CLASSIFICATION OF TRIPARTITE TENSORS WITH SMALL GEOMETRIC RANKS

A Dissertation

by

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ABSTRACT

Geometric Rank of tensors was introduced by Kopparty et al. as a useful tool to study algebraic complexity theory, extremal combinatorics and quantum information theory. This dissertation studies the classification of tripartite tensors with small geometric ranks. We introduce primitive tensors and compression tensors, which reduces the classification problem to finding all primitive tensors.

There are close relations between tripartite tensors with bounded geometric ranks and linear determinantal varieties with bounded codimensions. We study linear determinantal varieties with bounded codimensions, and prove upper bounds of the dimensions of the ambient spaces.

Using the results on linear determinantal varieties, we find all primitive tensors with geometric rank 1, 2 and 3 up to change of coordinates, find upper bounds of multilinear ranks of primitive tensors with geometric rank 4, and prove the existence of such upper bounds in general. Finally, we explicitly classify all tripartite tensors with geometric rank at most 1, 2 and 3.

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Contributors

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1. INTRODUCTION

Various types of ranks of tensors have been introduced and studied in numerous areas such as algebraic complexity, extremal combinatorics and quantum information theory. Slice rank arose in the study of the cap set problem [2], and it turned out to be helpful in the study of the sunflower problem [3]. Analytic rank was introduced by [4] in the context of Fourier analysis, and [5] showed it lower bounds slice rank and can replace slice rank in the resolution of cap set problem. In the study of random tensors, analytic rank also measures the bias of a tensor.

In arithmetic complexity of matrix multiplication, people want to know asymptotically how many arithmetic operations are required to multiply two matrices. More precisely, determine the exponent of matrix multiplication ω, the number such that two $n \times n$ matrices can be multiplied using $O(n^{\omega+\epsilon})$ scalars additions and multiplications for any $\epsilon > 0$. In the study of finding upper bounds on ω, Strassen introduced subrank which measures the “value" of a tensor [6], and the asymptotic version of subrank plays an important role in Strassen's laser method [7]. In quantum information theory, people study the convertibility of stochastic local operations and classical communications (SLOCC). It turned out that the rate of converting GHZ states to triples of EPR states via SLOCC equals to ω [8], which made subrank an interesting object to study in quantum information theory. [9] introduced quantum functional and studied its relations with asymptotic slice rank and asymptotic subrank. Subrank is mysterious and hard to compute, and geometric rank gives good upper bounds on subrank.

Geometric rank was introduced in [10] as an extension of analytic rank from finite fields to algebraically closed fields, and as a tool to find upper bounds on border subrank and lower bounds on slice rank. [11] took a step further studying geometric rank systematically, giving results on tensors with geometric rank at most 3. [12] showed that the partition rank is at most 2^{n-1} times of the geometric rank for n-part tensors. Putting different types of ranks in an increasing order, we have:

$$\begin{aligned}\text{Subrank} \leqslant \text{Border Subrank} \leqslant \text{Geometric Rank}\\ \leqslant \text{Partition Rank} \leqslant \text{Slice Rank} \leqslant \text{Multilinear Ranks} \leqslant \text{Rank.}\end{aligned}$$

Any tripartite tensor $T \in A \otimes B \otimes C := \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$ can be regarded as a trilinear function $T : A^* \times B^* \times C^* \to \mathbb{C}$. Its **geometric rank** is defined to be:

$$\mathrm{GR}(T) := \mathrm{codim}\{(\alpha, \beta) \in A^* \times B^* \mid T(\alpha, \beta, \gamma) = 0, \forall \gamma \in C^*\}.$$

Given $r \leqslant \min\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, let $\mathcal{GR}_r$ be the set of tensors with geometric rank at most r. Such sets of tensors are Zariski closed [10], and are important varieties in algebraic geometry that provides a new aspect to understand geometric structures of spaces of tensors. Our goal is to study $\mathcal{GR}_r$ systematically and classifies all tensors in $\mathcal{GR}_r$ up to changes of bases and permutations of A, B and C if possible.

In §3.1, we introduce primitive tensors and compression tensors (Definition 6) as helpful tools to classify tensors in $\mathcal{GR}_r$. We show that every tensor can be decomposed as a sum of a primitive tensor and compression tensor with certain geometric ranks (Lemma 9), which reduces the problem of classifying tensors in $\mathcal{GR}_r$ to finding all primitive tensors in $\mathcal{GR}_r$. We also show that the matrix multiplication tensors are either primitive or compression (Corrollary 8).

On the other hand, Proposition 2 reveals the close relations of geometric rank with spaces of matrices of bounded rank and linear determinantal varieties, which are classically studied objects in algebraic geometry. So we take advantage of previous researches to understand geometric rank better. We study spaces of matrices whose determinantal subvarieties have bounded codimensions in §3.2.

In §4, using results from §3, we find all primitive tensors in $\mathcal{GR}_r$ and conclude the classifications of all tensors in $\mathcal{GR}_r$, for $r = 1, 2, 3$. Besides, we find the upper bounds of multilinear ranks of primitive tensors with geometric rank 4 (Theorem 26), and prove the existence of such upper bounds for primitive tensors with arbitrary geometric rank (Theorem 28).

2. LITERATURE REVIEW

2.1 Geometric Rank

This section reviews some basic properties of geometric rank introduced in [10]. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be positive integers, and $A := \mathbb{C}^{\mathbf{a}}, B := \mathbb{C}^{\mathbf{b}}, C := \mathbb{C}^{\mathbf{c}}$.

Definition 1 ([10]). For a tensor $T \in A \otimes B \otimes C$, the **geometric rank** of T is

$$\mathrm{GR}(T) := \mathrm{codim}\{(\alpha, \beta) \in A^* \times B^* \,|\, T(\alpha, \beta, \gamma) = 0, \forall \gamma \in C^*\}.$$

T induces a linear map $T_A : A^* \to B \otimes C$. Omitting the subscripts when there is no ambiguity, $T(A^*) \subset B \otimes C$ is an $\mathbf{a}$-dimensional space of $\mathbf{b} \times \mathbf{c}$ matrices. For $0 \leqslant j \leqslant \min\{\mathbf{b}, \mathbf{c}\}$, let

$$A_j^* := \{\alpha \in A^* \,|\, \mathrm{rank}(T(\alpha)) \leqslant j\} \subset A^*.$$

Fixing bases $\{a_i\}_{i=1}^{\mathbf{a}}, \{b_j\}_{j=1}^{\mathbf{b}}$ and $\{c_k\}_{k=1}^{\mathbf{c}}$ of A, B and C, and the dual basis $\{\alpha_i\}_{i=1}^{\mathbf{a}}$ of A^* corresponding to $\{a_i\}_{i=1}^{\mathbf{a}}$, we often represent $T(A^*)$ by a general point $T(\sum x_i \alpha_i)$ of $T(A^*)$ in a matrix form. That is, $T(A^*)$ will be written as a $\mathbf{b} \times \mathbf{c}$ matrix whose entries are linear forms in variables x_i's. Then A_j^* is the subvariety determined by all $(j + 1) \times (j + 1)$ minors of $T(A^*)$.

The following proposition can be regarded as an equivalent definition of geometric rank.

Proposition 2 ([10]). $\mathrm{GR}(T) = \min_j(\mathrm{codim} A_j^* + j)$.

Proof. Let $V_{AB} := \{(\alpha, \beta) \in A^* \times B^* \,|\, T(\alpha, \beta, \gamma) = 0, \forall \gamma \in C^*\}$, so that $\mathrm{GR}(T) = \mathrm{codim} V_{AB}$. Let $\pi : V_{AB} \to A^*$ be the restriction of the first projection $A^* \times B^* \to A^*$. Note that for any $\alpha \in A^*$,

$$\pi^{-1}(\alpha) = \{\alpha\} \times \mathrm{LeftKernel}(T(\alpha)).$$

Therefore $\pi^{-1}(\alpha)$ is a linear space of dimension $\mathbf{b} - \mathrm{rank}(T(\alpha))$.

Since $V_{AB} = \bigcup_j \pi^{-1}(A_j)$, and for $\alpha \in A_j^* \backslash A_{j-1}^*$, $\pi^{-1}(\alpha)$ is a linear space of dimension $\mathbf{b} - j$, we obtain:

$$\mathrm{dim} V_{AB} = \max_j \mathrm{dim}(\pi^{-1}(A_j^*)) = \max_j (\mathbf{b} - j)\mathrm{dim}(A_j^*),$$

which proves the proposition. $\square$

From Definition 1 we see that geometric rank is symmetric in A and B factors, i.e., swapping A and B factors of any tensor $T \in A \otimes B \otimes C$ does not change the value of $\mathrm{GR}(T)$. Meanwhile by Proposition 2, geometric rank is symmetric in B and C factors. This implies that geometric rank is symmetric in all A, B and C factors, so we have:

Proposition 2' ([10])**.**

$$\mathrm{GR}(T) = \min_j(\mathrm{codim}A_j^* + j) = \min_j(\mathrm{codim}B_j^* + j) = \min_j(\mathrm{codim}C_j^* + j).$$

Lemma 3 ([10])**.** *(Subadditivity) Let* $S, T \in A \otimes B \otimes C$*. Then* $\mathrm{GR}(S+T) \leqslant \mathrm{GR}(S) + \mathrm{GR}(T)$.

Proof. Let V_{AB}^S, V_{AB}^T and $V_{AB}^{S+T} \subset A^* \times B^*$ be the subvarieties in the definition of geometric rank for S, T and $S+T$ respectively, i.e., $\mathrm{GR}(S) = \mathrm{codim}V_{AB}^S$, $\mathrm{GR}(T) = \mathrm{codim}V_{AB}^T$, and $\mathrm{GR}(S+T) = \mathrm{codim}V_{AB}^{S+T}$. Clearly $V_{AB}^{S+T} \supset V_{AB}^S \cap V_{AB}^T$. So

$$\mathrm{GR}(S+T) = \mathrm{codim}V_{AB}^{S+T} \leqslant \mathrm{codim}(V_{AB}^S \cap V_{AB}^T) \leqslant \mathrm{codim}V_{AB}^S + \mathrm{codim}V_{AB}^T.$$

$\square$

2.2 Multilinear Ranks and Slice Rank

For $T \in A \otimes B \otimes C = \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$, the **multilinear ranks** are $\mathrm{ml}_A(T) := \mathrm{rank}(T_A)$, $\mathrm{ml}_B(T) := \mathrm{rank}(T_B)$ and $\mathrm{ml}_C(T) := \mathrm{rank}(T_C)$. And the **slice rank** is $\mathrm{SR}(T) := \min\{\mathrm{ml}_A(T_1) + \mathrm{ml}_B(T_2) + \mathrm{ml}_C(T_3) \mid T = T_1 + T_2 + T_3\}$.

It is clear to see that $\mathrm{SR}(T) \leqslant \mathrm{ml}_A(T), \mathrm{ml}_B(T), \mathrm{ml}_C(T)$ by the definition of slice rank. Note that $\dim A_0^* = \mathbf{a} - \mathrm{ml}_A(T)$, so $\mathrm{GR}(T) \leqslant \mathrm{codim}(A_0^* + 0) = \mathrm{ml}_A(T)$. Similarly, by Proposition 2', $\mathrm{GR}(T) \leqslant \mathrm{ml}_A(T), \mathrm{ml}_B(T), \mathrm{ml}_C(T)$.

By the definition of slice rank, there exist T_1, T_2 and T_3 such that $T = T_1 + T_2 + T_3$ and

$$\mathrm{SR}(T) = \mathrm{ml}_A(T_1) + \mathrm{ml}_B(T_2) + \mathrm{ml}_C(T_3) \geqslant \mathrm{GR}(T_1) + \mathrm{GR}(T_2) + \mathrm{GR}(T_3) \geqslant \mathrm{GR}(T).$$

To summarize, we conclude $\mathrm{GR}(T) \leqslant \mathrm{SR}(T) \leqslant \mathrm{ml}_A(T), \mathrm{ml}_B(T), \mathrm{ml}_C(T)$.

2.3 Linear Determinantal Variety

For a linear space of matrices $E \subset A \otimes B := \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$, let E_r be the locus of matrices of rank at most r, for $r \leqslant \min\{\mathbf{a}, \mathbf{b}\}$. In other words, $\mathbb{P}E_r = \mathbb{P}E \cap \sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$, the intersection of $\mathbb{P}E$ with the r-th secant variety of the Segre variety. E_r is cut out by all $(r+1) \times (r+1)$ minors set theoretically, and is called a **linear determinantal variety** (see, e.g., [13, Ch. II]).

Let $H := A \otimes B$, then H_r is the affine cone of $\sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$ and is called a generic determinantal variety. The defining ideal $I(H_r)$ is prime and generated by all $(r+1) \times (r+1)$ minors [14], and $\mathrm{codim}(H_r) = (\mathbf{a} - r)(\mathbf{b} - r)$ [15]. Since $E_r = H_r \cap E$ is a linear section of H_r, $\mathrm{codim}_E(E_r) \leqslant (\mathbf{a} - r)(\mathbf{b} - r)$.

To study $\mathcal{GR}_r$, note that by definition $T(A_i^*)$ consists of matrices in $B \otimes C$ of rank at most i, so it is a linear determinantal variety. Since $\mathrm{codim}_{T(A^*)}(T(A_i^*)) = \mathrm{codim}_{A^*}(A_i^*)$, by Proposition 2 we need to find all linear spaces $E \subset B \otimes C$ satisfying $\mathrm{codim}_E(E_i) \leqslant r - i$ for $0 \leqslant i \leqslant r$.

2.4 Space of Matrices of Bounded Rank

A linear space of matrices $E \subset A \otimes B := \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$ is said to have **bounded rank r** if all matrices in E have rank at most r, i.e., $E_r = E$. There are two important classes of spaces of bounded rank – primitive spaces [16] and compression spaces [17]. E is **compression** if there exist $A' \subset A$ and $B' \subset B$ of dimension p and q, such that $E \subset A' \otimes B + A \otimes B'$ and $p + q = r$. E is **primitive** of bounded rank r if for any subspaces $A' \subset A$ or $B' \subset B$ of codimension 1, $E \not\subset A' \otimes B$ or $A \otimes B'$, and neither $E \cap (A' \otimes B)$ nor $E \cap (A \otimes B')$ has bounded rank $r - 1$.

Atkinson and Lloyd showed that there is no primitive space of bounded rank 1, and every space of bounded rank r that is not compression is a "sum" of a compression space and a primitive space [16].

Theorem 4 ([16]). *If E is a space of $\mathbf{a} \times \mathbf{b}$ matrices of bounded rank r then these exist a primitive space F of bounded rank s and integers $p, q \geqslant 0$ with $r = p + q + s$ such that E is equivalent to a* *space of the form*

$$\begin{pmatrix} * & * & * \\ * & F & 0 \\ * & 0 & 0 \end{pmatrix}$$

where the top left block has size $p \times q$.

Moreover, if E is primitive of bounded rank r, then at least one of the following holds:

(1) $\mathbf{a} = r + 1, \mathbf{b} \leqslant \frac{1}{2}r(r+1)$*;*

(2) $\mathbf{a} \leqslant \frac{1}{2}r(r+1), \mathbf{b} = r + 1$*;*

(3) for some integers $c, d \geqslant 2$ with $c + d = r$, $\mathbf{a} \leqslant c + 1 + \frac{1}{2}d(d+1), \mathbf{b} \leqslant d + 1 + \frac{1}{2}c(c+1)$.

Later all primitive spaces of bounded rank 2 and 3 were classified in [18] – the only primitive space of bounded rank 2 is the space of 3×3 skew-symmetric matrices. [17] recasted the study with sheaves and gave geometric interpretations of all primitive spaces of bounded rank 3 as matrices. The classifications of spaces of bounded rank 4 or higher is still unknown by far.

Theorem 5 ([17])**.** *A primitive space of bounded rank 3 is equivalent to either the space of the following form*

$$\begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix},$$

or its transpose, or one of its projections and their transposes.

Proposition 2 shows $\mathrm{GR}(T) \leqslant r$ if at least one of $T(A^*), T(B^*)$ and $T(C^*)$ has bounded rank r. In §4, we will see that when $r = 1$ and 2 this condition is necessary (??). But it fails to be necessary when $r = 3$ as there are two exceptions (Theorem 24).

2.5 Matrix Multiplication Tensor

In the study of arithmetic complexity of matrix multiplication, Strassen found that the number of additions and multiplications are required to multiply two matrices asymptotically is determined by the rank of matrix multiplication tensors [7].

For positive integers $e \leqslant h \leqslant l$, put $A = \mathbb{C}^{e\times h}, B = \mathbb{C}^{h\times l}$ and $C = \mathbb{C}^{l\times e}$. Then the **matrix multiplication tensor** $M_{\langle e,h,l\rangle}$ is defined by $M_{\langle e,h,l\rangle}(x,y,z) = \mathrm{Tr}(xyz)$ for $x \in A^*$, $y \in B^*$ and $z \in C^*$. We often write $M_{\langle n\rangle} := M_{\langle n,n,n\rangle}$. With proper choices of bases, $M_{\langle e,h,l\rangle}$ may be written as the block form:

$$M_{\langle e,h,l\rangle}(A^*) = \begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{pmatrix} \tag{2.1}$$

where D is a $e \times h$ block consisting of linearly independent entries and there are l copies of D in $M_{\langle e,h,l\rangle}(A^*)$.

Strassen gave a lower bound of the border subrank of $M_{\langle e,h,l\rangle}$, which is $eh - \lfloor (e+h-l)^2/4 \rfloor$ if $e+h \geqslant l$ and eh otherwise [7]. Recently [10] surprisingly found that the above lower bound equals to the geometric rank of $M_{\langle e,h,l\rangle}$, and consequently equals to the border subrank of $M_{\langle e,h,l\rangle}$ since geometric rank upper bounds border subrank.

3. METHOD

3.1 Primitive and Compression Tensors

For any $r \geqslant 1$, $\mathcal{GR}_r$ contains a large class of tensors – the set of tensors with slice rank at most r. If a tensor has slice rank at most r, then there exists $s + p + q = r$ such that for some bases of A, B and C, the only non-vanishing entries of T only appear in the first s columns, the first p rows and the first q pages. Therefore these tensors are easy to understand in the study of classifications, and we are only interested in tensors whose geometric ranks are less than slice ranks.

Definition 6. T is **compression of geometric rank r** if $\mathrm{GR}(T) = \mathrm{SR}(T) = r$. T is **primitive of geometric rank r** if it cannot be written as $T = X + Y$ with $\mathrm{GR}(X) = r - 1$ and $\mathrm{GR}(Y) = 1$.

The following lemma gives a direct way to determine whether a tensor is primitive in general.

Lemma 7. *Given T with $1 < \mathrm{GR}(T) = r < \mathrm{SR}(T)$, then T is not primitive if and only if $\exists i < r$ such that by a permutation of A, B and C, $\mathrm{codim}(A_i^*) = r - i$ and A_i^* has a component of maximal dimension that is contained in a hyperplane of A^*.*

Proof. Let $\{a_i\}_{i=1}^{\mathbf{a}}$ be a basis of A, and $\{\alpha_i\}_{i=1}^{\mathbf{a}}$ be the dual basis of A^*. Write $A' := \langle a_2, \cdots, a_{\mathbf{a}} \rangle$, so $A'^* = \langle \alpha_2, \cdots, \alpha_{\mathbf{a}} \rangle$.

($\Rightarrow$) T is not primitive if and only if we can decompose $T = X + Y$ with $\mathrm{GR}(X) = r - 1$ and $\mathrm{GR}(Y) = 1$. Since $\mathrm{GR}(Y) = 1$ if and only if $\mathrm{SR}(Y) = 1$, by permuting A, B and C assume $\mathrm{ml}_A(Y) = 1$, and by changing basis of A assume $Y \in \langle a_1 \rangle \otimes B \otimes C$.

Then $T = X' + Y'$ where $X' := T|_{A' \otimes B \otimes C}$ and $Y' := T|_{\langle a_1 \rangle \otimes B \otimes C}$. Since $X' = X|_{A' \otimes B \otimes C}$, $\mathrm{GR}(X') \leqslant \mathrm{GR}(X) = r - 1$. By subadditivity of geometric rank and $\mathrm{SR}(Y') = \mathrm{GR}(Y') = 1$, $\mathrm{GR}(X') = r - 1$. By (2) there exists $i \leqslant r - 1$ such that $\mathrm{codim}\{\alpha \in A^* \mid \mathrm{rank}(X'(\alpha)) \leqslant i\} \leqslant r - 1 - i$. Then $\{\alpha \in A^* \mid \mathrm{rank}(X'(\alpha)) \leqslant i\} \cap A'^* \subset A_i^*$ has codimension $r - i$ in A^* and is contained in a hyperplane.

($\Leftarrow$) Assume $\mathrm{codim}(A_i^*) = r - i$ and A_i^* has a component Z of maximal dimension contained in A'^*. Let X' and Y' be defined the same as above. By definition $\{\alpha \in A'^* \mid \mathrm{rank}(X'(\alpha)) \leqslant i\} \supset Z$

so has codimension at most $r-i$ in A^*, then its codimension is at most $r-1-i$ in A'^*. Since $X' \in A' \otimes B \otimes C$, $\mathrm{GR}(X') \leqslant r-1$. By $T = X' + Y'$ and subadditivity of geometric rank, $\mathrm{GR}(X') = r-1$ and $\mathrm{GR}(Y') = 1$. □

Corollary 8. *For positive integers $e \leqslant h \leqslant l$, $M_{\langle e,h,l\rangle}$ is primitive if $e \geqslant 2$ and $e+h \geqslant l$, and it is compression otherwise.*

Proof. By Theorem 6.1 of [10], $\mathrm{GR}(M_{\langle e,h,l\rangle}) = eh$ if $e+h \leqslant l$ or $e = 1$. Since $\mathrm{GR}(M_{\langle e,h,l\rangle}) \leqslant \mathrm{SR}(M_{\langle e,h,l\rangle}) \leqslant \mathrm{ml}_A(M_{\langle e,h,l\rangle}) = eh$, we have $\mathrm{GR}(M_{\langle e,h,l\rangle}) = \mathrm{SR}(M_{\langle e,h,l\rangle}) = eh$ and therefore $M_{\langle e,h,l\rangle}$ is compression.

Assume $e \geqslant 2$ and $e+h \geqslant l$. The component of the maximal dimension $Z \subset A_i$ is determined by all $k \times k$ minors of D, where $k = \min\{e, \lceil \frac{i+1}{l} \rceil\}$. By [19, Theorem 2.1], $\mathrm{codim}(A_i) = \mathrm{codim}(Z) = (e+1-k)(h+1-k)$. So (2) achieves minimum only at $i = \lceil \frac{e+h-l}{2} \rceil l$ and $\lfloor \frac{e+h-l}{2} \rfloor l$. Then $k > 1$ and Z is not contained in any hyperplane. □

Although we define the primitive and compression tensors as analogues of primitive and compression spaces of matrices, their relations are subtle.

By definition T is compression of $\mathrm{GR}(T) = r$ if at least one of $T(A^*), T(B^*)$ or $T(C^*)$ is a compression space of bounded rank r and none has bounded rank $r-1$. The converse is true only for $r \leqslant 2$, as $T := \sum_{i=1}^{m} (a_1 \otimes b_i \otimes c_i + a_i \otimes b_1 \otimes c_i + a_i \otimes b_i \otimes c_1)$ is compression of $\mathrm{GR}(T) = 3$ but $T(A^*), T(B^*)$ and $T(C^*)$ contain elements of full rank.

If T is primitive of $\mathrm{GR}(T) = r$ and $T(A^*)$ has bounded rank r, then $T(A^*)$ is primitive of bounded rank r (after deleting zero rows and columns). Similarly for $T(B^*)$ and $T(C^*)$. However T could be primitive when $T(A^*), T(B^*)$ and $T(C^*)$ do not have bounded rank r.

For example, $M_{\langle 2\rangle}$ is primitive of geometric rank 3 by Corollary 8. But since $M_{\langle 2\rangle}(A^*)$ can be written as the block diagonal form (2.1), generic matrices in $M_{\langle 2\rangle}(A^*)$ have full rank 4. Therefore $M_{\langle 2\rangle}(A^*)$ does not have bounded rank 3. For the same reason, $M_{\langle 2\rangle}(B^*)$ and $M_{\langle 2\rangle}(C^*)$ do not either.

There is no primitive space of bounded rank 1, and all primitive spaces bounded rank 2 and 3 are listed in [18, 17]. We check every such primitive space and conclude that for $r \leqslant 3$, if $T(A^*)$ is primitive of bounded rank r, then T is primitive of geometric rank r. It is not known if this property persists when $r > 3$, because the set of all primitive spaces of larger bounded rank are not classified yet.

Lemma 9. *If T is not compression (i.e., $\mathrm{GR}(T) < \mathrm{SR}(T)$), then there exist a primitive tensor T_p and a compression tensor T_c, such that $T = T_p + T_c$ and $\mathrm{GR}(T_p) + \mathrm{GR}(T_c) = \mathrm{GR}(T)$.*

Proof. If T is primitive, set $T_p = T$ and $T_c = 0$.

If T is not primitive, assume $\mathrm{GR}(T) = r$, then we can write $T = X_1 + Y_1$ such that $\mathrm{GR}(X_1) = r - 1$ and $\mathrm{GR}(Y_1) = 1$. Similarly, whenever X_i is not primitive or zero, we can write $X_i = X_{i+1} + Y_{i+1}$ such that $\mathrm{GR}(X_i) = r - i$ and $\mathrm{GR}(Y_1) = 1$. If all X_i's obtained this way are not primitive, we have a decomposition $T = Y_1 + \cdots + Y_r$ where each Y_i has geometric rank 1 so has slice rank 1. This implies $\mathrm{SR}(T) = r = \mathrm{GR}(T)$, contradicting the assumption $\mathrm{GR}(T) < \mathrm{SR}(T)$.

So there exists $n < r$ such that X_n is primitive, then we obtain $T = T_p + T_c$ where $T_p := X_n$ and $T_c := Y_1 + \cdots + Y_n$. Since $\mathrm{GR}(T_p) = r - n$ and $\sum \mathrm{GR}(Y_i) = \sum \mathrm{SR}(Y_i) = n$, by subadditivity of geometric rank and slice rank, $\mathrm{GR}(T_c) = \mathrm{SR}(T_c) = n$. Therefore T_c is compression. □

Example 10 (Above decomposition is not unique)**.** Let $T \in A \otimes B \otimes C = \mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^6$ be

$$\begin{aligned} T :=& a_1 \otimes (b_2 \otimes c_1 + b_3 \otimes c_2 + b_4 \otimes c_3) + a_2 \otimes (b_1 \otimes c_1 - b_3 \otimes c_4 - b_4 \otimes c_5) \\ &+ a_3 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4 - b_4 \otimes c_6) + a_4 \otimes (b_1 \otimes c_3 + b_2 \otimes c_5 + b_3 \otimes c_6) + a_5 \otimes b_5 \otimes c_6 \end{aligned}$$

where $\{a_i\}_{i=1}^5, \{b_j\}_{j=1}^5$ and $\{c_k\}_{k=1}^6$ are bases of A, B and C respectively. So

$$T(A^*) = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & x_1 & 0 & -x_2 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & x_5 \end{pmatrix}.$$

Let $X_1 := T|_{A\otimes B\otimes\langle c_1,\cdots,c_5\rangle}, Y_1 := T|_{A\otimes B\otimes\langle c_6\rangle}, X_2 := T|_{A\otimes\langle b_1,\cdots,b_4\rangle\otimes C}$ and $Y_2 := T|_{A\otimes\langle b_5\rangle\otimes C}$. Since $X_1(A^*)$ consists of the first 5 columns of $T(A^*)$ and $X_2(A^*)$ consists of the first 4 rows of $T(A^*)$, they are primitive spaces of bounded rank 3 (after deleting the zero columns and rows). So

X_1 and X_2 are primitive of geometric rank 3, and $T = X_1 + Y_1 = X_2 + Y_2$ gives two different decompositions satisfying the conditions in Lemma 9.

By Lemma 9, to classify the set of tensors of geometric rank at most r, it suffices to find all primitive tensors of geometric rank at most r.

3.2 Determinantal Varieties of Bounded Codimensions

As discussed in §2.3, to classify tensors with bounded geometric ranks, it suffices to classify linear spaces of matrices whose determinantal subvarieties have bounded codimensions. This section studies the properties of such spaces and try to classify them up to invertible row and column operations.

Let $E \subset \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} =: A \otimes B$ be a linear subspace of dimension $\mathbf{c}$. Fix a basis $\{e_i, 1 \leqslant i \leqslant \mathbf{c}\}$ of E and bases of A and B, then each e_i can be written as an $\mathbf{a} \times \mathbf{b}$ matrix. Similar to how we represent $T(A^*) \subset B \otimes C$ as a matrix of linear forms, E is represented by the matrix corresponding to a general point $\sum_i x_i e_i$ of E, i.e., $E = (y^i_j)_{1 \leqslant i \leqslant \mathbf{a}, 1 \leqslant j \leqslant \mathbf{b}}$, where each y^i_j is a linear form in the variables $x_1, \cdots, x_{\mathbf{c}}$. For two subspaces $F, F' \subset E$, let $F + F'$ denote the sum of the two corresponding matrices of linear forms.

Denote the $(i_1, \cdots, i_k) \times (j_1, \cdots, j_k)$ minor of E as $\Delta^{i_1, \cdots, i_k}_{j_1, \cdots, j_k}$ and $\Delta_k := \Delta^{12 \cdots k}_{12 \cdots k}$. Unless otherwise stated, the codimension of a subset always refers to the codimension in E or $\mathbb{P}E$.

3.2.1 Case $\operatorname{codim}(E_r) = 1$

This subsection studies the case $\operatorname{codim}(E_r) = 1$, i.e. all nonzero $(r+1) \times (r+1)$ minors of E has a common polynomial factor of degree at least 1.

Lemma 11. *Let $E \subset \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$, $r < \mathbf{a}, \mathbf{b}$ and $E_r \neq E$. If there exists a degree $r+1$ polynomial P dividing all $(r+1) \times (r+1)$ minors of E, then either P factors into a product of linear forms, or $E \subset \mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}$.*

Proof. The hypothesis that all $(r+1) \times (r+1)$ minors of E are equal up to scale is invariant under changes of bases in A and B, so we are allowed to perform invertible row and column operations.

Since $E_r \neq E$, there exists a nonzero $(r+1)\times(r+1)$ minor of E. By changes of bases we can assume $\Delta_{r+1} = P$. We further assume $\Delta_r, \cdots, \Delta_2, y_1^1$ are nonzero.

Write $E = (y_j^i)_{1\leqslant i\leqslant \mathbf{a}, 1\leqslant j\leqslant \mathbf{b}}$. Consider the the block consisting of the first $r+1$ rows and the first $r+2$ columns:

$$\begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 & y_{r+2}^1 \\ \vdots & & \vdots & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} & y_{r+2}^{r+1} \end{pmatrix}.$$

Let $I := (1, 2, \cdots, r+1)$. For $j \leqslant r+1$, expand the minor consisting all columns except the j-th along the last column, then we have

$$c_j P = \Delta^I_{I\setminus j, r+2} = \sum_{i=1}^{r+1} (-1)^{i+(r+2)-1} y_{r+2}^i \Delta_{I\setminus j}^{I\setminus i}$$

for some $c_j \in \mathbb{C}$. Thus,

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{r+1} \end{pmatrix} P = (-1)^{r+1} \left((-1)^i \Delta_{I\setminus j}^{I\setminus i} \right)_{j,i=1}^{r+1} \begin{pmatrix} y_{r+2}^1 \\ \vdots \\ y_{r+2}^{r+1} \end{pmatrix}.$$

For every $j \leqslant r+1$, multiply $(-1)^j$ to the j-th row,

$$(-1)^{r+1} \begin{pmatrix} (-1)^1 c_1 \\ \vdots \\ (-1)^{r+1} c_{r+1} \end{pmatrix} P = \left((-1)^{i+j} \Delta_{I\setminus j}^{I\setminus i} \right)_{j,i=1}^{r+1} \begin{pmatrix} y_{r+2}^1 \\ \vdots \\ y_{r+2}^{r+1} \end{pmatrix}. \tag{3.1}$$

Now $((-1)^{i+j}\Delta_{I\setminus j}^{I\setminus i})_{j,i=1}^{r+1}$ is the cofactor matrix of the transpose of $(y_j^i)_{i,j=1}^{r+1}$, whose determinant is $\Delta_{r+1} = P$ by assumption. So

$$(-1)^{r+1} \begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 \\ \vdots & & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} \end{pmatrix} \begin{pmatrix} -c_1 \\ \vdots \\ (-1)^{r+1} c_{r+1} \end{pmatrix} = \begin{pmatrix} y_{r+2}^1 \\ \vdots \\ y_{r+2}^{r+1} \end{pmatrix}.$$

Therefore the column vector $(y_{r+2}^1, \ldots, y_{r+2}^{r+1})^t$ is a linear combination of all column vectors appearing in the upper left $(r+1)\times(r+1)$ block of E, i.e. $(y_j^1, \cdots, y_j^{r+1})^t, 1 \leqslant j \leqslant r+1$. By adding linear combinations of the first $r+1$ columns to the $(r+2)$-th, we may make the first $r+1$ entries of the $(r+2)$-th column equal to zero. Similarly, we may make the all last $\mathbf{b}-r-1$ entries in the first $r+1$ rows equal to zero. By the same argument, we may do the same for the first $r+1$

columns. Then the matrix E becomes:

$$E' = \begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{y}_{r+2}^{r+2} & \cdots & \tilde{y}_{\mathbf{b}}^{r+2} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \tilde{y}_{r+2}^{\mathbf{a}} & \cdots & \tilde{y}_{\mathbf{b}}^{\mathbf{a}} \end{pmatrix}. \tag{3.2}$$

If $\tilde{y}_{r+1+j}^{r+1+i} = 0, \forall i, j > 0$, let A' be the space corresponding to the first $r+1$ rows of E' and B the first $r+1$ columns, then $E \subset A' \otimes B' = \mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}$.

If there exists a nonzero $\tilde{y}_{r+1+j}^{r+1+i}$, by changes of bases assume it is $\tilde{y}_{r+2}^{r+2}$. For $1 \leqslant i_1 < \cdots < i_r \leqslant r+1$, $1 \leqslant j_1 < \cdots < j_r \leqslant r+1$, the $(r+1) \times (r+1)$ minor $\Delta_{j_1,\cdots,j_r,r+2}^{i_1,\cdots,i_r,r+2} = \Delta_{j_1,\cdots,j_r}^{i_1,\cdots,i_r} \tilde{y}_{r+2}^{r+2}$ is a multiple of Δ_{r+1}. Hence all $r \times r$ minors of the upper left $(r+1) \times (r+1)$ block equal up to scale.

By assumption $\Delta_r \neq 0$. Adding a linear combination of the first r columns to the $(r+1)$-th column and a linear combination of the first r rows to the $(r+1)$-th row, we can set all entries in $(r+1)$-th column and row zero except the $(r+1, r+1)$-th entry. Since $\Delta_{r+1} \neq 0$, the $(r+1, r+1)$-th entry is nonzero, written as $\tilde{y}_{r+1}^{r+1}$. Then E' becomes:

$$E'' = \begin{pmatrix} y_1^1 & \cdots & y_r^1 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ y_1^r & \cdots & y_r^r & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{y}_{r+1}^{r+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \tilde{y}_{r+2}^{r+2} & \cdots & \tilde{y}_{\mathbf{b}}^{r+2} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \tilde{y}_{r+2}^{\mathbf{a}} & \cdots & \tilde{y}_{\mathbf{b}}^{\mathbf{a}} \end{pmatrix}.$$

Repeat the above process on the upper left $k \times k$ blocks consecutively for $k = r-1, r-2, \cdots, 2,$

then E'' becomes:

$$\begin{pmatrix} y_1^1 & & & & & & \\ & \tilde{y}_2^2 & & & & & \\ & & \ddots & & & & \\ & & & \tilde{y}_{r+1}^{r+1} & & & \\ & & & & \tilde{y}_{r+2}^{r+2} & \cdots & \tilde{y}_{\mathbf{b}}^{r+2} \\ & & & & \vdots & & \vdots \\ & & & & \tilde{y}_{r+2}^{\mathbf{a}} & \cdots & \tilde{y}_{\mathbf{b}}^{\mathbf{a}} \end{pmatrix}.$$

Therefore $\Delta_{r+1} = y_1^1 \tilde{y}_2^2 \cdots \tilde{y}_{r+1}^{r+1}$ which factors into a product of linear forms. □

Lemma 12. *Let $E \subset \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$, $1 \leqslant r \leqslant \min\{\mathbf{a}, \mathbf{b}\} - 2$ and $E \neq E_{r+1}$. If there exists a polynomial P of degree k dividing all $(r+1) \times (r+1)$ minors, then:*

(1) if $k > r/2 + 1$ and for any nonzero $(r+1) \times (r+1)$ minor Δ, P and Δ/P are coprime, then P is a product of linear forms;

(2) if r is even, $k = r/2 + 1$ and for any nonzero $(r+1) \times (r+1)$ minor Δ, P and Δ/P are coprime, then either P is a product of linear forms or $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2}$;

(3) if $r \geqslant 3$ is odd, $k = (r+1)/2$ and P is irreducible, then either $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{\mathbf{b}}$, $\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{r+2}$, $\mathbb{C}^{r+3} \otimes \mathbb{C}^{r+3}$, or up to changes of bases E has a nonsingular $(r+1) \times (r+1)$ block such that all $r \times r$ minors of it are multiples of P.

Proof. (1) and (2): Proof by induction on r. The base case $r = 1$ is trivial. Assume $r > 1$ and assume that (1) and (2) holds for all integers smaller than r.

Given any nonzero $(r+2) \times (r+2)$ minor of E, by changes of bases we can assume it is Δ_{r+2}, and we further assume $\Delta_{r+1} \neq 0$.

Write $\Delta_{r+1} =: PQ$ and for $j \leqslant r+1$, $\Delta^I_{I \setminus j, r+2} =: PQ_j$, where each of the polynomials Q and Q_j's either is zero or has degree $r + 1 - k$. Then similar to Lemma 11, we have

$$(-1)^{r+1} \begin{pmatrix} -Q_1 \\ \vdots \\ (-1)^{r+1} Q_{r+1} \end{pmatrix} P = \left((-1)^{i+j} \Delta^{I \setminus i}_{I \setminus j} \right)_{j,i=1}^{r+1} \begin{pmatrix} y^1_{r+2} \\ \vdots \\ y^{r+1}_{r+2} \end{pmatrix}.$$

Using the cofactor matrix, we obtain:

$$\frac{(-1)^{r+1}}{Q}\begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 \\ \vdots & & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1}\end{pmatrix}\begin{pmatrix} -Q_1 \\ \vdots \\ (-1)^{r+1}Q_{r+1}\end{pmatrix} = \begin{pmatrix} y_{r+2}^1 \\ \vdots \\ y_{r+2}^{r+1}\end{pmatrix}.$$

By adding a rational combination (where the coefficients are $(-1)^j Q_j/Q$'s) of the first $r+1$ columns to the $(r+2)$-th column, we can put the first $r+1$ entries of the $(r+2)$-th column zero. By the same argument, put the first $r+1$ entries of the last $\mathbf{b}-r-1$ columns zero. And we can do the similar rational row operations to eliminate first $r+1$ entries of the last $\mathbf{a}-r-1$ rows. Then E becomes E' of the form (3.2).

Since the $(1,\cdots,r+1,r+2)\times(1,\cdots,r+1,r+2)$ minor is not changed by adding rational multiples of the first $r+1$ rows and columns to the $(r+2)$-th row and $(r+2)$-th column respectively, $\tilde{y}_{r+2}^{r+2} = \frac{\Delta_{r+2}}{\Delta_{r+1}}$. On the other hand, $\tilde{y}_{r+2}^{r+2}$ has the form T/Q for some polynomial T of degree $k+1$ if not zero, because all coefficients appearing in the row and column operations above are $(-1)^j Q_j/Q$'s. Thus,

$$\frac{T}{Q} = \tilde{y}_{r+2}^{r+2} = \frac{\Delta_{r+2}}{\Delta_{r+1}} = \frac{\Delta_{r+2}}{PQ} = \frac{(\Delta_{r+2}/P)}{Q} \tag{3.3}$$

and $T = \Delta_{r+2}/P$.

Since P and Q are coprime, the fact P divides all $(r+1)\times(r+1)$ minors is preserved after performing the above rational row and column operations.

If there exists an $r\times r$ minor of the upper left $(r+1)\times(r+1)$ block that is not a multiple of P, by changes of bases assume this minor is Δ_r. P divides the minor $\Delta_{1\cdots r,r+2}^{1\cdots r,r+2} = \tilde{y}_{r+2}^{r+2}\Delta_r = T\Delta_r/Q$, so T is a multiple of P. Hence P^2 divides $\Delta_{r+2} = TP$. If $k > r/2+1$, P^2 has degree $> r+2$, then we must have $\Delta_{r+2} = 0$, contradicting to the assumption $\Delta_{r+2} \neq 0$. If r is even and $k = r/2+1$, Δ_{r+2} is a multiple of P^2. By the arbitrariness of the choice of the nonzero $(r+2)\times(r+2)$ minor of E, all $(r+2)\times(r+2)$ minors equal to P^2 up to scale. By Lemma 11, P factors completely or $E \subset \mathbb{C}^{r+2}\otimes\mathbb{C}^{r+2}$.

If all $r\times r$ minors of the upper left $(r+1)\times(r+1)$ block are multiples of P. By induction, apply (1) by replacing r with $r-1$ so P factors into a product of linear forms.

(3): Similar to above let $\Delta_{r+1} =: PQ$ and Δ_{r+2} are nonzero. Since P is irreducible of degree $k = (r+1)/2$, either P and Q are coprime, or Q equals to P up to scale. In the latter case, we can choose another nonzero $(r+1) \times (r+1)$ minor from the top left $(r+2) \times (r+2)$ block such that P and Q are coprime, unless all $(r+1) \times (r+1)$ minors in the top left $(r+2) \times (r+2)$ block are multiples of P^2.

If all $(r+1) \times (r+1)$ minors in the top left $(r+2) \times (r+2)$ block are multiples of P^2, applying Lemma 11 to the top left $(r+2) \times (r+2)$ block we can put E as

$$E' = \begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 & 0 & y_{r+3}^1 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} & 0 & y_{r+3}^{r+1} & \cdots \\ 0 & \cdots & 0 & y_{r+2}^{r+2} & y_{r+3}^{r+2} & \cdots \\ y_1^{r+3} & \cdots & y_{r+1}^{r+3} & y_{r+2}^{r+3} & y_{r+3}^{r+3} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Consider the $(r+1) \times (r+1)$ minors involving y_{r+2}^{r+2} and $r \times r$ minors of the upper left $(r+1) \times (r+1)$ block: P dividing all $(r+1) \times (r+1)$ minors implies P dividing all $r \times r$ minors from the first $r+1$ rows. Apply (2) by replacing r with $r-1$ to the submatrix consisting of the first $r+1$ rows. Since P is irreducible of degree $k > 1$, this submatrix is in some $\mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}$, we can put y_j^i zero for $i \leqslant r+1$ and $j \geqslant r+3$ by changing basis of A. For the same reason all y_i^j for $i \leqslant r+1$ and $j \geqslant r+3$ can be put zero too. Then E' becomes $E'' = \mathrm{diag}(B_1, B_2)$ where B_1 is a $(r+1) \times (r+1)$ block. If B_2 has an nonzero 2×2 minor, consider the $(r+1) \times (r+1)$ minors consisting of it and any $(r-1) \times (r-1)$ minor of B_1, applying (1) replacing r with $r-2$ we see P factors into linear forms which contradicts the irreducibility. Therefore B_2 has bounded rank 1, then $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{\mathbf{b}}$ or $\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{r+2}$.

Now assume P and Q are coprime. Similar to the proof above, if there exists an $r \times r$ minor of the upper left $(r+1) \times (r+1)$ block that is not a multiple of P, P^2 divides Δ_{r+2}. By the arbitrariness of choice of nonzero $(r+2) \times (r+2)$ minors, P^2 divides all $(r+2) \times (r+2)$ minors. As $k = (r+1)/2$ and P is irreducible, P^2 divides all $(r+2) \times (r+2)$ minors and we can apply (1) by replacing r with $r+1$, then we conclude $E \subset \mathbb{C}^{r+3} \otimes \mathbb{C}^{r+3}$.

Otherwise all $r \times r$ minors of the upper left $(r+1) \times (r+1)$ block are multiples of P. $\square$

Corollary 13. *Let* $E \subset \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$*,* $1 \leqslant r \leqslant \min\{\mathbf{a}, \mathbf{b}\} - 2$*, and* $\operatorname{codim}(E_r) = 1$ *and* $E \neq E_{r+1}$*. Then:*

1. E_r *does not contain any irreducible hypersurface of degree* $k > r/2 + 1$*;*
2. *if* r *is even and* E_r *contains an irreducible hypersurface of degree* $r/2+1$*, then* $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2}$.

Proof. 1. If E_r contains an irreducible hypersurface of degree k, there exists an irreducible polynomial P of degree k dividing all $(r+1) \times (r+1)$ minors. Then for any $(r+1) \times (r+1)$ minor Δ, Δ/P has degree less than k so must be coprime with P. By (1) of Lemma 12, P factors, contradicting to the irreducibility.

2. Similar to the proof of 1 except we apply (2) of Lemma 12. As P cannot be a product of linear forms due to irreducibility, we conclude $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2}$. $\square$

3.2.2 Case $\operatorname{codim}(E_1) = n$

Let $E^{\perp} := \{f \in A^* \otimes B^* \mid f(E) = 0\}$. Define the **index of degeneracy** of E to be one plus the maximum dimension of a linear space contained in $\mathbb{P}E^{\perp} \cap \operatorname{Seg}(\mathbb{P}A^* \times \mathbb{P}B^*)$, denoted as κ. Equivalently, κ is the largest number of entries in the same row or column of E that can be simultaneously put to zero by changing bases of A and B.

The subspace E is called **E1-generic** if $\kappa = 0$. We call this property E1-generic because it corresponds to the notion of 1-generic for spaces of matrices given by Eisenbud [19], which differs with the notion of 1-generic that is often used for tensors (cf. [20]). We list two results of E1-generic spaces of our interest below.

Theorem 14 (Corollary 3.3 and Theorem 2.1 of [19])**.** *Let* $m = \min\{\mathbf{a}, \mathbf{b}\}$*. If* $E \subset A \otimes B$ *is E1-generic, then:*

1. *for* $k \leqslant m-1$, $\operatorname{codim}(E_k) \geqslant \mathbf{a} + \mathbf{b} - 2k - 1$*;*
2. *if* $F \subset E$ *is a subspace with* $\operatorname{codim}(F) \leqslant m-1$*, then* $\operatorname{codim}_F(F_{m-1}) = (\mathbf{a} - m + 1)(\mathbf{b} - m + 1)$.

For generic determinantal varieties, i.e. when $E = A \otimes B$, one expects E_k has codimension $(\mathbf{a} - k)(\mathbf{b} - k)$. E1-generic does not means generic but implies the genericity to some extent – E_{m-1} has the expected codimension, and the codimension of E_k has a lower bound $\mathbf{a} + \mathbf{b} - 2k - 1$.

Proposition 15. *Let $n := \operatorname{codim}(E_1)$, then there exist $0 \leqslant j \leqslant n$ and a linear subspace $F \subset E$ of codimension j, such that either $F \subset \mathbb{C}^k \otimes \mathbb{C}^l$ for some $k + l \leqslant n + 3 - j$ and $k, l \geqslant 2$, or $j = n$ and F has bounded rank 1.*

Proof. First assume all nonzero 2×2 minor of E are irreducible. So if there is an entry $y^i_j = 0$, then either all entries in the i-th row or all entries in the j-th column are zero. By changes of bases in A and B, there exist integers $k, l \geqslant 2$, such that $y^i_j \neq 0$ if and only if $i \leqslant k, j \leqslant l$.

$$E = \begin{pmatrix} y^1_1 & \cdots & y^1_l & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ y^k_1 & \cdots & y^k_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then the upper left $k \times l$ block of E is E1-generic. By Theorem 14 if $k, l \geqslant 2$, $\operatorname{codim}(E_1) \geqslant k + l - 3$, so $k + l \leqslant n + 3$.

If there is a 2×2 minor of M that factors into the product of two linear forms ℓ_1, ℓ_2, write $F := \{\ell_1 = 0\}$ and $F' := \{\ell_2 = 0\}$, then $E_1 = F_1 \cup F'_1$. At least one of the two components has codimension n in E. Say it is F_1, then $\operatorname{codim}_F(F_1) \leqslant n - 1$.

Together with the irreducible case, we conclude that at least one of the following holds:

1. there exists a hyperplane $F \subset E$ such that $\operatorname{codim}_F(F_1) = n - 1$;
2. $E \subset \mathbb{C}^k \otimes \mathbb{C}^l$ such that $k + l \leqslant n + 3$ and $k, l \geqslant 2$.

Using induction on $\dim(E)$, we conclude. □

3.2.3 Case $\operatorname{codim}(E_2) = 1$

If $\operatorname{codim}(E_2) = 1$, then there must exist an irreducible polynomial P of degree $k \leqslant 3$ dividing all 3×3 minors of E. If $k = 1$, then E contains a hyperplane $\{P = 0\}$ which has bounded rank 2.

If $k = 3$, by Lemma 11 $E \subset \mathbb{C}^3 \otimes \mathbb{C}^3$.

When $k = 2$, by Corollary 13 we have $E \subset \mathbb{C}^4 \otimes \mathbb{C}^4$, which suffices us to assume $E \subset A \otimes B$ with $\dim(A) = \dim(b) = 4$. The following proposition finds all such spaces up to changes of bases in E, A and B.

Proposition 16. *Let $E \subset A \otimes B := \mathbb{C}^4 \otimes \mathbb{C}^4$. If there exists an irreducible polynomial S of degree 2 dividing all 3×3 minors of E, then at least one of the following holds:*

1. *E has bounded rank 3;*
2. *up to changes of bases in E, A and B, E is either skew-symmetric, or has the form a diagonal block matrix $\operatorname{diag}(X, X)$ where*
$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \textit{ or } \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$
depending on the rank of S.

We defer the proof to §3.3.

3.2.4 Case $\operatorname{codim}(E_r) \leqslant n$

A subspace $E \subset A \otimes B$ is said to be **concise** if the associated tensor $T \in E^* \otimes A \otimes B$ is concise. Equivalently, there does not exist changes of bases in A or B such that any column or row of E consists of only zero entries. This section studies upper bounds of $\mathbf{a}$ and $\mathbf{b}$ for concise spaces E satisfying $\operatorname{codim}(E_r) \leqslant n$.

Proposition 17. *For any positive integer r, n, there exist positive integers M_1, M_2, such that if there exists a concise space $E \subset A \otimes B := \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$ with $\operatorname{codim}(E_r) \leqslant n$, then at least one of the following holds:*

(1) $\mathbf{a}$ *or* $\mathbf{b} \leqslant M_1$*;*

(2) $\mathbf{a}, \mathbf{b} \leqslant M_2$*;*

(3) $\exists$ *a hyperplane $F \subset E$ such that* $\operatorname{codim}_F(F_r) \leqslant n - 1$*;*

(4) $\exists 1 \leqslant i \leqslant r$ *such that $E = H + H'$ where* $\operatorname{codim}(H'_{r-i}) \leqslant n$ *and $H' \subset \mathbb{C}^i \otimes B$ or $A \otimes \mathbb{C}^i$.*

Proof. Proof by induction on r. For $r = 1$, by Proposition 15 we can set $M_1 = 1$ and $M_2 = n+1$.For $r \geqslant 2$ we divide the problem into different cases by the value of κ.

1. Case $\kappa = 0$.

$\kappa = 0$ if and only if E is E1-generic. Since $E_r \neq E$, $\mathbf{a}, \mathbf{b} \geqslant r+1$. By Theorem 14, $\mathbf{a} + \mathbf{b} \leqslant n + 2r + 1$.

2. Case $\kappa = 1$.

We can put $y_1^1 = 0$ by changing bases. Then the $(\mathbf{a}-1) \times (\mathbf{b}-1)$ submatrix consisting of entries in the last $\mathbf{a}-1$ rows and the last $\mathbf{b}-1$ columns is either 1-generic, or has $\kappa = 1$ so we can put $y_2^2 = 0$. Repeat this procedure until the bottom right $(\mathbf{a}-k) \times (\mathbf{b}-k)$ submatrix is 1-generic. Then $E = \begin{pmatrix} C_{k\times k} & * \\ * & D_{s\times t} \end{pmatrix}$ where D is 1-generic, $s = \mathbf{a} - k, t = \mathbf{b} - k$ and

$$C = \begin{pmatrix} 0 & * & * & \cdots & * & * \\ * & 0 & * & \cdots & * & * \\ * & * & 0 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ * & * & * & \cdots & 0 & * \\ * & * & * & \cdots & * & 0 \end{pmatrix}.$$

If $k \geqslant r+1$, consider the submatrix C' consisting entries in the first $r+1$ rows and the last $k-1$ columns of C:

$$C'_{(r+1)\times(k-1)} = \begin{pmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ * & 0 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \\ * & * & \cdots & * & 0 \end{pmatrix}$$

By the definition of κ, all nonzero entries in the same row or column of C are linearly independent. Therefore C' is a codimension $r-1$ subspace of some 1-generic space in $\mathbb{C}^{k-1} \otimes \mathbb{C}^{r+1}$. By 2 of Theorem 14, all $(r+1) \times (r+1)$ minors of C' determines a subvariety of codimension $\geqslant k - r - 1$, so $k \leqslant n + r + 1$.

Now to find upper bounds for s and t. If $s = r$ and $t \geqslant r$, the submatrix consisting of entries in the last $s + 1$ rows and the last $t + 1$ columns is a codimension 1 subspace of a 1-generic space in $\mathbb{C}^{s+1} \otimes \mathbb{C}^{t+1}$. So by 2 of Theorem 14 again $t \leqslant n + r - 1$. Similarly if $t = r$ then $s \leqslant n + r - 1$.

If $s, t \geqslant r + 1$, the submatrix consisting of entries in the last s rows and the last $t + 1$ columns is 1-generic. By 1 of Theorem 14, $n \geqslant 2$ and $s + t \leqslant n + 2r$.

To put everything together, either $\mathbf{a} \leqslant n + 2r$, $\mathbf{b} \leqslant n + 2r$ or $\mathbf{a}, \mathbf{b} \leqslant 2n + 2r$.

3. Case $2 \leqslant \kappa \leqslant \max\{M_1(r-1, n), M_2(r-1, n)\}$.

Claim: there exist $M_i(r, n, g), i = 1, 2$ such that if E has $\kappa = g$ and satisfies the hypothesis of the proposition, then either $\mathbf{a}$ or $\mathbf{b} \leqslant M_1(r, n, g)$, or $\mathbf{a}, \mathbf{b} \leqslant M_2(r, n, g)$, or the condition (3) holds.

We will find $M_i = M_i(r, n, g)$ by induction on g. By the last case, we can set $M_1(r, n, 1) = n + 2r$ and $M_2(r, n, 1) = 2n + 2r$. Assume claim is true for spaces of $\kappa < g$.

Write $E = \begin{pmatrix} C_{k\times k} & * \\ * & D_{s\times t} \end{pmatrix}$ such that C has zeros on the diagonal and D is 1-generic. Let $M(r, n, g-1) := \max\{M_1(r, n, g-1), M_2(r, n, g-1)\}$. If $k > 2M(r, n, g-1) + 1$, then the submatrix C' consisting of entries in the first $M(r, n, g-1)+1$ rows and the last $M(r, n, g-1)+1$ columns of C is a space of $\kappa = g-1$. However by the definitions of $M_i(r, n, g-1)$'s, all $(r+1) \times (r+1)$ minors of C' determine of codimension $> n$ subset. Therefore $k \leqslant 2M(r, n, g-1) + 1$.

Now s and t has the same upper bound as the last case. So we can set $M_1(r, n, g) := 2M(r, n, g-1) + r$ and $M_1(r, n, g) := 2M(r, n, g-1) + r + n$ which proves the claim.

4. Case $\kappa \geqslant \max\{M_1(r-1, n), M_2(r-1, n)\} + 1$.

Choose bases and possibly take transpose so that all entries in the top left $\kappa \times \kappa'$ block of E are zero for some $1 \leqslant \kappa' \leqslant \kappa$. So

$$E = \begin{pmatrix} O_{\kappa\times\kappa'} & H \\ G & D_{s\times t} \end{pmatrix}. \tag{3.4}$$

We take the largest κ' so that the submatrix H is concise in $\mathbb{C}^{\kappa} \otimes \mathbb{C}^{t}$. By the definition of κ, G is 1-generic.

Consider the $(r+1) \times (r+1)$ minors consisting of any single entry of G and any $r \times r$ minor of H. We must have $\operatorname{codim}_H(H_{r-1}) \leqslant n$ unless condition (3) holds. Since $\kappa \geqslant \max\{M_1(r-1,n), M_2(r-1,n)\}+1$, $t \leqslant M_1(r-1,n)$.

If $\kappa' \leqslant r$, then $\mathbf{b} \leqslant M_1(r-1,n)+r$.

If $\kappa' \geqslant r+1 > s$, consider the $(r+1) \times (r+1)$ minors that are a product of an $s \times S$ minor of G and a $(r+1-s) \times (r+1-s)$ minor of H. Then either $\operatorname{codim}_G(G_{s-1}) \leqslant n$ or $\operatorname{codim}_H(H_{r-s}) \leqslant n$. The latter inequality implies condition (4) holds. The former inequality implies $\kappa' \leqslant n+s-1 \leqslant n+r-1$, then $\mathbf{b} \leqslant n+r-1+M_1(r-1,n)$.

If $\kappa', s \geqslant r+1$, then $\operatorname{codim}_G(G_r) \leqslant n$. By Theorem 14 $s+\kappa' \leqslant n+2r+1$. So $\mathbf{b} \leqslant n+r+M_1(r-1,n)$.

Since $M_i(r,n,g+1) \geqslant M_i(r,n,g)$ for $g \geqslant 0$, $M_i(r,n,g)$ takes the maximum at $g = g' := \max\{M_1(r-1,n), M_2(r-1,n)\}$. So we can put $M_1(n,r) = \max\{M_1(r,n,g'), n+r+M_1(r-1,n)\}$ and $M_2 = M_1(r,n,g')$ which proves the proposition. □

Corollary 18. *Let $E \subset A \otimes B$ be concise and satisfy $\operatorname{codim} E_2 = 2$. Then at least one of the following holds:*

(1) $\mathbf{a}$ *or* $\mathbf{b} \leqslant 6$*;*

(2) $\mathbf{a}, \mathbf{b} \leqslant 8$*;*

(3) $\exists$ *a hyperplane $F \subset E$ such that* $\operatorname{codim}_F F_2 \leqslant 1$*;*

(4) E has bounded rank 2.

Proof. For $\kappa = 0$ or 1, by the proof of Proposition 17 either $\mathbf{a}$, $\mathbf{b} \leqslant 6$, or $\mathbf{a}, \mathbf{b} \leqslant 8$.

For $\kappa = 2$ or 3, put E into the form 3.4. If the condition (3) does not hold, G and H are both 1-generic. Then by Theorem 14, $\mathbf{a}$ or $\mathbf{b} \leqslant 6$.

For $\kappa \geqslant 4$, H must have bounded rank 1 and $t = 1$. If $\kappa \leqslant 3$, then $\mathbf{b} \leqslant 5$. If $\kappa \geqslant 4$, G must have bounded rank 1 and $s = 1$, then E has bounded rank 2. □

3.3 Proof of Proposition 16

Before proving the proposition, we need the following lemma.

Lemma 19. *Let $E \subset A \otimes B := \mathbb{C}^2 \otimes \mathbb{C}^2$ be a matrix of linear forms in variables $x_1, \cdots, x_{\mathbf{c}}$. Define*

$$X_1 := \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, X_2 := \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

1. *If $\det E = \det X_1$, then $E = X_1$ up to changes of bases in A and B.*
2. *If $\det E = \det X_2$, then either $E = X_2$ or $E = X_2^t$ up to changes of bases in A and B.*

Proof of Proposition 16. Say $\Delta_3 = x_1 S$, then the upper left 3×3 submatrix must be of the form $x_1 Z + U$ where Z is a 3×3 matrix of complex numbers and U has bounded rank rank 2. Therefore up to changes of bases, U is either compression or skew-symmetric. Since $\Delta_3 \neq 0$, taking transpose if necessary, we can write the upper left 3×3 submatrix as one of the following forms:

$$\textbf{(i)} \begin{pmatrix} y_1^1 & y_2^1 & y_3^1 \\ y_1^2 & y_2^2 & y_3^2 \\ 0 & 0 & x_1 \end{pmatrix}, \textbf{(ii)} \begin{pmatrix} x_1 & 0 & y_3^1 \\ 0 & x_1 & y_3^2 \\ y_1^3 & y_2^3 & y_3^3 \end{pmatrix}, \textbf{(iii)} \begin{pmatrix} x_1 & y_2^1 & y_3^1 \\ -y_2^1 & x_1 & y_3^2 \\ -y_3^1 & -y_2^2 & x_1 \end{pmatrix}$$

For the rest of the proof, we will discuss each of the above cases.

Case (i). $S = \Delta_2 = y_1^1 y_2^2 - y_2^1 y_1^2$ is an irreducible quadratic polynomial, hence has Waring rank 3 or 4. Changing basis in E we can write $S = x_1 x_3 - (x_2)^2$ or $x_1 x_4 - x_2 x_3$ depending on rank(S). By Lemma 19 we can put the top left 2×2 block as the form X_1 or X_2.

S divides $\Delta_{123}^{134} = x_1 \Delta_{12}^{14}$ and $\Delta_{123}^{234} = x_1 \Delta_{12}^{24}$, therefore $(y_1^4, y_2^4) \in \mathrm{span}\{(y_1^1, y_2^1), (y_1^2, y_2^2)\}$. Adding multiples of the first and the second row to the 4-th, we can set y_1^4 and y_2^4 to zeros.

If $\Delta_{34}^{34} = 0$, the right bottom 2×2 submatrix has bounded rank 1, then E has bounded rank 3.

Assume $\Delta_{34}^{34} \neq 0$. Since S divides $\Delta_{234}^{234} = y_2^2 \Delta_{34}^{34} \neq 0$, Δ_{34}^{34} is a non-zero multiple of S, hence can be normalized to S. Apply Lemma 19 again, E has one of the following forms:

$$\begin{pmatrix} x_1 & x_2 & y_3^1 & y_4^1 \\ x_2 & x_3 & y_3^2 & y_4^2 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_2 & x_3 \end{pmatrix} \text{ if rank}(S) = 3, \text{ or } \begin{pmatrix} x_1 & x_2 & y_3^1 & y_4^1 \\ x_3 & x_4 & y_3^2 & y_4^2 \\ 0 & 0 & x_1 & y_4^3 \\ 0 & 0 & y_3^4 & x_4 \end{pmatrix} \text{ if rank}(S) = 4$$

where $(y_4^3, y_3^4) = (x_2, x_3)$ or (x_3, x_2).

We consider separately the two subcases **(i.1)** rank(S) = 3 and **(i.2)** rank(S) = 4. And we further divide subcase **(i.2)** into two situations: **(i.2.1)** $(y_4^3, y_3^4) = (x_2, x_3)$, and **(i.2.2)** $(y_4^3, y_3^4) = (x_3, x_2)$.

Subcase (i.1). Assume $\text{rank}(S) = 3$. Write $y_4^1 = l_2 + l_2'$, $y_3^2 = l_3 + l_3'$, $y_4^2 = l_4 + l_4'$, and $\Delta_{234}^{123} = (l + l')S$ where $l, l_i \in \text{span}\{x_1, x_2, x_3\}$ and $l', l_i' \in \text{span}\{x_4, \cdots, x_m\}$. Then

$$l'S = l'(x_1x_3 - (x_2)^2) = (x_3l_2' - x_2l_4')x_1 + (x_2l_3' - x_3l_1')x_2.$$

Comparing terms that are multiples of $(x_1)^2$, we see $l' = 0$, which forces all $l_i' = 0$, so $y_j^i \in \text{span}\{x_1, x_2, x_3\}$. Adding multiples of the first two rows and columns to the last two rows and columns, we can put $y_4^1 = y_3^2 = 0$.

Write $l = a_1x_1 + a_2x_2 + a_3x_3$. Then

$$(a_1x_1 + a_2x_2 + a_3x_3)(x_1x_3 - (x_2)^2) = lS = \Delta_{234}^{123} = -x_2x_3y_3^1 - x_1x_2y_4^2.$$

Comparing the terms of multiples of $(x_1)^2x_3, x_1(x_3)^2$ and $(x_2)^3$, we see all $a_i = 0$ so $l = 0$. Comparing the coefficients of the rest cubic monomials, we obtain $a_1 + b_3 = 0$ and $a_2 = a_3 = b_1 = b_2 = 0$. If $a_1 = 0$, E has the form $\text{diag}(X_1, X_1)$ where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$. If $a_1 \neq 0$, multiply a_1 to the first two column and the last two rows, subtract the 1st row from the 3rd, and add the 4th row to the 2nd, then E again has the form $\text{diag}(X_1, X_1)$.

Subcase (i.2). Assume $\text{rank}(S) = 4$.

(i.2.1). If $(y_4^3, y_3^4) = (x_2, x_3)$: using the notations in (i.1), write $y_j^i = l_k + l_k'$ and $\Delta_{234}^{123} = (l + l')S$. Then $l'S = x_2(l_3'x_2 - l_4'x_1) - x_4(l_1'x_2 - l_2'x_1)$. Comparing terms we see all $l_i' = 0$. Then by adding multiples of the first two columns to the third and fourth, then adding multiples of the first two rows to the first and second, we can make $y_3^1 \in \text{span}\{x_2, x_4\}, y_3^2 \in \text{span}\{x_2\}, y_3^1 \in \text{span}\{x_1, x_2, x_4\}$.

Since $\Delta_{234}^{123} = lS = x_2(y_3^2x_2 - y_4^2x_1) - x_4(y_3^1x_2 - y_4^1x_1)$, writing y_j^i into linear combinations of x_k's, we see that there is no x_2x_3. Thus $l = 0$, then comparing terms of the above equation, we have $y_3^1 = y_3^2 = 0$. By Δ_{134}^{123} and Δ_{234}^{123}, $(y_4^1, y_4^2)^t \in \operatorname{span}\{(x_1,x_3)^t,(x_2,x_4)^t\}$, so we can set $(y_4^1, y_4^2)^t = 0$ by adding multiples of the first two columns to the fourth.

Therefore, by changing bases E has the form $\operatorname{diag}(X_2, X_2)$ where $X_2 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$.

(i.2.2). If $(y_4^3, y_3^4) = (x_3, x_2)$: using the notations in (i.1), write $y_j^i = l_k + l'_k$ and $\Delta_{234}^{123} = (l+l')S$. Then $l'S = x_2(l'_3x_3 - l'_4x_1) - x_4(l'_1x_3 - l'_2x_1)$. Comparing terms we see $l' = -l'_3 = l'_2$ and $l'_1 = l'_4 = 0$. Then by adding multiples of the first two columns to the third and fourth, then adding multiples of the first two rows to the first and second, we can write $y_4^1 = l', y_3^2 = -l', y_3^1 = \sum_{i=1}^4 a_ix_i, y_4^2 = \sum_{i=1}^4 d_ix_i$.

S divides other 3×3 minors, which implies $y_3^1 = y_4^2 = 0$. Therefore the only nonzero entries of E are in the upper left 4×4 block of the form:

$$\begin{pmatrix} x_1 & x_3 & 0 & l' \\ x_2 & x_4 & -l' & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_3 & x_4 \end{pmatrix}.$$

Swapping the first two rows and the last two rows, then multiply -1 to the first 2 rows. E becomes skew-symmetric.

Case (ii). Here $S = x_1y_3^3 - y_3^2y_2^3 - y_3^1y_1^3$. Modify $y_j^1, y_j^2, y_1^i, y_2^i$, $3 \leqslant i,j \leqslant 4$ such that their expressions (as linear forms in x_i's) do not contain x_1.

If $y_3^3 = 0$ and $\exists i,j > 2$, such that $y_j^i \neq 0$, we may change bases such that $y_3^3 \neq 0$, so we have two cases $y_3^3 \neq 0$ or $y_j^i = 0$ for all $i,j > 2$.

If $y_3^3 \neq 0$, then consider $\Delta_{12j}^{12i} = x_1(x_1y_j^i - y_1^iy_j^1 - y_2^iy_j^2)$. We obtain $y_j^i = c_j^iy_3^3$ for all $i,j > 2$ and constants c_j^i. Changing bases again, we may set $y_4^3 = y_3^4 = 0$ and $y_4^4 = y_3^3$ or 0. There are 3 subcases: **(ii.1)** $y_4^4 = y_3^3 \neq 0$, **(ii.2)** $y_4^4 = 0, y_3^3 \neq 0$, and **(ii.3)** $y_3^3 = y_4^4 = 0$.

Subcase (ii.1). Assume $y_4^4 = y_3^3 \neq 0$. Then $\Delta_{\rho 34}^{\rho 34} = y_3^3(x_1y_3^3 - y_\rho^3y_3^\rho - y_\rho^4y_4^\rho), \forall \rho = 1,2.$

Together with Δ^{12i}_{12i}'s we get

$$y^3_1 y^1_3 + y^4_1 y^1_4 = y^3_2 y^2_3 + y^4_2 y^2_4 = y^3_1 y^1_3 + y^3_2 y^2_3 = y^4_1 y^1_4 + y^4_2 y^2_4$$

Hence $y^3_1 y^1_3 = y^4_2 y^2_4$, $y^4_1 y^1_4 = y^3_2 y^2_3$.

$\Delta^{\rho 34}_{\sigma 34} = y^3_3(-y^3_\sigma y^\rho_3 - y^4_\sigma y^\rho_4) = 0$ for $(\rho, \sigma) = (0, 1)$ or $(1, 0)$, and $\Delta^{12i}_{12j} = 0$ for $i \neq j$. We get

$$y^3_1 y^1_4 + y^3_2 y^2_4 = y^4_1 y^1_3 + y^4_2 y^2_3 = y^3_1 y^2_3 + y^4_1 y^2_4 = y^3_2 y^1_3 + y^4_2 y^1_4 = 0$$

In other words, denoting $Q := y^3_1 y^1_3 + y^4_1 y^1_4$, the following equations hold:

$$\begin{pmatrix} y^3_1 & y^3_2 \\ y^4_1 & y^4_2 \end{pmatrix} \begin{pmatrix} y^1_3 & y^1_4 \\ y^2_3 & y^2_4 \end{pmatrix} = \begin{pmatrix} y^1_3 & y^1_4 \\ y^2_3 & y^2_4 \end{pmatrix} \begin{pmatrix} y^3_1 & y^3_2 \\ y^4_1 & y^4_2 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

Then by changing bases E equals to the matrix whose upper left 4×4 block is one of the following, and all other entries are zeros:

$$\begin{pmatrix} x_1 & 0 & a & d \\ 0 & x_1 & c & b \\ b & -d & y^3_3 & 0 \\ -c & a & 0 & y^3_3 \end{pmatrix}$$

for some linear forms or zeros a, b, c, d, y^3_3.

Subcase (ii.2). Assume $y^4_4 = 0, y^3_3 \neq 0$. Since $\Delta_4 \neq 0$, there exist $\rho, \sigma = 1, 2$, such that y^4_σ and $y^\rho_4 \neq 0$. Then $\Delta^{124}_{\sigma 34}$ and $\Delta^{\rho 34}_{124}$ implies $\Delta^{12}_{34} = \Delta^{34}_{12} = 0$. Then change bases in the first two rows and columns, we get:

$$\begin{pmatrix} c_1 x_1 & c_2 x_1 & y^1_3 & y^1_4 \\ c_3 x_1 & c_4 x_1 & 0 & 0 \\ y^3_1 & 0 & y^3_3 & 0 \\ y^4_1 & 0 & 0 & 0 \end{pmatrix}$$

for some constants c_i. $\Delta^{123}_{234} = -c_4 x_1 y^1_4 y^3_3$ implies $c_4 = 0$. Then $\Delta^{123}_{123} = -c_2 c_3 (x_1)^2 y^3_3$ implies either c_2 or $c_3 = 0$, contradicting the hypothesis $\Delta_3 \neq 0$.

Subcase (ii.3). Assume $y^3_3 = y^4_4 = 0$. As $S = y^3_1 y^1_3 + y^3_2 y^2_3$ is irreducible, y^3_1, y^3_2 are linearly independent, and so are y^1_3, y^2_3. Choose bases such that y^1_1 and y^2_2 are not necessary x_1, and $y^3_1 = x_1, y^3_2 = x_2$. Since $\mathrm{rank}(S) \geqslant 3$, at least one of y^1_3, y^2_3 is linearly independent with x_1, x_2. Without loss of generality assume x_1, x_2, y_3^1 are linearly independent, then choose bases such that $y_3^1 = x_3$:

$$\begin{pmatrix} y_1^1 & 0 & x_3 & y_4^1 \\ 0 & y_1^1 & y_3^2 & y_4^2 \\ x_1 & x_2 & 0 & 0 \\ y_1^4 & y_2^4 & 0 & 0 \end{pmatrix}.$$

If $\Delta_{34}^{12} = 0$, $\begin{pmatrix} x_3 & y_4^1 \\ y_3^2 & y_4^2 \end{pmatrix}$ has bounded rank 1 so we can set either the fourth column to zero (then $E \subset \mathbb{C}^4 \otimes \mathbb{C}^3$), or $y_3^2 = y_4^2 = 0$ (then Δ_{123}^{123} is a product of linear forms, contradicting to irreducibility of S).

If $\Delta_{34}^{12} \neq 0$, by linear independence of $y_3^1 = x_3$ and y_3^2, and Δ_{34}^{12} is a nonzero multiple of $S = x_1 y_3^1 + x_2 y_3^2$, we can normalize the fourth column such that $(y_4^1, y_4^2)^t = (x_2, -x_1)^t$ and $y_j^\rho = 0, \forall j > 4, \rho = 1, 2$. By the same argument, we can set $(y_1^4, y_2^4) = (y_3^2, -x_3)^t$ and $y_\sigma^i = 0, \forall i > 4, \sigma = 1, 2$.

Then E has the form:

$$\begin{pmatrix} y_1^1 & 0 & x_3 & x_2 \\ 0 & y_1^1 & y_3^2 & -x_1 \\ x_1 & x_2 & 0 & 0 \\ y_3^2 & -x_3 & 0 & 0 \end{pmatrix},$$

which is skew-symmetric after permuting rows and columns.

Case (iii). Here $S = x_1^2 + (y_2^1)^2 + (y_3^1)^2 + (y_3^2)^2$ is irreducible, so $\mathrm{rank}(S) > 2$. We consider two subcases by whether x_1, y_2^1, y_3^1, y_3^2 are linearly independent.

Subcase (iii.1). Assume x_1, y_2^1, y_3^1, y_3^2 are linearly independent. We can choose basis of E such that $y_2^1 = x_2, y_3^1 = x_3, y_3^2 = x_4$. In order that S divides all 3×3 minors, $(y_4^1, y_4^2, y_4^3)^t$ must be a linear combination of $(x_1, -x_2, -x_3)^t$, $(x_2, x_1, -x_4)^t$, $(x_3, x_4, x_1)^t$, and $(x_4, -x_3, x_2)^t$. Changing the basis we can put $(y_4^1, y_4^2, y_4^3)^t = (x_4, -x_3, x_2)^t$. By the same argument, $(y_1^4, y_2^4, y_3^4) = (-x_4, +x_3, -x_2)$. Consider the 3×3 minors involving y_4^4, we see $y_4^4 = -x_1$.

Hence E has the form:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & x_4 & -x_3 \\ -x_3 & -x_4 & x_1 & x_2 \\ -x_4 & x_3 & -x_2 & x_1 \end{pmatrix}.$$

Then E is the the complex quaternion algebra $\operatorname{span}_{\mathbb{C}}\{\mathbb{1}, I, J, K\}/(I^2+\mathbb{1}, J^2+\mathbb{1}, K^2+\mathbb{1}, IJK+\mathbb{1})$ and the associated tensor of E is the structure tensor of the complex quaternion. Since the complex quaternion algebra is isomorphic to the matrix algebra $\mathrm{Mat}_{2\times 2}$, their structure tensors equal up to changes of bases. So E equals to $M_{\langle 2\rangle}$ up to changes of bases in E, A and B.

Subcase (iii.2). Assume x_1, y_2^1, y_3^1, y_3^2 are linearly dependent. The irreducibility of S implies three of them are linearly independent. $x_1 \neq 0$ since $\Delta_3 \neq 0$. By changing bases assume $y_2^1 = x_2, y_3^1 = x_3, y_3^2 = a_1x_1 + a_2x_2 + a_3x_3$ for $a_i \in \mathbb{C}$. Then $S = x_1^2 + x_2^2 + x_3^2 + (a_1x_1 + a_2x_2 + a_3x_3)^2$. If $\dim\langle x_i, y_4^i \mid i = 1, 2, 3\rangle \geqslant 5$, the submatrix consisting of the first 3 rows is a subspace of a 1-generic space of codimension $\leqslant 2$, then Theorem 14 implies contradiction. So $\dim\langle x_i, y_4^i \mid i = 1, 2, 3\rangle \leqslant 4$.

Adding first 3 columns to the 4th, we can set $y_4^1 = 0$ or x_4. Write $y_4^2 = \sum_i b_ix_i$ and $y_4^3 = \sum_i c_ix_i$, $\Delta_{124}^{123} = LS$,$\Delta_{134}^{123} = MS$ and $\Delta_{224}^{123} = NS$ for some linear forms $L = \sum_i l_ix_i, M, N$.

If $y_4^1 = x_4$:

$$\begin{aligned}\Delta_{124}^{123} =&((a_1b_4 + c_4)x_1^2 + (a_2 + c_4)x_2^2 + (1 + a_3b_4)x_1x_3 + (a_2b_4 + a_1)x_1x_2 + (a_3 - b_4)x_2x_3)x_4 \\ &+ (c_1 + a_1b_1)x_1^3 + c_2x_2^3 + (c_3 - b_2)x_2^2x_3 - b_3x_2x_3^2 + (c_1 + a_2b_2)x_1x_2^2 \\ &+ (c_2 + a_1b_2 + a_2b_1)x_1^2x_2 + (c_3 + a_1b_3 + a_3b_1)x_1^2x_3 + a_3b_3x_1x_3^2 + (a_2b_3 + a_3b_2 - b_1)x_1x_2x_3.\end{aligned}$$

Note that there is no $x_3^2x_4$ in Δ_{124}^{123}. This implies either $a_3^2 + 1 = 0$ or $l_4 = 0$. If $l_4 = 0$, those terms divisible by x_4 have the sum zero:

$$(a_1b_4 + c_4)x_1^2 + (a_2 + c_4)x2^2 + (1 + a_3b_4)x1x3 + (a_2b_4 + a_1)x_1x_2 + (a_3 - b_4)x_2x_3 = 0$$

which implies $a_3^2 + 1 = 0$. Hence $a_3^2 + 1 = 0$ no matter if $l_4 = 0$.

There is no $x_2^2x_4$ in Δ_{134}^{123}, thus by the same argument, we must have $a_2^2 + 1 = 0$.

Compare the coefficients of x_2^3 in equality $\Delta_{124}^{123} = LS$ and x_3^3 in $\Delta_{134}^{123} = MS$, we get

$$c_2 = l_2(1 + a_2^2) = 0 \text{ and } -b_3 = m_3(1 + a_3^2) = 0.$$

Compare the coefficients of $x_2^2 x_4$ in $\Delta_{124}^{123} = LS$ and $x_3^2 x_4$ in $\Delta_{134}^{123} = MS$, we get

$$(a_2 + c_4) = l_4(1 + a_2^2) = 0 \text{ and } (a_3 - b_4) = m_4(1 + a_3^2) = 0.$$

Compare the coefficients of $x_2 x_3 x_4$ in $\Delta_{124}^{123} = LS$ and $\Delta_{134}^{123} = MS$, we get

$$2a_2 a_3 l_4 = (a_3 - b_4) = 0 \text{ and } 2a_2 a_3 m_4 = a_2 + c_4 = 0.$$

Therefore $l_4 = m_4 = 0$. Then the coefficients of every monomial divisible by x_4 in Δ_{124}^{123} and Δ_{134}^{123} equals zero. We get $a_1 = a_2/a_3$ from Δ_{124}^{123} but $a_1 = a_2 a_3$ contradicting $a_3^2 = -1$.

If $y_4^1 = 0$: since there is no $x_3^2 x_4$ in Δ_{124}^{123}, either $a_3^2 + 1 = 0$ or $l_4 = 0$.

If $l_4 = 0$, then the coefficients of every monomial divisible by x_4 in Δ_{124}^{123} equal zero, which implies $b_4 = c_4 = 0$. Therefore there is no x_4 appearing in the first 3 rows, and by the same argument x_4 does not appear in the first 3 columns. If $\exists i, j > 3$, such that $y_j^i \notin \operatorname{span}\{x_1, x_2, x_3\}$, then we can change basis in E to set $y_j^i = x_4$. Write $\Delta_{12j}^{12i} = x_4(x_1^2 + x_2^2) + p(x_1, x_2, x_3)$ for some polynomial p. $S = S(x_1, x_2, x_3)$ dividing $\Delta_{12j}^{12i} \neq 0$ implies that S divides $x_1^2 + x_2^2$, contradicting to the irreducibility of quadratic polynomial S. If there is no such y_j^i, then $\dim(E) = 3$.

Therefore $a_3^2 + 1 = 0$. And by the same argument, since there is no $x_2^2 x_4$ in Δ_{134}^{123}, $a_2^2 + 1 = 0$. Compare the coefficients of $x_2^2 x_4$ in $\Delta_{124}^{123} = LS$ and $x_3^2 x_4$ in $\Delta_{134}^{123} = MS$, we get $c_4 = b_4 = 0$.

Then by the same argument as in the case $l_4 = 0$ we obtain $\dim(E) = 3$. $\square$

4. RESULTS

By Proposition 2, a tensor $T \in \mathcal{GR}_r$ if and only if there exists j such that

$$\mathrm{codim} A_j^* \leqslant r - j. \tag{4.1}$$

When $j = 0$, A_0^* is the kernel of $T_A : A^* \to B \otimes C$, so its codimension equals to $\mathrm{ml}_A(T)$. In other words, (4.1) holds for $j = 0$ if and only if $ml_A(T) \leqslant r$, so T is compression.

When $j = r$, (4.1) is equivalent to $A = A_r^*$, i.e., $T(A^*) \subset B \otimes C$ is a space of matrices of bounded rank r. Since spaces of bounded rank 1, 2 and 3 are classified, we can utilize those results on the classification of tensors with geometric rank 1, 2 and 3.

To classify tensors in $\mathcal{GR}_r$, we have to study every case of $1 < j < r$.

4.1 Geometric Rank 1

For $r = 1$, by the discussion above, $\mathcal{GR}_1$ consists of two classes of tensors: the tensors with $\mathrm{ml}_A(T) \leqslant 1$ and the tensors satisfying $T(A^*)$ is a space of bounded rank 1. Since there is no primitive spaces of matrices of bounded rank 1, all tensors in $\mathcal{GR}_1$ are compression.

Proposition 20. *There is no primitive tensors of geometric rank 1.*

Proposition 20 recovers the results on tensors with geometric rank 1 from [[11]].

Corollary 21 ([11])**.** $\mathrm{GR}(T) = 1$ *if and only if* $\mathrm{SR}(T) = 1$.

4.2 Geometric Rank 2

Theorem 22. *Up to change of coordinates and deleting zero columns and rows, there is exactly one primitive tensor of geometric rank 2 of the form:*

$$T(A^*) = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}.$$

Theorem 22 recovers the results on tensors with geometric rank 2 from [[11]].

Corollary 23 ([11])**.** *A tensor T has geometric rank at most 3 if and only if $T(A^*), T(B^*)$ or $T(C^*)$ has bounded rank 2.*

proof of Proposition 22 and Corollary 23. $\mathrm{GR}(T) \leqslant 2$ if and only if at least one of the following three cases holds:

(i) $\mathrm{codim} A_2^* = 0$;

(ii) $\mathrm{codim} A_1^* \leqslant 1$;

(iii) $\mathrm{codim} A_0^* \leqslant 2$.

Case (i) is equivalent to $T(A^*)$ has bounded rank 2. And the only primitive space of bounded rank 2 is the 3-dimensional space of 3×3 skew-symmetric matrices, whose corresponding tensor is primitive by Lemma 7. Case (iii) is equivalent to $\mathrm{ml}_A(T) \leqslant 2$, so T is compression.

Case (ii) implies all 2×2 minors of $T(A^*)$ equal up to scale. If all 2×2 minors are zero, then $T(A^*)$ has bounded rank 1 and T is compression by Proposition 20. Otherwise, assume there exists a quadratic polynomial P dividing all 2×2 minors. By Lemma 11, either $T(A^*) \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ which means T is compression, or $P = l_1 l_2$ for some linear forms l_1, l_2. Let $T' := T|_{\{l_1=0\}}$, then all 2×2 minors of $T'(A^*)$ are zero, so $T'(A^*)$ has bounded rank 1 and by Proposition 20 $\mathrm{SR}(T') = 1$. Therefore $\mathrm{SR}(T) \leqslant 2$ and T is compression.

To prove the second statement of the theorem, we see that case (i) implies $T(A^*)$ has bounded rank 2, case (iii) implies both $T(B^*)$ and $T(C^*)$ has bounded rank 2. Case (ii) implies $\mathrm{SR}(T) \leqslant 2$. If $\mathrm{SR}(T) = 1$ then at least two of $T(A^*)$, $T(B^*)$ and $T(C^*)$ has bounded rank 1. If $\mathrm{SR}(T) = 2$, by definition we can decompose $T = T_1 + T_2$ and without loss of generality assume $\mathrm{ml}_A(T_1) = 1$, $\mathrm{ml}_A(T_2)$ or $\mathrm{ml}_B(T_2) = 1$, then $T(C^*)$ has bounded rank 2. □

4.3 Geometric Rank 3

This section studies the structure of the set of tensors with geometric rank at most 3.

Theorem 24. *A tensor $T \in A \otimes B \otimes C$ has geometric rank at most 3 if and only if one of the following conditions holds:*

1. *$T(A^*)$, $T(B^*)$ or $T(C^*)$ is of bounded rank 3, or*

2. $\mathrm{SR}(T) \leqslant 3$*, or*

3. *up to changes of bases* $T = M_{\langle 2 \rangle}$*.*

If T *is primitive of geometric rank 3, then up to changes of bases and permutations of* A*,* B *and* C*, it is either the matrix multiplication tensor* $M_{\langle 2 \rangle}$ *or the tensor such that* $T(A^*)$ *is a space of* 4×4 *skew-symmetric matrices of dimension* $4, 5$ *or* 6*.*

Proof. By (2), $\mathrm{GR}(T) \leqslant 3$ if and only if at least one of the following three cases holds:

(i) $\mathrm{codim} A_3^* = 0$;

(ii) $\mathrm{codim} A_2^* \leqslant 1$;

(iii) $\mathrm{codim} A_1^* \leqslant 2$;

(iv) $\mathrm{codim} A_0^* \leqslant 3$.

Case (i): $\mathrm{codim} A_3^* = 0 \iff T(A^*)$ has bounded rank 3.

Case (ii): If $\mathrm{codim} A_2^* = 0$, then $\mathrm{GR}(T) = 2$, so $T(A^*), T(B^*)$ or $T(C^*)$ is of bounded rank 2.

When $\mathrm{codim} A_2^* = 1$, according to the discussion in §3.2.3 and Proposition 16, at least one of the following holds:

1. $T = T' + T''$ where $T'(A^*)$ is a space of bounded rank 2 and $\mathrm{ml}_A(T'') = 1$, so T is not primitive.
2. $T(A^*), T(B^*)$ or $T(C^*)$ has bounded rank 3;
3. up to changes of bases $T = M_{\langle 2 \rangle}$.

By classification of $\mathcal{GR}_2$, any non-primitive tensor of $\mathrm{GR} = 3$ is either compression or at least one of $T(A^*), T(B^*)$ and $T(C^*)$ has bounded rank 3.

Case (iii): By the discussion in §3.2.2, if there is a nonzero 2×2 minor that is a product of 2 linear forms, T is not primitive. If all nonzero 2×2 minors are irreducible, $T(A^*) \subset \mathbb{C}^2 \otimes \mathbb{C}^3$ or $\mathbb{C}^3 \otimes \mathbb{C}^2$, so has bounded rank 2.

By Theorem 5, if $T(A^*)$ is primitive spaces of bounded rank 3, then either $T(B^*)$ or $T(C^*)$ is 4×4 skew-symmetric.

Case (iv): $\mathrm{codim} A_0^* \leqslant 3 \iff \mathrm{ml}_A(T) \leqslant 3$. So either T is compression or $\mathrm{GR}(T) = 2$, which implies T cannot be primitive of geometric rank 3. And $\mathrm{ml}_A(T) \leqslant 3$ implies both $T(B^*)$ and $T(C^*)$ have bounded rank 3. $\square$

By classifications of $\mathcal{GR}_r$ for $r = 1, 2$ and 3, we summarize the following relations between geometric rank and slice rank:

Corollary 25. *1.* $\mathrm{GR}(T) = 1 \iff \mathrm{SR}(T) = 1$.

2. If $ml_A(T)$, $ml_B(T)$ or $ml_C(T) > 3$, then $\mathrm{GR}(T) = 2 \iff \mathrm{SR}(T) = 2$.

3. If at least one of $ml_A(T)$, $ml_B(T)$ and $ml_C(T) > 6$, or at least two of them > 4, then $\mathrm{GR}(T) = 3 \iff \mathrm{SR}(T) = 3$.

However we cannot draw any similar conclusion for $r \geqslant 4$. As a counter example, let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be defined as

$$T(A^*) := \begin{pmatrix} 0 & x_1 & x_2 & & & & \\ -x_1 & 0 & x_3 & & & & \\ -x_2 & -x_3 & 0 & & & & \\ & & & x_4 & x_5 & \cdots & x_m \\ & & & & x_4 & & \\ & & & & & \ddots & \\ & & & & & & x_4 \end{pmatrix}.$$

Then T is a direct sum of the primitive tensor of geometric rank 2 and a compression tensor of geometric rank 2. So $\mathrm{GR}(T) = 4$, $\mathrm{SR}(T) = 5$.

4.4 Geometric Rank 4 and in General

Theorem 26. *If $T \in A \otimes B \otimes C := \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$ is primitive of geometric rank 4, then either at least 2 of $\mathrm{ml}_A(T)$, $\mathrm{ml}_B(T)$ and $\mathrm{ml}_C(T)$ are at most 6, or all of them are at most 8.*

Proof. $\mathrm{GR}(T) \leqslant 4$ if and only if at least one of the following cases holds:

(1) $A_4^* = A^*$;

(2) $\mathrm{codim}(A_3^*) \leqslant 1$;

(3) $\mathrm{codim}(A_2^*) \leqslant 2$;

(4) $\operatorname{codim}(A_1^*) \leqslant 3$;

(5) $\operatorname{codim}(A_0^*) \leqslant 4$;

(1) $\iff$ $T(A^*)$ has bounded rank 4. T is primitive only if $T(A^*)$ is a primitive space of bounded rank 4. By Theorem 4, if a primitive space of bounded rank 4 has size $n_1 \times n_2$, then either $n_1 \leqslant 5$ and $n_2 \leqslant 10$, $n_1 \leqslant 10$ and $n_2 \leqslant 5$, or $n_1 \leqslant 6$ and $n_2 \leqslant 6$. So either $\mathrm{ml}_B(T) \leqslant 5$, or $\mathrm{ml}_C(T) \leqslant 5$, or $\mathrm{ml}_B(T), \mathrm{ml}_C(T) \leqslant 6$.

(2) $\iff$ there exists an irreducible polynomial P of degree $\geqslant 1$ dividing all 4×4 minors of $T(A^*)$.

(2.1) $\deg P = 1$: by Lemma 7 T is not primitive.

(2.2) $\deg P = 2$: By Lemma 12, up to changes of bases the upper left 4×4 submatrix of $T(A^*)$ has determinant equal to P^2 and P divides all 3×3 minors of the submatrix. Proposition 16 gives a classification of such 4×4 matrix. Since the determinant does not vanish, the submatrix cannot have bounded rank 3, so the submatrix is either skew-symmetric or has the form $\operatorname{diag}(X, X)$.

(2.2.i) Case $\operatorname{diag}(X, X)$: write $T(A^*)$ as the block form:

$$T(A^*) = \begin{pmatrix} X & 0 & E_1 \\ 0 & X & E_2 \\ D_1 & D_2 & F \end{pmatrix}$$

where X has determinant S, and E_i and D_j are $2 \times (\mathbf{c} - 4)$ and $(\mathbf{b} - 4) \times 2$ blocks.

For $1 \leqslant i \leqslant 2, 3 \leqslant k \leqslant 4, 5 \leqslant j, l \leqslant \mathbf{c}$, the minor $\Delta^{i34j}_{12kl} = \Delta^{ij}_{12}\Delta^{34}_{kl}$ is divisible by the irreducible quadratic polynomial S. Therefore either $S|\Delta^{ij}_{12}$ or $S|\Delta^{34}_{kl}$. By Lemma 11, either D_1 or E_2 can be put to 0 by adding first 2 rows or columns to the rest. By the same argument, either D_2 or E_1 can be put to 0.

If $D_1 = D_2 = 0$ or $E_1 = E_2 = 0$, S divides $\Delta^{13ij}_{13kl} = (y^1_1)^2\Delta^{ij}_{kl}$ for $i, j, k, l \geqslant 5$. So S divides all 2×2 minors of F. By Lemma 11 either $F \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ or F has bounded rank 1. Therefore $\mathrm{ml}_B(T)$ or $\mathrm{ml}_C(T) \leqslant 6$ and T is not concise.

If $D_1 = E_1 = 0$ or $D_2 = E_2 = 0$, without loss of generalities assume $D_1 = E_1 = 0$. Consider the minors $\Delta^{1ijk}_{1lst} = y^1_1 \Delta^{ijk}_{lst}$ for $i,j,k,l,s,t \geqslant 3$. So S divides all 3×3 minors of $G := \begin{pmatrix} X & E_2 \\ D_2 & F \end{pmatrix}$. By Lemma 16 either $G \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ or G has bounded rank 2. Therefore $\mathrm{ml}_B(T) \leqslant 6$ or $\mathrm{ml}_C(T) \leqslant 6$ and T is not concise.

(2.2.ii) Case skew-symmetric: permute the first 4 rows and columns to put $T(A^*)$ into the following form

$$T(A^*) = \left(\begin{array}{cc|cc|c} x_1 & 0 & a & d & \multirow{2}{*}{E_1} \\ 0 & x_1 & c & b & \\ \hline b & -d & e & 0 & \multirow{2}{*}{E_2} \\ -c & a & 0 & e & \\ \hline \multicolumn{2}{c|}{D_1} & \multicolumn{2}{c|}{D_2} & F \end{array}\right).$$

Adding the first two rows and columns to the rest, so that $y^1_i, y^2_i, y^i_1, y^i_2$ do not contain x_1 in their expression, for all i,j. $S = x_1 e - ab + cd$ divides all 4×4 minors of $T(A^*)$. Restricting to the subspace $\{x_1 = 0\}$, then $S' := -ab + cd$ divides all 4×4 minors of $T(A^*)|_{x_1=0}$.

If S' is irreducible, for $3 \leqslant i,k \leqslant 4$ and $j,l \geqslant 5$, consider the minors $\Delta^{12ij}_{12kl} = \Delta^{12}_{kl}\Delta^{ij}_{12}$ of $T(A^*)|_{x_1=0}$. Similar to case (i), either $D_1|_{x_1=0}$ or $E_1|_{x_1=0}$ can be put 0. Without loss of generality assume $D_1|_{x_1=0}$.

Now working on $T(A^*)$, entries in D_1 are multiples of x_1. Then by adding multiples of first two rows to the last $m-4$ rows we can put $D_1 = 0$. S dividing $\Delta^{12ij}_{12kl} = (x_1)^2\Delta^{ij}_{kl}$ for $i,j \geqslant 5, k,l \geqslant 3$ implies it divides all 2×2 minors of the $(m-4)\times(m-2)$ block $(D_2\, F)$. So either $(D_2\, F)$ has bounded rank 2 or $(D_2\, F) \subset \mathbb{C}^2 \otimes \mathbb{C}^2$. If by changing bases $(D_2\, F)$ has nonzero entries only in the first 2 rows, $\mathrm{ml}_B(T) \leqslant 6$. Otherwise, by changing bases we can put all nonzero entries of $(D_2\, F)$ in its first 2 or 3 column. Then consider the 4×4 minors involving one entry of $(D_2\, F)$ and 3×3 minors from the first 4 rows of $T(A^*)$. By Proposition 16, either $\mathrm{ml}_C(T) \leqslant 6$, or $ml_B(T), ml_C(T) \leqslant 7$.

If $S' = -ab + cd$ is reducible, by changing bases we can put the block $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $\begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix}$

and the same for $\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$. Then the upper left 4×4 block of $T(A^*)$ becomes

$$\begin{pmatrix} x_1 & 0 & 0 & d' \\ 0 & x_1 & c' & b' \\ b' & -d' & e & 0 \\ -c' & 0 & 0 & e \end{pmatrix}.$$

Then permuting rows and columns we get

$$\begin{pmatrix} 0 & 0 & x_1 & d' \\ 0 & 0 & -c' & b' \\ b' & -d' & e & 0 \\ c' & x_1 & 0 & e \end{pmatrix}.$$

By the same argument, we can put $D_1 = 0$. Then all 3×3 minors of $(D_2\, F)$ are divisible by S. By Proposition 16 either $(D_2\, F)$ has bounded rank 2 or $(D_2\, F) \subset \mathbb{C}^4 \otimes \mathbb{C}^4$. By the same argument as the previous case, either $\mathrm{ml}_B(T) \leqslant 6$, or $\mathrm{ml}_B(T), \mathrm{ml}_C(T) \leqslant 8$.

(2.3) $\deg P = 3$: by Lemma 12, either P factors into linear forms so T is not primitive, or $T(A^*)$ has bounded rank 4.

(3) By Proposition 18, if T is primitive then either $\mathrm{ml}_B \leqslant 6$, $\mathrm{ml}_C(T) \leqslant 6$ or $\mathrm{ml}_B(T), \mathrm{ml}_C(T) \leqslant 8$.

(4) By Proposition 15, if T is primitive then $\mathrm{ml}_B(T) + \mathrm{ml}_C(T) \leqslant 6$.

(5) $\iff \dim(T(A^*)) \leqslant 4 \Rightarrow \mathrm{SR}(T) \leqslant 4$.

Putting everything together, either $\mathrm{ml}_B(T) \leqslant 8$, or $\mathrm{ml}_C(T) \leqslant 8$, or $\mathrm{ml}_B(T), \mathrm{ml}_C(T) \leqslant 6$. Since geometric rank is invariant by permuting A, B and C, we also have:

- either $\mathrm{ml}_A(T) \leqslant 8$, or $\mathrm{ml}_C(T) \leqslant 8$, or $\mathrm{ml}_A(T), \mathrm{ml}_C(T) \leqslant 6$;
- either $\mathrm{ml}_A(T) \leqslant 8$, or $\mathrm{ml}_B(T) \leqslant 8$, or $\mathrm{ml}_A(T), \mathrm{ml}_B(T) \leqslant 6$.

By inclusion-exclusion argument, we conclude the theorem. □

Corollary 27. *If* $\mathrm{ml}_A(T)$, $\mathrm{ml}_B(T)$ *and* $\mathrm{ml}_C(T) > 8$, *then* $\mathrm{GR}(T) \leqslant 4$ *if and only if either* $\mathrm{SR}(T) \leqslant 4$, *or up to changes of bases* $T = T' + T''$ *where* T' *is the* $3 \times 3 \times 3$ *skew-symmetric tensor and* $\mathrm{SR}(T'') = 2$.

As a consequence of Proposition 17, we draw a general conclusion for primitive tensors of geometric rank r.

Theorem 28. *For all* r, *there exists a positive integer* N_r, *such that if* $T \in A \otimes B \otimes C$ *is primitive of geometric rank* r, *then at least two of* $\mathrm{ml}_A(T)$, $\mathrm{ml}_B(T)$ *and* $\mathrm{ml}_C(T)$ *are at most* N_r.

5. SUMMARY AND CONCLUSIONS

This dissertation studied geometric rank of tripartite tensors. To classify the set of tensors with geometric rank at most r, we introduced primitive tensors and compression tensors. We proved the existence of the primitive-compression decompositions for any tensors, which reduced the problem of classifying tensors in $\mathcal{GR}_r$ to finding all primitive tensors in $\mathcal{GR}_r$.

Lemma 9. *If T is not compression (i.e., $\mathrm{GR}(T) < \mathrm{SR}(T)$), then there exist a primitive tensor T_p and a compression tensor T_c, such that $T = T_p + T_c$ and $\mathrm{GR}(T_p) + \mathrm{GR}(T_c) = \mathrm{GR}(T)$.*

Then we found all primitive tensors in GR_r for $r = 1, 2, 3$:

1. **Proposition 20.** *There is no primitive tensors of geometric rank 1.*
2. **Theorem 22.** *Up to change of coordinates and deleting zero columns and rows, there is exactly one primitive tensor of geometric rank 2 of the form:*
$$T(A^*) = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}.$$
3. **Theorem 24.** *If T is primitive of geometric rank 3, then up to changes of bases and permutations of A, B and C, it is either the matrix multiplication tensor $M_{\langle 2\rangle}$ or the tensor such that $T(A^*)$ is a space of 4×4 skew-symmetric matrices of dimension $4, 5$ or 6.*

For $r = 4$ and in general, we found upper bounds on multilinear ranks of primitive tensors in $\mathcal{GR}_r$:

1. **Theorem 26.** *If $T \in A \otimes B \otimes C$ is primitive of geometric rank 4, then either at least 2 of $\mathrm{ml}_A(T)$, $\mathrm{ml}_B(T)$ and $\mathrm{ml}_C(T)$ are at most 6, or all of them are at most 8.*
2. **Theorem 28.** *For all r, there exists a positive integer N_r, such that if $T \in A \otimes B \otimes C$ is primitive of geometric rank r, then at least two of $\mathrm{ml}_A(T)$, $\mathrm{ml}_B(T)$ and $\mathrm{ml}_C(T)$ are at most N_r.*

Finally, using the above results on primitive tensors, we were able to classify the all tensors in GR_3:

Theorem 24. *A tensor $T \in A \otimes B \otimes C$ has geometric rank at most 3 if and only if one of the following conditions holds:*

1. *$T(A^*)$, $T(B^*)$ or $T(C^*)$ is of bounded rank 3, or*
2. *$\mathrm{SR}(T) \leqslant 3$, or*
3. *up to changes of bases $T = M_{\langle 2 \rangle}$.*

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