# QUANTUM COLORING OF QUANTUM GRAPHS

A Dissertation

by

PRIYANGA GANESAN

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|---|---|
| Chair of Committee: | Eric Rowell |
| Committee Members: | Michael Brannan |
| | Kenneth Dykema |
| | Andreas Klappenecker |
| | Thomas Schlumprecht |
| Head of Department: | Sarah Witherspoon |

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# ABSTRACT

Quantum graphs are an operator space generalization of classical graphs that have appeared in different branches of mathematics including operator systems theory, non-commutative topology and quantum information theory. In this work, we develop a notion of quantum coloring for quantum graphs using a non-local game with quantum inputs & classical outputs that generalizes the coloring game for classical graphs.

Using this game, we define chromatic numbers for quantum graphs in the various (quantum) models and show that they are a analogue of D.Stahlke's [53] entanglement assisted chromatic numbers and that the classical model is equivalent to Kim & Mehta's [37] strong chromatic numbers for non-commutative graphs. We demonstrate explicit quantum colorings of all quantum complete graphs and prove that every quantum graph has a finite quantum chromatic number (but not necessarily classical chromatic number). We also show that every quantum graph is 4-colorable in the algebraic model.

Further, we obtain five lower bounds for the classical and quantum chromatic number of quantum graphs using the spectrum of the quantum adjacency operator. These bounds are achieved by applying a combinatorial characterization of quantum graph coloring obtained from the winning strategies of the quantum-to-classical nonlocal coloring game. We generalize all the spectral estimates of Elphick & Wocjan [19] to the quantum graph setting and in particular, prove a quantum generalization of the Hoffman's bound. We also demonstrate the tightness of our bounds in the case of quantum complete graphs.

# DEDICATION

To my father *Dr. S. Ganesan* and mother *Mrs. G. Uma Maheswari*

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## CONTRIBUTORS AND FUNDING SOURCES


### Contributors

This work was supported by a dissertation committee consisting of Professor Michael Brannan [Ph.D advisor], who is currently in the Department of Mathematics at the University of Waterloo, and Professor Eric Rowell [committee chair], Professor Kenneth Dykema, Professor Thomas Schlumprecht of the Department of Mathematics at Texas A&M University and Professor Andreas Klappenecker of the Department of Computer Science and Engineering at Texas A&M University.

Chapter 3 and 4 is based on joint work with Michael Brannan and Samuel Harris [5] and chapter 5 is based on my single-author preprint [23]. All other work conducted for the dissertation was completed by the author independently.

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# NOTATIONS AND NOMENCLATURE

| | |
|---|---|
| $[n]$ | discrete set $\{1, 2, \ldots, n\}$ |
| $\lvert\cdot\rangle$ | a column vector |
| $\langle\cdot\rvert$ | (conjugate transpose) row vector |
| $\longleftrightarrow$ | one-to-one correspondence |
| $M_n$ | set of all $n \times n$ complex matrices |
| $D_n$ | set of all diagonal $n \times n$ complex matrices |
| $e_i$ | unit vector whose $i^{th}$ entry is 1 and all other entries are 0 |
| $e_{ij}$ | matrix unit whose $i^{th}$ row, $j^{th}$ column has entry $1$ and all other entries are 0 |
| $\mathrm{Tr}$ | natural trace, given by summing all diagonal terms of a matrix |
| $B(\mathcal{H})$ | algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ |
| $\sigma(A)$ | spectrum of an operator $A$ |
| $G$ | a classical graph |
| $\mathcal{G}$ | a quantum graph |
| $K_c$ | classical complete graph on $c$ vertices |
| $\chi$ | classical chromatic number |
| $\chi_q$ | quantum chromatic number |
| POVM | Positive operator valued measure |
| PVM | Projection valued measure |
| UCP | Unital completely positive |
| CPTP | Completely positive and trace preserving |

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# 1. Introduction

Graph coloring has been well-studied in mathematics since the eighteenth century, with widespread applications in day-to-day life, including scheduling problems, register allocation, radio frequency assignments and sudoku solutions [46]. Traditionally, the coloring of a graph refers to an assignment of labels (called colors) to the vertices of a graph such that no two adjacent vertices share the same color. The chromatic number of a graph is defined to be the minimum number of colors for which such an assignment is possible.

More recently, a quantum generalization of the chromatic number was introduced within the framework of non-local games in quantum information theory [7]. The quantum chromatic number of a graph is defined as the minimal number of colors necessary in a quantum protocol in which two separated players, who cannot communicate with each other but share an entangled quantum state, try to convince an interrogator with certainty that they have a coloring for the given graph. There are known examples of classical graphs whose quantum chromatic number is strictly smaller than its classical chromatic number [7, 41], thus exhibiting the power of quantum entanglement. Quantum coloring games of classical graphs have close connections to Tsirelson's conjecture and the Connes embedding problem and have been extensively studied in the past decade [41, 47, 48, 52]. Motivated by coloring problems and non-local games, we investigate the notion of quantum coloring for *quantum graphs* in this dissertation.

Quantum graphs are a non-commutative generalization of classical graphs that have attracted significant attention in recent years due to their intriguing connections to several areas of mathematics, physics and computer science. Within information theory, quantum graphs appear quite naturally in the theory of zero-error communication in the form of confusability graphs of quantum channels. If the channel at hand is a noisy classical channel, the confusability graph is a finite simple graph on the input alphabet whose edges indicate which letters can be confused after passing through the channel. If the communication channel is genuinely quantum, then the role of the confusability graph is played by a more general structure [14], namely a *quantum graph*.

The idea of a quantum graph first appeared in [20], and has thereafter emerged independently in other disguises. In information theory, quantum graphs were introduced as a quantum analogue of the confusability graph of classical channels [14]. A more general definition was proposed in the context of quantum relations [58], which describes a quantum graph as a reflexive and symmetric quantum relation on a finite dimensional von-Neumann algebra. In [43], an equivalent perspective on quantum graphs was developed in a categorical framework (of quantum sets & quantum functions) using a quantum adjacency matrix. This idea was further generalized to the non-tracial setting in [4]. In recent years, research in quantum graph theory has undergone vast developments and quantum graphs have been explored in the context of zero-error quantum information theory, quantum error correction, operator algebras, non-local games, quantum symmetries, non-commutative topology and other fields [6, 8, 17, 25, 35, 42, 57]. There have also been multiple studies on the coloring of quantum graphs [37, 44, 53, 54], leading to different variants of the chromatic number of a quantum graph, in both the classical and quantum sense.

Our goal is to study the quantum coloring of quantum graphs and develop a double quantization of the chromatic number, namely the *quantum chromatic number of a quantum graph*. We achieve this by introducing a quantum-to-classical *non-local game* that extends the notion of coloring and chromatic numbers from classical graphs to quantum graphs. We adopt this approach as non-local games provide a convenient framework in which one can exhibit the advantages of using quantum entanglement as a resource to accomplish certain tasks. The general setup of a (classical input, classical output) two player non-local game is given in terms of a tuple $(I, O, \lambda)$, where $I$ and $O$ are finite sets and $\lambda : O \times O \times I \times I \to \{0, 1\}$ is a predicate function which determines the rules of the game. The game is played by two cooperating players, Alice and Bob, and a verifier (Referee). Each round proceeds by the verifier (randomly) selecting a pair of questions $(x, y) \in I \times I$ and sending $x$ to Alice and $y$ to Bob. Alice and Bob then respond with answers $(a, b) \in O \times O$. The verifier declares the round won if $\lambda(a, b, x, y) = 1$ and declares it lost if $\lambda(a, b, x, y) = 0$. The term *non-local* refers to the fact that during each round, Alice and Bob are spatially separated and are unable to communicate; neither Alice nor Bob knows which questions/answers the other received/returned. This non-locality makes winning each round of the game (with high probability) generally very difficult. It is in these scenarios that "quantum strategies" (which make use of some shared entangled resource between Alice and Bob) can allow the players to drastically improve their performance by better correlating their behaviors.

Within mathematics, the theory of non-local games has led to some spectacular developments in the field of operator algebras. Most notable here is the work of Junge-Navascues-Palazuelos-Perez-Garcia-Scholz-Werner [33], T. Fritz [22] and N. Ozawa [45] connecting the Connes-Kirchberg conjecture to Tsirelson's correlation sets in quantum information. Very recently, Ji-Natarajan-Vidick-Wright-Yuen [32] used non-local games to provide a counterexample to the Connes-Kirchberg conjecture. Another recent and quite remarkable application of non-local games in mathematics is the work of Mančinska-Roberson [40] which uses a non-local game, called the graph isomorphism game, to provide a quantum interpretation of pairs of graphs that admit the same number of homomorphisms from planar graphs. In particular, the graph coloring game, which is an example of a *synchronous* non-local game, has led to many developments in the operator algebraic aspects of non-local games. Winning strategies for synchronous games turn out to be completely described in terms of traces on a certain $*$-algebra associated to the game [30], bringing to bear many powerful operator algebraic techniques in the theory of non-local games.

In the present work, we introduce a quantum input-classical output non-local game [definition 4.1.4] that captures the coloring problem for quantum graphs. The inputs for our quantum graph coloring game are elements from a suitably chosen basis for the operator space of the quantum graph. These inputs are *quantum* in the sense that they are tensor products, where one player receives the left leg of the tensor and the other receives the right leg. The players respond individually with classical outputs, namely color assignments. They win the round if their responses jointly satisfy a synchronicity condition and respect the adjacency structure of the quantum graph. This game generalizes the non-local coloring game for classical graphs and leads to chromatic numbers for quantum graphs in different mathematical models: $loc, q, qs, qa, qc, C^*, hered, alg$. We show that the winning strategies of this quantum-to-classical nonlocal coloring game also give rise to a neat combinatorial characterization of quantum graph coloring [theorem 4.1.7].

The chromatic numbers introduced in our framework connect nicely with other versions of chromatic numbers in the literature. In particular, we prove that it is a special case of Stahlke's [53] entanglement-assisted chromatic number for non-commutative graphs [theorem 4.1.7] and that they agree with Kim & Mehta's [37] strong chromatic numbers in the classical case [theorem 4.1.9]. We demonstrate explicit quantum colorings of all quantum complete graphs using unitary error basis tools [theorem 4.2.4]. Specifically, we deduce that every quantum graph has a finite quantum chromatic number, while its classical chromatic number is infinite, unless the graph itself is classical [theorem 4.2.9]. We also show interesting extensions of classical results in this framework. In particular, we show that the game algebra of the 4-coloring game for a quantum graph is always non-trivial, and hence every quantum graph is four-colorable in the algebraic model [theorem 4.2.10].

It is useful to estimate the chromatic numbers of quantum graphs as they are closely related to the zero-error capacity of quantum channels [14]. However, computing the chromatic number of a general graph is an NP-hard problem. In classical graph theory, inequalities involving the eigenvalues of the adjacency matrix are often used to estimate the chromatic number. We adapt a similar idea and obtain five lower bounds for the classical and quantum chromatic numbers of quantum graphs. We achieve this by associating a spectrum [definition 5.1.1] to the quantum graph using the quantum adjacency operator and applying the combinatorial characterization of quantum coloring. Using techniques from Elphick & Wocjan [19], we show that several well-known classical bounds on the chromatic number also hold true in the quantum graph setting. Notably, we generalize the Hoffman's bound to quantum graphs [theorem 5.2.2]. We also introduce quantum analogues for the edge number, Laplacian and signless Laplacian along the way [definition 5.3.4], and demonstrate the tightness of our bounds in the case of quantum complete graphs [5.7].

**This dissertation is organized as follows:**

Chapter 2 provides the necessary background on quantum graphs and the connections between different definitions. This is largely based on literature from the articles [4,43,56,58].

Chapter 3 develops the general theory of non-local games with quantum inputs and classical outputs, which is required for chapter 4. We study the associated correlations, the various quantum models which give rise to it and discuss a generalization of synchronous correlations in this setting.

Chapter 4 introduces the coloring game for quantum graphs using the framework of quantum-to-classical non-local games. We study the corresponding winning strategies, game *-algebra and the chromatic numbers arising in this context. We also present the colorings of quantum complete graphs and results on algebraic colorings here.

In Chapter 5, we obtain the spectral lower bounds for the chromatic numbers of quantum graphs, followed by an illustration of the bounds for quantum complete graphs.

Chapter 6 contains concluding remarks and directions for future research.

# 2. Quantum graphs

Quantum graphs can be defined in different ways, as mentioned in the introduction. In this chapter, we review some definitions and relevant results on quantum graphs. Specifically, we look at the connection between different perspectives to quantum graphs and develop a dictionary (table) that allows for translation of graph properties between the diverse formalisms.

## 2.1 Quantum graphs as operator spaces

One way to describe quantum graphs is as operator spaces satisfying a certain bimodule property [58]. This is a direct generalization of the non-commutative graphs considered by R. Duan, S. Severini and A. Winter in [14], and D. Stahlke in [53]. This approach is most convenient for studying quantum coloring problems as it generalizes the edge structure of a classical graph. We describe this formalism first:

**Definition 2.1.1.** Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M} \subseteq B(\mathcal{H})$ be a (non-degenerate) von Neumann algebra. Let $\mathcal{M}' := \{x \in B(\mathcal{H}) \mid ax = xa \text{ for all } a \in \mathcal{M}\}$ denote the commutant of $\mathcal{M}$. A *quantum graph* on $\mathcal{M}$ is an operator space $S \subseteq B(\mathcal{H})$ that is closed under adjoint and is a bimodule over $\mathcal{M}'$, that is $\mathcal{M}'S\mathcal{M}' \subseteq S$. We denote this quantum graph by the tuple $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$.

The intuition is that $S$ contains operators that represent edges in the graph, as illustrated by the following example.

**Example 2.1.2.** Let $G$ be a classical graph on $n$ vertices. One can identify the vertex set of $G$ with the algebra of diagonal matrices $D_n \subseteq M_n$, by identifying each vertex $i$ with the diagonal matrix $e_{ii} \in D_n$. Then, $S_G = span\{e_{ij} : (i,j) \text{ is an edge in } G\} \subseteq M_n$ is a quantum graph over $D_n$.

**Remark 2.1.3.** Indeed, an operator space $S \subseteq M_n$ is of the form $S = S_G$ for some classical graph $G$ if and only $S$ is a bimodule over the diagonal algebra $D_n$ [58]. Also, two reflexive classical graphs $G_1, G_2$ are isomorphic if and only if their corresponding operator systems $S_{G_1}, S_{G_2}$ are unitally completely order isomorphic [44].

A "purely quantum" example is the following one:

**Example 2.1.4.** Let $\mathcal{M} = M_2$ and $S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$. Then $(S, M_2, B(\mathbb{C}^2))$ is a quantum graph on $M_2$ that doesn't arise from any classical graph.

**Remark 2.1.5.** It can be shown that the operator space $S$ associated to a quantum graph $(S, \mathcal{M}, B(\mathcal{H}))$ is essentially independent of the representation of $\mathcal{M}$ [56].

Motivated by confusability graphs in information theory, quantum graphs are generally assumed to be reflexive ($I \in S$) and hence, $S$ is an operator system in $B(\mathcal{H})$. But for the purposes of graph coloring, it is also common to consider irreflexive quantum graphs, that is quantum analogues of graphs without loops.

**Definition 2.1.6.** A quantum graph $(S, \mathcal{M}, B(\mathcal{H}))$ is said to be *irreflexive* if $S \subseteq (\mathcal{M}')^{\perp}$.

In particular, an irreflexive quantum graph on $M_n$ (with the standard representation $M_n = B(\mathbb{C}^n)$) is simply a self-adjoint traceless operator subspace in $M_n$. This is sometimes used as the definition of non-commutative graphs in the literature [53].

## 2.2 Quantum graphs with a quantum adjacency matrix

In chapter 5, we will take advantage of an alternate (but equivalent) definition of a quantum graph, which involves quantizing the vertex set and the adjacency matrix. This formalism was first introduced in [43] using the language of special symmetric dagger Frobenius algebras, and was later generalized to the non-tracial case in [4, 42]. In this perspective, the non-commutative analogue of a vertex set is played by a C*-algebra, which also carries the structure of a Hilbert space. It is defined as follows:

**Definition 2.2.1** (Quantum set)**.** A quantum set is a pair $(\mathcal{M}, \psi)$, where $\mathcal{M}$ is a finite dimensional C*-algebra and $\psi : \mathcal{M} \to \mathbb{C}$ is a faithful state.

Using $\psi$, one can view $\mathcal{M}$ as a Hilbert space $L^2(\mathcal{M}) = L^2(\mathcal{M}, \psi)$ obtained from the GNS representation of $\mathcal{M}$ with respect to $\psi$. That is, $L^2(\mathcal{M})$ is the vector space $\mathcal{M}$ equipped with the inner product $\langle x, y\rangle = \psi(y^*x)$.

**Notation 1.** *Let $m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ denote the multiplication map and $m^*$ denote the adjoint of $m$ when viewed as a linear operator from $L^2(\mathcal{M}) \otimes L^2(\mathcal{M}) \to L^2(\mathcal{M})$. Further, we denote the unit of $\mathcal{M}$ by $\mathbb{1}$ and let $\eta : \mathbb{C} \to \mathcal{M}$ be the unit map $\lambda \mapsto \lambda\mathbb{1}$. The adjoint of $\eta$ (as an operator on Hilbert spaces) is denoted by $\eta^*$ and is equal to $\psi$.*

While there are many choices for a faithful state $\psi$ on $\mathcal{M}$, we will restrict our attention to $\delta$-forms, as done in [4].

**Definition 2.2.2.** For $\delta > 0$, a state $\psi : \mathcal{M} \to \mathbb{C}$ is called a *$\delta$-form* if $mm^* = \delta^2 I$.

**Example 2.2.3.** Let $X$ be a finite set and $\mathcal{M} = C(X)$ be the algebra of continuous complex valued functions on $X$ . Then the uniform measure $\psi(f) = \frac{1}{|X|}\sum_{x\in X} f(x)$ is a $\delta$-form on $C(X)$ with $\delta^2 = |X|$. In this case, $m^*$ is given by $e_i \mapsto |X|(e_i \otimes e_i)$, where $e_i$ is the characteristic function on the set $\{i\} \subseteq X$.

**Example 2.2.4.** Let $\mathcal{M}$ be $M_n$ equipped with the canonical normalized trace $\psi = \frac{1}{n}\operatorname{Tr}$. Then $m^*(e_{ij}) = n\sum_{k=1}^{n} e_{ik} \otimes e_{kj}$, and $\psi$ is an $n$-form on $M_n$.

The $\delta$-forms in the above examples are tracial, that is $\psi(xy) = \psi(yx)$ for all $x, y \in \mathcal{M}$. A tracial $\delta$-form on a finite dimensional C*-algebra is unique and has a nice form, which will be used in later sections. We recall this now:

**Proposition 2.2.5.** *Let $\mathcal{M}$ be a finite dimensional C*-algebra, decomposed as $\mathcal{M} \cong \bigoplus_{i=1}^{N} M_{n_i}$, where $N, n_1, n_2, \ldots, n_N$ are some positive integers. Then, there exists a unique tracial $\delta$-form on $\mathcal{M}$ given by*

$$\psi = \frac{1}{dim(\mathcal{M})}\bigoplus_{i=1}^{N} n_i \operatorname{Tr}(\cdot) \tag{2.2.0.1}$$

*In this case, $\delta^2 = dim(\mathcal{M})$ and the state $\psi$ is called the Plancheral trace. Moreover, $\psi = \frac{1}{\dim(\mathcal{M})}\operatorname{Tr}|_{\mathcal{M}}$ , where $\operatorname{Tr} : B(L^2(\mathcal{M})) \to \mathbb{C}$ is the canonical trace.*

A quantum set endowed with an additional structure of a quantum adjacency matrix yields a quantum graph.

**Definition 2.2.6** ([4])**.** Let $\mathcal{M}$ be a finite dimensional C*-algebra equipped with a $\delta$-form $\psi$. A self-adjoint linear map $A : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is called a *quantum adjacency matrix* if it satisfies the following conditions:

1. $m(A \otimes A)m^* = \delta^2 A$,

2. $(I \otimes \eta^* m)(I \otimes A \otimes I)(m^* \eta \otimes I) = A$.

The tuple $\mathcal{G} = (\mathcal{M}, \psi, A)$ is called an (undirected) quantum graph.

The quantum graph $(\mathcal{M}, \psi, A)$ is said to be *reflexive* if it further satisfies the condition $m(A \otimes I)m^* = \delta^2 I$ or is said to be *irreflexive* if it satisfies the condition $m(A \otimes I)m^* = 0$.

The motivation for the above definition comes from the commutative setting where $\mathcal{M} = C(X)$ and $\psi$ is the uniform measure on $X$. In this case, the quantum adjacency matrix $A : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ can be identified with a matrix in $M_{|X|}(\mathbb{C})$, and the operation $\delta^{-2} m(P \otimes Q)m^*$ is simply the schur product of the matrices $P$ and $Q$, given by entrywise multiplication. So, the first condition in definition 2.2.6 says that $A$ must be an idempotent with respect to Schur multiplication, which is equivalent to saying that $A$ has entries in $\{0, 1\}$. The second condition says $A = A^T$. If we drop the second condition in definition 2.2.6, it is called a *directed* quantum graph [6].

**Remark 2.2.7.** The self-adjointness of $A$ along with condition (2) in definition 2.2.6 implies that $A$ is also *-preserving [42], that is $Ax^* = (Ax)^*$ for all $x \in \mathcal{M}$.

Every quantum set can be easily equipped with an adjacency operator to obtain a quantum graph. An example is the quantum complete graph.

**Definition 2.2.8.** Let $(\mathcal{M}, \psi)$ be a quantum set. A reflexive *quantum complete graph* on $\mathcal{M}$ is defined by $A = \delta^2 \psi(\cdot)\mathbb{1}$. In the classical case, this gives the all 1s matrix and corresponds to the reflexive complete graph on $\dim(\mathcal{M})$ vertices.

An *irreflexive* quantum complete graph on $(\mathcal{M}, \psi)$ is defined by $A = \delta^2 \psi(\cdot)\mathbb{1} - I$.

There are several non-trivial examples of quantum graphs. In particular, [42] gives a concrete classification of all undirected reflexive quantum graphs on $M_2$, and [25] gives an example of a quantum graph, which is not quantum isomorphic to any classical graph.

### 2.3 Connection between different approaches to quantum graphs

While the two definitions of quantum graphs given in 2.1.1 and 2.2.6 emerged independently and offer unique insights into the structure of a quantum graph, these perspectives can be shown to be equivalent [43, section 7]. The goal of this section is to review this connection, so that one can take advantage of both the perspectives. In particular, we will be adopting the operator space formalism (definition 2.1.1) in chapter 4 to develop the coloring game and then utilizing the quantum adjacency matrix formalism (definition 2.2.6) in chapter 5 to achieve the spectral bounds.

**Notation 2.** *Let $\mathcal{M}^{op}$ denote the opposite algebra of $\mathcal{M}$ and ${}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M})))$ denote the set of completely bounded maps $P$ on $B(L^2(\mathcal{M}))$ with the property $P(axb) = aP(x)b$, for all $x \in B(L^2(\mathcal{M}))$, $a, b \in \mathcal{M}'$.*

The translation between the definitions 2.1.1 and 2.2.6 can be achieved using the following two correspondences. A detailed algebraic proof for these correspondences may be found in [39].

(1) There is a well known *-isomorphism between $\mathcal{M} \otimes \mathcal{M}^{op}$ and ${}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M})))$ in finite dimensions [16] given by

$$\mathcal{M} \otimes \mathcal{M}^{op} \quad \cong \quad {}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M}))) \tag{2.3.0.1}$$

$$x \otimes y^{op} \quad \longleftrightarrow \quad x(\cdot)y. \tag{2.3.0.2}$$

(2) There is a bijective correspondence between linear operators $A \in B(L^2(\mathcal{M}, \psi))$ and elements $p \in \mathcal{M} \otimes \mathcal{M}^{op}$, given by

$$A(x) := \delta^2(\psi \otimes I)p(x \otimes 1), \qquad p := \delta^{-2}(I \otimes A)m^*(\mathbb{1}) \tag{2.3.0.3}$$

Combining (1) and (2), we get a bijective correspondence between $B(L^2(\mathcal{M}))$, $\mathcal{M} \otimes \mathcal{M}^{op}$ and ${}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M})))$, which allows us to translate between the different perspectives to quantum graphs.

$$\begin{array}{ccccc} B(L^2(\mathcal{M})) & \longleftrightarrow & \mathcal{M} \otimes \mathcal{M}^{op} & \longleftrightarrow & {}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M}))) \\ A & \longleftrightarrow & p & \longleftrightarrow & P \end{array}$$

We summarize this connection now.

**Proposition 2.3.1.** *Let $\mathcal{M}$ be a finite dimensional C*-algebra, equipped with its tracial $\delta$-form $\psi$.*

1. *Given a quantum graph $(\mathcal{M}, \psi, A)$, define $P : B(L^2(\mathcal{M})) \to B(L^2(\mathcal{M}))$ as*

$$P(X) = \delta^{-2} m(A \otimes X) m^*. \tag{2.3.0.4}$$

   *Then, range$(P)$ is a self-adjoint operator subspace in $B(L^2(\mathcal{M}))$ that is a bimodule over $\mathcal{M}'$.*

2. *Given a quantum graph $(S, \mathcal{M}, B(L^2(\mathcal{M})))$, let $P : B(L^2(\mathcal{M})) \to B(L^2(\mathcal{M}))$ denote a self-adjoint $\mathcal{M}' - \mathcal{M}'$ bimodule projection with $range(P) = S$.*

   *That is, $P(axb) = aP(x)b$, for all $x \in B(L^2(\mathcal{M}))$, $a, b \in \mathcal{M}'$ and $P^2 = P = P^*$, where the adjoint is taken with respect to the trace inner product on $B(L^2(\mathcal{M}))$. (Such a $P$ always exists and is unique for the given $S$ [56].)*

   *Then, $A : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ defined by*

$$A(x) = \delta^2 (\psi \otimes I) P(x \otimes 1) \tag{2.3.0.5}$$

   *is a quantum adjacency matrix on $(\mathcal{M}, \psi)$. Here, $P$ is interpreted as an element of $\mathcal{M} \otimes \mathcal{M}^{op}$ using (2.3.0.2).*

*The expressions (2.3.0.4) and (2.3.0.5) are inverses of each other.*

We note here that the correspondence between $S$ and linear operator $A$ is not one-to-one in general since there are several different $\mathcal{M}' - \mathcal{M}'$ bimodule idempotents $P$ with the same range $S$. However, there is a unique *self-adjoint* linear operator $A$ for a given $S$, which corresponds to the unique *orthogonal* bimodule projection onto $S$. In this case, $A$ is also completely positive, which was used as an alternate definition of quantum adjacency matrix in [8].

**Remark 2.3.2.** To get an $\mathcal{M}' - \mathcal{M}'$ bimodule map $\tilde{P}$ with range $S$, one can use the standard trick of averaging over the unitary group. Begin with a linear map $P = P^2$ with range $S$ and define

$$\tilde{P} = \iint\limits_{\mathcal{U}(\mathcal{M}')\times\mathcal{U}(\mathcal{M}')} a^{-1}P(axb)b^{-1}\; da\; db,$$

where $\mathcal{U}(\mathcal{M}')$ is the set of unitaries in $\mathcal{M}'$ and $da\; db$ represents integration with respect to the Haar measure on the compact unitary group $\mathcal{U}(\mathcal{M}')$. Then, $\tilde{P}$ is a $\mathcal{M}' - \mathcal{M}'$ bimodule map with range $S$. If we need $\tilde{P}$ to be self-adjoint, then we begin with a self-adjoint $P$.

**Summary**: We conclude this chapter with a table translating the properties of a quantum graph in the different perspectives. Let $\mathcal{M}$ denote a finite dimensional C*-algebra with its tracial $\delta$-form $\psi$. A quantum graph on $(\mathcal{M}, \psi)$ can be described in three main ways:

1. As an operator space $S \subseteq B(L^2(\mathcal{M}))$ that is a bimodule over the commutant of $\mathcal{M}$, (or alternately as an $\mathcal{M}' - \mathcal{M}'$ bimodule projection $P$ with range $S$),
2. As an idempotent element $p$ in $\mathcal{M} \otimes \mathcal{M}^{op}$,
3. As a "Schur idempotent" linear operator $A : L^2(\mathcal{M}) \to L^2(\mathcal{M})$.

It follows from the table that for undirected quantum graphs:

$$\begin{aligned} P^2 = P = P^* \quad &\iff \quad p^2 = p = p^* \\ &\iff \quad A \text{ is Schur-idempotent and real} \\ &\iff \quad A \text{ is Schur-idempotent and self-adjoint.} \end{aligned}$$

**Notation 3.** *In the following table:*

- $p = \sum_{i=1}^{t} a_i \otimes b_i \in \mathcal{M} \otimes \mathcal{M}^{op}$ *and* $\sigma$ *denotes the swap map on* $\mathcal{M} \otimes \mathcal{M}^{op}$*,*
- $m$ *denotes the multiplication map on* $\mathcal{M} \otimes \mathcal{M}$ *and* $\eta : \mathbb{C} \to \mathcal{M}$ *denotes the unit map* $1 \mapsto \mathbb{1}$
- $T \in B(L^2(\mathcal{M}))$*,* $\xi \in L^2(\mathcal{M})$ *and* $x, y \in \mathcal{M}'$*.*

Table 2.1: Different approaches to a quantum graph on $(\mathcal{M}, \psi)$

| PROPERTY (CLASSICAL GRAPH) | AS AN OPERATOR SPACE | AS A BIMODULE MAP | AS A PROJECTION | AS A QUANTUM ADJACENCY MATRIX |
|---|---|---|---|---|
| $G = (V, E, A)$ | $(\mathcal{M}, {}_{\mathcal{M}'}S_{\mathcal{M}'})$ <br> $S \subseteq B(L^2(\mathcal{M}))$ | $(\mathcal{M}, P)$ <br> $P \in {}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M})))$ | $(\mathcal{M}, p)$ <br> $p \in \mathcal{M} \otimes \mathcal{M}^{op}$ | $(\mathcal{M}, A)$ <br> $A \in B(L^2(\mathcal{M}))$ |
| Bimodule structure $E \subseteq V \times V$ | $\mathcal{M}'S\mathcal{M}' \subseteq S$ | $P(xTy) = xP(T)y$ | $\sum_i a_i(xTy)b_i = x(\sum_i a_i T b_i)y$ | $m(A \otimes xTy)m^* = x(m(A \otimes T)m^*)y$ |
| Schur idempotent $A \in M_n(\{0,1\})$ | $A \in S$ | $P^2 = P$ | $p^2 = p$ | $m(A \otimes A)m^* = \delta^2 A$ |
| Reflexive $A_{ii} = 1, \ \forall i$ | $\mathcal{M}' \subseteq S$ | $P(I) = I$ | $m(p) = \mathbb{1}$ | $m(A \otimes I)m^* = \delta^2 I$ |
| Irreflexive $A_{ii} = 0, \ \forall i$ | $\mathcal{M}' \perp S$ | $P(I) = 0$ | $m(p) = 0$ | $m(A \otimes I)m^* = 0$ |
| Undirected $A = A^T$ | $S = S^*$ | $P^*(T) = P(T^*)^*$ | $\sigma(p) = p$ | $(I \otimes \eta^* m)(I \otimes A \otimes I)$ $(m^* \eta \otimes I) = A$ <br> Alternatively, $A(\xi^*) = [A^*(\xi)]^*$ |
| Self adjoint $A = A^*$ | | $P(T^*) = P(T)^*$ | $\sigma(p) = p^*$ | $A(\xi) = A^*(\xi)$ |
| Real ($A = \overline{A}$) | | $P^*(T) = P(T)$ | $p^* = p$ | $A(\xi^*) = (A(\xi))^*$ |
| Positivity ($A$ is completely positive) | | $P$ is positive ($P = G^*G$) | $p$ is positive ($p = g^*g$) | $A$ is completely positive |

# 3. Nonlocal games with quantum inputs and classical outputs

In recent years, the theory of non-local games has risen to a level of great prominence in quantum information theory and related parts of physics and mathematics. In this project, we are mainly interested in a non-local game called the *graph coloring game* and certain extensions of it. Before concentrating on the quantum graph coloring game, we first develop some general theory on non-local games with quantum questions and classical answers, which will be needed for our discussion. Such games have already been used in the two-output context of quantum XOR games [27, 51].

## 3.1 Quantum input - classical output correlations

This section develops the general theory of quantum input - classical output correlations and the various quantum models which give rise to such correlations.

Recall that in a two player non-local game on $n$-classical inputs and $c$-classical outputs, the main objects of study are the bipartite correlation sets $C(n,c) \subseteq \mathbb{R}^{n^2 \times c^2}$ which model the players' behavior. The elements of the correlation sets are conditional probabilities $[p(a,b|x,y)]$, namely the probability that the players Alice and Bob return answers $a$ and $b$ (respectively), given that they received questions $x$ and $y$ (respectively).

$$X = [p(a,b|x,y)]_{1 \le a,b \le c, 1 \le x,y \le n} \in C(n,c) \subseteq (D_n \otimes D_n)^{c^2}. \tag{3.1.0.1}$$

The correlations (behaviors) $X \in C(n,c)$ that are physically relevant are those which can be realized using a (quantum) strategy. That is, by Alice and Bob performing joint measurements on a quantum mechanical system prepared in some initial state. Mathematically, a quantum strategy involves two (finite-dimensional) Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, family of positive operator-valued measure (POVMs) $\{p_1^x, p_2^x, \dots p_n^x\} \subseteq B(\mathcal{H}_A)$ corresponding to each input $x$, similarly POVMs $\{q_1^y, q_2^y, \dots q_n^y\} \subseteq B(\mathcal{H}_B)$ for each input $y$ and a state $\chi \in \mathcal{H}_A \otimes \mathcal{H}_B$. From this data, one obtains a correlation $X \in C(n,c)$ via the formula

$$p(a,b|x,y) = \langle p_a^x \otimes q_b^y \chi, \chi \rangle \,. \tag{3.1.0.2}$$

The subset of all correlations obtainable from quantum strategies as above is denoted by $C_q(n,c)$. In a similar manner, one can define other classes of correlations $C_t(n,c)$ ($t$ = local, quantum spatial, quantum approximate, quantum commuting) that are built from the corresponding classes of strategies. A review of all of these models may be found in [36].

To expand this framework and allow for quantum questions, we replace the question set $[n]\times[n]$ with the set of quantum states on the bipartite system $\mathbb{C}^n \otimes \mathbb{C}^n$. Our idea of a quantum input - classical output game is as follows: the inputs are quantum inputs, in the sense that the referee initializes the state space $\mathbb{C}^n \otimes \mathbb{C}^n$, where Alice has access to the left copy and Bob has access to the right copy of $\mathbb{C}^n$. Alice and Bob are allowed to share a(n entanglement) resource space $\mathcal{H}$ in some prepared state $\chi$. After receiving the input $\varphi$ on $\mathbb{C}^n \otimes \mathbb{C}^n$, they can perform measurements on the triple tensor product $\mathbb{C}^n \otimes \mathcal{H} \otimes \mathbb{C}^n$, and respond to the referee with classical outputs based on their measurements.

Our goal now is to develop the analogous notion of the correlation set $C(n,c)$ and its various subclasses arising from quantum strategies. In the following, our approach is somewhat backwards, in that we first define the different strategies and afterwards consider the associated correlations.

**Definition 3.1.1.** We define the different strategies associated to a two-player game with quantum questions on $\mathbb{C}^n \otimes \mathbb{C}^n$ and classical answers in $\{1,2,...,c\}$ as follows:

1. A **quantum strategy**, or a $q$**-strategy**, is given by two finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, a POVM $\{P_1,...,P_c\}$ on $\mathbb{C}^n \otimes \mathcal{H}_A$, a POVM $\{Q_1,...,Q_c\}$ on $\mathcal{H}_B \otimes \mathbb{C}^n$, and a state $\chi \in \mathcal{H}_A \otimes \mathcal{H}_B$.

2. A **quantum spatial strategy**, or a $qs$**-strategy**, is given in the same way as a $q$-strategy, except that we no longer assume that $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional.

3. A **quantum commuting strategy**, or a $qc$-**strategy**, is given by a single Hilbert space $\mathcal{H}$, a POVM $\{P_1, ..., P_c\}$ on $\mathbb{C}^n \otimes \mathcal{H}$, a POVM $\{Q_1, ..., Q_c\}$ on $\mathcal{H} \otimes \mathbb{C}^n$, and a state $\chi \in \mathcal{H}$, with the property that $(P_a \otimes I_n)(I_n \otimes Q_b) = (I_n \otimes Q_b)(P_a \otimes I_n)$ for all $a, b$.

4. A **local strategy**, or a **classical strategy**, is a quantum commuting strategy with the property that the set of operators $P_{a,ij}$ and $Q_{b,k\ell}$ generate a commutative $C^*$-algebra, where $P_a = (P_{a,ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ with $P_{a,ij} \in \mathcal{B}(\mathcal{H})$ and $Q_b = (Q_{b,k\ell}) \in M_n(\mathcal{B}(\mathcal{H}))$ with $Q_{b,k\ell} \in \mathcal{B}(\mathcal{H})$, for $1 \leq a, b \leq c$ and $1 \leq i, j \leq n$.

**Remark 3.1.2.** When viewed as block matrices, the commutation relation $(P_a \otimes I_n)(I_n \otimes Q_b) = (I_n \otimes Q_b)(P_a \otimes I_n)$ is easily seen to be equivalent to the requirement that $[P_{a,ij}, Q_{b,kl}] = 0 \in \mathcal{B}(\mathcal{H})$ for each $a, b, i, j, k, l$. (See, e.g., [9] and [28].)

Suppose now that the referee initializes $\mathbb{C}^n \otimes \mathbb{C}^n$ in the state $\varphi$. For a quantum strategy, the probability that Alice outputs $a$ and Bob outputs $b$ is given by

$$p(a, b|\varphi) = \langle (P_a \otimes Q_b)(\varphi \odot \chi), \varphi \odot \chi \rangle, \tag{3.1.0.3}$$

where $\varphi \odot \chi$ represents the (permuted) state in $\mathbb{C}^n \otimes (\mathcal{H}_A \otimes \mathcal{H}_B) \otimes \mathbb{C}^n$ rather than on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes (\mathcal{H}_A \otimes \mathcal{H}_B)$. For a quantum commuting strategy, we simply replace $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}$ and $(P_a \otimes Q_b)$ with $(P_a \otimes I_n)(I_n \otimes Q_b)$. We note that this definition of the probability of outputs can easily be extended to other (e.g. mixed) states in $\mathbb{C}^n \otimes \mathbb{C}^n$ that may not be included in the definition of the game. This is because the probabilities corresponding to Alice and Bob's strategy are encoded entirely in the *correlation* associated to their strategy.

Also, it is useful to note that for a non-local game with classical inputs, the associated family of POVMs may be consolidated as a single operator (corresponding to each output) as follows:

$$P_a := \begin{bmatrix} p_a^{x_1} & 0 & \dots & 0 \\ 0 & p_a^{x_2} & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & p_a^{x_n} \end{bmatrix} \in D_n \otimes B(\mathcal{H}_A). \tag{3.1.0.4}$$

While in the case of quantum inputs, we have:

$$P_a := \begin{bmatrix} P_{a,11} & P_{a,12} & \dots & P_{a,1n} \\ P_{a,21} & P_{a,22} & \dots & P_{a,2n} \\ \vdots & & \ddots & \vdots \\ P_{a,n1} & P_{a,n2} & \dots & P_{a,nn} \end{bmatrix} \in M_n \otimes B(\mathcal{H}_A). \tag{3.1.0.5}$$

**Example 3.1.3.** Let us look at some special cases of $\langle P_a \otimes Q_b(\varphi \odot \chi), (\varphi \odot \chi)\rangle$ below:

1. When $\varphi = e_i \otimes e_j$:

$$p(a,b|\varphi) = \langle P_a \otimes Q_b(\varphi \odot \chi), (\varphi \odot \chi)\rangle = \langle P_{a,ii} \otimes Q_{b,jj}\chi, \chi\rangle .$$

   This may be interpreted as $p(a,b|i,j)$ of the classical input-classical output game corresponding to input $(i,j)$.

2. When $\varphi = (e_i \otimes e_j) + (e_k \otimes e_l)$:

$$p(a,b|\varphi) = \langle P_{a,ii} \otimes Q_{b,jj}\chi, \chi\rangle + \langle P_{a,kk} \otimes Q_{b,ll}\chi, \chi\rangle + \langle P_{a,ik} \otimes Q_{b,jl}\chi, \chi\rangle + \langle P_{a,ki} \otimes Q_{b,lj}\chi, \chi\rangle .$$

   So, when we have entangled input states, the off-diagonal entries of $P$ and $Q$ come into play.

3. If the bra and ket vectors are different:

$$\langle P_a \otimes Q_b(e_k \otimes e_l \otimes \chi), (e_i \otimes e_j \otimes \chi)\rangle = \langle P_{a,ik} \otimes Q_{b,jl}\chi, \chi\rangle$$

**Definition 3.1.4.** The **correlation** associated to the strategy $(P_1, ..., P_c, Q_1, ..., Q_c, \chi)$ with $n$-dimensional quantum inputs and $c$ classical outputs is given by the tuple

$$X := (X^{(a,b)}_{(i,j),(k,\ell)}) = ((\langle (P_{a,ij} \otimes Q_{b,k\ell})\chi, \chi\rangle)_{i,j,k,\ell})_{a,b} \in (M_n \otimes M_n)^{c^2}, \tag{3.1.0.6}$$

in the case when the entanglement resource space for Alice and Bob is of the form $\mathcal{H}_A \otimes \mathcal{H}_B$. In the case when their resource space is a single Hilbert space $\mathcal{H}$, we replace $P_{a,ij} \otimes Q_{b,k\ell}$ with $P_{a,ij}Q_{b,k\ell}$.

For $t \in \{loc, q, qs, qa, qc\}$, we let $C_t(n, c)$ denote the set of correlations with classical inputs and classical outputs in the $t$-model. Now, we define $\mathcal{Q}_t(n, c)$, the corresponding set of all correlations with quantum inputs and classical outputs in the $t$-model.

**Definition 3.1.5.** Keeping the analogy with the sets $C_t(n, k)$, let

1. $\mathcal{Q}_q(n, c)$ be the set of all quantum correlations.

   $$\mathcal{Q}_q(n, c) = \{(\langle (P_{a,ij} \otimes Q_{b,k\ell})\chi, \chi\rangle)_{\substack{1\le i,j,k,\ell \le n, \\ 1 \le a,b \le c}}\} \subseteq (M_n \otimes M_n)^{c^2}, \tag{3.1.0.7}$$

   where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert spaces,
   $P_{a,ij} \in \mathcal{B}(\mathcal{H}_A)$ are such that $P_a = (P_{a,ij}) \in M_n(\mathcal{B}(\mathcal{H}_A))$ are positive with $\sum_{a=1}^c P_a = I$,
   $Q_{b,k\ell} \in \mathcal{B}(\mathcal{H}_B)$ are such that $Q_b = (Q_{b,k\ell}) \in M_n(\mathcal{B}(\mathcal{H}_B))$ are positive with $\sum_{b=1}^c Q_b = I$,
   and $\chi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a state.

2. $\mathcal{Q}_{qa}(n, c)$ be the closure of $\mathcal{Q}_q(n, c)$ in the norm topology.

3. $\mathcal{Q}_{qs}(n, c)$ be the set of all quantum spatial correlations, where $\mathcal{H}_A$ and $\mathcal{H}_B$ may not be finite-dimensional.

4. $\mathcal{Q}_{qc}(n, c)$ be the set of all quantum commuting correlations of the above form, where we replace the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ with a single Hilbert space $\mathcal{H}$, and $P_{a,ij} \otimes Q_{b,k\ell}$ with $P_{a,ij}Q_{b,k\ell}$.

5. $\mathcal{Q}_{loc}(n,c)$ be the set of all quantum commuting correlations where $C^*(\{P_{a,ij}, Q_{b,k\ell} : 1 \leq a, b \leq c,\ 1 \leq i,j,k,\ell \leq n\})$ is a commutative $C^*$-algebra.

The sets $C_t(n,c)$ embed into $\mathcal{Q}_t(n,c)$ in a natural way, as shown below.

**Proposition 3.1.6.** *Let $t \in \{loc, q, qs, qa, qc\}$. Then $C_t(n,c)$ is affinely isomorphic to*

$$\{X \in \mathcal{Q}_t(n,c) : X^{(a,b)}_{(i,j),(k,\ell)} = 0 \text{ if } i \neq j \text{ or } k \neq \ell\} \subseteq \mathcal{Q}_t(n,c).$$

*Moreover, the compression map*

$$X \mapsto (\delta_{ij}\delta_{k\ell}X^{(a,b)}_{(i,j),(k,\ell)}) : \mathcal{Q}_t(n,c) \to C_t(n,c)$$

*is a continuous affine map.*

*Proof.* The claims follow from the observations that if $\{E_{a,x}\}$ is a collection of positive operators such that $\{E_{a,x}\}_{a=1}^c$ is a POVM in $\mathcal{B}(\mathcal{H})$ for each $1 \leq x \leq n$, then the operators $P_a := \bigoplus_{x=1}^n E_{a,x}$ define a POVM in $M_n(\mathcal{B}(\mathcal{H}))$. Similarly, if $\{Q_a\}_{a=1}^c$ is a POVM in $M_n(\mathcal{B}(\mathcal{H}))$, then setting $F_{a,x} = Q_{a,xx} \in \mathcal{B}(\mathcal{H})$, we see that $\{F_{a,x}\}_{a=1}^c$ is a POVM in $\mathcal{B}(\mathcal{H})$ for each $1 \leq x \leq n$. $\square$

We end this section by noting some properties of the correlation sets:

- $\mathcal{Q}_t(n,c)$ is convex for all $t \in \{loc, q, qs, qa, qc\}$.
- $\mathcal{Q}_{loc}(n,c), \mathcal{Q}_{qa}(n,c), \mathcal{Q}_{qc}(n,c)$ is closed in $(M_n \otimes M_n)^{c^2}$ and $\mathcal{Q}_{qa}(n,c) = \overline{\mathcal{Q}_{qs}(n,c)} = \overline{\mathcal{Q}_q(n,c)}$.
- We have the following chain of inclusions:

$$\mathcal{Q}_{loc}(n,c) \subseteq \mathcal{Q}_q(n,c) \subseteq \mathcal{Q}_{qs}(n,c) \subseteq \mathcal{Q}_{qa}(n,c) \subseteq \mathcal{Q}_{qc}(n,c).$$

We further note that all these containments are strict in general:

– $\mathcal{Q}_{loc}(2,2) \neq \mathcal{Q}_q(2,2)$ by the CHSH game [60, Chapter 3].

– $\mathcal{Q}_q(5,3) \neq \mathcal{Q}_{qs}(5,3)$ by a theorem of A. Coladangelo and J. Stark [10].

– $\mathcal{Q}_{qs}(5,2) \neq \mathcal{Q}_{qa}(5,2)$ by a theorem of K. Dykema, V.I. Paulsen and J. Prakash [15].

– $\mathcal{Q}_{qa}(n,c) \neq \mathcal{Q}_{qc}(n,c)$ for some (likely very large) values of $n$ and $c$ due to the negative resolution to Connes' embedding problem [32].

## 3.2 Quantum-to-classical disambiguation theorems

In this section, we discuss a quantum-to-classical version of the *disambiguation theorems*. That is, we show that all correlations in $\mathcal{Q}_t(n,c)$ can be achieved using projection-valued measures (PVMs) instead of the more general notion of POVMs.

We first show that POVMs in our context can be dilated to PVMs entry-wise.

**Lemma 3.2.1.** *Let $\mathcal{H}$ be a Hilbert space, and let $\{Q_a\}_{a=1}^c$ be a POVM in $\mathcal{B}(\mathcal{H})$. Then there is a PVM $\{P_a\}_{a=1}^c$ in $M_{c+1}(\mathcal{B}(\mathcal{H}))$ such that, if $E_{11}$ is the first diagonal matrix unit in $M_{c+1}$, then $(E_{11} \otimes I_{\mathcal{H}})P_a(E_{11} \otimes I_{\mathcal{H}}) = E_{11} \otimes Q_a$ for all $1 \leq a \leq c$.*

*Proof.* We define $V = \begin{pmatrix} Q_1^{\frac{1}{2}} \\ \vdots \\ Q_c^{\frac{1}{2}} \end{pmatrix} \in M_{c,1}(\mathcal{B}(\mathcal{H}))$. Then $V$ is an isometry, so

$$U = \begin{pmatrix} V & \sqrt{I - VV^*} \\ 0 & -V^* \end{pmatrix} \in M_{c+1}(\mathcal{B}(\mathcal{H}))$$

is a unitary. Define $P_a = U^*(E_{aa} \otimes I_{\mathcal{H}})U$ for $1 \leq a \leq c-1$, and define $P_c = U^*((E_{cc}+E_{c+1,c+1}) \otimes I_{\mathcal{H}})U$. Then $\{P_a\}_{a=1}^c$ is a PVM in $M_{c+1}(\mathcal{B}(\mathcal{H}))$. Write $U = (U_{k\ell})_{k,\ell=1}^{c+1}$ where each $U_{k\ell} \in \mathcal{B}(\mathcal{H})$. The $(1,1)$ entry of $P_a$ is given by

$$(P_a)_{11} = U_{a1}^* U_{a1} = (Q_a^{\frac{1}{2}})(Q_a^{\frac{1}{2}}) = Q_a,$$

as desired. $\square$

As a result of Proposition 3.2.1, we obtain the desired dilation property for POVMs over $M_n(\mathcal{B}(\mathcal{H}))$.

**Proposition 3.2.2.** *Let $\mathcal{H}$ be a Hilbert space, and let $q_{a,ij} \in \mathcal{B}(\mathcal{H})$ for $1 \leq i,j \leq n$ and $1 \leq a \leq c$ be such that $\{Q_a\}_{a=1}^c$ is a POVM in $M_n(\mathcal{B}(\mathcal{H}))$, where $Q_a = (q_{a,ij})$. Let $V : \mathcal{H} \to \mathcal{H}^{(c+1)}$ be the isometry sending $\mathcal{H}$ to the first direct summand of $\mathcal{H}^{(c+1)}$. Then there are operators $p_{a,ij} \in M_{c+1}(\mathcal{B}(\mathcal{H}))$ such that $\{P_a\}_{a=1}^c$ is a PVM in $M_n(M_{c+1}(\mathcal{B}(\mathcal{H})))$, where $P_a = (p_{a,ij})$, and $V^* p_{a,ij} V = q_{a,ij}$ for all $1 \leq i,j \leq n$ and $1 \leq a \leq c$.*

*Proof.* We can regard $\{Q_a\}_{a=1}^c$ as a POVM in $M_n(\mathcal{B}(\mathcal{H}))$. By Proposition 3.2.1, there is a PVM $\{S_a\}_{a=1}^c$ in $M_{c+1}(M_n(\mathcal{B}(\mathcal{H})))$ such that the $(1,1)$ entry of $S_a$ is $Q_a$. Performing a canonical shuffle $M_{c+1}(M_n(\mathcal{B}(\mathcal{H}))) \simeq M_n(M_{c+1}(\mathcal{B}(\mathcal{H})))$ [49, p. 97] on each $S_a$, we obtain operators $p_{a,ij} \in M_{c+1}(\mathcal{B}(\mathcal{H}))$ such that the $(1,1)$-entry of $p_{a,ij}$ is $q_{a,ij}$, and $P_a = (p_{a,ij}) \in M_n(M_{c+1}(\mathcal{B}(\mathcal{H})))$ are projections with $\sum_{a=1}^c P_a = I$, completing the proof. $\square$

**Remark 3.2.3.** In the case of classical inputs and outputs, one would consider $n$ POVMs in $\mathcal{B}(\mathcal{H})$ with $c$ outputs each. It is a standard fact that such systems of POVMs can be dilated to a system of $n$ PVMs with $c$ outputs on a larger Hilbert space, which remains finite-dimensional whenever $\mathcal{H}$ is finite-dimensional.

Alternatively, one can consider $n$ POVMs $\{p_a^x\}_{a=1}^c$ for $1 \leq x \leq n$ on $\mathcal{H}$ as a single POVM on $\mathbb{C}^n \otimes \mathcal{H}$ by setting $Q_a = p_a^{x_1} \oplus \cdots \oplus p_a^{x_n}$, as in (3.1.0.4). Then one applies Proposition 3.2.2 to obtain a single PVM in $M_n(\mathcal{H} \otimes \mathbb{C}^{c+1})$; however, the projections may no longer be block-diagonal, so they may not induce a family of $n$ PVMs in $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^{c+1})$. In the case that $n = 1$, one can dilate a POVM with $c$ outputs in $\mathcal{B}(\mathcal{H})$ to a PVM with $c$ outputs in $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^c)$, which is more optimal than Proposition 3.2.2. On the other hand, as soon as $n \geq 2$, the dilation of Proposition 3.2.2 will be more optimal, since the general dilation of $n$ POVMs to $n$ PVMs requires an inductive argument.

To prove our disambiguation results, instead of working with definition (3.1.0.6) directly, we will use a characterization of the correlation sets using states on some associated algebras and operator systems.

**Definition 3.2.4.** We first define some universal objects that encode the correlation sets:

1. Let $\mathcal{Q}_{n,c}$ be the universal operator system generated by $c$ sets of $n^2$ entries $q_{a,ij}$ with the property that the matrix $Q_a = (q_{a,ij})$ is positive in $M_n(\mathcal{Q}_{n,c})$ for each $1 \leq a \leq c$ and $\sum_{a=1}^{c} Q_a = I_n$.

2. Let $\mathcal{P}_{n,c}$ be the universal unital $C^*$-algebra generated by $c$ sets of $n^2$ entries $p_{a,ij}$ such that $P_a = (p_{a,ij})$ is an orthogonal projection in $M_n(\mathcal{P}_{n,c})$ for each $1 \leq a \leq c$ and $\sum_{a=1}^{c} P_a = I_n$.

The correlations $\mathcal{Q}_t(n,c)$ are directly related to states on certain operator system tensor products of $\mathcal{Q}_{n,c}$.

**Notation 4.** *Let $\otimes_{\min}, \otimes_{\max}, \otimes_c$ denote the minimal tensor product, maximal tensor product and commuting tensor product respectively.*

We first show the connection between $\mathcal{Q}_{n,c}$ and $\mathcal{P}_{n,c}$. In the following, we let $C^*_{env}(\mathcal{S})$ be the C*-envelope of an operator system $\mathcal{S}$, first shown to exist by M. Hamana [26].

**Proposition 3.2.5.** *Let $n, c \in \mathbb{N}$. Then, $C^*_{env}(\mathcal{Q}_{n,c})$ is canonically $*$-isomorphic to the universal $C^*$-algebra $\mathcal{P}_{n,c}$.*

*Proof.* Let $p_{a,ij}$ be the canonical generators of $\mathcal{P}_{n,c}$, for $1 \leq i,j \leq n$ and $1 \leq a \leq c$. Since $P_a = (p_{a,ij})$ is a projection in $M_n(\mathcal{P}_{n,c})$, it is positive. Since $\sum_{a=1}^{c} P_a = I_n$, there is a UCP map $\varphi : \mathcal{Q}_{n,c} \to \mathcal{P}_{n,c}$ such that $\varphi(q_{a,ij}) = p_{a,ij}$. If we represent $\mathcal{Q}_{n,c} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then by Proposition 3.2.2, there is a unital $*$-homomorphism $\pi : \mathcal{P}_{n,c} \to M_{c+1}(\mathcal{B}(\mathcal{H}))$ such that compressing to the first coordinate yields the map $p_{a,ij} \to q_{a,ij}$. Hence, $\varphi$ is a complete order isomorphism. This shows that $\mathcal{P}_{n,c}$ is a $C^*$-cover for $\mathcal{Q}_{n,c}$, in the sense that there is a unital complete order embedding of $\mathcal{Q}_{n,c}$ into $\mathcal{P}_{n,c}$, whose range generates $\mathcal{P}_{n,c}$ as a $C^*$-algebra.

By the universal property of the $C^*$-envelope [26], there is a unique, surjective unital $*$-homomorphism $\rho : \mathcal{P}_{n,c} \to C^*_{env}(\mathcal{Q}_{n,c})$ such that $\rho(p_{a,ij}) = q_{a,ij}$ for all $1 \leq i,j \leq n$ and $1 \leq a \leq c$. As each $P_a$ is a projection in $\mathcal{P}_{n,c}$, the matrix $Q_a = (q_{a,ij}) \in M_n(C^*_{env}(\mathcal{Q}_{n,c}))$ is a projection as well. We will show that $\rho$ is injective by constructing an inverse. We assume that $\mathcal{P}_{n,c}$ is faithfully represented as a $C^*$-algebra of operators on a Hilbert space $\mathcal{K}$. Then the map $\varphi : \mathcal{Q}_{n,c} \to \mathcal{P}_{n,c}$ above extends to a UCP map $\sigma : C^*_{env}(\mathcal{Q}_{n,c}) \to \mathcal{B}(\mathcal{K})$ by Arveson's extension theorem [2]. We let $\sigma = V^*\beta(\cdot)V$ be a minimal Stinespring representation of $\sigma$, where $V : \mathcal{K} \to \mathcal{L}$ is an isometry and $\beta : C^*_{env}(\mathcal{Q}_{n,c}) \to \mathcal{B}(\mathcal{L})$ is a unital $*$-homomorphism. With respect to the decomposition $\mathcal{L} = \mathcal{K} \oplus \mathcal{K}^\perp$, one has

$$\beta(q_{a,ij}) = \begin{pmatrix} \varphi(q_{a,ij}) & * \\ * & * \end{pmatrix} = \begin{pmatrix} p_{a,ij} & * \\ * & * \end{pmatrix}.$$

Thus, after a shuffle, one may write $\beta^{(n)}(Q_a) = (\beta(q_{a,ij}))$ as

$$\begin{pmatrix} \varphi^{(n)}(Q_a) & * \\ * & * \end{pmatrix} = \begin{pmatrix} P_a & * \\ * & * \end{pmatrix}.$$

As $Q_a$ is a projection in $M_n(C^*_{env}(\mathcal{Q}_{n,c}))$, so is $\beta^{(n)}(Q_a)$ in $M_n(\mathcal{B}(\mathcal{L}))$. But $P_a$ is a projection as well, so the off-diagonal blocks must be $0$. Therefore, reversing the shuffle yields

$$\beta(q_{a,ij}) = \begin{pmatrix} p_{a,ij} & 0 \\ 0 & * \end{pmatrix}.$$

Considering $\beta(q^*_{a,ij}q_{a,ij})$ and $\beta(q_{a,ij}q^*_{a,ij})$, it follows that the multiplicative domain of $\sigma$ contains $q_{a,ij}$ for each $1 \leq i, j \leq n$ and $1 \leq a \leq c$; as these elements generate $C^*_{env}(\mathcal{Q}_{n,c})$, $\sigma$ must be a $*$-homomorphism. Since $\rho$ and $\sigma$ are mutual inverses on the generators, they must be mutual inverses on the whole algebras. Hence, $\rho$ is injective, so that $C^*_{env}(\mathcal{Q}_{n,c}) \simeq \mathcal{P}_{n,c}$. $\square$

To prove our disambiguation results, we will use the following facts about $\mathcal{P}_{n,c}$. See [5, section 1] for more details.

1. $\mathcal{P}_{n,c}$ has the lifting property. That is, whenever $\mathcal{B}$ is a $C^*$-algebra, $\mathcal{J}$ is an ideal in $\mathcal{B}$, and $\varphi : \mathcal{P}_{n,c} \to \mathcal{B}/\mathcal{J}$ is a contractive completely positive map, then there exists a contractive completely positive lift $\widetilde{\varphi} : \mathcal{P}_{n,c} \to \mathcal{B}$ of $\varphi$.

2. $\mathcal{P}_{n,c}$ is residually finite-dimensional (RFD). That is, for any $x \in \mathcal{P}_{n,c} \setminus \{0\}$, there exists $k \in \mathbb{N}$ and a finite-dimensional representation $\pi : \mathcal{P}_{n,c} \to M_k$ with $\pi(x) \neq 0$.

3. $\mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$ is RFD as minimal tensor products of RFD $C^*$-algebras remain RFD.

4. The map $p_{a,ij} \mapsto p^{op}_{a,ji}$ extends to a unital $*$-isomorphism $\pi : \mathcal{P}_{n,c} \to \mathcal{P}^{op}_{n,c}$.

We first prove the fact that quantum commuting correlations with a finite-dimensional entanglement space must belong to $\mathcal{Q}_q(n,c)$.

**Lemma 3.2.6.** *Suppose that $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in \mathcal{Q}_{qc}(n,c)$ can be written as $X = (\langle P_{a,ij}Q_{b,k\ell}\chi, \chi\rangle)$, where $P_a = (P_{a,ij})$ and $Q_b = (Q_{b,k\ell})$ are positive in $M_n(\mathcal{B}(\mathcal{H}))$, $\sum_{a=1}^c P_a = \sum_{a=1}^c Q_a = I_n$, $[P_{a,ij}, Q_{b,k\ell}] = 0$ for all $i,j,k,\ell,a,b$ and $\chi \in \mathcal{H}$ is a unit vector. If $\mathcal{H}$ is finite-dimensional, then $X \in \mathcal{Q}_q(n,c)$.*

*Proof.* Let $\mathcal{A}$ be the $C^*$-algebra generated by the set $\{P_{a,ij} : 1 \leq a \leq c,\ 1 \leq i,j \leq n\}$ and let $\mathcal{B}$ be the $C^*$-algebra generated by the set $\{Q_{b,k\ell} : 1 \leq b \leq c,\ 1 \leq k,\ell \leq n\}$. Then $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-subalgebras of $\mathcal{B}(\mathcal{H})$, and every element of $\mathcal{A}$ commutes with every element of $\mathcal{B}$. By a theorem of Tsirelson [55], there are finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, an isometry $V : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B$, and unital $*$-homomorphisms $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_A)$ and $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{H}_B)$ such that $V^*(\pi(P_{a,ij}) \otimes \rho(Q_{b,k\ell}))V = P_{a,ij}Q_{b,k\ell}$ for all $a,b,i,j,k,\ell$. Defining the unit vector $\xi = V\chi \in \mathcal{H}_A \otimes \mathcal{H}_B$, we see that

$$X^{(a,b)}_{(i,j),(k,\ell)} = \langle(\pi(P_{a,ij}) \otimes \rho(Q_{b,k\ell}))\xi, \xi\rangle.$$

Therefore, $X \in \mathcal{Q}_q(n,c)$. □

Now, we can prove the disambiguation theorems for $\mathcal{Q}_t(n,c)$.

**Remark 3.2.7.** By Proposition 3.2.2, any element of $\mathcal{Q}_q(n,c)$ can be represented using a finite-dimensional tensor product framework $\mathcal{H}_A \otimes \mathcal{H}_B$ and PVMs $\{P_a\}_{a=1}^c$ on $\mathcal{H}_A$ and $\{Q_b\}_{b=1}^c$ on $\mathcal{H}_B$, respectively. This fact holds because, given a POVM $\{Q_b\}_{b=1}^c$ in $\mathcal{B}(\mathcal{H})$, the dilation in Proposition

3.2.2 is in $M_{c+1}(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{(c+1)})$; in particular, the Hilbert space remains finite-dimensional if $\mathcal{H}$ is finite-dimensional. Similarly, it is easy to see that all elements of $\mathcal{Q}_{qs}(n,c)$ can be represented using PVMs.

Next, we show that every element $\mathcal{Q}_{qa}(n,c)$ can be represented by PVMs, which arise from the minimal tensor product of $\mathcal{P}_{n,c}$.

**Theorem 3.2.8.** *Let $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in (M_n \otimes M_n)^{c^2}$. The following are equivalent:*

1. *$X$ belongs to $\mathcal{Q}_{qa}(n,c)$.*
2. *There is a state $s : \mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c} \to \mathbb{C}$ satisfying $s(p_{a,ij} \otimes p_{b,k\ell}) = X^{(a,b)}_{(i,j),(k,\ell)}$ for all $1 \leq a,b \leq c$ and $1 \leq i,j,k,\ell \leq n$.*
3. *There is a state $s : \mathcal{Q}_{n,c} \otimes_{\min} \mathcal{Q}_{n,c} \to \mathbb{C}$ satisfying $s(q_{a,ij} \otimes q_{b,k\ell}) = X^{(a,b)}_{(i,j),(k,\ell)}$ for all $1 \leq a,b \leq c$ and $1 \leq i,j,k,\ell \leq n$.*

*Proof.* We recall that the minimal tensor product of operator spaces (in particular, operator systems) is injective [34]. Since $\mathcal{Q}_{n,c} \subseteq \mathcal{P}_{n,c}$ via the mapping $q_{a,ij} \mapsto p_{a,ij}$, injectivity of the minimal tensor product shows that $\mathcal{Q}_{n,c} \otimes_{\min} \mathcal{Q}_{n,c} \subseteq \mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$ completely order isomorphically. Using the Hahn-Banach theorem, it then follows that (2) and (3) are equivalent.

If (1) holds, then $X$ is in $\mathcal{Q}_{qa}(n,c)$, so it is a pointwise limit of elements of $\mathcal{Q}_q(n,c)$. Since elements of $\mathcal{Q}_q(n,c)$ can be represented by PVMs, $X$ is a limit of elements which correspond to finite-dimensional tensor product representations of $\mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$, which are automatically continuous. Hence, (1) implies (2). Lastly, suppose that (2) holds. Since $\mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$ is RFD, a theorem of R. Exel and T.A. Loring [21] shows that $s$ is a $w^*$-limit of states $s_\lambda$ on $\mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$ whose GNS representations are finite-dimensional. Applying Lemma 3.2.6, each $s_\lambda$ applied to the generators $p_{a,ij} \otimes p_{b,k\ell}$ of $\mathcal{P}_{n,c} \otimes_{\min} \mathcal{P}_{n,c}$ yields an element $X_\lambda$ of $\mathcal{Q}_q(n,c)$; moreover, $\lim_\lambda X_\lambda = X$ pointwise. This shows that $X \in \overline{\mathcal{Q}_q(n,c)} = \mathcal{Q}_{qa}(n,c)$, which shows that (2) implies (1). □

To establish the same disambiguation theorem for $qc$-correlations, we will show that the *commuting* tensor product $\mathcal{Q}_{n,c} \otimes_c \mathcal{Q}_{n,c}$ is completely order isomorphic to the copy of $\mathcal{Q}_{n,c} \otimes \mathcal{Q}_{n,c}$

inside of $\mathcal{P}_{n,c} \otimes_{\max} \mathcal{P}_{n,c}$. We recall that, if $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then an element $Y$ in $M_n(\mathcal{S} \otimes_c \mathcal{T})$ is defined as positive in the commuting tensor product provided that $Y = Y^*$ and, whenever $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ are UCP maps with commuting ranges, then $(\varphi \cdot \psi)^{(n)}(Y)$ is positive in $M_n(\mathcal{B}(\mathcal{H}))$, where $\varphi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(\mathcal{H})$ is the linear map defined by $(\varphi \cdot \psi)(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in \mathcal{S}$ and $y \in \mathcal{T}$.

The next lemma is an adaptation of [29, Proposition 4.6].

**Lemma 3.2.9.** *Let $\mathcal{S}$ be an operator system. Then the canonical map $\mathcal{Q}_{n,c} \otimes_c \mathcal{S} \to \mathcal{P}_{n,c} \otimes_{\max} \mathcal{S}$ is a complete order embedding.*

*Proof.* Since $\mathcal{P}_{n,c}$ is a unital $C^*$-algebra, we have $\mathcal{P}_{n,c} \otimes_c \mathcal{S} = \mathcal{P}_{n,c} \otimes_{\max} \mathcal{S}$ [34, Theorem 6.7]. The canonical map $\mathcal{Q}_{n,c} \otimes_c \mathcal{S} \to \mathcal{P}_{n,c} \otimes_c \mathcal{S}$ is a tensor product of canonical inclusion maps, which are UCP. By functoriality of the commuting tensor product [34], the inclusion $\mathcal{Q}_{n,c} \otimes_c \mathcal{S} \to \mathcal{P}_{n,c} \otimes_c \mathcal{S}$ is UCP. Hence, it suffices to show that this map is a complete order embedding.

To this end, suppose that $Y = Y^* \in M_m(\mathcal{Q}_{n,c} \otimes \mathcal{S})$ is a positive element of $M_m(\mathcal{P}_{n,c} \otimes_c \mathcal{S})$. Let $\varphi : \mathcal{Q}_{n,c} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be UCP maps with commuting ranges; we will show that $(\varphi \cdot \psi)^{(m)}(Y)$ is positive in $M_m(\mathcal{B}(\mathcal{H}))$. For convenience, we define $Q_{a,ij} = \varphi(q_{a,ij})$. By Proposition 3.2.2, there is a unital $*$-homomorphism $\pi : \mathcal{P}_{n,c} \to M_{c+1}(\mathcal{B}(\mathcal{H}))$ such that the $(1,1)$ corner of $\pi(p_{a,ij})$ is $Q_{a,ij}$ for all $1 \leq a \leq c$ and $1 \leq i,j \leq n$. Moreover, for each $x \in \mathcal{P}_{n,c}$, each block of $\pi(x)$ in $\mathcal{B}(\mathcal{H})$ belongs to the $C^*$-algebra generated by the set $\{Q_{a,ij} : 1 \leq a \leq c,\, 1 \leq i,j \leq n\}$. We extend $\varphi$ to a UCP map on $\mathcal{P}_{n,c}$ by defining $\varphi(\cdot) = (\pi(\cdot))_{11}$. Define $\widetilde{\psi} : \mathcal{S} \to M_{c+1}(\mathcal{B}(\mathcal{H}))$ by $\widetilde{\psi}(s) = I_{c+1} \otimes \psi(s)$. Since $\psi(s)$ commutes with the range of $\varphi$, $\psi(s)$ must commute with the $C^*$-algebra generated by the range of $\varphi$. Hence, $\psi(s)$ commutes with every block of $\pi(p_{a,ij})$, for all $a, i, j$. By the multiplicativity of $\pi$, $\psi(s)$ commutes with the range of $\pi$. By definition of the commuting tensor product, this means that $\pi \cdot \widetilde{\psi} : \mathcal{P}_{n,c} \otimes_c \mathcal{S} \to M_{c+1}(\mathcal{B}(\mathcal{H}))$ is UCP; moreover, the $(1,1)$ block of $\pi \cdot \widetilde{\psi}$ is $\varphi \cdot \psi$. This means that $\varphi \cdot \psi$ is UCP on $\mathcal{P}_{n,c} \otimes_c \mathcal{S}$. Restricting to the copy of the algebraic tensor product $\mathcal{Q}_{n,c} \otimes \mathcal{S}$, it follows that $(\varphi \cdot \psi)^{(m)}(Y)$ is positive, making the canonical map $\mathcal{Q}_{n,c} \otimes_c \mathcal{S} \to \mathcal{P}_{n,c} \otimes_c \mathcal{S}$ a complete order embedding. $\square$

**Theorem 3.2.10.** *Let $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in (M_n \otimes M_n)^{c^2}$. The following are equivalent.*

1. *$X$ belongs to $\mathcal{Q}_{qc}(n,c)$.*

2. *There is a state $s : \mathcal{P}_{n,c} \otimes_{\max} \mathcal{P}_{n,c} \to \mathbb{C}$ satisfying $s(p_{a,ij} \otimes p_{b,k\ell}) = X^{(a,b)}_{(i,j),(k,\ell)}$ for all $1 \leq a,b \leq c$ and $1 \leq i,j,k,\ell \leq n$.*

3. *There is a state $s : \mathcal{Q}_{n,c} \otimes_c \mathcal{Q}_{n,c} \to \mathbb{C}$ satisfying $s(q_{a,ij} \otimes q_{b,k\ell}) = X^{(a,b)}_{(i,j),(k,\ell)}$ for all $1 \leq a,b \leq c$ and $1 \leq i,j,k,\ell \leq n$.*

*Proof.* Since $\mathcal{Q}_{qc}(n,c)$ is defined in terms of POVMs where Alice's entries commute with Bob's, we see that (1) is equivalent to (3). Based on two applications of Lemma 3.2.9, we see that $\mathcal{Q}_{n,c} \otimes_c \mathcal{Q}_{n,c}$ is completely order isomorphic to the image of $\mathcal{Q}_{n,c} \otimes \mathcal{Q}_{n,c}$ in $\mathcal{P}_{n,c} \otimes_{\max} \mathcal{P}_{n,c}$. Hence, (2) and (3) are equivalent. $\square$

When considering the quantum-to-classical graph coloring game, the local model will be of interest because of its link to the usual notion of a (classical) coloring of a quantum graph. It is helpful to note that all strategies in $\mathcal{Q}_{loc}(n,c)$ can be obtained using PVMs instead of just POVMs. A standard argument shows that limits of convex combinations of elements of $\mathcal{Q}_{loc}(n,c)$ represented by PVMs from abelian algebras can still be represented by PVMs from abelian algebras. With this fact in hand, we can prove the disambiguation theorem for $\mathcal{Q}_{loc}(n,c)$.

**Theorem 3.2.11.** *Let $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in (M_n \otimes M_n)^{c^2}$. The following are equivalent:*

1. *$X$ belongs to $\mathcal{Q}_{loc}(n,c)$;*

2. *There is a commutative $C^*$-algebra $\mathcal{A}$, a state $s$ on $\mathcal{A}$ and POVMs $\{P_1, ..., P_c\}$, $\{Q_1, ..., Q_c\} \subseteq M_n(\mathcal{A})$ such that*
$$X^{(a,b)}_{(i,j),(k,\ell)} = s(P_{a,ij} Q_{b,k\ell});$$

3. *There is a commutative $C^*$-algebra $\mathcal{A}$, a state $s$ on $\mathcal{A}$, and PVMs $\{P_1, ..., P_c\}, \{Q_1, ..., Q_c\} \subseteq M_n(\mathcal{A})$ such that*
$$X^{(a,b)}_{(i,j),(k,\ell)} = s(P_{a,ij} Q_{b,k\ell}).$$

*Proof.* Clearly (1) and (2) are equivalent by the definition of $\mathcal{Q}_{loc}(n,c)$. Since every PVM is a POVM, (3) implies (2). Hence, we need only show that (2) implies (3). Suppose that

$$X^{(a,b)}_{(i,j),(k,\ell)} = s(P_{a,ij}Q_{b,k\ell})$$

for a state $s$ on a commutative $C^*$-algebra $\mathcal{A}$ and a POVMs $P_1, ..., P_c$ and $Q_1, ..., Q_c$ in $M_n(\mathcal{A})$. Then $\mathcal{A} \simeq C(Y)$ for a compact Hausdorff space $Y$. The extreme points of the state space of $Y$ are simply evaluation functionals $\delta_y$ for $y \in Y$, which are multiplicative. Hence, $\delta_y^{(n)}(Q_a) \in M_n(\mathbb{C})$ defines a POVM with $c$ outputs in $M_n(\mathbb{C})$, where $\delta_y^{(n)} = \mathrm{id}_n \otimes \delta_y$. Recall that the extreme points of the set of positive contractions in a von Neumann algebra are precisely the projections in the von Neumann algebra. An easy application of this argument shows that the extreme points of the set of POVMs with $c$ outputs in a von Neumann algebra are precisely the PVMs with $c$ outputs. Hence, $\{\delta_y^{(n)}(Q_1), ..., \delta_y^{(n)}(Q_c)\}$ lies in the closed convex hull of the set of PVMs in $M_n(\mathbb{C})$ with $c$ outputs. Applying a similar argument to $\{\delta_y^{(n)}(P_1), ..., \delta_y^{(n)}(P_c)\}$, it follows that the correlation $(\delta_y(P_{a,ij}Q_{b,k\ell}))$ is a convex combination of elements of $\mathcal{Q}_{loc}(n,c)$ obtained by tensoring projections from $M_n(\mathbb{C})$. Taking the closed convex hull, we obtain the original correlation $X$. In this way, we can write $X$ using projection-valued measures, which shows that (2) implies (3). $\square$

In the next section, we look at the analogous class of synchronous correlations in our new framework.

### 3.3 Synchronous quantum input–classical output correlations

Recall that a correlation $P = (p(a,b|x,y)) \in C(n,k)$ is called *synchronous* if $p(a,b|x,x) = 0$ whenever $a \neq b$ [30]. In this section, we introduce a generalization of synchronous correlations to our quantum framework and characterize these synchronous correlations in terms of tracial states on $C^*$-algebras.

In the following considerations, we fix once and for all an orthonormal basis $\{e_1, ..., e_n\}$ for $\mathbb{C}^n$, but the results also hold for other bases using unitary transformations.

**Definition 3.3.1.** Let $S \subseteq [n]$. We define the **maximally entangled Bell state corresponding to**

$S$ as the vector

$$\varphi_S = \frac{1}{\sqrt{|S|}} \sum_{j \in S} e_j \otimes e_j.$$

**Definition 3.3.2.** Let $X \in \mathcal{Q}_t(n,c)$ be a correlation in $n$-dimensional quantum inputs and $c$ classical outputs, where $t \in \{loc, q, qs, qa, qc\}$. We say that $X$ is **synchronous** provided that there is a partition $S_1 \dot\cup \cdots \dot\cup S_\ell$ of $[n]$ with the property that, if $a \neq b$, then

$$p(a,b|\varphi_{S_r}) = 0 \text{ for all } 1 \leq r \leq \ell.$$

We define the subset

$$\mathcal{Q}_t^s(n,c) = \{X \in \mathcal{Q}_t(n,c) : X \text{ is synchronous}\}.$$

The following proposition gives a very useful description of synchronicity in terms of the entries of the matrices involved in the correlation.

**Proposition 3.3.3.** *Let* $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in \mathcal{Q}_t(n,c)$ *for* $t \in \{loc, q, qs, qa, qc\}$. *The following are equivalent:*

$$X \in \mathcal{Q}_t^s(n,c) \iff \frac{1}{n} \sum_{a=1}^{c} \sum_{i,j=1}^{n} X^{(a,a)}_{(i,j),(i,j)} = 1 \iff \sum_{i,j=1}^{n} X^{(a,b)}_{(i,j),(i,j)} = 0 \textit{ for } a \neq b. \qquad (3.3.0.1)$$

*Proof.* Suppose that $X$ can be represented using the PVMs $\{P_a\}_{a=1}^c$ in $\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $\{Q_b\}_{b=1}^c$ in $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ and the state $\chi \in \mathcal{H}$. We observe that, if $S \subseteq [n]$, then

$$\begin{aligned} p(a,b|\varphi_S) &= \frac{1}{|S|} \sum_{i,j \in S} \langle (P_a \otimes I_n)(I_n \otimes Q_b)(e_j \otimes \chi \otimes e_j), e_i \otimes \chi \otimes e_i \rangle \\ &= \frac{1}{|S|} \sum_{i,j \in S} \langle P_{a,ij} Q_{b,ij} \chi, \chi \rangle \\ &= \frac{1}{|S|} \sum_{i,j \in S} X^{(a,b)}_{(i,j),(i,j)}. \end{aligned}$$

Suppose that $X$ is synchronous, and let $S_1, ..., S_\ell$ be a partition of $[n]$ for which $p(a, b|\varphi_{S_r}) = 0$ whenever $a \neq b$ and $1 \leq r \leq \ell$. Then the above calculation shows that $\sum_{i,j \in S_r} X^{(a,b)}_{(i,j),(i,j)} = 0$ for all $r$. Summing over all $r$, it follows that $\sum_{i,j=1}^{n} X^{(a,b)}_{(i,j),(i,j)} = 0$ whenever $a \neq b$. Hence, (1) implies (3).

Next, we show that (3) implies (2). Notice that, for any $X \in \mathcal{Q}_{qc}(n, c)$,

$$\begin{aligned}\sum_{a,b=1}^{c} \sum_{i,j=1}^{n} X^{(a,b)}_{(i,j),(i,j)} &= \sum_{a,b=1}^{c} \sum_{i,j=1}^{n} \langle P_{a,ij} Q_{b,ij} \chi, \chi \rangle \\ &= \sum_{i,j=1}^{n} \left\langle \left( \sum_{a=1}^{c} P_{a,ij} \right) \left( \sum_{b=1}^{c} Q_{b,ij} \right) \chi, \chi \right\rangle \\ &= \sum_{i=1}^{n} \langle \chi, \chi \rangle = n,\end{aligned}$$

where we have used the fact that $\sum_{a=1}^{c} P_a = \sum_{b=1}^{c} Q_b = I_n$ implies that $\sum_{a=1}^{c} P_{a,ij} = \sum_{b=1}^{c} Q_{b,ij}$ is $I$ when $i = j$ and $0$ otherwise. Therefore,

$$\frac{1}{n} \sum_{i,j=1}^{n} X^{(a,b)}_{(i,j),(i,j)} = 1,$$

which shows that (2) holds.

Lastly, if (2) holds, then (1) immediately follows using the single-set partition $S = [n]$. □

**Remark 3.3.4.** In the case of a correlation $p(a, b|x, y) \in C_t(n, c)$ with $n$ classical inputs and $c$ classical outputs, using the $[n] = \{1\} \cup \{2\} \cup \cdots \cup \{n\}$, we see that any synchronous correlation in $C_t(n, c)$ is a synchronous correlation in the sense of the definition above. In this way, we see that

$$C_t^s(n, c) \subseteq \mathcal{Q}_t^s(n, c).$$

We now prove an analogue of [47, Theorem 5.5], namely that synchronous correlations with $n$-dimensional inputs and $c$ outputs arise from tracial states on the algebra generated by Alice's operators (respectively, Bob's operators). We will also see that, in any realization of a synchronous correlation, Bob's operators can be described naturally in terms of Alice's operators.

By a **realization** of $X \in \mathcal{Q}_{qc}(n,c)$, we simply mean a 4-tuple $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \mathcal{H}, \psi)$, where $\{P_a\}_{a=1}^c$ is a PVM on $\mathbb{C}^n \otimes \mathcal{H}$, $\{Q_b\}_{b=1}^c$ is a PVM on $\mathcal{H} \otimes \mathbb{C}^n$, $\psi$ is a state in $\mathcal{H}$, and $[P_a \otimes I_n, I_n \otimes Q_b] = 0$ for all $a, b$.

**Theorem 3.3.5.** *Let $X = (X_{(i,j),(k,\ell)}^{(a,b)}) \in \mathcal{Q}_{qc}^s(n,c)$. Let $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \mathcal{H}, \psi)$ be a realization of $X$. Then:*

1. *$Q_{a,ij}\psi = P_{a,ij}^*\psi$ for all $1 \leq a \leq c$ and $1 \leq i,j \leq n$.*
2. *The state $\rho = \langle (\cdot)\psi, \psi\rangle$ is a tracial state on the $C^*$-algebra $\mathcal{A}$ generated by $\{P_{a,ij} : 1 \leq a \leq c,\ 1 \leq i,j \leq n\}$, and on the $C^*$-algebra $\mathcal{B}$ generated by $\{Q_{b,k\ell} : 1 \leq b \leq c,\ 1 \leq k,\ell \leq n\}$.*

*Conversely, if $P_{a,ij}$ are operators in a tracial $C^*$-algebra $\mathcal{A}$ with a trace $\tau$, such that the operators $P_a = (P_{a,ij}) \in M_n(\mathcal{A})$ form a PVM with $c$ outputs, then $(\tau(P_{a,ij}P_{b,k\ell}^*))$ defines an element of $\mathcal{Q}_{qc}^s(n,c)$.*

*Proof.* Suppose $X \in \mathcal{Q}_{qc}^s(n,c)$, with realization $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \mathcal{H}, \psi)$. By Proposition 3.3.3, we have

$$\begin{aligned}
1 &= \frac{1}{n}\sum_{a=1}^{c}\sum_{i,j=1}^{n} X^{(a,a)}_{(i,j),(i,j)} && (3.3.0.2)\\
&= \frac{1}{n}\sum_{a=1}^{c}\sum_{i,j=1}^{n} \langle P_{a,ij}Q_{a,ij}\psi,\psi\rangle \\
&\leq \frac{1}{n}\sum_{a=1}^{c}\sum_{i,j=1}^{n} |\langle P_{a,ij}Q_{a,ij}\psi,\psi\rangle| && (3.3.0.3)\\
&= \frac{1}{n}\sum_{a=1}^{c}\sum_{i,j=1}^{n} |\langle Q_{a,ij}\psi,P^*_{a,ij}\psi\rangle| \\
&\leq \frac{1}{n}\sum_{a=1}^{c}\sum_{i,j=1}^{n} \|Q_{a,ij}\psi\|\|P^*_{a,ij}\psi\| && (3.3.0.4)\\
&\leq \frac{1}{n}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \|Q_{a,ij}\psi\|^2\right)^{\frac{1}{2}}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \|P^*_{a,ij}\psi\|^2\right)^{\frac{1}{2}} && (3.3.0.5)\\
&= \frac{1}{n}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \langle Q^*_{a,ij}Q_{a,ij}\psi,\psi\rangle\right)^{\frac{1}{2}}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \langle P_{a,ij}P^*_{a,ij}\psi,\psi\rangle\right)^{\frac{1}{2}} \\
&= \frac{1}{n}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \langle Q_{a,ji}Q_{a,ij}\psi,\psi\rangle\right)^{\frac{1}{2}}\left(\sum_{a=1}^{c}\sum_{i,j=1}^{n} \langle P_{a,ij}P_{a,ji}\psi,\psi\rangle\right)^{\frac{1}{2}}
\end{aligned}$$

Since $P_a$ and $Q_a$ are projections, the last line is equal to

$$\begin{aligned}
\frac{1}{n}\left(\sum_{a=1}^{c}\sum_{j=1}^{n}\langle Q_{a,jj}\psi,\psi\rangle\right)^{\frac{1}{2}}\left(\sum_{a=1}^{c}\sum_{i=1}^{n}\langle P_{a,ii}\psi,\psi\rangle\right)^{\frac{1}{2}} &= \frac{1}{n}\left(\sum_{j=1}^{n}\langle I_{\mathcal{H}}\psi,\psi\rangle\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\langle I_{\mathcal{H}}\psi,\psi\rangle\right)^{\frac{1}{2}}\\
&= \frac{1}{n}\cdot\sqrt{n}\cdot\sqrt{n}\\
&= 1.
\end{aligned}$$

Therefore, all of these inequalities are equalities. Then (3.3.0.3) implies that

$$X^{(a,a)}_{(i,j),(i,j)} \geq 0 \text{ for all } 1 \leq a \leq c,\ 1 \leq i,j \leq n.$$

The equality case of (3.3.0.4) shows that

$$Q_{a,ij}\psi = \alpha_{a,ij}P^*_{a,ij}\psi \text{ for some } \alpha_{a,ij} \in \mathbb{T}. \tag{3.3.0.6}$$

Then equation (3.3.0.6) yields

$$X^{(a,a)}_{(i,j),(i,j)} = \alpha_{a,ij}\langle P_{a,ij}P^*_{a,ij}\psi, \psi\rangle = \alpha_{a,ij}\|P^*_{a,ij}\psi\|^2.$$

Since $X^{(a,a)}_{(i,j),(i,j)} \geq 0$ and $\|P^*_{a,ij}\psi\|^2 \geq 0$, we either have $P^*_{a,ij}\psi = 0$ or $\alpha_{a,ij} = 1$. In either case, we obtain

$$Q_{a,ij}\psi = P^*_{a,ij}\psi,$$

as desired.

To prove (2), it suffices to show that it holds for $\mathcal{A} = C^*(\{P_{a,ij}\}_{a,i,j})$; a similar argument works for $\mathcal{B} = C^*(\{Q_{b,k\ell}\}_{b,k,\ell})$. Let $\rho : \mathcal{A} \to \mathbb{C}$ be the state given by $\rho(X) = \langle X\psi, \psi\rangle$. Let $W = P^{m_1}_{a_1,i_1j_1} \cdots P^{m_k}_{a_k,i_kj_k}$ be a word in $\{P_{a,ij}, P^*_{a,ij}\}_{a,i,j}$, where we denote by $P^{-1}_{a_\ell,i_\ell j_\ell}$ the operator $P^*_{a_\ell,i_\ell j_\ell}$ and let $m_\ell \in \{-1, 1\}$. We will first show that $W\psi = Q^{-m_k}_{a_k,i_kj_k} \cdots Q^{-m_1}_{a_1,i_1j_1}\psi$, where $Q^{-1}_{a_\ell,i_\ell j_\ell} := Q^*_{a_\ell,i_\ell j_\ell}$. Using the fact that $P_{a,ij}$ and $Q_{b,k\ell}$ $*$-commute for each $a, b, i, j, k, \ell$, we obtain

$$\begin{aligned} W\psi &= P^{m_1}_{a_1,i_1j_1} \cdots P^{m_k}_{a_k,i_kj_k}\psi \\ &= P^{m_1}_{a_1,i_1j_1} \cdots P^{m_{k-1}}_{a_{k-1},i_{k-1}j_{k-1}}Q^{-m_k}_{a_k,i_kj_k}\psi \\ &= Q^{-m_k}_{a_k,i_kj_k}(P^{m_1}_{a_1,i_1j_1} \cdots P^{m_{k-1}}_{a_{k-1},i_{k-1}j_{k-1}})\psi, \end{aligned}$$

and the desired equality easily follows by induction on $k$.

For $1 \le a \le c$ and $1 \le i, j \le n$,

$$\begin{aligned}
\rho(P_{a,ij}W) &= \langle P_{a,ij}W\psi, \psi\rangle \\
&= \langle P_{a,ij}(Q^{m_1}_{a_1,i_1j_1}\cdots Q^{m_k}_{a_k,i_kj_k})^*\psi, \psi\rangle \\
&= \langle (Q^{m_1}_{a_1,i_1j_1}\cdots Q^{m_k}_{a_k,i_kj_k})^* P_{a,ij}\psi, \psi\rangle \\
&= \langle P_{a,ij}\psi, Q^{m_1}_{a_1,i_1j_1}\cdots Q^{m_k}_{a_k,i_kj_k}\psi\rangle \\
&= \langle P_{a,ij}\psi, (P^{m_1}_{a_1,i_1j_1}\cdots P^{m_k}_{a_k,i_kj_k})^*\psi\rangle \\
&= \langle P_{a,ij}\psi, W^*\psi\rangle \\
&= \langle WP_{a,ij}\psi, \psi\rangle = \rho(WP_{a,ij}).
\end{aligned}$$

In the same way, $\rho(P_{a,ij}P_{b,k\ell}W) = \rho(WP_{a,ij}P_{b,k\ell})$. It follows by induction, linearity and continuity that $\rho$ is tracial on $\mathcal{A}$, as desired.

For the converse direction, we recall the standard fact that, if $\mathcal{A}$ is a unital $C^*$-algebra and $\tau$ is a trace on $\mathcal{A}$, then there is a state $s : \mathcal{A} \otimes_{\max} \mathcal{A}^{op} \to \mathbb{C}$ satisfying $s(x \otimes y^{op}) = \tau(xy)$ for all $x, y \in \mathcal{A}$. Thus, if $P_1, ..., P_c \in M_n(\mathcal{A})$ is a projection-valued measure, then

$$s(P_{a,ij} \otimes P^{op}_{b,k\ell}) = \tau(P_{a,ij}P_{b,k\ell})\, \forall 1 \le a, b \le c,\ 1 \le i, j, k, \ell \le n.$$

Applying the universal property of $\mathcal{P}_{n,c}$, we obtain a state $\gamma : \mathcal{P}_{n,c} \otimes_{\max} \mathcal{P}^{op}_{n,c} \to \mathbb{C}$ satisfying

$$\gamma(p_{a,ij} \otimes p^{op}_{b,k\ell}) = \tau(P_{a,ij}P_{b,k\ell}).$$

The map $p_{a,ij} \otimes p_{b,k\ell} \mapsto \tau(P_{a,ij}P_{b,\ell k}) = \tau(P_{a,ij}P^*_{b,k\ell})$ defines a state on $\mathcal{P}_{n,c} \otimes_{\max} \mathcal{P}_{n,c}$, and hence $X := \tau(P_{a,ij}P^*_{b,k\ell})$ defines an element of $\mathcal{Q}_{qc}(n, c)$. If $a \neq b$, then

$$\begin{aligned}
\sum_{i,j=1}^{n} X^{(a,b)}_{(i,j),(i,j)} &= \sum_{i,j=1}^{n} \tau(P_{a,ij}P^*_{b,ij}) \\
&= \mathrm{Tr} \otimes \tau(P_aP_b) = 0,
\end{aligned}$$

since $P_aP_b = 0$. By Proposition 3.3.3, $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in \mathcal{Q}^s_{qc}(n,c)$. $\square$

In light of Theorem 3.3.5, we may refer to a **synchronous $t$-strategy** $(\{P_a\}_{a=1}^c, \chi)$ when referring to a $t$-strategy $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \chi)$ where the associated correlation is synchronous.

**Corollary 3.3.6.** *Let $(X^{(a,b)}_{(i,j),(k,\ell)}) \in \mathcal{Q}^s_t(n,c)$ where $t \in \{loc, q, qs, qa, qc\}$. Then:*

1. $X^{(a,b)}_{(i,i),(j,j)} \geq 0$ *for all* $1 \leq a,b \leq c$ *and* $1 \leq i,j \leq n$.
2. $X^{(a,b)}_{(i,j),(k,\ell)} = \overline{X^{(a,b)}_{(j,i),(\ell,k)}}$.
3. *For any* $1 \leq a \neq b \leq c$ *and* $1 \leq i,j \leq n$, *we have*
$$\sum_{k=1}^n X^{(a,b)}_{(i,k),(j,k)} = \sum_{k=1}^n X^{(a,b)}_{(k,i),(k,j)} = 0.$$
4. *For any* $1 \leq i,j \leq n$, *we have*
$$\sum_{a=1}^c \sum_{k=1}^n X^{(a,a)}_{(i,k),(j,k)} = \sum_{a=1}^c \sum_{k=1}^n X^{(a,a)}_{(k,i),(k,j)} = \delta_{ij}.$$

*Proof.* By Theorem 3.3.5, we may choose projections $P_1, ..., P_c \in M_n(\mathcal{A})$, for a unital $C^*$-algebra $\mathcal{A}$, along with a tracial state $\tau$ on $\mathcal{A}$ such that

$$X^{(a,b)}_{(i,j),(k,\ell)} = \tau(P_{a,ij}P^*_{b,k\ell}) \text{ for all } 1 \leq a,b \leq c,\ 1 \leq i,j,k,\ell \leq n.$$

Since $P_a$ is a projection, it defines a positive element of $M_n(\mathcal{A})$. Compressing to any diagonal block preserves positivity, which implies that $P_{a,ii} \in \mathcal{A}^+$ for any $i$. Since $\tau$ is a trace, it follows that $\tau(P_{a,ii}P_{b,jj}) \geq 0$ for any $i,j,a,b$. Hence, (1) follows.

We note that (2) follows easily from the fact that $\tau$ is a trace and that, since $\tau$ is a state, one has $\tau(Y^*) = \overline{\tau(Y)}$ for all $Y \in \mathcal{A}$.

To show (3), we observe that

$$\begin{aligned}\sum_{k=1}^{n} X^{(a,b)}_{(i,k),(j,k)} &= \sum_{k=1}^{n} \tau(P_{a,ik}P^*_{b,jk}) \\ &= \sum_{k=1}^{n} \tau(P_{a,ik}P_{b,kj}) \\ &= \tau\left(\sum_{k=1}^{n} P_{a,ik}P_{b,kj}\right) \\ &= \tau((P_aP_b)_{ij}) = 0,\end{aligned}$$

since $P_aP_b = 0$. Similarly, $\sum_{k=1}^{n} X^{(a,b)}_{(k,i),(k,j)} = 0$ when $a \neq b$.

A similar argument establishes (4). Indeed, we have

$$\sum_{a=1}^{c}\sum_{k=1}^{n} X^{(a,a)}_{(i,k),(j,k)} = \sum_{a=1}^{c}\sum_{k=1}^{n} \tau(P_{a,ik}P_{a,kj}) = \tau\left(\sum_{a=1}^{c} P_{a,ij}\right),$$

and this latter sum is $\delta_{ij}$, since $\sum_{a=1}^{c} P_a = I$. The other equation in (4) follows similarly. □

**Remark 3.3.7.** It makes sense to discuss synchronicity of a strategy with respect to a different orthonormal basis $\mathbf{v} = \{v_1, ..., v_n\}$ of $\mathbb{C}^n$. In this case, a $qc$-strategy $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \chi)$ is said to be **synchronous with respect to** $\{v_1, ..., v_n\}$ if there is a partition $S_1 \cup \cdots \cup S_s$ of $[n]$ such that for each $r$ and $\varphi_{S_r,\mathbf{v}} := \frac{1}{\sqrt{|S_r|}}\sum_{j \in S_r} v_j \otimes v_j$, we have

$$p(a, b|\varphi_{S_r,\mathbf{v}}) = 0 \text{ if } a \neq b.$$

One can then write down an analogue of Theorem 3.3.5 in this context. Alternatively, one can simply let $\widetilde{P}_a = U^*P_aU$ and $\widetilde{Q}_b = U^*Q_bU$, where $U$ is the unitary satisfying $Ue_i = v_i$ for all $i$. Then applying Theorem 3.3.5 relates the entries of $\widetilde{Q}_a$ to the entries of $\widetilde{P}_a$, while showing that the state $\langle(\cdot)\chi, \chi\rangle$ is a trace on the algebra generated by the entries of the operators $\widetilde{Q}_a$ (respectively, $\widetilde{P}_a$). Since $P_a = U\widetilde{P}_aU^*$ and $Q_b = U\widetilde{Q}_bU^*$, the entries of $P_a$ (respectively, $Q_b$) are linear combinations of the entries of $\widetilde{P}_a$ (respectively, $\widetilde{Q}_b$), so it follows that the algebra generated by the entries of the operators $P_a$ (respectively, $Q_b$) is the same as the algebra generated by the entries of $\widetilde{P}_a$ (respectively, $\widetilde{Q}_b$).

It is helpful to describe the simplest ways to realize synchronous correlations. To that end, we spend the rest of this section describing the simplest realizations for $t \in \{loc, q, qs, qa\}$. We start with the case of $\mathcal{Q}^s_{loc}(n,c)$.

**Corollary 3.3.8.** *Let $X \in (M_n \otimes M_n)^{c^2}$. Then $X$ belongs to $\mathcal{Q}^s_{loc}(n,c)$ if and only if there is a unital, commutative $C^*$-algebra $\mathcal{A}$, a projection-valued measure $\{P_a\}_{a=1}^c \subseteq M_n(\mathcal{A})$ for $1 \leq a \leq c$, and a faithful state $\psi \in \mathcal{S}(\mathcal{A})$ such that, for all $1 \leq a, b \leq c$ and $1 \leq i, j, k, \ell \leq n$,*

$$X^{(a,b)}_{(i,j),(k,\ell)} = \psi(P_{a,ij} P^*_{b,k\ell}).$$

*Moreover, if $X$ is an extreme point in $\mathcal{Q}^s_{loc}(n,c)$, then we may take $\mathcal{A} = \mathbb{C}$.*

*Proof.* If $X \in \mathcal{Q}^s_{loc}(n,c)$, then by definition of loc-correlations, $X$ can be written using projection-valued measures $\{P_a\}_{a=1}^c$ and $\{Q_b\}_{b=1}^c$ in $M_n(\mathcal{B}(\mathcal{H}))$, along with a state $\chi \in \mathcal{H}$, such that $X^{(a,b)}_{(i,j),(k,\ell)} = \langle P_{a,ij} Q_{b,k\ell} \chi, \chi \rangle$ and the $C^*$-algebra $\mathcal{A}$ generated by the set of all entries $P_{a,ij}$ and $Q_{b,\ell}$ is a commutative $C^*$-algebra. Applying Theorem 3.3.5, we can write $X^{(a,b)}_{(i,j),(k,\ell)} = \psi(P_{a,ij} P^*_{b,k\ell})$, where $\psi(\cdot) = \langle (\cdot)\chi, \chi \rangle$. As this state is tracial, by replacing $\mathcal{A}$ with its quotient by the kernel of the GNS representation of $\psi$ if necessary, we may assume without loss of generality that $\psi$ is faithful, which establishes the forward direction. The converse follows by the converse of Theorem 3.3.5 and the definition of $\mathcal{Q}_{loc}(n,c)$.

To establish the claim about extreme points, we note that every element of $\mathcal{Q}_{loc}(n,c)$ is a limit of convex combinations of correlations arising from PVMs in $M_n(\mathbb{C})$. Evidently the set of elements of $\mathcal{Q}_{loc}(n,c)$ that have realizations using PVMs in $M_n(\mathbb{C})$ is a closed set. As $\mathcal{Q}_{loc}(n,c)$ is compact and convex, the converse of the Krein-Milman theorem shows that extreme points in $\mathcal{Q}_{loc}(n,c)$ must have a realization using PVMs in $M_n(\mathbb{C})$. Now, the proof of the forward direction of Theorem 3.3.5 shows that $\frac{1}{n} \sum_{a=1}^c \sum_{i,j=1}^n Y^{(a,b)}_{(i,j),(i,j)} \leq 1$ for any $Y \in \mathcal{Q}_{qc}(n,c)$. Moreover, this inequality is an equality if and only if $Y$ is synchronous, by Proposition 3.3.3. Hence, $\mathcal{Q}^s_{loc}(n,c)$

is a face in $\mathcal{Q}_{loc}(n,c)$, so extreme points in $\mathcal{Q}_{loc}^s(n,c)$ are also extreme points in $\mathcal{Q}_{loc}(n,c)$. This shows that $X$ has a realization using the algebra $\mathcal{A}=\mathbb{C}$. □

**Corollary 3.3.9.** *Let $X \in (M_n \otimes M_n)^{c^2}$. Then $X$ belongs to $\mathcal{Q}_q^s(n,c)$ if and only if there is a finite-dimensional $C^*$-algebra $\mathcal{A}$, a projection-valued measure $\{P_a\}_{a=1}^c \subseteq M_n(\mathcal{A})$ for $1 \le a \le c$, and a faithful tracial state $\psi \in \mathcal{S}(\mathcal{A})$ such that, for all $1 \le a,b \le c$ and $1 \le i,j,k,\ell \le n$,*

$$X_{(i,j),(k,\ell)}^{(a,b)} = \psi(P_{a,ij}P_{b,k\ell}^*).$$

*Moreover, if $X$ is an extreme point in $\mathcal{Q}_q^s(n,c)$, then we may take $\mathcal{A}=M_d$ for some $d$, and hence $\psi = \mathrm{Tr}_d$, where $\mathrm{Tr}_d$ is the normalized trace on $M_d$.*

*Proof.* If $X$ belongs to $\mathcal{Q}_q^s(n,c)$, then one can write $X = (\langle (P_{a,ij} \otimes Q_{b,k\ell})\chi, \chi\rangle)$ for projection-valued measures $\{P_a\}_{a=1}^c \subseteq M_n(\mathcal{B}(\mathcal{H}_A))$ and $\{Q_b\}_{b=1}^c \subseteq M_n(\mathcal{B}(\mathcal{H}_B))$ on finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, along with a unit vector $\chi \in \mathcal{H}_A \otimes \mathcal{H}_B$. By Theorem 3.3.5, we may write $X = \psi(P_{a,ij}P_{b,k\ell}^*)$ where $\psi$ is the (necessarily faithful) tracial state on the finite-dimensional $C^*$-algebra $\mathcal{A}$ generated by the set $\{P_{a,ij} : 1 \le a \le c,\ 1 \le i,j \le n\}$.

Conversely, if $X$ can be written as $X = (\psi(P_{a,ij}P_{b,k\ell}^*))$ for a projection-valued measure $\{P_a\}_{a=1}^c \subseteq M_n(\mathcal{A})$, where $\mathcal{A}$ is a finite-dimensional $C^*$-algebra with a faithful trace $\psi$ on $\mathcal{A}$, then the proof of Theorem 3.3.5 yields a finite-dimensional realization of $X$ as an element of $\mathcal{Q}_{qc}^s(n,c)$. By Lemma 3.2.6, we must have $X \in \mathcal{Q}_q^s(n,c)$.

Now, assume that $X$ is extreme in $\mathcal{Q}_q^s(n,c)$. Since $\mathcal{A}$ is finite-dimensional, it is $*$-isomorphic to $\bigoplus_{r=1}^m M_{k_r}$ for some $r$ and numbers $k_1,...,k_r \in \mathbb{N}$. Since $\psi$ is a trace on $\mathcal{A}$, there must be $t_1,...,t_m \ge 0$ such that $\sum_{r=1}^m t_r = 1$ and $\psi(\cdot) = \sum_{r=1}^m t_r \mathrm{Tr}_{k_r}(\cdot)$, where $\mathrm{Tr}_{k_r}$ is the normalized trace on $M_{k_r}$. Writing $P_{a,ij} = \bigoplus_{r=1}^m P_{a,ij}^{(r)}$ for each $1 \le a \le c$ and $1 \le i,j \le n$, where $P_{a,ij}^{(r)} \in M_{k_r}$, we have

$$X_{(i,j),(k,\ell)}^{(a,b)} = \sum_{r=1}^m t_r \mathrm{Tr}_{k_r}(P_{a,ij}^{(r)}(P_{b,k\ell}^{(r)})^*).$$

Since $P_a^{(r)} = (P_{a,ij}^{(r)}) \in M_n(M_{k_r})$ must define an orthogonal projection and $\sum_{a=1}^c P_a^{(r)} = I_n \otimes I_{k_r}$, it follows that $X^{(a,b)}_{r,(i,j),(k,\ell)} = \mathrm{Tr}_{k_r}(P^{(r)}_{a,ij}(P^{(r)}_{b,k\ell})^*) \in \mathcal{Q}^s_q(n,c)$, and $\sum_{r=1}^m t_r X^{(a,b)}_{r,(i,j),(k,\ell)} = X^{(a,b)}_{(i,j),(k,\ell)}$. Therefore, $X^{(a,b)}_{r,(i,j),(k,\ell)} = X^{(a,b)}_{(i,j),(k,\ell)}$ for each $r$. This shows that we may take $\mathcal{A}$ to be a matrix algebra, completing the proof. □

Next, we prove that $\mathcal{Q}^s_{qs}(n,c) = \mathcal{Q}^s_q(n,c)$, using a similar approach to [36].

**Theorem 3.3.10.** *For each $n, c \in \mathbb{N}$, we have $\mathcal{Q}^s_{qs}(n,c) = \mathcal{Q}^s_q(n,c)$.*

*Proof.* Let $X = (X^{(a,b)}_{(i,j),(k,\ell)}) \in \mathcal{Q}^s_{qs}(n,c)$, and write

$$X^{(a,b)}_{(i,j),(k,\ell)} = \langle (P_{a,ij} \otimes Q_{b,k\ell})\psi, \psi\rangle,$$

where $P_a = (P_{a,ij})$ is a projection in $\mathbb{C}^n \otimes \mathcal{H}_A$ for each $1 \leq a \leq c$, $Q_b = (Q_{b,ij})$ is a projection in $\mathcal{H}_B \otimes \mathbb{C}^n$ for each $1 \leq b \leq c$, $\sum_{a=1}^c P_a = I_{\mathbb{C}^n \otimes \mathcal{H}_A}$, $\sum_{b=1}^c Q_b = I_{\mathcal{H}_B \otimes \mathbb{C}^n}$, and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a state. We can arrange to have $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$. For example, if $\dim(\mathcal{H}_A) < \dim(\mathcal{H}_B)$, then we choose a Hilbert space $\mathcal{H}_C$ with $\dim(\mathcal{H}_A \oplus \mathcal{H}_C) = \dim(\mathcal{H}_B)$, and extend $P_a$ by defining $\widetilde{P}_{a,ij} = P_{a,ij} \oplus \delta_{ij} I_{\mathcal{H}_C}$. Then

$$\langle (\widetilde{P}_{a,ij} \otimes Q_{b,k\ell})\psi, \psi\rangle = \langle (P_{a,ij} \otimes Q_{b,k\ell})\psi, \psi\rangle = X^{(a,b)}_{(i,j),(k,\ell)}.$$

In this way, we may assume without loss of generality that $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.

We write down a Schmidt decomposition

$$\psi = \sum_{p=1}^{\infty} \alpha_p e_p \otimes f_p,$$

where $\{e_p\}_{p=1}^{\infty} \subseteq \mathcal{H}_A$ and $\{f_p\}_{p=1}^{\infty} \subseteq \mathcal{H}_B$ are orthonormal, and $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ are such that $\sum_{p=1}^{\infty} \alpha_p^2 = 1$. If one extends these orthonormal sets to orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and defines additional $\alpha_p$'s to be $0$, then after direct summing a Hilbert space on one side if necessary, we may assume that $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$ and that $\{e_r\}_{r\in I}$ is an orthonormal basis for $\mathcal{H}_A$, and $\{f_s\}_{s\in I}$ is an orthonormal basis for $\mathcal{H}_B$.

We rewrite the (at most) countable set $\{\alpha_q : \alpha_q \neq 0\} = \{\beta_v : v \in V\}$, where $V = \{1, 2, ...\}$ and $\beta_v > \beta_{v+1}$ for all $v \in V$. We define $K_v = \{e_q : \alpha_q = \beta_v\}$ and $L_v = \{f_q : \alpha_q = \beta_v\}$, and define subspaces $\mathcal{K}_v = \text{span}(K_v)$ and $\mathcal{L}_v = \text{span}(L_v)$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Since $\sum_{q=1}^{\infty} |\alpha_q|^2 = 1$, it follows that each $K_v$ and $L_v$ must be finite, so that $\mathcal{K}_v$ and $\mathcal{L}_v$ are finite-dimensional. We will show that each $\mathcal{K}_v$ is invariant for the operators $\{P_{a,ij} : 1 \leq a \leq c, 1 \leq i, j \leq n\}$, and that each $\mathcal{L}_v$ is invariant for the operators $\{Q_{b,k\ell} : 1 \leq b \leq c, 1 \leq k, \ell \leq n\}$. To this end, let $\omega$ be a primitive $c$-th root of unity, and define order $c$ unitaries $U = \sum_{a=1}^{c} \omega^a P_a \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}_A)$ and $V = \sum_{b=1}^{c} \omega^{-b} Q_b \in \mathcal{B}(\mathcal{H}_A \otimes \mathbb{C}^n)$. Since $X$ is synchronous, by Theorem 3.3.5, we know that

$$(I_{\mathcal{H}_A} \otimes Q_{a,ij}^*)\psi = (P_{a,ij} \otimes I_{\mathcal{H}_B})\psi \text{ and } (I_{\mathcal{H}_A} \otimes Q_{a,ij}Q_{b,ij}^*)\psi = (P_{b,ij}P_{a,ij}^* \otimes I_{\mathcal{H}_B})\psi.$$

Since $U_{ij}U_{ij}^* = \sum_{a,b=1}^{c} \omega^{a-b} P_{a,ij}P_{b,ij}^*$ and $V_{ij}V_{ij}^* = \sum_{a,b=1}^{c} \omega^{b-a} Q_{a,ij}Q_{b,ij}^*$, it follows that

$$(I_{\mathcal{H}_A} \otimes V_{ij}^*)\psi = (U_{ij} \otimes I_{\mathcal{H}_B})\psi \text{ and } (I_{\mathcal{H}_A} \otimes V_{ij}V_{ij}^*)\psi = (U_{ij}U_{ij}^* \otimes I_{\mathcal{H}_B})\psi.$$

Using this fact and the decomposition of $\psi$,

$$\alpha_q \langle U_{ij}e_q, e_p\rangle = \langle (U_{ij} \otimes I_{\mathcal{H}_B})\psi, e_p \otimes f_q\rangle = \langle (I_{\mathcal{H}_A} \otimes V_{ij}^*)\psi, e_p \otimes f_q\rangle = \alpha_p \langle V_{ij}^* f_p, f_q\rangle.$$

Since $U$ and $V$ are unitary, it follows that, for all $p$,

$$\sum_{i,j=1}^{n} \|U_{ij}^* e_p\|^2 = \sum_{i,j=1}^{n} \langle U_{ij}U_{ij}^* e_p, e_p\rangle = n \text{ and } \sum_{i,j=1}^{n} \|U_{ij}e_p\|^2 = \sum_{i,j=1}^{n} \langle U_{ij}^*U_{ij}e_p, e_p\rangle = n.$$

Similarly, $\sum_{i,j=1}^{n} \|V_{ij}^* f_q\|^2 = \sum_{i,j=1}^{n} \|V_{ij}f_q\|^2 = n$. Suppose that $q$ is such that $e_q \in K_1$. Then using the fact that $\alpha_q = \alpha_1$ and that $\alpha_p \le \alpha_1$ for all $p$ yields

$$\begin{aligned}
n|\alpha_1|^2 &= \sum_{i,j=1}^{n} |\alpha_1|^2 \|V_{ij}^* f_q\|^2 \\
&\ge \sum_{i,j=1}^{n} \sum_{p=1}^{\infty} |\alpha_p|^2 |\langle V_{ij}^* f_p, f_q\rangle|^2 \\
&= \sum_{i,j=1}^{n} \sum_{p=1}^{\infty} |\alpha_q|^2 |\langle U_{ij} e_q, e_p\rangle|^2 \\
&= |\alpha_1|^2 \sum_{i,j=1}^{n} \sum_{p=1}^{\infty} |\langle U_{ij} e_q, e_p\rangle|^2 \\
&= |\alpha_1|^2 \sum_{i,j=1}^{n} \sum_{p=1}^{\infty} |\langle U_{ij}^* e_p, e_q\rangle|^2 \\
&= |\alpha_1|^2 \sum_{i,j=1}^{n} \|U_{ij}^* e_q\|^2 \\
&= n|\alpha_1|^2.
\end{aligned}$$

Therefore, we must have equality at all lines. If $p$ is such that $e_p \notin K_1$, then since $\alpha_p < \alpha_1$, we must have $0 = \sum_{i,j=1}^{n} |\alpha_p|^2 |\langle V_{ij}^* f_p, f_q\rangle|^2 = \sum_{i,j=1}^{n} |\alpha_q|^2 |\langle U_{ij} e_q, e_p\rangle|^2$. Therefore, $\langle U_{ij} e_q, e_p\rangle = 0$ for each such $p$, which shows that $U_{ij} e_q \perp e_p$ for all $p$ with $e_p \notin K_1$. Since this happens whenever $\alpha_q = \alpha_1$, the subspace $\mathcal{K}_1$ must be invariant for every $U_{ij}$. By the same argument as above with the quantity $\sum_{i=1}^{n} |\alpha_1|^2 \|V_{ij} f_q\|^2$, it follows that $\mathcal{K}_1$ is invariant for every $U_{ij}^*$. Therefore, $\mathcal{K}_1$ is reducing for the operators $U_{ij}$, for all $1 \le i, j \le n$. A similar argument proves that $\mathcal{L}_1$ is reducing for the operators $V_{k\ell}$, for all $1 \le k, \ell \le n$.

Now, choose $q$ such that $e_q \in K_2$ and $f_q \in L_2$. If $\alpha_p > \alpha_q$, then $\alpha_p = \alpha_1$, so that $e_p \in K_1$ and $f_p \in K_1$. The above shows that $\langle U_{ij} e_q, e_p\rangle = 0$ and $\langle U_{ij}^* e_q, e_p\rangle = 0$, so that $U_{ij} e_q \perp \mathcal{K}_1$ and $U_{ij}^* e_q \perp \mathcal{K}_2$. Similarly, $V_{k\ell} f_q \perp \mathcal{L}_1$ and $V_{k\ell}^* f_q \perp \mathcal{L}_1$. Then using a similar string of inequalities as before, one obtains $U_{ij} e_q \perp e_p$ whenever $p$ is such that $e_p \notin K_2$ and $q$ is such that $e_q \in K_2$. Therefore, one finds that $\mathcal{K}_2$ is invariant for each $U_{ij}$. A similar argument shows that $\mathcal{K}_2$ is invariant for $U_{ij}^*$, so that $\mathcal{K}_2$ is reducing for $\{U_{ij} : 1 \le i, j \le n\}$. The same argument shows that $\{V_{k\ell} : 1 \le$

$k, \ell \leq n\}$ reduces $\mathcal{K}_2$.

It follows by induction that $\mathcal{K}_v$ is reducing for $\{U_{ij} : 1 \leq i, j \leq n\}$ for all $v$ and that $\mathcal{L}_v$ is reducing for $\{V_{k\ell} : 1 \leq k, \ell \leq n\}$ for all $v$. By construction of the unitaries $U$ and $V$, we know that

$$P_a = \frac{1}{c} \sum_{d=1}^{c} \omega^{-ad} U^d \text{ and } Q_b = \frac{1}{c} \sum_{d=1}^{c} \omega^{bd} V^d.$$

Therefore, $\mathcal{K}_v$ is reducing for each $P_{a,ij}$, and $\mathcal{L}_v$ is reducing for each $Q_{b,k\ell}$, as desired.

Finally, we will exhibit $X = (X_{(i,j),(k,\ell)}^{(a,b)})$ as a countable convex combination of elements of $\mathcal{Q}_q^s(n,c)$. One can regard elements of $\mathcal{Q}_q^s(n,c)$ as elements of $\mathbb{C}^{n^4 \times c^2}$, or as elements of $\mathbb{R}^{2(n^4 c^2)}$. Then by a countably infinite version of Carathéodory's Theorem [11], this will show that $X$ belongs to $\mathcal{Q}_q^s(n,c)$, which will complete the proof. (As mentioned in [36], this result from [11] holds even with non-closed convex sets, of which $\mathcal{Q}_q^s(n,c)$ is an example.)

For each $v \in V$, we let $d_v = \dim(\mathcal{K}_v) = \dim(\mathcal{L}_v) = |K_v| = |L_v|$, which is finite. Define the state

$$\psi_v = \frac{1}{\sqrt{d_v}} \sum_{p: e_p \in K_v} e_p \otimes f_p,$$

and define

$$P_{v,a,ij} = P_{a,ij}|_{\mathcal{K}_v} \text{ and } Q_{v,b,k\ell} = Q_{b,k\ell}|_{\mathcal{L}_v}.$$

Since $\mathcal{K}_v$ is reducing for $P_{a,ij}$, and since $P_a$ is a projection, the operator $P_{v,a} = (P_{v,a,ij})_{i,j=1}^n$ is a projection on $\mathbb{C}^n \otimes \mathcal{K}_v$. Similarly, $Q_{v,b} = (Q_{v,b,k\ell})_{k,\ell=1}^n$ is a projection on $\mathcal{L}_v \otimes \mathbb{C}^n$. Moreover, $\sum_{a=1}^c P_{v,a} = I_{\mathbb{C}^n} \otimes I_{\mathcal{K}_v}$ and $\sum_{b=1}^c Q_{v,b} = I_{\mathcal{L}_v} \otimes I_{\mathbb{C}^n}$. Therefore, the correlation

$$X_v = (X_{v,(i,j),(k,\ell)}^{(a,b)}) = (\langle (P_{v,a,ij} \otimes Q_{v,b,k\ell}) \psi_v, \psi_v \rangle)$$

belongs to $\mathcal{Q}_q(n,c)$ for each $v$. Set $t_v = \beta_v^2 d_v$. Then $t_v \geq 0$ and $\sum_{v \geq 1} t_v = \sum_{p=1}^{\infty} |\alpha_p|^2 = 1$.

Finally, for each $1 \le a, b \le c$ and $1 \le i, j \le n$, we compute

$$\begin{aligned} X^{(a,b)}_{(i,j),(k,\ell)} &= \langle (P_{a,ij} \otimes Q_{b,k\ell})\psi, \psi \rangle \\ &= \sum_v \sum_{p,q: e_p, e_q \in K_v} \beta_v^2 \langle (P_{a,ij} \otimes Q_{b,k\ell})(e_p \otimes f_p), e_q \otimes f_q \rangle \\ &= \sum_v \beta_v^2 d_v \langle (P_{v,a,ij} \otimes Q_{v,b,k\ell})\psi_v, \psi_v \rangle \\ &= \sum_v t_v X^{(a,b)}_{v,(i,j),(k,\ell)}. \end{aligned}$$

It follows that $X = \sum_v t_v X_v$. Since each $X_v \in \mathcal{Q}_q(n, c)$, it follows that $X \in \mathcal{Q}_q(n, c)$. Since $X$ is also synchronous, we obtain $X \in \mathcal{Q}_q^s(n, c)$, completing the proof. $\square$

For completeness, we close this section by stating that elements of $\mathcal{Q}_{qa}^s(n, c)$ can be represented using the trace on $\mathcal{R}^{\mathcal{U}}$ and projection-valued measures with $c$ outputs in $M_n(\mathcal{R}^{\mathcal{U}})$, where $\mathcal{R}^{\mathcal{U}}$ denotes an ultrapower of the hyperfinite $II_1$-factor $\mathcal{R}$ by a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. The proof can be found in [5, section 3].

**Theorem 3.3.11.** *Let $X = (X^{(a,b)}_{(i,j),(k,\ell)})$ be an element of $(M_n \otimes M_n)^{c^2}$. The following statements are equivalent:*

1. *$X$ belongs to $\mathcal{Q}_{qa}^s(n, c)$;*
2. *$X$ belongs to $\overline{\mathcal{Q}_q^s(n, c)}$;*
3. *There is a separable unital $C^*$-algebra $\mathcal{A}$, a PVM $\{P_1, ..., P_c\}$ in $M_n(\mathcal{A})$, and an amenable trace $\tau$ on $\mathcal{A}$ such that, for all $1 \le i, j, k, \ell \le n$ and $1 \le a, b \le c$,*

$$X^{(a,b)}_{(i,j),(k,\ell)} = \tau(P_{a,ij} P^*_{b,k\ell});$$

4. *There are elements $q_{a,ij}$ in $\mathcal{R}^{\mathcal{U}}$ such that $q_a = (q_{a,ij})$ are projections in $M_n(\mathcal{R}^{\mathcal{U}})$ with $\sum_{a=1}^c q_a = I_n$ and*

$$X^{(a,b)}_{(i,j),(k,\ell)} = \mathrm{Tr}_{\mathcal{R}^{\mathcal{U}}}(q_{a,ij} q^*_{b,k\ell}).$$

# 4. Quantum graph coloring game and chromatic numbers

The non-local classical graph coloring game on $c$ colors is described by a finite simple graph $G$, with input set $I = V(G)$ (the vertex set of $G$) and output set $O = \{1, 2, \ldots, c\}$. The goal of Alice and Bob in this game is to convince the referee that there exists a coloring of $G$ with $c$ colors. In particular, the rules of the game are determined by the following two requirements:

1. Alice and Bob's answers must be *synchronous*, meaning that if they receive the same vertex $x \in V(G)$, then they must return the color $a \in O$.

2. If the referee supplies an edge $(x, y) \in E(G)$ to Alice and Bob, then they must respond with different colors $a$ and $b$ $(a \neq b)$.

In this section, we aim to extend this coloring game to the setting of quantum graphs.

## 4.1 The quantum-to-classical graph coloring game

Throughout our discussion, we use the bimodule perspective of quantum graphs [definition 2.1.1] considered by N. Weaver. For our purposes, we refer to a **quantum graph** as a triple $(\mathcal{S}, \mathcal{M}, M_n)$, where

- $\mathcal{M}$ is a (non-degenerate) von Neumann algebra and $\mathcal{M} \subseteq M_n$;

- $\mathcal{S} \subseteq M_n$ is an operator system; and

- $\mathcal{S}$ is an $\mathcal{M}'$-$\mathcal{M}'$-bimodule with respect to matrix multiplication.

In our discussion below, one can just as well use the "traceless" version of quantum graphs along the lines of D. Stahlke [53]; i.e., one replaces the second condition with the condition that $\mathcal{S}$ is a self-adjoint subspace of $M_n$ with $\mathrm{Tr}(X) = 0$ for every $X \in \mathcal{S}$. This condition, combined with the bimodule property, would force $\mathcal{S} \subseteq (\mathcal{M}')^{\perp}$. Our use of the operator system approach is generally cosmetic: one can easily adapt the quantum-classical game to traceless self-adjoint operator spaces that are $\mathcal{M}'$-$\mathcal{M}'$-bimodules with respect to matrix multiplication.

We begin by exhibiting a certain orthonormal basis for $\mathcal{S}$ with respect to the (unnormalized) trace on $M_n$. It is from this (preferred) basis for $\mathcal{S}$ that we will extract our input states for the coloring game.

**Proposition 4.1.1.** *Let $\mathcal{K}_1, ..., \mathcal{K}_m$ be non-zero subspaces of $\mathbb{C}^n$ with $\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_m = \mathbb{C}^n$, such that $\mathcal{M}$ acts irreducibly on each $\mathcal{K}_r$. Let $E_r$ be the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{K}_r$, for each $1 \leq r \leq m$. Then there exists an orthonormal basis $\mathcal{F}$ of $\mathcal{S} \subseteq M_n$ with respect to the unnormalized trace, such that*

- $\frac{1}{\sqrt{\dim(\mathcal{K}_r)}} E_r \in \mathcal{F}$ *for each* $1 \leq r \leq m$*;*
- $\mathcal{F}$ *contains an orthonormal basis for* $\mathcal{M}'$*; and*
- *For each* $Y \in \mathcal{F}$*, there are unique* $r, s$ *with* $E_r Y E_s = Y$*.*

*Proof.* Since $\mathcal{M}$ acts irreducibly on $\mathcal{K}_r$, it follows that $E_r \in \mathcal{M}'$. Let $X$ be an element of $\mathcal{S}$. As $\mathcal{S}$ is an $\mathcal{M}'$-$\mathcal{M}'$-bimodule, it follows that $E_r X E_s \in \mathcal{S}$ for all $1 \leq r, s \leq m$. Moreover, since $\sum_{r=1}^m E_r = 1$, we have $X = \sum_{r,s=1}^m E_r X E_s$. Given $X, Y \in \mathcal{S}$, we have $\langle E_r X E_s, E_p Y E_q \rangle = 0$ whenever $r \neq p$ or $s \neq q$, where $\langle \cdot, \cdot \rangle$ is the inner product with respect to the unnormalized trace on $M_n$. We choose an orthonormal basis $\mathcal{F}_{r,s}$ for $E_r \mathcal{S} E_s$ with respect to this inner product as follows. We start with an orthonormal basis for $E_r \mathcal{M}' E_s$; if $r = s$, then we arrange for this orthonormal basis to contain $\frac{1}{\sqrt{\dim(\mathcal{K}_r)}} E_r$. Then we extend the orthonormal basis for $E_r \mathcal{M}' E_s$ to an orthonormal basis for $E_r \mathcal{S} E_s$. Also, if $X \in \mathcal{S} \cap (\mathcal{M}')^\perp$ and $Y \in \mathcal{M}'$, then

$$\langle E_r X E_s, Y \rangle = \mathrm{Tr}(Y^* E_r X E_s) = \mathrm{Tr}(X E_s Y^* E_r) = \langle X, E_r Y E_s \rangle = 0,$$

which shows that $E_r(\mathcal{S} \cap (\mathcal{M}')^\perp) E_s \perp \mathcal{M}'$. Then $\mathcal{F} = \bigcup_{r,s} \mathcal{F}_{r,s}$ is an orthonormal basis for $\mathcal{S}$, which evidently satisfies all three properties. $\square$

**Definition 4.1.2.** We call a basis for $\mathcal{S}$ satisfying Proposition 4.1.1 as a **quantum edge basis** for $(\mathcal{S}, \mathcal{M}, M_n)$.

Alternatively, one could arrange for a quantum edge basis for $\mathcal{S}$ to also contain a normalized system of matrix units for $\mathcal{M}'$, since a quantum edge basis must already contain the normalized diagonal matrix units. We will see in Theorem 4.1.7 that the game is independent of the quantum edge basis chosen.

Once an orthonormal basis for $\mathbb{C}^n$ has been fixed, one can define the inputs for the game using the following well-known correspondence between vectors in $\mathbb{C}^n \otimes \mathbb{C}^n$ and matrices in $M_n$. With respect to a basis $\{v_1, ..., v_n\}$, this correspondence is given by the assignment $v_i \otimes v_j \mapsto v_i v_j^*$, where $v_i v_j^*$ is the rank-one operator in $M_n$ such that $v_i v_j^*(x) = \langle x, v_j \rangle v_i$ for all $x \in \mathbb{C}^n$.

**Proposition 4.1.3.** *Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph with quantum edge basis $\mathcal{F}$. Let $\{v_1, ..., v_n\}$ be an orthonormal basis for $\mathbb{C}^n$ that can be partitioned into bases for the subspaces $\mathcal{K}_1, ..., \mathcal{K}_m$. For each $Y_\alpha \in \mathcal{F}$, write $Y_\alpha = \sum_{p,q} y_{\alpha,pq}\, v_p v_q^*$ for $y_{\alpha,pq} \in \mathbb{C}$. Then the set*

$$\left\{ \sum_{p,q} y_{\alpha,pq}\ v_p \otimes v_q \right\}_\alpha \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$$

*is orthonormal.*

*Proof.* This result immediately follows from the fact that the correspondence $v_i \otimes v_j \mapsto v_i v_j^*$ preserves inner products, when using the canonical inner product on $\mathbb{C}^n \otimes \mathbb{C}^n$ and the (unnormalized) Hilbert-Schmidt inner product on $M_n$. $\square$

With the notion of quantum edge bases in hand, we now define the coloring game for the quantum graph $(\mathcal{S}, \mathcal{M}, M_n)$ with $c$ classical colors.

**Definition 4.1.4.** Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph, and let $\{v_1, ..., v_n\}$ be a basis for $\mathbb{C}^n$ that can be partitioned into bases for the subspaces $\mathcal{K}_1, ..., \mathcal{K}_r$. The **quantum-to-classical graph coloring game** on $c$-colors, with respect to the basis $\{v_1, ..., v_n\}$ and the quantum edge basis $\mathcal{F}$, is defined as follows:

1. INPUTS: The inputs are of the form $\sum_{p,q} y_{\alpha,pq}\ v_p \otimes v_q$, where $Y_\alpha := \sum_{p,q} y_{\alpha,pq}\ v_p v_q^*$ is an element of $\mathcal{F}$.

2. OUTPUTS: The outputs are colors $a, b \in \{1, ..., c\}$.

3. There are two rules to the game:

   - Adjacency rule: If $Y_\alpha \perp \mathcal{M}'$, then Alice and Bob must respond with different colors; i.e., $a \neq b$.
   - Same vertex rule: If $Y_\alpha \in \mathcal{M}'$, then Alice and Bob must respond with the same color; i.e., $a = b$.

Notice that the second rule will include a synchronicity condition: the inputs corresponding to $\frac{1}{\sqrt{\dim(\mathcal{K}_r)}} E_r$ will arise in the second rule. We will see that the rule applied to these inputs will force Bob's projections to arise from Alice's projections; the rule applied to the other basis elements of $\mathcal{M}'$ will be what forces the projections to live in $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$, rather than $M_n \otimes \mathcal{B}(\mathcal{H})$.

While the above definition of the game seems heavily basis-dependent, we will see that the existence of winning strategies in the various models is independent of the basis $\{v_1, ..., v_n\}$, and independent of the quantum edge basis $\mathcal{F}$ chosen for $(\mathcal{S}, \mathcal{M}, M_n)$. This will be a direct consequence of Theorem 4.1.7.

The players may adopt different types of strategies $\{loc, q, qs, qa, qc\}$ to win this game. We would now like to obtain a combinatorial characterization of quantum graph coloring using the winning strategies for the coloring game. For this, we first review Kraus operators in the infinite-dimensional case. Recall that a von Neumann algebra $\mathcal{N}$ is **finite** if every isometry in $\mathcal{N}$ is a unitary; i.e., $v^*v = 1$ implies $vv^* = 1$ in $\mathcal{N}$. We choose to work with finite von Neumann algebras since they are always equipped with a normal tracial state. We will be dealing with the case when $\mathcal{N}$ is a finite von Neumann algebra equipped with a *faithful* normal trace $\tau$. One may always choose a faithful normal representation $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ such that $\tau(\cdot) = \langle (\cdot)\chi, \chi \rangle$ for some unit vector $\chi \in \mathcal{H}$.

Suppose that $\mathcal{L} \subseteq \mathcal{B}(\mathcal{K})$ is another von Neumann algebra with faithful normal trace $\rho$. If $\Phi : \mathcal{L} \to \mathcal{N}$ is a normal UCP map, then $\Phi_* : \mathcal{N}_* \to \mathcal{L}_*$ is a CPTP map. In our context, $\mathcal{L}$ will be a finite-dimensional von Neumann algebra, so a UCP map $\Phi : \mathcal{L} \to \mathcal{N}$ is automatically normal. One may choose $\mathcal{K}$ to be finite-dimensional and extend $\Phi$ to a UCP map from $\mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{H})$, which is still (automatically) normal. Then one may choose Kraus operators $F_i$ such that $\Phi(\cdot) = \sum_{i=1}^m F_i^*(\cdot)F_i$, where $m$ is either finite or countably infinite. In the latter case, the sum converges in the SOT*-topology. Then $\Phi_* : \mathcal{N}_* \to \mathcal{L}_* = \mathcal{L}$ can be written as

$$\Phi_*(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*.$$

The interested reader can consult [12] (and the references therein) for more information on these topics.

Now, we address some of the basis dependence of the game before the main theorem. The next lemma shows that, up to a unitary conjugation, the basis for $\mathbb{C}^n$ in Definition 4.1.4 does not matter.

**Lemma 4.1.5.** *Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph, and write $\mathbb{C}^n = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_m$, where $\mathcal{M}$ acts irreducibly on each $\mathcal{K}_r$. Let $\{v_1, ..., v_n\}$ be an orthonormal basis for $\mathbb{C}^n$ that can be partitioned into bases for the subspaces $\mathcal{K}_1, ..., \mathcal{K}_m$. Define $U \in M_n$ to be the unitary such that $Ue_i = v_i$ for all $i$, where $\{e_1, ..., e_n\}$ is another orthonormal basis for $\mathbb{C}^n$. Suppose that $X \in \mathcal{Q}_{qc}(n, c)$, and let $\{Y_\alpha\}_\alpha$ be a quantum edge basis for $(\mathcal{S}, \mathcal{M}, M_n)$. Then $X$ is a winning strategy for the coloring game for $((\mathcal{S}, \mathcal{M}, M_n), K_c)$ with respect to $\{Y_\alpha\}_\alpha$ if and only if $Z := (U \otimes U)^* X (U \otimes U)$ is a winning strategy for the coloring game for $((U^*\mathcal{S}U, U^*\mathcal{M}U, M_n), K_c)$ with respect to the quantum edge basis $\{U^* Y_\alpha U\}_\alpha$.*

*Proof.* Suppose that we can write $X = (\langle (P_a \otimes I_n)(I_n \otimes Q_b)(e_j \otimes \chi \otimes e_\ell), e_i \otimes \chi \otimes e_k \rangle)$, where $(\{P_a\}_{a=1}^c, \{Q_b\}_{b=1}^c, \chi)$ is a $qc$-strategy on a Hilbert space $\mathcal{H}$. Then

$$\langle (P_a \otimes I_n)(I_n \otimes Q_b)(v_j \otimes \chi \otimes v_\ell), v_i \otimes \chi \otimes v_k \rangle = \langle (U^* P_a U \otimes I_n)(I_n \otimes U^* Q_b U)(e_j \otimes \chi \otimes e_\ell), e_i \otimes \chi \otimes e_k \rangle.$$

In other words, the element $Z = (Z^{(a,b)}) := ((U \otimes U)^* X^{(a,b)} (U \otimes U))$ is a $qc$-correlation with respect to the basis $\{v_1, ..., v_n\}$. It is not hard to see that, if $\mathcal{F}$ is a quantum edge basis for $(\mathcal{S}, \mathcal{M}, M_n)$, then $U^*\mathcal{F}U$ is a quantum edge basis for $(U^*\mathcal{S}U, U^*\mathcal{M}U, M_n)$, since $U^*\mathcal{M}'U = (U^*\mathcal{M}U)'$ and the Hilbert-Schmidt inner product is invariant under unitary conjugation. Therefore, if $Y_\alpha = \sum_{p,q} y_{\alpha,pq} v_p v_q^*$ belongs to $\mathcal{F}$, then its associated input vector is $\sum_{p,q} y_{\alpha,pq} v_p \otimes v_q$. Then $U^* Y_\alpha U = \sum_{p,q} y_{\alpha,pq} U^* v_p v_q^* U$ has associated input vector $\sum_{p,q} y_{\alpha,pq} U^* v_p \otimes U^* v_q = \sum_{p,q} y_{\alpha,pq} e_p \otimes e_q$.

Therefore, the probability of Alice and Bob outputting $(a, b)$ given the input vector $\sum_{p,q} y_{\alpha,pq} v_p \otimes v_q$, with respect to the correlation $X$, is the same as the probability of outputting $(a, b)$ given the input vector $\sum_{p,q} y_{\alpha,pq} e_p \otimes e_q$, with respect to the correlation $Z$. As this equality occurs for any element of the quantum edge basis $\mathcal{F}$, the desired result follows. $\square$

**Remark 4.1.6.** The previous remark, along with the adjacency rule, forces any winning strategy to be synchronous with respect to the basis $\{v_1, ..., v_n\}$. Thus, in our main theorem, we may assume that we are dealing with a synchronous $t$-strategy $(\{P_a\}_{a=1}^c, \chi)$, where $\{P_a\}_{a=1}^c$ is a PVM and $\chi$ is a faithful normal tracial state on the von Neumann algebra generated by the entries of $\{P_a\}_{a=1}^c$. Note that conjugating $\{P_a\}_{a=1}^c$ by a unitary in $M_n$ does not change the von Neumann algebra generated by the entries of the operators $P_a$.

**Theorem 4.1.7.** *Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph, $c \in \mathbb{N}$, and let $t \in \{loc, q, qa, qc\}$. Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a (non-degenerate) finite von Neumann algebra, and $\chi \in \mathcal{H}$ be a unit vector such that $\tau = \langle (\cdot)\chi, \chi\rangle$ is a faithful (normal) trace on $\mathcal{N}$. Let $D_c$ be the set of all diagonal matrices in $M_c$. The following are equivalent:*

1. *There is a winning strategy $(\{P_a\}_{a=1}^c, \chi)$ from $\mathcal{N}$ for the coloring game for $(\mathcal{S}, \mathcal{M}, M_n)$ on $c$ colors with respect to any quantum edge basis.*

2. *There is a winning strategy $(\{P_a\}_{a=1}^c, \chi)$ from $\mathcal{N}$ for the coloring game for $(\mathcal{S}, \mathcal{M}, M_n)$ on $c$ colors with respect to some quantum edge basis.*

3. *There is a PVM $\{P_a\}_{a=1}^c$ in $\mathcal{M} \otimes \mathcal{N}$ satisfying the following:*

$$P_a((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1)P_a = 0 \, \text{for } 1 \le a \le c. \tag{4.1.0.1}$$

4. *There is a CPTP map* $\Phi : \mathcal{M} \otimes \mathcal{N}_* \to D_c$ *of the form* $\Phi(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*$ *such that*

$$F_i((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1_{\mathcal{N}})F_j^* \subseteq (D_c)^\perp \text{ for all i,j,} \tag{4.1.0.2}$$

*and*

$$F_i(\mathcal{M}' \otimes 1_{\mathcal{N}})F_j^* \subseteq D_c \text{ for all } i, j. \tag{4.1.0.3}$$

*Proof.* Clearly (1) implies (2). We will show that (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). Let $\{v_1, ..., v_n\}$ be an orthonormal basis for $\mathbb{C}^n$. Let $U$ be the unitary such that $Ue_i = v_i$ for all $i$. Suppose that we can establish (3) for the PVM $\{(U \otimes 1_{\mathcal{N}})^* P_a (U \otimes 1_{\mathcal{N}})\}_{a=1}^c$ and the quantum graph $(U^*\mathcal{S}U, U^*\mathcal{M}U, M_n)$. Using the fact that $(U^*\mathcal{M}U)' = U^*\mathcal{M}'U$, the condition in (3) can be written as

$$(U \otimes 1_{\mathcal{N}})^* P_a (U \otimes 1_{\mathcal{N}})((U^*\mathcal{S}U) \cap (U^*\mathcal{M}'U)^\perp \otimes 1_{\mathcal{N}})(U \otimes 1_{\mathcal{N}})^* P_b (U \otimes 1_{\mathcal{N}}) = 0 \text{ if } a \not\sim b.$$

It is not hard to see that $(U^*\mathcal{M}'U)^\perp = U^*(\mathcal{M}')^\perp U$, so that the above reduces to

$$(U \otimes 1_{\mathcal{N}})^* P_a ((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1_{\mathcal{N}}) P_b (U \otimes 1_{\mathcal{N}}) = 0.$$

Since $U$ is a unitary, we obtain the desired condition for $\{P_a\}_{a=1}^c$ with respect to the quantum graph $(\mathcal{S}, \mathcal{M}, M_n)$. Hence, we may assume without loss of generality that $v_i = e_i$ for all $i$.

Then, given a matrix $Y = \sum_{p,q} y_{pq} v_p v_q^*$ with associated unit vector $y = \sum_{p,q} y_{pq} v_p \otimes v_q$, the probability of Alice and Bob outputting $a$ and $b$ respectively, given $y$ and using the synchronous strategy $(\{P_a\}_{a=1}^c, \chi)$, is

$$\begin{aligned} p(a,b|y) &= \left\langle (P_{a,ij}P_{b,k\ell}^*)_{(i,j),(k,\ell)} \left( \sum_{p,q} y_{pq} v_p \otimes \chi \otimes v_q \right), \sum_{r,s} y_{rs} v_r \otimes \chi \otimes v_s \right\rangle \\ &= \left\langle \sum_{i,j,k,\ell} v_i \otimes P_{a,ij} y_{j\ell} P_{b,k\ell}^* \chi \otimes v_k, \sum_{r,s} y_{rs} v_r \otimes \chi \otimes v_s \right\rangle \\ &= \sum_{i,j,k,\ell} \left\langle P_{a,ij} y_{j\ell} P_{b,k\ell}^* \overline{y}_{ik} \chi, \chi \right\rangle \\ &= \sum_{i,j,k,\ell} \tau(P_{a,ij} y_{j\ell} P_{b,\ell k} \overline{y}_{ik}) \\ &= \operatorname{Tr} \otimes \tau \left( \left( \sum_{j,\ell} P_{a,ij} y_{j\ell} P_{b,\ell k} \right)_{i,k} (Y^* \otimes 1_{\mathcal{N}}) \right) \\ &= \operatorname{Tr} \otimes \tau (P_a (Y \otimes 1_{\mathcal{N}}) P_b (Y^* \otimes 1_{\mathcal{N}})) \\ &= \operatorname{Tr} \otimes \tau (P_a (Y \otimes 1_{\mathcal{N}}) P_b (Y^* \otimes 1_{\mathcal{N}}) P_a), \end{aligned} \tag{4.1.0.4}$$

where we have used the fact that $P_a$ is an orthogonal projection. Now, suppose that $\mathcal{F} = \{Y_\alpha\}_\alpha$ is a quantum edge basis for $(\mathcal{S}, \mathcal{M}, M_n)$, and suppose that $(\{P_a\}_{a=1}^c, \chi)$ is a winning strategy with respect to this quantum edge basis. If $Y_\alpha \in \mathcal{M}'$, then Equation 4.1.0.4 and faithfulness of the trace gives $P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_b = 0$ whenever $a \neq b$. Then

$$P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_a = \sum_{b=1}^c P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_b = P_a(Y_\alpha \otimes 1_{\mathcal{N}}) \left( \sum_{b=1}^c P_b \right) = P_a(Y_\alpha \otimes 1_{\mathcal{N}}).$$

Similarly, $P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_a = (Y_\alpha \otimes 1_{\mathcal{N}})P_a$. Hence, $P_a$ commutes with $Y_\alpha \otimes 1_{\mathcal{N}}$ whenever $Y_\alpha \in \mathcal{M}'$. This shows that $P_a \in (\mathcal{M}' \otimes 1_{\mathcal{N}})' \cap (M_n \otimes \mathcal{N}) = \mathcal{M} \otimes \mathcal{B}(\mathcal{H}) \cap (M_n \otimes \mathcal{N}) = \mathcal{M} \otimes \mathcal{N}$.

Similarly, if $Y_\alpha \perp \mathcal{M}'$, then the rules of the game and the faithfulness of the trace force $P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_a = 0$, which shows that (3) holds.

Now we show that (3) implies (4). If (3) holds, then there is a projection-valued measure $\{P_a\}_{a=1}^c$ in $\mathcal{M} \otimes \mathcal{N}$ such that $P_a(Y \otimes 1_{\mathcal{N}})P_a = 0$ for all $Y \in \mathcal{S} \cap (\mathcal{M}')^\perp$ and all $a$. Then the map $\Psi : D_c \to \mathcal{M} \otimes \mathcal{N}$ given by $\Psi(E_{kk}) = P_k$ is a unital $*$-homomorphism. Since $D_c$ is finite-dimensional, $\Psi$ is normal. Hence, we may find Kraus operators $F_1, F_2, \ldots$ in $\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^c)$ such that

$$\Psi(\cdot) = \sum_{i=1}^{m} F_i^*(\cdot)F_i,$$

where $m$ is either finite or $\aleph_0$. In the infinite case, these sums converge in the SOT$^*$-topology. Then $\Psi = \Theta^*$ for a CPTP map $\Theta : \mathcal{M}_* \otimes \mathcal{N}_* = \mathcal{M} \otimes \mathcal{N}_* \to D_c$ given by

$$\Theta(\cdot) = \sum_{i=1}^{m} F_i(\cdot)F_i^*.$$

Given $Y \in \mathcal{S}$, we set $Z_{a,b,i,j} = E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{bb}$. Notice that

$$Z_{a,b,i,j}Z_{a,b,i,j}^* = E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{bb}F_j(Y^* \otimes 1_{\mathcal{N}})F_i^* E_{aa},$$

so summing over $j$ (for fixed $i$, this sum will converge in the SOT*-topology) and using the fact that $\sum_{i=1}^{m} F_j^* E_{bb} F_j = \Psi(E_{bb}) = P_b$,

$$\sum_{j=1}^{m} Z_{a,b,i,j}Z_{a,b,i,j}^* = E_{aa}F_i(Y \otimes 1_{\mathcal{N}})P_b(Y^* \otimes 1_{\mathcal{N}})F_i^* E_{aa}.$$

Now set $W_{a,b,i} = E_{aa}F_i(Y \otimes 1_{\mathcal{N}})P_b$. Then $\sum_{j=1}^{m} Z_{a,b,i,j}Z_{a,b,i,j}^* = W_{a,b,i}W_{a,b,i}^*$, since $P_b$ is a projection. On the other hand,

$$W_{a,b,i}^* W_{a,b,i} = P_b(Y^* \otimes 1_{\mathcal{N}})F_i^* E_{aa}F_i(Y \otimes 1_{\mathcal{N}})P_b,$$

so summing over $i$ gives

$$\sum_{i=1}^{m} W_{a,b,i}^* W_{a,b,i} = P_b(Y^* \otimes 1_{\mathcal{N}})P_a(Y \otimes 1_{\mathcal{N}})P_b = (P_a(Y \otimes 1_{\mathcal{N}})P_b)^*(P_a(Y \otimes 1_{\mathcal{N}})P_b).$$

It follows that, if the latter quantity is zero, then $Z_{a,b,i,j} = 0$ for all $i, j$. By condition (3), if $Y \in \mathcal{S} \cap (\mathcal{M}')^{\perp}$, then $P_a(Y \otimes 1_{\mathcal{N}})P_a = 0$. This immediately implies that $E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{aa} = 0$.

Then

$$F_i(Y \otimes 1_{\mathcal{N}})F_j^* = \sum_{a,b} E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{bb} = \sum_a E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{aa} \in D_c^{\perp}.$$

Since each $P_a$ belongs to $\mathcal{M} \otimes \mathcal{N}$, $P_a$ commutes with $\mathcal{M}' \otimes 1_{\mathcal{N}}$. Therefore, $P_a(Y \otimes 1_{\mathcal{N}})P_a = 0$ for all $a$ and $Y \in \mathcal{M}'$. A consideration of the above equations, yields $E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{aa} = 0$ whenever $Y \in \mathcal{M}'$. In that case, we have

$$F_i(Y \otimes 1_{\mathcal{N}})F_j^* = \sum_{a,b} E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{bb} = \sum_a E_{aa}F_i(Y \otimes 1_{\mathcal{N}})F_j^* E_{aa} \in D_c,$$

which yields the second part of condition (4). Hence, (3) implies (4).

Lastly, suppose that (4) holds; we will obtain a winning strategy for the game. Suppose that $\Phi : \mathcal{M}_* \otimes \mathcal{N}_* \to D_c$ is a CPTP map of the form $\Phi(\cdot) = \sum_{i=1}^m R_i(\cdot)R_i^*$, such that $R_i(Y \otimes 1_{\mathcal{N}})R_j^* \in D_c^{\perp}$ for all $1 \leq i, j \leq m$ and $Y \in \mathcal{S} \cap (\mathcal{M}')^{\perp}$, and $R_i(Y \otimes 1_{\mathcal{N}})R_j^* \in D_c$ for all $Y \in \mathcal{M}'$. Then $\Phi^*(\cdot) = \sum_{i=1}^m R_i^*(\cdot)R_i$ defines a normal UCP map from $D_c$ to $\mathcal{M} \otimes \mathcal{N}$. Let $P_a = \Phi^*(E_{aa}) = \sum_{i=1}^m R_i^* E_{aa} R_i$ for each $1 \leq a \leq c$. Since $\Phi^*$ is UCP, $\{P_a\}_{a=1}^c$ is a POVM in $\mathcal{M} \otimes \mathcal{N}$. By considering the unitary $U$ sending $e_i$ to $v_i$ for each $i$, along with the POVM $\{U^* P_a U\}_{a=1}^c$, the quantum graph $(U^*\mathcal{S}U, U^*\mathcal{M}U, M_n)$ and the operators $R_iU$ if necessary, we may assume without loss of generality that $v_i = e_i$ for all $i$. We will show that $X_{(i,j),(k,\ell)}^{(a,b)} = (\tau(P_{a,ij}P_{b,k\ell}^*))$ defines a winning $t$-strategy for the quantum graph coloring game for $((\mathcal{S}, \mathcal{M}, M_n), K_c)$.

For $1 \leq a, b \leq c$, $1 \leq i, j \leq m$ and $Y \in \mathcal{S}$, we define $V_{a,b,i,j} = E_{aa}R_i(Y \otimes 1_{\mathcal{N}})R_j^* E_{bb}$. Then

$$\sum_{j=1}^m V_{a,b,i,j}V_{a,b,i,j}^* = \sum_{j=1}^m E_{aa}R_i(Y \otimes 1_{\mathcal{N}})R_j^* E_{bb} R_j(Y^* \otimes 1_{\mathcal{N}})R_i^* E_{aa} = E_{aa}R_i(Y \otimes 1_{\mathcal{N}})P_b(Y^* \otimes 1_{\mathcal{N}})R_i^* E_{aa},$$

since $P_b = \Phi^*(E_{bb})$. Therefore, $\sum_{j=1}^m V_{a,b,i,j}V_{a,b,i,j}^* = T_{a,b,i}^* T_{a,b,i}$ where $T_{a,b,i} = P_b^{\frac{1}{2}}(Y^* \otimes 1_{\mathcal{N}})R_i^* E_{aa}$.

Next, we examine the sum

$$\sum_{i=1}^{m} T_{a,b,i}T_{a,b,i}^{*} = \sum_{i=1}^{m} P_b^{\frac{1}{2}}(Y^* \otimes 1_{\mathcal{N}})R_i^* E_{aa} R_i (Y \otimes 1_{\mathcal{N}})P_b^{\frac{1}{2}} = P_b^{\frac{1}{2}}(Y^* \otimes 1_{\mathcal{N}})P_a(Y \otimes 1_{\mathcal{N}})P_b^{\frac{1}{2}}.$$

In the case when $Y \in \mathcal{M}'$, we have $V_{a,b,i,j} = 0$ whenever $a \neq b$, which implies that $P_b^{\frac{1}{2}}(Y^* \otimes 1_{\mathcal{N}})P_a(Y \otimes 1_{\mathcal{N}})P_b^{\frac{1}{2}} = 0$. It follows that $P_a^{\frac{1}{2}}(Y \otimes 1_{\mathcal{N}})P_b^{\frac{1}{2}} = 0$. Multiplying on the left by $P_a^{\frac{1}{2}}$ and on the right by $P_b^{\frac{1}{2}}$, we obtain $P_a(Y \otimes 1_{\mathcal{N}})P_b = 0$ whenever $Y \in \mathcal{M}'$ and $a \neq b$. The case when $Y = 1_{\mathcal{M}}$ shows that $P_aP_b = 0$ for $a \neq b$. Combining this orthogonality with the fact that $\{P_a\}_{a=1}^{c}$ is a POVM, we conclude that $\{P_a\}_{a=1}^{c}$ is a PVM. Similarly, if $Y \in \mathcal{S} \cap (\mathcal{M}')^{\perp}$, then by condition (4), $V_{a,a,i,j} = 0$ for all $a \in [c]$. The same calculation shows that $P_a(Y \otimes 1_{\mathcal{N}})P_a = 0$ in this case as well.

Therefore, using Equation (4.1.0.4), if $\{Y_\alpha\}_\alpha$ is a quantum edge basis for $(\mathcal{S}, \mathcal{M}, M_n)$, $Y_\alpha$ has associated unit vector $y_\alpha$ and $Y_\alpha \perp \mathcal{M}'$, then by equation (4.1.0.4),

$$p(a, a|y_\alpha) = \langle P_a(Y_\alpha \otimes 1_{\mathcal{N}})P_a, Y_\alpha \rangle = 0. \text{ for all } a.$$

If $Y_\alpha$ belongs to $\mathcal{M}$ with associated unit vector $y_\alpha$, then $p(a, b|y_\alpha) = \langle P_a(Y \otimes 1_{\mathcal{N}})P_b, Y \otimes 1_{\mathcal{N}} \rangle = 0$ as well. This shows that $(\{P_a\}_{a=1}^{c}, \chi)$ defines a winning strategy for the coloring game for $(\mathcal{S}, \mathcal{M}, M_n)$ on $c$ colors with respect to any quantum edge basis, completing the proof. □

**Notation 5.** *In the following discussion, we write $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{t} K_c$ to mean that there is a winning $t$-strategy for the graph coloring game for $(\mathcal{S}, \mathcal{M}, M_n)$ on $c$ colors.*

Here, $K_c$ is used to denote a classical complete graph on $c$ vertices, with the understanding that coloring a graph with $c$ classical colors is basically finding a graph homomorphism into the classical complete graph $(K_c)$ on $c$ vertices.

Later, we will also see other algebraic models of coloring $\{C^*, hered, alg\}$, which we include in the definition below, for convenience:

**Definition 4.1.8.** Let $t \in \{loc, q, qs, qa, qc, C^*, hered, alg\}$. Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph.

We define

$$\chi_t((\mathcal{S}, \mathcal{M}, M_n)) = \min\{c \in \mathbb{N} : (\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{t} K_c\},$$

and we define $\chi_t((\mathcal{S}, \mathcal{M}, M_n)) = \infty$ if $(\mathcal{S}, \mathcal{M}, M_n) \stackrel{t}{\not\to} K_c$ for all $c \in \mathbb{N}$.

We now show that our classical chromatic numbers for irreflexive non-commutative graphs coincide with Kim-Mehta's strong chromatic numbers [37].

**Theorem 4.1.9.** *Let $(S, M_n, M_n)$ be an irreflexive non-commutative graph (i.e. $\mathrm{Tr}(X) = 0$ for all $X \in S$). Then $c = \chi_{loc}((S, M_n, M_n))$ if and only if there exists an orthonormal basis $\{u_1, u_2, \dots u_n\}$ of $\mathbb{C}^n$ that can be partitioned into $c$ subsets $S_1, S_2, \dots, S_c$ such that $v_i v_j^* \in S^\perp$ for all $v_i, v_j \in S_l$ and $1 \le l \le c$.*

*Proof.* First, suppose $\{P_k\}_{k=1}^c \subseteq M_n$ is a $c$-coloring of $(S, M_n, M_n)$ in our sense. Since $P_k$ is an orthogonal projection, there exists an orthonormal basis $V_k$ for range$(P_k)$ such that $P_k = \sum_{u \in V_k} uu^*$. Then for each $X \in S$, we have

$$0 = P_k X P_k = \sum_{u,v \in V_k} uu^* X vv^* = \sum_{u,v \in V_k} (u^* X v) uv^*.$$

Since $\{uv^*\}_{u,v \in V_k}$ is a linearly independent subset, it follows that $u^* X v = 0$. That is $uv^* \perp X$, for all $u, v \in V_k$ and $X \in S$. Hence, $\{V_k\}_{k=1}^c$ is a strong $c$-coloring of $S$. Here we use the fact that $\sum_{k=1}^c range(P_k) = M_n$ as $\sum_{k=1}^c P_k = I$.

Conversely, let $S_k = \{v_1, v_2, \dots, v_t\} \subseteq \mathbb{C}^n$ be a subset of $S$ such that $v_i v_j^* \perp S$ for all $v_i, v_j \in S_k$. Define $P_k = \sum_{i=1}^t v_i v_i^* \in M_n \otimes \mathbb{C}$. Then

$$P_k S P_k = \sum_{i,j=1}^t v_i v_i^* S v_j v_j^* = \sum_{i,j=1}^t (v_i^* S v_j) v_i v_j^* = 0.$$

Hence, $\{P_k\}_{k=1}^c$ gives a $c$-coloring of $S$ in the $loc$ model. (Note that $\mathcal{M}' = I_n$.) $\square$

**Example 4.1.10.** Let

$$\mathcal{S} = \text{span}\,\{I,\, E_{ij} : i \neq j\} \subseteq M_n,$$

which is a quantum graph on $M_n$. It is known [37] that $\chi((\mathcal{S}, M_n, M_n)) = n$. Here, we will show that $\chi_{qc}((\mathcal{S}, M_n)) = n$ as well, which shows that $\chi_t((\mathcal{S}, M_n)) = n$ for any $t \in \{loc, q, qa, qc\}$.

Evidently the basis $\mathcal{F} = \{I, E_{ij} : i \neq j\}$ is a quantum edge basis for $(\mathcal{S}, M_n, M_n)$. Now, suppose that $P_1, ..., P_c$ are projections in $M_n(\mathcal{B}(\mathcal{H}))$ with $P_a(E_{k\ell} \otimes I)P_a = 0$ for all $1 \leq a \leq c$ and $1 \leq k \neq \ell \leq n$. A winning strategy in the $qc$-model for coloring $(\mathcal{S}, M_n)$ with $c$ colors would mean that there is a trace $\tau$ on the algebra generated by the $P_{a,ij}$'s and that

$$p(a, a|e_i \otimes e_j) = 0 \text{ if } i \neq j.$$

This implies that

$$\tau(P_{a,ii}P_{a,jj}^*) = 0 \text{ for all } i \neq j.$$

By taking a quotient by the kernel of the GNS representation of the trace, we may assume that $\tau$ is faithful. Then by faithfulness of $\tau$ and positivity of $P_{a,jj}$, we have $P_{a,ii}P_{a,jj} = 0$ for all $i \neq j$. Now, choose $i \neq j$. Notice that, for each $i$, the set $\{P_{a,ii}\}$ is a POVM on $\mathcal{H}$. Moreover, for any $a, b \in \{1, ..., c\}$,

$$p(a, b|e_i \otimes e_j) = \tau(P_{a,ii}P_{b,jj}^*) = \tau(P_{a,ii}P_{b,jj}).$$

Thus, the only information relevant to Alice and Bob winning the game is the correlation $(\tau(P_{a,ii}P_{b,jj}))_{a,b,i,j} \in C_{qc}^s(n, c)$. By faithfulness, this forces each $P_{a,ii}$ to be a projection. By the synchronous condition, the previous equation and faithfulness of the trace, we obtain

$$P_{a,ii}P_{a,jj} = 0 = P_{a,ii}P_{b,ii}$$

whenever $a \neq b$ and $i \neq j$. Therefore, $(\tau(P_{a,ii}P_{b,jj}))_{a,b,i,j} \in C_{qc}^{bs}(n, c)$; that is, the correlation is *bisynchronous* in the sense of [50]. By [50], we must have $c \geq n$. Therefore, $\chi_{qc}(\mathcal{S}, M_n, M_n) \geq n$. It follows that $\chi_t(\mathcal{S}, M_n, M_n) = n$ for every $t \in \{loc, q, qa, qc\}$.

Next, we would like to relate winning strategies for the coloring game to the entanglement-assisted coloring notion in the sense of D. Stahlke [53]. The following theorem will show that, in the $loc$ model, condition (4) of Theorem 4.1.7 is an analogue of Stahlke's notion of graph coloring [53] for a non-commutative graph (i.e. a quantum graph with $\mathcal{M} = M_n$), with an added assumption on the commutant of $\mathcal{M}$. A similar analogue holds in the $q$-model, with natural generalizations to the $qa$ and $qc$ models.

We observe that, if we start with a projection-valued measure $\{P_a\}_{a=1}^c$ whose block entries are in a tracial von Neumann algebra $(\mathcal{N}, \tau)$, where $\tau$ is faithful and normal, then either all four conditions of Theorem 4.1.7 are satisfied by the PVM, or none of the four conditions are satisfied. Notice that we needed to start with a PVM and a faithful trace for this to happen.

Using theorem 4.1.7 and the characterizations of synchronous correlations, we obtain the following theorem:

**Theorem 4.1.11.** *Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph and let $D_c$ be the set of all diagonal matrices in $M_c$.*

1. *$(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{loc} K_c$ if and only if there is a CPTP map $\Phi : \mathcal{M} \to D_c$ of the form $\Phi(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*$ such that*

$$F_i(\mathcal{S} \cap (\mathcal{M}')^\perp)F_j^* \subseteq (D_c)^\perp \text{ for all } i, j,$$

   *and*

$$F_i\mathcal{M}'F_j^* \subseteq D_c \text{ for all } i, j.$$

2. *$(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{q} K_c$ if and only if there exists $d \in \mathbb{N}$ and a CPTP map $\Phi : \mathcal{M} \otimes M_d \to D_c$ of the form $\Phi(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*$ such that*

$$F_i((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes I_d)F_j^* \subseteq (D_c)^\perp \text{ for all } i, j,$$

   *and*

$$F_i(\mathcal{M}' \otimes I_d)F_j^* \subseteq D_c \text{ for all } i, j.$$

*3.* $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{qa} K_c$ *if and only if there is a CPTP map* $\Phi : \mathcal{M} \otimes (\mathcal{R}^{\mathcal{U}})_* \to D_c$ *of the form* $\Phi(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*$ *such that*

$$F_i((\mathcal{S} \cap (\mathcal{M}')^{\perp}) \otimes 1_{\mathcal{R}^{\mathcal{U}}})F_j^* \subseteq (D_c)^{\perp} \text{ for all } i, j,$$

*and*

$$F_i(\mathcal{M}' \otimes 1_{\mathcal{R}^{\mathcal{U}}})F_j^* \subseteq D_c \text{ for all } i, j.$$

*4.* $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{qc} K_c$ *if and only if there is a von Neumann algebra* $\mathcal{N}$*, a faithful normal trace* $\tau$ *on* $\mathcal{N}$*, and a CPTP map* $\Phi : \mathcal{M} \otimes \mathcal{N}_* \to D_c$ *of the form* $\Phi(\cdot) = \sum_{i=1}^m F_i(\cdot)F_i^*$ *such that*

$$F_i((\mathcal{S} \cap (\mathcal{M}')^{\perp}) \otimes 1_{\mathcal{N}})F_j^* \subseteq (D_c)^{\perp} \text{ for all } i, j,$$

*and*

$$F_i(\mathcal{M}' \otimes 1_{\mathcal{N}})F_j^* \subseteq D_c \text{ for all } i, j.$$

*Proof.* We consider the case $t = loc$ first. If $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{loc} K_c$, then there is a winning $loc$-strategy for the coloring game on $(\mathcal{S}, \mathcal{M}, M_n)$ with $c$ colors. Since $\mathcal{Q}_{loc}^s(n, c)$ is convex and non-empty, one may obtain an extreme point in $\mathcal{Q}_{loc}^s(n, c)$ that wins the game with probability $1$. Applying Corollary 3.3.8, there is a realization of this correlation using a PVM $\{P_a\}_{a=1}^c$ in $\mathcal{M} = \mathcal{M} \otimes \mathbb{C}$. Then the result follows by condition (4) of Theorem 4.1.7 with $\mathcal{N} = \mathbb{C}$. The converse of (1) holds by condition (3) of Theorem 4.1.7.

The argument is similar for $t = q$. Indeed, if there is a winning strategy for the homomorphism game in the $q$-model, then an application of Corollary 3.3.9 shows that there is a winning $q$-strategy using an extreme point in $\mathcal{Q}_q^s(n, c)$, which can be realized using projections whose entries are in $M_d$, for some $d$. Then condition (4) of Theorem 4.1.7 with $\mathcal{N} = M_d$ yields the desired CPTP map. The converse, as before, holds by condition (3) of Theorem 4.1.7.

We note that (3) holds because of Theorem 3.3.11. Condition (4) is achieved using the following well-known trick: if $\mathcal{A}$ is a unital, separable $C^*$-algebra with tracial state $\tau$, and if $\pi_\tau : \mathcal{A} \to$

$\mathcal{B}(\mathcal{H}_\tau)$ is the GNS representation of $\tau$ with cyclic vector $\xi$, then $\pi_\tau(\mathcal{A})''$ is a finite von Neumann algebra and $\langle(\cdot)\xi,\xi\rangle$ is a faithful normal trace on $\pi_\tau(\mathcal{A})''$. We leave the details to the reader. $\square$

**Remark 4.1.12.** Our notion of quantum graph coloring is a special case of Stahlke's coloring [53] in the sense that we use unital *-homomorphisms, while Stahlke's notion uses more general UCP maps. This leads to the additional pushforward condition on the commutant in our case.

For synchronous games with classical inputs and classical outputs, J.W. Helton, K.P. Meyer, V.I. Paulsen and M. Satriano constructed a universal $*$-algebra for the game, generated by self-adjoint idempotents whose products were $0$ when the related pair of outputs was not allowed [30]. One can define a game $*$-algebra in our context as follows.

**Definition 4.1.13.** Let $(\mathcal{S},\mathcal{M},M_n)$ be a quantum graph and let $c\in\mathbb{N}$. The **game $*$-algebra** for the coloring game for $(\mathcal{S},\mathcal{M},M_n)$ on $c$ colors, denoted $\mathcal{A}(\text{Hom}((\mathcal{S},\mathcal{M},M_n),K_c))$, is the universal $*$-algebra generated by entries $\{p_{a,ij}:1\le a\le c,\ 1\le i,j\le n\}$ subject to the relations

- $p_a=(p_{a,ij})_{i,j}$ satisfies $p_a^2=p_a=p_a^*$ and $\sum_{a=1}^c p_a=I_n$, where $I_n$ is the $n\times n$ identity matrix;
- $p_a((\mathcal{S}\cap(\mathcal{M}')^\perp)\otimes 1)p_a=0$ for each $a$; and
- $p_a(\mathcal{M}'\otimes 1)p_b=0$ for each $a\neq b$.

We say that the algebra **exists** if $1\neq 0$ in the algebra.

As one might expect, we obtain the following characterizations of the various flavors of winning strategies for the coloring game in terms of $*$-homomorphisms of the game algebra. We follow notation 5 to state this result.

**Theorem 4.1.14.** *Let $(\mathcal{S},\mathcal{M},M_n)$ be a quantum graph and let $c\in\mathbb{N}$.*

1. $(\mathcal{S},\mathcal{M},M_n)\xrightarrow{loc}K_c \iff$ *there is a unital $*$-homomorphism $\mathcal{A}(\mathit{Hom}((\mathcal{S},\mathcal{M},M_n),K_c))\to\mathbb{C}$.*

2. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{q} K_c$ *if and only if there is a unital* $*$*-homomorphism* $\mathcal{A}(\mathit{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \to M_d$ *for some* $d \in \mathbb{N}$.

3. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{qa} K_c$ *if and only if there is a unital* $*$*-homomorphism* $\mathcal{A}(\mathit{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \to \mathcal{R}^{\mathcal{U}}$.

4. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{qc} K_c$ *if and only if there is a unital* $*$*-homomorphism* $\mathcal{A}(\mathit{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \to \mathcal{C}$*, where* $\mathcal{C}$ *is a tracial* $C^*$*-algebra.*

One can also define algebraic coloring, $C^*$-coloring and hereditary coloring of quantum graphs. Recall that a unital $*$-algebra $\mathcal{A}$ is said to be **hereditary** if, whenever $x_1, ..., x_n \in \mathcal{A}$ are such that $x_1^* x_1 + \cdots + x_n^* x_n = 0$, then $x_1 = x_2 = \cdots = x_n = 0$. If one defines $\mathcal{A}_+$ as the cone generated by all elements of the form $x^* x$ for $x \in \mathcal{A}$, then $\mathcal{A}$ being hereditary is equivalent to having $\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}$. Every unital $C^*$-algebra is hereditary as a unital $*$-algebra.

**Definition 4.1.15.** Let $(\mathcal{S}, \mathcal{M}, M_n)$ be a quantum graph and let $c \in \mathbb{N}$. We write

1. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{alg} K_c$ provided $\mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \neq 0$.
   (Note that the game algebra is non-trivial does not mean it has a non-trivial representation.)

2. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{C^*} K_c$ provided that there is a unital $*$-homomorphism

$$\pi : \mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \to \mathcal{B}(\mathcal{H}),$$

   for some Hilbert space $\mathcal{H}$. (Equivalently, by the Gelfand-Naimark theorem, one may simply require that the game algebra has a representation into some unital $C^*$-algebra.)

3. $(\mathcal{S}, \mathcal{M}, M_n) \xrightarrow{hered} K_c$ provided that there is a unital $*$-homomorphism from $\mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c))$ into a (non-zero) hereditary unital $*$-algebra.

With what we have established so far, we have the following sequence of implications for different types of colorings on a quantum graph $\mathcal{G} := (\mathcal{S}, \mathcal{M}, M_n)$:

$$\mathcal{G} \xrightarrow{loc} K_c \implies \mathcal{G} \xrightarrow{q} K_c \implies \mathcal{G} \xrightarrow{qa} K_c \implies \mathcal{G} \xrightarrow{qc} K_c \tag{4.1.0.5}$$
$$\implies \mathcal{G} \xrightarrow{C*} K_c \implies \mathcal{G} \xrightarrow{hered} K_c \implies \mathcal{G} \xrightarrow{alg} K_c \tag{4.1.0.6}$$

Due to the inclusions of the models, we always have

$$\chi_{loc}((\mathcal{S}, \mathcal{M}, M_n)) \geq \chi_q((\mathcal{S}, \mathcal{M}, M_n)) \geq \chi_{qa}((\mathcal{S}, \mathcal{M}, M_n)) \geq \chi_{qc}((\mathcal{S}, \mathcal{M}, M_n))$$
$$\geq \chi_{C^*}((\mathcal{S}, \mathcal{M}, M_n)) \geq \chi_{hered}((\mathcal{S}, \mathcal{M}, M_n)) \geq \chi_{alg}((\mathcal{S}, \mathcal{M}, M_n)).$$

It is known that some of the implications in (4.1.0.5) cannot be reversed:

- There are many examples of classical graphs $G$ with $\chi_q(G) < \chi_{loc}(G)$ [7, 41].
- Theorem 4.2.9 will show that $\chi_q((M_n, \mathcal{M}, M_n)) = \dim(\mathcal{M})$ but $\chi_{loc}((M_n, \mathcal{M}, M_n)) = \infty$, whenever $\mathcal{M}$ is non-abelian.
- The separation between $\chi_q$ and $\chi_{qa}$ follows from an earlier work for Zhengfeng Ji.
- $\chi_{alg}(K_5) = 4$ but $\chi_{hered}(K_5) \neq 4$ [30]. This result will also be generalized to quantum graphs later.

We conclude this section by showing that our notions of coloring reduce to the analogous types of coloring for classical graphs in the case when $(\mathcal{S}, \mathcal{M}, M_n)$ is a classical graph. Recall example 2.1.2 that, for a classical graph $G$ on $n$ vertices, the graph operator system $\mathcal{S}_G$ (or classical quantum graph) is defined as

$$\mathcal{S}_G = \text{span}\,(\{E_{ii} : 1 \leq i \leq n\} \cup \{E_{ij} : i \sim j \text{ in } G\}).$$

Note that $\mathcal{S}_G$ is naturally a quantum graph when viewed as a bimodule over the diagonal algebra $D_n = D_n' \subseteq M_n$.

**Corollary 4.1.16.** *Let $G$ be a classical graphs on $n$ vertices and $c \in \mathbb{N}$. Suppose that $t \in \{loc, q, qa, qc, C^*, hered, alg\}$. Then $G \xrightarrow{t} K_c$ if and only if $(\mathcal{S}_G, D_n, M_n) \xrightarrow{t} K_c$.*

*Proof.* We will show that the algebra $\mathcal{A}(\text{Hom}(G, K_c))$ from [30] is isomorphic to $\mathcal{A}(\text{Hom}((\mathcal{S}_G, D_n, M_n), K_c))$. The former algebra is the universal unital $*$-algebra generated by self-adjoint idempotents $e_{x,a}$ such that $\sum_{x=1}^n e_{x,a} = 1$, $e_{x,a}e_{x,b} = 0$ if $a \neq b$, and $e_{x,a}e_{y,b} = 0$ if $x \sim y$ in $G$ but $a = b$. Since $D_n = D_n'$, the latter algebra is the universal unital $*$-algebra generated by elements $p_{a,ij}$ such that $p_a = (p_{a,ij}) \in M_n(\mathcal{A})$ is a self-adjoint idempotent with $\sum_{a=1}^c p_a = I_n$, $p_a((\mathcal{S}_G \cap (D_n)^\perp) \otimes 1)p_b = 0$ whenever $a = b$, and $p_a(D_n \otimes 1)p_b = 0$ whenever $a \neq b$. Since $E_{ii} \in D_n$, using the fact that $p_a(D_n \otimes 1)p_b = 0$ for $a \neq b$, we obtain

$$p_a(E_{ii} \otimes 1)p_a = \sum_{b=1}^c p_a(E_{ii} \otimes 1)p_b = p_a(E_{ii} \otimes 1).$$

Similarly, $p_a(E_{ii} \otimes 1)p_a = (E_{ii} \otimes 1)p_a$, so that $E_{ii} \otimes 1$ commutes with $p_a$. It follows that $p_{a,ij} = 0$ whenever $i \neq j$. Since $p_a^2 = p_a = p_a^*$, we see that $p_{a,ii}^2 = p_{a,ii} = p_{a,ii}^*$. For $1 \leq a \leq c$ and $1 \leq x \leq n$, we define $q_{x,a} = p_{a,xx}$. Then $q_{x,a}$ is a self-adjoint idempotent and $\sum_{a=1}^c q_{x,a} = 1$ for all $1 \leq x \leq n$. Note that, if $x \sim y$ in $G$ but $a = b$, then

$$q_{x,a}q_{y,b} = p_{a,xx}p_{b,yy} = p_a(E_{xy} \otimes 1)p_b = 0,$$

since $E_{xy} \in \mathcal{S}_G \cap (D_n)^\perp$ and $a = b$. Similarly, if $a \neq b$, then $q_{x,a}q_{x,b} = p_a(E_{xx} \otimes 1)p_b = 0$ since $E_{xx} \in D_n$. By the universal property of $\mathcal{A}(\text{Hom}(G, K_c))$, there is a unital $*$-homomorphism $\pi : \mathcal{A}(\text{Hom}(G, K_c)) \to \mathcal{A}(\text{Hom}((\mathcal{S}_G, D_n, M_n), K_c))$ such that $\pi(e_{x,a}) = q_{x,a}$ for all $x, a$.

Conversely, in $\mathcal{A}(\text{Hom}(G, K_c))$, one can construct the $n \times n$ matrices $f_a = (f_{a,ij})$ with $f_{a,ij} = 0$ for $i \neq j$ and $f_{a,ii} = e_{a,i}$. Then evidently $f_a^2 = f_a = f_a^*$ and $\sum_{a=1}^c f_a = I_n$. Since $e_{x,a}e_{x,b} = 0$ for $a \neq b$, we see that $f_a(E_{xx} \otimes 1)f_b = 0$ if $a \neq b$. Since $D_n = \text{span}\,\{E_{xx} : 1 \leq x \leq n\}$, it follows that $f_a(D_n \otimes 1)f_b = 0$ for $a \neq b$. Similarly, it is not hard to see that $f_a(E_{xy} \otimes 1)f_b = 0$ whenever $x \sim y$ in $G$ but $a = b$. By the universal property, there is a unital $*$-homomorphism $\rho : \mathcal{A}(\text{Hom}((\mathcal{S}_G, D_n, M_n), K_c)) \to \mathcal{A}(\text{Hom}(G, K_c))$ such that $\rho(p_{a,ij}) = f_{a,ij}$. Evidently $\rho$ and

$\pi$ are mutual inverses on the generators, so we conclude that the algebras are $*$-isomorphic. The result follows. $\square$

**Remark 4.1.17.** As a consequence of Corollary 4.1.16, whenever $G$ is a classical graph, we have $\chi_t(G) = \chi_t((\mathcal{S}_G, D_n, M_n))$. This result is well known [48]. As $\chi_{loc}(G)$ is the (classical) chromatic number of a classical graph $G$, we sometimes use the notation $\chi((\mathcal{S}, \mathcal{M}, M_n))$ for $\chi_{loc}((\mathcal{S}, \mathcal{M}, M_n))$.

In the next section, we will show explicit computations for some of the quantum chromatic numbers and also prove that every quantum graph has a finite quantum chromatic number, but not necessarily classical chromatic number.

## 4.2 Quantum complete graphs and algebraic colorings

In this section, we consider *quantum complete graphs*; that is, graphs of the form $(M_n, \mathcal{M}, M_n)$, where $\mathcal{M} \subseteq M_n$ is a non-degenerate von Neumann algebra. We show that $\chi_t((M_n, \mathcal{M}, M_n)) = \dim(\mathcal{M})$ for all $t \in \{q, qa, qc, C^*, hered\}$. In contrast, we will see that $\chi_{loc}((M_n, \mathcal{M}, M_n))$ is finite if and only if $\mathcal{M}$ is abelian; in the case when $\mathcal{M}$ is abelian, we recover known results on colorings for the (classical) complete graph on $\dim(\mathcal{M})$ vertices. The algebraic model for colorings is known to be very wild. At the end of this section, we will extend a surprising result of [30]: in the algebraic model: that *any* quantum graph can be 4-colored.

We start with a simple proposition on unitary equivalence that we will use throughout this section.

**Proposition 4.2.1.** *Let $\mathcal{M} \subseteq M_n$ be a non-degenerate von Neumann algebra. Then there is a unitary $U \in M_n$ such that $U^* \mathcal{M} U = \bigoplus_{r=1}^m \mathbb{C} I_{n_r} \otimes M_{k_r}$. Moreover, for any $t \in \{loc, q, qa, qs, qc, C^*, hered, alg\}$, we have*

$$\chi_t((M_n, \mathcal{M}, M_n)) = \chi_t\left(M_n, \bigoplus_{r=1}^m \mathbb{C} I_{n_r} \otimes M_{k_r}, M_n\right).$$

*Proof.* The existence of the unitary $U$ is a consequence of the theory of finite-dimensional $C^*$-algebras. It is not hard to see that $(U^* \mathcal{M} U)' = U^* \mathcal{M}' U$. Now, an element $X \in M_n$ belongs to $\mathcal{M}'$ if and only if $\mathrm{Tr}(XY) = 0$ for all $Y \in \mathcal{M}'$. This statement is equivalent to having $\mathrm{Tr}((U^*XU)(U^*YU)) = 0$ for all $Y \in \mathcal{M}'$, since $U$ is unitary. It follows that $U^*(\mathcal{M}')^\perp U = (U^*\mathcal{M}'U)^\perp$.

Now, suppose that $\{P_a\}_{a=1}^c \subseteq M_n \otimes \mathcal{A}$ is a collection of self-adjoint idempotents summing to $I_n \otimes 1_\mathcal{A}$, where $\mathcal{A}$ is a unital $*$-algebra. Then it is evident that $P_a((\mathcal{M}')^\perp \otimes 1_\mathcal{A})P_a = 0$ if and only if $\widetilde{P}_a((U^*\mathcal{M}'U)^\perp \otimes 1_\mathcal{A})\widetilde{P}_a = 0$, where $\widetilde{P}_a = (U^* \otimes 1_\mathcal{A})P_a(U \otimes 1_\mathcal{A})$. Similarly, if $a \neq b$, then $P_a(\mathcal{M}' \otimes 1_\mathcal{A})P_b = 0$ if and only if $\widetilde{P}_a((U^*\mathcal{M}'U) \otimes 1_\mathcal{A})\widetilde{P}_b = 0$. Thus, there is a bijective correspondence between algebraic $c$-colorings of $(M_n, \mathcal{M}, M_n)$ and algebraic $c$-colorings of $(M_n, \bigoplus_{r=1}^m \mathbb{C}I_{n_r} \otimes M_{k_r}, M_n)$. This yields the equality of chromatic numbers for $t = alg$; the other cases are similar. □

The different chromatic numbers satisfy a certain monotonicity as well.

**Proposition 4.2.2.** *If $(\mathcal{S}, \mathcal{M}, M_n)$ and $(\mathcal{T}, \mathcal{M}, M_n)$ are quantum graphs with $\mathcal{S} \subseteq \mathcal{T}$, then*

$$\chi_t((\mathcal{S}, \mathcal{M}, M_n)) \leq \chi_t((\mathcal{T}, \mathcal{M}, M_n)).$$

*Proof.* We deal with the $t = alg$ case; all the other cases are similar. If $(\mathcal{T}, \mathcal{M}, M_n)$ has no algebraic coloring, then $\chi_{alg}((\mathcal{T}, \mathcal{M}, M_n)) = \infty$, so the desired result holds. Otherwise, let $\mathcal{A}$ be a (non-zero) unital $*$-algebra. Suppose that $\{P_a\}_{a=1}^c$ are self-adjoint idempotents in $M_n(\mathcal{A})$ such that $\sum_{a=1}^c P_a = I_n$, $P_a((\mathcal{T} \cap (\mathcal{M}')^\perp) \otimes 1_\mathcal{A})P_a = 0$ for all $a$, and $P_a(\mathcal{M}' \otimes 1_\mathcal{A})P_b = 0$ for all $a \neq b$. Then evidently $P_a((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1_\mathcal{A})P_a = 0$ as well, so the self-adjoint idempotents form an algebraic $c$-coloring of $(\mathcal{S}, \mathcal{M}, M_n)$. This shows that $\chi_{alg}((\mathcal{S}, \mathcal{M}, M_n)) \leq \chi_{alg}((\mathcal{T}, \mathcal{M}, M_n))$. The proof for the other models is the same. □

By Proposition 4.2.2, to establish that every quantum graph has a finite quantum coloring, it suffices to consider quantum complete graphs. First, we look at $(M_n, M_n, M_n)$, the quantum complete graph. While we will have an alternative quantum coloring of this quantum graph from Theorem 4.2.4, the protocol given in Theorem 4.2.3 is minimal for $(M_n, M_n, M_n)$ in terms of the dimension of the ancillary algebra. Moreover, it gives a foretaste of the protocol that we use for the quantum complete graph $(M_n, \mathcal{M}, M_n)$ when $\mathcal{M}$ is not isomorphic to a matrix algebra. The broad idea is to use unitary error basis and orthogonal projections associated with it.

**Theorem 4.2.3.** *Let $d, k \in \mathbb{N}$, and let $n = dk$. Let $\mathcal{M} = \mathbb{C}I_d \otimes M_k$. Then $\chi_q((M_n, \mathcal{M}, M_n)) \leq k^2$.*

*Proof.* We construct our projections from the canonical orthonormal basis for $\mathbb{C}^k \otimes \mathbb{C}^k$ that consists of maximally entangled vectors; that is, the basis of the form

$$\varphi_{a,b} = \frac{1}{\sqrt{k}} \sum_{p=0}^{k-1} \exp\left(\frac{2\pi i a(p+b)}{k}\right) e_{b+p} \otimes e_p,$$

where addition in the indices of the vectors is done modulo $k$. (See [24] for example.) We define projections in $\mathcal{M} \otimes \mathcal{M}$, for all $1 \leq a, b \leq n$, by

$$P_{(a,b)} = \frac{1}{k} \sum_{p,q=0}^{k-1} \exp\left(\frac{2\pi i a(p-q)}{k}\right) I_d \otimes E_{b+p,b+q} \otimes I_d \otimes E_{pq}.$$

Since the set $\{\varphi_{(a,b)}\}_{a,b=1}^n$ is orthonormal, it is not hard to see that $\{P_{(a,b)}\}_{a,b=1}^n$ is a family of mutually orthogonal projections. Moreover, $\sum_{a,b=1}^n P_{(a,b)} = I_d \otimes I_k \otimes I_d \otimes I_k$. With respect to $M_n$, $(\mathcal{M}')^\perp$ is spanned by elements of the form $E_{xy} \otimes E_{vw}$ and $E_{xy} \otimes (E_{vv} - E_{ww})$ for $1 \leq x, y \leq d$ and $1 \leq v, w \leq k$ with $v \neq w$. For $Y = E_{xy} \otimes E_{vw} \otimes (I_d \otimes I_k)$, one computes $P_{(a,b)} Y P_{(a,b)}$ and obtains

$$\frac{1}{k^2} \sum_{p,q,p',q'=0}^{k-1} \exp\left(\frac{2\pi i a(p+p'-q-q')}{k}\right) E_{xy} \otimes E_{b+p,b+q} E_{vw} E_{b+p',b+q'} \otimes I_d \otimes E_{pq} E_{p'q'}.$$

For a term in the above sum to be non-zero, one requires that $b + q = v$, $w = b + p'$, and $q = p'$. Equivalently, a term in the sum is non-zero only when $q = p'$ and $b + q = v = w$. Hence, if $v \neq w$, then the above sum is $0$. In the case when $v = w$, one obtains

$$\frac{1}{k^2} \sum_{p,q'=0}^{k-1} \exp\left(\frac{2\pi i a(p-q')}{k}\right) E_{xy} \otimes E_{b+p,b+q'} \otimes I_d \otimes E_{pq'}.$$

The above expression does not depend on $v$, so we conclude that, for all $1 \leq v, w \leq k$,

$$P_{(a,b)}(E_{xy} \otimes E_{vv} \otimes I_d \otimes I_k)P_{(a,b)} = P_{(a,b)}(E_{xy} \otimes E_{ww} \otimes I_d \otimes I_k)P_{(a,b)}.$$

This shows that $P_{(a,b)}(X \otimes I_d \otimes I_k)P_{(a,b)} = 0$ whenever $X = E_{xy} \otimes E_{vw}$ or $X = E_{xy} \otimes (E_{vv} - E_{ww})$ for $v \neq w$. As such elements span $(\mathcal{M}')^\perp$, we see that

$$P_{(a,b)}(X \otimes I_d \otimes I_k)P_{(a,b)} = 0 \, \forall X \in (\mathcal{M}')^\perp.$$

Finally, we show that $P_{(a,b)}(\mathcal{M}' \otimes I_d \otimes I_k)P_{(a',b')} = 0$ whenever $(a, b) \neq (a', b')$. If $Y \in \mathcal{M}'$, then $Y \otimes (I_d \otimes I_k)$ commutes with each $P_{(a,b)}$, since $P_{(a,b)} \in \mathcal{M} \otimes (I_d \otimes M_k)$. Therefore, if $(a, b) \neq (a', b')$, we have

$$P_{(a,b)}(Y \otimes (I_d \otimes I_k))P_{(a',b')} = P_{(a,b)}P_{(a',b')}(Y \otimes (I_d \otimes I_k)) = 0.$$

Putting all of these equations together, we see that there is a representation of the game algebra $\pi : \mathcal{A}(\text{Hom}((M_n, \mathcal{M}, M_n), K_{k^2})) \to \mathbb{C}I_d \otimes M_k \otimes M_k$. Therefore, $\chi_q((M_n, \mathcal{M}, M_n)) \leq k^2$, which yields the claimed result. $\square$

For a general complete quantum graph $(M_n, \mathcal{M}, M_n)$, we require a slightly different approach. The protocol in the previous proof is used in the context of quantum teleportation, and essentially arises from the use of a “shift and multiply" unitary error basis for $M_n$ [24, 59]. To give a $\dim(\mathcal{M})$-coloring for $(M_n, \mathcal{M}, M_n)$ in the $q$-model, we will use what we refer to as a “global shift and local multiply" framework.

**Theorem 4.2.4.** *Let $\mathcal{M}$ be a non-degenerate von Neumann algebra in $M_n$. For the quantum complete graph $(M_n, \mathcal{M}, M_n)$, we have $\chi_q((M_n, \mathcal{M}, M_n)) \leq \dim(\mathcal{M})$.*

*Proof.* Up to unitary equivalence in $M_n$, we may write $\mathcal{M} = \bigoplus_{r=1}^m (\mathbb{C}I_{n_r} \otimes M_{k_r})$, where $n = \sum_{r=1}^m n_r k_r$. We will exhibit a PVM in $\mathcal{M} \otimes M_d$, with $d = \text{lcm}(k_1, ..., k_m)$, satisfying the properties of a quantum coloring for $(M_n, \mathcal{M}, M_n)$. For notational convenience, we index our set of $\dim(\mathcal{M})$ colors by the triples $(s,a,b)$, where $1 \leq s \leq m$ and $0 \leq a,b \leq k_s - 1$. For $1 \leq r \leq m$ and $1 \leq i \leq k_r$, we define $P_{(s,a,b)} = \bigoplus_{r=1}^m I_{n_r} \otimes P_{(a,b)}^{(r,s)}$, where $P_{(a,b)}^{(r,s)} = (P_{(a,b),(i,j)}^{(r,s)})_{i,j=0}^{k_r-1} \in M_{k_r}(M_d)$ is given by

$$P_{(a,b),(i,j)}^{(r,s)} = \frac{\delta_{rs}}{k_r} \omega_{k_r}^{(i-j)a} I_{d_r} \otimes E_{i+b,j+b},$$

where $\omega_{k_r}$ is a primitive $k_r$-th root of unity and $d_r = \frac{d}{k_r}$. (Note that indices are computed modulo $k_r$.) By our choice of the operators $P_{(a,b)}^{(r,s)}$, we see that each $P_{(s,a,b)}$ belongs to $\mathcal{M} \otimes M_d$.

First, we show that $\sum_{s=1}^m \sum_{a,b=0}^{k_s-1} P_{(s,a,b)} = I_n \otimes I_d$. For each $1 \leq r \leq m$ and $0 \leq i,j \leq k_r - 1$,

$$\sum_{a,b=0}^{n_r-1} P_{(a,b),(i,j)}^{(r,r)} = \frac{1}{k_r} \sum_{a,b=0}^{n_r-1} \omega_{k_r}^{(i-j)a} I_{d_r} \otimes E_{i+b,j+b}.$$

If $i \neq j$, then the above sum over $a$ is $0$, for each value of $b$. If $i = j$, then the above sum simply becomes

$$\sum_{b=0}^{n_r-1} I_{d_r} \otimes E_{i+b,i+b} = I_{d_r} \otimes I_{n_r} = I_d.$$

Thus, $\sum_{a,b=0}^{n_r-1} P_{(a,b)}^{(r,r)} = I_{k_r} \otimes I_d$. Since $P_{(a,b)}^{(r,s)} = 0$ if $s \neq r$, it follows that $\sum_{s=1}^m \sum_{a,b=0}^{n_s-1} P_{(a,b)}^{(r,s)} = I_{k_r} \otimes I_d$ for each $1 \leq r \leq m$. As $P_{(s,a,b)} = \bigoplus_{r=1}^m I_{n_r} \otimes P_{(a,b)}^{(r,s)}$, we must have $\sum_{s=1}^m \sum_{a,b=0}^{n_s-1} P_{(s,a,b)} = I_n \otimes I_d$.

Next, we check that each $P_{(s,a,b)}$ is an orthogonal projection. By definition, it is easy to see that $P_{(s,a,b)}^* = P_{(s,a,b)}$ for all $s,a,b$. To compute $P_{(s,a,b)}^2$, we note that

$$P_{(s,a,b)}^2 = \bigoplus_{r=1}^m I_{n_r} \otimes (P_{(a,b)}^{(r,s)})^2,$$

so it suffices to show that each $P_{(a,b)}^{(r,s)}$ is an idempotent in $M_{k_r} \otimes M_d$. If $r \neq s$, then this is immediate.

In the other case, we have

$$\begin{aligned} P^{(r,r)}_{(a,b),(v,j)}P^{(r,r)}_{(a,b),(j,w)} &= \frac{1}{k_r^2}\omega_{k_r}^{(v-w)a} I_{d_r} \otimes E_{v+b,j+b}E_{j+b,w+b} \\ &= \frac{1}{k_r^2}\omega_{k_r}^{(v-w)a} I_{d_r} \otimes E_{v+b,w+b} \\ &= \frac{1}{k_r} P^{(r,r)}_{(a,b),(v,w)}. \end{aligned}$$

Since this happens for all $0 \leq v, w \leq k_r - 1$, it follows that $P^{(r,r)}_{(a,b),(v,w)} = \sum_{j=0}^{k_r-1} P^{(r,r)}_{(a,b),(v,j)}P^{(r,r)}_{(a,b),(j,w)}$. Therefore, $P^{(r,r)}_{(a,b)}$ is idempotent in $M_{k_r} \otimes M_d$, so $P_{(s,a,b)}$ is an orthogonal projection.

Now, we show that $P_{(s,a,b)}((\mathcal{M}')^\perp \otimes I_d)P_{(s,a,b)} = 0$ for all $a$. We note that $\mathcal{M}' = \bigoplus_{r=1}^m M_{n_r} \otimes \mathbb{C}I_{k_r}$. Hence, $(\mathcal{M}')^\perp$ is spanned by the canonical matrix units that do not reside in $\mathcal{M}'$, and elements from each $M_{n_r} \otimes M_{k_r}$ of the form $E_{ij} \otimes E_{vw}$ and $E_{ij} \otimes (E_{vv} - E_{ww})$, where $1 \leq i, j \leq n_r$, $0 \leq v, w \leq k_r - 1$, and $v \neq w$. By a consideration of blocks, if a matrix unit $E_{xy}$ does not belong to $\bigoplus_{r=1}^m M_{n_r} \otimes M_{k_r}$, then in $M_n \otimes M_d$, the element $P_{(s,a,b)}(E_{xy} \otimes I_d)P_{(s,a,b)}$ is a product of two entries from $P_{(s,a,b)}$, at least one of which will be $0$.

Next, we suppose that $0 \leq v, w \leq k_r - 1$ with $v \neq w$ and $1 \leq i, j \leq n_r$, and consider the matrix unit $E_{ij} \otimes E_{vw} \in M_{n_s} \otimes M_{k_s} \subset \bigoplus_{r=1}^m M_{n_r} \otimes M_{k_r}$. Since $P_{(s,a,b)} = \bigoplus_{r=1}^m I_{n_r} \otimes P^{(r,s)}_{(a,b)}$,

$$P_{(s,a,b)}(E_{ij} \otimes E_{vw} \otimes I_d)P_{(s,a,b)} = E_{ij} \otimes \sum_{k,\ell=0}^{k_r-1} P^{(s,s)}_{(a,b),(k,v)}P^{(s,s)}_{(a,b),(w,\ell)} = 0,$$

since $P^{(s,s)}_{(a,b),(k,v)}P^{(s,s)}_{(a,b),(w,\ell)} = 0$ for all $v \neq w$ in $\{0, ..., n_s - 1\}$. For the last case, we look at the element $E_{ij} \otimes (E_{vv} - E_{ww})$ in $M_{n_s} \otimes M_{k_s} \subset (\mathcal{M}')^\perp$, where $v \neq w$. Multiplying on the left and right by $P_{(s,a,b)}$ yields

$$P_{(s,a,b)}(E_{ij} \otimes (E_{vv} - E_{ww}) \otimes I_d)P_{(s,a,b)} = E_{ij} \otimes \sum_{k,\ell=0}^{k_r-1} (P^{(s)}_{(s,a,b),(k,v)}P^{(s)}_{(s,a,b),(v,\ell)} - P^{(s)}_{(s,a,b),(k,w)}P^{(s)}_{(s,a,b),(w,\ell)}) = 0,$$

since $P^{(s)}_{(s,a,b),(k,v)}P^{(s)}_{(s,a,b),(v,\ell)} = \frac{1}{k_r}P^{(s)}_{(s,a,b),(k,\ell)}$ for any $1 \leq s \leq m$ and $0 \leq k, v, \ell \leq k_r - 1$. Putting all of these facts together, we conclude that $\{P_{(s,a,b)} : 1 \leq s \leq m,\ 0 \leq a, b \leq n_s - 1\} \subset \mathcal{M} \otimes M_d$

is a quantum $\dim(\mathcal{M})$-coloring of $(M_n, \mathcal{M}, M_n)$, as desired. $\square$

**Remark 4.2.5.** We suspect that the ancillary algebra in the previous proof is the minimal choice, but are unable to prove this. In the case when $\mathcal{M} = M_n$, this is immediate, since having a PVM with $n^2$ outputs in $M_n \otimes M_f$, and with each projection non-zero, requires $f \geq n$.

Next, we will show that $\chi_{hered}((M_n, \mathcal{M}, M_n)) \geq \dim(\mathcal{M})$, which will show that, for every $t \in \{q, qa, qc, C^*, hered\}$, we have $\chi_t((M_n, \mathcal{M}, M_n)) = \dim(\mathcal{M})$. Moreover, we will show that $\dim(\mathcal{M})$-colorings of $(M_n, \mathcal{M}, M_n)$ in the hereditary model must arise from *trace-preserving* $*$-homomorphisms $\Psi : D_{\dim(\mathcal{M})} \to \mathcal{M} \otimes \mathcal{A}$. More precisely, we equip $D_{\dim(\mathcal{M})}$ with its canonical uniform trace $\psi_{D_{\dim(\mathcal{M})}}$ satisfying $\psi_{D_{\dim(\mathcal{M})}}(e_a) = \frac{1}{\dim(\mathcal{M})}$ for all $1 \leq a \leq \dim(\mathcal{M})$. We also equip the von Neumann algebra $\mathcal{M} \simeq \bigoplus_{r=1}^m \mathbb{C} I_{n_r} \otimes M_{k_r}$ with its canonical "Plancherel" trace given by

$$\psi_{\mathcal{M}} = \bigoplus_{r=1}^{m} \frac{k_r}{n_r \dim(\mathcal{M})} \operatorname{Tr}_{n_r k_r}(\cdot).$$

Then we will show that the $*$-homomorphism $\Psi$ satisfies the following trace covariance condition:

$$(\psi_{\mathcal{M}} \otimes \text{id})\Psi(x) = \psi_{D_{\dim(\mathcal{M})}}(x) 1_{\mathcal{A}} \qquad (x \in D_{\dim(\mathcal{M})}).$$

We thus establish that the hereditary coloring number for any complete quantum graph $(M_n, \mathcal{M}, M_n)$ is $\dim(\mathcal{M})$, and moreover, the above trace-preserving condition shows that any minimal hereditary coloring induces a quantum version of isomorphism between $(M_n, \mathcal{M}, M_n)$ and the complete graph $K_{\dim(\mathcal{M})}$ on $\dim(\mathcal{M})$ vertices. Here, the notion of a "quantum isomorphism" means a quantum isomorphism between quantum graphs in the sense of [4], when using an ancillary hereditary unital $*$-algebra $\mathcal{A}$. This result can be interpreted as a quantum analogue of the (classically obvious) fact that any minimal coloring of a complete graph $K_c$ is automatically a graph *isomorphism* $K_c \to K_c$.

We consider the case when $\mathcal{M} \simeq \mathbb{C} I_d \otimes M_k$ first.

**Lemma 4.2.6.** *Let $d, k \in \mathbb{N}$ and let $n = dk$. Consider the quantum graph $(M_n, \mathcal{M}, M_n)$ with*

$\mathcal{M} = \mathbb{C}I_d \otimes M_k$. *Let $\mathcal{A}$ be a unital $*$-algebra, and let $\{P_1, ..., P_c\} \in \mathcal{M} \otimes \mathcal{A}$ be a family of mutually orthogonal projections such that $\sum_{a=1}^c P_a = I_{dk} \otimes 1_{\mathcal{A}}$ and*

$$P_a(X \otimes 1_{\mathcal{A}})P_a = 0 \textit{ for all } X \in (\mathcal{M}')^{\perp}.$$

*Then for each $a$, the element $R_a = \frac{k}{d\dim(\mathcal{M})}(\mathrm{Tr}_{dk} \otimes id_{\mathcal{A}})(P_a)$ is a self-adjoint idempotent in $\mathcal{A}$, and $\sum_{a=1}^c R_a = k^2 1_{\mathcal{A}}$.*

*Proof.* Since $\mathcal{M} = I_d \otimes M_k$, we have $\mathcal{M}' = M_d \otimes I_k$ and $n = dk$. Now, let $1 \leq v, w \leq k$ with $x \neq y$, and let $1 \leq i, j \leq d$. Then $E_{ij} \otimes (E_{vv} - E_{ww})$ belongs to $(\mathcal{M}')^{\perp}$, so we must have

$$P_a(E_{ij} \otimes (E_{vv} - E_{ww}) \otimes 1_{\mathcal{A}})P_a = 0 \, \forall 1 \leq a \leq c.$$

Similarly, $E_{ij} \otimes E_{vw}$ is in $(\mathcal{M}')^{\perp}$, so

$$P_a(E_{ij} \otimes E_{vw} \otimes 1_{\mathcal{A}})P_a = 0 \, \forall 1 \leq a \leq c.$$

Note that $P_a \in \mathcal{M} \otimes \mathcal{A} = I_d \otimes M_k \otimes \mathcal{A}$, so $P_a = \sum_{p,q=1}^k \sum_{x=1}^d E_{xx} \otimes E_{pq} \otimes P_{a,x,pq}$, with the property that $P_{a,x,pq} = P_{a,y,pq}$ for any $1 \leq x, y \leq d$. For simplicity, we set $P_{a,pq} = P_{a,x,pq}$ for any $1 \leq x \leq d$. The quantity on the left of the above is exactly

$$\sum_{p,q=1}^k E_{ij} \otimes E_{pq} \otimes P_{a,pv}P_{a,wq}$$

so this says that $P_{a,pv}P_{a,wq} = 0$ and $P_{a,pv}P_{a,vq} = P_{a,pw}P_{a,wq}$. Now, since $P_a$ is a projection, we have $P_{a,pq} = \sum_{v=1}^k P_{a,pv}P_{a,vq} = kP_{a,pv}P_{a,vq}$ for all $p, q$. In particular, $P_{a,vv} = kP_{a,vv}^2$. By scaling, we see that $kP_{a,vv}$ is a self-adjoint idempotent. Similarly, since $P_{a,pv}P_{a,wq} = 0$ if $v \neq w$, we see that $P_{a,vv}P_{a,ww} = 0$. Therefore, $\{kP_{a,vv}\}_{v=1}^n$ is a collection of mutually orthogonal projections in $\mathcal{A}$.

Next, we set $R_a = \sum_{v=1}^k kP_{a,vv}$ for each $1 \leq a \leq c$. Then $R_a$ is a self-adjoint idempotent. We see that

$$\sum_{a=1}^{c} R_a = \sum_{a=1}^{c}\sum_{v=1}^{k} kP_{a,vv} = \sum_{v=1}^{k} k1_{\mathcal{A}} = k^2 1_{\mathcal{A}},$$

which completes the proof. $\square$

Now, we deal with the case of a general quantum complete graph.

**Theorem 4.2.7.** *Let $(M_n, \mathcal{M}, M_n)$ be a quantum complete graph. Let $\mathcal{A}$ be a hereditary $*$-algebra, and let $\{P_a\}_{a=1}^c \subseteq \mathcal{M} \otimes \mathcal{A}$ be a hereditary $c$-coloring of $(M_n, \mathcal{M}, M_n)$. Then $c \geq \dim(\mathcal{M})$. Moreover, if $c = \dim(\mathcal{M})$, then for each $1 \leq a \leq \dim(\mathcal{M})$ we have*

$$(\psi_{\mathcal{M}} \otimes id_{\mathcal{A}})(P_a) = \frac{1}{\dim(\mathcal{M})} 1_{\mathcal{A}}.$$

*Proof.* Up to unitary equivalence, we may write $\mathcal{M} = \bigoplus_{r=1}^m \mathbb{C}I_{n_r} \otimes M_{k_r}$. Then

$$\mathcal{M}' = \bigoplus_{r=1}^{m} M_{n_r} \otimes \mathbb{C}I_{k_r}.$$

Define $\mathcal{E}_r = 0 \oplus \cdots \oplus I_{n_r} \otimes I_{k_r} \oplus 0 \oplus \cdots \oplus 0$, which belongs to $\mathcal{M}' \cap \mathcal{M}$. Then defining $\widetilde{P}_a = (\mathcal{E}_r \otimes 1_{\mathcal{A}}) P_a (\mathcal{E}_r \otimes 1_{\mathcal{A}}) \in (\mathcal{E}_r \mathcal{M} \mathcal{E}_r) \otimes \mathcal{A}$, we obtain a family of mutually orthogonal projections whose sum is $\mathcal{E}_r$. Since $\mathcal{E}_r$ is central in $\mathcal{M}$, we see that $(\mathcal{E}_r \mathcal{M} \mathcal{E}_r)' = \mathcal{E}_r \mathcal{M}' \mathcal{E}_r$, while $\mathcal{E}_r M_n \mathcal{E}_r = M_{n_r k_r}$. It is evident that $X \in \mathcal{B}(\mathcal{E}_r \mathbb{C}^n) \cap (\mathcal{E}_r \mathcal{M}' \mathcal{E}_r)^{\perp}$ if and only if $X = \mathcal{E}_r X \mathcal{E}_r$ and $X \perp \mathcal{M}'$ in $M_n$. Therefore, for $X \in \mathcal{B}(\mathcal{E}_r \mathbb{C}^n) \cap (\mathcal{E}_r \mathcal{M}' \mathcal{E}_r)^{\perp}$ and $1 \leq a \leq c$, one has

$$\widetilde{P}_a (X \otimes 1_{\mathcal{A}}) \widetilde{P}_a = (\mathcal{E}_r \otimes 1_{\mathcal{A}}) P_a (\mathcal{E}_r X \mathcal{E}_r \otimes 1_{\mathcal{A}}) P_a (\mathcal{E}_r \otimes 1_{\mathcal{A}}) = 0,$$

using the fact that $\mathcal{E}_r X \mathcal{E}_r = X$ and $X$ belongs to $\mathcal{M}'$. Therefore, $\{\widetilde{P}_a\}_{a=1}^c$ is a hereditary coloring of the quantum complete graph $(M_{n_r k_r}, \mathcal{E}_r \mathcal{M} \mathcal{E}_r, M_{n_r k_r})$.

Since $\mathcal{E}_r \mathcal{M} \mathcal{E}_r = \mathbb{C}I_{n_r} \otimes M_{k_r}$, by Lemma 4.2.6, we see that $R_a^{(r)} := \frac{k_r}{n_r} (\mathrm{Tr}_{n_r k_r} \otimes \mathrm{id}_{\mathcal{A}})(\widetilde{P}_a)$ is a self-adjoint idempotent in $\mathcal{A}$ for each $1 \leq a \leq c$ and $1 \leq r \leq m$. Moreover, $\sum_{a=1}^c R_a^{(r)} = k_r^2 1_{\mathcal{A}}$.

Next, we claim that $R_a^{(r)} R_a^{(s)} = 0$ if $r \neq s$. To show this orthogonality relation, it suffices to show that $P_{a,xx}P_{a,yy} = 0$ whenever $P_{a,xx}$ is a block from $(\mathcal{E}_r\mathcal{M}\mathcal{E}_r) \otimes \mathcal{A}$ and $P_{a,yy}$ is a block from $(\mathcal{E}_s\mathcal{M}\mathcal{E}_s) \otimes \mathcal{A}$. If $x$ and $y$ are chosen in this way, then the matrix unit $E_{xy}$ in $M_n$ satisfies $\mathcal{E}_r(E_{xy})\mathcal{E}_s = E_{xy}$ and $\mathcal{E}_pE_{xy}\mathcal{E}_q = 0$ for all other pairs $(p, q)$. It is not hard to see that $E_{xy}$ belongs to $(\mathcal{M}')^\perp$, so that $P_a(E_{xy} \otimes 1_{\mathcal{A}})P_a = 0$. Considering the $(x, y)$-block of this equation gives $P_{a,xx}P_{a,yy} = 0$. It follows that $R_a^{(r)}R_a^{(s)} = 0$ for $r \neq s$.

Since $\{R_a^{(r)}\}_{r=1}^m$ is a collection of mutually orthogonal projections in $\mathcal{A}$, the element $R_a := \sum_{r=1}^m R_a^{(r)}$ is a self-adjoint idempotent in $\mathcal{A}$ for each $a$. Considering blocks, it is not hard to see that

$$\sum_{a=1}^c R_a = \sum_{a=1}^c \sum_{r=1}^m R_a^{(r)} = \sum_{r=1}^m k_r^2 1_{\mathcal{A}} = \dim(\mathcal{M})1_{\mathcal{A}}.$$

Since $R_a$ is a self-adjoint idempotent, so is $1_{\mathcal{A}} - R_a$. Their sum is given by

$$\sum_{a=1}^c (1_{\mathcal{A}} - R_a) = c1_{\mathcal{A}} - \sum_{a=1}^c R_a = (c - \dim(\mathcal{M}))1_{\mathcal{A}}.$$

It follows that $c \geq \dim(\mathcal{M})$, since the sum above is a sum of positives and $\mathcal{A}$ is hereditary.

Now, if $c = \dim(\mathcal{M})$, then the above sum of positives in $\mathcal{A}$ is $0$, which forces $1_{\mathcal{A}} - R_a = 0$ for all $a$. Hence, $R_a = 1_{\mathcal{A}}$. Since $R_a = \sum_{r=1}^m R_a^{(r)}$ and $R_a^{(r)} = \frac{k_r}{n_r}(\mathrm{Tr}_{n_rk_r} \otimes \mathrm{id}_{\mathcal{A}})(P_a)$, we see that

$$\sum_{r=1}^m \frac{k_r}{n_r}(\mathrm{Tr}_{n_rk_r} \otimes \mathrm{id}_{\mathcal{A}})(P_a) = 1_{\mathcal{A}}.$$

Therefore,

$$(\psi_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{A}})(P_a) = \sum_{r=1}^m \frac{k_r}{\dim(\mathcal{M})n_r}(\mathrm{Tr}_{n_rk_r} \otimes \mathrm{id}_{\mathcal{A}})(P_a) = \frac{1}{\dim(\mathcal{M})}1_{\mathcal{A}}.$$

□

**Remark 4.2.8.** In essence, Theorem 4.2.7 proves that any $q$-coloring of $(M_n, \mathcal{M}, M_n)$ with $\dim(\mathcal{M})$ colors induces a quantum isomorphism between the quantum graph $(M_n, \mathcal{M}, M_n)$ and the classical graph $K_{\dim(\mathcal{M})}$. This isomorphism occurs because any such coloring with ancillary algebra

$\mathcal{A}$ yields a (necessarily injective) unital $*$-isomorphism $\pi : D_{\dim(\mathcal{M})} \to \mathcal{M} \otimes \mathcal{A}$ satisfying the properties of a quantum graph homomorphism, with the additional property that $(\psi_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{A}}) \circ \pi = \pi \circ \psi_{D_{\dim(\mathcal{M})}}$.

In contrast to the case of $q$-colorings, the existence of a $loc$-coloring for a complete quantum graph is equivalent to the von Neumann algebra being abelian. This is imposed by the trace preserving property of the *-homomorphism.

**Theorem 4.2.9.** *Let $\mathcal{M} \subseteq M_n$ be a non-degenerate von Neumann algebra. Then $\chi_{loc}((M_n, \mathcal{M}, M_n))$ is finite if and only if $\mathcal{M}$ is abelian. In particular, if $\mathcal{M}$ is non-abelian, then $\chi((M_n, \mathcal{M}, M_n)) \neq \chi_q((M_n, \mathcal{M}, M_n))$.*

*Proof.* Suppose that there is a $c$-coloring of $(M_n, \mathcal{M}, M_n)$ in the $loc$-model. Up to unitary equivalence, we write $\mathcal{M} = \bigoplus_{r=1}^m \mathbb{C}I_{n_r} \otimes M_{k_r}$. We may choose projections $P_a \in \mathcal{M}$ such that $\sum_{a=1}^c P_a = I_n$ and $P_a((\mathcal{M}')^\perp)P_a = 0$ for all $a$. Let $R_a = \sum_{r=1}^m \frac{k_r}{n_r} \operatorname{Tr}_{n_r k_r}(P_a)$ as in the proof of the last theorem. Each $R_a$ is an idempotent in $\mathbb{C}$; hence, either $R_a = 0$ or $R_a = 1$. We know that $\sum_{a=1}^c R_a = \dim(\mathcal{M})$, so exactly $\dim(\mathcal{M})$ of the $R_a$'s are non-zero. Since $R_a$ is given by a trace on $\mathcal{M}$ which is faithful, having $R_a = 0$ implies that $P_a = 0$. Hence, by discarding any projections $P_a$ for which $R_a = 0$, we may assume without loss of generality that $R_a = 1$ for all $a$, and that $c = \dim(\mathcal{M})$.

Let $\mathcal{E}_r$ be the orthogonal projection onto the copy of $\mathbb{C}I_{n_r} \otimes M_{k_r}$ inside of $\mathcal{M} = \bigoplus_{r=1}^m \mathbb{C}I_{n_r} \otimes M_{k_r}$. Then, as before, the PVM $\{\mathcal{E}_r P_a \mathcal{E}_r\}_{a=1}^{\dim(\mathcal{M})}$ yields a classical $\dim(\mathcal{M})$-coloring for $(M_{n_r k_r}, \mathbb{C}I_{n_r} \otimes M_{k_r}, M_{n_r k_r})$. We will show that $k_r = 1$. By the same argument as above, by discarding values of $a$ for which $\frac{k_r}{n_r} \operatorname{Tr}_{n_r k_r}(\mathcal{E}_r P_a \mathcal{E}_r) = 0$, we may assume that there are exactly $k_r^2$ non-zero projections $\mathcal{E}_r P_a \mathcal{E}_r$ that yield a $k_r^2$-classical coloring for $(M_{n_r k_r}, \mathbb{C}I_{n_r} \otimes M_{k_r}, M_{n_r k_r})$. Set $\widetilde{P}_a = \mathcal{E}_r P_a \mathcal{E}_r$. By Theorem 4.2.7, for each $a$, we have $\frac{k_r}{n_r} \operatorname{Tr}_{n_r k_r}(\widetilde{P}_a) = 1$. Notice that $k_r \widetilde{P}_a = I_{n_r} \otimes k_r Q_a$ for some projection $k_r Q_a \in M_{k_r}$. Hence, $\operatorname{Tr}_{k_r}(k_r Q_a) = 1$. Let $\lambda_1, ..., \lambda_{k_r}$ be the eigenvalues of $k_r Q_a$ in $M_{k_r}$. Since each $\lambda_i \in \{0, 1\}$ and $\sum_{i=1}^{k_r} \lambda_i = \operatorname{Tr}_{k_r}(k_r Q_a) = 1$, there is exactly one $\lambda_i$ that is non-zero. Hence, $Q_a$ is rank one. The sum over all non-zero $Q_a$ gives $I_{k_r}$, and each $Q_a$ is rank one.

Hence, the number of $a$ for which $Q_a$ is non-zero must be $k_r$. Since we assumed that this number is $k_r^2$, we must have $k_r = k_r^2$. Since $k_r > 0$, we have $k_r = 1$. Since $r$ was arbitrary, we see that $\mathcal{M} = \bigoplus_{r=1}^m \mathbb{C}I_{n_r} \otimes M_{k_r} = \bigoplus_{r=1}^m \mathbb{C}I_{n_r}$ is abelian.

Conversely, suppose that $\mathcal{M}$ is abelian. Then the proof of Theorem 4.2.7 yields projections $P_a \in \mathcal{M} \otimes M_d$, where $d = \dim(\mathcal{M})$, such that $\sum_{a=1}^d P_a = I_n \otimes I_d$ and $P_a(X \otimes I_d)P_a = 0$ whenever $X \in (\mathcal{M}')^\perp$. Moreover, the projections obtained in this case satisfy $P_{a,ij}P_{b,k\ell} = P_{b,k\ell}P_{a,ij}$ for all $1 \leq a, b \leq d$ and $1 \leq i, j, k, \ell \leq n$. Thus, the entries of the projections $P_a$ must $*$-commute with each other, so the $C^*$-algebra they generate is abelian. Since there is a $d$-coloring for $(M_n, \mathcal{M}, M_n)$ with an abelian ancilla, this implies that $\chi_{loc}((M_n, \mathcal{M}, M_n)) \leq d$. □

Using the monotonicity of colorings and the results above on quantum complete graphs, we see that every quantum graph has a finite quantum coloring. As a result, we obtain the following algebraic four coloring theorem for quantum graphs, generalizing a result from [30]. The idea is simple: we have shown that $\mathcal{A}(\text{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_{\dim(\mathcal{M})})) \neq 0$ and it is known [30] that $\mathcal{A}(\text{Hom}(K_{\dim(\mathcal{M})}, K_4)) \neq 0$. Composing these two, we get the desired result.

**Theorem 4.2.10.** *Let $(\mathcal{S}, \mathcal{M}, M_n)$ be any quantum graph. Then $\chi_{alg}((\mathcal{S}, \mathcal{M}, M_n)) \leq 4$.*

*Proof.* Suppose that $\chi_{alg}(\mathcal{S}, \mathcal{M}, M_n) \leq c$ for some $c < \infty$. Then $\mathcal{A}(\text{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c))$ exists. We will let $p_1, ..., p_c$ be the canonical self-adjoint idempotents in the matrix algebra $M_n(\mathcal{A}(\text{Hom}((\mathcal{S}, \mathcal{M}, M_n),$ By [30], there is an algebraic homomorphism $K_c \to K_4$. Thus, there are self-adjoint idempotents $f_{a,v}$ in $\mathcal{A}(\text{Hom}(K_c, K_4))$ for $1 \leq a \leq c$ and $1 \leq v \leq 4$ such that $\sum_{v=1}^4 f_{a,v} = 1$ for all $a$ and $f_{a,v}f_{b,v} = 0$ whenever $a \neq b$. Define

$$q_{v,ij} = \sum_{a=1}^{c} p_{a,ij} \otimes f_{a,v} \in \mathcal{A}(\text{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \otimes \mathcal{A}(\text{Hom}(K_c, K_4)).$$

Then

$$\begin{aligned}\sum_{k=1}^{n} q_{v,ik}q_{v,kj} &= \sum_{k=1}^{n}\left(\sum_{a=1}^{c} p_{a,ik}\otimes f_{a,v}\right)\left(\sum_{b=1}^{c} p_{b,kj}\otimes f_{b,v}\right)\\ &= \sum_{k=1}^{n}\sum_{a,b=1}^{c} p_{a,ik}p_{b,kj}\otimes f_{a,v}f_{b,v}\\ &= \sum_{a=1}^{c}\sum_{k=1}^{n} p_{a,ik}p_{a,kj}\otimes f_{a,v}^2\\ &= \sum_{a=1}^{c} p_{a,ij}\otimes f_{a,v} = q_{v,ij}.\end{aligned}$$

Therefore, $q_v = (q_{v,ij})$ is an idempotent for each $v$. Similarly, one can see that $q_v^* = q_v$ (that is, $q_{v,ij}^* = q_{v,ji}$) and $\sum_{v=1}^{4} q_{v,ij}$ is 0 if $i \neq j$ and 1 if $i = j$. Let $X = (x_{ij}) \in M_n$. Letting $1 \otimes 1$ denote the unit in the tensor product of the game algebras,

$$q_v(X\otimes 1\otimes 1)q_w = \left(\sum_{k,\ell=1}^{n} q_{v,ik}x_{k\ell}q_{w,\ell j}\right)_{i,j} = \left(\sum_{k,\ell=1}^{n}\sum_{a,b=1}^{c} p_{a,ik}x_{k\ell}p_{b,\ell j}\otimes f_{a,v}f_{b,w}\right)_{i,j}. \quad (4.2.0.1)$$

If $X \in (\mathcal{M}')^{\perp}$ and $a = b$, then the above sum becomes

$$q_v(X\otimes 1\otimes 1)q_v = \left(\sum_{k,\ell=1}^{n}\sum_{a=1}^{c} p_{a,ik}x_{k\ell}p_{a,\ell j}\otimes f_{a,v}\right) = \sum_{a=1}^{c} p_a(X\otimes 1)p_a\otimes f_{a,v} = 0,$$

by definition of $\mathcal{A}(\mathrm{Hom}((\mathcal{S},\mathcal{M},M_n),K_c))$. If $X \in \mathcal{M}'$ and $a \neq b$, then $\sum_{k,\ell=1}^{n} p_{a,ik}x_{k\ell}p_{b,\ell j}$ is the $(i,j)$ entry of $p_a(X\otimes 1)p_b = 0$. Thus, if $v \neq w$, then Equation (4.2.0.1) reduces to

$$q_v(X\otimes 1\otimes 1)q_w = \left(\sum_{k,\ell=1}^{n}\sum_{a=1}^{c} p_{a,ik}x_{k\ell}p_{a,\ell j}\otimes f_{a,v}f_{a,w}\right)_{i,j} = 0,$$

since $f_{a,v}f_{a,w} = 0$ for $v \neq w$. Therefore, letting $r_{v,ij}$ be the canonical generators of $\mathcal{A}(\mathrm{Hom}((\mathcal{S},\mathcal{M},M_n),K_4))$, we obtain a unital $*$-homomorphism

$$\pi : \mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_4)) \to \mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_c)) \otimes \mathcal{A}(\mathrm{Hom}(K_c, K_r)),$$
$$r_{v,ij} \mapsto q_{v,ij}.$$

The latter algebra is non-zero, so $\mathcal{A}(\mathrm{Hom}((\mathcal{S}, \mathcal{M}, M_n), K_4)) \neq \{0\}$. Thus,

$$\chi_{alg}((\mathcal{S}, \mathcal{M}, M_n)) \leq 4.$$

$\square$

# 5. Spectral bounds for the chromatic number of quantum graphs

Chromatic numbers of quantum graphs are closely related to the zero-error capacity of quantum channels [14]. Hence, estimating these numbers can be useful for the development of zero-error quantum communication. To this end, we obtain lower bounds for the classical and quantum chromatic numbers of quantum graphs in this chapter.

Our approach uses the quantum adjacency operator, defined in [4, 43], to associate a spectrum to the given quantum graph. We use this spectrum and techniques adapted from [19] to achieve the bounds. The algebraic characterization of quantum coloring given by theorem 4.1.7 is very convenient for these purposes. We begin by recalling it for an irreflexive quantum graph.

**Definition 5.0.1.** Let $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$ be an irreflexive quantum graph. We say that there is a $c$-coloring of $\mathcal{G}$ if there exists a finite von-Neumann algebra $\mathcal{N}$ with a faithful normal trace and projections $\{P_a\}_{a=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ such that

1. $P_a^2 = P_a = P_a^*$, for $1 \leq a \leq c$,
2. $\sum_{i=1}^c P_a = I_{\mathcal{M}\otimes\mathcal{N}}$,

satisfying the following condition:

$$P_a(X \otimes I_{\mathcal{N}})P_a = 0, \; \forall X \in S \text{ and } 1 \leq a \leq c. \tag{5.0.0.1}$$

- If $\dim(\mathcal{N}) = 1$, we call it a *classical* $(loc)$ coloring of $\mathcal{G}$.
- If $\dim(\mathcal{N}) < \infty$, we call it a *quantum* $(q)$ coloring of $\mathcal{G}$.
- More generally, when $\mathcal{N}$ is a finite von-Neumann algebra (possibly infinite dimensional), it is called a *quantum commuting* $(qc)$ coloring of $\mathcal{G}$.

The projections $\{P_a\}_{a=1}^c$ are obtained from the winning strategies of the non-local quantum graph coloring game. In particular, when $\mathcal{M} = D_n$, we recover the usual classical and quantum coloring of classical graphs on $n$ vertices.

Using this, we can get a combinatorial definition for the chromatic numbers of quantum graphs.

**Definition 5.0.2.** Let $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$ be an irreflexive quantum graph, with $c$-coloring in the sense of definition 5.0.1.

- The *classical chromatic number* of $\mathcal{G}$, $\chi(\mathcal{G})$, is defined to be the least $c$ over all $\mathcal{N}$ with $\dim(\mathcal{N}) = 1$.
- The *quantum chromatic number* of $\mathcal{G}$, $\chi_q(\mathcal{G})$, is defined to be the least $c$ over all $\mathcal{N}$ with $\dim(\mathcal{N}) < \infty$.
- The *quantum commuting chromatic number* of $\mathcal{G}$, $\chi_{qc}(\mathcal{G})$, is defined to be the least $c$ over all finite von-Neumann algebra $\mathcal{N}$.

While definition 2.1.1 was used in chapter 4 for developing chromatic number of quantum graphs, definition 2.2.6 offers the advantage of associating a spectrum with the quantum graph, which is useful for estimating these chromatic numbers.

**Convention 1.** For the remainder of this dissertation, $\mathcal{M}$ denotes a finite dimensional C*-algebra equipped with its *tracial* $\delta$-form $\psi$, as given in 2.2.5. We assume that our quantum graph $(S, \mathcal{M}, B(L^2(\mathcal{M}, \psi)))$ is irreflexive. Further, $A$ always refers to the unique self-adjoint quantum adjacency matrix associated with $S$, as discussed in proposition 2.3.1. We denote this quantum graph by $\mathcal{G} = (\mathcal{M}, \psi, A, S)$.

Our spectral bounds can be summarized as follows:

**Theorem 5.0.3.** *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph, and let $\chi(\mathcal{G}), \chi_q(\mathcal{G})$ and $\chi_{qc}(\mathcal{G})$ denote the classical, quantum and quantum commuting chromatic numbers of $\mathcal{G}$ respectively. Then,*

$$1+\max\left\{\frac{\lambda_{\max}}{|\lambda_{\min}|}, \frac{\dim(S)}{\dim(S)-\dim(\mathcal{M})\gamma_{\min}}, \frac{s^{\pm}}{s^{\mp}}, \frac{n^{\pm}}{n^{\mp}}, \frac{\lambda_{\max}}{\lambda_{\max}-\gamma_{\max}+\theta_{max}}\right\} \leq \chi_{qc}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi(\mathcal{G}). \tag{5.0.0.2}$$

Here, $\lambda_{\max}, \lambda_{\min}$ denote the maximum and minimum eigenvalues of $A$; $s^+, s^-$ denote the sum of the squares of the positive and negative eigenvalues of $A$ respectively; $n^+, n^-$ are the number of positive and negative eigenvalues of $A$ including multiplicities; $\gamma_{\max}, \gamma_{\min}$ denote the maximum and minimum eigenvalues of the signless Laplacian operator (definition 5.3.4); and $\theta_{\max}$ denotes the maximum eigenvalue of the Laplacian operator (definition 5.3.4).

The key ingredient in proving these bounds is lemma 5.1.2 and 5.1.3, which holds true across the classical ($loc$), quantum ($q$) and quantum-commuting ($qc$) coloring models. With these, the proof of the corresponding bounds for classical graphs can essentially be adapted to our setting. For the sake of concreteness, we prove our results in the quantum coloring framework ($\dim(\mathcal{N}) < \infty$). However, by using an infinite dimensional version of proposition 5.1.4 ($\dim(\mathcal{N}) = \infty$), all the bounds can be directly transferred to the quantum commuting chromatic numbers as well. Since $\chi_{qc}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi(\mathcal{G})$, our estimates are also lower bounds for the classical chromatic number of quantum graphs.

### 5.1 Use of quantum adjacency matrix in coloring

We begin by defining the spectrum of a quantum graph.

**Definition 5.1.1.** Let $\mathcal{M}$ be a finite dimensional C*-algebra equipped with its tracial $\delta$-form $\psi$, and let $\mathcal{G} = (S, \mathcal{M}, B(L^2(\mathcal{M}, \psi)))$ be a (undirected) quantum graph on $(\mathcal{M}, \psi)$. The *spectrum of* $\mathcal{G}$ is defined to be the spectrum of the quantum adjacency operator $A$, defined by

$$A = \delta^{-2}(\psi \otimes I)P_S(I \otimes \mathbb{1}), \tag{5.1.0.1}$$

where $P_S$ is the orthogonal bimodule projection onto $S$.

Note that $A$ is *self-adjoint* and so, the spectrum of an undirected quantum graph is real. We now show the connection between quantum adjacency matrix and quantum graph coloring by generalizing some algebraic results in [19] to the quantum graph setting. The following lemma proves that "pinching" operation annihilates the quantum adjacency matrix and leaves the commutant of the quantum vertex set invariant. Throughout our discussion, we follow convention 1.

**Lemma 5.1.2.** *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph. If $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ is an arbitrary $c$-coloring of $\mathcal{G}$ in the sense of definition 5.0.1, then*

$$\sum_{k=1}^{c} P_k(A \otimes I_{\mathcal{N}})P_k = 0, \tag{5.1.0.2}$$

$$\sum_{k=1}^{c} P_k(E \otimes I_{\mathcal{N}})P_k = E \otimes I_{\mathcal{N}}, \;\; \forall E \in \mathcal{M}'. \tag{5.1.0.3}$$

*Proof.* We first show that $A \in S$. Recall that $A$ is given by (5.1.0.1), using the orthogonal bimodule projection onto $S$. Using the inverse relations (2.3.0.4) and (2.3.0.5), it can be shown that $P_S$ must be of the form $\delta^{-2}m(A \otimes (\cdot))m^*$. In particular, $P_S(A) = \delta^{-2}m(A \otimes A)m^* = A$ by the Schur idempotent property of $A$. So, $A \in range(P_S) = S$. Now, by (5.0.0.1), we get that $\sum_{k=1}^c P_k(A \otimes I_{\mathcal{N}})P_k = 0$.

Equation (5.1.0.3) follows from the fact that the projections $P_k \in \mathcal{M} \otimes \mathcal{N}$ commute with $E \otimes I_{\mathcal{N}} \in \mathcal{M}' \otimes \mathcal{N}'$, and $\sum_{k=1}^c P_k = I_{\mathcal{M}\otimes\mathcal{N}}$. □

The next lemma is a corresponding result for the "twirling" operation.

**Lemma 5.1.3.** *Suppose $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ is an irreflexive quantum graph and $\{P_k\}_{k=1}^c \subseteq \mathcal{M}\otimes\mathcal{N}$ is a $c$-coloring of $\mathcal{G}$ in the sense of definition 5.0.1. Define $U := \sum_{l=1}^c \omega^l P_l$, where $\omega = e^{2\pi i/c}$ is a $c^{th}$ root of unity. Then,*

$$\sum_{k=1}^{c} P_k(X \otimes I_{\mathcal{N}})P_k = \frac{1}{c}\sum_{k=1}^{c} U^k(X \otimes I_{\mathcal{N}})(U^*)^k, \quad \forall\, X \in B(L^2(\mathcal{M})). \tag{5.1.0.4}$$

*In particular,*

$$\sum_{k=1}^{c} U^k(A \otimes I_{\mathcal{N}})(U^*)^k = 0, \tag{5.1.0.5}$$

$$\sum_{k=1}^{c} U^k(E \otimes I_{\mathcal{N}})(U^*)^k = c\,(E \otimes I_{\mathcal{N}}), \;\; \forall E \in \mathcal{M}'. \tag{5.1.0.6}$$

*Proof.* Note that $U^* = \sum_{l=1}^c \omega^{-l} P_l$ since $\{P_l\}_{l=1}^c$ are self-adjoint. Also, the $k^{th}$ power of $U$ is given by

$$U^k = \sum_{l=1}^{c} \omega^{lk} P_l$$

as the projections $\{P_l\}_{l=1}^c$ are mutually orthogonal, that is $P_i P_j = 0$ if $i \neq j$. Now, for $X \in B(L^2(\mathcal{M}))$, we obtain:

$$\begin{aligned}
\sum_{k=1}^{c} U^k (X \otimes I_{\mathcal{N}}) (U^*)^k &= \sum_{k=1}^{c} \sum_{l,l'=1}^{c} \omega^{(l-l')k} P_l (X \otimes I_{\mathcal{N}}) P_{l'} \\
&= \sum_{l,l'=1}^{c} (\sum_{k=1}^{c} \omega^{(l-l')k}) P_l (X \otimes I_{\mathcal{N}}) P_{l'} \\
&= \sum_{l,l'=1}^{c} (c\ \delta_{l,l'}) P_l (X \otimes I_{\mathcal{N}}) P_{l'}, \text{ where } \delta_{l,l'} \text{ denotes the Krönecker delta} \\
&= c \sum_{l=1}^{c} P_l (X \otimes I_{\mathcal{N}}) P_l
\end{aligned}$$

Hence, we get the result. The rest follows from lemma 5.1.2. □

Next, we note some obvious properties of $A \otimes I_{\mathcal{N}}$ for future reference.

**Proposition 5.1.4.** *Suppose $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ is an irreflexive quantum graph and $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ is an arbitrary $c$-quantum coloring of $\mathcal{G}$ in the sense of definition 5.0.1. Assume that $2 \leq \dim(\mathcal{M}) < \infty$ and $\mathcal{N} \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, with $\dim(\mathcal{H}) = d$.*

*Define $\tilde{A} = A \otimes I_{\mathcal{N}}$. Then*

1. *$\tilde{A}$ is self-adjoint and has real eigenvalues.*
2. *The spectrum of $\tilde{A}$ has the same elements as the spectrum of $A$, but each with a multiplicity of $d$. In particular, the largest and smallest eigenvalue of $\tilde{A}$ coincide with the largest and smallest eigenvalue of $A$, respectively.*
3. *$\tilde{A} = \sum_{a,b=1}^{c} P_a \tilde{A} P_b$.*

4. *$\tilde{A}$ can be expressed as a block partitioned matrix* $\begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} & \dots & \widehat{A}_{1c} \\ \widehat{A}_{21} & \widehat{A}_{22} & \dots & \widehat{A}_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{A}_{c1} & \widehat{A}_{c2} & \dots & \widehat{A}_{cc} \end{bmatrix}$*, such that* $\widehat{A}_{ii} = 0$ *for all* $i \in [c]$*. In particular,* $\mathrm{Tr}(A) = \frac{1}{d}\mathrm{Tr}\left(\tilde{A}\right) = 0$.

*Proof.* The first two statements are evident since $A$ is self-adjoint and tensoring with identity only produces more copies of the same eigenvalues. The third statement follows from the fact that $\sum_{k=1}^{c} P_k = I_{\mathcal{M}\otimes\mathcal{N}}$.

To see the last statement, note that $\tilde{A}$ can be interpreted as a giant matrix over complex numbers as $\mathcal{M}$ and $\mathcal{N}$ are finite dimensional. Choose an orthonormal basis for $L^2(\mathcal{M}) \otimes \mathcal{H}$ such that all the projections $P_k$ are represented as diagonal matrices. Identify $\widehat{A}_{ab}$ with the matrix $P_a\tilde{A}P_b$. Then, we get the desired block partition. From (5.1.0.2), it follows that $\widehat{A}_{ii} = 0$ for $1 \le i \le c$. ☐

## 5.2 Hoffman's bound

One of the well-known spectral bounds in graph theory is the Hoffman's bound [31]. This is a lower bound on the chromatic number of a graph using the largest and smallest eigenvalues of the adjacency matrix. The classical bound is as follows: If $G$ is an irreflexive classical graph whose adjacency matrix $A$ has eigenvalues $\lambda_{\max} = \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n = \lambda_{\min}$, then

$$1 + \frac{\lambda_{\max}}{|\lambda_{\min}|} \le \chi(G). \tag{5.2.0.1}$$

We can prove a quantum version of this bound using the following result from linear algebra.

**Lemma 5.2.1.** *Let $A$ be a self-adjoint matrix, block partitioned as* $\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$. *Then,*

$$(n-1)\lambda_{\min}(A) + \lambda_{\max}(A) \leq \sum_{i=1}^{n} \lambda_{\max}(A_{ii}),$$

*where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and minimum eigenvalues of that matrix.*

*Proof.* We start with the case $n = 2$. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a normalized eigenvector ($\|x_1\|^2 + \|x_2\|^2 = 1$) corresponding to $\lambda_{\max}(A)$. Define $y = \begin{bmatrix} \frac{\|x_2\|}{\|x_1\|} x_1 \\ -\frac{\|x_1\|}{\|x_2\|} x_2 \end{bmatrix}$. Then, we have

$$\lambda_{\max}(A) + \lambda_{\min}(A) \leq \langle x| A |x\rangle + \langle y| A |y\rangle = \frac{\langle x_1| A_{11} |x_1\rangle}{\|x_1\|^2} + \frac{\langle x_2| A_{22} |x_2\rangle}{\|x_2\|^2} \leq \lambda_{\max}(A_{11}) + \lambda_{\max}(A_{22}).$$

The general case follows by induction on $n$. □

The generalization of Hoffman's bound to quantum graphs is as follows:

**Theorem 5.2.2.** *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph and $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\dim(\mathcal{M})} = \lambda_{\min}$ be all the eigenvalues of $A$. Then*

$$1 + \frac{\lambda_{\max}}{|\lambda_{\min}|} \leq \chi_q(\mathcal{G}). \tag{5.2.0.2}$$

*Proof.* Let $\{P_k\}_{k=1}^{c} \subseteq \mathcal{M} \otimes \mathcal{N}$ be a $c$-quantum coloring of $\mathcal{G}$ and $\tilde{A} = A \otimes I_{\mathcal{N}}$. Partition $\tilde{A}$ as $[\widehat{A}_{ab}]_{a,b=1}^{c}$, as in proposition 5.1.4. Applying lemma 5.2.1, we get

$$(c-1)\lambda_{\min}(\tilde{A}) + \lambda_{\max}(\tilde{A}) \leq \sum_{i=1}^{c} \lambda_{\max}(\widehat{A}_{ii}). \tag{5.2.0.3}$$

But $\widehat{A}_{ii} = 0$ for all $1 \leq i \leq c$. Hence equation (5.2.0.3) reduces to

$$(c-1)\lambda_{\min}(\tilde{A}) + \lambda_{\max}(\tilde{A}) \leq 0.$$

Recall that $\lambda_{\min}(\tilde{A}) = \lambda_{\min}(A)$ and $\lambda_{\max}(\tilde{A}) = \lambda_{\max}(A)$. So, we get $(c-1)\lambda_{\min}(A)+\lambda_{\max}(A) \leq 0$. On rearranging and taking minimum over all $c$, we get

$$1 + \frac{\lambda_{\max}(A)}{|\lambda_{\min}(A)|} \leq \chi_q(\mathcal{G}).$$

□

## 5.3 Lower bound using edge number

In this section, we prove a spectral lower bound on the quantum chromatic number using a quantum analogue for the number of edges in the graph.

For a classical graph $G$ with $n$ vertices and $m$ edges, it was shown [13] that

$$1 + \frac{2m}{2m - n\gamma_{\min}} \leq \chi(G), \tag{5.3.0.1}$$

where $\gamma_{\min}$ is the minimum eigenvalue of the signless Laplacian of $G$. To prove a generalization of this bound to arbitrary quantum graphs $(\mathcal{M}, \psi, A, S)$, we first introduce a quantum analogue for $m, n$ and $\gamma_{\min}$.

Recall that the degree matrix for classical graphs is a diagonal matrix obtained from the action of the adjacency matrix on the all 1s vector. This can be extended to quantum graphs as follows:

**Definition 5.3.1.** Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be a quantum graph and $\mathbb{1}$ denote the unit in $\mathcal{M}$. Then the *quantum degree matrix* of $\mathcal{G}$ is a linear operator $D \in B(L^2(\mathcal{M}))$ given by

$$D : \mathcal{M} \longrightarrow \mathcal{M} \text{ as } x \mapsto x(A\mathbb{1}), \forall x \in \mathcal{M}.$$

In other words, $D$ can be interpreted as $A\mathbb{1} \in \mathcal{M}$ viewed as an element of $B(L^2(\mathcal{M}))$ under the *right* regular representation.

**Remark 5.3.2.** The definition $D = A\mathbb{1}$ was also used in [8] and [42]. The only difference in our case is that we view $D$ under the *right* regular representation, instead of the usual left regular representation of $\mathcal{M}$. The advantage of using right regular representation is that $D$ then belongs to $\mathcal{M}'$.

Our next goal is to define a quantum analogue for the "number of edges" in the graph. To do that, we need the following result:

**Proposition 5.3.3.** *Let $\mathcal{M}$ be a finite dimensional C*-algebra, equipped with its tracial $\delta$-form $\psi$. If $(\mathcal{M}, \psi, A, S)$ is a quantum graph with degree matrix $D$, then,*

$$\mathrm{Tr}(D) = \delta^2\psi(A\mathbb{1}) = \dim(S). \tag{5.3.0.2}$$

*Proof.* Let $P_S : B(L^2(\mathcal{M})) \to B(L^2(\mathcal{M}))$ denote the orthogonal bimodule projection onto $S$. We can express $P_S$ as an element $\sum_{i=1}^t x_i \otimes y_i^{op} \in \mathcal{M} \otimes \mathcal{M}^{op}$, such that $P_S(a \otimes b) = \sum_{i=1}^t x_i a \otimes b y_i$, for all $a, b \in \mathcal{M}$ using the correspondence mentioned in remark 2.3.0.1. Now, $A = \delta^2(\psi \otimes I)P_S(I \otimes \eta)$ implies

$$A(\mathbb{1}) = \delta^2(\psi \otimes I)P_S(\mathbb{1} \otimes \mathbb{1}) = \delta^2(\psi \otimes I)(\sum_{i=1}^t x_i \otimes y_i) = \delta^2 \sum_{i=1}^t \psi(x_i)y_i. \tag{5.3.0.3}$$

Thus,

$$
\begin{aligned}
\psi(A\mathbb{1}) &= \psi(\delta^2 \sum_{i=1}^{t} \psi(x_i) y_i) = \delta^2 \sum_{i=1}^{t} \psi(x_i)\psi(y_i) \\
&= \delta^2 \sum_{i=1}^{t} \langle x_i, \mathbb{1}\rangle \langle y_i, \mathbb{1}\rangle \\
&= \delta^2 \sum_{i=1}^{t} \langle x_i \otimes y_i, \mathbb{1}\otimes\mathbb{1}\rangle = \delta^2 \left\langle \sum_{i=1}^{t} x_i \otimes y_i, \mathbb{1}\otimes\mathbb{1} \right\rangle \\
&= \delta^2 \langle P_S, I\rangle\,, \text{ when viewed as operators on } B(L^2(\mathcal{M})) \\
&= \delta^2 \frac{\mathrm{Tr}(P_S)}{\dim(B(L^2(\mathcal{M})))} \\
&= \dim(\mathcal{M}) \frac{\dim(S)}{\dim(\mathcal{M})^2} = \frac{\dim(S)}{\dim(\mathcal{M})}
\end{aligned}
$$

where we have used the fact that $\psi$ is a tracial state and $\delta^2 = \dim(\mathcal{M})$. Also, the trace on $B(L^2(\mathcal{M}))$ restricted to $\mathcal{M}$ (or $\mathcal{M}'$ by symmetry) is just $\dim(\mathcal{M})\psi$. So,

$$\mathrm{Tr}(D) = \dim(\mathcal{M})\ \psi(A\mathbb{1}).$$

Hence, $\mathrm{Tr}(D) = \dim(S)$. □

We now define quantum analogues of some classical quantities:

**Definition 5.3.4.** Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph with degree matrix $D$.

1. The *quantum vertex number* for $\mathcal{G}$ is defined to be $\dim(\mathcal{M})$.
2. The *quantum edge number* for $\mathcal{G}$ is defined to be $\dfrac{\mathrm{Tr}(D)}{2} = \dfrac{\dim(S)}{2}$.
3. The *Laplacian* of $\mathcal{G}$ is the linear operator $L = D - A \in B(L^2(\mathcal{M}))$.
4. The *signless Laplacian* of $\mathcal{G}$ is the linear operator $Q = D + A \in B(L^2(\mathcal{M}))$.

For a classical irreflexive graph $G = (V, E)$, these definitions clearly coincide with the usual values. In particular, if $\mathcal{G} = (S_G, D_{|V|}, M_{|V|})$, then the quantum vertex number is $|V|$ and the quantum edge number is $|E|$ since $2|E| = \sum_{v\in V} deg(v) = \mathrm{Tr}(D)$.

**Remark 5.3.5.** The quantum edge number need not be an integer in general. But for most purposes, we will only need $2m = \mathrm{Tr}(D) = \dim(S)$.

We are now ready to prove a quantum version of the spectral bound in (5.3.0.1).

**Theorem 5.3.6.** *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph . Then*

$$1 + \frac{2m}{2m - n\gamma_{\min}} \leq \chi_q(\mathcal{G}), \tag{5.3.0.4}$$

*where $m$ is the quantum edge number, $n$ is the quantum vertex number and $\gamma_{\min}$ is the minimum eigenvalue of the signless Laplacian of $\mathcal{G}$, in the sense of definition 5.3.4. More precisely,*

$$1 + \frac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}} \leq \chi_q(\mathcal{G}). \tag{5.3.0.5}$$

*Proof.* Let $\{P_k\}_{k=1}^{c} \subseteq \mathcal{M} \otimes \mathcal{N}$ be a $c$-quantum coloring of $\mathcal{G}$ and let $U$ be defined as in lemma 5.1.3. Then, (5.1.0.5) can be rearranged as $U^c(A \otimes I_{\mathcal{N}})(U^*)^c = -\sum_{k=1}^{c-1} U^k(A \otimes I_{\mathcal{N}})(U^*)^k$. Using $D - Q = -A$ and $U^c = I_{\mathcal{M}\otimes\mathcal{N}}$, we get

$$\begin{aligned}
A \otimes I_{\mathcal{N}} &= \sum_{k=1}^{c-1} U^k((D-Q) \otimes I_{\mathcal{N}})(U^*)^k \\
&= \sum_{k=1}^{c-1} U^k(D \otimes I_{\mathcal{N}})(U^*)^k - \sum_{k=1}^{c-1} U^k(Q \otimes I_{\mathcal{N}})(U^*)^k \\
&= (D \otimes I_{\mathcal{N}}) \sum_{k=1}^{c-1} U^k(U^*)^k - \sum_{k=1}^{c-1} U^k(Q \otimes I_{\mathcal{N}})(U^*)^k \\
&= (c-1)(D \otimes I_{\mathcal{N}}) - \sum_{k=1}^{c-1} U^k(Q \otimes I_{\mathcal{N}})(U^*)^k
\end{aligned}$$

where we have used the fact that $D \in \mathcal{M}'$ and hence $D \otimes I_{\mathcal{N}}$ commutes with $U \in \mathcal{M} \otimes \mathcal{N}$. Let $\mathcal{N}$ be represented in some $B(\mathcal{H})$ and let $u$ denote a unit vector in $\mathcal{H}$ such that $\langle u, u\rangle = 1$. Further, let $|\xi\rangle = \mathbb{1} \otimes u$ denote a column vector in $L^2(\mathcal{M}) \otimes \mathcal{H}$ and $\langle \xi|$ denote its corresponding conjugate row vector. Multiplying the left and right most sides of the above equation by $\langle \xi|$ from the left and by $|\xi\rangle$ from the right, we obtain

$$\langle\xi|\, A\otimes I_{\mathcal{N}}\,|\xi\rangle = (c-1)\,\langle\xi|\, D\otimes I_{\mathcal{N}}\,|\xi\rangle - \sum_{k=1}^{c-1}\langle\xi|\, U^k(Q\otimes I_{\mathcal{N}})(U^*)^k\,|\xi\rangle\,. \tag{5.3.0.6}$$

Now, $\langle\xi|\, A\otimes I_{\mathcal{N}}\,|\xi\rangle = \langle\mathbb{1}, A\mathbb{1}\rangle\,\langle u,u\rangle = \psi((A\mathbb{1})^*) = \psi(A\mathbb{1}) = \dfrac{\dim(S)}{\dim(\mathcal{M})}$, where we use the *-preserving property of $A$ (remark 2.2.7) and proposition 5.3.3. Similarly, $\langle\xi|D\otimes I_{\mathcal{N}}|\xi\rangle = \dfrac{\dim(S)}{\dim(\mathcal{M})}$. To estimate the last term, recall that eigenvalues are invariant under unitary conjugation and tensoring with identity only changes their multiplicity. So,

$$\begin{aligned}
\gamma_{\min} &= \min\left\{\langle w|\, Q\,|w\rangle : w\in L^2(\mathcal{M}), \langle w,w\rangle = 1\right\}\\
&= \min\left\{\langle v|Q\otimes I_{\mathcal{N}}|v\rangle : v\in L^2(\mathcal{M})\otimes\mathcal{H}, \langle v,v\rangle = 1\right\}\\
&= \min\left\{\langle v|U^k(Q\otimes I_{\mathcal{N}})(U^*)^k|v\rangle : v\in L^2(\mathcal{M})\otimes\mathcal{H}, \langle v,v\rangle = 1\right\}\\
&\le \langle\xi|U^k(Q\otimes I_{\mathcal{N}})(U^*)^k|\xi\rangle,\ \ \forall k\in[c].
\end{aligned}$$

Hence, (5.3.0.6) leads to

$$\frac{\dim(S)}{\dim(\mathcal{M})} \le (c-1)\frac{\dim(S)}{\dim(\mathcal{M})} - (c-1)\gamma_{\min}, \tag{5.3.0.7}$$

which upon rearranging yields $1 + \dfrac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}} \le c$. Taking minimum over all $c$ , we get the desired bound.

$\square$

## 5.4 Bound using the sum of square of eigenvalues

In [1], it was proved that for a classical graph $G$,

$$1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} \le \chi(G), \tag{5.4.0.1}$$

where $s^+$ is the sum of the squares of the positive eigenvalues of the adjacency matrix and $s^-$ is the sum of the squares of its negative eigenvalues. In this section, we show that the above bound also works in the setting of quantum graphs. We first recall the following result from linear algebra, whose proof can be found in [1].

**Lemma 5.4.1.** *Let* $X = [X_{ij}]_{i,j}^r$ *and* $Y = [Y_{ij}]_{i,j}^r$ *be two positive semidefinite matrices conformally partitioned. If* $X_{ii} = Y_{ii}$ *for* $1 \leq i \leq r$ *and* $XY = 0$*, then* $\operatorname{Tr}(X^*X) \leq (r-1)\operatorname{Tr}(Y^*Y)$.

We now adapt the proof of the classical bound in [1] to the quantum case.

**Theorem 5.4.2.** *Let* $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ *be an irreflexive quantum graph and* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\dim(\mathcal{M})}$ *be all the eigenvalues of* $A$*. Let* $s^+ = \sum_{\lambda_i > 0}(\lambda_i)^2$ *and* $s^- = \sum_{\lambda_i<0}(\lambda_i)^2$*. Then,*

$$1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} \leq \chi_q(\mathcal{G}). \tag{5.4.0.2}$$

*Proof.* Let $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ be a $c$-quantum coloring of $\mathcal{G}$. Further, let $\tilde{A} = A \otimes I_{\mathcal{N}}$ and let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t$ be all the eigenvalues of $\tilde{A}$. Consider a spectral decomposition of $\tilde{A}$,

$$\tilde{A} = \sum_{i=1}^{t} \mu_i (v_i v_i^*), \text{ where } v_i \in L^2(\mathcal{M}) \otimes \mathcal{H}, \tag{5.4.0.3}$$

and write $\tilde{A} = \tilde{B} - \tilde{C}$, where

$$\tilde{B} = \sum_{\mu_i > 0} \mu_i (v_i v_i^*) \qquad \tilde{C} = \sum_{\mu_i<0} -\mu_i (v_i v_i^*). \tag{5.4.0.4}$$

Suppose $N \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then,

$$\operatorname{Tr}\left(\tilde{B}^*\tilde{B}\right) = \sum_{\mu_i>0} \mu_i^2 = \dim(\mathcal{H})\ s^+ \text{ and } \operatorname{Tr}\left(\tilde{C}^*\tilde{C}\right) = \sum_{\mu_i<0} \mu_i^2 = \dim(\mathcal{H})\ s^-. \tag{5.4.0.5}$$

Partition $\tilde{A}$ as $[\widehat{A}_{ab}]_{a,b=1}^{c}$ as in proposition 5.1.4. Similarly, let

$$\tilde{B} = [\widehat{B}_{ab}]_{a,b=1}^{c} = \sum_{a,b=1}^{c} P_a \tilde{B} P_b \text{ and } \tilde{C} = [\widehat{C}_{ab}]_{a,b=1}^{c} = \sum_{a,b=1}^{c} P_a \tilde{C} P_b.$$

Now, $B$ and $C$ are positive semidefinite matrices that are conformally partitioned. Further, $\widehat{B}_{ii} = \widehat{C}_{ii}$ since $0 = P_i \tilde{A} P_i = P_i \tilde{B} P_i - P_i \tilde{C} P_i$ for all $1 \leq i \leq c$. Also $\tilde{B}\tilde{C} = \tilde{C}\tilde{B} = 0$. So, by lemma 5.4.1 and (5.4.0.5), it follows that $\dfrac{s^+}{s^-} \leq c-1$ and $\dfrac{s^-}{s^+} \leq c-1$. Taking minimum over all $c$, we get $1 + \max\left\{\dfrac{s^+}{s^-}, \dfrac{s^-}{s^+}\right\} \leq \chi_q(\mathcal{G})$. □

## 5.5 Inertial lower bound

In this section, our goal is to generalize the following inertial bound [18] to quantum graphs:

$$1 + \max\left\{\frac{n^+}{n^-}, \frac{n^-}{n^+}\right\} \leq \chi(G), \tag{5.5.0.1}$$

where $(n^+, n^0, n^-)$ is the inertia of $G$. We begin with defining the inertia of a quantum graph:

**Definition 5.5.1.** Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be a quantum graph and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\dim(\mathcal{M})}$ denote the eigenvalues of $A$. The *inertia* of $\mathcal{G}$ is the ordered triple $(n^+, n^0, n^-)$, where $n^+$, $n^0$ and $n^-$ are the numbers of positive, zero and negative eigenvalues of $A$ including multiplicities.

**Theorem 5.5.2.** *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph with inertia $(n^+, n^0, n^-)$. Then,*

$$1 + \max\left\{\frac{n^+}{n^-}, \frac{n^-}{n^+}\right\} \leq \chi_q(\mathcal{G}). \tag{5.5.0.2}$$

*Proof.* Let $\{P_k\}_{k=1}^{c} \subseteq \mathcal{M} \otimes \mathcal{N}$ be a $c$-quantum coloring of $\mathcal{G}$. Let $U$ be defined as in lemma 5.1.3 and $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be defined as in the proof of theorem 5.4.2. Then, we have

$$\sum_{k=1}^{c-1} U^k \tilde{B} (U^*)^k - \sum_{k=1}^{c-1} U^k \tilde{C} (U^*)^k = \sum_{k=1}^{c-1} U^k \tilde{A} (U^*)^k = -\tilde{A} = \tilde{C} - \tilde{B}, \tag{5.5.0.3}$$

Note that $\tilde{B}$ and $\tilde{C}$ are positive definite operators with $rank(\tilde{B}) = n^+$ and $rank(\tilde{C}) = n^-$.

Further let

$$P^+ = \sum_{\mu_i > 0} v_i v_i^* \text{ and } P^- = \sum_{\mu_i < 0} v_i v_i^*$$

denote the orthogonal projectors onto the subspaces spanned by the eigenvectors corresponding to the positive and negative eigenvalues of $\tilde{A}$ respectively. Observe that $\tilde{B} = P^+ \tilde{A} P^+$ and $\tilde{C} = -P^- \tilde{A} P^-$. Multiplying (5.5.0.3) by $P^-$ on both sides, we obtain:

$$P^- \sum_{k=1}^{c-1} U^k \tilde{B} (U^*)^k P^- - P^- \sum_{k=1}^{c-1} U^k \tilde{C} (U^*)^k P^- = C \tag{5.5.0.4}$$

Now we use the fact that if $X, Y$ are two positive definite matrices such that $X - Y$ is positive definite, then $rank(X) \geq rank(Y)$. By applying this to (5.5.0.4), we get

$$rank(P^- \sum_{k=1}^{c-1} U^k \tilde{B} (U^*)^k P^-) \geq rank(C).$$

Recall that the rank of a sum is less than or equal to the sum of the ranks of the summands, and that the rank of a product is less than or equal to the minimum of the ranks of the factors. So, we get $(c-1)n^+ \geq n^-$. Similarly, it can be shown that $(c-1)n^- \geq n^+$. Hence, $\max \left\{ \frac{n^+}{n^-}, \frac{n^-}{n^+} \right\} \leq c - 1$. Taking minimum over all $c$, we get the desired bound. □

## 5.6 Bound using maximum eigenvalue of the Laplacian and signless Laplacian

Let $L$ and $Q$ denote the Laplacian and signless Laplacian of $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ in the sense of definition 5.3.4. Further, let $\lambda_{\max}, \theta_{\max}$ and $\gamma_{\max}$ denote the largest eigenvalue of $A, L$ and $Q$ respectively. Then

$$1 + \frac{\lambda_{\max}}{\lambda_{\max} - \gamma_{\max} + \theta_{max}} \leq \chi_q(\mathcal{G}). \tag{5.6.0.1}$$

Like the previous cases, this bound can also be shown by adapting the classical proof [38] and applying lemma 5.1.3.

## 5.7 Application to quantum complete graphs

In this section, we illustrate the tightness of these bounds in the case of quantum complete graphs [definition 2.2.8]. Let $K_{\mathcal{M}}$ denote the irreflexive quantum complete graph on $(\mathcal{M}, \psi)$. The quantum adjacency matrix in this case is given by $A = \delta^2\psi(\cdot)\mathbb{1} - I$. For $x \in \mathcal{M}$, we have

$$\begin{aligned} A(x) &= \delta^2\psi(x)\mathbb{1} - I \\ &= (\dim \mathcal{M}) \langle x, \mathbb{1}\rangle \mathbb{1} - I \\ &= (\dim \mathcal{M}) P_{\mathbb{1}}(x) - I, \end{aligned}$$

where $P_{\mathbb{1}} : \mathcal{M} \to \mathcal{M}$ denotes the orthogonal projection onto 1, given by $x \mapsto \langle x, \mathbb{1}\rangle \mathbb{1}$. Since $P_{\mathbb{1}}$ is a rank-1 projection, its spectrum is precisely $\{0, 1\}$, where $0$ has a multiplicity of $\dim(\mathcal{M}) - 1$. Using functional calculus, we get

$$\sigma(A) = \{\dim(\mathcal{M}) - 1, \;\; -1\}, \tag{5.7.0.1}$$

where $-1$ has a multiplicity of $\dim(\mathcal{M}) - 1$. Similarly, we get

$$\sigma(Q) = \{2\dim(\mathcal{M}) - 2, \; \dim(\mathcal{M}) - 2\}, \tag{5.7.0.2}$$

where $\dim(\mathcal{M}) - 2$ has a multiplicity of $\dim(\mathcal{M}) - 1$, and

$$\sigma(L) = \{\dim(\mathcal{M}), 0\}, \tag{5.7.0.3}$$

where $\dim(\mathcal{M})$ has a multiplicity of $\dim(\mathcal{M}) - 1$.

Thus, for an irreflexive quantum complete graph, we have:

- $\lambda_{\max} = \dim \mathcal{M} - 1$, $\lambda_{\min} = -1$
- $\gamma_{\max} = 2\dim(\mathcal{M}) - 2$, $\gamma_{\min} = \dim \mathcal{M} - 2$
- $\theta_{\max} = \dim \mathcal{M}$

- $s^+ = (\dim(\mathcal{M}) - 1)^2$, $s^- = \dim(\mathcal{M}) - 1$
- $n^+ = 1$, $n^- = \dim(\mathcal{M}) - 1$
- $2m = \dim(\mathcal{M})^2 - \dim(\mathcal{M})$

On applying these to (5.0.0.2), we see that all the five spectral bounds give the same result, namely:

$$\dim(\mathcal{M}) \leq \chi_q(K_{\mathcal{M}}). \tag{5.7.0.4}$$

The reverse inequality $\chi_q(K_{\mathcal{M}}) \leq \dim(\mathcal{M})$ was proved in theorem 4.2.4, and $\chi_q(K_{\mathcal{M}}) = \dim(\mathcal{M})$. So, we conclude that all the bounds in (5.0.0.2) are tight in the case of quantum complete graphs.

# 6. Summary

We have introduced a quantum input-classical output non-local game that captures the coloring problem for quantum graphs. We showed that the coloring models arising in this context are a special case of D.Stahlke's entanglement assisted coloring of non-commutative graphs. We also proved that every quantum graph has a finite quantum coloring and is four-colorable in the algebraic model. Further, we developed a combinatorial characterization of quantum graph coloring and obtained lower bounds for the chromatic numbers of quantum graphs using the spectrum of the quantum adjacency operator.

**Future work**: We believe that quantum graph theory is a promising field of study. Below, we list some research questions for future work.

1. Motivated by our work on quantum chromatic numbers $(\chi_t)$, one could ask what is the quantum independence number $(\alpha_t)$ of a quantum graph and how it relates to $(\chi_t)$? In particular, it would be interesting to investigate if there is a $t$-Lovász type inequality for classical graphs and for general quantum graphs, where $t \in \{q, qs, qa, qc, ns\}$.

2. Finding two graphs that are $qc$-isomorphic but not $qa$-isomorphic would give another counterexample to Connes Embedding Problem. In particular, finding quantum graphs with the analogous property would be very interesting, and might be even easier to do since quantum graphs are more flexible than classical graphs.

3. It would also be interesting to find bounds that exhibit a separation between the different chromatic numbers of quantum graphs. Alternatively, investigating examples of quantum graphs that show a separation between these spectral bounds would also be helpful.

4. We conjecture that our quantum chromatic numbers are *equivalent* to Stahlke's entanglement-assisted chromatic numbers. It maybe possible to prove this using a recent dilation result from [3].

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