THE SECOND MOMENT OF DIRICHLET *L*-FUNCTIONS ALONG A COSET

A Dissertation

by

BRADFORD ALEXANDER GARCIA

Submitted to the Graduate and Professional School of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Matthew Young
Committee Members,	Matthew Papanikolas
	Riad Masri
	Raul Medina
Head of Department,	Andrea Bonito

August 2022

Major Subject: Mathematics

ABSTRACT

We compute the second moment of Dirichlet L-functions along a coset at the central point, achieving an asymptotic result in the q-aspect.

DEDICATION

This dissertation is dedicated to my family: Bonnie and Charles, Brice and Sarah, Brittany and Jacob, and Hannah and Benjamin. Thank you for your unconditional love and support throughout this time in my life.

ACKNOWLEDGMENTS

CONTRIBUTORS AND FUNDING SOURCES

Contributors

Funding Sources

TABLE OF CONTENTS

1. INTRODUCTION

Studying the analytic behavior of L-functions inside the critical strip has been a fruitful area of research in the field of number theory for many years due to the arithmetic information it can reveal. Two aspects of L-functions which are of particular interest are their power moments and their rate of growth along the critical line. A classic example of the latter type of result is the Weyl bound which states that the Riemann zeta function satisfies

$$\zeta(1/2 + it) \ll_\varepsilon t^{1/6+\varepsilon}. \tag{1.1}$$

When it comes to power moments, mathematicians such as Hardy and Littlewood in 1918 and Ingham in 1926 were studying moments of the Riemann zeta function. However, it wasn't until Selberg in 1946 that the field began to turn its attention to moments of Dirichlet L-functions. Selberg produced an asymptotic result for the second moment of Dirichlet L-functions with moduli q. What Selberg realized that many mathematicians have noticed since is that the true analogue of studying the Riemann zeta function in the so-called t-aspect is studying Dirichlet L-functions in the q-aspect. That is to say, an asymptotic result about $\zeta(1/2 + it)$ as $t \to \infty$ can lead to a similar result about the Dirichlet L-functions $L(1/2, \chi)$ as $q \to \infty$, where χ is a primitive character modulo q.

Heath-Brown published two papers of note in 1978. In [1], he studied the twelfth moment of the Riemann zeta function, finding that

$$\int_T^{2T} |\zeta(1/2 + it)|^{12} dt \ll_\varepsilon T^{2+\varepsilon}, \tag{1.2}$$

a result which easily recovers (1.1) while also proving that $|\zeta(1/2 + it)|$ cannot be *too large too often*. Two mathematicians whose work inspired my own looked at the q-analogue of (1.2) in

particular cases: Nunes investigated L-functions with smooth square-free moduli in [2] while Milićević and White explored L-functions with odd prime power moduli in [3]. The other 1978 paper by Heath-Brown more directly influenced my work. In [4], he spends some time examining sums of the form defined in (2.1). In particular, he produces Lemma 2.1.2. Along with some other tools such as an approximate functional equation, a dyadic partition of unity, and Poisson summation, this lemma turns out to be instrumental in bounding the second moment of Dirichlet L-functions along a coset, a connection that doesn't seem to have been made before now.

Pushing the work further, by utilizing a Postnikov-type formula in a different range of the dyadic partition of unity, an asymptotic result can be achieved. This asymptotic result is in fact my main theorem, but the application of a Postnikov-type formula introduces into the final result a_ψ, a constant which requires some pre-introduction. Eager readers can find a more detailed treatment in the statement and proof of Lemma 2.2.9, but for now, suffice it to say that, when ψ is a Dirichlet character modulo q and d is a positive integer with $d|q$ and $q|d^2$, a_ψ is the number $(\mod \frac{q}{d})$ such that $\psi(1 + dx) = e(\frac{a_\psi dx}{q})$, where $e(x) = e^{2\pi ix}$. Additionally, since $a_\psi \in \mathbb{Z}/(q/d)\mathbb{Z}$, let's choose once and for all that

$$0 < |a_\psi| < \frac{q}{2d}, \tag{1.3}$$

where the strict inequalities are later justified by Lemma 2.2.10.

1.1   Statement of Results

Thus, we arrive at the main theorem of my project, split into two cases for the sake of compactness and comparison. However, we will first develop a new notation to convey a hypothesis used frequently throughout this paper. Recalling that $\nu_p(\cdot)$ denotes the p-adic valuation, let a and b be positive integers. We will write "$a \prec b$" to mean that a and b share all of the same prime factors and, for any prime $p|b$, $\nu_p(a) < \nu_p(b)$. Similarly, we will write "$a \preceq b$" to mean that a and b share all of the same prime factors and, for any prime $p|b$, $\nu_p(a) \leq \nu_p(b)$. Note that $a \prec b$ implies that

$a|b$ and $a < b$, while $a \preceq b$ implies that $a|b$ and $a \leq b$.

Theorem 1.1.1. *Let q be a positive odd integer, ψ be a primitive even character modulo q, and d be a positive integer such that $d \prec q$ and $q \preceq d^2$. Then*

$$\sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} |L(1/2, \chi \cdot \psi)|^2 = MT + O\left(q^\varepsilon \left(d^{1/4}q^{1/4} + \frac{d}{q^{1/8}}\right)\right), \tag{1.4}$$

where MT is a main term defined by $MT = \mathcal{D} + \mathcal{A}$ with

$$\mathcal{D} = \frac{\varphi(d)}{2}\frac{\varphi(q)}{q}\left[\log(q) + c_0 + 2\theta(q)\right], \tag{1.5}$$

$$\mathcal{A} = \frac{\varphi(d)}{d}\sqrt{q}\,\cos\left(\frac{2\pi a_\psi}{q}\right)\frac{\sigma_0(|a_\psi|)}{\sqrt{|a_\psi|}}, \tag{1.6}$$

where c_0 is an absolute constant defined in (3.9), $\theta(q) = \sum_{p|q}\frac{\log p}{p-1}$, and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$.

Recall that $q \preceq d^2$ implies that $q \leq d^2$. Thus, $d^{1/4}q^{1/4} \leq \frac{d}{q^{1/8}}$, meaning that the error term in (1.4) can simplify to $O\left(dq^{-1/8+\varepsilon}\right)$. Moreover, an immediate consequence of this asymptotic result is that not all of the L-functions in this family can vanish at $s = 1/2$.

For the next theorem, we will have that $d|\frac{q}{d}$, so let b_ψ be the reduction of $a_\psi \pmod{d}$ such that

$$0 < |b_\psi| < \frac{d}{2}, \tag{1.7}$$

the strict inequalities again being allowed by the eventual Lemma 2.2.10.

Theorem 1.1.2. *Let q be a positive odd integer with $(q, 3) = 1$, ψ be a primitive even character modulo q, and d be a positive integer such that $d^2 \preceq q$ and $q \preceq d^3$. Then*

$$\sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} |L(1/2, \chi \cdot \psi)|^2 = MT' + O\left(q^\varepsilon \left(\frac{q^{1/2}}{d^{1/4}} + \frac{d}{q^{1/8}}\right)\right), \tag{1.8}$$

3

where MT' is a main term defined by $MT' = \mathcal{D} + \mathcal{A}'$ with $\mathcal{D}$ as defined in (1.5) and

$$\mathcal{A}' = \begin{cases} c_1 \; \varphi(d) \; \cos\left(\frac{2\pi(b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2)}{q}\right) \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}, & q \equiv 1 \pmod 4 \\ c_2 \; \varphi(d) \; \sin\left(\frac{2\pi(b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2)}{q}\right) \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}, & q \equiv 3 \pmod 4 \end{cases} \tag{1.9}$$

where c_1 and c_2 are real constants with $|c_1| = |c_2| = 1$ defined in (3.83) and (3.86), respectively.

Recall that $d^2 \preceq q$ implies that $d^2 \leq q$. Thus, $\frac{d}{q^{1/8}} \leq \frac{q^{1/2}}{d^{1/4}}$, meaning that the error term in (1.8) can simplify to $O\left(d^{-1/4}q^{1/2+\varepsilon}\right)$. Theorem 1.1.2 can be extended with little effort to q such that $(q,3) \neq 1$ if we also have that $q \preceq \frac{1}{3}d^3$. Note that, although $\mathcal{A}'$ can be negative, $|\mathcal{A}'| < \mathcal{D}$. Hence, Theorem 1.1.2 is truly an asymptotic result when $d \geq q^{2/5+\varepsilon}$, as this is when the main term is larger than the so-called error term. With this in mind, it's worth noting that this result is not vacuous as $\frac{2}{5} < \frac{1}{2}$. Furthermore, the asymptotic result we get when $d \geq q^{2/5+\varepsilon}$ again shows that not all of the L-functions in this family can vanish at $s = 1/2$.

The following is an upper bound on the second moment that serves as a nice warm-up to my main theorems.

Theorem 1.1.3. *Let q be a positive integer, let ψ be a primitive character modulo q, and let $d|q$. Then*

$$\sum_{\chi \; (\mathrm{mod}\; d)} |L(1/2, \chi \cdot \psi)|^2 \ll q^\varepsilon \left(d + \left(\frac{q}{d}\right)^{1/2}\right). \tag{1.10}$$

2. PRELIMINARIES

In this section, we will lay the groundwork and develop the tools necessary to prove the theorems just stated in the introduction.

2.1 Various Bounds & Evaluations

First, let's define the summation studied by Heath-Brown in [4] and cite the associated bound.

Definition 2.1.1. Let χ be a character (mod q), and let h and n be integers. Denote

$$S(q; \chi, h, n) = \sum_{m=0}^{q-1} \chi(m+h)\overline{\chi}(m)e(mn/q). \tag{2.1}$$

Lemma 2.1.2 (Heath-Brown, [4], Lemma 9)**.** *Suppose that q is odd, $q_0|q$, and $\varepsilon > 0$. Then*

$$\sum_{1 \le h \le A} \sum_{1 \le n \le B} |S(q; \chi, hq_0, n)| \ll (\sigma_0(q)\sigma_{-1/4}(q))^4 q^{1/2}\{ABq_0^{-1/2} + (qAq_0)^{1/4}(AB/q_0)^{\varepsilon}\}, \tag{2.2}$$

and

$$\sum_{1 \le h \le A} |S(q; \chi, hq_0, 0)| \ll (\sigma_0(q))^2 q_0 A. \tag{2.3}$$

Remark 2.1.3. Lemma 9 in [4] gives a bound of $|S(q; \chi, 4hq_0, n)|$ for general q. However, it can be seen by reading through the proof that the result holds for $|S(q; \chi, hq_0, n)|$ without the 4 if we add the condition that q be odd.

We next evaluate a particular type of quadratic exponential sum.

Lemma 2.1.4. *Let r be a positive odd integer. Let A, B be integers such that $(B, r) = 1$. Then*

$$\sum_{u \pmod r} e_r\left(Au + Bu^2\right) = e\left(-\frac{\overline{4B}A^2}{r}\right)\left(\frac{B}{r}\right)\varepsilon_r\sqrt{r}, \tag{2.4}$$

where $e_q(x) = e\left(\frac{x}{q}\right) = e^{\frac{2\pi i x}{q}}$, the bar notation indicates the multiplicative inverse modulo r, $\left(\frac{B}{r}\right)$

is the Jacobi symbol, and $\varepsilon_r = \begin{cases} 1, & r \equiv 1 \ (\mathrm{mod}\ 4) \\ i, & r \equiv 3 \ (\mathrm{mod}\ 4) \end{cases}$.

Proof. By completing the square and applying (3.38) from [5], we can see that

$$\sum_{u \ (\mathrm{mod}\ r)} e_r \left(Au + Bu^2 \right) = e_r \left(-\overline{4B} A^2 \right) \sum_{u \ (\mathrm{mod}\ r)} e_r \left(B \left(u + \overline{2B} A \right)^2 \right) \tag{2.5}$$

$$= e \left(-\frac{\overline{4B} A^2}{r} \right) \left(\frac{B}{r} \right) \varepsilon_r \sqrt{r}. \qquad \square$$

Another simple lemma to get us warmed up is the evaluation of the following integral using known Mellin transforms.

Lemma 2.1.5. *Let* k *be a non-zero integer and* s *be a complex number with* $-\frac{1}{2} < \Re(s) < \frac{1}{2}$. *Then*

$$\int_0^\infty x^{-s} e(kx) \frac{dx}{\sqrt{x}} = \frac{\Gamma(1/2 - s)}{(2\pi|k|)^{1/2-s}} \left(\cos\left(\frac{\pi}{2} \left(\frac{1}{2} - s \right) \right) + i \operatorname{sgn}(k) \sin\left(\frac{\pi}{2} \left(\frac{1}{2} - s \right) \right) \right), \tag{2.6}$$

where $\operatorname{sgn}(k) = \begin{cases} 1, & k > 0 \\ -1, & k < 0 \end{cases}$.

Proof. Let $I = \int_0^\infty x^{-s} e(kx) \frac{dx}{\sqrt{x}}$. Applying Euler's formula and rearranging the x results in

$$I = \int_0^\infty x^{1/2-s} (\cos(2\pi kx) + i \sin(2\pi kx)) \frac{dx}{x}. \tag{2.7}$$

Because k may be positive or negative, we will scale x by $\frac{1}{2\pi|k|}$, thus giving us

$$I = (2\pi|k|)^{s-1/2} \int_0^\infty x^{1/2-s} (\cos(x) + i \operatorname{sgn}(k) \sin(x)) \frac{dx}{x} \tag{2.8}$$

since cosine and sine are respectively even and odd. Splitting up the integral gets us

$$I = (2\pi|k|)^{s-1/2} \left(\int_0^\infty x^{1/2-s} \cos(x) \frac{dx}{x} + i \operatorname{sgn}(k) \int_0^\infty x^{1/2-s} \sin(x) \frac{dx}{x} \right). \tag{2.9}$$

6

These integrals can be recognized as the Mellin transforms of $\cos(x)$ and $\sin(x)$. The hypothesis that $-\frac{1}{2} < \Re(s) < \frac{1}{2}$ implies that $0 < \Re(1/2-s) < 1$, meaning we are in the region of convergence for both Mellin transforms. Therefore, applying the known evaluations of these Mellin transforms (as can be found in a source such as [6]) will conclude the proof. $\qquad \square$

As an immediate consequence of this, we have the following corollary.

Corollary 2.1.6. *Let k be a non-zero integer and s be a complex number with $-\frac{1}{2} < \Re(s) < \frac{1}{2}$. Then*

$$\int_0^\infty x^{-s} e(kx) \frac{dx}{\sqrt{x}} + \int_0^\infty x^{-s} e(-kx) \frac{dx}{\sqrt{x}} = 2 \frac{\Gamma(1/2 - s)}{(2\pi|k|)^{1/2-s}} \cos\left(\frac{\pi}{2}\left(\frac{1}{2} - s \right) \right) \qquad (2.10)$$

2.2 Postnikov

We now motivate a Postnikov formula by first defining a global analogue of the p-adic logarithm. For this, we need the notation $a|b^\infty$, which means that there exists a positive integer A such that $a|b^A$.

Definition 2.2.1. For positive odd integers d and q such that $d|q$ and $q|d^\infty$, define the formal power series in the indeterminate x,

$$\mathcal{L}_q(1 + dx) = \sum_{k=1}^\infty (-1)^{k+1} \frac{d^k}{k} \, x^k \in \mathbb{Q}[[x]]. \qquad (2.11)$$

The importance of the conditions that $d|q$ and $q|d^\infty$ is that they guarantee d and q will share all of the same prime factors. Thus, while the coefficients of this power series are certainly rational numbers, we can prove something stronger. We will show that all of these coefficients belong to the set

$$R_q = \{x \in \mathbb{Q} \text{ such that } \nu_p(x) \geq 0 \text{ for all } p|q\}. \qquad (2.12)$$

Note that R_q is a sub-ring of $\mathbb{Q}$ due to the p-adic valuation properties of $\nu_p(m \cdot n) = \nu_p(m) + \nu_p(n)$ and $\nu_p(m + n) \geq \min\{\nu_p(m), \nu_p(n)\}$. Another way to characterize the elements of the ring R_q is

to say that, for $x \in R_q$, if $x = \frac{a}{b}$ with $(a,b) = 1$, then $(b,q) = 1$. Therefore, there exists a ring homomorphism $\varphi : R_q \to \mathbb{Z}/q\mathbb{Z}$ given by

$$\varphi\left(\frac{a}{b}\right) = a\bar{b}, \tag{2.13}$$

where the bar notation indicates the multiplicative inverse modulo q. Let's now show why and how this can be applied to the coefficients of $\mathcal{L}_q(1 + dx)$.

Lemma 2.2.2. *Let d and q be positive odd integers such that $d|q$ and $q|d^\infty$. For any positive integer k and any prime p such that $p|q$,*

$$\nu_p(k) \leq \nu_p(d^{k-1}). \tag{2.14}$$

More generally, for any positive integer A and any prime p such that $p|q$, there exists a positive integer N such that $\nu_p(k) \leq \nu_p(d^{k-A})$ for all $k \geq N$.

Proof. We have $\nu_p(k) \leq \frac{\ln(k)}{\ln(p)} \leq k - 1 \leq \nu_p(d^{k-1})$, where the last inequality follows from the fact that d and q share all of the same prime factors. Now, (2.14) follows. In the more general case, $\nu_p(d^{k-A}) \geq k - A$ and $\nu_p(k) = O(\ln(k))$, so there will always exist a large enough choice of N such that $\nu_p(k) \leq \nu_p(d^{k-A})$ for all $k \geq N$. $\qquad\square$

Remark 2.2.3. While Lemma 2.2.2 only guarantees the existence of a positive integer N, the method of proof shows that a constructive candidate would be the minimal positive integer M such that $k - A \geq \ln(k)$ for all $k \geq M$. This minimal M can be found for a particular positive integer A using methods of calculus, and an example of future relevance is that $M = 5$ when $A = 3$.

Lemma 2.2.4. *We have $\mathcal{L}_q(1 + dx) \in R_q[[x]]$.*

Proof. If we write $\mathcal{L}_q(1 + dx) = \sum_{k=1}^{\infty} c_k x^k$ where $c_k = (-1)^{k+1}d^k/k$, then we need to show that $c_k \in R_q$ for all k. For any prime p such that $p|q$, (2.14) implies that $0 \leq \nu_p\left(\frac{d^{k-1}}{k}\right) < \nu_p(c_k)$, for any k. $\qquad\square$

8

Now that we have shown that $\mathcal{L}_q(1 + dx)$ lives in $R_q[[x]] \subseteq \mathbb{Q}[[x]]$, given the ring homomorphism φ from (2.13), there is an induced ring homomorphism $\overline{\varphi} : R_q[[x]] \to (\mathbb{Z}/q\mathbb{Z})[[x]]$ which maps

$$\overline{\varphi}(\mathcal{L}_q(1 + dx)) = \sum_{k=1}^{\infty} \varphi(c_k)x^k. \tag{2.15}$$

By abuse of notation, we may view $\mathcal{L}_q(1 + dx)$ as being in $(\mathbb{Z}/q\mathbb{Z})[[x]]$ by way of this reduction map. From this perspective, we may now observe that $\mathcal{L}_q(1 + dx)$ is not as infinite as it once seemed.

Lemma 2.2.5. *We have $\mathcal{L}_q(1 + dx) \in (\mathbb{Z}/q\mathbb{Z})[x]$.*

Proof. We wish to prove that $\mathcal{L}_q(1+dx)$ has only finitely many coefficients which are non-zero in $\mathbb{Z}/q\mathbb{Z}$. Suppose that $\{p|q : p \text{ is a prime}\} = \{p_1, \ldots, p_r\}$. Recall that $q|d^{\infty}$ implies that there exists a positive integer A such that $q|d^A$. By Lemma 2.2.2, we know that there exists a positive integer N_i for each $i = 1, \ldots, r$ such that $\nu_{p_i}(k) \leq \nu_{p_i}(d^{k-A})$ for all $k \geq N_i$. Hence, we can take $N = \max\{N_1, \ldots, N_r\}$, and it will follow that $\mathcal{L}_q(1 + dx) = \sum_{k=1}^{N}(-1)^{k+1} \dfrac{d^k}{k} \, x^k$ in $(\mathbb{Z}/q\mathbb{Z})[[x]]$. $\qquad \square$

This will open the door to discussing various properties of $\mathcal{L}_q(1 + dx)$ modulo q, such as the following periodicity and additivity properties. These lemmas will require two indeterminates, so we will embed $\mathcal{L}_q(1 + dx)$ into $(\mathbb{Z}/q\mathbb{Z})[x, y]$ in the obvious way.

Lemma 2.2.6. *We have $\mathcal{L}_q(1 + dx) = \mathcal{L}_q\left(1 + d(x + \frac{q}{d}y)\right)$ in $(\mathbb{Z}/q\mathbb{Z})[x, y]$.*

Proof. Using polynomial substitution, we may evaluate

$$\mathcal{L}_q\left(1 + d\left(x + \frac{q}{d}y\right)\right) = \sum_{k=1}^{\infty}(-1)^{k+1} \frac{d^k}{k}\left(x + \frac{q}{d}y\right)^k \tag{2.16}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(dx + qy)^k. \tag{2.17}$$

9

For $k \in \{1, 2, \dots\}$, applying the binomial theorem and isolating the $j = 0$ term gives

$$\frac{(-1)^{k+1}}{k}(dx + qy)^k = \frac{(-1)^{k+1}}{k} \sum_{j=0}^{k} \binom{k}{j}(dx)^{k-j}(qy)^j \tag{2.18}$$

$$= \frac{(-1)^{k+1}}{k}(dx)^k + q\left[\underbrace{(-1)^{k+1} \sum_{j=1}^{k} \binom{k}{j}\frac{d^{k-j}q^{j-1}}{k}x^{k-j}y^j}_{\in (\mathbb{Z}/q\mathbb{Z})[x,y]} \right]. \tag{2.19}$$

This final claim that the indicated expression is in $(\mathbb{Z}/q\mathbb{Z})[x, y]$ follows from the facts that $d|q$ and (2.14) holds for any prime p such that $p|q$, as well as our abuse of notation via (2.15). $\qquad \square$

Lemma 2.2.7. *We have $\mathcal{L}_q((1 + dx)(1 + dy)) = \mathcal{L}_q(1 + dx) + \mathcal{L}_q(1 + dy)$ in $(\mathbb{Z}/q\mathbb{Z})[x, y]$.*

See pages 79-80 of [7] for more details.

Proof. We have the well-known additive property for the real logarithm function: $\log((1 + dx)(1 + dy)) = \log(1 + dx) + \log(1 + dy)$. Hence, the power series expansions of these two expressions are equal wherever they both converge, meaning that all of their corresponding coefficients are equal. Thus, since $\mathcal{L}_q(1 + dx)$ matches the power series expansion of the real logarithm (reduced modulo q via (2.15)), this property also holds for $\mathcal{L}_q(1 + dx)$. $\qquad \square$

Thanks to Lemma 2.2.5, we will be able to substitute $\mathcal{L}_q(1 + dx)$ into an exponential function without ever introducing a notion of convergence but while maintaining the periodicity and additivity properties outlined in Lemmas 2.2.6 and 2.2.7. We now observe how $\mathcal{L}_q(1 + dx)$ behaves with respect to various moduli.

Lemma 2.2.8. *Let q be a positive odd integer and d be a positive integer such that $d|q$.*
(1) If $q|d^\infty$, then $\mathcal{L}_q(1 + dx) \equiv dx \pmod{(q, d^2)}$.
(2) If $(q, 3) = 1$ and $q|d^3$, then $\mathcal{L}_q(1 + dx) \equiv dx - \overline{2}(dx)^2 \pmod{q}$.

Proof. Recall that $\mathcal{L}_q(1 + dx) = \sum_{k=1}^{\infty} (-1)^{k+1} \dfrac{d^k}{k} x^k \in (\mathbb{Z}/q\mathbb{Z})[x]$. Expanding this out gives

$$\mathcal{L}_q(1 + dx) = dx - \frac{1}{2}(dx)^2 + \frac{1}{3}(dx)^3 - d^3 \underbrace{\left(\frac{1}{4}dx^4 - \frac{1}{5}d^2 x^5 + \dots \right)}_{\in (\mathbb{Z}/q\mathbb{Z})[x]}. \tag{2.20}$$

The claim that the tail of this series is still a polynomial with coefficients in $\mathbb{Z}/q\mathbb{Z}$ after factoring d^3 follows from Lemma 2.2.2, or more specifically the observations in Remark 2.2.3. Elaborating on the details, since $k - 3 \geq \ln(k)$ for all $k \geq 5$, we may choose $N = 5$ from Lemma 2.2.2. However, since q is odd, $\frac{1}{4} = \overline{2}^2$ in $\mathbb{Z}/q\mathbb{Z}$, so we could also pick $N = 4$. Lastly, because $d|q$ overall and either $q|d^\infty$ in (1) or $q|d^3$ in (2), d and q share all of the same prime factors, so it must be that either $(q, 3) = 1$ or $3|d$. Therefore, by their respective hypotheses, both statements follow immediately. $\square$

We are now ready to state and prove our Postnikov formula.

Lemma 2.2.9. *Let q be a positive odd integer and d be a positive integer such that $d|q$ and $q|d^\infty$. There exists a unique group homomorphism $a : \widehat{(\mathbb{Z}/q\mathbb{Z})^*} \to \mathbb{Z}/(q/d)\mathbb{Z}$, $\psi \mapsto a_\psi$, such that a Postnikov-type formula holds: for each Dirichlet character ψ modulo q and $x \in \mathbb{Z}$ we have*

$$\psi(1 + dx) = e_q(a_\psi \mathcal{L}_q(1 + dx)). \tag{2.21}$$

Proof. Consider the reduction modulo d map

$$(\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/d\mathbb{Z})^*, \tag{2.22}$$

and denote its kernel by K. Since d and q share all of the same prime factors, $K = \left\{ 1 + dx : x \pmod{\frac{q}{d}} \right\}$, so $|K| = \frac{q}{d}$. Consider the map $f : K \to S^1$ defined by

$$f(1 + dx) = e_q(\mathcal{L}_q(1 + dx)). \tag{2.23}$$

11

The function f is well-defined by Lemmas 2.2.5 and 2.2.6. Furthermore, f is a group homomorphism by Lemma 2.2.7. Thus, f is a character on K, and we claim that f has order $\frac{q}{d}$ in $\widehat{K}$. Indeed, if p is a prime such that $p|\frac{q}{d}$, then we have $f(1+dx)^{q/dp} = e_q\left(\frac{q}{dp}\mathcal{L}_q(1+dx)\right) = e_{dp}\left(\mathcal{L}_q(1+dx)\right) = e_{dp}(dx) = e(\frac{x}{p})$, by Lemma 2.2.8 since $dp|(q,d^2)$. Hence, $\widehat{K}$ is cyclic and generated by f. Therefore, every element of $\widehat{K}$ is of the form f^a for some $a \pmod{\frac{q}{d}}$. In particular, ψ is a character on $(\mathbb{Z}/q\mathbb{Z})^*$, so restricting ψ to K makes it an element of $\widehat{K}$. Thus, there exists a unique $a_\psi \pmod{\frac{q}{d}}$ such that $\psi|_K = f^{a_\psi}$, which is equivalent to the Postnikov formula. $\qquad\square$

Having formally introduced a_ψ, we will need the following lemma in order to conclude that a_ψ is coprime to q/d as a consequence of ψ being primitive.

Lemma 2.2.10. *Let q be a positive odd integer, ψ be a primitive character modulo q, and d be a positive integer such that $d|q$ and $q|d^\infty$. Then $(a_\psi, \frac{q}{d}) = 1$.*

Proof. Firstly, if $d = q$, then a_ψ is trivially 0 and the conclusion holds. When $d \neq q$, for the sake of contradiction, assume that all of the hypotheses hold but $(a_\psi, \frac{q}{d}) > 1$. Thus, there exists a prime p such that $p|a_\psi$ and $p|\frac{q}{d}$. Note that d and q share all the same prime factors, so $p|\frac{q}{d}$ implies that $p^2|q$, which further implies that $q|(\frac{q}{p})^2$. Thus, $(q, (\frac{q}{p})^2) = q$. Using Lemma 2.2.9 and statement (1) of Lemma 2.2.8, both with $\frac{q}{p}$ playing the role of d, observe that, for $y \in \mathbb{Z}$,

$$\psi\left(1 + \frac{q}{p}y\right) = e_q\left(a_\psi\mathcal{L}_q\left(1 + \frac{q}{p}y\right)\right) \tag{2.24}$$

$$= e_q\left(a_\psi\left(\frac{q}{p}y\right)\right) \tag{2.25}$$

$$= e\left(\frac{a_\psi}{p}y\right) = 1. \tag{2.26}$$

Moreover, for any $r \in \mathbb{Z}$ such that $(r, q) = 1$, we have $\psi\left(r + \frac{q}{p}y\right) = \psi(r)\psi\left(1 + \frac{q}{p}y\bar{r}\right) = \psi(r)$. Therefore, ψ has a smaller periodicity of q/p, contradicting the assumption that ψ is a primitive character modulo q. $\qquad\square$

2.3 Miscellaneous Lemmas

In the course of this paper, we encounter sums of the form

$$\mathcal{S}_{q,d}(\psi,k) := \sum_{u \ (\mathrm{mod}\ \frac{q}{d})} \psi(1+du)e_q(dku). \tag{2.27}$$

Using Postnikov, we can evaluate such sums given similar conditions to the ones we've enforced thus far.

Lemma 2.3.1. *Let q be a positive odd integer with $(q,3)=1$, ψ be a primitive character modulo q, and d be a positive integer such that $d^2|q$ and $q|d^3$. Also, let k be an integer. Then*

$$\mathcal{S}_{q,d}(\psi,k) = e\left(\frac{\overline{2a_\psi}(k+a_\psi)^2}{q}\right)\left(\frac{-\overline{2}a_\psi}{q/d^2}\right)\varepsilon_{q/d^2}\sqrt{q} \tag{2.28}$$

if $k \equiv -a_\psi \ (\mathrm{mod}\ d)$, and $\mathcal{S}_{q,d}(\psi,k) = 0$ otherwise. In particular, $\mathcal{S}_{q,d}(\psi,0) = 0$.

Proof. By Lemma 2.2.9, we have

$$\mathcal{S}_{q,d}(\psi,k) = \sum_{u \ (\mathrm{mod}\ \frac{q}{d})} e_q(dku + a_\psi\mathcal{L}_q(1+du)). \tag{2.29}$$

By statement (2) of Lemma 2.2.8, this can truncate as

$$\mathcal{S}_{q,d}(\psi,k) = \sum_{u \ (\mathrm{mod}\ \frac{q}{d})} e_q\left(dku + a_\psi\left(du - \overline{2}(du)^2\right)\right) \tag{2.30}$$

$$= \sum_{u \ (\mathrm{mod}\ \frac{q}{d})} e_{q/d}\left((k+a_\psi)u - \overline{2}a_\psi du^2\right). \tag{2.31}$$

Since q/d^2 is an integer by hypothesis, we may shift the sum by q/d^2 to reveal that either this sum

vanishes or else $k \equiv -a_\psi \pmod{d}$. Then,

$$\mathcal{S}_{q,d}(\psi, k) = \sum_{u \ (\text{mod } \frac{q}{d})} e_{q/d^2}\left(\left(\frac{k+a_\psi}{d}\right)u - \overline{2}a_\psi u^2\right) \tag{2.32}$$

$$= d \sum_{u \ (\text{mod } \frac{q}{d^2})} e_{q/d^2}\left(\left(\frac{k+a_\psi}{d}\right)u - \overline{2}a_\psi u^2\right). \tag{2.33}$$

Because ψ is primitive, Lemma 2.2.10 gives us that $(a_\psi, \frac{q}{d}) = 1$, and so $a_\psi \neq 0$ since $d^2|q$ implies that $d|\frac{q}{d}$. Firstly, this tells us that $\mathcal{S}_{q,d}(\psi, 0) = 0$ since $0 \equiv -a_\psi \pmod{d}$ would contradict a_ψ being coprime to q/d. Secondly, this allows us to apply Lemma 2.1.4, thus concluding the proof. $\qquad\square$

The following approximate functional equation for the product of two Dirichlet L-functions is a variation of a theorem found in [5].

Lemma 2.3.2 (Iwaniec-Kowalski, [5], Theorem 5.3). *Let χ be a primitive even character modulo q. Let $G(s)$ be any function which is holomorphic and bounded in the strip $-4 < Re(s) < 4$, even, and normalized by $G(0) = 1$. Then*

$$L(1/2, \chi)L(1/2, \overline{\chi}) = 2 \sum_{m,n \geq 1} \frac{\chi(m)\overline{\chi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right) \tag{2.34}$$

where $V(x)$ is a smooth function defined by

$$V(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \frac{\gamma(1/2 + s)^2}{\gamma(1/2)^2} x^{-s} ds \tag{2.35}$$

and

$$\gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right). \tag{2.36}$$

This next lemma encompasses the opening moves to prove both Theorem 1.1.1 and Theorem 1.1.2.

Lemma 2.3.3. *Let q be a positive odd integer, ψ be a primitive even character modulo q, and d be*

14

a positive integer such that $d \prec q$. Then

$$\sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} |L(1/2, \chi \cdot \psi)|^2 = \varphi(d) \sum_{\pm} \sum_{\substack{m \equiv \pm n \pmod{d} \\ (mn, q) = 1}} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right), \qquad (2.37)$$

with V as defined in (2.35).

Proof. Let $\mathcal{M} = \displaystyle\sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} |L(1/2, \chi \cdot \psi)|^2$. Since $\overline{L(1/2, \chi)} = L(1/2, \overline{\chi})$,

$$\mathcal{M} = \sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} L(1/2, \chi \cdot \psi) L(1/2, \overline{\chi \cdot \psi}). \qquad (2.38)$$

The character $\chi \cdot \psi$ is primitive modulo q because ψ is primitive modulo q and $d \prec q$, meaning d has all of the same prime factors as q but with a strictly smaller power of each. More obviously, $\chi \cdot \psi$ is even since it is the product of two even characters. Thus, applying Lemma 2.3.2 results in

$$\mathcal{M} = \sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} 2 \sum_{m,n \geq 1} \frac{\chi(m)\psi(m)\overline{\chi}(n)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right). \qquad (2.39)$$

As a common trick to detect the parity of a character, observe that

$$\frac{\chi(1) + \chi(-1)}{2} = \begin{cases} 1, & \chi \text{ is even} \\ 0, & \chi \text{ is odd} \end{cases}. \qquad (2.40)$$

Hence, it follows that

$$\mathcal{M} = \sum_{\pm} \sum_{m,n \geq 1} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right) \sum_{\chi \pmod{d}} \chi(m)\overline{\chi}(\pm n). \qquad (2.41)$$

Finally, using the orthogonality relation for Dirichlet characters, we may conclude the proof. $\quad\square$

In the course of proving both Theorem 1.1.1 and Theorem 1.1.2, it will be necessary to explain

15

where each main term comes from. For instance, the subsequent lemma shows where $\mathcal{D}$ in (1.5) originates from.

Lemma 2.3.4. *Let q be a positive integer, and let V be as defined in (2.35). Then*

$$\sum_{\substack{(n,q)=1}} \frac{1}{n} V\left(\frac{n^2}{q}\right) = \frac{\varphi(q)}{q} \left[\frac{1}{2}\log(q) + \gamma_0 + \gamma'(1/2) + \theta(q)\right] + O(q^{-1/2+\varepsilon}), \tag{2.42}$$

where γ_0 is Euler's constant.

Proof. Applying the definition of V gives us

$$\sum_{\substack{(n,q)=1}} \frac{1}{n} V\left(\frac{n^2}{q}\right) = \frac{1}{2\pi i} \int_{(1)} \zeta_q(1+2s) q^s \frac{G(s)}{s} \frac{\gamma(1/2+s)^2}{\gamma(1/2)^2} ds, \tag{2.43}$$

where $\zeta_q(u) = \zeta(u) \cdot \prod_{p|q}\left(1 - \frac{1}{p^u}\right)$. To evaluate this asymptotically, we want to shift the contour to the line $(-1/2 + \varepsilon)$. We will pick up a residue at $s = 0$ which we can calculate by computing the Laurent expansion of each factor in the integrand. Firstly,

$$\zeta(1+2s) = \frac{1}{2s} + \gamma_0 + O(s). \tag{2.44}$$

Secondly, denote $\eta_q(s) = \prod_{p|q}\left(1 - \frac{1}{p^{1+2s}}\right)$. Then $\eta_q(0) = \prod_{p|q}\left(1 - \frac{1}{p}\right) = \frac{\varphi(q)}{q}$, and by logarithmic differentiation, $\frac{\eta'_q}{\eta_q}(0) = 2\sum_{p|q} \frac{\log(p)}{p-1} = 2\theta(q)$. Hence,

$$\eta_q(s) = \eta_q(0)\left(1 + s\frac{\eta'_q(0)}{\eta_q(0)} + O(s^2)\right) \tag{2.45}$$

$$= \frac{\varphi(q)}{q}\left(1 + 2s\theta(q) + O(s^2)\right). \tag{2.46}$$

We also have that

$$q^s = 1 + s\log(q) + O(s^2) \tag{2.47}$$

16

and

$$\frac{G(s)}{s} = \frac{1}{s} + O(s) \tag{2.48}$$

since G is even and $G(0) = 1$. Lastly,

$$\frac{\gamma(1/2 + s)^2}{\gamma(1/2)^2} = 1 + 2s\gamma'(1/2) + O(s^2). \tag{2.49}$$

Combining all of these Laurent expansions, we can find the Laurent expansion of the integrand in (2.43) about $s = 0$. We can then find the coefficient of the s^{-1} term, this being the residue at $s = 0$. After shifting the contour to the line $(-1/2 + \varepsilon)$, we can trivially bound the integral by $O(q^{-1/2+\varepsilon})$, thus concluding the proof. $\qquad \square$

3. PROOF OF THEOREM 1.1.2

The focus of this section will be proving Theorem 1.1.2. The reason for passing over Theorem 1.1.1 is that the proof is very similar, except in some ways which make it simpler, as will be outlined in Section 4.1. Likewise, the proof of Theorem 1.1.3 will essentially use a subset of the tools used to prove Theorem 1.1.2, so we will only sketch a proof in Section 4.2.

3.1 Diagonal Term

Since $d^2 \preceq q$ implies that $d \prec q$, we may apply Lemma 2.3.3 to

$$\mathcal{M} = \mathcal{M}(\psi) := \sum_{\substack{\chi \pmod{d} \\ \chi \text{ even}}} |L(1/2, \chi \cdot \psi)|^2, \tag{3.1}$$

which will immediately bring us to

$$\mathcal{M} = \varphi(d) \sum_{\pm} \sum_{\substack{m \equiv \pm n \pmod{d} \\ (mn,q)=1}} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right). \tag{3.2}$$

As per the law of trichotomy, we will decompose $\mathcal{M}$ into three terms:

$$\mathcal{M} = \mathcal{M}_{m=n} + \mathcal{M}_{m>n} + \mathcal{M}_{m<n}, \tag{3.3}$$

where

$$\mathcal{M}_{m=n} := \varphi(d) \sum_{\pm} \sum_{\substack{n \equiv \pm n \pmod{d} \\ (n,q)=1}} \frac{1}{n} V\left(\frac{n^2}{q}\right), \tag{3.4}$$

$$\mathcal{M}_{m>n} = \mathcal{M}_{m>n}(\psi) := \varphi(d) \sum_{\pm} \sum_{\substack{m>n\geq 1 \\ m \equiv \pm n \pmod{d}}} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right), \tag{3.5}$$

$$\mathcal{M}_{m<n} = \mathcal{M}_{m<n}(\psi) := \varphi(d) \sum_{\pm} \sum_{\substack{n>m\geq 1 \\ m \equiv \pm n \pmod{d}}} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} V\left(\frac{mn}{q}\right). \tag{3.6}$$

Lemma 3.1.1. *Let q be a positive odd integer and d be a positive integer such that $d \preceq q$. Then*

$$\mathcal{M}_{m=n} = \frac{\varphi(d)}{2} \frac{\varphi(q)}{q} \left[\log(q) + 2(\gamma_0 + \gamma'(1/2)) + 2\theta(q)\right] + O(dq^{-1/2+\varepsilon}). \tag{3.7}$$

Proof. Observe that we cannot simultaneously have $n \equiv -n \pmod{d}$ and $(n, q) = 1$ since q is odd and d has all of the same prime factors as q. Therefore, the "diagonal term" contribution from $m = n$ gives

$$\mathcal{M}_{m=n} = \varphi(d) \sum_{(n,q)=1} \frac{1}{n} V\left(\frac{n^2}{q}\right). \tag{3.8}$$

Applying Lemma 2.3.4 will conclude the proof. □

Referring back to $\mathcal{D}$ from (1.5), note that it has just been revealed that

$$c_0 = 2(\gamma_0 + \gamma'(1/2)). \tag{3.9}$$

3.2   Remaining Setup

As for the remaining main term and error terms, let's focus on the case where $m > n$ since interchanging m and n amounts to only a conjugation in the character ψ. That is to say, we have the symmetry

$$\mathcal{M}_{m<n}(\psi) = \mathcal{M}_{m>n}\left(\overline{\psi}\right). \tag{3.10}$$

Applying a dyadic partition of unity to (3.5) results in

$$\mathcal{M}_{m>n} = \sum_{M,N \text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \sum_{\pm} \sum_{\substack{m>n\geq 1 \\ m\equiv \pm n \ (\text{mod } d)}} \psi(m)\overline{\psi}(n)W_{M,N}(m,n) \qquad (3.11)$$

where

$$W_{M,N}(m,n) = \sqrt{\frac{MN}{mn}} V\left(\frac{mn}{q}\right) \eta\left(\frac{m}{M}\right) \eta\left(\frac{n}{N}\right) \qquad (3.12)$$

with $W^{(j,k)}(m,n) \ll_{j,k} M^{-j}N^{-k}$ and $supp(W_{M,N}(m,n)) \subseteq [M, 2M] \times [N, 2N]$.

Let's first consider the terms with $m \equiv n \ (\text{mod } d)$, that is

$$\mathcal{M}^{+}_{m>n} := \sum_{M,N \text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \sum_{\substack{m>n\geq 1 \\ m\equiv n \ (\text{mod } d)}} \psi(m)\overline{\psi}(n)W_{M,N}(m,n). \qquad (3.13)$$

Thus, we may parametrize $m = n + dl$ with $l \geq 1$ since $m > n$. Applying this substitution, we denote

$$\mathcal{B}^{+}_{m>n}(M,N) := \sum_{l\geq 1}\sum_{n\geq 1} \psi(n+dl)\overline{\psi}(n)W_{M,N}(n+dl,n), \qquad (3.14)$$

so that

$$\mathcal{M}^{+}_{m>n} = \sum_{M,N \text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \mathcal{B}^{+}_{m>n}(M,N). \qquad (3.15)$$

3.3   Off-diagonal Terms When Far Apart

We will eventually split the dyadic summations of $\mathcal{M}^{+}_{m>n}$ into two ranges depending on whether M and N are nearby or far apart. These ranges will be defined in Section 3.5, but in the meantime, we will develop two different methods, each of which we will be applied in one of the two ranges. Let's start with the method which will be useful when M and N are far apart.

We wish to apply Poisson summation to the outer sum of $\mathcal{B}^{+}_{m>n}(M,N)$. However, first we will

observe some properties of the function

$$W_{n,d}^+(l) := W_{M,N}(n + dl, n), \tag{3.16}$$

namely that

$$supp(W_{n,d}^+(l)) \subseteq \left[\frac{M-n}{d}, \frac{2M-n}{d}\right] \tag{3.17}$$

and

$$W^{(j)}(l) \ll_j \left(\frac{M}{d}\right)^{-j}. \tag{3.18}$$

Once we apply Poisson summation, we will end up with the Fourier transform of this W function, so let's now record some properties of $\widehat{W}$ for future use.

Proposition 3.3.1. *Let q and A be positive integers. If $W(x)$ is a function with support in an interval of length A that satisfies $W^{(j)}(x) \ll_j A^{-j}$, then $\left|\widehat{W}(x)\right| \ll A$ and $\widehat{W}(x)$ decays rapidly for $x \gg \frac{q^\varepsilon}{A}$ and any $\varepsilon > 0$.*

These properties follow directly from integration by parts and the given hypotheses, so the proof will be omitted. Now, let's set our goal for this section.

Lemma 3.3.2. *With $\mathcal{B}_{m>n}^+(M, N)$ as defined in (3.14), we have*

$$\mathcal{B}_{m>n}^+(M, N) = \mathcal{A}_{m>n}^+(M, N) + O\left(\frac{Nq^{1/2+\varepsilon}}{d}\right), \tag{3.19}$$

where

$$\mathcal{A}_{m>n}^+(M, N) = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \, e\left(\frac{\overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right) \sum_{n|b_\psi} \widehat{W}_{n,d}^+\left(\frac{-b_\psi d}{nq}\right). \tag{3.20}$$

Proof. First, reversing the order of summation in (3.14) and applying Poisson summation in the

variable l gives us

$$\mathcal{B}^+_{m>n}(M, N) = \sum_{n \geq 1} \overline{\psi}(n) \left(\frac{d}{q} \sum_{j \in \mathbb{Z}} \sum_{u \ (\mathrm{mod} \ \frac{q}{d})} \psi(n + du)\, e\left(\frac{dju}{q} \right) \widehat{W}^+_{n,d}\left(\frac{dj}{q} \right) \right). \qquad (3.21)$$

Scaling u by n results in

$$\mathcal{B}^+_{m>n}(M, N) = \frac{d}{q} \sum_{\substack{n \geq 1 \\ (n,q)=1}} \sum_{j \in \mathbb{Z}} \widehat{W}^+_{n,d}\left(\frac{dj}{q} \right) \sum_{u \ (\mathrm{mod} \ \frac{q}{d})} \psi(1 + du)\, e\left(\frac{djnu}{q} \right), \qquad (3.22)$$

where the innermost sum can be recognized as $\mathcal{S}_{q,d}(\psi, jn)$ from (2.27). Hence, applying Lemma 2.3.1 yields

$$\mathcal{B}^+_{m>n}(M, N) = \left(\frac{-\overline{2}a_\psi}{q/d^2} \right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \sum_{\substack{n \geq 1 \\ jn \equiv -a_\psi \ (\mathrm{mod} \ d)}} \sum_{j \neq 0} \widehat{W}^+_{n,d}\left(\frac{dj}{q} \right) e\left(\frac{\overline{2a_\psi}(jn + a_\psi)^2}{q} \right), \qquad (3.23)$$

where the condition $(n, q) = 1$ is now accounted for by $jn \equiv -a_\psi \ (\mathrm{mod} \ d)$. Since $a_\psi \in \mathbb{Z}/(q/d)\mathbb{Z}$ and $d | \frac{q}{d}$, let $a_\psi \equiv b_\psi \ (\mathrm{mod} \ d)$ for $b_\psi \in \mathbb{Z}/d\mathbb{Z}$ such that

$$0 < |b_\psi| < \frac{d}{2}, \qquad (3.24)$$

where the inequalities can be strict thanks to Lemma 2.2.10. The contribution from $jn = -b_\psi$ gives us $\mathcal{A}^+_{m>n}(M, N)$, while we will use ET to denote the remaining terms. Thus, we currently have

$$|ET| \leq \frac{d}{\sqrt{q}} \sum_{\substack{n \geq 1 \\ jn \equiv -a_\psi \ (\mathrm{mod} \ d) \\ jn \neq -b_\psi}} \sum_{j \neq 0} \left| \widehat{W}^+_{n,d}\left(\frac{dj}{q} \right) \right|. \qquad (3.25)$$

Recalling (3.17) and (3.18), Proposition 3.3.1 tells us that the overall contribution to ET from $|j| > \frac{q^{1+\varepsilon}}{M}$ can easily be accounted for by $O\left(q^{-2022}\right)$. We also have that the variable n naturally satisfies $n \ll N$. Therefore, if we let $k = jn$, then the non-negligible contribution comes from $|k| \ll \frac{Nq^{1+\varepsilon}}{M}$, and for each k, the number of ways to factor $k = jn \neq 0$ is at most $k^{\varepsilon'} \leq q^\varepsilon$. Hence,

applying the bound from Proposition 3.3.1, we arrive at

$$ET \ll q^{-2022} + q^{\varepsilon}\left(\frac{M}{\sqrt{q}} \sum_{\substack{0<|k|\ll\frac{Nq^{1+\varepsilon}}{M} \\ k\equiv-a_{\psi} \pmod{d} \\ k\neq-b_{\psi}}} 1\right) \ll \frac{Nq^{1/2+\varepsilon}}{d}. \tag{3.26}$$

$\square$

At (3.13), we began considering only the terms with $m \equiv n \pmod{d}$. If we similarly define

$$\mathcal{M}^-_{m>n} := \sum_{M,N\text{dyadic}}\sum \frac{\varphi(d)}{\sqrt{MN}} \sum_{\substack{m>n\geq 1 \\ m\equiv-n \pmod{d}}} \psi(m)\overline{\psi}(n)W_{M,N}(m,n), \tag{3.27}$$

and parametrize $m = -n + dl$ so that

$$\mathcal{M}^-_{m>n} = \sum_{M,N\text{dyadic}}\sum \frac{\varphi(d)}{\sqrt{MN}}\mathcal{B}^-_{m>n}(M,N), \tag{3.28}$$

with

$$\mathcal{B}^-_{m>n}(M,N) = \sum_{l\geq 1}\sum_{n\geq 1}\psi(-n+dl)\overline{\psi}(n)W_{M,N}(-n+dl,n), \tag{3.29}$$

then it's just a matter of bookkeeping to see that a very similar result is possible. This time,

$$W^-_{n,d}(l) := W_{M,N}(-n+dl,n) \tag{3.30}$$

has the same derivative bounds and has support in an interval of the same length as $W^+_{n,d}(l)$, so Proposition 3.3.1 will have the same effect, and when we apply Lemma 2.27 this time, it will be to $\mathcal{S}_{q,d}(\psi,-jn)$. Therefore, it is without proof that we state the following lemma.

Lemma 3.3.3. *With $\mathcal{B}^-_{m>n}(M,N)$ as defined in (3.29), we have*

$$\mathcal{B}^-_{m>n}(M,N) = \mathcal{A}^-_{m>n}(M,N) + O\left(\frac{Nq^{1/2+\varepsilon}}{d}\right), \tag{3.31}$$

where

$$\mathcal{A}^-_{m>n}(M,N) = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \, e\left(\frac{\overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right) \sum_{n|b_\psi} \widehat{W}^-_{n,d}\left(\frac{b_\psi d}{nq}\right). \tag{3.32}$$

3.4   Off-diagonal Terms When Nearby

Returning to (3.14), we now wish to consider the complementary range where M and N are nearby. To do this, we'll first define the function

$$W^+_{dl}(n) := W_{M,N}(n + dl, n), \tag{3.33}$$

observing that

$$supp(W^+_{dl}(n)) \subseteq [N, 2N] \tag{3.34}$$

and

$$W^{(j)}(n) \ll_j N^{-j}. \tag{3.35}$$

Now, let's state our objective for this section.

Lemma 3.4.1. *With $\mathcal{B}^+_{m>n}(M,N)$ as defined in (3.14), we have*

$$\mathcal{B}^+_{m>n}(M,N) \ll q^\varepsilon \left(\frac{Mq^{1/2}}{d^{3/2}} + \frac{M^{1/4}N}{q^{1/4}}\right). \tag{3.36}$$

Proof. Applying Poisson summation to (3.14) in the variable n gives us

$$\mathcal{B}^+_{m>n}(M,N) = \sum_{l\geq1} \left(\frac{1}{q}\sum_{k\in\mathbb{Z}}\sum_{u \pmod q} \psi(u+dl)\overline{\psi}(u)\, e\left(\frac{ku}{q}\right) \widehat{W}^+_{dl}\left(\frac{k}{q}\right)\right) \tag{3.37}$$

$$= \frac{1}{q}\sum_{l\geq1}\sum_{k\in\mathbb{Z}} \widehat{W}^+_{dl}\left(\frac{k}{q}\right) \sum_{u \pmod q} \psi(u+dl)\overline{\psi}(u)\, e\left(\frac{ku}{q}\right). \tag{3.38}$$

Recalling (3.34) and (3.35), Proposition 3.3.1 tells us that $\left|\widehat{W}^+_{dl}(x)\right| \ll N$ and $\widehat{W}^+_{dl}(x)$ decays

rapidly for $x \gg \frac{q^\varepsilon}{N}$. Thus, the overall contribution to $\mathcal{B}^+_{m>n}(M, N)$ from $|k| > \frac{q^{1+\varepsilon}}{N}$ can be accounted for by $O\left(q^{-2022}\right)$. Applying the bound for $\widehat{W}^+_{dl}$ and restricting our summations simplifies things to

$$\mathcal{B}^+_{m>n}(M, N) \ll q^{-2022} + \frac{N}{q} \sum_{1 \le l \ll \frac{M}{d}} \sum_{0 \le |k| \ll \frac{q^{1+\varepsilon}}{N}} \left| \sum_{u \,(\mathrm{mod}\ q)} \psi(u + dl)\overline{\psi}(u)\, e\left(\frac{ku}{q}\right) \right|, \qquad (3.39)$$

since the variable l naturally satisfies $l \ll \frac{M}{d}$, just as $m \ll M$. The innermost sum can be recognized as $S(q; \psi, dl, k)$ from (2.1). Thanks to a symmetry in both l and k, we may apply Lemma 2.1.2, first extracting the $k = 0$ term and then bounding the rest. This results in

$$\mathcal{B}^+_{m>n}(M, N) \ll MNq^{-1+\varepsilon} + \frac{N}{q} \sum_{1 \le |l| \ll \frac{M}{d}} \sum_{1 \le |k| \ll \frac{q^{1+\varepsilon}}{N}} \left| \sum_{u \,(\mathrm{mod}\ q)} \psi(u + dl)\overline{\psi}(u)\, e\left(\frac{ku}{q}\right) \right| \qquad (3.40)$$

$$\ll q^\varepsilon \left(\frac{Mq^{1/2}}{d^{3/2}} + \frac{M^{1/4}N}{q^{1/4}} + \frac{MN}{q} \right). \qquad (3.41)$$

Finally, since $M \le q$, we have that $\frac{MN}{q} \le \frac{M^{1/4}N}{q^{1/4}}$, so we can drop the last term. $\qquad \square$

Out of a convenience which will become evident in Section 3.5, we'd like to include $\mathcal{A}^+_{m>n}(M, N)$ in this previous result, in spite of the fact that it does not naturally manifest using the methods of this section.

Lemma 3.4.2. *With $\mathcal{B}^+_{m>n}(M, N)$ as defined in (3.14), we have*

$$\mathcal{B}^+_{m>n}(M, N) = \mathcal{A}^+_{m>n}(M, N) + O\left(q^\varepsilon \left(\frac{Mq^{1/2}}{d^{3/2}} + \frac{M^{1/4}N}{q^{1/4}} \right) \right), \qquad (3.42)$$

where

$$\mathcal{A}^+_{m>n}(M, N) = \left(\frac{-\overline{2}a_\psi}{q/d^2} \right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}}\, e\left(\frac{\overline{2a_\psi}(a_\psi - b_\psi)^2}{q} \right) \sum_{n | b_\psi} \widehat{W}^+_{n,d}\left(\frac{-b_\psi d}{nq} \right). \qquad (3.43)$$

Proof. Notice that

$$|\mathcal{A}^+_{m>n}(M,N)| \leq \frac{d}{\sqrt{q}} \sum_{n|b_\psi} \left| \widehat{W}^+_{n,d}\left(\frac{-b_\psi d}{nq}\right) \right|.$$ (3.44)

Recalling from (3.17), (3.18), and Proposition 3.3.1 that $\left| \widehat{W}^+_{n,d}\left(\frac{-b_\psi d}{nq}\right) \right| \ll \frac{M}{d}$, we get that

$$\mathcal{A}^+_{m>n}(M,N) \ll q^\varepsilon \left(\frac{M}{q^{1/2}}\right)$$ (3.45)

since $\sigma_0(x) \ll x^\varepsilon$ and $b_\psi < q$. Finally, observe that $d^2 \preceq q$ implies $d \leq q^{1/2}$, so we will always have that $\frac{M}{q^{1/2}} < \frac{Mq^{1/2}}{d^{3/2}}$. Therefore, it is a valid choice to add $\mathcal{A}^+_{m>n}(M,N)$ to the conclusion of Lemma 3.4.1, thus concluding this proof. $\square$

Just like last time, we can follow through the proofs of Lemmas 3.4.1 and 3.4.2 accounting for the negative in $\mathcal{B}^-_{m>n}(M,N)$. We would define

$$W^-_{dl}(n) := W_{M,N}(-n+dl, n),$$ (3.46)

which has the same derivative bounds and has support in an interval of the same length as $W^+_{dl}(n)$, and we'd instead recognize $S(q;\psi, -dl, k)$ when applying Lemma 2.1.2. Therefore, we will again state this similar lemma without proof.

Lemma 3.4.3. *With $\mathcal{B}^-_{m>n}(M,N)$ as defined in (3.29), we have*

$$\mathcal{B}^-_{m>n}(M,N) = \mathcal{A}^-_{m>n}(M,N) + O\left(q^\varepsilon \left(\frac{Mq^{1/2}}{d^{3/2}} + \frac{M^{1/4}N}{q^{1/4}}\right)\right),$$ (3.47)

where

$$\mathcal{A}^-_{m>n}(M,N) = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \, e\left(\frac{\overline{2a_\psi}(a_\psi-b_\psi)^2}{q}\right) \sum_{n|b_\psi} \widehat{W}^-_{n,d}\left(\frac{b_\psi d}{nq}\right).$$ (3.48)

26

3.5 Combining $\mathcal{M}^+_{m>n}$ and $\mathcal{M}^-_{m>n}$

As defined, and as the notation should suggest,

$$\mathcal{M}_{m>n} = \sum_\pm \mathcal{M}^\pm_{m>n}. \tag{3.49}$$

Then it's also true that

$$\mathcal{M}_{m>n} = \sum_{M,N\text{dyadic}} \sum \frac{\varphi(d)}{\sqrt{MN}} \sum_\pm \mathcal{B}^\pm_{m>n}(M,N). \tag{3.50}$$

However, before discussing how $\mathcal{M}^+_{m>n}$ and $\mathcal{M}^-_{m>n}$ will combine, we will first need to focus on a smaller piece of the puzzle.

Lemma 3.5.1. *With $W^+_{n,d}(l)$ as defined in (3.16) and $W^-_{n,d}(l)$ as defined in (3.30), we have that*

$$\sum_{M,N\text{dyadic}} \frac{1}{\sqrt{MN}} \sum_{n|b_\psi} \left(\widehat{W}^+_{n,d}\left(\frac{-b_\psi d}{nq}\right) + \widehat{W}^-_{n,d}\left(\frac{b_\psi d}{nq}\right) \right) = \frac{\sqrt{q}}{d} \frac{e\left(-\frac{b_\psi}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}. \tag{3.51}$$

Proof. Let

$$\mathcal{W} = \sum_{M,N\text{dyadic}} \frac{1}{\sqrt{MN}} \sum_{n|b_\psi} \left(\widehat{W}^+_{n,d}\left(\frac{-b_\psi d}{nq}\right) + \widehat{W}^-_{n,d}\left(\frac{b_\psi d}{nq}\right) \right). \tag{3.52}$$

Retracing the definitions from (3.16) and (3.30) back to (3.12), we have

$$W^\pm_{n,d}(l) = W_{M,N}(\pm n + dl, n) = \sqrt{\frac{MN}{(\pm n + dl)n}} V\left(\frac{(\pm n + dl)n}{q}\right) \eta\left(\frac{\pm n + dl}{M}\right) \eta\left(\frac{n}{N}\right). \tag{3.53}$$

After applying the Fourier transforms and evaluating the dyadic partition of unity, we arrive at

$$\mathcal{W} = \sum_{n|b_\psi} \int_{-n/d}^\infty \frac{1}{\sqrt{(n+dy)n}} V\left(\frac{(n+dy)n}{q}\right) e\left(\frac{b_\psi dy}{nq}\right) dy$$

$$+ \sum_{n|b_\psi} \int_{n/d}^\infty \frac{1}{\sqrt{(-n+dy)n}} V\left(\frac{(-n+dy)n}{q}\right) e\left(-\frac{b_\psi dy}{nq}\right) dy. \tag{3.54}$$

27

We next scale y by n in both integrals to produce

$$\mathcal{W} = \sum_{n|b_\psi} \int_{-1/d}^{\infty} \frac{1}{\sqrt{1+dy}} V\left(\frac{n^2(1+dy)}{q}\right) e\left(\frac{b_\psi dy}{q}\right) dy$$

$$+ \sum_{n|b_\psi} \int_{1/d}^{\infty} \frac{1}{\sqrt{-1+dy}} V\left(\frac{n^2(-1+dy)}{q}\right) e\left(-\frac{b_\psi dy}{q}\right) dy. \quad (3.55)$$

We now make the substitution $qx = 1 + d \cdot y$ in the first integral and the substitution $qx = -1 + d \cdot y$ in the second, resulting in

$$\mathcal{W} = \frac{\sqrt{q}}{d} e\left(\frac{-b_\psi}{q}\right) \sum_{n|b_\psi} \int_0^{\infty} V(n^2 x)\left(e\left(b_\psi x\right) + e\left(-b_\psi x\right)\right) \frac{dx}{\sqrt{x}}. \quad (3.56)$$

Applying the definition of $V(x)$ (found in (2.35)) and reversing the order of integration gives

$$\mathcal{W} = \frac{\sqrt{q}}{d} e\left(\frac{-b_\psi}{q}\right) \sum_{n|b_\psi} \int_0^{\infty} \left(\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \frac{\gamma(1/2+s)^2}{\gamma(1/2)^2} (n^2 x)^{-s} ds\right) \left(e\left(b_\psi x\right) + e\left(-b_\psi x\right)\right) \frac{dx}{\sqrt{x}}$$

$$(3.57)$$

$$= \frac{\sqrt{q}}{2\pi i d} e\left(\frac{-b_\psi}{q}\right) \sum_{n|b_\psi} \int_{(1)} \frac{G(s)}{s} \frac{\gamma(1/2+s)^2}{\gamma(1/2)^2} n^{-2s} \left(\int_0^{\infty} x^{-s}\left(e\left(b_\psi x\right) + e\left(-b_\psi x\right)\right) \frac{dx}{\sqrt{x}}\right) ds.$$

$$(3.58)$$

We can shift the contour of the outer integral to the line $(1/4)$ so that $-\frac{1}{2} < \Re(s) < \frac{1}{2}$, allowing us to apply Corollary 2.1.6 and resulting in

$$\mathcal{W} = \frac{\sqrt{q}}{\pi i d} e\left(\frac{-b_\psi}{q}\right) \sum_{n|b_\psi} \int_{(1/4)} \frac{G(s)}{s} \frac{\gamma(1/2+s)^2}{\gamma(1/2)^2} n^{-2s} \frac{\Gamma(1/2-s)}{(2\pi|b_\psi|)^{1/2-s}} \cos\left(\frac{\pi}{2}\left(\frac{1}{2}-s\right)\right) ds.$$

$$(3.59)$$

Applying the definition of $\gamma(s)$ (found in (2.36)) and rearranging some terms will produce

$$\mathcal{W} = \frac{\sqrt{q} e\left(\frac{-b_\psi}{q}\right)}{\pi i d \sqrt{2\pi|b_\psi|}\Gamma(1/4)^2} \int_{(1/4)} G(s)\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)^2 \Gamma(1/2-s) \cos\left(\frac{\pi}{4}-\frac{\pi s}{2}\right) \sum_{n|b_\psi} \left(\frac{2|b_\psi|}{n^2}\right)^s \frac{ds}{s}.$$

$$(3.60)$$

28

Using the formulas

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z-1/2} \Gamma(z)\Gamma(z+1/2) \tag{3.61}$$

for $2z = \frac{1}{2} - s$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{3.62}$$

for $z = \frac{3}{4} - \frac{s}{2}$, and

$$\cos(z) = \sin(z + \pi/2) \tag{3.63}$$

for $z = \frac{\pi}{4} - \frac{\pi s}{2}$, it can be shown that this integrand is an even function, thanks as well to a symmetry in the pairs of divisors of b_ψ. Therefore, twice the value of this integral is the same as $2\pi i$ times the residue at $s = 0$. Therefore, since $G(0) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, this concludes the proof. $\square$

Using Lemmas 3.3.2, 3.3.3, 3.4.2, and 3.4.3, we will now begin to combine our cases.

Lemma 3.5.2. *With $\mathcal{M}_{m>n}(\psi)$ as defined in (3.5), we have*

$$\mathcal{M}_{m>n}(\psi) = \mathcal{A}_{m>n}(\psi) + O\left(q^\varepsilon \left(\frac{q^{1/2}}{d^{1/4}} + \frac{d}{q^{1/8}}\right)\right), \tag{3.64}$$

where

$$\mathcal{A}_{m>n}(\psi) = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2}\ \varphi(d)\ \frac{e\left(\frac{\overline{2a_\psi}(a_\psi - b_\psi)^2 - b_\psi}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}. \tag{3.65}$$

Proof. We will begin by following through on something that was previously alluded to, that is splitting the dyadic summations of $\mathcal{M}_{m>n}$ into two ranges depending on whether M and N are nearby or far apart. We will now reveal that the cutoff for these two ranges is $M = d^{1/2}N$. Therefore, starting at (3.50), we have

$$\mathcal{M}_{m>n} = \sum_{\substack{M,N\text{dyadic} \\ M \geq d^{1/2}N}} \frac{\varphi(d)}{\sqrt{MN}} \sum_{\pm} \mathcal{B}^\pm_{m>n}(M,N) + \sum_{\substack{M,N\text{dyadic} \\ M < d^{1/2}N}} \frac{\varphi(d)}{\sqrt{MN}} \sum_{\pm} \mathcal{B}^\pm_{m>n}(M,N). \tag{3.66}$$

Since we are in the case where $m > n$, the first term is when M and N are far apart, so we will apply Lemmas 3.3.2 and 3.3.3, and the second term is when M and N are nearby, so we will apply

Lemmas 3.4.2 and 3.4.3. Doing so will result in

$$\mathcal{M}_{m>n} = \sum_{\substack{M,N\text{dyadic} \\ M \geq d^{1/2}N}} \frac{\varphi(d)}{\sqrt{MN}} \left\{ \sum_{\pm} \mathcal{A}^{\pm}_{m>n}(M,N) + O\left(\frac{Nq^{1/2+\varepsilon}}{d}\right) \right\}$$

$$+ \sum_{\substack{M,N\text{dyadic} \\ M < d^{1/2}N}} \frac{\varphi(d)}{\sqrt{MN}} \left\{ \sum_{\pm} \mathcal{A}^{\pm}_{m>n}(M,N) + O\left(q^{\varepsilon}\left(\frac{Mq^{1/2}}{d^{3/2}} + \frac{M^{1/4}N}{q^{1/4}}\right)\right) \right\}. \quad (3.67)$$

Distributing into each pair of curly brackets, it becomes clear that we can reassemble the dyadic sums in the first terms while keeping them split in the second, yielding

$$\mathcal{M}_{m>n} = \mathcal{A}_{m>n} + \sum_{\substack{M,N\text{dyadic} \\ M \geq d^{1/2}N}} O\left(\frac{N^{1/2}q^{1/2+\varepsilon}}{M^{1/2}}\right) + \sum_{\substack{M,N\text{dyadic} \\ M < d^{1/2}N}} O\left(q^{\varepsilon}\left(\frac{M^{1/2}q^{1/2}}{N^{1/2}d^{1/2}} + \frac{N^{1/2}d}{M^{1/4}q^{1/4}}\right)\right),$$

$$(3.68)$$

where

$$\mathcal{A}_{m>n} = \sum_{M,N\text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \sum_{\pm} \mathcal{A}^{\pm}_{m>n}(M,N). \quad (3.69)$$

It remains to be shown that $\mathcal{A}_{m>n} = \mathcal{A}_{m>n}(\psi)$, but first we will choose the summands which respectively maximize each of the dyadic sums in (3.68), giving us

$$\mathcal{M}_{m>n} = \mathcal{A}_{m>n} + O\left(q^{\varepsilon}\left(\frac{q^{1/2}}{d^{1/4}} + \frac{d}{q^{1/8}}\right)\right), \quad (3.70)$$

recalling for the last term that $m > n$ and $MN \ll q$. Now, focusing on $\mathcal{A}_{m>n}$, we currently have

$$\mathcal{A}_{m>n} = \sum_{M,N\text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \left(\frac{-\overline{2a_{\psi}}}{q/d^2}\right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \, e\left(\frac{\overline{2a_{\psi}}(a_{\psi} - b_{\psi})^2}{q}\right) \sum_{n|b_{\psi}} \widehat{W}^{+}_{n,d}\left(\frac{-b_{\psi}d}{nq}\right)$$

$$+ \sum_{M,N\text{dyadic}} \frac{\varphi(d)}{\sqrt{MN}} \left(\frac{-\overline{2a_{\psi}}}{q/d^2}\right) \varepsilon_{q/d^2} \frac{d}{\sqrt{q}} \, e\left(\frac{\overline{2a_{\psi}}(a_{\psi} - b_{\psi})^2}{q}\right) \sum_{n|b_{\psi}} \widehat{W}^{-}_{n,d}\left(\frac{b_{\psi}d}{nq}\right). \quad (3.71)$$

Therefore, applying Lemma 3.5.1 will show that $\mathcal{A}_{m>n} = \mathcal{A}_{m>n}(\psi)$, thus concluding the proof.

$\square$

Recall the symmetry described in (3.10). By this simple fact, we get the following lemma for free.

Lemma 3.5.3. *With $\mathcal{M}_{m<n}(\psi)$ as defined in (3.6), we have*

$$\mathcal{M}_{m<n}(\psi) = \mathcal{A}_{m<n}(\psi) + O\left(q^\varepsilon\left(\frac{q^{1/2}}{d^{1/4}} + \frac{d}{q^{1/8}}\right)\right), \tag{3.72}$$

where

$$\mathcal{A}_{m<n}(\psi) = \left(\frac{-\overline{2}a_{\overline{\psi}}}{q/d^2}\right)\varepsilon_{q/d^2}\ \varphi(d)\ \frac{e\left(\frac{\overline{2a_{\overline{\psi}}}(a_{\overline{\psi}} - b_{\overline{\psi}})^2 - b_{\overline{\psi}}}{q}\right)\sigma_0(|b_{\overline{\psi}}|)}{2}\frac{\sigma_0(|b_{\overline{\psi}}|)}{\sqrt{|b_{\overline{\psi}}|}}. \tag{3.73}$$

3.6 Combining $\mathcal{M}_{m>n}$ and $\mathcal{M}_{m<n}$

From Lemma 2.2.9, it follows that the general relationship between a_ψ and $a_{\overline{\psi}}$ is that

$$a_{\overline{\psi}} \equiv -a_\psi \ (\mathrm{mod}\ q/d). \tag{3.74}$$

However, recalling the range of a_ψ fixed back in (1.3), we will actually have

$$a_{\overline{\psi}} = -a_\psi. \tag{3.75}$$

These same properties will be inherited by b_ψ thanks to the range fixed in (3.24), so crucially,

$$b_{\overline{\psi}} = -b_\psi. \tag{3.76}$$

Thus, from Lemmas 3.5.2 and 3.5.3, we get that

$$\mathcal{M}_{m>n} + \mathcal{M}_{m<n} = \mathcal{A}' + O\left(q^\varepsilon\left(\frac{q^{1/2}}{d^{1/4}} + \frac{d}{q^{1/8}}\right)\right), \tag{3.77}$$

31

where

$$\mathcal{A}' = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2} \; \varphi(d) \; \frac{e\left(\frac{\overline{2a_\psi}(a_\psi - b_\psi)^2 - b_\psi}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}} + \left(\frac{\overline{2}a_\psi}{q/d^2}\right) \varepsilon_{q/d^2} \; \varphi(d) \; \frac{e\left(\frac{b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}.$$
(3.78)

Since any odd square must be $1 \pmod 4$, $q \equiv q/d^2 \pmod 4$, so

$$\varepsilon_q = \varepsilon_{q/d^2}.$$
(3.79)

On the other hand, the Jacobi symbol $\left(\frac{\cdot}{p}\right)$ is multiplicative, and $\left(\frac{-1}{p}\right) = \varepsilon_p^2$, so

$$\left(\frac{-\overline{2}a_\psi}{q/d^2}\right) = \left(\frac{-1}{q/d^2}\right)\left(\frac{\overline{2}a_\psi}{q/d^2}\right) = \varepsilon_{q/d^2}^2\left(\frac{\overline{2}a_\psi}{q/d^2}\right) = \varepsilon_q^2\left(\frac{\overline{2}a_\psi}{q/d^2}\right).$$
(3.80)

Therefore, if $q \equiv 1 \pmod 4$, then

$$\mathcal{A}' = \left(\frac{\overline{2}a_\psi}{q/d^2}\right) \varphi(d) \; \frac{e\left(\frac{b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right) + e\left(-\frac{b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}$$
(3.81)

$$= \left(\frac{\overline{2}a_\psi}{q/d^2}\right) \varphi(d) \cos\left(\frac{2\pi(b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2)}{q}\right) \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}.$$
(3.82)

Referring back to (1.9), this shows that

$$c_1 = \left(\frac{\overline{2}a_\psi}{q/d^2}\right).$$
(3.83)

If instead $q \equiv 3 \pmod 4$, then

$$\mathcal{A}' = \left(\frac{-\overline{2}a_\psi}{q/d^2}\right) \varphi(d) \; \frac{ie\left(\frac{b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right) - ie\left(-\frac{b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2}{q}\right)}{2} \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}$$
(3.84)

$$= \left(\frac{\overline{2}a_\psi}{q/d^2}\right) \varphi(d) \sin\left(\frac{2\pi(b_\psi - \overline{2a_\psi}(a_\psi - b_\psi)^2)}{q}\right) \frac{\sigma_0(|b_\psi|)}{\sqrt{|b_\psi|}}.$$
(3.85)

Referring again back to (1.9), this shows that

$$c_2 = \left(\frac{\overline{2}a_\psi}{q/d^2} \right).$$

(3.86)

Thus, the $\mathcal{A}'$ we found here perfectly matches the $\mathcal{A}'$ in (1.9). Therefore, combining this with Lemma 3.1.1, we have proved Theorem 1.1.2.

4. SKETCHING THE PROOFS OF REMAINING THEOREMS

4.1 A Sketch of the Proof of Theorem 1.1.1

In the proof of Theorem 1.1.1, the earliest difference in comparison to the proof of Theorem 1.1.2 is in Section 3.3. The condition that $q \preceq d^2$ means that, after applying the Postnikov formula from Lemma 2.2.9 in Lemma 2.3.1, we can use statement (1) of Lemma 2.2.8 rather than statement (2). This will eliminate the need for Lemma 2.1.4, and we will instead get that $jn \equiv -a_\psi \pmod{q/d}$. Thus, since $\frac{q}{d} \leq d$ in this theorem, there is no need to define b_ψ. From here on out, the answer will look a little different, but the methods are the same, if not simpler. The new cutoff in the proof of Lemma 3.5.2 is $Mq^{1/2} = Nd^{3/2}$, and $\mathcal{M}_{m>n}$ combines with $\mathcal{M}_{m<n}$ without breaking into cases. This concludes the sketch of this proof.

4.2 A Sketch of the Proof of Theorem 1.1.3

Because Theorem 1.1.3 does not fix the parity of ψ and sums over all χ from the beginning, we would never need to apply the trick in (2.40). Thus, we would not have the " $\pm$ " to worry about throughout the proof, though the function $V(x)$ in Lemma 2.3.2 would now depend on the parity of $\chi \cdot \psi$. Lemma 3.1.1 still holds, but since we only care about bounding the second moment, a corollary would be that

$$\mathcal{M}_{m=n} \ll dq^\varepsilon. \tag{4.1}$$

From Lemma 3.4.1, we know how to bound $\mathcal{B}^+_{m>n}(M, N)$ and thus $\mathcal{M}^+_{m>n}(\psi)$ in (3.15), that is

$$\mathcal{M}^+_{m>n}(\psi) = q^\varepsilon \sum_{M,N\text{dyadic}} \sum O\left(\frac{M^{1/2}q^{1/2}}{N^{1/2}d^{1/2}} + \frac{N^{1/2}d}{M^{1/4}q^{1/4}}\right). \tag{4.2}$$

This bound performs best when M and N are balanced, so in the proof of Theorem 1.1.3, we can decide in the beginning to only apply one dyadic partition of unity and choose the summand which maximizes that dyadic sum, which amounts to letting $M = N = q^{1/2}$ here. Therefore, since we

won't have separate cases for " $\pm$ " in this proof, we have

$$\mathcal{M}_{m>n}(\psi) \ll q^{\varepsilon} \left(\frac{q^{1/2}}{d^{1/2}} + \frac{d}{q^{1/8}} \right), \tag{4.3}$$

which when combined with (4.1) and the symmetry (3.10), concludes the sketch of this proof.

REFERENCES

[1] D. R. Heath-Brown, "The twelfth power moment of the riemann-function," *Quart. J. Math.*, vol. 29, no. 4, p. 443–462, 1978.

[2] R. M. Nunes, "The twelfth moment of dirichlet l-functions with smooth moduli," *Int. Math. Res. Not.*, vol. 2021, no. 12, p. 9180–9202, 2021.

[3] D. Milićević and D. White, "Twelfth moment of dirichlet l-functions to prime power moduli," *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, vol. 22, no. 4, p. 1879–1898, 2021.

[4] D. R. Heath-Brown, "Hybrid bounds for dirichlet l-functions," *Invent. Math.*, vol. 47, no. 2, pp. 149–170, 1978.

[5] H. Iwaniec and E. Kowalski, *Analytic number theory*. Providence, RI: American Mathematical Society, 2004.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. San Diego, CA: Academic Press, Inc., sixth ed., 2000.

[7] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*. New York, NY: Springer-Verlag, second ed., 1984.