(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1} - \mathbf{2m}^{\mathbf{E}_\mathbf{I}}, \mathbf{0})$ are given in Proposition 5.18. When some $|q_i|$ is even, the following proposition shows that the leading terms in the asymptotics of the other Fourier coefficients cancel out with each other.

Proposition 5.23. *Suppose the assumptions in Proposition 5.12 hold. Further suppose there exists $i_0 \in I$ such that $|q_{i_0}|$ is even. Then for every pair $(\mathbf{E}_\mathbf{I}', \mathbf{s}', \mathbf{A}_{\zeta_\mathbf{I}}', \mathbf{0})$ and $(\mathbf{E}_\mathbf{I}'', \mathbf{s}'', \mathbf{A}_{\zeta_\mathbf{I}}'', \mathbf{0})$ with $E_{i_0}' = -1, E_{i_0}'' = 1$, $E_i' = E_i''$ for all $i \in I \smallsetminus \{i_0\}$, $(s_{i_0}', A_{\zeta_{i_0}}') = (\tilde{s}_{i_0}^+, -2\tilde{m}_{i_0}^+)$, $(s_{i_0}'', A_{\zeta_{i_0}}'') = (\tilde{s}_{i_0}^-, -2\tilde{m}_{i_0}^-)$*

and $(s'_i, A'_{\zeta_i}) = (s''_i, A''_{\zeta_i})$ *for all* $i \in I \smallsetminus \{i_0\}$, *we have*

$$\widehat{f_r}^{\mathbf{E}_I}(\mathbf{s}', \mathbf{A}'_{\zeta_{\mathbf{I}}}, \mathbf{0}) = C'_r \frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm \mathbb{T}_{(M\smallsetminus L, \boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)}) + \sqrt{-1}\mathrm{CS}(M^{(r)})\right)} \left(1 + O\Big(\frac{1}{r}\Big)\right)$$

and

$$\widehat{f_r}^{\mathbf{E}_I}(\mathbf{s}'', \mathbf{A}''_{\zeta_{\mathbf{I}}}, \mathbf{0}) = -C'_r \frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm \mathbb{T}_{(M\smallsetminus L, \boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)}) + \sqrt{-1}\mathrm{CS}(M^{(r)})\right)} \left(1 + O\Big(\frac{1}{r}\Big)\right)$$

for some sequence of complex number C'_r *with norm* 1.

Proof. We first study the asymptotics of $\widehat{f_r}^{\mathbf{E}_I}(\mathbf{s}', \mathbf{A}'_{\zeta_{\mathbf{I}}}, \mathbf{0})$. Note that by Proposition 4.1,

$$\begin{aligned}\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}'_{\mathbf{I}}, \mathbf{A}'_{\zeta_I}, \boldsymbol{B}) &= \frac{r^{|I|+c}\big(\prod_{i\in I} E_i\big)}{2^{|I|+c}\pi^{|I|+c}} \\ &\times \int_{D_H} (-1)^{\sum_{i\in I} A'_{i,\zeta_i}} \phi_r\left(\mathbf{s}_I, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}}\left(W_r^{\mathbf{E}_I}(\mathbf{s}'_{\mathbf{I}}, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - \sum_{i\in I} 2\pi A'_{i,\zeta_i}\alpha_{i,\zeta_i}\right)} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi},\end{aligned}$$

By Lemma 5.20, 5.21 and a direct computation, we can write

$$\begin{aligned}&W_r^{\mathbf{E}_I}(\mathbf{s}', \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - \sum_{i\in I} 2\pi A'_{i,\zeta_i}\alpha_{i,\zeta_i} \\ =& G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) - 2\pi^2 \sum_{i\in I} A'_{\zeta_i} + \sum_{i\in I} 2\pi\beta_i\Big(-E_i J_i(s'_i) - \frac{p_i}{q_i}\Big) + \sum_{i\in I}\pi^2\left(K_i(s'_i) + \frac{p'_i}{q_i}\right) + \frac{4\pi^2}{r^2} h_I.\end{aligned}$$

Moreover, by Lemma 5.21 and Lemma 2.18 (2), since $\tilde{s}_i^{\pm} = s_i^{\pm} + \frac{q_i}{2} \pmod{q_i}$, we have

$$J_i(\tilde{s}_i^{\pm}) - J_i(\tilde{s}_i) = \mp \frac{p'_i}{q_i} \pmod{\mathbb{Z}}.$$

Thus, similar to the proof of Proposition 4.3, we can write

$$\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}'_{\mathbf{I}}, \mathbf{A}'_{\zeta_I}, \mathbf{0}) = \frac{Y'(\mathbf{E}_I) r^{|I|+c}}{2^{|I|+c}\pi^{|I|+c}} \int_{D_H} \phi_r\left(\mathbf{s}^{\mathbf{E}_I}, \boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}\right) e^{\frac{r}{4\pi\sqrt{-1}} G_r^{E_I}(\boldsymbol{\alpha}_{\xi_I}, \boldsymbol{\zeta})} d\boldsymbol{\alpha}_{\zeta_I} d\boldsymbol{\xi}, \tag{5.73}$$

where

$$Y'(\mathbf{E}_I) = -(-1)^{\sum_{i\in I}\left(\frac{p'_i}{q_i}+E_iJ_i(s'_i)\right)+|I|}\left(\prod_{i\in I}E_i\right)e^{\frac{r\pi}{4\sqrt{-1}}\sum_{i\in I}\left(-2A'_{\zeta_i}+K_i(s'_i)+\frac{p'_i}{q_i}\right)}, \tag{5.74}$$

$$\begin{aligned}&\phi_r(\mathbf{s}_I,\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = \psi(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\\ &\times e^{\sqrt{-1}\left(\sum_{i\in I}\left(\frac{p'_i}{q_i}(\beta_i-\pi)+\frac{p_i}{q_i}(\alpha_{i,\zeta_i}-\pi)+\frac{E_i(\alpha_{i,\zeta_i}+\beta_i-2\pi)}{q_i}\right)+\sum_{i\in I}a_{i,0}\beta_i+\sum_{i\in I}\left(\frac{\iota_i}{2}\right)\alpha_{i,\zeta_i}+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)\alpha_j\right)},\end{aligned}$$

and

$$\begin{aligned}G_r^{\boldsymbol{E}_I}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}) = &\sum_{i\in I}\left[-\left(\frac{p'_i}{q_i}+a_{i,0}\right)(\beta_i-\pi)^2 - \frac{p_i(\alpha_{i,\zeta_i}-\pi)^2+2E_i(\beta_i-\pi)(\alpha_{i,\zeta_i}-\pi)}{q_i}\right]\\ &-\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)(\alpha_j-\pi)^2 - \sum_{i\in I}\frac{\iota_i}{2}(\alpha_{i,\zeta_i}-\pi)^2\\ &+\sum_{s=1}^{c}U_r(\alpha_{s_1},\dots,\alpha_{s_6},\xi_s) + \Big(\sum_{i=1}^{n}\frac{\iota_i}{2}\Big)\pi^2.\end{aligned}$$

By (5.73), Proposition 5.12 and 5.13,

$$\begin{aligned}&\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}'_{\mathbf{I}},\mathbf{A}'_{\zeta_I},\mathbf{0})\\ &= \frac{Y'(\mathbf{E}_I)r^{|I|+c}}{2^{|I|+c}\pi^{|I|+c}}\left(\frac{2}{r}\right)^c\left(\frac{2\pi}{r}\right)^{\frac{|I|+c}{2}}(4\pi\sqrt{-1})^{\frac{|I|+c}{2}}\\ &\qquad\frac{C^{\mathbf{E}_I}(\mathbf{z}^{\boldsymbol{E}_I})}{\sqrt{-\left(\prod_{i\in I}q_i\right)\det\mathrm{Hess}(G^{\mathbf{E}_I})(\mathbf{z}^{\mathbf{E_I}})}}e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})+\sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}}))}\Big(1+O\Big(\frac{1}{r}\Big)\Big)\\ &= C'_r\frac{e^{\frac{1}{2}\sum_{k=1}^n\mu_k\mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}}e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)})+\sqrt{-1}\mathrm{CS}(M^{(r)})\right)}\left(1+O\Big(\frac{1}{r}\Big)\right),\end{aligned}$$

where $C'_r = Y'(\mathbf{E}_I)2^{-|I|+c}r^{\frac{|I|-c}{2}}(-1)^{-\frac{rc}{2}+\frac{|I|+c}{4}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}$

By the same argument, we have

$$\begin{aligned}&\widehat{f_r^{\mathbf{E}_I}}(\mathbf{s}''_{\mathbf{I}}, \mathbf{A}''_{\zeta_I}, \mathbf{0})\\&= C''_r \frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm \mathbb{T}_{(M\smallsetminus L,\boldsymbol{\Upsilon})}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)})+\sqrt{-1}\mathrm{CS}(M^{(r)})\right)}\left(1+O\Big(\frac{1}{r}\Big)\right),\end{aligned}$$

where $C''_r = Y''(\mathbf{E}_I)2^{-|I|+c} r^{\frac{|I|-c}{2}}(-1)^{-\frac{rc}{2}+\frac{|I|+c}{4}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}$ with

$$Y''(\mathbf{E}_I) = -(-1)^{\sum_{i\in I}\left(\frac{p'_i}{q_i}+E_i J_i(s''_i)\right)+|I|}\left(\prod_{i\in I} E_i\right) e^{\frac{r\pi}{4\sqrt{-1}}\sum_{i\in I}\left(-2A''_{\zeta_i}+K_i(s''_i)+\frac{p'_i}{q_i}\right)}. \tag{5.75}$$

To prove the proposition, it suffices to study the ratio of C'_r and C''_r. Note that from (5.74) and (5.75),

$$\frac{C'_r}{C''_r} = \frac{Y'(\mathbf{E}_I)}{Y''(\mathbf{E}_I)} = -(-1)^{-(J_{i_0}(\tilde{s}^+_{i_0})+J_{i_0}(\tilde{s}^-_{i_0}))} e^{\frac{r\pi}{4\sqrt{-1}}(K_{i_0}(s'_{i_0})-K_{i_0}(s''_{i_0})+4(\tilde{m}'_{i_0}-\tilde{m}''_{i_0}))}. \tag{5.76}$$

From Lemma 2.18 (2), we know that

$$J_{i_0}(s^+_{i_0}) \equiv -J_{i_0}(s^-_{i_0}) \pmod 2,$$

From Lemma 5.21, we know that $\tilde{s}^{\pm}_{i_0} = s^{\pm}_{i_0} + \frac{q_{i_0}}{2} \pmod{q_{i_0}}$. Thus, we have $J_{i_0}(\tilde{s}^+_{i_0}) - J_{i_0}(s^+_{i_0}) \equiv 1 \pmod 2$, $J_{i_0}(\tilde{s}^-_{i_0}) - J_{i_0}(s^+_{i_0}) \equiv 1 \pmod 2$ and

$$J_{i_0}(\tilde{s}^+_{i_0})+J_{i_0}(\tilde{s}^-_{i_0}) = \big(J_{i_0}(\tilde{s}^+_{i_0})-J_{i_0}(s^+_{i_0})\big)+\big(J_{i_0}(\tilde{s}^-_{i_0})-J_{i_0}(s^+_{i_0})\big)+\big(J_{i_0}(s^+_{i_0})+J_{i_0}(s^-_{i_0})\big) \equiv 0 \pmod 2.$$

This implies that

$$(-1)^{E_i(J_{i_0}(\tilde{s}^+_{i_0})+J_{i_0}(\tilde{s}^-_{i_0}))} = 1. \tag{5.77}$$

From the definition of K in Lemma 2.18 (3), we get

$$\begin{aligned}&K_{i_0}(\tilde{s}^+_{i_0}) - K_{i_0}(\tilde{s}^-_{i_0}) + 4(\tilde{m}^+_{i_0} - \tilde{m}^-_{i_0})\\ =&\frac{4C_{i_0,\xi_{i_0}-1}}{q_{i_0}}(\tilde{s}^+_{i_0} + \tilde{s}^-_{i_0} + 1 + K_{i_0,\xi_{i_0}-1})(\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0}) + 4(\tilde{m}^+_{i_0} - \tilde{m}^-_{i_0}).\end{aligned} \tag{5.78}$$

Besides, from the definition of I and (5.55),

$$I_{i_0}(\tilde{s}^+_{i_0}) + I_{i_0}(\tilde{s}^-_{i_0}) = -2C_{i_0,\xi_{i_0}-1}(\tilde{s}^+_{i_0} + \tilde{s}^-_{i_0} + 1 + K_{i_0,\xi_{i_0}-1}) = 2q_{i_0}(\tilde{m}^+_{i_0} + \tilde{m}^-_{i_0}). \tag{5.79}$$

From (5.78) and (5.79), we have

$$K_{i_0}(\tilde{s}^+_{i_0}) - K_{i_0}(\tilde{s}^-_{i_0}) + 4(\tilde{m}^+_{i_0} - \tilde{m}^-_{i_0}) = 4\big((-\tilde{m}^+_{i_0} - \tilde{m}^-_{i_0})(\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0}) + \tilde{m}^+_{i_0} + \tilde{m}^-_{i_0}\big).$$

In particular,

$$e^{\frac{r\pi}{4\sqrt{-1}}(K_{i_0}(s'_{i_0}) - K_{i_0}(s''_{i_0}) + 4(\tilde{m}'_{i_0} - \tilde{m}''_{i_0}))} = (-1)^{(\tilde{m}^+_{i_0} + \tilde{m}^-_{i_0})(\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0} - 1)}.$$

From Lemma 5.21, we know that $\tilde{s}^{\pm}_{i_0} = s^{\pm}_{i_0} + \frac{q_{i_0}}{2} \pmod{q_{i_0}}$. Since q_{i_0} is even, we have

$$\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0} = \big(\tilde{s}^+_{i_0} - s^+_{i_0}\big) - \big(\tilde{s}^-_{i_0} - s^-_{i_0}\big) + \big(s^+_{i_0} - s^-_{i_0}\big) \equiv s^+_{i_0} - s^-_{i_0} \pmod{2}.$$

From Lemma 2.18 (1), we know that $s^+_{i_0} - s^-_{i_0} \equiv p'_{i_0} \pmod{q_{i_0}}$. Moreover, since $p_{i_0}p'_{i_0} + q_{i_0}q'_{i_0} = 1$ and q_{i_0} is even, p'_{i_0} must be odd. Altogether, we have

$$\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0} - 1 \equiv s^+_{i_0} - s^-_{i_0} - 1 \equiv 0 \pmod{2}$$

and

$$e^{\frac{r\pi}{4\sqrt{-1}}(K_{i_0}(s'_{i_0}) - K_{i_0}(s''_{i_0}) + 4(\tilde{m}'_{i_0} - \tilde{m}''_{i_0}))} = (-1)^{(\tilde{m}^+_{i_0} + \tilde{m}^-_{i_0})(\tilde{s}^+_{i_0} - \tilde{s}^-_{i_0} - 1)} = 1. \tag{5.80}$$

From (5.76), (5.77) and (5.80), we get $C_r'' = -C_r'$. This completes the proof. $\square$

5.7 Estimate of error term and the proof of the main theorems

The following proposition shows that the error term in Proposition 4.2 is negligible compared to the leading Fourier coefficient.

Proposition 5.24. *There exists $\delta_0 > 0$ such that if*

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \smallsetminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}),$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H \smallsetminus D_{\delta_0}}$ is the closure of $D_H \smallsetminus D_{\delta_0}$, then there exists $\epsilon' > 0$ such that the error term in Proposition 4.2 is less than $O(e^{\frac{r}{4\pi}(\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) - \epsilon')})$.

Proof. For a fixed $\boldsymbol{\alpha}_J = (\alpha_j)_{j \in J}$, let

$$M_{\boldsymbol{\alpha}_J} = \max \left\{ \sum_{s=1}^{c} 2V(\alpha_{s_1}, \dots, \alpha_{s_6}, \xi_s) \,\middle|\, (\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \partial \mathrm{D_H} \cup \big(\mathrm{D_A} \smallsetminus \mathrm{D_H}\big) \right\}$$

where V is as defined in (5.2). Then by [5, Sections 3 & 4],

$$M_{\boldsymbol{\alpha}_J} < 2cv_8.$$

Besides, we know that $\mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \leq 2cv_8$ and equality holds if and only if

$$(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) = (\pi, \dots, \pi, \frac{7\pi}{4}, \dots, \frac{7\pi}{4}).$$

As a result, we can choose $\delta_0 > 0$ sufficiently small so that

$$M_{\boldsymbol{\alpha}_J} < \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \smallsetminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}).$$

The result follows from the fact that the error terms in Proposition 4.2 contains those $g_r^{\mathbf{E}_I}(\mathbf{s}_I, \mathbf{m}_{\zeta_I}, \mathbf{k})$ with $(\mathbf{m}_{\zeta_I}, \mathbf{k}) \in \mathrm{D_H} \cup \big(\mathrm{D_A} \smallsetminus \mathrm{D_H}\big)$. $\square$

Lemma 5.25. *There exists $\delta > 0$ such that if $|\mathrm{H}(u_k)| < \delta$ for all $k = 1, \ldots, n$, then we have $\mathbf{z}^{\boldsymbol{E}_I} \in D_{\delta_0, \mathbb{C}}$ and*

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \setminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\},$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H \setminus D_{\delta_0}}$ is the closure of $D_H \setminus D_{\delta_0}$.

Proof. Note that by Proposition 5.11, we have

$$|\alpha_i^* - \pi| = \frac{|\mathrm{H}(u_i)|}{2} < \frac{\delta}{2}.$$

Moreover, $\{\xi_s\}_{s=1}^c$ depends continuously on $\{\alpha_k\}_{k=1}^n$ with $\xi_s(\pi, \ldots, \pi) = \frac{7\pi}{4}$ for all $s = 1, \ldots, c$. Altogether, by choosing $\delta > 0$ sufficiently small, we have $\mathbf{z}^{\boldsymbol{E}_I} \in D_{\delta_0, \mathbb{C}}$. Besides, $\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})$ depends continuously on $\{\mathrm{H}(u_k)\}_{k=1}^n$ and is equal to $2cv_8$ when $\mathrm{H}(u_k) = 0$ for $k = 1, \ldots, n$. Moreover,

$$2cv_8 > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \setminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\}.$$

By choosing $\delta > 0$ sufficiently small, we have

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \setminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\}.$$

$\square$

Lemma 5.26. *There exists $\epsilon > 0$ such that whenever $\theta_i, \theta_j \in [0, \epsilon)$ for all $i \in I$ and $j \in J$, we have $\mathbf{z}^{\boldsymbol{E}_I} \in D_{\delta_0, \mathbb{C}}$ and*

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}) \in \overline{D_H \setminus D_{\delta_0}}} \mathrm{Im}\, \tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\},$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I}, \boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H \setminus D_{\delta_0}}$ is the closure of $D_H \setminus D_{\delta_0}$.

Proof. First, when $\beta_i = \alpha_j = \pi$ for all $i \in I, j \in J$, we have $(\theta_1, \ldots, \theta_n) = \mathbf{0} = (0, \ldots, 0)$,

$\mathbf{z}^{\boldsymbol{E}_I} = (\pi, \dots, \pi, \frac{7\pi}{4}, \dots, \frac{7\pi}{4}) \in D_{\delta_0,\mathbb{C}}$ and

$$\mathrm{Vol}(M_{L_{\mathbf{0}}}) = 2cv_8 > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\overline{D_H\smallsetminus D_{\delta_0}}} \mathrm{Im}\,\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\}.$$

By continuity, there exists $\epsilon > 0$ such that if $\{\beta_i\}_{i\in I}$ and $\{\alpha_j\}_{j\in J}$ are all in $(\pi-\epsilon,\pi+\epsilon)$, then the critical point $\mathbf{z}^{\mathbf{E}_I}$ of $G^{\mathbf{E}_I}$ in Proposition 5.11 lies in $D_{\delta_0,\mathbb{C}}$, and $\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}})$ is sufficiently close to $\mathrm{Vol}(M_{L_{\mathbf{0}}}) = 2cv_8$ so that

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\overline{D_H\smallsetminus D_{\delta_0}}} \mathrm{Im}\,\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\}.$$

□

Lemma 5.27. *There exists $\epsilon > 0$ and $C > 0$ such that whenever $\theta_j \in [0,\epsilon)$ for all $j \in J$, $|p_i| + |q_i| > C$ and $\theta_i \in [0,\pi)$ for all $i \in I$, we have $\mathbf{z}^{\boldsymbol{E}_I} \in D_{\delta_0,\mathbb{C}}$ and*

$$\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) > \max\left\{ \max_{(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})\in\overline{D_H\smallsetminus D_{\delta_0}}} \mathrm{Im}\,\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi}), 2cv_8 - 4\pi\delta_0 \right\},$$

where $\tilde{U}(\boldsymbol{\alpha}_{\zeta_I},\boldsymbol{\xi})$ is defined in (5.8) and $\overline{D_H\smallsetminus D_{\delta_0}}$ is the closure of $D_H\smallsetminus D_{\delta_0}$.

Proof. Let $\delta > 0$ be the constant in Lemma 5.25. For each $k \in \{1,2,\dots,n\}$, recall that the generalized Dehn filling invariant of the logarithmic holonomy $\mathrm{H}(u_k)$ around $0 \in \mathbb{C}$ is defined by sending 0 to $\infty \in \mathbb{R}^2 \cup \{\infty\} = \mathbb{S}^2$ and sending $\mathrm{H}(u_k) \neq 0$ to the unique pair $(p_k,q_k) \in \mathbb{S}^2$ satisifying

$$p_k\mathrm{H}(u_k) + q_k\mathrm{H}(v_k) = 2\pi\sqrt{-1}.$$

It is well-known that the generalized Dehn filling invariant gives a local homeomorphism from an open neighborhood of $(0,\dots,0) \in \mathbb{C}^n$ to an open neighborhood around $(\infty,\dots,\infty) \in (\mathbb{S}^2)^n$ by sending the logarithmic holonomies $(\mathrm{H}(u_1),\dots,\mathrm{H}(u_n))$ to the generalized Dehn filling invariants $((p_1,q_1),\dots,(p_n,q_n))$ (see e.g. Corollary 15.2.17 and Proposition 15.3.1 in [32]). In particular, there exists $C > 0$ such that whenever $|p_k| + |q_k| > C$ for all $k = 1,\dots,n$, we have $|\mathrm{H}(u_k)| < \delta$

for all $k = 1, \dots, n$.

Note that for $i \in I$ with $|p_i| + |q_i| > C$ and $\theta_i \in (0, \pi)$, the equation $p_i \mathrm{H}(u_i) + q_i \mathrm{H}(v_i) = \theta\sqrt{-1}$ implies that

$$\Big(\frac{2\pi p_i}{\theta_i}\Big)\mathrm{H}(u_i) + \Big(\frac{2\pi q_i}{\theta_i}\Big)\mathrm{H}(v_i) = 2\pi\sqrt{-1}.$$

In particular, the generalized Dehn invariant of $\mathrm{H}(u_i)$ is given by $(\frac{2\pi p_i}{\theta_i}, \frac{2\pi q_i}{\theta_i})$, which satisfies

$$\Big|\frac{2\pi p_i}{\theta_i}\Big| + \Big|\frac{2\pi q_i}{\theta_i}\Big| = (|p_i| + |q_i|)\Big(\frac{2\pi}{\theta_i}\Big) > |p_i| + |q_i| > C.$$

Besides, for $j \in J$, if the cone angle $\theta_j \in (0, 2\pi/C)$, then the equation $\mathrm{H}(u_j) = \theta_j\sqrt{-1}$ implies that

$$\Big(\frac{2\pi}{\theta_j}\Big)\mathrm{H}(u_j) = 2\pi\sqrt{-1}.$$

In particular, the generalized Dehn invariant of $\mathrm{H}(u_j)$ is given by $(\frac{2\pi}{\theta_j}, 0)$, which satisfies

$$\Big|\frac{2\pi}{\theta_j}\Big| > C.$$

As a result, whenever $\theta_j \in [0, \frac{2\pi}{C})$ for all $j \in J$, $|p_i| + |q_i| > C$ and $\theta_i \in [0, \pi)$ for all $i \in I$, we have $|\mathrm{H}(u_k)| < \delta$ for all $k = 1, \dots, n$. The results follow from Lemma 5.25. $\square$

Proof of Theorem 1.5, 1.6 and 1.7. By Lemma 5.25, 5.26 and 5.27, the assumptions in Proposition 5.12, 5.19, 5.22, 5.23 and 5.24 are satisfied. Thus, by Proposition 3.4, Proposition 4.2, Proposition 5.18, Proposition 5.19, Proposition 5.22, Proposition 5.23 and Proposition 5.24, we have

$$\begin{aligned}
&\mathrm{RT}_r(M, L, (\mathbf{n}_I, \mathbf{m}_J))\\
=&Z_r\Big(\sum_{\mathbf{E}_I} \widehat{f}_r(\mathbf{s}^{\mathbf{E}_I}, \mathbf{1} - \mathbf{2m}^{\mathbf{E_I}}, \mathbf{0})\Big)\left(1 + O\left(\frac{1}{r}\right)\right)\\
=&C\frac{e^{\frac{1}{2}\sum_{k=1}^n \mu_k \mathrm{H}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\smallsetminus L, \mathbf{m})}([\rho_{M_{L_{\boldsymbol{\theta}}}}])}} e^{\frac{r}{4\pi}\Big((\mathrm{Vol}(M_{L_{\boldsymbol{\theta}}}) + \sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol{\theta}}}))\Big)}\Big(1 + O\Big(\frac{1}{r}\Big)\Big),
\end{aligned} \tag{5.81}$$

where

$$C = \frac{(-1)^{\sum_{i\in I}(\zeta_i+1+\sum_{l=1}^{\zeta_i}a_{i,l})}(\sqrt{-1})^{\sum_{i\in I}\frac{\zeta_i-1}{2}}(-1)^{-\frac{rc}{2}+\sum_{i\in I}\left(a_{i,0}+\frac{\iota_i}{2}\right)+\sum_{j\in J}\left(a_{j,0}+\frac{\iota_j}{2}\right)}}{\sqrt{-1}^{\sum_{i\in I}\zeta_i-c}}$$
$$\times e^{\frac{\pi\sqrt{-1}}{r}\sum_{i\in I}\sum_{l=1}^{\zeta_i-1}a_{i,l}-\frac{r\pi\sqrt{-1}}{4}(\sum_{i\in I}(a_{i,0}+a_{i,\zeta_i})+\sum_{j\in J}a_{j,0})+\sigma(\tilde{L}_{\mathrm{FSL},I}\cup L')(\frac{3}{r}+\frac{r+1}{4})\sqrt{-1}\pi}$$
$$\times e^{\frac{r\pi}{4\sqrt{-1}}\sum_{i\in I}\left(4m_i^+-2+K_i(s_i^+)+\frac{p_i'}{q_i}\right)}$$

is a quantity of norm 1 independent of the geometric structure on M. □

Proof of Theorem 1.8 and 1.9. From (5.81), we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{4\pi}{r}\log \mathrm{RT}_r(M,L,(\mathbf{n}_I,\mathbf{m}_J))\\
&=\mathrm{Vol}(M_{L_{\boldsymbol\theta}})+\sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol\theta}})-2c\pi^2\sqrt{-1}+\pi^2\sqrt{-1}\Big(\sum_{i\in I}(a_{i,0}+a_{i,\zeta_i})+\sum_{j\in J}a_{j,0})+\sigma(\tilde{L}_{\mathrm{FSL},I}\cup L')\Big)\\
&\quad-\pi^2\sqrt{-1}\sum_{i\in I}\left(4m_i^+-2+K_i(s_i^+)+\frac{p_i'}{q_i}\right)\\
&=\mathrm{Vol}(M_{L_{\boldsymbol\theta}})+\sqrt{-1}\,\mathrm{CS}(M_{L_{\boldsymbol\theta}})\pmod{\pi^2\sqrt{-1}\mathbb{Z}},
\end{aligned}$$

where in the last equality we apply Lemma 2.18 (3). □

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