ON LEFT-INVARIANT RANK 2 DISTRIBUTIONS WITH ZERO GENERALIZED WILCZYNSKI INVARIANTS ON 6-DIMENSIONAL LIE GROUPS

A Thesis

by

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ABSTRACT

The thesis is devoted to the study of the fundamental system of invariants of bracket-generating rank 2 distributions, i.e. fields of planes, on n-dimensional manifolds (shortly, $(2, n)$-distributions). Distributions are considered modulo the natural action of the group of diffeomorphisms. In the case of $n = 5$, the minimal n for which there are locally nonequivalent maximally nonholonomic $(2, n)$-distributions, E. Cartan constructed the fundamental invariant called the *Cartan tensor* and showed that vanishing of this invariant implies that the distribution is locally equivalent to the maximally symmetric one. In a series of works in the 2000s A . Agrachev, B. Doubrov, and I. Zelenko discovered the relation between local geometry of $(2, n)$-distributions and classical geometry of self-dual curves in $(2n-7)$-dimensional projective space. In particular, from their theory, it follows that a collection of $n - 4$ fundamental invariants for such self-dual curves, constructed by E. Wilczynski in 1905 gives rise to the new invariants of $(2, n)$-distributions, called the *generalized Wilczynski invariants*. I. Zelenko proved that in the case of $n = 5$, wherein there is only one generalized Wilczynski invariant, it coincides with the Cartan tensor. Further, B. Doubrov and I. Zelenko showed that there exists exactly one, up to local equivalence, maximally symmetric bracket-generating $(2, n)$-distribution with the 5-dimensional cube, called the *symplectically flat distribution*, and for such distribution, all generalized Wilczynski invariants vanish identically. The natural question is whether or not the symplectically flat distribution is the only, up to local equivalence, distribution with this property. For $n = 5$ the answer to this question is positive due to E. Cartan. The main result of the thesis is that the answer is also positive in the case of left-invariant maximally nonholonomic $(2, 6)$-distributions on Lie groups. We also give an example of a left-invariant distribution with both Wilczynski invariants equal to zero on a 6-dimensional Lie group which is not isomorphic to the nilpotent Lie group on which the symplectically flat distribution naturally lives. Among other new results, we clarify the algebraic structure of the generalized Wilczynski invariants and their dependence on the Tanaka symbol of a distribution.

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1. INTRODUCTION

1.1 Equivalence problem for distributions

A rank l vector distribution D on an n-dimensional manifold M is, by definition a vector subbundle of the tangent bundle TM of M with l-dimensional fibers. Equivalently, for every point q_0 the exists a neighborhood U and a tuple $\{X_1, \dots X_l\}$ of vector fields in U such that

$$D(q) = \text{span}\{X_1, \dots X_l\}, \quad \forall q \in U \tag{1.1}$$

The tuple of vector fields $\{X_1, \dots X_l\}$ satisfying (1.1) is called the *local basis* of the distribution D. Informally, for each point, $q \in M$, a $l-$dimensional subspace $D(q)$ of the tangent space T_qM is chosen, and $D(q)$ smoothly depends on q.

Two vector distributions D_1 and D_2 are called *equivalent*, if there exists a diffeomorphism $F : M \to M$ such that

$$F_*D_1(1) = D_2(F(q))), \quad \forall q \in M, \tag{1.2}$$

where F_* stands for the push-forward of the diffeomorphism F defined on the tangent bundle TM. Two germs of vector distributions D_1 and D_2 at point $q_0 \in M$ are called *locally equivalent*, if there exist neighborhoods U and $\tilde{U}$ and a diffeomorphism $F : U \to \tilde{U}$ such that

$$F_*D(q) = D_2(F(q)), \forall q \in U;$$
$$F(q_0) = q_0$$

In other words, the equivalence (local equivalence) of distributions is the equivalence relations defined by the natural action (i.e., by push-forward) of the group (pseudo-group) of diffeomorphisms (local differomorphism) on the set of distributions (germs of distributions).

The way to solve the corresponding equivalence problem for distributions (germs of distributions), i.e., to determine whether or not two distributions (germs of distributions) are equivalent (locally equivalent), is to construct and analyze their invariants under the equivalence, i.e., the quantities which are preserved under the natural action of the group of diffeomorphism (the presidio-group of local diffeomorphisms).

A basic example of a discrete invariant of distribution D at q is the *small growth vector* at q, which can be obtained by making iterative Lie brackets of the vector fields tangent to the distribution. In more detail, we construct the filtration of the tangent bundle TM

$$D \subset D^2 \subset \dots D^r \subset \dots \subset TM \tag{1.3}$$

with

$$D^2 = D + [D, D], \dots, D^{j+1} = D^j + [D, D^j], \tag{1.4}$$

where

$$[D, D^j] = \operatorname{span}\{[X, Y] : X \in D, Y \in D^j\} \tag{1.5}$$

and D^j is called the *$j-$th power of the distribution D.*

The *small growth vector* at q, which can be obtained is a tuple of integers

$$(\dim D(q), \dim D^2(q), \dots) \tag{1.6}$$

of the distribution D. From the naturality of the Lie brackets of vector fields, i.e., the fact that $F_*[X, Y] = [F_*X, F_*Y]$, it follows that the small growth vector is the invariant of the equivalence problem.

If there exists r such that $D^r = TM$, we say that D is a bracket-generating distribution. And the minimal such r is called the *degree of nonholonomy* of the distribution D. By Rashevskii-Chow theorem [7, 17], if D is bracket generating distribution on a connected manifold M, then any two points of M can be connected by a curve which is the concatenation of smooth curves tangent to D at each point.[1]

[1] In fact, it can be shown [15] that they can be shown by a smooth curve.

In general, for $j > 1$, the dimensions of $D^j(q)$ vary as one varies the point q. A distribution D is called *equiregular* at a point q_0 if there is a neighborhood U of q_0 in M for each $j > 0$ the dimensions of subspaces $D^j(q)$ are constant for all $q \in U$, i.e., the small growth vector of D at every point $q \in U$ is the same. We will say that a distribution is an $(\ell_1, \ell_2, \ldots, \ell_r)$*-distribution*, where $\ell_r = n$ if its small growth vector at every point is equal to $(\ell_1, \ell_2, \ldots, \ell_r)$.

From now on, we will consider equiregular and bracket-generating distributions. These assumptions are not quite restrictive. Indeed, regarding equiregularity, since for every positive integer j, the map $q \to \dim D^j(q)$

from M to $\mathbb{Z}^+$ is lower semi-continuous, the set of all regular points is generic on M so that any distribution is equiregular in a neighborhood of a generic point. Furthermore, if a distribution is equiregular but not bracket generating in some neighborhood U, then there exists a positive integer r such that D^r is a proper involutive subbundle of TU and the distribution D is bracket generating on each integral submanifold of D^r in U. So, we can restrict ourselves to this integral submanifold instead of U.

The small growth vector is usually a very rough invariant of distributions, i.e., it determines a distribution up to equivalence only in very few cases, as uequiangularvalence classes of distributions are determined by functional invariants. Let us make a rough estimate for the number of functional parameters in the equivalence problem under consideration. Recall that the Grassmannian $\mathrm{G}r_n(l)$ of n-dimensional subspaces of $\mathbb{R}^n$ is an $(n-l)l$-dimensional manifold: Indeed, fix an l-dimensional subspace $\Lambda_0 \in \mathrm{G}r_n(l)$ and a complimentary $(n-l)$-dimensional subspace Λ_∞. Denote by $\Lambda_\infty^\pitchfork$ the subset of $\mathrm{G}r_n(l)$, consisting of all l-dimensional subspaces of $\mathbb{R}^n$ transversal to Λ_∞. Then any $\Lambda \in \Lambda_\infty^\pitchfork$ can be seen as a graph of the unique linear map $L_\Lambda : \Lambda_0 \to \Lambda_\infty$. Hence, fixing bases on Λ_0 and Λ_∞, one can assign to any $\Lambda \in \Lambda_\infty^\pitchfork$ the $(n-l) \times l$-matrix A_Λ representing the linear map L_Λ in these bases. The set $\Lambda_\infty^\pitchfork$ and the map $\Lambda \mapsto A_\Lambda$ defines a coordinate chart, and the collection of these charts defines the structure of smooth $(n-l)l$-dimensional manifold on $\mathrm{G}r_n(l)$.

This implies that in a fixed coordinate system on M, in order to describe an (l, n)-distribution

D, one needs $l(n-l)$ functions, and these functions are independent in general. A coordinate change is given by n functions, so generically, we can normalize only n functions among those $l(n-l)$ functions describing D. This means that $d(l,n) := l(n-l)-n$ functions cannot be normalized.

Although these arguments are not quite rigorous, in all cases when the quantity $d(l,n)$ is non-positive, i.e., when according to these arguments the functional invariants are not expected, the generic germs of (l,n)-distributions are equivalent one to each others. Except the trivial cases of $l=0$ and $l=n$, the quantity $d(l,n)$ is non-positive only in the following three case:

1. $l=1$, i.e D is a line distribution. In this case $d(1,n)=-1$. By the standard theorem on the rectification of vector fields without stationary points, all line distributions are locally equivalent to the distribution generated by the vector field $\frac{\partial}{\partial x_1}$ in some local coordinates $(x_1,\ldots,x_n)$.

2. $l=n-1$, i.e. D is a corank 1 distribution. In this case $d(n-1,n)=-1$. In this case, the distribution D is the kernel of a nonzero 1-form α defined up to a multiplication by a nonzero function, and for any $q\in M$ the form $d\alpha(q)|_{D(q)}$ is well defined up to a multiplication by a constant. By the classical Darboux theorem, if the rank of the form $d\alpha(q)|_{D(q)}$ is constant and equal to $2r$, then there exists a local coordinate system $(x_1,\ldots,x_r,p_1,\ldots,p_r,u,v_1,\ldots v_{n-1-2r}$ such that the distribution D is the kernel of the 1-dorm

$$du - p_1 d\,x_1 - \ldots - p_r d\,x_r. \tag{1.7}$$

   In the case when the rank $2r$ of the form $d\alpha(q)|_{D(q)}$ is maxima,l possible, i.e. when $r = \left[\dfrac{n-1}{2}\right]$, the distribution D is called *contact if n is odd and quasi-contact or even-contact if n is even. Obviously, germs of contact and quasi-contact distributions are generic among germs of corank 1 distributions and by the Darboux theorem, all such distributions are locally equivalent to each other. Note that from the formula* (1.7)*, it follows that in the case of* *odd $n = 2k+1$ contact distributions are locally equivalent, to the natural distribution on the space $J^1(\mathbb{R}^k, \mathbb{R})$ of the first jets of scalar functions $u(x_1 others, x_k)$ of k variables, with $p_i = \frac{\partial u}{\partial x_i}$.*

3. *$l = 2$ and $n = 4$, i.e. D is a distribution of planes on a 4-dimensional manifold. Assume that at a point q, the small growth vector is $(2, 3, 4)$, i.e., the iterative Lie brackets of sections of D grow in the maximal possible way. Note that if Then, by the Engel theorem, there exists a local coordinate system (t, x_0, x_1, x_2) such that D is the intersection of kernels of the 1-forms $dx_0 - x_1 dt$ and $dx_1 - x_2 dt$. Note that this is a natural distribution on the space $J^2(\mathbb{R}, \mathbb{R}^2)$ of 2-jets of functions from $\mathbb{R}$ to $\mathbb{R}^2$.*

Note that in the first two cases, $d(l, n) = -1$ and the analogous statements are valid for more refined objects such as vector fields and 1-forms that are described by one more function in a fixed coordinate system compared to line distributions and corank 1 distributions respectively. Therefore by the analogous counting arguments functional invariants are not expected also for these more refined structures and moreover, the normal forms of items 1 and 2 above are also valid for them in generic position.

For all other pairs, (l, n) generic germs of distribution have functional invariants and therefore there are non-equivalent germs of distributions with the same small growth vector.

1.2 Problem statement and main results

The case $l = 2$ and $n = 5$ is the smallest dimensional case when the functional invariants are expected, and it was treated by E. Cartan in 1910 [5]. To every such distribution, Cartan assigned an invariant homogeneous polynomial of degree 4 on each plane $D(q)$. He called it the *covariant binary biquadratic form*. We call it *Car n's tensor*). Moreover, Cartan proved that a $(2, 3, 5)$-distribution has the vanishing Cartan tensor if and only if it is locally equivalent to a specific distribution. The germ of this distribution, up to local equivalence, will be called the *Cartan flat distribution*. In modern language, the latter result is a particular case of the general theory of harmonic curvatures in parabolic geometry ([21, 6, 23]).

To describe the Cartan flat distribution, describe first a class of rank 2 distributions in $\mathbb{R}^n$ that come from so-called underdetermined ODEs of the type

$$z'(x) = F\big(x, y(x), \dots, y^{(n-3)}(x), z(x)\big), \tag{1.8}$$

for two functions $y(x)$ and $z(x)$. Setting $p_i = y^{(i)}$, $0 \le i \le n-3$, with each equation (1.8) one can associate the rank 2 distribution on $\mathbb{R}^n$ with coordinates $(x, p_0, \dots, p_r, z)$ defined as the annihilator of the following $n-2$ 1-forms:

$$\begin{aligned} &dp_i - p_{i+1}dx,\ \ 0 \le i \le n-4, \\ &dz - F(x, p_0, \dots, p_{n-3}, z)dx. \end{aligned} \tag{1.9}$$

The Cartan flat distribution is represented by the distribution associated with the underdetermined ODE $z'(x) = \big(y''(x)\big)^2$.

Recall that a vector field X is called an infinitesimal symmetry of a distribution D if $[X, D] \subset D$ or ,equivalently, if the local flow generated by X preserves D. Cartan proved that the Cartan flat distribution is the maximally symmetric distribution among all $(2,3,5)$-distribution having 14-dimensional algebra of infinitesimal symmetries isomorphic to split real form of the exceptional simple Lie algebra G_2.

In 2006 in [25] I. Zelenko constructed the analog of Cartan's tensors of bracket-generating rank 2 distributions on n-diemnsional manifold for every $n \ge 5$ using the ideas from Optimal Control theory, namely studying certain dynamics in the cotangent bundle along the foliation of so-called abnormal extremals (singular curves) of distributions, see alsosections2 below. In this way, the relation between local geometry of $(2,n)$-distributions with 5-dimensional D^3 and differential geometry of curve in Lagrangian Grassmannians was found (see also section 3 below) and the first nontrivial invariant of such curves gave the invariant of such distributions, called the *fundamental ifrm*. Zelenko has shown in [26] that in the case $n = 5$, his fundamental form coincides with the Cartan tensor.

Later B. Doubrov and i. Zelenko ([11],[13], using the classical E. Wilczynski's theory of self-dual curves in projective spaces ([22]), constructed $n-4$ invariants of rank 2 distribution, called the *generalized Wilczynski invariant* on n-dimensional manifolds satisfying certain genericity assumption called maximality of class (see also sections 4, 5 below). The very first generalized Wilczynski invariant coincides with Zelenko's fundamental form. Doubrov and Zelenko also show in [10] that among all rank 2 distributions of maximal class with 5-dimensional D^3 on an n dimensional manifold, $n \geq 5$ the maximally symmetric ones are isomorphic to the one associated with with the underdetermined ODE $z'(x) = \big(y''(x)\big)^2$, generalizing Cartan's results in the case of $n = 5$. This distribution will be called *symplectically flat* as the cotangent bundle and the language of symplectic geometry is crucial in this theory, and in order to make a distinction with the notion of flat distribution with the given Tanaka symbol, see subsection 6.1 below. Moreover, the symplectically flat distribution has all $n-4$ generalized Wilczynski invariants equal to zero.

The following question is central to the current thesis:

Main question *Whether or not from the fact that all $n-4$ generalized Wilczynski invariants vanish, it follows that the rank 2 distribution with 5-dimensional D^3 on an n dimensional manifold is equivalent to the symplectically flat one in the case $n = 6$?*

By the above, for $n = 5$ the answer to this question is positive. In [11], the positive answer to this question was given for rank 2 distributions associated with underdetermined ODEs of the type

$$z'(x) = F\big(x, y(x), \dots, y^{(n-3)}(x)\big), \quad F_{y^{(n-3)}y^{(n-3)}} \neq 0 \tag{1.10}$$

i.e. when F in (1.8) is independent of z (the condition $F_{y^{(n-3)}y^{(n-3)}} \neq 0$ is needed to ensure that $\mathrm{rank} D^3 = 5$.

The following reformulation of the aforementioned result from [11] will be crucial for us here:

Theorem 1.2.1. *[11, Proposition 2.1 and Theorem 5.1] If a $(2,3,5,6)$-distribution D has an infinitesimal symmetry X lying in D^3 but not in D^2 [2] Then, because both Wilczynski invariants of D*

[2]In fact, in [11][Proposition 2.1] the result is formulated for a rank 2 distribution on n-dimensional manifold with $n \geq 5$ and $\dim D^3 = 5$ under additional assumption that the factorization by the foliation of integral curves of X

vanish, it follows that the distribution is equivalent to the symplectically flat distribution.

The main result of this thesis is that the answer to the main question is positive in the case of left-invariant $(2,3,5,6)$-distribution on Lie groups (see Theorem 8.1.1). In section 9, we also give an example of a left-invariant distribution with both Wilczynski invariants equal to zero on a 6-dimensional Lie group which is not isomorphic to the nilpotent Lie group on which the symplectically flat distribution naturally lives as a left-invariant one. Among other new results, we show in Theorem 6.2.2 that a Wilczynski invariant of a distribution D at a point q is not zero if it is nonzero for the flat distribution corresponding to the Tanaka symbol of D at q.

sends D to the Goursat distribution on the quotient manifold. Note that for $n = 6$, this condition is equivalent to the one given in the present formulation

2. ABNORMAL EXTREMALS

The main idea of [25], based on earlier works [2, 1, 4], for the construction of invariants of distributions was to use the notion of abnormal extremals, coming from the Pontryagin Maximum Principle (PMP) in Optimal Control [16, 3]. Given a distribution D, a *horizontal* (or *admissible*) curve of D is an absolutely continuous curve $\alpha : [0, T] \to M$ which is tangent to D at almost every point, i.e., $\alpha'(t) \in D(\gamma(t))$ for almost every $t \in [0, T]$. By the aforementioned Rashevskii-Chow theorem, the set $\mathrm{Hor}_D(q_0, q_1)$ of the horizontal curves of a bracket -generating distribution D on a connected manifold M connecting two given points q_0 and q_1 of M is nonempty. Defining on this set of curves any cost functional of integral type, for example, a length with respect to a Riemannian metric on M, we get an optimal control problem with constraints given by the distribution D and fixed initial and terminal points: among all curves from $\mathrm{Hor}_D(q_0, q_1)$ to find a curve on which the chosen cost functional attains its minimum. Such a curve is called a *minimizer* of the optimal control problem.

The Pontryagin Maximum Principle gives the necessary condition for a curve to be the minimizer of such optimal control problem for all pairs of initial and terminal points (q_0, q_1): the minimizers are described as projections of special curves in the cotangent bundle T^*M, called the Pontryagin extremals.

There are two types of Pontryagin extremals: *normal* and *abnormal*. Normal extremals correspond to a nonzero Lagrange multiplier near the cost functional, while abnormal extremals correspond to a zero Lagrange multiplier near the cost functional and so they do not depend on the cost functional but on the distribution D itself. So the abnormal extremals can be described purely geometrically in D without referring to any optimal control problem.

Below we restrict ourselves to this geometric description without relating it to the Pontraygin Maximum Principle, as it is enough for our purposes. For this, we need to recall the construction of the canonical symplectic structure on the cotangent bundle T^*M. Let $\pi : T^*M \to M$ denote the canonical projection The *tautological Liouville* 1*-form* s on T^*M is defined as follows: if

$\lambda = (p, q) \in T^*M$, where $q \in M$ and $p \in T_q^*M$, and $v \in T_\lambda T^*M$, then

$$s(\lambda)(v) := p\big(\pi_*(v)\big). \tag{2.1}$$

Given a differential 2-form ω on a manifold N its kernel $\mathrm{Ker}\omega_z$ at a point $z \in N$ is defined as follows:

$$\mathrm{Ker}\ \omega_z = \{X \in T_zN, \omega_z(X, Y) = 0, \forall Y \in T_zN\}. \tag{2.2}$$

A differential 2-form ω is called nondegenerate at a point z if $\mathrm{Ker}\,\omega_z = 0$ and it is degenerate otherwise. Recall that a symplectic manifold is a manifold endowed with a closed nondegenerate differential 2-form, called a symplectic structure.

Remark 2.1.1. *Note that if* $\dim N$ *is odd, then a* 2*-form is degenerate at any point. This is a consequence of the fact from linear algebra that the determinant of a skew-symmetric matrix of odd size is zero. So a symplectic manifold has to be of an even dimension. Also, note that if the dimension of* N *then the condition for a* 2*-form is nondegenerate is generic.*

It turns out that the exterior derivative of the tautological 1-form s:

$$\sigma := ds. \tag{2.3}$$

is nondegenerate. The form σ is called the *canonical symplectic structure on* T^*M and it makes T^*M symplectic manifold.

Further, the filtration (1.3) on TM defines the dual filtration on T^*M. To define the latte, we call $(D^j)^\perp \subset T^*M$ the *annihilator of the* j*th power* D^l, i.e

$$(D^j)^\perp = \{\lambda \in T^*M : \lambda = (p, q), p(v) = 0,\ \forall v \in D^j(q)\} \tag{2.4}$$

In particular, since $D^1 = D$ the annihilator $D^\perp$ of the distribution D itself is in fact $(D^1)^\perp$. Note that $D^\perp$ is a codimension l submanifold of T^*M. We will say that a curve $\gamma : [0, t] \to T^*M$

is nontrivial if $\gamma([0, T]$ is not one point.

Definition 2.1.2. *An abnormal extremal of the distribution D is an absolutely continuous nontirival curve $\gamma : [0, T] \to T^*M$ such that the following two conditions hold*

$$\gamma(t) \in D^{\perp} \quad \forall t \in [0.T], \tag{2.5}$$

$$\dot{\gamma}(t) \in \mathrm{Ker}(\sigma|_{D^{\perp}})_{\gamma(t)} \quad \textit{a.e.} \quad t \in [0, T], \tag{2.6}$$

where $\sigma|_{D^{\perp}}$ is the restriction of the canonical form σ to the annihilator D^{perp} of D.

By the last sentence of Remark (2.1.1), if l is even, then one expects that $\mathrm{Ker}(\sigma|_{D^{\perp}})_{\lambda} = 0$ for a generic point $\lambda \in D^{perp}$ so there is no abnormal extremal passing through such a point and the locus of abnormal extremals belong to a proper (stratified) submanifold of $D^{\perp}$.

To understand this more quantitatively, let us describe Definition 2.1.2 in terms of a local basis $\{X_1, \ldots, X_l\}$ of the distribution D. For this, we need more constructions from elementary symplectic geometry.

A Hamiltonian is any smooth function on T^*M. To any Hamiltonian H one can assign a vector field $\vec{H}$ on T^*M as the unique vector field satisfying

$$i_{\vec{H}}\sigma = -dH, \tag{2.7}$$

where $i_{\vec{H}}$ denotes the operation of the interior product, $(i_{\vec{H}})\,\sigma_{\lambda}(Y) := \sigma(\vec{H}, Y)$ for all $Y \in T_{\lambda}T^*M$. The existence and uniqueness of $\vec{H}$ satisfying (2.7) is the direct consequence of the nondegeneracy of σ. The vector field $\vec{H}$ is called the *Hamiltonian vector field* associated with the Hamiltonian H. Given two Hamiltonians H_1 and H_2, the *Poisson brackets* $\{H_1, H_2\}$ is another Hamiltonian defined by

$$\{H_1, H_2\} := \overrightarrow{H_1}(H_2) = d\,H_2\big(\overrightarrow{H_1}\big). \tag{2.8}$$

Now given a vector field X on M define the Hamiltonian $H_X : T^*M \to \mathbb{R}$, the *quasi-impulse* of

X) as follows H_X:

$$H_X(\lambda) = p\Big(X_i(q)\Big), \quad \text{where } \lambda = (p, q), q \in M, p \in T_q^*M \tag{2.9}$$

It is easy to see that the corresponding Hamiltonian vector field $\overrightarrow{H_X}$ satisfies $\pi_* \overrightarrow{H_X} = X$. The vector field is called the *Hamiltonian lift* of X. The Hamiltonian lifts satisfy the following important natural property:

$$\overrightarrow{H_X}(H_Y) = d\, H_Y\big(\overrightarrow{H_X}\big) = \{H_X, H_Y\} = H_{[X,Y]}. \tag{2.10}$$

In particular, the Hamiltonian lift establishes the isomorphism between the Lie algebra of vector fields on M with the product given by the Lie brackets and the Lie algebra of quasi-impulses with the product given by the Poisson brackets.

Now, let $\{X_1, \ldots, X_l\}$ be a local basis of distribution D and let

$$u_i := H_{X_i}. \tag{2.11}$$

Then

$$D^\perp = \{\lambda \in T^*M : u_1(\lambda) = \ldots = u_l(\lambda) = 0\}. \tag{2.12}$$

From this , using (2.7), it is easy to show that

$$\mathrm{Ker}(\sigma|_{(D)^\perp})_\lambda = \mathrm{span}\{\vec{u_1}\big(\lambda\big), \ldots, \vec{u_l}\big(\lambda\big)\}, \quad \forall \lambda \in D^\perp. \tag{2.13}$$

This implies the following

Lemma 2.1.3. *An absolutely continuous nontrivial curve $\gamma : [0, T] \to T^*M$ is an abnormal extremal of D if and only if condition* (2.5) *holds and*

$$\dot\gamma(t) \in \mathrm{span}\{\vec{u_1}\big(\gamma(t)\big), \ldots, \vec{u_l}\big(\gamma(t)\big)\} \quad \textit{a.e.} \quad t \in [0, T], . \tag{2.14}$$

Now apply Lemma 2.1.3 to the case $\operatorname{rank} D = 2$. From now on we assume that $\operatorname{rank} D^2 = 3$ The following notations will be convenient in the sequel

$$W_D = (D^2)^\perp \backslash (D^3)^\perp \tag{2.15}$$

$$\mathcal{C}(\lambda) = \operatorname{Ker}(\sigma|_{D^\perp})_\lambda \cap T_\lambda (D^2)^\perp, \quad \lambda \in (D^2)^\perp \tag{2.16}$$

Proposition 2.1.4. *An abnormal extremal of a rank* 2 *distribution* D *lies in* $D^{2\perp}$*. Moreover, if* $\lambda \in W_D$*, then* $\dim \mathcal{C}(\lambda) = 1$*, and a curve lying in* W_D *is an abnormal extremal if and only if it is an integral curve of the rank* 1 *distribution* $\mathcal{C}$*.*

Proof. if γ is an abnormal extremal then by (2.14) there exist absolutely continuous functions $\lambda_1(t)$ and $\lambda(t)$ such that

$$\dot{\gamma}(t) = \lambda_1(t)\vec{u_1}\big(\gamma(t)\big) + \lambda_2(t)\vec{u_2}\big(\gamma(t)\big) \quad \text{a.e.} \quad t \in [0, T], . \tag{2.17}$$

By (2.5) and (2.12) for a.e. t

$$\dot{\gamma}(t) \in T_{\gamma(t)} D^\perp = \{v \in T_{\gamma(t)} T^*M : du_1(v) = du_2(v) = 0\}. \tag{2.18}$$

Hence, substituting (2.17) into (2.18) and (2.10), we get

$$\begin{aligned} d\,u_1(\alpha_1(t)\vec{u_1}\big(\gamma(t)\big) + \alpha_2(t)\vec{u_2}\big(\gamma(t)\big) &= -\alpha_2(t) H_{[X_1,X_2]}\big(\gamma(t)\big) = 0 \\ d\,u_2(\alpha_1(t)\vec{u_1}\big(\gamma(t)\big) + \alpha_2(t)\vec{u_2}\big(\gamma(t)\big) &= \alpha_1(t) H_{[X_1,X_2]}\big(\gamma(t)\big) = 0 \end{aligned} \tag{2.19}$$

Hence, if $\dot{\gamma}(t) \neq 0$, at least one of $\alpha_i\big(\gamma(t)\big)$ is not zero, so (2.19) implies that

$$H_{[X_1,X_2]}\big(\gamma(t)\big) = 0. \tag{2.20}$$

so $\gamma(t) \in (D^2)^\perp$. From nontriviality and absolute continuity of γ it follows that γ lies $(D^2)^\perp$.

Further, if $v \in \mathcal{C}(\lambda)$ where $\mathcal{C}(\lambda)$ is as in (2.16), then there exist scalars α_1 and α_2 such that

$$v = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2, \tag{2.21}$$

$$dH_{[X_1, X_2]}(v) = 0. \tag{2.22}$$

Substituting (2.21) to (2.22) and using (2.10), we get

$$\alpha_1 H_{\big[X_1,[X_1,X_2]\big]}(\lambda) + \alpha_2 H_{\big[X_2,[X_1,X_2]\big]}(\lambda) = 0 \tag{2.23}$$

Since by our assumptions $\lambda \notin (D^3)^\perp$, either $H_{\big[X_1,[X_1,X_2]\big]}(\lambda)$ or $H_{\big[X_2,[X_1,X_2]\big]}(\lambda)$ is not zero, so $\mathcal{C}(\lambda)$ is 1-dimensional and in fact

$$\mathcal{C}(\lambda) = \operatorname{span}\Big\{ H_{\big[X_1,[X_1,X_2]\big]}(\lambda)\vec{u}_2(\lambda) - H_{\big[X_2,[X_1,X_2]\big]}(\lambda)\vec{u}_1(\lambda) \Big\}. \tag{2.24}$$

The last statement of the proposition follows from the fact that an abnormal extremal must be tangent to the rank 1 distribution $\mathcal{C}$ at every point of W_D, which follows from the previous statement of the proposition that we proved. $\square$

Definition 2.1.5. *The rank 1 distribution $\mathcal{C}$ defined by* (2.16) *(or , equivalently, by* (2.24)*) is called the characteristic line distribution of D on W_D. The foliation of abnormal extremals generated by the characteristic line distribution $\mathcal{C}$ is called the characteristic foliation of W_D. The leaves of this foliation are called regular abnormal extremals.*

Remark 2.1.6. Note that for $\lambda \in W_D$ the characteristic line distribution satisfies

$$\mathcal{C}(\lambda) = \operatorname{Ker}(\sigma|_{(D^2)^\perp})_\lambda, \quad \lambda \in W_D \tag{2.25}$$

Indeed, if $v \in \mathrm{Ker}(\sigma|_{(D^2)^\perp})_\lambda$ then there exist scalars α_1 and α_2 such that

$$v = \alpha_1 \vec{u}_1(\lambda) + \alpha_2 \vec{u}_2(\lambda) + \alpha_3 \overrightarrow{H}_{[X_1,X_2]}(\lambda), \tag{2.26}$$

$$du_1(v) = du_2(v) = 0 \tag{2.27}$$

$$dH_{[X_1,X_2]}(v) = 0. \tag{2.28}$$

Plugging (2.26) into (2.27) and using (2.10) and the fact that $W_d \subset (D^2)^\perp$, we get

$$\alpha_3 H_{\left[X_1,[X_1,X_2]\right]}(\lambda) = \alpha_3 H_{\left[X_2,[X_1,X_2]\right]}\big(\lambda\big) = 0. \tag{2.29}$$

Since by our assumptions $\lambda \notin (D^3)^\perp$, either $H_{\left[X_1,[X_1,X_2]\right]}(\lambda)$ or $H_{\left[X_2,[X_1,X_2]\right]}(\lambda)$ is not zero, so equation (2.29) implies that $\alpha_3 = 0$, so $v \in \mathrm{Ker}(\sigma|_{D^\perp})_\lambda$ and therefore $v \in \mathcal{C}(\lambda)$. So, we proved that $\mathrm{Ker}(\sigma|_{(D)^2\perp})_\lambda \subset \mathcal{C}(\lambda)$. the reverse inclusion is trivial, so we get (2.25).

3. JACOBI CURVES OF ABNORMAL EXTREMALS

3.1 Construction of Jacobi curve

Note that in addition to the characteristic foliation of abnormal extremals the manifold W_D is endowed with the structure of a fiber bundle. In this section to any abnormal extremal γ of a rank 2 distribution on W_D, we assign a special curve in a Lagrangian Grassmannian, called the *Jacobi curve of* γ. Speaking informally, the Jacobi curve of an abnormal extremal γ describes the dynamics of the fibers of W_D along γ. This curve is in fact recovered from a special curve in projective space. We will refer to it here as the *derived Jacobi curve of* γ. Invariants of curves in projective spaces under the action of the projective linear group were constructed by Wilczynski in [22]. Since all construction is coordinate-free, Wilczynski invariants of the derived Jacobi curve of each regular abnormal extremal produce invariants of the original distribution called the *generalized Wilczynski invariants*. These invariants are the main object of study in this thesis.

We start with some preliminaries. Recall that the Euler field $\vec{e}$ on a vector bundle (and in particular on T^*M) is the vector field so that its flow consists of a one-parametric family of homotheties in its fibers. In more details, let δ_s be the flow of homotheties on the fibers of T^*M:

$$\delta_s(p,q) = (e^s p, q), \quad q \in M, p \in T_q^*M \tag{3.1}$$

and let

$$\vec{e}(\lambda) = \frac{d}{ds}\delta_s(\lambda)|_{s=0}. \tag{3.2}$$

Directly from the definition the tautological Liouville 1-form s (see (2.1)) and the canonical symplectic form sigma (see (2.3)) it is easy to show that

$$i_{\vec{e}}\sigma = s. \tag{3.3}$$

Consider the following corank 1 distribution $\widetilde{\Delta}$ on T^*M:

$$\widetilde{\Delta} := \mathrm{Ker} s = \{v \in T_\lambda T^*M : s(v) = 0\} \tag{3.4}$$

Since the form σ is nondegenerate, the kernel of $\sigma|_{\widetilde{\Delta}}$ is one-dimensional at every point. Moreover by (3.3)

$$\mathrm{Ker}\ \sigma|_{\widetilde{\Delta}} = \mathrm{span}\{\vec{e}\} \tag{3.5}$$

Restrict the distribution $\widetilde{\Delta}$ from T^*M to $(D^2)^\perp$, i.e. define

$$\overline{\Delta} := \widetilde{\Delta} \cap T\mathbb{P}(D^2)^\perp \tag{3.6}$$

Lemma 3.1.1. *The following relation holds:*

$$\mathrm{Ker}\big(\sigma|_{\overline{\Delta}}\big)_\lambda = \mathrm{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\}. \quad \forall \lambda \in W_D \tag{3.7}$$

Proof. Obviously, $\overline{\Delta}$ is a corank 1 distribution in $(D^2)^\perp$. Therefore, since by (2.25) the kernel of $\big(\sigma|_{(D^2)^\perp}\big)_\lambda$ is 1-dimensional, the kernel of $\big(\sigma|_{\overline{\Delta}}\big)_\lambda$ is 2-dimensional

Further, since by (2.24)

$$\pi_* \mathcal{C}(\lambda) \in D\big(\pi(\lambda)\big), \quad \forall \lambda \in W_D. \tag{3.8}$$

and $W_D \subset D^\perp$, we have that $s_\lambda\big(\mathcal{C}(\lambda)\big) = 0$, so

$$\mathcal{C}(\lambda) \in \overline{\Delta}(\lambda). \tag{3.9}$$

This together with (2.25) implies that

$$C(\lambda) \in \mathrm{Ker}\big(\sigma|_{\overline{\Delta}}\big)_\lambda. \tag{3.10}$$

Finally, $\vec{e}(\lambda)$ is tangent to $(D^2)^\perp$ as the latter is preserved by its flow of homotheties δ_s. Besides, since $\vec{e}$ is vertical, i.e. $\pi_*(\vec{e}) = 0$, $\vec{e}(\lambda) \in \overline{\Delta}(\lambda)$. Thus, the latter together with (3.5) implies that

$$\vec{e}(\lambda) \in \operatorname{Ker}\big(\sigma|_{\overline{\Delta}}\big)_\lambda. \tag{3.11}$$

From the fact $C(\lambda)$ and $\vec{e}(\lambda)$ are linearly independent and 2-dimensionality of $\operatorname{Ker}\big(\sigma|_{\overline{\Delta}}\big)_\lambda$ we get (3.7). □

One can lift our original distribution D to the distribution $\widehat{D}$ by the push-forward of the canonical projection π restricted to W_D, namely, let

$$\widehat{D}(\lambda) := \pi^* D(\lambda) = \{v \in T_\lambda W_D : \pi_* v \in D(\pi(\lambda))\} \tag{3.12}$$

By arguments similar to the proof of (3.9) we have that

$$\widehat{D} \subset \overline{\Delta} \tag{3.13}$$

and also that

$$\sigma|_{\widehat{D}}(\lambda) = 0, \quad \forall \lambda \in W_D. \tag{3.14}$$

Moreover, by Lemma 3.1.1 we have that σ induces the symplectic (i.e. nondegenerate skew-symmetric) form ω on $\overline{\Delta}/\mathrm{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\}$ and relation (3.14) implies that the space $\widehat{D}/\mathrm{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\}$ is *isotropic* with respect to ω, i.e. the restriction of ω to it is zero. Moreover, counting the dimensions,

$$\begin{aligned} &\dim\, \overline{\Delta}/\mathrm{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\} = 2n - 6, \\ &\dim\, \widehat{D}/\mathrm{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\} = n - 3, \end{aligned} \tag{3.15}$$

so the latter space is the *Lagrangian subspace* of the former, i.e. it is a half-dimensional isotropic subspace w.r.t. to the form ω.

Now we are ready to define the Jacobi curve for a regular abnormal extremal γ. There exists a neighborhood O_γ of γ such that the quotient space

$$N = O_\gamma/(\text{the characteristic foliation on } O_\gamma) \tag{3.16}$$

has the natural structure of a smooth manifold.

Consider a canonical projection

$$\Phi : W_D \to N \tag{3.17}$$

to the quotient manifold. From (3.10) it follows that

$$\Delta := \Phi_*(\overline{\Delta}) \tag{3.18}$$

is a well-define corank 1 distribution on N with the defining 1-form α such that $\Pi^*\alpha = s|_{O_\gamma}$. Note that $\dim N$ is odd (and equal to $2n - \operatorname{rank} D^2 - 1 = 2n - 4$), and Δ is an even-contact distribution (see item (2) in the list in the Introduction), i.e $d\,\alpha|_\Delta$ has the one-dimensional kernel, which is in fact generated by $\Phi_*\vec{e}$. The line distribution $\operatorname{span}\{\Phi_*\vec{e}\}$ is well defined rank 1 distribution on N as the the distribution $\operatorname{span}\{\mathcal{C}(\lambda), \vec{e}(\lambda)\}$ is involutive by (3.7). Thus, the vector space

$$\mathcal{S}_\gamma := \Delta(\gamma)/\operatorname{span}\{\Phi_*\vec{e}(\lambda)\}, \quad \lambda \in \gamma \tag{3.19}$$

is endowed with the natural symplectic form ω_γ induced by $d\,\alpha|_\Delta$. Given a point $\lambda \in \gamma$ define

$$J_\gamma(\lambda) := \Phi_*(\widehat{D}(\lambda))/\operatorname{span}\{\Phi_*\vec{e}\}. \tag{3.20}$$

From (3.14) the space $J_\gamma(\lambda)$ is isotropic with respect the form ω_γ. Note that by counting dimensions by analogy with (3.15) we get

$$\dim \mathcal{S}_\gamma = 2n - 6, \quad \dim\ J_\gamma(\lambda) = n - 3, \quad \lambda \in \gamma, \tag{3.21}$$

so $J_\gamma(\lambda)$ is a Lagrangian subspace of W_γ. The set of all Lagrangian subspaces of a linear symplectic space $\mathcal{S}_\gamma$ is called the *Lagrangian Grassmannian* of $\mathcal{S}_\gamma$ and is denoted by $\mathrm{LG}(\mathcal{S}_\gamma)$. The Lagrangian Grassmannian (similar to Grassmannians) is a homogeneous space (of a symplectic group modulo a subgroup preserving a fixed Lagrangian subspace) and therefore is endowed with the natural structure of a manifold.

Definition 3.1.2. *Given a regular abnormal extremal γ the curve $\lambda \mapsto J_\gamma(\lambda), \lambda \in \gamma$ in the Lagrangian Grassmannian $\mathcal{S}_\gamma$ is called the Jacobi curve of γ.*

Since an abnormal have is unparametrized, i.e. does a priori not have a distinguish parametrization, its Jacobi curve is an unparametrized curve as well.

3.2 Osculating flag

Jacobi curves of regular abnormal extremals of rank 2 distributions are not arbitrary and satisfy very special properties. These properties can be described via the process of *osculation*. In more detail, the Jacobi curve J_γ produces the curve of flags in $\mathcal{S}_\gamma$ via a series of osculations and skew-orthogonal complements:

$$\ldots \subset J_\gamma^{(-\nu)} \subset \ldots \subset J_\gamma^{(0)} = J_\gamma \subset J_\gamma^{(1)} \subset \ldots \subset J_\gamma^{(\nu)} \subset \ldots \subset \mathcal{S}_\gamma, \tag{3.22}$$

where

1. $J_\gamma^{(i)}$ with $i \geq 0$ is the *i-th osculating space of the curve J_γ at* λ defined as follows: Let $\varphi : \gamma \to \mathbb{R}$ be a parametrization of γ with $\varphi(\lambda) = 0$. Here by a parametrization of a curve γ, we mean a local coordinate map pf $gamma$, considered as a one-dimensional manifold. Look on $J_\gamma(\cdot)$ as a tautological vector bundle over itself, i.e. the bundle over $J_\gamma(\cdot)$ with the fiber over the point $J_\gamma(t)$ being vector space $J_\gamma(t)$. Let $\Gamma(J_\gamma)$ be the space of sections of this bundle. Define

$$J_\gamma^{(i)}(\lambda) = \mathrm{span}\{\frac{d^j}{d\tau^j}\ell\big(\varphi^{-1}(\tau)\big)|_{\tau=0} : \ell \in \Gamma\big(J_\gamma\big), 0 \leq j \leq i\}. \tag{3.23}$$

It is easy to see the right-hand side of (3.23) is independent of the choice of the parametrization $\varphi : \gamma \to \mathbb{R}$ with $\varphi(\lambda) = 0$.

2. $J_\gamma^{(-i)} := \left(J_\gamma^{(i)}\right)^\angle$, the skew-symmetric complement of $J_\gamma^{(i)}$.

As shown in [25, 10], for rank 2 distributions,

$$\dim J_\gamma^{(i+1)} - \dim J_\gamma^{(i)} \leq 1. \tag{3.24}$$

$$J_\gamma^{(-1)}(\lambda) = \Phi_*(\mathcal{V}(\lambda)), \quad \lambda \in \gamma, \tag{3.25}$$

where

$$\mathcal{V}(\lambda) = \operatorname{Ker} d\pi(\lambda) \cap T_\lambda(D^2)^\perp \tag{3.26}$$

is the vertical distribution of the bundle W_D, i.e. the distribution consisting of the tangent spaces to the fibers of the bundle $\pi : W_D \to M$.

The curve J_γ is called *regular* if the subspaces $J_\gamma(\lambda)$ do not belong to a fixed hyperplane of $\mathcal{S}_\gamma$, Hence, for generic $\lambda \in \gamma$ the following three mutually equivalent conditions hold :

1. $J_\gamma^{(n-3)}(\lambda) = \mathcal{S}_\gamma$;

2. $\dim J_\gamma^{(i)} = i + n - 3$ for $3 - n \leq i \leq n - 3$;

3. $\dim J_\gamma^{(4-n)} = 1$, i.e. near λ, $\bar{\lambda} \mapsto J_\gamma^{(4-n)}(\bar{\lambda})$, $\bar{\lambda} \in \gamma$, is the curve in the projective space $\mathbb{P}\mathcal{S}_\gamma$

Besides, if one of these three conditions holds then

$$J_\gamma^{(4-n+i)}(\lambda) = \left(J_\gamma^{(4-n)}\right)^{(i)}(\lambda), \ i \geq 0, \tag{3.27}$$

where $\left(J_\gamma^{(4-n)}\right)^{(i)}$ is the i-th osculating space of the curve $J_\gamma^{(4-n)}$ at λ defined by the analogy with (3.23) (just replacing J_γ with $J_\gamma^{(4-n)}$ in that formula). In other words, the whole osculating flag (3.22) is recovered from the curve in projective space given by $J_\gamma^{(4-n)}$.

Definition 3.2.1. *A smooth curve J in a projective space $\mathbb{P}\mathcal{W}$ of an $(k+1)$-dimensional vector space W is called convex if the kth osculating space is equal to $\mathcal{W}$ at any point.*

Let $\mathcal{R}_D \subset W_D$, the *Jacobi regularity locus of D*, be the set of $\lambda \in W_D$ such that the germ at λ of $\bar{\lambda} \mapsto J_\gamma^{(4-n)}(\bar{\lambda}), \bar{\lambda} \in \gamma$ is a convex curve in the projective space $\mathbb{P}\mathcal{S}_\gamma$, where γ is the abnormal extremal passing through λ. If $\lambda \in \mathcal{R}_D$, then the the germ at λ of $\bar{\lambda} \mapsto J_\gamma^{(4-n)}(\bar{\lambda}), \bar{\lambda} \in \gamma$ will be called the *derived Jacobi curve of γ attached to λ.*

The rank 2 distribution D is of *maximal class* at the point q if

$$\mathcal{R}_D(q) := \mathcal{R}_D \cap \pi^{-1}(q) \tag{3.28}$$

is not empty. As shown in [25] all $(2,3,5)$ and $(2,3,5,6)$ distributions are of maximal class at every point. Moreover, examples of rank 2 bracket generating distribution with $\dim D^3 = 5$, which are not of maximal class at generic points are not known.

Any invariant of the derived Jacobi curve with respect to a projective linear group acting on $\mathbb{P}W_\gamma$ (with the action induced by the standard action of $GL(\mathcal{S}_\gamma)$ on $\mathcal{S}_\gamma$) will produce the invariant of the original distribution D. The construction of these invariants are discussed in the net section

4. WILCZYNSKI INVARIANTS OF CONVEX CURVES IN PROJECTIVE SPACE

4.1 Laguerre-Forsyth's Canonical Forms

Let J be a convex curve in a projective space $\mathbb{P}V$ of a $(k+1)$-dimensional vector space V. First, fix a parametrization on J: $t \mapsto J(t)$ (in this section a parametrization of a curve $J$is an injective immersion for an interval of $\mathbb{R}$ to J).

As before, we consider J as a tautological bundle over itself and denote by $\Gamma(J)$ the space of the smooth section of this bundle. Let $E \in \Gamma(J)$ be a nowhere zero section of J $E(t) \in J(t)$. From the convexity assumption $E(t), E'(t), \ldots, E^{(k)}(t)$ constitute a basis of V, i.e. there exists the unique collection of functions $B_0(t), \ldots, B_k(t)$ such that

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k} B_i(t)\frac{d^i}{dt^i}E(t). \tag{4.1}$$

If $\widetilde{E}$ is another nowhere zero section of J, $\widetilde{E}(t) \in J(t)$, then there exists a nonzero scalar-valued function $\lambda(t)$ such that $\widetilde{E}(t) = \lambda(t)E(t)$. This function is called the *transition scaling between sections E and $\widetilde{E}$*. If $\widetilde{B}_0(t), \ldots, \widetilde{B}_k(t)$ is the collection of functions such that

$$\frac{d^{k+1}}{dt^{k+1}}\widetilde{E}(t) = \sum_{i=0}^{k} \widetilde{B}_i(t)\frac{d^i}{dt^i}\widetilde{E}(t). \tag{4.2}$$

then by direct calculations

$$\widetilde{B}_k(t) = B_k(t) + (k+1)\frac{\lambda'(t)}{\lambda(t)}. \tag{4.3}$$

So, we can make $\widetilde{B}_k(t) \equiv 0$ by choosing the scaling $\lambda(t)$ so that it satisfies the following linear ordinary differential equation:

$$\lambda'(t) = -\frac{1}{k+1}B_k(t)\lambda(t). \tag{4.4}$$

In this case (4.1) becomes

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-1} B_i(t)\frac{d^i}{dt^i}E(t). \tag{4.5}$$

A section E, satisfying

$$B_k(t) \equiv 0, \tag{4.6}$$

is called a *canonical section of the curve J w.r.t. the chosen parametrization* $t \to J$. If E and $\widetilde{E}$ are two canonical sections with respect to the same parametrization then both B_k and $\widetilde{B}_k$ are zero identically and (4.4) become $\lambda'(t) = 0$, i.e. $\lambda(t)$ is a nonzero constant. In other words,*a canonical section with respect to the given parametrization is defined up to a constant scaling*.

Further, assume that

$$\tau = \varphi(t) \tag{4.7}$$

and we consider a reparametrization $\tau \to J(\varphi^{-1}(\tau)$ of J. Assume that E is a canonical section with respect to the parameter t and $\bar{E}$ is a canonical section with respect to the parameter τ. Let $\bar{B}_0(t), \dots, \bar{B}_k(t)$ be the collection satisfying the decomposition (4.1) with E replaced by $\bar{E}$ and t replaced by tau (note that by the assumption $\bar{B}_k \equiv 0$. Then by direct computations, first,

$$\bar{E}\big(\varphi(t)\big) = C(\varphi'(t))^{k/2}E(t) \tag{4.8}$$

for a nonzero constant C, and second we have the following transformation rule for B_{k-1} under reparametrization (4.7):

$$\bar{B}_{k-1}(\varphi(t))\big(\varphi'(t)\big)^2 = \left(B_{k-1}(t) + \frac{(k+1)(k+2)}{12}\mathbb{S}(\varphi(t)\right), \tag{4.9}$$

where

$$\mathbb{S}(\varphi) := \frac{\varphi^{(3)}}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 \tag{4.10}$$

is the *Schwarzian derivative* of function φ. So we can make $\bar{B}_{k-1}(t) \equiv 0$ by choosing the reparametrization (4.7) so that it satisfies the following ordinary differential equation:

$$\mathbb{S}(\varphi) = -\frac{12}{(k+1)(k+2)}B_{k-1}. \tag{4.11}$$

It is a classical fact that φ is a solution of the ordinary differential equation

$$\mathbb{S}(\varphi) = \rho \tag{4.12}$$

for a given function ρ if and only of $\varphi = \frac{y_1}{y_2}$, where y_1 and y_2 form a fundamental set of solutions of the linear homogeneous second order ordinary differential equation

$$y'' + \rho y = 0.$$

This implies that (4.11) has a solution at least in a neighborhood of a given time moment and all solutions are in fact defined up to a Möbius transformation, i.e., if φ and $\tilde{\varphi}$ are solutions of (4.12) then there exist constants a, b, c, d, $ad - bc \neq 0$ so that

$$\tilde{\varphi} = \frac{a\varphi + b}{c\varphi + d}. \tag{4.13}$$

This implies that the set of parametrizations for which for a canonical section we have that

$$B_{k-1}(t) \equiv 0. \tag{4.14}$$

is not empty, and all such parametrizations are defined up to a Monious transformation as in (4.13). These special parametrizations are called *projective parametrizations* of J and we say that they define the *canonical projective structure* on the convex curve J.

If t is a projective parametrization and E is a canonical section with respect to it, the moving frame $(E(t), E'(t), \ldots, E^{(k)}(t))$ satisfies

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-2} B_i(t)\frac{d^i}{dt^i}E(t), \tag{4.15}$$

i.e. $B_k = B_{k-1} = 0$, compared to (4.1). The decomposition (4.15) is called the *Laguerre -Forsyth canonical form* of the moving frame $(E(t), E'(t), \ldots, E^{(k)}(t))$. Equations (4.8) and (4.13) show that the group of all transformations of the pairs (parametrization,canonical section) preserving the Laguerre-Forsyth form consists of transformations of the form

$$\big(t, E(t)\big) \mapsto \left(\frac{at+b}{ct+d}, \frac{\alpha}{(ct+d)^k}E\left(\frac{at+b}{ct+d}\right)\right), \quad ad - bc = 1, \alpha \neq 0. \tag{4.16}$$

This group of transformation is in fact isomorphic to $GL_2(\mathbb{R})$.

4.2 Wilczynski invariants: original approach

Now assume that t and τ, $\tau = \varphi(t)$, are two projective parametrizations and $B_{k-2}(t)$ and $\bar{B}_{k-2}(t)$ are the corresponding coefficients near $(k-2)$nd derivative in the decomposition (4.15). Then from (4.15), applied both for t and τ, it follows immediately that

$$B_{k-2}(t) = \bar{B}_{k-2}\big(\varphi(t)\big)\big(\varphi'(t)\big)^3, \tag{4.17}$$

or, equivalently

$$B_{k-2}(t)dt^3 = \bar{B}_{k-2}(\tau)d\tau^3 \tag{4.18}$$

This means that the form

$$\mathcal{W}_1 = (k-2)!B_{k-2}(dt)^3 \tag{4.19}$$

considered as a special cubic polynomial on the tangent line to every point of the curve of J, or, shortly, a special section of the line bundle $\mathrm{Sym}^3(T^*J)$, is independent of the choice of the projective parametrization.[1] This form is called the *Wilczynski invariant of degree* 3.

More generally, Wilczynski [22, p.32, equation (48)] found that the following section of the line bundle $\mathrm{Sym}^{i+2}(T^*J)$

$$\mathcal{W}_i(t) \stackrel{def}{=} \frac{(i+1)!}{(2i+2)!}\left(\sum_{j=1}^{i}\frac{(-1)^{j-1}(2i-j+3)!(k-i+j-2)!}{(i+2-j)!(j-1)!}B^{(j-1)}_{k-2-i+j}(t)\right)(dt)^{i+2} \tag{4.20}$$

on J does not depend on the choice of the projective parameter. We call this form i*th Wilczynski invariant*, where $1 \le i \le k-1$.

To get (4.20) Wilczynski used the Sophus Lie theory of finding the invariants of objects under the action of Lie groups. In our case, the action is of $GL_2(\mathbb{R})$ and is given by (4.16). He tried to find the ith Wilcyznski invariant in the form of a linear combination of $\{B^{(j-1)}_{k-2-i+j}(t)dt^{i+2}\}_{j=1}^{i}$, i.e. in the form

$$\left(\sum_{j=1}^{i}\alpha_{i,j}B^{(j-1)}_{k-2-i+j}(t)\right)(dt)^{i+2}. \tag{4.21}$$

for some tuple of constants $\{\alpha_{i,j}\}_{j=1}^{i}$ and to determined for which tuple of $\{\alpha_{i,j}\}_{j=1}^{i}$ the quantity in (4.21) is transformed as in (4.17) (with the factor $\big(\phi'(t)\big)^3$ replaced by $\big(\phi'(t)\big)^i$). Following the Lie idea, instead of looking at how the quantity (4.21) transforms under the action of the Lie group $GL_2(\mathbb{R})$ given by (4.16), it is more convenient to look at how the infinitesimal version of the quantity (4.21) transforms under the induced action of the Lie algebra $\mathrm{lg}_2(\mathbb{R})$. This lead to a system of $i-1$ independent linear equations for the tuple $\{\alpha_j\}_{j=1}^{i}$ and

$$\alpha_{i,j} = \frac{(i+1)!}{(2i+2)!}\frac{(-1)^{j-1}(2i-j+3)!(k-i+j-2)!}{(i+2-j)!(j-1)!}, \quad j=1,\dots i \tag{4.22}$$

as in (4.20) is a generator of the line of solutions of this linear system.

Remark 4.2.1. *Note that by* (4.20) *if for a projective parameter* $B_{k-2} \equiv \ldots \equiv B_{k-2-i} = 0$, *then*

[1] the coefficient $(k-2)!$ in (4.19) is not important, as it is just a choice of normalization so that it will be consistent to a more general formula (4.20).

the first i Wilcynski invariants vanish and the ith Wilczynski invariant $\mathcal{W}_i$ satisfies

$$\mathcal{W}_i(t) = (k-1-i)! B_{k-1-i}(t) dt^{i+2}. \tag{4.23}$$

In an arbitrary (not necessarily projective) parameter t the ith Wicyinski invariant $\mathcal{W}_i(t)$ can be written in the form

$$\mathcal{W}_i(t) = A_i(t)\, dt^{i+2} \tag{4.24}$$

for some function A_i called the *density* of the ith Wilczynski invariant with respect to the parameter t.

4.3 Wilczynski invariants: $\mathfrak{sl}_2$epresentation theory approach

In practice, and in particular, in section 5 initial parametrization on a curve in projective space is not projective and instead of making reparametrization to the projective one it is more convenient to express the density of the Wilczinski invariant with respect to the initial parameter in terms of the coefficient of (4.5) i.e. including the coefficient B_{k-1} and its derivative. In order to do this, it is more convenient to use an alternative approach by means of the representation of the Lie algebra $\mathfrak{sl}_2$ is given in [18, 19, 9].

Below we briefly describe this approach. Recall that $sl_2(\mathbb{R}$ is the Lie algebra of traceless 2×2 matrices. Choose the basis $\{H, X, Y\}$ of $sl_2(\mathbb{R}$ as follows:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.25}$$

Then we have the following commutation relations

$$[H, X] = -2X, \quad [H, Y] = 2Y, \quad [X, Y] = H, \tag{4.26}$$

Any basis of $\mathfrak{sl}_2$ satisfying the commutation relations (4.26) is called $\mathfrak{sl}_2$-*triple*. It is well known ([14, chapter 11]) that for every integer k the space $\mathrm{Sym}^k(\mathbb{R}^2)$ with the action of $\mathrm{sl}_2(\mathbb{R})$, induced from the standard action $\mathrm{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$, is the unique, up to an isomorphism, irreducible $\mathfrak{sl}_2(\mathbb{R})$-module of dimension $k+1$. One can choose a basis $\{e_1,\dots e_{k+1}\}$ of $\mathrm{Sym}^k(\mathbb{R}^2)$ such that the induced action of H, X, and Y on $\mathrm{Sym}^k(\mathbb{R}^2)$ is as follows :

$$\begin{aligned}
&H.e_i=(k-2i)e_i, \quad i=1,\dots,k+1.\\
&X.e_{k+1}=0, \quad X.e_i=e_{i+1}, \qquad i=1,\dots,k\\
&Y.e_1=0, \quad Y.e_i=i(k-i+1)e_{i-1}, \quad i=2,\dots,k+1.
\end{aligned} \tag{4.27}$$

In the sequel, we denote the matrices representing actions of H,X and Y form (4.27) in the basis $\{e_1,\dots e_{k+1}\}$ by $\mathcal{H}_k$, $\mathcal{X}_k$, and $\mathcal{Y}_k$, respectively. The main result of Se-Ashi theory can be formulated as follows:

Theorem 4.3.1. *Given a convex curve J in a projective space $\mathbb{P}V$ of a $(k+1)$-dimensional vector space V, a parametrization $t\mapsto J(t)$ of J, and a canonical section E of J with respect to the parameter t, there exist the unique moving frame $E_1(t)=E(t)$, $E_2(t),\dots E_{k+1}(t)$ satisfying the following structure equation:*

$$\big(E_1'(t),\dots,E_{k+1}'(t)\big)=\big(E_1(t),\dots,E_{k+1}(t)\big)\Big(\mathcal{X}_k+\rho(t)\mathcal{Y}_k+\sum_{i=2}^{k}\Theta_{i-1}(t)(\mathcal{Y}_k)^i\Big). \tag{4.28}$$

Moreover, the forms $\Theta_i(t)\,dt^{i+2}$, $j=1,\dots,k-1$ are well-defined, i.e. independent of a parametrization elements of $\mathrm{Sym}^{i+2}(T^*J)$ *and there exist the universal nonzero constant $C_{i,k}$ and the universal polynomials $P_{i,k}$ of $i-3$ variable without free terms such that the density $A_i(t)$ of the Wilczynski invariant from* (4.24) *w.r.t. the same parameter t is expressed in terms of $\Theta_1,\dots\Theta_i$ as follows*

$$A_i(t)=C_{i,k}\Theta_i(t)+Pi_{i,k}\big(\Theta_1(t),\dots,\Theta_{i-1}(t)\big). \tag{4.29}$$

Remark 4.3.2. The forms $\Theta_i(t)dt^{i+2}$ will be called *the ith Se-Ashi invariant* Since $C_{i,k}\neq 0$ in (4.29) one can recover all Se-Ashi invariants from the Wilczynski invariants and vice versa.

The main advantage of (4.28) is that it defines Se-Ashi forms immediately in an arbitrary parameter without reduction to projective parameters. Projective parameters correspond to the case of $\rho(t) \equiv 0$ in (4.28).

To get an explicit form of (4.29) and then the formula for the Wilczynski invariants in terms of coefficients $\{B_i(t)\}_{i=0}^{k-1}$ in arbitrary parameter, one can proceed as follows:

1. Express the moving frame $E(t), E'(t), \dots E'(k)(t)$ in terms of $E_1(t), \dots E_{k+1}(t)$ and $(\rho(t), \{\Theta_i\}_{i=1}^{k-1})$, using (4.28).

2. Use the relation between moving frame from the previous item and (4.5) to express $(\rho(t), \{\Theta_i(t)\}_{i=1}^{k-1})$ in terms of $\{B_i(t)\}_{i=0}^{k-1}$.

3. For $\rho \equiv 0$ (which is equivalent to $B_{k-1} \equiv 0$,) the previous item will give the relation between $\{\Theta_i(t)\}_{i=1}^{k-1}$ and $\{B_i(t)\}_{i=0}^{k-2}$, and therefore via (4.20) the explicit forms of the formula (4.29)

4. For arbitrary ρ using the formulas obtained in item (2) and (3), one get the expression for the density $A_i(t)$ of Wilczynski invariants in the arbitrary parameter in terms of $\{B_i(t)\}_{i=0}^{k-1}$.

As the direct consequence of Theorem 4.3.1 and Remark 4.3.2 we get the following

Corollary 4.3.3. *The density $A_i(t)$ of the ith Wilczinsky invariant is a polynomial with respect to the collection of functions $\{B_i(t)\}_{i=0}^{k-1}$ from the decomposition* (4.5) *and their derivatives (possibly of high order).*

The explicit results of the implementation of the strategy given in Remark 4.3.2 in particular cases needed for this thesis will be given in the next section 4.4.

Remark 4.3.4. *In fact, using either Wiczynski or Se-Ashi technique one can show that the statement of Corollary 4.3.3 is valid also for the colleciton of function $\{B_i(t)\}_{i=0}^{k-1}$ from the decomposition* (4.1).

4.4 Self-dual curves in projective space

As follows from section 3.2 the derived Jacobi curve of an abnormal extremal, even if it is convex, is not an arbitrary curve in projective space but it comes from the process of osculations/

skew-symmetric complement with respect to the symplectic form given in the even-dimensional ambient vector space. Such convex curves are self-dual in the following sense: Given a convex curve J in $\mathbb{P}V$ the dual curve J^* in $\mathbb{P}V^*$ consists of lines in $\mathbb{P}V^*$ annihilating the hyperplanes $J^{(k-1)}$ obtained from J by the osculation of order $k-1$. The curve J is called *self-dial* if it is equivalent to its dual, i.e. there exists an isomorphism $L : V \to V^*$ such that

$$LJ(t) = J^*(t). \tag{4.30}$$

In fact, it can be shown that if $\dim V = k + 1 = 2m$ then J is self-dual if and only if there exists the unique, up to a multiplication by a constant, symplectic form ω on V, such that the curve $J^{(m-1)}$ of $(m-1)$st osculating subspaces of J is Lagrangian w.r.t. ω. The relation between the form ω and the isomorphism satisfying (4.30) is given by

$$\omega(x, y) = Lx(y). \tag{4.31}$$

Theorem 4.4.1. *[22, Chapter 2, §5] The curve is self-dual if and only if all Wilczynski invariants of odd degree vanish.*

In particular, by Remark 4.2.1 first nontrivial Wilczynski invariant is of degree 4: and in the projective parameter

$$\mathcal{W}_2 = (k-3)! B_{k-3}(t) dt^4.$$

The main reason why Theorem 4.4.1 holds is that in order that the matrices in the structure equation (4.28) must belong to the corresponding symplectic algebra, which implies vanishing of Θ_i (and therefore of A_i) for odd i.

According to (3.21), for derived Jacobi curves of an abnormal extremal $\gamma \subset \mathcal{R}_D$ of rank 2 distribution on n-dimensional manifold the dimension of the ambient vector space is $2(n-3)$. From now one set

$$m := n - 3. \tag{4.32}$$

Then $k = 2m - 1$.

Remark 4.4.2. *Note that for $n = 5$, $k = 3$, and so by Theorem* (4.4.1)*, $\mathcal{W}_2$ is the only possibly nonzero Wilzcinski invariant. If $n = 6$, then $k = 5$ and there are two possibly nonzero invariants $\mathcal{W}_2$ and $\mathcal{W}_4$.*

Implementing the procedure of Remark 4.3.2 one can get by direct computation that in the arbitrary parameter t we have the following density for potentially nonzero Wilczinski invariants for self-dual curves in the $2m-1$-diemsnioal projective space in the case $m = 2$ and $m = 3$:

1. $m = 2$ (the case of $(2, 3, 5)$ distributions, see also [26])

$$A_2 = B_0 + \frac{9}{100}(B_2)^2 - \frac{3}{10}B_2''; \tag{4.33}$$

2. $m = 3$ (the case of $(2, 3, 5, 6)$ distributions)

$$A_2 = 2\left(B_2 + \frac{37}{175}(B_4)^2 - \frac{9}{5}B_4''\right), \tag{4.34}$$

$$A_4 = B_0 + \frac{1}{441}B_2B_4 + \frac{178}{15435}(B_4)^3 - \frac{5}{18}B_2'' - \tag{4.35}$$

$$\frac{5}{441}(B_4')^2 - \frac{59}{441}B_4B_4'' + \frac{37}{7}B_4^{(4)}. \tag{4.36}$$

Remark 4.4.3. *The formula for the density of the first possible nonzero Wilczynski invariants of a self-dual curve for the general $m \geq 2$ is*

$$A_1 = (2m-2)!\left(\frac{1}{(2m-2)(2m-3)}B_{2m-4} + \frac{(10m+7)}{20(4m^2-1)m}B_{2m-2}^2 - \frac{3}{20}B_{2m-2}''\right) \tag{4.37}$$

Finally, in the sequel, we need the following

Remark 4.4.4. *If J is a self-dual curve, $t \to J(t)$ is a parametrization and ω. Then it can be shown (see [27]) that a section E is canonical if and only if corresponding symplectic form as in*

(4.31), *then a section* E *is canonical if and only if* $\omega\left(\frac{d^m}{dt^m}E(t), \frac{d^{m-1}}{dt^{m-1}}E(t)\right) \equiv const$ *Moreover, in our case, the ambient space* $\mathcal{S}_\gamma$ *of the derived Jacobi curve* J_γ *is endowed with the symplectic structure and not a symplectic structure* $\omega = \omega_\gamma$*, up to a multiplication by a constant. Therefore one can "normalize" further the canonical section by imposing that*

$$\left|\omega\left(\frac{d^m}{dt^m}E(t), \frac{d^{m-1}}{dt^{m-1}}E(t)\right)\right| \equiv 1. \tag{4.38}$$

The section satisfying 4.38 will be called the ĉanonical section of the self-dual curve in the projective space with the ambient symplectic space. This canonical section is defined up to multiplication by ± 1 *and can be found as follows. Assume that* $\widetilde{E}$ *is some nowhere zero section, and*

$$\alpha(t) = \left|\omega\left(\frac{d^m}{dt^m}\widetilde{E}(t), \frac{d^{m-1}}{dt^{m-1}}\widetilde{E}(t)\right)\right|. \tag{4.39}$$

Then

$$E(t) = \pm\alpha(t)^{-1/2}\widetilde{E}(t) \tag{4.40}$$

is the canonical section in the sense of (4.38).

5. GENERALIZED WILCZYNSKI INVARIANTS OF RANK 2 DISTRIBUTIONS

Now we are ready to look closely on the invariant of a rank 2 distribution defined by taking the $2i$th Wilczinski invariant of rank 2 distributions of the derived Jacobi curves of abnormal extremals.

First, fix a local basis of the distribution D and let

$$X_3 := [X_1, X_2], \quad X_4 := [X_1, X_3], \quad X_5 := [X_2, X_3] \tag{5.1}$$

Using the notations introduced in (2.9) let

$$u_i := H_{X_i}, \quad i = 1, \dots 5. \tag{5.2}$$

If we assign to the local basis $\{X_1, X_2\}$ of D the vector field $\vec{h}_{X_1,X_2}$ on $(D^2)^\perp$ as follows:

$$\vec{h}_{X_1,X_2} := u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1, \tag{5.3}$$

then by (2.24), using the new notation in (5.2), for every $\lambda \in W_D$ the charactersitic line distribution $\mathcal{C}$ is generated by $\vec{h}_{X_1,X_2}$.

Let, as before, $\mathcal{R}_D \subset W_D$ be the Jacobi regularity locus of D , i.e. the set of $\lambda \in W_D$ such that the germ of $J_\gamma(\lambda)$ at λ is convex, where γ is the abnormal extremal passing through λ. For any $\lambda \in \mathcal{R}_D$, let $\mathcal{W}_{2i}^\lambda$ be the $2i$th Wilczynski invariants of the Jacobi curve J_γ at λ, $1 \leq i \leq n-4$. Recall that by constructions $\mathcal{W}_{2i}^\lambda$ is a degree $2(i+1)$ homogeneous function on the tangent line to γ at λ.

Define the following real-valued function on $\mathcal{R}_D$

$$A_{2i}^{X_1,X_2}(\lambda) := \mathcal{W}_{2i}^\lambda\big(\vec{h}_{X_1,X_2}(\lambda)\big). \tag{5.4}$$

Note that by our constructions if $t \mapsto J_\gamma(e^{t\vec{h}_{X_1,X_2}}\lambda)$ is a parametrization of J_γ, where $e^{t\vec{h}_{X_1,X_2}}$

denotes the flow generated by the vector field $\vec{h}_{X_1,X_2}$, then $A_i^{X_1,X_2}(\lambda)$ is nothing but the density of the $2i$th Wilcynski invariant of this curve w.r.t. the parametrization t at $t=0$.

Let us check how $A_{2i}^{X_1,X_2}(\lambda)$ is transformed under the change of the local basis of D.

Lemma 5.1.1. *If $(\widetilde{X}_1,\widetilde{X}_2)$ is another basis of the distribution D, $(\widetilde{X}_1,\widetilde{X}_2)=(X_1,X_2)T$, $T\in GL_2(\mathbb{R})$, then*

$$\vec{h}_{\widetilde{X}_1,\widetilde{X}_2}(\lambda)=\big(\det T\big(\pi(\lambda)\big)\big)^2\left(\pi(\lambda)\right)\vec{h}_{X_1,X_2}(\lambda). \tag{5.5}$$

Proof. Assume that $\widetilde{X}_3,\widetilde{X}_4$, and $\widetilde{X}_5$ are defined from $\widetilde{X}_1$, and $\widetilde{X}_2$ similar to (5.1). Then by direct computations

$$\widetilde{X}_3=\det T\,X_3 \mod D \tag{5.6}$$

$$(\widetilde{X}_4,\widetilde{X}_5)=\det T(X_4,X_5)T \mod D^2. \tag{5.7}$$

Let $\tilde{u}_i=H_{\widetilde{X}_i}$, $i=1,\dots,5$. Since $u_1=u_2=u_3=0$ on $(D^2)^\perp$, (5.7) implies that

$$(\vec{\tilde{u}}_1,\vec{\tilde{u}}_2)(\lambda)=(\vec{u}_1,\vec{u}_2)\lambda)T\big(\pi(\lambda)\big),\quad \lambda\in(D^2)^\perp \tag{5.8}$$

$$(\tilde{u}_4,\tilde{u}_5)(\lambda)=\det T\big(\pi(\lambda)\big)(u_4,u_5)T\big(\pi(\lambda)\big),\quad \lambda\in(D^2)^\perp \tag{5.9}$$

From (5.3),

$$\vec{h}_{\widetilde{X}_1,\widetilde{X}_2}=\tilde{u}_4\vec{\tilde{u}}_2-\tilde{u}_5\vec{\tilde{u}}_1 \tag{5.10}$$

Plugging (5.8) and (5.9) into (5.10) we get (5.5). $\square$

Lemma 5.1.1 and the homogeneity of $\mathcal{W}_{2i}$ implies that

$$A_{2i}^{\widetilde{X}_1,\widetilde{X}_2}(\lambda)=\det T\big(\pi(\lambda)\big)^{4(i+1)}A_{2i}^{X_1,X_2}(\lambda) \tag{5.11}$$

i.e., the restriction $A_{2i}^{X_1,X_2}$ to the fiber $\mathcal{R}_D(q)$ of the bundle $\mathcal{R}_D$ over M (as in (3.28)) is the well-defined function on $\mathcal{R}_D$, up to the multiplication on a positive constant. We call it *the ith generalized Wilczynski of D at the point q.*

In the sequel, for shortness, we will use $\vec{h} := \vec{h}_{X_1,X_2}$ and $A_{2i} := A_{2i}^{X_1,X_2}$.

We call the set

$$\mathrm{Sing}_D := (D^2)^\perp \backslash \mathcal{R}_D \tag{5.12}$$

the *Jacobi singularity locus of* D. We also set $\mathrm{Sing}_D(q) = \mathrm{Sing}_D \cap \pi^{-1}(q)$.

Proposition 5.1.2. *If the rank* 2 *distribution is of maximal class at a point* $q \in M$*, then* $A_{2i}|_{(D^2)^\perp(q)}$ *is a degree* $2(i+1)$ *homogeneous rational function on* $(D^2)^\perp(q)$ *with singularities lying in* $\mathrm{Sing}_D(q)$

.

Proof. First note that from (5.3) it follows that of δ_s is the homothety as in (3.1)

$$\pi_* \vec{h}\big(\delta_s(\lambda)\big) := e^s \pi_* \vec{h}(\lambda) \tag{5.13}$$

This together with the fact the $2i$th Wilczinski invariant of the derived Jacobi curve J_γ is homogeneous of degree $2(i+1)$ on the tangent line of the Jacobi curve implies that $A_{2i}|_{(D^2)^\perp(q)}$ is a degree $2(i+1)$ homogeneous function (which is well- defined on $\mathcal{R}_D(q)$)).

Now we prove that this function is rational. Let N be the space of abnormal extremals near an abnormal extremal γ, as defined in (3.16) and Φ be as (3.17). For calculation of the generalized Wilczynski invariants, it is more convenient to work with special filtration of $T(D^2)^\perp$ induced by the osculating flag (3.22). This filtration is given by

$$\mathcal{J}^{(i)}(\lambda) := (\Phi_*)^{-1}\left(J_\gamma^{(i)}(\lambda)\right), \quad \lambda \in \gamma, \quad i \in \mathbb{Z} \tag{5.14}$$

$$\mathcal{V}^{(i)}(\lambda) := \mathcal{J}^{(i)}(\lambda) \cap \mathcal{V}(\lambda) \tag{5.15}$$

In particular,

$$\mathcal{J}^{(0)}(\lambda) = \widehat{D}(\lambda), \tag{5.16}$$

$$\mathcal{J}^{(-1)}(\lambda) = \mathcal{V}(\lambda) \oplus \mathcal{C}(\lambda) \tag{5.17}$$

by (3.20) and (3.25), respectively. Further, for any $\lambda \in \gamma$ the $T_\gamma N$ is naturally identified with $T_\lambda(D^2)^\perp/\mathcal{C}(\lambda)$,

$$T_\gamma N \cong T_\lambda(D^2)^\perp/\mathcal{C}(\lambda) \tag{5.18}$$

and under this identification (3.20), (5.16), and (5.14) imply that

$$J^{(i)}_\gamma\big(e^{t\vec{h}}\lambda\big) := e^{-t\vec{h}}_*\left(\mathcal{J}^{(i)}\big(e^{t\vec{h}}\lambda\big)\right)/\mathrm{span}\{\vec{h}(\lambda),\vec{e}(\lambda\}. \tag{5.19}$$

Here, as before, $e^{t\vec{h}}$ denotes the flow generated by the vector field $\vec{h}$.

Recall that the operation of Lie derivative $\mathrm{Lie}_{\vec{h}}$ along the vector field $\vec{h}$ restricted to vector fields is defined as follows:

$$\mathrm{Lie}_{\vec{h}}Y(\lambda) = \frac{d}{dt}e^{-t\vec{h}}_*Y\big(e^{t\vec{h}}\lambda\big)|_{t=0} \tag{5.20}$$

for any vector field Y on $(D^2)^\perp$ and it coincides with the operation $\mathrm{ad}\vec{h}$ of taking Lie brackets $[\vec{h},Y]$, i.e.

$$\mathrm{Lie}_{\vec{h}}Y = \mathrm{ad}\,\vec{h}\,Y. \tag{5.21}$$

Therefore the operation $\frac{d}{dt}$ on sections of parametrized curves $J^{(i)}$ translates to the operation $\mathrm{ad}\,\vec{h}$ on appropriate sections of $\mathcal{J}^{(i)}$. In particular, (3.27) implies that

$$\mathcal{J}^{(4-n+i)}(\lambda) = (\mathrm{ad}\,\vec{h})^i\left(\mathcal{J}^{(4-n)}(\lambda)\right) = (\mathrm{ad}\,\vec{h})^i\left(\mathcal{V}^{(4-n)}(\lambda)\right) \quad \lambda \in \mathcal{R}_D,\ i \geq 0. \tag{5.22}$$

For a section $\widetilde{E}$ of the derived Jacobi curve $t \mapsto J^{(4-n)}(e^{t\vec{h}})$ (as long as $e^{t\vec{h}}\lambda \in \mathcal{R}_D$) there exist the unique $\mathrm{mod}\ \mathrm{span}\{\vec{e}(e^{t\vec{h}}\lambda)\}$ vector $\widetilde{\mathcal{E}}(e^{t\vec{h}}\lambda) \in \mathcal{V}^{(4-n)}(e^{t\vec{h}}\lambda)$ such that

$$\Phi_*(\widetilde{\mathcal{E}}) = \widetilde{E} \tag{5.23}$$

or, equivalently, under the identification (5.18),

$$\widetilde{E}(t) = e_*^{-t\vec{h}}\widetilde{\mathcal{E}}\big(e^{t\vec{h}}\lambda\big) \quad \mathrm{mod\,span}\left\{\vec{h}(\lambda), \vec{e}(\lambda)\right\} \tag{5.24}$$

Consequently, by (5.21)

$$\widetilde{\mathcal{E}}^{(i)}(0) = (\mathrm{ad}\,\vec{h})^i\widetilde{\mathcal{E}}(\lambda) \quad \mathrm{mod\,span}\left\{\vec{h}(\lambda), \vec{e}(\lambda)\right\}. \tag{5.25}$$

Vice versa any section $\widetilde{\mathcal{E}}$ of $\mathcal{V}^{(4-n)}$ (which is a rank 2 distribution $\mathcal{R}_D$) which is nowhere colinear to the Euler field $\vec{e}$ defines nowhere section of Jacobi curves of abnormal extremals lying in $\mathcal{R}_D$ through the relation (5.23). Hence, given such a section $\widetilde{\mathcal{E}}$ the decomposition (4.2) translates to the decomposition

$$(\mathrm{ad}\vec{h})^{2m}\widetilde{\mathcal{E}}(\lambda) = \sum_{i=0}^{2m-1}\widetilde{\mathcal{B}}_i(\lambda)(\mathrm{ad}\vec{h})^i\widetilde{\mathcal{E}}(\lambda) \tag{5.26}$$

for a collection of functions $\{\widetilde{\mathcal{B}}_i(\lambda)\}_{i=0}^{2m-1}$ well -defined on $\mathcal{R}_D$. Moreover, by Remark (4.3.4) the generalized Wilczynski invariant A_{2i} is a polynomial with respect to the collection of functions $\{\mathcal{B}_i\}_{i=0}^{2m-1}$ from the decomposition (5.26) and their (iterative) directional derivatives in the direction of $\vec{h}$.

Further, extend the collection $(X_1, \dots, X_5)$ to a local frame $(X_1, X_2, \dots, X_n)$ of TM. Then there exist functions c_{ij}^k, $1 \leq i, j, k \leq n$ such that

$$[X_i, X_j] = \sum_{k=1}^{6} c_{ij}^k X_k. \tag{5.27}$$

These functions are called the *structure functions associated with* $\{X_i\}i = 1^n$. Let $u_i := H_{X_i}$ be the quasi-impulse of X_i as defined in (2.9).

Let $\overline{X_i}$ be the vector field on T^*M defined as follows

$$\begin{aligned} &\pi_*(\overline{X_i}) = X_i, \\ &du_j(\bar{X}_i) = 0, \quad 1 \leq i, j \leq n \end{aligned} \tag{5.28}$$

The tuple

$$\{\overline{X_1}, \dots, \overline{X_n}, \partial_{u_1}, \dots, \partial_{u_n}\} \tag{5.29}$$

forms the local moving frame of T^*M.

The Hamiltonian lift $\vec{u}_i$ of the vector field X_i can be written in this frame as follows:

$$\vec{u_i} = \overline{X_i} + \sum_{j=1}^{n} \vec{u_i}(u_j)\partial_{u_j} = \overline{X_i} + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ij}^k u_k \partial_{u_j}, \tag{5.30}$$

In particular, we have

$$\begin{aligned} \vec{h} = u_4\vec{u}_2 - u_5\vec{u}_1 = u_4\overline{X_2} - u_5\overline{X_1} + \\ \sum_{j=1}^{n}\sum_{k=1}^{n}(c_{2j}^k u_k u_4 - c_{1j}^k u_k u_5)\partial_{u_j}. \end{aligned} \tag{5.31}$$

Further by (5.22) for $i = n-4$ $(= (m-1))$ and (5.16)

$$\mathcal{V}(\lambda) = \operatorname{span}\{(\operatorname{ad}\vec{h})^i \widetilde{\mathcal{E}}(\lambda)\}_{i=0}^{m-2} \mod \operatorname{span}\left\{\vec{h}(\lambda), \vec{e}(\lambda)\right\} \tag{5.32}$$

where $\mathcal{V}$ is as (3.26). Since

$$\mathcal{V}(\lambda) = \operatorname{span}\{\partial_{u_4}(\lambda), \dots, \partial_{u_n}(\lambda)\} \tag{5.33}$$

relation (5.32) implies that the vector field $\widetilde{E}$ can be chosen such that it has rational components with respect to the frame (5.29). Since by (5.31) the vector field $\vec{h}$ has the same property, then all vector fields of the form $(\operatorname{ad}\vec{h})^i\widetilde{\mathcal{E}}$ satisfy this property as well, and so by (5.26) all functions $\widetilde{\mathcal{B}}_i$ are rational on the fibers of $(D^2)^\perp$ (and smooth on $\mathcal{R}_D$). The statement of proposition follows from this and the last sentence of the previous paragraph. $\square$

Remark 5.1.3. The proof of Proposition 5.1.2 can be performed without relying on Remark (4.3.4) (which formally does not follow from Theorem (4.3.1) but Corollary 4.3.3, which gives conclusions on canonical sections ones instead of arbitrary one. Translating the conclusions of Remark (4.4.4) from curves in projective space to distributions based on the relation (5.24), we get that if $\widetilde{\mathcal{E}}$ is a section of $\mathcal{V}^{(4-n)}$ on distribution $\mathcal{R}_D$) which is nowhere colinear to the Euler field $\vec{e}$ and by analogy with t(4.39) and (4.40)

$$\alpha(\lambda) = \left|\sigma\left((\mathrm{ad}\vec{h})^m\widetilde{\mathcal{E}}(\lambda), (\mathrm{ad}\vec{h})^{m-1}\widetilde{\mathcal{E}}(\lambda)\right)\right| \tag{5.34}$$

where σ is the canonical symplectic form on T^*M as defines in (2.3). Then

$$\mathcal{E} = \pm\alpha^{-1/2}\widetilde{\mathcal{E}} \tag{5.35}$$

gives the canonical sections of the derived Jacobi curve after application of Φ_* similar to (5.23). This procedure defines the unique , up to multiplication by ± 1 and $\operatorname{mod}\operatorname{span}\{\vec{e}\}$ section of $\mathcal{V}^{(4-n)}$, which will be also the *canonical section* associated to the local basis (X_1, X_2). In this case, similar to (4.5) we get the following decomposition

$$(\mathrm{ad}\vec{h})^{2m}\mathcal{E}(\lambda) = \sum_{i=0}^{2m-2}\mathcal{B}_i(\lambda)(\mathrm{ad}\vec{h})^i\mathcal{E}(\lambda), \tag{5.36}$$

i.e. $\mathcal{B}_{2m-1} \equiv 0$ compared to (5.26).

Even choosing $\widetilde{\mathcal{E}}$ with rational components, (5.35) implies the components of $\mathcal{E}$ are in general not rational, as $\alpha(\lambda)^{1/2}$, in general, is not rational. Nevertheless, the differences of powers of α appearing in $(\mathrm{ad}\vec{h})^i\mathcal{E}$ is always integer ,so the coefficients $\{\mathcal{B}_i(\lambda)\}_{i=0}^{2m-2}$ are rational and one completes the proof of Proposition 5.1.2by referring to Corollary 4.3.3.

Finally, we can translate the formulas (4.33), (4.34), (4.35), and (4.37) for the first two nontrivial Wilczynski invariants of self-dual curves in projective space to the generalized Wilczynski invariants of rank 2 distributions we get the following:

If $\{\mathcal{B}_i(\lambda)\}_{i=0}^{2m-2}$ is a collection of functions from decomposition (5.36) then the first nontrivial

Generalized Wilczynski invariant satisfies (compare with (4.37)

$$A_2 = (2m-2)!(\frac{1}{(2m-2)(2m-3)}B_{2m-4}+\frac{10m+7}{20(4m^2-1)m}(B_{2m-2})^2-\frac{3}{20}(\mathrm{ad}\,\vec{h})^2 B_{2m-2}), \quad (5.37)$$

In particular,

1. for $m = 2$ (the case of $(2,3,5)$ distributions, compare with (4.33)) we have

$$A_2 = \mathcal{B}_0 + \frac{9}{100}(\mathcal{B}_2)^2 - \frac{3}{10}(\mathrm{ad}\,\vec{h})^2\mathcal{B}_2. \quad (5.38)$$

2. for $m = 3$ (the case of $(2,3,5,6)$ distributions, compare with (4.34)) we have:

$$A_2 = 2(B_2 + \frac{37}{175}(B_4)^2 - \frac{9}{5}(\mathrm{ad}\,\vec{h})^2 B_4), \quad (5.39)$$

For the second nontrivial generalized Wilczinski invariant A_4 we need the formula only in the case of the smallest dimension it appears, i.e. for $(2,3,5,6)$ and it is s follows (com[are to (4.35)):

$$\begin{aligned} A_4 = B_0 + \frac{1}{144}B_2B_4 + \frac{178}{15435}(B_4)^3 - \frac{5}{18}(\mathrm{ad}\,\vec{h})^2 B_2 - \\ \frac{5}{441}((\mathrm{ad}\,\vec{h})B_4)^2 - \frac{59}{441}B_4(\mathrm{ad}\,\vec{h})^2 B_4 + \frac{37}{7}(\mathrm{ad}\,\vec{h})^4 B_4. \end{aligned} \quad (5.40)$$

6. QUAIHOMOGENEOUS DECOMPOSITION OF GENERALIZED WILCZYNSKI INVARIANTS AND THE ROLE OF TANAKA SYMBOL

While by Proposition 5.1.2 the generalized Wilczynski invariants A_{2i} are homogeneous rational functions on the fibers in the usual sense, we can say more about its algebraic structure by introducing natural quasi-weights (or multi-weights) respecting the natural filtration on the fibers of $(D^2)^\perp$. In this case the terms of maximal possible quasi-weight of the numerator and denominator of A_{2i} correspond to the so-called Tanaka symbol of distributions. Tanaka symbol is another basic invariant of an equivariant distribution, which in general is more subtle than its small growth vector, and depending on the situation can be discrete or continuous, but again usually it does not determine the equivalence class of distributions. All our construction so far through abnormal extremals do not depend on Tanaka symbols and this is the big advantage of this approach as in general Tanaka symbols are impossible to classify. However, such a basic invariant as Tanaka symbol inevitably encoded in the generalized Wilczynski invariants. The next subsection is a detour on Tanaka symbol required in the sequel.

6.1 Tanaka symbols of distributions

Let D be an equiregular bracket-generating distribution with a degree of nonholonomy r. Set

$$\mathfrak{m}_{-1}(q) := D(q), \quad \mathfrak{m}_{-j}(q) := D^j(q)/D^{j-1}(q), \ \forall j > 1 \tag{6.1}$$

and consider the graded space

$$\mathfrak{m}(q) = \bigoplus_{j=-\mu}^{-1} \mathfrak{m}_j(q), \tag{6.2}$$

associated with the filtration (1.3).

The space $\mathfrak{m}(q)$ is endowed with the natural structure of a graded Lie algebra, i.e. with the natural Lie product $[\cdot, \cdot]$ such that

$$[\mathfrak{m}_{i_1}(q), \mathfrak{m}_{i_2}(q)] \subset \mathfrak{m}_{i_1+i_2}(q), \tag{6.3}$$

defined as follows:

Let $\mathfrak{p}_j : D^j(q) \mapsto \mathfrak{m}_{-j}(q)$ be the canonical projection to a factor space. Take $Y_1 \in \mathfrak{m}_{-i_1}(q)$ and $Y_2 \in \mathfrak{m}_{-i_2}(q)$. To define the Lie bracket $[Y_1, Y_2]$ take a local section $\widetilde{Y}_1$ of the distribution D^{i_1} and a local section $\widetilde{Y}_2$ of the distribution D^{i_2} such that

$$\mathfrak{p}_{i_1}\big(\widetilde{Y}_1(q)\big) = Y_1, \quad \mathfrak{p}_{i_2}\big(\widetilde{Y}_2(q)\big) = Y_2. \tag{6.4}$$

It is clear from definitions of the spaces D^j that $[Y_1, Y_2] \in \mathfrak{m}_{i_1+i_2}(q)$. Then set

$$[Y_1, Y_2] := \mathfrak{p}_{i_1+i_2}\big([\widetilde{Y}_1, \widetilde{Y}_2](q)\big). \tag{6.5}$$

It can be shown ([20, 24]) that the right-hand side of (6.5) does not depend on the choice of sections $\widetilde{Y}_1$ and $\widetilde{Y}_2$.

Definition 6.1.1. *The graded Lie algebra* $\mathfrak{m}(q)$ *from* (6.2) *is called the symbol of the distribution* D *at the point* q.

By constructions, it is clear that the Lie algebra $\mathfrak{m}(q)$ is nilpotent and generated by $\mathfrak{m}_{-1}(q)$. A $\mathbb{Z}_{-}$-graded nilpotent Lie algebra, generated by its -1 component is called the *fundamental graded Lie algebra*. Given a fundamental graded Lie algebra $\mathfrak{m}$ we say that a distribution D has *constant symbol* $\mathfrak{m}$ (or it is of *constant type* $\mathfrak{m}$), if the Tanaka symbol of d at every point is isomophic to $\mathfrak{m}$ (as graded Lie algebras). A distirbution $D_{\mathfrak{m}}$ is called the *flat distribution of constant type* $\mathfrak{m}$ if it is locally equivalent (at every point) to the left-invariant distribution $\widehat{D}$ on the simply connected Lie group with the Lie algebra $\mathfrak{m}$ and the identity e , such that this left-invariant distribution is equal to $\mathfrak{m}_{-1}$ at e.

Example 6.1.2. $(2, 3, 5)$ **distributions** Let D be a rank 2 distribution in $\mathbb{R}^5$ with small growhth vector $(2, 3, 5)$. Such distributions were treated by E. Cartan in his famous work [5]. In this case $\mathfrak{m}(q) = \mathfrak{m}_{-1}(q) \oplus \mathfrak{m}_{-2}(q) \oplus \mathfrak{m}_{-3}(q)$ with $\dim \mathfrak{m}_{-1}(q) = \dim \mathfrak{m}_{-3}(q) = 2$ and $\dim \mathfrak{m}_{-2}(q) = 1$. Choose a basis X_1, X_2 of $\mathfrak{m}_{-1}(q)$ and set

$$X_3 = [X_1, X_2], \quad X_4 = [X_1, X_3], \quad X_5 = [X_2, X_3]. \tag{6.6}$$

Then by assumptions on the small growth vector X_1, X_2, X_3, X_4, and X_5 constitute a basis of $\mathfrak{m}(q)$ and the products in (6.6) are the only nonzero products among elements of this basis, taking into account slew-symmetricity. Therefore, the symbol $\mathfrak{m}(q)$ at any point q is isomorphic to the step 3 free nilpotent Lie algebra with two generators. The flat distribution with this symbol is isomorphic to the Cartan distribution as define in subsection 1.2

Example 6.1.3. $(2, 3, 5, 6)$**-distributions.** Let D be a $(2, 3, 5, 6)$-distribution. Let us classify all possible Tanaka symbols in this case. Note that $g^{-2}(q) = D^{-2}(q)/D^{-1}(q)$ is a line. Fixing a generator Z of this one has a well defined linear map $\operatorname{ad} Z : \mathfrak{m}_{-1} \mapsto \mathfrak{m}_{-3}$ given by $\operatorname{ad} Z(Y) := [Z, Y]$, $Y \in \mathfrak{m}_{-1}(q)$. Since in the considered case $\dim \mathfrak{m}_{-1}(q) = \dim \mathfrak{m}_{-3}(q) = 2$, the linear map $\operatorname{ad} Z$ is an isomorphism. Moreover, since Z is defined up to a multiplication by a nonzero constant, then $\mathrm{ad} Z$ defines the identification between $\mathfrak{m}_{-1}(q)$ and $\mathfrak{m}_{-3}(q)$ up to a multiplication by a nonzero constant. Furthermore, the Lie product on $\mathfrak{m}(q)$ defines the linear map from $\mathfrak{m}_{-1}(q) \otimes \mathfrak{m}_{-3}(q)$ to $\mathfrak{m}_{-4}(q)$, which, using the identification between $\mathfrak{m}_{-1}(q)$ and $\mathfrak{m}_{-3}(q)$, gives the linear map from $\mathfrak{m}_{-1}(q) \otimes \mathfrak{m}_{-1}(q)$ to $\mathfrak{m}_{-4}(q)$, defined up to a multiplication by a nonzero constant. Since the latter space is one- dimensional, this map is nothing but a bilinear form on $\mathfrak{m}_{-1}(q)$ defined by a multiplication by a nonzero constant. More precisely, choose a generator W of $\mathfrak{m}_{-4}(q)$, then there exist a bilinear form B such that

$$\Big[Y_1, [Z, Y_2]\Big] = B(Y_1, Y_2)W. \tag{6.7}$$

Moreover, from constructions and Jacobi identity, this bilinear form is symmetric. Indeed, for any

Y_1 and Y_2 in $\mathfrak{m}_{-1}$ we have

$$B(Y_1,Y_2)W = \big[Y_1,[Z,Y_2]\big]W = \big[[Y_1,Z],Y_2\big]W = \big[Y_2,[Z,Y_1]\big]W = B(Y_2,Y_1)W,$$

which implies that $B(Y_1,Y_2) = B(Y_2,Y_1)$. Here for getting the second equality we use the Jacobi identity and the fact that $[Y_1,Y_2]$ and Z are collinear because they belong to $\mathfrak{m}_{-2}$, which is one-dimensional.

The symmetric bilinear form B form (6.7) or, more precisely, its conformal class is called the *canonical symmetric form of the* $(2,3,5,6)$*-distribution* D.

Obviously, the rank and the signature of the corresponding quadratic form $Y \mapsto B(Y,Y)$ are invariants of the Tanaka symbol of D at q. Moreover, the symbol is uniquely determined by the rank and the signature of the quadratic form $Y \mapsto B(Y,Y)$. Indeed, we can choose a basis $X_1,\dots X_6$ of $\mathfrak{m}(q)$ such that $\mathfrak{m}_{-1} = \langle X_1,X_2\rangle$, vectors X_3,X_4, and X_5 are as in (6.6), and the bilinear form B on $\mathfrak{m}_{-1}(q)$, given by (6.7) with $Z = X_3$ and $W = X_6$, satisfies

$$B(X_1,X_1) = 1, B(X_1,X_2) = 0, B(X_2,X_2) = \varepsilon, \quad \varepsilon \in \{-1,0,1\}.$$

or, equivalently, the only (possibly) nonzero Lie products of vectors $X_1,\dots X_6$, in addition to (6.6) and taking into account skew-symmetricity, are

$$[X_1,X_4] = X_6, [X_2,X_5] = \varepsilon X_6, \quad \varepsilon \in \{-1,0,1\} \tag{6.8}$$

Thus in this case we have exactly 3 non-isomorphic symbols $\mathfrak{m}_\epsilon$ depending on the values of ε:

1. *elliptic*, when $\varepsilon = 1$, or ,equivalently, the quadratic form $B(Y,Y)$ is positive definite,

2. *hyperbolic*, when $\varepsilon = -1$, or ,equivalently, the quadratic form $B(Y,Y)$ is non-degenerate and sign indefinite, and

3. *parabolic*, when $\varepsilon = 0$ or, equivalently, the quadratic form $B(Y,Y)$ is degenerate but not equal to zero. The kernels of the form B at q is a distinguished rank 1 subdistribution on

each fiber $D(q)$ and if X_2 is the local basis of this rank 1 distribution, then

$$[X_2, D^3] \subset D^3. \tag{6.9}$$

It is easy to see that the t distribution with parabolic symbol is isomorphic to the symplectically flat $(2, 3, 5, 6)$ distribution as defined in subsection 1.2

6.2 Quasi-weights and quasihomogeneity

Assume that an equiregular rank 2 bracket generating distribution D has the small growth vector $(j_1, j_2, ldots, j_r)$ with $j_1 = 2$, $j_2 = 3$, $j_3 = 5$, $j_r = n$. Set $j_0 := 0$. Fix a local frame $(X_1, X_2, \ldots, X_n)$ of TM which is adapted to the weak derived flag, i.e.

$$D^k = \langle X_1, \ldots, X_{j_k} \rangle, \quad 1 \leq k \leq 4.$$

Let c_{ij}^k be the structure functions of this frame as defined in (5.27). We also denote by $\theta^1, \ldots \theta^j$ the dual coframe of the frame $(X_1, X_2, \ldots, X_n)$ defined by $\theta^j(X_i) = \delta_i^j$, where δ_i^j is the Kronecker symbol.

From the procedure of calculations of the generalized Wilczynski invariants described in the proof of Proposition 5.1.2 it follows that the generalized Wilczynski invariants are rational functions of u_i, c_{ij}^k and their (iterative) directional derivatives in the direction of vector fields X_1 and X_2. It is useful to define quasi-weights wt to X_i's, u_i's , c_{ij}^k, their directional derivatives and products of those as follows:

1. $\mathrm{wt}(X_i) = k$ if $j_{k-1} < i \leq j_k$;

2. $\mathrm{wt}(u_i) = \mathrm{wt}(\vec{u_i}) = -\mathrm{wt}(\partial u_i) = -\mathrm{wt}\theta^i := \mathrm{wt}(X_i)$;

3. $\mathrm{wt}(c_{ij}^k) := \mathrm{wt}(X_i) + \mathrm{wt}(X_j) - \mathrm{wt}(X_k)$.

4. The quasi-weight of the product of two quasihomogeneous objects is equivalent to the sum of the individual quasi-weights of each object. This applies to both regular multiplication,

the directional derivative along a vector field, and wedge product of forms. The exterior differential does not change the weight.

Remark 6.2.1. Note that $\mathrm{wt}(c_{ij}^k) \geq 0$ as $[D^i, D^j] \subset D^{i+j}$ and c_{ij}^k with $\mathrm{wt}(c_{ij}^k) > 0$ do not contribute to the Tanaka symbol and therefore to the calculation of the Wilczynski invariant of the flat distribution corresponding the Tanaka symbol of D at the given point. The same is true for the derivatives of c_{ij}^k $\mathrm{wt}(c_{ij}^k) = 0$, as such c_{ij}^k are considered constant for the flat distribution (note that these derivative have positive quasi-weight in our setting).

We say that an object (a polynomial, a vector field, a differential form) is quasi-homogeneous (counting the quasi-weights of structure functions and their derivatives) if it is a sum of terms of the same quasi-weight wt. Rules (1)-(4) define quasi-homogeneous components of functions, vector fields polynomial and differential forms. Note that in this setting the quasi-weight of a constant function is equal to zero.

For example,

$$\mathrm{wt}(\vec{h}) = \mathrm{wt}(u_4)\mathrm{wt}(\vec{u_2}) = \mathrm{wt}(u_5)\mathrm{wt}(\vec{u_1}) = 3 + 1 = 4 \tag{6.10}$$

Note also that the tautological Liouville form s on T^*M defined by (2.1) can be written as

$$s = \sum_{i=1}^{n} u_i \pi^* \theta^i \tag{6.11}$$

Therefore s has a quasi-weight 0. Consequently, the canonical symplectic form $\sigma = ds$ on T^*M has quasi-weight zero as well.

Since $\vec{u}_j$ and $\vec{h}$ are quasi-homogeneous, from the arguments in the proof of Proposition 5.1.2 a section $\widetilde{\mathcal{E}}$ of $\mathcal{J}^{(4-n}$ nowhere colinear to the Euler field, can be taken as quasi-homogeneous. Consequently, all vectors fields $\mathrm{ad}\vec{h})^i\widetilde{\mathcal{E}}$ are quasi-homogeneous.

Therefore, the canonical section $\mathcal{E}(\lambda)$ associated with the local basis (X_1, X_2) defined in Remark 5.1.3 and all vector fields $(\mathrm{ad}\vec{h})^i\mathcal{E}$ are quasihomogeneous as well. Now let us determine their quasi-weight.

First, by (6.10) the quasi-weights of $\mathrm{ad}\vec{h})^i\mathcal{E}$ form the arithmetic progression with the difference between any two consecutive numbers equal to 4. Second, by (4.39) and (5.35)

$$\left|\sigma\left((\mathrm{ad}\vec{h})^m\mathcal{E}(\lambda), (\mathrm{ad}\vec{h})^{m-1}\mathcal{E}(\lambda)\right)\right| \equiv 1 \tag{6.12}$$

Since the quasi-eight of a constant is zero as well as of the form σ we get that

$$\mathrm{wt}(\mathrm{ad}\vec{h})^m\mathcal{E}(\lambda) = -\mathrm{wt}(\mathrm{ad}\vec{h})^{m-1}\mathcal{E}(\lambda),$$

which together with the fact that their difference is equal to 4 implies that

$$\mathrm{wt}(\mathrm{ad}\vec{h})^m\mathcal{E}(\lambda) = -\mathrm{wt}(\mathrm{ad}\vec{h})^{m-1}\mathcal{E}(\lambda) = 2,$$

and, more generally,

$$\mathrm{wt}(\mathrm{ad}\vec{h})^i\mathcal{E}(\lambda) = -\mathrm{wt}(\mathrm{ad}\vec{h})^{m-1}\mathcal{E}(\lambda) = 2 - 4m + 4i. \tag{6.13}$$

This implies that the weight of the functions $\mathcal{B}_i$ from the decomposition (5.36) satisfy

$$\mathrm{wt}\mathcal{B}_i = \mathrm{wt}\left(\mathrm{ad}\vec{h})^{2m}\mathcal{E}\right) - \mathrm{wt}\left(\mathrm{ad}\vec{h})^i\mathcal{E}\right) = 8m - 4i \tag{6.14}$$

The Wilczynski invariantis A_{2i} is quasi-homogeneous and has the same weigh as $\mathcal{B}_{2m-2-i}$, i.e.

$$\mathrm{wt}A_{2i} = \mathrm{wt}(B_{2m-2-2i}) = 8(i+1). \tag{6.15}$$

So far in the calculation of quasi-weight, we counted the quasi-weights of structure functions and their derivatives. If we will count the quasi- weight of u_i's only the numerator and the denominator of the ith generalized Wilczynski invariant A_{2i} of a distribution D at a point $q \in M$ will become not quasi-homogeneous and can be decomposed into quasihomogeneous components. The quasi-homogeneous components of maximal possible weight will depend on c_{ij}^k with weight zero, i.e. according to Remark 6.2.1, these components will be the same as if one calculates the ith generalized Wilczynski invariant of the flat distribution corresponding the Tanaka symbol of D at q. So, if the monomial containing c_{ij}^k with weight zero in theor coefficient does not appear, it means that the ith generalized Wilczynski invariant of the flat distribution corresponding the Tanaka symbol of D at q vanishes, but if such terms appear it means that the ith generalized Wilczynski invariant of the flat distribution corresponding the Tanaka symbol of D at q does not vanishes these terms will have the maximal quasi-weight in u_i's and will appear for all distributions with the same Tanaka symbol at q. Hence, we proved the following

Theorem 6.2.2. *The ith generalized Wilczynski invariant of a distribution D at a point q is not zero if it is nonzero for the flat distribution corresponding to the Tanaka symbol of D at q.*

6.3 Applications to (2,3,5) distributions

For $n = 5$ and $(2,3,5)$-distributions $(D^3)^\perp(q) = 0$ and by (3.25), we $J^{(4-n)} = \mathcal{V}$, where, as before $\mathcal{V}$, is the tangent to the fiber of $(D^2)^\perp$ (see (3.26)). This implies that the Jacobi singularity locus $\mathrm{Sing_D}$ is empty which implies that The only one nontrivial generalized Wilczynski invariant A_2 is a homogeneous degree 4 polynomial on the fibers and can be computed using the formula (5.38) using (5.36) and the fact that the canonical section w.r.t. to $\vec{h}$ satisfies:

$$\mathcal{E}(\lambda) = \gamma_4(\lambda)\partial_{u_4} + \gamma_5(\lambda)\partial_{u_5}, \text{ where } \gamma_4(\lambda)u_5 - \gamma_5(\lambda)u_4 \equiv 1$$

For example, one can take $\mathcal{E} = \frac{1}{u_5}\partial_{u_4} or - \frac{1}{u_4}\partial_{u_5}$, see [25, 26] for more details.

In the considered case one can look at the generalized Wilczinski invariant in a slightly different way. Note that for every $v \in D(q)$ there exist the unique, up to homothety, $\lambda \in (D^2)^\perp(q)$ and the unique $\widehat{v} \in \mathcal{C}(\lambda)$ such that $\pi_*\widehat{v} = v$. The map

$$v \mapsto \mathcal{W}_2^\lambda(\widehat{v}) \tag{6.16}$$

is a well-defined degree 4 homogeneous function (and in fact a polynomial by the previous paragraph) on $D(q)$, called the *tangential generalize Wilczynski invariant of a* $(2,3,5)$*- distribution*.

Theorem 6.3.1. *[26] The tangential generalized Wilczynski invariant of a* $(2,3,5)$*- distribution coincides, up to a universal constant multiple, with its Cartan tensor.*

From this and (6.16) it follows that for $(2,3,5)$-distributions vanishing the generalized Wilczynski invariant is equivalent to the vanishing of its Cartan tensor and so the answer to the main question posed in the Introduction is positive, i.e. vanishing the generalized Wilczynski invariant implies that $(2,3,5)$ is locally equivalent to the Cartan flat one (not etha in this case there is only on tanaka symbol).

7. APPLICATIONS TO (2,3,5,6)-DISTRIBUTIONS

One can choose a local basis (X_1, X_2) of the distribution such that it can be extended to a local frame $X_1, \dots, X_6)$ of TM so that depending on their Tanaka symbol in addition to (6.6)

$$X_6 = [X_1, X_4] \tag{7.1}$$

$$[X_2, X_5] = \varepsilon X_6 \mod D^3,\ \varepsilon \in \{-1, 0, 1\}. \tag{7.2}$$

Using (5.31) we have

$$[\vec{h}, \partial_{u_6}] = u_5 \partial u_4 - \varepsilon u_4 \partial u_5 \in \mathcal{V} \tag{7.3}$$

From this and (5.32) it is not difficult to show that

$$\mathcal{V}^{(-2)}(\lambda) = \mathrm{span}\{\partial_{u_6}(\lambda), \vec{e}(\lambda)\}. \tag{7.4}$$

The relation (7.4) has the following coordinate-free interpretation:Let $q = \pi(\lambda)$ and $(D^i)^p erp(q) = (D^i)^\perp \cap \pi^{-1}(q)$. Then $\mathcal{V}^{(-2)}(\lambda)$ is equal to the plane in $(D^2)^p erp(q)$ passing through th e origin and the line through λ which is parallel to $(D^3)^p erp(q)$.

Lemma 7.1.1. *The following relation holds*

$$|\sigma(ad\vec{h}^3(\partial_{u_6}), ad\vec{h}^2(\partial_{u_6})| = (\varepsilon u_4^2 + u_5^2)^2 \tag{7.5}$$

and the canonical section $\mathcal{E}$ the canonical section associated to the chosen local basis (X_1, X_2) is , up to multiple on $\pm$ and modulo $\mathrm{span}\{\vec{e}(\lambda)\}$, equal to

$$\mathcal{E} = \frac{1}{\varepsilon u_4^2 + u_5^2} \partial_{u_6}. \tag{7.6}$$

Proof. By direct calculation,

$$(\mathrm{ad}\vec{h})^2(\partial_{u_6}) = -u_5\overline{X_2} - \varepsilon u_4\overline{X_1} \mod \mathcal{V} = \tag{7.7}$$

$$- u_5\vec{u}_2 - \varepsilon u_4\vec{u}_1 + (\varepsilon u_4^2 + u_5^2)\partial_{u_3} \mod \mathcal{V}, \tag{7.8}$$

$$(\mathrm{ad}\vec{h})^3((\partial_{u_6})) = (\varepsilon u_4^2 + u_5^2)\overline{X_3} \mathrm{mod}\, \widehat{D} = \tag{7.9}$$

$$(\varepsilon u_4^2 + u_5^2)\left(\vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2}\right) \mod \widehat{D}. \tag{7.10}$$

where $\widehat{D}$ is as in (3.12). Substituting (7.8) and (7.10) into the left-hand side of (7.5) and using (2.7) and (2.8) one gets the right-hand side of (7.5). Then (7.6) follows from (5.35). □

Formula (7.6) has the following more intrinsic interpretation: Let B be canonical symmetric form of the distribution D, defined by (6.7) Define

$$Q(\lambda) := B\left(\pi_*\vec{h}(\lambda), \pi_*\vec{h}(\lambda)\right). \tag{7.11}$$

Then in the chosen local frame $Q(\lambda) = \varepsilon u_4^2 + u_5^2$, so (7.6) can be written as

$$\mathcal{E} = \frac{1}{Q}\partial_{u_6}. \tag{7.12}$$

While the function Q depends on the choice of the local basis (X_1, X_2) its zero level set is independent of this choice.

Corollary 7.1.2. *The Jacobi singularity locus* Sing_D *of a* $(2,3,5,6)$*-distribution* D *satisfies*

$$\mathrm{Sing}_D = \{\lambda \in W_D : Q(\lambda) = 0\}, \tag{7.13}$$

i.e. $\lambda \in \mathrm{Sing}_D$ *if and only if the charcteristic line* $\mathcal{C}(\lambda)$ *is projected to a null (isotropic) line of* B *and* $\pi(\lambda)$*. In particular, any* $(2,3,5,6)$*-distribution is of maximal class at every point.*

Further, from (7.12) and formulas (5.39) and (5.40) , we have the following

Proposition 7.1.3. *The i generalized Wilczynski invariant A_{2i}, $i = 1, 2$, of a $2, 3, 5, 6)$-distribution has the form*

$$A_{2i} = \frac{P_i}{Q^{2(i+1)}}, \quad i = 1, 2. \tag{7.14}$$

where P_i is a polynomial.

It is easy to analyze the quasi-weight (counting the structure functions) and the usual degree of the polynomial P_i:

1. From (6.15) and the fact that $\mathrm{wt}\,(Q) = 6$, it follows that

$$\mathrm{wt}(P_i) = \mathrm{wt}(A_{2i}) + 2(i+1)\mathrm{wt}(Q) = 20(i+1), \quad i = 1, 2; \tag{7.15}$$

2. If $\deg$ denotes the usual degree, then $\deg A_{2i} = 2(i+1)$ and $\deg Q = 2$. Consequently.

$$\deg P_i = \deg A_{2i} + 2(i+1)\deg Q = 6(i+1). \tag{7.16}$$

Lemma 7.1.4. *Polynomials P_1 and P_2 defined in* (7.14) *can be written in the following form form*

$$\begin{aligned} P_1 &= \sum_{i=0}^{4} g_k(u_4, u_5)u_6^k, \\ P_2 &= \sum_{i=0}^{6} q_k(u_4, u_5)u_6^k, \end{aligned} \tag{7.17}$$

where g_k and q_k are polynomials in u_4 and u_5 of degrees $12-k$ and $18-k$, respectively. Moreover, the weight of structure functions appearing ion g_k and q_k is equal to $4-k$ and $6-k$ respectively.

Proof. Assume that the monomial $u_4^l u_5^j u_6^k$ appears in P_i. Then since $\mathrm{wt} u_6 = 4$ and $\mathrm{wt} u_4 = \mathrm{wt} u_5 = 3$ by (7.15) and (7.16) it follows that

$$3(l+j) + 4k \leq 20(i+1), \tag{7.18}$$

$$l + j + k = 6(i+1) \tag{7.19}$$

Subtracting 3 times (7.19) from (7.18) we get $k \leq 2(i+1)$. This implies (7.17). The degrees of g_k and q_k are calculated from (7.16). The weights of structure functions in g_k and q_k are equal to

$$20(i+1) - 3(l+j) - 4k \stackrel{(7.19)}{=} 2(i+1) - k \tag{7.20}$$

with $i = 1$ for g_k and $i = 2$ for q_k. This completes the proof of the lemma. □

In general, the denominator in (7.14) is not canceled. However, if the characteristic distribution $\mathcal{C}$ is tangent to the zero level sets of Q, then the generalized Wilczynski invariants A_2 and A_4 are polynomials.

In particular, by direct computations using (7.6), (5.31), (6.6), (6.8),(5.36), (5.39), and (5.40) one can show that for the flat distributions with given Tanaka symbol

$$A_1 = -\varepsilon^2 \frac{54}{175} u_6^4, \quad A_2 = -\frac{1354}{15435} \varepsilon u_6^6, \quad \varepsilon \in \{-1, 0, 1\}. \tag{7.21}$$

This shows that for the flat $(2,3,5,6)$- distribution with parabolic Tanaka symbol (i.e. when $\varepsilon = 0$) both generalized Wilczinski invariants vanish identically, for the flat $(2,3,5,6)$- distribution with elliptic or hyperbolic Tanaka symbol bith generalized Wilczynski invariants are not zero. This together with Theorem 6.2.2 implies the following

Corollary 7.1.5. *For a (2,3,5,6)-distribution with an elliptic or hyperbolic symbol at a given point both Wilczynski invariants do not vanish identically at this point.*

The last sentence of Lemma 7.1.4 implies the weight of any structure function appearing in g_4 and q_6 is zero, i.e. g_4 and q_6 are the same as in the case of the flat distribution with given Tanaka symbol:

$$g_4 = -\frac{54}{175} \varepsilon^2 Q^4, \quad q_6 = -\frac{1354}{15435} \varepsilon Q^6. \tag{7.22}$$

In particular, if the symbol of a distribution at a point is parabolic then

$$g_4 = 0, \quad q_6 = 0. \tag{7.23}$$

Corollary 7.1.5 implies that in order to address the main question of the Introduction we must concentrate on $(2, 3, 5, 6)$-distributions with a parabolic Tanaka symbol at every point.

In this case, we can take a local basis (X_1, X_2) of the distribution D so that X_2 generates the kernel of the canonical symmetric form B. We can complete it to a local frame $\{X_i\}_{i=1}^6$ on M by

$$\begin{aligned} X_3 &= [X_1, X_2], \quad X_4 = [X_1, X_3], \\ X_5 &= [X_2, X_3], \quad X_6 = [X_1, X_4]. \end{aligned} \tag{7.24}$$

We call the frame $\{X_i\}_{i=1}^6$ satisfying (7.24), the *standard extension of the local basis* $\{X_1, X_2\}$ *of* D with X_2 generating the kernel of B.

Remark 7.1.6. *From* (7.24)*, using Jacobi identity, it follows that*

$$[X_1, X_5] = [X_2, X_4] \tag{7.25}$$

Indeed,

$$\begin{aligned} &[X_1, X_5] = \big[X_1, [X_2, X_3]\big] = \big[[X_1, X_2], X_3\big] + \big[X_2, [X_1, X_3]\big] = \\ &[X_3, X_3] + \big[X_2, X_4\big] = [X_2, X_4]. \end{aligned} \tag{7.26}$$

By direct calculation one can show that in the parabolic case, in addition to (7.23) we have

$$g_3 = 0. \tag{7.27}$$

Let $r_{k,j}$ be the coefficients of the monomial $u_4^{12-k-j}u_5^j$ in g_k,

$$g_k = \sum_j r_{k,j} u_4^{12-k-j} u_5^j \tag{7.28}$$

The most general change of the local basis (X_1, X_2) preserving the property that X_2 generates the kernel of the canonical symmetric form B is

$$\widetilde{X}_1 = a_{11}X_1 + a_{12}X_2, \quad \widetilde{X}_2 = a_{22}X_2 \tag{7.29}$$

for some functions a_{11}, a_{22}, a_{12} on an open set of M where the local basis (X_1, X_2) is defined. Then plugging this to (7.24), one gets by direct computations:

$$\begin{aligned} \widetilde{X}_3 &= a_{11}a_{22}X_3 \mod D, \\ \widetilde{X}_4 &= a_{11}a_{22}(a_{11}X_4 + a_{12}X_5) \mod D^2, \\ \widetilde{X}_5 &= a_{11}a_{22}^2X_5 \mod D^2, \\ \widetilde{X}_6 &= a_{11}^3a_{22}X_6 \mod D^3. \end{aligned} \tag{7.30}$$

Consequently, on $(D^2)^\perp$

$$\begin{aligned} \tilde{u}_4 &= a_{11}a_{22}(a_{11}u_4 + a_{12}u_5), \\ \tilde{u}_5 &= a_{11}a_{22}^2u_5, \\ \widetilde{u}_6 &= a_{11}^3a_{22}u_6 \mod \text{a linear form in } u_4 \text{ and } u_5. \end{aligned} \tag{7.31}$$

Using the transformation rules (7.31) one can get invariants of $(2, 3, 5, 6)$- distribution with parabolic Tanaka symbol from (7.17) and (7.28). In particular, (7.23), (7.27), and (7.31) imply that the coefficient of the monomial in the polynomial g_2 with the maximal possible degree of u_5, i.e. the coefficient $r_{2,8}$ is the relative invariant of the distribution, i.e. it is either 0 or nonzero, independently of the choice of a local frame. In fact, By direct but tedious computations one can prove

Lemma 7.1.7. *The coefficients $r_{2,j}$ vanish for $0 \leq j \leq 7$ and*

$$r_{2,8} = \frac{4032}{25}(c_{25}^4)^2. \tag{7.32}$$

In particular, if the first generalized Wilczynski invariant vanishes then

$$c_{25}^4 \equiv 0. \tag{7.33}$$

Remark 7.1.8. *The fact that the structure function c_{25}^4 is the relative invariant of a $2,3,5,6)$-distribution with the parabolic symbol can be checked directly without referring to the calculations of the first generalized Wilczynski invariant: it is easy to show using* (7.30) *that under the change of local basis as in* (7.29) *c_{25}^4 is transformed to $\dfrac{a_{22}}{a_{11}}c_{25}^4$.*

8. LEFT-INVARIANT DISTRIBUTIONS ON LIE GROUPS WITH ZERO WILCZYNSKI INVARIANTS ARE LOCALLY FLAT

The following theorem is the main result of this thesis:

Theorem 8.1.1. *A left-invariant* $(2,3,5,6)$*-distribution on a 6-dimensional Lie group* G *with both generalized Wilczynski invariants equal to zero is locally equivalent to the flat distribution with parabolic Tanaka symbol (equivalently, to the symplectically flat* $(2,3,5,6)$*-distribution).*

Proof. First of all, by Corollary 7.1.5 such distribution D must have the parabolic Tanaka symbol at every point. We can take a basis $\{X_1, X_2\}$ of D consisting of left-invariant vector fields such that X_2 generates the kernel of the canonical symmetric form B on D. We can extend it to the frame on M as in (7.24). Since Lie brackets of left-invariant vector fields are left-invariant, this frame consists of left-invariant vector fields, and all structure functions of the frame, defined by (5.27), are constant.

The main idea is to show that if D satisfies the assumptions of Theorem 8.1.1, then it satisfies the assumptions of Theorem 1.2.1, i.e. that D has an infinitesimal symmetry X lying in D^3 but not in D^2. Recall that

$$D^3 = \operatorname{span}\{X_1, \dots X_5\}. \tag{8.1}$$

Since any right-invariant vector field commutes with any left-invariant vector field and D is left-invariant, any right-invariant vector field is an infinitesimal symmetry of D. Therefore, it is enough to show that there exists a right-invariant vector field lying in D^3, but not in D^2. By(8.1) a right-invariant vector field X lies in D^3 if and only if the following relation holds:

$$(R_g)_* X \in \operatorname{span}\big\{\{(L_g)_* X_i\}_{i=1}^5\big\}, \forall g \in G \tag{8.2}$$

where L_g and R_g are the left and right translations by an element g or, equivalently, by applying $(L_{g^{-1}})_*$ and using the definition of the adjoint representation Ad of the group G on its Lie algebra given by $Adg := (L_g)_*(R_{g^{-1}})_*$ we get

$$\mathrm{Ad}_g(X) \in \mathrm{span}\big\{\{X_i\}_{i=1}^5\big\}, \forall g \in G \tag{8.3}$$

which is equivalent to is infinitesimal version

$$\mathrm{ad}Y(X) \in \mathrm{span}\big\{\{X_i\}_{i=1}^5\big\}, \forall Y \in \mathrm{Lie}\, G, \tag{8.4}$$

where $\mathrm{Lie}\, G$ denotes the Lie algebra of G. The latter can be equivalently reformulated as

$$\mathrm{Im}(\mathrm{ad}X) \subset \mathrm{span}\big\{\{X_i\}_{i=1}^5\big\}, \tag{8.5}$$

where $\mathrm{Im}(\mathrm{ad}X)$ stands for the image of the operator $\mathrm{ad}X$. Let us show that under the assumption such X exists.By (8.1) We look for X in the form

$$X = \sum_{i=1}^{4} \alpha_i X_i + X_5. \tag{8.6}$$

By assuming that the coefficient of X_5 is nonzero (and so can be made into 1 by scaling) we ensure that X does not lie in D^2.

First, by (6.9),

$$[X, X_2] \in \mathrm{span}\big\{\{X_i\}_{i=1}^5\big\}. \tag{8.7}$$

Second, by (6.9) again

$$[X, X_1] = \alpha_4 X_6 \,\mathrm{mod}\ \mathrm{span}\big\{\{X_i\}_{i=1}^5\big\}$$

So, to get (8.5) we need

$$\alpha_4 = 0. \tag{8.8}$$

Further, using the Jacobi identity (see (9.8), the definition of structure function given by (5.27), relations (7.24), and the fact that X_2 is the kernel of the canonical symmetric form, we get

$$[X_3, X_5] = [[X_1, X_2], X_5] = [[X_1, X_5], X_2] + [X_1, [X_2, X_5]] = \tag{8.9}$$

$$\Big(\sum_{i=1}^{5}(c_{15}^i c_{i2}^6 + c_{1i}^6 c_{25}^i)\Big) X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\} = c_{25}^4 X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\}. \tag{8.10}$$

By Lemma 7.1.7 vanishing the first Wilczynski invariant implies that $c_{25}^4 = 0$, so the last relation implies

$$[X_3, X_5] \in \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\}. \tag{8.11}$$

So, plugging (8.6) into $[X, X_3]$ and using (8.11) and (8.8) we get that

$$[X, X_3] = \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\}. \tag{8.12}$$

By the same arguments

$$[X, X_5] = \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\}. \tag{8.13}$$

Next,

$$[X, X_4] = (\alpha_1 + \alpha_3 c_{34}^6 - c_{45}^6) X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\}, \tag{8.14}$$

$$[X, X_6] = (\alpha_1 c_{16}^6 + \alpha_2 c_{26}^6 + \alpha_3 c_{36}^6 + c_{56}^6) X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\} \tag{8.15}$$

So we have to solve the following system of linear equations w.r.t. α_1, α_2, and α_3:

$$\begin{aligned} &\alpha_1 + \alpha_3 c_{34}^6 = c_{45}^6, \\ &\alpha_1 c_{16}^6 + \alpha_2 c_{26}^6 + \alpha_3 c_{36}^6 = -c_{56}^6. \end{aligned} \tag{8.16}$$

Case 1: Either $c_{26}^6 \neq 0$ **or** $c_{36}^6 \neq c_{16}^6 c_{34}^6$. In this case system (8.16) has a solution, as the matrix of the system, i.e.

$$\begin{pmatrix} 1 & 0 & c_{34}^6 \\ c_{16}^6 & c_{26}^6 & c_{36}^6 \end{pmatrix}$$

has rank 2.

Case 2: Assume that

$$c_{26}^6 = 0 \tag{8.17}$$

$$c_{36}^6 = c_{16}^6 c_{34}^6. \tag{8.18}$$

In this case for solvability of system (8.16) it is enough to prove that the following matrix

$$\begin{pmatrix} 1 & 0 & c_{34}^6 & c_{45}^6 \\ c_{16}^6 & 0 & c_{36}^6 & -c_{56}^6 \end{pmatrix} \tag{8.19}$$

has rank 1. First from Jacobi identities,(8.7), and (7.24) we get

$$\begin{aligned} &[X_3, X_6] = [[X_1, X_2], X_6] = [[X_1, X_6], X_2] + [[X_1, [X_2, X_6]] = \\ &(-c_{16}^6 c_{26}^6 + c_{16}^6 c_{26}^6 + c_{26}^4) X_6 \text{ span}\big\{\{X_i\}_{i=1}^5\big\} = c_{26}^4 X_6 \text{ span}\big\{\{X_i\}_{i=1}^5\big\}, \end{aligned} \tag{8.20}$$

i.e.

$$c_{26}^4 = c_{36}^3. \tag{8.21}$$

Second, from Jacobi identities, (8.11), (8.21), (8.17), and (7.24) it follows that

$$\begin{aligned} &[X_5, X_6] = [[X_2, X_3], X_6] = [[X_2, X_6], X_3] + [[X_2, [X_3, X_6]] = \\ &(c_{26}^4 c_{43}^6 + c_{26}^6 c_{36}^6) X_6 \bmod \text{span}\big\{\{X_i\}_{i=1}^5\big\} = -c_{36}^6 c_{34}^6 \bmod \ \text{span}\big\{\{X_i\}_{i=1}^5\big\}, \end{aligned} \tag{8.22}$$

i.e., taking also into account (8.18),

$$c_{56}^6 = -c_{36}^6 c_{34}^6 = -c_{16}^6 (c_{34}^6)^2. \tag{8.23}$$

Further, use the Jacobi identity, (7.24), and (7.25), to get

$$\begin{aligned}&[X_3, X_4] = [[X_1, X_2], X_4] = [[X_1, X_4], X_2] + [X_1, [X_2, X_4]] = \\ &(c_{15}^4 - c_{26}^6) X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\},\end{aligned} \tag{8.24}$$

i.e., taking into account (8.17), we get

$$c_{34}^6 = c_{15}^4 - c_{26}^6 = c_{15}^4 \tag{8.25}$$

Finally, using Jacobi identity, (7.25)(8.17) and (8.25) we get

$$\begin{aligned}&[X_4, X_5] = [X_4, [X_2, X_3]] = [[X_4, X_2], X_3] + [X_2, [X_4, X_3]] = \\ &-[[X_1, X_5], X_3] - [X_2, [X_3, X_4]] = -c_{15}^4 c_{34}^6 + c_{26}^6 c_{34^6} \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\} = \\ &(c_{34}^6)^2 X_6 \mod \operatorname{span}\big\{\{X_i\}_{i=1}^5\big\},\end{aligned} \tag{8.26}$$

i.e.,

$$c_{45}^6 = (c_{34}^6)^2. \tag{8.27}$$

Combining formulas (8.18), (8.23), and (8.27) we get that in the nonzero matrix (8.19) the second row is equal to the first row multiplied by c_{16}^6, so this matrix has rank 1, which completes the proof of the theorem. $\square$

9. AN EXAMPLE OF A LEFT-INVARIANT (2,3,5,6)-DISTRIBUTION WITH ZERO WILCZYNSKI INVARIANTS ON A SOLVABLE LIE GROUP

The goal of this section is to give an example of a left-invariant distribution with both Wilczynski invariants equal to zero on a 6-dimensional Lie group whose Lie algebra is not isomorphic to the parabolic Tanaka symbol. Then by Theorem 8.1.1 a local diffeomorphism establishing the local equivalence of this distribution with the flat distribution of the parabolic Tanaka symbol does not preserve the group operation.

As already mentioned before, Corollary 7.1.5 such distribution D must have the parabolic Tanaka symbol. Again take a basis $\{X_1, X_2\}$ of D consisting of left-invariant vector fields such that X_2 generates the kernel of the canonical symmetric form B on D. We can extend it to the frame on M as in (7.24). Since Lie brackets of left-invariant vector fields are left-invariant, this frame consists of left-invariant vector fields, and all structure functions of the frame, defined by (5.27), are constant.

Now assume that

$$[X_1, X_5] = [X_2, X_4] = c_{15}^5 X_5, \quad c_{15}^5 \neq 0, \tag{9.1}$$

$$[X_2, X_6] = c_{26}^5 X_5, \tag{9.2}$$

$$[X_1, X_6] \in \operatorname{span}\{X_2, X_4\}. \tag{9.3}$$

Proposition 9.1.1. *Given an arbitrary constant c_{26}^5 and an arbitrary nonzero constant c_{15}^5 there exists the unique Lie algebra on a vector space spanned by $X_1, \ldots, X_6$ with Lie brackets satisfying* (7.24) *and* (9.1)-(9.3). *The only nontrivial Lie brackets on the elements of the basis $X_1, \ldots, X_6$, in* *addition to the ones from* (7.24) *and* (9.1)-(9.3) *and taking onto account skew-symmetricity are*

$$[X_1, X_6] = -\left(c^5_{26} - (c^5_{15})^2\right)^2 X_2 + (2c^5_{26} - (c^5_{15})^2)X_4, , \tag{9.4}$$

$$[X_3, X_4] = \left((c^5_{15})^2 - c^5_{26}\right) X_5, \tag{9.5}$$

$$[X_3, X_6] = c^5_{15}\left((c^5_{15})^2 - c^5_{26}\right) X_5, \tag{9.6}$$

$$[X_4, X_6] = \left(c^5_{26} - (c^5_{15})^2\right)^2 X_5 \tag{9.7}$$

Proof. The proof is by verification of all possible Jacobi identities

$$\big[[X_i, X_j], X_k\big] = \big[[X_i, X_k], X_j]\big] + \big[X_i, [X_j, X_k]\big], \tag{9.8}$$

for $i, j, k \in \{1, 2, \ldots, 6\}$. Indeed,

1. Relation (9.5) is obtain from (9.8) for $i = 1$, $j = 2$, and $k = 4$, using (7.25), (9.1), and the fact that $X_3 = [X_1, X_2]$.

2. Applying (9.8) for $i = 1$, $j = 2$, and $k = 6$, then using the fact that $X_3 = [X_1, X_2]$ and relations (9.2) and (9.3), we get

$$[X_3, X_6] = c^5_{15}\left(c^5_{26} - c^4_{16})\right) X_5 \tag{9.9}$$

3. Applying (9.8) for $i = 1$, $j = 3$, and $k = 4$, then using that $[X_1, X_3] = X_4$, $[X_1, X_4] = X_6$, and (9.5), (9.9) we get that

$$c^4_{16} = 2c^5_{26} - (c^5_{15})^2. \tag{9.10}$$

   Substituting this into (9.9) we get (9.6).

4. Applying (9.8) for $i = 1$, $j = 3$, and $k = 6$, then using that $[X_1, X_3] = X_4$ and relations

(9.3), (9.6) we get that

$$[X_4, X_6] = \Big(c_{16}^2 + 2\big(c_{26}^5 - (c_{15}^5)^2\big)^2\Big) X_5. \tag{9.11}$$

5. Applying (9.8) for $i = 1$, $j = 4$, and $k = 6$, then using the fact that $[X_1, X_4] = X_6$ and the relations (9.3), (7.25), (9.11), we get that

$$c_{16}^2 = -\big(c_{26}^5 - (c_{15}^5)^2\big)^2. \tag{9.12}$$

   This together with (9.10) and (9.3) implies (9.4). Further, substituting (9.12) into (9.11) we get (9.7).

All other Jacobi identities hold automatically. This completes the proof of the proposition. □

Further, note that using scaling $X_1 \to \alpha_1 X_1$ and $X_2 - > \alpha_2 X_2$ and extending the new local basis in a standard way as in (7.24) we get that

$$c_{15}^5 \to \alpha_1 c_{15}^5, \quad c_{26}^5 \to \alpha_1^2 c_{26}^5.$$

Consequently, under assumption that $c_{15}^5 \neq 0$ we can make

$$c_{15}^5 = 1. \tag{9.13}$$

by appropriate scaling. Denote by $c := c_{26}^5$ the remaining parameter. So, we get the left-invariant distribution D_c with the local basis of left-invariant vector fields $\{X_1, X_2\}$ on the Lie group with the Lie algebra spanned by $(X_1, \ldots, X_6)$ so that the only nontrivial brackets, up to skew-symmetricity, are (7.24) and

$$
\begin{aligned}
&[X_1, X_5] = [X_2, X_4] = X_5, \\
&[X_2, X_6] = cX_5, \\
&[X_1, X_6] = -(c-1)^2 X_2 + (2c-1)X_4, \\
&[X_3, X_4] = (1-c) X_5, \\
&[X_3, X_6] = (1-c) X_5, \\
&[X_4, X_6] = (c-1)^2 X_5.
\end{aligned}
\tag{9.14}
$$

Theorem 9.1.2. *The distribution D_c has both vanishing Wilczynski invariant if and only if $c = 3$.*

Proof. We follow the algorithm for the calculations of the Wilczynski invariants. By (5.30) in this case $\vec{h}$ on $(D^2)^\perp$ has the form

$$
\begin{aligned}
&\vec{h} = u_4\vec{u}_2 - u_5\vec{u}_1 = u_4(\bar{X}_2 + u_5\partial_{u_4} + cu_5\partial_{u_6}) - u_5(\bar{X}_1 + u_6\partial_{u_4} + u_5\partial_{u_5} + (2c-1)u_4\partial_{u_6}) = \\
&u_4\bar{X}_2 - u_5\bar{X}_1 + (u_4u_5 - u_5u_6)\partial_{u_4} - u_5^2\partial_{u_5} + (1-c)u_4u_5\partial_{u_6}.
\end{aligned}
\tag{9.15}
$$

Let $\mathcal{E}$ be as in (7.6). Then by direct computation

$$\mathrm{ad}\vec{h}\mathcal{E} \quad = \frac{1}{u_5}\partial_{u_4} + \frac{2}{u_5}\partial_{u_6} \tag{9.16}$$

$$\mathrm{ad}\vec{h}^2\mathcal{E} \quad = -\frac{1}{u_5}\bar{X}_2 + 2\partial_{u_4} + (c+1)\partial_{u_6} \tag{9.17}$$

$$\mathrm{ad}\vec{h}^3\mathcal{E} \quad = -3\bar{X}_2 + \bar{X}_3 + (c-1)u_5\partial_{u_4} + 2(c-1)u_5\partial_{u_6} \tag{9.18}$$

$$\mathrm{ad}\vec{h}^4\mathcal{E} \quad = -(c-1)u_5\bar{X}_2 + 3u_5\bar{X}_3 - u_5\bar{X}_4 + u_4\bar{X}_5 + (c-1)(c-3)u_5^2\partial_{u_6} \tag{9.19}$$

$$\mathrm{ad}\vec{h}^5\mathcal{E} \quad = (c-1)u_5^2\bar{X}_2 + (c-4)u_5^2\bar{X}_3 - 2u_5^2\bar{X}_4 + (2u_4u_5 - u_5u_6)\bar{X}_5 \tag{9.20}$$

$$+ u_5^2\bar{X}_6 + (c-1)(c-3)u_5^2(\partial_{u_4} - 2\partial_{u_6}) \tag{9.21}$$

$$\mathrm{ad}\vec{h}^6\mathcal{E} \quad = (c-3)\left(-6(c-1)u_5^4\partial_{u_4} + (c-1)(c+5)u_5^4\partial_{u_6} - 3u_5^3X_3 - 3u_5^3X_4 + 3u_4u_5^2X_5\right) \tag{9.22}$$

In particular, if $c = 3$ we have $\mathrm{ad}\vec{h}^6\mathcal{E} = 0$ which implies that both Wilczynski invariants vanish in this case.

It remains to show that for $c \neq 3$ at least one of Wilczynski's invariants does not vanish. For this, using (9.16), one can show by direct computations that the coefficients $\mathcal{B}_i$ $i = 0, \ldots, 4$, from the decomposition (5.36) satisfy[1]:

$$\begin{gathered} \mathcal{B}_4 = 3(c-3)u_5^2, \quad \mathcal{B}_3 = -12(c-3)u_5^3, \quad \mathcal{B}_2 = -3(c-13)(c-3)u_5^4 \\ \mathcal{B}_1 = 12(c-7)(c-3)u_5^5, \quad \mathcal{B}_0 = (c-13)(c-7)(c-3)u_5^6 \end{gathered} \tag{9.23}$$

Substituting this into (5.39) and (5.40) and using (9.15) we get

$$A_2 = -\frac{24}{175}(c-3)(16c-13)u_5^4 \tag{9.24}$$

$$A_4 = \frac{2}{5145}(c-3)(3321c^2 - 32176c + 4635889)u_5^6 \tag{9.25}$$

so that A_2 vanish if and only of $c = 3$ or $c = \frac{13}{16}$, but A_4 does not vanish in the latter case. This

[1] Note that $\mathcal{B}_5 = 0$ because $\mathcal{E}$ is the canonical section of the Jacobi curves

completes the proof of the theorem. □

Finally, note that for $c = 3$ (and in fact for any c) the Lie algebra with the product rules given by (7.24) and (9.14) is not nilpotent [2] and therefore is not isomorphic to the parabolic Tanaka symbol.

[2] Note that it is solvable.

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