ON FINDING A SORT OF REGULAR VALUE OF POSITIVE DEFINITE DIAGONALIZED SYMMETRIC MATRICES

A Thesis

by

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ABSTRACT

We claim that given restrictions over multiple Banach Spaces (and usually one Hilbert Space) with a common arbitrary defining variable constant, we can attain a unique numerically ordered regular value of any n-dimensional *Sym*$^*(n, \mathbb{R})$ with dimension 1 using methods of Optimization.

Through this, we hope to assist in constraint based research in infinite dimensions in the field of Differential Geometry.

We will, throughout this paper, study the existence and uniqueness of a two variable solution towards a Foundation for this regular value formulation of $X \in$ Diagonalized *Sym*$^*(n, \mathbb{R}) \subset$ *Sym*$^*(n, \mathbb{R}) \subset \mathbf{M}(n, \mathbb{R})$, briefly study the equivalence of two formulations that we conjecture under further restrictions on our variable m represent the same Foundation aforementioned, and look over several numerical examples to accompany our study.

DEDICATION

To my mother, father, uncle Amit, aunt Orit, aunt Sue, grandmother Bubie, brothers and sister, and finally to my late grandfathers and grandmother, Saaba, Zadie, and Safta.

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NOMENCLATURE

Banach Space	A complete vector space with a norm.
Hilbert Space	A vector space with an inner product such that the inner product of an element with itself is the respective Banach norm squared.
Sym*	The set of Symmetric Positive Definite Matrices where a Symmetric Matrix holds the same values on it's upper triangle as it's lower triangle reflected across the diagonal. A Positive Definite Matrix is then a matrix such that every one of it's Eigenvalues is greater than zero, i.e. $x^T Ax > 0$ for all $A \in$ *Sym** and x arbitrary.
DSym*	Diagonalized Symmetric Positive Definite Matrices, where the only values (positive) therein are the Eigenvalues ordered from least to greatest along the diagonal.
ℓ^1	Banach Space of sequences whose series is absolutely convergent; i.e., $\sum_{n=1}^{\infty} \lvert x_n \rvert < \infty$.
ℓ^2	Hilbert Space of square-summable sequences whose squared series is absolutely convergent, i.e., $\sum_{n=1}^{\infty} \lvert x_n \rvert^2 < \infty$.
ℓ^p	Banach Space of p-power-summable sequences whose p-power series is absolutely convergent, i.e., $\sum_{n=1}^{\infty} \lvert x_n \rvert^p < \infty$.
Ψ_A	The ordered matrix obtained through a column by column stack of Eigenvectors of A ordered in accordance to their relative size.
Δ	Delta is the symbol used to represent the product of all the coordinates divided by the radius in spherical polar coordinates, i.e. $\prod_{i=1}^{n} \frac{x_i}{r}$.
Ω	Omega is the symbol used to represent the sum of all the coordinates divided by the radius in spherical polar coordinates, i.e. $\sum_{i=1}^{n} \frac{x_i}{r}$.

Φ_i	Phi$_i$ is the symbol used to represent the sum of all the coordinates taken to the i^{th} power divided by the radius to the i^{th} power in spherical polar coordinates, i.e. $\sum_{i=1}^{n}\left(\frac{x_i}{r}\right)^i$

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1. INTRODUCTION AND SETUP

1.1 Setup, Questions, & Justification

Firstly, we should understand what the definition of a diagonalized *Sym*$^*(n, \mathbb{R})$ is: In order to calculate our regular value we must apply two fundamental constraint first. Namely, the following,

$$\det(X) = \prod_{i=1}^{n} x_i = 1 \quad \text{where } x_i \text{ represents diagonal elements of } X \tag{1.1}$$

and

$$0 < x_1 \leq \cdots \leq x_n. \tag{1.2}$$

What then constitutes a regular value of *Sym*$^*(n, \mathbb{R})$ (positive [or negative] definite symmetric matrices)? Are *Sym*$^*(n, \mathbb{R})$ matrices diagonalizable?[1] Furthermore, how should we define the diagonalization function over *Sym*$^*(n, \mathbb{R})$? Why do positive (negative) definite Symmetric Matrices necessitate positive (or negative) real valued eigenvalues?

1.1.1 Differential Geometry Intro

To start things off here, we must define a function from *Sym*$^*(n, \mathbb{R})$ to *DSym*$^*(n, \mathbb{R})$ in the following manner

$$\Phi : \mathit{Sym}^*(n, \mathbb{R}) \longrightarrow \mathit{DSym}^*(n, \mathbb{R}) \quad \text{where} \quad \Phi(A) = \Psi_A^{-1} A \Psi_A = \Psi_A^T A \Psi_A. \tag{1.3}$$

Here Ψ_A is the ordered matrix obtained through the Eigenvectors of A ordered in accordance to their relative size. An eigenvector x, as is commonly known, is a vector such that $Ax = \lambda x$ for eigenvalues λ, or for our purposes, x_i, where the matrix making up Ψ_A is a column by column stack of these eigenvectors.

Differential Geometry Grunt Work — Regularity

For $A \in$ *Sym*$^*(n, \mathbb{R})$ and $B \in T_A$*Sym*$^*(n, \mathbb{R})$, we have, for $\alpha : (-\varepsilon, \varepsilon) \longrightarrow$ *Sym*$^*(n, \mathbb{R})$ given by $t \longmapsto A + tB$,

$$d\Phi_A(B) = (\Phi \circ \alpha)'(t)\Big|_{t=0} = \left[(\Psi_*(t))^T(A+tB)\Psi_*(t)\right]'\Big|_{t=0} \tag{1.4}$$

$$= (\Psi_*'(t))^T A\Psi_*(t) + (\Psi_*(t))^T A\Psi_*'(t) + (\Psi_*'(t))^T tB\Psi_*(t) + (\Psi_*(t))^T B\Psi_*(t) + (\Psi_*(t))^T tB\Psi_*'(t)\Big|_{t=0}$$

$$= (\Psi_*'(0))^T A\Psi_*(0) + (\Psi_*(0))^T A\Psi_*'(0) + (\Psi_*(0))^T B\Psi_*(0)$$

$$= \Psi_B^T A\Psi_A + \Psi_A^T A\Psi_B + \Psi_A^T B\Psi_A \tag{1.5}$$

Differential Geometry Grunt Work — Lie Algebra

So the Lie Algebra over *DSym*$^*(n, \mathbb{R})$, which we'll denote by $\mathfrak{dsym}^*(\mathbb{R}_n^+) \cong d\Phi_I(\mathbb{R}) \cong$ ker$d\Phi_I$, is then defined as

$$\Psi_B^T I\Psi_I + \Psi_I^T I\Psi_B + \Psi_I^T B\Psi_I = 0$$

$$\Longrightarrow B = -\Psi_I\Psi_B^T - \Psi_B\Psi_I^T = -(\Psi_B^T + \Psi_B) \tag{1.6}$$

and the question arises as to what significance this holds and how diagonal matrices with determinant one (or possibly negative one in association with negative definite symmetric matrices with odd dimension) arise as regular values.

1.1.2 Differential Geometry Justification

We begin by taking the inverse of the det$\circ\Phi$ function by first writing them separately as follows

$$\text{det}^{-1} : \mathbb{R} \longrightarrow Sym(n, \mathbb{R})$$

$$\Phi^{-1} : \textit{DSym}^*(n, \mathbb{R}) \longrightarrow \textit{Sym}^*(n, \mathbb{R}) \quad \text{where} \quad \Phi^{-1}(D) = \Psi_A\Phi(A)\Psi_A^T$$

where the inverse is then

$$\Phi^{-1} \circ \det{}^{-1} : \mathbb{R} \longrightarrow Sym^{*}(n, \mathbb{R}). \tag{1.7}$$

Given that the determinant must be 1 (or -1) we are attempting to determine whether

$$(\det \circ \Phi)^{-1}(1) \quad \text{and} \quad (\det \circ \Phi)^{-1}(-1) \tag{1.8}$$

in the case of odd negative definite symmetric matrices constitutes a Submanifold. Here, I will argue graphically via 2 Dimensional substrata that it deterministically does.

1.1.3 Differential Geometry Justification — Graph

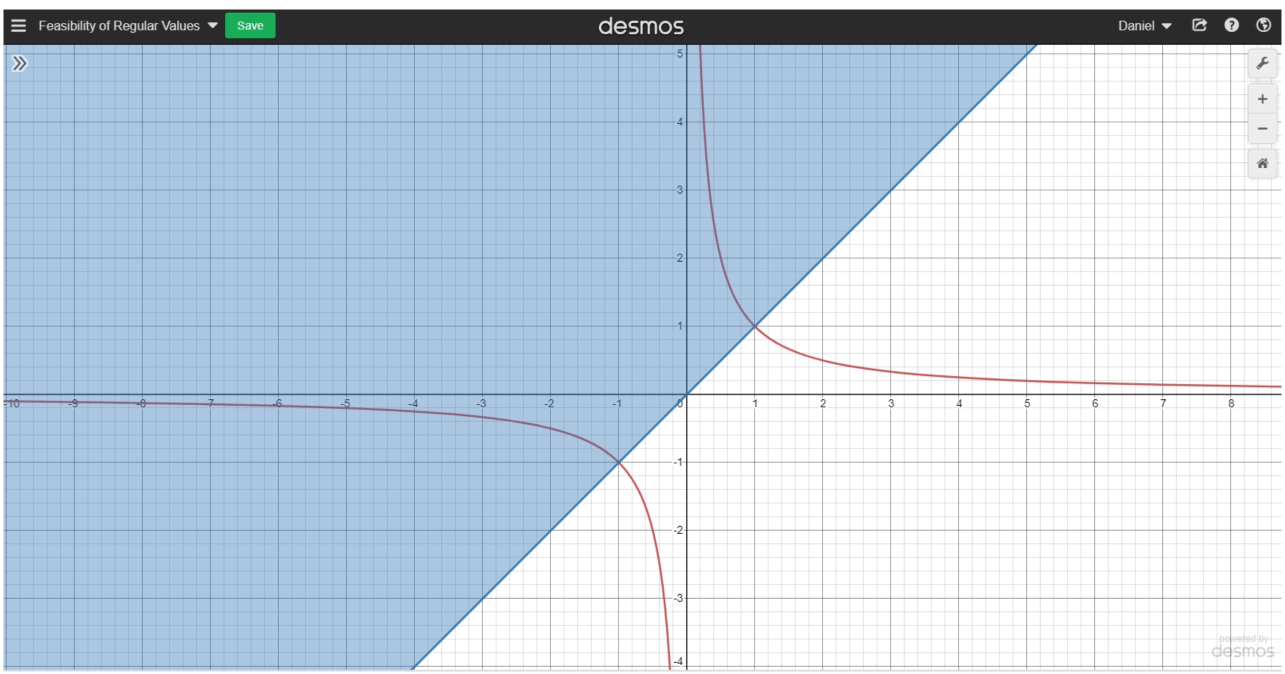

Figure 1.1: Here we see a 2 Dimensional basic representation that provides a basis primarily for the intersection of the conditions $\prod_{i=1}^{n} x_i = 1$ and $x_1 \leq \ldots \leq x_n$ where the second condition is graphically ornamental but important to visualize; so, then, we see with this graphical representation that (given the $Sym(n, \mathbb{R})$ determinants are dense in the real numbers) every local neighborhood of the Submanifold that represents this intersection is a regular value. This extends to n Dimensions and Submanifolds therein.

2. OPTIMIZATION PROBLEM DERIVATION AND EXPOSITION

2.1 Optimization Problem Intro

2.1.1 Introduction to Optimization

We then start by delineating the number of Dimensions comprising our solution we're working with as some finite positive real $\mathbb{R}^n$. Each variable, in turn, will presuppose one of the n diagonals of our *DSym*$^*(n, \mathbb{R})$ matrix. We now, for a change of pace, define our constraints over this Inequality that allows us to choose $n \in \mathbb{N}$ and $m \in \mathbb{R}^+$ and retrieve a unique (Claim!) n-sized list of Banach spaces that generate a working and functional solution.

2.1.2 Setup for Optimization

These constraints are defined as follows: These enumerable spaces will then be Banach spaces ℓ^p with integers p such that $1 \leq p \leq n-2$ represents an exhaustive list of all integers represented in our first constraint problem. Alternatively, we may use Optimization methods to arrive at the same result (Conjecture!) using ulterior methods of Optimization and Banach spaces ℓ^q with integers q such that $1 \leq q \leq n-1$ represents a similar exhaustive list of integers represented in our second presented constraint problem.

2.2 Problem Statement

2.2.1 Optimization Problem (Weak)

Because we do not have preference for specific components and they are all symmetric, without loss of generality, we may assume we have numerically ordered $x = \{x_i\}_{i=1}^n \in \mathbb{R}_n^+$ such that $0 < x_1 \leq \ldots \leq x_n$

$$\prod_{i=1}^{n} x_i = 1 \qquad \left\{ \|x\|_{\ell^p}^p = m^{\frac{1}{n-p}} \right\}_{p=1}^{n-3} \qquad \|x\|_{\ell^{n-2}}^{n-2} \leq \sqrt{m} \tag{2.1}$$

Notably, a special case here where we do not need the last inequality is in our 3 Dimensional case, and therefore omit the inequality there as it is unnecessary. We say that a point $x \in \mathbb{R}_n^+$ is feasible if it satisfies the above ordering, product constraint, and Banach constraints, and that

$$J(x) := J(x_1, \ldots, x_n) = x_1 \tag{2.2}$$

in association with previously given ordering constraints and

$$x^* = \arg \min_{x \text{ feasible}} J(x) \tag{2.3}$$

is a solution to our problem for $m^{\frac{1}{n-1}} \geq n$ then we have a unique set for each m. And what we will attempt to show is whether there is a uniquely defined solution as we had claimed there was for each m.

2.2.2 Optimization Problem (Strong)

Alternatively, we can have the following formulas for $x = \{x_i\}_{i=1}^n \in \mathbb{R}_n^+$ such that for $0 < x_1 \leq \ldots \leq x_n$ as before

$$\prod_{i=1}^{n} x_i \geq 1 \qquad \left\{ \|x\|_{\ell^q}^q \leq m^{\frac{1}{n-q}} \right\}_{q=1}^{n-2} \tag{2.4}$$

are convex as the intersection of convex sets and we are seeking to minimize the distance to

$$\|x\|_{\ell^{n-1}} = m \qquad \text{for } m^{\frac{1}{n-1}} \geq n. \tag{2.5}$$

Conjecture 1 (Weak and Strong Equivalence)**.** *These 2 problems, weak and strong, are equivalent.*

Proof. Possibly foolish. Left to the reader's discretion. □

2.3 Justification and Equivalence of Optimization Problems

2.3.1 Justification

Something that may be concerning at this point is whether the sets in both Optimization Problems retain some nonempty intersection over the feasible sets. Namely over

$$\prod_{i=1}^{n} x_i = 1 \qquad \left\{ \|x\|_{\ell^p}^p = m^{\frac{1}{n-p}} \right\}_{p=1}^{n-3} \qquad \|x\|_{\ell^{n-2}}^{n-2} \leq \sqrt{m} \tag{2.6}$$

$$\prod_{i=1}^{n} x_i \geq 1 \qquad \left\{ \|x\|_{\ell^q}^q \leq m^{\frac{1}{n-q}} \right\}_{q=1}^{n-2} \tag{2.7}$$

we have that $x_1 + \cdots + x_n = m^{\frac{1}{n-1}}$ and for $2 \leq p < n-2$ as well that $x_1^p + \cdots + x_n^p = m^{\frac{1}{n-p}}$. As I will discuss in the next several sections, for the second problem, we have intersection.

2.3.2 Equivalence

This intersection happens because at the boundary of the convex set, we have algebraically that

$$\|x\|_{\ell^1}^p - \|x\|_{\ell^p}^p = m^{\frac{p}{n-1}} - m^{\frac{1}{n-p}} = m^{\frac{p(n-p)}{(n-1)(n-p)}} - m^{\frac{n-1}{(n-1)(n-p)}} > 0 \tag{2.8}$$

iff $p(n-p) > n-1$ iff $n > p+1$ which is one of our conditions for p. This proves that the ℓ^1 and ℓ^p spaces all intersect. In order to complete the proof, we'll need to compute whether arbitrary p and q spaces intersect for $p < q$. So we take

$$\|x\|_{\ell^p}^{pq} - \|x\|_{\ell^q}^{pq} = m^{\frac{q}{n-p}} - m^{\frac{p}{n-q}} = m^{\frac{q(n-q)}{(n-p)(n-q)}} - m^{\frac{p(n-p)}{(n-p)(n-q)}} > 0 \tag{2.9}$$

iff $q(n-q) > p(n-p)$ iff $(q-p)n > q^2 - p^2$ iff $n > q+p$. This is true for $2 < q, p \leq n-2$. As it turns out, I conjecture without proof that (2.6) and (2.7) are equivalent and provide equations (2.8) and (2.9) as proof. Here, we have then established existence of the first problem (2.6).

2.4 Convexity

2.4.1 Introduction to Convexity

In this and the following slides, we would like to show rigorously that the set defined by

$$F(x) = \left\{ x \in \mathbb{R}^n : \prod_{k=1}^{n} x_k \geq 1, 0 < x_1, \cdots, x_n \right\} \text{ is convex.} \tag{2.10}$$

$$\text{To do this, we let } f(x) = \left\{ x \in \mathbb{R}^n : \frac{1}{\prod_{k=1}^{n} x_k}, \text{ for } 0 < x_1, \cdots, x_n \right\}. \tag{2.11}$$

To show that $f(x)$ is convex, we only have to show that its second derivative is a symmetric positive definite matrix at any point $x = (x_1, \cdots, x_n)^T$, with $0 < x_1, \cdots, x_n$.

2.4.2 Convexity Formulas

We then have the following identities

$$f_{x_i} = \frac{-1}{x_i^2 \prod_{k \neq i} x_k} \tag{2.12}$$

$$f_{x_i x_j} = \frac{1}{x_i^2 x_j^2 \prod_{k \neq i,j} x_k} = \frac{1}{\prod_{k=1}^{n} x_k^3} x_i x_j \prod_{k \neq i,j} x_k^2 \tag{2.13}$$

$$f_{x_i x_i} = \frac{2}{x_i^3 \prod_{k \neq i} x_k} = \frac{2 \prod_{k \neq i} x_i^2}{\prod_{k=1}^{n} x_k^3} \tag{2.14}$$

$$\nabla^2 f(x) = \frac{1}{\prod_{k=1}^{n} x_k^3} \left\{ \left[x_i x_j \prod_{k \neq i,j} x_k^2 \right]_{1 \leq i,j \leq n} + \text{diag} \left(\prod_{k \neq i} x_k^2 \right) \right\} \tag{2.15}$$

2.4.3 Convexity Exposition

For any vector $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)^T \in \mathbb{R}^n$ with $\overline{x} \neq \theta$, we have that

$$\overline{x}^T \nabla^2 f(x) \overline{x} = \frac{1}{\prod_{k=1}^{n} x_k^3} \left\{ \overline{x}^T \left[x_i x_j \prod_{k \neq i,j} x_k^2 \right] \overline{x} + \overline{x}^T \text{diag} \left(\prod_{k \neq i} x_k^2 \right) \overline{x} \right\} \tag{2.16}$$

$$= \frac{1}{\prod_{k=1}^{n} x_k^3} \left\{ \left(\sum_{i=1}^{n} \overline{x}_i \prod_{k=1}^{n} x_k \right)^2 + \sum_{i=1}^{n} \overline{x}_i^2 \prod_{k \neq i} x_k^2 \right\} > 0, \tag{2.17}$$

since $\prod_{k=1}^{n} x_k^2 > 0$ and $\overline{x} \neq \theta$. Thus $f(x)$ is convex. Then $\forall\, r > 0$, the set

$$\left\{ x \in \mathbb{R}^n : \prod_{k=1}^{n} x_i \geq r, 0 < x_1, \cdots, x_n \right\} = \left\{ x \in \mathbb{R}^n : f(x) \leq \frac{1}{r}, 0 < x_1, \cdots, x_n \right\} \tag{2.18}$$

is closed and convex, even though this convex set is unbounded, since all other balls defined by Banach norms in (2.7) are closed, bounded, and convex.

We also note that the point $x = (1, \ldots, 1)$ satisfies all the inequality constraints in (2.7) with $\prod_{k=1}^{n} x_i = 1,\ \|x\|_{\ell^q}^q \leq m^{\frac{1}{n-q}}$ for all $q = 1, \ldots, n-2$ and $m \geq n^{n-1}$. We conclude that the feasible set defined by inequalities in (2.7) is a closed and bounded convex set and therefore a compact set. Next we note that the objective function $J(x) = \|x\|_{\ell^{n-1}}^{n-1}$ is continuous thus attains its minimum at the feasible set: That is, we have established the existence of the second alternate problem.

3. NUMERICAL EXAMPLES DISCRETELY SOLVED IN THREE AND BRIEFLY SKETCHED IN FOUR DIMENSIONS

3.1 Three Dimensional Discrete Solution — Introduction

3.1.1 Three Dimensional Figure

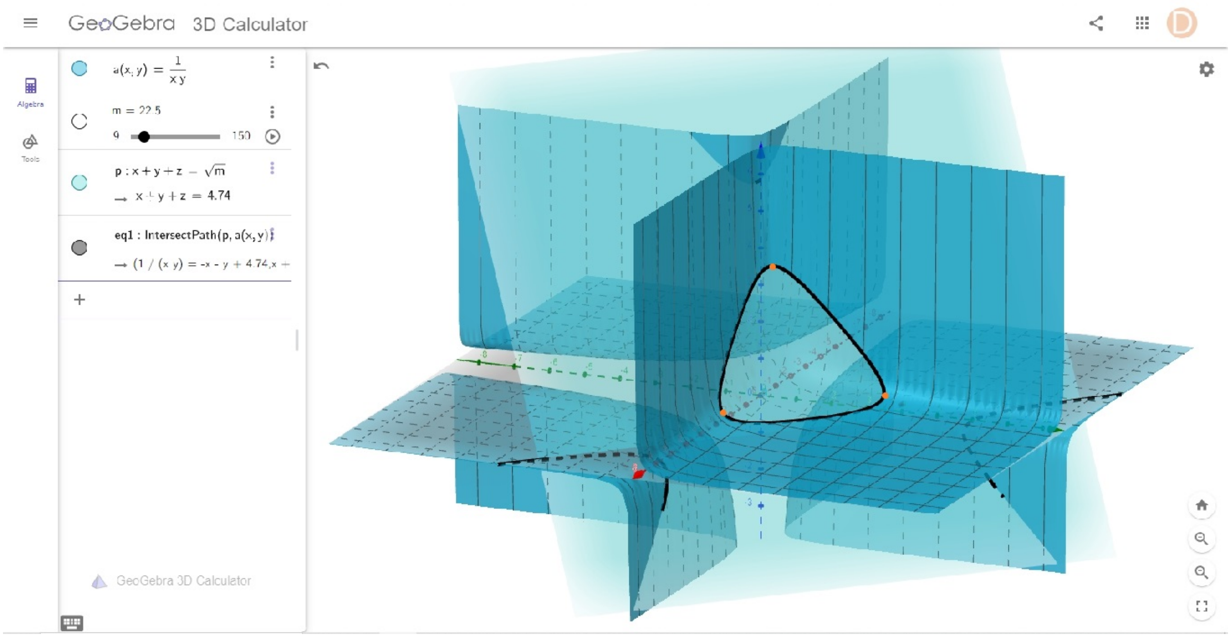

Figure 3.1: Here, we can clearly see a delimited solution for a three Dimensional Basis of regular values of the Diagonalization process with respect to the Special Orthogonal group, with x, y, and z solution pairs marked in orange in direct correspondence with the solution below.

3.1.2 Three Dimensional Intuition

In order to study whether there is a unique solution, we prove the particular case for 3 dimensions first to get some idea of the intuition behind this concept. We note that in this archetypal simple case, we have, $\vec{x} = (x, y, z)$ given $0 < x \leq y \leq z$,

$$xyz = 1 \qquad\qquad x + y + z = \sqrt{m} \geq n \qquad\qquad x^* = \arg \min_{\vec{x}\text{ feasible}} x \tag{3.1}$$

and we would like to find an ordered $\vec{x}$ such that such a unique functional solution exists at the extremum. So we then take

$$x = r\sin(\varphi)\cos(\theta) \qquad y = r\sin(\varphi)\sin(\theta) \quad z = r\cos(\varphi) \tag{3.2}$$

as our spherical polar coordinates and find that

$$r^3(\sin^2(\varphi)\cos(\varphi)\sin(\theta)\cos(\theta)) = r^3\Delta = 1 \quad \text{so that} \quad r^3 = \frac{1}{\Delta} \tag{3.3}$$

$$r(\sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta) + \cos(\varphi)) = r\Omega = \sqrt{m} \quad \text{so that} \quad r = \frac{\sqrt{m}}{\Omega} \tag{3.4}$$

$$\text{We then find that we have} \quad r = \frac{1}{\Delta^{\frac{1}{3}}} = \frac{\sqrt{m}}{\Omega}.$$

This gives us 2 systems of equations (3.3) and (3.4) with three variables r, φ, and θ (remember that m is given arbitrarily as a defining constant of our inequality) under our previous condition that one of the variables must attain a minimum. Without loss of generality, we again assume the set is ordered as $0 < x \leq y \leq z$ and take x to be the variable we are minimizing.

3.2 Three Dimensional Discrete Solution — Solution

3.2.1 Three Dimensional Solution — part 1

We then use some hand-drawn calculations as follows

$$x + \frac{1}{xz} + z = \sqrt{m} \tag{3.5}$$

and taking the derivative with respect to z we have,

$$x' - \frac{x'z + x}{x^2z^2} + 1 = 0 \tag{3.6}$$

or

$$x'(z) = \frac{\frac{x}{(xz)^2} - 1}{1 - \frac{z}{(xz)^2}} = \frac{x(1 - xz^2)}{z(x^2 z - 1)} = \frac{D_1}{D_2} = 0. \tag{3.7}$$

This must then either mean that $x'(z)$ maintains an extremum when either $D_1 = 0$ or $D_2 = 0$ rendering $x'(z) = 0$ and $x'(z)$ as undefined respectively. We start with the second case, namely $D_2 = 0$, or

$$x^2 z = 1 \implies x = \sqrt{\frac{1}{z}} \text{ and } y = \sqrt{\frac{1}{z}} \tag{3.8}$$

via plug and play since $y = \frac{1}{xz}$ here. Now, taking

$$2\sqrt{\frac{1}{z}} + z = \sqrt{m} \tag{3.9}$$

we can solve for a cubic polynomial equation of z and x represented as a function of z as follows

$$z^{\frac{3}{2}} - \sqrt{m} z^{\frac{1}{2}} + 2 = 0 \tag{3.10}$$

with $A = 1$, $B = 0$, $C = \sqrt{m}$, and $D = 2$, and using Wolfram Alpha over cubic systems of equations, we find the solution to this problem to then be $x = y = \frac{1}{\sqrt{z}}$ and

$$z = \left(\frac{\left(\sqrt{3m^{\frac{3}{2}} - 81} - 9\right)^{\frac{2}{3}} + \sqrt[3]{3}\sqrt{m}}{3^{\frac{2}{3}} \sqrt[3]{\sqrt{3m^{\frac{3}{2}} - 81} - 9}} \right)^2 \tag{3.11}$$

where we take m under the constraints previously discussed to retain a permissible answer.

3.2.2 Three Dimensional Solution — part 2

We then move on to the first case, namely $D_1 = 0$, or

$$xz^2 = 1 \implies x = \frac{1}{z^2} \text{ and } y = z \tag{3.12}$$

via plug and play since $y = \frac{1}{xz}$ here. Now, taking

$$\frac{1}{z^2} + 2z = \sqrt{m} \tag{3.13}$$

we can solve for a cubic polynomial equation of z and x represented as a function of z as follows

$$2z^3 - \sqrt{m}z^2 + 1 = 0 \tag{3.14}$$

with $A = 2$, $B = \sqrt{m}$, $C = 0$, and $D = 1$, and using Wolfram Alpha over cubic systems of equations, we find the solution to this problem to then be $x = \frac{1}{z^2}$ and

$$z = \frac{1}{6}\left(\frac{m}{\sqrt[3]{m^{\frac{3}{2}} + 6\sqrt{3}\sqrt{m^{\frac{3}{2}} - 27} - 54}} + \sqrt[3]{m^{\frac{3}{2}} + 6\sqrt{3}\sqrt{m^{\frac{3}{2}} - 27} - 54} + \sqrt{m}\right) \tag{3.15}$$

where we take m under the constraints previously discussed to retain a permissible answer. Now the only way to rigorously test whether either of these answers is our answer is to compare them to one another graphically as follows in Figure 3.2.

This completes our proof of uniqueness with respect to a three dimensional regular value of our Diagonalized Symmetric Positive Definite Matrix given some arbitrary chosen defining constant. Of course, we will consider more general cases for $\mathbb{R}^n$.

3.3 Four Dimensional Solution — Introduction

3.3.1 Four Dimensional Introduction

In a similar method as previously applied, within an understandably weaker form of discrete solution, under a four-dimensional split in our Diagonalized Symmetric Positive Definite Matrices, we have the three equations initially separate from the Weierstrass Maximization Theorem[2] as follows,

$$xyzw = 1 \qquad x + y + z + w = \sqrt[3]{m} \geq 4 \qquad x^2 + y^2 + z^2 + w^2 = \sqrt{m}. \tag{3.16}$$

Figure 3.2: Here, we can directly compare the two answers and find that only when $x'(z)$ is undefined do we have an answer based on our constraint such that x^* is minimized and z^* is maximized by their direct relationship. When $x'(z) = 0$, we in fact achieve a maximum x^* and a minimum z^* such that $0 < x^* \leq y^* \leq z^*$.

Taking spherical polar coordinates again, we retain the following equations

$$x = r\sin(\varphi)\cos(\theta)\cos(\phi) \qquad y = r\sin(\varphi)\cos(\theta)\sin(\phi) \tag{3.17}$$

$$z = r\sin(\varphi)\sin(\theta) \qquad w = r\cos(\varphi). \tag{3.18}$$

The first and second equations then become

$$r^4\sin^3(\varphi)\cos(\varphi)\cos^2(\theta)\sin(\theta)\cos(\phi)\sin(\phi) = r^4\Delta = 1 \tag{3.19}$$

and

$$r(\sin(\varphi)\cos(\theta)\cos(\phi) + \sin(\varphi)\cos(\theta)\sin(\phi) + \sin(\varphi)\sin(\theta) + \cos(\varphi)) = r\Omega = \sqrt[3]{m}. \tag{3.20}$$

3.4 Four Dimensional Solution — Existence and Uniqueness

3.4.1 Four Dimensional Existence

The second equation then simplifies r to

$$r^2 = \sqrt{m} \tag{3.21}$$

which then translates the system into

$$\Omega^{12} = \frac{1}{\Delta} = m, \tag{3.22}$$

which is a system of three equations (3.19), (3.20), and (3.21) with four unknowns r, φ, θ, and ϕ. If you also consider the minimality of our smallest variable x using the aforementioned Optimization method utilizing compactness and the Weierstrass Maximization Theorem[2], then we clearly have Existence of our Solution in Four Dimensions.

3.4.2 Four Dimensional Uniqueness

Taking $y = y(x, w)$ and $z = z(x, w)$, we have

$$x^2 + y^2(x, w) + z^2(x, w) + w^2 = \sqrt{m}. \tag{3.23}$$

Taking the derivative $\frac{d}{dw}(3.23)$ such that $x = x(w)$ we obtain the result

$$2xx' + 2y(x, w)(y_x(x, w)x' + y_w(x, w)) + 2z(x, w)(z_x(x, w)x' + z_w(x, w)) + 2w = 0 \tag{3.24}$$

$$\implies x'(x + y(x, w)y_x(x, w) + z(x, w)z_x(x, w)) + y(x, w)y_w(x, w) + z(x, w)z_w(x, w) + w = 0$$

$$x' = -\frac{y(x, w)y_w(x, w) + z(x, w)z_w(x, w) + w}{x + y(x, w)y_x(x, w) + z(x, w)z_x(x, w)} = \frac{E_1}{E_2} = 0, \tag{3.25}$$

which either results in $E_1 = 0$ with $x' = 0$ or $E_2 = 0$ with x' as undefined.

Solving $E_1 = 0$ for $x = x(w)$ then plugging it into (3.23) we have that

$$x^2(w) + y^2(x(w), w) + z^2(x(w), w) + w^2 = \sqrt{m}. \tag{3.26}$$

Solve it for w_1^*, then $x_1^* = x_1(w_1^*)$, $y_1^* = y_1(x_1^*, w_1^*)$, $z_1^* = z_1(x_1^*, w_1^*)$. For x' not defined we solve $E_2 = 0$ for $x = x_2(w)$, then plug it in to (3.23) to return the equation

$$x^2(w) + y^2(x(w), w) + z^2(x(w), w) + w^2 = \sqrt{m}. \tag{3.27}$$

Again, solve it for w_1^*, then $x_1^* = x_1(w_1^*)$, $y_1^* = y_1(x_1^*, w_1^*)$, $z_1^* = z_1(x_1^*, w_1^*)$. Comparing x_1^* with x_2^*, the smaller feasible value will be the minimum x-value.

3.4.3 Four Dimensional Conjecture

Conjecture

Conjecture 2 (Four Dimensions)**.**

$$\arg\max w^* \quad \Longrightarrow \quad \arg\min x^* = x(w^*)$$

$$\textit{and} \arg\min x^* \quad \Longrightarrow \quad \arg\max w^* = w(x^*)$$

$$\textit{with } y^* = z^* \textit{ for } n = 4.$$

This completes our sketch of a complete proof of uniqueness with respect to a four dimensional regular value of our Diagonalized Symmetric Positive Definite Matrix given some arbitrary chosen defining constant. But how does it behave when the dimension is greater than or equal to five? Is the solution still unique?

4.1 Five Dimensional Optimization

4.1.1 Five Dimensional Optimization — Introduction

We start by looking at a similar case to the Four Dimensional case, but in Five Dimensions. This leaves us with the following equations, with different variables for the sake of clarity, with $0 < a \leq b \leq c \leq d \leq e$

$$abcde = 1 \qquad a+b+c+d+e = \sqrt[4]{m} \qquad a^2+b^2+c^2+d^2+e^2 = \sqrt[3]{m} \tag{4.1}$$

$$a^3+b^3+c^3+d^3+e^3 \leq \sqrt{m} \qquad x^* = \arg \min_{x \text{ feasible}} a \tag{4.2}$$

We look again at this series of equations within the confines of polar coordinates, and immediately notice that

$$r^2 = \sqrt[3]{m} \implies r = \sqrt[6]{m} \implies m = r^6 \text{ giving us a fixed radius.} \tag{4.3}$$

This gives us one less variable to worry about through a direct normative relationship between the radius r and our given variable m.

4.1.2 Five Dimensional Optimization — Variables

We then take several known functional formulas, for context, as follows

$$\Delta = \sin^4(\theta_1)\cos^3(\theta_2)\cos^2(\theta_3)\cos(\theta_4)\cos(\theta_1)\sin(\theta_2)\sin(\theta_3)\sin(\theta_4) \tag{4.4}$$

$$\begin{aligned}\Omega = {} & \sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_4) \\ & + \sin(\theta_1)\cos(\theta_2)\sin(\theta_3) + \sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\end{aligned} \tag{4.5}$$

$$\Phi = \big(\sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4)\big)^3 + \big(\sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_4)\big)^3$$
$$+ \big(\sin(\theta_1)\cos(\theta_2)\sin(\theta_3)\big)^3 + \big(\sin(\theta_1)\sin(\theta_2)\big)^3 + \big(\cos(\theta_1)\big)^3 \tag{4.6}$$

4.1.3 Five Dimensional Optimization — Problem

We then arrive at the following equations naturally given our initial conditions and an easily verifiable conversion to polar coordinates:

$$r^5\Delta = 1 \tag{4.7}$$
$$r\Omega = m^{\frac{1}{4}} = r^{\frac{3}{2}} \quad \Longrightarrow \quad \Omega = r^{\frac{1}{2}} \tag{4.8}$$
$$r^3\Phi \leq m^{\frac{1}{2}} = r^3 \quad \Longrightarrow \quad \Phi \leq 1 \tag{4.9}$$

Since we've already narrowed down a relationship for r in terms of m, we are only left with four angular variables. The equation $\Phi = 1$ leaves us with three dependent angular variables, of our choosing. Furthermore, the two equations $\Omega^2 = \Delta^{-\frac{1}{5}} = r$ with fixed r gives us a single angular variable that is dependent.

We can then take the maximum over e or the minimum over a in order to yield the same result. Given polar coordinates

$$a = r\sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4) \qquad b = r\sin(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_4) \tag{4.10}$$

$$c = r\sin(\theta_1)\cos(\theta_2)\sin(\theta_3) \qquad d = r\sin(\theta_1)\sin(\theta_2) \qquad e = r\cos(\theta_1) \tag{4.11}$$

and since three of these angles are given by our pre-disclosed formulas, we only need to focus on a single angle. Suppose that angle is θ_1 for our purposes. Then we can presume beyond a reasonable doubt that our conclusion holds and our claim is correct. Namely, that our solution is unique!

4.2 Nine Dimensional Optimization

4.2.1 Nine Dimensional Optimization — Introduction

We then attempt to find a unique solution to a nine Dimensional system to further prove our point. Skipping past the usual introduction and referring to earlier parts of my Thesis for reference to our fundamental form, we have the following functional variables given that our radius r is fixed by the normative Hilbert space and our angular variables are given by $\{\phi_i\}_{i=1}^{8}$ and omitted for clarity.

4.2.2 Nine Dimensional Optimization — Variables

($\cos_t$ or $\sin_s$ for integers t, s is meant to represent t or s consecutive trig functions with incrementally differing angular variables where $\cos^r$ or $\sin^u$ still represents appropriate exponentials)

$$\begin{aligned}
\Delta =& \sin^8\cos^7\cos^6\cos^5\cos^4\cos^3\cos^2\cos \\
& \cos\ \ \sin\ \ \sin\ \ \sin\ \ \sin\ \ \sin\ \ \sin\ \ \sin \qquad (4.12)\\
\Omega =& \sin\cos_7+\sin\cos_6\sin+\sin\cos_5\sin+\sin\cos_4\sin \\
& +\sin\cos_3\sin+\sin\cos_2\sin+\sin\cos\sin+\sin_2+\cos \qquad (4.13)\\
\Phi_i =& \big(\sin\cos_7\big)^i+\big(\sin\cos_6\sin\big)^i+\big(\sin\cos_5\sin\big)^i+\big(\sin\cos_4\sin\big)^i \\
& +\big(\sin\cos_3\sin\big)^i+\big(\sin\cos_2\sin\big)^i+\big(\sin\cos\sin\big)^i+\big(\sin_2\big)^i+\big(\cos\big)^i \qquad (4.14)
\end{aligned}$$

for integers $3 \leq i \leq 7$.

4.2.3 Nine Dimensional Optimization — Problem

We then have the following intuitive equations using the previously defined functional variables Δ, Ω, and $\big\{\Phi_i\big\}_{i=3}^{7}$:

$$r^9\Delta = 1 \qquad r\Omega = \sqrt[8]{m} \qquad r^2 = \sqrt[7]{m} \qquad \Big\{r^i\Phi_i = m^{\frac{1}{n-i}}\Big\}_{i=3}^{6} \qquad r^7\Phi_7 \leq m^{\frac{1}{n-7}} \qquad (4.15)$$

which when simplified leads us to

$$r = \Delta^{-\frac{1}{9}} = \Omega^{\frac{4}{3}} = \sqrt[14]{m} = \left\{ \Phi_i^{\frac{n-i}{14-in+i^2}} \right\}_{i=3}^{6} \geq \Phi_7^{\frac{n-7}{63-7n}}. \tag{4.16}$$

5. CONCLUSION

5.1 General Solution — Introduction

This then, like clockwork, perspires into eight known equations over nine unknowns. We can then assume the Optimization function which minimizes our smallest Banach space's (with respect to our Geometric Integral) size. With this, we have proven uniqueness once again, which leads us to our motivation for a formal generalized solution to induce uniqueness. Our general solution is complete with $n-4$ given Φ_i equations and two Ω and Δ equations with one more equation marginalizing our resultant r in an effort toward normalisation with respect to our choice of m. The following is a completion of, and thereby a display of, the final form of our solution which renders a replete set of regular values for our diagonalized Special Orthogonal Matrix with just two input variables, dimension n and arbitrary m within bounds.

5.2 General Solution

For $3 \leq i \leq n-3$, we have

$$\Delta^{-\frac{1}{n}} = r \tag{5.1}$$

$$\Omega^{\frac{n-1}{n-3}} = r \tag{5.2}$$

$$\sqrt[2(n-2)]{m} = r \tag{5.3}$$

$$\left\{ \Phi_i^{\overline{\frac{n-i}{(2-i)n+i^2-4}}} \right\}_i = r \tag{5.4}$$

$$\Phi_{n-2} \leq 1 \tag{5.5}$$

Along with our minimization this should yield uniqueness!

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