UNIFORM ACCELERATION RADIATION WITH A PROCA FIELD

A Dissertation

by

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Submitted to the Graduate and Professional School of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2022

Major Subject: Mathematics

ABSTRACT

We conduct a theoretical investigation of radiation from a uniformly accelerating particle coupled to a Proca field. The fact that a Proca field allows non-conservation of charge facilitates the use of regularizations where the charge changes in time. Using several classical methods, we obtain expressions for the field, and expressions for the rate of emitted energy and particle number. We also obtain general expressions for the quantity of radiation when the charge is an arbitrary function of time, both when the charge is at rest, and when it is uniformly accelerating.

Separately, in the context of the longstanding puzzle of the equivalence principle and radiation, we carry out an analysis of electromagnetic radiation in the case where observer and charge both have distinct accelerations. We confirm the result of Hirayama and others, that the rate of radiation is proportional to the square of the difference of the accelerations.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Stephen Fulling (advisor), Igor Zelenko, and Dean Baskin of the Department of Mathematics, and Katrin Becker of the Department of Physics & Astronomy.

Collaborative discussion was provided by Andre G. S. Landulfo of Federal University of ABC, George E. A. Matsas of São Paulo State University, Kimball A. Milton of the University of Oklahoma, Gerard Kennedy of the University of Southampton, and Timothy Bates, student at Texas A&M.

All other work conducted for the dissertation was completed by the student independently.

Funding Sources

Graduate study was supported by a teaching assistantship from Texas A&M University.

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1. INTRODUCTION

A uniformly accelerated charged particle emits radiation, in accordance with the Larmor formula $P = \frac{2}{3}\frac{q^2a^2}{4\pi}$. However, historically and presently, several aspects of this situation have been found to be puzzling.

- The Abraham-Lorentz expression for the radiation reaction force, $\frac{2}{3}\frac{q^2}{4\pi}(\dot{a}^\mu - v^\mu a_\nu a^\nu)$ vanishes identically for a uniformly accelerated charge, leading to the conclusion that there should be no radiation, in contradiction to the Larmor formula.

- By the equivalence principle, a co-accelerated observer would interpret a uniformly accelerated charge as being at rest in a uniform gravitational field. Such an observer would not expect to see radiation.

- There is uncertainty about the meaning of radiation and its locality.

While ways of resolving these and other issues have been proposed, they remain controversial, and efforts at clarification are ongoing by many authors. The purpose of this dissertation is to provide an additional tool for these efforts by analyzing a variation of the problem, in which one supposes the field has a nonzero mass, namely a Proca field. The main advantage of a Proca field, for this application, is that its source need not satisfy conservation of charge. This facilitates several regularization/limiting procedures, including allowing the charge to oscillate with nonzero frequency, or to appear and disappear after a finite interval of time.

This dissertation is organized as follows. Section 2.1 provides historical background to the problem. Sections 2.2 - 2.4 define and explain the system being studied, namely a Proca field with a uniformly accelerating source. The remaining sections of chapter 2 each outline a method of calculating and quantifying the field and its radiation.

Sections 3.1 - 3.3 provide the specific results of the methods introduced in the previous chapter. Sections 3.4 - 3.6 introduce variations on those methods which give further results quantifying radiation.

Chapter 4 discusses the standalone topic of the generalization of the Larmor formula for nonzero observer acceleration, and its relationship to the equivalence principle. It does not consider the Proca field, but only the usual electromagnetic field. This chapter serves as an illustration of some of the issues of acceleration and radiation.

2. BACKGROUND AND METHODS

The issue of radiation from a uniformly accelerated charge has been investigated by numerous authors. Principal sources for this dissertation include Ren and Weinberg [1], and Landulfo, Fulling, and Matsas [2], who consider a massless scalar field; Boulware [3], and Higuchi, Matsas, and Sudarsky [4, 5], who consider the electromagnetic field; and Castiñeiras et al. [6], who consider the Proca field, although with a different emphasis from this dissertation.

2.1 Historical highlights

The electromagnetic field of a uniformly accelerating charge is explicitly known, and the first to calculate it was Born [7] in 1909. In 1955, Bondi and Gold [8] updated this solution to account for the singularity on the boundary of the particle's causal future. In 1920, Pauli [9] argued that such a charge emits no radiation, based on the fact that the magnetic field vanishes at $t = 0$. The view that a uniformly accelerating charge does not radiate was further popularized by Feynman [10] in 1962, evoking the (mistaken) idea that the radiated power should equal the work done by the Abraham-Lorentz force. Fulton and Rohrlich [11] in 1960, and later papers by Rohrlich [12, 13], performed the calculations showing that the uniformly accelerated charge does radiate for inertial observers, but not for co-accelerated observers, making apparent that the notion of radiation is observer-dependent. Other contributions to the discussion of uniform acceleration and the equivalence principle include Kovetz and Tauber [14], Ginzburg [15], and Pauri and Vallisneri [16]. Boulware's contribution [3] in 1980 is also significant for emphasizing the role of horizons for accelerated observers. In 2010, Rowland [17] clarified the question of energy conservation and the relationship to the Abraham-Lorentz force, showing that radiated power is not the work done by the radiation reaction, but comes from energy stored in the field. Finally for completeness it is important to mention the generalization of the Abraham-Lorentz force to curved spacetime, given by DeWitt and Brehme [18] and Hobbs [19], as well as the generalization of the Larmor formula to gravitational fields and general accelerated observers, given by Kretzschmar and Fugmann [20, 21] and Hirayama [22, 23].

2.2 Proca field

Here and throughout, this dissertation uses metric signature $-, +, +, +$ and units where $c = \hbar = 1$. The Proca equation for a vector field A^μ, with mass m and source J^μ, is

$$-\nabla_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu \tag{2.1}$$

where $F^{\mu\nu}$ is the field strength tensor given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{2.2}$$

Taking the divergence of the Proca equation yields the divergence constraint

$$m^2 \nabla_\mu A^\mu = \nabla_\mu J^\mu \tag{2.3}$$

Notice that, in contrast to the electromagnetic field, a Proca field is permitted to have non-conserved source. Equation (2.1) is equivalent to the vector Klein-Gordon equation

$$(-\nabla^2 + m^2) A^\nu = \tilde{J}^\nu \tag{2.4}$$

in conjunction with the constraint (2.3). Note that there is an altered source

$$\tilde{J}^\nu = J^\nu - \frac{1}{m^2} \nabla^\nu \nabla_\mu J^\mu \tag{2.5}$$

Equation (2.4) by itself implies that

$$(-\nabla^2 + m^2)(m^2 \nabla_\nu A^\nu - \nabla_\nu J^\nu) = 0 \tag{2.6}$$

Thus, by the uniqueness of solutions to hyperbolic equations, if the divergence constraint is satisfied on a space-like hypersurface, then it is satisfied throughout spacetime. In particular, for a source compactly supported in time, the retarded solution to (2.4) automatically satisfies the divergence constraint.

Alternatively to modifying the source, if V^μ is a solution to the vector Klein-Gordon equation (2.4) with right-hand side J^μ, then

$$A^\mu = V^\nu - \frac{1}{m^2}\nabla^\nu\nabla_\mu V^\mu \tag{2.7}$$

is a solution to the Proca equation with the same source J^μ.

Let Σ be a space-like hypersurface with future-directed unit normal vector n^μ. Solutions of the homogeneous Proca equation carry an (indefinite) inner product

$$\langle A_1, A_2\rangle = i\int_\Sigma n_\nu(\overline{A}_{1\,\mu}F_2^{\nu\mu} - A_{2\,\mu}\overline{F}_1^{\nu\mu})\,d\sigma \tag{2.8}$$

where $d\sigma$ is the measure on Σ with respect to its Riemannian metric. This expression is independent of the choice of Σ because the vector field in parentheses has divergence zero.

Inevitably one is interested in taking the zero mass limit of the Proca field in order to shed light on the electromagnetic case. The mathematical details of the zero mass limit have been worked out in [24]. The upshot is that the zero mass limit of a Proca solution is the corresponding solution to Maxwell's equations, provided that the source is conserved and the observable in question is gauge-invariant.

2.3 Rindler coordinates

In special relativity, the notion of (linear) uniform acceleration means having proper acceleration a^μ of constant magnitude $a = \sqrt{a^\mu a_\mu}$, and constant spatial direction parallel to the velocity.

The resulting motion is hyperbolic, shown in figure 2.1.

$$z(t) = \sqrt{t^2 + \frac{1}{a^2}} \tag{2.9}$$

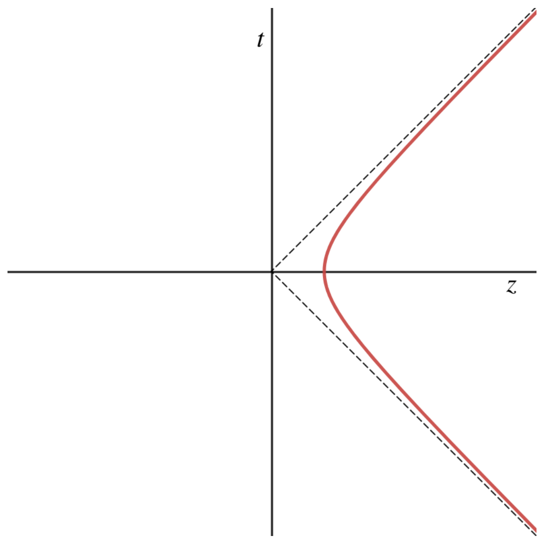

Figure 2.1: The trajectory of a particle (red) in uniform acceleration

In flat spacetime, we consider coordinates (τ, ξ, x, y) that are adapted to a family of uniformly accelerated observers. They are related to the Cartesian coordinates by

$$t = \frac{e^{a\xi}}{a} \sinh(a\tau) \qquad z = \frac{e^{a\xi}}{a} \cosh(a\tau) \tag{2.10}$$

The coordinate basis transformations are

$$\partial_\tau = a(z\partial_t + t\partial_z) \qquad \partial_\xi = a(t\partial_t + z\partial_z) \tag{2.11}$$

$$\partial_t = ae^{-2a\xi}(z\partial_\tau - t\partial_\xi) \qquad \partial_z = ae^{-2a\xi}(z\partial_\xi - t\partial_\tau) \tag{2.12}$$

$$d\tau = ae^{-2a\xi}(zdt - tdz) \qquad d\xi = ae^{-2a\xi}(zdz - tdt) \tag{2.13}$$

$$dt = a(td\xi + zd\tau) \qquad dz = a(zd\xi + td\tau) \tag{2.14}$$

These coordinates cover the wedge of spacetime $z > |t|$ (the Rindler wedge). We will also sometimes employ the auxiliary coordinate $\chi = e^{a\xi}$. The Rindler coordinate system is shown in figure 2.2.

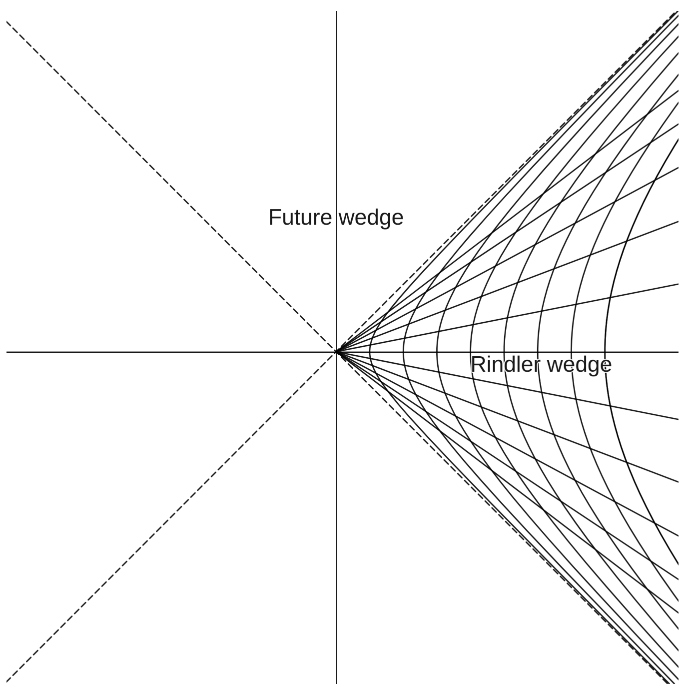

Figure 2.2: The Rindler coordinate system in the z-t plane

In these coordinates, the metric takes the form

$$g = e^{2a\xi}(-d\tau^2 + d\xi^2) + dx^2 + dy^2 \tag{2.15}$$

Translation in the τ direction is a time-translation symmetry of the Rindler spacetime. The orbits of constant x, y, ξ form a family of worldlines all having uniform acceleration. Those located at $\xi = 0$ are privileged in that their proper time coincides with their τ coordinate, and their proper acceleration is a. These coordinates are adapted to the symmetries of a uniformly accelerated source. They also capture the appropriate notion of a "co-accelerated" reference frame. τ-stationary observers are rigid with respect to each other, and their natural notions of time translation and simultaneity are those associated with the τ coordinate. Note that this notion of co-accelerated frames is not one in which the observers have equal proper acceleration. The observers have different proper acceleration, but they remain a constant proper distance from each other. The distinction is the same as in the well-known Bell's spaceship paradox [25] concerning two spaceships with a string tied between them.

The Klein-Gordon equation in Rindler coordinates takes the form

$$e^{-2a\xi}(\partial_\tau^2\phi - \partial_\xi^2\phi) - \partial_x^2\phi - \partial_y^2\phi + m^2\phi = 0 \tag{2.16}$$

A family of positive τ-frequency solutions is given by the Rindler modes, indexed by frequency $\omega \geq 0$ and transverse momentum $\mathbf{k}_\perp = (k_x, k_y)$. $k_\perp = \sqrt{k_x^2 + k_y^2}$, and $\mathbf{x}_\perp = (x, y)$.

$$\phi^{\omega,\mathbf{k}_\perp} = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} K_{i\omega/a}\left(\sqrt{k_\perp^2 + m^2}\frac{e^{a\xi}}{a}\right) e^{i(\mathbf{k}_\perp\cdot\mathbf{x}_\perp - \omega\tau)} \tag{2.17}$$

These solutions are orthogonal and normalized in the sense determined by equation (2.26) below.

2.4 The source

We are interested in a uniformly accelerated charge. In Rindler coordinates, this corresponds to a source

$$J = q\,\delta(\xi)\delta(x)\delta(y)\,\partial_\tau \tag{2.18}$$

where ∂_τ is the τ coordinate basis vector. For some parts of the calculation it will be helpful to regularize this source by letting the charge be a function of time, $q(\tau)$. Because the radiation is proportional to q^2, for any regularization it is appropriate to renormalize by a factor of $\sqrt{\int q^2\,d\tau}$. For example, much of the following uses the regularization where the charge oscillates with frequency E, and/or appears and disappears outside a finite time interval. Expressly,

$$J = \begin{cases} \sqrt{2}\,q\,\cos(E\tau)\delta(\xi)\delta(x)\delta(y)\,\partial_\tau, & |\tau| < T \\ 0 & \text{else} \end{cases} \tag{2.19}$$

The regularization involving oscillation is discussed in [5].

2.5 The method of Unruh modes

This section considers Rindler modes and Unruh modes as solutions of the homogeneous Proca equation, normalized with respect to the inner product (2.8). The classical solution can be expanded in these modes. The method is the same as that in [2]. There are three independent polarizations for the Proca field, worked out in detail in [6]. These modes, labeled $\mathrm{I}, \mathrm{II}, \mathrm{III}$, are defined in the Rindler wedge, and are expressed here in the $d\tau, d\xi, dx, dy$ dual basis. Letting $\rho = \sqrt{k_\perp^2 + m^2}$,

$$V_\mu^{R,\mathrm{I},\omega,\mathbf{k}_\perp} = \frac{1}{k_\perp}(0,\ 0,\ k_y\phi^{\omega,\mathbf{k}_\perp},\ -k_x\phi^{\omega,\mathbf{k}_\perp}) \tag{2.20}$$

$$V_\mu^{R,\mathrm{II},\omega,\mathbf{k}_\perp} = \frac{1}{\rho}(\partial_\xi\phi^{\omega,\mathbf{k}_\perp},\ -i\omega\phi^{\omega,\mathbf{k}_\perp},\ 0,\ 0) \tag{2.21}$$

$$V_\mu^{R,\mathrm{III},\omega,\mathbf{k}_\perp} = \frac{1}{m}\left(\frac{-i\omega k_\perp}{\rho}\phi^{\omega,\mathbf{k}_\perp},\ \frac{k_\perp}{\rho}\partial_\xi\phi^{\omega,\mathbf{k}_\perp},\ \frac{ik_x\rho}{k_\perp}\phi^{\omega,\mathbf{k}_\perp},\ \frac{ik_y\rho}{k_\perp}\phi^{\omega,\mathbf{k}_\perp}\right) \tag{2.22}$$

where $\phi^{\omega,\mathbf{k}_\perp}$ are the functions given in (2.17). There are also Rindler modes in the left wedge given by

$$V_\mu^{L,\lambda,\omega,\mathbf{k}_\perp}(t,z,x,y) = \overline{V_\mu^{R,\lambda,\omega,\mathbf{k}_\perp}(-t,-z,x,y)} \tag{2.23}$$

The Unruh modes [26] are linear combinations of V_μ^L and V_μ^R, which are positive frequency in the Minkowski sense:

$$W_\mu^{1,\lambda,\omega,\mathbf{k}_\perp} = \frac{V_\mu^{R,\lambda,\omega,\mathbf{k}_\perp} + e^{-\pi\omega/a}\,\overline{V_\mu^{L,\lambda,\omega,-\mathbf{k}_\perp}}}{\sqrt{1 - e^{-2\pi\omega/a}}} \tag{2.24}$$

$$W_\mu^{2,\lambda,\omega,\mathbf{k}_\perp} = \frac{V_\mu^{L,\lambda,\omega,\mathbf{k}_\perp} + e^{-\pi\omega/a}\,\overline{V_\mu^{R,\lambda,\omega,-\mathbf{k}_\perp}}}{\sqrt{1-e^{-2\pi\omega/a}}} \tag{2.25}$$

They are orthonormal in the sense of inner product (2.8):

$$\langle W^{\sigma,\lambda,\omega,\mathbf{k}_\perp},\, W^{\sigma',\lambda',\omega',\mathbf{k}'_\perp}\rangle = \delta_{\sigma\sigma'}\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp) \tag{2.26}$$

One may calculate the retarded solution by expanding in Unruh modes. This will make use of the so-called causal propagator $E = A - R$, the difference of the advanced and retarded propagators. Thus in the future wedge $t > |z|$, $R(\tilde{J}) = -E(\tilde{J})$. So we have

$$R(\tilde{J}) = -\sum_{\sigma,\lambda}\int_0^\infty d\omega \int d^2k_\perp \langle W^{\sigma,\lambda,\omega,\mathbf{k}_\perp}, E(\tilde{J})\rangle\, W^{\sigma,\lambda,\omega,\mathbf{k}_\perp} + \text{c.c.} \tag{2.27}$$

Furthermore, for any solution f of the homogeneous Klein-Gordon equation, there is the identity

$$\langle f, E(\tilde{J})\rangle = i\int \bar{f}\tilde{J}\, d^4x \tag{2.28}$$

so we may calculate the expansion coefficients as

$$i\int \overline{W_\mu^{\sigma,\lambda,\omega,\mathbf{k}_\perp}}\,\tilde{J}^\mu\, d^4x \tag{2.29}$$

Here follows a derivation of identity (2.28). Let f be a solution of the homogeneous Klein-Gordon equation, and let J be compactly supported. Let Σ_1 and Σ_2 be spacelike hypersurfaces in the past and future of the support of J.

$$\int fJ\, d^4x = \frac{1}{2}\int f(-\nabla^2 + m^2)(A+R)J\, d^4x \tag{2.30}$$

This may be integrated by parts twice, moving the derivatives to f, at the cost of boundary terms on Σ_1 and Σ_2:

$$
\begin{aligned}
= \frac{1}{2}\Bigg(\int_{\Sigma_2} n_2^\mu f\,(-\nabla_\mu RJ)\; d\sigma_2 - \int_{\Sigma_1} n_1^\mu f\,(-\nabla_\mu AJ)\; d\sigma_1 \\
- \int_{\Sigma_2} n_2^\mu (-\nabla_\mu f) RJ\; d\sigma_2 + \int_{\Sigma_1} n_1^\mu (-\nabla_\mu f) AJ\; d\sigma_1 \\
+ \int ((-\nabla^2 + m^2) f)(A+R) J\; d^4x \Bigg)
\end{aligned}
\tag{2.31}
$$

The last integral vanishes because f is a solution. Rearranging,

$$
\begin{aligned}
&= \frac{1}{2}\left(\int_{\Sigma_1} n_1^\mu (f \nabla_\mu AJ - AJ \nabla_\mu f)\; d\sigma_1 - \int_{\Sigma_2} n_2^\mu (f \nabla_\mu RJ - RJ \nabla_\mu f)\; d\sigma_2 \right) \\
&= \langle f, E(J) \rangle
\end{aligned}
\tag{2.32}
$$

2.6 The method of the retarded propagator

Another approach to calculating the retarded solution is to directly integrate the source against the retarded propagator, as in [3]. We will take $E = 0$ so the source is conserved. If we have a conserved source following a worldline $r(\lambda)$ with velocity vector $v^\mu(\lambda)$, then we may compute the retarded solution as

$$
A^\mu(x) = q \int v^\mu(\lambda) R(x, r(\lambda))\; d\lambda \tag{2.33}
$$

where

$$
R(x,y) = \frac{1}{2\pi} \theta(x^0 - y^0) \left(\delta(-s^2) - \theta(-s^2) \frac{m J_1(m\sqrt{-s^2})}{2\sqrt{-s^2}} \right) \tag{2.34}
$$

is the Klein-Gordon retarded propagator, and

$$
-s^2 = -(x-y)^2 \tag{2.35}
$$

In principle, the idea behind equation (2.7) could adapt this method to non-conserved sources. Then we would have

$$A^\mu(x) = \int q(\lambda) v^\mu(\lambda) R(x, r(\lambda))\, d\lambda - \frac{1}{m^2}\nabla^\mu \int \dot{q}(\lambda) R(x, r(\lambda))\, d\lambda \tag{2.36}$$

This idea has not been pursued here, as the relevant integrals are unwieldy.

2.7 General form of the stress-energy tensor

One way to quantify the presence of radiation is with the stress-energy tensor. This section will use cylindrical Rindler coordinates $\tau, \chi, x_\perp, \phi$, for which the metric takes the form

$$g = -\chi^2 d\tau^2 + \frac{1}{a^2} d\chi^2 + dx_\perp^2 + x_\perp^2 d\phi^2 \tag{2.37}$$

The stress-energy tensor of the Proca field is

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha + m^2 A_\mu A_\nu - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} m^2 g_{\mu\nu} A_\alpha A^\alpha \tag{2.38}$$

In the Rindler wedge, the retarded solution for the uniformly accelerated charge has the general form

$$A = u(\chi, x_\perp) d\tau \tag{2.39}$$

Consequently, the field strength tensor takes the form

$$F = u_\chi d\chi \wedge d\tau + u_\perp dx_\perp \wedge d\tau \tag{2.40}$$

where $u_\chi = \frac{\partial u}{\partial \chi}$ and $u_\perp = \frac{\partial u}{\partial x_\perp}$. The components of the stress-energy tensor may be computed in terms of these functions:

$$T_{\tau\tau} = \frac{1}{2}(a^2 u_\chi^2 + u_\perp^2 + m^2 u^2) \tag{2.41}$$

$$T_{\tau i} = 0 \qquad \text{for } i \neq \tau \tag{2.42}$$

$$T_{\chi\chi} = \frac{1}{2a^2\chi^2}(-a^2u_\chi^2 + u_\perp^2 + m^2u^2) \tag{2.43}$$

$$T_{\perp\perp} = \frac{1}{2\chi^2}(a^2u_\chi^2 - u_\perp^2 + m^2u^2) \tag{2.44}$$

$$T_{\phi\phi} = \frac{x_\perp^2}{2\chi^2}(a^2u_\chi^2 + u_\perp^2 + m^2u^2) \tag{2.45}$$

$$T_{\phi i} = 0 \qquad \text{for } i \neq \phi \tag{2.46}$$

$$T_{\chi\perp} = -\frac{1}{\chi^2}u_\chi u_\perp \tag{2.47}$$

As an example, these components may be explicitly computed in the massless case, $m = 0$, for which

$$u = -\frac{qa}{4\pi}\frac{1+\chi^2+a^2x_\perp^2}{\sqrt{(1+\chi^2+a^2x_\perp^2)^2 - 4\chi^2}} \tag{2.48}$$

$$u_\chi = \frac{-qa\chi(1-\chi^2+a^2x_\perp^2)}{\pi((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^{3/2}} \tag{2.49}$$

$$u_\perp = \frac{2qa^3x_\perp\chi^2}{\pi((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^{3/2}} \tag{2.50}$$

$$\begin{aligned} T_{\tau\tau} &= \frac{q^2a^4\chi^2((1-\chi^2+a^2x_\perp^2)^2 + 4a^2x_\perp^2\chi^2)}{2\pi^2((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^3} \\ &= \frac{q^2a^4\chi^2}{2\pi^2((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^2} \end{aligned} \tag{2.51}$$

$$T_{\perp\perp} = -a^2T_{\chi\chi} = \frac{q^2a^4((1-\chi^2+a^2x_\perp^2)^2 - 4a^2x_\perp^2\chi^2)}{2\pi^2((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^3} \tag{2.52}$$

$$T_{\chi\perp} = \frac{2q^2a^4\chi x_\perp(1-\chi^2+a^2x_\perp^2)}{\pi^2((1+\chi^2+a^2x_\perp^2)^2-4\chi^2)^3} \tag{2.53}$$

These results agree with those computed in [3], except for $T_{\chi\chi}$. We believe the corresponding expression to $T_{\chi\chi}$ in [3] to be incorrect, as may be verified by checking the tracelessness condition for the massless case.

The general forms for the future wedge are similar. Using the coordinates

$$t = \frac{\theta}{a}\cosh(a\zeta) \qquad z = \frac{\theta}{a}\sinh(a\zeta) \tag{2.54}$$

for which the metric takes the form

$$g = -\frac{1}{a^2}d\theta^2 + \theta^2 d\zeta^2 + dx_\perp^2 + x_\perp^2 d\phi^2 \tag{2.55}$$

the field has the form

$$A = u(\theta, x_\perp)d\zeta \tag{2.56}$$

Then,

$$T_{\zeta\zeta} = \frac{1}{2}(-a^2u_\theta^2 + u_\perp^2 + m^2u^2) \tag{2.57}$$

$$T_{\zeta i} = 0 \qquad \text{for } i \neq \zeta \tag{2.58}$$

$$T_{\theta\theta} = \frac{1}{2a^2\theta^2}(a^2u_\theta^2 + u_\perp^2 + m^2u^2) \tag{2.59}$$

$$T_{\perp\perp} = \frac{1}{2\theta^2}(a^2u_\theta^2 + u_\perp^2 - m^2u^2) \tag{2.60}$$

$$T_{\phi\phi} = \frac{x_\perp^2}{2\theta^2}(a^2u_\theta^2 - u_\perp^2 - m^2u^2) \tag{2.61}$$

$$T_{\phi i} = 0 \qquad \text{for } i \neq \phi \tag{2.62}$$

$$T_{\theta\perp} = \frac{1}{\theta^2}u_\theta u_\perp \tag{2.63}$$

2.8 Classical particle number

Another way of quantifying radiation is using the classical particle number [27]. Despite the terminology of particles, this is a classical notion. Classical fields have a conserved current, whose time component can be thought of as the classical particle density. This is most well known perhaps in the case of the Klein-Gordon field, but it is true for other fields as well. In our case, the classical particle number can be defined as

$$N = \langle KR\tilde{J}, KR\tilde{J}\rangle \tag{2.64}$$

where $KR\tilde{J}$ denotes the Minkowski-positive frequency part of the retarded solution. Since the Unruh modes are Minkowski-positive frequency, this is given by

$$KR\tilde{J} = -\sum_{\sigma}\int d\omega d^2k_\perp \left(i\int \overline{W_\mu^{\sigma,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu\, d^4x\right) W^{\sigma,\mathrm{II},\omega,\mathbf{k}_\perp} \tag{2.65}$$

2.9 Quantum field theory

For completeness, and to complement the work of other authors, it is useful to consider the problem from the point of view of quantum field theory. The results of this section are generic for quantum fields with a classical source. The case of a massless scalar field is covered in [2], and the results carry over to the Proca field with minimal changes.

The quantum field theory of the Proca field with a classical source can be solved exactly. Let $A_{\text{ret}} = R\tilde{J}$ and $A_{\text{adv}} = A\tilde{J}$ denote the retarded and advanced solutions. Then the quantum field can be written as

$$\hat{A} = A_{\text{ret}}\hat{I} + \hat{A}_{\text{in}} = A_{\text{adv}}\hat{I} + \hat{A}_{\text{out}} \tag{2.66}$$

The single-particle Hilbert space is the space of positive-frequency solutions to the homogeneous field equation which are $\langle,\rangle$-square-integrable. The full Hilbert space is the Fock space constructed out of the single-particle space. Each element f of the single-particle space has an associated annihilation operator $\hat{a}(f)$ and its adjoint creation operator $\hat{a}^\dagger(f)$. The "in" representation above is associated with operators $\hat{a}_{\text{in}}(f)$ and vacuum state $|0_{\text{in}}\rangle$, and the "out" representation is associated with operators $\hat{a}_{\text{out}}(f)$ and vacuum state $|0_{\text{out}}\rangle$. Note the operators $\hat{a}(f)$ and $\hat{a}^\dagger(f)$ are only defined for positive frequency f. The quantum field encodes the linear map from test functions to operators:

$$\langle K\hat{A}_{\text{in}}, f\rangle = \hat{a}^\dagger_{\text{in}}(f) \tag{2.67}$$

where, as above, K denotes the positive-frequency part of a field. Likewise for the "out" representation:

$$\langle K\hat{A}_{\text{out}}, f\rangle = \hat{a}^\dagger_{\text{out}}(f) \tag{2.68}$$

The two representations are connected by the S-matrix. In particular

$$|0_{\text{in}}\rangle = \hat{S}|0_{\text{out}}\rangle \tag{2.69}$$

We have

$$\begin{aligned}
\hat{S} &= \exp\left[-i\int \hat{A}_{\text{out}}\tilde{J}\, d^4x\right] \\
&= \exp\left[-i\int (K\hat{A}_{\text{out}})\tilde{J}\, d^4x - i\int (K\hat{A}_{\text{out}})^\dagger\tilde{J}\, d^4x\right] \\
&= \exp\left[-\langle (K\hat{A}_{\text{out}})^\dagger, E\tilde{J}\rangle - \langle K\hat{A}_{\text{out}}, E\tilde{J}\rangle\right] \\
&= \exp\left[\hat{a}_{\text{out}}(KE\tilde{J}) - \hat{a}^\dagger_{\text{out}}(KE\tilde{J})\right]
\end{aligned} \tag{2.70}$$

using the identity (2.28). Note the inner products project the positive frequency part of $E\tilde{J}$. Now applying $e^{a+b} = e^a e^b e^{-\frac{1}{2}[a,b]}$, as well as

$$\left[\hat{a}_{\text{out}}(KE\tilde{J}), \hat{a}^\dagger_{\text{out}}(KE\tilde{J})\right] = \|KE\tilde{J}\|^2\hat{I} \tag{2.71}$$

we find

$$\hat{S} = e^{-\hat{a}^\dagger_{\text{out}}(KE\tilde{J})}e^{\hat{a}_{\text{out}}(KE\tilde{J})}e^{-\frac{1}{2}\|KE\tilde{J}\|^2} \tag{2.72}$$

Consequently,

$$|0_{\text{in}}\rangle = e^{-\frac{1}{2}\|KE\tilde{J}\|^2} e^{-\hat{a}^\dagger_{\text{out}}(KE\tilde{J})}|0_{\text{out}}\rangle \tag{2.73}$$

a coherent state, an eigenstate of $\hat{a}_{\text{out}}(f)$, with eigenvalue $-\langle f, E\tilde{J}\rangle$,and an eigenstate of $K\hat{A}_{\text{out}}$ with eigenvalue $-KE\tilde{J}$. The field expectation value is

$$\langle 0_{\text{in}}|\hat{A}_{\text{out}}|0_{\text{in}}\rangle = -E\tilde{J} \tag{2.74}$$

so that

$$\langle 0_{\text{in}}|\hat{A}|0_{\text{in}}\rangle = A_{\text{ret}} \tag{2.75}$$

The total particle number operator is

$$\hat{N} = \langle (K\hat{A}_{\text{out}})^\dagger, K\hat{A}_{\text{out}}\rangle \tag{2.76}$$

Its expectation value is

$$\langle 0_{\text{in}}|\hat{N}|0_{\text{in}}\rangle = \|KE\tilde{J}\|^2 \tag{2.77}$$

This is then the same expression as the classical particle number.

The results above refer to the exact solution of the quantum field theory. The calculation of tree-level perturbation theory for the Proca field with uniformly accelerated source has been carried out in [6]. That paper employed an unnecessary regularization, introducing a dipole at distance $L \to \infty$ to compensate for lack of charge conservation. This was presumably done so as to match with previous work on the electromagnetic field. As we know, charge conservation is not necessary for the Proca field. The results of [6] are unaffected by simply omitting the compensating dipole. Their results agree with the classical particle number results obtained here in equations (3.29) and (3.51).

3.1 The method of Unruh modes

Proceeding from section 2.5, we can expand the retarded solution in the future wedge with Unruh modes. Recalling equation (2.29), the expansion coefficients can be computed as

$$i \int \overline{W_{\mu}^{\sigma,\lambda,\omega,\mathbf{k}_{\perp}}} \tilde{J}^{\mu} \, d^4x \tag{3.1}$$

Conveniently, the second term of $\tilde{J}$ representing non-conservation of charge, may be neglected here, because when it is integrated by parts, we find a $\nabla^{\mu}\overline{W}_{\mu}$, which vanishes because the Unruh mode is a solution of the homogeneous Proca equation.

The expansion coefficients may now be calculated, using regularization (2.19). First, we see that $i \int \overline{W_{\mu}^{\sigma,\mathrm{I},\omega,\mathbf{k}_{\perp}}} J^{\mu} \, d^4x = 0$, because J has only a τ component. Next,

$$\begin{aligned} i \int \overline{W_{\mu}^{1,\mathrm{II},\omega,\mathbf{k}_{\perp}}} \tilde{J}^{\mu} \, d^4x &= \\ = \frac{iq\sqrt{2}}{\sqrt{1-e^{-2\pi\omega/a}}} \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} K'_{i\omega/a}(\rho/a) \int_{-T}^{T} \cos(E\tau) e^{i\omega\tau} \, d\tau &= \\ \frac{iq\sqrt{2}}{\sqrt{1-e^{-2\pi\omega/a}}} \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} K'_{i\omega/a}(\rho/a) \left[\frac{\sin((E-\omega)T)}{E-\omega} + \frac{\sin((E+\omega)T)}{E+\omega} \right] \end{aligned} \tag{3.2}$$

In the limit $T \to \infty$,

$$\frac{\sin((E \pm \omega)T)}{E \pm \omega} \to \pi\delta(E \pm \omega) \tag{3.3}$$

and since we are only considering frequencies $\omega \geq 0$, we may retain only the $\delta(\omega - E)$, and replace ω with E.

$$i \int \overline{W_{\mu}^{1,\mathrm{II},\omega,\mathbf{k}_{\perp}}} \tilde{J}^{\mu} \, d^4x = \frac{iq}{2\pi\sqrt{a}} \sqrt{e^{\pi E/a}} K'_{iE/a}(\rho/a) \delta(\omega - E) \tag{3.4}$$

Likewise,

$$
\begin{aligned}
i\int \overline{W_\mu^{2,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x =& e^{-\pi\omega/a}\, i\int \overline{W_\mu^{1,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x \\
=& \frac{iq}{2\pi\sqrt{a}}\frac{1}{\sqrt{e^{\pi E/a}}}K'_{iE/a}(\rho/a)\delta(\omega-E)
\end{aligned}
\tag{3.5}
$$

And the coefficients for the type III modes also,

$$
\begin{aligned}
i\int \overline{W_\mu^{1,\mathrm{III},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x = \\
-\frac{q\sqrt{2}}{\sqrt{1-e^{-2\pi\omega/a}}}\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}\frac{\omega k_\perp}{m\rho}K_{i\omega/a}(\rho/a)\int_{-T}^{T}\cos(E\tau)e^{i\omega\tau}\, d\tau \\
\to -\frac{q}{2\pi\sqrt{a}}\sqrt{e^{\pi E/a}}\frac{Ek_\perp}{m\rho}K_{iE/a}(\rho/a)\delta(\omega-E)
\end{aligned}
\tag{3.6}
$$

Likewise,

$$
\begin{aligned}
i\int \overline{W_\mu^{2,\mathrm{III},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x =& -e^{-\pi\omega/a}\, i\int \overline{W_\mu^{1,\mathrm{III},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x \\
=& \frac{q}{2\pi\sqrt{a}}\frac{1}{\sqrt{e^{\pi E/a}}}\frac{Ek_\perp}{m\rho}K_{iE/a}(\rho/a)\delta(\omega-E)
\end{aligned}
\tag{3.7}
$$

Now in the limit $E\to 0$, the type III coefficients vanish, and we are left with only

$$
\begin{aligned}
i\int \overline{W_\mu^{1,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x =& i\int \overline{W_\mu^{2,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu \, d^4x \\
=& -\frac{iq}{2\pi\sqrt{a}}K_1(\rho/a)\delta(\omega)
\end{aligned}
\tag{3.8}
$$

using the fact that $K_0' = -K_1$.

We now look in the future wedge and introduce coordinates ζ and η using

$$
t = \frac{e^{a\eta}}{a}\cosh(a\zeta) \qquad z = \frac{e^{a\eta}}{a}\sinh(a\zeta)
\tag{3.9}
$$

Using these coordinates, in the future wedge the zero-frequency Unruh modes take the form

$$W^{1,\mathrm{II},0,\mathbf{k}_\perp} = W^{2,\mathrm{II},0,\mathbf{k}_\perp} = \frac{1}{\sqrt{32\pi^2 a}} H_1^{(2)}\left(\rho\frac{e^{a\eta}}{a}\right) e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} d\zeta \tag{3.10}$$

Therefore, following equation (2.27), we have

$$R(\tilde{J}) = \left[\frac{iq\sqrt{2}}{8\pi^2 a}\int_0^\infty k_\perp dk_\perp \int_0^{2\pi} d\phi\, K_1(\rho/a) H_1^{(2)}\left(\rho\frac{e^{a\eta}}{a}\right) e^{ik_\perp x_\perp \cos\phi} + \text{c.c.}\right] d\zeta \tag{3.11}$$

where polar coordinates in $\mathbf{k}_\perp$ have been used. Finally, using

$$H_1^{(2)} = J_1 - iY_1 \tag{3.12}$$

and

$$\int_0^{2\pi} e^{ik_\perp x_\perp \cos\phi} d\phi = 2\pi J_0(k_\perp x_\perp) \tag{3.13}$$

we have

$$R(\tilde{J}) = \left[\frac{q}{\sqrt{2}\pi a}\int_0^\infty k_\perp K_1(\rho/a) Y_1\left(\rho\frac{e^{a\eta}}{a}\right) J_0(k_\perp x_\perp) dk_\perp\right] d\zeta \tag{3.14}$$

This gives us an expression for the Proca field of the uniformly accelerated source.

For the purposes of studying radiation, one would like to plug this expression into the the formulas for components of the stress-energy tensor described in section 2.7. Unfortunately, any further simplification of the integral does not appear to be possible, and the lack of a simple algebraic expression makes further calculations unfeasible. More importantly, unlike the electromagnetic field, the Proca field radiation can travel in the interior of the future lightcone. This means the field at any point depends on the entire past history of the source and not just a single retarded time, so it is not possible to isolate the radiation emitted from a single time on the particle's worldline, and there isn't a clear choice of surface to integrate over. Consequently, this method does not fully illuminate the question of radiation. Similar challenges apply to the result of the next section. A way of moving beyond these challenges is to employ the methods of sections 3.3 and 3.4, in which actual expressions for the rate of radiation are obtained.

3.2 The method of the retarded propagator

Continuing from section 2.6, we calculate the retarded solution in the Rindler wedge using the method of [3]. In Minkowski coordinates t, z, x, y, our source has worldline

$$r(\lambda) = \left(\frac{1}{a}\sinh(a\lambda),\ \frac{1}{a}\cosh(a\lambda),\ 0,\ 0\right) \tag{3.15}$$

with velocity

$$v^{\mu} = (\cosh(a\lambda),\ \sinh(a\lambda),\ 0,\ 0) \tag{3.16}$$

The spacetime interval between $r(\lambda)$ and the field point $x = \left(\frac{\chi}{a}\sinh(a\tau),\ \frac{\chi}{a}\cosh(a\tau),\ \mathbf{x}_{\perp}\right)$ is

$$-s^2 = \frac{2\chi}{a^2}\cosh(a(\tau-\lambda)) - \frac{1}{a^2}(\chi^2+1) - x_{\perp}^2 \tag{3.17}$$

Then we have

$$\begin{aligned} A_{\tau} &= \frac{\partial t}{\partial \tau}A_t + \frac{\partial z}{\partial \tau}A_z \\ &= q\int(-\chi\cosh(a\tau)\cosh(a\lambda) + \chi\sinh(a\tau)\sinh(a\lambda))R(-s^2)d\lambda \\ &= -q\chi\int\cosh(a(\tau-\lambda))R(-s^2)d\lambda \end{aligned} \tag{3.18}$$

$$\begin{aligned} A_{\chi} &= \frac{\partial t}{\partial \chi}A_t + \frac{\partial z}{\partial \chi}A_z \\ &= q\int(-\sinh(a\tau)\cosh(a\lambda) + \cosh(a\tau)\sinh(a\lambda))R(-s^2)d\lambda \\ &= -q\int\sinh(a(\tau-\lambda))R(-s^2)d\lambda \end{aligned} \tag{3.19}$$

These integrals can be evaluated by changing the variable of integration to $u = -s^2$. The measure changes to

$$du = \frac{2\chi}{a}\sinh(a(\tau-\lambda))d\lambda = a\sqrt{\left(u + \frac{1}{a^2}(\chi^2+1) + x_{\perp}^2\right)^2 - \frac{4\chi^2}{a^4}}\,d\lambda \tag{3.20}$$

Then A_χ becomes

$$A_\chi = \frac{qa}{4\pi\chi}\left(\int \delta(u)du - \int_0^\infty \frac{mJ_1(m\sqrt{u})}{2\sqrt{u}}du\right) \tag{3.21}$$

Upon the further variable change $w = m\sqrt{u}$, the second integral becomes $\int_0^\infty J_1(w)dw = 1$, so all together

$$A_\chi = 0 \tag{3.22}$$

More complicated is A_τ

$$\begin{aligned} A_\tau &= -\frac{qa}{4\pi}\int \frac{\left(u + \frac{1}{a^2}(\chi^2+1) + x_\perp^2\right)\left(\delta(u) - \theta(u)\frac{mJ_1(m\sqrt{u})}{2\sqrt{u}}\right)}{\sqrt{\left(u + \frac{1}{a^2}(\chi^2+1) + x_\perp^2\right)^2 - \frac{4\chi^2}{a^4}}}du \\ &= -\frac{qa}{4\pi}\left(\frac{1+\chi^2+a^2x_\perp^2}{\sqrt{(1+\chi^2+a^2x_\perp^2)^2 - 4\chi^2}} - \int_0^\infty \frac{\left(\frac{w^2a^2}{m^2} + 1 + \chi^2 + a^2x_\perp^2\right)J_1(w)}{\sqrt{\left(\frac{w^2a^2}{m^2} + 1 + \chi^2 + a^2x_\perp^2\right)^2 - 4\chi^2}}\,dw\right) \end{aligned} \tag{3.23}$$

3.3 Classical particle number

Proceeding from section 2.8. As found earlier, the Unruh mode coefficients for the source regularized with finite T are

$$\begin{aligned} i\int \overline{W_\mu^{1,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu\, d^4x &= e^{\pi\omega/a}\, i\int \overline{W_\mu^{2,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu\, d^4x \\ &= \frac{iq\sqrt{2}}{\sqrt{1-e^{-2\pi\omega/a}}}\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}K'_{i\omega/a}(\rho/a)\left[\frac{\sin(\omega T)}{\omega}\right] \end{aligned} \tag{3.24}$$

where $\rho = \sqrt{k_\perp^2 + m^2}$. So,

$$KR\tilde{J} = -\sum_\sigma \int_0^\infty d\omega \int d^2k_\perp \left(i\int \overline{W_\mu^{\sigma,\mathrm{II},\omega,\mathbf{k}_\perp}}\tilde{J}^\mu\, d^4x\right)\, W^{\sigma,\mathrm{II},\omega,\mathbf{k}_\perp} \tag{3.25}$$

By the orthonormality of the Unruh modes,

$$\begin{aligned}&\langle KR\tilde{J}, KR\tilde{J}\rangle = \\ &\frac{q^2}{2\pi^4 a}\int d^2k_\perp d\omega\Big(1+e^{-2\pi\omega/a}\Big)\frac{\sinh(\pi\omega/a)}{1-e^{-2\pi\omega/a}}\big|K'_{i\omega/a}(\rho/a)\big|^2\left(\frac{\sin(\omega T)}{\omega}\right)^2\end{aligned}\tag{3.26}$$

In the limit of large T,

$$\left(\frac{\sin(\omega T)}{\omega}\right)^2 \to \pi T\delta(\omega)\tag{3.27}$$

$$\begin{aligned}\langle KR\tilde{J}, KR\tilde{J}\rangle &\to \frac{q^2}{2\pi^4 a}\int d^2k_\perp\big|K_1(\rho/a)\big|^2\pi T \\ &= \frac{q^2 aT}{\pi^2}\int_0^\infty dx\, x\left|K_1\left(\sqrt{x^2+\frac{m^2}{a^2}}\right)\right|^2\end{aligned}\tag{3.28}$$

Divide by the total time $2T$ to get the rate of particles:

$$\begin{aligned}\text{Rate} &= \frac{q^2 a}{2\pi^2}\int_0^\infty dx\, x\left|K_1\left(\sqrt{x^2+\frac{m^2}{a^2}}\right)\right|^2 \\ &= \frac{q^2 a}{4\pi^2}\frac{m^2}{a^2}\left(K_0(m/a)K_2(m/a)-K_1(m/a)^2\right)\end{aligned}\tag{3.29}$$

This result agrees with the expression obtained in [6] using tree-level quantum field theory. The zero-mass limit diverges logarithmically, as expected in a typical infrared divergence of a massless field. If considered in terms of its spectrum as a function of $k_\perp$, the zero-mass limit of this result agrees with the result for the electromagnetic field found in [5].

3.4 The method of Minkowski modes

This section will repeat the analysis of sections 3.1 and 3.3 using Minkowski modes (plane waves). These modes will be indexed by wave vector $\mathbf{k} = (k_x, k_y, k_z)$. Also let $k_\perp$ and ρ be as before, but now let $\omega = \sqrt{\rho^2 + k_z^2}$ be the Minkowski frequency. The polarization modes to be used are

$$U_\mu^{\mathrm{I},\mathbf{k}} = \frac{1}{k_\perp}\frac{1}{\sqrt{2\omega(2\pi)^3}}\,(0,\, k_y,\, -k_x, 0)\, e^{ik_\mu x^\mu}\tag{3.30}$$

$$U_\mu^{\mathrm{II},\mathbf{k}} = \frac{1}{k_\perp}\frac{1}{\sqrt{k_z^2+k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\left(0,\, k_xk_z,\, k_yk_z,\, -k_\perp^2\right)e^{ik_\mu x^\mu} \tag{3.31}$$

$$U_\mu^{\mathrm{III},\mathbf{k}} = \frac{\omega}{m}\frac{1}{\sqrt{k_z^2+k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\left(-\frac{k_z^2+k_\perp^2}{\omega},\, k_x,\, k_y,\, k_z\right)e^{ik_\mu x^\mu} \tag{3.32}$$

They are orthonormal:

$$\langle U^{\lambda,\mathbf{k}}, U^{\lambda',\mathbf{k}'}\rangle = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}') \tag{3.33}$$

A convenient regularization for this section is $q(\tau) = qe^{-\epsilon\cosh(a\tau)}$ with the intention of letting $\epsilon \to 0$, so that

$$J = qe^{-\epsilon\cosh(a\tau)}\;\delta(\xi)\delta(x)\delta(y)\,\partial_\tau \tag{3.34}$$

In the Minkowski coordinate basis,

$$\partial_\tau = (e^{a\xi}\cosh(a\tau),\, 0,\, 0,\, e^{a\xi}\sinh(a\tau)) \tag{3.35}$$

Thus in order to calculate the coefficients $i\int \overline{U_\mu}J^\mu\, d^4x$, we need the following integral identities:

$$\int_{-\infty}^{\infty} e^{-i(A\cosh(t)+B\sinh(t))}\sinh(t)\,dt = 2iB\,\frac{K_1(\sqrt{B^2-A^2})}{\sqrt{B^2-A^2}} \tag{3.36}$$

$$\int_{-\infty}^{\infty} e^{-i(A\cosh(t)+B\sinh(t))}\cosh(t)\,dt = -2iA\,\frac{K_1(\sqrt{B^2-A^2})}{\sqrt{B^2-A^2}} \tag{3.37}$$

where in this case,

$$A = \frac{k_z}{a} - i\epsilon \qquad B = -\frac{\omega}{a} \tag{3.38}$$

The result is

$$i\int \overline{U_\mu^{\mathrm{I}}}J^\mu\, d^4x = 0 \tag{3.39}$$

$$i\int \overline{U_\mu^{\mathrm{II}}}J^\mu\, d^4x = -\frac{2q}{a^2}\frac{k_\perp\omega}{\sqrt{k_z^2+k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\frac{K_1(u)}{u} \tag{3.40}$$

$$i\int \overline{U_\mu^{\mathrm{III}}}J^\mu\, d^4x = \frac{2q}{a^2}\frac{1}{\sqrt{k_z^2+k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\frac{K_1(u)}{u}\left(k_zm + \frac{i\epsilon a}{m}(k_z^2+k_\perp^2)\right) \tag{3.41}$$

where

$$u = \sqrt{\frac{\rho^2}{a^2} + \frac{2i\epsilon k_z}{a} + \epsilon^2} \tag{3.42}$$

Now, as in section 3.3, the total classical particle number can be found as

$$\begin{aligned} dN &= \sum_\lambda \left| i \int \overline{U_\mu^\lambda} J^\mu \, d^4x \right|^2 \, d^3\mathbf{k} \\ &= \frac{4q^2}{a^4} \frac{1}{2\omega(2\pi)^3} \, \left| \frac{K_1(u)}{u} \right|^2 \left(\rho^2 + \frac{\epsilon^2 a^2}{m^2} (k_z^2 + k_\perp^2) \right) \, d^3\mathbf{k} \end{aligned} \tag{3.43}$$

This result expresses the Minkowski-frequency dependence of the emitted particles. The second term, proportional to $\frac{\epsilon^2 a^2}{m^2}$, has conceptual subtleties. It arises only from polarization mode III, and is associated with radiation resulting from the time-dependence of the magnitude of the charge, as discussed further in section 3.6. It makes no contribution to dN in the limit $\epsilon \to 0$, but it also diverges if the limit $m \to 0$ is taken first. It will be neglected for the remainder of this section. Further discussion follows at the end of this section. The ϵ inside of u may not be neglected, as it is needed for the convergence of the k_z integral.

The integral over k_z can be done as follows. To make the contour of integration more explicit, parameterize the variable u as

$$u = \sqrt{v^2 + \frac{\rho^2}{a^2} + \epsilon^2} + iv \tag{3.44}$$

such that

$$k_z = \frac{a}{\epsilon} \, v \sqrt{v^2 + \frac{\rho^2}{a^2} + \epsilon^2} \tag{3.45}$$

and

$$dk_z = \frac{a}{\epsilon} \frac{|u|^2}{\sqrt{v^2 + \frac{\rho^2}{a^2} + \epsilon^2}} \, dv \tag{3.46}$$

In terms of the parameter v, the frequency ω becomes

$$\omega = \frac{a}{\epsilon} \sqrt{(v^2 + \epsilon^2) \left(v^2 + \frac{\rho^2}{a^2} \right)} \tag{3.47}$$

With this change of variable, we have

$$dN = \frac{4q^2}{a^4}\frac{\rho^2}{2(2\pi)^3}\frac{\left|K_1\left(\sqrt{v^2+\frac{\rho^2}{a^2}+\epsilon^2}+iv\right)\right|^2}{\sqrt{(v^2+\epsilon^2)\left(v^2+\frac{\rho^2}{a^2}\right)\left(v^2+\frac{\rho^2}{a^2}+\epsilon^2\right)}}\,dvd^2k_\perp \tag{3.48}$$

At this point we are safe to neglect the instances of ϵ^2 that merely make an adjustment to the mass. The divergence is controlled by the factor $1/\sqrt{v^2+\epsilon^2}$. In the limit $\epsilon \to 0$, it behaves as

$$\frac{1}{\sqrt{v^2+\epsilon^2}} \to 2\ln(1/\epsilon)\delta(v) \tag{3.49}$$

The factor of $\ln(1/\epsilon)$ is proportional to the effective total proper time T_{prop}, which should be divided to get the rate of particles. More accurately, T_{prop} should be found by integrating the regularization $q(\tau)^2$:

$$T_{\text{prop}} = \int_{-\infty}^{\infty} e^{-2\epsilon\cosh(a\tau)}\,d\tau = \frac{2}{a}K_0(2\epsilon) \approx \frac{2}{a}\ln(1/\epsilon) \tag{3.50}$$

Thus the rate of particles is

$$\text{Rate} = \frac{N}{T_{\text{prop}}} = \frac{q^2}{4\pi^3 a}\int |K_1(\rho/a)|^2\,d^2k_\perp \tag{3.51}$$

This agrees with equation (3.29).

The benefit of calculating the particle number via the Minkowski frequency spectrum is it gives an alternative method for computing the total energy in the field, using $dE = \omega dN$. Using equation (3.48),

$$dE = \frac{4q^2}{a^4}\frac{\rho^2}{2(2\pi)^3}\frac{a}{\epsilon}\frac{\left|K_1\left(\sqrt{v^2+\frac{\rho^2}{a^2}}+iv\right)\right|^2}{\sqrt{v^2+\frac{\rho^2}{a^2}}}\,dvd^2k_\perp \tag{3.52}$$

The energy is proportional to effective total inertial time T_{inert}, rather than proper time. This can be found in terms of ϵ by integrating the regularization $q(t)^2$:

$$T_{\text{inert}} = \int_{-\infty}^{\infty} e^{-2\epsilon\sqrt{1+a^2t^2}}\, dt = \frac{2}{a} K_1(2\epsilon) \approx \frac{1}{a\epsilon} \tag{3.53}$$

Thus the energy in the field is

$$E = \frac{q^2 T_{\text{inert}}}{4\pi^3 a^2} \int \rho^2 \frac{\left| K_1\left(\sqrt{v^2 + \frac{\rho^2}{a^2}} + iv\right)\right|^2}{\sqrt{v^2 + \frac{\rho^2}{a^2}}}\, dv d^2k_\perp \tag{3.54}$$

The integral may be expressed somewhat more simply via a further change of variables

$$r \sin\theta = \rho/a \qquad r\cos\theta = v \tag{3.55}$$

giving

$$E = \frac{q^2 a^2 T_{\text{inert}}}{2\pi^2} \int_0^{\pi} \int_{\frac{m}{a\sin\theta}}^{\infty} r^3 \sin^3\theta \left|K_1(r(1 + i\cos\theta))\right|^2\, dr d\theta \tag{3.56}$$

There is a mysterious discrepancy in the zero-mass limit of this expression. At $m = 0$, the double integral evaluates to $\pi/4$. This would give a radiation rate of $\frac{q^2a^2}{8\pi}$, different from the expected Larmor value of $\frac{q^2a^2}{6\pi}$. This could be due to the fact that the zero-mass limit of a Proca field does not correspond to an electromagnetic solution if the source is not conserved. There may be a subtlety in the simultaneous $T \to \infty$, $m \to 0$ limit, as further discussed at the end of section 3.6. However, we do not believe that neglecting the $\frac{\epsilon^2 a^2}{m^2}$ term in equation (3.43) can account for this discrepancy, because it represents polarization mode III radiation, which would not be present at all for the electromagnetic field.

3.5 Regularizing the acceleration

This section presents an alternative regularization method. Instead of regularizing the magnitude of the charge, one may regularize the magnitude of the acceleration. That is, we change the source's trajectory so that its proper acceleration goes to zero in the past and future. As ever, the specific choice is one of convenience. Consider the following family of hyperbolic trajectories in Minkowski coordinates.

$$z(t) = v\sqrt{t^2 + \frac{1}{a^2}} \tag{3.57}$$

The parameter $v \in [0, 1]$ represents the initial and final limiting speed of the particle. The velocity vector, in Minkowski coordinates, is

$$v^\mu = \sqrt{\frac{t^2 + a^{-2}}{(1-v^2)t^2 + a^{-2}}}\left(1, 0, 0, \frac{vt}{\sqrt{t^2 + a^{-2}}}\right) \tag{3.58}$$

The square of the proper acceleration is

$$a^\mu a_\mu = \frac{v^2}{a^4((1-v^2)t^2 + a^{-2})^3} \tag{3.59}$$

whose maximum value is $v^2 a^2$. The current source is

$$J^\mu = q\left(1, 0, 0, \frac{vt}{\sqrt{t^2 + a^{-2}}}\right)\delta\left(z - v\sqrt{t^2 + \frac{1}{a^2}}\right)\delta(x)\delta(y) \tag{3.60}$$

The calculation of the Minkowski mode coefficients proceeds similarly to section 3.4. The results are:

$$i\int \overline{U^{\mathrm{I}}_\mu} J^\mu \, d^4x = 0 \tag{3.61}$$

$$i\int \overline{U^{\mathrm{II}}_\mu} J^\mu \, d^4x = -\frac{2qv}{a^2}\frac{k_\perp \omega}{\sqrt{k_z^2 + k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\frac{K_1(u)}{u} \tag{3.62}$$

$$i\int \overline{U^{\mathrm{III}}_\mu} J^\mu \, d^4x = \frac{2qv}{a^2}\frac{k_z m}{\sqrt{k_z^2 + k_\perp^2}}\frac{1}{\sqrt{2\omega(2\pi)^3}}\frac{K_1(u)}{u} \tag{3.63}$$

where now

$$u = \frac{1}{a}\sqrt{\rho^2 + (1-v^2)k_z^2} \tag{3.64}$$

The particle number is

$$dN = \frac{q^2 v^2}{4\pi^3 a^4}\frac{\rho^2}{\omega}\left|\frac{K_1(u)}{u}\right|^2 \, d^3k \tag{3.65}$$

We may change integration variable from $k_\perp$ to

$$r = \sqrt{\rho^2 + (1 - v^2)k_z^2} \tag{3.66}$$

The effect is that the remaining variable k_z is only integrated between $\pm\sqrt{(r^2 - m^2)/(1 - v^2)}$.

$$N = \frac{q^2 v^2}{2\pi^2 a^2} \int_m^\infty \frac{|K_1(r/a)|^2}{r^2} \int_{-\sqrt{(r^2-m^2)/(1-v^2)}}^{\sqrt{(r^2-m^2)/(1-v^2)}} \frac{r^2 - (1 - v^2)k_z^2}{\sqrt{r^2 + v^2 k_z^2}} \, dk_z \, r dr \tag{3.67}$$

This may be straightforwardly computed, but the k_z integration is even cleaner for the energy.

$$dE = \omega \, dN = \frac{q^2 v^2}{2\pi^2 a^2} \left(r^2 - (1 - v^2)k_z^2\right) \frac{|K_1(r/a)|^2}{r^2} \, dk_z \, r dr \tag{3.68}$$

$$\begin{aligned} E &= \frac{q^2 v^2}{3\pi^2 a^2 \sqrt{1 - v^2}} \int_m^\infty \sqrt{r^2 - m^2} \left(2 + \frac{m^2}{r^2}\right) |K_1(r/a)|^2 \, r dr \\ &= \frac{q^2 v^2 a}{3\pi^2 \sqrt{1 - v^2}} \int_{m/a}^\infty \sqrt{x^2 - \frac{m^2}{a^2}} \left(2 + \frac{m^2}{a^2 x^2}\right) |K_1(x)|^2 \, x dx \end{aligned} \tag{3.69}$$

It is tempting to divide this result by the effective total time over which the source is accelerating, to get a rate of radiation. Unfortunately, it is unclear what is a suitable definition of effective total time. Even if we assume the radiation is proportional to $a^\mu a_\mu$ (which is evidently not precisely true for $m \neq 0$), the time distribution in equation 3.59 has long tails. So any choice of cutoff point or other measure of width would be somewhat arbitrary. Nevertheless, we can approximately say of the effective total time that

$$T \propto \frac{1}{a\sqrt{1 - v^2}} \tag{3.70}$$

Consequently, the average power

$$P \propto q^2 v^2 a^2 \tag{3.71}$$

is finite in the $v \to 1$ limit.

3.6 Radiation from a charge changing in time

As mentioned previously, the source of the Proca field need not be conserved. An important feature of Proca radiation is that a charge whose magnitude changes in time can radiate, even if the charge is at rest. First, the radiation from an oscillating charge will be calculated in order to give a qualitative picture of radiation from time-dependent charges. Consider a particle at rest with charge

$$q(t) = qe^{-i\omega t} \tag{3.72}$$

A real field can be obtained at the end by taking real parts. The field is spherically symmetric, and the retarded scalar potential is found to be

$$\phi = \frac{q}{4\pi r} e^{i(kr-\omega t)} \tag{3.73}$$

where

$$k = \sqrt{\omega^2 - m^2} \tag{3.74}$$

Note that this expression remains valid for $|\omega| < m$, in which case the field falls off exponentially, approaching the well-known Yukawa potential when $\omega = 0$. The magnetic field $\mathbf{B}$ vanishes. The electric field and vector potential are radial and satisfy

$$\nabla \cdot \mathbf{E} = -m^2\phi \qquad \dot{\mathbf{E}} = m^2\mathbf{A} \tag{3.75}$$

so they are given by

$$\mathbf{E} = -\frac{qm^2}{4\pi r^2}\left(\frac{r}{ik} + \frac{1}{k^2}\right) e^{i(kr-\omega t)}\ \hat{r} \tag{3.76}$$

$$\mathbf{A} = \frac{q}{4\pi}\left(\frac{\omega}{rk} + \frac{i\omega}{r^2k^2}\right) e^{i(kr-\omega t)}\ \hat{r} \tag{3.77}$$

After taking real parts, the Poynting vector is

$$\begin{aligned}\mathbf{S} &= m^2\phi\mathbf{A} \\ &= \frac{q^2m^2}{16\pi^2 r}\cos(kr-\omega t)\left(\frac{\omega}{rk}\cos(kr-\omega t)-\frac{\omega}{r^2k^2}\sin(kr-\omega t)\right)\,\hat{r} \\ &\to \frac{q^2m^2}{16\pi^2r^2}\frac{\omega}{k}\cos^2(kr-\omega t)\,\hat{r}\end{aligned} \tag{3.78}$$

taking the radiation-zone limit and retaining only the $1/r^2$ term. The average radiated power is then

$$P = \frac{q^2m^2}{8\pi}\frac{\omega}{\sqrt{\omega^2-m^2}} \tag{3.79}$$

For frequencies less than m, the field is exponentially damped, and there is no radiation. This has the qualitative implication for more general time dependence that if the charge is turned on and off slowly enough, there should be no contribution to the radiation from this effect.

For more general time dependence, the field can be difficult to calculate, but the radiated energy and particle number can be found from the Minkowski-mode amplitudes, similarly to previous sections. For a particle at rest, the only polarization to contribute is U^{III}.

$$i\int \overline{U^{\mathrm{III}}_\mu} J^\mu \, d^4x = \frac{-i}{m}\frac{\sqrt{k_z^2+k_\perp^2}}{\sqrt{2\omega(2\pi)^3}}\,\tilde{q}(\omega) \tag{3.80}$$

where

$$\tilde{q}(\omega) = \int_{-\infty}^{\infty} q(t)e^{i\omega t}\,dt \tag{3.81}$$

Then, the particle number is

$$\begin{aligned}dN &= \left| i\int \overline{U^{\mathrm{III}}_\mu} J^\mu \, d^4x\right|^2 \, d^3\mathbf{k} \\ &= \frac{1}{m^2}\frac{k_z^2+k_\perp^2}{16\pi^3\omega}\,|\tilde{q}(\omega)|^2\,d^3\mathbf{k}\end{aligned} \tag{3.82}$$

Utilizing the spherical symmetry and letting $k^2 = k_z^2 + k_\perp^2$,

$$N = \frac{1}{4\pi^2 m^2} \int_0^\infty \frac{k^4}{\sqrt{k^2+m^2}} \left| \tilde{q}\left(\sqrt{k^2+m^2}\right) \right|^2 \, dk \tag{3.83}$$

Likewise the radiated energy is

$$dE = \omega \, dN = \frac{1}{m^2} \frac{k_z^2 + k_\perp^2}{16\pi^3} \left|\tilde{q}(\omega)\right|^2 \, d^3\mathbf{k} \tag{3.84}$$

$$E = \frac{1}{4\pi^2 m^2} \int_0^\infty k^4 \left| \tilde{q}\left(\sqrt{k^2+m^2}\right) \right|^2 \, dk \tag{3.85}$$

For example, consider a Gaussian $q(t) = qe^{-t^2/2\sigma^2}$. In this case, $\tilde{q}(\omega) = q\sqrt{2\pi}\sigma \, e^{-\frac{1}{2}\sigma^2\omega^2}$. Then,

$$\begin{aligned} N &= \frac{q^2\sigma^2 \, e^{-m^2\sigma^2}}{2\pi m^2} \int_0^\infty \frac{k^4}{\sqrt{k^2+m^2}} \, e^{-\sigma^2 k^2} \, dk \\ &= \frac{q^2}{8\pi} \, e^{-\frac{1}{2}m^2\sigma^2} \left(m^2\sigma^2 K_0\left(\frac{1}{2}m^2\sigma^2\right) + (1-m^2\sigma^2) K_1\left(\frac{1}{2}m^2\sigma^2\right) \right) \end{aligned} \tag{3.86}$$

$$\begin{aligned} E &= \frac{q^2\sigma^2 \, e^{-m^2\sigma^2}}{2\pi m^2} \int_0^\infty k^4 \, e^{-\sigma^2 k^2} \, dk \\ &= \frac{3q^2 e^{-m^2\sigma^2}}{16\sqrt{\pi} m^2 \sigma^3} \end{aligned} \tag{3.87}$$

The radiation is exponentially suppressed at $m^2\sigma^2 \gg 1$. This confirms the intuition that this mechanism makes negligible contribution to the radiation if the charge's magnitude varies slowly enough (slower than m). This result also illustrates the subtlety of the simultaneous $m \to 0$, $T \to \infty$ limit. If we fix m and send $\sigma \to \infty$, we get $E \to 0$, the 'right' answer. But if we fix σ and send $m \to 0$, we get $E \to \infty$.

3.7 Uniform acceleration with general time dependence

This section considers the combined effects of uniform acceleration and a time-dependent charge. The setup and strategy are similar to section 3.1, but allowing the charge to be a more general function of time. As it was not possible to obtain a workable expression for the field even in the case where the regularization was removed, calculating the field is all the more unfeasible with a time-dependent charge. However, a general expression for the particle number can be obtained from the Unruh-mode coefficients.

In Rindler coordinates, the source is

$$J = q(\tau)\, \delta(\xi)\delta(x)\delta(y)\, \partial_\tau \tag{3.88}$$

where the charge $q(\tau)$ is a function of τ, its proper time. This function may be Fourier-expanded

$$q(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{q}(E) e^{-iE\tau}\, dE \tag{3.89}$$

$q(\tau)$ is real. Suppose further that it is time-symmetric, so that $q(\tau) = \overline{q(\tau)} = q(-\tau)$. Consequently, we may rewrite this as

$$q(\tau) = \frac{1}{\pi}\int_{0}^{\infty} \tilde{q}(E)\cos(E\tau)\, dE \tag{3.90}$$

This representation is convenient because we have already obtained the Unruh-mode coefficients for cosine time-dependence in equations (3.4)-(3.7). Restating them here,

$$\langle W^{1,\mathrm{II}}_{\cos}\rangle = \frac{i}{2\pi\sqrt{2a}}\sqrt{e^{\pi E/a}}K'_{iE/a}(\rho/a)\,\delta(\omega - E) \tag{3.91}$$

$$\langle W^{2,\mathrm{II}}_{\cos}\rangle = \frac{i}{2\pi\sqrt{2a}}\sqrt{e^{-\pi E/a}}K'_{iE/a}(\rho/a)\,\delta(\omega - E) \tag{3.92}$$

$$\langle W^{1,\mathrm{III}}_{\cos}\rangle = \frac{1}{2\pi\sqrt{2a}}\sqrt{e^{\pi E/a}}\frac{Ek_\perp}{m\rho}K_{iE/a}(\rho/a)\,\delta(\omega - E) \tag{3.93}$$

$$\langle W^{2,\mathrm{III}}_{\cos}\rangle = \frac{1}{2\pi\sqrt{2a}}\sqrt{e^{-\pi E/a}}\frac{Ek_\perp}{m\rho}K_{iE/a}(\rho/a)\,\delta(\omega - E) \tag{3.94}$$

By linearity, the Unruh-mode coefficients for the general time-dependence are

$$\begin{aligned} i\int \overline{W^{\sigma,\lambda}_\mu}\tilde{J}^\mu\, d^4x =& \frac{1}{\pi}\int_0^\infty \tilde{q}(E)\,\langle W^{\sigma,\lambda}_{\cos}\rangle\, dE \\ =& \frac{1}{\pi}\,\tilde{q}(\omega)\,\langle W^{\sigma,\lambda}_{\cos}\rangle_{E=\omega} \end{aligned} \tag{3.95}$$

As before, these square to give the particle number.

$$dN = \frac{|\tilde{q}(\omega)|^2}{4\pi^4 a} \cosh(\pi\omega/a) \left(\left| K'_{i\omega/a}(\rho/a) \right|^2 + \frac{\omega^2 k_\perp^2}{m^2 \rho^2} \left| K_{i\omega/a}(\rho/a) \right|^2 \right) \, d\omega d^2 k_\perp \tag{3.96}$$

4. A FAMILY OF RINDLER FRAMES: A MASSLESS CASE STUDY

This chapter consists of a computation for the massless field, along with conceptual discussion, as an illustration of what is known about radiation in accelerated frames. It concerns the generalization of the Larmor formula when both observer and source have different and independent accelerations.

4.1 Background

The relationship between acceleration (Larmor) radiation and the equivalence principle of relativity is an old paradox in classical electromagnetism. The main questions are of two types:

1. A charged particle with nonzero proper acceleration may be interpreted by a comoving observer as being at rest in a gravitational field. Does such an observer observe the emission of radiation?

2. A charged particle undergoing inertial motion may be interpreted by an accelerating observer as being in free fall in a gravitational field. Does such an observer observe the emission of radiation?

Both questions remain controversial, but many regard the authoritative resolution to have been given by Rohrlich [13], who answered no to question 1 and yes to question 2. If we set aside issues of conceptual interpretation, both questions may be framed as well-posed problems of classical field theory, and answered by explicit calculation of the electromagnetic field in the appropriate coordinate system. The result is the solution given by Rohrlich.

In this classical field theory context, the resolution of the apparent paradox is that the notion of radiation is observer-dependent. As will be reflected in the following calculations, this observer-dependence arises from two sources. The first is the observer's notion of energy, which depends on their state of motion in a manner well-understood even at the level of special relativity. For non-inertial motions and coordinate systems, and if the energy is distributed over a region of space, the observer's instantaneous velocity is no longer sufficient to define their notion of energy. In general spacetimes, what is needed for this is a timelike Killing vector field, generating the observer's notion of time-translation symmetry.

The second source of observer-dependence of radiation is the observer's notion of simultaneity. Radiation, as a *rate* of energy, also depends on how the observer does their time-slicing, or what events they regard as simultaneous. For inertial motions, the relativities of energy and simultaneity cancel each other under Lorentz transformations, and that is why the rate of radiation is Lorentz-invariant. But as shown by the calculations of Rohrlich, the cancellation does not persist for more general motion. These two inputs, stationarity and simultaneity, are both needed to define radiation. They are given by the data of a Killing field and a time function. We see that the concept of an observer cannot simply be quantified by the observer's worldline, but requires as input a frame of reference extending throughout a spacetime region.

As mentioned in section 2.1, Kretzschmar and Fugmann [20, 21] , as well as Hirayama [22, 23], have given the generalization of the Larmor formula to the case where observer and charge have arbitrary velocities and accelerations. The classic questions about the equivalence principle and radiation in accelerated frames may be treated as special cases of this. Suppose the particle has velocity v^μ and proper acceleration a^μ, and suppose the worldlines of the observer's reference frame have velocity u^μ and proper acceleration g^μ. Let $g = \sqrt{g_\mu g^\mu}$. Let

$$h^\mu_{\ \nu} = \delta^\mu_{\ \nu} + v^\mu v_\nu \tag{4.1}$$

be the projection orthogonal to v^μ. Define the Hirayama acceleration vector as

$$\alpha^\mu = h^\mu_{\ \nu}(a^\nu - g^\nu - g u^\nu) \tag{4.2}$$

All these quantities are to be evaluated at the retarded time, that is, at the point of emission. The general result is that the radiated power is proportional to $\alpha^\mu \alpha_\mu$.

The purpose of this chapter is to reproduce this result, via explicit calculation, for the case where $v^\mu = u^\mu$ and a^μ and g^μ are collinear. At the time of performing this calculation, the author was unaware of the work of Kretzschmar, Fugmann, and Hirayama.

Rindler coordinates describe how Minkowski spacetime looks to a uniformly accelerated observer. The main technical tool of this chapter is to use a family of Rindler coordinate systems, parameterized by their z-displacement, and by their proper acceleration. This way, the possibilities for displacement between observer and charge, and for their two independent accelerations, are at least partly parameterized.

4.2 The field and stress-energy of a point charge in arbitrary motion

This section consists of standard textbook material which can be found, for example, in [28] chapter 5. We will use signature $-, +, +, +$, and employ dot product notation for the four-dimensional dot product. That is, $A \cdot B = A_\mu B^\mu$. Consider a particle of charge q in Minkowski spacetime, with four-velocity v^μ and proper four-acceleration a^μ. Let $R^\mu = x^\mu - x_0^\mu$ be the four-dimensional displacement vector, where x_0^μ is the location of the particle. The Lienard-Wiechert potential of the point charge in arbitrary motion is

$$A_\nu = \frac{q v_\nu}{4\pi v \cdot R} \tag{4.3}$$

where everything is evaluated at the retarded time, or to put it another way, on the future lightcone, characterized by $R \cdot R = 0$. The field tensor is

$$F_{\mu\nu} = \frac{q}{4\pi (v \cdot R)^2} \left[R_\mu (a_\nu - f v_\nu) - R_\nu (a_\mu - f v_\mu) \right] \tag{4.4}$$

where

$$f = \frac{1 + a \cdot R}{v \cdot R} \tag{4.5}$$

The stress-energy tensor is

$$T_{\mu\nu} = \frac{q^2}{(4\pi)^2 (v \cdot R)^4} \left[R_\mu R_\nu (a^2 - f^2) + R_\mu (a_\nu - f v_\nu) + R_\nu (a_\mu - f v_\mu) + \frac{1}{2} g_{\mu\nu} \right] \tag{4.6}$$

The acceleration of the charge need not be uniform. The field on the future lightcone of an instant on the particle's worldline depends only on v^μ and a^μ at that instant. We may choose coordinates for the inertial frame so that the charge passes through the origin and is instantaneously at rest, with its acceleration in the z-direction. Also in the inertial frame, let

$$x_\perp = \sqrt{x^2 + y^2} \tag{4.7}$$

$$r = \sqrt{x_\perp^2 + z^2} \tag{4.8}$$

With the inertial frame coordinates so chosen, we have

$$R^\mu = r\hat{t} + z\hat{z} + x_\perp \hat{x}_\perp \tag{4.9}$$

$$a^\mu = a\hat{z} \tag{4.10}$$

$$v^\mu = \hat{t} \tag{4.11}$$

$$v \cdot R = -r \tag{4.12}$$

$$a \cdot R = az \tag{4.13}$$

The parameter a may be positive or negative. The idea is to fix the origin as the point of emission we are interested in. We will look at the field on the future light cone of the origin. The behavior of the charge at other points of its trajectory does not matter in this context, because we are using a massless field. The emission point is fixed at the origin, and the observer's coordinate system will be moved around.

4.3 A family of displaced Rindler frames

Introduce a family of Rindler frames with coordinates χ and τ, and parameters χ_0 and g, taken to be positive. The coordinates are related to the inertial coordinates by

$$t = \frac{\chi}{g}\sinh(g\tau) \qquad z + \frac{\chi_0}{g} = \frac{\chi}{g}\cosh(g\tau) \tag{4.14}$$

and cover the wedge of spacetime where $z + \frac{\chi_0}{g} > |t|$. This setup is depicted in figure 4.1. In these coordinates, the metric is

$$g_{\mu\nu} = -\chi^2 d\tau^2 + \frac{d\chi^2}{g^2} + dx^2 + dy^2 \tag{4.15}$$

representing a uniform gravitational field. Rindler time τ is a time coordinate for which the metric is invariant. Among τ-stationary observers, those at $\chi = 1$ are privileged in that their proper time coincides with the Rindler time τ. The worldlines of τ-stationary observers are in uniform acceleration, although with differing values of proper acceleration. Privileged observers at $\chi = 1$ have proper acceleration equal to g. The origin of the inertial coordinate system, where the charge is located, in the Rindler coordinates is at $\chi = \chi_0$. Thus χ_0 represents the vertical displacement of the charge in the Rindler spacetime, and also represents a redshift factor between the observer and the charge.

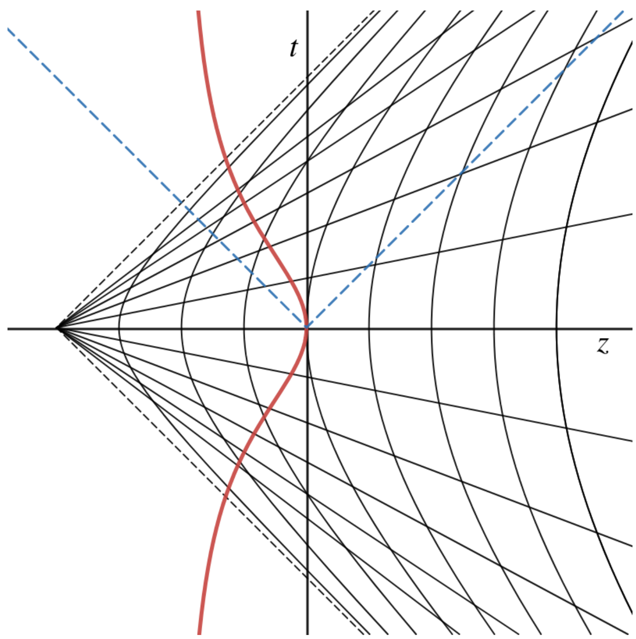

Figure 4.1: The shifted Rindler coordinate system, with an example source trajectory (red), and the future lightcone of the origin (blue)

The τ symmetry of Rindler spacetime is associated with a Killing field

$$\xi^\mu = g\left(z + \frac{\chi_0}{g}\right)\hat{t} + gt\hat{z} \tag{4.16}$$

which gives us the Rindler observers' notion of conserved energy.

The 3-dimensional geometry of a constant τ hypersurface is Euclidean. The future lightcone of the origin intersected with such a surface turns out to be a sphere centered at

$$\chi = \chi_0 \cosh(g\tau) \tag{4.17}$$

with radius

$$\rho = \frac{\chi_0}{g}\sinh g\tau \tag{4.18}$$

On this sphere,

$$t = r = \frac{\chi\rho}{\chi_0} \tag{4.19}$$

The geometry of the lightcone sphere as seen in Rindler space is shown in figure 4.2. We may introduce a polar angle θ on the lightcone sphere, and express the other coordinates in terms of ρ and θ:

$$x_\perp = \rho\sin\theta \tag{4.20}$$

$$z = \frac{g\rho}{\chi_0}\left(\rho + \sqrt{\rho^2 + \frac{\chi_0^2}{g^2}}\cos\theta\right) \tag{4.21}$$

$$r = \frac{g\rho}{\chi_0}\left(\sqrt{\rho^2 + \frac{\chi_0^2}{g^2}} + \rho\cos\theta\right) \tag{4.22}$$

The unit normal vector to the lightcone sphere is

$$\begin{aligned}\hat{n} &= \cos\theta\,\hat{\chi} + \sin\theta\,\hat{x}_\perp \\ &= \frac{g\rho\cos\theta}{\chi_0}\,\hat{t} + \frac{g\cos\theta}{\chi_0}\sqrt{\rho^2 + \frac{\chi_0^2}{g^2}}\,\hat{z} + \sin\theta\,\hat{x}_\perp\end{aligned} \tag{4.23}$$

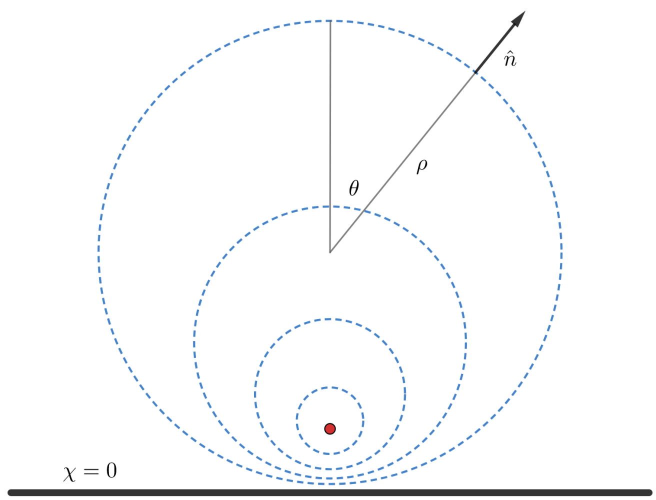

Figure 4.2: The future lightcone of the origin at successive times τ as seen in Rindler space, shown in the x-χ plane

4.4 Evaluating the Rindler energy flux on the light-sphere

We may now calculate the flux of Rindler energy through the lightcone sphere as

$$S = -T_{\mu\nu}\xi^\mu\hat{n}^\nu \tag{4.24}$$

The needed ingredients are the following dot products:

$$R \cdot \xi = -r\chi_0 \tag{4.25}$$

$$R \cdot \hat{n} = \rho \tag{4.26}$$

$$\hat{n} \cdot \xi = 0 \tag{4.27}$$

$$(a - fv) \cdot \hat{n} = \frac{\rho \cos\theta}{r\chi_0}(a\chi_0 - g) \tag{4.28}$$

$$(a - fv) \cdot \xi = -\frac{1}{r}(gz + \chi_0 - az\chi_0 - agx_\perp^2) \tag{4.29}$$

Putting these together in the stress tensor gives

$$\begin{aligned} S = &-\frac{q^2}{(4\pi)^2 r^4}\Big[-\frac{\rho\chi_0}{r}(a^2x_\perp^2 - 2az - 1) - \rho\cos\theta(a\chi_0 - g) \\ &\qquad - \frac{\rho}{r}(gz + \chi_0 + az\chi_0 - agx_\perp^2)\Big] \\ = &-\frac{q^2\rho}{(4\pi)^2 r^5}(a\chi_0 - g)(-ax_\perp^2 + z - r\cos\theta) \\ = &\frac{q^2\rho}{(4\pi)^2 r^5}(a\chi_0 - g)\left(ax_\perp^2 - \frac{g\rho^2\sin^2\theta}{\chi_0}\right) \\ = &\frac{q^2\rho^3\sin^2\theta}{(4\pi)^2\chi_0 r^5}(a\chi_0 - g)^2 \end{aligned} \tag{4.30}$$

At this point several qualitative features are apparent.

- In the $g \to 0$ limit, the radiation has the same features as Larmor radiation.
- An inertial charge, with $a = 0$, does radiate for accelerated observers.
- A Rindler-stationary charge is one with $a\chi_0 = g$, and does not radiate.

Recall that a may be positive or negative, accounting for whether the observer and charge accelerate in the same or opposite direction. However χ_0 must be positive, since otherwise the future lightcone of the origin does not intersect the observer's Rindler wedge.

4.5 Digression on the interpretation of the conservation equation

In arbitrary spacetimes, a conservation law is expressed locally by the equation

$$\nabla_\mu J^\mu = 0 \tag{4.31}$$

Suppose we have a static metric of the form

$$g_{\mu\nu} = -u(x)dt^2 + g^{(3)}_{\mu\nu}(x) \tag{4.32}$$

where $g^{(3)}$ is a metric on 3-dimensional space independent of t. The Rindler spacetime is of this form, but also others like Schwarzschild. The vector field J^μ decomposes as $(J^t, \overrightarrow{J})$. Then the conservation equation takes the form

$$\frac{\partial}{\partial t}J^t + \frac{1}{\sqrt{u}}\nabla \cdot (\sqrt{u}\overrightarrow{J}) = 0 \tag{4.33}$$

where the second term involves the 3-dimensional divergence with respect to $g^{(3)}$. This can be rewritten

$$\frac{\partial}{\partial t}(\sqrt{u}J^t) + \nabla \cdot (\sqrt{u}\overrightarrow{J}) = 0 \tag{4.34}$$

The interpretation is that the conserved quantity has density $\sqrt{u}J^t$ and flux $\sqrt{u}\overrightarrow{J}$. The volume factor $\sqrt{u}$ must be included, even though it does not appear in $g^{(3)}$. This result is sufficient for the purpose of the present Rindler spacetime calculation. It is generalized in section 4.7 to the case where the Killing vector need not be hypersurface-orthogonal.

4.6 Results

In Rindler spacetime, the volume factor is χ, making the flux

$$\chi S = \frac{q^2\rho^2\sin^2\theta}{(4\pi)^2 r^4}(a\chi_0 - g)^2 \tag{4.35}$$

This may now be integrated over the light-sphere to give the power

$$
\begin{aligned}
P &= \int \chi S \, \rho^2 d\Omega \\
&= \frac{q^2(a\chi_0 - g)^2}{(4\pi)^2} \int_0^\pi \frac{\rho^4}{r^4} \sin^2\theta \, 2\pi \sin\theta \, d\theta \\
&= \frac{q^2(a\chi_0 - g)^2}{8\pi} \int_0^\pi \left(\sqrt{1 + \left(\frac{g\rho}{\chi_0}\right)^2} + \frac{g\rho}{\chi_0}\cos\theta \right)^{-4} \sin^3\theta \, d\theta
\end{aligned}
\tag{4.36}
$$

The θ integral is equal to $4/3$ for all values of ρ, so the final result is independent of ρ:

$$
\begin{aligned}
P &= \frac{2}{3} \, \frac{q^2(a\chi_0 - g)^2}{4\pi} \\
&= \frac{2}{3} \, \frac{q^2\chi_0^2}{4\pi} \left(a - \frac{g}{\chi_0} \right)^2
\end{aligned}
\tag{4.37}
$$

We may interpret this result in the language of Hirayama as in (4.2), and verify that it is in agreement with the general formula found in [22, 23]. a^μ and v^μ are as given in equations (4.10) and (4.11). In the Rindler frame, the τ-stationary worldlines have velocity $u^\mu = \frac{1}{\chi}\xi^\mu$, and proper acceleration

$$
g^\mu = u^\nu \nabla_\nu u^\mu = \frac{g^2}{\chi^2} \left(t\hat{t} + (z + \chi_0/g)\hat{z} \right) \tag{4.38}
$$

These must be evaluated at the origin, giving $u^\mu = \hat{t}$ and $g^\mu = \frac{g}{\chi_0}\hat{z}$. Since $u^\mu = v^\mu$, the Hirayama acceleration vector simplifies to

$$
\alpha^\mu = a^\mu - g^\mu = \left(a - \frac{g}{\chi_0} \right) \hat{z} \tag{4.39}
$$

and the radiation is indeed proportional to $\alpha^\mu \alpha_\mu$. The factor of χ_0^2 may be interpreted as a redshift factor between the source, located at $\chi = \chi_0$, and the observer, located at $\chi = 1$. The power is doubly redshifted to account for both the redshift of the energy, and also the time dilation.

4.7 Conservation laws in a spacetime with a Killing vector and a time function

This section generalizes the result of section 4.5 in a coordinate-independent manner, dropping the assumption that the Killing vector be hypersurface-orthogonal. If a spacetime is equipped with a stationary frame of reference in the form of a Killing vector and a time function, this structure allows the equation $\nabla_\mu J^\mu = 0$ to be interpreted by stationary observers as a local conservation law for some conserved quantity, in a manner compatible with the observers' notions of time-translation and simultaneity. Consider a spacetime equipped with a timelike Killing vector field ξ^μ and a time function t that is compatible in the sense that its Lie derivative is equal to 1:

$$\pounds_\xi(t) = 1 \tag{4.40}$$

In particular, this means

$$\pounds_\xi(\nabla_\mu t) = 0 \tag{4.41}$$

so the foliation of constant-t hypersurfaces is preserved under the transformation generated by ξ^μ. This data represents a notion of time-translation and simultaneity. It allows us to interpret spacetime as spatial manifold Σ with a time evolution. Let $h_{\mu\nu}$ be the restriction of the metric to the constant-t surface Σ. $h_{\mu\nu}$ is preserved under time evolution, and as a Riemannian metric, endows Σ with its own geometry and covariant derivatives. We seek to interpret the conservation equation in a manner compatible with this picture.

An arbitrary vector field J^μ uniquely decomposes as

$$J^\mu = J^0 \xi^\mu + j^\mu \tag{4.42}$$

where

$$J^0 = J^\mu \nabla_\mu t \tag{4.43}$$

$$j^\mu \nabla_\mu t = 0 \tag{4.44}$$

meaning that j^μ is tangent to the surfaces of constant t. We may then calculate

$$\nabla_\mu J^\mu = \pounds_\xi J^0 + \nabla_\mu j^\mu \tag{4.45}$$

However, the second term is not the same as the divergence of j^μ with respect to the metric $h_{\mu\nu}$ on Σ. The derivative operator D on Σ associated with $h_{\mu\nu}$ is the orthogonal projection onto Σ of the operator ∇. Hence the divergence we are interested in is

$$D_\mu j^\mu = \nabla_\mu j^\mu + n_\mu n^\nu \nabla_\nu j^\mu \tag{4.46}$$

where n^μ is the unit normal vector to Σ. Using the fact that $j^\mu n_\mu = 0$, this equals

$$= \nabla_\mu j^\mu - j^\mu n^\nu \nabla_\nu n_\mu \tag{4.47}$$

Now introduce a function f defined by

$$n_\mu = f \nabla_\mu t \tag{4.48}$$

Now we have

$$\begin{aligned} D_\mu j^\mu &= \nabla_\mu j^\mu - j^\mu n^\nu \nabla_\nu (f \nabla_\mu t) \\ &= \nabla_\mu j^\mu - j^\mu n^\nu f \nabla_\nu \nabla_\mu t \end{aligned} \tag{4.49}$$

Since ∇ is torsion-free, this equals

$$\begin{aligned} &= \nabla_\mu j^\mu - j^\mu n^\nu f \nabla_\mu \nabla_\nu t \\ &= \nabla_\mu j^\mu - j^\mu n^\nu f \nabla_\mu \left(\frac{n_\nu}{f} \right) \end{aligned} \tag{4.50}$$

Finally, since $n^\nu n_\nu = -1$,

$$\begin{aligned} &= \nabla_\mu j^\mu + j^\mu f \nabla_\nu (1/f) \\ &= f \nabla_\mu \left(\frac{j^\mu}{f} \right) \end{aligned} \tag{4.51}$$

This result holds for arbitrary vector fields tangent to Σ. Hence all together,

$$\nabla_\mu J^\mu = \pounds_\xi J^0 + \frac{1}{f} D_\mu (f j^\mu) \tag{4.52}$$

Note that

$$\frac{1}{f} = n^\mu \nabla_\mu t = \sqrt{\nabla_\mu t \nabla^\mu t} \tag{4.53}$$

and consequently, $\pounds_\xi(f) = 0$. The conservation equation can be rewritten

$$0 = \pounds_\xi (f J^0) + D_\mu (f j^\mu) \tag{4.54}$$

By using Gauss's law in Σ, we have the interpretation that there is a conserved quantity of density $f J^0$ and flux $f j^\mu$.

A uniformly accelerated charge coupled to a Proca field emits radiation. An expression for the Proca field resulting from such a source is given by

$$A = \left[\frac{q}{\sqrt{2}\pi a} \int_0^\infty k_\perp K_1(\rho/a) Y_1 \left(\rho \frac{e^{a\eta}}{a} \right) J_0(k_\perp x_\perp) dk_\perp \right] d\zeta \tag{5.1}$$

A calculation of the rate of emission of the classical particle number gives the result.

$$\text{Rate} = \frac{q^2 a}{2\pi^2} \int_0^\infty \left| K_1 \left(\sqrt{x^2 + \frac{m^2}{a^2}} \right) \right|^2 x \, dx \tag{5.2}$$

This result is plotted in figure 5.1 as a function of m/a. The methods of Unruh mode expansion and Minkowski mode expansion both obtain this result, and it also agrees with the results of quantum mechanics.

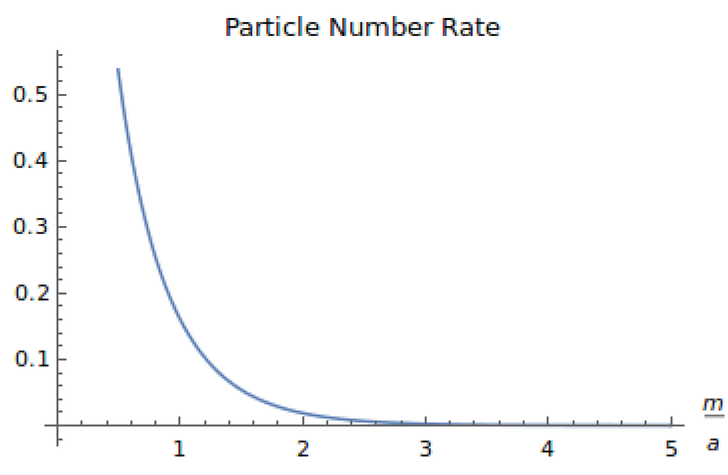

Figure 5.1: The emitted particle number rate, equation (5.2), in units of $\frac{q^2 a}{2\pi^2}$

The Minkowski mode expansion can also be used to obtain an expression for the emitted energy, which is

$$E = \frac{q^2 a^2 T_{\text{inert}}}{2\pi^2} \int_0^\pi \int_{\frac{m}{a \sin\theta}}^\infty r^3 \sin^3\theta \left| K_1(r(1 + i\cos\theta)) \right|^2 \, dr d\theta \tag{5.3}$$

This result is plotted in figure 5.2 as a function of m/a.

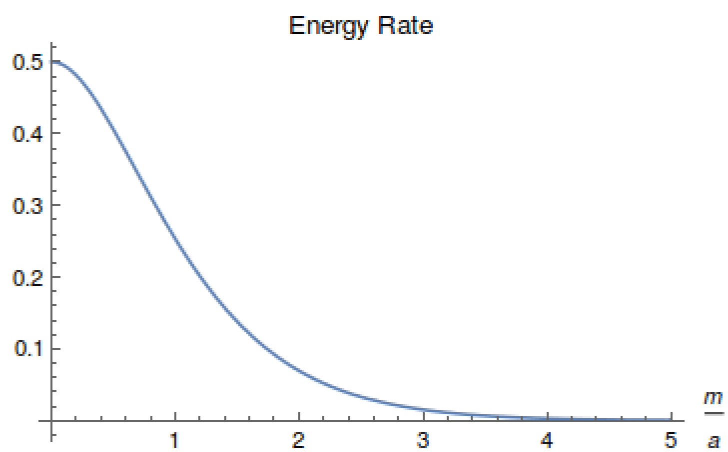

Figure 5.2: The emitted power, equation (5.3), in units of $\frac{q^2 a^2}{4\pi}$

While these results refer to the long-time limit, in which the charge is conserved, the calculational methods utilize regularization of the source such that the charge is not conserved. That this is possible is a feature of the Proca field.

An alternative to regularizing the charge is to regularize its acceleration. If we consider a hyperbolic trajectory with limiting velocity $v < 1$, the result for the radiated energy is

$$E = \frac{q^2 v^2 a}{3\pi^2 \sqrt{1 - v^2}} \int_{m/a}^\infty \sqrt{x^2 - \frac{m^2}{a^2}} \left(2 + \frac{m^2}{a^2 x^2}\right) \left| K_1(x) \right|^2 \, x \, dx \tag{5.4}$$

We can also consider Proca radiation from a charge at rest whose magnitude changes in time.

General expressions for the emitted particle number and energy in this case are

$$N = \frac{1}{4\pi^2 m^2} \int_0^\infty \frac{k^4}{\sqrt{k^2+m^2}} \left| \tilde{q}\left(\sqrt{k^2+m^2}\right) \right|^2 \, dk \tag{5.5}$$

$$E = \frac{1}{4\pi^2 m^2} \int_0^\infty k^4 \left| \tilde{q}\left(\sqrt{k^2+m^2}\right) \right|^2 \, dk \tag{5.6}$$

The results confirm that such radiation can be neglected in the limit where the charge varies slowly enough.

If the charge is both uniformly accelerating and changing in time, a general expression for the spectrum of the emitted particle number is found to be

$$dN = \frac{|\tilde{q}(\omega)|^2}{4\pi^4 a} \cosh(\pi\omega/a) \left(\left| K'_{i\omega/a}(\rho/a) \right|^2 + \frac{\omega^2 k_\perp^2}{m^2 \rho^2} \left| K_{i\omega/a}(\rho/a) \right|^2 \right) \, d\omega d^2 k_\perp \tag{5.7}$$

Finally, a case study for the electromagnetic field confirms Hirayama's generalization of the Larmor formula for accelerating observers. In the context of a certain family of Rindler frames, with source acceleration a and observer acceleration g, the radiated power is found to be

$$P = \frac{2}{3} \frac{q^2 \chi_0^2}{4\pi} \left(a - \frac{g}{\chi_0} \right)^2 \tag{5.8}$$

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