SECONDARY HIGHER INVARIANTS AND CYCLIC COHOMOLOGY FOR GROUPS OF POLYNOMIAL GROWTH

A Dissertation

by

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ABSTRACT

Given a complete n-dimensional Riemannian manifold X admitting a suitable action by a group Γ, each Γ-equivariant elliptic differential operator D gives rise to a higher index class $\mathrm{Ind}_\Gamma(D)$. A central strength of this algebraic invariant lies in the ability to encode important "symmetries" of X, with its construction involving differential geometry, K-theory, functional analysis, and C^*-algebraic notions. When a primary invariant such as $\mathrm{Ind}_\Gamma(D)$ is trivial, even finer geometrical and topological information can be obtained through the analysis of naturally occurring secondary higher invariants. An intrinsic issue is the general difficulty in computation of such invariants; generally speaking, the efficacy of the tools and techniques applied in order to reduce this computability difficulty depend crucially on the structure of the group Γ.

The content of this thesis concerns higher index theory in the setting of complete closed spin manifolds M with finitely generated and virtually nilpotent fundamental groups. A common approach is to pair the given secondary higher invariant with cyclic cohomology classes associated to the group algebra $\mathbb{C}\Gamma$. The immediate question which arises is determining when this pairing can be rigorously well-defined; one aspect to be addressed is purely algebraic topological in nature, while another main difficulty involves norm estimates in functional calculus and subtle convergence issues. The first contribution of the thesis is to show that with respect to a virtually nilpotent group Γ every cyclic cocycle class on $\mathbb{C}\Gamma$ has a representative of polynomial growth. This cohomological growth condition is essential to proving that every cyclic cocycle class extends continuously from $\mathbb{C}\Gamma$ to certain geometric C^*-algebras, and provides the foundation for showing that under certain curvature assumptions and for $\pi_1(M)$ virtually nilpotent, the explicit integral formula describing a higher analogue of Lott's delocalized eta invariant converges absolutely and is well-defined. We also use a determinant map construction of de la Harpe and Skandalis– adapted by Xie and Yu– to prove that if Γ is of polynomial growth then there is a well defined pairing between delocalized cyclic cocycles and K-theory classes of C^*-algebraic secondary higher invariants. When this K-theory class is that of a higher rho invariant of an invertible differential operator we show this pairing is precisely the aforementioned higher analogue of Lott's delocalized eta invariant. As an application of this equivalence we provide a delocalized higher Atiyah-Patodi-Singer index theorem for compact spin manifolds with boundary, equipped with a positive scalar metric.

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1. INTRODUCTION*

Given a Fredholm operator $T : X \longrightarrow Y$ between two Banach spaces the classic index theory for Fredholm operators provides an integer valued analytic index

$$\mathrm{ind}(T) = \dim\ker(T) - \dim\mathrm{coker}(T)$$

which is invariant under perturbations of T by compact operators. The non-vanishing of $\mathrm{ind}(T)$ is thus an obstruction to invertibility of a Fredholm operator T. When T is an elliptic differential operator with X and Y smooth vector bundles over a smooth closed manifold M the work of Atiyah and Singer [4] showed the equivalence between $\mathrm{ind}(T)$ and the often more tractable topological index (see (4.51) of Section 4.4).

Let M be a complete n-dimensional Riemannian manifold with a discrete group G acting on it properly and cocompactly by isometries. Each G-equivariant elliptic differential operator D on M gives rise to a higher index class $\mathrm{Ind}_G(D)$ in the K-theory group $K_n(C_r^*(G))$ of the reduced group C^*-algebra $C_r^*(G)$. Higher index classes are invariant under homotopy, and being an obstruction to the invertibility of D, are often referred to as primary invariants. Higher index theory provides a far-reaching generalization of the Fredholm index by taking into consideration the symmetries of the underlying spaces; in particular, if M is a complete compact Riemannian manifold with an associated Dirac-type operator D, a higher index theory intrinsically involves the fundamental group $\pi_1(M)$. The higher index theory plays a fundamental role in the studies of many important open problems having relations to geometry and topology, such as the Novikov conjecture, the Baum-Connes conjecture, and the Gromov-Lawson-Rosenberg conjecture.

A secondary higher invariant– so called due to its natural appearance upon the vanishing

*Part of this chapter has been reprinted with permission from "S.A.K.A. John. Secondary Higher Invariants and Cyclic Cohomology for Groups of Polynomial Growth. *Journal of Noncommutative Geometry*, (accepted), 2021 " of a primary invariant such as $\mathrm{Ind}_G(D)$– was developed by Lott [31] within the framework of noncommutative differential forms, for manifolds with virtually nilpotent fundamental groups and D invertible. Lott's work was heavily inspired by the work of Bismut and Cheeger on eta forms [8], which naturally arise in the index theory for families of manifolds with boundary [2]. Lott's higher eta invariant, despite being defined by an explicit integral formula of noncommutative differential forms is unfortunately difficult to compute in general. To reduce the computability difficulty and make this second higher invariant more applicable to problems in geometry and topology one needs to pair it with the cyclic cohomology of the group algebra. The delocalized eta invariant of Lott [32] can be formally thought of as precisely such a pairing with respect to traces (see the formula (4.2)).

In Definition 2.22 and Definition 2.23 we provide a precise definition of the cyclic cohomology groups $HC^*(\mathbb{C}G)$ and their delocalized counterparts $HC^*(\mathbb{C}G, \mathrm{cl}(\gamma))$ with respect to the group algebra $\mathbb{C}G$ and conjugacy classes $\mathrm{cl}(\gamma)$. Given a delocalized cyclic cocycle class $[\varphi_\gamma] \in HC^*(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma))$ of any degree, where the conjugacy class $\mathrm{cl}(\gamma)$ is not trivial, a higher analogue of Lott's delocalized eta invariant $\eta_{[\varphi_\gamma]}(\widetilde{D})$ is given in Definition 4.2; the explicit formula for $\eta_{[\varphi_\gamma]}(\widetilde{D})$ is described in terms of the transgression formula for Connes-Chern character [12, 15]. The natural problem which arises is in determining when this *delocalized higher eta invariant* can actually be rigorously well-defined and involves some subtle convergence issues which depend crucially on the growth conditions of the cyclic cocycles; in addition, it is essential to prove that the pairing is independent of the choice of representative of any given cocycle class. As a necessary prelude, the first half of this thesis is concerned with establishing the first main result (see Corollary 3.7), namely that for a virtually nilpotent group Γ every delocalized cyclic cocycle class on the group algebra has a representative of polynomial growth.

An analogous result was proven by Chen, Wang, Xie, and Yu [11] under the assumptions of Γ being a word hyperbolic group; while some of their techniques carry over without much fuss, the overall proofs require distinctly separate ingredients. By Z_γ denote the centralizer of

γ, by $\gamma^{\mathbb{Z}}$ denote the cyclic group generated by γ, and by N_γ denote the quotient group $Z_\gamma/\gamma^{\mathbb{Z}}$. It is not hard to show that N_γ is virtually nilpotent if Γ is; moreover, it is proven by Ji [23] that virtually nilpotent groups are of polynomial cohomology, hence the group cohomology $H^*(N_\gamma, \mathbb{C})$ is polynomially bounded. The essential ingredient in proving Corollary 3.7 is thus the construction of an explicit morphism to delocalized cyclic cohomology which preserves polynomial growth. Using the long exact sequence of periodic cyclic cohomology involving the (delocalized) Connes periodicity operator [12] and combining this with a cohomological dimension result of Ji [24], we prove that $H^*(N_\gamma, \mathbb{C})$ does not contribute to the delocalized cyclic cohomology of $\mathbb{C}\Gamma$ if γ is of infinite order. We are thus able to construct a rational isomorphism between cohomology groups of a certain augmented complex of delocalized cyclic cocycles and the group cohomology $H^*(Z_\gamma, \mathbb{C})$. The main technical difficulty lies in exploiting the equivalence between singular cohomology $H^*(B\Gamma, \mathbb{C})$ of universal classifying spaces and group cohomology $H^*(\Gamma, \mathbb{C})$ in order to prove that the relevant rational isomorphisms have geometrically significant analogues which explicitly can be shown to preserve polynomial growth of cocycle classes.

We are thus able to show that whenever M possesses a finitely generated and virtually nilpotent fundamental group the higher analogue pairing of Lott's higher eta invariant with (delocalized) cyclic cocycles is well-defined, under the conditions that the Dirac operator on $\widetilde{M}$ is invertible– or more generally has a spectral gap at zero.

Theorem 1.1. *Let M be a closed odd-dimensional spin manifold equipped with a positive scalar metric g, and fundamental group which is is virtually nilpotent. Denoting by $\widetilde{D}$ the associated lift of the Dirac operator D to the universal cover $\widetilde{M}$, the higher delocalized eta invariant $\eta_{[\varphi_\gamma]}(\widetilde{D})$ converges absolutely for every $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma))$. Moreover, if*

$$S_\gamma^* : HC^{2m}(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma)) \longrightarrow HC^{2m+2}(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma))$$

denotes the delocalized Connes periodicity operator, then $\eta_{[S_\gamma\varphi_\gamma]}(\widetilde{D}) = \eta_{[\varphi_\gamma]}(\widetilde{D})$.

When the higher index class of an operator is trivial– given a specific trivialization– a secondary index theoretic invariant naturally arises through a C^*-algebraic approach. For example, consider the associated Dirac operator on the universal covering $\widetilde{M}$ of a closed, n-dimensional spin manifold M equipped with a positive scalar curvature metric g. The Lichnerowicz formula (see (4.1) of Section 4.1) asserts that the Dirac operator on $\widetilde{M}$ is invertible [29], and so $\mathrm{Ind}_G(D)$ must necessarily be trivial. In this case, there is a natural C^*-algebraic secondary invariant $\rho(\widetilde{D}, \widetilde{g})$ introduced by Higson and Roe [20, 21, 22], called the higher rho invariant (there is a essentially similar invariant originally defined by Weinberger [44]). This higher rho invariant is an obstruction to the inverse of the Dirac operator being local, and describes a class belonging to the group $K_n(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$, where $\pi_1(M)$ is the fundamental group of M. As mentioned before, such a secondary index theoretic invariant often plays an important role in problems in geometry and topology (cf. [45, 46, 49]). The precise description of the geometric C^*-algebra $C^*_{L,0}(\widetilde{M})^{\pi_1(M)}$ is provided in Definition 2.15, and the particular construction of the higher rho invariant is given at the beginning of Section 4.2. In the case that $\pi_1(M)$ is a virtually nilpotent group we provide– using the construction of the determinant map of [50]– in Section 4.2 an explicit formula (see Definition 4.11)for a pairing of C^*-algebraic secondary invariants and delocalized cyclic cocycles of the group algebra is realized. Moreover, in the particular instance that $[u] \in K_1(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$ is the K-theory class of the higher rho invariant $\rho(\widetilde{D}, \widetilde{g})$, then the pairing is given explicitly in terms of the higher delocalized eta invariant $\eta_{[\varphi_\gamma]}(\widetilde{D})$.

Theorem 1.2. *Let M be a closed odd-dimensional spin manifold with virtually nilpotent fundamental group, then every delocalized cyclic cocycle $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma))$ induces a natural map*

$$\tau_{[\varphi_\gamma]} : K_1(C_{L,0}(\widetilde{M})^{\pi_1(M)}) \longrightarrow \mathbb{C}$$

If M has positive scalar curvature metric g then $\tau_{[\varphi_\gamma]}(u)$ converges absolutely. When $[u] =$

$\rho(\widetilde{D},\widetilde{g})$ *is the* K*-theory class of the higher rho invariant there is an equivalence*

$$\tau_{[\varphi_\gamma]}(\rho(\widetilde{D},\widetilde{g})) = (-1)^m \eta_{[\varphi_\gamma]}(\widetilde{D})$$

The above theorem holds in the more general case that for the Dirac operator D on M, the associated lift $\widetilde{D}$ to the universal cover $\widetilde{M}$ is invertible. Showing that the map $\tau_{[\varphi_\gamma]}$ is well defined occupies the majority of Section 4.2; in particular, the extension of φ_γ from the group algebra to the localization algebra requires the existence of a certain smooth dense subalgebra of $C_r^*(\pi_1(M))$ introduced by Connes and Moscovici [16]. In [50], Xie and Yu established such a pairing between delocalized cyclic cocycles of degree $m = 0$– delocalized traces– and classes $[u]$ belonging to $K_1(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$, under the assumption that the relevant conjugacy class has polynomial growth. Later, in [11], under the assumption that $\pi_1(M)$ is a hyperbolic group, this construction was extended to allow for a pairing between delocalized cyclic cocycles of all degrees and the K-theory classes $[u] \in K_n(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$. In the hyperbolic case, convergence of $\tau_{[\varphi_\gamma]}$ relies on the properties of Puschnigg's [39] smooth dense subalgebra in an essential way.

The map $\tau_{[\varphi_\gamma]}$ allows for a constructive and explicit approach to a higher delocalized Atiyah-Patodi-Singer index theorem. In Section 4.4 we prove a direct relationship between pairings of K-theory classes $[u] \in K_n(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$ with $\tau_{[\varphi_\gamma]}$, and the pairing of classes $\partial[p]$ with respect to the delocalized Connes-Chern character map [12, 15] (see (4.52) for the explicit expression used here), where p is an idempotent and

$$\partial : K_n(C^*(\widetilde{M})^{\pi_1(M)}) \longrightarrow K_{n-1}(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$$

is the usual K-theory connecting map. Combined with Theorem 1.2 this provides the following version of a higher delocalized Atiyah-Patodi-Singer index theorem.

Theorem 1.3. *Let* W *be a compact spin manifold with boundary, equipped with a scalar curvature metric* g *which is positive on* ∂W*, and fundamental group which is virtually nilpotent.*

Denote by $\widetilde{D}_W$ and $\widetilde{D}_{\partial W}$ the lifted Dirac operators on $\widetilde{W}$ and its boundary, respectively.

$$\mathsf{ch}_{[\varphi_\gamma]}\left(\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W)\right) = \frac{(-1)^{m+1}}{2}\eta_{[\varphi_\gamma]}(\widetilde{D}_{\partial W})$$

for any $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\pi_1(M), \mathrm{cl}(\gamma))$, where $\mathsf{ch}_{[\varphi_\gamma]}$ is the delocalized Connes-Chern character map which pairs cyclic cocycles with the K-theoretic index class.

There have been various versions of a higher Atiyah-Patodi-Singer theorem in the literature, such as [26, 27, 17] and [43]. The form of the this result strongly mirrors that conjectured by Lott [31, Conjecture 1], and is essentially a general case of that proven by Xie and Yu [50, Proposition 5.3] for zero dimensional cyclic cocycles. In Section 4.4 we provide the basic background of the original APS index theorem, and show how the above theorem is specifically related to it. See the discussion following [11, Theorem 7.3] for more details on the relationships and differences of the above theorem with other existing results of higher APS index theorems.

This thesis is organized as follows. Section 2.1 provides a brief review of topological K-theory of C^*-algebras and construction of the index map. In Section 2.2 we provide the properties of the geometric C^*-algebras which shall be used throughout, as well as detail the construction of important smooth dense sub-algebras. Section 2.3 is concerned primarily with providing the definition of cyclic cohomology and detailing the relationship between this and cohomology for groups; in addition we recall an essential construction for explicit representative of cyclic cocycle classes. In Section 3.1 we review the long exact sequence of periodic cyclic cohomology involving the (delocalized) Connes periodicity operator (see Definition 3.2 and (3.4)); combining this with a cohomological dimension result we prove a necessary torsion argument. We are thus able to construct a rational isomorphism between cohomology groups of a certain complex of cyclic cocycles and group cohomology of a particular subgroup of $\pi_1(M)$. Using the universal classifying description of group cohomology and the previous rational isomorphism, the entirety of Section 3.2 is devoted to proving that every delocalized cyclic cocycle has a representative of polynomial growth. In Section 4.1, given a delocalized cyclic cocycle of polynomial growth we define a higher analogue of Lott's delocalized eta invariant and prove it converges for invertible elliptic operators. In Section 4.2, we first review a construction of Higson and Roe's higher rho invariant as an explicit K-theory class. We provide an explicit formula for the pairing between C^*-algebraic secondary invariants and delocalized cyclic cocycles of the group algebra for virtually nilpotent groups, and prove it is well-defined. In particular, in the case that the secondary invariant is a K-theoretic higher rho invariant of an invertible elliptic differential operator, we show in Section 4.3 that this pairing is precisely the higher delocalized eta invariant of the given operator. In Section 4.4, we use the determinant map of the previous section to determine a pairing between delocalized cyclic cocycles and C^*-algebraic Atiyah-Patodi-Singer index classes for manifolds with boundary, when the fundamental group of the given manifold is virtually nilpotent.

2. PRELIMINARIES*

In all that follows we will take M to be a closed odd-dimensional spin manifold, which is equipped with a positive scalar metric g. By D we denote the Dirac operator associated to M, and analogously by $\widetilde{D}$ the associated lift to the universal cover $\widetilde{M}$. By $\Gamma = \pi_1(M)$ we refer to a countable discrete finitely generated group which is also the fundamental group of M. Given $\gamma \in \Gamma$ the centralizer of γ will be denoted $Z_\Gamma(\gamma)$, or if there is no confusion as to the group Γ, by Z_γ; likewise, if $\gamma^{\mathbb{Z}}$ is the cyclic group generated by γ, then the quotient group $Z_\gamma/\gamma^{\mathbb{Z}}$ will be denoted by N_γ. By $\mathbb{C}\Gamma$ and $\mathbb{Z}\Gamma$ we mean the group algebra with complex coefficients and the group ring with integer coefficients, respectively.

We recall that a finitely generated discrete group Γ comes equipped with a length function l_S with respect to some given symmetric generating set $S \subset \Gamma$.

$$l_S(g) = \min\{c \in \mathbb{N} : \exists s_1, \ldots, s_c \in S, s_1 \cdots s_c = g\} \tag{2.1}$$

There exists an associated word metric $d_S(g,h) = ||g^{-1}h||; = l_S(gh)$ which is left-invariant with respect to the group action. More importantly, since the metric spaces (Γ, S) and (Γ, T) are quasi-isomorphic for any choice of generating sets S and T, we are able to ignore this choice when dealing with the word metric (or length function); henceforth we will merely refer to *the* length function l_Γ or *the* word metric d_Γ. Thanks to the work of Gromov [18] we will also use the terminology “virtually nilpotent" and “group of polynomial growth" interchangeably throughout, depending on the circumstance. Recall that a virtually nilpotent group is one which possesses a finite index normal nilpotent subgroup, and that Γ is of

*Part of this chapter has been reprinted with permission from “S.A.K.A. John. Secondary Higher Invariants and Cyclic Cohomology for Groups of Polynomial Growth. *Journal of Noncommutative Geometry*, (accepted), 2021" polynomial growth if there exists positive integer constants C_0 and m such that

$$|\{g \in \Gamma : ||g|| \leq n\}| \leq C_0(n+1)^m \quad \forall n \in \mathbb{N} \tag{2.2}$$

2.1 Topological K-theory

The material provided in this section is all fairly standard and can be found in various forms in texts such as that of Atiyah [1], Willett and Yu [47], Park [36], and Blackadder [9]. In 1957 Alexander Grothendieck used the notion of isomorphism classes (*Klasse*) of coherent sheaves on an algebraic variety X in order to formulate his Grothendieck-Riemann-Roch theorem. From the Abelian monoid formed by the sheaves under the direct sum operation, one then defines a group using isomorphism classes of sheaves as generators of the group, subject to a relation that identifies any extension of two sheaves with their sum. The resulting *Grothendieck group* $K_0(X)$ is the universal group completion of the monoid, which can be concretely realized through the formal adding of inverses. The topological analogue to this algebraic notion was developed soon after by Atiyah and Hirzebruch who developed a K-theory based on vectors bundles over compact Hausdorff spaces, the resulting extraordinary cohomology theory being known as topological K-theory.

Definition 2.1. For a compact Hausdorff space X, let $\mathsf{Vect}_{\mathbb{C}}(X)$ denote the commutative monoid of isomorphism classes of finite dimensional complex vector bundles over X, with binary operation being the Whitney sum. We define $K^0_{top}(X) = K_0(\mathsf{Vect}_{\mathbb{C}}(X))$ to be the associated Grothendieck group completion.

This thesis is concerned with the question of complex vector bundles only, though one can transfer much of the basic theory verbatim to the case of real or symplectic bundles. Moreover, the operation of tensoring vector bundles gives $K^0_{top}(X)$ a commutative ring structure, and thus K^0_{top} describes a contravariant functor from the homotopy category of topological spaces to the category of commutative rings.

Definition 2.2. The reduced K-theory group $\widetilde{K}^0_{top}(X)$ is the Grothendieck completion arising from stably isomorphic vector bundle classes, or alternatively as the kernel of the map $i^* : K^0_{top}(X) \longrightarrow K^0_{top}(\{x_0\}) \cong \mathbb{Z}$ induced by the inclusion $i : x_0 \hookrightarrow X$ of a basepoint.

The higher reduced K-groups can thus be defined according to $\widetilde{K}^0_{top}(S^n X)$, where $S^n X$ is the n'th reduced suspension (this coincides with the unreduced suspension for CW complexes) of X. The celebrated periodicity theorem of Bott extends this definition to the positive reduced K-groups by asserting that $\widetilde{K}^{-n}_{top}(X) \cong \widetilde{K}^{-n-2}_{top}(X)$. Since $K^{-n}_{top}(X)$ can be simply defined as $\widetilde{K}^{-n}_{top}(X \sqcup \{x_0\})$, this further shows that for the case of complex coefficients there exist only two distinct groups: $K^0_{top}(X)$ and $K^1_{top}(X)$.

We now turn to recalling some basic notions concerning the K-theory of C^*-algebras, by first extending the topological K-theory to that of commutative C^*-algebras. The Swan-Serre theorem states that there is a one-to-one correspondence between (finite dimensional) vector bundles over X and (finitely generated) projective modules of $C(X)$, the algebra of complex-valued continuous functions on X. Passing to isomorphism classes, there is a natural definition for the topological K-theory groups of $C(X)$.

$$K_n^{top}(C(X)) := K_n(\mathsf{Proj}(\mathsf{C}(\mathsf{X}))) \cong K_n(\mathsf{Vect}_{\mathbb{C}}(X)) := K^n_{top}(X)$$

The famous theorem of Gelfand asserts that the category of compact Hausdorff topological spaces is contravariantly equivalent [1] to the category of commutative C^*-algebras, and the equivalence functor $\hat{\varphi}$ is called the Gelfand-transform. In particular, if $\mathcal{A}$ is a commutative C^*-algebra, then $\hat{\varphi} : \mathcal{A} \longrightarrow C(X)$ is an isomorphism for some choice of compact Hausdorff space X, and allows us to simply define $K_n(\mathcal{A}) := K_n^{top}(\hat{\varphi}(\mathcal{A}))$.

Definition 2.3. Let $\mathcal{A}$ be a C^*-algebra and $M_n(\mathcal{A})$ be the $n \times n$ matrix ring with entries in $\mathcal{A}$. There is an embedding of $M_n(\mathcal{A})$ in $M_{n+1}(\mathcal{A})$ given by $a_n \mapsto \left(\begin{smallmatrix} a_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and generally the matrix direct sum defines a binary operation $M_n(\mathcal{A}) \times M_k(\mathcal{A}) \longrightarrow M_{n+k}(\mathcal{A})$. We thus

[1] This contravariant relationship is the reason why we have adopted a reversal of subscript and superscript for the topological K-theory of $C(X)$ as opposed to that of X itself.

define the direct limit object

$$M_\infty(\mathcal{A}) = \varinjlim_{n\in\mathbb{N}} M_n(\mathcal{A}) = \left(\bigsqcup_{n\in\mathbb{N}} M_n(\mathcal{A})\right) / \sim$$

where $a_k \sim a_i$ if and only if there exists m such that $a_k \oplus 0_{m-k} = a_i \oplus 0_{m-i} \in M_m(\mathcal{A})$.

Remark 2.4. *There is a one-to-one correspondence between the collection of isomorphism classes of finitely generated projective modules over the C^*-algebra $\mathcal{A}$, and the collection of Murray-von Neumann (or homotopy) equivalent idempotents in $M_\infty(\mathcal{A})$.*

The above remark, coupled with the Swan-Serre theorem, provides an alternative definition of $K_0(\mathcal{A})$ in terms of equivalence classes of idempotents, and thus suggest a concrete method of expanding the notion of K_0 to non-commutative C^*-algebras.

Definition 2.5. Let $\mathcal{A}$ be any unital C^*-algebra and $\mathsf{Idem}(M_\infty(\mathcal{A}))$ denote the Abelian monoid formed by the collection of all equivalence classes of idempotents in $M_\infty(\mathcal{A})$ with direct sum as a binary operation. The Grothendieck group completion of $\mathsf{Idem}(M_\infty(\mathcal{A}))$ is the K-theory group $K_0(\mathcal{A})$.

Definition 2.6. Let $\mathcal{A}$ be a C^*-algebra and $S\mathcal{A}$ the suspension C^*-algebra defined as $S\mathcal{A} = \{f \in C([0,1],\mathcal{A}) \mid f(0) = f(1) = 0\}$. The group $K_1(\mathcal{A})$ is defined to be $K_0(S\mathcal{A})$, and similarly $K_n(\mathcal{A}) = K_0(S^n\mathcal{A})$, where $S^n\mathcal{A}$ is obtained by taking the n-fold suspension of $\mathcal{A}$.

Mirroring Definition 2.3, there is an alternative way of constructing K_1 through invertible matrix elements and which is often more useful for concrete computations.

Definition 2.7. Let $\mathrm{GL}_n(\mathcal{A})$ denote the $n \times n$ matrix group of invertibles with entries in a unital C^*-algebra $\mathcal{A}$, and $\mathrm{GL}_n(\mathcal{A})_0$ denote the component containing the identity I_n. There is an embedding of $\mathrm{GL}_n(\mathcal{A})$ in $\mathrm{GL}_{n+1}(\mathcal{A})$ given by $u_n \mapsto \left(\begin{smallmatrix} u_n & 0 \\ 0 & 1 \end{smallmatrix}\right)$; generally the matrix direct sum defines a binary operation $\mathrm{GL}_n(\mathcal{A}) \times \mathrm{GL}_k(\mathcal{A}) \longrightarrow \mathrm{GL}_{n+k}(\mathcal{A})$. We thus define–

analogously for $\mathrm{GL}_\infty(\mathcal{A})_0$– the direct limit object

$$\mathrm{GL}_\infty(\mathcal{A}) = \varinjlim_{n\in\mathbb{N}} \mathrm{GL}_n(\mathcal{A}) = \left(\bigsqcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathcal{A})\right) / \sim$$

where $u_k \sim u_i$ if and only if there exists m such that $u_k \oplus I_{m-k} = u_i \oplus I_{m-i} \in \mathrm{GL}_m(\mathcal{A})$.

Proposition 2.8. *There is a natural isomorphism between $K_0(S\mathcal{A})$ and the quotient group $\mathrm{GL}_\infty(\mathcal{A})/\mathrm{GL}_\infty(\mathcal{A})_0$.*

K-theory satisfies all the usual properties one would expect from a cohomology theory; particularly, every short exact sequence $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ of C^*-algebras gives rise to a long exact sequence of K-theory groups. It is, however, important to note that the Bott periodicity theorem asserts that the Bott map $\beta : K_0(\mathcal{A}) \longrightarrow K_2(\mathcal{A})$ is an isomorphism for every C^*-algebra. Hence, we obtain a six-term long exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{C}) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_2} \\ K_1(\mathcal{C}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{I}) \end{array} \tag{2.3}$$

with connecting maps ∂_i. Here $\partial_2 := \mathrm{Ind}_2 \circ \beta$ and $\partial_1 := \mathrm{Ind}$ is the *index map* referred to in the introduction; note that $\mathrm{Ind}_2 : K_2(\mathcal{C}) \longrightarrow K_1(\mathcal{I})$ is being understood as the index map $\mathrm{Ind} : K_1(S\mathcal{C}) \longrightarrow K_0(S\mathcal{I})$. Using the definition $K_1(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})/\mathrm{GL}_\infty(\mathcal{A})_0$, we can provide a concrete description of the index map in terms of invertibles and projections.

Definition 2.9. Consider the short exact sequence $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ of unital C^*-algebras. Given a cohomology class $[u] \in K_1(\mathcal{C})$ with $u \in \mathrm{GL}_n(\mathcal{C})$, there exists an invertible $v \in \mathrm{GL}_\infty(\mathcal{B})$ such that v is a lift of the element $\left(\begin{smallmatrix} 0_n & -u^{-1} \\ u & 0_n \end{smallmatrix}\right) \in \mathrm{GL}_{2n}(\mathcal{C})$. We define $\mathrm{Ind}[u] \in K_0(\mathcal{I})$ to be the cohomology class of a formal difference of idempotents.

$$\mathrm{Ind}[u] := \left[v \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix} v^{-1} \right] - \left[\begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix} \right]$$

If $\mathcal{I}$ is an ideal of $\mathcal{B}$ and $\mathcal{C}$ is the quotient C^*-algebra, then the lift element v can be made very explicit in terms of u. There are also important representations for the index map which are geometrical in nature or involve use of functional calculus (see [47, Section 2.8]).

2.2 Geometric C^*-algebras and Smooth Dense Sub-algebras

Let X be a proper metric space, and $C_0(X)$ the algebra of continuous functions on X which vanish at infinity. An X-module H_X is a separable Hilbert space equipped with a $*$-representation $\pi : C_0(X) \longrightarrow \mathcal{B}(H_X)$ into the algebra of bounded operators on H_X, and is called non-degenerate if the $*$-representation of $C_0(X)$ is non-degenerate. If no non-zero function $f \in C_0(X)$ acts a compact operator under this $*$-representation, then we call H_X a standard X-module.

Definition 2.10. Recall that an operator T acting on a Hilbert space $\mathcal{H}$ belongs to the algebra of compact operators $\mathcal{K} \subset \mathcal{B}(\mathcal{H})$ if the image under T of every bounded subset has compact closure.

(i) Let $T \in \mathcal{B}(H_X)$ be a bounded linear operator acting on H_X, then T is locally compact if for all $f \in C_0(X)$ both fT and Tf are compact operators. We similarly call T pseudo-local if the weaker condition, $[T, f] = TF - fT$ is a compact operator for all $f \in C_0(X)$, is satisfied.

(ii) Again assume that T belongs to $\mathcal{B}(H_X)$; the propagation of T is defined to be

$$\sup\{d(x, y) : (x, y) \in \mathsf{Supp}(T)\}$$

where $\mathsf{Supp}(T)$ denotes the support of T, which is the set

$$\{(x, y) \in X \times X : \exists f, g \in C_0(X) \text{ such that } gTf = 0 \text{ and } f(x) \neq 0, g(y) \neq 0\}^c$$

If we further impose that H_X is a standard and non-degenerate X-module, then there exist important constructions of certain geometric C^*-algebras. The first two of these, described in Definition 2.11 were introduced by Roe in [40], and the coarse homotopy invariance of their K-theory was subsequently proven by Higson and Roe [19].

Definition 2.11. The C^*-algebra generated by all locally compact operators with finite propagation in $\mathcal{B}(H_X)$, is the Roe algebra of X and is denoted by $C^*(X)$. If we instead consider the C^*-algebra generated by all pseudo-local operators with finite propagation in $\mathcal{B}(H_X)$, then we obtain a related algebra $D^*(X)$. In fact, $D^*(X)$ is a subalgebra of the multiplier algebra $\mathcal{M}(C^*(X))$– which is the largest unital C^*-algebra containing $C^*(X)$ as an ideal.

Definition 2.12. Let $\mathsf{prop}(T)$ denote the propagation of an operator $T \in \mathcal{B}(H_X)$. The localization algebras $C_L^*(X)$ and $D_L^*(X)$ introduced by Yu [51] are defined as the C^*-algebras generated by S_1 and S_2 respectively, where f is bounded and uniformly norm-continuous

$$S_1 = \Big\{ f : [0, \infty) \longrightarrow C^*(X) | \lim_{t \longrightarrow \infty} \mathsf{prop}(f(t)) = 0 \Big\}$$

$$S_2 = \Big\{ f : [0, \infty) \longrightarrow D^*(X) | \lim_{t \longrightarrow \infty} \mathsf{prop}(f(t)) = 0 \Big\}$$

Once again $D_L^*(X)$ is a subalgebra of the multiplier algebra $\mathcal{M}(C_L^*(X))$. The kernel of the evaluation map $\mathsf{ev} : C_L^*(X) \longrightarrow C^*(X)$ defined by $\mathsf{ev}(f) = f(0)$ is an ideal of $C_L^*(X)$, and is itself a C^*-algebra which we denote by $C_{L,0}^*(X)$. Analogously we also define the C^*-algebra $D_{L,0}^*(X)$ as the kernel of $\mathsf{ev} : D_L^*(X) \longrightarrow D^*(X)$.

It follows that the Roe algebra and its localization fit into a short exact sequence– analogously for $D^*(X)$– which give rise to a six term K-theoretic long exact sequence with connecting map ∂, and for which $i = 0, 1 (\text{mod } 2)$ by Bott periodicity.

$$0 \longrightarrow C_{L,0}^*(X) \hookrightarrow C_L^*(X) \xrightarrow{\mathsf{ev}} C^*(X) \longrightarrow 0 \tag{2.4}$$

$$\begin{array}{ccccc} K_i(C^*_{L,0}(X)) & \longrightarrow & K_i(C^*_L(X)) & \longrightarrow & K_i(C^*(X)) \\ \partial\big\uparrow & & & & \big\downarrow\partial \\ K_{i-1}(C^*(X)) & \longleftarrow & K_{i-1}(C^*_L(X)) & \longleftarrow & K_{i-1}(C^*_{L,0}(X)) \end{array} \tag{2.5}$$

Assuming that a group G acts properly and cocompactly on X by isometries, we can equip H_X with a covariant unitary representation of G, which we will denote by ϖ. Explicitly, if $g \in G, f \in C_0(X)$ and $v \in H_X$

$$\varpi(g)(\pi(f)v) = \pi(f^g)(\varpi(g)v)$$

where $f^g(x) = f(g^{-1}x)$. We call the system (H_X, π, ϖ) a covariant system.

Definition 2.13. Suppose that H_X is a standard and non-degenerate X-module, and G acts on X properly and cocompactly. Moreover, for each $x \in X$ the action of the stabilizer group G_x on H_X is isomorphic to the action of G_x on $l^2(G_x) \otimes \mathcal{H}$ for some infinite dimensional Hilbert space $\mathcal{H}$, where G_x acts trivially on $\mathcal{H}$ and by translations on $l^2(G_x)$. Under these conditions a covariant system (H_X, π, ϖ) is called *admissible*.

If it is not necessary to emphasize the representations we shall simply refer to the admissible system (H_X, π, ϖ) by H_X, and describe it as an admissible (X, G)-module.

Remark 2.14. *For every locally compact metric space X which admits a proper and cocompact isometric action of G, there exists an admissible covariant system (H_X, π, ϖ).*

Definition 2.15. Consider a locally compact metric space X which admits a proper and cocompact isometric action of G, and fix some admissible (X, G)-module H_X. The G-equivariant Roe algebra $C^*(X)^G$ is the completion in $\mathcal{B}(H_X)$ of the $*$-algebra $\mathbb{C}[X]^G$ of all G-invariant locally compact operators with finite propagation in $\mathcal{B}(H_X)$. Replacing G-invariant locally compact operators with G-invariant pseudo-local operators we similarly obtain $D^*(X)^G$. The G-equivariant localization algebras $C^*_L(X)^G$ and $D^*_L(X)^G$ are defined as the C^*-algebras generated by S_1 and S_2 respectively, where f is bounded and uniformly norm-continuous

$$S_1 = \Big\{ f : [0,\infty) \longrightarrow C^*(X)^G | \lim_{t\longrightarrow\infty} \mathsf{prop}(f(t)) = 0 \Big\}$$

$$S_2 = \Big\{ f : [0,\infty) \longrightarrow D^*(X)^G | \lim_{t\longrightarrow\infty} \mathsf{prop}(f(t)) = 0 \Big\}$$

Analogous to Definition 2.12 we can also define the ideals $C^*_{L,0}(X)^G$ and $D^*_{L,0}(X)^G$ as the kernels of the evaluation map.

The equivariant Roe algebra– analogously for $D^*(X)^G$– fits into similar short exact sequence as did the original Roe algebra

$$0 \longrightarrow C^*_{L,0}(X)^G \hookrightarrow C^*_L(X)^G \xrightarrow{\mathsf{ev}} C^*(X)^G \longrightarrow 0$$

An especially useful consequence of the cocompact action of G on X is that there exists a $*$-isomorphism between $C^*_r(G) \otimes \mathcal{K}$ and $C^*(X)^G$, where $C^*_r(G)$ is the reduced group C^*-algebra of G.

Remark 2.16. *The geometric C^*-algebras defined in Definition 2.12 and Definition 2.15 are all unique up to isomorphism, independent of the choice of H_X is a standard and non-degenerate X-module. Likewise the G-equivariant versions are also, up to isomorphism, independent of the choice of admissible (X, G)-module H_X.*

Let Γ and M be as described above; we turn our attention to construction of two important smooth dense subalgebras of $C^*_r(\Gamma) \otimes \mathcal{K} \cong C^*(\widetilde{M})^\Gamma$, the first of which is essentially a slight modification of Connes and Moscovici's [16].

Definition 2.17. Fixing a basis of $L^2(M)$, the algebra $\mathscr{R}$ of smooth operators on M can be identified with the algebra of matrices $(a_{ij})_{i,j\in\mathbb{N}}$ satisfying

$$\sup_{i,j\in\mathbb{N}} i^k j^l |a_{ij}| < \infty \quad \forall k, l \in \mathbb{N}$$

Consider the unbounded operators $\Delta_1 : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ and $\Delta_2 : \ell^2(\Gamma) \longrightarrow \ell^2(\Gamma)$ defined on basis elements according to

$$\Delta_1(\delta_j) = j\delta_j,\ j \in \mathbb{N} \qquad \text{and} \qquad \Delta_2(g) = ||g|| \cdot g,\ g \in \Gamma$$

Denoting by I the identity operator and with $[\cdot,\cdot]$ being the usual commutator bracket, we have unbounded derivations $\partial(T) = [\Delta_2, T]$ of operators $T \in \mathcal{B}(\ell^2(\Gamma))$ and unbounded derivations $\widetilde{\partial}(T) = [\Delta_2 \otimes I, T]$ of operators $T \in \mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N}))$. Define an algebra

$$\mathscr{B}(\widetilde{M})^\Gamma = \{A \in C_r^*(\Gamma) \otimes \mathcal{K} :\ \widetilde{\partial}^k(A) \circ (I \otimes \Delta_1)^2 \text{ is bounded } \forall k \in \mathbb{N}\}$$

The crucial property of $\mathscr{B}(\widetilde{M})^\Gamma$ is that it contains $\mathbb{C}\Gamma \otimes \mathscr{R}$ as a dense subalgebra, is itself a smooth dense subalgebra of $C^*(\widetilde{M})^\Gamma$, and is thus closed under holomorphic functional calculus. Moreover, $\mathscr{B}(\widetilde{M})^\Gamma$ is a Fréchet algebra under the sequence of seminorms $\{||\cdot||_{\mathscr{B},k} : k \in \mathbb{N}\}$, where $||A||_{\mathscr{B},k} = ||\widetilde{\partial}^k(A) \circ (I \otimes \Delta_1)^2||_{op}$ is the operator norm of $\widetilde{\partial}^k(A) \circ (I \otimes \Delta_1)^2$.

Definition 2.18. We define a kind of localization algebra $\mathscr{B}_L(\widetilde{M})^\Gamma$ associated to $\mathscr{B}(\widetilde{M})^\Gamma$, which by construction is a smooth dense subalgebra of $C_L^*(\widetilde{M})^\Gamma$ and thus is closed under holomorphic functional calculus.

$$\mathscr{B}_L(\widetilde{M})^\Gamma = \{f \in C_L^*(\widetilde{M})^\Gamma :\ f \text{ is piecewise smooth w.r.t } t\ ,\, f(t) \in \mathscr{B}(\widetilde{M})^\Gamma\ \forall t \in [0,\infty)\}$$

and also define $\mathscr{B}_{L,0}(\widetilde{M})^\Gamma$ to be the kernel of the usual evaluation map $\mathsf{ev} : \mathscr{B}_L(\widetilde{M})^\Gamma \longrightarrow \mathscr{B}(\widetilde{M})^\Gamma$ defined by $\mathsf{ev}(f) = f(0)$.

Proposition 2.19. *The inclusions $\mathscr{B}_L(\widetilde{M})^\Gamma \hookrightarrow C_L^*(\widetilde{M})^\Gamma$ and $\mathscr{B}_{L,0}(\widetilde{M})^\Gamma \hookrightarrow C_{L,0}^*(\widetilde{M})^\Gamma$ induce isomorphisms on K-theory*

$$K_i(\mathscr{B}_L(\widetilde{M})^\Gamma) \cong K_i(C_L^*(\widetilde{M})^\Gamma) \qquad K_i(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma) \cong K_i(C_{L,0}^*(\widetilde{M})^\Gamma)$$

Proof. If $\mathcal{B}$ is a smooth dense subalgebra of a C^*-algebra $\mathcal{A}$, then the inclusion map $i : \mathcal{B} \hookrightarrow \mathcal{A}$ induces an isomorphism on K-theory.. $\square$

Using the construction of Xie and Yu [50, Equation (10)] we now look at the second kind of smooth dense subalgebra of $C^*(\widetilde{M})^\Gamma$, this time working more directly with $\widetilde{M}$. Let A belong to the algebra $C^\infty(\widetilde{M} \times \widetilde{M})$ of smooth functions on $\widetilde{M} \times \widetilde{M}$, and assume that A is both Γ-invariant and of finite propagation. Explicitly, we mean that

$$A(gx, gy) = A(x, y) \quad \forall g \in \Gamma$$

$$\exists R > 0 \text{ such that } A(x, y) = 0,\ \forall (x, y) \in \widetilde{M} \times \widetilde{M} \text{ satisfying } d_{\widetilde{M}}(x, y) > R$$

Definition 2.20. Denote by $\mathscr{L}(\widetilde{M})^\Gamma$ the convolution algebra of $A \in C^\infty(\widetilde{M} \times \widetilde{M})$ which are both Γ-invariant and of finite propagation. The action of $\mathscr{L}(\widetilde{M})^\Gamma$ on $L^2(\widetilde{M})$ is according to

$$(Af)(x) = \int_{\widetilde{M}} A(x, y) f(y)\, dy \qquad \text{for} \qquad A \in \mathscr{L}(\widetilde{M})^\Gamma,\ f \in L^2(\widetilde{M})$$

Denote by $\hat{\rho} : \widetilde{M} \longrightarrow [0, \infty)$ the distance function $\hat{\rho}(x) = \hat{\rho}(x, y_0)$ for some fixed point $y_0 \in \widetilde{M}$, with ρ being the modification of $\hat{\rho}$ near y_0 to ensure smoothness. The multiplication operator T_ρ thus acts as an unbounded operator on $L^2(\widetilde{M})$, according to $(T_\rho f)(x) = \rho(x) f(x)$. Using the commutator bracket we can define a derivation $\widetilde{\partial} = [T_\rho, \cdot] : \mathscr{L}(\widetilde{M})^\Gamma \longrightarrow \mathscr{L}(\widetilde{M})^\Gamma$.

$$\mathscr{A}(\widetilde{M})^\Gamma = \{A \in C^*(\widetilde{M})^\Gamma :\ \widetilde{\partial}^k(A) \circ (\Delta + 1)^{n_0} \text{ is bounded } \forall k \in \mathbb{N}\}$$

where Δ is the Laplace operator on $\widetilde{M}$, and n_0 is a fixed integer greater than $\dim(M)$. The associated norm is given by $||A||_{\mathscr{A},k} = ||\widetilde{\partial}^k(A) \circ (\Delta + 1)^{n_0}||_{op}$, which is the operator norm of $\widetilde{\partial}^k(A) \circ (\Delta + 1)^{n_0}$.

The same proof of Connes and Moscovici [16, Lemma 6.4] shows that $\mathscr{A}(\widetilde{M})^\Gamma$ is closed under holomorphic functional calculus, and contains $\mathscr{L}(\widetilde{M})^\Gamma$ as a subalgebra. Before proceeding to define the generalized higher eta invariant in Section 4.1 we first recall a necessary extension of $\mathscr{A}(\widetilde{M})^\Gamma$, by introducing bundles. Consider the bundle on $\widetilde{M} \times \widetilde{M}$ given by $\mathsf{End}(\mathcal{S}) = p_1^*(\mathcal{S}) \otimes p_2^*(\mathcal{S}^*)$, where $p_i : \widetilde{M} \times \widetilde{M} \longrightarrow \widetilde{M}$ are the obvious projection maps, with $\mathcal{S}$ and $\mathcal{S}^*$ being the spinor bundle on $\widetilde{M}$ and its dual bundle, respectively. Consider the set $C^\infty(\widetilde{M} \times \widetilde{M}, \mathsf{End}(\mathcal{S}))$ of all smooth sections of the bundle $\mathsf{End}(\mathcal{S})$ on $\widetilde{M} \times \widetilde{M}$, and note that there exists a natural diagonal action of Γ on $\mathsf{End}(\mathcal{S})$. Thus, we can construct $\mathscr{L}(\widetilde{M}, \mathcal{S})^\Gamma$ as the convolution algebra of all Γ-invariant finite propagation elements of $C^\infty(\widetilde{M} \times \widetilde{M}, \mathsf{End}(\mathcal{S}))$. Let $L^2(\widetilde{M}, \mathcal{S})$ denote the the space of L^2-sections of $\mathcal{S}$ over $\widetilde{M}$; there is an action of $\mathscr{L}(\widetilde{M}, \mathcal{S})^\Gamma$ on $L^2(\widetilde{M}, \mathcal{S})$

$$(Af)(x) = \int_{\widetilde{M}} A(x,y) f(y)\, dy \qquad A \in \mathscr{L}(\widetilde{M}, \mathcal{S})^\Gamma \quad f \in L^2(\widetilde{M}, \mathcal{S}) \tag{2.6}$$

Now since $L^2(\widetilde{M}, \mathcal{S})$ is an admissible $(\widetilde{M}, \mathcal{S})$-module we can construct the Γ-equivariant Roe algebra $C^*(\widetilde{M}, \mathcal{S})^\Gamma$ associated to it; however by Remark 2.16 Roe algebras are up to isomorphism independent of the choice of admissible module. Thus, we will also denote by $C^*(\widetilde{M})^\Gamma$ the Γ-equivariant Roe algebra constructed with respect to $L^2(\widetilde{M}, \mathcal{S})$.

Definition 2.21. Let $\widetilde{D}$ be the Dirac operator on $\widetilde{M}$, and fix some integer $n_0 > \dim M$, then

$$\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma = \{A \in C^*(\widetilde{M})^\Gamma : \ \widetilde{\partial}^k(A) \circ (\widetilde{D}^{2n_0} + 1) \text{ is bounded } \forall k \in \mathbb{N}\}$$

where $\widetilde{\partial} = [T_\rho, \cdot]$ is the derivation on $\mathscr{L}(\widetilde{M}, \mathcal{S})^\Gamma$ if we take T_ρ to be the multiplication operator on $L^2(\widetilde{M}, \mathcal{S})$. The algebras $\mathscr{A}_L(\widetilde{M}, \mathcal{S})^\Gamma$ and $\mathscr{A}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma$ are defined analogously to those in Definition 2.18. The associated norm is given by $||A||_{\mathscr{A},\mathcal{S},k} = ||\widetilde{\partial}^k(A) \circ (\widetilde{D}^{2n_0} + 1)||_{op}$, which is the operator norm of $\widetilde{\partial}(A)^k \circ (\widetilde{D}^{2n_0} + 1)$.

If there is no cause for confusion, we shall remove the explicit spinor notation and simply denote the above norm on $\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma$ by $||A||_{\mathscr{A},k}$. We end this section with a brief reminder of the notion of projective tensor product $\mathcal{A}^{\hat{\otimes}_\pi^m}$ with respect to any of the $*$-algebras constructed above. If $\mathcal{A} \otimes \mathcal{B}$ is the algebraic tensor product, then recall that the projective tensor product

$\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ is the completion of $\mathcal{A}\otimes\mathcal{B}$ with respect to the projective cross norm

$$\pi(x) = \inf\left\{\sum_{i=1}^{n_x} ||A_i||_{\mathcal{A}}||B_i||_{\mathcal{B}}:\ x = \sum_{i=1}^{n_x} A_i\otimes B_i\right\} \tag{2.7}$$

where $||\cdot||_{\mathcal{A}}$ denotes the norm on $\mathcal{A}$. We will denote the norm on $\mathcal{A}^{\hat{\otimes}_\pi^m}$ by $||\cdot||_{\mathcal{A}^{\hat{\otimes}m}}$ and usually write elements of $\mathcal{A}^{\hat{\otimes}_\pi^m}$ as $A_1\hat{\otimes}\cdots\hat{\otimes}A_m$.

2.3 Cyclic and Group Cohomology

Definition 2.22. Denote by $C^n(\mathbb{C}\Gamma)$ the cyclic module consisting of all $(n+1)$-functionals $f:(\mathbb{C}\Gamma)^{\otimes n+1}\longrightarrow\mathbb{C}$ together with maps $d_i:(\mathbb{C}\Gamma)^{\otimes n+1}\longrightarrow(\mathbb{C}\Gamma)^{\otimes n}$ defined according to

$$d_i(a_0\otimes\cdots\otimes a_n) = a_0\otimes\cdots\otimes a_{i-1}\otimes(a_ia_{i+1})\otimes a_{i+2}\otimes a_n \qquad \text{for} \quad 0\le i<n$$

$$d_n(a_0\otimes\cdots\otimes a_n) = (a_na_0)\otimes a_1\otimes\cdots\otimes a_{n-1}$$

and a cyclic operator $\mathfrak{t}$, where $\mathfrak{t}f(a_0\otimes\cdots\otimes a_n) = (-1)^nf(a_n\otimes a_0\cdots\otimes a_{n-1})$. Define the coboundary differential $b: C^n(\mathbb{C}\Gamma)\longrightarrow C^n(\mathbb{C}\Gamma)$ by $b=\sum_{i=0}^{n+1}(-1)^i\delta^i$, where δ^i is the dual to d_i; that is $\langle\delta^if,a\rangle = \langle f,d_i(a)\rangle$. Hence we have

$$(bf)(a_0\otimes\cdots\otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^if(a_0\otimes\cdots\otimes(a_ia_{i+1})\otimes a_n)+(-1)^{n+1}f(a_{n+1}a_0\otimes a_1\otimes\cdots\otimes a_n)$$

The cohomology of the complex $(C^n(\mathbb{C}\Gamma),b)$ is the cyclic cohomology $HC^*(\mathbb{C}\Gamma)$.

Definition 2.23. Fix $\gamma\in\Gamma$ and denote by $(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))^{\otimes n+1}$ the subcomplex of $(\mathbb{C}\Gamma)^{\otimes n+1}$ spanned by all elements $(g_0,\ldots,g_n)\in\Gamma^{n+1}$ satisfying $g_0\cdots g_n\in\mathrm{cl}(\gamma)$, where $\mathrm{cl}(\gamma)$ is the conjugacy class of γ. This gives rise to a cyclic submodule $C^n(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$ of $C^n(\mathbb{C}\Gamma)$ which comprises the collection of functionals which vanish on $(g_0,\ldots,g_n)$ if $g_0\cdots g_n\notin\mathrm{cl}(\gamma)$. The coboundary differential b preserves this cyclic subcomplex, and we thus denote the cohomology of $(C^n(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$ by $HC^*(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$.

Definition 2.24. By $H^*(N_\gamma, \mathbb{C})$ we are referring to the groups $\mathsf{Ext}^*_{\mathbb{Z}N_\gamma}(\mathbb{Z}, \mathbb{C})$ defined over the projective $\mathbb{Z}N_\gamma$-resolution of $\mathbb{Z}$. Namely, consider the resolution

$$\cdots \longrightarrow \mathbb{Z}N_\gamma^{k+1} \xrightarrow{\partial_k} \mathbb{Z}N_\gamma^k \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} \mathbb{Z}N_\gamma \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0$$

where, if $\hat{h_i}$ denotes a deleted entry, ∂_k acts on the basis elements according to

$$\partial_k(h_0, \ldots, h_k) = \sum_{i=0}^{k} (-1)^k (h_0, \ldots, \widehat{h_i}, \ldots, h_k)$$

Dropping the $\mathbb{Z}$ term and applying the contravariant functor $\mathsf{Hom}_{N_\gamma}(-, \mathbb{C})$ to this resolution produces a cochain complex with coboundary differential $\hat{b}$

$$\cdots \xleftarrow{\hat{b}} \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^k, \mathbb{C}) \xleftarrow{\hat{b}} \cdots \xleftarrow{\hat{b}} \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma, \mathbb{C}) \xleftarrow{\hat{b}} 0$$

$$(\hat{b}\phi)(h_0, \cdots, h_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \phi(h_0, \ldots, \widehat{h_i}, \ldots, h_{k+1})$$

The cohomology of this complex is defined to be the group cohomology $H^*(N_\gamma, \mathbb{C})$.

Note that every cochain $\phi \in H^n(N_\gamma, \mathbb{C})$ satisfies the "homogeneous" condition: that is, for every $h \in N_\gamma$, $h\phi(h_0, \ldots, h_n) = \phi(hh_0, \ldots, hh_n)$. It will be extremely useful to replace the standard cochain complex with the sub-complex of homogeneous skew-cochains

$$\varphi(\sigma(h_0, h_1, \ldots, h_n)) = \varphi(h_{\sigma(0)}, h_{\sigma(1)}, \ldots, h_{\sigma(n)}) = \mathrm{sgn}(\sigma)\varphi(h_0, h_1, \ldots, h_n) \quad \forall \sigma \in S_{n+1} \quad (2.8)$$

where S_{n+1} is the symmetric group on $n+1$ letters. It is an immediate consequence of this definition that $\varphi(h_0, \ldots, h_n)$ vanishes whenever $h_i = h_j$ for $i \neq j$; just take σ to be the permutation satisfying $\sigma(i) = j, \sigma(j) = i$, and which fixes all other indices. Define the map

$F : \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^n, \mathbb{C}) \longrightarrow \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^n, \mathbb{C})$ according to

$$(F\phi)(h_0, \ldots, h_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\phi(\sigma(h_0, \ldots, h_n)) \tag{2.9}$$

Proposition 2.25. *For every $\phi \in \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^n, \mathbb{C})$ the cochain $F\phi$ is a skew cochain φ.*

Proof. Let σ_0 be any fixed even permutation– that is σ_0 is decomposable as an even number of 2-cycles, hence $\operatorname{sgn}(\sigma_0) = 1$. Since left multiplication of any group on itself is a free and transitive action, it follows that for each σ there exists a unique τ_σ such that $\sigma_0\tau_\sigma = \sigma$.

$$(F\phi)(\sigma_0(h_0, \ldots, h_n)) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\phi(\sigma_0\sigma(h_0, \ldots, h_n))$$

$$= \frac{1}{(n+1)!} \sum_{\sigma_0\tau_\sigma \in S_{n+1}} \operatorname{sgn}(\sigma_0\tau_\sigma)\phi(\sigma_0\sigma(h_0, \ldots, h_n))$$

$$= \frac{1}{(n+1)!} \sum_{\tau_\sigma \in S_{n+1}} \operatorname{sgn}(\tau_\sigma)\phi(\sigma(h_0, \ldots, h_n))$$

Since $\operatorname{sign}(\sigma_0)\operatorname{sign}(\tau_\sigma) = \operatorname{sign}(\sigma_0\tau_\sigma) = \operatorname{sgn}(\sigma)$ and σ_0 is an even permutation then τ_σ must have the same parity as σ. It follows that

$$(F\phi)(\sigma_0(h_0, \ldots, h_n)) = \frac{1}{(n+1)!} \sum_{\tau_\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\phi(\sigma(h_0, \ldots, h_n)) = (F\phi)(h_0, \ldots, h_n)$$

Follow the same argument, if σ_0 is an odd permutation then again for each σ there exists a unique τ_σ such that $\sigma_0\tau_\sigma = \sigma$. However, since $\operatorname{sgn}(\sigma_0) = -1$ it follows that τ_σ must possess opposite parity to σ, hence

$$(F\phi)(\sigma_0(h_0, \ldots, h_n)) = \frac{1}{(n+1)!} \sum_{\tau_\sigma \in S_{n+1}} -\operatorname{sgn}(\sigma)\phi(\sigma(h_0, \ldots, h_n)) = -(F\phi)(h_0, \ldots, h_n)$$

□

Lemma 2.26. *The induced map $F^* : H^*(N_\gamma, \mathbb{C}) \longrightarrow H^*(N_\gamma, \mathbb{C})$ is an isomorphism.*

Proof. That F induces an isomorphism on cohomology (with real or complex coefficients) follows if we can show $F \simeq \mathrm{Id}$ as chain complex maps. First a straightforward calculation proves that F is a chain complex map, in the sense that the following diagram commutes for all n.

$$\begin{array}{ccc} \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^{n+1},\mathbb{C}) & \xleftarrow{\hat{b}} & \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^{n},\mathbb{C}) \\ \downarrow F & & \downarrow F \\ \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^{n+1},\mathbb{C}) & \xleftarrow{\hat{b}} & \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^{n},\mathbb{C}) \end{array}$$

$$(\hat{b}\circ F\phi)(h_0,\ldots,h_{n+1}) = \sum_{i=0}^{n+1}(-1)^i(F\phi)(h_0,\ldots,\widehat{h_i},\ldots,h_{n+1})$$

$$= \frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}\mathrm{sgn}(\sigma)\sum_{i=0}^{n+1}(-1)^i\phi(\sigma(h_0,\ldots,\widehat{h_i},\ldots,h_{n+1}))$$

$$= \frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}\mathrm{sgn}(\sigma)(\hat{b}\phi)(\sigma(h_0,\ldots,h_{n+1})) = (F\circ\hat{b}\phi)(h_0,\ldots,h_{n+1})$$

Now, F is chain homotopic to Id on each $\mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^n,\mathbb{C})$ if there exists a sequence of maps $\{p_k \mid p_k : \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^k,\mathbb{C}) \longrightarrow \mathsf{Hom}_{N_\gamma}(\mathbb{Z}N_\gamma^{k-1},\mathbb{C})\}$ such that $F - \mathrm{Id} = b\circ p_n + p_{n+1}\circ b$. For ease of notation denote $\mathbf{h}_{n-1} = (h_0,\ldots,h_{n-1})$; we will define

$$(p_n\phi)(\mathbf{h}_{n-1}) = \frac{(-1)^n}{(n+1)!}\sum_{\sigma\in S_{n+1}}\mathrm{sgn}(\sigma)\phi(\sigma(\mathbf{h}_{n-1},eh_{n-1})) - (-1)^n\phi(\mathbf{h}_{n-1},eh_{n-1})$$

where $eh_{n-1} = h_{n-1}$ denotes a copy of h_{n-1} inserted into the n'th position. For further ease of notation we will denote $(h_0,\ldots,\widehat{h_i},\ldots,h_n,eh_n)$ by $(\mathbf{h}_{n,\hat{i}},eh_n)$ for $i\le n$.

$$(b\circ p_n\phi)(\mathbf{h}_n) = \sum_{i=0}^{n}(-1)^i(p_n\phi)(h_0,\ldots,\widehat{h_i},\ldots,h_n)$$

$$= \sum_{i=0}^{n}(-1)^i\left(\frac{(-1)^n}{(n+1)!}\sum_{\sigma\in S_{n+1}}\mathrm{sgn}(\sigma)\phi(\sigma(\mathbf{h}_{n,\hat{i}},eh_n)) - (-1)^n\phi(\mathbf{h}_{n,\hat{i}},eh_n)\right)$$

$$(p_{n+1}\circ b\phi)(\mathbf{h}_n) = \frac{(-1)^{n+1}}{(n+2)!}\sum_{\sigma\in S_{n+1}}\mathrm{sgn}(\sigma)(b\phi)(\sigma(\mathbf{h}_n,eh_n)) - (-1)^{n+1}(b\phi)(\mathbf{h}_n,eh_n)$$

$$= \sum_{i=0}^{n+1} (-1)^i \left(\frac{(-1)^{n+1}}{(n+2)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\sigma(\mathbf{h}_{n,\hat{i}}, eh_n)) - (-1)^{n+1} \phi(\mathbf{h}_{n,\hat{i}}, eh_n) \right)$$

Using the fact that by Proposition 2.25 the expressions

$$\frac{(-1)^n}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\mathbf{h}_{n,\hat{i}}, eh_n) \quad \text{and} \quad \frac{(-1)^{n+1}}{(n+2)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\mathbf{h}_{n,\hat{i}}, eh_n)$$

vanish for all $i \leq n-1$ since $h_n = eh_n$, we thus have the reduced identities

$$(b \circ p_n \phi)(\mathbf{h}_n) = \frac{(-1)^n (-1)^n}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\sigma(\mathbf{h}_{n,\hat{n}}, eh_n)) - \sum_{i=0}^{n} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n)$$

$$(p_{n+1} \circ b\phi)(\mathbf{h}_n) = \frac{1}{(n+2)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \Big((-1)^{2n+2} \phi(\sigma(\mathbf{h}_n, \widehat{eh_n})) + (-1)^{2n+1} \phi(\sigma(\mathbf{h}_{n,\hat{n}}, eh_n)) \Big)$$

$$- \sum_{i=0}^{n+1} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n)$$

$$= \sum_{i=0}^{n+1} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n) + \frac{1}{(n+2)!} \sum_{\sigma \in S_{n+1}} 0 = \sum_{i=0}^{n+1} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n)$$

where we have used the fact $(\mathbf{h}_{n,\hat{n}}, eh_n) = (\mathbf{h}_n, \widehat{eh_n})$. Moreover, it is readily apparent that both these tuples are also equal to $\mathbf{h}_n$; it follows that $(b \circ p_n \phi + p_{n+1} \circ b\phi)(\mathbf{h}_n)$ simplifies to exactly the expression for $(F - \mathrm{Id})\phi(\mathbf{h}_n)$

$$\frac{(-1)^{2n}}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\sigma(\mathbf{h}_{n,\hat{n}}, eh_n)) + \sum_{i=0}^{n+1} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n) - \sum_{i=0}^{n} (-1)^{i+n} \phi(\mathbf{h}_{n,\hat{i}}, eh_n)$$

$$= \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) \phi(\sigma(\mathbf{h}_n)) - \phi(\mathbf{h}_n) = (F - \mathrm{Id})\phi(\mathbf{h}_n)$$

□

For the remainder of this paper, when referring to group cohomology it will be with respect to the subcomplex of skew cochains. The following splitting of cyclic (co)-homology was proven by Burghelea [10] using topological arguments, and Nistor [35] provided a later algebraic proof.

Theorem 2.27.

$$HC^*(\mathbb{C}\Gamma) \cong \prod_{\mathrm{cl}(\gamma)} HC^*(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$$

Moreover there exist isomorphisms with group (co)-homology

$$HC^*(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \cong \begin{cases} H^*(N_\gamma, \mathbb{C}) & \gamma \text{ is of infinite order} \\ H^*(N_\gamma, \mathbb{C}) \otimes_{\mathbb{C}} HC^*(\mathbb{C}) & \gamma \text{ is of finite order} \end{cases}$$

Definition 2.28. Fix a group element γ with conjugacy class $\mathrm{cl}(\gamma)$, and let $C^k(\Gamma, Z_\gamma, \gamma)$ be the collection of all multilinear forms $\alpha : \Gamma^{k+1} \longrightarrow \mathbb{C}$ satisfying

$$\alpha(g_{\sigma(0)}, g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \mathrm{sgn}(\sigma)\alpha(g_0, g_1, \ldots, g_k) \quad \forall \sigma \in S_{k+1}$$

$$\alpha(zg_0, zg_1, \ldots, zg_k) = \alpha(g_0, g_1, \ldots, g_k) \quad \forall z \in Z_\gamma$$

$$\alpha(\gamma g_0, g_1, \ldots, g_k) = \alpha(g_0, g_1, \ldots, g_k)$$

The coboundary map $\hat{b} : C^k(\Gamma, Z_\gamma, \gamma) \longrightarrow C^{k+1}(\Gamma, Z_\gamma, \gamma)$ gives rise to a cochain complex $(C^n(\Gamma, Z_\gamma, \gamma), \hat{b})$, the cohomology of which we will denote by $H^*(\mathcal{C}, \mathbb{C})$

$$\cdots \xleftarrow{\hat{b}} C^{k+1}(\Gamma, Z_\gamma, \gamma) \xleftarrow{\hat{b}} C^k(\Gamma, Z_\gamma, \gamma) \xleftarrow{\hat{b}} \cdots \xleftarrow{\hat{b}} C^0(\Gamma, Z_\gamma, \gamma) \xleftarrow{\hat{b}} 0$$

$$(\hat{b}\phi)(g_0, \cdots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \phi(g_0, \ldots, \widehat{g_i}, \ldots, g_{k+1})$$

Recall that a cyclic cocycle of $(C^n(\mathbb{C}\Gamma), b)$ is a functional φ which belongs to the kernel $ZC^n(\mathbb{C}\Gamma)$ of the coboundary differential. If $\mathrm{cl}(\gamma)$ is non-trivial we call the cyclic cocycles φ_γ of $(C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ *delocalized* cyclic cocycles. Following the example of Lott [31, Section 4.1] we can construct explicit representations of any delocalized cyclic cocycle as follows:

associated to each $\alpha \in H^*(\mathcal{C}, \mathbb{C})$ define

$$\varphi_{\alpha,\gamma}(g_0, g_1, \ldots, g_n) = \begin{cases} 0 & \text{if } g_0 g_1 \cdots g_n \notin \mathrm{cl}(\gamma) \\ \alpha(h, hg_0, \ldots, hg_0 g_1 \cdots g_{n-1}) & \text{if } g_0 g_1 \cdots g_n = h^{-1}\gamma h \end{cases} \tag{2.10}$$

By multilinearity of α, it is immediate that given $a_i = \sum_{g_i \in \Gamma} c_{g_i} \cdot g_i$ in the group algebra

$$\varphi_{\alpha,\gamma}(a_0 \otimes \cdots \otimes a_n) = \sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} c_{g_0} \cdots c_{g_n} \varphi_{\alpha,\gamma}(g_0, g_1, \ldots, g_n)$$

It is also apparent that the property $\varphi_{\alpha,\gamma}(\gamma g_0, g_1, \ldots, g_k) = \alpha(g_0, g_1, \ldots, g_k)$ generalizes to any element of $\gamma^{\mathbb{Z}}$, that is for any $r \in \mathbb{Z}$– it suffices to consider $r \geq 0$– we have

$$\varphi_{\alpha,\gamma}(\gamma^r g_0, g_1, \ldots, g_k) = \varphi_{\alpha,\gamma}(\gamma^{r-1} g_0, g_1, \ldots, g_k) = \cdots = \varphi_{\alpha,\gamma}(g_0, g_1, \ldots, g_k)$$

If $\mathcal{A}$ is a unital algebra such that φ_γ admits an extension to $\mathcal{A}$ then we define a unitized version of the cyclic cocycle. Let $\mathcal{A}^+$ be the algebra formed from adjoining a unit to $\mathcal{A}$, then the homomorphism $(A, \lambda) \longmapsto (A + \lambda 1_{\mathcal{A}}, \lambda)$ provides an isomorphism between $\mathcal{A}^+$ and $\mathcal{A} \oplus \mathbb{C}1$. For any $\varphi \in ZC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ we define

$$\overline{\varphi}_\gamma(\overline{A}_0 \hat{\otimes} \cdots \hat{\otimes} \overline{A}_n) = \varphi_\gamma(A \hat{\otimes} \cdots \hat{\otimes} A_n) \quad \text{where} \quad \overline{A}_i = (A_i, \lambda_i) \in \mathcal{A}^+ \tag{2.11}$$

and as shown in [15, Chapter 3.3] the condition $b\varphi_\gamma = 0$ still holds.

Remark 2.29. *With respect to the delocalized cyclic cocycle representations $\varphi_{\alpha,\gamma}$ there is an elementary way to move between $\alpha(g_0, g_1, \ldots, g_n)$ and the normalized form*

$$\alpha(h, hg_0, \ldots, hg_0 g_1 \cdots g_{n-1})$$

which clearly vanishes if $g_i = e$ for any $0 \leq i \leq n-1$. For each $y \in \mathrm{cl}(\gamma)$ fix some $h^y \in \Gamma$ such that $(h^y)^{-1}\gamma h^y = y$. In particular, the elements $y_0 = g_0 g_1 \cdots g_n$ and $y_i = g_i \cdots g_n g_0 \cdots g_{i-1}$

all belong to $\mathrm{cl}(\gamma)$ *for* $1 \leq i \leq n$*, since by hypothesis* $y_0 \in \mathrm{cl}(\gamma)$*, and direct computation shows that* $y_i = (g_0 \cdots g_{i-1})^{-1} y_0 (g_0 \cdots g_{i-1})$*.*

The map F *defined by* $F(g_i) = h^{y_0}(g_i \cdots g_n)^{-1} y_i$ *induces a map on* $H^*(\mathcal{C}, \mathbb{C})$ *according to* $F^*[\alpha] = [\alpha \circ F]$ *where since* $g_0 g_1 \cdots g_n = y_0 = (h^{y_0})^{-1} \gamma h^{y_0}$

$$(\alpha \circ F)(g_0, g_1, \ldots, g_n) = \alpha(F(g_0), F(g_1), \ldots, F(g_n))$$

$$= \alpha(h^{y_0}(g_0 \cdots g_n)^{-1} y_0, h^{y_0}(g_1 \cdots g_n)^{-1} y_1, \ldots, h^{y_0} g_n^{-1} y_n)$$

$$= \alpha(h^{y_0}(g_0 \cdots g_n)^{-1}(g_0 \cdots g_n), h^{y_0}(g_1 \cdots g_n)^{-1} g_1 \cdots g_n g_0, \ldots, h^{y_0} g_n^{-1} g_n g_0 \cdots g_{n-1})$$

$$= \alpha(h^{y_0}, h^{y_0} g_0, \ldots, h^{y_0} g_0 \cdots g_{n-1})$$

This property of F *carries over to* $\varphi_{\alpha,\gamma}$ *acting on the group algebra* $\mathbb{C}\Gamma$ *by extending* F *linearly.*

3. CYCLIC COHOMOLOGY OF POLYNOMIAL GROWTH GROUPS*

The convergence properties of the integrals defining the pairing of delocalized cyclic cocycles with higher invariants depend crucially on the growth conditions of the cyclic cocycles. This in turn is linked to the growth properties of conjugacy classes of Γ, in particular it is proven in [23] that polynomial growth groups are of polynomial cohomology– with respect to coefficients in $\mathbb{C}$.

Definition 3.1. The group Γ is of polynomial cohomology if for any $[\phi] \in H^*(\Gamma, \mathbb{C})$ there exists (a skew cocycle) $\varphi \in Z(\mathsf{Hom}_\Gamma(\mathbb{Z}\Gamma^*, \mathbb{C}), \hat{b})$ such that $[\varphi] = [\phi]$, and φ is of polynomial growth. That is, φ satisfies the following bound for positive integer constants R_φ and k

$$|\varphi(g_0, g_1, \dots, g_n)| \leq R_\varphi(1 + ||g_0||)^{2k}(1 + ||g_1||)^{2k} \cdots (1 + ||g_n||)^{2k} \tag{3.1}$$

By Remark 2.29 it follows that any normalized group cocycle α also has polynomial growth if the non-normalized version does, since

$$|\alpha(h, hg_0, \dots, hg_0g_1 \cdots g_{n-1})| = |\alpha(F(g_0), F(g_1), \dots, F(g_n))|$$

The splitting of delocalized cyclic cohomology as shown in Theorem 2.27 provides an abstract isomorphism between group cohomology and cyclic cohomology, but we desire an explicit construction of this mapping, so as to prove that polynomial growth group cocycles are mapped to polynomial growth cyclic cocycles. This shall be proven in Section 3.2 through the use of the classifying space approach to group cohomology, while in the section immediately following we show that our attention can be restricted to the case where γ is a torsion element.

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3.1 Cohomological Dimension and Connes Periodicity Map

In the next section our results depend crucially on $H^*(N_\gamma, \mathbb{C})$ not contributing to the delocalized cyclic cohomology of $\mathbb{C}\Gamma$ whenever γ is of infinite order; in this section we make this notion precise. For any unital associative algebra A over a field containing $\mathbb{Q}$– hence particularly for the group algebra $\mathbb{C}\Gamma$– the cyclic and Hochschild homology fit into a long exact sequence

$$\cdots \longrightarrow HH^n(A) \longrightarrow HC^{n-1}(A) \xrightarrow{S^*} HC^{n+1}(A) \longrightarrow HH^{n+1}(A) \longrightarrow \cdots \tag{3.2}$$

where S is the Connes periodicity operator introduced in [12, II.1]. We will use the explicit construction in terms of maps of complexes that is provided in [30, Chapter 2], and so provide the following expression for S when $A = \mathbb{C}\Gamma$. Let b^* be the homomorphism induced by the boundary map b, and define a map $\beta : HC^*(\mathbb{C}\Gamma) \longrightarrow HC^{*+1}(\mathbb{C}\Gamma)$ according to

$$(\beta\varphi)(g_0, g_1, \ldots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i i(\delta^i\varphi)(g_0, g_1, \ldots, g_{k+1}) \tag{3.3}$$

Hence $(\beta b)^* : HC^*(\mathbb{C}\Gamma) \longrightarrow HC^{*+2}(\mathbb{C}\Gamma)$ and similarly for the map induced by $b\beta$. Dual to the result given in [30, Theorem 2.2.7] we have that for any cohomology class $[\varphi] \in HC^n(\mathbb{C}\Gamma)$ its image under the periodicity operator[1] is

$$S^*[\varphi] = [S\varphi] \in HC^{n+2}(\mathbb{C}\Gamma) \quad \text{where} \quad S = \frac{1}{(n+1)(n+2)}(\beta b + b\beta) \tag{3.4}$$

$$b\beta = \sum_{0 \le i < j \le n+2} (-1)^{i+j}(j-i)\delta^i\delta^j \qquad \beta b = \sum_{0 \le i < j \le n+2} (-1)^{i+j}(i-j+1)\delta^i\delta^j \tag{3.5}$$

Definition 3.2. The delocalized Connes periodicity operator S_γ is obtained by the restric-

[1] Note that our choice of constant differs from that of Connes [12] due to the constants involved in the definition (see Equation (4.52)) of the Connes-Chern character

tion of S to the sub-complex $(C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$.

$$S_\gamma^* : HC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow HC^{n+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$$

In the construction of the group cohomology $H^*(G, \mathbb{C}) = \mathsf{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{C})$ the minimal length of the projective $\mathbb{Z}G$-resolution over $\mathbb{Z}$ is called the cohomological dimension of the group. Denoting this by $\mathrm{cd}_{\mathbb{Z}}(G)$, it is immediate from the definition that if $\mathrm{cd}_{\mathbb{Z}}(G) = n$ then $H^k(G, \mathbb{C}) = 0$ for all $k > n$. If we consider projective $\mathbb{Q}G$-resolutions instead, then there is notion of rational cohomology and rational cohomological dimension. Recall that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, hence tensoring with $\mathbb{Q}$ preserves exactness, and $\mathrm{cd}_{\mathbb{Q}}(G)$ denotes the minimal length of the projective resolution defining the groups

$$H^*(G, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathsf{Ext}^*_{\mathbb{Q}G}(\mathbb{Q}, \mathbb{C})$$

When G is a group of polynomial growth then it belongs to the class $\mathcal{C}$ of groups satisfying the following conditions (see [24, Section 4])

(i) G is of finite rational cohomological dimension.

(ii) The rational cohomological dimension of $N_g = Z_G(g)/g^{\mathbb{Z}}$ is finite whenever g is not a torsion element.

It is now important to note that if γ is of infinite order then $\mathrm{cl}(\gamma)$ is torsion free, for otherwise there exists $g^{-1}\gamma g \in \mathrm{cl}(\gamma)$ of finite order, and thus $e = (g^{-1}\gamma g)^k = g^{-1}\gamma^k g$, which implies $\gamma^k = e$. In this torsion free case the nilpotency of S_γ with respect to the long exact sequence

$$\begin{aligned} \cdots \longrightarrow HH^{n-1}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow HC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \xrightarrow{S^*} HC^{n+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \\ \longrightarrow HH^{n+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow \cdots \end{aligned} \tag{3.6}$$

follows from the proof of [24, Theorem 4.2], and so we have that $HC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) = 0$ for

$n > 0$ and γ of infinite order.

Lemma 3.3. *If Γ is of polynomial growth, then N_γ is of polynomial cohomology.*

Proof. Since Γ is of polynomial growth, then a theorem of Gromov [18] states that Γ is virtually nilpotent. We take N to be a normal nilpotent subgroup of finite index, and so $Z_\gamma \cap N \leq N$ is finitely generated and nilpotent. The short exact sequence

$$1 \longrightarrow Z_\gamma \cap N \longrightarrow Z_\gamma \longrightarrow \Gamma/N \longrightarrow 1$$

shows that $Z_\gamma \cap N$ is of finite index in Z_γ, hence Z_γ is finitely generated and admits a word length function l_{Z_γ} which is bounded by l_Γ. It follows that Z_γ is of polynomial growth with respect to any word length function on it. Taking the quotient by a central cyclic group preserves polynomial growth, and thus by [23, Corollary 4.2] N_γ is of polynomial cohomology. □

It should be made explicit that in the above proof we also obtained that Z_γ was of polynomial cohomology– or equivalently that $H^*(Z_\gamma, \mathbb{C})$ is polynomial bounded. Extending the notion of polynomial cohomology to that of delocalized cyclic cocycles, we call $HC^*(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ polynomially bounded if every cohomology class admits a representative $\varphi_\gamma \in (C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ which is of polynomial growth.

Lemma 3.4. *There is a rational isomorphism between the cohomology group $H^n(\mathcal{C}, \mathbb{C})$ (see Definition 2.28) and $H^n(Z_\gamma, \mathbb{C})$ for $n \geq 1$.*

Proof. We begin with an alteration of the complex $((C^n(\Gamma, Z_\gamma, \gamma), \hat{b})$ constructed in Definition 2.28, by removing the third condition: $\alpha(\gamma g_0, g_1, \ldots, g_n) = \alpha(g_0, g_1, \ldots, g_n)$. This produces a larger cochain complex which shall be denoted by $((D^n(\Gamma, Z_\gamma, \gamma), \hat{b})$, and there is a natural inclusion map $\imath : ((C^n(\Gamma, Z_\gamma, \gamma), \hat{b}) \hookrightarrow ((D^n(\Gamma, Z_\gamma, \gamma), \hat{b})$. In the other direction we consider an "averaging" map $\mathcal{R} : ((D^n(\Gamma, Z_\gamma, \gamma), \hat{b}) \longrightarrow ((C^n(\Gamma, Z_\gamma, \gamma), \hat{b})$ similar to that from [11, Theorem 5.2], defined according to

$$(\mathcal{R}\alpha)(g_0, g_1, \dots, g_n) = \sum_{r_0, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0} g_0, \gamma^{r_1} g_1, \dots, \gamma^{r_n} g_n) \tag{3.7}$$

where $\mathrm{ord}(\gamma)$ is the order of γ. By the above results of this section we only need to concern ourselves with γ being a torsion element, and thus the above sum is finite, so $\mathcal{R}$ is well defined. To show that $\mathcal{R}$ is a surjective map it is first necessary to prove that $(\mathcal{R}\alpha)$ actually belongs to the complex $((C^n(\Gamma, Z_\gamma, \gamma), \hat{b})$; namely if γ is torsion, then $\gamma^{\mathbb{Z}} = \{e, \gamma, \dots, \gamma^{\mathrm{ord}(\gamma)-1}\} = \gamma \cdot \gamma^{\mathbb{Z}}$ and

$$(\mathcal{R}\alpha)(\gamma g_0, g_1, \dots, g_n) = \sum_{r_0, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0+1} g_0, \gamma^{r_1} g_1, \dots, \gamma^{r_n} g_n)$$

$$\sum_{r_0+1, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0+1} g_0, \gamma^{r_1} g_1, \dots, \gamma^{r_n} g_n) = (\mathcal{R}\alpha)(g_0, g_1, \dots, g_n)$$

It is similarly straightforward to prove that the following diagram commutes

$$\begin{array}{ccc} D^{n+1}(\Gamma, Z_\gamma, \gamma) & \xleftarrow{\hat{b}} & D^n(\Gamma, Z_\gamma, \gamma) \\ \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \mathcal{R}} \\ C^{n+1}(\Gamma, Z_\gamma, \gamma) & \xleftarrow{\hat{b}} & C^n(\Gamma, Z_\gamma, \gamma) \end{array}$$

and so $\mathcal{R}$ is indeed a chain complex map. Explicitly, we have

$$(\hat{b} \circ \mathcal{R}\alpha)(g_0, g_1, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i (\mathcal{R}\alpha)(g_0, \dots, \widehat{g_i}, \dots, g_{n+1})$$

$$= \sum_{i=0}^{n+1} (-1)^i \sum_{r_0, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0} g_0, \dots, \widehat{\gamma^{r_1} g_1}, \dots, \gamma^{r_n} g_n)$$

$$= \sum_{r_0, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} \sum_{i=0}^{n+1} (-1)^i \alpha(\gamma^{r_0} g_0, \dots, \widehat{\gamma^{r_i} g_i}, \dots, \gamma^{r_{n+1}} g_{n+1})$$

$$= \sum_{r_0, r_1, \dots, r_n = 1}^{\mathrm{ord}(\gamma)} (\hat{b}\alpha)(\gamma^{r_0} g_0, \gamma^{r_1} g_1, \dots, \gamma^{r_{n+1}} g_{n+1}) = (\mathcal{R} \circ \hat{b}\alpha)(g_0, g_1, \dots, g_{n+1})$$

We now prove that the composition $\mathcal{R}\circ\imath : ((C^n(\Gamma, Z_\gamma,\gamma),\hat{b}) \longrightarrow ((C^n(\Gamma, Z_\gamma,\gamma),\hat{b})$ is rationally equivalent to the identity map Id_C, and thus by extension that $\mathcal{R}$ is a rational surjection. Using the property $\alpha(\gamma^r g_0, g_1,\dots,g_n) = \alpha(g_0,g_1,\dots,g_n)$, for any $\alpha\in C^n(\Gamma, Z_\gamma,\gamma)$

$$(\mathcal{R}\circ\imath\alpha)(g_0,g_1,\dots,g_n) = \sum_{r_0,r_1,\dots,r_n=1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0}g_0,\gamma^{r_1}g_1,\dots,\gamma^{r_n}g_n)$$

$$= \sum_{r_0=1}^{\mathrm{ord}(\gamma)}\sum_{r_1,\dots,r_n=1}^{\mathrm{ord}(\gamma)} \alpha(\gamma^{r_0}g_0,\gamma^{r_1}g_1,\dots,\gamma^{r_n}g_n)$$

$$= \frac{\mathrm{ord}(\gamma)(\mathrm{ord}(\gamma)+1)}{2}\sum_{r_1,\dots,r_n=1}^{\mathrm{ord}(\gamma)} \alpha(g_0,\gamma^{r_1}g_1,\dots,\gamma^{r_n}g_n)$$

For convenience denote the coefficient $\mathrm{ord}(\gamma)(\mathrm{ord}(\gamma)+1)/2$ by A, then by repeated shifting of the g_0 element the above sum becomes

$$= (-1)A\sum_{r_1=1}^{\mathrm{ord}(\gamma)}\sum_{r_2,\dots,r_n=1}^{\mathrm{ord}(\gamma)} \alpha(g_1,g_0,\gamma^{r_2}g_2,\dots,\gamma^{r_n}g_n)$$

$$= (-1)A^2\sum_{r_2,\dots,r_n=1}^{\mathrm{ord}(\gamma)} \alpha(g_1,g_0,\gamma^{r_2}g_2,\dots,\gamma^{r_n}g_n) = \cdots = (-1)^n A^{n+1}\alpha(g_1,\dots,g_n,g_0)$$

$$= (-1)^n(-1)^n A^{n+1}\alpha(g_0,g_1,\dots,g_n) = A^{n+1}\alpha(g_0,g_1,\dots,g_n)$$

It thus follows that as maps from $(C^n(\Gamma, Z_\gamma,\gamma),\hat{b})\otimes_{\mathbb{Z}}\mathbb{Q}$ to itself, we have the equality

$$(\mathcal{R}\circ\imath)\otimes_{\mathbb{Z}}\frac{1}{A^{n+1}} = Id_C\otimes_{\mathbb{Z}} 1 \tag{3.8}$$

which establishes the isomorphism on cohomology $H^*(\mathcal{C},\mathbb{C})\otimes_{\mathbb{Z}}\mathbb{Q} \overset{\mathcal{R}}{\cong} H^*(\mathcal{D},\mathbb{C})\otimes_{\mathbb{Z}}\mathbb{Q}$. The desired result now follows from Nistor's [35, Section 2.7] application of spectral sequences to prove that $H^*(\mathcal{D},\mathbb{C})$ is isomorphic to the group cohomology $H^*(Z_\gamma,\mathbb{C})$. □

3.2 Classifying Space Construction

The isomorphism $H^*(\mathcal{D}, \mathbb{C}) \cong H^*(Z_\gamma, \mathbb{C})$ mentioned in Lemma 3.4 unfortunately does not provide an explicit way to realize preservation of polynomial cohomology. It is, however, easy to show that as defined in Lemma 3.4 the map $\mathcal{R}$ behaves as desired in this respect.

Proposition 3.5. *The map* $(\mathcal{R} \otimes_{\mathbb{Z}} 1/A^{n+1})^*$ *preserves polynomial growth.*

Proof. It suffices to show that if α is of polynomial growth then so is $\mathcal{R}\alpha$, which follows directly from γ having finite order.

$$|(\mathcal{R}\alpha)(g_0, g_1, \ldots, g_n)| = \left| \sum_{r_0, r_1, \ldots, r_n = 1}^{\text{ord}(\gamma)} \alpha(\gamma^{r_0} g_0, \gamma^{r_1} g_1, \ldots, \gamma^{r_n} g_n) \right|$$

$$\leq \sum_{r_0, r_1, \ldots, r_n = 1}^{\text{ord}(\gamma)} |\alpha(\gamma^{r_0} g_0, \gamma^{r_1} g_1, \ldots, \gamma^{r_n} g_n)| \leq \sum_{r_0, r_1, \ldots, r_n = 1}^{\text{ord}(\gamma)} R_\alpha (1 + ||\gamma^{r_0} g_0||)^{2k} \cdots (1 + ||\gamma^{r_n} g_n||)^{2k}$$

$$\leq (\text{ord}(\gamma))^{n+1} \max_{r_0, r_1, \ldots, r_n \in \{1, 2 \ldots, \text{ord}(\gamma)\}} R_\alpha (1 + ||\gamma^{r_0} g_0||)^{2k} \cdots (1 + ||\gamma^{r_n} g_n||)^{2k}$$

$$= (\text{ord}(\gamma))^{n+1} R_\alpha (1 + ||\gamma^{m_0} g_0||)^{2k} \cdots (1 + ||\gamma^{m_n} g_n||)^{2k}$$

□

Theorem 3.6. *For every nontrivial conjugacy class* $\text{cl}(\gamma)$ *for* γ *of finite order, if* $H^*(Z_\gamma, \mathbb{C})$ *is of polynomial cohomology then* $H^*(\mathcal{D}, \mathbb{C})$ *is polynomially bounded.*

Proof. Consider the simplicial set $E_*\Gamma$, the nerve of Γ, with simplices $E_k\Gamma := \Gamma \times \Gamma^k$ and relations

$$d_i(g_0, \ldots, g_k) = \begin{cases} (g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_k) & 0 \leq i \leq k-1 \\ (g_0, g_1 \ldots, g_{k-1}) & i = k \end{cases}$$
$$s_j(g_0, \ldots, g_k) = (g_0, \ldots, g_j, e, g_{j+1}, \ldots, g_k) \tag{3.9}$$
$$(g_0, \ldots, g_{k-1}, g_k) g = (g_0 g, \ldots, g_{k-1}, g_k) \ \forall g \in \Gamma$$

The last equation defines a free Γ-action on the nerve according to right multiplication. The contractible space $E\Gamma = |E_*\Gamma|$– which is a locally finite CW-complex– is the geometrical realization of the nerve under the relations

$$|E_*\Gamma| = \bigsqcup_{k\geq 0} E_k\Gamma \times \triangle^k / \sim \qquad \begin{array}{ll} (x, \delta_i t) \sim (d_i x, t) & \text{for } x \in E_k\Gamma, t \in \triangle^{k-1} \\ (x, \sigma_j t) \sim (s_j x, t) & \text{for } x \in E_k\Gamma, t \in \triangle^{k+1} \end{array} \tag{3.10}$$

where $\triangle^k = \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} : t_i \in [0,1], \sum_{i=0}^k t_i = 1\}$ is the standard k-simplex with degeneracy maps σ_j and face maps δ_i. Since right multiplication is a free action, it is immediate that Γ acts freely on $E\Gamma$, thus we define the classifying space $B\Gamma = |B_*\Gamma|$ to be the orbit space $E\Gamma/\Gamma$; the simplices of $B_*\Gamma$ are thus of the form $E_k\Gamma/\Gamma$. More generally, it is useful to view $E\Gamma$ as a the universal principal bundle $p : E\Gamma \longrightarrow B\Gamma$, which provides the relation of balanced products

$$B\Gamma = E\Gamma \times_\Gamma \Gamma = \{(v, w) \in E\Gamma \times \Gamma/((v, gw) \sim (vg, w))\}$$

Applying the above simplicial construction to Z_γ we can similarly construct the universal principal Z_γ-bundle $p : EZ_\gamma \longrightarrow BZ_\gamma$. The geometric realization is functorial, hence any group homomorphism induces a homomorphism on $E\Gamma$ at the simplex level. Since Γ is a discrete group then Z_γ is *admissible* as a subgroup, and there is the principal Z_γ-bundle $q : \Gamma \longrightarrow \Gamma/Z_\gamma$ where q is the quotient map. In particular, since $p : E\Gamma \longrightarrow B\Gamma$ is a universal principal Γ-bundle

$$p' : E\Gamma \times_{Z_\gamma} \Gamma \longrightarrow E\Gamma \times_\Gamma (\Gamma/Z_\gamma) = (Z_\gamma \curvearrowright E\Gamma)/Z_\gamma$$

is a universal principal Z-bundle, where $Z_\gamma \curvearrowright E\Gamma$ has the same nerve $E_*\Gamma$, except with the third relation in (3.9) replaced by a Z_γ group action

$$(g_0, \ldots, g_{k-1}, g_k)z = (g_0 z, \ldots, g_{k-1}, g_k) \; \forall z \in Z_\gamma$$

By universality of the principal bundle there is a homotopy equivalence between BZ_γ and $(Z_\gamma \curvearrowright E\Gamma)/Z_\gamma$ as CW-complexes, which implies equivalence of (co)homology groups. (See [33] for a more detailed discussion on classifying spaces and fibre bundles)

Every point v in an oriented m-simplex V_m of $E\Gamma$ can be expressed as a formal sum, using barycentric coordinates and the vertices $\mathbf{g}_i = (e, \ldots, g_i, \ldots, e)$.

$$\sum_{i=0}^{m} t_i = 1 \quad \text{and} \quad v = \sum_{i=0}^{m} t_i \mathbf{g}_i \qquad\qquad V_m = g_0[g_1|\cdots|g_m]$$

The particular notation for the m-simplex is important in emphasizing the group action on the first coordinate. As we shall see shortly, this is also useful when describing boundary maps of simplicial chain complexes. First consider the CW-complex $B\Gamma$ as the image of $E\Gamma$ under the projection map p acting on the nerve by deletion of the first coordinate. Hence, an oriented m-simplex V_m of $B\Gamma$ is of the form $p(V_m) = [g_1|\cdots|g_m]$; hence we construct the following chain complex, with $C_m(B\Gamma)$ the free module generated by the basis of m-simplexes.

$$\cdots \longrightarrow C_m(B\Gamma) \xrightarrow{\partial} C_{m-1}(B\Gamma) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(B\Gamma) \xrightarrow{\partial} 0$$

$$c_\alpha \in \mathbb{C}, \quad \sum_\alpha c_\alpha p(V_m)_\alpha \in C_m(B\Gamma) \qquad \partial = \sum_{i=0}^{m} (-1)^i d_i$$

Thus, explicitly expressing the boundary map action shows that $\partial(p(V_m)_\alpha)$ is equal to

$$p(g_0 g_1[g_2|\cdots|g_m]) + \sum_{i=1}^{m-1} (-1)^i p(g_0[g_1|\cdots|g_i g_{i+1}|\cdots|g_m]) + (-1)^m p(g_m g_0[g_1|\cdots|g_{m-1}])$$

$$= [g_2|\cdots|g_m] + \sum_{i=1}^{m-1} (-1)^i [g_1|\cdots|g_i g_{i+1}|\cdots|g_m] + (-1)^m [g_1|\cdots|g_{m-1}]$$

Dualizing, the associated simplicial cochain complex $C_i^*(B\Gamma) := \mathsf{Hom}(C_i(B\Gamma), \mathbb{C})$ is obtained, with coboundary map $b = \sum_{i=0}^{n+1} \delta^i$; recall that $\langle \delta^i f, v \rangle = \langle f, d_i(v) \rangle$.

$$0 \longrightarrow C_0^*(B\Gamma) \xrightarrow{b} C_1^*(B\Gamma) \xrightarrow{b} \cdots \xrightarrow{b} C_m^*(B\Gamma) \xrightarrow{b} \cdots$$

The simplicial cohomology groups $H^*(B\Gamma, \mathbb{C})$ of this complex give precisely the same characteristic classes as the group cohomology $H^*(\Gamma, \mathbb{C})$. More usefully, the expression of bundles in the language of balanced products allows us to similarly obtain equivalences

$$H^*(EZ_\gamma \times_{Z_\gamma} Z_\gamma, \mathbb{C}) = H^*(BZ_\gamma, \mathbb{C}) = H^*(Z_\gamma, \mathbb{C})$$

$$H^*(E\Gamma \times_\Gamma (\Gamma/Z_\gamma), \mathbb{C}) = H^*((Z_\gamma \curvearrowright E\Gamma)/Z_\gamma, \mathbb{C}) = H^{Z_\gamma{}^*}(\Gamma, \mathbb{C}) = H^*(\mathcal{D}, \mathbb{C})$$

It follows that exhibiting an explicit Z_γ-equivariant map $\psi : E\Gamma \times_\Gamma (\Gamma/Z_\gamma) \longrightarrow EZ_\gamma \times_{Z_\gamma} Z_\gamma$ which induces a polynomial growth preserving isomorphism on cohomology

$$\psi^* : H^*(EZ_\gamma \times_{Z_\gamma} Z_\gamma, \mathbb{C}) \longrightarrow H^*(E\Gamma \times_\Gamma (\Gamma/Z_\gamma), \mathbb{C}) \tag{3.11}$$

also provides an isomorphism $\psi^* : H^*(Z_\gamma, \mathbb{C}) \longrightarrow H^*(\mathcal{D}, \mathbb{C})$ preserving polynomial cohomology. It is of course necessary to be precise by what is meant by polynomial cohomology in the context of the classifying space construction. If for every class $[\varphi] \in H^m(B\Gamma, \mathbb{C})$ there exists a representative $\widetilde{\varphi} \in (C_m^*(B\Gamma), b)$ which is of polynomial growth, then $H^*(B\Gamma, \mathbb{C})$ is polynomial cohomology. Viewing $\widetilde{\varphi}$ as a function on basis elements, it is of polynomial growth given

$$\begin{aligned} \widetilde{\varphi} = \sum_\alpha c_\alpha p(V_m)_\alpha = \sum_\alpha c_\alpha p(g_{0_\alpha}[g_{1_\alpha}|\cdots|g_{m_\alpha}]) := \widetilde{\varphi}(g_{0_\alpha}, g_{1_\alpha}, \ldots, g_{m_\alpha}) = c_\alpha \\ |\widetilde{\varphi}(g_{0_\alpha}, g_{1_\alpha}, \ldots, g_{m_\alpha})| = |c_\alpha| \le R_{\widetilde{\varphi}}(1 + ||g_{0_\alpha}||)^{2k}(1 + ||g_{1_\alpha}||)^{2k} \cdots (1 + ||g_{m_\alpha}||)^{2k} \end{aligned} \tag{3.12}$$

where $R_{\widetilde{\varphi}}$ and k are positive integer constants. Recall that the word length function l_Z is bounded above by l_Γ, hence we shall view Z_γ as metrically embedded in Γ. Fix some generating set S of Γ and also fix an ordering for S, then every $g \in \Gamma$ can be lexicographically ordered; denote by $\mathsf{lex}(\Gamma)$ the lexicographic ordering of Γ under that of S. We introduce a map $f : \Gamma \longrightarrow Z_\gamma$ defined as the following minimizing $z \in \mathsf{lex}(\Gamma) \cap Z_\gamma$.

$$f(g) := \min_{z \in \mathsf{lex}(\Gamma) \cap Z_\gamma} \left\{ \min_{z \in Z_\gamma} \{ d_\Gamma(z, g) \leq ||g|| \} \right\} \tag{3.13}$$

Due to the lexicographic ordering this provides a unique element $f(g) \in Z_\gamma$. Such an element always exists, since there is at least the option $z = e$; in particular, it is a direct consequence that $f(z) = z$ for $z \in Z_\gamma$. Moreover, using the fact that left multiplication of any group on itself is a free and transitive action it follows that f is Z_γ-equivariant, since if we fix $z_0 \in Z_\gamma$

$$f(z_0 g) := \min_{z \in \mathsf{lex}(\Gamma) \cap Z_\gamma} \left\{ \min_{z \in Z_\gamma} \{ d_\Gamma(z, z_0 g) \leq ||z_0 g|| \} \right\}$$

$$= \min_{z_0 z \in \mathsf{lex}(\Gamma) \cap Z_\gamma} \left\{ \min_{z_0 z \in Z_\gamma} \{ d_\Gamma(z_0 z, z_0 g) \leq ||z_0 g|| \} \right\}$$

$$= \min_{z_0 z \in \mathsf{lex}(\Gamma) \cap Z_\gamma} \left\{ \min_{z_0 z \in Z_\gamma} \{ d_\Gamma(z, g) \leq ||z_0 g|| \} \right\} =: z_0 f(g)$$

It is similarly possible to express a Z_γ-invariant map $\tilde{f} : \Gamma / Z_\gamma \longrightarrow Z_\gamma$ in terms of the map f constructed above. We recall that for $h, g \in \Gamma$

$$d_\Gamma(hZ_\gamma, g) = \min_{z \in Z_\gamma} \{ d_\Gamma(hz, g) \} \qquad \text{and} \qquad d_\Gamma(h_0 Z_\gamma, h_1 Z_\gamma) = \min_{z, z' \in Z_\gamma} \{ d_\Gamma(h_0 z, h_1 z') \}$$

and thus define $\tilde{f}(hZ_\gamma)$ to be the value of $f(hz)$ present in the minimization

$$\min_{z \in Z_\gamma} \{ d_\Gamma(hz, f(hz)) \}$$

By the bijection between cosets of Γ / Z_γ and conjugacy classes of γ arising from the map $hZ_\gamma \longmapsto h^{-1} \gamma h$ we ensure well-definedness of $\tilde{f}$. The map ψ acts on $E\Gamma \times_\Gamma (\Gamma / Z_\gamma)$ according to

$$\psi(x, hZ_\gamma) = \psi\left(\sum_i t_i \mathbf{g}_i, hZ_\gamma\right) = \left(\sum_i t_i(e, \ldots, f(g_i), \ldots, e), \tilde{f}(hZ_\gamma)\right) \tag{3.14}$$

From this we can show that ψ is Z_γ-equivariant on $E\Gamma$ and Z_γ-invariant in the second argument. Pick any $(z, z') \in Z_\gamma \times Z_\gamma$, then converting the right Γ-action on $E\Gamma$ to a left one

$$\psi((z, z')(x, hZ_\gamma)) = \psi(z^{-1}x, z'hZ_\gamma) = \psi\left(\sum_i t_i(z^{-1}\mathbf{g}_i), z'hZ_\gamma\right)$$

$$= \left(\sum_i t_i(z^{-1}, \ldots, f(z^{-1}g_i), \ldots, z^{-1}), \tilde{f}(z'hZ_\gamma)\right)$$

$$= \left(\sum_i t_i(z^{-1}e, \ldots, z^{-1}f(g_i), \ldots, z^{-1}e), \tilde{f}(hZ_\gamma)\right)$$

$$= \left(\sum_i t_i \cdot z^{-1}(e, \ldots, f(g_i), \ldots, e), \tilde{f}(hZ_\gamma)\right) = (z, e) \cdot \psi(x, hZ_\gamma)$$

By the definition provided by equation (3.12) the map $\psi : E\Gamma \times_\Gamma (\Gamma/Z_\gamma) \longrightarrow EZ_\gamma \times_{Z_\gamma} Z_\gamma$ preserves polynomial growth if there exists a positive integer r and constant $K > 0$ such that

$$(\rho, d_\Gamma)_p\left(\psi(x, h_0 Z_\gamma), \psi(y, h_1 Z_\gamma)\right) \leq K \cdot \left[(\rho, d_\Gamma)_p\left((x, h_0 Z_\gamma), (y, h_1 Z_\gamma)\right)\right]^r \tag{3.15}$$

This ensures that for any change $\lambda|\phi|$ in the value of a cyclic cocycle representative of the class $[\varphi] \in H^*(EZ_\gamma \times_{Z_\gamma} Z_\gamma, \mathbb{C})$ there exists a representative of $\psi^*[\varphi] \in H^*(E\Gamma \times_\Gamma (\Gamma/Z_\gamma), \mathbb{C})$ whose absolute value $|\psi^*\phi|$ changes no more than $K(\lambda|\phi|)^r$. As a preliminary necessity to proving this property of ψ, we recall that a path metric can be put on $E\Gamma$ such that for x, y not in the same connected component $\rho(x, y) = 1$, and otherwise if x and y are joined by a union of paths $\bigcup_{l=1}^k \alpha_l$, where each α_l belongs to a single simplex

$$\rho(x, y) = \inf_{\alpha_l} \sum_{l=1}^{k} \text{length}(\alpha_l)$$

Hence there exists points v and v' in this simplex such that length$(\alpha_l) = \rho(v, v')$; the metric ρ is defined as $\min\{\rho_1, \rho_2\}$, where

$$\rho_1\left(\sum_i t_i\mathbf{g}_i, \sum_i t'_i\mathbf{g}_i\right) = \sum_i |t_i - t'_i|||g_i|| \qquad \rho_2\left(\sum_i t_i\mathbf{g}_i, \sum_i t'_i\mathbf{g}_i\right) = \sum_{i,j} t_i t'_j d_\Gamma(g_i, g_j)$$

The metric placed on Γ/Z_γ will be the usual word metric d_Γ, and we assign to $E\Gamma \times_\Gamma (\Gamma/Z_\gamma)$ the p-product metric $(\rho, d_\Gamma)_p$, for $p \in [1, \infty)$. By the left Γ-invariance of d_Γ, and the fact that $(xg, w) \sim (x, gw)$ it follows that $(\rho, d_\Gamma)_p$ is also left Γ-invariant. Without loss of generality we may take x and y to belong to the same connected component, and since the collection of left cosets partition Γ, we assume that h_0Z_γ is distinct from h_1Z_γ. The proof of the inequality (3.15) thus follows immediately from the definition of ψ: explicitly, $(\rho_1, d_\Gamma)_p\,(\psi(x, h_0Z_\gamma), \psi(y, h_1Z_\gamma))$ has the expression

$$\left\|\rho_1\left(\sum_{l=1}^{k}\sum_{i_l} t_{i_l} f(\mathbf{g}_{i_l}), \sum_{l=1}^{k}\sum_{i_l} t'_{i_l} f(\mathbf{g}_{i_l})\right), d_\Gamma(\tilde{f}(h_0Z_\gamma), \tilde{f}(h_1Z_\gamma))\right\|_p$$

$$= \left\|\sum_{l=1}^{k}\sum_{i_l,j_l} t_{i_l}t'_{j_l} d_\Gamma(f(g_{i_l}), f(g_{j_l})), d_\Gamma(\tilde{f}(h_0Z_\gamma), \tilde{f}(h_1Z_\gamma))\right\|_p$$

$$\leq \left\|\sum_{l=1}^{k}\sum_{i_l,j_l} 4t_{i_l}t'_{j_l} d_\Gamma(g_{i_l}, g_{j_l}), 4d_\Gamma(h_0Z_\gamma, h_1Z_\gamma)\right\|_p$$

The inequality in the last line follows from the bound $d_\Gamma(f(g_i), f(g_j) \leq 4d_\Gamma(g_i, g_j)$ proven below in Proposition 3.8 and the analogous one for $\tilde{f}$. Similarly, with respect to the metric ρ_2 we have

$$\left\|\rho_2\left(\sum_{l=1}^{k}\sum_{i_l} t_{i_l} f(\mathbf{g}_{i_l}), \sum_{l=1}^{k}\sum_{i_l} t'_{i_l} f(\mathbf{g}_{i_l})\right), d_\Gamma(\tilde{f}(h_0Z_\gamma), \tilde{f}(h_1Z_\gamma))\right\|_p$$

$$= \left\|\sum_{l=1}^{k}\sum_{i_l} |t_{i_l} - t'_{i_l}|||f(g_{i_l})||, d_\Gamma(\tilde{f}(h_0Z_\gamma), \tilde{f}(h_1Z_\gamma))\right\|_p$$

$$\leq \left|\left|\sum_{l=1}^{k}\sum_{i_l} 2|t_{i_l} - t'_{i_l}|||g_{i_l}||, 4d_\Gamma(h_0 Z_\gamma, h_1 Z_\gamma)\right|\right|_p$$

The factor of 2 present in the last line stems from the fact that

$$||f(g_i)|| = d_\Gamma(f(g_i), e) \leq d_\Gamma(f(g_i), g_i) + d_\Gamma(g_i, e) \leq d_\Gamma(e, g_i) + d_\Gamma(g_i, e) = 2||g_i||$$

In combination with the result for ρ_1 this proves the inequality (3.15) for $K = 4$ and $r = 1$. □

Corollary 3.7. *Every delocalized cyclic cocycle class* $[\varphi_\gamma] \in HC^*(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ *has a representative* $\varphi_{\alpha,\gamma}$ *of polynomial growth, hence* $H^*(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ *is polynomially bounded.*

Proof. Since Γ is of polynomial growth, then the proof of Lemma 3.3 asserts that $H^*(Z_\gamma, \mathbb{C})$ is of polynomial cohomology. Furthermore, if we denote by $\mathcal{R}^{-1}$ the inverse isomorphism to that constructed in Lemma 3.4, then by Proposition 3.5 and Theorem 3.6 there exists a chain of isomorphisms which preserve polynomial cohomology for all $n \geq 1$.

$$H^n(Z_\gamma, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{(\psi \otimes_{\mathbb{Z}} 1)^*} H^n(\mathcal{D}, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{(\mathcal{R}^{-1} \otimes_{\mathbb{Z}} 1/A^{n+1})^*} H^n(\mathcal{C}, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

The desired result now follows from recalling that by Definition 2.28 there exists an explicit representation $\varphi_{\alpha,\gamma} \in (C^n(\Gamma, Z_\gamma, \gamma), \hat{b})$ for each $\varphi_\gamma \in (C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$. □

Proposition 3.8. *Let* $f : \Gamma \longrightarrow Z_\gamma$ *and* $\tilde{f} : \Gamma/Z_\gamma \longrightarrow Z_\gamma$ *be as described in Theorem 3.6, then both have a Lipschitz constant of 4.*

Proof. Given distinct $g_i, g_j \in \Gamma$ with word representations $g_i = s_{i_1}s_{i_2}\cdots s_{i_k}$ and $g_j = s_{j_1}s_{j_2}\cdots s_{j_k}$

$$d_\Gamma(g_i, g_j) = ||g_i g_j^{-1}|| = l_\Gamma(s_{i_1}s_{i_2}\cdots s_{i_k - m}s_{j_k - m}^{-1}\cdots s_{j_2}^{-1}s_{j_1}^{-1}) = j_k + i_k - 2m$$

where m is the cancellation length. By definition of f provided above in (3.13) there exist lexicographically minimal $z_i = f(g_i)$ and $z_j = f(g_j)$ such that $d_\Gamma(z_i, g_i)$ and $d_\Gamma(z_j, g_j)$ are minimized. By the properties of the metric, it is immediate that

$$d_\Gamma(z_i, z_j) \leq d_\Gamma(z_i, g_i) + d_\Gamma(g_i, g_j) + d_\Gamma(g_j, z_j) \leq i_k + (j_k + i_k - 2m) + j_k \leq 2(i_k + j_k)$$

On the other hand, expressing z_i and z_j as the words $z_i = s'_{i_1} s'_{i_2} \cdots s'_{i_{k'}}$ and $z_j = s'_{j_1} s'_{j_2} \cdots s'_{j_{k'}}$

$$d_\Gamma(z_i, g_i) = ||z_i g_i^{-1}|| = l_\Gamma(s'_{i_1} s'_{i_2} \cdots s'_{i_{k'}-b} s^{-1}_{i_k - b} \cdots s^{-1}_{i_2} s^{-1}_{i_1}) = i_{k'} + i_k - 2b$$

$$d_\Gamma(g_j, z_j) = ||g_j z_j^{-1}|| = l_\Gamma(s_{j_1} s_{j_2} \cdots s_{j_k - a} s'^{-1}_{j_{k'}-a} \cdots s'^{-1}_{j_2} s'^{-1}_{j_1}) = j_{k'} + j_k - 2a$$

We can thus make use of the decomposition $z_i z_j^{-1} = z_i g_i^{-1} g_i g_j^{-1} g_j z_j^{-1}$ and obtain the reduced word expression for $z_i z_j^{-1}$ as

$$s'_{i_1} \cdots s'_{i_{k'}-b} s^{-1}_{i_k - b} \cdots s^{-1}_{i_1} s_{i_1} \cdots s_{i_k - m} s^{-1}_{j_k - m} \cdots s^{-1}_{j_1} s_{j_1} \cdots s_{j_k - a} s'^{-1}_{j_{k'}-a} \cdots s'^{-1}_{j_1}$$

$$= s'_{i_1} s'_{i_2} \cdots s'_{i_{k'}-b} s^{-1}_{i_k - b} \cdots s^{-1}_{i_k - m - 1} s_{j_k - m - 1} \cdots s_{j_k - a} s'^{-1}_{j_{k'}-a} \cdots s'^{-1}_{j_2} s'^{-1}_{j_1}$$

where– without loss of generality– we have assumed that $m \geq a, b$. It follows that

$$l_\Gamma(z_i z_j^{-1}) = i_{k'} - b + (i_k - b - i_k + m + 1) + (j_k - a - j_k + m + 1) + j_{k'} - a$$

$$= i_{k'} + j_{k'} - 2a - 2b + 2m + 2$$

However, since the identity element is always a possible choice for z_i we know that $i_{k'} + i_k - 2b \leq i_k$, from which it follows that $i_{k'} - 2b \leq 0$, and analogously $j_{k'} + j_k - 2a \leq j_k$ implies that $j_{k'} - 2a \leq 0$; this provides the bound $d_\Gamma(z_i, z_j) \leq 2m + 2$. We thus have the two following cases:

(i) If $m \geq i_k + j_k$ then $d_\Gamma(g_i, g_j) = j_k + i_k - 2m \geq \frac{1}{2}(j_k + i_k)$, and the bound $d_\Gamma(z_i, z_j) \leq 2(i_k + j_k)$ provides: $d_\Gamma(z_i, z_j) \leq 4 d_\Gamma(g_i, g_j)$

(ii) If $m \leq i_k + j_k$ then $d_\Gamma(g_i, g_j) = j_k + i_k - 2m \geq 2m$, and the bound $d_\Gamma(z_i, z_j) \leq 2m + 2$ provides: $d_\Gamma(z_i, z_j) \leq 2 d_\Gamma(g_i, g_j)$

An analogous result is obtained for $\tilde{f}$ using its definition with respect to f and the inequalities already proven for f. The proof follows along the exact same lines, explicitly providing

$$\min_{z,z' \in Z_\gamma} \{d_\Gamma(f(h_0 z), f(h_1 z'))\} \leq 4 \min_{z,z' \in Z_\gamma} \{d_\Gamma(h_0 z, h_1 z')\}$$

$$d_\Gamma(\tilde{f}(h_0 Z_\gamma), \tilde{f}(h_1 Z_\gamma)) \leq 4 d_\Gamma(h_0 Z_\gamma, h_1 Z_\gamma)$$

□

When choosing a representative of polynomial growth for the purposes of the following sections, it is paramount that this choice is independent of representative in a way that respects polynomial growth. Explicitly, let $[\varphi_\gamma]$ be a delocalized cyclic cocycle class with polynomial growth representatives $\psi_{\gamma,1}, \psi_{\gamma,2} \in (C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$. Since by definition the group $H^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ is the quotient

$$ZC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))/BC^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$$

then $\psi_{\gamma,1}$ and $\psi_{\gamma,2}$ being cohomologous implies existence of a delocalized cyclic cocycle ϕ belonging to $(C^{n-1}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ such that $\psi_{\gamma,1} - \psi_{\gamma,2} = b\phi$.

Remark 3.9. *If $[\varphi_\gamma] \in H^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ has polynomial growth representatives $\psi_{\gamma,1}$ and $\psi_{\gamma,2}$, then there exists a cyclic cocycle ϕ of polynomial growth such that $\psi_{\gamma,1} - \psi_{\gamma,2} = b\phi$.*

Proof. For any length function l on Γ it is proven by Ji [23, Theorem 2.23] that the inclusion $i : \mathbb{C}\Gamma \hookrightarrow S_1^l(\Gamma)$ induces an isomorphism between the Schwartz cohomology ${}^sH_1^n(\Gamma, \mathbb{C}) = H^n(S_1^l(\Gamma), \mathbb{C})$ and the group cohomology $H^n(\Gamma, \mathbb{C})$ for Γ a discrete countable group of polynomial growth. In particular, we have ${}^sH_l^n(Z_\gamma, \mathbb{C}) \cong H^n(Z_\gamma, \mathbb{C})$ and so Proposition 3.5 along with the strategy of Theorem 3.6 provides for a polynomial growth preserving map such that ϕ is the image of a representative of an element in ${}^sH_l^n(Z_\gamma, \mathbb{C})$. □

4. PAIRING OF CYCLIC COHOMOLOGY CLASSES IN ODD DIMENSION*

4.1 Delocalized Higher Eta Invariant

Let $\widetilde{D}$ be the Dirac operator lifted to $\widetilde{M}$, denote by $\mathcal{S}$ the associated spinor bundle, and by $\nabla : C^\infty(\widetilde{M}, \mathcal{S}) \longrightarrow C^\infty(\widetilde{M}, T^*\widetilde{M} \otimes \mathcal{S})$ the connection on $\mathcal{S}$. Since M has positive scalar curvature $\kappa > 0$ associated to $\widetilde{g}$, then Lichnerowicz's formula [29]

$$\widetilde{D}^2 = \nabla\nabla^* + \frac{\kappa}{4} \tag{4.1}$$

implies that $\widetilde{D}$ is invertible. Moreover, $\widetilde{D}$ is a self-adjoint elliptic operator, and so possesses a real spectrum: $\sigma(\widetilde{D}) \subset \mathbb{R}$. The invertibility condition particularly provides existence of a spectral gap at 0, which will be necessary in ensuring convergence of the integral introduced in Definition 4.2. This will be a higher analogue of the delocalized eta invariant Lott [32] introduced in the case of 0-dimensional cyclic cocycles– that is for traces. Given a non-trivial conjugacy class $\mathrm{cl}(\gamma)$ of the fundamental group $\Gamma = \pi_1(M)$, Lott's delocalized eta invariant can be formally defined as the pairing between Lott's higher eta invariant and traces.

$$\eta_{\mathrm{tr}_\gamma}(\widetilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{tr}_\gamma \left(\widetilde{D} e^{-t^2 \widetilde{D}^2} \right) \tag{4.2}$$

Here the trace map $\mathrm{tr} : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ continuously extends to a suitable smooth dense subalgebra of $C_r^*(\Gamma)$ to which $\widetilde{D} e^{-t^2 \widetilde{D}^2}$ belongs. Generally, if $\mathcal{F}$ is a fundamental domain of $\widetilde{M}$ under the action of Γ, then for Γ-equivariant kernels $A \in C^\infty(\widetilde{M} \times \widetilde{M})$

$$\mathrm{tr}_\gamma(A) = \sum_{g \in \mathrm{cl}(\gamma)} \int_{\mathcal{F}} A(x, gx)\, dx \tag{4.3}$$

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Under the assumption of hyperbolicity or polynomial growth of the conjugacy class of γ, Lott [32] showed convergence of the above integral. Invertibility of $\widetilde{D}$ is in general a necessary condition for this convergence, as was shown by the construction of a divergent counterexample by Piazza and Schick [37, Section 3]. However, it was proven by Chen, Wang, Xie and Yu [11, Theorem 1.1] that as long as the spectral gap of $\widetilde{D}$ is sufficiently large, then $\eta_{\mathrm{tr}_\gamma}(\widetilde{D})$ converges absolutely, and does not require any restriction on the fundamental group of the manifold.

Since we shall have occasion to use their properties often, we shall briefly recall the most important aspects of the space $S(\mathbb{R})$ of Schwartz functions. By definition, f belongs to $S(\mathbb{R})$ if $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a smooth function such that for every $k, m \in \mathbb{N}$

$$\lim_{|x| \longrightarrow \infty} x^k \frac{d^m}{dx^m}(f(x)) = 0$$

This implies that f is bounded with respect to the family of semi-norms

$$||f||_{k,m} = \sup_{x \in \mathbb{R}} \left| x^k \frac{d^m}{dx^m}(f(x)) \right| \tag{4.4}$$

Moreover, the Fourier transform $f \longmapsto \hat{f}$ is an automorphism of the Schwartz space, thus $\hat{f} \in S(\mathbb{R})$ for every Schwartz function f, where

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \tag{4.5}$$

Lemma 4.1. *If Φ is a Schwartz function and $\widetilde{D}$ is the lifted Dirac operator associated to $\widetilde{M}$, then $\Phi(\widetilde{D}) \in \mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma$.*

Proof. This is proven as Proposition 4.6 in [50] □

Definition 4.2. For any delocalized cyclic cocycle class $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ the delocalized higher eta invariant of $\widetilde{D}$ with respect to $[\varphi_\gamma]$ is defined as

$$\eta_{\varphi_\gamma}(\widetilde{D}) := \frac{m!}{\pi i}\int_0^\infty \eta_{\varphi_\gamma}(\widetilde{D},t)\, dt \tag{4.6}$$

where $\eta_{\varphi_\gamma}(\widetilde{D},t) = \varphi_\gamma((\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D}))\hat{\otimes}((u_t(\widetilde{D})-\mathbb{1})\hat{\otimes}(u_t^{-1}(\widetilde{D})-\mathbb{1}))^{\hat{\otimes}m})$ and

$$F_t(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{tx} e^{-s^2}\, ds \qquad u_t(x) = e^{2\pi i F_t(x)} \qquad \dot{u}_t(x) = \frac{d}{dt}u_t(x)$$

Note that the arguments of $\eta_{\varphi_\gamma}(\widetilde{D},t)$ all belong to $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$, since $u_t(x)-1, u_t^{-1}(x)-1$ and $\dot{u}_t(x)u_t^{-1}(x)$ are all Schwartz functions. In particular, we have the simplification

$$\dot{u}_t(x)u_t^{-1}(x) = 2\pi i\left(\frac{d}{dt}F_t(x)\right)e^{2\pi i F_t(x)}e^{-2\pi i F_t(x)} = 2\pi i\left(\frac{d}{dt}F_t(x)\right) = 2i\sqrt{\pi}xe^{-t^2x^2}$$

It is also useful to consider the representation of the delocalized higher eta invariant in terms of smooth Schwartz kernels, namely if L_i is an element of the convolution algebra $\mathscr{L}(\widetilde{M},\mathcal{S})^\Gamma$, the action of φ_γ on $\mathbb{C}\Gamma$ can be extended to $\mathscr{L}(\widetilde{M},\mathcal{S})^\Gamma$ by– abusing notation a little we will denote the Schwartz kernel of L_i by L_i also– defining $\varphi_\gamma(L_0\hat{\otimes}L_1\hat{\otimes}\ldots\hat{\otimes}L_n)$ to be

$$\sum_{g_0g_1\cdots g_n\in \mathrm{cl}(\gamma)} \varphi_\gamma(g_0,\ldots,g_n)\int_{\mathcal{F}^{n+1}} \mathsf{tr}\left(\prod_{i=0}^n L_i(x_i, g_ix_{i+1})\right) dx_0\cdots dx_n \quad : \ x_{n+1}=x_0 \tag{4.7}$$

where $\mathcal{F}$ is the fundamental domain of $\widetilde{M}$ under the action of $\Gamma=\pi_1(M)$, and tr denotes the pointwise matrix trace, not to be confused with the trace norm $||\cdot||_{tr}$ for trace class operators. Denoting by $a_t(x,y), b_t(x,y)$ and $k_t(x,y)$ the Schwartz kernels of the operators $u_t(\widetilde{D})-\mathbb{1}, u_t^{-1}(\widetilde{D})-\mathbb{1}$ and $\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D})$ respectively, then $\eta_{\varphi_\gamma}(\widetilde{D},t)$ is given by

$$\sum_{g_0g_1\cdots g_{2m}\in \mathrm{cl}(\gamma)} \varphi_\gamma(\mathbf{g}_{2m})\int_{\mathcal{F}^{2m+1}} \mathsf{tr}\left(k_t(x_0,g_0x_1)\prod_{i=1}^{2m-1} a_t(x_i,g_ix_{i+1})b_t(x_{i+1},g_{i+1}x_{i+2})\right) d\mathbf{x}_{2m}$$

$$\mathbf{g}_{2m} := (g_0,\ldots,g_{2m}) \qquad d\mathbf{x}_{2m} := dx_0\cdots dx_{2m} \tag{4.8}$$

In this form we can better exploit the properties of $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$, in order to prove that $\eta_{\varphi_\gamma}(\widetilde{D})$ converges for $\widetilde{D}$ invertible and Γ of polynomial growth. The first step is proving extension of delocalized cyclic cocycles on the smooth dense subalgebra $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$, in terms of kernel operators. Since the fundamental group of a manifold has a cocompact, isometric, and properly discontinuous action on the universal cover, by the Švarc-Milnor lemma [42, 34] there is a quasi-isometry $f:\pi_1(M)\longrightarrow\widetilde{M}$; for every $g,h\in\Gamma$ there exists $K\geq 1,\ell\geq 0$ such that

$$d_\Gamma(g,h)-K\ell\leq Kd_{\widetilde{M}}(f(g),f(h))\leq K^2d_\Gamma(g,h)+K\ell$$

and for every $y\in\widetilde{M}$ there exists $g_y\in\Gamma$ such that $d_{\widetilde{M}}(f(g_y),y)\leq\ell$. In particular, we may fix some $p\in\widetilde{M}$ and define $f(g)=gp$; moreover, restricting our attention to points belonging to $\mathcal{F}$ the value of (K,ℓ) can be taken to be $(1,\mathrm{diam}(\mathcal{F}))$ since each orbit is cobounded.

We note that in [11, Section 8], under the assumption that $\pi_1(M)$ is of polynomial growth, the authors used the techniques of [13, 14] to establish that the above definition of the delocalized higher eta invariant agrees with Lott's higher eta invariant [31, Section 4.4 & 4.6] up to a constant.

Theorem 4.3. *Let $\Gamma=\pi_1(M)$ and $\varphi_\gamma\in(C^n(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$ be a delocalized cyclic cocycle of polynomial growth, then φ_γ extends continuously on the algebra $(\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma)^{\hat{\otimes}_\pi^{n+1}}$.*

Proof. Denote by $\hat{\rho}:\widetilde{M}\longrightarrow[0,\infty)$ the distance function $\hat{\rho}(x)=\hat{\rho}(x,y_0)$ for some fixed point $y_0\in\widetilde{M}$, with ρ being the modification of $\hat{\rho}$ near y_0 to ensure smoothness. Let $B\in\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$ and recall that we have the norm $||B||_{\mathscr{A},k}=||\widetilde{\partial}^k(B)\circ(\widetilde{D}^{2n_0}+1)||_{op}$; for any $f\in L^2(\widetilde{M},\mathcal{S})$ the Sobolev embedding theorem provides existence of some constant C such that

$$|B(f)(x)|\leq C||(1+\widetilde{D}^{2n_0})B(f)||_{L^2(\widetilde{M},\mathcal{S})}\leq C||(\widetilde{D}^{2n_0}+1)B||_{op}||f||_{L^2(\widetilde{M},\mathcal{S})}$$

In particular, since f is arbitrary, the bound $||B(x,\cdot)||_{L^2(\widetilde{M},\mathcal{S})}\leq C||(\widetilde{D}^{2n_0}+1)B||_{op}$ shows that taking the supremum over all $(x,y) \in \widetilde{M} \times \widetilde{M}$ the Schwartz kernel

$$\widetilde{\partial}^k(B)(x,y) = (\rho(x) - \rho(y))^k B(x,y)$$

has operator norm bounded by $||B||_{\mathscr{A},k}$, hence it is a uniformly bounded continuous function for all $k \in \mathbb{N}$. Now view $B(x,y)$ as a matrix acting on the spinors $f(y)$, where each section f has a representation as a matrix in the complex Clifford algebra $\mathbb{C}\ell_{\dim(M)}$. If I is the identity matrix, then by the Holder inequality for Schatten p-norms

$$|\mathsf{tr}(B(x,y))| \leq ||B(x,y)||_{tr} = ||B(x,y)I||_{tr} \leq ||B(x,y)||_{op}||I||_{tr} < 2^{n_0}||B(x,y)||_{op}$$

Since all points of $\widetilde{M}$ belong to some orbit of the fundamental domain we have the bound $|\rho(x_i) - \rho(x_{i+1})| \leq \text{diam}(\mathcal{F})$, and by quasi-isometry of Γ and the universal cover, we have (taking a family of quasi-isometries $f_i(g) = gx_i$)

$$|\rho(x_{i+1}) - \rho(gx_{i+1})| \geq d_\Gamma(e,g) - \text{diam}(\mathcal{F}) = ||g|| - \text{diam}(\mathcal{F})$$

From this, an application of the reverse triangle inequality provides the bound

$$|\rho(x_i) - \rho(gx_{i+1})| = |\rho(x_i) - \rho(x_{i+1}) + \rho(x_{i+1}) - \rho(gx_{i+1})|$$

$$\geq |\rho(x_{i+1}) - \rho(gx_{i+1})| - |\rho(x_i) - \rho(x_{i+1})| \geq ||g|| - \text{diam}(\mathcal{F}) - \text{diam}(\mathcal{F})$$

Denoting the matrix norm $||\cdot||_{op}$ by $|\cdot|$, the boundedness properties of the Schwartz kernel implies existence of a constant $C_k > 0$ such that for each $k \in \mathbb{N}$

$$C_k|\widetilde{\partial}^{3k}(B_i)(x_i, gx_{i+1})|^2 = (\rho(x_i) - \rho(gx_{i+1}))^{6k}|B_i(x_i, gx_{i+1})|^2 \geq (1 + ||g||)^{6k}|B_i(x_i, gx_{i+1})|^2$$

We will use the explicit representation $\varphi_{\alpha,\gamma}$ of φ_γ, and for ease of notation, we shorten the argument of α by writing $\alpha(\mathbf{g}_n)$; we wish to prove convergence of the following sum.

$$\varphi_{\alpha,\gamma}(B_0\hat{\otimes}B_1\hat{\otimes}\cdots\hat{\otimes}B_n) = \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} \alpha(\mathbf{g}) \int_{\mathcal{F}^{n+1}} \mathsf{tr}\left(\prod_{i=0}^{n} B_i(x_i, g_i x_{i+1})\right)\, dx_0\cdots dx_n \quad (4.9)$$

From the fact that α is of polynomial growth, and using the above inequalities coupled with Cauchy-Schwartz, $|\varphi_{\alpha,\gamma}(B_0\hat{\otimes}B_1\hat{\otimes}\cdots\hat{\otimes}B_n)|$ is bounded above by

$$\sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} |\alpha(\mathbf{g}_n)| \int_{\mathcal{F}^{n+1}} \left|\mathsf{tr}\left(\prod_{i=0}^{n} B_i(x_i, g_i x_{i+1})\right)\right|\, dx_0\cdots dx_n$$

$$\leq \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} 2^{n_0}|\alpha(\mathbf{g}_n)| \int_{\mathcal{F}^{n+1}} \left|\prod_{i=0}^{n} B_i(x_i, g_i x_{i+1})\right|\, dx_0\cdots dx_n$$

$$\leq \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} R_\alpha \prod_{i=0}^{n}(1+||g_i||)^{2k} \prod_{i=0}^{n}\left(\int_{\mathcal{F}^2} |B_i(x_i, g_i x_{i+1})|^2\, dx_i dx_{i+1}\right)^{1/2}$$

$$= \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} R_\alpha \prod_{i=0}^{n}\left(\int_{\mathcal{F}^2} (1+||g_i||)^{4k}|B_i(x_i, g_i x_{i+1})|^2\, dx_i dx_{i+1}\right)^{1/2}$$

$$\leq \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} R_\alpha C_k^{1/2} \prod_{i=0}^{n}\left((1+||g_i||)^{-2k}\int_{\mathcal{F}^2} |\widetilde{\partial}^{2k}(B_i)(x_i, g_i x_{i+1})|^2\, dx_i dx_{i+1}\right)^{1/2}$$

$$\leq R_\alpha C_k^{1/2} \prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}\int_{\mathcal{F}^2} |\widetilde{\partial}^{2k}(B_i)(x_i, g_i x_{i+1})|^2\, dx_i dx_{i+1}\right)^{1/2}$$

For each g_i the integral over the fundamental domain is finite, since $\widetilde{\partial}^{2k}(B_i)(x_i, g_i x_{i+1})$ is uniformly bounded; explicitly there exists a constant Λ_k such that for $g_0g_1\cdots g_n \in \mathrm{cl}(\gamma)$ the above product is bounded above by

$$\leq R_\alpha C_k^{1/2} \prod_{i=0}^{n} \left(\sum_{g_i \in \Gamma} (1+||g_i||)^{-2k} \int_{\mathcal{F}^2} \sup_{(x_i, g_i x_{i+1}) \in \mathcal{F} \times \mathcal{F}} |\widetilde{\partial}^{2k}(B_i)(x_i, g_i x_{i+1})|^2 \, dx_i dx_{i+1} \right)^{1/2}$$
$$R_\alpha C_k^{1/2} \prod_{i=0}^{n} \left(\sum_{g_i \in \Gamma} (1+||g_i||)^{-2k} \int_{\mathcal{F}^2} \Lambda_k^2 ||B_i||^2_{\mathscr{A},k} \, dx_i dx_{i+1} \right)^{1/2}$$
$$\leq R_\alpha C_k^{1/2} \mathrm{diam}(\mathcal{F}) \Lambda_k \prod_{i=0}^{n} \left(\sum_{g_i \in \Gamma} (1+||g_i||)^{-2k} ||B_i||^2_{\mathscr{A},k} \right)^{1/2} \tag{4.10}$$

where we will denote $R_{\alpha,k} = R_\alpha C_k^{1/2} \mathrm{diam}(\mathcal{F}) \Lambda_k$. Moreover, due to Γ being of polynomial growth, there exists k_i such that

$$(1+||g_i||)^{-2k_i} |\{g_i \in \Gamma : ||g_i|| \leq c\}| < \frac{1}{c^2}$$

It follows that each of the sums in the final expression (4.10) are finite for sufficiently large k, and thus so is any finite product of them. Now, by construction $\mathscr{L}(\widetilde{M})^\Gamma$ is a smooth dense sub-algebra of $\mathscr{A}(\widetilde{M})^\Gamma$, and this relationship also extends when considering their projective tensor products. We have just proven that $\varphi_{\alpha,\gamma}$ is continuous on $(\mathscr{A}(\widetilde{M})^\Gamma)^{\hat{\otimes}_\pi^{n+1}}$; to obtain the desired result it suffices to prove that for operators $B_0, \ldots, B_n \in \mathscr{L}(\widetilde{M})^\Gamma$, whenever $\sigma \in S_{n+1}$ is a cyclic shift

$$\mathrm{sgn}(\sigma) \varphi_{\alpha,\gamma}(B_0 \hat{\otimes} B_1 \hat{\otimes} \cdots \hat{\otimes} B_n) = \varphi_{\alpha,\gamma}(B_{\sigma(0)} \hat{\otimes} B_{\sigma(1)} \hat{\otimes} \cdots \hat{\otimes} B_{\sigma(n)}) \tag{4.11}$$

By application of the Fubini-Tonelli theorem, and since $\varphi_{\alpha,\gamma}$ is a cyclic cocycle on $\mathbb{C}\Gamma$, we obtain that $\varphi_{\alpha,\gamma}(B_{\sigma(0)} \hat{\otimes} B_{\sigma(1)} \hat{\otimes} \cdots \hat{\otimes} B_{\sigma(n)})$ is equal to

$$\sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} \mathrm{sgn}(\sigma) \varphi_{\alpha,\gamma}(\mathbf{g}_n) \int_{\mathcal{F}^{n+1}} \mathsf{tr} \left(\prod_{i=0}^{n} B_{\sigma(i)}(x_{\sigma(i)}, g_{\sigma(i)} x_{\sigma(i+1)}) \right) \, dx_{\sigma(0)} \cdots dx_{\sigma(n)}$$

$$= \operatorname{sgn}(\sigma) \sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} \varphi_{\alpha,\gamma}(\mathbf{g}_n) \int_{\mathcal{F}^{n+1}} \mathsf{tr}\left(\prod_{i=0}^{n} B_i(x_i, g_i x_{i+1}) \right) \, dx_0 \cdots dx_n$$

the latter expression clearly being the definition of $\operatorname{sgn}(\sigma)\varphi_{\alpha,\gamma}(B_0 \hat{\otimes} B_1 \hat{\otimes} \cdots \hat{\otimes} B_n)$. □

The following technical result is one which we will have occasion to use often, both in the remainder of this section and elsewhere.

Proposition 4.4. *For any collection of Schwartz functions $f_0, f_1, \ldots, f_n \in S(\mathbb{R})$ and any delocalized cyclic cocycle $\varphi_\gamma \in (C^n(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ of polynomial growth*

$$\lim_{t \longrightarrow 0} \varphi_\gamma(f_0(t\widetilde{D}) \hat{\otimes} f_0(t\widetilde{D}) \hat{\otimes} \cdots \hat{\otimes} f_n(t\widetilde{D})) = 0$$

Proof. Fix some $t \neq 0$ and consider the Schwartz functions $f_{i,t}(x) = f_i(tx)$ for $1 \leq i \leq n$; since the Fourier transform is an automorphism of $S(\mathbb{R})$ there exists $g_{i,t} \in S(\mathbb{R})$ such that $\hat{g}_{i,t} \equiv f_{i,t}$. Using the change of variables $x = y/t$, and the definition from (4.5), we obtain

$$f_{i,t}(\xi) = \hat{g}_{i,t}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{i,t}(x) e^{-i\xi x} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_i(tx) e^{-i\xi y/t} \, \frac{dy}{t}$$

$$= \frac{1}{2\pi t} \int_{-\infty}^{\infty} g_i(y) e^{-iy(\xi/t)} \, dy = \frac{\hat{g}_i(\xi/t)}{t} = f_i(t\xi)$$

Now since each of these functions is Schwartz, (4.4) asserts that the following limit exists and is finite; in particular, $\hat{g}_i(t\xi) \longrightarrow 0$ faster than any inverse power of t as $t \longrightarrow \infty$, from which it follows that

$$\lim_{t \longrightarrow 0} f_{i,t}(\xi) = \lim_{t \longrightarrow 0} f_i(t\xi) = \lim_{t \longrightarrow 0} \frac{\hat{g}_i(\xi/t)}{t} = \lim_{t \longrightarrow \infty} t \cdot \hat{g}_i(t\xi) = 0$$

Turning to functional calculus, by Lemma 4.1 each $f_i(t\widetilde{D})$ belongs to $\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma$, and the spectral gap at 0 of $\widetilde{D}$ ensures that $||f_i(t\widetilde{D})||_{\mathscr{A},k}$ converges to 0 as $t \longrightarrow 0$. From (4.10) of

Theorem 4.3 it follows that there exists a positive constant $R_{\alpha,k}$ such that

$$\lim_{t\longrightarrow 0} |\varphi_\gamma(f_0(t\widetilde{D})\hat{\otimes} f_0(t\widetilde{D})\hat{\otimes}\cdots\hat{\otimes} f_n(t\widetilde{D}))|$$

$$\leq \lim_{t\longrightarrow 0} R_{\alpha,k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}||f_i(t\widetilde{D})||^2_{\mathscr{A},k}\right)^{1/2} = 0$$

□

Lemma 4.5. *Let* $[\varphi_\gamma]\in HC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$*, then if* $\widetilde{D}$ *is invertible and* φ_γ *is of polynomial growth, then* $\eta_{\varphi_\gamma}(\widetilde{D})$ *converges absolutely.*

Proof. The higher delocalized eta invariant can be split into two integrals, as follows.

$$\eta_{\varphi_\gamma}(\widetilde{D}) := \frac{m!}{\pi i}\int_0^\infty \eta_{\varphi_\gamma}(\widetilde{D},t)\,dt = \frac{m!}{\pi i}\left(\int_0^1 \eta_{\varphi_\gamma}(\widetilde{D},t)\,dt + \int_1^\infty \eta_{\varphi_\gamma}(\widetilde{D},t)\,dt\right)$$

For the first integral, absolute convergence follows from Theorem 4.3 and Proposition 4.4, using the Schwartz kernel expression.

$$\int_0^1 |\eta_{\varphi_\gamma}(\widetilde{D},t)|\,dt \leq \sup_{t\in[0,1]} |\varphi_\gamma((\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D}))\hat{\otimes}((u_t(\widetilde{D})-\mathbb{1})\hat{\otimes}(u_t^{-1}(\widetilde{D})-\mathbb{1}))^{\hat{\otimes}m})| < \infty \quad (4.12)$$

For the second integral we use the fact that $\widetilde{D}$ is invertible to observe that the spectrum $\sigma(\widetilde{D})$ of $\widetilde{D}$, has finite spectral radius $r>0$. By the spectral mapping theorem for holomorphic functional calculus the operator $f(\widetilde{D}) = e^{-\widetilde{D}^2}$ has spectrum $f(\sigma(\widetilde{D})) = \sigma(f(\widetilde{D}))$, from which it follows that the spectral radius of $f(\widetilde{D})$ is equal to e^{-r^2}. Moreover, since $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$ is closed under holomorphic functional calculus the spectral radius of $f(\widetilde{D})$ is unchanged when viewed as an operator in $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$. By Gelfand's formula we have

$$e^{-r^2} = \lim_{n\longrightarrow\infty}\left(||e^{-n\widetilde{D}^2}||_{\mathscr{A},k}\right)^{\frac{1}{n}}$$

so for $t \in [n, n+1)$ there exists n large enough such that for some constants $C_0, C_1 > 0$

$$||C_0 e^{-t\widetilde{D}^2}||_{\mathscr{A},k} \leq ||e^{-n\widetilde{D}^2}||_{\mathscr{A},k} \leq e^{-nr^2/2} \leq C_1 e^{-tr^2/2} \tag{4.13}$$

It is useful to work in the unitization $(\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+$, for which $\overline{\varphi}_\gamma$ is well defined and continuous on the projective tensor product $((\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+)^{\hat{\otimes}_\pi^{2m+1}}$ by Theorem 4.3. Using the explicit expression $(\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D})) = 2i\sqrt{\pi}\widetilde{D}e^{-t^2\widetilde{D}^2}$ and the fact that $\mathbb{1} \in (\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+$, we follow an argument similar to that of [11, Proposition 3.30]. Firstly, note that

$$\begin{aligned}
&\varphi_\gamma((\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D}))\hat{\otimes}((u_t(\widetilde{D}) - \mathbb{1})\hat{\otimes}(u_t^{-1}(\widetilde{D}) - \mathbb{1}))^{\hat{\otimes}m}) \\
&= \overline{\varphi}_\gamma((\dot{u}_t(\widetilde{D})u_t^{-1}(\widetilde{D}))\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}) \\
&= \overline{\varphi}_\gamma(2i\sqrt{\pi}\widetilde{D}e^{-t^2\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}) \\
&= \overline{\varphi}_\gamma(2i\sqrt{\pi}\widetilde{D}e^{-\widetilde{D}^2} \times e^{-(t^2-1)\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}) \\
&= 2i\sqrt{\pi} \cdot \overline{\varphi}_\gamma(\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m} \times (e^{-(t^2-1)\widetilde{D}^2}\hat{\otimes}1^{\hat{\otimes}2m}))
\end{aligned}$$

Now we consider the inclusion map $\imath : (\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+ \longrightarrow ((\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+)^{\hat{\otimes}_\pi^{2m+1}}$ defined according to $\imath(\overline{B}) = \overline{B}\hat{\otimes}1^{\hat{\otimes}2m}$; by definition of the projective cross norm this mapping is an isometry.

$$||\imath(\overline{B})||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} = ||\overline{B}\hat{\otimes}1^{\hat{\otimes}2m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} = ||\overline{B}||_{\mathscr{A}^+,k}$$

In particular, taking the projective tensor norm of the argument of the cyclic cocycle gives us the following bound for large enough t, where C_0, C_1 and r are as defined above.

$$\begin{aligned}
&||(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D}))\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} \\
&= |2i\sqrt{\pi}|\cdot||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}\times(e^{-(t^2-1)\widetilde{D}^2}\hat{\otimes}1^{\hat{\otimes}2m})||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} \\
&\leq |2i\sqrt{\pi}|\cdot||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k}||e^{-(t^2-1)\widetilde{D}^2}\hat{\otimes}1^{\hat{\otimes}2m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} \\
&= 2\sqrt{\pi}||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k}||e^{-(t^2-1)\widetilde{D}^2}||_{\mathscr{A}^+,k} \\
&\leq (2\sqrt{\pi})\left(\frac{C_1}{C_0}e^{-(t^2-1)r^2/2}\right)||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k} \\
&= \left(C_2e^{-(t^2-1)r^2/2}\right)||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(u_t(\widetilde{D})\hat{\otimes}u_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k}
\end{aligned} \tag{4.14}$$

By the inclusion isometry and (4.10) of Theorem 4.3, for large enough k

$$|\varphi_\gamma(\overline{B}_0\hat{\otimes}\overline{B}_1\hat{\otimes}\cdots\hat{\otimes}\overline{B}_n)| \leq R_{\alpha,k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}||\overline{B}_i||^2_{\mathscr{A}^+,k}\right)^{1/2}$$

$$= R_{\alpha,k}\prod_{i=0}^{n}||\overline{B}_i||_{\mathscr{A}^+,k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}\right)^{1/2}$$

$$= R_{\alpha,k}||\overline{B}_0\hat{\otimes}\overline{B}_1\hat{\otimes}\cdots\hat{\otimes}\overline{B}_n||_{(\mathscr{A}^+)^{\hat{\otimes}n+1},k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}\right)^{1/2}$$

$$= R_{\alpha,k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}||\overline{B}_0\hat{\otimes}\overline{B}_1\hat{\otimes}\cdots\hat{\otimes}\overline{B}_n||^2_{(\mathscr{A}^+)^{\hat{\otimes}n+1},k}\right)^{1/2}$$

The absolute convergence of $\int_1^\infty \eta_{\varphi_\gamma}(\widetilde{D},t)\,dt$ follows from this bound along with (4.14).

$$\int_1^\infty \eta_{\varphi_\gamma}(\widetilde{D},t)\,dt = \int_1^\infty |\overline{\varphi}_\gamma((\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D}))\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m})|\,dt$$

$$\leq \int_1^\infty R_{\alpha,k}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}\frac{\left(C_2e^{-(t^2-1)r^2}\right)}{(1+||g_i||)^{2k}}||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||^2_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k}\right)^{1/2}\,dt$$

$$\left[\sup_{t\in[1,\infty)}\prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}(1+||g_i||)^{-2k}||\widetilde{D}e^{-\widetilde{D}^2}\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}||^2_{(\mathscr{A}^+)^{\hat{\otimes}2m+1},k}\right)^{1/2}\right.$$
$$\left.\times\int_1^\infty R_{\alpha,k}C_2e^{-(t^2-1)r^2/2}\,dt\right]<\infty$$

□

Theorem 4.6. *The higher delocalized eta invariant is independent of the choice of cocycle representative. Explicitly, if* $[\varphi_\gamma]=[\phi_\gamma]\in HC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$, *then* $\eta_{\varphi_\gamma}(\widetilde{D})=\eta_{\phi_\gamma}(\widetilde{D})$

Proof. By hypothesis, φ_γ and ϕ_γ are cohomologous via a coboundary $b\varphi$ belonging to $BC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$. By the results of Section 3.2 we can assume $\varphi\in(C^{2m-1}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$ to be a skew cochain of polynomial growth, and it suffices to prove that $\eta_{b\varphi}(\widetilde{D})=0$. Here it is useful to work in the unitization $(\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma)^+$ and thus we will use the pairing described by Connes for $K_1(\mathcal{A}^+)$ and odd cyclic cocycles.

$$\varphi((u_t(\widetilde{D})-\mathbb{1}\hat{\otimes}u_t^{-1}(\widetilde{D})-\mathbb{1})^{\hat{\otimes}m})=\overline{\varphi}((\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m})\tag{4.15}$$

By computing the derivative of this unitized cyclic cocycle we wish to obtain a transgression formula $HC^{2m-1}((\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma)^+,\mathrm{cl}(\gamma))\longmapsto HC^{2m}(\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma,\mathrm{cl}(\gamma))$. In particular, we will prove the transgression formula of [11, eq(3.23)]

$$m\eta_{b\varphi}(\widetilde{D},t)=\frac{d}{dt}\overline{\varphi}((\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m})\tag{4.16}$$

Direct computation of the right hand side of the above equation gives us

$$\sum_{j=0}^{m-1}\overline{\varphi}((\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}j}\hat{\otimes}\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-j-1})$$
$$+\sum_{j=0}^{m-1}\overline{\varphi}((\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}j}\hat{\otimes}\bar{u}_t(\widetilde{D})\hat{\otimes}\dot{\bar{u}}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-j-1})$$

We now use the relation $0=\frac{d}{dt}(\bar{u}_t^{-1}\bar{u}_t)=(\dot{\bar{u}}_t^{-1})\bar{u}_t+(\dot{\bar{u}}_t)\bar{u}_t^{-1}$ so the term $\dot{\bar{u}}_t^{-1}(\widetilde{D})$ can be rewritten: $-\bar{u}_t^{-1}(\widetilde{D})\dot{u}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})$. Considering the first summand, since $\overline{\varphi}_\gamma$ is a cyclic cocycle, we may apply the cyclic operator $\mathfrak{t}$– shifting the last $\bar{u}_t(\widetilde{D})$ term – and changing the sign by a factor of $(-1)^{2m+1} = -1$. With this in mind the above sums simplify to

$$\sum_{j=0}^{m-1} \overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})$$
$$+\sum_{j=0}^{m-1} \overline{\varphi}(\bar{u}_t(\widetilde{D})\hat{\otimes} - \bar{u}_t^{-1}(\widetilde{D})\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})$$
$$= m\overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})$$
$$-m\overline{\varphi}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-2}(\widetilde{D})\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})$$

Now we compute $b\overline{\varphi}$, making the important reminder that by Remark 2.29 and the discussion preceding it, we can normalize $\overline{\varphi}$ so that it vanishes on the unit $\mathbb{1} \in (\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+$.

$$(b\overline{\varphi})(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m}) =$$
$$\overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})$$
$$+(-1)^{2m}\overline{\varphi}(\bar{u}_t^{-1}(\widetilde{D})\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1}\hat{\otimes}\bar{u}_t(\widetilde{D}))$$
$$+\sum_{i=0}^{m-1}(-1)^{2i+1}\Big[\overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}i}$$
$$\hat{\otimes}\bar{u}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-i-1}\Big]$$
$$+\sum_{i=0}^{m-1}(-1)^{2i+2}\Big[\overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}i}\hat{\otimes}\bar{u}_t(\widetilde{D})$$
$$\hat{\otimes}\bar{u}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-i-1})\Big]$$

By definition, $\bar{u}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D}) = \bar{u}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D}) = \mathbb{1}$ so the latter two sums vanish, leaving

$$\begin{gathered}\overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1}) \\ +\overline{\varphi}(\bar{u}_t^{-2}(\widetilde{D})\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1}\hat{\otimes}\bar{u}_t(\widetilde{D})) \\ = \overline{\varphi}(\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1}) \\ -\overline{\varphi}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-2}(\widetilde{D})\dot{\bar{u}}_t(\widetilde{D})\hat{\otimes}(\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m-1})\end{gathered}$$

Comparing to the computation of the derivative, we obtain (4.16) as desired. Ignoring the constant $\frac{m!}{\pi i}$ in the definition of the delocalized higher eta invariant and integrating both sides with respect to t

$$m\eta_{b\varphi}(\widetilde{D}) = m \lim_{T\longrightarrow\infty} \int_{1/T}^{T} \eta_{b\varphi}(\widetilde{D},t)\, dt = m \lim_{T\longrightarrow\infty} \int_{1/T}^{T} \frac{d}{dt}\overline{\varphi}((\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes}m})\, dt$$

$$= m \lim_{T\longrightarrow\infty} \varphi((u_T(\widetilde{D}) - \mathbb{1}\hat{\otimes}u_T^{-1}(\widetilde{D}) - \mathbb{1})^{\hat{\otimes}m} - m \lim_{T\longrightarrow 0} \varphi((u_T(\widetilde{D}) - \mathbb{1}\hat{\otimes}u_T^{-1}(\widetilde{D}) - \mathbb{1})^{\hat{\otimes}m})$$

The splitting of the limit in the final line is justified by the absolute convergence of the integral. By Proposition 4.4 we know that $\lim_{T\longrightarrow 0} \varphi((u_T(\widetilde{D}) - \mathbb{1}\hat{\otimes}u_T^{-1}(\widetilde{D}) - \mathbb{1})^{\hat{\otimes}m}) = 0$. To deal with the case $T \longrightarrow \infty$ we recall some properties of holomorphic functional calculus. Take $x \in (0,\infty)$, then we have the limits

$$\lim_{T\longrightarrow\infty} u_T(x) - 1 = \lim_{T\longrightarrow\infty} \exp\left(\frac{2\pi i}{\sqrt{\pi}} \int_{-\infty}^{Tx} e^{-s^2}\, ds\right) - 1 = \exp\left(\frac{2\pi i}{\sqrt{\pi}} \cdot \sqrt{\pi}\right) - 1 = e^{2\pi i} - 1 = 0$$

$$\lim_{T\longrightarrow\infty} u_T(x) - 1 = \lim_{T\longrightarrow\infty} \exp\left(\frac{2\pi i}{\sqrt{\pi}} \int_{-\infty}^{Tx} e^{-s^2}\, ds\right) - 1 = \exp\left(\frac{2\pi i}{\sqrt{\pi}} \cdot 0\right) - 1 = e^{0} - 1 = 0$$

and in particular the convergence to 0 is uniform on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$ for any $\varepsilon > 0$. The exact same results holds true when considering the function $u_T^{-1}(x) - 1$. In both cases the invertibility of $\widetilde{D}$– and hence existence of a spectral gap at 0– means that both $u_T(\widetilde{D}) - \mathbb{1}$ and $u_T^{-1}(\widetilde{D}) - \mathbb{1}$

also converge in the $||\cdot||_{\mathscr{A},k}$ norm to 0. It thus follows from the bounds of Theorem 4.3 that

$$\lim_{T\longrightarrow\infty} |\varphi((u_T(\widetilde{D}) - \mathbb{1}\hat{\otimes}u_T^{-1}(\widetilde{D}) - \mathbb{1})^{\hat{\otimes}m}| = 0 \tag{4.17}$$

□

Proposition 4.7. *Let $S_\gamma^* : HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow HC^{2m+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ be the delocalized Connes periodicity operator, then $\eta_{[\varphi_\gamma]}(\widetilde{D}) = \eta_{[S_\gamma\varphi_\gamma]}(\widetilde{D})$ for every $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$.*

Proof. We may assume by Corollary 3.7 that φ_γ is of polynomial growth; by the definition of S_γ the cocycle $S_\gamma\varphi_\gamma$ is also of polynomial growth. Since our expression for S_γ coincides with that of [11, Definition 3.32] the result follows from [11, Proposition 3.33] □

4.2 Delocalized Pairing of Higher Rho Invariant

To begin with, we recall the assumptions put on the spin manifold M, namely that it is closed and odd dimensional with a positive scalar curvature metric g. Let $\widetilde{D}$ be the Dirac operator lifted to the universal cover $\widetilde{M}$, s a section of the spinor bundle $\mathcal{S}$, and $\nabla : C^\infty(\widetilde{M}, \mathcal{S}) \longrightarrow C^\infty(\widetilde{M}, T^*\widetilde{M} \otimes \mathcal{S})$ the connection on $\mathcal{S}$. Since $\widetilde{M}$ has positive scalar curvature $\kappa > 0$ associated to $\widetilde{g}$, then Lichnerowicz's formula shows that $\widetilde{D}$ is invertible, hence there exists a spectral gap at 0. Since $\widetilde{D}$ is an elliptic essentially self-adjoint operator, using a suitable normalizing function ψ, the operator $\psi(\widetilde{D})$ is bounded pseudo-local and self-adjoint. In particular, to emphasize the relationship with the delocalized higher eta invariant it is particularly useful that for $t \in (0, \infty)$ we consider

$$\psi(tx) = \frac{2}{\sqrt{\pi}} \int_0^{tx} e^{-s^2}\, ds \tag{4.18}$$

Since there exist a spectral gap at 0 with respect to $\widetilde{D}$, the limit $||\lim_{t\longrightarrow 0} \psi(\widetilde{D}/t)||_{op}$ exists and converges to $||(\widetilde{D}|\widetilde{D}|^{-1})||_{op}$, where $\widetilde{D}|\widetilde{D}|^{-1} = \mathrm{signum}(\widetilde{D})$. Define an operator $H_0 = \frac{1}{2}(\mathbb{1} + \widetilde{D}|D|^{-1})$ and let $\{\phi_{s,j}\}$ be a partition of unity subordinate to the Γ-invariant locally finite open cover $\{U_{s,j}\}_{s,j\in\mathbb{N}}$ of $\widetilde{M}$. For each s we take $\mathrm{diam}(U_{s,j}) < \frac{1}{s}$, and so for $t \geq 0$ we follow the construction of [48, Section 2.3] to form the operator

$$H(t) = \sum_j (s+1-t)\phi_{s,j}^{1/2} H_0 \phi_{s,j}^{1/2} + (t-s)\phi_{s+1,j}^{1/2} H_0 \phi_{s+1,j}^{1/2} \ : t \in [s, s+1] \tag{4.19}$$

Since the support $\mathrm{supp}(\phi_{s,j})$ of each member of the partition of unity is a subset of $U_{s,j}$ the propagation of $H(t)$ tends to 0 as $t \longrightarrow \infty$. Together with $H(t)$ being a pseudo-local, self-adjoint bounded operator, this gives us that $H(t) \in D^*(\widetilde{M}, \mathcal{S})^\Gamma$; moreover, due to the choice of χ we have $H'(t), H(t)^2 - \mathbb{1} \in C^*(\widetilde{M}, \mathcal{S})^\Gamma$. Moreover, as $H(t)$ is a projection, the path of invertibles

$$S = \{u(t) = \exp(2\pi i H(t)) |\ t \in [0, \infty)\}$$

belong to $(C^*(\widetilde{M}, \mathcal{S})^\Gamma)^+$, and since $\exp(2\pi i \cdot \mathrm{signum}(x)) = 1$ for any $x \neq 0$ we have by construction that $u(0) = \mathbb{1}$. It follows that u belongs to the kernel of the evaluation map

$$\mathsf{ev} : (C_L^*(\widetilde{M}, \mathcal{S})^\Gamma)^+ \longrightarrow (C^*(\widetilde{M}, \mathcal{S})^\Gamma)^+$$

and so the path S gives rise to a K-theory class $[u] \in K_1(C_{L,0}^*(\widetilde{M}, \mathcal{S})^\Gamma)$, which is by definition the higher rho invariant $\rho(\widetilde{D}, \widetilde{g})$ of Higson and Roe [20, 21, 22]. Before defining the pairing between cyclic cocycles and the higher rho invariant it is useful to introduce a few technical notions which will be needed later on. By Proposition 2.19 any class of invertible $[u] \in K_1(C_{L,0}^*(\widetilde{M}, \mathcal{S})^\Gamma)$ is directly equivalent to a class of invertible $[u] \in K_1(\mathscr{B}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)$, and we also recall that the (localized)-equivariant Roe algebra is independent of the choice of admissible module, hence we will work within the framework of $\mathscr{B}(\widetilde{M})^\Gamma$. The following notion of a *local* map comes from [50, Definition 3.3]

Definition 4.8. Consider the unitization $(\mathscr{B}(\widetilde{M})^\Gamma)^+$ of the algebra $\mathscr{B}(\widetilde{M})^\Gamma$, and its suspension $S\mathscr{B}(\widetilde{M})^\Gamma$. If $\mathcal{A}$ is a C^*-algebra recall that the suspension $S\mathcal{A}$ is defined as

$$\{f \in C([0,1], \mathcal{A}) \mid f(0) = f(1) = 0\}$$

Identify S^1 with the quotient space $[0,1]/(0 \sim 1)$, and call an element $f \in S\mathscr{B}(\widetilde{M})^\Gamma$ invertible if it is a piecewise smooth loop $f : S^1 \longrightarrow (\mathscr{B}(\widetilde{M})^\Gamma)^+$ of invertible elements satisfying $f(0) = f(1) = \mathbb{1}$. The map f is *local* if there exists $f_L \in S\mathscr{B}_L(\widetilde{M})^\Gamma$ such that the following hold

(i) $f_L : S^1 \longrightarrow (\mathscr{B}_L(\widetilde{M})^\Gamma)^+$ is a loop of invertible elements satisfying $f_L(0) = f_L(1) = \mathbb{1}$.

(ii) f is the image of f_L under the evaluation map $\mathsf{ev} : S\mathscr{B}_L(\widetilde{M})^\Gamma \longrightarrow S\mathscr{B}(\widetilde{M})^\Gamma$

Recall that identifying the Bott generator b as the class $[e^{2\pi i\theta}] \in K_1(C_0(\mathbb{R}))$ the Bott periodicity map β provides the following relationship between idempotents of a C^*-algebra $\mathcal{A}$ and invertibles of the suspension

$$\beta : K_0(\mathcal{A}) \longrightarrow K_1(S\mathcal{A}) \qquad \beta[p] = [bp + (1-p)] \tag{4.20}$$

Combining this with the Baum-Douglas geometric description of K-homology we obtain the following result concerning the propagation properties of local loops which is essentially the same as [50, Lemma 3.4], and we refer the reader to the proof given in that paper.

Lemma 4.9. *If $f \in S\mathscr{B}(\widetilde{M})^\Gamma$ is a local invertible then for any $\varepsilon > 0$ there exists an idempotent $p \in \mathscr{B}(\widetilde{M})^\Gamma$ such that $\mathsf{prop}(p) \leq \varepsilon$ and $f(\theta)$ is homotopic to the element $\psi(\theta) = e^{2\pi i\theta}p + (1-p)$ through a piecewise smooth family of invertible elements.*

We now go through the process of assigning to any class $[u] \in K_1(C^*_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)$ a special representative which will enable the calculations later on in this section. Making use of the results proved in [50, Proposition 3.5], there exist a piecewise smooth path of invertible elements $h(t) \in \mathscr{B}(\widetilde{M})^\Gamma$ connecting $u(1)$ and $e^{2\pi i \frac{E(1)+1}{2}}$ where the operator $E : [1,\infty) \longrightarrow D^*(\widetilde{M})^\Gamma$ has uniformly bounded operator norm and satisfies

$$\lim_{t\longrightarrow\infty} \mathsf{prop}(E(t)) = 0 \qquad E'(t) \in \mathscr{B}(\widetilde{M})^\Gamma \qquad E(t)^2 - 1 \in \mathscr{B}(\widetilde{M})^\Gamma \qquad E^*(t) = E(t)$$

where there exists a twisted Dirac operator $\widetilde{D}$ over a *spin*^{c} manifold, along with smooth normalizing function $\chi : \mathbb{R} \longrightarrow [-1,1]$ such that $E(t) = \chi(\widetilde{D}/t)$. We can thus define the *regularized representative* of u to be

$$w(t) = \begin{cases} u(t) & 0 \leq t \leq 1 \\ h(t) & 1 \leq t \leq 2 \\ e^{2\pi i \frac{E(t-1)+1}{2}} & t \geq 2 \end{cases} \tag{4.21}$$

Moreover, by [25, Theorem 3.8] and [50, Proposition 3.5] if v is another such representative then there exist a family of piecewise smooth maps $\{E_s\}_{s\in[0,1]}$ belonging to $D^*_{L,0}(\widetilde{M})^\Gamma$ and having the same properties as E. In particular, the propagation of $E_s(t)$ goes to zero uniformly in s as $t \longrightarrow \infty$, and

$$\frac{\partial}{\partial t} E_s(t) \in \mathscr{B}(\widetilde{M})^\Gamma \qquad E_s(t)^2 - 1 \in \mathscr{B}(\widetilde{M})^\Gamma \qquad E_s(t)^* = E_s(t)$$

Furthermore there exists piecewise smooth family of invertibles $\{v_s\}_{s\in[0,1]}$ belonging to $(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)^+$ and which satisfy

(i) $v_0(t) = w(t)$ for $t \in [0,\infty)$, and $v_1(t) = v(t)$ for all $t \notin (1,2)$

(ii) $v_s(t) = \exp(2\pi i \frac{E_s(t-1)+1}{2})$ for all $t \geq 2$

(iii) $v_1 v^{-1} : [1,2] \longrightarrow (\mathscr{B}(\widetilde{M})^\Gamma)^+$ is a local loop of invertible elements.

Definition 4.10. Let $a_i = \sum_{g_i\in\Gamma} c_{g_i} \cdot g_i$ be an element of the group algebra $\mathbb{C}\Gamma$, and ω_i belong to the algebra $\mathscr{R}$ of smooth operators on a closed oriented Riemannian manifold. Denoting $W_i = a_i \otimes \omega_i$, the action of φ_γ on $\mathbb{C}\Gamma$ can be extended to $\mathbb{C}\Gamma \otimes \mathscr{R}$ by

$$\varphi_\gamma(W_0 \hat{\otimes} W_1 \hat{\otimes} \cdots \hat{\otimes} W_n) = \mathsf{tr}(\omega_0\omega_1\cdots\omega_n) \cdot \varphi_\gamma(a_0, a_1, \ldots, a_n)$$

Definition 4.11. Given $[\rho(\widetilde{D}, \widetilde{g})] \in K_1(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)$ with w being its regularized representative, associated to each delocalized cyclic cocycle $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ the determinant map τ_{φ_γ} is defined by

$$\tau_{\varphi_\gamma}(\rho(\widetilde{D},\widetilde{g})) := \frac{1}{\pi i}\int_0^\infty \overline{\varphi}_\gamma(\widetilde{\mathsf{ch}}(w(t),\dot{w}(t))\, dt$$

$$\widetilde{\mathsf{ch}}(w,\dot{w}) = (-1)^m\,(m-1)!\sum_{j=1}^{m}\Big((w^{-1}\hat{\otimes}w)^{\hat{\otimes}j}\hat{\otimes}(w^{-1}\dot{w})\hat{\otimes}(w^{-1}\hat{\otimes}w)^{\hat{\otimes}(m-j)}\Big)$$

We remark that the definition of $\widetilde{\mathsf{ch}}$ is directly modeled upon the secondary odd Chern character pairing invertibles of $\mathrm{GL}(N,\mathbb{C})$ and traces (see, for example, [28, Section 1.2]). Moreover, by the property of cyclic cocycles this expression can be simplified so that our coefficients exactly resemble that of the delocalized higher eta invariant. By the action of the cyclic operator $\mathfrak{t}$ we obtain from applying the $n = 2(m-j)+1$ fold composition $\mathfrak{t}^n$ for each j, that the integrand can be written as

$$\frac{(-1)^m\,(m-1)!}{\pi i}\sum_{j=1}^{m}(-1)^{2m(2m-2j+1)}\overline{\varphi}_\gamma\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)$$

from which it follows that there is the simplified expression for the determinant map

$$\tau_{\varphi_\gamma}(\rho(\widetilde{D},\widetilde{g})) := \frac{(-1)^m m!}{\pi i}\int_0^\infty \overline{\varphi}_\gamma\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)\,dt \tag{4.22}$$

It is not at all obvious why the above pairing is well defined, the resolving of this doubt occupying the remainder of this section.

Theorem 4.12. *Let $\Gamma = \pi_1(M)$ and $\varphi_\gamma \in (C^n(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$ be a delocalized cyclic cocycle of polynomial growth, then φ_γ extends continuously on the algebra $(\mathscr{B}(\widetilde{M})^\Gamma)^{\hat{\otimes}_\pi^{n+1}}$.*

Proof. Again using the explicit representation for delocalized cyclic cocycles, we show that $\varphi_{\alpha,\gamma}$ extends to a continuous multi-linear map on $(\mathscr{B}(\widetilde{M})^\Gamma)^{\hat{\otimes}_\pi^{n+1}}$. By fixing a basis, let $B_k \in \mathscr{B}(\widetilde{M})^\Gamma$ be represented by the matrix $(\beta^k_{ij})_{i,j\in\mathbb{N}}$ with $\beta^k_{ij} \in C^*_r(\Gamma)$. We wish to prove convergence of

$$
\begin{aligned}
\varphi_{\alpha,\gamma}(B_0 \hat{\otimes} B_1 \hat{\otimes} \cdots \hat{\otimes} B_n) &= \varphi_{\alpha,\gamma}\left(\sum_{g_0 \in \Gamma} B_0(g_0) \cdot g_0 \hat{\otimes} \cdots \hat{\otimes} \sum_{g_n \in \Gamma} B_n(g_n) \cdot g_n \right) \\
&= \sum_{g_0 \in \Gamma} \sum_{g_1 \in \Gamma} \cdots \sum_{g_n \in \Gamma} \mathsf{tr}(B_0(g_0) B_1(g_1) \cdots B_n(g_n)) \cdot \varphi_{\alpha,\gamma}(g_0, g_1, \cdots, g_n) \\
&= \sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} \mathsf{tr}(C(g_0, \ldots, g_n)) \cdot \alpha(h, hg_0, \ldots, hg_0 g_1 \cdots g_{n-1})
\end{aligned}
\tag{4.23}
$$

Straight forward matrix multiplication gives the product $C = (c_{ij})_{i,j \in \mathbb{N}}$ as having entries

$$
c_{ij}(g_0, \ldots, g_n) = \sum_{k_{n-1} \in \mathbb{N}} \cdots \sum_{k_1 \in \mathbb{N}} \sum_{k_0 \in \mathbb{N}} \left(\beta^0_{ik_0}(g_0) \beta^1_{k_0 k_1}(g_1) \cdots \beta^n_{k_{n-1} j}(g_n) \right)
$$

For ease of notation, we shorten the argument of α by writing $\alpha(\mathbf{g})$; in addition we suppress the argument of the functions c_{ij}. Taking the desired trace in the above expression (4.23), and using the fact that α is of polynomial growth, we obtain the inequality

$$
\begin{aligned}
|\varphi_{\alpha,\gamma}(B_0 \hat{\otimes} B_1 \hat{\otimes} \cdots \hat{\otimes} B_n)| &\leq \sum_{j \in \mathbb{N}} \sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} |c_{jj}| |\alpha(\mathbf{g})| \\
&\leq \sum_{j \in \mathbb{N}} \sum_{g_0 g_1 \cdots g_n \in \mathrm{cl}(\gamma)} R_\alpha (1 + ||g_0||)^{2k} (1 + ||g_1||)^{2k} \cdots (1 + ||g_n||)^{2k} |c_{jj}|
\end{aligned}
\tag{4.24}
$$

for some positive constant R_α. Next, considering the following inequality for $|c_{jj}|$

$$
\sum_{j \in \mathbb{N}} |c_{jj}| \leq \sum_{k_0, \ldots, k_{n-1}, j \in \mathbb{N}} \left| \left(\beta^0_{jk_0}(g_0) \cdots \beta^n_{k_{n-1} j}(g_n) \right) \right| \leq \prod_{i=0}^{n} \left(\sum_{k_{i-1}, k_i \in \mathbb{N}} |\beta^i_{k_{i-1} k_i}(g_i)|^2 \right)^{1/2}
$$

where $j = k_{-1} = k_n$. Substituting the above final product into the second line of (4.24),

$|\varphi_{\alpha,\gamma}(B_0\hat{\otimes}B_1\hat{\otimes}\cdots\hat{\otimes}B_n)|$ is bounded above by

$$\sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)} R_\alpha \prod_{i=0}^{n}(1+||g_i||)^{2k}\prod_{i=0}^{n}\left(\sum_{k_{i-1},k_i\in\mathbb{N}}|\beta^i_{k_{i-1}k_i}(g_i)|^2\right)^{1/2}$$

$$= R_\alpha \sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)}\prod_{i=0}^{n}\left(\sum_{k_{i-1},k_i\in\mathbb{N}}|\beta^i_{k_{i-1}k_i}(g_i)|^2\right)^{1/2}((1+||g_i||)^{4k})^{1/2}$$

$$= R_\alpha \prod_{i=0}^{n}\sum_{g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)}\left(\sum_{k_{i-1},k_i\in\mathbb{N}}(1+||g_i||)^{4k}|\beta^i_{k_{i-1}k_i}(g_i)|^2\right)^{1/2}$$

$$\leq R_\alpha \prod_{i=0}^{n}\left(\sum_{g_i\in\Gamma}\sum_{k_{i-1},k_i\in\mathbb{N}}(1+||g_i||)^{4k}|\beta^i_{k_{i-1}k_i}(g_i)|^2\right)^{1/2} \quad : \; g_0g_1\cdots g_n\in\mathrm{cl}(\gamma)$$

In particular, the proof of [16, Lemma 6.4] shows that each of the double sums in the final expression are in fact bounded by the norm $||B_i||_{\mathscr{B},k}$ hence finite for all k, and thus so is any finite product of them. Since $\mathbb{C}\Gamma\otimes\mathscr{R}$ is a smooth dense sub-algebra of $\mathscr{B}(\widetilde{M})^\Gamma$ and $\varphi_{\alpha,\gamma}$ has been proven to be continuous on $\mathscr{B}(\widetilde{M})^\Gamma$, it suffices to prove that for operators $W_0,\ldots,W_n\in\mathbb{C}\Gamma\otimes\mathscr{R}$

$$\mathrm{sgn}(\sigma)\varphi_{\alpha,\gamma}(W_0\hat{\otimes}W_1\hat{\otimes}\cdots\hat{\otimes}W_n)=\varphi_{\alpha,\gamma}(W_{\sigma(0)}\hat{\otimes}W_{\sigma(1)}\hat{\otimes}\cdots\hat{\otimes}W_{\sigma(n)}) \tag{4.25}$$

whenever $\sigma\in S_{n+1}$ is a cyclic shift. Write $W_i=a_i\otimes\omega_i$, then using the fact that the trace is invariant under cyclic shifts and that $\varphi_{\alpha,\gamma}$ is a cyclic cocycle on $\mathbb{C}\Gamma$

$$\varphi_{\alpha,\gamma}(W_{\sigma(0)}\hat{\otimes}W_{\sigma(1)}\hat{\otimes}\cdots\hat{\otimes}W_{\sigma(n)})=\mathsf{trace}(\omega_{\sigma(0)}\omega_{\sigma(1)}\cdots\omega_{\sigma(n)})\cdot\varphi_{\alpha,\gamma}(a_{\sigma(0)},a_{\sigma(1)},\ldots,a_{\sigma(n)})$$

$$=\mathsf{trace}(\omega_0\omega_1\cdots\omega_n)\cdot\mathrm{sgn}(\sigma)\varphi_{\alpha,\gamma}(a_0,a_1,\ldots,a_n)$$

$$=\mathrm{sgn}(\sigma)\varphi_{\alpha,\gamma}(a_0\otimes\omega_0,a_1\otimes\omega_1,\ldots,a_n\otimes\omega_n)=\mathrm{sgn}(\sigma)\varphi_{\alpha,\gamma}(W_0\hat{\otimes}W_1\hat{\otimes}\cdots\hat{\otimes}W_n)$$

□

Proposition 4.13. *Let w be a regularized representative of some class $[u] \in K_1(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)$. For all $t \geq 2$ there exists a finite propagation operator $\varpi \in \mathscr{B}(\widetilde{M})^\Gamma$ such that for every $k > 0$ and any $\varepsilon > 0$*

$$||w(t) - \varpi(t)||_{\mathscr{B},k} < C_k/t^3 \qquad \mathsf{prop}(\varpi(t)) < \varepsilon$$

whenever $t \geq \max\{e^2, 6r\}$, where C_k and r are positive constants .

Proof. For real valued x, finite $r > 0$ and $z = \pi i(x+1) \in B_r(0)$ define the bounded holomorphic function $f(z) = e^{2\pi i \frac{x+1}{2}}$. Consider the Taylor series expansion of $f(z)$ centered at the origin, and for each $m \in \mathbb{N}$ define

$$\begin{gathered} P_m(z) = \sum_{k=0}^{m} \frac{\left(2\pi i \frac{x+1}{2}\right)^k}{k!} \qquad R_m(z) = \sum_{k=m+1}^{\infty} \frac{\left(2\pi i \frac{x+1}{2}\right)^k}{k!} \\ A_m(t) = P_m(G(t)) = P_m\left(2\pi i \frac{E(t-1)+1}{2}\right) \end{gathered} \tag{4.26}$$

We know that $E(t)$ has compact real spectrum which is symmetric around $\lambda = 0$; in particular the spectral radius $\mathrm{rad}(\sigma(E(t)))$ is bounded above by $||E(t)||_{op}$. It is clear that the same holds true for $G(t)$, so choose $r > \sup_{t\geq 1}\{||G(t)||_{op}\}$ so that $\sigma(G(t)) \subset B_r(0)$, then $f \in H^\infty(B_r(0))$ and since $\mathscr{B}(\widetilde{M})^\Gamma$ is closed under holomorphic functional calculus, the usual remainder bound

$$|R_m(z)| \leq c\frac{|z|^{m+1}}{(m+1)!} \quad \text{if} \quad |f^{(m+1)}(z)| \leq c, \forall z \in B_r(0)$$

has a functional equivalent with respect to every $||\cdot||_{\mathscr{B},k}$. In particular for every $k > 0$, there exists a constant $C_{k,t} > 0$ such that

$$||w(t) - A_m(t)||_{\mathscr{B},k} = ||R_m(G(t))||_{\mathscr{B},k} \leq C_{k,t}||R_m||_\infty = C_{k,t} \sup_{z\in B_r(0)} |R_m(z)|$$

$$\leq C_{k,t} \cdot c \sup_{z \in B_r(0)} \frac{|z|^{m+1}}{(m+1)!} \leq C_{k,t} \cdot c\frac{r^{m+1}}{(m+1)!} = C_{k,t}\frac{r^{m+1}}{(m+1)!}$$

Note that we can take $c = 1$ since $|f^{(m+1)}(z)| = |e^z| = |e^{i(\pi x+\pi)}| = 1$. Suppose that

$m \geq \max\{e^2, 6r\}$, then by applying Stirling's approximation

$$C_{k,t}\frac{r^{m+1}}{(m+1)!} \leq C_{k,t}\frac{r^{m+1}}{\sqrt{2\pi}(m+1)^{m+3/2}e^{-(m+1)}} < \frac{C_{k,t}}{m^3}$$

Since the exponential is an entire function its power series converges uniformly on the compact set $\overline{B}_r(0)$, and thus we can choose a finite $C_k \geq \sup_{t\in[2,\infty)}\{C_{k,t}\}$. For all $t \geq 2$ we define the operator $\varpi(t)$ according to

$$\varpi(t) = \sum_{m=2}^{\infty} 1_{[N_m,N_{m+1})}(t) \cdot A_m(t) \tag{4.27}$$

where $N_2 = 2$ and the constants $N_{m+1} > N_m \geq m$ depend only on the propagation of E. From the definition of $E(t)$ the propagation tends to 0 as $t \longrightarrow \infty$; hence for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\mathsf{prop}(E(t)) < \varepsilon/N_\varepsilon$ whenever $t \geq N_m$ for sufficiently large m. Recall that if S and T are bounded operators on some module H_X then

$$\mathsf{prop}(ST) \leq \mathsf{prop}(S) + \mathsf{prop}(T) \qquad \mathsf{prop}(S+T) \leq \max\{\mathsf{prop}(S), \mathsf{prop}(T)\}$$

For $m > 2$ it is possible to set each N_m large enough such that if $t \in [N_m, N_{m+1})$ then $m \leq N_\varepsilon \leq N_m \leq t$; thus, by the definition of $\varpi(t)$, we have the following result.

$$\text{prop}(\varpi(t+1)) = \text{prop}(A_m(t+1)) = \text{prop}\left(\sum_{k=0}^{m}\frac{\left(2\pi i\frac{E(t)+1}{2}\right)^k}{k!}\right) \leq \text{prop}\left(\frac{\left(2\pi i\frac{E(t)+1}{2}\right)^m}{m!}\right)$$

$$= \text{prop}\left(\frac{E(t)+1}{2}\right)^m \leq m\cdot\text{prop}\left(\frac{E(t)+1}{2}\right) \leq m\cdot\text{prop}(E(t)) < \frac{m\varepsilon}{N_\varepsilon} \leq \varepsilon$$

□

Corollary 4.14. *Let w be a regularized representative of some class $[u] \in K_1(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)$. For all $t \geq 2$ there exists a finite propagation operator $\varpi \in \mathscr{B}(\widetilde{M})^\Gamma$ such that for every $k > 0$*

and any $\varepsilon > 0$

$$||w^{-1}(t) - \varpi^{-1}(t)||_{\mathscr{B},k} < C_k/t^3 \qquad \mathsf{prop}(\varpi^{-1}(t)) < \varepsilon$$

whenever $t \geq \max\{e^2, 6r\}$, *where* C_k *and* r *are positive constants .*

Proof. Take $f(z) = e^{-2\pi i \frac{x+1}{2}}$ and apply the same argument as above. □

For each member of the family of invertibles $\{v_s\}_{s\in[0,1]}$ the above also leads to finite propagation operators $\varpi_s(t)$ and $\varpi_s^{-1}(t)$ having analogous properties. The following technical result will be of similar importance in establishing well-definedness of the determinant map.

Remark 4.15. *For* $\varphi_\gamma \in C^n((\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ *and* $B_0, \ldots, B_n \in \mathscr{B}(\widetilde{M}, \mathcal{S})^\Gamma$ *– or equivalently for* $A_0, \ldots, A_n \in \mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma$ *– there exists* $\varepsilon_{\widetilde{M}}$ *which depends only on* M *such that*

$$\varphi_\gamma(B_0 \hat{\otimes} B_1 \hat{\otimes} \cdots \hat{\otimes} B_n) = 0$$

whenever $\mathsf{prop}(B_i) < \varepsilon_{\widetilde{M}}$ *for each* $0 \leq i \leq n$.

Theorem 4.16. *Let* $[u] \in K_1(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)$ *and* w *be a regularized representative of* u, *then the determinant map* τ_φ *converges absolutely for any* $\varphi_\gamma \in (C^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$ *of polynomial growth.*

Proof. Using the simplified expression of Equation (4.22) we can write $\tau_\varphi(u)$ as

$$\begin{aligned} &\frac{(-1)^m m!}{\pi i} \int_0^2 \varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (w^{-1}(t) \hat{\otimes} w(t))^{\hat{\otimes} m} \right) \, dt \\ +&\frac{(-1)^m m!}{\pi i} \int_2^\infty \varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (w^{-1}(t) \hat{\otimes} w(t))^{\hat{\otimes} m} \right) \, dt \end{aligned} \tag{4.28}$$

The first integral is finite due to the uniform boundedness of $||B||_{\mathscr{B},k}$ and the results of Theorem 4.12, being bounded above by

$$2R_\alpha \frac{m!}{\pi i} \sup_{t\in[0,2]} ||w(t)||_{\mathscr{B},k} ||w^{-1}(t)||_{\mathscr{B},k} ||w^{-1}(t)\dot{w}(t)||_{\mathscr{B},k} < \infty$$

By Proposition 4.13 and its corollary there exist operators $\varpi(t)$ and $\varpi^{-1}(t)$ belonging to $(\mathscr{B}(\widetilde{M})^{\Gamma})^{+}$ such that for any $\varepsilon > 0$ there exists t large enough such that

$$\mathsf{prop}(\varpi(t)), \mathsf{prop}(\varpi^{-1}(t)) < \varepsilon$$

By basic distribution of tensors over addition and mutlilinearity of cyclic cocycles– ignoring the constant– we obtain the following expansion for the second integral of (4.28).

$$\begin{aligned} &\int_2^\infty \varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (w^{-1}(t) - \varpi^{-1}(t) \hat{\otimes} w(t))^{\hat{\otimes} m} \right) \, dt \\ &+ \int_2^\infty \varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (w^{-1}(t) \hat{\otimes} w(t) - \varpi(t))^{\hat{\otimes} m} \right) \, dt \\ &+ \int_2^\infty \varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (\varpi^{-1}(t) \hat{\otimes} \varpi(t))^{\hat{\otimes} m} \right) \, dt \end{aligned} \tag{4.29}$$

Since $t \geq 2$ the description of the regularized representative gives us that $w^{-1}(t)\dot{w}(t) = \pi i E'(t-1)$, the propagation of which tends to 0 as $t \longrightarrow \infty$. By Remark 4.15 it follows that the integrand $\varphi_\gamma \left((w^{-1}(t)\dot{w}(t)) \hat{\otimes} (\varpi^{-1}(t) \hat{\otimes} \varpi(t))^{\hat{\otimes} m} \right) = 0$ once the propagation of all these operators is less than some $\varepsilon_{\widetilde{M}}$, which occurs for large enough t. Hence there exists some t_ε such that the last integral of (4.29) is bounded above by

$$\int_2^{t_\varepsilon} \pi i \left| \varphi_\gamma \left(E'(t-1) \hat{\otimes} (\varpi^{-1}(t) \hat{\otimes} \varpi(t))^{\hat{\otimes} m} \right) \right| \, dt$$

which is finite from the bounds of Theorem 4.12. Similarly, by Proposition 4.13 there exists some constant $r > 0$ such that for $t \geq 6r$

$$||\varpi(t) - w(t)||_{\mathscr{B},k} \text{ and } ||\varpi(t)^{-1} - w^{-1}(t)||_{\mathscr{B},k} < \frac{C_k}{t^3}$$

The norm boundedness of all $B \in (\mathscr{B}(\widetilde{M})^{\Gamma})^{+}$ and the results of Theorem 4.12 provide existence of $M_k, N_k > 0$ such that the second integral of (4.29) is bounded by

$$\int_2^{6r} |\varphi_\gamma\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t) - \varpi(t))^{\hat{\otimes}m}\right)| \, dt$$

$$+\int_{6r}^{\infty} |\varphi_\gamma\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t) - \varpi(t))^{\hat{\otimes}m}\right)| \, dt$$

$$\leq R_\alpha \sup_{t\in[2,6r]} ||w^{-1}(t)\dot{w}(t)||_{\mathscr{B},k}||w(t)||^m_{\mathscr{B},k}||\varpi(t) - w(t)||^m_{\mathscr{B},k}$$

$$+R_\alpha \int_{6r}^{\infty} ||w^{-1}(t)\dot{w}(t)||_{\mathscr{B},k}||w(t)||^m_{\mathscr{B},k}||\varpi(t) - w(t)||^m_{\mathscr{B},k} \, dt$$

$$= (6r-2)R_\alpha N_k + R_\alpha \int_{6r}^{\infty} ||w^{-1}(t)\dot{w}(t)||_{\mathscr{B},k}||w(t)||^m_{\mathscr{B},k}||\varpi(t) - w(t)||^m_{\mathscr{B},k} \, dt$$

$$\leq (6r-2)R_\alpha N_k + R_\alpha \int_{6r}^{\infty} M_k^2||\varpi(t) - w(t)||^m_{\mathscr{B},k} \, dt$$

The finiteness of the above integral follows directly from

$$\int_{6r}^{\infty} M_k^2||\varpi(t) - w(t)||_{\mathscr{B},k} \, dt < M_k^2 C_k \int_{6r}^{\infty} \frac{1}{t^{3m}} \, dt$$

An exact replica of this argument applied to $w^{-1}(t) - \varpi^{-1}(t)$ shows that the first integral of (4.29) is finite, hence finishes our proof. □

The following three results show that $\tau_\varphi([u])$ is independent of the choice of regularized representatives, with Theorem 4.19 providing the proof that the replacement of u_1 by w through the path of local loops behaves as intended.

Lemma 4.17. *Let $[u] \in K_1(\mathscr{B}^*_{L,0}(\widetilde{M})^\Gamma)$ with v and w both being regularized representatives, and $\{v_s\}_{s\in[0,1]}$ the associated family of piecewise smooth invertibles. For any delocalized cyclic cocycle $\varphi_\gamma \in (C^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)), b)$*

$$\frac{\partial}{\partial s}\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m}\right) = \frac{\partial}{\partial t}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m}\right)$$

Proof. Working in the unitization $(\mathscr{B}^*(\widetilde{M})^\Gamma)^+$ and noting that every invertible element in

$(\mathscr{B}^*(\widetilde{M})^\Gamma)^+$ can be viewed as one in $(\mathscr{B}^*_{L,0}(\widetilde{M})^\Gamma)$, for ease of notation the variable t is suppressed; that is we will write w as opposed to $w(t)$ unless necessary. We might as well prove the vanishing of the following generalized difference

$$\begin{aligned}&\frac{\partial}{\partial s}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)\\ -&\frac{\partial}{\partial t}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)\end{aligned} \tag{4.30}$$

Now, by definition every cyclic cocycle φ_γ belongs to the kernel of the boundary map b. Through tedious but straightforward computations we will show that the following double sum gives precisely the expression for (4.30).

$$\begin{aligned}0&=\sum_{k=0}^{m}(b\varphi_\gamma)\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\\ &=\sum_{k=0}^{m}\sum_{l=0}^{2m+1}(\delta^l\varphi_\gamma)\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\end{aligned} \tag{4.31}$$

Firstly, looking at (4.30), explicit computation of the $\frac{\partial}{\partial s}\varphi_\gamma(\cdots)$ term gives

$$\begin{aligned}&\sum_{i=0}^{j-1}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(\partial_s v_s^{-1}\hat{\otimes}v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j-i-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)\\ +&\sum_{i=0}^{j-1}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\hat{\otimes}\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j-i-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m-j)}\right)\\ &+\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_s\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)\\ &+\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(\partial_s v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)\\ +&\sum_{i=0}^{m-j-1}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(\partial_s v_s^{-1}\hat{\otimes}v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j-i-1)}\right)\\ +&\sum_{i=0}^{m-j-1}\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\hat{\otimes}\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j-i-1)}\right)\end{aligned}$$

Now, for every fixed t we use the relation $0=\partial_s(v_s^{-1}v_s)=(\partial_s v_s^{-1})v_s+(\partial_s v_s)v_s^{-1}$ so the term $\partial_s v_s^{-1}(t)$ can be rewritten: $-v_s^{-1}(t)(\partial_s v_s(t))v_s^{-1}(t)$. Moreover, since $\varphi_{\alpha,\gamma}$ is a cyclic cocycle, application of the cyclic operator $\mathfrak{t}$ only affects the value by $(-1)^{2m}$; in this case, the action of $\mathfrak{t}^{2m-2j+1}$ has the effect of shifting the $v_s^{-1} \cdot \partial_t v_s$ term to the left j places. With this in mind, and by making the index shift $i \mapsto i-j$, the latter two sums become exactly as the first two, providing the intended simplified form.

$$\begin{aligned}
&\frac{\partial}{\partial s}\varphi_\gamma \left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right) = \\
&-2\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right) \\
&+2\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_s v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right) \\
&+\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_s\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right) \\
&-\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}((v_s^{-1}\cdot\partial_t v_s)(v_s^{-1}\cdot\partial_s v_s))\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)
\end{aligned} \tag{4.32}$$

We will drop the coefficient of 2 in the rest of this proof since it is inconsequential; analogous arguments to those preceding (4.32) provide us with the similar result

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi_\gamma \left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(n-j)}\right) = \\
&-2\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right) \\
&+2\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right) \\
&+\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}(v_s^{-1}\cdot\partial_t\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right) \\
&-\varphi_\gamma\left((v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}j}\hat{\otimes}((v_s^{-1}\cdot\partial_s v_s)(v_s^{-1}\cdot\partial_t v_s))\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-j)}\right)
\end{aligned} \tag{4.33}$$

It is clear that the last two terms of (4.32) and (4.33) are identical, since $\partial_s\partial_t = \partial_t\partial_s$. Taking the difference of these two equations gives the expanded form of (4.30)

$$\begin{aligned}
&-\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right)\\
&\qquad+\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_s v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right)\\
&+\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right)\\
&\qquad-\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right)
\end{aligned}\tag{4.34}$$

We now turn our attention to the double sum expression in (4.31). Using the fact that $\varphi_\gamma(\overline{A}_0\hat{\otimes}\cdots\hat{\otimes}\overline{A}_n)$ vanishes if $\overline{A}_i=\mathbb{1}$ for any $\overline{A}_i\in\mathcal{A}^+$, fixing any k the image of $\varphi_\gamma(\cdots)$ under δ^l vanishes for indices $l\neq 0,2m+1$ as follows

$$(-1)^l\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}\mathbb{1}\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right):\ l\leq 2k-1$$

$$(-1)^l\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}v_s^{-1}\hat{\otimes}\mathbb{1}\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\right):\ l\geq 2k+3$$

$$(-1)^l\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}\mathbb{1}\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\right):\ l\geq 2k+2$$

$$(-1)^l\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}v_s^{-1}\hat{\otimes}\mathbb{1}\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right):\ l\leq 2k-2$$

The first two expressions are for l odd, and the latter two for l an even integer; there is the obvious caveat that each expression only holds if neither $k-1$ or $m-k-1$ are negative. It is then a direct consequence that for $k\neq 0,m$ we have the surviving terms

$$\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\ :\quad l=0$$

$$\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\ :\quad l=2k$$

$$-\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\right)\ :\quad l=2k+1$$

$$-\varphi_\gamma\left(\partial_s v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\hat{\otimes}v_s^{-1}\right)\ :\quad l=2m+1$$

A bit more care is necessitated for the index $k=0$ but the surviving terms are exactly the same except for the cases $l=0=2k$; analogously the only difference when $j=m$ is for $l=2k+1=2m+1$.

$$(\delta^l\varphi_\gamma)(\cdots)=\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m}\right)\ :\quad l=0,k=0$$
$$(\delta^l\varphi_\gamma)(\cdots)=-\varphi_\gamma\left((v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m}\right)\ :\quad l=2m+1,k=m$$

The above two terms clearly cancel in the total sum, hence (4.31) is equal to

$$\begin{aligned}
&-\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-1)}\right)\\
&-\varphi_\gamma\left(\partial_s v_s\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-1)}\hat{\otimes}v_s^{-1}\right)\\
&+\sum_{k=1}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\\
&-\sum_{k=1}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\right)\\
&+\sum_{k=1}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k-1}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k)}\right)\\
&-\sum_{k=1}^{m-1}\varphi_\gamma\left(\partial_s v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}k}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-k-1)}\hat{\otimes}v_s^{-1}\right)\\
&+\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\right)\\
&+\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}m-1}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\right)
\end{aligned}$$

Collecting like terms and changing the dummy variable k to i, the above simplifies to

$$\begin{aligned}&\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i-1}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i)}\right)\\&-\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s\cdot v_s^{-1})\hat{\otimes}v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\right)\\&\quad+\sum_{i=0}^{m-1}\varphi_\gamma\left((v_s^{-1}\cdot\partial_s v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i-1}\hat{\otimes}v_s^{-1}\hat{\otimes}\partial_t v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i)}\right)\\&\quad-\sum_{i=0}^{m-1}\varphi_\gamma\left(\partial_s v_s\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}i}\hat{\otimes}(v_s^{-1}\cdot\partial_t v_s)\hat{\otimes}(v_s^{-1}\hat{\otimes}v_s)^{\hat{\otimes}(m-i-1)}\hat{\otimes}v_s^{-1}\right)=0\end{aligned}$$

By applying the cyclic operator the necessary number of times– changing the sign by a factor of $(-1)^{2mk}=1$– we obtain precisely (4.34), proving $\frac{\partial}{\partial s}\varphi_\gamma(\cdots)-\frac{\partial}{\partial t}\varphi_\gamma(\cdots)=0$. □

Corollary 4.18. *Let $[u]\in K_1(\mathscr{B}^*_{L,0}(\widetilde{M})^\Gamma)$ with v and w both being regularized representatives and $\{v_s\}_{s\in[0,1]}$ the associated family of piecewise smooth invertibles, then $\tau_\varphi(v_1)=\tau_\varphi(w)$ for any polynomial growth $\varphi_\gamma\in(C^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$.*

Proof. Taking a double integral $\int_0^T\int_0^1\, ds\, dt$ over the left-hand derivative in the equality of Lemma 4.17 provides the chain of equalities

$$\begin{aligned}&\int_0^T\int_0^1\frac{\partial}{\partial s}\varphi_\gamma\left((v_s(t)^{-1}\partial_t v_s(t))\hat{\otimes}(v_s^{-1}(t)\hat{\otimes}v_s(t))^{\hat{\otimes}m}\right)\,ds\,dt\\&\quad=\int_0^T\varphi_\gamma\left((v_s(t)^{-1}\partial_t v_s(t))\hat{\otimes}(v_s^{-1}(t)\hat{\otimes}v_s(t))^{\hat{\otimes}m}\right)\Big|_0^1\,dt\\&\quad=\int_0^T\varphi_\gamma\left((v_1(t)^{-1}\partial_t v_1(t))\hat{\otimes}(v_1^{-1}(t)\hat{\otimes}v_1(t))^{\hat{\otimes}m}\right)\,dt\\&\quad\quad-\int_0^T\varphi_\gamma\left((w(t)^{-1}\partial_t w(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)\,dt\end{aligned}\tag{4.35}$$

where in the last integral we have used the fact $v_0(t)=w(t)$. Integrating the other side of the derivative equality shows that the above expression is also equal to

$$
\begin{aligned}
&\int_0^1 \int_0^T \frac{\partial}{\partial t}\varphi_\gamma\left((v_s(t)^{-1}\partial_s v_s(t))\hat{\otimes}(v_s^{-1}(t)\hat{\otimes}v_s(t))^{\hat{\otimes}m}\right)\, dt\, ds \\
&\quad = \int_0^1 \varphi_\gamma\left((v_s(t)^{-1}\partial_s v_s(t))\hat{\otimes}(v_s^{-1}(t)\hat{\otimes}v_s(t))^{\hat{\otimes}m}\right)\Big|_0^T\, ds \\
&\quad = \int_0^1 \varphi_\gamma\left((v_s(T)^{-1}\partial_s v_s(T))\hat{\otimes}(v_s^{-1}(T)\hat{\otimes}v_s(T))^{\hat{\otimes}m}\right)\, ds \\
&\quad\ - \int_0^1 \varphi_\gamma\left((v_s(0)^{-1}\partial_s v_s(0))\hat{\otimes}(v_s^{-1}(0)\hat{\otimes}v_s(0))^{\hat{\otimes}m}\right)\, ds
\end{aligned}
\tag{4.36}
$$

The last integral vanishes since $v_s(0) \equiv v_s^{-1}(0) \equiv \mathbb{1}$ for all $s \in [0,1]$, implying that $\varphi_\gamma\left((v_s(0)^{-1}\partial_s v_s(0))\hat{\otimes}(v_s^{-1}(0)\hat{\otimes}v_s(0))^{\hat{\otimes}m}\right) = 0$. Multiplying by $\frac{(-1)^m m!}{\pi i}$ then taking the limit as T goes to infinity in (4.35) we clearly obtain the expression $\tau_\varphi(v_1) - \tau_\varphi(w)$. It immediately follows from doing the same procedure to (4.36) that

$$
\begin{aligned}
\tau_\varphi(v_1) - \tau_\varphi(w) &= \lim_{T\longrightarrow\infty} \frac{(-1)^m m!}{\pi i}\int_0^T \varphi_\gamma\left((v_1(t)^{-1}\partial_t v_1(t))\hat{\otimes}(v_1^{-1}(t)\hat{\otimes}v_1(t))^{\hat{\otimes}m}\right)\, dt \\
&\quad - \lim_{T\longrightarrow\infty} \frac{(-1)^m m!}{\pi i}\int_0^T \varphi_\gamma\left((w(t)^{-1}\partial_t w(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)\, dt \\
&= \frac{(-1)^m m!}{\pi i}\int_0^1 \lim_{T\longrightarrow\infty} \varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(v_s^{-1}(T)\hat{\otimes}v_s(T))^{\hat{\otimes}m}\right)\, ds
\end{aligned}
$$

where our interchanging of limits and integrals is easily checked to be justified. It thus only remains to prove that for all s, uniformly with respect to the norm $||\cdot||_{\mathscr{B},k}$

$$
\lim_{T\longrightarrow\infty} \varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(v_s^{-1}(T)\hat{\otimes}v_s(T))^{\hat{\otimes}m}\right) = 0 \tag{4.37}
$$

By basic distribution of tensors over addition and mutlilinearity of cyclic cocycles the above decomposes as

$$\lim_{T\longrightarrow\infty}\varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(v_s^{-1}(T)-\varpi^{-1}(T)\hat{\otimes}v_s(T))^{\hat{\otimes}m}\right)$$
$$+\lim_{T\longrightarrow\infty}\varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(\varpi^{-1}(T)\hat{\otimes}v_s(T)-\varpi(T))^{\hat{\otimes}m}\right)$$
$$+\lim_{T\longrightarrow\infty}\varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(\varpi^{-1}(T)\hat{\otimes}\varpi(T))^{\hat{\otimes}m}\right)$$

By Proposition 4.13 and its corollary there exist operators $\varpi(t),\varpi^{-1}(t)\in(\mathscr{B}(\widetilde{M})^\Gamma)^+$ such that for any $\varepsilon>0$ there exists t large enough such that the following hold.

$$||\varpi(t)-v_s(t)||_{\mathscr{B},k}\ ,\ ||\varpi(t)^{-1}-v_s^{-1}(t)||_{\mathscr{B},k}<\frac{C_k}{t^3}\ ,\ \mathsf{prop}(\varpi(t))\ ,\ \mathsf{prop}(\varpi^{-1}(t))<\varepsilon$$

We may assume that $T\geq 2$ so the description of the regularized representative gives $v_s^{-1}(T)\cdot\partial_s v_s(T)=\pi i\partial_s E_s(T)$, the propagation of which tends to 0 as $T\longrightarrow\infty$. Thus for large enough T the propagation of all the operators is less than some $\varepsilon_{\widetilde{M}}$ and by Remark 4.15

$$\lim_{T\longrightarrow\infty}\varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(\varpi^{-1}(T)\hat{\otimes}\varpi(T))^{\hat{\otimes}m}\right)=0$$

By the results of Theorem 4.12 and the norm boundedness of all $B\in(\mathscr{B}(\widetilde{M})^\Gamma)^+$

$$\lim_{T\longrightarrow\infty}\left|\varphi_\gamma\left((v_s^{-1}(T)\partial_s v_s(T))\hat{\otimes}(v_s^{-1}(T)-\varpi^{-1}(T)\hat{\otimes}v_s(T))^{\hat{\otimes}m}\right)\right|$$

$$\leq R_\alpha\lim_{T\longrightarrow\infty}||v_s^{-1}(T)\partial_s v_s(T)||_{\mathscr{B},k}||\varpi(t)^{-1}-v_s^{-1}(T)||^m_{\mathscr{B},k}||v_s(T)||^m_{\mathscr{B},k}$$
$$\leq R_\alpha C_k^m\lim_{T\longrightarrow\infty}\frac{||v_s^{-1}(T)\partial_s v_s(T)||_{\mathscr{B},k}||v_s(T)||^m_{\mathscr{B},k}}{T^{3m}}=0$$

An exact replica of this argument holds for the term involving $v_s^{-1}(T)-\varpi^{-1}(T)$. □

Theorem 4.19. *Let $[u]\in K_1(\mathscr{B}_{L,0}(\widetilde{M})^\Gamma)$ with v and w both being regularized representatives, then $\tau_\varphi(v)=\tau_\varphi(w)$ for any polynomial growth $\varphi_\gamma\in(C^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma)),b)$.*

Proof. We have proven in the above corollary that $\tau_\varphi(v_1)=\tau_\varphi(w)$. By construction $v_1(s)=$

$v(s)$ for all $s \in \mathbb{R}_{\geq 0} \setminus (1,2)$, with $v_1 v^{-1} : [1,2] \longrightarrow (\mathscr{B}(\widetilde{M})^\Gamma)^+$ being a local loop of invertible elements. Thus there exists a local invertible $f : S^1 \longrightarrow (\mathscr{B}(\widetilde{M})^\Gamma)^+$, which by Lemma 4.9 is homotopic to $e^{2\pi i\theta}P(t) + (\mathbb{1} - P(t))$ for some idempotent $P(t)$, such that $v(s)$ and $v_1(s)$ differ by $f(\theta)$ as elements in $K_1(\mathscr{B}(\widetilde{M})^\Gamma)$.

$$\tau_\varphi(v) = \tau_\varphi(v_1) + \tau_\varphi(v_1^{-1}v) = \tau_\varphi(w) + \tau_\varphi(v_1 v^{-1}) = \tau_\varphi(w) + \tau_\varphi(f_L)$$

$$= \tau_\varphi(w) + \frac{(-1)^m m!}{\pi i} \int_0^\infty \int_0^1 \overline{\varphi}_\gamma \left((f^{-1}(\theta)\dot{f}(\theta)) \hat{\otimes} (f^{-1}(\theta) \hat{\otimes} f(\theta))^{\hat{\otimes} m} \right) \, d\theta \, dt$$

Here $\dot{f}(\theta) = 2\pi i e^{2\pi i\theta} P(t)$ refers to the derivative with respect to θ; moreover

$$\begin{aligned} &(e^{2\pi i\theta}P(t) + (\mathbb{1} - P(t)))(e^{-2\pi i\theta}P(t) + (\mathbb{1} - P(t)) \\ &= e^{2\pi i\theta - 2\pi i\theta}P^2(t) + e^{2\pi i\theta}P(t)(\mathbb{1} - P(t)) + e^{-2\pi i\theta}P(t)(\mathbb{1} - P(t)) + (\mathbb{1} - P(t))^2 \\ &= P^2(t) + \mathbb{1} - 2P(t) + P^2(t) = P(t) + \mathbb{1} - 2P(t) + P(t) = \mathbb{1} \end{aligned}$$

which implies that $f^{-1}(\theta)$ is equal (homotopic) to $(e^{-2\pi i\theta}P(t) + (\mathbb{1} - P(t)))$.

$$\begin{aligned} &\int_0^1 \overline{\varphi}_\gamma \left((f(\theta)^{-1}\dot{f}(\theta)) \hat{\otimes} (f^{-1}(\theta) \hat{\otimes} f(\theta))^{\hat{\otimes} m} \right) \, d\theta \\ &= \int_0^1 \overline{\varphi}_\gamma \left((f^{-1}(\theta)\dot{f}(\theta)) \hat{\otimes} (e^{-2\pi i\theta}P(t) + (\mathbb{1} - P(t)) \hat{\otimes} e^{2\pi i\theta}P(t) + (\mathbb{1} - P(t)))^{\hat{\otimes} m} \right) \, d\theta \\ &= \int_0^1 \overline{\varphi}_\gamma \left(2\pi i P(t) \hat{\otimes} ((e^{-2\pi i\theta} - 1)P(t) + \mathbb{1} \hat{\otimes} (e^{2\pi i\theta} - 1)P(t) + \mathbb{1})^{\hat{\otimes} m} \right) \, d\theta \end{aligned}$$

For ease of notation set $r_\theta = e^{2\pi i\theta} - 1$ and $c_\theta = e^{-2\pi i\theta} - 1$; using multilinearity of cyclic cocycles and the fact that $\overline{\varphi}_\gamma$ vanishes on the unit, the integrand simplifies to

$$\begin{aligned}
&\overline{\varphi}_\gamma\left(2\pi i P(t)\hat{\otimes}((e^{-2\pi i\theta}-1)P(t)\hat{\otimes}(e^{2\pi i\theta}-1)P(t))^{\hat{\otimes}m}\right)\\
&+\sum_{j=0}^{m-1}\overline{\varphi}_\gamma\left(2\pi i P(t)\hat{\otimes}(c_\theta P(t)\hat{\otimes}r_\theta P(t))^{\hat{\otimes}j}\hat{\otimes}\mathbb{1}\hat{\otimes}r_\theta P(t)+\mathbb{1}\hat{\otimes}(r_\theta P(t)+\mathbb{1}\hat{\otimes}_\theta P(t)+\mathbb{1})^{\hat{\otimes}m-j-1}\right)\\
&+\sum_{j=0}^{m-1}\overline{\varphi}_\gamma\left(2\pi i P(t)\hat{\otimes}(c_\theta P(t)\hat{\otimes}r_\theta P(t))^{\hat{\otimes}j}\hat{\otimes}c_\theta P(t)\hat{\otimes}\mathbb{1}\hat{\otimes}(c_\theta P(t)+\mathbb{1}\hat{\otimes}r_\theta P(t)+\mathbb{1})^{\hat{\otimes}m-j-1}\right)\\
&=2\pi i(e^{-2\pi i\theta}-1)^m(e^{2\pi i\theta}-1)^m\varphi_\gamma\left(P(t)^{\hat{\otimes}2m+1}\right)+0+0
\end{aligned}\tag{4.38}$$

Vanishing of the double integral now follows from Remark 4.15 and the fact that for all $\varepsilon>0$ there exists t_ε large enough such that $\mathsf{prop}(P(t_\varepsilon))\leq\varepsilon$. □

Theorem 4.20. *The determinant map pairing the higher rho invariant is independent of the choice of delocalized cyclic cocycle representative. Explicitly, if* $[\varphi_\gamma]=[\phi_\gamma]\in HC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$, *then* $\tau_{\varphi_\gamma}(\rho(\widetilde{D},\widetilde{g}))=\tau_{\phi_\gamma}(\rho(\widetilde{D},\widetilde{g}))$

Proof. By the previous results we are able to fix a regularized representative w, and by hypothesis, φ_γ and ϕ_γ are cohomologous via a coboundary $b\varphi\in BC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$. We obtain an identical transgression formula as was calculated in Theorem 4.6, (4.16)

$$m(b\overline{\varphi})\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)=\frac{d}{dt}\overline{\varphi}((w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m})\tag{4.39}$$

Using the simplified definition of the determinant map, provided by (4.22)

$$m\tau_{b\varphi}(\rho(\widetilde{D},\widetilde{g})):=\frac{(-1)^m m!}{\pi i}\int_0^\infty m(b\overline{\varphi}_\gamma)\left((w^{-1}(t)\dot{w}(t))\hat{\otimes}(w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m}\right)\,dt\tag{4.40}$$

Thus by the transgression formula $\tau_{b\varphi}(\rho(\widetilde{D},\widetilde{g}))$ is equal (up to a constant) to

$$\lim_{t\longrightarrow\infty}\overline{\varphi}((w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m})-\lim_{t\longrightarrow 0}\overline{\varphi}((w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m})\tag{4.41}$$

By construction $w(0)=\mathbb{1}$, and $\overline{\varphi}$ vanishes on $\mathbb{1}$, thus $\lim_{t\longrightarrow 0}\overline{\varphi}((w^{-1}(t)\hat{\otimes}w(t))^{\hat{\otimes}m})=0$.

Moreover, for all $t \geq 2$ some smooth normalizing function χ can be chosen such that

$$w(t) = \exp\left(2\pi i \frac{E(t)+1}{2}\right) = \exp\left(2\pi i \frac{\chi(\widetilde{D}/t)+1}{2}\right)$$

By the proof of [11, Proposition 6.7] the smooth normalizing function χ can be chosen such that its distributional Fourier transform has compact support. In particular, this compact support implies $\hat{\chi}$ belongs to the space $S'(\mathbb{R})$ of tempered distributions, and hence so do all its derivatives. Moreover, since the Fourier transform is a linear automorphism on $S'(\mathbb{R})$ we also have that $\chi \in S'(\mathbb{R})$ along with derivatives of all orders. Now viewing χ as a smooth *function* we thus have for all $k \in \mathbb{N}$ that the derivative $\chi^{(k)}(x)$ grows no faster than some polynomial $p_k(x)$; that is there exists an integer N_k along with $C_k > 0$ satisfying

$$|\chi^{(k)}(x)| \leq C_k(1+|x|^2)^{N_k}$$

Denote $\psi(x) = \exp\left(2\pi i \frac{\chi(x)+1}{2}\right) - 1 = e^{i(\pi\chi(x)+\pi)} - 1$, then using power series expansion

$$\psi(x) = \sum_{k=0}^{\infty} \frac{i^k(\pi\chi(x)+\pi)^k}{k!} - 1 = \sum_{k=1}^{\infty} \frac{i^k(\pi\chi(x)+\pi)^k}{k!}$$

If $\psi^{(k)}(x)$ denotes the k'th derivative, then we have the following expression for $\psi^{(k)}(x)$

$$(\pi i)^k(\chi'(x))^k\psi(x) + \sum_{2m_2+\cdots+(k-1)m_{k-1}=k} C_{m_2,\ldots,m_{k-1}} \prod_{l=2}^{k-1}(\chi^l(x))^{m_l}\psi(x) + (\pi i)\chi^{(k)}(x)\psi(x)$$

where $0 \leq m_l \leq k$. By periodicity of $e^{i\theta}$ it suffices to replace $\pi\chi(x)+\pi$ with $\pi(1-|\chi(x)|)$.

$$|x^m\psi^k(x)| \leq |x|^m \left(\sum_{m_1+2m_2+\cdots+km_k=k} C_{m_1,\ldots,m_k} \prod_{l=1}^{k} |(\chi^l(x))|^{m_l}|\psi(x)|\right)$$

$$\leq |x|^m \left(\sum_{m_1+2m_2+\cdots+km_k=k} C_{m_1,\ldots,m_k} \prod_{l=1}^{k} C_k^{m_l}(1+|x|^2)^{N_l m_l} \sum_{n=1}^{\infty} \frac{|\pi i^n|(1-|\chi(x)|)^n}{n!}\right)$$

Now, by continuity of $\chi(x)$ and the fact that it is an odd function we have that for every $1 > \varepsilon > 0$ there exists N_ε such that $1 - |\chi(x)| < \varepsilon$ whenever $|x| \geq N_\varepsilon$. Since the class of K-theory representative is independent of the class of smooth normalizing function, without loss of generality assume that $\lim_{x \longrightarrow \pm\infty} \chi(x)$ converges to ± 1 fast enough so we can always choose an $N_{\varepsilon,m}$ large enough such that for each m

$$\varepsilon < \frac{1}{|x|^{m+1}(1+|x|^2)^{N_1 m_1}} \quad \text{if} \quad |x| \geq N_{\varepsilon,m}$$

From this it immediately follows that ψ is a Schwartz function, and thus by Proposition 4.4 $\lim_{t \longrightarrow \infty} \overline{\varphi}((w^{-1}(t)\hat{\otimes} w(t))^{\hat{\otimes} m}) = 0$ □

Proposition 4.21. *Let* $S_\gamma^* : HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow HC^{2m+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ *be the delocalized Connes periodicity operator, then* $\tau_{[S_\gamma \varphi_\gamma]}(\rho(\widetilde{D}, \widetilde{g})) = \tau_{[\varphi_\gamma]}(\rho(\widetilde{D}, \widetilde{g}))$ *for every* $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$.

Proof. The proof exactly mirrors that of Proposition 4.7. □

4.3 Proof of Theorem 1.2

The following discussion and the proof of Proposition 4.22 closely align with the proof of Theorem 4.3 in [50]. We first recall the construction at the beginning of the previous section of a representative of $\rho(\widetilde{D}, \widetilde{g})$ using the path of invertibles

$$S = \{U(t) = \exp(2\pi i H(t)) |\ t \in [0, \infty)\}$$

This construction can be altered by using the following smooth normalizing function ψ, where F_t is as defined in the construction of Lott's higher eta invariant.

$$\psi(t^{-1}x) = F_{1/t}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/t} e^{-s^2}\, ds \qquad t > 0 \tag{4.42}$$

The invertibility of the Dirac operator $\widetilde{D}$ implies that $\psi(t^{-1}D)$ converges in operator norm to $\frac{1}{2}(\mathbb{1} + \widetilde{D}|\widetilde{D}|^{-1})$ as $t \to 0$. Since ψ is a smooth normalizing function the operator $\psi(\widetilde{D})^2 - \mathbb{1}$

is locally compact, hence $e^{2\pi i F_t(\widetilde{D})} \equiv \mathbb{1}$ modulo locally compact operators. Moreover, ψ can be approximated by smooth normalizing functions with compactly supported distributional Fourier transforms, hence the inverse Fourier transform relation

$$\psi(\widetilde{D}) = \int_{-\infty}^{\infty} \hat{\psi}(s) e^{is\widetilde{D}} \, ds \tag{4.43}$$

is well defined, and by finite propagation property of the wave operator $e^{is\widetilde{D}}$ it follows that the path $U \in (C^*_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$ defined by

$$U(t) = U_t(\widetilde{D}) = e^{2\pi i \psi(\widetilde{D}/t)}; t \in (0, \infty) \qquad U_0 \equiv \mathbb{1} \tag{4.44}$$

can be uniformly approximated by paths of invertible elements with finite propagation. In totality, we have that U is an invertible element of $(C^*_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$ and gives rise to a class in $K_1(C^*_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)$.

Proposition 4.22. *The element U is invertible in $(\mathscr{A}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$.*

Proof. Firstly, we note that by the proof of Theorem 4.20 the function $U_t(x) - 1$ is a Schwartz function, and so by Lemma 4.1, $U_t(\widetilde{D}) - \mathbb{1}$ belongs to $\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)$ for all $t \in [0, \infty)$. Being of the form $e^{f(\widetilde{D})}$ this further proves that $U_t(\widetilde{D}) \in (\mathscr{A}(\widetilde{M}, \mathcal{S})^\Gamma)^+$ is invertible for all t. By definition of the localization algebra, to prove that U is an invertible element of $(\mathscr{A}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$ it suffices to show that $U - \mathbb{1}$ is a piecewise smooth function on the half-line. By the description of $F_{1/t}$ this is immediate for all $t \in (0, \infty)$, hence we only need to show smoothness at $t = 0$ with respect to the Frechet topology generated by the seminorms $||\cdot||_{\mathscr{A},k}$. By the smoothness and boundedness of $F_{1/t}(x)$, an application of the fundamental theorem of calculus gives

$$U_t(\widetilde{D}) - \mathbb{1} = \exp(2\pi i \psi(\widetilde{D}/t)) - 1 = \exp\left(\int_0^t 2\pi i \frac{d}{ds} \psi(\widetilde{D}/s) \, ds\right) - \mathbb{1} \tag{4.45}$$

Denoting by $\dot{\psi}(s)$ the derivative with respect to s, for $t \in (0, \infty)$ the usual power series expansion and sub-multiplicative property of norms in Banach algebras gives

$$||U_t(\widetilde{D}) - \mathbb{1}||_{\mathscr{A},k} = \exp\left(\int_0^t 2\pi i\dot{\psi}(\widetilde{D}/s)\, ds\right) - \mathbb{1} = \left|\left|\sum_{n=1}^{\infty} \frac{1}{n!}\left(\int_0^t 2\pi i\dot{\psi}(\widetilde{D}/s)\, ds\right)^n\right|\right|_{\mathscr{A},k}$$

$$\leq \sum_{n=1}^{\infty} \frac{(2\pi)^n}{n!} \left|\left|\left(\int_0^t \dot{\psi}(\widetilde{D}/s)\, ds\right)\right|\right|_{\mathscr{A},k}^n \leq \sum_{n=1}^{\infty} \frac{(2\pi)^n}{n!} \left|\left|\left(\frac{1}{\sqrt{\pi}}\frac{-2}{s^2}\widetilde{D}e^{-\widetilde{D}^2/s^2}\right)\Big|_0^t\right|\right|_{\mathscr{A},k}^n$$

Following a similar line of argumentation to that on [50, Page 21], as in Lemma 4.5 we use the fact that $\widetilde{D}$ is invertible– along with $\mathscr{A}(\widetilde{M},\mathcal{S})^\Gamma$ being closed under holomorphic functional calculus– to observe that for each seminorm Gelfand's formula asserts

$$e^{-r^2} = \lim_{n\longrightarrow\infty}\left(||e^{-n\widetilde{D}^2}||_{\mathscr{A},k}\right)^{\frac{1}{n}}$$

This implies that for $1/s^2 \in [n, n+1)$ there exists n large enough such that for some positive constants C_0 and C_1

$$||C_0 e^{-\widetilde{D}^2/s^2}||_{\mathscr{A},k} \leq ||e^{-n\widetilde{D}^2}||_{\mathscr{A},k} \leq e^{-nr^2/2} \leq C_1 e^{-r^2/2s^2} \tag{4.46}$$

Thus for $s > 0$ sufficiently small and for some positive constant C_2 we obtain the bound

$$\left|\left|\frac{1}{\sqrt{\pi}}\frac{-2}{s^2}\widetilde{D}e^{-\widetilde{D}^2/s^2}\right|\right|_{\mathscr{A},k} = \left|\left|\frac{1}{\sqrt{\pi}}\frac{-2}{s^2}\widetilde{D}e^{-\widetilde{D}^2}\cdot e^{-(1/s^2-1)\widetilde{D}^2}\right|\right|_{\mathscr{A},k}$$

$$\leq \left|\left|\frac{1}{\sqrt{\pi}}\frac{-2}{s^2}\widetilde{D}e^{-\widetilde{D}^2}\right|\right|_{\mathscr{A},k} \left|\left|e^{-(1/s^2-1)\widetilde{D}^2}\right|\right|_{\mathscr{A},k} = \frac{-2}{s^2\sqrt{\pi}}\left|\left|\widetilde{D}e^{-\widetilde{D}^2}\right|\right|_{\mathscr{A},k}\left|\left|e^{-(1/s^2-1)\widetilde{D}^2}\right|\right|_{\mathscr{A},k}$$

$$\leq \frac{-2}{s^2\sqrt{\pi}}C_2\left|\left|e^{-(1/s^2-1)\widetilde{D}^2}\right|\right|_{\mathscr{A},k} \leq \frac{-2}{s^2\sqrt{\pi}}C_2C_1e^{-r^2/2s^2}$$

This final term converges to 0 as $s \to 0$, hence from the above calculations we obtain– taking $t > 0$ sufficiently small to begin with– the bound

$$\lim_{t\longrightarrow 0}||U_t(\widetilde{D}) - \mathbb{1}||_{\mathscr{A},k} \leq \lim_{t\longrightarrow 0}\sum_{n=1}^{\infty}\frac{(2\pi)^n}{n!}\left(\frac{-2}{t^2\sqrt{\pi}}C_2C_1e^{-r^2/2t^2} - \lim_{s\longrightarrow 0}\frac{-2}{s^2\sqrt{\pi}}C_2C_1e^{-r^2/2s^2}\right)^n$$

$$= \lim_{t \longrightarrow 0} \sum_{n=1}^{\infty} \frac{(2\pi)^n}{n!} \left(\frac{-2}{t^2\sqrt{\pi}} C_2 C_1 e^{-r^2/2t^2} \right)^n = \lim_{t \longrightarrow 0} \exp\left(\frac{1}{t^2} C e^{-r^2/2t^2} \right) - 1 = 0$$

Since $U_0(\widetilde{D}) = \mathbb{1}$ it follows that $U - \mathbb{1}$ is continuous with respect to the family of seminorms; the same holds true for all orders of its derivatives according to the expansion

$$\frac{d}{dt}(U_t(x) - 1) = \sum_{m_1 + 2m_2 + \cdots + km_k = k} C_{m_1, \ldots, m_k} \prod_{l=1}^{k} C_k^{m_l} \left(\frac{d^l}{dt^l} \psi(x/t) \right)^{m_l} (U_t(x) - 1) \tag{4.47}$$

where $0 \leq m_l \leq k$. Thus $U - \mathbb{1}$ is smooth with respect to $|| \cdot ||_{\mathscr{A},k}$ which proves that $U \in (\mathscr{A}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$ as desired. □

Corollary 4.23. *Let* $\varphi_\gamma \in (C^{2m}\mathbb{C}\Gamma, \mathrm{cl}(\gamma), b)$ *be of polynomial growth, with* $\widetilde{D}$ *being invertible, then the integral*

$$\int_0^\infty \overline{\varphi}_\gamma \left(\dot{U}_t(\widetilde{D}) U_t^{-1}(\widetilde{D}) \hat{\otimes} (U_t(\widetilde{D}) \hat{\otimes} U_t^{-1}(\widetilde{D}))^{\hat{\otimes} m} \right) dt$$

converges absolutely.

Proof. This is a direct consequence of Lemma 4.5. □

The construction of a regularized representative of $\rho(\widetilde{D}, \widetilde{g})$ involves the choice of some smooth normalizing function χ with compactly supported distributional Fourier transform $\hat{\chi}$, such that $E(t) = (\chi(\widetilde{D}/t) + \mathbb{1})/2$ and E has the properties outlined on page 60. Adapting the argument preceding [11, Proposition 6.11] we can thus construct a path w

$$w(t) = \begin{cases} U(t) & 0 \leq t \leq 1 \\ e^{2\pi i((2-t)\psi(\widetilde{D}) + (t-1)E(1))} & 1 \leq t \leq 2 \\ e^{2\pi i E(t-1)} & t \geq 2 \end{cases} \tag{4.48}$$

which defines a regularized representative. On the other hand, by definition, U is its own regularized representative and the equality of $[U]$ and $[w]$ as K-theory classes in $K_1(C^*_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)$ follows from their being homotopic in $(\mathscr{B}_{L,0}(\widetilde{M}, \mathcal{S})^\Gamma)^+$. Explicitly, we have the homotopy induced by the family of invertibles $h_s : s \in [0,1]$ defined by

$$h_s(t) = \begin{cases} U(t) & 0 \leq t \leq 1 \\ e^{2\pi i((2-t)\psi(\widetilde{D})+(t-1)(sE(1)+(1-s)\psi(\widetilde{D})))} & 1 \leq t \leq 1+s \\ e^{2\pi i(sE(t-1)+(1-s)\psi(\frac{\widetilde{D}}{t-1}))} & t \geq 1+s \end{cases} \tag{4.49}$$

Thus by Corollary 4.23 and the proofs of Lemma 4.17 and Corollary 4.18 we obtain that

$$\tau_{[\varphi_\gamma]}(\rho(\widetilde{D},\widetilde{g})) = \tau_{[\varphi_\gamma]}(w) = \tau_{\varphi_\gamma}(U)$$

$$= \frac{(-1)^m m!}{\pi i} \int_0^\infty \overline{\varphi}_\gamma \left(U_t(\widetilde{D})^{-1} \dot{U}_t(\widetilde{D}) \hat{\otimes} (U_t^{-1}(\widetilde{D}) \hat{\otimes} U_t(\widetilde{D}))^{\hat{\otimes} m} \right) \, dt$$

$$= \frac{(-1)^m m!}{\pi i} \int_0^\infty \overline{\varphi}_\gamma \left(\dot{\bar{u}}_t(\widetilde{D}) \bar{u}_t^{-1}(\widetilde{D}) \hat{\otimes} (\bar{u}_t(\widetilde{D}) \hat{\otimes} \bar{u}_t^{-1}(\widetilde{D}))^{\hat{\otimes} m} \right) \, dt = (-1)^m \eta_{[\varphi_\gamma]}(\widetilde{D})$$

where $\bar{u}_t = U_{1/t}$ and we have used the substitution $u_t \longleftrightarrow u_t^{-1}$.

4.4 Delocalized Higher Atiyah-Patodi-Singer Index Theorem

Consider smooth vector bundles V_1 and V_2 over a compact orientable smooth manifold M– without boundary– and an elliptic differential operator $D : V_1 \longrightarrow V_2$ which acts on the smooth sections of these vector bundles. Since every such D has a pseudo inverse it is a Fredholm operator, with *analytical index* defined by

$$\mathrm{ind}(D) = \dim\ker(D) - \dim\ker(D^*) \tag{4.50}$$

Let us also recall the *topological index* of D with respect to a cohomological formula

$$\int_M \mathsf{ch}(D)\mathsf{Td}(T^*M \otimes \mathbb{C}) \tag{4.51}$$

where $\mathsf{Td}(T^*M \otimes \mathbb{C})$ is the Todd class of the complexified tangent bundle of M, and $\mathsf{ch}(D)$ is the Thom isomorphism pullback of a particular Chern class associated to D. The original version of the Atiyah-Singer index theorem [4] was proven through use of cobordism theory, and asserts that for a compact manifold without boundary the topological index of D is equal to its analytical index. A more powerful K-theoretic approach [5, 6] was later provided, with the resulting formula for the topological index shown to be equivalent to the aforementioned cohomological one. Using this K-theory framework, the Atiyah-Patodi-Singer index theorem [2, 3] generalizes the equality of topological and analytical indexes to include manifolds with boundary, under satisfaction of certain global boundary conditions. By considering $\mathrm{Ind}_G(D)$ rather than $\mathrm{ind}(D)$, this further admits a kind of (delocalized) higher analogue, in our case modeled on that of Lott [31] (see the relationship between equations (1) and (66)).

Prior to stating and proving the delocalized version of a higher Atiyah-Patodi-Singer index theorem we will first exhibit a necessary relationship between the determinant map τ_{φ_γ} of the previous section and the Connes-Chern character map. In the remainder of this section we will work within the restriction of even dimensional cyclic cocycles and under the condition of a compact spin manifold M having fundamental group Γ of polynomial growth. In particular, given a delocalized cyclic cocycle $\varphi_\gamma \in (C^{2m}\mathbb{C}\Gamma, \mathrm{cl}(\gamma), b)$ we will define the φ_γ-component of the Connes-Chern character of an idempotent $p \in \mathscr{B}(\widetilde{M})^\Gamma$ according to that of [30, Chapter 8]

$$\mathsf{ch}_{\varphi_\gamma}(p) := \frac{(-1)^m(2m)!}{m!}\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right) \tag{4.52}$$

Firstly let us prove the usual well-definedness properties.

Proposition 4.24. *Let $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$, then the $[\varphi_\gamma]$-component of the Connes-Chern character*

$$\mathsf{ch}_{[\varphi_\gamma]} : K_0(\mathscr{B}(\widetilde{M})^\Gamma) \longrightarrow \mathbb{C}$$

is well defined, particularly being independent of the choices of cocycle representative and K-theory class representative.

Proof. Since φ_γ can be chosen to be of polynomial growth and $p \in (\mathscr{B}(\widetilde{M})^\Gamma)^+$ then Theorem 4.12 asserts that the formula for the φ_γ-component of the Connes-Chern character makes sense. Suppose that φ_γ and ϕ_γ belong to the same cohomology class in $HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$; by hypothesis, φ_γ and ϕ_γ are cohomologous via a coboundary $b\varphi \in BC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$. Independence with respect to cyclic cocycle representatives thus follows from showing that $\mathsf{ch}_{b\varphi}(p) = 0$ for any idempotent p. A direct computation gives

$$\begin{aligned}(b\varphi)\left(p^{\hat{\otimes}2m+1}\right) &= (-1)^{2m}\varphi\left(p^2\hat{\otimes}p^{\hat{\otimes}2m-1}\right) + \sum_{i=0}^{2m-1}(-1)^i\varphi\left(p^{\hat{\otimes}i}\hat{\otimes}p^2\hat{\otimes}p^{\hat{\otimes}2m-i-1}\right) \\ &= \varphi\left(p^{\hat{\otimes}2m}\right) + m\left(\varphi\left(p^{\hat{\otimes}2m}\right) - \varphi\left(p^{\hat{\otimes}2m}\right)\right) = \varphi\left(p^{\hat{\otimes}2m}\right) = 0\end{aligned} \tag{4.53}$$

since by the definition of the cyclic operator $\mathfrak{t}\varphi\left(p^{\hat{\otimes}2m}\right) = (-1)^{2m-1}\varphi\left(p^{\hat{\otimes}2m}\right)$. Let us now turn our attention to proving that if $p_0, p_1 \in (\mathscr{B}(\widetilde{M})^\Gamma)^+$ belong to the same class in $K_0(\mathscr{B}(\widetilde{M})^\Gamma)$ then $\mathsf{ch}_{[\varphi_\gamma]}(p_0) = \mathsf{ch}_{[\varphi_\gamma]}(p_1)$. By hypothesis there exists a piecewise smooth family of idempotents $p_t : t \in (0,1)$ connecting p_0 and p_1, which allows for the usual trick of taking the derivative.

$$\begin{aligned}\frac{d}{dt}\varphi_\gamma\left(p_t^{\hat{\otimes}2m+1}\right) &= \sum_{i=0}^{2m}\varphi_\gamma\left(p_t^{\hat{\otimes}i}\hat{\otimes}\dot{p}_t\hat{\otimes}p_t^{\hat{\otimes}2m-i}\right) = \sum_{i=0}^{2m}\mathfrak{t}^{2m+1-i}\varphi_\gamma\left(p_t^{\hat{\otimes}i}\hat{\otimes}\dot{p}_t\hat{\otimes}p_t^{\hat{\otimes}2m-i}\right) \\ &= \sum_{i=0}^{2m}\varphi_\gamma\left(\dot{p}_t\hat{\otimes}p_t^{\hat{\otimes}2m}\right) = (2m+1)\varphi_\gamma\left(\dot{p}_t\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\end{aligned} \tag{4.54}$$

On the other hand since φ_γ belongs to the kernel of the boundary map b, we have

$$\begin{aligned}
0 &= (b\varphi_\gamma)\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m+1}\right)\\
&= \varphi_\gamma\left((\dot{p}_t p_t^2 - p_t\dot{p}_t p_t)\hat{\otimes}p^{\hat{\otimes}2m}\right) + (-1)^{2m+1}\varphi_\gamma\left((p_t\dot{p}_t p_t - p_t^2\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\\
&\quad + \sum_{i=1}^{2m}(-1)^i\varphi_\gamma\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}i-1}\hat{\otimes}p_t^2\hat{\otimes}p_t^{\hat{\otimes}2m-i}\right)\\
&= \varphi_\gamma\left((\dot{p}_t p_t - p_t\dot{p}_t p_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right) - \varphi_\gamma\left((p_t\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\\
&\quad + \sum_{i=1}^{2m}(-1)^i\varphi_\gamma\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\\
&= \varphi_\gamma\left((\mathbb{1} - p_t)\dot{p}_t p_t\hat{\otimes}p_t^{\hat{\otimes}2m}\right) - \varphi_\gamma\left(p_t\dot{p}_t(p_t - \mathbb{1})\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\\
&\quad + m\left(\varphi_\gamma\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right) - \varphi_\gamma\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m}\right)\right)\\
&= \varphi_\gamma\left(((\mathbb{1} - p_t)\dot{p}_t p_t + p_t\dot{p}_t(\mathbb{1} - p_t))\hat{\otimes}p_t^{\hat{\otimes}2m}\right)
\end{aligned}$$

Since p_t and $(\mathbb{1} - p_t)$ are orthogonal idempotents, the Peirce decomposition for associative unital algebras– for the non-unital case the element $(\mathbb{1} - p_t)$ is formally viewed as satisfying $(\mathbb{1} - p_t)\mathscr{B}(\widetilde{M})^\Gamma = \{B - p_t B : \ B \in \mathscr{B}(\widetilde{M})^\Gamma\}$– asserts that $(\mathscr{B}(\widetilde{M})^\Gamma)^+$ splits as

$$p_t(\mathscr{B}(\widetilde{M})^\Gamma)^+ p_t \oplus p_t(\mathscr{B}(\widetilde{M})^\Gamma)^+(\mathbb{1} - p_t) \oplus (\mathbb{1} - p_t)(\mathscr{B}(\widetilde{M})^\Gamma)^+ p_t \oplus (\mathbb{1} - p_t)(\mathscr{B}(\widetilde{M})^\Gamma)^+(\mathbb{1} - p_t)$$

which means that $\dot{p}_t$ is decomposable as

$$p_t\dot{p}_t p_t + p_t\dot{p}_t(\mathbb{1} - p_t) + (\mathbb{1} - p_t)\dot{p}_t + (\mathbb{1} - p_t)\dot{p}_t(\mathbb{1} - p_t)$$

However, since p_t is an idempotent it follows that $\dot{p}_t = (\dot{p_t^2}) = \dot{p}_t p_t + p_t\dot{p}_t$, hence

$$p_t\dot{p}_t p_t = p_t(\dot{p}_t p_t + p_t\dot{p}_t)p_t = p_t^2\dot{p}_t p_t + p_t\dot{p}_t p_t^2 = 2p_t\dot{p}_t p_t \implies p_t\dot{p}_t p_t = 0$$

$$(\mathbb{1} - p_t)\dot{p}_t(\mathbb{1} - p_t) = (\mathbb{1} - p_t)(\dot{p}_t p_t + p_t\dot{p}_t)(\mathbb{1} - p_t)$$

$$= (\mathbb{1} - p_t)\dot{p}_t p_t(\mathbb{1} - p_t) + (\mathbb{1} - p_t)p_t\dot{p}_t(\mathbb{1} - p_t)$$

$$= (\mathbb{1} - p_t)\dot{p}_t 0 + 0\dot{p}_t(\mathbb{1} - p_t) = 0$$

This means that $\dot{p}_t = p_t\dot{p}_t(\mathbb{1} - p_t) + (\mathbb{1} - p_t)\dot{p}_t$, hence we obtain a transgression formula

$$HC^{2m}((\mathscr{B}(\widetilde{M})^\Gamma), \mathrm{cl}(\gamma)) \longmapsto HC^{2m+1}(\mathscr{B}(\widetilde{M})^\Gamma, \mathrm{cl}(\gamma))$$

$$\begin{aligned} \frac{d}{dt}\varphi_\gamma\left(p_t^{\hat{\otimes}2m+1}\right) &= (2m+1)\varphi_\gamma\left(\dot{p}_t\hat{\otimes}p_t^{\hat{\otimes}2m}\right) \\ = (2m+1)(b\varphi_\gamma)&\left((\dot{p}_t p_t - p_t\dot{p}_t)\hat{\otimes}p_t^{\hat{\otimes}2m+1}\right) = 0 \end{aligned} \tag{4.55}$$

The desired result now follows immediately from integration

$$0 = \int_0^1 \frac{(-1)^m(2m)!}{m!}\frac{d}{dt}\varphi_\gamma\left(p_t^{\hat{\otimes}2m+1}\right)\, dt = \mathsf{ch}_{[\varphi_\gamma]}(p_1) - \mathsf{ch}_{[\varphi_\gamma]}(p_0)$$

□

Proposition 4.25. *Let $S_\gamma^* : HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma)) \longrightarrow HC^{2m+2}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$ be the delocalized Connes periodicity operator, then $\mathsf{ch}_{[\varphi_\gamma]} = \mathsf{ch}_{[S_\gamma\varphi_\gamma]}$ for every $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$.*

Proof. Recalling the definition of the map β as given in (3.4) of Section 3.1 it is straightforward to compute the action of βb and $b\beta$ as refers to the Connes-Chern character. It is not difficult to see that $(\beta \circ b\varphi_\gamma)\left(p^{\hat{\otimes}2m+3}\right)$ vanishes, since it equals

$$\begin{aligned} &\sum_{i=1}^{2m+1}(-1)^i i(b\varphi_\gamma)\left(p^{\hat{\otimes}i}\hat{\otimes}p^2\hat{\otimes}p^{\hat{\otimes}2m+1-i}\right) + (-1)^{2m+2}(2m+2)(b\varphi_\gamma)\left(p^2\hat{\otimes}p^{\hat{\otimes}2m+1}\right) \\ &= \sum_{i=1}^{2m+2}(-1)^i i(b\varphi_\gamma)\left(p^{\hat{\otimes}2m+2}\right) = (m+1)(b\varphi_\gamma)\left(p^{\hat{\otimes}2m+2}\right) \\ &= (m+1)\left[\varphi_\gamma\left(p^2\hat{\otimes}p^{\hat{\otimes}2m}\right) + \sum_{i=1}^{2m}(-1)^i\varphi_\gamma\left(p^{\hat{\otimes}i}\hat{\otimes}p^2\hat{\otimes}p^{\hat{\otimes}2m-i}\right) + (-1)^{2m+1}\varphi_\gamma\left(p^2\hat{\otimes}p^{\hat{\otimes}2m}\right)\right] \\ &= (m+1)\left[\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right) - \varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right) + 2m\left(\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right) - \varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)\right)\right] = 0 \end{aligned}$$

Following the same basic computational arguments as above we obtain

$$(b\circ\beta\varphi_\gamma)\left(p^{\hat{\otimes}2m+3}\right)=\sum_{i=0}^{2m+2}(-1)^i(\beta\varphi_\gamma)\left(p^{\hat{\otimes}2m+2}\right)=(\beta\varphi_\gamma)\left(p^{\hat{\otimes}2m+2}\right)$$
$$=\sum_{i=1}^{2m+1}(-1)^i i\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)=-(2m+1)\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)+\sum_{i=1}^{m}(2i-(2i-1))\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)$$
$$=-(2m+1)\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)+m\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)=-(m+1)\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)$$

Using the relation $S_\gamma=\frac{1}{(2m+1)(2m+2)}(\beta b+b\beta)$ as in Definition 3.2 we obtain the desired result.

$$\mathsf{ch}_{[S_\gamma\varphi_\gamma]}(p)=\frac{(-1)^{m+1}(2m+2)!}{(m+1)!}(S_\gamma\varphi_\gamma)\left(p^{\hat{\otimes}2m+3}\right)$$
$$=\frac{(-1)^{m+1}(2m+2)!}{(2m+1)(2m+2)(m+1)!}\left(\beta\circ b\varphi_\gamma\left(p^{\hat{\otimes}2m+3}\right)+(b\circ\beta\varphi_\gamma)\left(p^{\hat{\otimes}2m+3}\right)\right)$$
$$=0+\frac{(-1)^m(m+1)(2m)!}{(m+1)!}\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)=\frac{(-1)^m(2m)!}{m!}\varphi_\gamma\left(p^{\hat{\otimes}2m+1}\right)=\mathsf{ch}_{[\varphi_\gamma]}(p)$$

□

Lemma 4.26. *Let $[\varphi_\gamma]\in HC^{2m}(\mathbb{C}\Gamma,\mathrm{cl}(\gamma))$, then the following diagram commutes*

$$\begin{array}{ccc} K_1(C_{L,0}(\widetilde{M})^\Gamma) & \xrightarrow{\tau_{[\varphi_\gamma]}} & \mathbb{C} \\ \partial\uparrow & & \uparrow\times(-2) \\ K_0(C^*(\widetilde{M})^\Gamma) & \xrightarrow{\mathsf{ch}_{[\varphi_\gamma]}} & \mathbb{C} \end{array}$$

Proof. By Proposition 2.19 we know that the K-theory of $C^*(\widetilde{M})^\Gamma$ coincides with that of $\mathscr{B}(\widetilde{M})^\Gamma$, and likewise with respect to the localization algebras. Thus we can view every element of $K_0(C^*(\widetilde{M})^\Gamma)$ as a formal difference of two idempotents belonging to $(\mathscr{B}(\widetilde{M})^\Gamma)^+$. Each idempotent $p\in\mathscr{B}(\widetilde{M})^\Gamma$ defines an element $F\in\mathscr{B}_L(\widetilde{M})^\Gamma$

$$F(t)=\begin{cases}(1-t)p & t\in[0,1]\\ 0 & t\in(1,\infty)\end{cases}\tag{4.56}$$

If $\partial : K_0(C^*(\widetilde{M})^\Gamma) \longrightarrow K_1(C_{L,0}(\widetilde{M})^\Gamma)$ denotes the K-theoretical connecting map, then $\partial[p] = [u]$ defines a K-theory class of invertibles in $K_1(\mathscr{B}^*_{L,0}(\widetilde{M})^\Gamma)$, where $u(t) = e^{2\pi i F(t)}$ for $t \in [0,\infty)$. The proof now follows by along the same lines as for the calculations in [11, Proposition 7.2]. Invertiblity of u is clear, with $u^{-1}(t) = e^{-2\pi i F(t)}$; to show that $u \in (\mathscr{B}^*_{L,0}(\widetilde{M})^\Gamma)^+$ we only need to prove that u belongs to the kernel of the evaluation map, that is $u(0) = \mathbb{1}$.

$$u(t) = e^{2\pi i F(t)} = \sum_{n=0}^{\infty} \frac{(2\pi i)^n (1-t)^n p^n}{n!} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(2\pi i)^n (1-t)^n p^n}{n!}$$

$$\mathbb{1} + \sum_{n=1}^{\infty} \frac{(2\pi i)^n (1-t)^n p}{n!} = \mathbb{1} + \left(\sum_{n=0}^{\infty} \frac{(2\pi i)^n (1-t)^n}{n!} - 1 \right) p = \mathbb{1} + \left(e^{2\pi i(1-t)} - 1\right) p$$

This expression for $u(t)$ makes it clear that $u(0) = \mathbb{1}$, and moreover we obtain

$$u^{-1}(t)\dot{u}(t) = \left(\mathbb{1} + \left(e^{-2\pi i(1-t)} - 1\right) p\right) \left(2\pi i e^{-2\pi i(1-t)} p\right) = -2\pi i p \tag{4.57}$$

Since $\overline{\varphi}_\gamma$ is multilinear and vanishes on the unit, following the arguments detailed in the second half of Theorem 4.19 gives

$$\begin{aligned}
\tau_{[\varphi_\gamma]}(u) &= \frac{(-1)^m m!}{\pi i} \int_0^\infty \overline{\varphi}_\gamma \left((u^{-1}(t)\dot{u}(t)) \hat{\otimes} (u^{-1}(t) \hat{\otimes} u(t))^{\hat{\otimes} m} \right) \, dt \\
&= \frac{(-1)^m m!}{\pi i} \int_0^1 \overline{\varphi}_\gamma \left((u^{-1}(t)\dot{u}(t)) \hat{\otimes} (u^{-1}(t) \hat{\otimes} u(t))^{\hat{\otimes} m} \right) \, dt \\
&= \frac{(-1)^m m!}{\pi i} \int_0^1 \overline{\varphi}_\gamma \left(-2\pi i p \hat{\otimes} \left(\left(\mathbb{1} + \left(e^{-2\pi i(1-t)} - 1\right) p\right) \hat{\otimes} \left(\mathbb{1} + \left(e^{2\pi i(1-t)} - 1\right) p\right)\right)^{\hat{\otimes} m} \right) \, dt \\
&= \frac{(-1)^m m!}{\pi i} \int_0^1 \overline{\varphi}_\gamma \left(-2\pi i p \hat{\otimes} \left(\left(e^{-2\pi i(1-t)} - 1\right) p \hat{\otimes} \left(e^{2\pi i(1-t)} - 1\right) p\right)^{\hat{\otimes} m} \right) \, dt \\
&= (-1)^m (-2) m! \varphi_\gamma \left(p^{\hat{\otimes} 2m+1} \right) \int_0^1 \left(e^{-2\pi i(1-t)} - 1\right)^m \left(e^{2\pi i(1-t)} - 1\right)^m \, dt
\end{aligned} \tag{4.58}$$

At this junction we will make a short combinatorial detour in order to more easily evaluate the integral; setting $z = 2\pi i(1-t)$ we have the decomposition

$$(e^z - 1)^m (e^{-z} - 1)^m = \left(\sum_{k_1=0}^{m} (-1)^{m-k_1} \binom{m}{k_1} e^{k_1 z}\right) \left(\sum_{k_2=0}^{m} (-1)^{m-k_2} \binom{m}{k_2} e^{-k_2 z}\right) = \sum_{n=0}^{2m} c_n$$

$$\begin{aligned} c_n &= \sum_{k=0}^{n} (-1)^{m-k} (-1)^{m+k-n} \binom{m}{k} \binom{m}{n-k} e^{kz} e^{-(n-k)z} \\ &= \sum_{k=0}^{n} (-1)^{2m-n} \binom{m}{k} \binom{m}{n-k} e^{(2k-n)z} \end{aligned}$$

It is now straightforward to express the above integral as

$$\int_0^1 \left(e^{-2\pi i(1-t)} - 1\right)^m \left(e^{2\pi i(1-t)} - 1\right)^m \; dt$$

$$= \int_0^1 \sum_{n=0}^{2m} \sum_{k=0}^{n} (-1)^{2m-n} \binom{m}{k} \binom{m}{n-k} e^{(2k-n)2\pi i(1-t)} \; dt$$

$$= \sum_{n=0, n\neq 2k}^{2m} \sum_{k=0}^{n} (-1)^{2m-n} \binom{m}{k} \binom{m}{n-k} \frac{e^{(2k-n)2\pi i(1-t)}}{(n-2k)2\pi i}\Bigg|_0^1 + \sum_{k=0}^{2m} (-1)^{2m-2k} \binom{m}{k} \binom{m}{2k-k} e^0$$

$$= \sum_{n=0, n\neq 2k}^{2m} \sum_{k=0}^{n} (-1)^{2m-n} \binom{m}{k} \binom{m}{n-k} \frac{1-1}{(n-2k)2\pi i} + \sum_{k=0}^{2m} \binom{m}{k} \binom{m}{k} = \sum_{k=0}^{2m} \binom{m}{k}^2$$

Using the fact that $\binom{m}{k} = 0$ whenever $k > m$, and applying the Chu-Vandermonde identity, we obtain the desired relationship

$$\tau_{[\varphi_\gamma]}(u) = (-1)^m (-2) m! \varphi_\gamma \left(p^{\hat{\otimes} 2m+1}\right) \sum_{k=0}^{m} \binom{m}{k}^2 = (-1)^m (-2) m! \varphi_\gamma \left(p^{\hat{\otimes} 2m+1}\right) \binom{2m}{m}$$

$$= \frac{(-1)^m (-2) m! (2m)!}{(m!)^2} \varphi_\gamma \left(p^{\hat{\otimes} 2m+1}\right) = (-2) \mathsf{ch}_{[\varphi_\gamma]}(p)$$

□

Now we shall set up the necessary preliminaries for a delocalized version of the Atiyah-

Patodi-Singer index theorem. To begin with, let W be a compact n-dimensional spin manifold with boundary $\partial W = M$ which is closed, and naturally is an $n-1$-dimensional spin manifold. Moreover, W is endowed with a Riemannian metric g which has product structure near M and is of positive scalar curvature metric when restricted to M. Let $\widetilde{D}_W$ be the Dirac operator lifted to the universal cover $\widetilde{W}$, $\widetilde{g}$ be the metric lifted to $\widetilde{W}$, and by $\partial\widetilde{W} = \widetilde{M}$ denote the lifting of M with respect to the covering map $p : \widetilde{W} \longrightarrow W$. As shown in [48, Section 3] the operator $\widetilde{D}_W$ defines a higher index $\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W) \in K_n(C^*(\widetilde{W})^{\pi_1(W)})$, and as we have already detailed in Section 4.2 in the case of $n-1$ being odd, the Dirac operator $\widetilde{D}_M$ defines a higher rho invariant $\rho(\widetilde{D}_M, \widetilde{g})$ in $K_{n-1}(C^*_{L,0}(\widetilde{M})^{\pi_1(W)})$. Recall that every equivariant coarse map $f : X \longrightarrow Y$ induces a homomorphism $C(f) : C^*(X)^G \longrightarrow C^*(Y)^G$, which itself induces a functorial map $K(f)$ on the K-theory. Clearly the lifted inclusion map $\widetilde{\imath} : \widetilde{M} \hookrightarrow \widetilde{W}$ is equivariantly coarse and so gives rise to a natural homomorphism

$$K(\widetilde{\imath}) : K_{n-1}(C^*_{L,0}(\widetilde{M})^{\pi_1(W)}) \longrightarrow K_{n-1}(C^*_{L,0}(\widetilde{W})^{\pi_1(W)}) \tag{4.59}$$

We will denote the image of $\rho(\widetilde{D}_M, \widetilde{g})$ under this map to also be $\rho(\widetilde{D}_M, \widetilde{g})$.

Theorem 4.27 (Delocalized APS Index Theorem)**.** *Let W be a compact even dimensional spin manifold with closed boundary $\partial W = M$, and endowed with a Riemannian metric g which has product structure near M and is of positive scalar curvature metric when restricted to M. If $\pi_1(W)$ is countable discrete, finitely generated, and of polynomial growth*

$$\mathsf{ch}_{[\varphi_\gamma]}\left(\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W)\right) = \frac{(-1)^{m+1}}{2}\eta_{[\varphi_\gamma]}(\widetilde{D}_M)$$

for any $[\varphi_\gamma] \in HC^{2m}(\mathbb{C}\Gamma, \mathrm{cl}(\gamma))$

Proof. The proof of Lemma 4.26 did not depend on the dimension or boundary structure of M, the only necessity being that $\widetilde{M}$ admit a proper and co-compact isometric action of Γ

(see Definition 2.15); thus the following diagram also commutes.

$$\begin{array}{ccc} K_1(C_{L,0}(\widetilde{W})^{\pi_1(W)}) & \xrightarrow{\tau_{[\varphi_\gamma]}} & \mathbb{C} \\ \partial \uparrow & & \uparrow \times(-2) \\ K_0(C^*(\widetilde{W})^{\pi_1(W)}) & \xrightarrow{\mathsf{ch}_{[\varphi_\gamma]}} & \mathbb{C} \end{array} \tag{4.60}$$

Moreover, since $\dim(W) = n$ is even, by [38, Theorem 1.14] and [48, Theorem A] the image of the higher index under the connecting map is

$$\partial \left(\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W) \right) = \rho(\widetilde{D}_M, \widetilde{g}) \in K_{n-1}(C^*_{L,0}(\widetilde{W})^{\pi_1(W)}) \cong K_1(C^*_{L,0}(\widetilde{W})^{\pi_1(W)}) \tag{4.61}$$

Coupling this identity with the main result of Section 4.3 we obtain

$$\begin{aligned} -2\mathsf{ch}_{[\varphi_\gamma]} \left(\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W) \right) &= \tau_{[\varphi_\gamma]} \left(\partial \left(\mathrm{Ind}_{\pi_1(W)}(\widetilde{D}_W) \right) \right) \\ &= \tau_{[\varphi_\gamma]} \left(\rho(\widetilde{D}_M, \widetilde{g}) \right) = (-1)^m \eta_{[\varphi_\gamma]}(\widetilde{D}_M) \end{aligned} \tag{4.62}$$

□

5. CONCLUSIONS

The results proven– and techniques employed– in this thesis depend heavily upon $\pi_1(M)$ being a virtually nilpotent group, parallel to the work of [11], which establishes analogous results for $\pi_1(M)$ a hyperbolic group. A major reason why these classes of groups are so frequently the subject of study throughout the years of higher manifold theory is due to the vast body of work which can be drawn upon in exploiting their geometric and group theoretical properties. One natural extension to be considered, if staying within this framework, is to ask the following:

Question 1. *Are there analogues to Theorem 1.1 and Theorem 1.2 when considering split extensions $\pi_1(M) = G_1 \rtimes_f G_2$ in the case where one of G_1 or G_2 is a hyperbolic group, and the other is virtually nilpotent?*

For the simplest case we can consider (an inner direct product) $\pi_1(M) = G_1 \times G_2$, which naturally arises for M having the structure of a product of two Riemannian manifolds. The immediate issue is algebraic topological, in that to ensure that every cyclic cocycle has a representative of polynomial growth rate we need to consider group cohomology $H^n(G_1 \times G_2, \mathbb{C})$ and classifying spaces of products. Since our interest is in complex coefficients, the Künneth formula decomposition for group cohomology reduces to its simplest form

$$H^n(G_1 \times G_2, \mathbb{C}) \cong \bigoplus_{i+j=n} \big(H^i(G_1, \mathbb{C}) \otimes H^j(G_2, \mathbb{C})\big) \tag{5.1}$$

On the classifying space front, it follows from the functoriality of $B : G \longrightarrow BG$ for topological groups that $B(G_1 \times G_2)$ is homotopically equivalent as a CW-complex to $B(G_1) \times B(G_2)$. In a paper currently under preparation we have made use of this structure to directly generalize the combined approaches of this thesis and [11] with respect to cyclic cocycle growth rate. However, it is not entirely clear what the correct choice of smooth dense subalgebra of $C_r^*(G_1 \times G_2) \otimes \mathcal{K}$ should be, since we naively have need of both a Puschnigg construction [39] in addition to that of Connes and Moscovici [16]. If we wish to depart from the relatively well-known world of virtually nilpotent and hyperbolic groups, there is a class of geometrically important groups which arises as a candidate for investigation. The Baumslag-Solitar groups [7] are easily described according to the presentation $BS(m,n) = \langle x, y |\ x^{-1}y^m x = y^n \rangle$. These groups have been instructive in a considerable number of geometric and algebraic counterexamples; for example, a remarkable result of Shalen [41] showed that if $|m| \neq |n|$ there exists no non-degenerate morphism $\phi : BS(m,n) \longrightarrow \pi_1(M)$ for M any connected and orientable 3-manifold. All Baumslag-Solitar groups (excepting the case $|m| = |n| = 1$) are of exponential growth (hence not virtually nilpotent) and none of them can be subgroups of hyperbolic groups, thus a priori we cannot expect to be able to directly modify the arguments specific to the virtually nilpotent or hyperbolic cases.

Question 2. *Is it possible to establish conditions on M which allow for defining a pairing between cyclic cohomology and the higher rho invariant, given $\pi_1(M) = BS(m,n)$?*

As a final note, I will mention that there is a direct proof of Proposition 4.7 using just the definitions of (3.4) and (3.5) and explicit computation of the action on cyclic cocycles: the case of $m = 0$ is easily verifiable.

$$\begin{aligned}
\eta_{[S_\gamma\varphi_\gamma]}(\widetilde{D}) &:= \frac{1}{2\pi i}\int_0^\infty (S_\gamma\overline{\varphi}_\gamma)(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))\,dt = \\
&\frac{1}{2\pi i}\int_0^\infty \frac{-1}{2}(\beta\circ b\overline{\varphi}_\gamma)(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))\,dt \\
&+\frac{1}{2\pi i}\int_0^\infty -(b\circ\beta\overline{\varphi}_\gamma)(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})\hat{\otimes}\bar{u}_t(\widetilde{D})\hat{\otimes}\bar{u}_t^{-1}(\widetilde{D}))\,dt \\
&= \frac{1}{2\pi i}\int_0^\infty 0\,dt + \frac{2}{2\pi i}\int_0^\infty \overline{\varphi}_\gamma(\dot{\bar{u}}_t(\widetilde{D})\bar{u}_t^{-1}(\widetilde{D})) =: \eta_{[\varphi_\gamma]}(\widetilde{D})
\end{aligned}$$

However, I decided to not include the several pages of calculations due to their unenlightening nature and a tendency for very important parity mistakes to be hidden in the morass of summations and cancellations, somewhat in the vein of Lemma 4.17.

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