LOCAL COHOMOLOGY, MULTIGRADINGS AND POLYHEDRAL COMBINATORICS

A Dissertation

by

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ABSTRACT

The goal of combinatorial commutative algebra is to study the interplay between commutative algebra and various subfields of combinatorics such as enumerative combinatorics and discrete geometry. Among the central objects in combinatorial commutative algebra are square-free monomial ideals and semigroup rings. This dissertation examines monomial ideals in affine semigroup rings.

The two main results in this dissertation are as follows. First, we give a combinatorial characterization for not only the Cohen–Macaulay $\mathbb{Z}^d$-graded module of affine semigroup rings but also the quotients of polynomial rings by cellular binomial ideals. This criterion involves vanishing homology of finitely many polyhedral cell complexes. These polyhedral cell complexes are derived from degree spaces, a space of all multigradings with special finite topology. Our main contribution is to construct degree spaces corresponding to $\mathbb{Z}^d$-graded modules.

Next, we elucidate a hidden duality between the local cohomologies of simplicial affine semigroup rings by extending the Ishida complex as well as using a Hochster-type Hilbert series formula. The extensions of the Ishida complex allow us to calculate local cohomology of the given module with all possible radical monomial ideal supports. With our degree spaces, we calculate the Hilbert series of both the local cohomology of the simplicial affine semigroup ring $\Bbbk[Q]$ with a monomial ideal I support and that of the quotient $\Bbbk[Q]/I$ with the maximal monomial ideal support. Finally, we showed that there is a 1-1 correspondence of grains between these two local cohomologies such that whose cohomologies of the chaffs are dual.

DEDICATION

To my mother, my father, and my friend Peter Lee Seunghun.

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$\mathbb{N}$	The set of non-negative integers
d	A natural number denoting the dimension of an object
$n, m, \cdots$	Some natural numbers
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$	The ring of integers, the field of rational numbers (resp. real numbers)
$\Bbbk$	A field of any characteristic
$u, v, w, \cdots$	Integer vectors which might denote $\mathbb{Z}^d$-degrees.
$\lVert u \rVert$	The Euclidean norm of a vector u
A	A generating set (as $d \times n$ integer matrix) of an affine semigroup
Q	An affine semigroup
$\mathcal{H}(Q)$	The set of holes of Q
$\Bbbk[Q]$	An affine semigroup ring
$\mathscr{P}$	A polyhedron
$\mathcal{H}$	A hyperplane
$\mathcal{A}$	A hyperplane arrangement
$\mathfrak{r}$ (resp. $\mathfrak{R}$)	A region (resp. cumulative region) of the given hyperplane arrangement
$\mathfrak{r}(\mathcal{A})$ (resp. $\mathfrak{R}(\mathcal{A})$)	The set of regions (resp. cumulative regions) of $\mathcal{A}$
$\mathbb{R}_{\geq 0}(Q)$	The polyhedral cone over Q
Δ	A polyhedral complex
$F, G, H, \cdots$	Faces of a polyhedral complex
$\mathcal{F}(\mathscr{P})$	The face lattice of $\mathscr{P}$
$\mathrm{RelInt}(\mathscr{P})$	The relative interior of $\mathscr{P}$
$\mathfrak{m}$	The (graded) maximal ideal
I_Δ	The monomial radical ideal corresponding to a polyhedral complex Δ
I	An ideal of an affine semigroup ring
M	A module over a ring
E	An injective module of a ring
$\mathrm{Ass}(M)$	The set of associated primes of M
$\mathcal{I}$	A monomial ideal of a graded ring
$\tilde{\mathcal{C}}\,(M)$	The (singular, cellular, or simplicial) chain complex of M
$L(M)$	The (generalized) Ishida complex of M
$H^n_T(M)$	The n-th local cohomology of M supported at I
$\mathrm{Hilb}(x, M)$	The Hilbert series of a module M in terms of the variable x
K	A transverse section of a polyhedron
$\widehat{F}$	A face of a $\mathscr{P}$ which corresponding to the face F of its K
$\underline{\mathrm{deg.\,p}}(M)$	The set of all degree pairs of a graded module M
$\overline{\mathrm{deg.\,p}}(M)$	The set of all overlap classes of degree pairs of M
$\bigcup \deg(M)$	The set of all degrees of M and its graded localizations
$\mathcal{G}$	The set of grains
ζ	The set of cellular variables

1. INTRODUCTION AND BACKGROUND

1.1 Introduction

The goal of combinatorial commutative algebra is to study the interplay between commutative algebra and various subfields of combinatorics such as enumerative combinatorics and discrete geometry. Among the central objects in combinatorial commutative algebra are square-free monomial ideals and semigroup rings. This dissertation examines monomial ideals in affine semigroup rings.

The two main results in this dissertation are as follows. First, we give a combinatorial characterization for the Cohen–Macaulay $\mathbb{Z}^d$-graded modules of affine semigroup rings by constructing degree pairs (Chapter 2). This criterion involves vanishing homology of finitely many polyhedral cell complexes. This result was achieved by generalizing the notion of standard pairs, which was originally devised by [52] for monomial ideals of polynomial rings, to the semigroup context and then applying these new standard pairs to study the Ishida complex which is used to compute local cohomology. We produce a Hochster-type formula for the Hilbert series of local cohomology of $\mathbb{Z}^d$-graded modules of affine semigroup rings, expressed as a finite sum of rational functions arising from the Betti numbers of finitely many polyhedral objects.

Next, we provides a new way to provide a finite sum of rational functions as the Hilbert series of local cohomologies of the quotients of polynomial rings by cellular binomial ideals with an ideal support whose contraction is a monomial ideal in an affine semigroup rings (Chapter 3). This work was done by generalizing Ishida complex so that it can work for the quotients of polynomial rings by cellular binomial ideals.

From these main results, we have several applications of the main results (Chapter 4). First of all, in Section 4.1, we provides an alternative proof of the classification of Cohen–Macaulay affine semigroup rings, which was firstly proved by [54]. Next, in Section 4.2, we also generalized the Hochster's theorem [27], stating about the Hilbert series of local cohomologies of Stanley-Reisner rings with the graded maximal ideal supports. Our generalized Hochster's theorem provides a finite sum of rational functions as Hilbert series' of local cohomologies of the quotients of (not necessarily normal) affine simplicial rings by any monomial ideals with any radical monomial ideal supports.

Lastly, in Section 4.3 we elucidate a hidden duality between the local cohomologies of Stanley-Reisner rings by extending the Ishida complex as well as using a Hochster-type formula (Section 4.3). To see this, fix a Stanley-Reisner ring $\Bbbk[\mathbf{t}]/I_\Delta$ for a radical monomial ideal I_Δ of the polynomial ring $\Bbbk[\mathbf{t}]$ corresponding to a simplicial complex Δ. Previously, [29] reported that

$$H^i_{\mathfrak{m}}(\Bbbk[\mathbf{t}]/I_\Delta) = 0 \iff H^{\dim \Bbbk[\mathbf{t}]-i}_{I_\Delta}(\Bbbk[\mathbf{t}]) = 0,$$

where $H^i_{\mathfrak{m}}(\Bbbk[\mathbf{t}]/I_\Delta)$ is the i-th local cohomology of the Stanley-Reisner ring $\Bbbk[\mathbf{t}]/I_\Delta$ supported on the maximal monomial ideal $\mathfrak{m}$ and $H^{\dim \Bbbk[\mathbf{t}]-i}_{I_\Delta}(\Bbbk[\mathbf{t}])$ is the $(\dim \Bbbk[\mathbf{t}] - i)$-th local cohomology of the polynomial ring $\Bbbk[\mathbf{t}]$ supported on the radical monomial ideal I_Δ. From the original formula of Hochster, the Hilbert series of $H^\bullet_{\mathfrak{m}}(\Bbbk[\mathbf{t}]/I_\Delta)$ can be decomposed as a finite sum over components G_F, for each $F \in \Delta$ from the decomposition of standard monomials of $\Bbbk[\mathbf{t}]/I_\Delta$ and its localizations by all monomial prime ideals. On the other hand, there is a decomposition of the degree sets of the extended Ishida complex of $\Bbbk[\mathbf{t}]$ supported on I_Δ, where the components are equal to sets of lattice points from orthants of $\Bbbk[\mathbf{t}]$ arising from localizations by faces of d-simplex. Therefore, we may label each component as $\mathfrak{r}_F$ for each F in the d-simplex. We find that for each face F of the d-simplex,

$$H^i_{\mathfrak{m}}(\Bbbk[\mathbf{t}]/I_\Delta)_\alpha \cong H^{\dim \Bbbk[\mathbf{t}]-i}_{I_\Delta}(\Bbbk[\mathbf{t}])_\beta$$

for any $\alpha \in G_F$ and $\beta \in \mathfrak{r}_{F^c}$.

Every reader who are interested in doing research on further topics might like the last section, Section 4.4, which contain some open problems related to these main results and applications. We hope the readers enjoy this dissertation.

1.2 Background

Throughout this dissertation, assume all rings are commutative rings with unity. We let $\Bbbk$ denote a field and $S = \Bbbk[x_1, x_2, \cdots, x_d]$ be a polynomial ring over $\Bbbk$ with d variables. Also, $\mathbb{N}$ is the set of natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, 2, \cdots\}$ and use $[n]$ to denote $\{1, 2, \cdots, n\}$. To abbreviate our notation, for an integer vector $u \in \mathbb{N}^d$, we use x^u to denote the monomial $x_1^{u_1}x_2^{u_2}\cdots x_n^{u_d}$. Also, we assume that readers are familiar with basic homological algebras over category theory. Please refer [42], [12], [46], and [56] if readers are interested in more foundational work, .

1.2.1 Polyhedral geometry

Let $\mathbb{R}$ be the set of all real numbers. Given the d-dimensional real space $\mathbb{R}^d$, pick a vector $u \in \mathbb{R}^d$. Then, a *hyperplane* $\mathcal{H}_{u,c}$ corresponding to u is an affine $\mathbb{R}^{d-1}$ subspace consisting of vectors whose dot product with u is a constant c. Likewise, we set $\mathcal{H}_{u,c}^+$ (resp. $\mathcal{H}_{u,c}^-$) to be a set consisting of vectors whose dot product with u is greater than (resp. less than) c. We call these $\mathcal{H}_{u,c}^+$ or $\mathcal{H}_{u,c}^-$ as *open halfspaces*. Closure of open half spaces are called *closed halfspaces*. For example, the closure $\overline{\mathcal{H}_{u,c}^+}$ of an open half space $\mathcal{H}_{u,c}^+$ is equal to $\mathcal{H}_{u,c}^+ \cup \mathcal{H}_{u,c}$.

Definition 1.2.1. A *polyhedron* $\mathscr{P}$ is an intersection of finitely many closed half spaces. Especially, a *(convex) polytope* is a polyhedron which is bounded.

Among all possible polyhedrons, we are interested in specific types of polyhedrons, called *polyhedral cones*.

Definition 1.2.2. A *polyhedral cone* is a polyhedron $\mathscr{P}$ which is a cone, such that $\lambda u \in \mathscr{P}$ for any $\lambda \geq 0 \in \mathbb{R}$ and $u \in \mathscr{P}$.

Boundaries of a polyhedron also can be decomposed into several sub-polyhedrons, called *faces*.

Definition 1.2.3. A nonempty subset F of a polyhedral cone $\mathscr{P}$ is called a *face* if there exists $v \in \mathbb{R}^d$ such that the dot product of elements in F with v achieves its maximum value. In other words,

$$F = \text{face}_v(\mathscr{P}) := \{a \in \mathscr{P} \mid v\dot{a} \geq \langle v, u\rangle \text{ for all } u \in \mathscr{P}\}.$$

Especially, a zero-dimensional face is called a *vertex*. A *supporting hyperplane* H of a face F is a hyperplane such that $H \cap \mathscr{P} = F$. In such case, a vector u which is normal to H and $\langle u, x\rangle \leq 0$ (resp.$\langle u, x\rangle \geq 0$) for all $x \in \mathscr{P}$ is called *outer normal vector* (resp. *inner normal vector*).

When u is an outer normal vector of a supporting hyperplane of a face F, then $F = \text{face}_u(\mathscr{P})$. Also, we regard $\emptyset$ as a face of every polyhedron. Now we define the concept of *relative interior*.

Definition 1.2.4. An *affine hull* of a set $A \subset \mathbb{R}^d$ of real vectors is the set of all real linear combinations $\sum_{i=1}^m \lambda_i u_i$ for which $\sum_{i=1}^m \lambda_i = 1$ and $u_i \in A$ for any $m \in \mathbb{N}$. The *relative interior* $\text{RelInt}(\mathscr{P})$ of $\mathscr{P}$ is the interior of $\mathscr{P}$ with respect to the affine hull of $\mathscr{P}$.

For a face F of $\mathscr{P}$, $\text{RelInt}(F)$ is equal to the set of all points in F that do not lie in any other proper faces of $\mathscr{P}$.

Definition 1.2.5. The *dimension* $\dim \mathscr{P}$ of a nonempty polyhedron $\mathscr{P}$ is defined to be the dimension of its affine hull.

Conventionally, $\dim(\emptyset) := -1$.

Definition 1.2.6. A polyhedron is *pointed* if it has only one zero-dimensional face.

Also, we want to inform that the set $\mathcal{F}(\mathscr{P})$ of all faces of a polyhedral cone $\mathscr{P}$ has a special structure.

Definition 1.2.7. A *poset* is a set with partial orders. A *join* of two elements of a poset is the least upper bound of those two elements. A *meet* of two elements of a poset is the greatest lower bound of those two elements. A *lattice* is a poset such that every pair of elements has its join and meet in the poset.

Definition 1.2.8. A *transverse section* K of a polyhedron $\mathscr{P}$ is a polytope generated by a hyperplane $\mathcal{H}$ intersecting with $\mathscr{P}$ such that $K = \mathscr{P} \cap \mathcal{H}$ and $\mathcal{H}$ meets all unbounded faces of $\mathscr{P}$.

Lemma 1.2.9 ([59, Proposition 1.12, Exercise 2.19].)**.** *A transverse section of a polyhedral cone always exists, and its face lattice is order-isomorphic to* $\mathcal{F}(\mathscr{P}) \setminus \{\varnothing\}$

Theorem 1.2.10. *Suppose* $\mathcal{F}(\mathscr{P})$ *is a set of all faces of a pointed polyhedral cone* $\mathscr{P}$ *ordered by inclusion. Then,* $\mathcal{F}(\mathscr{P})$ *is a lattice.*

Proof. [59, Theorem 2.7] shows $\mathcal{F}(\mathscr{P})$ is a lattice if $\mathscr{P}$ is a polytope. Now apply Lemma 1.2.9. □

From this fact, we construct a set of polyhedra with special structures, called complex.

Definition 1.2.11. A *polyhedral complex* Δ is a collection of polyhedra satisfying conditions below;

- If $F \in \Delta$ and $G \in \mathcal{F}(F)$, then $G \in \Delta$.
- For $F, G \in \Delta$, then $F \cap G$ is a common face of both F and G.

Two polyhedral complexes are *combinatorially equivalent* if they are order-ismorphic as posets. The union of all polyhedra in a polyhedral complex Δ is called *realization* of Δ.

Thus, for any polyhedron $\mathscr{P}$, $\mathcal{F}(\mathscr{P})$ is also regarded as a polyhedral complex. Indeed, some subsets of polyhedral complex also form a new polyhedral complex with some special constructions.

Definition 1.2.12. Given a polyhedral complex Δ with a vertex V, the set

$$\Delta/V := \{F \in \Delta \mid F \subset V\}$$

is called the *vertex figure* of $\mathscr{P}$ at V. Likewise, for any face $F \in \Delta$, the set $\Delta/F := \{G \in \Delta \mid G \supseteq F\}$ is called the *link* of Δ at F.

Notes that the vertex figure is a special case of the link when the given face is zero-dimensional.

Theorem 1.2.13 ([15] [59, p.54])**.** *A link is order-isomorphic to a polyhedral complex.*

Proof. Given a realized polyhedral complex $\mathscr{P}$ embedded on $\mathbb{R}^d$ with its vertex V, choose a sphere S^{d-1} centered at V such that every nonempty face of $\mathscr{P}$ containing V is not inside of the sphere. By identifying $S^{d-1} \cap \mathscr{P}$ as a subset of $\mathbb{R}^{d-1}$ via a standard chart of S^{d-1}, one can conclude that this is a realization of the polyhedral complex mapped by $F \mapsto F \cap S^{d-1}$ for all $F \in \Delta/V$. One can easily show that the realization of a link can be obtained by taking vertex figure consecutively for the vertices of the given face. $\square$

Definition 1.2.14. The *dual* (or *polar*) polyhedral complex Δ^{op} of a polyhedral complex Δ is a poset whose elements are the same as Δ, whose order is given by reverse inclusion.

The dual polyhedral complex is also a polyhedral complex [59, Corollay 2.14]. Now, we adopt a notion of combinatorial connectedness.

Definition 1.2.15. Let $\max(\Delta)$ be the set of maximal elements of Δ, and let $\bigcap \max(\Delta)$ be a set of faces which are intersections of maximal faces of Δ as a poset. Δ is *m-combinatorially connected* for $m := \min_{F \in \bigcap \max(\Delta)} \dim F$.

This is a much finer notion of connectedness than the usual topological n-connectedness (See [23] for the notion of topological connectedness). For example, a simplicial complex consisting of two triangles sharing an edge is 1-combinatorially connected but contractible (infinitely-connected) in the sense of topological n-connectedness.

Lemma 1.2.16 ([38])**.** *Every vertex figure of m-combinatorially connected polyhedral complex Δ is at least $(m-1)$-combinatorially connected. If $m \geq 1$, for any vertex $V \in \Delta$, the realization of Δ/V is contractible.*

Proof. If $m = 0$, there is nothing to prove. Let $m \geq 1$. Given a vertex V, let F be the intersection of all maximal faces of Δ containing V. Then, $\dim F \geq m$. All maximal faces of Δ/V are inherited from those maximal faces of Δ containing V, hence Δ/V is $(\dim F - 1)$-combinatorially connected, and $\dim F - 1 \geq m - 1$.

For the second statement, let $X = \{G \in \Delta \mid G \supseteq V\}$ as a set of faces of Δ. If we identify X as union of its elements, then X is homotopic to F. Thus, collapse each maximal face in X

containing V continuously to F. Moreover, we claim $X \setminus V$ is homotopic to $F \setminus V$. To see this, let S_V be a sphere centered at V and generating the vertex figure of Δ and F on its surface. The homotopy from X to F restricted on S_V gives the homotopy between vertex figures of X and F over V if $\dim F \geq 1$. $\square$

Lemma 1.2.17 ([38]). *Given a d-dimensional polyhedral complex Δ homeomorphic to a disk D^d, let V be its vertex in the interior of the realization of Δ. Then Δ/V is homeomorphic to S^{d-1}. Otherwise, if V is in the boundary of the realization of Δ, then Δ/V is homeomorphic to D^{d-1}.*

This lemma follows from investigating the intersection with S^d centered at V.

We change gears and rigorously introduce the notion of a specific operation on polytopes known as “cutting." Given a polytope $\mathscr{P}$ embedded in $\mathbb{R}^d$ and a face $F = \text{face}u(\mathscr{P})$, where u is a normal vector of F, we define the operation of cutting F from $\mathscr{P}$ as follows: we choose a supporting hyperplane $\mathcal{H}u, c$ of F, where c is the distance from the origin to $\mathcal{H}_{u,c}$, and then remove the portion of $\mathscr{P}$ lying on one side of $\mathcal{H}_{u,c}$.

Definition 1.2.18. The *separated polytope* $\mathscr{P} \setminus F$ is the polytope defined as the intersection of $\mathscr{P}$ and the outer half space of H'_F.

For example, if F is a vertex, then the separated polytope $\mathscr{P} \setminus F$ is combinatorially equivalent to the vertex figure $\mathscr{P}/F$. However, the link is generally not the same as a separated polytope when F is positive-dimensional. We remark that $\mathcal{F}(\mathscr{P} \setminus F)$ does not depend on the choice of H'_F.

1.2.2 Basic algebraic topology and Alexander duality

In this subsection, we introduce various complexes such as simplicial complexes and CW complexes, which are fundamental objects in algebraic topology. These complexes allow us to study the properties of spaces by assigning algebraic structures to them. In addition, we introduce the simplicial, polyhedral, CW, and singular homology and cohomology, which are important tools for distinguishing topological spaces up to homotopy equivalence. Homology and cohomology are algebraic invariants that capture the essential features of a space and provide a way to compare and classify spaces based on their structure. Finally, we introduce the combinatorial and topological versions of Alexander duality and its related notions. Alexander duality is a powerful tool for computing the cohomology of a space in terms of the homology of its complement, and it has many applications in geometry, topology, and combinatorics. By understanding these concepts, we can develop a deeper understanding of the topological structure of the polyhedral complexes underlying $\mathbb{Z}^d$-graded modules.

First of all, we introduce a notion of simplicial complex.

Definition 1.2.19 ([23, p.103])**.** The *d-simplex* $[v_0, \cdots, v_d]$ is the smallest convex set in $\mathbb{R}^{d+1}$ containing $d+1$ points $v_0, \cdots, v_d$ that do not lie in a proper hyperplane. A *simplicial complex* is a polyhedral complex consisting of simplices.

Simplicial complexes and polyhedral complexes are special cases of *cell complexes*, defined as below.

Definition 1.2.20 ([35, Ch.5])**.** A space X with a finite set Γ (whose elements are called *cells*) of subsets of X is (finite regular) *cell complex* if

- X is union of all elements of Γ,
- the subsets $e \in \Gamma$ are pairwise disjoint,
- for each $e \in \Gamma$, $e \neq \emptyset$, there exists a homeomorphism from a closed i-dimensional disk $D^i := \{x \in \mathbb{R}^i \mid \|x\| \leq 1\}$ to the closure $\overline{e}$ of e which maps the open disk of D^i to e.
- $\emptyset \in \Gamma$.

Let Γ^i be the subset of all elements of Γ whose closure is homeomorphic to the i-dimensional disk D^i. An element of Γ^i has *dimension* i. Conventionally, as we did in the polyhedral complex, $\Gamma^{-1} := \{\emptyset\}$. Also, the *dimension* of Γ is $\max\{i \mid \Gamma^i \neq \emptyset\}$. Then, we require the following property to be held.

- If $e \in \Gamma^i$ and $e' \in \Gamma^{i-2}$ and $e' \subset e$, then there exists exactly two *cells* $e_1, e_2 \in \Gamma^{i-1}$ such that $e \supset e_1 \supset e'$ and $e \supset e_2 \supset e'$.

Indeed, the last property was derived from the first four properties [23].

Definition 1.2.21. An *incidence function* $\epsilon : \Gamma \times \Gamma \to \{0, \pm 1\}$ is a function satisfying the following properties;

- For $(e, e') \in \Gamma^i \times \Gamma^{i-1}$, $\epsilon(e, e') \in \{0, \pm 1\}$
- $\epsilon(e, e') \neq 0 \iff e'$ is a face of e.
- $\epsilon(e, \emptyset) = 1$ for all 0-cells e.
- For $e \in \Gamma^i$ and $e' \in \Gamma^{i-2}$ with $(i-1)$ cells e_1, e_2 such that

$$e \supset e_1 \supset e' \text{ and } e \supset e_2 \supset e',$$

$$\epsilon(e, e_1)\epsilon(e_1, e') + \epsilon(e, e_2)\epsilon(e_2, e') = 0.$$

Lemma 1.2.22 ([23, Lemma IV.7.1]). *If Γ is a cell complex, then an incidence function on Γ exists.*

From this, we may define a chain complex as follow.

Definition 1.2.23. The *augmented oriented chain complex* of d-dimensional cell complex Γ is a chain complex

$$\tilde{\mathcal{C}}(\Gamma) : 0 \to C_{d-1} \to C_{d-2} \to \cdots C_0 \to C_{-1} \to 0$$

where we set

$$C_i := \bigoplus_{e \in \Gamma^i} \mathbb{Z}e \text{ and } \partial(e) = \sum_{e^i \in \Gamma^{i-1}} \epsilon(e, e')e' \text{ for } e' \in \Gamma^i.$$

Likewise, the *augmented oriented cochain complex* is the chain complex whose differential map is reverted. The *i-th cellular reduced homology* $H_{i,\mathrm{CW}}(\Gamma)$ is defined as $\ker C_i / \operatorname{im} C_{i+1}$. Likewise, the *$i$-th reduced cellular cohomology* $H^{i,\mathrm{CW}}(\Gamma)$ is defined as $\ker C_i / \operatorname{im} C_{i-1}$ from the augmented oriented cochain complex.

One can easily show that the cochain complex is the same as $\mathrm{Hom}(\tilde{\mathcal{C}}(\Gamma), \mathbb{Z})$. Also, if Γ is from simplicial complex, we specify the homology as a simplicial (co)homology using the notation $H_{i,\mathrm{Simp}}(\Gamma)$ or $H_{i,\mathrm{Simp}}(\Gamma)$.

Now, we introduce the notion of *singular homology*.

Definition 1.2.24 ([23]). A *realized d-simplex* Δ^d is a set in $\mathbb{R}^{d+1}$ such that

$$\Delta^d := \{u \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} u_i = 1\}.$$

A *singular d-simplices* in a space X is a continuous map from a realized simplex Δ^d to X. Let Γ^d be a set of all singular d-simplices in a space X. Since Δ^d has a natural cell complex structure, there is an incidence function ϵ for Δ^d. Fix it for all singular simplices. Then, the *augmented oriented (singular) chain complex* of X is a chain complex

$$\tilde{\mathcal{C}}(X) : 0 \to C_{d-1} \to C_{d-2} \to \cdots C_0 \to C_{-1} \to 0$$

where we set

$$C_i := \bigoplus_{e \in \Gamma^i} \mathbb{Z}e \text{ and } \partial(e) = \sum_{e^i \in \Gamma^{i-1}} \epsilon(e, e')e' \text{ for } e' \in \Gamma^i.$$

Likewise, the *augmented oriented (singular) cochain complex* is the chain complex whose differential map is reverted. The *i-th singular reduced homology* $H_i(X)$ is defined as $\ker C_i / \operatorname{im} C_{i+1}$. Likewise, the *$i$-th reduced singular cohomology* $H^i(\Gamma)$ is defined as $\ker C_i / \operatorname{im} C_{i-1}$ from the augmented oriented (singular) cochain complex.

It is known that the singular (co)homology and cellular (co)homology coincides.

Theorem 1.2.25 ([11, Theorem 6.2.3], [23, Lemma IV.4.2]). *Suppose that Γ be a finite regular cell complex of X. Then, $H_{i,CW}(\Gamma) = H_i(X)$ and $H^{i,CW}(\Gamma) = H^i(X)$ for all i.*

Now, we are ready to state a duality which we want to apply for Chapter 3, called Alexander duality.

Theorem 1.2.26 (Alexander duality [7, 8, 23]). *For any compact locally contractible nonempty proper topological subspace K of a $(d+1)$-dimensional sphere S^d,*

$$\tilde{H}_i(S^d \setminus K) \cong \tilde{H}^{d-i-1}(K).$$

For any polyhedral subcomplex Δ of the boundary of a polytope $\mathscr{P}$,

$$\tilde{H}_i(\Delta) \cong \tilde{H}^{d-i-3}(\Delta^*)$$

where $\Delta^* := (\mathcal{F}(\mathscr{P}) \setminus \Delta)^{op}$ is the Alexander dual *of Δ, which is a subcomplex of the dual polytope $\mathscr{P}^{op}$.*

Proof. To support the first statement, we refer the reader to [23, Theorem 3.44]. As for the second statement, the relevant reference can be found in [7, 8]. $\square$

Definition 1.2.27. Given a polytope $\mathscr{P}$ and its proper polyhedral subcomplex Δ, the *abstract dual polyhedral complex* is the set $\Delta^{*,\mathrm{op}} := (\mathcal{F}(\mathscr{P}) \setminus \Delta)^{\mathrm{op}}$ with the partial order reversed. Let $\tilde{\mathcal{C}}\,(\mathscr{P})$ be the reduced chain cell complex of $\mathscr{P}$. Indeed, this is exactly the same as the reduced cochain cell complex of $\mathscr{P}^{\mathrm{op}}$. Let $\tilde{\mathcal{C}}\,(\Delta^{*,\mathrm{op}})_{\mathrm{unmoved}}$ be a chain complex which deletes components (and componentwise maps) corresponding to faces not in $\Delta^{*,\mathrm{op}}$ from $\tilde{\mathcal{C}}\,(\mathscr{P})$. We call $\tilde{\mathcal{C}}\,(\Delta^{*,\mathrm{op}})_{\mathrm{unmoved}}$ the *unmoved Alexander dual chain complex* of $\Delta^{*,\mathrm{op}}$.

In other words, as a poset, $\Delta^{*,\mathrm{op}}$ is order isomorphic to $(\mathcal{F}(\mathscr{P}) \setminus \Delta)$. However, we intentionally use this name to emphasize the fact that Δ^* is either an abstract polyhedral complex. Also, $\tilde{\mathcal{C}}\,(\Delta^{*,\mathrm{op}})_{\mathrm{unmoved}}$ is a well-defined chain complex, since it coincides with a reduced chain complex of the Alexander dual of Δ (with reversed indices) for homology or cohomology. Moreover, for any $i \in \mathbb{Z}$,

Corollary 1.2.28. $H_i(\tilde{\mathcal{C}}\,(\Delta)) \cong H_{i+1}(\tilde{\mathcal{C}}\,(\Delta^{*,op})_{unmoved})$ *and* $H^i(\tilde{\mathcal{C}}\,(\Delta)) \cong H^{i+1}(\tilde{\mathcal{C}}\,(\Delta^{*,op})_{unmoved})$.

Proof. From Theorem 1.2.26, observe that $H^{d-i-3}(\tilde{\mathcal{C}}\,(\Delta^*)) \cong H_{i+1}(\tilde{\mathcal{C}}\,(\Delta^{*,\mathrm{op}})_{\mathrm{unmoved}})$ by comparing degrees of their chain complexes. The cohomology case is similar. $\square$

1.2.3 Hyperplane arrangements

Definition 1.2.29. The *hyperplane arrangement* $\mathcal{A} := \{\mathcal{H}_{u_1,c_1}, \cdots, \mathcal{H}_{u_m,c_m}\}$ of a polyhedron $\mathscr{P}$, or sometimes called *H-representation of $\mathscr{P}$*, is the collection of supporting hyperplanes of the facets of $\mathscr{P}$ in $\mathbb{R}^d$. A hyperplane arrangement is *linear* if all hyperplanes in the arrangement contain the origin.

We make the convention that $\bigcap_{i=1}^m \mathcal{H}_{u_i}^+ = \mathscr{P}$. Also, it is easily observed that a hyperplane arrangement is linear if and only if $c_i = 0$ for all $i \in [m]$.

Definition 1.2.30. A *region* $\mathfrak{r}$ of $\mathcal{A}$ is a connected component of $\mathbb{R}^d - \bigcup_{\mathcal{H}\in\mathcal{A}} \mathcal{H}$. $\mathfrak{r}(\mathcal{A})$ refers to the collection of all regions of $\mathcal{A}$.

Suppose $\mathcal{A}$ consists of a minimal number of hyperplanes which generate a rational polyhedral cone $\mathscr{P}$. Then $\mathcal{A}$ is linear and all regions in $\mathfrak{r}(\mathcal{A})$ are unbounded rational polyhedral cones. Thus, we may omit c_i from the index of hyperplanes such that $\mathcal{A} := \{\mathcal{H}_{u_1}, \cdots, \mathcal{H}_{u_m}\}$. Then, every region $\mathfrak{r}$ can be expressed as

$$\mathfrak{r}_S := \left(\bigcap_{i\in[m]\smallsetminus S} \mathcal{H}_i^+\right) \cap \left(\bigcap_{i\in S} \mathcal{H}_i^-\right) \smallsetminus \bigcup_{i=1}^m \mathcal{H}_i$$

for a subset $S \subseteq [m]$ where $\mathcal{H}_i^-$ is the complement of $\mathcal{H}_i^+$. In other words, a region is labeled by the collection of hyperplanes whose positive half space contains it.

Definition 1.2.31 ([4, 18])**.** The *poset of regions* $\mathfrak{r}(\mathcal{A})$ of a hyperplane arrangment$\mathcal{A}$ is a set consisting of all regions with the partial order by reverse inclusion;

$$\mathfrak{r}_{S_1} \leq \mathfrak{r}_{S_2} \text{ if } S_1 \supseteq S_2.$$

This definition coincides with that of [4, 18] regarding $\mathscr{P} = \mathfrak{r}_\emptyset$ as the base region.

Lemma 1.2.32 ([18, Lemma 1.3])**.** *There is a canonical embedding $\mathcal{F}(\mathscr{P}) \to \mathfrak{r}(\mathcal{A})$ which send a face F to the set of indices of hyperplanes containing F.*

Since we are interested in regions partitioning $\mathbb{R}^d$ along with the set of degrees of standard monomials of localizations, we modify the definition of $\mathfrak{r}_S$ as follows, to include boundaries:

Definition 1.2.33 (Modified definition of 1.2.30)**.** A *region* of a index set S $\mathfrak{r}_S$ is a closure of the (original) region defined as follow;

$$\mathfrak{r}_S := \left(\bigcap_{i \in [m] \smallsetminus S} \mathcal{H}_i^+ \right) \cap \left(\bigcap_{i \in S} \mathcal{H}_i^- \right).$$

Likewise, we induce a notion of cumulative region, which is a union of all regions which are less than the given region.

Definition 1.2.34. A *cumulative region* $\mathfrak{R}_S = \left(\bigcap_{i \in [m] \smallsetminus S} \mathcal{H}_i^+ \right)$ is the union of all regions less then $\mathfrak{r}_S$. The *poset of cumulative regions* $\mathfrak{R}(\mathcal{A})$ is a set of all cumulative regions ordered by inclusion.

By definition, $\mathfrak{R}(\mathcal{A}) \cong \mathfrak{r}(\mathcal{A})$ as posets.

Lemma 1.2.35. [51] *If $\mathcal{A}$ is a linear hyperplane arrangement, then all regions in $\mathfrak{r}(\mathcal{A})$ are unbounded.*

Example 1.2.36.

1. Let $\mathscr{P} = \mathbb{R}_{\geq 0} A$ with $A = \left[\begin{smallmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \end{smallmatrix}\right]$. Then $\mathscr{P}$ is a 2-dimensional cone with facets (rays) $\mathbb{R}_{\geq 0} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\mathbb{R}_{\geq 0} \left[\begin{smallmatrix} 1 \\ a_{n-1} \end{smallmatrix}\right]$. Hence $\mathcal{A} = \{\mathcal{H}_1 := \mathbb{R} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \mathcal{H}_2 := \mathbb{R} \left[\begin{smallmatrix} 1 \\ a_{n-1} \end{smallmatrix}\right]\}$. Since $\mathscr{P}$ is a homogenization of the 1-simplex, $\mathfrak{r}(\mathcal{A}), \mathfrak{R}(\mathcal{A})$ and $\mathcal{F}(Q)$ are all isomorphic as posets.

2. Let

$$u_1 = (0, 1, 0)^t, u_2 = (-1, 0, 1)^t, u_3 = (0, -1, 1)^t, u_4 = (1, 0, 0)^t$$

and a hyperplane arrangement $\mathcal{A} = \{\mathcal{H}_{u_1}, \mathcal{H}_{u_2}, \mathcal{H}_{u_3}, \mathcal{H}_{u_4}\}$. Then, the intersection of its positive half spaces forms a polyhedral cone $\mathscr{P}$ generated by rays

$$(v_1, v_2, v_3, v_4) = \left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right].$$

Index its faces as follow.

$$F_1 := \langle v_2, v_2 \rangle, \qquad F_2 := \langle v_2, v_3 \rangle, \qquad F_3 := \langle v_3, v_4 \rangle, \qquad F_4 := \langle v_4, v_1 \rangle$$

where $\langle u_1, \cdots, u_m \rangle$ is an abbreviation of the span of $\{u_1, \cdots, u_m\}$ by non-negative real numbers. Then, for any face F, label F by the subset of $\{1, 2, 3, 4\}$ whose corresponding facet contains F. For example, the ray generated by v_1 is indexed by $\{1, 4\}$. Then we have the desired injection from $\mathcal{F}(\mathscr{P})$ to $\mathfrak{r}(\mathcal{A})$ by sending a face F to its index. This relationship is depicted in Figure 1.1. Note that this is **not** a bijection; which is explained in Example 1.2.60 Number 2.

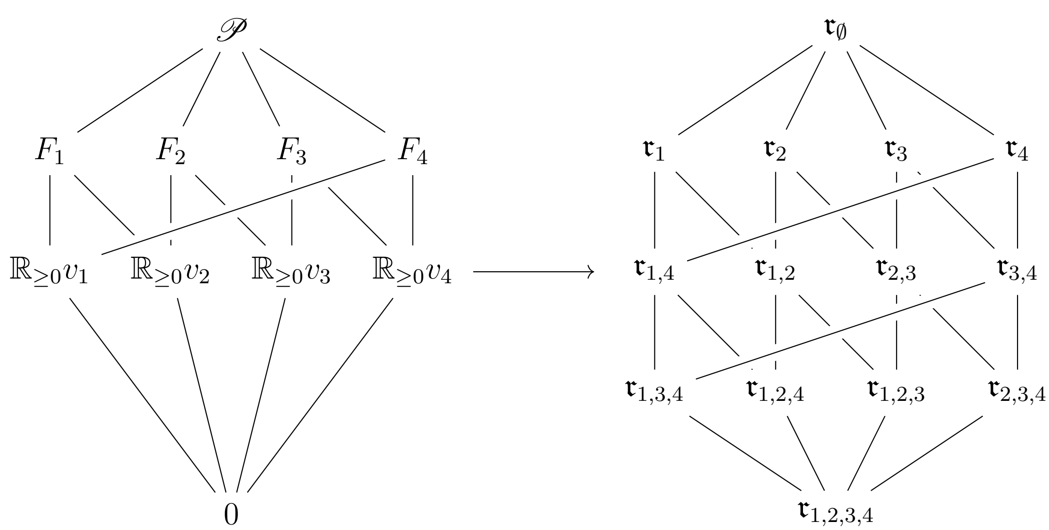


Figure 1.1: Hasse diagrams of $\mathcal{F}(\mathscr{P})$ and $\mathfrak{r}(\mathcal{A})$ in Example 1.2.36 Number 2

1.2.4 Affine semigroup rings

Throughout this article, $A = \{v_1, \ldots, v_n\} \subset \mathbb{Z}^d \smallsetminus \{0\}$ is a fixed finite set of nonzero lattice points, called a *configuration*. We may abuse notation and use A to also denote the $d \times n$ integer matrix whose columns are $v_1, \ldots, v_n$.

Definition 1.2.37. An *affine semigroup* Q generated by A is a finitely submonoid of $\mathbb{Z}^d$ defined as $Q = \mathbb{N}A$, consisting of all nonnegative linear integral combinations of the elements of A. The set $\mathbb{R}_{\geq 0}Q = \mathbb{R}_{\geq 0}A$ of nonnegative real combinations of elements of Q (or A) is a polyhedral cone, called the *underlying cone of* Q. The *dimension* of an affine semigroup Q is defined to be dimension of its underlying cone.

We assume that Q is *strongly convex*, meaning that $\mathbb{R}_{\geq 0}Q$ do not contain a line. Like a ring, it has a natural ideal structure as follow.

Definition 1.2.38. A subset T of an affine semigroup Q is called an *ideal* if $Q + T \subseteq T$. For any subset S of Q, *the ideal* $\langle S \rangle$ *generated by* S is the smallest ideal in Q that contains S. An ideal T is *prime* if for any two elements $u, v \in Q$, $u + v \in T$ implies $u \in T$ or $v \in T$.

Affine semigroups inherit some properties from its underlying cone.

Definition 1.2.39. A subset of Q is called a *face* of Q if its complement is a prime ideal; the collection of faces of Q is denoted by $\mathcal{F}(Q)$.

Lemma 1.2.40 ([42, Lemma 7.12])**.** *There is a one to one correspondence between $\mathcal{F}(\mathbb{R}_{\geq 0}Q) \smallsetminus \{\emptyset\}$ and $\mathcal{F}(Q)$, given by intersecting the faces of $\mathbb{R}_{\geq 0}Q$ with Q.*

Also notes that if K is a transverse section of $\mathbb{R}_{\geq 0}Q$, then $\mathcal{F}(K)$ is bijective to $\mathcal{F}(Q)$ from Lemma 1.2.9. This fact is used heavily on this dissertation.

Likewise,

Lemma 1.2.41. *Let Q is a pointed affine semigroup. The set of radical ideals has a 1-1 correspondence with the set of subcomplexes of $\mathcal{F}(Q)$.*

This was already known when Q is normal; see [31, Exercise 20.44], [57, p.348-349], and [41, p.117]. We introduce proof of Lemma 1.2.41 for future reference. The proof is very similar in the case of the normal affine semigroup.

Proof of Lemma 1.2.41. Given a subcomplex Δ of $\mathcal{F}(Q)$, let $I_\Delta := \{u \in Q : u \notin \mathbb{N}F$ for all $F \in \Delta\}$. We claim that I_Δ is an ideal. Indeed, if there exist $v \in I_\Delta$ and $w \in Q$ such that $v + w \in \mathbb{N}F$ for some $F \in \Delta$, then $v \in \mathbb{N}F$, a contradiction. Also, I_Δ is radical; suppose that there exist $v \in Q$ and $N \in \mathbb{N}$ such that $Nv \in I_\Delta$ but $v \notin I_\Delta$. This implies $v \in \mathbb{N}F$ for some $F \in \Delta$. Therefore, $Nv = v + \cdots + v \in \mathbb{N}F$, a contradiction.

Conversely, let T be a radical ideal. Define $\Delta := \{F \in \mathcal{F}(Q) : \mathrm{RelInt}(\mathbb{R}_{\geq 0}F) \cap T = \emptyset\}$. It suffices to show that Δ is a subcomplex of $\mathcal{F}(Q)$ and $T = I_\Delta$. Suppose G be a face of $F \in \Delta$. If $\mathrm{RelInt}(\mathbb{R}_{\geq 0}G) \cap T \neq \emptyset$, for any $u \in \mathrm{RelInt}(\mathbb{R}_{\geq 0}G) \cap T$ and $v \in \mathrm{RelInt}(\mathbb{R}_{\geq 0}F) \cap \mathbb{N}F$, $v + u \in \mathrm{RelInt}(\mathbb{R}_{\geq 0}F) \cap T$, a contradiction. Therefore $G \in \Delta$. To see $I_\Delta = T$, observes that $I_\Delta \supseteq T$ first. Conversely, for any $u \in I_\Delta$, $u \in \mathrm{RelInt}(\mathbb{R}_{\geq 0}F) \cap \mathbb{N}F$ for some $F \in \mathcal{F}(Q) \smallsetminus \Delta$. Fix $v \in \mathrm{RelInt}(\mathbb{N}F) \cap T$. Since $u, v \in \mathbb{Q}_{\geq 0}Q$, there exists N such that $Nu - v \in Q$. This shows $u \in T$. $\square$

Also, we adopt the notion of divisibility as below.

Definition 1.2.42. We emphasize that, throughout this article, divisibility refers to the ring $\Bbbk[\mathbb{N}A]$, and not to $\Bbbk[x^{\pm}]$. To be completely precise, $x^{u'} \mid x^u$ means that $u - u' \in \mathbb{N}A$. We abuse terminology, and also state that u' *divides* u in this case.

Definition 1.2.43. If F is a *facet* (a codimension one face) of Q, we define its *primitive integral support function* $\varphi_F : \mathbb{R}^d \to \mathbb{R}$ by the following properties:

1. φ_F is linear,
2. $\varphi_F(\mathbb{Z}^d) = \mathbb{Z}$,
3. $\varphi_F(v_i) \geq 0$ for $i = 1, \ldots, n$,
4. $\varphi_F(v_i) = 0$ if and only if $v_i \in H$.

Primitive integral support functions give a measure of how far a point is from a facet of Q: if $u \in \mathbb{Z}^d$, $\varphi_F(u)$ is the number of hyperplanes parallel to $\mathbb{R}F$ that pass through integer points, and lie between u and $\mathbb{R}F$, with a sign to indicate whether u is on the side of $\mathbb{R}F$ that contains $\mathbb{R}_{\geq 0}Q$.

Definition 1.2.44. The *relative interior* of a face of Q is defined to be the intersection of the relative interior of the corresponding face of $\mathbb{R}_{\geq 0}Q$ with Q.

If the context is clear, we may abuse notation and refer to a subset $F \subset A$ as a face of $\mathbb{N}A$, to indicate that $\mathbb{N}F$ is a face of $\mathbb{N}A$.

Definition 1.2.45. The set $\mathcal{H}(Q) := (\mathbb{Z}Q \cap \mathbb{R}_{\geq 0}Q) \smallsetminus Q$ is called the set of *holes* of Q. If an affine semigroup contains no holes, it is said to be *normal* or *saturated*. In case of saturated affine semigroup Q,

$$\mathbb{R}_{\geq 0}Q \cap \mathbb{Z}Q = \mathbb{N}A$$

Indeed, we are interested in not only affine semigroups but also affine semigroup rings defined as below.

Definition 1.2.46. An *affine semigroup ring* $\Bbbk[Q] = \Bbbk[x^{v_1}, \cdots, x^{v_n}]$ is a subring of the *Laurent polynomial ring* $\Bbbk[x^{\pm}] := \Bbbk[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$.

Since we assume that A is of rank d, $\Bbbk[Q]$ is a d-dimensional $\Bbbk$-algebra. Also, as a ring, $\Bbbk[Q]$ is Noetherian.

Affine semigroup rings are Noetherian, as they are quotients of polynomial rings. This can be restated as a version of Dickson's Lemma.

Lemma 1.2.47. *Let S be a nonempty subset of Q such that no two elements of S are comparable with respect to divisibility. Then S is finite.*

Proof. By contradiction, assume that S contains an infinite sequence $\{u_i\}_{i=1}^{\infty}$. Consider $I_j = \langle x^{u_1}, \ldots, x^{u_j}\rangle$ for $j \geq 1$. Since $x^u \in I_j$ if and only if $x^{u_i} | x^u$ for some $1 \leq i \leq j$, we see that $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is an infinite ascending chain, which contradicts Noetherianity of $\Bbbk[Q]$. □

Lemma 1.2.48. *There is a natural bijection between the elements of an affine semigroup Q and the monomials of the corresponding affine semigroup ring $\Bbbk[Q]$.*

Proof. For an element $u \in Q$, send it to $x^u \in \Bbbk[Q]$. □

This establishes a one to one correspondence between monomial ideals of $\Bbbk[Q]$ and ideals of Q. If T is an ideal of Q, we denote the corresponding monomial ideal of $\Bbbk[Q]$ by I; more precisely,

$$I = \langle x^v \mid v \in T \rangle.$$

Thus,

Lemma 1.2.49. *If F is a face of Q, the ideal*

$$P_F := \langle x^v \mid v \notin F \rangle$$

is a corresponding to the complement of F is a prime monomial ideal of $\Bbbk[Q]$; all prime monomial ideals of $\Bbbk[Q]$ arise in this way.

Now we are ready to see that localizations of affine semigroup by an additively closed set corresponds to those of affine semigroup rings by a multiplicatively closed monomial sets.

Definition 1.2.50. A set $S \subseteq Q$ is called *additively closed* if it contains 0 and is closed under addition. The *localization* $Q - \mathbb{N}S$ of Q by an additively closed set S is defined as $Q - \mathbb{N}S := Q + \mathbb{Z}S$.

The localization of Q by S is equal to the localization of Q by the minimal additively closed set containing S whose complement is a prime ideal

Lemma 1.2.51 ([14, Lemma 1.1])**.** *Let $S \subseteq \Bbbk[Q]$ be a set of monomials that is multiplicatively closed and let $\mathbb{N}F$ be the minimal face of Q containing $\{v \in Q \mid x^v \in S\}$. Then,*

$$S^{-1}\,\Bbbk[Q] \cong \Bbbk[Q - \mathbb{N}F].$$

Definition 1.2.52. If $T \subset Q$ is an ideal corresponding to the monomial ideal I in $\Bbbk[Q]$, and F is a face of Q, the *localization of I at the prime ideal P_F*, denoted I_F is corresponding to the ideal $T_F := T - \mathbb{N}F$ of the semigroup $Q - \mathbb{N}F$.

Moreover, localization also induces a reversed injective map (i.e., contravariant functor) from faces of underlying polyhedral cone.

Lemma 1.2.53. *The maps*

$$\begin{aligned} \mathcal{F}(Q - \mathbb{N}F) \to \{G \in \mathcal{F}(Q) \mid G \supset F\} \qquad & \textit{given by} \qquad \mathbb{N}G' \mapsto \mathbb{N}(G' \cap A) \\ \{G \in \mathcal{F}(Q) \mid G \supset F\} \to \mathcal{F}(Q - \mathbb{N}F) \qquad & \textit{given by} \qquad \mathbb{N}G \mapsto \mathbb{N}G - \mathbb{N}F \end{aligned}$$

are bijective.

Proof. It suffices to show that $\mathcal{F}(Q - \mathbb{N}F)$ and $\{G \in \mathcal{F}(Q) \mid G \supseteq F\}$ are in bijection. Fix $G \in \mathcal{F}(Q)$. Let w be an outer normal vector so that $G = \text{face}_w(Q)$. Recall that the absolute maximum of the functional $\langle w, - \rangle$ on $\mathbb{R}_{\geq 0}Q$ is zero since every face of Q contains the origin.

We claim

$$\text{face}_w(Q - \mathbb{N}F) = \begin{cases} \varnothing & \text{if } F \not\subseteq G \\ G \cup (-F) & \text{if } F \subseteq G. \end{cases}$$

If F is not a face of G, there exists a nonzero element $f \in F \smallsetminus G$ such that $\langle w, f \rangle < \langle w, g \rangle = 0$ for any $g \in G$. Since $\langle w, -mf \rangle$ diverges when $m \to \infty$, $\text{face}_w(Q - \mathbb{N}F) = \varnothing$. If F is a face of G, $G \cup (-F) \subseteq face_w(Q - \mathbb{N}F)$. Pick $v \in face_w(Q - \mathbb{N}F) \cap (Q - \mathbb{N}F)$. Then, $v = u - f$ for some $u \in Q$ and $f \in \mathbb{N}F$. Since $0 = \langle w, v \rangle = \langle w, u \rangle$, $u \in \mathbb{N}G$. Thus, $v \in G \cup (-F)$. $\square$

Indeed, by thinking Q as a grading, we can regard $\Bbbk[Q]$ as a graded ring.

Definition 1.2.54. Given a commutative monoid Q, if we can decompose a ring R into a direct sums

$$R = \bigoplus_{u \in Q} R_u \text{ satisfying } R_u R_v \subseteq R_{u+v} \text{ for all } u, v \in Q,$$

we say R is a *graded*, or Q-*graded ring.* Likewise, given a module M over Q-graded ring A, if we can decompose a module M into a direct sums

$$M = \bigoplus_{u \in Q} M_u \text{ satisfying } R_u M_v \subseteq M_{u+v} \text{ for all } u, v \in Q,$$

we say M is a *graded*, or Q-*graded module* over A.

Definition 1.2.55. A graded module $M = \oplus_{u \in Q} M_u$ is *finely graded* if $\dim_{\Bbbk} M_u \leq 1$ for all degrees $v \in \mathbb{Z}^d$.

For any face $F \in \mathcal{F}(Q)$, $\Bbbk[Q - \mathbb{N}F]$ is *finely* $\mathbb{Z}^d$-graded as follows.

Lemma 1.2.56. $\dim_{\Bbbk}(\Bbbk[Q - \mathbb{N}F])_v = 1$ *if* $v \in Q - \mathbb{N}F$*. Otherwise,* $\dim_{\Bbbk}(\Bbbk[Q - \mathbb{N}F])_v = 0$.

Proof. As $\Bbbk[Q - \mathbb{N}F] \subseteq \Bbbk[\mathbb{Z}Q]$, it suffices to show that $\dim_{\Bbbk}(\Bbbk[\mathbb{Z}Q])_v \leq 1$. If $v \notin \mathbb{Z}Q$, then $\Bbbk[\mathbb{Z}Q]_v = \{0\}$, otherwise $\Bbbk[\mathbb{Z}Q]_v = \operatorname{span}_{\Bbbk}\{x^v\}$. □

Definition 1.2.57. An affine semigroup Q is *pointed* if its corresponding cone $\mathbb{R}_{\geq 0}Q$ is a pointed polyhedron.

Also, localization affects the structure of regions as below.

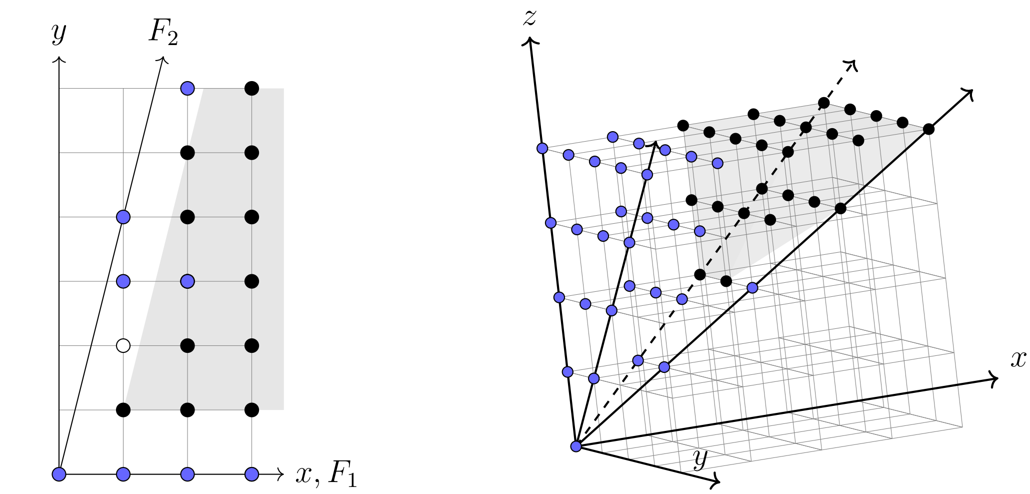
Proposition 1.2.58. *Given a face* $F \in \mathcal{F}(Q)$*, let* S *be a set of indices of hyperplanes whose half-space contains* F*. Then,* $\mathbb{R}_{\geq 0}(Q - \mathbb{N}F) = \mathfrak{R}_S$*.*

Proof. From $\mathbb{Z}F \cup Q \subset \mathfrak{R}_S$, $\mathbb{R}_{\geq 0}(Q - \mathbb{N}F) \subseteq \mathfrak{R}_S$. Conversely, for any $x \in \mathfrak{R}_S$, $\langle w_i, x\rangle \geq 0$ when $i \in S$. Pick $f \in \operatorname{RelInt}(\mathbb{N}F)$ and let $x' = x + \left(\sum_{i \notin S} a_i\right) f$ where a_i is a non-negative real number such that $\langle w_i, x + a_i f\rangle \geq 0$. Then, $x = x' + (x - x')$ with $x' \in \mathbb{R}_{\geq 0}Q$ and $(x - x') \in \operatorname{span}(F)$. Thus, $\mathbb{R}_{\geq 0}(Q - \mathbb{N}F) = \mathbb{R}_{\geq 0}(\mathbb{Z}F \cup Q) \supseteq \mathfrak{R}_S$. □

Definition 1.2.59. A *category of* Q $\mathbf{Cat}_Q$ is a poset containing all localizations of Q ordered by inclusion.

Below, we describe all posets that arise in this section using a commutative diagram.

$$\mathcal{F}(Q) \xrightarrow[\cong]{Q - \mathbb{N}(-)} \mathbf{Cat}_Q \xhookrightarrow{\mathbb{R}_{\geq 0}(-)} \mathfrak{R}(\mathcal{A}) \xrightarrow{\cong} \mathfrak{r}(\mathcal{A}) \xhookrightarrow{\phi} (2^{\mathcal{A}})^{\text{op}}$$



(a) $Q = \mathbb{N}\left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{smallmatrix}\right]$ and $I = \langle \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \rangle$ (b) $Q = \mathbb{N}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $I = \left\langle \begin{bmatrix} 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{bmatrix} \right\rangle$

Figure 1.2: Monomials and ideals in affine semigroups

Note that the embedding $\mathcal{F}(Q) \to \mathfrak{r}(\mathcal{A})$ in [18, Lemma 1.3] is split into the diagram above. Moreover, all posets are indexed by a subposet of $(2^{\mathcal{A}})^{op}$, a poset of subsets of $\mathcal{A}$ by reverse inclusion. The inclusion map ϕ returns the set of indices of positive half-spaces containing the given element.

Example 1.2.60 (Continuation of Example 1.2.36)**.**

1. (Monomial curves) Let $Q = \mathbb{N}A$ with $A = \left[\begin{smallmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_1 & \cdots & a_{n-1} \end{smallmatrix}\right]$ such that $0 < a_1 < \cdots < a_{n-1}$ are relatively prime integers. If $0, a_1, a_2, \ldots, a_{n-1}$ are consecutive integers, then $\Bbbk[Q]$ is the coordinate ring of a rational normal curve; otherwise, $\Bbbk[Q]$ is not normal.

   Figure 1.2a illustrates the example where $a_1 = 1, a_2 = 3, a_3 = 4$. Elements of the semigroup Q are represented by filled dots. Since $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ is a hole of Q (in fact, it is the only hole of Q), it is depicted as an empty circle. Let $T = \langle \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \rangle$. The elements of T are colored black, while the elements in Q but not in T are colored blue. This includes $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$, which is not in T because $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ is a hole of Q.

   Lastly, notes that $\mathcal{F}(Q), \mathbf{Cat}_Q, \mathfrak{R}(\mathcal{A}), \mathfrak{r}(\mathcal{A})$, and $2^{\mathcal{A}}$ are all isomorphic as a poset, depicted in a Table 1.1.

$\mathcal{F}(Q)$	$\mathbf{Cat}_Q$	$\mathfrak{R}(\mathcal{A})$	$\mathfrak{r}(\mathcal{A})$	$2^{\mathcal{A}}$
0	Q	$\mathscr{P}$	$\mathscr{P} = \mathcal{H}_1^{(+)} \cap \mathcal{H}_2^{(+)}$	$\{\mathcal{H}_1, \mathcal{H}_2\}$
F_1	$Q - \mathbb{N}F_1$	$\mathcal{H}_1^{(+)} = \{y \geq 0\}$	$\mathcal{H}_1^{(+)} \cap \mathcal{H}_2^{(-)} = \{y \geq 0, y > a_{n-1}x\}$	$\{\mathcal{H}_1\}$
F_2	$Q - \mathbb{N}F_2$	$\mathcal{H}_2^{(+)} = \{y \leq a_{n-1}x\}$	$\mathcal{H}_1^{(-)} \cap \mathcal{H}_2^{(+)} = \{y < 0, y \leq a_{n-1}x\}$	$\{\mathcal{H}_2\}$
Q	$\mathbb{Z}^2$	$\mathbb{R}^2$	$\mathcal{H}_1^{(-)} \cap \mathcal{H}_2^{(-)} = \{y < 0, y > a_{n-1}x\}$	$\varnothing$

Table 1.1: Description of $\mathcal{F}(Q), \mathbf{Cat}_Q, \mathfrak{R}(\mathcal{A}), \mathfrak{r}(\mathcal{A})$, and $2^{\mathcal{A}}$ of a Monomial Curve in Example 1.2.60

2. (Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$) Let $Q = \mathbb{N}A$ with $A = \left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right]$. The affine semigroup ring $\Bbbk[Q]$ is isomorphic to $\Bbbk[z, xz, yz, xyz] \cong \Bbbk[a, b, c, d]/\langle ac - bd\rangle$. The exponent vectors of monomials in the ideal $I = \langle x^2z^2, x^2yz^2, x^2y^3z^3, x^3y^3z^3\rangle \subset \Bbbk[Q]$ are depicted by black dots in Figure 1.2b. Notes that the underlying polyhedron of Q is equal to the polyhedral cone in Example 1.2.36 Number 2.

   Observe that $\mathbb{R}_{\geq 0}(Q - F_i) = \mathfrak{R}_{\{i\}}$ for any $i \in [n]$ and

$$\mathbb{R}_{\geq 0}(Q - \langle v_1\rangle) = \mathfrak{R}_{1,4}, \qquad \mathbb{R}_{\geq 0}(Q - \langle v_2\rangle) = \mathfrak{R}_{1,2},$$
$$\mathbb{R}_{\geq 0}(Q - \langle v_3\rangle) = \mathfrak{R}_{2,3}, \qquad \mathbb{R}_{\geq 0}(Q - \langle v_4\rangle) = \mathfrak{R}_{3,4}.$$

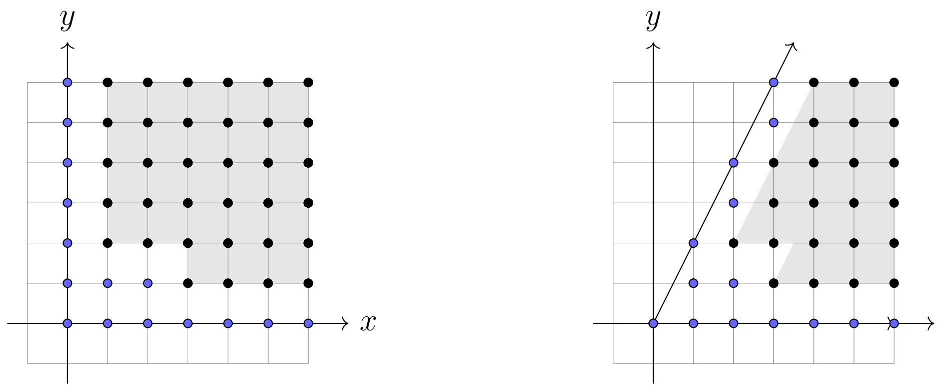
   Thus, $\mathfrak{R}(\mathcal{A}) \supsetneq \mathbb{R}_{\geq 0}(\mathbf{Cat}_Q)$, for example, because localization cannot generate affine semigroups in $\mathfrak{R}_{1,2,3}$. This illustrates the nontrivial injection $\mathfrak{r}(\mathcal{A}) \cong \mathfrak{R}(\mathcal{A}) \supsetneq \mathcal{F}(Q) \cong \mathbf{Cat}_Q$, described in Figure 1.1.

   More examples are as below.

Example 1.2.61.

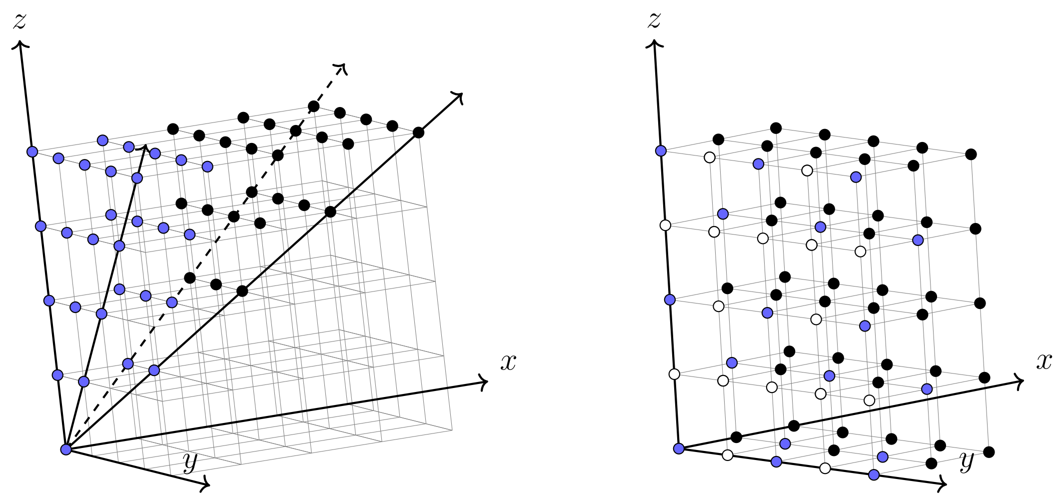
1. Let A be a $d \times d$ identity matrix. Then $\mathbb{N}A = \mathbb{N}^d$ and consequently $\Bbbk[\mathbb{N}A] \cong \Bbbk[x_1, \cdots, x_d]$. A face of $\mathbb{N}A$ is a set of all nonnegative integral combinations of a subset of (the columns of) A. In Figure 1.3a, the shaded region represents the monomial ideal $I = \langle x^3y, xy^2\rangle \subset \Bbbk[x, y]$.

2. Let $A = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right]$. Then $\mathbb{N}A$ is a saturated semigroup, and $\Bbbk[\mathbb{N}A] \cong \Bbbk[x, xy, xy^2]$ is a normal semigroup ring, a subring of $\Bbbk[x, y]$. Figure 1.3b illustrates the ideal $\langle x^2y^2, x^3y\rangle \subseteq \Bbbk[\mathbb{N}A]$.



(a) $\langle x^3y, xy^2\rangle$ (shaded region) in $\Bbbk[x,y]$ (b) $\langle x^2y^2, x^3y\rangle$ (shaded region) in $\Bbbk[x,xy,xy^2]$

Figure 1.3: Examples of ideals in two-dimensional affine semigroup rings



(a) $\langle x^2z^2, x^2yz^2, x^2yz^2\rangle$ in $\Bbbk[z,xz,yz,xyz]$ (b) $\langle x, xyz, xyz^2\rangle$ in $\Bbbk[x,xy,xz,xyz,y^2,z^2]$

Figure 1.4: Examples of ideals in three-dimensional affine semigroup rings

3. Let $A = \left[\begin{smallmatrix} 0&1&0&1\\ 0&0&1&1\\ 1&1&1&1\end{smallmatrix}\right]$. In this case, $\Bbbk[\mathbb{N}A] \cong \Bbbk[z,xz,yz,xyz]$ is a saturated affine semigroup ring. We depict the ideal $\langle x^2z^2, x^2yz^2, x^2yz^2\rangle \subset \Bbbk[\mathbb{N}A]$ in Figure 1.4a.

4. Let $A = \left[\begin{smallmatrix} 0&0&1&1&1&1\\ 2&0&0&1&0&1\\ 0&2&0&0&1&1\end{smallmatrix}\right]$. Then $\Bbbk[\mathbb{N}A] \cong \Bbbk[x,xy,xz,xyz,y^2,z^2]$ is a subring of $\Bbbk[x,y,z]$. In this case, $\mathbb{N}A$ is not saturated, and the set of holes $\mathcal{H}(\mathbb{N}A)$ is $\{(a,b,0) \mid (a,b) \in \mathbb{N}^2 \smallsetminus (2\mathbb{N}\times 2\mathbb{N})\}$. In Figure 1.4b, the see the ideal $\langle x, xyz, xyz^2\rangle \subset \Bbbk[\mathbb{N}A]$. Holes are represented by white circles.

1.2.5 Local cohomology

Local cohomology is defined as a derived functor of a special fuctor, called *torsion functor*. For the readers who are not familiar with such concepts, please consult with [56].

Definition 1.2.62. Given a commutative ring R, its module M, and its ideal I, *I-torsion submodule* $\Gamma_I(M)$ of M is a module consisting of all elements of M which are annihilated by some power of I. In other words,

$$\Gamma_I = \bigcup_{n \in \mathbb{N}} (0 :_M I^n).$$

Lemma 1.2.63 ([9, p.2])**.** *Γ_I is an endofunctor on the category of R-modules.*

Definition 1.2.64. For $i \in \mathbb{N}$, the *i-th local cohomology with respect to I* $H^i_I(-)$ is the i-th right derived functor of Γ_I. For an R-module M, the *i-th local cohomology module of M with respect to I* is $H^i_I(M)$, the application of $H^i_I(-)$ on M.

Lemma 1.2.65 ([9, Remark 1.2.3])**.** *$H^i_I(-) = H^i_{\sqrt{I}}(-)$, where $\sqrt{I}$ is the radical of I.*

When R is Noetherian, there is another way of calculating the local cohomology via so-called *Čech complex.*

Definition 1.2.66 (Čech Complex)**.** Given a Noetherian ring R, suppose $I = \langle r_1, \cdots, r_n \rangle$ for some elements $r_i \in R$. The *Čech complex* is a chain complex

$$C^\bullet : 0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n \to 0$$

such that C^k is a direct sum of localizations $R_{\prod_{i \in \sigma} f_i}$ of R by a multiplicative set generated by an element $\prod_{i \in \sigma} f_i$ where σ is a k-subset of $[n]$, and its differential is defined summandwisely by direct sum of

$$R_{\prod_{i \in \sigma} f_i} \to R_{\prod_{i \in \sigma \cup \{j\}} f_i} := (-1)^{(\text{position of } j \text{ in ascending order})-1} \text{nat}.$$

where nat is a canonical map given by localization.

Theorem 1.2.67 ([22,47])**.** *Given R-module M, $H^i(M \otimes_R C^\bullet) \cong H^i_I(M)$.*

If R is an affine semigroup ring, then there is more combinatorially-friendly way of calculating the local cohomology when the supporting ideal is graded maximal ideal [30]. Given a pointed affine semigroup Q with transverse section K, there is a canonical isomorphism $\widehat{-} : \mathcal{F}(K) \to \mathcal{F}(Q)$ given as follows: $\widehat{F}$ is the minimal face of Q such that $\mathbb{R}_{\geq 0}\widehat{F} \supseteq \mathbb{R}_{\geq 0}F$. Since Q is pointed, it has a unique zero-dimensional face, namely the origin, which corresponds to the (-1)-dimensional face $\varnothing$ of K. As a CW complex, K has an incidence function $\epsilon : \bigoplus_{i=-1}^{d-1} \mathcal{F}(K)^i \times \mathcal{F}(K)^{i+1} \to \{0, \pm 1\}$ that has a nonzero value when two faces are incident. Now we are ready to state the definition of Ishida complex.

Definition 1.2.68 ([30])**.** Let $\mathfrak{m}$ be the maximal monomial ideal of $\Bbbk[Q]$. The set of all k-dimensional faces in $\mathcal{F}(K)$ is denoted by $\mathcal{F}(K)^k$. Let $L^\bullet$ be the chain complex

$$L^\bullet : 0 \longrightarrow L^0 \xrightarrow{\partial} L^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L^d \xrightarrow{\partial} 0, \qquad L^k := \bigoplus_{F \in \mathcal{F}(K)^{k-1}} \Bbbk[Q - \mathbb{N}\widehat{F}]$$

where the differential $\partial : L^k \to L^{k+1}$ is induced by the componentwise map $\partial_{F,G}$ with $F \in \mathcal{F}(K)^{k-1}$, $G \in \mathcal{F}(K)^k$ such that

$$\partial_{F,G} : \Bbbk[Q - \mathbb{N}\widehat{F}] \to \Bbbk[Q - \mathbb{N}\widehat{G}] \text{ to be } \begin{cases} 0 & \text{if } F \not\subset G \\ \epsilon(F,G) \cdot \text{nat} & \text{if } F \subset G \end{cases}$$

with nat, the canonical injection $\Bbbk[Q - \mathbb{N}\widehat{F}] \to \Bbbk[Q - \mathbb{N}\widehat{G}]$ when $F \subseteq G$. We say that $L^\bullet \otimes_{\Bbbk[Q]} M$ is the *Ishida complex of a $\Bbbk[Q]$-module M supported at the maximal monomial ideal.*

The cohomology of the Ishida complex of M supported at the maximal monomial ideal is isomorphic to the local cohomology of M supported at the maximal monomial ideal.

Theorem 1.2.69 ([30, Theorem 6.2.5])**.** *For any $\Bbbk[Q]$-module M, and all $k \geq 0$,*

$$H^k_{\mathfrak{m}}(M) \cong H^k(L^\bullet \underset{\Bbbk[Q]}{\otimes} M).$$

Example 1.2.70 (Continuation of Example 1.2.60)**.**

1. Given $Q/T = \mathbb{N}\left[\begin{smallmatrix}1&1&1&1\\0&1&3&4\end{smallmatrix}\right]/\langle[\begin{smallmatrix}1\\1\end{smallmatrix}]\rangle$, let $S = \Bbbk[Q]/I$, where I is a monomial ideal corresponding to T. The transverse section of $\mathbb{R}_{\geq 0}Q$ is a line segment with vertices $F_1 = [\begin{smallmatrix}1\\0\end{smallmatrix}]$ and $F_2 = [\begin{smallmatrix}1\\4\end{smallmatrix}]$ respectively. Thus, the corresponding Ishida complex of S with the maximal ideal support is

$$L^{\bullet} : 0 \to S \to S_x \oplus S_{xy^4} \to 0 \to 0$$

2. Given $Q/T = \mathbb{N}\left[\begin{smallmatrix}0&1&1&0\\0&0&1&1\\1&1&1&1\end{smallmatrix}\right]/\left\langle\left[\begin{smallmatrix}2&2&2&3\\0&1&3&3\\2&2&3&3\end{smallmatrix}\right]\right\rangle$, let $S = \Bbbk[Q]/I$, where I is a monomial ideal corresponding to T. $\mathbb{R}_{\geq 0}Q$'s transverse section is rectangular. Due to the fact that all other localizations of S are zero except for the localizations by monomial prime ideals corresponding to $\widehat{u_1} := \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right], \widehat{u_4} := \left[\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right]$, and $F_4 := \left[\begin{smallmatrix}0&0\\0&1\\1&1\end{smallmatrix}\right]$, the Ishida complex of S with the maximal ideal support is as follows;

$$L^{\bullet} : 0 \to S \to S_z \oplus S_{yz} \to S_{z,yz} \to 0$$

1.2.6 Cohen–Macaulayness

One of the primary concerns of using local cohomology is to determine whether a given module or ring is *Cohen–Macaulay* or not. To define what a Cohen-Macaulay ring is, we need to start from the notion of regular sequences. Intuitively, a regular sequence is a generalization of the notion of linear independence from vector spaces to modules over a ring. A Cohen-Macaulay ring is a local ring in which all maximal sequences of elements are regular, and it is a fundamental concept in commutative algebra with applications in algebraic geometry and algebraic topology. Local cohomology provides a powerful tool for studying the Cohen-Macaulay property of a module or ring and plays an essential role in many areas of mathematics.

Definition 1.2.71. Given a commutative ring R, an *R-regular sequence* $(r_1, r_2, \cdots, r_m)$ is a sequence of elements of R such that r_i is not a zero-divisor on $R/(r_1, r_2, \cdots, r_{i-1}) \neq 0$ for $i = 1, 2, \cdots, m$. The *depth*, or *depth of R with respect to an ideal I* is the supremum of lengths of all R-regular seq. from the ideal T. The *dimension* of R is the supremum of lengths of all chain of prime ideals. R is *Cohen–Macaulay* if for any maximal ideal $\mathfrak{m}$ of R, the depth of $R_{\mathfrak{m}}$ with respect to the ideal $\mathfrak{m}R_{\mathfrak{m}}$ is equal to $\dim R$.

Especially, we are interested in when the ring is graded.

Definition 1.2.72 ([11, Definition 1.5.13])**.** Given a graded commutative ring R, a graded ideal $\mathfrak{m}$ of R is $*$-*maximal* if every graded ideal that properly contains $\mathfrak{m}$ equals R. The ring R is called $*$-*local* if it has a unique $*$-maximal ideal $\mathfrak{m}$.

Example 1.2.73. Every pointed affine semigroup rings are $*$-local.

Lemma 1.2.74 ([11, Exercise 2.1.27 (c)])**.** *Given a $*$-local ring R, R is Cohen–Macaulay if and only if depth $(\mathfrak{m}^* R_{\mathfrak{m}^*}, R_{\mathfrak{m}^*}) = \dim R$.*

Thus, Cohen–Macaulayness can be judged by two informations, the depth and dimension of the module. Indeed, local cohomology is a mathematical objects only let us know what is the depth and dimension of the given module.

Theorem 1.2.75 ([11, Theorem 3.5.7])**.** *Given a Noetherian local ring R with its maximal ideal $\mathfrak{m}$, let M be a finite R-module of depth $(\mathfrak{m}, M) = t$ and dimension d. Then,*

- $H^i_{\mathfrak{m}}(M) = 0$ *for $i < t$ and $i > d$.*
- $H^t_{\mathfrak{m}}(M) \neq 0$ *and* $H^d_{\mathfrak{m}}(M) \neq 0$.

Corollary 1.2.76 ([11, Remark 3.6.18, Theorem 3.5.6])**.** *If R is $*$-local, then let $H^i_{\mathfrak{m}}(M)$ is the injective limit of $\operatorname{Ext}^i_R(R/\mathfrak{m}^k, M)$ on the category of graded modules. Then, $H^i_{\mathfrak{m}}(M) \cong H^i_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$.*

Hence, in case of modules over $*$-local ring, the Čech complex and Ishida complex gives us the desired local cohomology. One important corollary of this result is

Corollary 1.2.77. *A module M over a $*$-local ring is Cohen–Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all i except $i = \dim M$.*

Proof. This is just came from the fact that depth is equal to the dimension when M is Cohen–Macaulay. $\square$

Example 1.2.78. $R := \mathbb{K}[x, y]$ is Cohen–Macaulay since (x, y) forms a R-sequence which implies that depth is at least 2. Since depth is always less than the dimension of R, which is 2, we can assure that depth is equal to the dimension. If one calculate the local cohomology of R, then it has only nonzero module on index 2.

However, $R := \mathbb{K}[x^2, x^3, xy]/(x^4y, x^4y^2)$ is not Cohen–Macaulay since it has no nonzero divisors, which implies that the depth is 0, while $(x^2, x^3) \subseteq (x^2, x^3, xy)$ implies that $\dim R = 1$.

1.2.7 Lattice binomial ideals and cellular binomial ideals

In algebraic combinatorics and algebraic geometry, binomial ideals are a class of ideals generated by binomials, which are polynomial equations with two terms. They have many applications in various areas of mathematics, such as algebraic statistics, coding theory, and algebraic cryptography. In this context, we introduce families of binomial ideals, which are sets of binomial ideals that share certain properties or characteristics.

Definition 1.2.79. Given a d-dimensional polynomial ring $\Bbbk[x_1, \cdots, x_d]$ over a field $\Bbbk$, a *binomial* is a polynomial having at most two terms. A *binomial ideal* is an ideal generated by binomials.

We are interested in three types of binomial ideals, *lattice ideals*, *toric ideals*, and *cellular binomial ideals*.

Definition 1.2.80. Let L_ρ be a subgroup of $\mathbb{Z}^d$. A *partial character* $\rho : L_\rho \to \Bbbk^*$ on $\mathbb{Z}^d$ is a group homomorphism, where $\Bbbk^*$ is the multiplicative group of $\Bbbk$. The *lattice ideal* $I(\rho)$ corresponding to ρ is the ideal in $\Bbbk[x]$ defined as

$$I(\rho) := \langle x^u - \rho(u - v)x^v \mid u - v \in L_\rho \rangle \subset \Bbbk[x_1, \ldots, x_n]. \tag{1.1}$$

We remark that in [19], these lattice ideals are denoted by $I(\rho)_+$, while $I(\rho)$ is used for lattice ideals in Laurent polynomial rings. Since we do not need the more general context, we use (1.1) for economy in the notation.

Definition 1.2.81. The *saturation* $(L_\rho)_{\text{sat}}$ of a lattice L_ρ is $(L_\rho)_{\text{sat}} := (\mathbb{Q} \otimes_{\mathbb{Z}} L_\rho) \cap \mathbb{Z}^d$. A lattice is *saturated* if $L_\rho = (L_\rho)_{\text{sat}}$.

One important property of a lattice ideal is that it is prime if and only if it is saturated, when $\Bbbk$ is algebraically closed.

Theorem 1.2.82 ([19, Theorem 2.1.c.] [19, Corollary 2.5])**.** *If $\Bbbk$ is algebraically closed, then $I(\rho)$ is prime if and only if L_ρ is saturated. Furthermore, the associated primes of a lattice ideal are lattice ideals corresponding to the saturation of the underlying lattice.*

Indeed, there is a notion of generalized lattice ideal, which is called a *cellular binomial ideal*.

Definition 1.2.83. An ideal $I \subset \Bbbk[x]$ is *cellular* if all variables are either nonzero divisors modulo I or nilpotent modulo I. The nonzero divisor variables are known as the *cellular variables* of I. Let $\zeta \subset [n]$ be the set of all indices of cellular variables of a cellular binomial ideal I, then I is ζ*-cellular*.

Lattice ideal are special cases of cellular binomial ideals, where all variables are cellular. Also,

Theorem 1.2.84 ([20, Theorem 2.6])**.** *$I \cap \Bbbk[\mathbb{N}^\zeta]$ is a lattice binomial ideal.*

The following result can be found in [19, Section 6] and also in [20, Corollary 3.5].

Theorem 1.2.85. *Let I be a ζ-cellular binomial ideal in $\Bbbk[x]$. The associated primes of I are the ideals $\Bbbk[x] \cdot P + \langle x_i \mid i \in \zeta^c \rangle$, where $P \subset \Bbbk\left[\mathbb{N}^\zeta\right]$ runs over the associated primes of lattice ideals of the form $(I : m) \cap \Bbbk\left[\mathbb{N}^\zeta\right]$, for monomials $m \in \Bbbk\left[\mathbb{N}^{\zeta^c}\right]$.*

Lastly, the most specific example of lattice ideals are some prime lattice ideals called *toric*.

Definition 1.2.86. A *toric ideal* is the prime lattice ideal $I(\chi)$ corresponding to the trivial character $\chi : L_\chi \to \{1\} \subset \Bbbk^*$.

Toric ideals are important since all affine semigroup rings can be constructed via taking quotient of polynomial rings by toric ideals.

Theorem 1.2.87 ([42, Theorem 7.3])**.** *An* affine semigroup Q *is the quotient of* $\mathbb{N}^d$ *by the equivalence relation* $\sim_{L_\chi}$ *given by* $u \sim_{L_\chi} v \iff u - v \in L_\chi$. *A quotient of polynomial ring by corresponding prime lattice ideal* $\Bbbk[x]/I(\chi)$ *is isomorphic to an affine semigroup ring* $\Bbbk[Q]$.

2. DEGREE SPACE

2.1 Multidegrees and localization for graded $\Bbbk[\mathbb{N}A]$-modules

Let $Q = \mathbb{N}A$ be an (not-necessarily pointed) affine semigroup defined in Definition 1.2.37. Lemma 1.2.51 shows that the faces of Q govern the localizations of any $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module M by monomials. In this section, we examine the effect of localization on the supporting multidegrees of M, defined as follows.

Definition 2.1.1 ([39, Definition 3.1 and 3.2], [40, Definition 3.1])**.** Let M be a $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module.

1. The *degree set* of M is defined to be

$$\deg(M) := \{u \in \mathbb{Z}^d \mid M_u \neq 0\}.$$

2. Especially, let $\mathcal{I}$ be a monomial ideal in $\Bbbk[Q]$. The *standard monomials* of $\mathcal{I}$ are the monomials in $\Bbbk[Q]$ that do not belong to $\mathcal{I}$. We denote

$$\operatorname{std}(\mathcal{I}) = \{a \in Q \mid t^a \notin \mathcal{I}\}. \tag{2.1}$$

   In other words, $\operatorname{std}(\mathcal{I}) = \deg\left(\Bbbk[Q]/\mathcal{I}\right)$

3. A *proper pair* of M is a pair (u, F) where $u \in Q$ and $F \in \mathcal{F}(Q)$ such that $u + \mathbb{N}F \subseteq \deg(M)$.

4. If (u, F) and (v, G) are proper pairs, we say $(u, F) < (v, G)$ if $u + \mathbb{N}F \subseteq v + \mathbb{N}G$. A proper pair (u, F) of M is called a *degree pair* of M if it is maximal among proper pairs in this partial order.

5. We say that (u, F) *divides* (v, G) if there is $w \in \mathbb{N}A$ such that $u + w + \mathbb{N}F \subset v + \mathbb{N}G$. (see Notation 1.2.42) to the pairs of Q.)

This is the natural extension of the original definition of standard pairs from [52], although the partial order is reversed. Notes that the definition of standard pair, which was first introduced by [52]. Also we note that overlapping is a special case of divisibility, which means that divisibility is not an antisymmetric relation, and therefore not a partial order on pairs. This difficulty is resolved if we extend the definition of divisibility to overlap classes of pairs.

Lemma 2.1.2. *Suppose (u, F) divides (v, G). If (u', F) overlaps (u, F) and (v', G) overlaps (v, G), then (u', F) divides (v', G). We conclude that divisibility is a well-defined relation on the overlap classes of pairs of Q. Moreover, divisibility is a partial order on such overlap classes.*

Proof. Since (u, F) and (u', F) overlap, we may choose $w_1 \in \mathbb{N}F$ such that $u' + w_1 \in u + \mathbb{N}F$, which implies that $u' + w_1 + \mathbb{N}F \subseteq u + \mathbb{N}F$. As (u, F) divides (v, G), there is $w_2 \in \mathbb{N}A$ such that $u + w_2 + \mathbb{N}F \subseteq v + \mathbb{N}G$. But then $u' + w_1 + w_2 + \mathbb{N}F \subset v + \mathbb{N}G$. Finally, select $w_3 \in \mathbb{N}G$ such that $v + w_3 \in v' + \mathbb{N}G$. Then $u' + w_1 + w_2 + w_3 + \mathbb{N}F \subseteq v' + \mathbb{N}G$. It follows that divisibility is well defined on overlap classes of pairs of Q. Showing that divisibility is a partial order, in this case, is similarly straightforward. $\square$

Lemma 2.1.3 ([40, Lemma 3.2])**.** *Any finitely generated $\mathbb{Z}^d$-graded $\mathbb{k}[Q]$-module M has finitely many degree pairs. Therefore, it has finitely many overlap classes.*

Proof. Let $0 = M_0 \subset M_1 \subset \cdots M_l = M$ be a chain of submodules of M such that $M_{i-1}/M_i \cong \mathbb{k}[\mathbb{N}A]/P_i$ where P_i is a graded prime ideal of $\mathbb{k}[\mathbb{N}A]$.

Thus, $\deg(M_i/M_{i-1}) = u_i + \mathbb{N}F_i$ for some $u_i \in \mathbb{Z}^d$ and a face F_i corresponding to P_i. To see that

$$\deg(M) = \deg(M_0 \oplus M_1/M_0 \oplus \cdots \oplus M_l/M_{l-1}) = \bigcup_{i=1}^{l} u_i + \mathbb{N}F_i,$$

note that for any homogeneous element $m \in M$, there exists i such that $\overline{m} \in M_i/M_{i-1}$ is nonzero. Hence, $\deg(m) \in u_i + \mathbb{N}F_i$. Conversely, we may lift any graded element in the direct sum to M. This says we have a finite pair cover, from here to finitely many standard pairs as demonstrated by [39, Theorem 3.16].

In case of the overlap classes, the statement is clear since the cardinality of the set of all overlap classes are bounded by that of the set of all degree pairs. $\square$

Indeed, to make a relation between degree pairs on localizations well-defined, we adopt a notion of *overlap class*.

Definition 2.1.4. Two degree pairs (u, F) and (v, F) with the same face F *overlap* if the intersection $(u + \mathbb{N}F) \cap (v + \mathbb{N}F)$ is nonempty.

Lemma 2.1.5. *Overlapping is an equivalence relation.*

Proof. Reflexivity and symmetry hold via those property of the intersection. To see transitivity, let $(u, F) \sim (v, F)$ and $(v, F) \sim (w, F)$. By the definition, there exists $v_1 + f_1 \in (u+\mathbb{N}F) \cap (v+\mathbb{N}F)$ and $v_2 + f_2 \in (v+\mathbb{N}F) \cap (w+\mathbb{N}F)$ for some $f_1, f_2 \in \mathbb{N}F$. Since both vectors are in $(v+\mathbb{N}F)$, we may rewrite it as $v_1 + f_1 = v + f_1'$ and $v_2 + f_2 = v + f_2'$ for some $f_1', f_2' \in \mathbb{N}F$. Now, one can see that $v + f_1' + f_2'$ is in both $(u + \mathbb{N}F)$ and $(w + \mathbb{N}F)$ simultaneously. This shows $(u + \mathbb{N}F) \cap (w + \mathbb{N}F)$ is nonempty. $\square$

Definition 2.1.6. The *overlap class* $[u, F]$ is the equivalence class containing the degree pair (u, F). We define $\operatorname{deg.p}(M)$ (resp. $\overline{\operatorname{deg.p}}(M)$) as the set of all (resp. overlap classes of) degree pairs of M.

Example 2.1.7 (Standard pairs and void pairs)**.**

1. Let I be a monomial ideal of $\Bbbk[\mathbb{N}A]$. The degree pairs of $M = \Bbbk[\mathbb{N}A]/I$ are the *standard pairs* of I introduced in [52] and generalized in [39, Definition 3.1 and 3.2].

2. Given an affine semigroup $Q := \mathbb{N}A$, the *saturation of* Q is $Q_{\text{sat}} = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}Q$. It is known that the affine semigroup ring corresponding to the saturation of of Q is the normalization of $\Bbbk[Q]$. The set of *holes* of Q is defined to be the difference $Q_{\text{sat}} \smallsetminus Q$.

   The set of holes of Q is also the degree set $\deg(\Bbbk[Q_{\text{sat}}]/\Bbbk[Q])$. As the $\Bbbk[Q]$-module $M = \Bbbk[Q_{\text{sat}}]/\Bbbk[Q]$ is finitely generated [48][§3. Proposition16] by Noether's normalization lemma, applying Lemma 2.1.3 provides an alternative algebraic proof of the well-known combinatorial result [25]. Namely, the set of holes of a semigroup Q is a finite union of translates of faces of Q. Later on, we refer to degree pairs of M as *void pairs*.

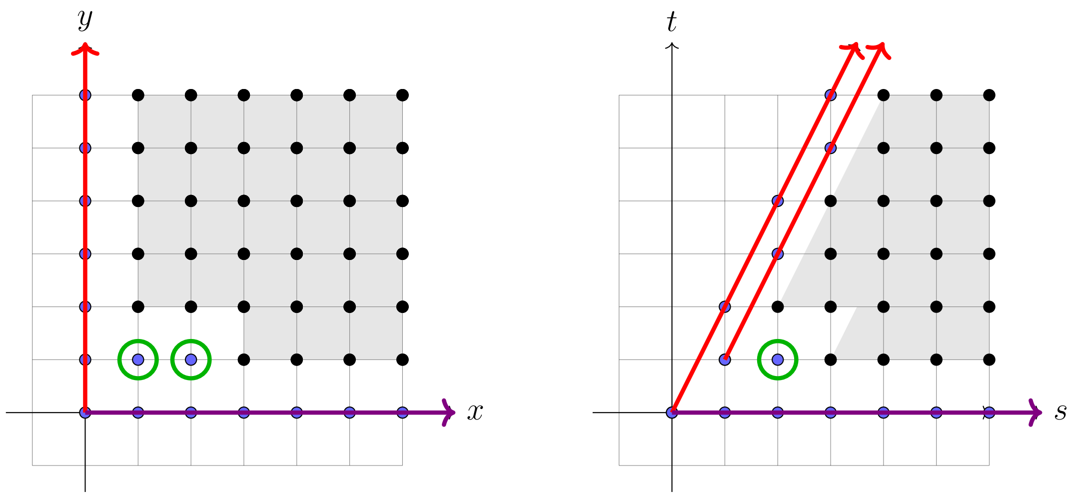
Example 2.1.8 (Continuation of Example 1.2.61)**.**

1. Consider $\langle x^3y, xy^2\rangle$ in $\Bbbk[x,y]$. Denote $F = \{(1,0)\}$, $G = \{(0,1)\}$, and $O = \varnothing$. These subsets of the (columns of) A respectively span the nonnegative x-axis, the nonnegative y-axis, and the origin, which are the proper faces of $\mathbb{R}_{\geq 0}A$. Our ideal has four degree pairs, $((0,0),F)$, $((0,0),G)$, $((1,1),O)$, and $((2,1),O)$, depicted in Figure 2.1a using thick lines. In this case, $((1,1),O)$ divides $((2,1),O)$. Thus there are three maximal degree pairs with respect to divisibility. There are no overlapping degree pairs.

2. Now, look $\langle x^2y^2, x^3y\rangle$ in $\Bbbk[x, xy, xy^2]$. This ideal also has four degree pairs: $((0,0),G)$, $((1,1),G)$, $((0,0),F)$, and $((2,1),O)$ depicted in Figure 2.1b. Here $F = \{(1,0)\}$, $G = \{(1,2)\}$, and $O = \varnothing$ correspond to the proper faces of the cone $\mathbb{R}_{\geq 0}A$. The degree pair $((0,0),G)$ divides $((1,1),G)$, and we again have three maximal degree pairs with respect to divisibility. There are no overlapping degree pairs.

3. This example illustrates that overlapping degree pairs can occur even if the semigroup ring is normal. Consider $\langle x^2z^2, x^2yz^2, x^2yz^2\rangle$ in $\Bbbk[z, xz, yz, xyz]$. Let $F = \{(0,0,1),(0,1,1)\}$, which gives the face of $\mathbb{R}_{\geq 0}A$ whose linear span is the yz-plane. In this case, we have three degree pairs $((0,0,0),F)$ (a blue region in Figure 2.2a), $((1,0,1),F)$ (a yellow region in Figure 2.2a), and $((1,1,1),F)$ (a red region in Figure 2.2a). The degree pairs $((1,0,1),F)$ and $((1,1,1),F)$ overlap. In this case, there are two overlap classes of degree pairs. As the pair $((0,0,0),F)$ divides (the overlap class of) $((1,0,1),F)$, we have only one overlap class which is maximal with respect to divisibility.

4. We now work with $\langle x, xyz, xyz^2\rangle$ in $\Bbbk[x, xy, xz, xyz, y^2, z^2]$. Note again that this semigroup ring is not normal. Let $F = \{(0,0,2),(0,2,0)\}$, be the face of $\mathbb{R}_{\geq 0}A$ whose linear span is the



(a) degree pairs of $\mathcal{I} = \langle x^3y, xy^2 \rangle$ in $\Bbbk[x, y]$ (b) degree pairs of $\langle x^2y^2, x^3y \rangle$ in $\Bbbk[x, xy, xy^2]$

Figure 2.1: Degree pairs in two-dimensional affine semigroup rings

yz-plane, and let $G = \{(0, 2, 0)\}$ the face whose linear span is the y-axis. In this case, our monomial ideal has three degree pairs: $((0, 0, 0), F)$ (yellow points in Figure 2.2b), $((1, 0, 1), F)$ (red points in Figure 2.2b), and $((1, 1, 0), G)$ (blue points in Figure 2.2b). We note that $((1, 1, 0), G)$ cannot divide $((1, 0, 1), F)$. It follows there are two degree pairs that are maximal with respect to divisibility. A feature of this example is that the Zariski closure of the set $(1, 0, 1) + \mathbb{N}F$ contains $(1, 1, 0) + \mathbb{N}G$, a situation that does not occur for degree pairs of monomial ideals in polynomial rings.

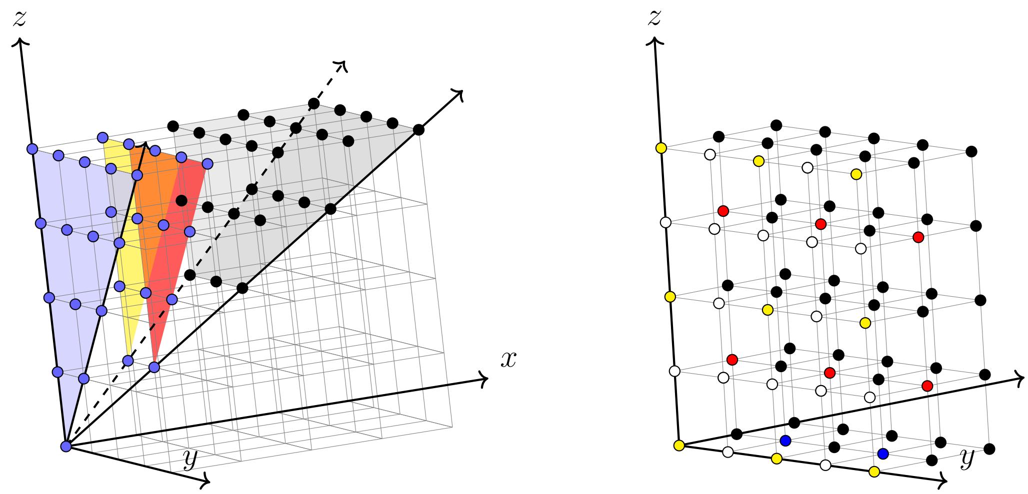
Example 2.1.9 (Continuation of Example 1.2.70)**.**

1. Degree pairs of $M := \Bbbk[x, xy, xy^3, xy^4]/\langle xy \rangle$ are

$$\text{(green) } \left(\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right], \varnothing\right), \text{(blue) } \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_1\right), \text{(red) } \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_2\right), \left(\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], F_2\right), \left(\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right], F_2\right).$$

   In Figure 2.3a, these are indicated by a green (dotted) circle, a blue (dashed) line, and red (straight) lines. Each of the degree pairs forms an overlap class.

2. Let $A := \left[\begin{smallmatrix}0&1&1&0\\0&0&1&1\\1&1&1&1\end{smallmatrix}\right]$. Degree pairs of $M := \Bbbk\left[\mathbb{N}A\right]/I$ where $I = \langle x^2z^2, x^2yz^2, x^2y^3z^3, x^3y^3z^3 \rangle$



(a) degree pairs of $\langle x^2z^2, x^2yz^2, x^2yz^2\rangle$ (b) degree pairs of $\langle x, xyz, xyz^2\rangle$

Figure 2.2: degree pairs in three-dimensional affine semigroup rings

is a monomial ideal of $\Bbbk[z, xz, xyz, yz] \cong \Bbbk[\mathbb{N}A]$ are

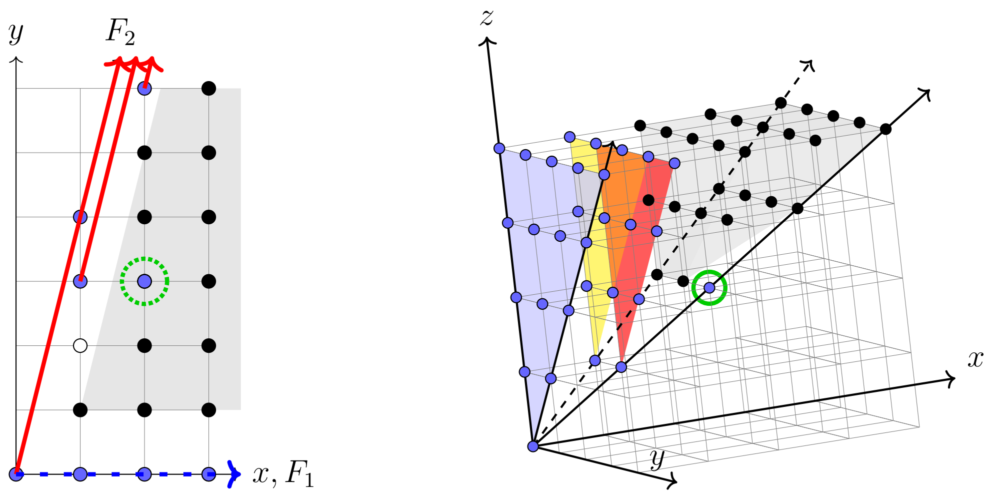
$$\text{(green)} \left(\begin{bmatrix}2\\2\\2\end{bmatrix}, \varnothing\right), \text{(blue)} \left(\begin{bmatrix}0\\0\\0\end{bmatrix}, F_4\right), \text{(yellow)} \left(\begin{bmatrix}1\\0\\1\end{bmatrix}, F_4\right), \text{(red)} \left(\begin{bmatrix}1\\1\\1\end{bmatrix}, F_4\right).$$

In Figure 2.3b, these are indicated by a green circle, a blue triangle (in zy-plane), a yellow triangle, and a red triangle (in $x = 1$ plane), respectively. As illustrated in Figure 2.3b, the yellow and red triangles represented by $\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, F_4\right)$ and $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, F_4\right)$ overlap. Hence, the overlap classes of I are

$$\left\{\left(\begin{bmatrix}2\\2\\2\end{bmatrix}, \varnothing\right)\right\}, \left\{\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, F_4\right)\right\}, \left\{\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, F_4\right), \left(\begin{bmatrix}1\\1\\1\end{bmatrix}, F_4\right)\right\}.$$

Notably, the union $\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + \mathbb{N}F_4\right) \cup \left(\begin{bmatrix}1\\1\\1\end{bmatrix} + \mathbb{N}F_4\right)$ is a subset of translates of faces represented by standard pairs of I_{F_4}, the localization of I by a face F_4.

We assert that localization by a monomial prime ideal P_F (defined in Lemma 1.2.49) generates an injective map between sets of overlap classes of degree pairs, $\overline{\deg.\,\mathrm{p}}(M)$ and $\overline{\deg.\,\mathrm{p}}(M_{P_F})$. For economy of notation,



(a) $Q = \mathbb{N}\left[\begin{smallmatrix}1&1&1&1\\0&1&3&4\end{smallmatrix}\right]$ and $\deg(I) = \langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \rangle$ (b) $Q = \mathbb{N}\begin{bmatrix}0&1&0&1\\0&0&1&1\\1&1&1&1\end{bmatrix}$ and $\deg(I) = \left\langle \begin{bmatrix}2&2&2&3\\0&1&3&3\\2&2&3&3\end{bmatrix} \right\rangle$

Figure 2.3: Degree pairs of $\Bbbk[Q]/I$

Definition 2.1.10. Let M_F be the *localization of* M *by a monomial prime ideal* P_F *corresponding to a face* $F \in \mathcal{F}(Q)$.

To begin, we see that the face lattice $\mathcal{F}(Q - \mathbb{N}F)$ of the affine semigroup $Q - \mathbb{N}F$ arising from localization by the face F can be identified as a subset of the face lattice $\mathcal{F}(Q)$ as follows.

As a consequence of Lemma 1.2.53, we can express any face of $Q - \mathbb{N}F$ as $\mathbb{N}G - \mathbb{N}F$ for some face $G \in \mathcal{F}(Q)$ such that $G \supset F$. Likewise, $(u, G \cup (-F))$ denotes a degree pair of a $\Bbbk[Q - \mathbb{N}F]$-module M_F.

Our next step is to show that each degree pair of a localization of M can be lifted to a degree pair of M.

Lemma 2.1.11. *Suppose $G \supseteq F \in \mathcal{F}(Q)$. Given a degree pair $(u, G \cup (-F))$ of M_F, there exists $u' \in \deg(M)$ such that $(u', G \cup (-F)) = (u, G \cup (-F))$ and (u', G) is a degree pair of M.*

By abuse of notation, let $x^{-\infty} = 0 \in \Bbbk[Q]$.

Proof. Assume that $\{m_1, m_2, \cdots, m_l\}$ is a minimal generating set of M with $\deg(m_i) = u_i$. Select an appropriate $f \in \mathbb{N}F$ so that $u + f \in Q$. Let $w_i = u + f - u_i$ if $w_i \in \deg(Q)$ and

$x^{w_i} m_i \neq 0$ or $w_i = -\infty$ otherwise. Set $m := \left(\sum_{i=1}^{l} x^{w_i} m_i\right) \in M$. Then, $m/x^f \in M_F$ is a nonzero homogeneous element of degree u; otherwise no element of M_F with degree u can be generated. Hence, m is a nonzero homogeneous element of degree $u + f$. Also, $(\deg(m), G)$ is a proper pair of M, otherwise, if $u + f + g \notin \deg(M)$, then no element of M_F with degree $u + f + g$ exists. Thus, a degree pair (u', G') exists that contains $(\deg(m), G)$ with $G' \supseteq G$. We may assume that $m' = \sum_{i=1}^{l} x^{w'_i} m_i$ is of order u' with $w_i = u' - u_i \in Q$ or $w_i = -\infty$. Since $\deg(m) = u' + g'$ for some $g' \in \mathbb{N}G'$ and $w'_i \neq -\infty$ if $w_i \neq -\infty$, $x^g m' = m$.

Furthermore, we claim $G' = G$. Suppose not, then we can have $w \in \mathbb{N}G' \smallsetminus \mathbb{N}G$ such that $u + w \notin \deg(M_F)$ by the maximality of $(u, G \cup (-F))$. Thus, $x^w \cdot m/x^f = 0$, which implies $x^{w+g'} \cdot m' = 0$, contradicting the fact that $w + g' + u' \in \deg(M)$. Hence, $g' \in \mathbb{N}G$.

Finally, we assert $(u', G \cup (-F)) = (u, G \cup (-F))$. Indeed $u = u' + g' - f$ indicates that $u \in u' + \mathbb{N}(G \cup (-F))$, implying $(u', G \cup (-F)) > (u, G \cup (-F))$. Also, $(u', G \cup (-F))$ is a proper pair; otherwise, we would not have an element whose degree is in $u + \mathbb{N}(G \cup (-F))$, contradiction. These two degree pairs are same due to the maximality of $(u, G \cup (-F))$. $\square$

The choice of u' is not unique; see Example 2.1.16(2). Fortunately, their overlap class is uniquely determined.

Lemma 2.1.12. *Suppose $G \supseteq F \in \mathcal{F}(Q)$. Given two overlapping degree pairs $(u, G \cup (-F))$ and $(v, G \cup (-F))$ of M_F, let u' and v' be degrees of $\deg(M)$ chosen by Lemma 2.1.11. Then, (u', G) and (v', G) overlap.*

Proof. From $(u, G \cup (-F)) = (u', G \cup (-F))$ and $(v, G \cup (-F)) = (v', G \cup (-F))$, there exists $g_u, g_v \in \mathbb{N}G$, $f_u, f_v \in \mathbb{N}F$ such that

$$u' + g_u - f_u = u + g_v - f_v.$$

Hence, $u' + g_u + f_v = u + g_v + f_u \in \deg(M_F)$. Again, $u' + g_u + f_v \in \deg(M)$ when a set of minimal generators is fixed and a similar construction in the proof of Lemma 2.1.11 is used. $\square$

As a consequence of the lemma above, we obtain the desired injective map between sets of overlap classes under localization. We provide a new notation to describe this map.

Definition 2.1.13. Given an overlap class $[u, F] \in \overline{\deg.\mathrm{p}}(M)$, let $\bigcup[u, F] := \bigcup_{(v,F)\in[u,F]} v + \mathbb{N}F$.

In other words, $\bigcup[u, F]$ is the union of all translates of faces represented by degree pairs in $[u, F]$.

Theorem 2.1.14. *Suppose $F \subseteq G \subseteq H$ are faces of Q. Given an overlap class $[u, H \cup (-G)] \in \overline{\deg.\mathrm{p}}(M_G)$, there exists a unique overlap class $[u', H \cup (-F)] \in \overline{\deg.\mathrm{p}}(M_F)$ such that*

$$\bigcup[u', H \cup (-F)] = \left(\bigcup[u, H \cup (-G)]\right) \cap \deg(M_F).$$

We denote this injection $[u', H \cup (-F)] = \mathrm{res}_{G,F}([u, H \cup (-G)])$, and call it restriction. *These restriction maps satisfy that for any $F \subseteq G, G' \subseteq H$, $\mathrm{res}_{H,G} \circ \mathrm{res}_{G,F} = \mathrm{res}_{H,F} = \mathrm{res}_{H,G'} \circ \mathrm{res}_{G',F}$.*

Proof of Theorem 2.1.14. It is sufficient to show that the map is defined in the case $F = 0$. For a given overlap class $[u, H] \in \overline{\deg.\mathrm{p}}(M_G)$, let $\mathrm{res}_{G,0}([u, H]) := [u', H]$ where u' is determined by Lemma 2.1.11. As demonstrated in Lemma 2.1.12, $\mathrm{res}_{G,0}$ is well-defined and injective. Moreover, by analogy to the construction of elements of M_F with degrees in $\bigcup[u, H]$ in the proof of Lemma 2.1.11, $\bigcup[u, H] = (\bigcup[u', H]) \cap \deg(M)$. Associativity is clear from the definition. $\square$

Finally, we provide a statement about void pairs.

Corollary 2.1.15. *Let $\{F_i\}_{i=1}^m$ be the set of all facets of a (not-necessarily pointed) affine semigroup Q. Let $M := \Bbbk[Q_{sat}]/\Bbbk[Q]$. If $Q \neq \bigcap_{i=1}^m Q - \mathbb{N}F_i$, then there exists a void pair (u, F) such that F is not a facet.*

Proof. By the previous results, if the only void pairs Q arise from facets, then $Q = \bigcap_{i=1}^m Q - \mathbb{N}F_i$. In other words, from $Q \neq \bigcap_{i=1}^m Q - \mathbb{N}F_i$, M is nonzero. Moreover, if all void pairs arise from facets, then all degrees from holes are outside of $\bigcap_{i=1}^m Q - \mathbb{N}F_i$, hence $Q = \bigcap_{i=1}^m Q - \mathbb{N}F_i$, a contradiction. $\square$

Example 2.1.16 (Continuation of Example 2.1.9)**.**

1. Given $M = \Bbbk\left[\mathbb{N}\left[\begin{smallmatrix}1&1&1&1\\0&1&3&4\end{smallmatrix}\right]\right]/\left\langle x^{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}\right\rangle$, all overlap classes of $\deg(M)$ are singletons. Indeed,

$$\begin{aligned}
\operatorname{deg.p}(M) &= \{\text{(green) } (\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right], \varnothing), \text{(blue) } (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_1), \text{(red) } (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_2), (\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], F_2), (\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right], F_2)\} \\
\operatorname{deg.p}(M_{F_1}) &= \{\text{(blue) } (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_1 \cup (-F_1))\} \\
\operatorname{deg.p}(M_{F_2}) &= \{\text{(red) } (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], F_2 \cup (-F_2)), (\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], F_2 \cup (-F_2)), (\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right], F_2 \cup (-F_2))\}\,.
\end{aligned}$$

   This shows two injections $\overline{\operatorname{deg.p}}(I_{F_1}) \hookrightarrow \overline{\operatorname{deg.p}}(I) \hookleftarrow \overline{\operatorname{deg.p}}(I_{F_2})$.

2. Given $M = \Bbbk\left[\mathbb{N}\left[\begin{smallmatrix}0&1&1&0\\0&0&1&1\\1&1&1&1\end{smallmatrix}\right]\right]/\left\langle x^{\left[\begin{smallmatrix}2&2&2&3\\0&1&3&3\\2&2&3&3\end{smallmatrix}\right]}\right\rangle$, the set of degree pairs of ideals in each localizations are as follows.

$$\begin{aligned}
\operatorname{deg.p}(M) &= \left\{\text{(green) } \left(\begin{bmatrix}2\\2\\2\end{bmatrix}, \varnothing\right), \text{(blue) } \left(\begin{bmatrix}0\\0\\0\end{bmatrix}, F_4\right), \text{(yellow) } \left(\begin{bmatrix}1\\0\\1\end{bmatrix}, F_4\right), \text{(red) } \left(\begin{bmatrix}1\\1\\1\end{bmatrix}, F_4\right)\right\} \\
\operatorname{deg.p}(M_G) &= \left\{\text{(blue) } \left(\begin{bmatrix}0\\0\\0\end{bmatrix}, F_4 \cup (-G)\right), \text{(orange) } \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, F_4 \cup (-G)\right)\right\}
\end{aligned}$$

   for any $G \in \{u_1, u_4, F_4\}$. Indeed,

$$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, F_4 \cup (-F_4)\right) = \left(\begin{bmatrix}1\\0\\1\end{bmatrix}, F_4 \cup (-F_4)\right) = \left(\begin{bmatrix}1\\1\\1\end{bmatrix}, F_4 \cup (-F_4)\right).$$

   This is an example of how the selection of u' in Lemma 2.1.11 is not unique. Nonetheless, we have an injection $\overline{\operatorname{deg.p}}(I_G) \to \overline{\operatorname{deg.p}}(I)$ by sending the orange overlap class to the overlap class consisting of yellow and red standard pairs.

2.2 Algebraic properties of finely graded $\mathbb{Z}^d$-graded modules*

In this section we develop the theory of degree pairs in the context of monomial ideals in affine semigroup rings. We then use degree pairs to describe primary and irreducible decompositions of a (finitely generated) finely graded $\mathbb{Z}^d$-graded module, and to compute multiplicities of associated primes.

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2.2.1 Primary Decomposition

Our goal now is to use degree pairs to give a primary decomposition of a (finitely generated) finely graded $\mathbb{Z}^d$-graded module M in $\Bbbk[Q]$ with graded primary components. This is achieved in Theorem 2.2.3, whose proof we break into several steps.

First, we give a necessary condition for a (finitely generated) finely graded $\mathbb{Z}^d$-graded module to be primary.

Proposition 2.2.1. *Let M be a (finitely generated) finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module. If all the degree pairs of M belong to the same face F of Q, then $\mathcal{I}$ is P_F-primary.*

Proof. This proof has two parts. We first show that P_F is an associated prime of M, and then show that no other prime is associated.

Since M is $\mathbb{Z}^d$-homogeneous, so are all of its associated primes, which means that the only possible associated primes are of the form P_G for some face G of $\mathbb{R}_{\geq 0}Q$. Moreover, a prime P_G is associated to M if and only if $(M : x^u) = P_G$ for some monomial $x^u \in \Bbbk[Q]$, $u \in Q$.

Note that the assumption on the degree pairs means that, for any $v \in \deg(M)$, we have $v + \mathbb{N}F \subset \deg(M)$. In ideal-theoretic terms, this means that $\mathrm{Ann}_{\Bbbk[Q]}(m) \subseteq P_F$ for any graded element $m \in M$ whose degree is v.

Let (u, F) be degree pair of M whose overlap class $[u, F]$ is maximal with respect to divisibility. We claim that if $v \in u + \mathbb{N}F$, then $\mathrm{Ann}_{\Bbbk[Q]}(m) = P_F$. To see this, let $m \in M$ be a graded element whose degree is v and $w \in Q \smallsetminus \mathbb{N}F$. If $x^w \cdot m \in M$, then $v + w$ belongs to $u' + \mathbb{N}F$ for some degree pair (u', F) of M (all degree pairs of M belong to F). As $w \notin \mathbb{N}F$, this contradicts the maximality of $[u, F]$. We conclude that if $w \in Q \smallsetminus \mathbb{N}F$, then $x^w \cdot m = 0$, so that $x^w \in \mathrm{Ann}_{\Bbbk[Q]}(m) \supset P_F$. We already knew the reverse inclusion, therefore $\mathrm{Ann}_{\Bbbk[Q]}(m) = P_F$, which shows that P_F is associated to M.

To complete the proof, we show that, if the overlap class of (u, F) is not maximal with respect to divisibility and $v \in u + \mathbb{N}F$, then $\mathrm{Ann}_{\Bbbk[Q]}(m)$ with $\deg(m) = v$ is not prime. Since $[u, F]$ is not maximal, there is a degree pair (u', F) of $\mathcal{I}$, whose overlap class is maximal with respect to divisibility, and such that (u, F) divides (u', F). In particular, there is $w \in Q$ such that $v + w \in u' + \mathbb{N}F$. Note that $w \notin \mathbb{N}F$ as (u, F) and (u', F) are not in the same overlap class. Since (u', F) is a degree pair of M, follows that $x^w \notin \operatorname{Ann}_{\Bbbk[Q]}(m)$. By the previous argument, however, since $w \in Q \smallsetminus \mathbb{N}F$ and $v + w \in u' + \mathbb{N}F$, we have $x^w \in \operatorname{Ann}_{\Bbbk[Q]}(x^w \cdot m)$, which implies $x^{2w} \in \operatorname{Ann}_{\Bbbk[Q]}(m)$. We conclude that $\operatorname{Ann}_{\Bbbk[Q]}(m)$ is not prime. $\square$

The converse of Proposition 2.2.1 holds and is proved by exhibiting a primary decomposition (Theorem 2.2.3). Our next step is to construct a P_F-primary ideal, which is later shown to be a valid choice for a P_F-primary component of M.

Proposition 2.2.2. *Let M be a (finitely generated) finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]-$module, and let F be a face of Q such that M has a degree pair belonging to F. Set*

$$C_F = \left\{ m \in M \;\middle|\; \begin{array}{l} \deg(m) \in Q \textit{ divides some element of } u + \mathbb{N}F \textit{ for some degree pair} \\ (u, F) \textit{ of } M \textit{ whose overlap class is maximal with respect to divisibility} \end{array} \right\}.$$

Then C_F is a submodule of M, which is P_F-primary.

Proof. The first assertion is equivalent to the following statement, whose proof is straightforward: if $\deg(m) := v$ does not divide any degree arising from the maximal overlap classes, and $w \in Q$, then $v + w$ cannot divide any such degree either.

It remains to be shown that C_F is P_F-primary. By Proposition 2.2.1, it is enough to show that all degree pairs of C_F belong to the face F. To see this, we first observe that if $m \in S$ with $\deg(m) = v$, then $x^w \cdot m \in S$ for all $w \in \mathbb{N}F$. This implies that (v, F) is a proper pair for all $v \in \deg(S)$.

To finish the proof, we check that C_F has no proper pairs of the form (v, G), where G strictly contains F. This is a consequence of the following claim:

> If $m \in C_F$ with $\deg(m) = v$ and $w \in Q \smallsetminus \mathbb{N}F$, there is a positive integer k such that $x^{kw} \cdot m = 0$.

To prove the claim note that, as $w \in Q \setminus \mathbb{N}F$, there is a facet H of Q such that H contains F and $\varphi_H(w) > 0$ (see Definition 1.2.43).

Since H contains F, φ_H is constant on each set $u + \mathbb{N}F$. Moreover, if (u, F) and (u', F) are overlapping degree pairs of M, then the value of φ_H on $u+\mathbb{N}F$ equals the value of φ_H on $u'+\mathbb{N}F$. Now, by Lemma 2.1.3 there are finitely many maximal overlap classes of degree pairs. This implies that there is a positive integer N which is an upper bound for the values that φ_H attains on these classes. It follows that for any graded elements in C_F, the value of φ_H on its exponent is at most N. In particular, $\varphi_H(v) \leq N$. Since $\varphi_H(w) > 0$, we may choose a sufficiently large k so that $\varphi_H(v + kw) = \varphi_H(v) + k\varphi_H(w) > N$. It follows that $x^{kw} \cdot m = 0$, as was claimed. □

Theorem 2.2.3. *Let M be a (finitely generated) finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module. Let*

$$\mathscr{S} = \{F \textit{ face of } Q \mid M \textit{ has a degree pair belonging to } F\}.$$

For $F \in \mathscr{S}$, let C_F be as in Proposition 2.2.2. Then $0 = \cap_{F\in\mathscr{S}} C_F$ is an irredundant primary decomposition of M.

Consequently,

1. *P_F is associated to M if and only if M has a degree pair that belongs to F.*

2. *M is P_F-primary if and only if all degree pairs of M belong to F.*

Proof. By Proposition 2.2.2 it is enough to show that $0 = \cap_{F\in\mathscr{S}} C_F$. We claim that if a graded element m has nonzero degree, then $m \notin \cap_{F\in\mathscr{S}} C_F$, or equivalently, $m \notin C_F$ for some $F \in \mathscr{S}$. To see this, since $\deg(m) = v \neq 0$, there is a degree pair (u, F) of M such that $v \in u + \mathbb{N}F$. But then $m \notin C_F$ by the construction of C_F. □

Example 2.2.4 (Continuation of Examples 1.2.61 and 2.1.8)**.**

1. Recall that $M := \Bbbk[x, y]/\langle x^3y, xy^2\rangle$ has three maximal overlap classes of degree pairs as follow; $((0, 0), F)$, $((0, 0), G)$, and $((2, 1), O)$. In the notation of Proposition 2.2.2 and Theorem 2.2.3, $C_F = \Bbbk[x, y]/\langle y\rangle$, $C_G = \Bbbk[x, y]/\langle x\rangle$ and $C_O = \Bbbk[x, y]/\langle x^3, y^2\rangle$, yielding the primary

decomposition (in terms of ideals)

$$\langle x^3y, xy^2\rangle = \langle y\rangle \cap \langle x\rangle \cap \langle x^3, y^2\rangle.$$

Or, in terms of modules, $0 = C_F \cap C_G \cap C_O$.

2. Figure 2.4 depicts the primary decomposition of $\langle x^2y^2, x^3y\rangle \subset \Bbbk[x, xy, xy^2]$. Ideals are indicated using shaded regions, degree pairs are illustrated using thick lines and circles.

3. In this case, the ideal $\langle x^2z^2, x^2yz^2, x^2yz^2\rangle \subset \Bbbk[z, xz, yz, xyz]$ under consideration is P_F-primary.

4. A primary decomposition of $\langle x, xyz, xyz^2\rangle \subset \Bbbk[x, xy, xz, xyz, y^2, z^2]$ is depicted in Figure 2.5. Exponents of monomials in the ideal are colored black. Other colors are used to indicate monomials from the same degree pair.

2.2.2 Irreducible Decomposition

We now address the irreducible decomposition of (finitely generated) finely graded $\mathbb{Z}^d$-graded modules over semigroup rings using degree pairs. While the existence of monomial irreducible decomposition of monomial ideals in semigroup rings is known [42, Corollary 11.5, Proposition 11.41], an effective combinatorial description of such a decomposition was missing from the literature before this work. As a side note, we recall that monomial ideals in semigroup rings can be viewed as binomial ideals in polynomial rings, and mention that binomial ideals do not in general have irreducible decompositions into binomial ideals [32].

In order to decide whether a (finitely generated) finely graded $\mathbb{Z}^d$-graded module is irreducible, one must examine socles. That is the gist of the following result.

Theorem 2.2.5 ([55, Proposition 3.14])**.** *Let $(R, \mathfrak{m})$ be a local noetherian ring and let M be a (finitely generated) finely graded R-module. Let $\mathfrak{p}$ be an associated prime of M, and denote its residue field by K. Let N be the submodule of M whose elements are annihilated by $\mathfrak{p}$. The number* *of $\mathfrak{p}$-primary components in an irredundant irreducible decomposition of the null submodule of M is the dimension of the localization $N_{\mathfrak{p}}$ as a K-vector space.*

We are now able to determine whether a monomial ideal in $\Bbbk[A]$ is irreducible.

Corollary 2.2.6. *Suppose that a finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module M is a P_F-primary. The number of overlap classes of degree pairs of M that are maximal with respect to divisibility equals the number of components in an irredundant irreducible decomposition of M. In particular, M is irreducible if and only if it has a single overlap class of degree pairs that is maximal with respect to divisibility.*

Proof. Recall that by Theorem 2.2.3, all degree pairs of M belong to F. The proof of Proposition 2.2.1 shows that, in this situation, the submodule of M whose elements are annihilated by P_F is spanned as a $\Bbbk$-vector space by the monomials m such that $\deg(m) = v \in u + \mathbb{N}F$ for some degree pair (u, F) whose overlap class is maximal with respect to divisibility. After localization at P_F, this module becomes a vector space over the residue field whose dimension equals the number of overlap classes of degree pairs that are maximal with respect to divisibility. This assertion follows from the following observations: $v \in a + \mathbb{N}F$ and $v' \in u' + \mathbb{N}F$ where (u, F) and (u', F) are overlapping degree pairs, then $v - v' \in \mathbb{Z}F$, so that $m - m'$ where $\deg(m) = v$ and $\deg(m') = v'$ is a unit after localization at P_F. Note also that a linear combination of monomials with coefficients in the residue field can only be zero if the pairwise differences of the exponents of the monomials belong to $\mathbb{Z}F$. Now the desired result follows from Theorem 2.2.5. $\square$

By Theorem 2.2.3, in order to perform irreducible decompositions of monomial ideals, it is enough to do it for primary monomial ideals.

Proposition 2.2.7. *Let M be a (finitely generated) finely graded $\mathbb{Z}^d$-graded P_F-primary $\Bbbk[Q]$-module, and let $[v_1, F], \ldots, [v_\ell, F]$ be the maximal overlap classes of degree pairs of $\mathcal{I}$ with respect to divisibility. For each $1 \leq i \leq \ell$, let*

$$T_i = \{w \in v + \mathbb{N}F \mid (v, F) \textit{ is a degree pair of } M \textit{ whose overlap class divides } [v_i, F]\}.$$

Then T_i is the set of degrees of a submodule J_i. Moreover J_i is irreducible, and $0 = J_1 \cap \cdots \cap J_\ell$ is an irredundant irreducible decomposition of M.

Proof. The arguments that proved Proposition 2.2.2 show that J_i is a module all of whose degree pairs belong to F. By construction, $[v_i, F]$ is the unique overlap class of degree pairs of J_i that is maximal with respect to divisibility. It follows that J_i is irreducible by Corollary 2.2.6. The decomposition $0 = \cap_{i=1}^{\ell} J_i$ is verified in the same way as the primary decomposition in Theorem 2.2.3. □

We emphasize that Theorem 2.2.3 and Proposition 2.2.7 can be combined to produce an irredundant irreducible decomposition of a (finitely generated) finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module in terms of its degree pairs.

Example 2.2.8. The primary decompositions in Example 2.2.4 are also irredundant irreducible decompositions. We now give two more examples for non-normal two-dimensional semigroup rings. In the first one, the primary decomposition of Theorem 2.2.3 is already irreducible, in the second one, the primary decomposition is not irreducible.

(i) Let $Q = \mathbb{N}\left[\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 0 & 0 \end{smallmatrix}\right]$, and consider $\mathcal{I} = \langle x^3y^2, x^5y \rangle \subset \Bbbk[xy, xy^2, x^2, x^3] \cong \Bbbk[Q]$. The irreducible decomposition arising from Proposition 2.2.7 is depicted in Figure 2.6. We point out that the set of holes $\mathcal{H}(Q)$ of Q is $\{(2,0),(3,1)\}$ so that the shaded region in Figure 2.6a contains some elements of $\Bbbk[Q]/\mathcal{I}$.

(ii) Let $Q = \mathbb{N}\left[\begin{smallmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]$, and consider $\mathcal{I} = \langle y^2, xy^2 \rangle \subset \Bbbk[x^2, xy, y] \cong \Bbbk[Q]$. The irreducible decomposition of $\mathcal{I}$ arising from Proposition 2.2.7 is depicted in Figure 2.7. In this case, if $F = \{(2,0)\}$, the color yellow is used for the degree pair $((0,0), F)$, the color red for $((1,1), F)$ and blue for $((0,1), F)$. Note that $((0,0), F)$ and $((1,1), F)$ are maximal with respect to divisibility and do not overlap because $(1,0) \notin Q$.

2.3 Algorithms for finding and using degree pairs of quotients of affine semigroup rings by monomial ideals*

We now turn to concrete methods to compute degree pairs and use degree pairs to produce primary and irreducible decompositions for quotients by monomial ideals in an affine semigroup ring. The algorithms outlined in this dissertation are based on three important facts.

1. The complete face lattice of the cone $\mathbb{R}_{\geq 0}Q$ can be computed if Q is given. This includes finding the primitive integral support functions (Definition 1.2.43) for all the facets of $\mathbb{R}_{\geq 0}A$.
2. A (homogeneous or inhomogeneous) system of linear equations and inequalities with integer coefficients can be solved, in the sense that there exist algorithms to find the coordinatewise minimal solutions and free variables.
3. There are algorithms to compute degree pairs for monomial ideals in polynomial rings.

We emphasize that solving linear systems over the integers is a fundamental problem in many areas and continues to be the focus of much research, especially in convex and discrete optimization; finding the faces of a cone is an important basic question in discrete geometry. There are many approaches to carry out the computational tasks mentioned above.

Relevant questions that can be easily stated as systems of linear equations and inequalities include the following. Given $u \in \mathbb{Z}^d$, and F a finite subset of $\mathbb{Z}^d$. Does u belong to $\mathbb{Z}F$? Does u belong to $\mathbb{N}F$? With these in hand and knowledge of the faces of $\mathbb{R}_{\geq 0}Q$, we can determine, given two pairs (u, F) and (v, G) of Q, whether $(u, F) < (v, G)$, whether (u, F) divides (v, G), and whether (u, F) and (v, F) overlap.

In what follows, for F a face of Q, we use $\mathbb{N}^F$ to denote $\mathbb{N}^{|F|}$ with coordinates indexed by the elements of F. If G is another face of Q, and $F \subset G$, then we consider the natural inclusion $\mathbb{N}^F \subset \mathbb{N}^G$ where elements of $\mathbb{N}^F$ are considered as elements of $\mathbb{N}^G$ whose coordinates indexed by $G \setminus F$ are zero.

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The following result is the main building block for computing degree pairs in Theorem 2.3.5. Its proof is inspired by ideas from [25].

Theorem 2.3.1. *Let $v, v' \in Q$ and let G, G' be faces of Q such that $G \cap G' = G$. There exists an algorithm to compute a finite collection C of pairs over faces of G such that*

$$(v + \mathbb{N}G) \smallsetminus (v' + \mathbb{N}G') = \cup_{(u,F)\in C}(u + \mathbb{N}F).$$

Proof. Consider the set

$$\{u \in \mathbb{N}^G \mid b + G \cdot u \in (b' + \mathbb{N}G')\}. \tag{2.2}$$

Since $G \subseteq G'$, this is the set of the exponents of the monomials in a monomial ideal J in $\Bbbk[\mathbb{N}^G] = \Bbbk[y_j \mid a_i \in G]$. Observe that

$$\begin{aligned}(b + \mathbb{N}G) \smallsetminus (b' + \mathbb{N}G') &= \{b + G \cdot v \mid v \in \mathbb{N}^G \text{ does not belong to the set (2.2)}\} \\ &= \{b + G \cdot v \mid y^v \notin J\} \end{aligned} \tag{2.3}$$

Our goal is thus to find the standard monomials of J. First we determine minimal generators for J, which are the coordinatewise minimal elements of (2.2).

Now consider

$$\{(u, w) \in \mathbb{N}^G \times \mathbb{N}^{G'} \mid b + G \cdot u = b' + G' \cdot w\}. \tag{2.4}$$

Note that the set (2.2) is the projection onto the first component of the set (2.4).

Let $\bar{u}$ be a coordinatewise minimal element of (2.2). Then there is $\bar{w} \in \mathbb{N}^{G'}$ such that $(\bar{u}, \bar{w})$ belongs to (2.4). Let (u, w) be a coordinatewise minimal element of (2.4) that is coordinatewise less than or equal to $(\bar{u}, \bar{w})$. It follows that u belongs to (2.2) and is coordinatewise less than or equal to $\bar{u}$ so that $u = \bar{u}$. This shows that the coordinatewise minimal elements of (2.2) are the projections of the coordinatewise minimal elements of (2.4). Since the set (2.4) is the set of integer solutions of a system of linear equations and inequalities defined over $\mathbb{Z}$, its coordinatewise minimal elements can be computed. That there are finitely many such elements follows from

Dickson's Lemma.

Since we now know the generators of the monomial ideal J, we can compute its standard pairs and write

$$(b + \mathbb{N}G) \smallsetminus (b' + \mathbb{N}G') = \cup_{(u,\sigma)\in \mathrm{deg.p}(J)}(b + G \cdot u + \mathbb{N}\{a_i \mid i \in \sigma\})$$

We use the convention that the degree pairs of $J \subset \Bbbk[\mathbb{N}^G]$ are of the form (u, σ) where $u \in \mathbb{N}^G$ and $\sigma \subset \{i \mid a_i \in G\}$. By definition, the fact that (u, σ) is a degree pair of J implies that $y^u \prod_{i\in\sigma} y_i^{\lambda_i} \notin J$ for all $\lambda_i \in \mathbb{N}$, $i \in \sigma$.

It only remains to be proved is that if (u, σ) is a degree pair of J then $\{a_i \mid i \in \sigma\}$ is a face of G.

Let (u, σ) be a degree pair of J, and let F be the smallest face of G such that $\mathbb{N}\{a_i \mid i \in \sigma\}$ meets the relative interior of $\mathbb{R}_{\geq 0}F$. Let $\sum_{i\in\sigma} \lambda_i a_i$ be an element of the relative interior of F with $\lambda_i \in \mathbb{N}$ for $i \in \sigma$, and set $\lambda \in \mathbb{N}^G$ whose ith coordinate is λ_i if $i \in \sigma$ and 0 otherwise. Then $y^{u+N\lambda} \notin J$ for all $N \in \mathbb{N}$. Now let $a = \sum_{a_i\in F} \mu_i a_i \in \mathbb{N}F$, with the $\mu_i \in \mathbb{N}$, and let $\mu \in \mathbb{N}^G$ whose ith coordinate is μ_i if $a_i \in F$ and 0 otherwise, so that $a = G \cdot \mu$. Since $\sum_{i\in\sigma} \lambda_i a_i$ is in the relative interior of $\mathbb{R}_{\geq 0}F$, we may choose N large enough that $NG \cdot \lambda - a \in \mathbb{N}F$, and we may write $G \cdot (N\lambda - \mu) = G \cdot \nu$ with $\nu \in \mathbb{N}^F \subset \mathbb{N}^G$. But then $G \cdot (\nu + \mu) = G \cdot (N\lambda)$, and as $b + G \cdot (u + N\lambda) \notin b' + \mathbb{N}G'$ (because $y^{u+N\lambda} \notin J$), we have $b + G \cdot (u + \nu + \mu) \notin b' + \mathbb{N}G'$, which in turn implies that $y^{u+\mu} \notin J$. It follows that $(u, \{i \mid a_i \in F\})$ is a proper pair of J. Since (u, σ) is a degree pair of J, we conclude that $\sigma = \{i \mid a_i \in F\}$. □

We need two more auxiliary results for the computation of standard pairs in Theorem 2.3.5. Here is the first one.

Lemma 2.3.2. *Let F be a face of Q and let $a \in Q$. There exists an algorithm to compute the minimal elements (with respect to divisibility) of the set $(a + \mathbb{R}F) \cap Q$.*

Proof. Recall that the primitive integral support functions of the facets of Q (Definition 1.2.43) are linear forms with integer coefficients, and which can be computed. Then $b \in (a + \mathbb{R}F)$ if and only if $\varphi_H(b) = \varphi_H(a)$ for all facets H of Q containing F. The elements of $(a + \mathbb{R}F) \cap Q$ that are minimal with respect to divisibility are the elements of the form $A \cdot u$, where u is a coordinatewise minimal element of:

$$\{u \in \mathbb{N}^n \mid \varphi_H(A \cdot u) = 0 \text{ for all } H \text{ facet of } A, H \supseteq F\}. \tag{2.5}$$

This set is given by integer linear equations and inequalities, and its coordinatewise minimal elements can be computed. $\square$

Definition 2.3.3. Let $\mathcal{I}$ be a monomial ideal in $\Bbbk[Q]$. Recall the notation $\operatorname{std}(\mathcal{I})$ introduced in (2.1.1). A *cover* of the standard monomials of $\mathcal{I}$ is a finite collection C of pairs of Q such that

$$\operatorname{std}(\mathcal{I}) = \cup_{(a,F)\in C}(a + \mathbb{N}F).$$

Proposition 2.3.4. *Let $\mathcal{I}$ be a monomial ideal in $\Bbbk[Q]$. There is an algorithm whose input is a cover of the standard monomials of $\mathcal{I}$, and whose output is the set of degree pairs of $\mathcal{I}$.*

Proof. Let C_0 be a cover of the standard monomials of $\mathcal{I}$. Then all the elements of C_0 are proper pairs of $\mathcal{I}$. Note that for $(a, F) \in C$, if $b \in Q$ divides Q, then (b, F) is also a proper pair of $\mathcal{I}$.

For each $(a, F) \in C_0$, use Lemma 2.3.2 to compute the minimal elements with respect to divisibility of $(a + \mathbb{R}F) \cap Q$, and replace (a, F) by the collection of pairs (b, F), where b is a minimal element of $(a+\mathbb{R}F)\cap Q$ that divides a. In this way we obtain another collection of pairs C_1, which is also a cover of the standard monomials of $\mathcal{I}$.

Next, given (a, F) in C_1, and G a face of Q that is not strictly contained in F, we can determine algorithmically whether (a, G) is a proper pair of $\mathcal{I}$, as follows. First, if C_1 contains no pairs of the form $(b, G') \in C_1$ with $G' \supseteq G$, then (a, G) is not proper. Otherwise, find whether there is a pair $(b, G') \in C_1$ with $G' \supseteq G$ such that $(a+\mathbb{N}G) \smallsetminus (b+\mathbb{N}G') \subsetneq a+\mathbb{N}G$. If no such pair exists, (a, G) is not proper. (To see this, note that if (a, G) is proper, the elements of $a + \mathbb{N}G$ are exponents of standard monomials. Since C_1 is a cover of standard monomials, $a + \mathbb{N}G = (a + \mathbb{N}G) \cap (\cup_{(b,F)\in C_1}(b+\mathbb{N}F)) = \cup_{(b,F)\in C_1}\big((a+\mathbb{N}G)\cap(b+\mathbb{N}F)\big)$. Each intersection $(a+\mathbb{N}G)\cap(b+\mathbb{N}F)$ is a finite union of sets $c + \mathbb{N}F'$ where $F' \subset G \cap F \subset G$. If all the intersections involve faces that are strictly contained in G, then we have written $a + \mathbb{N}G$ as a **finite** union of sets $c + \mathbb{N}F'$ with F' strictly contained in G, which is impossible for dimension reasons.)

If such a pair exists, $(a + \mathbb{N}G) \smallsetminus (b + \mathbb{N}G')$ is a union of sets of the form $a' + \mathbb{N}G''$ where G'' is a proper face of G, so we reduce to verifying whether the pairs (a', G'') in the union are proper pairs of $\mathcal{I}$. This yields an iterative procedure to determine whether (a, G) is proper.

If $(a, F) \in C_1$, replace (a, F) by all pairs of the form (a, G), where G is not strictly contained in F, (a, G) is proper for $\mathcal{I}$, and G is maximal with this property. We obtain a finite collection of pairs C_2, which is still a cover for the standard monomials of $\mathcal{I}$. Now apply to C_2 the same procedure we used to go from C_0 to C_1 to construct a new cover C_3, and apply to C_3 the same procedure we applied to C_1, to get a new cover C_4.

We claim that repeating this process yields, after finitely many iterations, a cover C which is stable under the given operations. To see this, first note that our procedure replaces a proper pair by a collection of proper pairs, all of which are greater than or equal to the original pair with respect to the partial order $<$. Next, we recall that the set of proper pairs of $\mathcal{I}$ has finitely many elements that are maximal with respect to $<$, namely the degree pairs (Lemma 2.1.3). Finally, $<$-chains that are bounded above are finite, because of the strong convexity assumption on $\mathbb{R}_{\geq 0}A$. From these observations it follows that our procedure arrives at a stable cover C after finitely many steps.

The stable cover C has the following properties

- If $(a, F) \in C$ and F' is a face of Q that strictly contains F, then (a, F') is not a proper pair of $\mathcal{I}$.
- If $(a, F) \in C$ and (a, G) is a proper pair of $\mathcal{I}$, then C contains a pair (a, G') such that $G' \supseteq G$.

We claim that C contains the degree pairs of $\mathcal{I}$.

Let (b, G) be a degree pair of $\mathcal{I}$, and let $(a, F) \in C$ such that $b \in a + \mathbb{N}F$. Note that every element of $a + \mathbb{N}G$ divides some element of $b + \mathbb{N}G$. This implies that (a, G) is a proper pair of $\mathcal{I}$ since (b, G) is proper. By construction of C, C contains a pair (a, F') such that F' contains G. Then $b + \mathbb{N}G \subset a + \mathbb{N}F'$. Since (a, F') is proper, and (b, G) is standard, we must have $(a, F') = (b, G)$, which means that $(b, G) \in C$.

Thus, in order to obtain the degree pairs of $\mathcal{I}$, we select the elements of C that are maximal with respect to $<$. $\square$

We are now ready to compute degree pairs.

Theorem 2.3.5. *There exists an algorithm whose input is the set of (monomial) generators of a monomial ideal $\mathcal{I}$ in $\Bbbk[Q]$, and whose output is the set of degree pairs of $\mathcal{I}$. Moreover, the overlap classes of standard pairs can be computed, and those that are maximal with respect to divisibility can be given.*

Proof. Suppose that $\mathcal{I} = \langle t^b \rangle$. Then the set of standard monomials of $\mathcal{I}$ is $Q \smallsetminus (b + Q)$, and we can compute the standard pairs of $\mathcal{I}$ using Theorem 2.3.1 and Proposition 2.3.4.

If $\mathcal{I} = \langle t^{b_1}, t^{b_2} \rangle$, first compute the degree pairs of $\langle t^{b_1} \rangle$ and for each such degree pair (a, F), compute $(a + \mathbb{N}F) \smallsetminus (b_2 + Q)$. This yields a cover of the standard monomials of $\mathcal{I}$, which can be massaged using Proposition 2.3.4 to obtain the degree pairs of $\mathcal{I}$.

In general, if the degree pairs of $\langle t^{b_1}, \dots, t^{b_\ell} \rangle$ are known, then we may use the same idea to compute the standard pairs of $\langle t^{b_1}, \dots, t^{b_\ell}, t^{b_{\ell+1}} \rangle$.

Finally, finding the overlap classes and determining the maximal ones with respect to divisibility can be done by finding whether certain linear systems of equations and inequalities have integer solutions. $\square$

Remark 2.3.6. Having computed the (overlap classes of) degree pairs of $\mathcal{I}$, the associated primes of $\mathcal{I}$ and their corresponding multiplicities can be computed by inspection.

Example 2.3.7 (Continuation of Example 1.2.61(2))**.** Recall $\mathcal{I} = \langle x^2y^2, x^3y \rangle \subset \Bbbk[x, xy, xy^2] \cong \Bbbk[Q]$ for $A = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right]$. In this case, we illustrate how to compute the degree pairs of $\mathcal{I}$ using the method described in Theorem 2.3.5.

First we apply Theorem 2.3.1 to $Q \smallsetminus ((2,2) + Q)$, to obtain a cover of standard monomials for $\mathcal{I}_0 = \langle x^2y^2 \rangle$. In this case, the set (2.4) turns out to be

$$\{(u, w) \in \mathbb{N}^A \times \mathbb{N}^A \mid b + A \cdot u = (2,2)^t + A \cdot w\} = \{(u, w) \in \mathbb{N}^A \times \mathbb{N}^A \mid A(u - w) = (2,2)^t\}.$$

A straightforward calculation shows that this set is the same as

$$\{(u,w) : u - w = (0,2,0) \text{ or } (1,0,1)\} = \{(u+w,w) : u = (0,2,0) \text{ or } (1,0,1), w \in \mathbb{N}^A\}.$$

It follows that the minimal solutions we are looking for are $(0,2,0)$ and $(1,0,1)$. We see that the ideal $J \subset \Bbbk[\mathbb{N}^A] = \Bbbk[z_1, z_2, z_3]$ in the proof of Theorem 2.3.1 equals $\langle z_2^2, z_1 z_3\rangle$ and its degree pairs are

$$\big((0,0,0), \{(0,0,1)\}\big),\ \big((0,1,0), \{(0,0,1)\}\big),\ \big((0,0,0), \{(1,0,0)\}\big), \text{ and } \big((0,1,0), \{(1,0,0)\}\big).$$

Thus,

$$Q \smallsetminus \left((2,2)^t + Q\right) = \mathbb{N}\{(1,2)\} \cup \left((1,1) + \mathbb{N}\{(1,2)\}\right) \cup \mathbb{N}\{(1,0)\} \cup \left((1,1) + \mathbb{N}\{(1,0)\}\right).$$

These form a cover of the standard monomials of $\langle x^2y^2\rangle$, and it is easily checked that this is the cover by degree pairs of the standard monomials of $\langle x^2y^2\rangle$.

Now, to find the degree pairs of $\mathcal{I}$, we compute $b + \mathbb{N}F \smallsetminus \big((3,1) + Q\big)$ for each $b + \mathbb{N}F$ where $(b, F) \in \operatorname{deg.p}(\langle x^2y^2\rangle)$. Note that the sets $\mathbb{N}\{(1,2)\}$, $(1,1) + \mathbb{N}\{(1,2)\}$, and $\mathbb{N}\{(1,0\}$ have an empty intersection with $(3,1) + Q$. Hence we only need to compute $\big((1,1) + \mathbb{N}\{(1,0)\}\big)\big) \smallsetminus \big((3,1) + Q\big)$.

We apply Theorem 2.3.1 to do this, yielding $J = \langle z_1^2\rangle$ in $\Bbbk[z_1]$. This ideal has two degree pairs $(0, \varnothing)$ and $(1, \varnothing)$. It follows that

$$\big((1,1) + \mathbb{N}\{(1,0)\}\big) \smallsetminus \big((3,1) + Q\big) = \{(0,0), (1,1)\}$$

Thus, we have a cover of standard monomials of $\mathcal{I}$ is given by

$$\{\big((0,0), \{(1,2)\}\big),\ \big((1,1), \{(1,2)\}\big),\ \big((0,0), \{(1,0)\}\big),\ \big((0,0), \varnothing\big), \big((1,1), \varnothing\big)\}$$

We next use Proposition 2.3.4 to remove $((0,0),\varnothing)$. Finally, the set of degree pairs of $\mathcal{I}$ is

$$\big((0,0),\{(1,2)\}\big),\ \big((1,1),\{(1,2)\}\big),\ \big((0,0),\{(1,0)\}\big),\ \text{ and } \big((1,1),\varnothing\big),$$

as depicted in Figure 2.1b.

We now know how to compute the degree pairs of a monomial ideal $\mathcal{I}$ in $\Bbbk[Q]$. It is natural to try to reverse the process and find generators of a monomial ideal whose degree pairs are given.

Theorem 2.3.8. *There exists an algorithm whose input is the set of degree pairs of a monomial ideal in* $\Bbbk[Q]$ *and whose output is a set of generators for this ideal.*

Proof. If the degree pairs of a monomial ideal $\mathcal{I}$ are known, we can determine which overlap classes are maximal with respect to divisibility (among all pairs belonging to the same face). For each degree pair (a, F) in such an overlap class, we can compute $a + \mathbb{N}F \smallsetminus \big(\cup_{(b,G)\in \mathrm{deg.p}(\mathcal{I}) | G\neq F}\ b + \mathbb{N}G\big)$ as a union over pairs (α, F') of sets $a + \mathbb{N}F'$. For each such pair (α, F') and each a_i mid $a_i \in A \smallsetminus F$, $t^{a_i}t^{\alpha} \in \mathcal{I}$ by construction (use maximality of the overlap class). Let J_1 be the ideal generated by all monomials $t^{a_i}t^{\alpha}$ obtained in this way. Then $J_1 \subset \mathcal{I}$.

Compute the degree pairs of J_1. If they coincide with the degree pairs of $\mathcal{I}$, then $J_1 = \mathcal{I}$ and we are done.

Otherwise, pick maximal overlap classes of degree pairs of J_1 that are not degree pairs of $\mathcal{I}$, remove all degree pairs of $\mathcal{I}$, and use this to find elements of $\mathcal{I}$ that do not belong to J_1. Obtain an ideal $J_2 \supsetneq J_1$.

Repeat this procedure. Since $\Bbbk[Q]$ is Noetherian, this process must arrive at $\mathcal{I}$ in a finite number of steps. $\square$

Remark 2.3.9. We observe that degree pairs can be used to compute intersections of monomial ideals. If $\mathcal{I}$ and J are monomial ideals in $\Bbbk[Q]$, then the union of the collections of degree pairs of $\mathcal{I}$ and J is a cover for the standard monomials of $\mathcal{I} \cap J$. Applying Proposition 2.3.4 yields the degree pairs of $\mathcal{I} \cap J$, and we can compute generators using Theorem 2.3.8.

Example 2.3.10. Recall Example 2.2.8 (ii). In this case, the degree pairs are $\big((0,0),\{(2,0)\}\big)$, $\big((0,1),\{(2,0)\}\big)$, and $\big((1,1),\{(2,0)\}\big)$; we wish to recover the generators of the ideal from this information, using the method from Theorem 2.3.8. If we start with the degree pair $\big((0,1),\{(2,0)\}\big)$, we obtain the ideal $J_1 = \langle xy^2, x^2y^2\rangle$. Using $\big((0,0),\{(2,0)\}\big)$ next, we find the monomials y^2, xy^2, which generate $\mathcal{I}$.

We can now compute irreducible decompositions of monomial ideals in $\Bbbk[Q]$.

Theorem 2.3.11. *There exists an algorithm whose input is the set of degree pairs of a monomial ideal $\mathcal{I}$ in $\Bbbk[Q]$, and whose output is an irreducible decomposition for $\mathcal{I}$.*

Proof. Given the degree pairs of $\mathcal{I}$, we can determine the associated primes of $\mathcal{I}$. If P_F is associated to $\mathcal{I}$, let $[\bar{a}, F]$ be an overlap class of degree pairs of $\mathcal{I}$ that is maximal with respect to divisibility (among overlap classes belonging to F).

Let (a, F) be a degree pair of $\mathcal{I}$ whose overlap class is $[\bar{a}, F]$. Define

$$\bigcup_{(a,F)\in[\bar{a},F]} \{(u,v,w) \in \mathbb{N}^A \times \mathbb{N}^A \times \mathbb{N}^F \mid A\cdot u + A\cdot v = a + F\cdot w\}. \tag{2.6}$$

Note that u belongs to the projection of (2.6) onto the first factor if and only if $A \cdot u$ divides an element of $a + \mathbb{N}F$ where $(a, F) \in [\bar{a}, F]$. Using Theorem 2.2.3 and Proposition 2.2.7, we see that these are (exponents of) the standard monomials in a valid irreducible component of $\mathcal{I}$.

Adapting the method from Theorem 2.3.1, we can find a cover of the standard monomials of this irreducible component. Proposition 2.3.4 yields the corresponding standard pairs, and Theorem 2.3.8 provides generators. $\square$

Remark 2.3.12. We can adapt the proof of Theorem 2.3.11 to compute primary components. Alternatively, we can compute the irreducible components first, and then use Remark 2.3.9 to intersect all irreducible components associated to the same prime, yielding the corresponding primary component.

2.4 `StdPairs`: Implementation of the algorithms in `SageMath` *

We present `StdPairs`, a `SageMath` library to compute standard pairs of a monomial ideal over a pointed (non-normal) affine semigroup ring. Moreover, `StdPairs` provides the associated prime ideals, the corresponding multiplicities, and an irredundant irreducible primary decomposition of a monomial ideal. The library expands on the `standardPairs` function on `Macaulay2` over polynomial rings, and is based on algorithms from [39]. We also provide methods that allow the outputs from this library to be compatible with the `Normaliz` package of `Macaulay2` and `SageMath`.

We remark that our notation here differs from existing notation for standard pairs computation over polynomial rings in `Macaulay2`. Over the polynomial ring $\Bbbk[x_1, x_2, \cdots, x_n]$, a pair is a tuple (x^u, V) where x^u is a monomial $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ for some integer vector $u = (a_1, a_2, \cdots, a_n)$ and V is a set of variables [28, 52]. From the viewpoint of affine semigroup rings, the polynomial ring is a special case when the underlying affine semigroup is generated by an $n \times n$ identity matrix $\mathbf{I}$. Since the cone $\mathbb{R}_{\geq 0}\mathbf{I}$ is a simplicial cone, i.e., every subset of rays form a face, we may interpret V as a face. The following example shows the different notations for the standard pairs of a monomial ideal $I = \langle x^{\left[\begin{smallmatrix}1\\3\\1\end{smallmatrix}\right]}, x^{\left[\begin{smallmatrix}1\\2\\2\end{smallmatrix}\right]}, x^{\left[\begin{smallmatrix}0\\3\\2\end{smallmatrix}\right]}, x^{\left[\begin{smallmatrix}0\\2\\3\end{smallmatrix}\right]}\rangle$ in the polynomial ring $\Bbbk[x_1, x_2, x_3]$.

In `Macaulay2`,

```
i1 : R = QQ[x,y,z];

i2 : I = monomialIdeal(x*y^3*z, x*y^2*z^2, y^3*z^2, y^2*z^3)

                            3       2 2    3 2    2 3
o2 = monomialIdeal (x*y z, x*y z , y z , y z )


o2 : MonomialIdeal of R
```

---

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```
i3 : standardPairs I

o3 = {{1, {x, z}}, {y, {x, z}}, {1, {x, y}}, {z, {y}},
      2           2 2
     {y z, {x}}, {y z , {}}}

o3 : List
```

whereas in the given library `StdPairs` in `SageMath`,

```
sage: from stdpairs import *
sage: A = matrix(ZZ,[[1,0,0],[0,1,0],[0,0,1]])
sage: Q = AffineMonoid(A)
sage: M = matrix(ZZ,[[1,1,0,0],[3,2,3,2],[1,2,2,3]])
sage: I = MonomialIdeal(M,Q)
sage: I.standard_cover()
{(1,): [([[0], [0], [1]]^T,[[0], [1], [0]])],
 (0, 2): [([[0], [1], [0]]^T,[[1, 0], [0, 0], [0, 1]]),
  ([[0], [0], [0]]^T,[[1, 0], [0, 0], [0, 1]])],
 (0, 1): [([[0], [0], [0]]^T,[[1, 0], [0, 1], [0, 0]])],
 (): [([[0], [2], [2]]^T,[[], [], []])],
 (0,): [([[0], [2], [1]]^T,[[1], [0], [0]])]}
```

`()`, `(0,)`, `(0, 2)`, `(0, 1)`, and `(1,)` in `StdPairs` of `SageMath` are indices of columns of the matrix `A`. These denote `{}`, `{x}`, `{x, y}`, `{x, z}`, and `{y}` respectively in `Macaulay2`. Thus, for example, a pair `([[0], [0], [1]]^T,[[0], [1], [0]])` represents `{z, {y}}`, `([[0], [0], [0]]^T,[[1, 0], [0, 0], [0, 1]])` represents `{1, {x, z }}`, and so on. Therefore, this example shows that `StdPairs` is consistent with `Macaulay2`.

2.4.1 Classes in **`StdPairs`**

We implement three classes related to affine semigroups, semigroup ideals, and proper pairs respectively. This implementation is based on `SageMath` 9.1 with `Python` 3.7.3. and `4ti2` package. Detailed usage and examples of each method or object can be found by the command <method_name>? in `SageMath` or `https://byeongsuyu.github.io/StdPairs/`, the documentation of `StdPairs` made by the `Sphinx` package.

2.4.1.1 Class `AffineMonoid`

This class is constructed by using an integer matrix $\mathbf{A}$. The name follows the convention of `SageMath` which distinguishes monoid from semigroup. In `SageMath`, $\mathbf{A}$ can be expressed as a 2-dimensional `NumPy.ndarray` type or an integer matrix of `SageMath`. For example,

```
sage: from stdpairs import *
sage: A = matrix(ZZ,[[1,2],[0,2]])
sage: Q = AffineMonoid(A)
```

generates Q as a type of `AffineMonoid`. This class has several methods as explained below.

- `Q.gens()` returns a matrix generating an affine monoid Q as `NumPy.ndarray` type. This may not be a minimal generating set of Q.
- `Q.mingens()` returns a minimal generating matrix of an affine monoid of Q.
- `Q.poly()` returns a real cone $\mathbb{R}_{\geq 0}Q$ represented as a type of `Polyhedron` in `SageMath`. If one generates Q with `True` parameter, i.e.,

  ```
  sage: from stdpairs import *
  sage: A = matrix(ZZ,[[1,2],[0,2]])
  sage: Q = AffineMonoid(A,is_normaliz=True)
  ```

  then `Q.poly()` is of a class of `Normaliz` integral polyhedron. This requires `PyNormaliz` package. See [34] for more details.

- `Q.face_lattice()` returns a finite lattice containing all faces of the affine semigroup. A face in the lattice is saved as a tuple storing column numbers of generators **A**. This lattice is of type of `Finite Lattice Poset` in `SageMath`. For example,

```
sage: Q.face_lattice()
Finite lattice containing 5 elements
sage: Q.face_lattice().list()
[(-1,), (), (0,), (1,), (0, 1)]
```

- `Q.index_to_face()` returns a `dictionary` type object whose keys are tuples denoting indices of column vectors consisting of faces, and whose items are corresponding faces of `Q.poly()`. For example,

```
sage: Q.index_to_face()
{(-1,): A -1-dimensional face of a Polyhedron in ZZ^2,
 (): A 0-dimensional face of a Polyhedron in ZZ^2
 defined as the convex hull of 1 vertex,
 (0,): A 1-dimensional face of a Polyhedron in ZZ^2
 defined as the convex hull of 1 vertex and 1 ray,
 (1,): A 1-dimensional face of a Polyhedron in ZZ^2
 defined as the convex hull of 1 vertex and 1 ray,
 (0,1): A 2-dimensional face of a Polyhedron in ZZ^2
  defined as the convex hull of 1 vertex and 2 rays}
```

- `Q.index_of_face(matrix face)` returns a face as a tuple of indices of column vectors of a generator **A** corresponding to a given submatrix `face` of **A**. For example,

```
sage: M = matrix(ZZ,[[2],[2]])
sage: Q.index_of_face(M)
(1,)
```

`face` should be a submatrix of `Q.gens()` which form a face.

- `Q.face(tuple index)` returns a face as a submatrix of a generator $\mathbf{A}$ corresponding to a given tuple `index`. For example,

```
sage: Q.face((1,))
array([[2],
       [2]])
```

- `Q.integral_support_vectors()` return a `dictionary` type object whose keys are tuples denoting faces and whose items are integral support functions of facets containing $\mathbf{F}$ as a vector form. An *integral support function* $\phi_{\mathbf{H}}$ of a facet $\mathbf{H}$ is a linear function $\phi_{\mathbf{H}} : \mathbb{R}^d \to \mathbb{R}$ such that $\phi_{\mathbf{H}}(\mathbb{Z}^d) = \mathbb{Z}$, $\phi_{\mathbf{H}}(u) \geq 0$ for all column vectors u of generators A, and $\phi_{\mathbf{H}}(u) = 0$ if and only if $u \in \mathbf{H}$. By linearity, $\phi_{\mathbf{H}}(u) = v \cdot u$ for some rational vector v. We call v as an *integral support vector*. Each item of `Q.integral_support_vectors()` is a matrix as `NumPy.ndarray` type whose rows are integral support vectors of facets containing the given face. For example,

```
sage: Q.integral_support_vectors()
{(): array([[ 0,  1],
        [ 1, -1]]),
 (0,): array([[0, 1]]),
 (1,): array([[ 1, -1]]),
 (0, 1): array([], dtype=int64)}
```

In this code, `()` denoting 0 has two integral support vectors, since it is an intersection of two facets `(0,)` and `(1,)`, while `(0,1)` has no such integral support vectors since it is not a proper face but the affine semigroup itself. See [39, Definition 2.1] for the precise definition of a (primitive) integral support function.

- `Q.is_empty()` returns a boolean value indicating whether Q is a trivial affine semigroup or not. A *trivial affine semigroup* is an empty set as an affine semigroup.

- `Q.is_pointed()` returns a boolean value indicating whether Q is a pointed affine semigroup or not.

- `Q.is_element(vector` v`)` returns nonnegative integral inhomogeneous solutions (minimal integer solutions) of $\mathbf{A}\mathbf{x} = v$ using `zsolve` in [1]. If v is not an element of an affine semigroup Q, then it returns an empty matrix. v should be a `NumPy.ndarray` type 2-Dimensional object with one column, or a matrix of `SageMath` with only one column.

- `Q.save_txt()` returns a string containing information of `Q`. This can be loaded again using `txt_to_affinemonoid(string info)`, which will be explained in Subsection 2.4.1.4.

- `Q.save(string path)` saves the given object `Q` as binary file on the given path. This can be loaded again using `load(path)`, pre-existing global function of `SageMath`.

Moreover, one can directly compare affine semigroups using the equality operator == in `SageMath`.

2.4.1.2 Class `MonomialIdeal`

This class is constructed by an affine semigroup Q and generators of an ideal as a matrix form, say $\mathbf{M}$, which is a 2-dimensional `NumPy.ndarray` object or an integer matrix of `SageMath`. For example,

```
sage: M = matrix(ZZ,[[4,6],[4,6]])
sage: I = MonomialIdeal(M,Q)
sage: I
An ideal whose generating set is
[[4]
 [4]]
```

As shown in the example above, this class stores only minimal generators of the ideal. The attributes and methods are explained below.

- `I.gens()` returns the minimal generators of I as a `NumPy.ndarray` type object.

- `I.ambient_monoid()` returns the ambient affine semigroup of I.

- `I.standard_cover(verbose = True)` returns the *standard cover* of I. This is a `dictionary` object whose keys are faces and whose items are list consisting of `ProperPair` type objects whose face is equal to the corresponding key. `ProperPair` object will be explained in Subsection 2.4.1.3. The definition of the standard cover will be given in Subsection 2.4.2.1. Users can check progress of computation if `verbose=False`.

- `I.overlap_classes()` returns a `dictionary` object whose keys are tuples denoting faces and whose items are list of lists representing overlap classes of I. An *overlap class* of an ideal I is a set of standard pairs such that their representing submonoids intersect nontrivially.

- `I.maximal_overlap_classes()` returns all maximal overlap classes of I with divisibility. An overlap class is *maximal with divisibility* if every pair in the overlap class can divides only pairs in itself. See [39, Section 3] for the detail.

- `I.irreducible_decomposition()` returns a list of components of the irredundant irreducible primary decomposition of I.

- `I.associated_primes()` returns all associated prime ideals of I as a `dictionary` type. In other words, the function returns a dictionary whose keys are faces of the affine semigroup as `tuple` and whose values are associated prime ideals corresponding to the face in its key.

- `I.multiplicity(ideal P or face` **F**`)` returns a multiplicity of I over the given associated prime P. Since there is a one-to-one correspondence between monomial prime ideals and faces of an affine semigroup, this method takes the face **F** (as a tuple) corresponding to a prime ideal P as a valid input instead.

- `I.is_element(vector` v`)` returns nonnegative integral inhomogeneous solutions (minimal integer solutions) of $\mathbf{Ax} = v - u$ for each generator u of I using `zsolve` in [1]. If v is an element of ideal, then it returns a list $[\mathbf{x}, u]$ for some generator u such that $u + \mathbf{Ax}^T = v$. Otherwise, it returns an empty matrix. v should be a `NumPy.ndarray` type 2-Dimensional object with one column, or a matrix of `SageMath` with only one column.

- `I.is_standard_monomial(vector` v`)` returns a boolean value indicating whether the given vector v is a standard monomial or not.

- `I.is_principal()` returns a boolean value indicating whether I is principal or not. Likewise, `I.is_empty()`, `I.is_irreducible()`, `I.is_primary()`, `I.is_prime()`, and `I.is_radical()` return a boolean value indicating whether I has the properties implied by their name or not.

- `I.radical()` returns the radical of I as an `MonomialIdeal` object.

- `I.intersect(J)` returns an intersection of two ideals I and J as a `MonomialIdeal` object. Likewise, addition $+$, multiplication $*$, and comparison == are defined between two objects. The following example shows an addition of two monomial ideals in `SageMath`.

```
sage: I = MonomialIdeal(matrix(ZZ,[[4,6],[4,6]]),Q)
sage: J = MonomialIdeal(matrix(ZZ,[[5],[0]]),Q)
sage: I.intersect(J)
An ideal whose generating set is
[[9]
 [4]]
sage: I+J
An ideal whose generating set is
[[5 4]
 [0 4]]
```

```
sage: I*J
An ideal whose generating set is
[[9]
 [4]]
```

- `I.save_txt()` returns a string which can be used to recover the object I and their pre-calculated properties without calculation. In other words, it contains the generators of I as well as its standard cover, overlap classes, associated primes, and irreducible primary decompositions if they were calculated. This can be loaded again using `txt_to_monomialideal(string info)`, which will be explained in Subsection 2.4.1.4.

- `I.save(string path)` saves the given object `I` as binary file on the given path. This can be loaded again using `load(path)`, pre-existing global function of `SageMath`.

2.4.1.3 *Class* `ProperPair`

A proper pair $(u, \mathbf{F})$ of an ideal I can be declared in `SageMath` by specifying an ideal I, a standard monomial u as a matrix form (or `NumPy` 2D array), and a face $\mathbf{F}$ as a tuple. If $(u, \mathbf{F})$ is not proper, then `SageMath` calls a `ValueError`. The following example shows two ways of defining a proper pair.

```
sage: import numpy as np
sage: I = MonomialIdeal(matrix(ZZ,[[4,6],[4,6]]),Q)
sage: PP = ProperPair(np.array([2,0])[np.newaxis].T,(0,),I)
sage: PP
([[2], [0]]^T,[[1], [0]])
sage: QQ = ProperPair(np.array([2,0])[np.newaxis].T,(0,),I,
....: properness =True)
sage: QQ
([[2], [0]]^T,[[1], [0]])
```

The second line tests whether the pair is a proper pair of the given ideal I or not before generating `PP`. However, the fourth line with `properness=Tue` generates `QQ` without checking whether `QQ` is proper pair of I or not. Use the third parameter with `True` only if the generating pair is proper a priori. In any case, each `PP` and `QQ` denotes proper pair whose initial monomial is $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and whose face is $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

The attributes and methods are explained below. We assume that `PP` denotes a proper pair $(u, \mathbf{F})$.

- `PP.monomial()`, `PP.face()`, and `PP.ambient_ideal()` return the initial monomial u (as `NumPy` 2D array), the face $\mathbf{F}$ (as a `tuple`), and its ambient ideal (as an object of `AffineMonoid`) respectively.

- `PP.is_maximal()` returns a boolean value indicating whether the given pair is maximal with respect to the divisibility of proper pairs of the ambient ideal. If PP is generated without testing whether its monomial is in the given ideal I or not, this methods raise warning instead of returning a boolean value.

- `PP.is_element(vector` v`)` returns nonnegative integral inhomogeneous solutions (minimal integer solutions) of $u + \mathbf{F}\mathbf{x} = v$ using `zsolve` in [1]. If v is not an element of the submonoid $u + \mathbb{N}\mathbf{F}$, then it returns an empty matrix.

- Like `AffineMonoid` or `MonomialIdeal`, one can directly compare proper pairs using the equality operator == in `SageMath`.

2.4.1.4 Global functions

- `prime_ideal(tuple face, AffineMonoid Q)` returns a prime ideal of the given `AffineMonoid` object Q corresponding to the tuple object `face` as an object of the class `MonomialIdeal`.

```
sage: prime_ideal((1,),Q)
```

```
An ideal whose generating set is
[[1]
 [0]]
```

- `div_pairs(pair PP, pair QQ)` returns a matrix whose column $\mathbf{u}$ is a minimal solution of $u+\mathbf{u}+\mathbb{N}\mathbf{F} \subseteq v+\mathbb{N}\mathbf{G}$ if $PP=(u,\mathbf{F})$ and $QQ=(v,\mathbf{G})$. The returned value is a nonempty matrix if and only if a pair PP divides a pair QQ. For example, suppose two pairs PP and QQ are $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]+\mathbb{N}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]+\mathbb{N}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ respectively. Then,

```
sage: I = MonomialIdeal(matrix(ZZ,[[4,6],[4,6]]),Q)
sage: PP = ProperPair(matrix(ZZ,[[2],[0]]),(0,),I)
sage: QQ = ProperPair(matrix(ZZ,[[3],[0]]),(0,),I)
sage: div_pairs(PP,QQ)
[1]
[0]
sage: div_pairs(QQ,PP)
[0]
[0]
```

since $(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]+\mathbb{N}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) \supseteq (\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]+\mathbb{N}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$.

- `txt_to_affinemonoid(string info)` (resp. `txt_to_monomialideal(string info)`) loads an `AffineMonoid` object (resp. a `MonomialIdeal` object) in the specially created string variable `info`, which is generated by `AffineMonoid.save_txt()` (resp. `MonomialIdeal.save_txt()`) methods. These are useful for users who want to avoid repeating calculation which was previously done. For example, the ideal J in below

```
sage: I = MonomialIdeal(matrix(ZZ,[[4,2],[4,0]]),Q)
sage: I.standard_cover()
{(): [([[1], [0]]^T,[[], []]),
```

```
  ([[0], [0]]^T,[[], []]),
  ([[3], [2]]^T,[[], []]),
  ([[2], [2]]^T,[[], []])]}
sage: J=txt_to_monomialideal(I.save_txt())
sage: J.standard_cover()
{(): [([[1], [0]]^T,[[], []]),
  ([[0], [0]]^T,[[], []]),
  ([[3], [2]]^T,[[], []]),
  ([[2], [2]]^T,[[], []])]}
```

is a new `MonomialIdeal` object, however it does not need time to calculate its standard cover, since pre-calculated information of the standard cover was stored in `I.save_txt()` and transferred to J.

- `pair_difference(ProperPair PP, ProperPair QQ)` is a global function decomposing $PP \setminus QQ$ as a finite union of pairs. See Theorem 2.4.1 and subsequent arguments for details.

- `from_macaulay2(string var_name)` and `to_macaulay2(MonomialIdeal I)` are global functions used for communicating with `Macaulay2` objects. See Section 2.4.3 for details.

2.4.2 Implementation of an algorithm finding standard pairs

2.4.2.1 Case 1: Principal Ideal

A *cover* of standard monomials of an ideal I is a set of proper pairs of I such that the union of all subsemigroups $u + \mathbb{N}\mathbf{F}$ corresponding to an element $(u, \mathbf{F})$ of the cover is equal to the set of all standard monomials. The *standard cover* of an ideal I is a cover of I whose elements are standard pairs. The standard cover of a monomial ideal I is unique by the maximality of standard pairs among all proper pairs of I. A key idea in [39, Section 4] is to construct covers containing all standard pairs. Once a cover is obtained, we can then produce the standard cover.

The following result helps to compute the standard cover in the special case of a principal ideal.

Theorem 2.4.1 ([39, Theorem 4.1]). *Let $v, v' \in \mathbb{N}\mathbf{A}$ and let $\mathbf{G}, \mathbf{G}'$ be faces of A such that $\mathbf{G} \cap \mathbf{G}' = \mathbf{G}$. There exists an algorithm to compute a finite collection C of pairs over faces of $\mathbf{G}$ such that*

$$(v + \mathbf{G}) \smallsetminus (v' + \mathbf{G}') = \cup_{(u,\mathbf{F}) \in C} (u + \mathbf{F}).$$

The *pair difference* of the pairs $(v, \mathbf{G})$ and $(v', \mathbf{G}')$ is a finite collection of pairs over faces of $\mathbf{G}$ given by Theorem 2.4.1.

Corollary 2.4.2. *Given a principal ideal $I = \langle v \rangle$, the pair difference of pairs $(0, \mathbf{A})$ and $(v, \mathbf{A})$ is the standard cover of I.*

Proof. Theorem 2.4.1 implies that the pair difference is a cover of I. To see it is the standard cover, suppose that the ambient affine semigroup is generated by $\mathbf{A} = \left[u_1 \cdots u_n \right]$. Let (w, F) be a proper pair in the pair difference. Without loss of generality, we assume that $F = \left[u_1 \cdots u_m \right]$ for some $m < n$ by renumbering indices. By the proof of Theorem 2.4.1 in [39], $w = A \cdot u$ where $x^u \in \Bbbk[\mathbb{N}^n]$ is a standard monomial such that $(x^u, \{x_1, \cdots, x_m\})$ is a standard pair of some monomial ideal J in $\Bbbk[\mathbb{N}^n]$.

Suppose that there exists $(d, \mathbf{G})$ such that $F \subseteq \mathbf{G}$ and $d + g = w$ for some $g \in G$. Since $d \in \mathbb{N}\mathbf{A}$, $d = Aw'$ for some $w' \in \mathbb{N}^n$. Since A is pointed, w' is coordinatewisely less than u. Thus, $(x^{w'}, \{x_1, \cdots, x_m\})$ contains $(x^u, \{x_1, \cdots, x_m\})$. Lastly, $(x^{w'}, \{x_1, \cdots, x_m\})$ is a proper pair of J, otherwise, there exists $x^v \in \Bbbk[x_1, \cdots, x_m] \subseteq \Bbbk[\mathbb{N}^n]$ such that $x^{w'+v} \in J$. Then, $x^g x^{w'+v} \in J \implies x^{u+v} \in J \cap (x^u, \{x_1, \cdots, x_m\}) = \emptyset$ leads to a contradiction.

Thus, by maximality of the standard pair, $w' = u$. This implies $d = w$. Moreover, $G = F$, otherwise there exists $j \in \{1, 2, \cdots, n\} \smallsetminus \{1, \cdots, m\}$ such that $x^u x_j^l \notin J$ for any l, which implies that $(x^u, \{x_1, \cdots, x_m, x_j\})$ is a proper pair of J strictly containing a standard pair $(x^u, \{x_1, \cdots, x_m\})$ of J, a contradiction. $\square$

Theorem 2.4.1 is implemented as a method `pair_difference(`$(v, \mathbf{F})$`, `$(v', \mathbf{F}')$`)` in the library `StdPairs`. Two input arguments should be of type `ProperPair`. It returns the pair difference of pairs $(v, \mathbf{F})$ and $(v', \mathbf{F}')$ with `dictionary` type, called `Cover`. `Cover` classifies pairs by its faces. For example, the code below shows the pair difference of pairs $(0, \mathbf{A})$ and $((0,2), \mathbf{A})$, which are

$$\left(0, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right), \text{ and } \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right).$$

```
sage: from stdpairs import *
sage: Q = AffineMonoid(matrix(ZZ, [[2,0,1],[0,1,1]]))
sage: I = MonomialIdeal(matrix(ZZ,0),Q)
sage: C = ProperPair(np.array([[0,0]]).T, (0,1,2), I )
sage: D = ProperPair(np.array([[0,2]]).T, (0,1,2), I )
sage: print(pair_difference(C,D))
{(0,): [([[1], [2]]^T,[[2], [0]]), ([[1], [1]]^T,[[2], [0]]),
([[0], [1]]^T,[[2], [0]]), ([[0], [0]]^T,[[2], [0]])]}
```

By Corollary 2.4.2, it is the standard cover of the ideal $I = \langle (0,2) \rangle$ in an affine semigroup $\mathbb{N}\left[\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$.

`pair_difference(`$(v, \mathbf{F})$`, `$(v', \mathbf{F}')$`)` uses `standardPairs` of `Macaulay2` internally to find standard pairs of a polynomial ring, which is implemented by [28]. Briefly, this method `pair_difference(`$(v, \mathbf{F})$`, `$(v', \mathbf{F}')$`)` calculates minimal solutions of the integer linear system

$$\begin{bmatrix} F & -F' \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = v' - v$$

using `zsolve` in `4ti2`. The solutions form an ideal J of a polynomial ring in the proof of Theorem 2.4.1 on `Macaulay2`. Next, `standardPairs` derives standard pairs of J. Lastly, the method `pair_difference` constructs proper pairs based on the standard pairs of J, and classifies the proper pairs based on their faces and returns the pair difference.

2.4.2.2 Case 2: General ideal

[39, Proposition 4.4] gives an algorithm to find the standard cover of non-principal monomial ideals.

Proposition 2.4.3 ([39, Proposition 4.4])**.** *Let I be a monomial ideal in $\Bbbk[\mathbb{N}\mathbf{A}]$. There is an algorithm whose input is a cover of the standard monomials of I, and whose output is the standard cover of I.*

According to the proof of [39, Proposition 4.4], this is achieved by repeating the procedures below.

1. Input: C_0, an initial cover of I.

2. For each $(u, F) \in C_0$, find minimal solutions of $(u + \mathbb{R}F) \cap \mathbb{N}\mathbf{A}$ using the primitive integral support functions. (See [39, Lemma 4.2] for the detail.)

   - If $v_1, v_2, \cdots, v_m$ are minimal solutions of $(u + \mathbb{R}F) \cap \mathbb{N}\mathbf{A}$, construct pairs such as $(v_1, F), (v_2, F), \cdots, (v_m, F)$ and store them in the attribute C_1.

3. For each pair $(v, F) \in C_1$, construct (v, G) for any face G which is not strictly contained in F. If (v, G) is a proper pair of I, save (v, G) on the attribute C_2.

4. If C_0 is equal to C_2, done. Otherwise, set $C_0 := C_2$ and repeat the above process.

`_czero_to_cone(`C_0, I`)` method in the hidden module `_stdpairs` of `StdPairs` implements Number 2 to return C_1. It calls a method named `_minimal_holes(vector` u`, face F, affine semigroup A)` internally, which is the implementation of Lemma 4.2. `_cone_to_ctwo(`C_1, I`)` method implements Number 3 of the above item list. Since the constructor method of the class `ProperPair` checks whether the pair is proper or not, the method `_cone_to_ctwo(`C_1, I`)` tries to construct proper pairs as an attribute in `SageMath` and records it if it is successful.

Now we are ready to find the standard cover of a general ideal I whose minimal generators are $\langle v_1, \cdots, v_n \rangle$. One can find standard pairs as in [39, Theorem 4.5] described below.

1. Find the standard cover C of $\langle v_1 \rangle$ using pair difference.

2. For $i = 2$ to n:

   (a) For each pair (v, F) in C, replace it with elements of the pair difference of pairs (v, F) and (v_i, A). After this process C is a cover of an ideal $\langle v_1, v_2, \cdots, v_i \rangle$.

   (b) Using an algorithm of Proposition 2.4.3 find the standard cover C' of $\langle v_1, v_2, \cdots, v_i \rangle$.

   (c) Replace C with C'.

3. Return C.

The returned value C is now the standard cover of I.

`StdPairs` implements [39, Theorem 4.5] as a hidden method `_standard_pairs(`I`)`. This method has an input I whose type is `MonomialIdeal`. It returns a cover whose type is `dictionary`, classifying standard pairs by its face. For example, the code below shows that the standard cover of an ideal generated by $\begin{bmatrix} 2&2&2 \\ 0&1&2 \\ 2&2&2 \end{bmatrix}$ in an affine semigroup $\mathbb{N}\mathbf{A} = \mathbb{N}\begin{bmatrix} 0&1&1&0 \\ 0&0&1&1 \\ 1&1&1&1 \end{bmatrix}$ is

$$\left\{ \left(0, \begin{bmatrix} 0&0 \\ 0&1 \\ 1&1 \end{bmatrix}\right), \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0&0 \\ 0&1 \\ 1&1 \end{bmatrix}\right), \text{ and } \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0&0 \\ 0&1 \\ 1&1 \end{bmatrix}\right) \right\}.$$

```
sage: from stdpairs import *
sage: Q=AffineMonoid(matrix(ZZ,[[0,1,1,0],[0,0,1,1],[1,1,1,1]]))
sage: I=MonomialIdeal(matrix(ZZ,[[2,2,2],[0,1,2],[2,2,2]]),Q)
sage: I.standard_cover()
{(0, 3): [([[1], [0], [1]]^T,[[0, 0], [0, 1], [1, 1]]),
  ([[1], [1], [1]]^T,[[0, 0], [0, 1], [1, 1]]),
  ([[0], [0], [0]]^T,[[0, 0], [0, 1], [1, 1]])]}
```

2.4.3 Compatibility with `Normaliz` package in `SageMath` and `Macaulay2`

`Normaliz` is a package in `SageMath` and `Macaulay2` for finding Hilbert bases of rational cones and its normal affine monoid [13]. `StdPairs` has methods translating classes in Section 2.4.1 into objects in the `Normaliz` package. If an affine semigroup $\mathbb{N}\mathbf{A}$ is *normal*, i.e., $\mathbb{N}\mathbf{A} = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0} A$, then this translation works well. However, if it is not normal, then this translates $\mathbb{N}\mathbf{A}$ into its saturation described in Section 2.4.1.

For `SageMath`, one can have a polyhedron over $\mathbb{Z}$ with the `Normaliz` package in `SageMath` by adding an argument `True` on the constructor of `AffineMonoid`. For example, the code below gives an `AffineMonoid` class attribute Q whose attribute `Q.poly()` is a polyhedron over $\mathbb{Z}$ with `Normaliz`. Therefore, you can use all methods on `Normaliz` object. For example,

```
sage: from stdpairs import *
sage: Q=AffineMonoid(matrix(ZZ, [[0,1,1,0],[0,0,1,1],[1,1,1,1]]),
is_normaliz=True)
sage: Q.poly().hilbert_series([0,0,1])
(t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
```

For `Macaulay2`, `to_macaulay2( MonomialIdeal I)` returns a dictionary storing attribute of `Macaulay2` computations. This dictionary contains an affine semigroup ring, a list of generators of an ideal, and a list of standard pairs in `Macaulay2`. For example,

```
sage: from stdpairs import *
sage: Q=AffineMonoid(matrix(ZZ,[[0,1,1,0],[0,0,1,1],[1,1,1,1]]))
sage: I=MonomialIdeal(matrix(ZZ,[[2,2,2],[0,1,2],[2,2,2]]),Q)
sage: S=to_macaulay2(I)
sage: S
{'AffineSemigroupRing': ZZ[c, a*c, a*b*c, b*c]
```

```
 monomial subalgebra of PolyRing,
 'MonomialIdeal':   2 2   2   2   2 2 2
 {a c , a b*c , a b c }

 List,
 'StandardCover': {{1, {c, b*c}}, {a*c, {c, b*c}},
 {a*b*c, {c, b*c}}}

 List}
```

Moreover, `Macaulay2` objects `AffineSemigroupRing`, `MonomialSubalgebra`, and `list` of standard cover can be accessible via `macaulay.eval(string)` method with string `R`, `I`, and `SC`. For instance, the example below shows how to access such `Macaulay2` objects.

```
sage: macaulay2.eval('R')
ZZ[c, a*c, a*b*c, b*c]

monomial subalgebra of PolyRing
sage: macaulay2.eval('I')
  2 2   2   2   2 2 2
{a c , a b*c , a b c }

List
sage: macaulay2.eval('SC')
{{1, {c, b*c}}, {a*c, {c, b*c}},
{a*b*c, {c, b*c}}}

List
```

In `Macaulay2`, a type `MonomialSubalgebra` in the `Normaliz` package may correspond to an affine semigroup ring. Since `Normaliz` has no attributes for a monomial ideal of the type `MonomialSubalgebra`, the ideal is stored as a list of its generators. The standard cover of I is also sent to `Macaulay2` as a nested list, similar to the output of the method `standardPairs` in `Macaulay2`.

Conversely, `from_macaulay( Macaulay2` S`)` translates `monomialSubalgebra` object S of `Macaulay2` into an `AffineMonoid` object in `StdPairs`. For example,

```
sage: R = macaulay2.eval('ZZ[x,y,z]')
sage: macaulay2.needsPackage('"Normaliz"')
Normaliz
sage: macaulay2.eval('S=createMonomialSubalgebra
{x^2*y, x*z, z^3}')
    2         3
ZZ[x y, x*z, z ]


monomial subalgebra of ZZ[x..z]
sage: Q=from_macaulay2('S')
sage: Q
An affine semigroup whose generating set is
[[2 1 0]
 [1 0 0]
 [0 1 3]]
```

2.5 Degree space with degree pair topology

We now construct the *degree space* of M,

$$\bigcup \deg(M) := \bigcup_{F \in \mathcal{F}(Q)} \deg(M_F),$$

a topological space formed by gluing all degree pairs of localizations of a module. This structure enables us to simultaneously record all degrees resulting from localizations. Moreover, the minimal open sets of $\bigcup \deg(M)$, called *grains*, partition all degrees belonging to a fixed collection of localizations. A grain's *chaff* is the poset of all localizations that contain the given grain. Together with Subsection 1.2.5, these tools yield a Hochster-type formula for the Hilbert series of the local cohomology of M in Section 2.6.

Definition 2.5.1 (Degree space and degree pair topology)**.** The *degree space* $\bigcup \deg(M)$ of a (finitely generated) finely graded $\Bbbk[Q]$-module M is the union of $\deg(M_F)$ for all faces F of $\mathcal{F}(Q)$. The *degree pair topology* is the smallest topology on $\bigcup \deg(M)$ such that for any face $F \in \mathcal{F}(Q)$ and for an overlap class $[u, G \cup (-F)] \in \overline{\deg.\mathrm{p}}(M_F)$, the set $\bigcup[u, G \cup (-F)]$ is both open and closed.

Definition 2.5.2 (Grain and chaff)**.** In the degree pair topology, we refer to a minimal nonempty open set as a *grain* of $\bigcup \deg(M)$. Let $\mathcal{G}(M)$ be the set of all such grains. The *chaff* of a grain G, D_{G}, is defined as the collection of all localizations of M containing G.

These names are inspired by the agricultural metaphors of Grothendieck; we bundle degree pairs on $\bigcup \deg(M)$ and thresh (topologize) them in order to obtain grains. Chaff is a layer of grain that provides information about the grain's containment in certain localizations.

Remark 2.5.3. The sectors and sector partition introduced in [37] are almost the same the grains and chaff used in this article. The main differences are the topological context, and that grains actually refine the sector partition.

Lemma 2.5.4. *The degree pair topology has finitely many open sets.*

Proof. From Lemma 2.1.3, $\overline{\deg.\mathrm{p}}(M_F)$ is finite. Also, $\mathcal{F}(Q)$ is finite. Finally, a subbase including all overlap classes and their complements over all localizations is used to generate the topology. Hence the topology has finitely many open sets. $\square$

Our next result is that the grain set $\mathcal{G}(Q/T)$ partitions the degrees of a module.

Proposition 2.5.5. *$\mathcal{G}(M)$ partitions $\bigcup \deg(M)$ and is therefore a basis of the degree pair topology.*

Proof. It suffices to show that $\mathcal{G}(M)$ partitions $\bigcup \deg(M)$. First of all, for any two elements S and S' of $\mathcal{G}(M)$, $S \cap S' = \varnothing$. Otherwise, S and S' cannot be minimal nonempty opens, a contradiction. To see that $\mathcal{G}(M)$ covers $\bigcup \deg(M)$, suppose $u \in \deg(M_F)$ for some face F. Let

$$S := \left(\bigcap_{\substack{u \in [v,G] \in \overline{\deg.\mathrm{p}}(M_{F'}) \\ F' \in \mathcal{F}(Q)}} \left(\bigcup [v,G] \right) \right) \cap \left(\bigcap_{\substack{u \notin [v,G] \in \overline{\deg.\mathrm{p}}(M_{F'}) \\ F' \in \mathcal{F}(Q)}} \left(\bigcup [v,G] \right)^c \right).$$

This is a nonempty open set since $u \in S$. We claim that $S \in \mathcal{G}(M)$. Suppose not; then there exists an open set $S' \subsetneq S$. By the property of the subbase, we may let $S' \subseteq S \cap (\bigcup [v,G]) \subsetneq S$ or $S' \subseteq S \cap (\bigcup [v,G])^c \subsetneq S$ for some overlap class $[v,G]$. If $S' \subseteq S \cap (\bigcup [v,G]) \subsetneq S$ holds, then $(\bigcup [v,G])^c$ contains u, thus $(\bigcup [v,G])^c \cap S = S$ by the construction of S. This implies that $S' = \varnothing$. If $S' \subseteq S \cap (\bigcup [v,G])^c \subsetneq S$ holds, then $S \cap (\bigcup [v,G]) = S$ implies $S' = \varnothing$. In both cases, S' is empty, a contradiction. $\square$

Example 2.5.6 (Continuation of Example 2.1.16)**.**

1. Given $M = \Bbbk[\mathbb{N}\left[\begin{smallmatrix}1&1&1&1\\0&1&3&4\end{smallmatrix}\right]]/\langle xy \rangle$, $\bigcup \deg(M)$ is the union of integral points in $y = 4x, y = 4x-1, y = 4x-2, y = 0$ and $\{(2,2)\}$. Moreover, 10 grains

   (red) $\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right] + \mathbb{N}F_2, \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] + \mathbb{N}F_2, \left[\begin{smallmatrix}2\\6\end{smallmatrix}\right] + \mathbb{N}F_1$, (blue) $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \mathbb{N}F_1$,(cyan) $\left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right] + \mathbb{N}(-F_1)$,

   (orange) $\left[\begin{smallmatrix}-1\\4\end{smallmatrix}\right] + \mathbb{N}(-F_2), \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right] + \mathbb{N}(-F_2), \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + \mathbb{N}(-F_2)$,(yellow) $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, (green) $\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$

   are depicted in Figure 2.8a. Two grains with the same color have the same chaff. Indeed, for the given grain G with color from Figure 2.8a,

   (red) $D_{\mathsf{G}} := \{0, F_2\}$, (blue) $D_{\mathsf{G}} := \{0, F_1\}$, (cyan) $D_{\mathsf{G}} := \{F_1\}$,

   (orange) $D_{\mathsf{G}} := \{F_2\}$, (green) $D_{\mathsf{G}} := \{0\}$, (yellow) $D_{\mathsf{G}} := \{0, F_1, F_2\}$.

Note that $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ is a hole filled by the localization with respect to F_2, and therefore lies only in the degree pair of M_{F_2}.

2. Given $M = \Bbbk\left[\mathbb{N}\left[\begin{smallmatrix}0&1&1&0\\0&0&1&1\\1&1&1&1\end{smallmatrix}\right]\right] / \langle x^2z^2, x^2yz^2, x^2y^3z^3, x^3y^3z^3\rangle$, $\bigcup \deg(M)$ consists of the yz-plane ($x = 0$), its translation $x = 1$, and the point $(2, 2, 2)$. 10 grains

$$\text{(red) } \left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right] + \mathbb{N}F_4, \left(\left\{\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]\right\} + \mathbb{N}F_4\right), \text{(blue) } \left[\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right] + \mathbb{N}\left[\begin{smallmatrix}-u_1\\u_4\end{smallmatrix}\right]^t, \left[\begin{smallmatrix}1\\0\\-1\end{smallmatrix}\right] + \mathbb{N}\left[\begin{smallmatrix}-u_1\\u_4\end{smallmatrix}\right]^t, \text{(violet) } \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$$
$$\text{(cyan) } \left[\begin{smallmatrix}0\\-1\\-1\end{smallmatrix}\right] + \mathbb{N}\left[\begin{smallmatrix}u_1\\-u_4\end{smallmatrix}\right]^t, \left[\begin{smallmatrix}1\\-1\\-1\end{smallmatrix}\right] + \mathbb{N}\left[\begin{smallmatrix}u_1\\-u_4\end{smallmatrix}\right]^t, \text{(orange) } \left[\begin{smallmatrix}0\\-1\\-2\end{smallmatrix}\right] - \mathbb{N}F_4, \left[\begin{smallmatrix}1\\-1\\-2\end{smallmatrix}\right] - \mathbb{N}F_4, \text{(green) } \left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]$$

are depicted in $x = 1$ and $x = 0$ planes of Figure 2.8b except $\left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]$. Note that $\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$ is not a monomial of Q but that of $Q - \mathbb{N}u_1$ or $Q - \mathbb{N}u_4$, and therefore it lies in the intersection of two degree pairs that came from M_{u_1} and M_{u_4} respectively. For the given grain G with color as in the figures, the chaffs are as follows.

$$\text{(red) } D_{\mathsf{G}} := \{0, u_1, u_4, F_4\} \qquad \text{(blue) } D_{\mathsf{G}} := \{u_1, F_4\}, \qquad \text{(cyan) } D_{\mathsf{G}} := \{u_4, F_4\},$$
$$\text{(orange) } D_{\mathsf{G}} := \{F_4\}, \qquad \text{(green) } D_{\mathsf{G}} := \{0\}, \qquad \text{(violet) } D_{\mathsf{G}} := \{u_1, u_4, F_4\}.$$

2.6 Hochster-type formula of Hilbert series using grains

We derive a Hochster-type formula for the Hilbert series of the local cohomology of a (finitely generated) finely graded $\mathbb{Z}^d$-graded $\Bbbk[Q]$-module M. Quotients of affine semigroup rings by monomial ideals are an important example.

Definition 2.6.1 ([40, Definition 5.1])**.** Let M be a (finitely generated) finely graded graded $\Bbbk[Q]$-module. Given an element $u \in \mathbb{Z}^d$, let M_u be the degree-u graded piece of M. Let K be the transverse section of Q from Subsection 1.2.5. The degree-u *graded piece of K over M* is the subset of the face lattice of K given by

$$K_u := \{F \in \mathcal{F}(K) \mid u \in \deg(M_F)\}.$$

Likewise, denote $\widehat{K_u} := \{\widehat{F} \mid F \in K_u\}$ the degree-u *graded piece of the affine semigroup* Q *over* M. To align with the Ishida complex, the homological degree of the reduced chain complex of K_u must be shifted as follows:

$$\tilde{\mathcal{C}}\left(K_u\right) : 0 \longrightarrow C^d \xrightarrow{\partial} C^{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^0 \xrightarrow{\partial} 0, \quad C^k := \bigoplus_{\substack{F \in K_u \\ \dim F = k-1}} \Bbbk\, F$$

with the differential

$$\partial(F) := \sum_{\substack{G \in K_u \\ \dim G = k-1}} \epsilon(F, G) G \text{ for } \dim F = k,$$

where ϵ is an incidence function inherited from K and $\Bbbk\, F$ is a 1-dimensional $\Bbbk$-vector space having $\{F\}$ as a basis.

Lemma 2.6.2. *$\tilde{\mathcal{C}}\left(K_u\right)$ is well-defined and* $\left(L^\bullet \otimes_{\Bbbk[Q]} M\right) = \operatorname{Hom}_\Bbbk(\tilde{\mathcal{C}}\left(K_u\right), \Bbbk)$.

Proof. For any $k \in \mathbb{N}$,

$$\left(L^k \underset{\Bbbk[Q]}{\otimes} M\right)_u = \left(\bigoplus_{F \in \mathcal{F}(K)^{k-1}} M_{\widehat{F}}\right)_u = \bigoplus_{\substack{F \in \mathcal{F}(K)^{k-1} \\ M_{\widehat{F}} \neq 0}} \left(M_{\widehat{F}}\right)_u \cong \bigoplus_{\substack{F \in \mathcal{F}(K)^{k-1} \\ u \in \deg(M_{\widehat{F}})}} \Bbbk\, F = \bigoplus_{\substack{F \in K_u \\ \dim F = k-1}} \Bbbk\, F,$$

which is equal to C^k. Apply the functor $\operatorname{Hom}_\Bbbk(-, \Bbbk)$ to obtain $\tilde{\mathcal{C}}\left(K_u\right)$. Since differentials in $L^\bullet \otimes_{\Bbbk[Q]} M$ are $\Bbbk$-linear, their images under $\operatorname{Hom}_\Bbbk(-, \Bbbk)$ agree with differentials in $\tilde{\mathcal{C}}\left(K_u\right)$. □

Furhtermore, if two elements of $\mathbb{Z}^d$ are in the same grain, their graded pieces of the Ishida complex coincide.

Lemma 2.6.3. *For any $u \in \mathsf{G} \in \mathcal{G}\left(M\right)$, $\widehat{K_u} = D_\mathsf{G}$. Thus, $\tilde{\mathcal{C}}\left(K_u\right) = \tilde{\mathcal{C}}\left(K_v\right)$ if $u, v \in \mathsf{G}$. If there is no grain containing u, then $\tilde{\mathcal{C}}\left(K_u\right) = 0$.*

Proof. By Proposition 2.5.5, if there is no grain containing u, then $u \notin \deg(M_F)$ for any $F \in \mathcal{F}(Q)$, so $\tilde{\mathcal{C}}\left(K_u\right) = 0$. $\widehat{K_u} = D_\mathsf{G}$ is clear from the definition of chaff. □

As a consequence of the previous result, we may use the notation $\tilde{\mathcal{C}}\left(K_\mathsf{G}\right) := \tilde{\mathcal{C}}\left(K_u\right)$ for the grain G containing u. $\tilde{\mathcal{C}}\left(K_\mathsf{G}\right)$ coincides with the chain complex of $K_\mathsf{G} = D_\mathsf{G}$. Since $\mathcal{G}(M)$ is finite, according to Lemma 2.5.4, the Hilbert series of the local cohomology of M is a finite sum over cohomologies of chaffs as follows.

Theorem 2.6.4 (Hochster-type formula for the Ishida complex)**.** *The multi-graded Hilbert series for the local cohomology of a graded module M with support at the maximal ideal $\mathfrak{m}$ is*

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), \mathbf{t}) = \sum_{\mathsf{G} \in \mathcal{G}(M)} \dim_{\Bbbk} H^i(\operatorname{Hom}_{\Bbbk}(\tilde{\mathcal{C}}\left(K_{\mathsf{G}}\right), \Bbbk)) \sum_{u \in \mathsf{G}} \mathbf{t}^u.$$

Proof.

$$\begin{aligned}
\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), \mathbf{t}) &= \sum_{u \in \mathbb{Z}^d} \dim_{\Bbbk} \left(H^i_{\mathfrak{m}}(M)\right)_u \mathbf{t}^u = \sum_{u \in \mathbb{Z}^d} \dim_{\Bbbk} \left(H^i(L^\bullet \underset{\Bbbk[Q]}{\otimes} M)\right)_u \mathbf{t}^u \\
&= \sum_{u \in \bigcup \deg(M)} \dim_{\Bbbk} H^i(\operatorname{Hom}_{\Bbbk}(\tilde{\mathcal{C}}\left(K_u\right), \Bbbk))\mathbf{t}^u \qquad \text{(Lemma 2.6.2)} \\
&= \sum_{\mathsf{G} \in \mathcal{G}(M)} \sum_{u \in \mathsf{G}} \dim_{\Bbbk} H^i(\operatorname{Hom}_{\Bbbk}(\tilde{\mathcal{C}}\left(K_{\mathsf{G}}\right), \Bbbk))\mathbf{t}^u \quad \text{(Lemma 2.6.3)} \\
&= \sum_{\mathsf{G} \in \mathcal{G}(M)} \dim_{\Bbbk} H^i(\operatorname{Hom}_{\Bbbk}(\tilde{\mathcal{C}}\left(K_{\mathsf{G}}\right), \Bbbk)) \left(\sum_{u \in \mathsf{G}} \mathbf{t}^u\right)
\end{aligned}$$

□

The Hilbert series in Theorem 2.6.4 is a finite sum involving generating functions of lattice points in polyhedra. To conclude these generating functions are rational, the underlying cone must be pointed [2, 3].

Corollary 2.6.5. *If Q is pointed, the Hilbert series in Theorem 2.6.4 can be expressed as a (formal) sum of rational functions.*

In the non-pointed case, Hochster-type formulas are not necessarily given by rational functions. For example, the Hilbert series of the Laurent polynomial ring $\Bbbk[x, x^{-1}]$ cannot be expressed as a rational function, since $\frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0$, which is different from the formal sum $\sum_{i \in \mathbb{Z}} x^i$.

Vanishing of local cohomology is a standard way to detect whether a ring is Cohen–Macaulay.

Theorem 2.6.6 (Combinatorial Cohen–Macaulay criterion)**.** *Given a pointed affine semigroup ring $\Bbbk[Q]$ and a monomial ideal I, $\Bbbk[Q]/I$ is Cohen–Macaulay ring if and only if every chaff of grains in $\mathcal{G}(\Bbbk[Q]/I)$ is either acyclic or (-1)-dimensional in homological index $l := \dim \Bbbk[Q]/I$.*

Proof. According to Theorem 2.6.4, $\mathrm{Hilb}(H^i_{\mathfrak{m}}(\Bbbk[Q]/I), \mathbf{t})$ is zero for all i except $i = l$ if and only if all chaffs are either acyclic or (-1)-dimensional at index l. Thus, $\Bbbk[Q]/I$ is a Cohen–Macaulay module over $\Bbbk[Q]$ if and only if all chaffs are either acyclic or (-1)-dimensional at index l. Since the maximal ideal of $\Bbbk[Q]/I$ is the image of $\mathfrak{m}$, it is a Cohen–Macaulay ring if and only if it is a Cohen–Macaulay module over $\Bbbk[Q]$ by [11, Theorem 3.5.7]. Finally, apply argument in Corollary 1.2.76 and 1.2.77 using the fact that $\Bbbk[Q]$ is a $*$-local ring with a unique homogeneous maximal ideal. More precisely, the given fact implies that $H^\bullet_{\mathfrak{m}}(M) \cong H^\bullet_{\mathfrak{m}\Bbbk[Q]_{\mathfrak{m}}}(M_{\mathfrak{m}})$ and $M_{\mathfrak{m}}$ is Cohen–Macaulay if and only if $H^\bullet_{\mathfrak{m}\Bbbk[Q]_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is nonzero when $\bullet = \dim M$ for any $\Bbbk[Q]$-module M [11, Remark 3.6.18, Theorem 3.5.7]. $\square$

Example 2.6.7 (Continuation of Example 1.2.70)**.**

1. As illustrated in Example 2.5.6(1), $Q/I = \mathbb{N}\left[\begin{smallmatrix}1&1&1&1\\0&1&3&4\end{smallmatrix}\right] / \langle\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\rangle$ has six distinct colored chaffs. We take the unions of grains of the same color and color-code these unions. The following table summarizes their rational generating functions.

$$\text{(red) } f_{\mathrm{r}} := \frac{xy^4 + xy^3 + x^2y^6}{1 - xy^4} \qquad \text{(blue) } f_{\mathrm{b}} := \frac{x}{1-x}, \qquad \text{(cyan) } f_{\mathrm{c}} := \frac{1}{x-1},$$
$$\text{(orange) } f_{\mathrm{o}} := \frac{1 + xy^3 + x^2y^6}{xy^4 - 1}, \qquad \text{(green) } f_{\mathrm{g}} := x^2y^3, \qquad \text{(yellow) } f_{\mathrm{y}} := 1$$

where $x := \mathbf{t}^{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$ and $y := \mathbf{t}^{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}$. Thus, $\tilde{\mathcal{C}}(K_G)$ is a member of one of three chain complexes below.

$$\tilde{\mathcal{C}}(K_y) : 0 \to \Bbbk \to \Bbbk^2 \to 0 \qquad \tilde{\mathcal{C}}(K_r), \tilde{\mathcal{C}}(K_b) : 0 \to \Bbbk \to \Bbbk \to 0$$
$$\tilde{\mathcal{C}}(K_g) : 0 \to \Bbbk \to 0 \to 0 \qquad \tilde{\mathcal{C}}(K_c), \tilde{\mathcal{C}}(K_o) : 0 \to 0 \to \Bbbk \to 0$$

As a result,

$$\mathrm{Hilb}(H^0_{\mathfrak{m}}(S), \{x, y\}) = f_g \qquad \mathrm{Hilb}(H^1_{\mathfrak{m}}(S), \{x, y\}) = f_y + f_c + f_o.$$

2. As illustrated in Example 2.5.6(2), $Q/I = \mathbb{N}\left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right] / \left\langle \left[\begin{smallmatrix} 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{smallmatrix}\right] \right\rangle$ has six distinct colored chaffs. As before, we take unions of grains of the same color and index these unions according to their color. Their rational generating functions are as follows:

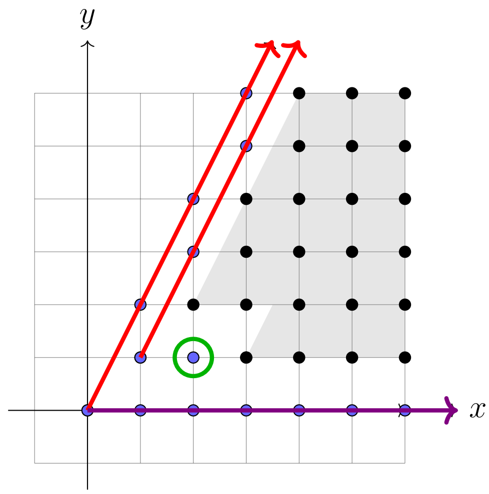
$$\text{(red) } f_r := \frac{1+x}{(1-z)(1-yz)} - x, \quad \text{(blue) } f_b := \frac{1+x}{(z-1)(1-yz)}, \quad \text{(green) } f_g := (xyz)^2,$$
$$\text{(orange) } f_o := \frac{1+x}{(z-1)(yz-1)}, \quad \text{(cyan) } f_c := \frac{1+x}{(1-z)(yz-1)}, \quad \text{(violet) } f_v := x$$

where $x := \mathbf{t}^{\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]}$, $y := \mathbf{t}^{\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]}$, and $z := \mathbf{t}^{\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]}$. We may classify $\tilde{\mathcal{C}}(K_G)$ as follows:

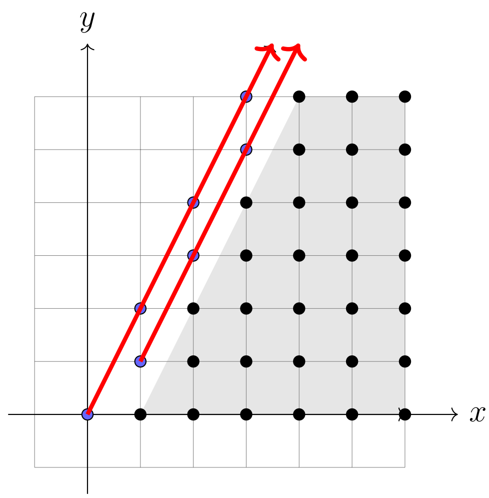
$$\tilde{\mathcal{C}}(K_r) : 0 \to \Bbbk \to \Bbbk^2 \to \Bbbk \to 0 \qquad \tilde{\mathcal{C}}(K_b), \tilde{\mathcal{C}}(K_c) : 0 \to 0 \to \Bbbk \to \Bbbk \to 0$$
$$\tilde{\mathcal{C}}(K_g) : 0 \to \Bbbk \to 0 \to 0 \to 0 \qquad \tilde{\mathcal{C}}(K_v) : 0 \to 0 \to \Bbbk^2 \to \Bbbk \to 0$$
$$\tilde{\mathcal{C}}(K_o) : 0 \to 0 \to 0 \to \Bbbk \to 0$$

Hence,

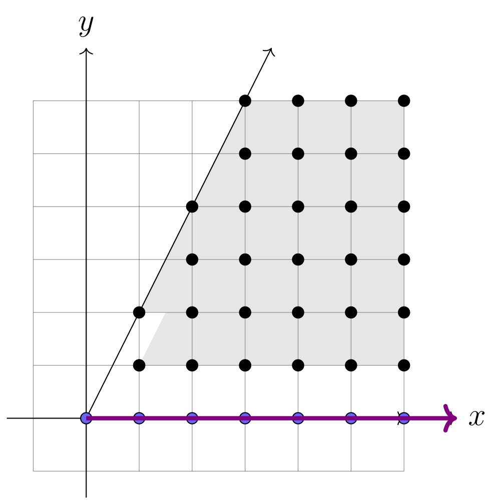
$$\mathrm{Hilb}(H^0_{\mathfrak{m}}(S), \{x, y\}) = f_g \qquad \mathrm{Hilb}(H^1_{\mathfrak{m}}(S), \{x, y\}) = f_v \qquad \mathrm{Hilb}(H^2_{\mathfrak{m}}(S), \{x, y\}) = f_o.$$



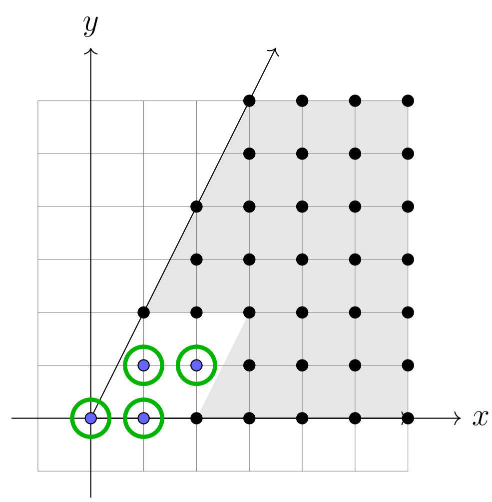
(a) degree pairs of $\langle x^2y^2, x^3y\rangle$



(b) degree pairs of $C_G = \langle x\rangle$

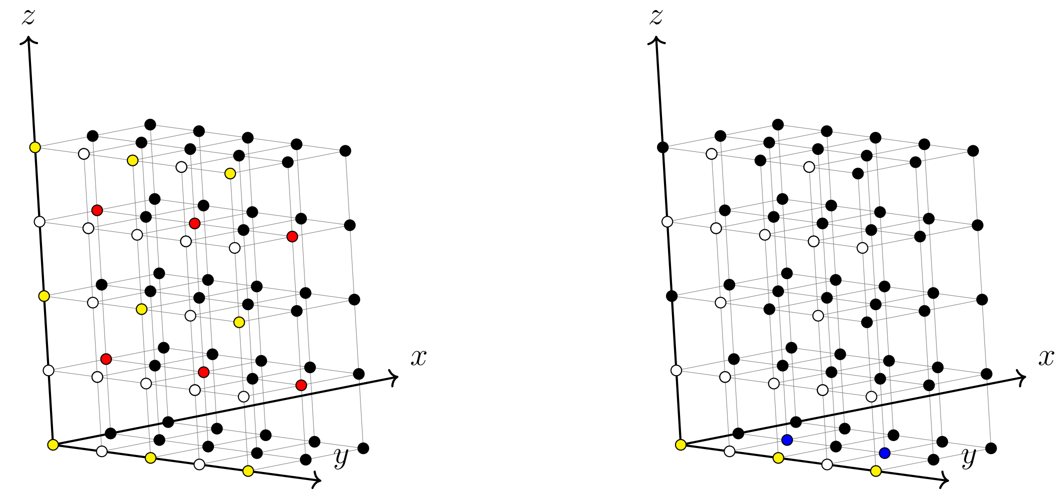


(c) degree pairs of $C_F = \langle xy, xy^2\rangle$



(d) degree pairs of $C_O = \langle x^2, xy^2\rangle$

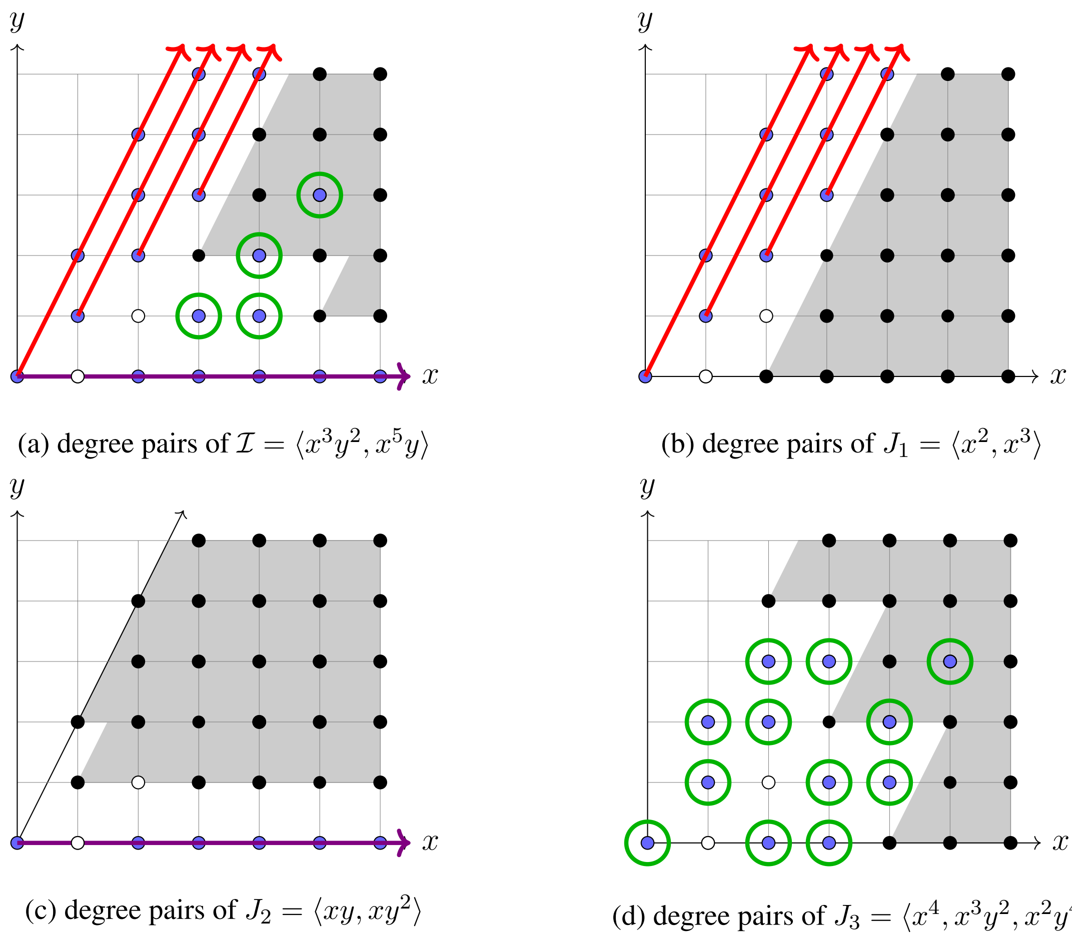
Figure 2.4: A primary decomposition of $\langle x^2y^2, x^3y\rangle$ in $\Bbbk[x, xy, xy^2]$



(a) $C_F = \langle x, xy, xyz, xyz^2 \rangle$

(b) $C_G = \langle x, xz, z^2, xyz \rangle$

Figure 2.5: A primary decomposition of $\langle x, xyz, xyz^2 \rangle$ in $\Bbbk\left[x, xy, xz, xyz, y^2, z^2\right]$



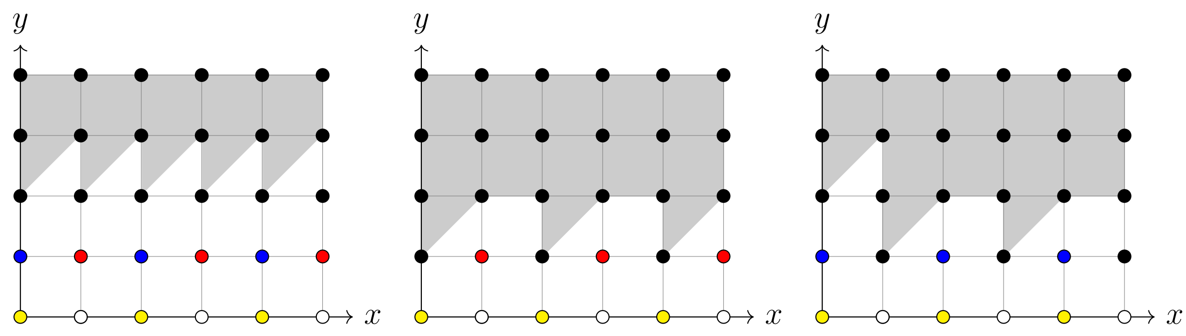
(a) degree pairs of $\mathcal{I} = \langle x^3y^2, x^5y \rangle$

(b) degree pairs of $J_1 = \langle x^2, x^3 \rangle$

(c) degree pairs of $J_2 = \langle xy, xy^2 \rangle$

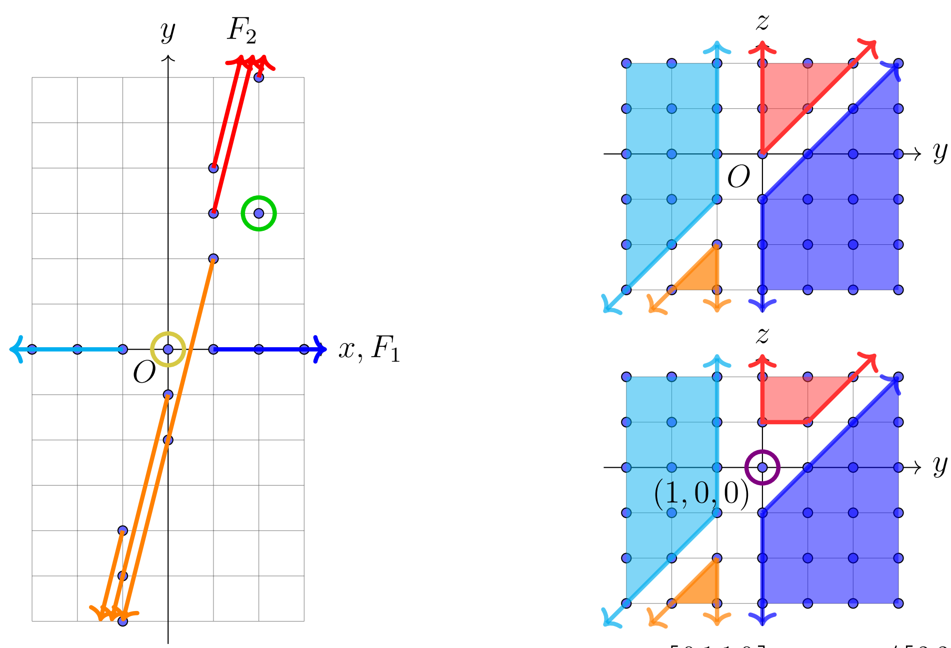
(d) degree pairs of $J_3 = \langle x^4, x^3y^2, x^2y^4 \rangle$

Figure 2.6: An irredundant irreducible decomposition of $\mathcal{I} = J_1 \cap J_2 \cap J_3$ in $\Bbbk[xy, xy^2, x^2, x^3]$.



(a) degree pairs of $\mathcal{I} = \langle y^2, xy^2 \rangle$ (b) degree pairs of $J_1 = \langle y \rangle$ (c) degree pairs of $J_2 = \langle xy, y^2 \rangle$

Figure 2.7: An irredundant irreducible decomposition of $\mathcal{I} = J_1 \cap J_2$ in $\Bbbk[x^2, y, xy]$.



(a) $Q = \mathbb{N}\left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{smallmatrix}\right]$ and $I = \langle \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \rangle$ (b) $Q = \mathbb{N}\left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right]$ and $I = \left\langle \left[\begin{smallmatrix} 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{smallmatrix}\right] \right\rangle$

Figure 2.8: $\bigcup \deg(Q/T)$ with grains

3. GENERALIZED ISHIDA COMPLEX

Suppose $\Bbbk$ is algebraically closed. In this chapter, we generalize the Ishida complex to compute the local cohomology of the quotient of a polynomial ring by a lattice ideal or ζ-cellular binomial ideal I by choosing a specific minimal prime toric ideal and its corresponding affine semigroup Q. Indeed, one may grade $\Bbbk[x]/I$ using the contraction map to the quotient $\Bbbk[\mathbb{N}A] = \Bbbk[Q]$ generated by the minimal prime toric ideal. In this case, if we choose the supporting monomial ideal as a extension of a radical monomial ideal of $\Bbbk[Q]$, then one can calculate local cohomology supported at the chosen ideal using generalized Ishida complex, which we will introduce below.

3.1 Generalized Ishida complex

Definition 3.1.1 (A-grading)**.** Given a lattice ideal I (resp. ζ-cellular binomial ideal I), pick a minimal associated prime ideal J of I (resp. of $I \cap \Bbbk[\mathbb{N}^\zeta]$). Since $\Bbbk$ is algebraically closed, J is a binomial prime ideal, and the quotient $\Bbbk[\mathbb{N}^\zeta]/J$ is isomorphic (by rescaling the variables) to an affine semigroup ring $\Bbbk[Q]$ with $Q = \mathbb{N}A$. The natural projection map $\Bbbk[x]/I \to \Bbbk[x]/J \cong \Bbbk[Q]$ (resp. $\Bbbk[x]/I \to \Bbbk[x]/(J + \langle x_i : i \in \zeta^c\rangle) \cong \Bbbk[Q]$) induces A-*grading* on $\Bbbk[x]/I$; in other words, for any monomial $\overline{x^u} \in \Bbbk[x]/I$, the A-*degree* of $\overline{x^u}$ is $\deg_A(\overline{x^u}) = A \cdot u$ (resp. $A \cdot u^\zeta$, where $u^\zeta := (u_i \in u; i \in \zeta)$).

We specify the A-grading on $\Bbbk[x]/I$ using the triple (I, J, A) unless the minimal prime J or the generators of affine semigroup $\mathbb{N}A$ are understood in context.

Definition 3.1.2. Let I_Δ be the radical monomial ideal of $\Bbbk[Q]$ associated to a subcomplex $\Delta \subset \mathcal{F}(Q)$. Then, $\sqrt{I_\Delta \cdot \Bbbk[x]/I}$ denotes the contraction of I_Δ via $\Bbbk[x]/I \to \Bbbk[x]/J \cong \Bbbk[Q]$.

To compute the local cohomology of a $\Bbbk[x]/I$-module supported on $\sqrt{I_\Delta \cdot \Bbbk[x]/I}$, we construct a (generalized) Ishida complex below.

Definition 3.1.3 (Generalized Ishida complex)**.** Let K_{I_Δ} be a transverse section of the polyhedron $\mathbb{R}_{\geq 0}\{u \in \mathbb{Z}^\zeta \mid x^u \in I_\Delta\}$ with the canonical isomorphism $\widehat{-} : \mathcal{F}(K_{I_\Delta}) \to \mathcal{F}(Q)$ where $\widehat{F}$ is the minimal face of Q such that $\mathbb{R}_{\geq 0}\widehat{F} \supseteq \mathbb{R}_{\geq 0}F$. The set of all k-dimensional faces in $\mathcal{F}(K_{I_\Delta})$ is denoted by $\mathcal{F}(K_{I_\Delta})^k$. Also, $(\Bbbk[x]/I)_{\widehat{F}}$ refers to the localization of $\Bbbk[x]/I$ by the multiplicative set consisting of all monomials in $\Bbbk[\mathbb{N}^\zeta]$ whose A-graded degrees are in $\mathbb{N}\widehat{F}$.

Let $L^\bullet$ be the chain complex

$$L^\bullet : 0 \longrightarrow L^0 \xrightarrow{\partial} L^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L^d \xrightarrow{\partial} 0, \quad L^k := \bigoplus_{F \in \mathcal{F}(K_{I_\Delta})^{k-1}} (\Bbbk[x]/I)_{\widehat{F}}$$

where the differential $\partial : L^k \to L^{k+1}$ is induced by a componentwise map $\partial_{F,G}$ with two faces $F \in \mathcal{F}(K_{I_\Delta})^{k-1}$, $G \in \mathcal{F}(K_{I_\Delta})^k$ such that

$$\partial_{F,G} : (\Bbbk[x]/I)_{\widehat{F}} \to (\Bbbk[x]/I)_{\widehat{G}} \text{ to be } \begin{cases} 0 & \text{if } F \not\subset G \\ \epsilon(F,G) \cdot \mathrm{nat} & \text{if } F \subset G \end{cases}$$

with nat, the canonical injection $(\Bbbk[x]/I)_{\widehat{F}} \to (\Bbbk[x]/I)_{\widehat{G}}$ when $F \subseteq G$. We say that $L^\bullet \otimes_{\Bbbk[x]/I} M$ is the *Ishida complex of a* $(\Bbbk[x]/I)$*-module* M *supported at the radical monomial ideal* $\sqrt{I_\Delta \cdot \Bbbk[x]/I}$.

The following theorem is the main result in this section. The proof is adapted from [11]. The key ingredients areLemma 3.1.5 andLemma 3.1.6 which are given later.

Theorem 3.1.4. *For any* $\Bbbk[x]/I$*-module* M*, and all* $k \geq 0$*,*

$$H^k_{I_\Delta \cdot \Bbbk[x]/I}(M) \cong H^k_{\sqrt{I_\Delta \cdot \Bbbk[x]/I}}(M) \cong H^k(L^\bullet \underset{\Bbbk[x]/I}{\otimes} M).$$

The first step in our proof is to verify that the zeroth homology of the generalized Ishida complex computes torsion.

Proof. This follows from Lemma 3.1.5 and Lemma 3.1.6 together with the fact that all the summands of the components of the Ishida complex are flat, thus $- \otimes_{\Bbbk[x]/I} L^\bullet$ is an exact functor. $\square$

Lemma 3.1.5. $H^0(L \otimes_{\Bbbk[x]/I} M) \cong (0 :_M I_\Delta^\infty)$.

Proof. It suffices to show that

$$\sqrt{\left\langle \bigcup_{F\in\mathcal{F}(K_{I_\Delta})^0} \{\overline{x^u} \in \Bbbk[\mathbb{N}^\zeta]/(I\cap\Bbbk[\mathbb{N}^\zeta]) : \deg_A(u) \in \mathrm{RelInt}(\mathbb{N}\widehat{F})\} \right\rangle} = \sqrt{I_\Delta \cdot \Bbbk[x]/I}.$$

The generators of the left hand-side ideal might be not the same as those of the multiplicative sets inducing localization of components in L^1 when I is not toric. However, they admit the same radical ideal in $\Bbbk[x]/I$.

First, let $\overline{x^u} \in \Bbbk[x]/I$ be an element whose A-degree is in $\mathbb{N}\widehat{F}$ for a vertex F of K_{I_Δ}. The canonical map $\Bbbk[x]/I \to \Bbbk[x]/J \cong \Bbbk[\mathbb{N}A]$ sends $\overline{x^u}$ to $x^{\deg_A(u)} \in \Bbbk[\mathbb{N}A]$ where $\deg_A(u) \in \mathrm{RelInt}(\mathbb{N}\widehat{F})$. Since $\widehat{F} \notin \Delta$, $x^{\deg_A(u)} \in I_\Delta$, we have that $\overline{x^u} \in \sqrt{I_\Delta \cdot \Bbbk[x]/I}$ by the correspondence between polyhedral subcomplexes and the radical monomial ideals.

Conversely, let $f \in \Bbbk[\mathbb{N}^\zeta]$ be a preimage of a monomial in $I_\Delta \subset \Bbbk[\mathbb{N}A]$ and $g \in \Bbbk[x]/I$ be an A-homogeneous element of $\Bbbk[x]/I$. Then, $f = \overline{x^u}$ for some $u \in \mathbb{N}^\zeta$ such that $Au \in \mathbb{N}\overline{F}$ for some $F \in \mathcal{F}(K_{I_\Delta})$. If $\dim F = 0$, fg is in the left hand-side of the equation. Suppose $\dim F > 0$; then F has vertices $\{v_1, \ldots, v_m\}$. Then, u is a linear combination of elements of $\mathbb{N}\widehat{v_i}$ over $\mathbb{Q}$, say $u = c_1u_1 + \cdots + c_mu_m$ where $u_i \in \mathbb{N}\widehat{v_i}$ and $c_i \in \mathbb{Q}$. Multiplying u by a suitable number N, we may assume that $c_i \in \mathbb{N}$. Then, $f^N = \prod_{i=1}^m (f_i)^{c_i}$ where $f_i = \overline{x^{u_i}}$, which implies that f^Ng^N is in the left hand-side, thus fg is in the left hand-side. □

To complete the proof of our main result, we need to check that the generalized Ishida complex is exact on injectives. This is stated in the following lemma, which requires three auxiliary results.

Lemma 3.1.6. *If M is an injective $\Bbbk[x]/I$-module, then $L^\bullet \otimes_{\Bbbk[x]/I} M$ is exact.*

Proof. It suffices to check the case when M is an injective indecomposable module $E(\Bbbk[x]/P)$ over a prime ideal P containing I. Let A_i be the i-th column of A. The set $F := \{A_i \mid x_iE(\Bbbk[x]/P) \cong E(\Bbbk[x]/P)\}$ is called the *face corresponding to* P. Lemma 3.1.7 shows that this is indeed a face of $\mathbb{N}A$. Lemma 3.1.9 shows that $L \otimes_{\Bbbk[x]/I} E(\Bbbk[x]/P)$ is exact using Lemma 3.1.8.

□

We start verifying our proposed face is a face.

Lemma 3.1.7. *Given a monomial prime ideal P containing I, $F' := \{A_i \mid x_i E(\Bbbk[x]/P) \cong E(\Bbbk[x]/P)\}$ is a face of $\mathbb{N}A$.*

Proof. Since $\mathrm{Ass}(E(\Bbbk[x]/P)) = \{P\}$ and the module is indecomposable, $x_i E(\Bbbk[x]/P)$ is either 0 if $x_i \in P$ or $E(\Bbbk[x]/P)$ if $x_i \notin P$. Now suppose that the given set F' is not a face; then there exists a minimal face F whose relative interior intersects with the relative interior of F'. Pick $A_j \in F \setminus F'$. Then the corresponding variable x_j induces the zero map on $E(\Bbbk[x]/P)$. If A_j is not in the relative interior of F, let $f \in \mathrm{RelInt}(\mathbb{N}F)$ so that $f = \sum_{A_i \in F'} c_i A_i$ for some $c_i \in \mathbb{N}$. Choose a suitable $N_1 \in \mathbb{N}$ such that $N_1 f = \sum_{A_i \in F'} c_i' A_i + dA_j$ for some nonnegative $c_i' \in \mathbb{Q}$ and $d \in \mathbb{Q}_{>0}$. By construction, $N_1 f$ is in the lattice (L_ρ, ρ). Thus, there exists $N_2 > N_1$ such that $N_2 f = \sum_{A_i \in F'} d_i A_i + d' A_j$ where $d_i, d \in \mathbb{N}$, $d > 0$. But then $x^{N_2 f} E(\Bbbk[x]/P) = 0$, which is a contradiction. If A_j is in the relative interior of F, a similar argument gives another contradiction. $\square$

The following is necessary to prove exactness.

Lemma 3.1.8. *Given a face $F \in \mathcal{F}(\mathbb{N}A)$, $K_{I_\Delta}^{\cap F} := K_{I_\Delta} \cap \mathbb{R}_{\geq 0} F$ is a face of K_{I_Δ}.*

Proof. If $F = \mathbb{N}A$, then the statement is clear. Assume $\dim F < \dim \mathbb{N}A$. First, we claim that for any $G \in \mathcal{F}(K_{I_\Delta})$, $G \subseteq \mathbb{R}_{\geq 0} F$ if and only if $\widehat{G} \subseteq F$. One direction follows straight from the definition of $\widehat{G}$. Conversely, assume that $\widehat{G} \not\subseteq F$. Then, $\mathrm{RelInt}(\mathbb{R}_{\geq 0}\widehat{G}) \cap \mathbb{R}_{\geq 0} F = \emptyset$ implies $\mathrm{RelInt}(G) \cap \mathbb{R}_{\geq 0} F = \emptyset$. Therefore $G \not\subseteq \mathbb{R}_{\geq 0} F$. This claim shows that $K_{I_\Delta}^{\cap F}$ is the union of all faces $G \in \mathcal{F}(K_{I_\Delta})$ such that $\widehat{G} \subseteq F$. Thus $K_{I_\Delta}^{\cap F}$ can be regarded as a realized subcomplex of $\mathcal{F}(K_{I_\Delta}^{\cap F}) := \{G \in \mathcal{F}(K_{I_\Delta}) : \widehat{G} \subseteq F\}$.

Next, we claim that $\mathcal{F}(K_{I_\Delta}^{\cap F})$ has a unique maximal element. Suppose not; let G_1 and G_2 be two distinct faces of $\mathcal{F}(K_{I_\Delta}^{\cap F})$ of maximal dimension. Then, $F \supseteq \widehat{G_1} + \widehat{G_2}$ implies that there is a face $\widehat{G_1} \vee \widehat{G_2} \in \mathcal{F}(\mathbb{N}A)$ such that $F \supseteq \widehat{G_1} \vee \widehat{G_2}$, the join of the two faces. Since G_1 and G_2 are distinct and the same dimension, $G_1 \vee G_2 \neq G_1$ or G_2. Therefore $G_1 \vee G_2 \supsetneq G_1 \cup G_2$. Hence, $\mathbb{R}_{\geq 0}(\widehat{G_1} \vee \widehat{G_2}) \supseteq \mathbb{R}_{\geq 0}(\widehat{G_1 \vee G_2})$, which implies $F \supseteq \widehat{G_1} \vee \widehat{G_2} \supseteq \widehat{G_1 \vee G_2}$. Hence,

$G_1 \vee G_2 \subseteq \mathbb{R}_{\geq 0}F$, therefore $G_1 \vee G_2 \in \mathcal{F}(K_{I_\Delta}^{\cap F})$, contradicting the maximality of G_1 and G_2. We conclude $\mathcal{F}(K_{I_\Delta}^{\cap F})$ has a unique maximal element, say H.

Lastly, we claim that $\mathcal{F}(H) = \mathcal{F}(K_{I_\Delta}^{\cap F})$, which implies $K_{I_\Delta}^{\cap F} = H$. For any $G \in \mathcal{F}(K_{I_\Delta}^{\cap F})$, let $G' := G \vee H$ in $\mathcal{F}(K_{I_\Delta})$. Then, $\emptyset \neq \operatorname{RelInt}(\widehat{G'}) \cap \mathbb{R}_{\geq 0}(\widehat{G} \cup \widehat{H}) \subseteq \operatorname{RelInt}(\widehat{G'}) \cap \mathbb{R}_{\geq 0}F \implies \widehat{G'} \subseteq F \implies G' \in \mathcal{F}(K_{I_\Delta}^{\cap F})$. By the maximality of H, $G' = H$, which implies $G \subseteq H$. $\square$

Lemma 3.1.9. *Given a monomial prime ideal P whose corresponding face is F, for $k \geq 1$,*

$$L^k \underset{\Bbbk[x]/I}{\otimes} E(\Bbbk[x]/P) = \bigoplus_{G \in \mathcal{F}\left(K_{I_\Delta}^{\cap F}\right)^{k-1}} E(\Bbbk[x]/P) \cong \operatorname{Hom}_{\mathbb{Z}}\left(\tilde{\mathcal{C}}\left(K_{I_\Delta}^{\cap F}\right)(-1), E\left(\Bbbk[x]/P\right)\right)$$

where $\tilde{\mathcal{C}}\left(K_{I_\Delta}^{\cap F}\right)$ is the reduced chain complex of $K_{I_\Delta}^{\cap F}$ as a CW complex.

Proof. Lemma 3.1.7 shows that for any $F, G \in \mathcal{F}(\mathbb{N}A)$,

$$E(\Bbbk[x]/P) \underset{\Bbbk[x]/I}{\otimes} (\Bbbk[x]/I)_G = \begin{cases} 0 & \text{if } G \not\subseteq F \\ E(\Bbbk[x]/P) & \text{if } G \subseteq F \end{cases}.$$

If F is not the image of a face in K_{I_Δ}, then no sub-face of F is the image of a face of K_{I_Δ}. Otherwise, there is a face $G \subseteq F$ containing an unbounded face of $\mathbb{R}_{\geq 0}\{u \in \mathbb{Z}^\zeta \mid x^u \in I_\Delta\}$. Then by the correspondence between radical monomial ideals and subcomplexes of $\mathcal{F}(Q)$, $\operatorname{RelInt}(F)$ contains an element of an unbounded face of $\mathbb{R}_{\geq 0}\{u \in \mathbb{Z}^\zeta \mid x^u \in I_\Delta\}$, a contradiction. Therefore, no images of faces are subsets of F. This implies that $K_{I_\Delta}^{\cap F} = 0$ and $L^k \otimes_{\Bbbk[\mathbb{N}A]} E(\Bbbk[x]/P) = 0$ for $k \geq 1$.

Otherwise, $F := \widehat{F'}$ for some $F' \in \mathcal{F}(K_{I_\Delta})$. Since $\widehat{G'} \subseteq F$ if and only if $G' \in \mathcal{F}(K_{I_\Delta}^{\cap F})$ byLemma 3.1.8, so the first equality holds.

Now observe that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, E(\Bbbk[x]/P)) \cong E(\Bbbk[x]/P)$ as a $\Bbbk[\mathbb{N}A]$-module. Thus,

$$L^\bullet \underset{\Bbbk[\mathbb{N}A]}{\otimes} E(\Bbbk[x]/P) = \bigoplus_{G\in\mathcal{F}(K^{\cap F}_{I_\Delta})^{\bullet-1}} E(\Bbbk[x]/P) \cong \bigoplus_{G\in\mathcal{F}(K^{\cap F}_{I_\Delta})^{\bullet-1}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, E(\Bbbk[x]/P))$$
$$\cong \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{G\in\mathcal{F}(K^{\cap F}_{I_\Delta})^{\bullet-1}} \mathbb{Z}, E(\Bbbk[x]/P)\right) = \mathrm{Hom}_{\mathbb{Z}}\left(\tilde{\mathcal{C}}\left(K^{\cap F}_{I_\Delta}\right)(-1), E(\Bbbk[x]/P)\right).$$

□

3.2 Local cohomology with monomial support for cellular binomial ideals

In this section we express the Hilbert series of the local cohomology with monomial support of $\Bbbk[x]/I$ as a (formal) finite sum of rational functions when I is a lattice ideal (Theorem 3.2.5) or a cellular binomial ideal (Theorem 3.2.6). As a corollary, we provide a generalization of Reisner's criterion to the context of cellular binomial ideals, which gives a Cohen-Macaulay characterization for $\Bbbk[x]/I$ in terms of the cohomology of finitely many chain complexes (Corollary 3.2.8). Let (I, J, A) be a tuple consisting of a lattice ideal (resp. cellular binomial ideal), a minimal prime ideal J of $I \cap \Bbbk[\mathbb{N}^\zeta]$, and the corresponding affine semigroup $\mathbb{N}A = Q$. Then J is also a prime lattice ideal and we may assume after rescaling the variables that $J = I(\xi)$ is toric, with lattice $L_\xi = (L_\rho)_{\text{sat}}$. Let $T := L_\xi/L_\rho$ be the corresponding torsion abelian group, then

$$\mathbb{Z}^d/L_\rho \cong T \oplus \mathbb{Z}A.$$

We may induce a fine grading of $\Bbbk[x]/I$ by $T \oplus \mathbb{Z}A$ as follows: for any $\overline{x^u} \in \Bbbk[x]/I$ for some $u \in \mathbb{Z}^d$, $\deg_{T,A}(\overline{x^u}) := (u + L_\rho, A \cdot u)$. Here we use $\overline{x^u}$ to indicate the image of $x^u \in \Bbbk[x]$ in $\Bbbk[x]/I$.

Let $\mathcal{I}$ be a monomial ideal of an affine semigroup ring $\Bbbk[Q]$. In this article, we extend the notion of degree space defined on Definition 2.5.1 further.

Definition 3.2.1. Suppose that $\mathcal{A}$ is the hyperplane arrangement consisting of the supporting hyperplanes of the facets of $\mathbb{R}_{\geq 0}Q$. Let $\mathfrak{r}(\mathcal{A})$ be the set of regions of $\mathcal{A}$. Then, for any region $\mathfrak{r} \in \mathfrak{r}(\mathcal{A})$, regard $\mathfrak{r} \cap (\mathbb{Z}^d \smallsetminus \bigcup \deg(\Bbbk[Q]/\mathcal{I}))$ as a space with the trivial topology.

The *extended degree space* $\mathbb{Z}Q$ of $\mathcal{I}$ is the disjoint union $\mathbb{Z}Q = (\bigcup \deg(\Bbbk[Q]/\mathcal{I})) \cup \big(\bigcup_{\mathfrak{r} \in \mathfrak{r}(\mathcal{A})} \mathfrak{r} \cap (\mathbb{Z}^d \smallsetminus \bigcup \deg(\Bbbk[Q]/\mathcal{I})\big)$ as a topological space. (As a set, $\mathbb{Z}Q$ equals $\mathbb{Z}^d$.) Lastly, let $\mathcal{G}(\mathbb{Z}Q)$ (resp. $\mathcal{G}(\Bbbk[Q]/\mathcal{I})$) be the collection of all grains of the extended (resp. original) degree space of $\mathcal{I}$.

As we did in Definition 2.5.2, we use the same terms *grain* and *chaff* to denote grains of the extended degree space, and its graded parts of the given Ishida complex.

Lemma 3.2.2. *$\mathcal{G}(\mathbb{Z}Q)$ is finite.*

Proof. This is came from the fact that $\mathcal{G}(\Bbbk[Q]/\mathcal{I})$ is finite [40, Lemma 4.3] and $\mathfrak{r}(\mathcal{A})$ is finite. $\square$

Definition 3.2.3. Given $t := u + L_\rho$, the *monomial ideal corresponding to t $\mathcal{I}_t$* is the following ideal in $\Bbbk[Q]$, defined as

$$\mathcal{I}_t := \langle x^{A \cdot u} \mid \deg_{T,A}(\overline{x^u}) = (t, A \cdot u) \text{ and } \overline{x^u} \in \Bbbk[x]/\mathcal{I} \rangle.$$

For an open set $O \in \mathcal{G}(\Bbbk[x]/\mathcal{I}_t)$, let $\tilde{\mathcal{C}}\,(O)$ be the graded part of the generalized Ishida complex associated to the element $\overline{x^u}$ whose torsion degree is t and whose A-degree is $Au \in O$. This is well-defined regardless of choice of $\overline{x^u}$, as is stated below.

Lemma 3.2.4. *If $\overline{x^u}$ and $\overline{x^v}$ with the same torsion degree are in the same grain of the extended degree space of $\mathcal{I}_t$, then their corresponding graded parts of the Ishida complex coincide.*

Proof. If neither $A \cdot u$ nor $A \cdot v$ are in the original degree space of $\mathcal{I}_t$, they must be in the same region of the hyperplane arrangement. Hence, for any localization by a face F, either both degrees are $\deg((\mathcal{I}_t) \cdot \Bbbk[\mathbb{N}A - \mathbb{N}F])$ or they do not belong to the same localization.

If both are in the original degree space, let $[u, F]$ be an overlap class whose degree set $\bigcup[u, F]$ contains $A \cdot u$ and $A \cdot v$, and such that F is minimal with this property. Then $\overline{x^u}$ and $\overline{x^v}$ do not appear in the localization of $\Bbbk[x]/I$ by a multiplicative set generated by variables corresponding to a proper face of F. Conversely, if there is no overlap class $[u, G]$ containing $A \cdot v$, this means that

$A \cdot v$ appears on every localization of Q by faces containing G. This completely determines the graded parts of the Ishida complex. $\square$

Theorem 3.2.5. *Given a lattice ideal I, define Q as before. The multi-graded Hilbert series for the ith local cohomology module of $\Bbbk[x]/I$ supported on the inverse image of the radical monomial ideal $I_\Delta \subset \Bbbk[Q]$ with respect to the $T \otimes \mathbb{Z}A$-grading is*

$$\operatorname{Hilb}(H^i_{I_\Delta}(\Bbbk[x]/I), \mathbf{t}) = \sum_{t \in T} \sum_{O \in \mathcal{G}(\Bbbk[\mathbb{N}A]/\mathcal{I}_t)} \dim H^i(\tilde{\mathcal{C}}\,(O)\,; \Bbbk) \sum_{\substack{u \in \mathbb{Z}^d \\ Au \in O}} x^{\overline{u}}.$$

Proof. This is a direct consequence of Lemma 3.2.4. $\square$

Since T is finite and $\mathcal{G}(\Bbbk[x]/\mathcal{I}_t)$ is finite for all $t \in T$, the sum is finite. Moreover, each grain $O \in \mathcal{G}(\Bbbk[x]/\mathcal{I}_t)$ is the set of lattice points in a convex polyhedron, so that $\sum_{\substack{u \in \mathbb{Z}^d \\ Au \in O}} x^{\overline{u}}$ can be written as a rational function [2, 3].

For the case of a ζ-cellular binomial ideal (I, J, A), let M be the multiplicative set consisting of monomials on the nilpotent variables of $\Bbbk[x]/I$. Then,

Theorem 3.2.6. *Given a cellular binomial ideal I, the multi-graded Hilbert series for the local cohomology module of $\Bbbk[x]/I$ supported on the image of the radical monomial ideal $I_\Delta \subset \Bbbk[Q]$ with respect to the $\bigoplus_{m \in M}(T_m \oplus \mathbb{Z}A_m)$-grading is*

$$\operatorname{Hilb}(H^i_{I_\Delta}(\Bbbk[x]/I), \mathbf{t}) = \sum_{m \in M} \sum_{t \in T_m} \sum_{O \in \mathcal{G}(\Bbbk[\mathbb{N}A_m]/I_t)} \dim H^i(\tilde{\mathcal{C}}\,(O)\,; \Bbbk) \sum_{\substack{u \in \mathbb{Z}^d \\ A_m \cdot u^\zeta \in O}} \overline{x^u}.$$

Proof. For each $m \in M$, $(I : m) \cap \Bbbk[\mathbb{N}^\zeta]$ is a lattice ideal containing $I \cap \Bbbk[\mathbb{N}^\zeta]$. Hence, according to Theorem 1.2.85, we may pick an associated prime ideal J_m of $I \cap \Bbbk[\mathbb{N}^\zeta]$ whose extension $J_m \Bbbk[x] + \langle x_i \mid i \in \zeta^c \rangle$ is the associated prime containing J. Moreover, the quotient $\Bbbk[x]/(J_m \Bbbk[x] + \langle x_i \mid i \in \zeta^c \rangle)$ is isomorphic to an affine semigroup ring $\Bbbk[\mathbb{N}A_m]$ for some integer matrix A_m. The canonical projection $\Bbbk[x]/J \to \Bbbk[x]/(J_m \Bbbk[x] + \langle x_i \mid i \in \zeta^c \rangle)$ induces a monoid map $\mathbb{N}A \to \mathbb{N}A_m$. By letting T_m be the quotient of the saturation of the lattice L_m corresponding to $(I : m) \cap \Bbbk[\mathbb{N}^\zeta]$

by L_m, we have a fine grading of $\Bbbk[x]/I$ by the abelian group

$$\bigoplus_{m \in M} (T_m \oplus \mathbb{Z}A_m)$$

via $\deg_{M,T,A}(\overline{x^u}) = (u^{\zeta^c}, u^\zeta + L_m, A_m \cdot u^\zeta)$. Thus, for a fixed $m \in M$ and a torsion $t \in T_m$, let

$$I_t := \langle x^{A_m \cdot u^\zeta} \mid \deg_{M,T,A}(\overline{x^u}) = (\deg(m), t, A \cdot u^\zeta) \text{ and } \overline{x^u} \in \Bbbk[x]/I \rangle.$$

Using the same arguments as for lattice ideals, we know that two elements whose degrees are in the same grain $O \in \mathcal{G}(\Bbbk[\mathbb{N}A_m]/I_t)$ have the same graded part of the Ishida complex $\tilde{\mathcal{C}}\,(O)$. $\square$

Again, this is a finite sum of rational functions. Corollary 3.2.8 gives us the equivalent of Reisner's criterion for cellular binomial ideals, providing a characterization of Cohen-Macaulayness in terms of the cohomology of finitely many polyhedral complexes. First we need an auxilliary result.

Lemma 3.2.7. *$\tilde{\mathcal{C}}\,(O)$ is the cochain complex of a polyhedral complex.*

Proof. It suffices to show that the nontrivial top dimensional part of $\tilde{\mathcal{C}}\,(O)$ is $\Bbbk^1$; suppose not; then there exists distinct maximal faces $\widehat{F_1}$ and $\widehat{F_2}$ of Q such that $\deg_A(\overline{x^{u_1}}) \in \widehat{F_1}$ and $\deg_A(\overline{x^{u_2}}) \in \widehat{F_2}$ for some distinct $\overline{x^{u_1}}$ and $\overline{x^{u_2}}$ with $u_1, u_2 \in O$. Then, the degree of the product $\overline{x^{u_1}} \cdot \overline{x^{u_2}}$ lies in the relative interior of a face $\widehat{G}$ in $\mathcal{F}(Q)$ which is a minimal face containing both $\widehat{F_1}$ and $\widehat{F_2}$, contradicting the maximality of $\widehat{F_1}$ and $\widehat{F_2}$. $\square$

Corollary 3.2.8. *Let I be a cellular binomial ideal. Then $\Bbbk[x]/I$ is Cohen–Macaulay if and only if $H^i(\tilde{\mathcal{C}}\,(O)\,;\Bbbk) = 0$ for all $i \neq$ dim$(\Bbbk[x]/I)$ and for all $O \in \mathcal{G}(\Bbbk[\mathbb{N}A_m]/I_t)$, $m \in M, t \in T_m$.*

Example 3.2.9. Let L be the following lattice in $\mathbb{Z}^4$ and L_{sat} its saturation.

$$L := \left\langle \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 1 \\ 5 \end{pmatrix} \right\rangle, L_{\text{sat}} := \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\rangle.$$

The torsion group $T := L/L_{\text{sat}}$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. We represent T as follows

$$T = \left\{ e = \overline{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \xi = \overline{\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}}, \xi^2 = \overline{\begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix}}, \xi^3 = \overline{\begin{pmatrix} -1 \\ 3 \\ 0 \\ -3 \end{pmatrix}}, \xi^4 = \overline{\begin{pmatrix} 0 \\ 4 \\ -2 \\ -4 \end{pmatrix}} \right\}$$

using GRevLex term order in Macaulay2.

In this case $Q := (\mathbb{Z}^4/L_{\text{sat}}) \cap \mathbb{N}^4 \cong \mathbb{N}\left[\begin{smallmatrix} 3 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$. As usual $\mathbb{Z}Q = \mathbb{Z}^4/L_{\text{sat}}$. On the polynomial ring $\mathbb{k}[a,b,c,d]$, the lattice ideals corresponding to L and L_{sat} are

$$I_L := \langle a^2 - c^3, acd^5 - b^5 \rangle \text{ and } I_{\text{sat}} := \langle bc - ad, cd^2 - b^2, c^2d - ab, c^3 - a^2 \rangle.$$

Hence, the Ishida complex supported on the maximal monomial ideal is

$$0 \to \mathbb{k}[x]/I_L \to (\mathbb{k}[x]/I_L)_{a,c} \oplus (\mathbb{k}[x]/I_L)_d \to (\mathbb{k}[x]/I_L)_{a,b,c,d} \to 0.$$

Here, we see that for any $0 \leq i$,

$$C^\bullet_{t,\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]} : 0 \to \mathbb{K}^{j+1} \to \mathbb{K}^{j+1+5} \to \mathbb{K}^5 \to 0$$

when $0 \leq j < 5$ and

$$C^\bullet_{t,\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]} : 0 \to \mathbb{K}^5 \to \mathbb{K}^{10} \to \mathbb{K}^5 \to 0$$

when $5 \leq j$ for any $t \in T$. Since there is no non-top cohomology, we conclude that $\mathbb{k}[x]/I_L$ is Cohen-Macaulay.

4. APPLICATIONS AND OPEN PROBLEMS

4.1 Alternative classification of Cohen–Macaulay affine semigroup rings

In this section we concentrate on the special case when $I = 0$. We give a new criterion using grains to detect whether $\Bbbk[Q]$ is Cohen–Macaulay and give an alternative proof of the Cohen–Macaulay condition in [54]. To begin, we recall a celebrated theorem of Hochster [26] when Q is normal, and prove it yet again with our methods.

Theorem 4.1.1 ([26])**.** *If Q is normal, $\Bbbk[Q]$ is Cohen–Macaulay.*

Definition 4.1.2 ([11])**.** Given a polyhedron $\mathscr{P}$ in $\mathbb{R}$-vector space V, let $u, v \in V$ two distinct points. If $[u, v]$ does not contain a point $v' \in \mathscr{P}$ with $v' \neq v$, we say that v is *visible* from u. A subset S is *visible* if every $v \in S$ is visible.

Proposition 4.1.3 ([11, Proposition 6.3.1])**.** *Given a polytope $\mathscr{P}'$, a contractible polyhedral subcomplex is formed by the set of all visible points from $u \in V \smallsetminus \mathscr{P}'$.*

We refer to this polyhedral subcomplex as the *u-visible subpolytope* of $\mathscr{P}'$.

Proof. Let $\mathcal{A}$ be the hyperplane arrangement generated by hyperplanes in the H-representation of $\mathbb{R}_{\geq 0}Q$. We claim that

$$\mathcal{G}(\Bbbk[Q]) = \{\mathfrak{r}_S \cap \mathbb{Z}Q \mid \mathfrak{r}_S \in \mathfrak{r}(\mathcal{A})\}. \tag{4.1}$$

If the equality (4.1) holds, for the given nonempty S and a point $u \in \mathfrak{r}_S \cap \mathbb{Z}Q$, construct a hyperplane $\mathcal{H}$ containing u and transversally intersecting $\mathbb{R}_{\geq 0}Q$. The transverse section K is then realized as a poytope $\mathcal{H} \cap \mathbb{R}_{\geq 0}Q$. Thus, the chaff of a grain can be defined as a subset of $\mathcal{F}(K)$ that is not visible from u. Due to the contractibility of both K and u-visible subpolytope of K, the chain complex over the chaff is contractible via the long exact sequence of cohomology. Thus, except the top dimension, the chaff of any grain has vanishing homology. This argument essentially paraphrases the proof of [11, Theorem 6.3.4].

To prove (4.1) recall that the poset of regions $\mathfrak{r}(\mathcal{A})$ partitions $\bigcup \deg(\Bbbk[Q]) = \mathbb{Z}Q$. We may use induction over the cardinality of S to determine that each grain is of the form $\mathfrak{r}_S \cap \mathbb{Z}Q$. Assume that the hyperplane arrangement $\mathcal{A}$ consists of the elements $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m$, and that F_i is the facet supported by $\mathcal{H}_i$ for $i \leq m$. Start with $|S| = m$; $\mathfrak{r}_S \cap \mathbb{Z}Q$ is a grain since $(0, Q)$ is the unique degree pair of Q and $0 + Q \subset 0 + (Q - \mathbb{N}F)$ for any face F.

To use induction, suppose we showed that $\mathfrak{r}_S \cap \mathbb{Z}Q$ with $|S| \leq m - i$ is a grain. Then, for any S with cardinality $m-(i+1)$, we claim $\mathfrak{r}_S \cap \mathbb{Z}Q = (\bigcap_{i \in S}(Q - \mathbb{N}F_i)) \smallsetminus \left(\bigcup_{T \supsetneq S} \mathfrak{r}_T\right) \cap \mathbb{Z}Q$. Indeed, $(\bigcap_{i \in S}(Q - \mathbb{N}F_i)) = \mathfrak{R}_S \cap \mathbb{Z}Q$ by the definition of the cumulative regions and the normality of Q. Then, the righthand side $(\bigcap_{i \in S}(Q - \mathbb{N}F_i)) \smallsetminus \left(\bigcup_{T \supsetneq S} \mathfrak{r}_T\right) \cap \mathbb{Z}Q$ is nothing more than the construction of $\mathfrak{r}_S$ from $\mathfrak{R}_S$. Furthermore, for each $T \supsetneq S$,$\mathfrak{r}_T \cap \mathbb{Z}Q$ is a grain by inductive hypothesis. This shows the proposed one-to-one correspondence between grains and regions in $\mathfrak{r}(\mathcal{A})$.

Now pick a grain $\mathfrak{r}_S \cap \mathbb{Z}Q$. Then $\mathfrak{r}_S \subseteq \mathfrak{R}_T \cap \mathbb{Z}Q$ if and only if $T \subseteq S$. Hence, its chaff can be identified as a subset of faces in $\mathcal{F}(Q)$ whose corresponding localizations contain $\mathfrak{r}_S \cap \mathbb{Z}Q$. $\square$

When Q is not normal, we need the chaffs and grains of the module $\Bbbk[Q_{\text{sat}}]/\Bbbk[Q]$ to determine the chaffs of grains of $\Bbbk[Q]$. To distinguish two chaffs and grains from different modules,

Definition 4.1.4. We refer to the grains and chaffs of the module $\Bbbk[Q_{\text{sat}}]/\Bbbk[Q]$ as *void grains and void chaffs*, respectively, in accordance with the conventions in Example 2.1.7(2).

Theorem 4.1.5. $\Bbbk[Q]$ *is Cohen–Macaulay if and only if every grain consisting of void grains has vanishing homology except in top dimension.*

Proof. If a grain G has a degree which exists in $\bigcup \deg(Q_{\text{sat}})$, then we may apply the same argument of Theorem 4.1.1 to show that the homology of D_{G} vanishes except for the top dimension. Hence, the only grains we need to investigate the homology of their chaff are a grain consisting of holes. $\square$

Now we are prepared to give an alternative proof of the main result of [54]. Let $F_1, \cdots, F_m$ be facets of a pointed affine semigroup Q. Let $\widetilde{Q} := \bigcap_{i=1}^{m}(Q - \mathbb{N}F_i)$. For any nonempty subset S of

$\{1, 2, \cdots, m\}$, let $\mathsf{G}_S := \bigcap_{i \notin S}(Q - \mathbb{N}F_i) \smallsetminus \bigcup_{j \in S}(Q - \mathbb{N}F_j)$. Let π_S be the simplicial complex of nonempty subsets I of S such that $\bigcap_{i \in I} F_i$ is a nonempty face of Q. By abuse of notation, we identify the face lattice $\mathcal{F}(\pi_S)$ of π_S as a subset $\{\bigcap_{i \in I} F_i \in \mathcal{F}(Q) \smallsetminus \{\varnothing\} \mid I \in \pi_S\} \cup \{\varnothing\}$ of $\mathcal{F}(Q)$. We say π_S is *acyclic* if its reduced homology group is zero for all indices.

Theorem 4.1.6 (Main theorem in [54]). *$\Bbbk[Q]$ is a Cohen–Macaulay ring if and only if (1) $\widetilde{Q} = Q$ and (2) for every $S \subseteq \{1, 2, \cdots, m\}$ with $\mathfrak{R}_S \in \mathfrak{r}(\mathcal{A})$, π_S is acyclic.*

Proof. Note that for any $u \in \bigcup \deg(\Bbbk[Q]) \smallsetminus \bigcup_{i=1}^m \deg(Q - \mathbb{N}F_i)$, $K_u = \{Q\}$, which therefore only contributes to the $(\dim Q)$-th local cohomology. Thus, to prove the conditions above imply Cohen–Macaulayness, it suffices to show that for any $u \in \bigcup_{i=1}^m Q - \mathbb{N}F_i$, $\tilde{\mathcal{C}}\,(K_u)$ is exact. Since G_S partitions $\bigcup \deg(\Bbbk[Q])$, assume $u \in G_S$ for some proper subset of $\{1, 2, \cdots, m\}$. Then, for any $F \in \mathcal{F}(\pi_S)^c$, $F \not\subset F_i$ for any $i \in S$, since $F_i \in \pi_S$. Thus, $F = \bigcap_{i \in J} F_i$ for some $J \subset \{1, 2, \cdots, m\} \smallsetminus S$ implies that $Q - \mathbb{N}F$ contains u. Conversely, for any face G of $\bigcap_{i \in S} F_i$, $u \notin Q - \mathbb{N}G$. Hence $K_u = \mathcal{F}(\pi_S)^c$ by identifying $\mathcal{F}(\pi_S)$ as a subset of $\mathcal{F}(Q)$. In this identification, π_S is isomorphic to a polyhedral subcomplex of the transverse section K of $\mathbb{R}_{\geq 0}Q$. Hence, the complements $\mathcal{F}(\pi_S)^c$ form a polyhedral subcomplex of the dual polytope K^{dual}. Apply Alexander duality to conclude that K_u is acyclic. This proves that $\Bbbk[Q]$ is Cohen–Macaulay in accordance with Theorem 2.6.6.

Conversely, suppose π_S is not acyclic. Pick $u \in \mathsf{G}_S$ such that $K_u = \mathcal{F}(\pi_S)^c$. Now Alexander duality ensures that $\tilde{\mathcal{C}}\,(\mathcal{F}(\pi_S)^c)$ has nontrivial cohomology at i-th index which is less than $(\dim \mathbb{Q})$. Also, if $Q' \neq Q$, Corollary 2.1.15 gives a void pair (v, F) with $\dim F \leq \dim Q - 2$. Let S be a set of indices of hyperplanes containing F. By Theorem 1.2.58 there exists $u \in (v + \mathbb{N}(F \cup (-F))) \cap \mathfrak{r}_S$. Hence, $K_u = (Q/F) \smallsetminus F := \{G \in \mathcal{F}(Q) \mid Q \supseteq G \supsetneq F\}$ is combinatorially equivalent to the polytope $(Q/F)^{\text{dual}}$ without its relative interior. Hence, $(\dim F)$-th homology of K_u is nonzero, as is the $(\dim Q - \dim F)$-th local cohomology. Thus, the u-graded part of the Ishida complex admits nonzero local cohomology with an index less than $\dim Q$, indicating that the semigroup ring is not Cohen–Macaulay. $\square$

Example 4.1.7 (Continuation of Example 2.6.7)**.** Both cases are not Cohen–Macaulay due to the presence of nonzero 0-th local cohomology.

Example 4.1.8 (Examples of non-normal affine semigroup rings)**.**

1. (A 3-dimensional non-Cohen–Macaulay affine semigroup ring) Let $Q := \mathbb{N}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & 1 \end{bmatrix}$. Index the rays, facets, and hyperplanes as follows

$$\begin{array}{cccc} \langle u_1 \rangle := \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle & \langle u_2 \rangle := \left\langle \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\rangle & \langle u_3 \rangle := \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle & \langle u_4 \rangle := \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \\ F_1 := \langle u_1, u_2 \rangle & F_2 := \langle u_2, u_3 \rangle & F_3 := \langle u_3, u_4 \rangle & F_4 := \langle u_4, u_1 \rangle \\ \mathcal{H}_1 := \{y = 0\} & \mathcal{H}_2 := \{2x - 2y + z = 0\} & \mathcal{H}_3 := \{x - y = 0\} & \mathcal{H}_4 := \{x - z = 0\} \end{array}$$

The Hasse diagram for the region of posets is identical to that in Figure 4.2. Moreover, the set of holes of the affine semigroup, $\mathcal{H}(Q)$, is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \mathbb{N}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ which lies in the xz-plane. This is because $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ acts as a barrier to the spread of holes in the relative interior of Q. According to Theorem 2.1.14, Q and $Q - \mathbb{N}u_2$ are the only non-normal affine semigroups that arise as a result of localization. Thus, the space of holes $\mathrm{Holes}(Q)$ equals $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, which is consistent with the set of holes of $Q - \mathbb{N}u_2$. Using Figure 4.2, $\mathrm{Holes}(Q)$ is decomposed into three possible sets in Table 4.1. Notes that $\mathcal{H}_{1,2,3,4} = \bigcap_{i=1,3,4}(Q - \mathbb{N}u_i) \smallsetminus Q$ and $\mathcal{H}_{1,2} = (Q - \mathbb{N}F_1) \cap (Q - \mathbb{N}F_2) \smallsetminus (Q - \mathbb{N}u_2)$, $\mathcal{H}_{1,2,3,4}$ and $\mathcal{H}_{1,2}$ form grains.

$\mathcal{H}_{1,2,3,4}$	$\mathcal{H}_{1,2,3}$	$\mathcal{H}_{1,2}$
$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbb{N}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbb{Z}_{\leq -2}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Table 4.1: Decomposition of $\mathrm{Holes}(Q)$.

On the other hand,

$$\mathcal{H}_{1,2,3} \subsetneq \mathsf{G} := (Q - \mathbb{N}u_3) \smallsetminus (Q \cup (Q - \mathbb{N}u_2) \cup \mathcal{H}_{1,2,3,4})$$

shows that $\mathcal{H}_{1,2,3}$ is a part of the grain G, whereas the remaining elements of G come from the region $\mathfrak{r}_{2,3}$. Thus,

$$\begin{aligned}\mathcal{G}(Q/T) = \{&\mathcal{H}_{1,2,3,4}, \mathcal{H}_{1,2}, Q, (Q - \mathbb{N}u_2) \cap \mathfrak{r}_{1,2}, (\mathfrak{r}_{2,3} \cap (Q - \mathbb{N}u_3)) \cup \mathcal{H}_{1,2,3}\} \\ &\cup \{\mathbb{Z}^3 \cap \mathfrak{r}_S \mid S \in \text{index}(\mathfrak{r}(\mathcal{A})) \text{ such that } S \neq \{1,2\}, \{2,3\}, \{1,2,3,4\}\},\end{aligned}$$

where $\text{index}(\mathfrak{r}(\mathcal{A}))$ denotes the set of all indices of elements of $\mathfrak{r}(\mathcal{A})$. Hence, it suffices to check whether chaffs

$$D_{\mathcal{H}_{1,2,3,4}} = \{u_i, F_j, Q\}_{\substack{i=1,3,4 \\ j=1,2,3,4}} \text{ and } D_{\mathcal{H}_{1,2}} = \{F_1, F_2, Q\}$$

have vanishing homology. Since $D_{\mathcal{H}_{1,2}}$ produces a non-zero second homology, the affine semigroup ring $\Bbbk[Q]$ is not Cohen–Macaulay.

2. (4-dimensional non-normal Cohen–Macaulay affine semigroup ring [10, Exercise 6.4]) Assume P is a simplex with the vertices $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, and $(0,0,5)$. $\mathbb{Z}^3$ is the smallest lattice that contains vertices of P. The *polytopal affine monoid* $M(P)$ [10, Definition 2.18] associated P is the affine semigroup $Q := \mathbb{N}A$ where $A = \{(1,u) : u \in \mathbb{Z}^3 \cap P\}$. In this example,

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_{18} \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&2 \\ 0&0&0&0&0&0&1&1&1&1&2&2&3&0&0&0&1&0 \\ 0&1&2&3&4&5&0&1&2&3&0&1&0&0&1&2&0&0 \end{bmatrix}.$$

$\mathbb{R}_{\geq 0}Q$ is a simplicial polyhedron with tetrahedral transverse section. We index its facets and hyperplanes as follows

$$\begin{aligned} F_1 &:= \langle u_6, u_{13}, u_{18}\rangle & F_2 &:= \langle u_1, u_7, u_{11}, u_{13}, u_{14}, u_{17}, u_{18}\rangle \\ F_3 &:= \langle u_1, \cdots, u_6, u_{14}, \cdots, u_{16}, u_{18}\rangle & F_4 &:= \langle u_1, \cdots, u_{13}\rangle \\ \mathcal{H}_1 &:= \{(30, -15, -10, -6) \cdot t = 0\} & \mathcal{H}_2 &:= \{(0,0,0,1) \cdot t = 0\} \\ \mathcal{H}_3 &:= \{(0,0,1,0) \cdot t = 0\} & \mathcal{H}_4 &:= \{(0,1,0,0) \cdot t = 0\} \end{aligned}$$

Since the transverse section K is a tetrahedron, we can index faces as intersections of facets uniquely. For example, each of the rays can be denoted as follows.

$$F_{2,3,4} := \langle u_1 \rangle \qquad F_{1,3,4} := \langle u_6 \rangle \qquad F_{1,2,4} := \langle u_{13} \rangle \qquad F_{1,2,3} := \langle u_{18} \rangle .$$

According to the HASE package [33], the affine semigroup $Q = \mathbb{N}A$ contains holes $\mathcal{H}(Q) := \left[\begin{smallmatrix} 2 & 1 & 2 & 4 \end{smallmatrix}\right]^t + \mathbb{N}F_1$. Thus, the space of holes $\operatorname{Holes}(Q)$ is $\left[\begin{smallmatrix} 2 & 1 & 2 & 4 \end{smallmatrix}\right]^t + \mathbb{Z}F_1$. We can decompose $\operatorname{Holes}(Q)$ into void grains using the hyperplane arrangement as follows:

$$\mathcal{H}_S := \left\{ \left[\begin{smallmatrix} 2+a+b+c \\ 1+2a \\ 2+3b \\ 4+5c \end{smallmatrix}\right] \mid a \in \operatorname{sgn}_4(S), b \in \operatorname{sgn}_3(S), c \in \operatorname{sgn}_2(S) \right\}$$

where $\operatorname{sgn}_i(S) := \begin{cases} \mathbb{N} & \text{if } i \in S \\ \mathbb{Z} \smallsetminus \mathbb{N} & \text{if } i \notin S \end{cases}$ for all $\{1\} \subseteq S \subseteq \{1,2,3,4\}$. There are two types of grains indexed by $2^{\{1,2,3,4\}}$ that emerge from iterative intersections of affine semigroups. For every S with $\{1\} \subseteq S \subseteq \{1,2,3,4\}$, the union $\mathsf{G}_{S \smallsetminus \{1\}} := \mathcal{H}_S \cup \left(\mathfrak{r}_{S\smallsetminus\{1\}} \cap \left(Q - \mathbb{N}F_{S\smallsetminus\{1\}}\right)\right)$ generates a grain of the first type, whereas $\mathsf{G}_S := \mathfrak{r}_S \cap (Q - \mathbb{N}F_S)$ generates a grain of the second type. Since there is no grain composed entirely of holes, all chaffs have vanishing homology except the top dimension. Therefore, $\Bbbk[Q]$ is Cohen–Macaulay.

4.2 Generalized Hochster's theorem for (non-normal) simplicial affine semigroup rings

In this section, we generalize the well-known Hochster's theorem [11, Theorem 5.3.8] for the quotients of (non-normal) simplicial affine semigroup rings by monomial ideals. The original theorem was stated for the Stanley-Reisner rings only. To begin with, let (I, I, A) be a prime lattice ideal I whose corresponding affine semigroup $Q = \mathbb{N}A$.

Definition 4.2.1. An affine semigroup Q is *simplicial* if the transverse section K of the polyhedral cone $\mathbb{R}_{\geq 0}Q$ is a d-simplex.

Assume that $\mathbb{N}A$ from (I, I, A) is simplicial. Let $\mathcal{A} = \{\mathcal{H}_i\}_{i=1}^d$ be the minimal hyperplane arrangement of $\mathbb{R}_{\geq 0}Q$. Label the faces of K by their supporting facets; for a facet $\mathcal{H}_i \cap K$, use the label $[d] \smallsetminus \{i\}$. Hence, the zero-dimensional face (of the transverse section) that does not lie in the hyperplane $\mathcal{H}_i$ is indexed by $\{i\}$. From the natural isomorphism $\mathcal{F}(Q) \cong \mathcal{F}(K)$, we may label faces of Q using $2^{[d]} = \mathcal{F}(K)$.

In this notation, for a face $F \in \mathcal{F}(Q)$, $\mathfrak{r}_F$ defined in Subsection 1.2.3 by regarding F as a subset of $[d]$, is a region generated by the positive half spaces containing F as a face of Q. This labeling is induced by the observation that the poset of regions $\mathfrak{r}(\mathcal{A})$ is equal to the face lattice $\mathcal{F}(K)$. This also agrees with the labeling of $\mathfrak{r}_F$ in Subsection 1.2.3; $\mathfrak{r}_F$ is the region contained in the positive half spaces of $\mathcal{H}_i$ where $i \in F$. For example, if F is the zero-dimensional face corresponding to x_i, then $\mathfrak{r}_F = \mathfrak{r}_{[d]\smallsetminus\{i\}}$.

Let I_Δ be a radical monomial ideal of $\Bbbk[Q] \cong \Bbbk[x]/I$ corresponding to a proper subcomplex Δ of K and a proper face $F \in \mathcal{F}(Q)$. In this setting, we always label all grains of the original degree space $\bigcup \deg(\Bbbk[Q]/I_\Delta)$ using a pair of faces as in the following result.

Definition 4.2.2. We denote $\max(\Delta)$ the collection of facets of Δ.

Lemma 4.2.3. *For every grain $S \in \mathcal{G}(\Bbbk[Q]/I)$ of the degree space $\bigcup \deg(\Bbbk[Q]/I_\Delta)$ there exists a unique pair of faces (F, G) such that $S \subseteq (Q - \mathbb{N}F) \cap \mathfrak{r}_F$ and $G \in \{G' \in \bigcap \max(\Delta) \mid G' \supseteq G \supseteq F\}$. We use this pair to label S.*

Proof. Given a face $F \in \mathcal{F}(Q)$, let $\mathcal{G}(Q/I_\Delta)_F$ be the set of all grains of the degree space $\bigcup \deg(\Bbbk[x]/I_\Delta)$ contained in $\mathfrak{r}_F$. Recall that $\bigcap \max(\Delta)_F$ is the collection of faces in $\bigcap \max(\Delta)$ containing F. We claim that $\mathcal{G}(Q/I_\Delta)_F$ and $\bigcap \max(\Delta)_F$ are in bijection.

Recall that all overlap classes (for I_Δ) are of the form $[0, G]$ for some face $G \in \max(\Delta)$. Hence, when $F = \tilde{0}$, the corresponding grain inside of $\mathfrak{r}_\emptyset \cap Q = Q$ is obtained by intersection and complement of faces. If $F \notin \Delta$, then the localization is zero, thus the statement is vacuously true. Suppose $F \gneq \tilde{0} \in \Delta$. The extension $(I_\Delta)_F$ of the ideal on $\Bbbk[Q - \mathbb{N}F]$ is still radical and its overlap classes are of form $[0, G \cup (-F)]$ where $[0, G]$ is an overlap class of $\Bbbk[Q]/I_\Delta$ for some face $G \supseteq F$. Thus, every grain in $\mathcal{G}(Q/I_\Delta)_F$ is obtained by intersecting open sets of the form $\bigcup [0, G \cup (-F)] \smallsetminus \left(\bigcup_{F' \subset G} \mathfrak{r}_{F'}\right)$. Thus $\mathcal{G}(Q/I_\Delta)_F$ is labeled by $\bigcap \max(\Delta)_F$. $\square$

We are now ready to study graded pieces of local cohomology modules.

Theorem 4.2.4. *Let u be an element (degree) in a grain S indexed by (F, G) with $F \neq G$. Then, $H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_u = 0$ for all i. If u is a degree in a grain indexed by a pair of the same face (F, F),*

$$H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_u \cong \tilde{H}^{i-\dim F-1}_{simp}((K/F) \cap \Delta)$$

where $\tilde{H}^{\bullet}_{simp}(-)$ means the reduced simplicial cohomology.

When our affine semigroup ring is the polynomial ring, the above reduces to the very well known formulas for local cohomology of Stanley–Reisner rings using homology of links.

Proof of Theorem 4.2.4. Pick a grain S that corresponds to (F, G). We know that $\deg(S) \subset \Bbbk[Q - H]$, where $H \in (F/G)$. We may assume $G = \bigcap_{i=1}^m G_i$ for some $G_i \in \max(\Delta)$ containing G. Then S is contained in an overlap class of I_Δ whose face is G_i. From Definition 3.1.3, the u-graded part of the Ishida complex is equal to the (shifted) chain complex of $\Delta(G_1, \cdots, G_m)/F$ where $\Delta(G_1, \cdots, G_m)$ is a subcomplex of Δ such that its maximal faces are $G_1, \cdots, G_m$. When $F \neq G$, Lemma 1.2.16 shows that $\Delta(G_1, \cdots, G_m)/F$ is contractible. Otherwise, $\Delta(G_1, \cdots, G_m)/F = \Delta \cap (K/F)$. □

We remark that this theorem holds more generally, for any affine semigroup Q whose poset of regions $\mathfrak{r}(\mathcal{A})$ is in bijection with $\mathcal{F}(Q)$.

Corollary 4.2.5. *Lemma 4.2.3 and Theorem 4.2.4 hold when Q is an affine semigroup such that $\mathcal{F}(Q)$ is in bijection to the poset of regions $\mathfrak{r}(\mathcal{A})$ of the hyperplane arrangement of $\mathbb{R}_{\geq 0}Q$.*

Proof. The property we used in the proof of Lemma 4.2.3 is that for any u, there exists a unique minimal face F such that $u \in Q - F$. This property holds if and only if $\mathfrak{r}(\mathcal{A})$ is in bijection to $\mathcal{F}(Q)$. □

Example 4.2.6 (Counterexample: Segre Embedding, continued from Example 2.5.6(2))**.** We consider the affine semigroup, $Q = \mathbb{N}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, which is depicted in Figure 4.1. Let u_i be the i-th

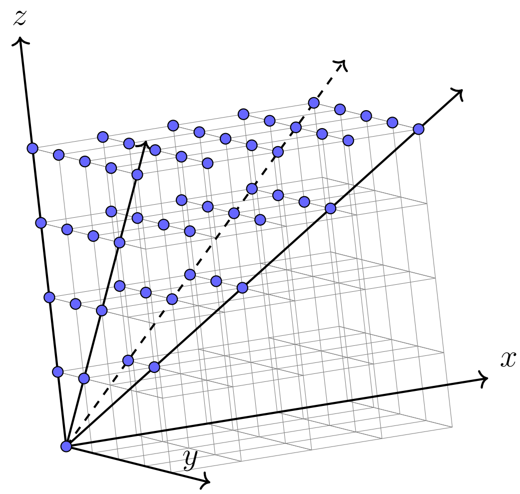


Figure 4.1: Degrees of $Q = \mathbb{N}\left[\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right]$

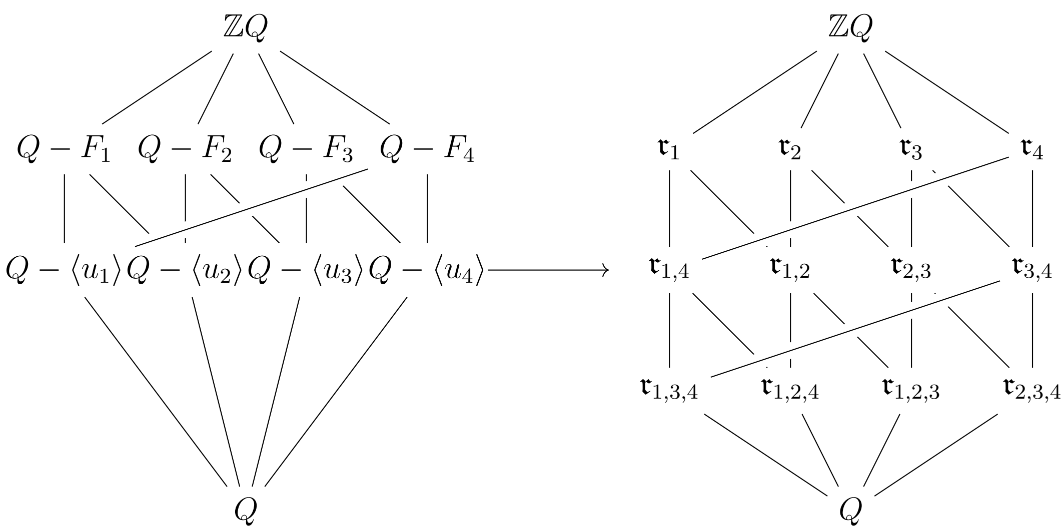


Figure 4.2: Hasse diagrams of $\mathbf{Cat}_Q$ and $\mathfrak{r}(\mathcal{A})$ of the Segre embedding

column of $\left[\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right]$. Denote the facets F_i and the hyperplane arrangement $\mathcal{A} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$ as below.

$$F_1 := \langle u_1, u_2 \rangle, \qquad F_2 := \langle u_2, u_3 \rangle, \qquad F_3 := \langle u_3, u_4 \rangle, \qquad F_4 := \langle u_4, u_1 \rangle$$
$$\mathcal{H}_1^{(+)} := \{y > 0\}, \qquad \mathcal{H}_2^{(+)} := \{z > x\}, \qquad \mathcal{H}_3^{(+)} := \{z > y\}, \qquad \mathcal{H}_4^{(+)} := \{x > 0\}.$$

For any face F, label F by the subset of $\{1, 2, 3, 4\}$ whose corresponding facet contains F. For example, $\langle u_1 \rangle$ is indexed by $\{1, 4\}$. Then we have the desired injection from $\mathcal{F}(Q)$ to $\mathfrak{r}(\mathcal{A})$ by sending a face F to $\mathfrak{r}_F := \mathbb{R}_{\geq 0}(Q - \mathbb{N}F) \setminus \bigcup_{G<F} \mathbb{R}_{\geq 0}(Q - \mathbb{N}G)$. This relationship is depicted in Figure 4.2. Note that this is **not** a bijection; for example,

$$(0, 1, 0)^t = (1, 1, 1)^t - (1, 0, 1)^t = (0, 1, 1)^t - (0, 0, 1)^t$$

is in both $Q - \langle u_1 \rangle$ and $Q - \langle u_2 \rangle$ but not in Q. Hence, $(0, 1, 0)^t \in \mathfrak{r}_{1,2,4}$. Therefore, we may not directly apply Corollary 4.2.5. However, still we may apply Corollary 3.2.8 to calculate its local cohomology by investigating the graded parts of the generalized Ishida complex corresponding to those “hidden" regions, i.e., regions in the cokernel of the map $\mathcal{F}(Q) \to \mathfrak{r}(\mathcal{A})$.

4.3 Duality between local cohomologies on simplicial affine semigroup rings

In this section, we relate the local cohomology of an affine semigroup ring $\Bbbk[Q]$ supported on a radical monomial ideal I_Δ to the local cohomology supported on the maximal ideal of the quotient of $\Bbbk[Q]/I_\Delta$. In the case of Stanley–Reisner rings, it is straightforward to determine the vanishing of such cohomologies using available formulas for (Hilbert series of) local cohomology due to Hochster and to Terai [53]. Throughout this section, we use the same notation as in Section 4.2.

4.3.1 Duality of graded local cohomologies

Let $\Bbbk[Q]$ be an affine semigroup ring of dimension $d := \dim \Bbbk[Q]$ as defined in Section 4.2. Suppose that Q has no hidden regions, i.e., the hyperplane arrangment $\mathcal{A}$ consisting of minimal supporting hyperplanes of $\mathbb{R}_{\geq 0}Q$ has poset of regions $\mathfrak{r}(\mathcal{A})$ canonically bijective to $\mathcal{F}(Q)$ [18, Lemma 1.3]. For example, this is the case when $\mathbb{R}_{\geq 0}Q$ is a cone over a simplex. Let I_Δ be a monomial radical ideal. Let K be the transverse section of $\mathbb{R}Q$ with its index sets defined in Section 4.2. Then, there is a duality between the local cohomology of $\Bbbk[Q]/I_\Delta$ with the maximal ideal $\mathfrak{m}$ support and the local cohomology of $\Bbbk[Q]$ with I_Δ-support as follow.

Theorem 4.3.1. *Given a face $F \in \mathcal{F}(Q)$, let S_F be the grain in the original degree space* *indexed by (F, F) from Lemma 4.2.3 if it exists. If such grain does not exists, let S_F be $\mathfrak{r}_F \cap (\bigcup \deg(\Bbbk[\mathbb{N}A]/J_\Delta))$. For any $u \in S_F$ and $v \in \mathfrak{r}_{F^c} \cap \mathbb{Z}Q$,*

$$(H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta))_u \cong (H^{d-i}_{I_\Delta}(\Bbbk[Q]))_v$$

We will present the proof of this theorem after demonstrating all of the lemmas required for its verification.

Some duality still holds in the case when hidden regions exist (only degrees outside of hidden regions are involved), but the statement is complicated and not very enlightening. The result is false for degrees corresponding to hidden regions, as seen in the following example.

Example 4.3.2 (Continuation of Example 4.2.6)**.** Let $I_\Delta = \langle x^{\left[\begin{smallmatrix}0\\1\\2\end{smallmatrix}\right]}, x^{\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right]}, x^{\left[\begin{smallmatrix}2\\1\\2\end{smallmatrix}\right]}\rangle$. It is a radical monomial ideal such that whose corresponding subcomplex Δ has F_1 and F_3 as facets. Then, for the grade $(0,1,0)^t$, notes that $x^{(0,1,0)^t} \in I_\Delta \cdot \Bbbk[Q - \mathbb{N}\langle u_1\rangle] \cap I_\Delta \cdot \Bbbk[Q - \mathbb{N}\langle u_2\rangle]$ and $x^{(0,1,0)^t} \notin \Bbbk[Q - \mathbb{N}\langle u_3\rangle] \cup \Bbbk[Q - \mathbb{N}\langle u_4\rangle] \cup \Bbbk[Q - \mathbb{N}F_3]$. This shows that the graded piece of $L^\bullet_{\mathfrak{m}} \otimes_{\Bbbk[Q]} \Bbbk[Q]/I_\Delta$, the Ishida complex of $\Bbbk[Q]/I_\Delta$ supported at the monomial maximal ideal $\mathfrak{m}$, is

$$L^\bullet_{\mathfrak{m}} \underset{\Bbbk[Q]}{\otimes} \Bbbk[Q]/I_\Delta : 0 \to 0 \to 0 \to 0 \to 0$$

On the other hand, the transverse section of $\Bbbk[Q]I_\Delta$ is also a rectangle whose vertices are embedded into F_2 or F_4 respectively. Because $(0,1,0)^t \in \mathfrak{r}_{1,2,4}$, $x^{(0,1,0)^t} \in \Bbbk[Q - \mathbb{N}F_i]$ when $i = 1, 2, 4$ only. The graded piece of $L^\bullet_{I_\Delta} \otimes_{\Bbbk[Q]} \Bbbk[Q]$, the Ishida complex of $\Bbbk[Q]$ supported at I_Δ, is therefore

$$L^\bullet_{I_\Delta} \underset{\Bbbk[Q]}{\otimes} \Bbbk[Q] : 0 \to \Bbbk^4 \to \Bbbk^3 \to \Bbbk \to 0$$

Consequently,

$$\left(H^j_{I_\Delta}(\Bbbk[Q])\right)_{\deg=(0,1,0)^t} = \begin{cases} \Bbbk & j = 1,2 \\ 0 & o.w. \end{cases}, \quad \left(H^j_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)\right)_{\deg=(0,1,0)^t} = 0 \text{ for all } j.$$

This contradicts the duality when the affine semigroup contains hidden regions.

From the example, we see that the theorem holds only when the affine semigroup has no hidden region. Therefore, an important example of affine semigroup rings where Theorem 4.3.1 is applicable is simplicial affine semigroup rings.

Corollary 4.3.3. *Theorem 4.3.1 holds when Q is a simplicial affine semigroup.*

Proof. Since simplicial affine semigroups have no hidden regions, the proof follows directly from that of Theorem 4.3.1. □

Note that $F^c := \{i \in [n] \mid i \notin F\}$ is a well-defined face since it either exists on both $\mathcal{F}(Q)$ and $\mathfrak{r}(\mathcal{A})$ simultaneously or on neither.

To proceed, assume that I_Δ is neither 0 nor $\Bbbk[Q]$. $\mathcal{G}(\Bbbk[Q])$ consists of $\mathfrak{r}_F \cap \mathbb{Z}Q$ for each face $F \in \mathcal{F}(Q)$. Given the transverse section K_{I_Δ} of $\mathbb{R}_{\geq 0} I_\Delta$, let $(K_{I_\Delta})_{\mathfrak{r}_{F^c} \cap \mathbb{Z}Q} = \{F \in K_{I_\Delta} \mid \widehat{F} \supset F^c\}$ be the set of faces of K_{I_Δ} whose corresponding faces in $\mathcal{F}(Q)$ contain F^c. Then, $\mathscr{P}_{F^c} := \mathcal{F}(K_{I_\Delta}) \smallsetminus (K_{I_\Delta})_{\mathfrak{r}_{F^c} \cap \mathbb{Z}Q}$ form a polyhedral complex since $G \supseteq G' \in (K_{I_\Delta})_{\mathfrak{r}_{F^c} \cap \mathbb{Z}Q}$ implies $G \in (K_{I_\Delta})_{\mathfrak{r}_{F^c} \cap \mathbb{Z}Q}$. Consequently, the graded part of the local cohomology of $\Bbbk[Q]$ with I_Δ-support is determined as below.

Lemma 4.3.4. *For a degree $u \in \mathfrak{r}_{F^c} \cap \mathbb{Z}Q$,*

$$H^i_{I_\Delta}(\Bbbk[Q])_u \cong (H^{i-1}_{\mathit{Ishida}}(\tilde{\mathcal{C}}\,(\mathscr{P}_{F^c})_{\mathit{unmoved}})) \cong (\tilde{H}^{i-2}_{\mathit{CW}}(\mathscr{P}_{F^c})).$$

Proof. The u-graded part of the Ishida complex with I_Δ-support consists of components whose localizations are by faces in $(K_{I_\Delta})_{\mathfrak{r}_{F^c} \cap \mathbb{Z}Q}$. Hence, the u-graded part of the Ishida complex is equal to the unmoved Alexander dual chain complex of $\mathscr{P}_{F^c}$. Therefore, the first isomorphism is from Corollary 1.2.28. The second isomorphism is from the difference between homological degrees of $\mathscr{P}_{F^c}$ at the Ishida complex and those of $\mathscr{P}_{F^c}$ at the CW chain complex. $\square$

By construction, $\mathscr{P}_{F^c}$ is a polyhedral complex consisting of faces of K_{I_Δ} whose corresponding faces of K induce localizations not containing the region $\mathfrak{r}_{F^c} \cap \mathbb{Z}Q$. From a topological viewpoint, $\mathscr{P}_{F^c}$ is obtained in two steps; first, cutting out all faces of $\max(\Delta)$ from K to yield K_{I_Δ}. Next, cut out all faces whose corresponding faces of K inducing localizations containing $\mathfrak{r}_{F^c}$ from K_{I_Δ} to get $\mathscr{P}_{F^c}$. Recall that *cut* is defined rigorously at Definition 1.2.18 in Subsection 1.2.1. We claim that interchanging these two procedures results in a topological space homotopic to $|\mathscr{P}_{F^c}|$.

Let K^{F^c} be a simplicial complex whose dual is $(2^{[d]}/F^c)$ on the simplex K. In other words, $K^{F^c} := 2^{[d]} \smallsetminus (2^{[d]}/F^c)$. K^{F^c} is the result of cutting out all faces from K whose corresponding faces of K induces localization containing $\mathfrak{r}_{F^c}$. Now, we cut out all the maximal faces of Δ from $|K^{F^c}|$ if they exist. We claim that the union of those faces cut by this process is the *closure* $\overline{((K/F) \cap \Delta)}$ defined by $\overline{((K/F) \cap \Delta)} := |\{\sigma \in 2^{[n]} \mid \sigma \in \tau \text{ for some } \tau \in (K/F) \cap \Delta\}|$. This is because faces contained in the relative interior of K^{F^c} as a topological space are faces containing F. Hence, cutting a maximal face of $\max(\Delta)$ not containing F does not change the combinatorial connectedness of the K^{F^c}. This argument proves the lemma below.

Lemma 4.3.5. *If $F \neq K$, $|\mathscr{P}_{F^c}|$ is homotopic to the $|K^{F^c}| \smallsetminus \overline{((K/F) \cap \Delta)}$.*

Furthermore,

Lemma 4.3.6. *If $F \notin \bigcap \max(\Delta)$, then $(H^\bullet_{I_\Delta}(\Bbbk[Q]))_v = 0$ for any $v \in \mathfrak{r}_{F^c} \cap \mathbb{Z}Q$.*

Proof. From Lemma 4.3.5, it suffices to show that $|K^{F^c}| \smallsetminus \overline{((K/F) \cap \Delta)}$ is contractible for any $F \notin \bigcap \max(\Delta)$. First, suppose that F belongs to Δ but not to $\bigcap \max(\Delta)$. Then there exists a minimal face $G \in \bigcap \max(\Delta)$ containing F. Now, G contains the boundary of K^{F^c}, thus cutting G from K^{F^c} does not change its contractibility. Next, suppose that $F \notin \Delta$. Then, no faces of $\bigcap \max(\Delta)$ contain F, thus if a face of $\bigcap \max(\Delta)$ intersects K^{F^c}, then the intersection lies on the boundary of K^{F^c} as a topological space. This keeps $|K^{F^c}| \smallsetminus \overline{((K/F) \cap \Delta)}$ contractible. $\square$

Note that in the case when $F \neq \tilde{0}$ or K, K^{F^c} is homeomorphic to the ball D^{d-2} since it excludes all interior elements of $K = 2^{[d]}$ and "punctures" the boundary of K. If $F = \tilde{0}$, then K^{F^c} is homeomorphic to a sphere S^{d-2}. If $F = K$, then K^{F^c} is an empty set as a (-1)-dimensional polyhedral complex.

Now we are ready to show that there is a homotopic image of $\mathscr{P}_{F^c}$ which is the dual of $(K/F) \cap \Delta$ in a sphere $S^{(d-1)-\dim F-1}$. Recall that $(K/F) \cap \Delta$ as a section can be seen as a subspace of sphere $S^{(d-1)-\dim F-1}$ by taking vertex figures iteratively as mentioned in Subsection 1.2.1.

Lemma 4.3.7. *$\mathscr{P}_{F^c}$ is homotopic to $S^{(d-1)-\dim F-1} \smallsetminus (K/F) \cap \Delta$ where $(K/F) \cap \Delta$ as a $((d-1) - \dim F)$-dimensional polyhedral complex realized in $S^{(d-1)-\dim F-1}$.*

Proof. If $F = \tilde{0}$, then $\mathscr{P}_{F^c}$ is combinatorially equivalent to the empty set as a polytope and $(K/F) \cap \Delta \cong S^{d-2}$. Also, if $F = K$, then $\Delta = K$, thus $\mathscr{P}_{F^c} = \mathcal{F}(K) \smallsetminus \{K\}$ and $(K/F) \cap \Delta = \{K\}$. Hence the statement holds for these two cases.

If F is nonempty, not maximal nor minimal in $\mathcal{F}(K)$, recall that K^{F^c} is a simplicial complex homotopic to D^{d-2} having F in its relative interior. We claim that $K^{F^c} \smallsetminus \overline{((K/F) \cap \Delta)}$ is homotopic to its image on S^{d-3}. To see this, pick a vertex v in F and take a sphere S^{d-3} centered at v but not containing any other vertices. By translation, assume v is the origin of $\mathbb{R}^{d-2}$ embedded in $\mathscr{P}_{F^c}$. This induces a canonical homotopy map from punctured $\mathbb{R}^{d-2}$ to the sphere S^{d-3} restricted to $K^{F^c} \smallsetminus \overline{((K/F) \cap \Delta)}$ giving the desired homotopy. Lastly, use Lemma 4.3.5 and Lemma 1.2.17 to conclude that taking vertex figure on K^{F^c} preserves its image over a polyhedral complex homeomorphic to D^{d-2} if $\dim F \geq 1$, or to S^{d-3} if $\dim F = 0$. Iterate this for the other vertices in F to complete the argument. $\square$

Corollary 4.3.8. *For any $i \in \mathbb{Z}$, $\tilde{H}^i_{\mathit{CW}}(\mathscr{P}_{F^c}) \cong \tilde{H}^{((d-1)-\dim F-1)-i-1}_{\mathit{simp}}((K/F) \cap \Delta)$.*

Proof. Lemma 4.3.7 shows $\tilde{H}^i_{\mathrm{CW}}(\mathscr{P}_{F^c}) \cong \tilde{H}^i_{\mathrm{CW}}(S^{(d-1)-\dim F-1} \smallsetminus (K/F) \cap \Delta)$. Hence, from the topological Alexander duality and the isomorphism between simplicial homology and cohomology,

$$\begin{aligned}\tilde{H}^i_{\text{CW}}(\mathscr{P}_{F^c}) &\cong \tilde{H}_{\text{CW},((d-1)-\dim F-1)-i-1}((K/F)\cap\Delta) \cong \tilde{H}_{\text{simp},((d-1)-\dim F-1)-i-1}((K/F)\cap\Delta)\\ &\cong \tilde{H}^{((d-1)-\dim F-1)-i-1}_{\text{simp}}((K/F)\cap\Delta).\end{aligned}$$

□

Corollary 4.3.9. *Let $u \in S_F$ where $S_F \in \mathcal{G}(\Bbbk[x]/I_\Delta)$ indexed by (F,F). Then, for any $v \in \mathfrak{r}_{F^c} \cap \mathbb{Z}Q$,*

$$H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_u \cong H^{d-i}_{I_\Delta}(Q)_v.$$

$$\begin{aligned}H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_u &\underbrace{\cong}_{Theorem\ 4.2.4} \tilde{H}^{i-\dim F-1}_{\text{simp}}((K/F)\cap\Delta)\\ &\underbrace{\cong}_{Corollary\ 4.3.8} \tilde{H}^{(d-1)-\dim F-1-(i-\dim F-1)-1}_{\text{CW}}(\mathscr{P}_{F^c})\\ &\underbrace{\cong}_{Lemma\ 4.3.4} H^{(d-1)-\dim F-1-(i-\dim F-1)-1+2}_{I_\Delta}(\Bbbk[Q])_v \cong H^{d-i}_{I_\Delta}(\Bbbk[Q])_v.\end{aligned}$$

We are finally ready to prove the main result of this section.

Proof of Theorem 4.3.1. If $\Delta = K$, then $I_\Delta = 0$. Hence $H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta) = 0$ except for $i = d$, also $H^i_0(\Bbbk[Q]) = 0$ except for $i = 0$. Moreover, by computation, $H^d_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta) = \Bbbk[\mathfrak{r}_\emptyset \cap \bigcup \deg(\Bbbk[x])]$ and $H^0_0(\Bbbk[Q]) = \Bbbk[Q] = \Bbbk[\mathfrak{r}_{[d]} \cap \bigcup \deg(\Bbbk[x])]$. Therefore, the desired duality holds.

If $\Delta = 0$, then $I_\Delta = \mathfrak{m}$. Hence, from the generalized Ishida complex $H^i_{\mathfrak{m}}(\Bbbk[Q]/\mathfrak{m}) = 0$ except for $i = 0$; in case of $i = 0$, $H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta) = \Bbbk = \Bbbk[\{0\} \cap \bigcup \deg(\Bbbk[x])]$. Also, $H^i_{\mathfrak{m}}(\Bbbk[Q]) = 0$ except for $i = d$, and $H^d_{\mathfrak{m}}(\Bbbk[Q]) = \Bbbk[\mathfrak{R}_\emptyset \cap \bigcup \deg(\Bbbk[x])]$. Thus, the duality holds for this case.

For all other cases, Δ is a proper subcomplex of K. Theorem 4.2.4 shows that grains with index (F,F) for all $F \in \bigcap \max(\Delta)$ may have nonzero local cohomology. Also, Lemma 4.3.6 shows that grains $\mathfrak{R}_{F^c} \cap \mathbb{Z}Q$ for any $F \in \bigcap \max(\Delta)$ may have nonzero local cohomology. Lastly,

d	Δ	I_Δ	$\deg_{\mathbb{Z}^d} H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)$	$\deg_{\mathbb{Z}^d} H^i_{I_\Delta}(\Bbbk[Q])$
1	1-sim	0	$(\emptyset, \mathfrak{r}_\emptyset)$	$(\mathfrak{r}_1, \emptyset)$
1	$\emptyset$	$\mathfrak{m}$	$(\{0\}, \emptyset)$	$(\emptyset, \mathfrak{r}_\emptyset)$
2	2-sim	0	$(\emptyset, \emptyset, \mathfrak{r}_\emptyset)$	$(\mathfrak{r}_{1,2}, \emptyset, \emptyset)$
2	pt	$\langle x \rangle$	$(\emptyset, \mathfrak{r}_2, \emptyset)$	$(\emptyset, \mathfrak{r}_1, \emptyset)$
2	pt	$\langle y \rangle$	$(\emptyset, \mathfrak{r}_1, \emptyset)$	$(\emptyset, \mathfrak{r}_2, \emptyset)$
2	2 pts	$\langle xy \rangle$	$(\emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \{0\}, \emptyset)$	$(\emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_\emptyset, \emptyset)$
2	$\emptyset$	$\mathfrak{m}$	$(\{0\}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_\emptyset)$
3	3-sim	0	$(\emptyset, \emptyset, \emptyset, \mathfrak{r}_\emptyset)$	$(\mathfrak{r}_{1,2,3}, \emptyset, \emptyset, \emptyset)$
3	2-sim(yz)	$\langle x \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_1, \emptyset)$	$(\emptyset, \mathfrak{r}_{2,3}, \emptyset, \emptyset)$
3	2-sim(xz)	$\langle y \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_2, \emptyset)$	$(\emptyset, \mathfrak{r}_{1,3}, \emptyset, \emptyset)$
3	2-sim(xy)	$\langle z \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_3, \emptyset)$	$(\emptyset, \mathfrak{r}_{1,2}, \emptyset, \emptyset)$
3	(xz,yz)	$\langle xy \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_{1,2}, \emptyset)$	$(\emptyset, \mathfrak{r}_{2,3} \cup \mathfrak{r}_{1,3} \cup \mathfrak{r}_3, \emptyset, \emptyset)$
3	(xy,yz)	$\langle xz \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_3 \cup \mathfrak{r}_{1,3}, \emptyset)$	$(\emptyset, \mathfrak{r}_{2,3} \cup \mathfrak{r}_{1,2} \cup \mathfrak{r}_2, \emptyset, \emptyset)$
3	(xy,xz)	$\langle yz \rangle$	$(\emptyset, \emptyset, \mathfrak{r}_2 \cup \mathfrak{r}_3 \cup \mathfrak{r}_{2,3}, \emptyset)$	$(\emptyset, \mathfrak{r}_{1,3} \cup \mathfrak{r}_{1,2} \cup \mathfrak{r}_1, \emptyset, \emptyset)$
3	(xy,yz,xz)	$\langle xyz \rangle$	$(\emptyset, \emptyset, (\bigcup_{\substack{i=\{1\},\{2\},\{3\},\\ \{1,2\},\{1,3\},\{2,3\}}} \mathfrak{r}_i) \cup \{0\}), \emptyset)$	$(\emptyset, (\bigcup_{i \neq \{1,2,3\}} \mathfrak{r}_i), \emptyset, \emptyset)$
3	(x,yz)	$\langle xy, xz \rangle$	$(\emptyset, \mathfrak{r}_{2,3} \cup \{0\}, \mathfrak{r}_1, \emptyset)$	$(\emptyset, \mathfrak{r}_{2,3}, \mathfrak{r}_1 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(y,xz)	$\langle xy, yz \rangle$	$(\emptyset, \mathfrak{r}_{1,3} \cup \{0\}, \mathfrak{r}_2, \emptyset)$	$(\emptyset, \mathfrak{r}_{1,3}, \mathfrak{r}_2 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(z,xy)	$\langle xz, yz \rangle$	$(\emptyset, \mathfrak{r}_{1,2} \cup \{0\}, \mathfrak{r}_3, \emptyset)$	$(\emptyset, \mathfrak{r}_{1,2}, \mathfrak{r}_3 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(y,z)	$\langle x, yz \rangle$	$(\emptyset, \mathfrak{r}_{2,3} \cup \mathfrak{r}_{1,2} \cup \{0\}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_3 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(x,z)	$\langle y, xz \rangle$	$(\emptyset, \mathfrak{r}_{1,3} \cup \mathfrak{r}_{1,2} \cup \{0\}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_2 \cup \mathfrak{r}_3 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(x,y)	$\langle z, yz \rangle$	$(\emptyset, \mathfrak{r}_{1,3} \cup \mathfrak{r}_{2,3} \cup \{0\}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_2 \cup \mathfrak{r}_1 \cup \mathfrak{r}_\emptyset, \emptyset)$
3	(z)	$\langle x, y \rangle$	$(\emptyset, \mathfrak{r}_{1,2}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_3, \emptyset)$
3	(x)	$\langle y, z \rangle$	$(\emptyset, \mathfrak{r}_{2,3}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_1, \emptyset)$
3	(y)	$\langle x, z \rangle$	$(\emptyset, \mathfrak{r}_{1,3}, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_2, \emptyset)$
3	(x,y,z)	$\langle xy, xz, yz \rangle$	$(\emptyset, \mathfrak{r}_{2,3} \cup \mathfrak{r}_{1,3} \cup \mathfrak{r}_{1,2} \cup \{0\}^2, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_3 \cup \mathfrak{r}_\emptyset^2, \emptyset)$
3	$\emptyset$	$\mathfrak{m}$	$(\{0\}, \emptyset, \emptyset, \emptyset)$	$(\emptyset, \emptyset, \emptyset, \mathfrak{r}_\emptyset)$

Table 4.2: Table of local cohomologies over a simplicial affine semigroup ring $\Bbbk[Q]$ when whose dimension d is 1,2, or 3.

Corollary 4.3.9 provides the desired local cohomology. $\square$

Example 4.3.10. In Table 4.2, we summarize the degrees of local cohomology over a simplicial affine semigroup ring $\Bbbk[Q]$ (with $\dim \Bbbk[Q] = 1, 2,$ or 3) and with a radical monomial ideal J_Δ from a simplicial complex generated by $\{x\}$ (resp. $\{x, y\}$ or $\{x, y, z\}$) corresponding to the variables of $\Bbbk[Q]$.

For example, the 12th row of the table illustrates that, when $\Bbbk[Q]$ is a 3-dimensional simplicial affine semigroup ring, and Δ is a simplicial complex consisting of two edges $\overline{xy}$ and $\overline{yz}$, then its corresponding radical monomial ideal is $\langle xy \rangle$, thus for any $\overline{x^{\vec{u}}} \in \Bbbk[Q]/J_\Delta$ with $\vec{u} \in \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_{1,2}$ and $x^{\vec{v}} \in \Bbbk[Q]$ with $\vec{v} \in \mathfrak{r}_{2,3} \cup \mathfrak{r}_{1,3} \cup \mathfrak{r}_3$,

$$H^2_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_{\vec{u}} \cong H^{3-2}_{J_\Delta}(\Bbbk[Q]/I_\Delta)_{\vec{v}}$$

and for all other $\vec{u}$ and $\vec{v}$,

$$H^i_{\mathfrak{m}}(\Bbbk[Q]/I_\Delta)_{\vec{u}} = H^i_{J_\Delta}(\Bbbk[Q]/I_\Delta)_{\vec{v}} = 0.$$

4.4 Open problems

4.4.1 Finding a combinatorial Gorenstein criterion for quotients of affine semigroup rings

There are four important classes of Noetherian graded rings with the unique homogeneous maximal ideals, which form the following chain of inclusions: Cohen–Macaulay rings $\supset$ Gorenstein rings $\supset$ complete intersection rings $\supset$ regular rings. For each type of ring, except Gorenstein rings, there is a combinatorial characterization discerning whether a certain quotient $\mathbb{K}[Q]/I$ of affine semigroup ring by a monomial ideal has the given property or not. For example, $\mathbb{K}[Q]/I$ is regular if and only if $\mathbb{K}[Q]/I$ has only one degree pair isomorphic to $\mathbb{N}^d$ for some d [36][Theorem 14.4]. Also, [21] and the $\mathbb{Z}Q$-graded Koszul complex over $\mathbb{K}[Q]/I$ give such a combinatorial characterization of complete intersection rings among affine semigroup rings or quotients of them by monomial ideals.

Hence, it seems natural to ask for a combinatorial Gorenstein criterion for quotients of affine semigroup rings by monomial ideals. Currently, combinatorial characterizations of Gorenstein affine semigroup rings [10, 54] and Gorenstein Stanley-Reisner rings [27, 49] are known. The local cohomology of canonical modules of quotients of affine semigroup rings may answer this question. The challenge of this problem is to find a finite decomposition of the canonical module compatible with the Ishida complex.

4.4.2 Characterizing local cohomology modules with infinite dimensional socles

The *support* of a graded module M is a set of multidegrees d whose corresponding graded piece M_d is nonzero. The degree pair topology shows that the supports of local cohomology modules $H^\bullet_J(\mathbb{K}[Q]/I)$ are covered by grains whose chaff is non-exact, i.e. has topological holes. The supports of $H^\bullet_J(\mathbb{K}[Q]/I)$ are unions and set differences of sets of lattice points in polyhedral objects, since grains are exactly those sets. On the other hand, the degrees of the socle of a module M can be regarded as lattice points on faces whose outer normal vectors equal the generators of Q. Hence, one may ask whether one can find a combinatorial condition to determine whether $H^\bullet_J(\mathbb{K}[Q]/I)$ has an infinite-dimensional socle [42][Problem 13.18]. This problem was inspired by Hartshorne's counter-example for Grothendieck's conjecture.

4.4.3 Class groups of non-normal toric varieties

Degree pairs may help to study class groups of non-normal toric varieties. Indeed, we constructed *void pairs* which organize all non-lattice points of the polyhedral cone $\mathbb{R}_{\geq 0}Q$ in terms of translations of faces [40]. Void pairs might allow one to calculate monomial fractional ideals of non-normal affine semigroup rings. If this was true, then we could calculate the class group of non-normal toric varieties by generalizing the fact that the class group of normal toric varieties is the direct sum of the class group of the field and that of the corresponding normal monoid [10][Theorem 4.60].

4.4.4 Irreducible resolution of quotients of non-normal affine semigroup rings

An ideal W of $\mathbb{K}[Q]$ is *irreducible* if W cannot be expressed as an intersection of two distinct ideals [42]. The *irreducible resolution* of a module M is an exact sequence $0 \to M \to \overline{W}^0 \to \cdots$ such that each $\overline{W}^i$ is a direct sum of quotients of $\mathbb{K}[Q]$ by irreducible ideals. Unless Q is normal, there is no known algorithm for constructing irreducible resolutions. Although an algorithm for constructing irreducible decomposition of normal affine semigroup rings was provided in [24], this is currently not implemented. A general algorithm was provided in [39] which is implemented in [58], but irreducible resolutions are not yet developed in this context. Void pairs introduced in Section 4.4.3 seem to give a promising approach towards finding an implementable algorithm for irreducible resolutions of monomial ideals over non-normal affine semigroup rings for `Macaulay2` or `SageMath`.

4.4.5 Classifying acyclic chaffs using hyperplane arrangements

Chaffs can be described using the *poset of regions* generated by hyperplanes of $\mathbb{R}_{\geq 0}Q$. The face lattice of $\mathbb{R}_{\geq 0}Q$ is embedded in this poset [5]. For example, every grain of the zero ideal of an affine semigroup ring has chaff equivalent to the intersection of the upper set of the poset of regions and the face lattice of the cone of the corresponding affine semigroup.

Despite the fact that the situation when the poset of regions is a lattice is well studied [16, 43–45], it is still unclear whether the intersection between the upper set of the poset of regions and the face lattice of cone as a sub-lattice of the poset of regions is acyclic or not. In the spirit of the characterization of f-vectors of simplicial complexes [17, 50] in relation to Kalai's conjecture [6], this question may shed lights on the combinatorial decomposition of polyhedral complexes.

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