UNIFORM ACCELERATION RADIATION

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RESEARCH COMPLIANCE CERTIFICATION

I, Timothy Bates, certify that all research compliance requirements related to this Undergraduate Research Scholars thesis have been addressed with my Research Faculty Advisor prior to the collection of any data used in this final thesis submission.

This project did not require approval from the Texas A&M University Research Compliance & Biosafety office.

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ABSTRACT

Uniform Acceleration Radiation

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Radiation of a uniformly accelerating source is an enduring problem in the physics literature that gives rise to multiple apparent paradoxes. Also in the literature there exist several notions for defining radiation. Here we focus on exploring the Poynting flux radiation, which is defined with respect to an observer's proper timelike Killing vector, of a source in a massless scalar field. We fill in details regarding the vanishing divergence condition of the stress tensor and verify explicitly that the divergence vanishes on the past horizon. Using a convenient set of coordinates, we plot vector fields of the Minkowski Poynting vector flux to gain insight into the flow of energy of the system. We then address implications to the equivalence principle paradox for uniformly accelerating sources, and argue that the qualitative equivalence principle provides the correct argument to 'save' the equivalence principle for scalar electrodynamics. We also provide a criterion, which would quantitatively verify this, if proven.

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Contributors

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1. INTRODUCTION

1.1 Historical Background

The situation of a uniformly accelerating charge (hyperbolic motion)[1] has been debated for decades. Since it was first considered by Max Born in 1909 [1], many volumes have been written on the subject, in which several "paradoxes" arise. The first paradox is that in hyperbolic motion, the radiation reaction force disappears, yet the Larmor formula, $P = -2q^2a^2/(3c^3)$, still predicts radiation. Born had originally concluded that there was no radiation with other notable figures, such as Pauli in 1921, returning the same result [2]. Bondi & Gold in 1955 [3] and later Fulton & Rohrlich in 1961 [4] provided more detailed calculations which resulted in the first major works concluding that a uniformly accelerating charge radiates. While the majority view today is that such a charge radiates, with many figures including Rohrlich, Ginzburg, and Schwinger [4][5][6] supporting this view, there are also distinguished names, including Pauli [2] and Feynman [7], who have called this into question, with Singal [8] still maintaining the view that there is no radiation.

Even if one accepts that there is radiation, there is further debate on which observers may detect radiation. For example, inertial observers observe radiation while co-accelerating observers do not [9][10]. Even inertial observers inside the R region may not observe radiation [10][11].

Further, different ways of defining radiation exist in the literature. In traditional classical field theory one uses the stress energy tensor to define radiation via the Poynting flux integrated over all angles. There are also methods, motivated by quantum field theory which use number operators to compute a quantity that can be interpreted as radiation. Number operators are not uniquely defined; Higuchi and Matsas give a classical particle number definition for radiation [12] and Kialka et al. provide two classical particle number operators [13] generalized to Klein-Gordon fields. One might ask whether these definitions yield congruent results.

A characteristic of hyperbolic motion is that there are two surfaces that separates Minkowski

[1]Uniform acceleration is defined as when the acceleration 4-vector is constant: $a^\mu a_\mu = g^2$. We consider rectilinear uniform acceleration; the motion in question takes the shape of a hyperbola on the z–t plane depicted in Figure 1.1

space into four natural regions[2]. The first surface, $H_p = \overline{L \cup P} \cap \overline{R \cup F}$, is called the past horizon, since any signal sent from the charge cannot be sent to $L \cup P$. The second surface, $H_f = \overline{L \cup F} \cap \overline{R \cup P}$, is called the future horizon, since no signal can be sent from $F \cup L$ to the charge. Figure 1.1 depicts the horizons and the 4 regions. In quantum field theory, the horizon of a uniformly accelerating observer is associated with the famous Fulling-Davies-Unruh effect [14] which predicts a thermal bath of particles from the Rindler horizon of an accelerating observer. Though derived originally for a quantum system, there is work done to demonstrate classical analogs to the Unruh effect [12][15] which may be useful in answering some of the 'paradoxes'.

The next 'paradox' arises from the principle of equivalence because uniform acceleration is associated to uniform gravitational fields. Thus one can ask whether a charge sitting on a table in some gravitational field radiates. One can also ask whether a charge in gravitational free fall radiates. There are some who say that the equivalence principle cannot be extended to electromagnetic phenomenon. Rohrlich, Ginzburg, Boulware, and Pauri & Vallisneri [9][11][16][17] have addressed this concluding that the principle of equivalence may be saved and demonstrates consistent experiments; however, there is still not a full consensus entirely over which experiments report radiation. Of course, Singal argues that no contradiction arises with the principle of equivalence as he maintains that there is no radiation from uniformly accelerated charges to begin with [18] [8].

Lastly, there is confusion over the interpretation of the 'Schott Term'. Fulton, Rohrlich and Ginzburg argue that this is key to understanding how radiation is possible while also satisfying energy conservation [4][5]. Rowland elaborates and makes the case that Singal is incorrect in his conclusion that there is no radiation by not accounting for the Schott term [19].

1.2 The Scope of This Paper

The case of a uniformly accelerating source has given rise to much debate and confusion around what we consider radiation and who can observe it. It asks us to analyze the notions and definition that we use to analyze these problems and by studying this problem we can hope

[2] $R = \{p \in \mathcal{M} : z+t > 0, z-t < 0\}$, $L = \{p \in \mathcal{M} : z+t < 0, z-t > 0\}$, $F = \{p \in \mathcal{M} : z+t > 0, z-t > 0\}$, $P := \{p \in \mathcal{M} : z+t < 0, z-t < 0\}$, where $\mathcal{M}$ is Minkowski space, and $p = (t, x, y, z) \in \mathcal{M}$

to clarify what we mean by these concepts including radiation, energy, principle of equivalence, Schott term, radiation reaction, etc.

With all of these ‘paradoxes’ to explore, this paper narrows its scope to the issue of defining radiation and the effects of these definitions on our understanding of the equivalence principle paradox. We explore these paradoxes in the context of a scalar source in a massless scalar field. We begin in section 2 by clarifying our notions of energy, and provide definitions of radiation. Next in section 3, we calculate the field and radiation of a scalar source in a massless scalar field. In section 4, we apply these results to a thought experiment concerning the equivalence principle. The paper also explores the question for what occurs for a source with the charge being turned on (and off) at some finite points.

We adopt the metric signature $(-,+,+,+)$ and use natural units[3]. Unless otherwise stated, t denotes Minkowski time, λ denotes Rindler time, and s denotes proper time.

[3] $c = \hbar = 1$

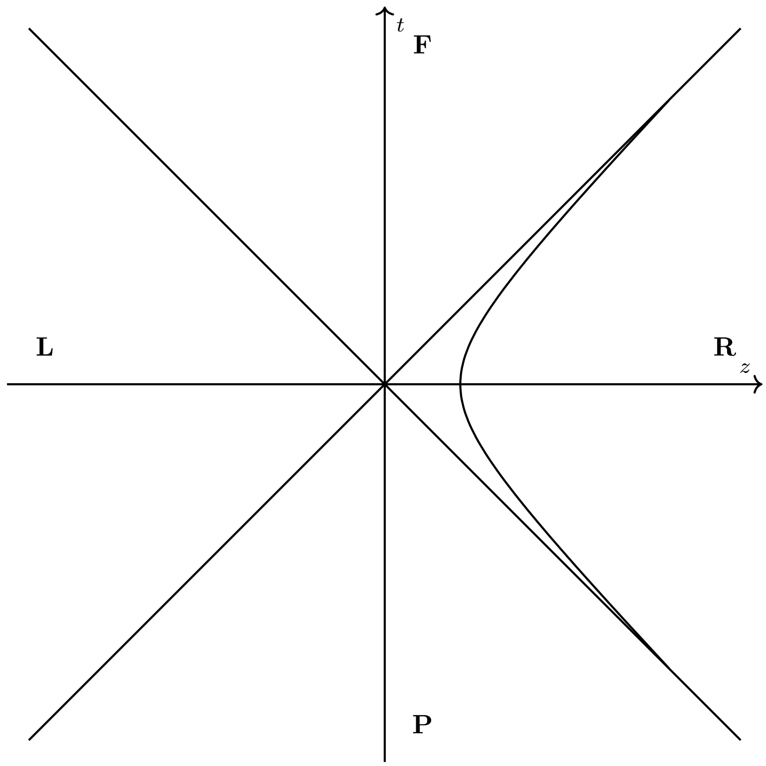

Figure 1.1: Diagram of hyperbolic motion with the 4 labeled quadrants

2. THE DEFINITIONS OF RADIATION AND THEIR KILLING VECTORS

Here we introduce Rindler coordinates, followed by a definition of energy. We then introduce several definitions of radiation which can be applied to inertial and accelerating observers.

2.1 Rindler Coordinates

Uniform acceleration is naturally described by a system of coordinates called Rindler coordinates. The line element for a Rindler coordinate frame is given by

$$ds^2 = -Z^2 d\lambda^2 + dZ^2 + dx^2 + dy^2 \qquad \text{(Eq. 2.1)}$$

Ignoring momentarily the transverse dimensions, we can think of these coordinates as 'hyperbolic polar coordinates' as any fixed Z is assigned to some hyperbola of fixed spacetime distance from the origin similar to how r in polar coordinates is assigned to a circle of fixed distance from the origin. λ is associated to how far one has travelled along that hyperbola, similar to how θ in polar coordinates tells one how far to move around the circle.

The transformations from standard Minkowski coordinates to Rindler coordinates are given by

$$t = \frac{e^{a\xi}}{a}\sinh(a\lambda), \quad z = \frac{e^{a\xi}}{a}\cosh(a\lambda) \qquad \text{(Eq. 2.2)}$$

The line element for this version of the coordinates is given by

$$ds^2 = e^{2a\xi}(-d\lambda^2 + d\xi^2) + dx^2 + dy^2 \qquad \text{(Eq. 2.3)}$$

One can see that Eq. 2.1 and Eq. 2.3 are equivalent when one sets $Z = e^{a\xi}$. We can visualize these coordinates in a spacetime diagram given by Figure 2.1.

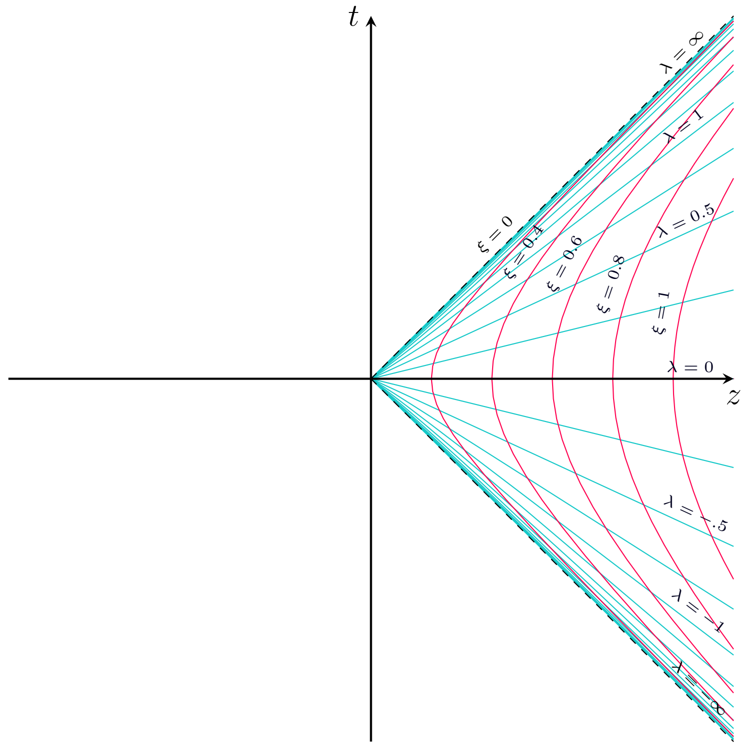

Figure 2.1: Rindler Coordinate Grid

The red hyperbolae are curves of constant ξ, and the blue lines from the origin are curves of constant λ. Rindler coordinates are defined only in the interior of the two dashed lines which represent the Rindler horizons.

2.2 The Definition of Energy

Before defining radiation, we need to define what exactly is being radiated. This requires a notion of energy. We define energy as the conserved quantity that is associated to a timelike Killing vector field. Killing vectors fields correspond to the vanishing of the Lie derivative of the metric; in other words, the metric is left unchanged under translation by a Killing vector. As the metric is invariant under an infinitesimal change generated by a Killing vector, these isometries are also called infinitesimal generators of symmetries (since the group of isometries may represent the symmetries of a system). These generators are associated via Noether's theorem to a conserved quantity. For inertial observers, the time translation vector field ∂_t[1] is a Killing vector (as the Minkowski metric is independent of time) that corresponds to the conserved quantity of energy. In Rindler space (ie for a uniformly accelerating observer) the notion of Energy is different. Rindler energy is the conserved quantity associated with the Killing field generated by the timelike Killing vector

$$\partial_\lambda = z\partial_t + t\partial_z = \frac{1}{2}[v\partial_v - u\partial_u] \quad \text{(Eq. 2.4)}$$

where λ is Rindler time[2]. Notice that this is also the Lorentz boost Killing vector in Minkowski space. Important to note here is that an arbitrarily accelerating frame as opposed to uniformly accelerating does not have a timelike Killing vector. As such the notion of a conserved energy does not exist for that frame. If the background spacetime is flat then we can still associate conserved quantities to the ambient flat Minkowski frame (hence we can see the privileged role that inertial observers play). Such a Killing vector also exists for the time coordinate of a uniformly accelerating observer in flat space and in the exterior of a static black hole, where in fact it is the only timelike Killing vector of such a frame.

Differentiating between the Minkowski and Rindler energies should be emphasized to demonstrate that 'energy' is a notion attached to reference frames. Throughout this paper, quantities corresponding to Minkowski (inertial) observers will be denoted with a subscript M, (ex. Q_M) and quantities corresponding to Rindler observers will be denoted with a subscript R (ex. Q_R).

2.3 The Definitions of Radiation

2.3.1 Poynting Flux

In this section we construct what we call a Poynting flux. Historically, the Poynting vector was first introduced in the 19th century to quantify the direct energy flux of an electromagnetic field. Integrating this flux over a spacelike surface yields power; this is naturally interpreted as electromagnetic radiation. Moving beyond the electromagnetic case, we can use the same machinery to capture the energy flux of a scalar field. The energy-momentum information is stored

[1] Here ∂_t is a coordinate vector, a notation used often in differential geometry

[2] Here u and v are null coordinates defined in appendix A

within the stress-energy tensor, but still requires a tool to identify what parts precisely correspond to energy flux. $T_{\alpha\beta}u^{\alpha}n^{\beta}$ captures the momentum density along the n^{β} direction for the observer associated with u^{α}, a timelike unit vector. If in place of u^{α}, we use a timelike Killing vector for which we have an associated energy, we can capture the energy density flowing through n^{β} for this observer. To this end, recall that associated to an observers notion of energy, one identifies a timelike Killing vector. By contracting this Killing vector with the stress energy tensor we have captured the energy density components. To quantify the flow we contract with a spacelike vector that is normal to a constant τ surface. The result is naturally interpreted as energy flux per unit area in the outward normal direction. Integrating this across all spatial angles yields power which quantifies the energy flow out of a region. The form of the Poynting flux is given as

$$S = -T_{\mu\nu}\xi^{\mu}\hat{n}^{\nu} \qquad \text{(Eq. 2.5)}$$

where ξ^{μ} is the Killing vector, $\hat{n}^{\nu}$ is a unit normal vector to the surface of integration, and $T_{\mu\nu}$ is the stress tensor.

2.3.2 Poynting Flux for an Inertial Observer

The timelike Killing field associated to an inertial observer is the inertial time translation Killing vector ∂_t. We contract ∂_t as well as a spacelike normal vector with the stress tensor: $\partial_t T^{\mu\nu} = T^{0\nu}$. To obtain the radiated power from the Poynting vector, we choose a constant t hypersurface (required by the conservation of energy) and obtain

$$\mathscr{R}_M = \int \hat{n}_{\nu}\xi_{\mu}T^{\mu\nu} = \int \hat{n}_{\nu}T^{0\nu} = \int ds_j T^{0j} \qquad \text{(Eq. 2.6)}$$

This gives us a notion of Energy flux out of a region surrounding the charge. Now this is just a special case for a more broad definition of radiation.

2.3.3 *Poynting Flux for a Rindler Observer*

Recall that the notion of energy can change for different classes of observers. While prior we had that the killing field $\xi^\mu = \partial_t$, now we consider a stationary observer in a Rindler frame whose notion of energy is associated to the Rindler time Killing vector $\xi^\mu = \partial_\lambda$. Now $\hat{n}^\nu$ is a unit normal vector to the surface of constant Rindler time $\Sigma(\lambda)$. This gives us a notion of energy flux out of a constant Rindler time hypersurface. The power radiated is then given by

$$\mathscr{R}_R = \int_{\Sigma(\lambda)} ZS \qquad \text{(Eq. 2.7)}$$

where $\Sigma(\lambda)$ denotes a spacelike surface where λ is constant. Z makes an appearance since it is the coefficient of the volume form.

2.3.4 *Classical Particle Number (Massless)*

We should remark that in quantum theory, the use of number operators to establish a presence of radiation is standard. Number operators have an interpretation of finite number of quanta of some particular field. It is important to note that these operators are not unique and different methods can return different number operators. Higuchi and Matsas [12] define a classical analog to the quantum number operator for a massless scalar field.

$$N_M = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 k}[\tilde{j}(k,\mathbf{k})]^2 \qquad \text{(Eq. 2.8)}$$

This is obtained by taking the Energy and dividing by frequency k. They also establish a number operator for Rindler observers; first introducing a Rindler energy quantity,

$$\begin{aligned} E_R =& -\frac{1}{2}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy [\phi^* \partial_\tau^2 \phi - \partial_\tau^* \partial_\tau \phi] \\ =& \pi \int_0^\infty d\omega \int d^2\mathbf{k}_\perp |\tilde{j}_R(\omega,\omega,\mathbf{k}_\perp)|^2 \end{aligned} \qquad \text{(Eq. 2.9)}$$

and following by dividing by Rindler mode frequency to obtain a Rindler number operator

$$N_R = \pi \int_0^\infty \frac{d\omega}{\omega} \int d^2\mathbf{k}_\perp |\tilde{j}_R(\omega, \omega, \mathbf{k}_\perp)|^2 \qquad \text{(Eq. 2.10)}$$

As mentioned previously, these operators can be defined in multiple ways. Kiałka et al. [13] define 2 particle number operators. Below is equation (19) from [13],

$$\hat{N}_2(t, V) = i \int_V d^3x \hat{\phi}^-(t, \mathbf{x}) \overleftrightarrow{\partial_t} \hat{\phi}^+(t, \mathbf{x}) \qquad \text{(Eq. 2.11)}$$

The advantage of this number operator is that it is presented in a form which may be applied to Minkowski and Rindler observers.

3. SCALAR RADIATION

In this section we compute the field of a uniformly accelerating source and calculate its radiation using the Poynting vector for inertial observers and Rindler observers. We mention the divergence-free condition of the stress tensor, which is computed in the appendix.

3.1 Background Scalar Field

The following discussion is filling in details of the treatment given by Ren & Weinberg [10]. Let ϕ be a massless scalar field, where the Lagrangian for a massless scalar field is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \qquad \text{(Eq. 3.1)}$$

It follows from the principle of stationary action that for a free massless scalar field, $\Box\phi = 0$. When coupled to a scalar source ρ given by

$$\rho = q\delta(\xi)\delta(x)\delta(y) \qquad \text{(Eq. 3.2)}$$

the field can be calculated with the retarded Green's function,

$$G_{ret}(x, x_s) = \theta(x^0 - x_s^0)\delta([x - x_s(s)]^2) \qquad \text{(Eq. 3.3)}$$

where x_s is a point on the trajectory of the source, and x is an observation point. The field is solved in the following form,

$$\phi(x) = \frac{q}{2\pi}\int_{-\infty}^{\infty} ds\theta(x^0 - x_s^0(s))\delta([x - x_s(s)]^2) \qquad \text{(Eq. 3.4)}$$

The theta function can be taken out of integral as $\theta(t + z)$, and we change limits of integration to

$$\phi(x) = \frac{q}{2\pi}\theta(t + z)\int_{-\infty}^{x_s^0(s)} ds\delta([x - x_s(s)]^2) \qquad \text{(Eq. 3.5)}$$

The argument of the delta function represents the squared spacetime interval from the source to observation point. It ensures that the field information travels on a null trajectory from the retarded point on the source of the charge. We have from distribution theory the identity

$$\int \delta(f(x)) = \sum_i \frac{\delta(x)}{|f'(x_0)|} \tag{Eq. 3.6}$$

Evaluating Δs^2 and $(\Delta s^2)'$,

$$|x - x_s(s)|^2 = -\left(t - \frac{1}{a}\sinh(as)\right)^2 + x^2 + y^2 + \left(z - \frac{1}{a}\cosh(as)\right)^2 \tag{Eq. 3.7a}$$

$$= -t^2 + \frac{2t}{a}\sinh(as) + \rho^2 + z^2 - \frac{2z}{a}\cosh(as) + \frac{1}{a^2} \tag{Eq. 3.7b}$$

$$(\Delta s^2)' = 2t\cosh(as) - 2z\sinh(as) \tag{Eq. 3.8}$$

Setting Eq. 3.7a equal to 0 and simplifying we obtain

$$t^2 - \rho^2 - z^2 - \frac{1}{a^2} = \frac{2t}{a}\sinh(as) - \frac{2z}{a}\cosh(as) \tag{Eq. 3.9}$$

$$\frac{a}{2}\left(-\chi^2 - \frac{1}{a^2}\right) = -\frac{e^{a\xi}}{a}\cosh(a(\tau - s)) \tag{Eq. 3.10}$$

Using the coordinates in [11], $Z = e^{a\xi}/a$, we have some useful identities

$$\chi^2 = Z^2 + \rho^2 \tag{Eq. 3.11}$$

$$\cosh(a(\tau - s)) = [\chi^2 + a^{-2}]\frac{a^2}{2e^{a\xi}} \tag{Eq. 3.12}$$

which we can use to solve for R,

$$R = e^{a\xi}\sinh(a(\tau - s)) = \frac{e^{a\xi}}{a^2}\sqrt{\cosh^2(a(\tau - s)) - 1} \tag{Eq. 3.13}$$

$$R = \frac{a}{2}\sqrt{\left(\chi^2 + \frac{1}{a^2}\right)^2 - \frac{4e^{2a\xi}}{a^4}} = \frac{a}{2}\sqrt{\left(Z^2 + \rho^2 + \frac{1}{a^2}\right)^2 - \frac{4Z^2}{a^2}} \tag{Eq. 3.14}$$

Eq. 3.14 is (III.16) in [11] which is equivalent to (2.3) in [10]. It then follows that the field is given by

$$\phi(x) = \frac{q}{4\pi R}\theta(t+z) \qquad \text{(Eq. 3.15)}$$

which is precisely (2.2) in [10]. The geometry of R corresponds to the radius of a distorted sphere centered at the source. The theta function ensures that the field is zero in the P and L regions of the Minkowski diagram. As a consequence we have a horizon on the $t = -z$ plane.

The use of Rindler coordinates for this calculation might suggest that this equation is no longer well defined in region F, but using the coordinate chart defined in section II of $[11]$, we can obtain the same result in region F. In region F we have

$$ds^2 = -(-a^2 Z^2 d\tau^2 + dZ^2) + dx^2 + dy^2 \qquad \text{(Eq. 3.16)}$$

and $z = Z\sinh(a\tau)$, $t = \cosh(a\tau)$. Making the appropriate changes and repeating the same procedure as before, we once again obtain

$$R = \frac{a}{2}\sqrt{\left(Z^2 + \rho^2 + \frac{1}{a^2}\right)^2 - \frac{4Z^2}{a^2}} \qquad \text{(Eq. 3.17)}$$

3.2 Radiation for Inertial Observers

3.2.1 Stress Energy Tensor

The stress energy tensor captures the information of energy-momentum of a field in a volume. A well defined field satisfies $\nabla_\alpha T^{\alpha\beta} = 0$ everywhere except the source; this is equivalent to local conservation of energy-momentum. We verify that in fact this vanishes, but to avoid disrupting the flow in the current exposition, we place this calculation in Appendix B using the coordinates defined in Appendix A.

To compute the ‘Poynting vector’ radiation, we must calculate the stress energy tensor

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi \qquad \text{(Eq. 3.18)}$$

The form of $T^{\mu\nu}$ was computed for inertial observers in [10] in the interior of R and F as

$$T^{\mu\nu} = \frac{q^2 a^2}{16\pi^2}\left[\frac{1}{R^4}\left(x^\mu x^\nu - \frac{1}{2}\eta^{\mu\nu}X^2\right) + \frac{1}{R^6}\left(a^{-2}\rho^\mu\rho^\nu + \frac{1}{2}\left(X^2 - a^{-2}\right)\left(x^\mu\rho^\nu + \rho^\mu x^\nu\right) - \rho^2 x^\mu x^\nu\right)\right] \tag{Eq. 3.19}$$

where $\rho^\mu = (0, x, y, 0)$. We computed the components of the stress tensor explicitly in null coordinates in the appendix.

3.2.2 Radiation

The radiated power was computed by Ren & Weinberg. They expressed T^{tj} in (2.14) of [10] as

$$T^{tj} = \frac{q^2}{16\pi^2}\left[\frac{a^2\cos^2\theta}{r^2} + \frac{a\cos\theta}{r^3}\right]\mathbf{r}^j \tag{Eq. 3.20}$$

The $\cos(\theta)$ tells us that the energy flow is greater along the plane that is coplanar to the hyperbolic motion. The opposite is true in the electromagnetic case, where there is a $\sin(\theta)$ instead tells us that the energy flow is greater along the plane perpendicular directions. Integrating this over a sphere of radius $r = t$ (which corresponds to the intersection of a constant Minkowski time hypersurface and the future light-cone from a point on the trajectory)

$$\mathscr{R}_M = \int dS_j T^{tj} = \frac{q^2}{16\pi^2}\int d\Omega\left[a^2\cos^2\theta + \frac{a\cos\theta}{r}\right] \tag{Eq. 3.21}$$

The terms in the stress tensor on the order of $1/r^2$ correspond to the emitted portions of the field while the terms of order $1/r^3$ correspond to energy that is bound to the source. Performing the integration yields

$$\mathscr{R}_M = \frac{q^2 a^2}{12\pi} \tag{Eq. 3.22}$$

We thus have a nonzero radiated power for inertial observers. Perhaps remarkably, this result is independent of both the considered point on the trajectory, and the time t chosen for the integration. This is known to be the case for the electromagnetic situation [20][11].

3.3 Radiation for Rindler Observers

There are several ways to address the radiation question for Rindler observers. [10] transformed the stress tensor Eq. 3.19 into Rindler coordinates and found that the $T^{\tau j}$ components vanished, thus there cannot be a Rindler Poynting flux. This concludes that Rindler observers who are stationary with respect to the uniformly accelerating charge observe no radiation: $\mathscr{R}_R = 0$. It remains to answer the question for what a Rindler observer may detect from an inertial charge: $\mathscr{R}_R$. One possible treatment may be by considering the field generated by an inertial charge, and transforming coordinates to the frame of a supported observer [16].

Another more general approach is the method of Hirayama in [21][22] where he considered the question of radiation for an electromagnetic charge undergoing an arbitrary acceleration (not necessarily uniform) with respect to a class of Rindler observers. The conclusion of Hirayama was that given a Rindler observer, the radiation of a charge according to Rindler energy is related to their relative acceleration. The result (Eq. 36 of [22]) (modified to our notation) is given by

$$\mathscr{R} = \frac{2q^2}{3}\alpha^\mu \alpha_\mu(-\bar{\xi}^\alpha v_\alpha)d\tau \qquad \text{(Eq. 3.23)}$$

where τ is proper time of the charge, $\alpha^\mu = h^\mu_\nu[a^\nu - w^\nu]$, $w^\mu = (g^\alpha g_\alpha)^{1/2}u^\mu + g^\mu$, where a and v are the acceleration and velocity of the charge, g and u are the acceleration and velocity of the observer fixed in the frame at the retarded point, and h^μ_ν is a projector orthogonal to v^μ. The bar over ξ indicates the Killing vector field evaluated at the point of emission. We note that $\mathscr{R} \equiv (a-g)^2$ is the square of the magnitude of the relative acceleration.

Moving back to the scalar case, we simplify this problem first by assuming $u^\mu = v^\mu$ and that the accelerations are coplanar. The field for an arbitrary accelerating source is given in Eq. 3.24.

$$\phi = \frac{-q}{4\pi(R^\mu v_\mu)} \qquad \text{(Eq. 3.24)}$$

We then find $\partial_\mu \phi$ given in Eq. 3.25

$$\partial_\mu \phi = \frac{q}{4\pi R^2}\left(\frac{1 + a^\alpha R_\alpha}{R} R_\mu - v_\mu\right) \tag{Eq. 3.25}$$

where $R = R^\alpha v_\alpha$. Let $f = (1 + a^\alpha R_\alpha)/R$, then the stress tensor components $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi$ are represented by

$$T_{\mu\nu} = \frac{q^2}{16\pi^2 R^4}\left[\left(f^2 R_\mu R_\nu - 2fR_\mu v_\nu + v_\mu v_\nu\right) + \frac{1}{2} g_{\mu\nu}\left(2fR + 1\right)\right] \tag{Eq. 3.26}$$

Next we would like to find $S = -T_{\mu\nu}\xi^\mu n^\nu$, where ξ is Rindler time Killing vector. We notice that the term in the expression of the stress tensor with $g_{\mu\nu}$ vanishes in S as ξ and $\hat{n}$ are orthogonal: $g_{\mu\nu}\xi^\mu \hat{n}^\nu = \xi_\mu n^\mu = 0$. It remains to find

$$S = -\frac{q^2}{16\pi^2 R^4}\xi^\mu n^\nu \left[f^2 R_\mu R_\nu - 2fR_\mu v_\nu + v_\mu v_\nu\right] \tag{Eq. 3.27}$$

Since our goal is to find $\mathscr{R}_M$ for an inertial charge, we can simplify this by letting $a = 0$.

$$S = -\frac{q^2}{16\pi^2 R^4}\xi^\mu n^\nu \left[\frac{R_\mu R_\nu}{R^2} - \frac{2R_\mu v_\nu}{R} + v_\mu v_\nu\right] \tag{Eq. 3.28}$$

Performing the integration of S over a sphere of constant Rindler time remains to be done in future work. . We should note here that in the electromagnetic case, this is resolved[1] and gives a power radiated $\mathscr{R}_R = \frac{2q^2 g^2}{12\pi}$. For the scalar case we would expect $\mathscr{R}_R = \frac{q^2 g^2}{12\pi}$.

3.4 Flow of Energy

In appendix B, we fill in some details regarding how the divergence of the stress tensor vanishes on the horizon. To understand the conservation law further, we can look at the flow of $S^\mu = -g^{\mu\alpha} T_{\alpha\beta}\xi^\beta$. The following figures were graphed using matplotlib. Each graph is a projection of these vector fields onto some plane. We set $a = .001$, so that the trajectory of the

[1] Luther D. Rinehart, private communication. See also [21]

charge is pushed horizontally far to the right on the z–t plane (in the u–v plane this would be to the upper left) in order to avoid having the trajectory intersect the plots. The vector fields for the Minkowski case are displayed in the figures: Figure 3.1, Figure 3.2, Figure 3.3, and Figure 3.4.

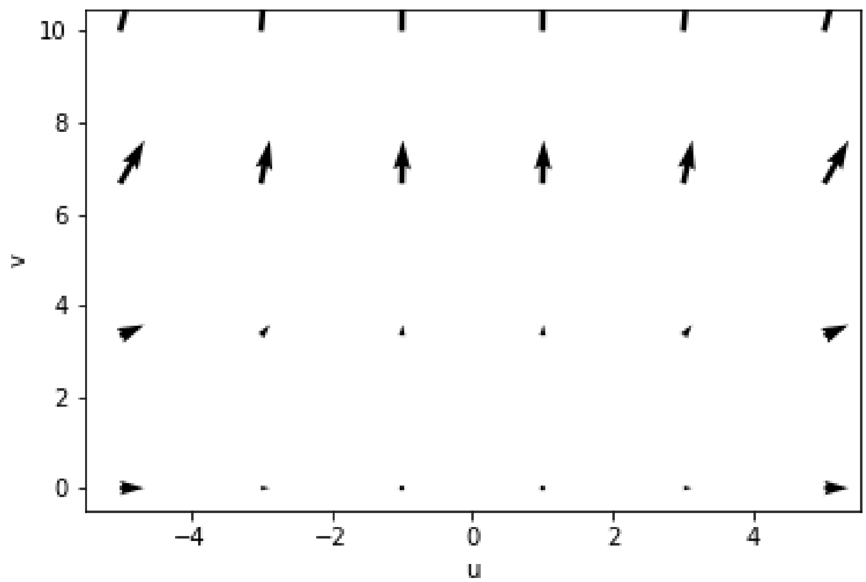

Figure 3.1: Minkowski Poynting Vector Field: x = y = 0

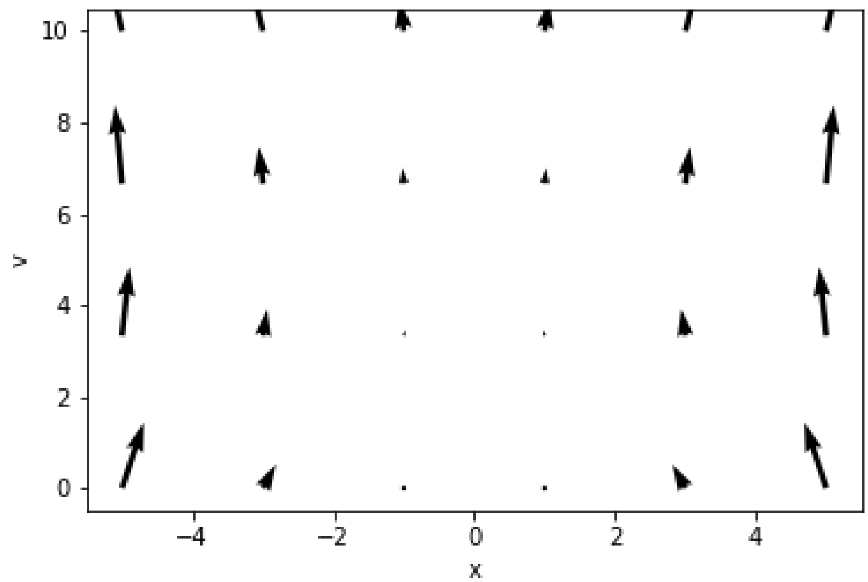

Figure 3.2: Minkowski Poynting Vector Field: u = -5, y = 0

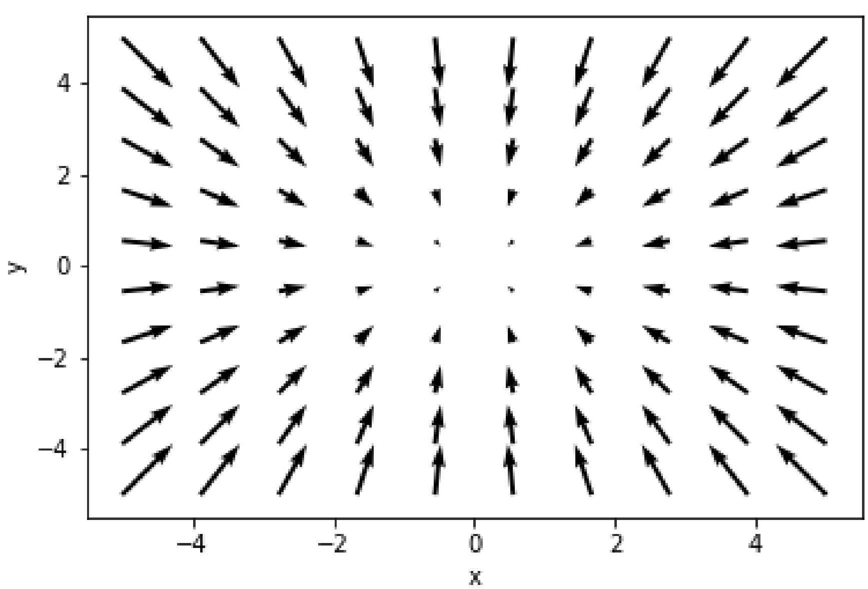

Figure 3.3: Minkowski Poynting Vector Field: u = -4, v = 3

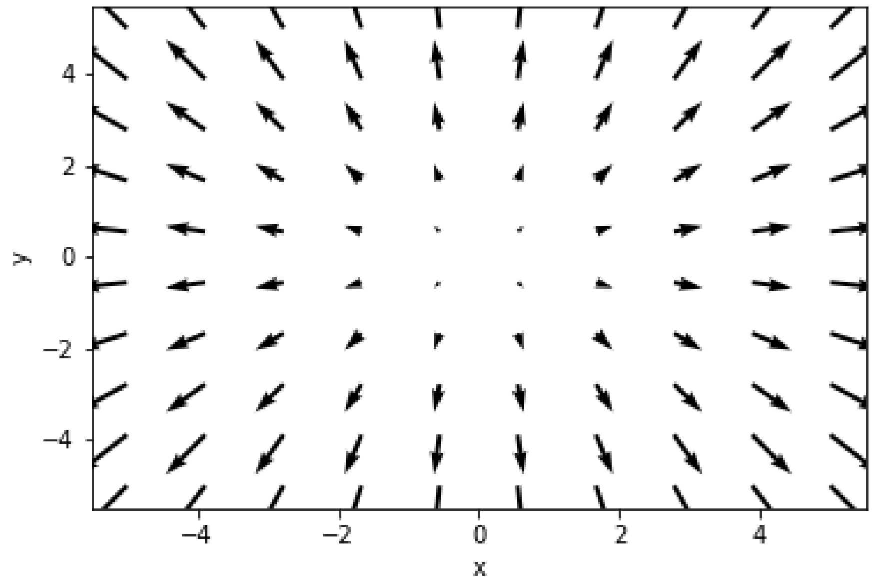

Figure 3.4: Minkowski Poynting Vector Field: u=-2, v=3

In the past region, when u is negative with absolute value greater than v, the flux moves inward to the plane of the source from the transverse directions, seen in Figure 3.3. When we move u closer to the future horizon ($u = 0$), the flux starts moving outward to the transverse directions, seen in Figure 3.4. When $|u| > |v|$, we have a inward flux, and when $|v| > |u|$, we have outward flux. This is because at these points the sum of components inside the S^x and S^y expressions change sign. How does this look on the z–t plane? $-u = v$ in the R region is the line $t = 0$, so when t is negative, we have inward flux. When t is positive, we have outward flux. Figure 3.2 captures flow fixed for $u = -5$. We can again see clearly that when $v < |u|$, the flow is in to the $x = 0$ plane, and when $v > |u|$, the flow is outward. Notice from the previous conclusion, that on the future horizon, $u = 0$, the flow is directed outward. We can also note, that on the past horizon, v = 0, the flow is directed inward from the transverse directions.

In Figure 3.1, the flow near the future horizon is rightward moving, in the direction of increasing v. Along the past horizon, the flux is leftward moving, in the direction of increasing u.

4. PRINCIPLE OF EQUIVALENCE

The equivalence principle is a fundamental postulate in Einstein's general theory of relativity. In this section we give several statements for the equivalence principle, and clarify the relevant statement in the context of acceleration radiation. We explore a thought experiment in which an observer and a source are placed in a uniform gravitational field. We ask whether in 4 different combinations of supporting or dropping the observer and supporting or dropping the source, whether the observer detects radiation from the source. If the results of the above experiments can reveal the presence of the gravitational field (if for example a supported source always radiates with respect to supported and falling observers), then this would be a paradox to the principle of equivalence. Can we reach a consensus about the observation of classical radiation and the equivalence principle? Up until now it appears that consensus is growing, and several papers [9][16][17] have been written on this, giving heuristic arguments which are building towards a consensus.

4.1 Definition of Equivalence Principle

The equivalence principle is the principle which glues special relativistic frames to gravity. Viewing local gravity effects as acceleration, and global effects as the curvature of spacetime leads one to adopt the manifold framework of general relativity. Different texts give different statements for the equivalence principle, some not necessarily equivalent. Three formulations of the equivalence principle are presented here.

4.1.1 Weak Equivalence Principle

Sean Carroll's textbook on gravity states the weak equivalence principle as " the motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime." [23]. Gravitational effects are seen as acceleration affects on small scales. Locally, it becomes impossible to distinguish between gravitation and acceleration. Take, for example, the famous Einstein elevator thought experiment, in which a person inside an elevator that is falling has no way to tell whether they are in free space, or whether they are in a gravitational field. Likewise, when the elevator is stopped, they have no way to conclude whether they are supported in some gravitational field, or whether their elevator is being accelerated by some external agent (e.g., a rocket engine) outside of the elevator. However, this statement makes no mention of how we would expect electrodynamics to behave. The following, 'strong' equivalence principle is one possible answer to this.

4.1.2 Strong Equivalence Principle

The strong equivalence principle, as the name suggests, is a much stronger statement. It states that the laws of physics in local frames reduce to special relativistic laws of physics; however, this prescription is much stronger than we need, so instead we move on to the 'qualitative equivalence principle'.

4.1.3 Qualitative (Rohrlich) Equivalence Principle

The latter is too strong and not entirely helpful in understanding the paradox at hand, while the former does not yet fully give insight into the nature of acceleration radiation. Fulling & Wilson [24] coined the name qualitative equivalence principle to describe an equivalence principle which captures the relevant physics for uniform acceleration. It is, in a sense, a corollary of the WEP where gravitational effects are interpreted as acceleration; the important of the qualitative equivalence principle result here is that observers in static frames in gravitational fields are qualitatively equivalent to uniformly accelerating frames.

4.2 The Supported Frame

The important feature of the qualitative equivalence principle rests in the fact that it allows us to treat supported frames in a static gravitational fields as equivalent to uniformly accelerating frames. We can introduce some more background by asking for a general frame of a stationary observer in a uniform gravitational field. Such a frame will be necessary for the physics of supported observers. The general line element was produced in [9][16]. By the qualitative equivalence principle, the Rindler metric describes such a supported frame in a static homogeneous gravitational field. We then obtain the following line element.

$$ds^2 = -g^2 Z^2 d\lambda^2 + dZ^2 + dx^2 + dy^2 \qquad \text{(Eq. 4.1)}$$

Since the supported frame is expressed using the familiar Rindler coordinates, which can be transformed to Minkowski coordinates, it follows that the physics of scalar radiation for supported observers can be calculated using precisely this approach. This treatment was done for electromagnetic fields in [9].

4.3 The 4 Cases

In this section we describe the thought experiment that has motivated much of this work. This classical thought experiment has been addressed in [9][17][24]. The following 4 cases are outlined in [9] and will be outlined here as well. Suppose we are in some homogeneous gravitational field. In this system we have a platform which is stationarily supported. One can think of this as a table supported on the surface of the earth (locally looks like a homogeneous gravitational field). We then consider a source and an observer in this system. We consider 4 cases, depicted in Figure 4.1, for observers and sources and ask the question of whether the observer will observe radiation.

1) Supported charge and observer

2) Free falling charge and supported observer

3) Supported charge and free falling observer

4) Free falling charge and observer

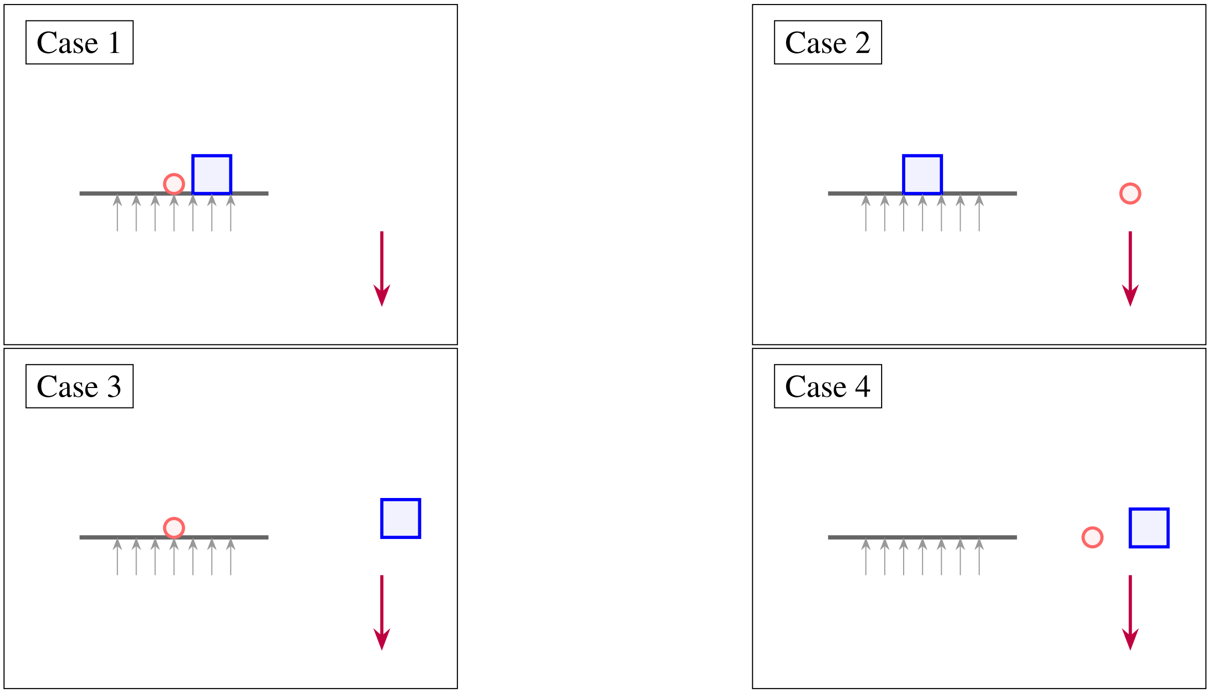

Figure 4.1: 4 cases of the equivalence principle paradox. The red circle denotes a charge and the Blue box, the detector

The essence of the argument of [9] is that in order to properly interpret the 4 cases in accordance with the equivalence principle quantitatively, we apply our special relativistic laws of electrodynamics to situations where the observer is free falling. Free falling observers are described in accordance with the equivalence principle to be frames where we can use the usual Minkowski model; thus we invoke the results for calculations concerning uniform acceleration of a source as seen by an inertial observer. Case 3 then reports radiation, and case 4 reports no radiation. How then does one calculate radiation for the supported observer experiments? The answer is that supported observers should be treated as Rindler observers, since the qualitative principle of equivalence equates local gravitation effects as acceleration affects. The constructions for Rindler observers now serve as tools that one uses to make quantitative predictions for these experiments. Pauri & Vallisneri in the electromagnetic case conclude that case 1 reports no radiation, and case 2 reports radiation in the affirmative. Thus for cases 2 & 3 where there is relative acceleration (in the sense that the two worldlines do not belong to the same Rindler congruence), radiation is detected, and in cases where there is no relative acceleration, no radiation is detected. This is precisely what one would expect qualitatively from the equivalence principle. We may ask whether these conclusions for electromagnetic fields can be achieved analogously for a scalar source.

In the fourth case, here a charge and observer are in free fall. The charge appears to be unaccelerated with respect to the observer. Using the result from section 3, $\mathscr{R}_M = 0$. Thus, there is no radiation observed in this case. In case 3 we once again we have a local inertial frame. The supported charge will appear to be uniformly accelerated in accordance with the qualitative principle of equivalence. From the result in section 3, $\mathscr{R}_M = \frac{q^2 g^2}{12\pi^2}$. It then follows that such an observer detects radiation. In the first 2 cases, we have to do computations with respect to that observer's notion of radiation. In this case we have a supported observer which from previous discussion we know is a Rindler observer; thus, we use the physics for Rindler observers in this case. We find that indeed for case 1, $\mathscr{R}_R = 0$, there is no radiation, and in case 2, we would need to perform the computation at the end of section 3.3 to verify quantitatively that this indeed will predict radiation.

5. CONCLUSION

5.1 Concluding Remarks

In conclusion, we have demonstrated several interesting results regarding radiation for scalar sources. We began by going through some careful definitions regarding notions of energy, settling on the Poynting flux as our primary object of study. We then examined a scalar source in a massless scalar field, derived some of its properties, and computed radiation for 2 classes of observers. In doing this, we filled in some details, namely computing explicitly the field for a uniformly accelerating source, the divergence of the stress tensor, and vector fields for Poynting vector flux. We then applied these results to the equivalence principle paradox and gave an argument which resolves the paradox by using the qualitative equivalence principle, and gives a criterion which if proven would also verify this quantitatively.

5.2 Further Areas of Study

The immediate next step would be to finish the computation of section 3.3 in order to completely understand the nature of Rindler radiation as well as resolving the associated application to the equivalence principle. One feature of scalar fields is that there is no analog to the continuity equation from electromagnetic theory. In other words, the source does not need to be conserved, so this can allow us to consider finite time scalar sources with possible interesting properties; consequences of which we still have yet to explore. We also have many more results that we can study regarding the flow of energy in this system. At present, we only explored a region of R close to the horizons, far from the charge. It may be insightful to look at the flow in the transverse directions on the other side of the charge. There is also room to find a more satisfactory answer to the balance of energy in this system.

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APPENDIX A: NULL COORDINATES IN 4D

Working with the divergencelessness of the stress tensor, it becomes convenient to consider a coordinate system which we call 'null coordinates'. Let $u = t - z$, $v = t + z$, $x = x$, $y = y$. u and v are null lines which on a spacetime diagram correspond to the rightward and leftward lightrays depicted in Figure A.1.

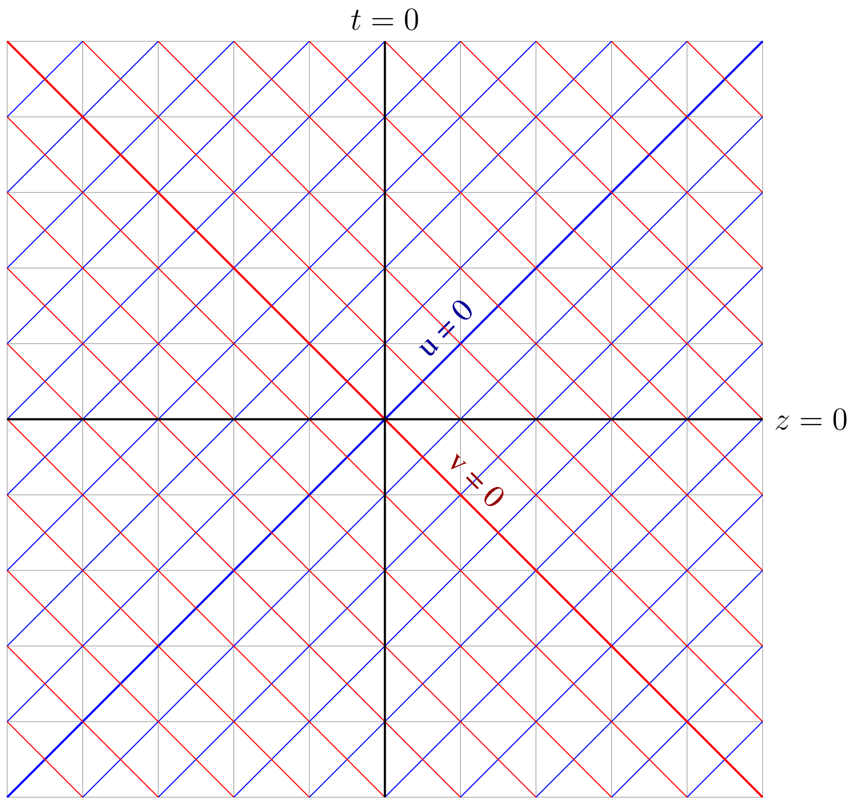

Figure A.1: Null Coordinates

Transforming back to usual Minkowski coordinates is $t = \frac{u+v}{2}$, $z = \frac{v-u}{2}$. We then have that $dt = \frac{1}{2}(du + dv)$, $dz = \frac{1}{2}(dv - du)$. We can note that $z^2 - t^2 = -uv$, thus the line element for null coordinates is given by

$$ds^2 = -dudv + dx^2 + dy^2 \qquad \text{(Eq. A.1)}$$

The metric tensor then takes the following covariant and contravariant forms.

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(Eq. A.2)}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(Eq. A.3)}$$

We remark that raising and lowering indices switches the place of u and v.

Assume ϕ is a massless free scalar field in a 4 dimensional flat spacetime. The independent stress tensor components for a massless scalar field given by $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi$ in null coordinates are

$$T_{uu} =\partial_u\phi\partial_u\phi \quad \text{(Eq. A.4a)}$$

$$T_{vv} =\partial_v\phi\partial_v\phi \quad \text{(Eq. A.4b)}$$

$$T_{uv} =\frac{1}{4}((\partial_x\phi)^2 + (\partial_y\phi)^2) \quad \text{(Eq. A.4c)}$$

$$T_{ux} =\partial_u\phi\partial_x\phi \quad \text{(Eq. A.4d)}$$

$$T_{uy} =\partial_u\phi\partial_y\phi \quad \text{(Eq. A.4e)}$$

$$T_{vx} =\partial_v\phi\partial_x\phi \quad \text{(Eq. A.4f)}$$

$$T_{vy} =\partial_v\phi\partial_y\phi \quad \text{(Eq. A.4g)}$$

$$T_{xx} =\partial_x\phi\partial_x\phi - \frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2 - 4\partial_u\phi\partial_v\phi) \quad \text{(Eq. A.4h)}$$

$$T_{yy} =\partial_y\phi\partial_y\phi - \frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2 - 4\partial\phi_u\partial_v\phi) \quad \text{(Eq. A.4i)}$$

$$T_{xy} =\partial_x\phi\partial_y\phi \quad \text{(Eq. A.4j)}$$

which can be confirmed by simple computation. In 2 dimensions [24] found that $T_{uv} = 0$ and thus the divergencelessness condition boiled down to confirming that $\partial_u T^{uu}$ and $\partial_v T^{vv}$ vanished demonstrating that they were independent, T_{uu} and T_{vv} could be interpreted as rightward and leftward flux. In 4 dimensions we note that this does not hold since T_{uv} does not vanish, thus indicating that the transverse coordinates relate the variation of $T_{\mu\nu}$ between the rightward and leftward rays. In appendix B, we compute explicitly the divergenceless condition for the uniformly accelerating source in 4 dimensions.

APPENDIX B: DIVERGENCE OF $T_{\mu\nu}$

The conservation law requires the divergence of $T_{\mu\nu}$ to vanish everywhere except at the worldline of the source. The stress tensor divergence components will be a sum of theta function and delta function terms. The terms in the divergence involving $\theta(v)$ must vanish because the field, $\phi(x) = \frac{q}{4a\pi R}\theta(v)$, is a solution of the wave equation. Algebraically, the reason for this vanishing is the following identity.

$$(\partial^\mu \partial_\mu R)R - 2\partial^\mu R \partial_\mu R = 0 \qquad \text{(Eq. B.1)}$$

The question remains whether the divergence vanishes on the horizon $v = 0$ as there are terms of $\delta(v)$. We can consider each component of the divergence.

Calculation: $T_{u\mu}$ Simplified

$$T_{uu} = \frac{q^2}{16\pi^2 R^4}(\partial_u R)^2 \theta(v) \qquad \text{(Eq. B.2a)}$$

$$T_{uv} = \frac{q^2((\partial_x R)^2 + (\partial_y R)^2)\theta(v)}{64\pi^2 R^4} \qquad \text{(Eq. B.2b)}$$

$$T_{ux} = \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\theta(v) \qquad \text{(Eq. B.2c)}$$

$$T_{uy} = \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\theta(v) \qquad \text{(Eq. B.2d)}$$

Indeed $\partial^\mu T_{u\mu}$ vanishes as the theta function terms add to zero since ϕ satisfies the wave equation, and we are left with $\frac{q^2}{16\pi^2 R^4}(\partial_u R)^2 \delta(v)$ which is zero since $\partial_u R = 0$ when $v = 0$.

Calculation: $T_{v\mu}$ Simplified

$$T_{vv} = \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)^2\theta(v)}{R^4} - \frac{2(\partial_v R)\delta(v)\theta(v)}{R^3} + \frac{1}{R^2}\delta(v)\delta(v)\right] \quad \text{(Eq. B.3a)}$$

$$T_{vu} = \frac{q^2((\partial_x R)^2 + (\partial_y R)^2)\theta(v)}{64\pi^2 R^4} = \frac{q^2}{16\pi^2}\left[\frac{((\partial_x R)^2 + (\partial_y R)^2)}{4R^4}\right]\theta(v) \quad \text{(Eq. B.3b)}$$

$$T_{vx} = \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_x R)}{R^4}\theta(v) - \frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right] \quad \text{(Eq. B.3c)}$$

$$T_{vy} = \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_y R)}{R^4}\theta(v) - \frac{(\partial_y R)}{R^3}\delta(v)\theta(v)\right] \quad \text{(Eq. B.3d)}$$

Theta function terms will sum to 0 by wave equation. The term that is a product of delta functions will vanish since there is a factor of $\partial_u R$ that vanishes. We are left with terms which are delta functions or products of theta and delta functions which since $\delta(v)\theta(v) = \frac{1}{2}\delta(v)$ means we can consider a grand sum of delta functions which are

$$\partial_u\left[2\frac{(\partial_v R)}{R^3}\right]\delta(v) - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4}\right]\delta(v) + \partial_x\left[\frac{-(\partial_x R)}{2R^3}\right]\delta(v) + \partial_y\left[\frac{-(\partial_y R)}{2R^3}\right]\delta(v) \quad \text{(Eq. B.4)}$$

differentiating the terms and pulling out the delta function

$$\sum \ldots\delta(v) = \Bigg(-2\left[\frac{-(\partial_{uv}R)R + (\partial_v R)(\partial_u R)}{R^4}\right] - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4}\right] \quad \text{(Eq. B.5a)}$$

$$- \frac{1}{2}\left[\frac{(\partial_{xx}R)R - 3(\partial_x R)^2}{R^4}\right] - \frac{1}{2}\left[\frac{(\partial_{yy}R)R - 3(\partial_y R)^2}{R^4}\right]\Bigg)\delta(v)$$

$$= -\frac{1}{2}\Bigg(4\left[\frac{-(\partial_{uv}R)R + (\partial_v R)(\partial_u R)}{R^4}\right] + \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{R^4}\right] \quad \text{(Eq. B.5b)}$$

$$+ \left[\frac{(\partial_{xx}R)R - 3(\partial_x R)^2}{R^4}\right] + \left[\frac{(\partial_{yy}R)R - 3(\partial_y R)^2}{R^4}\right]\Bigg)\delta(v)$$

Evaluating this at $v = 0$, we know that $\partial_u R$ is zero, so

$$\sum ...\delta(v) = \frac{-1}{2R^4}\left((-4(\partial_{uv}R) + (\partial_{xx}R) + (\partial_{yy}R))R - 2((\partial_x R)^2 + (\partial_y R)^2\right)\delta(v) \quad \text{(Eq. B.6)}$$

Since we are able to add in terms that are equal to zero,

$$\sum ...\delta(v) = \frac{-1}{2R^4}\left((\partial^\mu \partial_\mu R)R - 2(-4(\partial_u R)(\partial_v R) + (\partial_x R)^2 + (\partial_y R)^2\right)\delta(v) \quad \text{(Eq. B.7a)}$$

$$\frac{-1}{2R^4}\left((\partial^\mu \partial_\mu R)R - 2(\partial^\mu R \partial_\mu R\right)\delta(v) \quad \text{(Eq. B.7b)}$$

It follows that this vanishes since Eq. B.1 is zero.

Calculation: $T_{x\mu}$ Simplified

$$T_{xu} = \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\theta(v) \quad \text{(Eq. B.8a)}$$

$$T_{xv} = \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_x R)}{R^4}\theta(v) - \frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right] \quad \text{(Eq. B.8b)}$$

$$T_{xx} = \frac{q^2}{16\pi^2}\left(\left[\frac{(\partial_x R)^2 - (\partial_y R)^2 + 4(\partial_v R)(\partial_u R)}{2R^4}\right]\theta(v) - \frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right) \quad \text{(Eq. B.8c)}$$

$$T_{xy} = \frac{q^2}{16\pi^2 R^4}(\partial_x R)(\partial_y R)\theta(v) \quad \text{(Eq. B.8d)}$$

$(-2\partial_v T_{xu})$ yields a delta function term $\frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\delta(v)$ which at $v = 0$ is zero since $\partial_u R$ vanishes. T_{xy} will not give rise to any delta function terms. The only delta function terms will be

$$\sum ...\delta(v) = 2\partial_u\left(\frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right) - \partial_x\left(\frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right) \quad \text{(Eq. B.9a)}$$

$$= 2\left[\frac{(\partial_{ux}R - \partial_{xu}R)}{R^3} - 3\frac{(\partial_u R \partial_x R - \partial_x R \partial_u R)}{R^4}\right]\theta(v)\delta(v) \quad \text{(Eq. B.9b)}$$

$$= 0$$

Indeed vanishes.

Calculation: $T_{y\mu}$ Simplified

$$T_{yu} = \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\theta(v) \tag{Eq. B.10a}$$

$$T_{yv} = \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_y R)}{R^4}\theta(v) - \frac{(\partial_y R)}{R^3}\delta(v)\theta(v)\right] \tag{Eq. B.10b}$$

$$T_{yx} = \frac{q^2}{16\pi^2 R^4}(\partial_x R)(\partial_y R)\theta(v) \tag{Eq. B.10c}$$

$$T_{yy} = \frac{q^2}{16\pi^2}\left(\left[\frac{(\partial_y R)^2 - (\partial_x R)^2 + 4(\partial_v R)(\partial_u R)}{2R^4}\right]\theta(v) - \frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right) \tag{Eq. B.10d}$$

The argument is identical to the vanishing for $\partial^x T_{\mu x}$, so indeed $\partial^y T_{\mu y}$ vanishes.