ORBIT EQUIVALENCE INVARIANTS AND EXAMPLES

A Dissertation

by

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ABSTRACT

Orbit equivalence is an equivalence relation on measurable actions of groups that's been studied since the 1950's. It has connections to many areas of mathematics including descriptive set theory, percolation theory, ergodic theory, representation theory, von neumann algebras, and geometric group theory. In joint work with Robin Tucker-Drob, we show inner amenable groupoids have fixed price 1. This simultaneously generalizes and unifies two well known results on cost from the literature, namely, (1) a theorem of Kechris stating that every ergodic p.m.p. equivalence relation admitting a nontrivial asymptotically central sequence in its full group has cost 1, and (2) a theorem of Tucker-Drob stating that inner amenable groups have fixed price 1. We later study coamenable inclusions of inner amenable groupoids to generalize a result from the setting of groups. We also prove several equivalent conditions to amenability of an action of a groupoid. In additional joint work with Robin Tucker-Drob, we study wreath products up to orbit equivalence and show that $C_2 \wr \mathbb{F}_2$ is orbit equivalent to $C_n \wr \mathbb{F}_2$. In order to accomplish this, we introduce and study the notion of cofinitely equivariant maps. We also prove some examples of rigidity in this setting.

DEDICATION

To my father Wiesław Wróbel and my friend Mehrzad Monzavi

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1. INTRODUCTION AND BACKGROUND

1.1 Groups

Unless otherwise mentioned, Γ is always assumed to be a discrete countable group.

1.1.1 Amenable groups

Amenable groups were first introduced by von Neumann[38] in relation to the Banach-Tarski paradox. Since then, they've naturally turned up in many areas of mathematics and analysis.

Definition 1.1.1. A group Γ is said to be **amenable** if there exists a finitely additive probability measure(mean) $m : \mathcal{P}(\Gamma) \to [0, 1]$ such that $m(gA) = m(A)$ for every $g \in \Gamma$ and $A \subseteq \Gamma$. We call such a measure an **invariant mean**.

There are many equivalent characterizations of amenability, and I will list several in the next section.

Example 1.1.2. Some examples of amenable groups include

- finite groups
- abelian groups
- solvable groups
- groups of subexponential growth.

Amenability is closed under passing to subgroups, quotients, direct limits, and group extensions as von Neumann showed in his original paper.

Example 1.1.3. The canonical example of a nonamenable group is the free group on 2 generators $\mathbb{F}_2 = \langle a, b \rangle$. This can be seen due to the fact that $\mathbb{F}_2$ admits a paradoxical decomposition $\mathbb{F}_2 = \{e\} \sqcup \bigsqcup_{i \in \{a,b,a^{-1},b^{-1}\}} T_i = T_a \sqcup aT_{a^{-1}} = T_b \sqcup bT_{b^{-1}}$ where

$$T_i = \{g \in \mathbb{F}_2 | g = iw \text{ where } iw \text{ is a reduced word}\}.$$

It is immediate to check that this decomposition prevents the existence of invariant mean.

1.1.2 Amenable actions of groups

Strictly speaking, von Neumann's paper on amenability actually introduced amenable actions of groups on discrete spaces.

Definition 1.1.4. An action $\Gamma \curvearrowright X$ on a discrete space is called **amenable** if there exists a finitely additive probability measure $m : \mathcal{P}(X) \to [0,1]$ such that $m(gA) = m(A)$ for every $g \in \Gamma$ and $A \subseteq X$. Again, we call m in this case an Γ-invariant mean.

Remark 1.1.5. Γ is amenable if and only if the action $\Gamma \curvearrowright \Gamma$ by left translation is amenable.

We now list several of the most well-known characterizations of amenability.

Proposition 1.1.6. *Let $\Gamma \curvearrowright X$. TFAE*

1. *X admits a Γ-invariant mean*

2. *there exists a norm-1 positive sequence $f_n \in \ell^1(X)$ such that for every $g \in \Gamma$*

$$\|f_n - g \cdot f_n\| \to 0$$

3. *there exists a sequence of finite subsets $F_n \subseteq X$ such that for every $g \in \Gamma$*

$$\frac{|F_n \triangle g F_n|}{|F_n|} \to 0$$

*A sequence satisfying condition 2 or 3 is known as a **Reiter sequence** or **Følner sequence**, respectively.*

The following is a folklore lemma whose origins go back to von Neumann's original paper. It has appeared in many iterations over the years, strengthened slightly each time. We present a relatively simple version.

Lemma 1.1.7. *Assume there are two actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ of a countable group Γ. Suppose the action $\Gamma \curvearrowright X$ is amenable. If the action of the stabilizer $\Gamma_x \curvearrowright Y$ is an amenable action for every $x \in X$, then $\Gamma \curvearrowright Y$ is amenable.*

Proof. Let μ be a Γ-invariant mean on X. Take $X_0 \subseteq X$ containing exactly one point in each Γ-orbit. Let m_x be a Γ_x-invariant mean on Y for every $x \in X_0$. Now, extend this assignment by $m_{g\cdot x} = g \cdot m_x$. This assignment is well-defined since m_x is Γ_x-invariant. We define a mean m on Y by $m(A) = \int_X m_x(A)d\mu(x)$ and for $g \in \Gamma$, we calculate

$$\begin{aligned} m(g \cdot A) &= \int_X m_x(g \cdot A)d\mu(x) \\ &= \int_X m_{g^{-1}\cdot x}(A)d\mu(x) = \int_X m_x(A)d\mu(g \cdot x) \\ &= \int_X m_x(A)d\mu(x) = m(A) \end{aligned}$$

to get that m is Γ-invariant. □

Proposition 1.1.8. *Every action $\Gamma \curvearrowright X$ of an amenable group is amenable.*

Proof. Push forward the measure from the group Γ to each orbit by choosing a transversal $X_0 \subseteq X$. □

We also take the opportunity to define here coamenability of subgroups.

Definition 1.1.9. An inclusion $\Lambda \leq \Gamma$ is called **coamenable** if the action on the set of left cosets $\Gamma \curvearrowright \Gamma/\Lambda$ by multiplying on the left is amenable.

1.1.3 Inner amenable groups

The story of inner amenability dates back to the introduction of property Gamma in the setting of von Neumann algebras[37]. Property Gamma was used by them to distinguish between the group von Neumann algebras of S_∞ and $\mathbb{F}_2$. Eventually, in association with property Gamma, Effros introduced the notion of inner amenability of a group and showed that if the group von

Neumann algebra $L(\Gamma)$ has property Gamma, then the group Γ is inner amenable[12]. Vaes was able to show very recently that the converse does not hold[48].

Definition 1.1.10. A group is said to be **inner amenable** if there exists a mean $m : \mathcal{P}(\Gamma) \to [0, 1]$ that satisfies

- (conjugation-invariant) $m(gAg^{-1}) = m(A)$ for every $g \in \Gamma$ and $A \subseteq \Gamma$
- (diffuse) $m(D) = 0$ if $D \subseteq \Gamma$ is finite.

Clearly diffuseness prohibits any finite group from being inner amenable.

Example 1.1.11. Every infinite amenable group Γ is inner amenable. In fact, such a group admits a mean that is both left- and right- invariant by considering the group $\Gamma \times \Gamma$ which is amenable. Therefore, the action $\Gamma \times \Gamma \curvearrowright \Gamma$ where the first coordinate acts by left translation and the second coordinate acts by right translation is amenable and admits an invariant mean.

Example 1.1.12. Other examples include

- groups with infinite center
- infinite direct sums
- groups whose von Neumann algebra has property Gamma
- products with an inner amenable group
- dynamical alternating groups of top. free actions of amenable groups on the Cantor set[31]

Proposition 1.1.13. *Let $\Lambda \leq \Gamma$ be a coamenable inclusion of groups. If Λ is inner amenable, then Γ is inner amenable.*

Proof. Take a mean m_Λ witnessing inner amenability of Λ as well as a mean $m_{\Gamma/\Lambda}$ witnessing coamenability of the inclusion. We treat m_Λ as defined on the entirety of Γ letting $m_\Lambda(\Gamma \setminus \Lambda) = 0$.

Define a different mean on Γ by $m(A) := \int_{\Gamma/\Lambda} m_\Lambda(g^{-1}Ag) m_{\Gamma/\Lambda}(g\Lambda)$. This is well-defined since it is not dependent on the choice of cosets by conjugation-invariance. Now, let $h \in \Gamma$ and check

$$m(h^{-1}Ah) = \int_{\Gamma/\Lambda} m_\Lambda(g^{-1}h^{-1}Ahg) m_{\Gamma/\Lambda}(g\Lambda) = \int_{\Gamma/\Lambda} m_\Lambda(g^{-1}h^{-1}Ahg) m_{\Gamma/\Lambda}(hg\Lambda) = m(A)$$

to get conjugation-invariance of m. Diffuseness of m follows immediately since we're integrating the diffuse mean m_Λ. $\square$

A detailed study of the structure of inner amenable groups can be found in [47].

1.2 Orbit equivalence

Orbit equivalence has been studied under several names since the 1950's. Some of the earlier results included work of Singer that described the connection between orbit equivalence of actions and isomorphism of the associated group measure space constructions. The setting when discussing orbit equivalence is that of a group Γ acting in a measure-preserving manner on a standard probability space.

Definition 1.2.1. We say two probability measure preserving(pmp) actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are **orbit equivalent** if there exists a measure isomorphism $f : X_0 \to Y_0$ between conull subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ that sends orbits to orbits, i.e. $f(\Gamma \cdot x) = \Lambda \cdot f(x)$ for almost every $x \in X_0$.

This is a much weaker equivalence relation on actions of countable groups than conjugacy. In particular, different groups may admit orbit equivalent actions. Various invariants of orbit equivalence have been found since it was introduced, including ergodicity, strong ergodicity, hyperfiniteness, ℓ^2-Betti numbers and cost among others.

Definition 1.2.2. We say that two groups Γ, Λ are **orbit equivalent** if they admit free measure preserving actions that are orbit equivalent.

Further information about orbit equivalence and all the topics in this section can be found in the following excellent books and surveys[17][28][29][8].

1.2.1 Antirigidity results

There are several well-understood orbit equivalence classes of groups. Firstly, it's elementary to see all finite groups of a given cardinality form an orbit equivalence class. However, Dye contributed the first major result in this direction.

Theorem 1.2.3 (Dye's Theorem[11])**.** *Every pair of ergodic free pmp actions of $\mathbb{Z}$ is orbit equivalent.*

Several years later, Ornstein and Weiss proved a major flexibility result.

Theorem 1.2.4 (Ornstein-Weiss[39])**.** *Let Γ be amenable. Every free ergodic pmp action of Γ is orbit equivalent to an action of $\mathbb{Z}$.*

In particular, all amenable groups are in the same orbit equivalence class. A little bit of work shows that amenability is an invariant of orbit equivalence and hence, amenable groups form their own class.

1.2.2 Cost

Cost is a $[0, \infty]$-valued invariant of probability measure preserving(pmp) orbit equivalence relations that was first introduced by Levitt[35] and significantly developed by Gaboriau[18][19][21]. We think of these equivalence relations as being measurable subsets of $(X, \mu) \times (X, \mu)$ where (X, μ) is a standard probability space. They now have two projection maps p_1, p_2 onto the first and second coordinates respectively. By work of Feldman-Moore, every countable pmp equivalence relation arises as the orbit equivalence relation of a pmp action of some countable group[15]. We call a subset $\mathcal{G}$ of $X \times X$ symmetric if $(x, y) \in \mathcal{G} \implies (y, x) \in \mathcal{G}$.

Definition 1.2.5. A symmetric subset $\mathcal{G} \subseteq \mathcal{R}$ of an equivalence relation $\mathcal{R}$ is called a **graphing of** $\mathcal{R}$ if the equivalence relation $E_{\mathcal{G}} = \{(x, y) |$ there exists a finite sequence $x_0 = x, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \mathcal{G}\}$ generated by $\mathcal{G}$ is equal to $\mathcal{R}$.

Definition 1.2.6. Let $\mathcal{G}$ be a graphing. The **cost of a graphing** is denoted and defined as $C_\mu(\mathcal{G}) := \frac{1}{2} \int_X |p_1^{-1}(x)| d\mu(x)$.

Definition 1.2.7. Let $\mathcal{R}$ be a countable pmp equivalence relation. The **cost of the equivalence relation** is $C_\mu(\mathcal{R}) := \inf\{C_\mu(\mathcal{G}) | \mathcal{G}$ is a graphing of $\mathcal{R}\}$.

This also defines cost of a pmp action as the cost of the associated orbit equivalence relation. It is clear to see that two actions that are orbit equivalent must have the same cost. It can be shown that free pmp actions of infinite groups must have cost at least 1.

Example 1.2.8. We give an example of calculation of cost. Let θ_1, θ_2 be mutually irrational. Consider the free action $\mathbb{Z}^2 \curvearrowright (S^1, \mu)$ acting on the circle with the Haar measure by $(n, m) \cdot x = n\theta_1 + m\theta_2 + x \mod 1$. Fix $\varepsilon > 0$ and take $X \subseteq S^1$ to be a set of measure less than ε. Now define $\mathcal{G}_\varepsilon := \{(y, x) | y = (1, 0) \cdot x$ or $(x \in X$ and $y = (0, 1) \cdot x)\}$. The set $\mathcal{G}_\varepsilon$ is a graphing of the orbit equivalence relation of the action, since the actions of $(0, 1)$ and $(1, 0)$ commute and the action of $(0, 1)$ is ergodic. It is easy to calculate that $C_\mu(\mathcal{G}_\varepsilon) < 1 + \varepsilon$. This along with the fact that free pmp actions of infinite groups must have cost ≥ 1 tells us that $C_\mu(\mathbb{Z}^2 \curvearrowright (S^1, \mu)) = 1$.

This particular example and Ornstein-Weiss tells us that every free pmp action of an infinite amenable group has cost 1. However, the first case where cost was calculated and distinguished groups up to orbit equivalence is the following.

Theorem 1.2.9 (Gaboriau[19])**.** *Let $\mathbb{F}_n$ be the free group on n generators. The cost of every free pmp action of $\mathbb{F}_n$ is n.*

In particular, this implies that $\mathbb{F}_2$ and $\mathbb{F}_3$ do not admit orbit equivalent actions, and hence, are not orbit equivalent which was previously unknown. Cost also has some relationship with previously studied quantities such as the first ℓ^2-Betti number $\beta_1^{(2)}$.

Theorem 1.2.10 (Gaboriau[20])**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a free pmp action. Then $\beta_1^{(2)}(\Gamma) + 1 \leq C_\mu(\Gamma \curvearrowright (X, \mu))$.*

It is an open question whether the inequality can be replaced by equality, although it is known in many cases.

1.2.3 Bernoulli superrigidity

We first define the notion of a cocycle.

Definition 1.2.11. Let $\Gamma \curvearrowright (X, \mu)$ be an action of a countable group and let Λ be a discrete countable group. A **cocycle** of the action into Λ is a function $\alpha : \Gamma \times X \to \Lambda$ that satisfies $\alpha(gh, h \cdot x)\alpha(h, x) = \alpha(gh, x)$.

In 2007, Popa proved his remarkable cocycle superrigidity theorem using the techniques of deformation rigidity.

Theorem 1.2.12 (Popa[44])**.** *Let Γ be discrete countable group with property (T). For every p.m.p. action $\Gamma \curvearrowright Y$ of Γ, the associated Bernoulli extension $Y \otimes A^\Gamma \to Y$ satisfies that every measurable cocycle $\omega : \Gamma \times Y \otimes A^\Gamma \to L$ taking values in a discrete countable group L is cohomologous to a cocycle which descends to Y.*

This implies several very useful corollaries.

Corollary 1.2.13 (Cocycle Superrigidity)**.** *Let $\Gamma \curvearrowright A^\Gamma$ be a Bernoulli shift of a property (T) group. Then for every cocycle $\omega : \Gamma \times A^\Gamma \to L$ taking values in a discrete countable group L, there exists a homomorphism $\rho : \Gamma \to L$ and measurable map $\theta : A^\Gamma \to L$ such that $c(\gamma, x) = \theta(\gamma x)\rho(\gamma)\theta(x)^{-1}$.*

Corollary 1.2.14 (OE Rigidity)**.** *Let $\alpha : \Gamma \curvearrowright A^\Gamma$ be a Bernoulli shift of a property (T) group that has no nontrivial finite normal subgroups. Then the orbit equivalence class of α consists of actions that are conjugate to α.*

Bowen and Tucker-Drob give the name **Bernoulli superrigid** to groups that satisfy the conclusion of Popa's theorem and showed that Bernoulli superrigidity is an invariant of measure equivalence of groups[5].

We give some simple examples of Bernoulli superrigid groups and more can be found in the literature.

Example 1.2.15 (Popa[44][45]). Γ is Bernoulli superrigid if there is an infinite normal subgroup $N \triangleleft \Gamma$ such that one of the following holds

- (Γ, N) has relative property (T)
- $N = H \times K$ where H is infinite and K is nonamenable.

2. COST OF INNER AMENABLE GROUPOIDS*

2.1 Introduction

Cost is an $[0, \infty]$-valued invariant of p.m.p. orbit equivalence relations that was first introduced by Levitt[35] and significantly developed by Gaboriau[18][19][21]. By work of Connes-Feldman-Weiss[7], every aperiodic amenable equivalence relation has cost 1. Kechris showed in ([28], Theorem 8.1) that the existence of a nontrival asymptotically central sequence in the full group of an ergodic p.m.p. equivalence relation implies the equivalence relation has cost 1. More exposition on the many results of cost theory can be found in the surveys of Kechris and Miller[29] and Furman[17].

A discrete p.m.p. groupoid $\mathcal{G}$ is said to have fixed price if every principal extension of $\mathcal{G}$ has the same cost. Gaboriau proved that free groups have fixed price and asked whether every group has fixed price. Since then, fixed price has been shown for several large classes of groups, including finite, infinite amenable[39], strongly treeable[19], and inner amenable groups[47] amongst many others. Recently, Hutchcroft and Pete showed that Kazhdan groups have a principal extension with cost 1[25], but it is an open question whether these groups have fixed price.

Inner amenable groups were first introduced by Effros[12] in relation with property Gamma of a von Neumann algebra. Examples of inner amenable groups include infinite amenable groups, groups with infinite center, and groups admitting a ergodic p.m.p. action which is stable in the sense of Jones and Schmidt [26]. Kerr and Tucker-Drob have shown that dynamical alternating groups associated to topologically free actions of amenable groups on the Cantor set are inner amenable, and they use this to exhibit uncountably many pairwise nonisomorphic, finitely generated simple nonamenable inner amenable groups[31]. Recently, Kida and Tucker-Drob defined inner amenability for discrete p.m.p. groupoids[32] and showed that the action groupoid associated to a compact p.m.p. action of an inner amenable group is inner amenable. They also note that not

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every free p.m.p. action of an inner amenable group gives an inner amenable action groupoid. In particular, the action groupoid associated to the Bernoulli shift of any nonamenable group is not inner amenable[32, Corollary 6.3].

Theorem 2.1.1. *Assume $\varphi : R \to \mathcal{G}$ is a principal groupoid extension of an inner amenable groupoid $\mathcal{G}$. Then $C_{\mu_R^0}(R) = 1$.*

Specializing this to the case of countable groups recovers Tucker-Drob's result that inner amenable groups have fixed price 1, and specializing to the case of equivalence relations recovers Kechris's theorem that equivalence relations with a nontrivial asymptotically central sequence in their full group have cost 1 (since these equivalence relations are shown to be inner amenable in [32]).

In order to prove Theorem 2.1.1, we make heavy use of a generalization of von Neumann's notion of amenable actions, from the setting of groups to the setting of groupoids. We obtain the following key structural result along the way, generalizing a result from [47] which only applied to groups.

Theorem 2.1.2. *If $\mathcal{G}$ is an inner amenable groupoid, and $\mathcal{H} \leq \mathcal{G}$ is a nowhere amenable subgroupoid, then there is a groupoid $\mathcal{K}$ such that $\mathcal{H}$ is q-normal in $\mathcal{K}$ and $\mathcal{K}$ is q-normal in $\mathcal{G}$.*

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2.2 Preliminaries

2.2.1 Groupoids

Definition 2.2.1. A **groupoid** $\mathcal{G}$ is a small category in which every morphism is an isomorphism. We refer to the set of objects as the unit space, written as $\mathcal{G}^0$. There are source $s_\mathcal{G} : \mathcal{G} \to \mathcal{G}^0$ and range $r_\mathcal{G} : \mathcal{G} \to \mathcal{G}^0$ maps that send an element of the groupoid to its source and range, respectively, and an inclusion map $i_\mathcal{G} : \mathcal{G}^0 \to \mathcal{G}$, that sends a unit to the identity morphism at that unit. When there is no confusion, we will drop the subscripts on s, r, and i, and we will identify $\mathcal{G}^0$ with its image in $\mathcal{G}$ under i.

We say a groupoid $\mathcal{G}$ is **principal** if the map $g \mapsto (r(g), s(g))$ is injective.

An equivalence relation R on a set X is naturally a principal groupoid, with unit space X, source and range maps the right and left projections respectively, and composition given by the rule $(z, y)(y, x) = (z, x)$. Moreover, each principal groupoid is naturally isomorphic as a groupoid to an equivalence relation via the map $g \mapsto (r(g), s(g))$. Given this transparent equivalence of categories, we will freely and frequently identify principal groupoids with their associated equivalence relation.

Definition 2.2.2. A **discrete Borel groupoid** is a groupoid where both $\mathcal{G}$ and $\mathcal{G}^0$ are standard Borel spaces, the source, range, and inclusion maps s, r, and i are all Borel, s and r are countable-to-one, and the multiplication and inverse maps are Borel.

Definition 2.2.3. A **discrete p.m.p. groupoid** is a pair $(\mathcal{G}, \mu^0_{\mathcal{G}})$ where $\mathcal{G}$ is a discrete Borel groupoid and $\mu^0_{\mathcal{G}}$ is a Borel probability measure on $\mathcal{G}^0$ satisfying $\int_{\mathcal{G}^0} c^r_x d\mu^0_{\mathcal{G}} = \int_{\mathcal{G}^0} c^s_x d\mu^0_{\mathcal{G}}$ where c^r_x and c^s_x refer to the counting measure on $r^{-1}(x)$ and $s^{-1}(x)$ respectively. Set $\mu^1_{\mathcal{G}} := \int_{\mathcal{G}^0} c^r_x d\mu^0_{\mathcal{G}} = \int_{\mathcal{G}^0} c^s_x d\mu^0_{\mathcal{G}}$ to be this σ-finite measure on $\mathcal{G}$.

Again, we will drop the subscript on the measures when there is no cause for confusion.

Definition 2.2.4. A subset $A \subseteq \mathcal{G}^0$ is said to be $\mathcal{G}$-**invariant** if $\mu^0(\mathcal{G} \cdot A \triangle A) = 0$ where $\mathcal{G} \cdot A = \{x \in \mathcal{G}^0 \mid \exists g \in \mathcal{G} \text{ with } s(g) \in A \text{ and } r(g) = x\}$.

Definition 2.2.5. A discrete p.m.p. groupoid $\mathcal{G}$ is called **ergodic** if every $\mathcal{G}$-invariant subset $A \subseteq \mathcal{G}^0$ is μ^0-null or conull.

Example 2.2.6. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group on a standard probability space (X, μ). We define the discrete p.m.p. groupoid $\mathcal{G} = G \ltimes X$ with underlying set $G \times X$ and unit space $\mathcal{G}^0 = X$, with the groupoid operation $(g, h \cdot x)(h, x) := (gh, x)$. A groupoid that arises through such a process is called an **action groupoid**. If this action is ergodic, so is the groupoid it generates.

Definition 2.2.7. An **extension** of a discrete p.m.p. groupoid $\mathcal{G}$ is a discrete p.m.p. groupoid $\mathcal{H}$ together with a measure preserving groupoid homomorphism $\phi : (\mathcal{H}, \mu^1_{\mathcal{H}}) \to (\mathcal{G}, \mu^1_{\mathcal{G}})$. We call this a **principal extension** of $\mathcal{G}$ if the groupoid $\mathcal{H}$ is principal.

As explained in [5], the category of extensions of $\mathcal{G}$ is equivalent to the category of p.m.p. actions of $\mathcal{G}$.

Example 2.2.8. Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action. Then the map $\phi : G \ltimes (X, \mu) \to G$ defined by $(g, x) \mapsto g$ is a principal groupoid extension.

More detail about groupoid extensions can be found in [5] and [32].

Definition 2.2.9. A **measurable bisection** of a discrete p.m.p. groupoid $\mathcal{G}$ is a Borel subset σ of $\mathcal{G}$ such that the restrictions $r|_\sigma$ and $s|_\sigma$ are each bijections of σ with a conull subset of $\mathcal{G}^0$. A subset σ of $\mathcal{G}$ is called a **partial measurable bisection** if $r|_\sigma, s|_\sigma$ are only assumed to be injections.

Definition 2.2.10. The **full group** of a discrete p.m.p. groupoid $\mathcal{G}$ is the set, denoted by $[\mathcal{G}]$, of all measurable bisections. The **pseudogroup** of $\mathcal{G}$ is the set, denoted by $[[\mathcal{G}]]$, of all partial bisections. We identify two partial bisections σ_1 and σ_2 if their symmetric difference $\sigma_1 \triangle \sigma_2$ is μ^1-null. The full group admits a complete separable metric, namely $d(\sigma_1, \sigma_2) := \mu^1(\sigma_1 \triangle \sigma_2)$.

For subsets $A, B \subseteq \mathcal{G}$ define $A^{-1} := \{g^{-1} | g \in A\}$ and $AB := \{gh | g \in A,\ h \in B,$ and $s(g) = r(h)\}$. The full group and the full pseudogroup of $\mathcal{G}$ are then a group and inverse semigroup respectively, under these operations. For $g \in \mathcal{G}$ and $A \subset \mathcal{G}$ we also define $gA := \{g\}A$.

For a groupoid $\mathcal{G}$, and subset $A \subseteq \mathcal{G}^0$, we let $\mathcal{G}_A := \{g \in \mathcal{G} \mid r(g), s(g) \in A\}$. Fix a partial measurable bisection $\sigma \in [[\mathcal{G}]]$. For $g \in \mathcal{G}_{r(\sigma)}$, define the conjugate of g by σ, denoted g^σ, to be the unique element of $\sigma^{-1} g \sigma$. Likewise, for $D \subseteq \mathcal{G}$, define $D^\sigma = \{g^\sigma \mid g \in D\}$. For a function $f : \mathcal{G} \to \mathbb{C}$, define $f^\sigma : \mathcal{G} \to \mathbb{C}$ by

$$f^\sigma(g) = \begin{cases} f(g^{\sigma^{-1}}) & \text{if } g \in \mathcal{G}_{s(\sigma)} \\ 0 & \text{otherwise} \end{cases}$$

2.2.2 Actions of groupoids and amenability

Definition 2.2.11. A **locally countable fibered space** over a standard measure space (X, μ) consists of a standard Borel space W along with a countable-to-one Borel map $p : W \to X$. For $A \subseteq X$ we define $W^A := p^{-1}(A)$ and set $W^x := W^{\{x\}}$. We also define $\nu(A) := \int_X |W^x \cap A| d\mu$.

If $\mathcal{G}$ is a discrete p.m.p. groupoid and W is a locally countable fibered space over $\mathcal{G}^0$, then we define $\mathcal{G} * W := \{(g, w) \in \mathcal{G} \times W \mid s(g) = p(w)\}$.

Note that a discrete p.m.p. groupoid, together with either its source or range map, is a locally countable fibered space over $\mathcal{G}^0$, with $\nu_r = \nu_s = \mu^1$.

Definition 2.2.12. A **(left) Borel action of a p.m.p. groupoid** $\mathcal{G}$ **on a locally countable fibered space** $p : W \to \mathcal{G}^0$ is a Borel map $\alpha : \mathcal{G} * W \to W$, such that

(1) $\alpha(g, w) \in W^{r(g)}$ for each $g \in \mathcal{G}$ and $w \in W^{s(g)}$,

(2) for each $g \in \mathcal{G}$ the map $\alpha_g : w \mapsto \alpha(g, w)$, is a bijection from $W^{s(g)}$ to $W^{r(g)}$, and

(3) $\alpha_g \alpha_h = \alpha_{gh}$ whenever $s(g) = r(h)$

where we denote $\mathcal{G} * W = \{(g, w) | s(g) = p(w)\}$.

We will also simply write gw for $\alpha(g, w)$. For subsets $A \subseteq \mathcal{G}$ and $V \subseteq W$, denote $AV := \{gw \mid g \in A \text{ and } w \in V\}$. For $g \in \mathcal{G}$ and $V \subseteq W$, we define $gV := \{g\}V$.

Example 2.2.13. The **left translation action** λ, of a groupoid $\mathcal{G}$ on itself is defined, for $g, h \in \mathcal{G} * \mathcal{G} = \{(g, h) | s(g) = r(h)\}$, by $\lambda(g, h) := gh$.

Definition 2.2.14. A **measurable section** of a locally countable fibered space $p : W \to (X, \mu)$ is a Borel subset σ of W such that the restriction $p|_\sigma$, of p to σ, is a bijection of σ with a conull subset of X. A subset σ of W is called a **partial measurable section** if $p|_\sigma$ is only assumed to be injective.

Suppose we have an action of a discrete p.m.p. groupoid $\mathcal{G}$ on a locally countable fibered space $p : W \to \mathcal{G}^0$. Let $g \in \mathcal{G}$ and let $\sigma \subseteq W$ be a partial measurable section. We say that g fixes σ if $\varnothing \neq g\sigma \subseteq \sigma$. Notice that, in this case, the set $g\sigma$ contains a single point, so we will abuse notation and use $g\sigma$ to also denote this point.

Definition 2.2.15. The **stabilizer** of a partial section σ is defined to be the set

$$\mathcal{G}_\sigma := \{g \in \mathcal{G} \mid g \text{ fixes } \sigma\} = \{g \mid \varnothing \neq g\sigma \subseteq \sigma\}.$$

Definition 2.2.16. Let $\mathcal{G} \curvearrowright W$ be a Borel action of a discrete p.m.p. groupoid $\mathcal{G}$ on a locally countable fibered space W over $\mathcal{G}^0$. The action is called **amenable** if there exists a sequence of Borel functions $(f_n)_{n\in\mathbb{N}} : W \to [0,1]$ such that

1. $\sum_{w\in W^x} f_n(w) = 1$ for μ^0-almost every $x \in \mathcal{G}^0$ and all $n \in \mathbb{N}$
2. $\sum_{w\in W^{r(g)}} |f_n(w) - f_n(g^{-1}w)| \to 0$ as $n \to \infty$ for μ^1-almost every $g \in \mathcal{G}$.

Notice that this generalizes von Neumann's notion of amenable action when $\mathcal{G}$ is actually a group.

Remark 2.2.17. If a sequence f_n is $\mathcal{G}$-asymptotically invariant as in item (2), then, by the bounded convergence theorem, it is asymptotically invariant under the full group, i.e. $\|f_n - \sigma f_n\|_1 \to 0$ for $\sigma \in [\mathcal{G}]$. Conversely, if a sequence is asymptotically invariant under the full group, then a subsequence is $\mathcal{G}$-asymptotically invariant as in item (2).

Definition 2.2.18. A **mean** on (W,ν) is a norm one positive linear functional on $L^\infty(W,\nu)$.

We'll often treat means as finitely additive probability measures by letting $m(A) := m(1_A)$ for $A \subseteq W$ a ν-measurable subset.

Definition 2.2.19. Let $m \in (L^\infty(W,\nu))^*$ be a mean on a fibered space $p : W \to \mathcal{G}^0$. We say that m is **equidistributed** if for every measurable set $A \subseteq \mathcal{G}^0$,

$$m(W^A) = \mu^0(A).$$

Proposition 2.2.20. *Let $\mathcal{G}$ be a discrete p.m.p. groupoid that acts on the fibered space $p : W \to \mathcal{G}^0$. Assume the action admits an equidistributed mean $m \in (L^\infty(W,\nu))^*$ such that $m(\sigma A) = m(A)$ for every measurable set A and $\sigma \in [\mathcal{G}]$. Then the action $\mathcal{G} \curvearrowright W$ is amenable.*

Proof. Associate to every non-negative unit vector $f \in L^1(W,\nu)$ a function $p_f \in L^1(\mathcal{G}^0,\mu^0)$ defined by

$$p_f(x) := \sum_{w\in W^x} f(w).$$

Since m is equidistributed, given a net (f_i) converging to m, the net (p_{f_i}) weak-* converges to the function 1.

The proof of the following claim follows the proof given in [32, Claim 3.19].

Claim 2.2.21. *Let $f \in L^1(W, \nu)$ be a non-negative function with $\|f\|_1 = 1$. Then there exists a non-negative $g \in L^1(W, \nu)$ which satisfies $p_g = 1_{\mathcal{G}^0}$ and $\|f - g\|_1 = \|p_f - 1_{\mathcal{G}^0}\|_1$.*

Proof of Claim. Let $f_0 := f$. We proceed by transfinite induction on countable ordinals α to define a non-negative function $f_\alpha \in L^1(W, \nu)$ satisfying, for all $\beta < \alpha$:

1. $\|f_\beta - f_\alpha\|_1 = \|p_{f_\alpha} - p_{f_\beta}\|_1$
2. For almost every $x \in \mathcal{G}^0$, if $p_{f_\beta}(x) \leq 1$, then $p_{f_\beta}(x) \leq p_{f_\alpha}(x) \leq 1$
3. For almost every $x \in \mathcal{G}^0$, if $p_{f_\beta}(x) \geq 1$, then $p_{f_\beta}(x) \geq p_{f_\alpha}(x) \geq 1$
4. If $\|p_{f_\beta} - 1_{\mathcal{G}^0}\|_1 > 0$, then $\|p_{f_\alpha} - 1_{\mathcal{G}^0}\|_1 < \|p_{f_\beta} - 1_{\mathcal{G}^0}\|_1$, and if $p_{f_\beta} = 1_{\mathcal{G}^0}$, then $f_\alpha = f_\beta$.

If α is a limit ordinal, take an increasing sequence $\beta_1 < \beta_2 < \ldots$ such that $\alpha = \sup_i \beta_i$. The sequence $(p_{f_{\beta_i}})_i \subseteq L^1(\mathcal{G}^0, \mu^0)$ is Cauchy by properties (2) and (3). By property (1), the sequence $(f_{\beta_i})_i \subseteq L^1(W, \nu)$ is Cauchy. Let f_α be the limit point of this sequence.

If α is a successor ordinal and $p_{f_{\alpha-1}} = 1_{\mathcal{G}^0}$, then let $f_\alpha := f_{\alpha-1}$.

If α is a successor ordinal and $p_{f_{\alpha-1}} \neq 1_{\mathcal{G}^0}$, then there exists $\varepsilon > 0$ such that the sets $A_0 := \{x | p_{f_{\alpha-1}}(x) < 1 - \varepsilon\}$ and $A_1 := \{x | p_{f_{\alpha-1}}(x) > 1 + \varepsilon\}$ both have positive measure. For $i \in \{0, 1\}$, find partial measurable sections $C_i \subseteq W^{A_i}$ with $\nu(C_0) = \nu(C_1) > 0$ and $\varepsilon' := \inf\{f_{\alpha-1}|_{C_1}\} > 0$. By letting $\delta := \min\{\varepsilon, \varepsilon'\}$ and $f_\alpha := f_{\alpha-1} + \delta(1_{C_0} - 1_{C_1})$, we get a function with the required properties.

By property (4), there exists a countable ordinal κ such that $p_{f_\kappa} = 1_{\mathcal{G}^0}$, and setting $g := f_\kappa$ finishes the proof of the claim. □

Now, for a fixed finite collection $\Delta \subseteq [\mathcal{G}]$ and $\varepsilon > 0$, we consider the convex set $\{(f - \delta f)_{\delta \in \Delta} \times (p_f - 1_{\mathcal{G}^0}) \mid \|f\|_1 = 1, f \geq 0\} \subset L^1(W, \nu)^\Delta \times L^1(\mathcal{G}^0, \mu^0)$. By the Hahn-Banach theorem, this set contains 0 in its weak closure. So, by Mazur's Theorem, 0 is in the norm closure of our set.

By the claim, for every finite collection of sections $\Delta \subseteq [\mathcal{G}]$ and $\varepsilon > 0$, there exists a positive norm one function f such that $p_f = 1_{\mathcal{G}^0}$ and $\max_{\delta\in\Delta} \|f - \delta f\|_1 \leq \varepsilon$. Fix a countable dense subset $(\delta_k) \subseteq [\mathcal{G}]$. Take f_n as above satisfying $\max_{1\leq k\leq n} \|f_n - \delta_k f_n\|_1 \leq \frac{1}{n}$ and $p_{f_n} = 1_{\mathcal{G}^0}$. The functions f_n satisfy item (1) in the definition of amenability because $p_{f_n} = 1_{\mathcal{G}^0}$.

Let $\delta \in [\mathcal{G}]$ and $\varepsilon > 0$. There exists I such that $\mu^1(\delta\triangle\delta_I) < \frac{\varepsilon}{4}$ since (δ_k) is dense. Let $N > \max(\frac{2}{\varepsilon}, I)$. For $n > N$,

$$\|f_n - \delta f_n\|_1 \leq \|f_n - \delta_I f_n\|_1 + 2\mu^1(\delta\triangle\delta_I) < \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} < \varepsilon.$$

By remark 2.2.17, a subsequence of (f_n) satisfies item (2) in the definition of amenability and hence the action $\mathcal{G} \curvearrowright W$ is amenable. □

Definition 2.2.22. A discrete p.m.p. groupoid $\mathcal{G}$ is called **amenable** if the left translation action of $\mathcal{G}$ on itself is amenable.

In the case when $\mathcal{G}$ is an equivalence relation, this corresponds with the notion of amenability in the category of equivalence relations[29]. A study of amenable groupoids can be found in [1].

Definition 2.2.23. A groupoid $\mathcal{G}$ is called **nowhere amenable** if for every positive measure subset $A \subseteq \mathcal{G}^0$, the groupoid $\mathcal{G}_A = \{g \in \mathcal{G} \mid s(g), r(g) \in A\}$ is nonamenable.

In [32], Kida and Tucker-Drob introduced the following generalization of inner amenable groups.

Definition 2.2.24. A discrete p.m.p. groupoid $\mathcal{G}$ is called **inner amenable** if there exists a mean $m \in (L^\infty(\mathcal{G}, \mu^1))^*$ such that

(i) $m(\mathcal{G}_A) = \mu^0(A)$ for every μ^0-measurable $A \subseteq \mathcal{G}^0$

(ii) $m(A^\sigma) = m(A)$ for every μ^0-measurable $A \subseteq \mathcal{G}^0$ for every $\sigma \in [\mathcal{G}]$

(iii) $m(D) = 0$ for every μ^1-measurable $D \subseteq \mathcal{G}$ with $\mu^1(D) < \infty$

(iv) $m(A) = m(A^{-1})$ for every μ^0-measurable $A \subseteq \mathcal{G}^0$

2.2.3 Q-normality and cost

Definition 2.2.25. Fix a discrete p.m.p. groupoid $\mathcal{G}$. A subset $A \subseteq \mathcal{G}$ is said to **generate** $\mathcal{G}$ if the union $\langle A \rangle := \bigcup_{n \in \mathbb{N}} (A \cup A^{-1})^n$ is a μ^1-conull subset of $\mathcal{G}$.

Definition 2.2.26. A subset of a discrete p.m.p. groupoid $A \subseteq \mathcal{G}$ is called **aperiodic** if for almost every $x \in \mathcal{G}^0$, the set $s^{-1}(x) \cap A$ is infinite.

Sorin Popa introduced the notion of q-normality in [43].

Definition 2.2.27. A subgroupoid $\mathcal{H} \leq \mathcal{G}$ is **q-normal** in $\mathcal{G}$ if there exists a countable collection of partial sections $\Sigma \subset [[\mathcal{G}]]$ generating $\mathcal{G}$ such that for every $\sigma \in \Sigma$, the set $\mathcal{H}^\sigma \cap \mathcal{H}$ is aperiodic on $s(\sigma)$.

Proposition 2.2.28. *If the groupoid $\mathcal{H}$ is q-normal in $\mathcal{G}$ and $\varphi : \mathcal{K} \to \mathcal{G}$ is a groupoid extension of $\mathcal{G}$, then $\varphi^{-1}(\mathcal{H})$ is q-normal in $\mathcal{K}$.*

Proof. Let Σ be a countable collection of partial sections of $\mathcal{G}$ witnessing the q-normality of $\mathcal{H}$ in $\mathcal{G}$. Let $\Sigma' := \{\varphi^{-1}(\sigma) \mid \sigma \in \Sigma\}$. Notice Σ' is a collection of sections of $\mathcal{K}$ that generate $\mathcal{K}$. Now $\varphi^{-1}(\sigma)\varphi^{-1}(\mathcal{H})\varphi^{-1}(\sigma)^{-1} \cap \varphi^{-1}(\mathcal{H}) \supseteq \varphi^{-1}(\sigma\mathcal{H}\sigma^{-1} \cap \mathcal{H})$. This along with the fact that groupoid extensions are surjective lets us check q-normality of $\varphi^{-1}(\mathcal{H})$ in $\mathcal{K}$. $\square$

Proposition 2.2.29. *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be an extension of an amenable groupoid $\mathcal{G}$. Then $\mathcal{H}$ is amenable.*

Proof. If (f_n) is a sequence witnessing the amenability of $\mathcal{G}$, then the sequence $(f_n \circ \varphi)$ witnesses the amenability of $\mathcal{H}$. $\square$

Definition 2.2.30. The **cost** of a principal discrete p.m.p. groupoid R is defined

$$C_{\mu^0}(R) = \inf_{A \text{ generating } R} \mu^1(A).$$

Proposition 2.2.31. *Let $S \leq R$ be principal discrete p.m.p. groupoids. If S is q-normal in R then $C_{\mu^0_R}(R) \leq C_{\mu^0_R}(S)$.*

This is proved in Proposition A.2 in [47] and was first observed by Furman in [19].

2.3 A folklore lemma in the groupoid setting

Definition 2.3.1. An action of a discrete countable group G on a space X is called **amenable** if there exists a finitely additive probability measure $m : X \to [0, 1]$ that is invariant under the group action.

The following folklore lemma's origins go back to von Neumann's original paper which introduced the notion of amenability [38]. A proof can be found in chapter 1.

Lemma 2.3.2. *Let a discrete countable group G act on X and Y. Suppose the action $G \curvearrowright X$ is amenable. If the action of the stabilizer $G_x \curvearrowright Y$ is an amenable action for every $x \in X$, then $G \curvearrowright Y$ is amenable.*

We generalize this lemma in the following manner.

Lemma 2.3.3. *Let $\mathcal{G}$ be a discrete p.m.p. groupoid. Fix actions of $\mathcal{G}$ on the locally countable Borel fibered spaces $p : W \to \mathcal{G}^0$ and $q : V \to \mathcal{G}^0$. Suppose $\mathcal{G} \curvearrowright W$ is an amenable action. Suppose we have a countable collection of nontrivial partial measurable sections $\Sigma = \{\sigma_i\}_{i\in\mathbb{N}}$ of W with the following properties*

- $(\mathcal{G}\sigma_i) \cap (\mathcal{G}\sigma_j) = \varnothing$ *if* $i \neq j$
- $\bigsqcup_{i\in\mathbb{N}}(\mathcal{G}\sigma_i) = W$
- *the restricted action* $\mathcal{G}_{\sigma_i} \curvearrowright q^{-1}(p(\sigma_i))$ *is amenable for all* $i \in \mathbb{N}$.

Then the action $\mathcal{G} \curvearrowright V$ is amenable.

Proof. Fix (f_n) an amenability sequence for the action $\mathcal{G} \curvearrowright W$ and denote $f_n|_{W^x}$ by f_n^x.

Pick a sequence of Borel subsets $(W_n)_{n\in\mathbb{N}} \subseteq W$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathcal{G}^0$

- $|W_n \cap W^x|$ is finite
- $f_n^x(W_n \cap W^x) > 1 - 2^{-n}$.

We define

$$\hat{f}_n(w) = \begin{cases} \frac{f_n(w)}{f_n(W_n \cap W^{p(w)})} & w \in W_n \\ 0 & \text{otherwise} \end{cases}$$

by restricting supports and renormalizing. The sequence $(\hat{f}_n)$ also witnesses the amenability of the action $\mathcal{G} \curvearrowright W$ and has finite support on each fiber. By replacing f_n by $\hat{f}_n$, we may therefore assume that each of the functions f_n is supported on W_n.

For $w \in W$, we denote by σ_w the unique element σ of Σ such that $w \in \mathcal{G}\sigma$. Now, we would like to find a measurable function $\phi : W \to \mathcal{G}$ such that $r(\phi(w)) = p(w)$ and $\{w\} = \phi(w)\sigma_w$. Consider the map $\phi' : \mathcal{G} \times \Sigma \to W$ defined by $(g, \sigma) \mapsto g\sigma$. This is surjective by the hypothesis on Σ and countable-to-one since $\mathcal{G}$ has countable fibers. By the Lusin-Novikov Uniformization Theorem[27], there is an injective Borel map $\phi^* : W \to \mathcal{G} \times \Sigma$ with $\phi'(\phi^*(w)) = w$. By composing with a projection to $\mathcal{G}$, this is measurable and we get the map ϕ we were looking for. By abuse of notation we identify $\phi(w)\sigma_w$ with the point w it contains.

For $g \in \mathcal{G}$ and $w \in W^{s(g)}$, we let $h_{g,w} := \phi(gw)^{-1} g \phi(w)$. Notice that $h_{g,w} \in \mathcal{G}_{\sigma_w}$ since

$$\begin{aligned} h_{g,w}\sigma_w = \phi(gw)^{-1} g \phi(w)\sigma_w = \phi(gw)^{-1}(gw) &= \phi(gw)^{-1}(\phi(gw)\sigma_{gw}) \\ &= id_{s(\phi(gw))}\sigma_{gw} \in \sigma_{gw} = \sigma_w \end{aligned}$$

Let $(D_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite μ^1-measure subsets of $\mathcal{G}$ which exhaust the space. For each $\sigma \in \Sigma$, we pick a sequence $a_{\sigma,n}$ witnessing the amenability of the action $\mathcal{G}_\sigma \curvearrowright q^{-1}(p(\sigma))$. We may choose these sequences in such a way that there exists a sequence $(D'_n)_{n\in\mathbb{N}}$ satisfying the following

- $\dfrac{\mu^1(D'_n \cap D_i)}{\mu^1(D_i)} > 1 - 2^{-n}$ for all $i \leq n$
- for every g in D'_n and for every w in $W^{s(g)} \cap W_n$

$$\left\| a_{\sigma_w,n}^{r(h_{g,w})} - h_{g,w} a_{\sigma_w,n}^{s(h_{g,w})} \right\|_{\ell^1(V^{r(h_{g,w})})} < 2^{-n} \tag{2.1}$$

This is accomplished as follows. First take an amenability sequence $\hat{a}_{\sigma,n}$ for the action $\mathcal{G}_\sigma \curvearrowright q^{-1}(p(\sigma))$. Here, for a given $g \in \mathcal{G}$ and $n \in \mathbb{N}$, the set $W^{s(g)} \cap W_n$ is finite.

Fix an element $g \in \mathcal{G}$. Let $N(n,g)$ be the least integer such that for all $N \geq N(n,g)$,

$$\left\| \hat{a}_{\sigma_w,N}^{r(h_{g,w})} - h_{g,w}\hat{a}_{\sigma_w,N}^{s(h_{g,w})} \right\|_{\ell^1(V^{r(h_{g,w})})} \leq 2^{-n} \text{ for every } w \in W^{s(g)} \cap W_n.$$

Such an $N(n,g)$ exists because $h_{g,w} \in \mathcal{G}_{\sigma_w}$ and by definition of $\hat{a}_{\sigma,n}$ being an amenability sequence. For each $c \in \mathbb{N}$, define $\hat{D}_n(c) := \{g \in D_n \mid N(n,g) < c\}$. The sets $\hat{D}_n(c)$ increase to D_n as $c \to \infty$. Fix c_n such that $\frac{\mu^1(\hat{D}_n(c_n) \cap D_i)}{\mu^1(D_i)} > 1 - 2^{-n}$ for every $i \leq n$. Define $a_{\sigma,n} := \hat{a}_{\sigma,c_n}$ and $D'_n := \hat{D}_n(c_n)$.

Now, for each $n \in \mathbb{N}$, define $\xi_n(v) := \sum_{w \in W^{q(v)}} a_{\sigma_w,n}(\phi(w)^{-1}v) f_n^{q(v)}(w)$. This is defined since $\phi(w)^{-1}v \in q^{-1}(p(\sigma_w))$. We show the sequence $(\xi_n)_{n\in\mathbb{N}}$ witnesses the amenability of the action $\mathcal{G} \curvearrowright V$. We first check it satisfies item (2) in the definition of amenability. Let $g \in \mathcal{G}$ and let $x = s(g)$ and $y = r(g)$.

$$\|\xi_n^y - g\xi_n^x\|_{\ell^1(V^y)} = \sum_{v \in V^y} \left| \sum_{w \in W^y} a_{\sigma_w,n}(\phi(w)^{-1}v) f_n^y(w) - \sum_{w \in W^x} a_{\sigma_w,n}(\phi(w)^{-1}(g^{-1}v)) f_n^x(w) \right|$$

$$\leq \sum_{v \in V^y} \left| \sum_{w \in W^y} a_{\sigma_w,n}(\phi(w)^{-1}v) f_n^y(w) - \sum_{w \in W^x} a_{\sigma_w,n}(\phi(gw)^{-1}v) f_n^x(w) \right| \tag{2.2}$$

$$+ \sum_{v \in V^y} \left| \sum_{w \in W^x} a_{\sigma_w,n}(\phi(gw)^{-1}v) f_n^x(w) - \sum_{w \in W^x} a_{\sigma_w,n}(\phi(w)^{-1}g^{-1}v) f_n^x(w) \right| \tag{2.3}$$

Let's first look at eq. 2.2 now and bound it by rewriting it as follows.

$$\begin{aligned}
&\sum_{v\in V^y}\left|\sum_{w\in W^y} a_{\sigma_w,n}(\phi(w)^{-1}v)f_n^y(w) - \sum_{w\in W^x} a_{\sigma_w,n}(\phi(gw)^{-1}v)f_n^x(w)\right| \\
&\quad= \sum_{v\in V^y}\left|\sum_{w\in W^y} a_{\sigma_w,n}(\phi(w)^{-1}v)f_n^y(w) - \sum_{w\in W^y} a_{\sigma_w,n}(\phi(w)^{-1}v)f_n^x(g^{-1}w)\right| \\
&\quad\le \sum_{v\in V^y}\sum_{w\in W^y}\left|a_{\sigma_w,n}(\phi(w)^{-1}v)\right|\left|f_n^y(w) - f_n^x(g^{-1}w)\right| \\
&\quad= \sum_{w\in W^y}|f_n^y(w) - gf_n^x(w)| = \|f_n^y - gf_n^x\|_{\ell^1(W^y)} \longrightarrow 0 \text{ as } n\to\infty
\end{aligned}$$

Now, we find a bound for eq. 2.3.

$$\begin{aligned}
&\sum_{v\in V^y}\left|\sum_{w\in W^x}\left[a_{\sigma_w,n}(\phi(gw)^{-1}v) - a_{\sigma_w,n}(h_{g,w}^{-1}\phi(gw)^{-1}v)\right]f_n^x(w)\right| \\
&\quad\le \sum_{v\in V^y}\sum_{w\in W^x}|a_{\sigma_w,n}(\phi(gw)^{-1}v) - h_{g,w}a_{\sigma_w,n}(\phi(gw)^{-1}v)|f_n^x(w) \\
&\quad= \sum_{w\in W^x\cap W_n} f_n^x(w)\left\|a_{\sigma_w,n}^{r(h_{g,w})} - h_{g,w}a_{\sigma_w,n}^{s(h_{g,w})}\right\|_{\ell^1(V^{r(h_{g,w})})} \le 2^{-n}
\end{aligned}$$

We show for μ^1-almost every g, this last inequality holds when n is large enough. Let $E_{i,k} := D_i \setminus D'_k$. Notice that $\mu^1(E_{i,k}) \le 2^{-k}\mu^1(D_i)$ for $k \ge i$ and so $\sum_k \mu^1(E_{i,k}) < \infty$. Thus, $\mu^1(\limsup_k E_{i,k}) = 0$ and, in fact, $\mu^1(\bigcup_i \limsup_k E_{i,k}) = 0$. By assumption 2.1, the set $\bigcup_i \limsup_k E_{i,k}$ contains exactly the $g \in \mathcal{G}$ where the inequality fails to hold for infinitely many n. Thus, we get that item (2) in the definition of amenability is satisfied for the sequence (ξ_n).

It remains to check item (1). By the definition of ξ_n, we have

$$\xi_n^x(v) = \sum_{w\in W^x} f_n^x(w)[\phi(w)a_{\sigma_w,n}^{s(\phi(w))}(v)].$$

The measure $\phi(w)a_{\sigma_w,n}^{s(\phi(w))}$ is a probability measure since it is a pushforward of a probability measure. The function ξ_n^x is a convex combination of probability measures, and therefore, is a probability measure. $\square$

2.4 Cost of inner amenable groupoids

We generalize the following structural result about inner amenable groupoids from the setting of groups[47, Theorem 8].

Theorem 2.4.1. *If $\mathcal{G}$ is an inner amenable groupoid, and $\mathcal{H} \leq \mathcal{G}$ is a nowhere amenable subgroupoid, then there is a groupoid $\mathcal{K}$ such that $\mathcal{H}$ is q-normal in $\mathcal{K}$ and $\mathcal{K}$ is q-normal in $\mathcal{G}$. Moreover, $\mathcal{K}$ can be chosen so that, for every $n \in \mathbb{N}$, the groupoid $\mathcal{K} \cap \mathcal{K}^{\sigma_1} \cap \ldots \cap \mathcal{K}^{\sigma_n}$ is aperiodic for all bisections $\sigma_1, \ldots, \sigma_n \in [\mathcal{G}]$.*

Proof. By Lusin-Novikov[27], there is a countable subgroup $H \leq [\mathcal{H}]$ of the full group that generates $\mathcal{H}$. We can then define the action groupoid $\tilde{\mathcal{H}} := H \ltimes \mathcal{G}^0$ which comes with a natural surjective groupoid homomorphism $\varphi : \tilde{\mathcal{H}} \to \mathcal{H}$ that satisfies $s(\varphi(h,x)) = x$ and $r(\varphi(h,x)) = hx$.

We define two different actions of $\tilde{\mathcal{H}}$ on $\mathcal{G}$. The first being the action by conjugation $\alpha : \tilde{\mathcal{H}} \curvearrowright \mathcal{G}$ where $(h, r(g)) \cdot g = \varphi(h, r(g)) g \varphi(h, s(g))^{-1}$. For clarity in the rest of this proof, we use $\cdot$ to denote the conjugation action α. Fix a countable collection of measurable partial sections Σ of the range map r for the conjugation action such that $\tilde{\mathcal{H}} \cdot \sigma' \cap \tilde{\mathcal{H}} \cdot \sigma'' = \varnothing$ for $\sigma' \neq \sigma''$ and $\mathcal{G} = \bigsqcup_{\Sigma} \tilde{\mathcal{H}} \cdot \sigma$. The second action will be by left translation $\lambda : \tilde{\mathcal{H}} \curvearrowright \mathcal{G}$ where $(h, r(g))g = \varphi(h, r(g))g$.

Let $A_\sigma \subseteq dom(\sigma)$ be the unique maximal (mod null) set such that $(\mathcal{H} \cap \mathcal{H}^\sigma)_{A_\sigma}$ is nowhere amenable. Define $\Sigma_A = \{\sigma \cap s^{-1}(A_\sigma) \mid \sigma \in \Sigma\}$ and define $\mathcal{K}$ to be the groupoid generated by $\mathcal{H} \cup \Sigma_A$. Also, define $\Sigma_B = \{\sigma \setminus \mathcal{K} | \sigma \in \Sigma\}$ and note that $\mathcal{G} \setminus \mathcal{K} = \tilde{\mathcal{H}} \cdot \Sigma_B$ by the assumptions on Σ and since $\mathcal{K}$ is invariant under the action of $\tilde{\mathcal{H}}$. If $\tau \in \Sigma_A$ then $\mathcal{H} \cap \mathcal{H}^\tau$ is nowhere amenable on $s(\tau)$ and, in particular, aperiodic on $s(\tau)$. And if $h \in H$, then it's immediate that $\mathcal{H} \cap \mathcal{H}^h = \mathcal{H}$ is aperiodic. So we get that $\mathcal{H}$ is q-normal in $\mathcal{K}$.

Since $\mathcal{G}$ is inner amenable, there is a mean m on $\mathcal{G}$ as in definition 2.2.24. In particular, this mean is equidistributed with respect to the map r. We proceed by contradiction to show $m(\mathcal{K}) = 1$.

Assume not. So $m(\mathcal{G} \setminus \mathcal{K}) > 0$. The mean m is no longer necessarily equidistributed with respect to $r : \mathcal{G} \setminus \mathcal{K} \to \mathcal{G}^0$. However, we can define a finite Borel measure $\mu_{\mathcal{G}\setminus\mathcal{K}}$ on $\mathcal{G}^0$ by $\mu_{\mathcal{G}\setminus\mathcal{K}}(A) := m(r^{-1}(A) \setminus \mathcal{K})$ which is absolutely continuous with respect to μ^0.

Claim 2.4.2. *$\mu_{\mathcal{G}\setminus\mathcal{K}}$ is countably additive.*

Proof of claim. The mean m is assume to be equidistributed, so for any measurable partition $P = (A_1, \dots, A_k, \dots)$ of $\mathcal{G}^0$

$$1 = \sum_k \mu^0(A_k) = \sum_k m(r^{-1}(A_k))$$

which implies that for measurable $D_k \subseteq r^{-1}(A_k)$

$$m(\bigsqcup_k D_k) = \sum_k m(D_k).$$

Letting $D_k = r^{-1}(A_k) \setminus \mathcal{K}$, we see that $\mu_{\mathcal{G}\setminus\mathcal{K}}$ is countably additive. $\square$

This $\mu_{\mathcal{G}\setminus\mathcal{K}}$ and μ^0 are both $\mathcal{H}$-conjugation invariant, so the Radon-Nikodym derivative $f := \frac{d\mu_{\mathcal{G}\setminus\mathcal{K}}}{d\mu^0}$ is also $\mathcal{H}$-conjugation invariant. Additionally, f is bounded above by 1 and not almost everywhere equal to 0 by our assumption that $m(\mathcal{G} \setminus \mathcal{K}) > 0$. Therefore, we may find a small enough $\varepsilon > 0$ such that the $\mathcal{H}$-conjugation invariant set $W^0 := \{x | f(x) > \varepsilon\}$ has μ^0-positive measure. Let $W := r^{-1}(W^0) \setminus \mathcal{K}$. We define a conjugation-invariant positive linear functional $m_W \in (L^\infty(r^{-1}(W^0), \mu^1))^*$ by $m_W(D) := \int_W \frac{1_D}{f} dm$. This is still in $(L^\infty)^*$ since f is bounded above and below. Let $(f_k = \sum c_i^k 1_{A_i^k})$ be a non-decreasing sequence of simple functions converging to $\frac{1}{f}$ in L^∞. The mean m_W is now equidistributed with respect to the restricted map $r_W : W \to \mathcal{G}^0$, since for $A \subseteq W^0$,

$$\begin{aligned}
\mu^0(A) &= \int_A \frac{1}{f} d\mu_{\mathcal{G}\setminus\mathcal{K}} = \lim_k \int_A f_k d\mu_{\mathcal{G}\setminus\mathcal{K}} \\
&= \lim_k \sum_i c_i^k \mu_{\mathcal{G}\setminus\mathcal{K}}(A_i^k \cap A) = \lim_k \sum_i c_i^k m(r_W^{-1}(A_i^k \cap A)) \\
&= \lim_k \sum_i c_i^k \int_W 1_{r_W^{-1}(A_i^k) \cap r_W^{-1}(A)} dm = \lim_k \int_W f_k 1_{r_W^{-1}(A)} dm \\
&= \int_W \frac{1_{r_W^{-1}(A)}}{f} dm = m_W(r_W^{-1}(A))
\end{aligned}$$

By $\mathcal{H}$-invariance of $\mathcal{K}$ and W^0, we have a restricted conjugation action of $\tilde{\mathcal{H}} \curvearrowright W$ which we refer to by $\tilde{\alpha}$. Now, define a new mean $\tilde{m}_W := \frac{m_W}{m_W(W^0)}$ by renormalizing m_W. The mean $\tilde{m}_W$ satisfies the assumptions of proposition 2.2.20 for the action $\tilde{\alpha}$ and, therefore, there exists a sequence h_n witnessing amenability of $\tilde{\alpha} : \tilde{\mathcal{H}}_{W^0} \curvearrowright W$.

The left translation action $\lambda : \tilde{\mathcal{H}}_{W^0} \curvearrowright W \subseteq \mathcal{G}$ is nonamenable since $\mathcal{H}$ is nowhere amenable. Define $\Sigma_W := \{\tau \cap W | \tau \in \Sigma_B\}$. For $\tau \in \Sigma_W$, denote by $\tilde{\mathcal{H}}_\tau \subseteq \tilde{\mathcal{H}}_{W^0}$ the stabilizer of the section τ with respect to the action α. Let $\lambda_\tau : \tilde{\mathcal{H}}_\tau \curvearrowright r^{-1}(r(\tau))$ be the action λ restricted to $\tilde{\mathcal{H}}_\tau$. If this action is nonamenable, then the action $\varphi(\tilde{\mathcal{H}}_\tau) \curvearrowright r^{-1}(r(\tau))$ by left translation is nonamenable. But then, the action $\varphi(\tilde{\mathcal{H}}_\tau) \curvearrowright \varphi(\tilde{\mathcal{H}}_\tau)$ by left translation is nonamenable. Indeed, otherwise we can use the amenability sequence from $\varphi(\tilde{\mathcal{H}}_\tau)$ to show the action $\varphi(\tilde{\mathcal{H}}_\tau) \curvearrowright r^{-1}(r(\tau))$ is amenable. So the groupoid $\varphi(\tilde{\mathcal{H}}_\tau)$ is nonamenable if the action λ_τ is nonamenable.

By lemma 2.3.3, we know there exists a $\tau \in \Sigma_W$ such that $\varphi(\tilde{\mathcal{H}}_\tau)$ is nonamenable. Since $\tilde{\mathcal{H}}_\tau$ stabilizes τ under conjugation, $\varphi(\tilde{\mathcal{H}}_\tau) \subseteq \mathcal{H}^\tau$. So, $\mathcal{H} \cap \mathcal{H}^\tau \supseteq \varphi(\tilde{\mathcal{H}}_\tau)$ is nonamenable. We may find a positive measure set A_τ such that $(\mathcal{H} \cap \mathcal{H}^\tau)_{A_\tau}$ is nowhere amenable. Recall, $\tau = (\sigma \setminus s^{-1}(A_\sigma)) \cap W$ for some $\sigma \in \Sigma$. The sets A_σ and A_τ are disjoint. However, the groupoid $(\mathcal{H} \cap \mathcal{H}^\sigma)_{A_\sigma \cup A_\tau}$ is nowhere amenable which contradicts maximality of A_σ. Hence, $m(K) = 1$.

The conjugated groupoid $\mathcal{K}^\sigma := \{k^\sigma \mid k \in \mathcal{K}\}$ will still have mean 1 since m is conjugation invariant. So, $\mathcal{K} \cap \mathcal{K}^{\sigma_1} \cap \ldots \cap \mathcal{K}^{\sigma_n}$ will have mean 1 in $\mathcal{G}$ for all bisections $\sigma_1, \ldots, \sigma_n \in [\mathcal{G}]$. Since m is diffuse and equidistributed, this means that $\mathcal{K} \cap \mathcal{K}^{\sigma_1} \cap \ldots \cap \mathcal{K}^{\sigma_n}$ is aperiodic and hence $\mathcal{K}$ is q-normal in $\mathcal{G}$. □

Now we prove the main theorem.

Theorem 2.4.3. *Assume $\varphi : R \to \mathcal{G}$ is a principal groupoid extension of an inner amenable groupoid $\mathcal{G}$. Then $C_{\mu_R^0}(R) = 1$.*

Proof. By looking at the ergodic decomposition of $\mathcal{G}$, it suffices to deal with the groupoid $\mathcal{G}$ being ergodic. We prove this in two cases.

Assume first that the associated equivalence relation $R_\mathcal{G} := \{(r(g), s(g)) \mid g \in \mathcal{G}\}$ is finite. By ergodicity, $\mathcal{G}^0 = \{x_1, \ldots, x_n\}$. This means that by [32], the isotropy group $\mathcal{G}_{\{x_i\}} := \{g \in$

$\mathcal{G} \mid s(g = r(g) = x_i)\}$ is an inner amenable group. By [47], this has fixed price 1. Consider now the groupoid $\mathcal{G}' := \bigsqcup_{1 \leq i \leq n} \mathcal{G}_{\{x_i\}}$. Any principal extension of $\mathcal{G}'$ is a union of exactly n ergodic components each of which is a principal extension of some copy of $\mathcal{G}_{\{x_i\}}$ and so generated by a set of measure $1/n + \varepsilon$. So, $\mathcal{G}'$ has fixed price 1 and $C_{\mu_R^0}(\varphi^{-1}(\mathcal{G}')) = 1$. By propositions 2.2.28 and proposition 2.2.31, we get $C_{\mu_R^0}(R) = 1$.

If the underlying equivalence relation $R_{\mathcal{G}}$ is instead aperiodic, fix $\varepsilon > 0$ and define $\mathcal{H}' \leq \mathcal{G}$ as the maximal(mod null) ergodic amenable subgroupoid using Zorn's Lemma. The set of amenable ergodic subgroupoids is nonempty since $R_{\mathcal{G}}$ is aperiodic. Now, let $A \subseteq \mathcal{G} \backslash \mathcal{H}'$ with $0 < \mu^1(A) < \varepsilon$. Define $\mathcal{H} := \langle \mathcal{H}' \cup A \rangle$ so $\mathcal{H}$ is nonamenable. In particular, since $\mathcal{H}$ is ergodic nonamenable, it is nowhere amenable. Now, by theorem 2.4.1 and proposition 2.2.28, there exists $\mathcal{K}$ such that $\varphi^{-1}(\mathcal{H})$ is q-normal in $\varphi^{-1}(\mathcal{K})$ and $\varphi^{-1}(\mathcal{K})$ is q-normal in R.

Notice that $C_{\mu_R^0}(\varphi^{-1}(\mathcal{H}')) = 1$ since $\varphi|_{\varphi^{-1}(\mathcal{H}')}$ is a principal groupoid extension of an amenable groupoid. Therefore, $C_{\mu_R^0}(\varphi^{-1}(\mathcal{H})) < 1 + \varepsilon$. Last, proposition 2.2.31 gives us

$$C_{\mu_R^0}(R) \leq C_{\mu_R^0}(\varphi^{-1}(\mathcal{H})) < 1 + \varepsilon$$

proving the theorem. $\square$

3. COAMENABILITY

3.1 Introduction

Von Neumann introduced the notion of an amenable action in his seminal paper on paradoxicality[38]. In the setting of groups, an action of a countable discrete group $\Gamma \curvearrowright X$ is said to be amenable if the set X admits a finitely additive probability measure that is invariant under the action of the group. It is elementary to see that every action of an amenable group is amenable in this sense. However, the last 20 or so years have seen a prominent surge of progress when discussing amenable actions of nonamenable groups.

One particular example of amenable action comes up in the definition of coamenability. An inclusion of groups $H \subseteq G$ is coamenable if the action $G \curvearrowright G/H$ by left multiplication is amenable. Coamenability has been extensively studied in the setting of groups and even in the setting of von Neumann algebras[23][13][36][40][24]. It is well-known that if a group contains a coamenable inner amenable subgroup, then the group must be inner amenable. A similar theorem dealing with property Gamma was recently shown in the setting of von Neumann algebras by Bannon-Marrakchi-Ozawa[2]. Inner amenability was recently introduced to the setting of measured groupoids in the work of Kida and Tucker-Drob[32]. In this vein, we define the notion of coamenability in the setting of measured groupoids and show how it interacts with inner amenability.

Theorem 3.1.1. *Let $\mathcal{H} \subseteq \mathcal{G}$ be a coamenable inclusion of measured groupoids. If $\mathcal{H}$ is inner amenable, then $\mathcal{G}$ is inner amenable.*

We also give several equivalent characterizations of amenability of actions of groupoids, several of whose origins are drawn from the study of amenability of measured groupoids which can be found in Anantharaman-Renault[1].

3.2 Amenable actions

For this chapter, recall the preliminaries about groupoids from Chapter 1. The ambient setting will be an ergodic discrete pmp groupoid $\mathcal{G}$ acting with a Borel action on a locally countable Borel fibered space $p : W \to \mathcal{G}^0$. Recall the following definition from the previous chapter.

Definition 3.2.1. The action $\mathcal{G} \curvearrowright W$ is **amenable** if there exists a sequence of Borel functions $f_n : W \to [0, 1]$ that satisfies the following properties

- $\|f_n^x\|_{\ell^1} = 1$ for almost every x
- $\|f_n^{r(g)} - g \cdot f_n^{s(g)}\|_{\ell^1} \to 0$ for almost every g

where $f_n^x : W^x \to \mathbb{R}$ is the function f_n restricted to the fiber over x.

We require the following definitions before continuing.

Definition 3.2.2. A **Reiter sequence** for $\mathcal{G} \curvearrowright W$ is a sequence of Borel functions $f_n : W \to [0, 1]$ such that

- $\|f_n\|_{L^1(W,\nu)} = 1$
- $\|f_n - \sigma \cdot f_n\|_{L^1(W,\nu)} \to 0$ for each $\sigma \in [\mathcal{G}]$

Definition 3.2.3. A $[\mathcal{G}]$**-invariant mean** for $\mathcal{G} \curvearrowright W$ is a norm one positive linear functional $m \in (L^\infty(W, \nu))^*$ such that $m(\sigma \cdot A) = m(A)$ for every $\nu-$measurable set A and $\sigma \in [\mathcal{G}]$.

We now give several equivalent conditions to amenability of an action.

Proposition 3.2.4. *Let $\mathcal{G}$ be an ergodic discrete pmp groupoid and let $\mathcal{G} \curvearrowright W$. TFAE*

1. *$\mathcal{G} \curvearrowright W$ is amenable*
2. *W admits a Reiter sequence*
3. *W admits a $[\mathcal{G}]$-invariant mean*

4. *there exists* $P : L^\infty(W,\nu) \to L^\infty(\mathcal{G}^0,\mu)$ *such that* $P(\phi \cdot F) = \phi \cdot P(F)$ *for* $\phi \in [\mathcal{G}]$ *and* $P(\mathbb{1}_W) = \mathbb{1}_{\mathcal{G}^0}$

Proof. ($1 \implies 2$) Take the functions (f_n) witnessing amenability. These satisfy $\|f_n\|_1 = \int_X \|f_n^x\| d\mu = \int 1 d\mu = 1$. Since $f_n - \sigma \cdot f_n$ is a uniformly bounded sequence converging to 0, the norm converges to 0 by bounded convergence theorem.

($2 \implies 3$) Let f_n be the Reiter sequence. Fix a free ultrafilter ω. Define $m(A) := \lim_\omega \int \sum_{\xi \in p^{-1}(x)} f_n(\xi)\chi_A d\mu(x)$. It is immediate that m is a mean.

$$\begin{aligned}
m(\sigma \cdot A) &= \lim_\omega \int_{x\in X} \sum_{\xi \in p^{-1}(x)} f_n(\xi)\chi_{\sigma\cdot A}(\xi) d\mu(x) \\
&= \lim_\omega \int_{x\in X} \sum_{\xi \in p^{-1}(x)} f_n(\xi)\chi_A(\sigma^{-1}\xi) d\mu(x) \\
&= \lim_\omega \int_{x\in X} \sum_{\xi \in p^{-1}(\sigma^{-1}x)} f_n(\sigma\xi)\chi_A(\xi) d\mu(x) \\
&= \lim_\omega \int_{\sigma x\in X} \sum_{\xi \in p^{-1}(x)} \sigma^{-1}\cdot f_n(\xi)\chi_A(\xi) d\mu(\sigma x) \\
&= \lim_\omega \int_{x\in X} \sum_{\xi \in p^{-1}(x)} \sigma^{-1}\cdot f_n(\xi)\chi_A(\xi) d\mu(x)
\end{aligned}$$

So $m(\sigma \cdot A) - m(A) = \lim_\omega \int_{x\in X} \sum_{\xi\in p^{-1}(x)} (\sigma^{-1}\cdot f_n - f_n)(\xi)\chi_A(\xi) d\mu(x)$. Here $\sigma^{-1}\cdot f_n - f_n$ converges to 0 in norm by assumption and so, in fact, it converges weakly in $L^1(W,\nu)$. Therefore, $m(\sigma \cdot A) - m(A) = 0$.

($3 \implies 4$) For $F \in L^\infty(W,\nu)$ define $\mu_F(A) := \int_{W_A} F\, dm$.

Claim 3.2.5. μ_F *is countably additive.*

Proof of Claim. Since $\mathcal{G}$ is ergodic, for any measurable sets $A, B \subset \mathcal{G}^0$ such that $\mu(A) = \mu(B)$, there exists $\sigma \in [\mathcal{G}]$ such that $\sigma A = B$. The measure m is $[\mathcal{G}]$-invariant, so $m(W_A) = m(W_B) = \mu(A)$. So, $m(W_{\sqcup A_n}) = \sum m(W_{A_n})$ if A_n are disjoint and

$$\mu_F(\sqcup A_n) = \int_{W_{\sqcup A_n}} F\, dm = \int_{\sqcup W_{A_n}} F\, dm = \sum_n \int_{W_{A_n}} F\, dm = \sum_n \mu_F(A_n).$$

giving countable additivity. $\square$

Note that $\mu_F << \mu$, so there exists a Radon-Nikodym derivative $\frac{d\mu_F}{d\mu}$. We define $P(F) := \frac{d\mu_F}{d\mu}$ such that $\mu_F(A) = \int_A P(F)d\mu$. Here, $P(\mathbb{1}) = \mathbb{1}$ since $\mu_{\mathbb{1}}(A) = \int \mathbb{1}_{W_A} dm = \mu(A)$ as noted in the proof of the above claim. For $\phi \in [\mathcal{G}]$ and for every measurable $A \subseteq \mathcal{G}^0$, we see that

$$\begin{aligned}\int_A P(\phi \cdot F)d\mu &= \mu_{\phi \cdot F}(A) \\ &= \int_{W_A} \phi \cdot F(\xi)dm(\xi) \\ &= \int_{W_A} F(\phi^{-1} \cdot \xi)dm(\xi) \\ &= \int_{\phi^{-1} \cdot W_A} F(\xi)dm(\phi \cdot \xi) \\ &= \int_{W_{\phi^{-1} \cdot A}} F(\xi)dm(\xi) \\ &= \mu_F(\phi^{-1} \cdot A) \\ &= \int_{\phi^{-1} \cdot A} P(F)(x)d\mu(x) \\ &= \int_A P(F)(\phi^{-1} \cdot x)d\mu(\phi^{-1} \cdot x) \\ &= \int_A \phi \cdot P(F)d\mu\end{aligned}$$

So, $P(\phi \cdot F) = \phi \cdot P(F)$.

($4 \implies 3$) Given $[\mathcal{G}]$-equivariant $P : L^\infty(W,\nu) \to L^\infty(\mathcal{G}^0, \mu)$. Define $m(A) := \int P(\mathbb{1}_A)d\mu$ for $A \subseteq W$. Here, $m(W) = \int P(\mathbb{1})d\mu = \int \mathbb{1}\|d\mu = 1$. For $\phi \in [\mathcal{G}]$ and ν-measurable $A \subseteq W$

$$\begin{aligned}m(\phi \cdot A) &= \int P(\mathbb{1}_{\phi \cdot A})d\mu \\ &= \int P(\phi \cdot \mathbb{1}_A)d\mu \\ &= \int \phi \cdot P(\mathbb{1}_A)(x)d\mu(x) \\ &= \int P(\mathbb{1}_A)(\phi^{-1} \cdot x)d\mu(x)\end{aligned}$$

$$= \int P(\mathbb{1}_A) d\mu$$

$$= m(A)$$

$(3 \implies 1)$ Fix a $[\mathcal{G}]$-invariant mean $m \in (L^\infty(W, \nu))^*$. By ergodicity, this is equidistributed. Now, this implication is the content of proposition 2.2.20. $\square$

3.3 Coamenability

Throughout this section, $\mathcal{H}$ and $\mathcal{G}$ will be ergodic discrete pmp groupoids. We define the fibered space $p : \mathcal{H} \setminus \mathcal{G} \to \mathcal{G}^0$ where set of elements in $\mathcal{H} \setminus \mathcal{G}$ is $\{\mathcal{H}g \mid g \in \mathcal{G}\}$ and $p(\mathcal{H}g) := s(g)$. There is no particular reason we use right cosets and you get an equivalent definition by considering left cosets.

Definition 3.3.1. Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. We say this inclusion is **coamenable** if the action $\mathcal{G} \curvearrowright \mathcal{H} \setminus \mathcal{G}$ by right multiplication is amenable.

Definition 3.3.2. A discrete p.m.p. groupoid $\mathcal{G}$ is called **inner amenable** if there exists a sequence of measurable functions $h_n : \mathcal{G} \to [0, 1]$ such that

(i) $\|1_{\mathcal{G}_A} h_n\|_{L^1(\mathcal{G},\mu^1)} \to \mu^0(A)$ as $n \to \infty$ for every μ^0-measurable $A \subseteq \mathcal{G}^0$

(ii) $\|h_n^\sigma - h_n\|_{L^1(\mathcal{G},\mu^1)} \to 0$ as $n \to \infty$ for every $\sigma \in [\mathcal{G}]$

(iii) $\|1_D h_n\|_{L^1(\mathcal{G},\mu^1)} \to 0$ as $n \to \infty$ for every μ^1-measurable $D \subseteq \mathcal{G}$ with $\mu^1(D) < \infty$

(iv) $\sum_{\gamma \in s^{-1}(x)} h_n(\gamma) = 1 = \sum_{\gamma \in r^{-1}(x)} h_n(\gamma)$ for μ^0-a.e. $x \in \mathcal{G}^0$ and every n

Such a sequence of functions is referred to as an inner amenability sequence.

Condition (i) is referred to as the balanced condition and ends up surprisingly important in many proofs about these groupoids.

Definition 3.3.3. Given a discrete p.m.p. groupoid $\mathcal{G}$. A net of measurable functions $\xi_i : \mathcal{G} \to [0, 1]$ is said to be

- **conjugation invariant** if $\|\xi_i^\sigma - \xi_i\|_{L^1(\mathcal{G},\mu^1)} \to 0$ for every $\sigma \in [\mathcal{G}]$

- **diffuse** if $\|1_D\xi_i\|_{L^1(\mathcal{G},\mu^1)} \to 0$ for every μ^1-measurable $D \subseteq \mathcal{G}$ with $\mu^1(D) < \infty$

- **nontrivial** if $\sum_{\gamma\in s^{-1}(x)} \xi_i(\gamma) = 1$ for μ^0-a.e. $x \in \mathcal{G}^0$ for every i

Remark 3.3.4. In order to check inner amenability for an ergodic groupoid, it is enough to check there exists a conjugation invariant, diffuse, and nontrivial net[32].

Definition 3.3.5. For a subgroupoid $\mathcal{H} \leq \mathcal{G}$, we say a collection of elements $(\sigma_i)_{i\in\mathbb{N}} \subset [\mathcal{G}]$ is a **system of coset representatives** if

- $\mathcal{H}\sigma_i \cap \mathcal{H}\sigma_j = \emptyset$ if $i \neq j$

- $\bigsqcup \mathcal{H}\sigma_i = \mathcal{G}$

i.e. if it partitions $\mathcal{H} \setminus \mathcal{G}$.

Lemma 3.3.6. *Let $\mathcal{H}$ be an ergodic subgroupoid of a discrete pmp groupoid $\mathcal{G}$. Then there exists a system (σ_i) of coset representatives.*

Proof. We will construct σ_κ by induction.

Starting with $m = 0$, consider $R^m = R^m_\kappa := (\mathcal{G} \setminus \cup_{0\leq i<\kappa}\mathcal{H}\sigma_i)\,|_{X\setminus\cup_{0\leq j<m}s(\delta_j)\times X\setminus\cup_{0\leq j<m}r(\delta_j)}$. The range and source maps r, s are countable to one Borel maps when restricted to R^m. By Lusin-Novikov uniformization, there exists a partition $R^m = \sqcup_j R_j$ into countably many disjoint sets such that r is injective on R_j.

If $\mu(s(R)) > 0$, there exists J such that $\mu(s(R_J)) > 0$ since $s(R) = \cup_j s(R_j)$. Use Lusin-Novikov to get a complete section δ_m of $s|_{R_j}$. Note that both r and s are injective on $\cup_{k<m}\delta_k$.

If $\mu(s(R)) = 0$, then set $\sigma_\kappa = \sqcup_{k<m}\delta_k$. By measure exhaustion, this step will occur at some countable ordinal m and so σ_κ is a countable union of Borel sets mod null. This is bijective onto $s(R^0_\kappa)$ and $r(R^0_\kappa)$ under the maps s and r, respectively. Note that $\mathcal{H}\sigma_i \cap \mathcal{H}\sigma_j = \emptyset$ for $i < j$ by construction.

All that's left is to show that $r(R_n^0)$ and $s(R_n^0)$ either both have full or both have null measure. In the first case, the argument runs as is and we get a new element σ_n in our system. In the second case, $\mathcal{G} = \cup \mathcal{H}\sigma_i$ and so we are done. In order to proceed, we show $r(R_n^0)$ is invariant and, hence, either full or null measure by ergodicity of $\mathcal{H}$.

Let $A := r^{-1}(r(R_\kappa^0)^C)$. Note that $A \subseteq \sqcup_{i<\kappa}\mathcal{H}\sigma_i$, and so $\mathcal{H} \cdot A \subseteq \sqcup_{i<\kappa}\mathcal{H}\sigma_i$. By definition, $R_\kappa^0 \cap \sqcup_{i<\kappa}\mathcal{H}\sigma_i = \emptyset \implies R_\kappa^0 \cap \mathcal{H} \cdot A = \emptyset \implies A \cap \mathcal{H} \cdot R_\kappa^0 = \emptyset$. This means that

$$\forall h \in \mathcal{H}\, \forall \xi \in R_\kappa^0\, (h\xi \notin A)$$

and hence

$$\forall h \in \mathcal{H}\, \forall \xi \in R_\kappa^0\, (h \cdot r(\xi) = r(h\xi) \notin r(R_\kappa^0)^C).$$

Therefore, $r(R_\kappa^0)$ is invariant. Similarly, $s(R_\kappa^0)$ is invariant. Since $\mathcal{G}$ is pmp, $0 < \int_{\mathcal{G}^0} c_x^r(R_\kappa^0)d\mu = \int_{\mathcal{G}^0} c_x^s(R_\kappa^0)d\mu$ and so $\mu(s(R_\kappa^0)) = 0$ if and only if $\mu(r(R_\kappa^0)) = 0$. $\square$

It is worth noting that this lemma is quite false without the assumption of ergodicity.

3.4 Proof of main theorem

We now prove the main theorem. Much of the technical difficulties in the proof arise due to the fact that we define our ultimate inner amenability sequence in terms of a system of coset representatives and this sytem of representatives changes when we conjugate by an element of the full group.

Theorem 3.4.1. *Let $\mathcal{H}$ be an ergodic inner amenable coamenable subgroupoid of $\mathcal{G}$. Then $\mathcal{G}$ is inner amenable.*

Proof. Let h_n witness the inner amenability of $\mathcal{H}$. We treat h_n as defined on all of $\mathcal{G}$ by defining it to be zero on the complement of $\mathcal{H}$. By assumption, the action $\mathcal{G} \curvearrowright \mathcal{H} \setminus \mathcal{G}$ is amenable so take a sequence f_n witnessing the amenability.

Fix a set of coset representatives $\Sigma = (\sigma_i)$ for $\mathcal{H} \leq \mathcal{G}$. Define a map $n : \mathcal{G} \to \mathbb{N}$ by $g \in \mathcal{H}\sigma_{n(g)}$. This map is well-defined by the definition of coset representative.

Let $\varepsilon > 0$. Choose a finite collection of bisections $T = \{\tau_i\} \in [\mathcal{G}]$ and a set $\mu^1(D) < \infty$. We will find a function $\xi \in L^1(\mathcal{G})$ such that

1. $\|\xi^\tau - \xi\|_1 \leq \varepsilon$ for every $\tau \in T$
2. $\|1_D \xi\|_1 \leq \varepsilon$
3. $\sum_{s(g)=x} \xi(g) = 1$ for almost every $x \in \mathcal{G}^0$

Take $f = f_n \in L^1(\mathcal{H} \setminus \mathcal{G})$ with n large enough such that

- $\sum_{p(\mathcal{H}\gamma)=x} f(\mathcal{H}\gamma) = 1$ for every $x \in \mathcal{G}^0$
- $\|\tau \cdot f - f\| \leq \frac{\varepsilon}{3}$ for every $\tau \in T$

Fix a finite subset of the coset representatives $\Delta \subseteq \Sigma$ such that $\int_{(\mathcal{H}\setminus\mathcal{G})\setminus(\cup_\Delta \mathcal{H}\delta)} f \leq \frac{\varepsilon}{100}$. Take $h = h_n \in L^1(\mathcal{H})$ with n large enough such that for every $\tau \in T$ for every $\delta \in \Delta$

- $\sum_{s(\gamma)=x} h(\gamma) = 1$ for every $x \in \mathcal{G}^0$
- $\|\mathbb{1}_{D^\delta} h\|_1 \leq \frac{\varepsilon}{100|\Delta|}$
- $\int_x \sum_{s(g)=x} \left| h(\eta_{k(\delta x)} g \eta_{k(\delta x)}^{-1}) - \sum_j h(\eta_{k(\delta x)} B^j_{k(\delta x)} g (B^j_{k(\delta x)})^{-1} \eta^{-1}_{k(\delta x)}) \right| \leq \frac{\varepsilon}{100|\Delta|}$
- $\int_x \sum_{s(g)=x} \left| h(\sigma_{n(\delta x)} g \sigma_{n(\delta x)}^{-1}) - \sum_i h(\sigma_{n(\delta x)} B^{n(\delta x)}_i g (B^{n(\delta x)}_i)^{-1} \sigma^{-1}_{n(\delta x)}) \right| \leq \frac{\varepsilon}{100|\Delta|}$
- $\int_x \sum_{s(g)=x} \left| h(\rho^j_{k(\delta x)} \sigma_j B^j_{k(\delta x)} g (B^j_{k(\delta x)})^{-1} \sigma_j^{-1} (\rho^j_{k(\delta x)})^{-1}) - h(\sigma_j B^j_{k(\delta x)} g (B^j_{k(\delta x)})^{-1} \sigma_j^{-1}) \right|$
  $\leq \frac{\varepsilon}{100|\Delta||J^\tau_\delta|}$ for every $j \in J^\tau_\delta$
- $\int_x \sum_{s(g)=x} \left| h((\rho^{n(\delta x)}_i)^{-1} \eta_i B^{n(\delta x)}_i g (B^{n(\delta x)}_i)^{-1} \eta_i^{-1} \rho^{n(\delta x)}_i) - h(\eta_i B^{n(\delta x)}_i g (B^{n(\delta x)}_i)^{-1} \eta_i^{-1}) \right|$
  $\leq \frac{\varepsilon}{100|\Delta||I^\tau_\delta|}$ for every $i \in I^\tau_\delta$

where $\eta_i = \sigma_i \tau$ and k is defined by $\gamma \in \mathcal{H}\eta_{k(\gamma)}$. Here $\mathcal{G}^0 = \sqcup B^j_{k(\delta x)}$ where $B^j_i := \{x \in \mathcal{G}^0 | \mathcal{H}\sigma_j x = \mathcal{H}\eta_i x\}$. We also let $\rho^j_i \in [\mathcal{H}]$ such that $\rho^j_i \sigma_j B^j_i = \eta_i B^j_i$. We choose $J^\tau_\delta \subseteq \mathbb{N}$ beforehand to be a finite set such that

$$\sum_{j \in \mathbb{N} \setminus J^\tau_\delta} \int_x \sum_{s(g)=x} h(\rho^j_{k(\delta x)} \sigma_j B^j_{k(\delta x)} g (B^j_{k(\delta x)})^{-1} \sigma_j^{-1} (\rho^j_{k(\delta x)})^{-1}) \leq \frac{\varepsilon}{100|\Delta|}$$

and

$$\sum_{j\in\mathbb{N}\setminus J_\delta^\tau}\int_x\sum_{s(g)=x}h(\sigma_j B^j_{k(\delta x)}g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1})\leq\frac{\varepsilon}{100|\Delta|}$$

with I_δ^τ chosen mutatis mutandi.

It may not be immediately obvious that such an h exists. Looking carefully, there are only finitely many conditions to satisfy as we vary all the parameters so we just need to check that each condition is satisfied eventually. The first and second conditions follow immediately from conditions (iv) and (iii) in the definition of inner amenability. Similarly, the fifth and sixth points are directly satisfied due to condition (ii). The third and fourth points are less obviously satisfied due to condition (i). Indeed, note that if we have an inner amenability sequence h_i and $\mathcal{G}^0=\sqcup A_n$, then $\lim_i\|h_i\mathbb{1}_G\|=\mu(\mathcal{G}^0)=\sum_n\mu(A_n)=\sum_n\lim_i\|h_i\mathbb{1}_{G_{A_n}}\|=\lim_i\sum_n\|h_i\mathbb{1}_{G_{A_n}}\|$. Here the last interchange is by dominated convergence theorem. So

$$\forall\varepsilon\,\forall\text{ partition }\sqcup A_n=\mathcal{G}^0\,\exists\,h_i\text{ such that }\|h_i\|=_\varepsilon\sum_n\|h_i\mathbb{1}_{\mathcal{G}_{A_n}}\|$$

In particular, since h_i are positive, $\int_x\sum_{s(g)=x}\mathbb{1}_{\mathcal{G}\setminus(\sqcup\mathcal{G}_{A_n})}h_i(g)\leq\varepsilon$ which is exactly what the last condition is requesting.

We define ξ by

$$\xi(g):=\sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=s(g)}}h(\sigma_{n(\gamma)}g\sigma_{n(\gamma)}^{-1})f(\mathcal{H}\gamma)$$

This is immediately well-defined since $n(g)=n(hg)$ for $s(h)=r(g)$ and $h\in\mathcal{H}$. Why? Since $g\in\mathcal{H}\sigma_{n(g)}$, so $hg\in h\mathcal{H}\sigma_{n(g)}\subset\mathcal{H}\sigma_{n(g)}$.

We first check ξ satisfies 3. Fix $x\in\mathcal{G}^0$

$$\begin{aligned}\sum_{s(g)=x}\xi(g)&=\sum_{s(g)=x}\sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x}}h(\sigma_{n(\gamma)}g\sigma_{n(\gamma)}^{-1})f(\mathcal{H}\gamma)\\&=\sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x}}\sum_{s(g)=x}h(\sigma_{n(\gamma)}g\sigma_{n(\gamma)}^{-1})f(\mathcal{H}\gamma)\end{aligned}$$

$$= \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} \sum_{s(u)=\sigma_{n(\gamma)}x} h(u) f(\mathcal{H}\gamma)$$
$$= \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} f(\mathcal{H}\gamma) = 1$$

Now, we check that ξ satisfies 2. We have D, Δ and ε from before.

$$\begin{aligned}
\|1_D\xi\|_1 &= \int_{x\in\mathcal{G}^0} \sum_{\substack{s(g)=x \\ g\in D}} \xi(g)\, d\mu^0 \\
&= \int_{x\in\mathcal{G}^0} \sum_{\substack{s(g)=x \\ g\in D}} \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma)\, d\mu^0 \\
&= \int_{x\in\mathcal{G}^0} \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} \sum_{\substack{s(g)=s(\gamma) \\ g\in D}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma)\, d\mu^0 \\
&= \int_{\mathcal{H}\gamma\in\mathcal{H}\backslash\mathcal{G}} \sum_{\substack{s(g)=s(\gamma) \\ g\in D}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma)\, d\nu_{\mathcal{H}\backslash\mathcal{G}} \\
&= \sum_{\delta\in\Sigma} \int_{x\in\mathcal{G}^0} \sum_{\substack{s(g)=s(\delta x) \\ g\in D}} h(\sigma_{n(\delta x)} g \sigma_{n(\delta x)}^{-1}) f(\mathcal{H}\delta x)\, d\mu^0 \\
&= \sum_{\delta\in\Delta} \int_{x\in\mathcal{G}^0} \sum_{\substack{s(g)=x \\ g\in D}} h(\delta g \delta^{-1}) f(\mathcal{H}\delta x)\, d\mu^0 + \sum_{\delta\in\Sigma\backslash\Delta} \int_{x\in\mathcal{G}^0} \sum_{\substack{s(g)=x \\ g\in D}} h(\delta g \delta^{-1}) f(\mathcal{H}\delta x)\, d\mu^0 \\
&\leq \sum_{\delta\in\Delta} \int_{x\in\mathcal{G}^0} \sum_{\substack{s(u)=\delta x \\ u\in D^\delta}} h(u) f(\mathcal{H}\delta x)\, d\mu^0 + \sum_{\delta\in\Sigma\backslash\Delta} \int_{x\in\mathcal{G}^0} \sum_{s(g)=x} h(\delta g \delta^{-1}) f(\mathcal{H}\delta x)\, d\mu^0 \\
&\leq \sum_{\delta\in\Delta} \int_{x\in\mathcal{G}^0} \sum_{\substack{s(u)=\delta x \\ u\in D^\delta}} h(u)\, d\mu^0 + \sum_{\delta\in\Sigma\backslash\Delta} \int_{x\in\mathcal{G}^0} f(\mathcal{H}\delta x)\, d\mu^0 \\
&\leq |\Delta| \max_{\delta\in\Delta} \|\mathbb{1}_{D^\delta} h\| + \frac{\varepsilon}{100} < \varepsilon
\end{aligned}$$

Last, we check that ξ satisfies 1. Unfortunately, this ends up being rather involved. We have

$T = \{\tau_i\}$ and ε from before. Fix $\tau \in T$ and let $\eta_i := \sigma_i \tau$ and $k : \mathcal{G} \to \mathbb{N}$ be defined by $g \in \mathcal{H}\eta_{k(g)}$.

$$\begin{aligned}
\|\xi^\tau - \xi\|_1 &= \int_x \sum_{s(g)=x} \left|\xi(\tau g \tau^{-1}) - \xi(g)\right| d\mu^0 \\
&= \int_x \sum_{s(g)=x} \left| \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=s(\tau g \tau^{-1})}} h(\sigma_{n(\gamma)} \tau g \tau^{-1} \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma) - \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=s(g)}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma) \right| d\mu^0 \\
&\leq \int_x \sum_{s(g)=x} \left| \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma\tau)=x}} h(\eta_{k(\gamma\tau)} g \eta_{k(\gamma\tau)}^{-1}) f(\mathcal{H}\gamma) - \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) f(\mathcal{H}\gamma) \right| d\mu^0 \\
&\quad + \int_x \sum_{s(g)=x} \left| \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) f(\mathcal{H}\gamma) - \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma) \right| d\mu^0 \\
&= \int_x \sum_{s(g)=x} \left| \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) \left(f(\mathcal{H}\gamma\tau^{-1}) - f(\mathcal{H}\gamma)\right) \right| d\mu^0 \\
&\quad + \int_x \sum_{s(g)=x} \left| \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} \Big(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \Big) f(\mathcal{H}\gamma) \right| d\mu^0 \\
&\leq \int_x \sum_{\substack{\mathcal{H}\gamma \in \mathcal{H}\backslash\mathcal{G} \\ s(\gamma)=x}} \sum_{s(g)=x} h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) \left| f(\mathcal{H}\gamma\tau^{-1}) - f(\mathcal{H}\gamma) \right| d\mu^0 \\
&\quad + \int_x \sum_{s(g)=x} \left| \sum_{\substack{\delta \in \Delta \\ \gamma := \delta(x)}} \Big(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \Big) f(\mathcal{H}\gamma) \right| d\mu^0 \\
&\quad + \int_x \sum_{\substack{\delta \in \Sigma\backslash\Delta \\ \gamma := \delta(x)}} \sum_{s(g)=x} \left| h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \right| f(\mathcal{H}\gamma) d\mu^0 \\
&\leq \int_{\mathcal{H}\gamma} \left| f(\mathcal{H}\gamma\tau^{-1}) - f(\mathcal{H}\gamma) \right| d\nu_{\mathcal{H}\backslash\mathcal{G}} \\
&\quad + \int_x \sum_{s(g)=x} \left| \sum_{\substack{\delta \in \Delta \\ \gamma := \delta(x)}} \Big(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \Big) f(\mathcal{H}\gamma) \right| d\mu^0
\end{aligned}$$

$$+\int_x \sum_{\substack{\delta\in\Sigma\setminus\Delta \\ \gamma:=\delta(x)}} 2f(\mathcal{H}\gamma)d\mu^0$$

$$\leq \frac{\varepsilon}{3} + \int_x \sum_{s(g)=x} \left| \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \left(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \right) f(\mathcal{H}\gamma) \right| d\mu^0 + 2\frac{\varepsilon}{100}$$

Claim 3.4.2. $\int_x \sum_{s(g)=x} \left| \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \left(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \right) f(\mathcal{H}\gamma) \right| d\mu^0 \leq \frac{\varepsilon}{3}$

Proof of Claim. Let $B_i^j := \{x \in \mathcal{G}^0 \mid \mathcal{H}\sigma_j x = \mathcal{H}\eta_i x\}$. Define $\rho_i^j \in [\mathcal{H}]$ by

$$\rho_i^j \sigma_j B_i^j = \eta_i B_i^j$$

Now we consider the inside of our integral for a fixed g and $x = s(g)$

$$F(g,x) := \left| \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \left(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) \right) f(\mathcal{H}\gamma) \right|$$

$$\leq \left| \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \left(h(\eta_{k(\gamma)} g \eta_{k(\gamma)}^{-1}) - \sum_j h(\eta_{k(\gamma)} B_{k(\gamma)}^j g (B_{k(\gamma)}^j)^{-1} \eta_{k(\gamma)}^{-1}) \right) f(\mathcal{H}\gamma) \right| \tag{3.1}$$

$$+ \left| \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \left(\sum_j h(\rho_{k(\gamma)}^j \sigma_j B_{k(\gamma)}^j g (B_{k(\gamma)}^j)^{-1} \sigma_j^{-1} (\rho_{k(\gamma)}^j)^{-1}) - \sum_j h(\sigma_j B_{k(\gamma)}^j g (B_{k(\gamma)}^j)^{-1} \sigma_j^{-1}) \right) f(\mathcal{H}\gamma) \right| \tag{3.2}$$

$$+ \left| \sum_j \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} h(\sigma_j B_{k(\gamma)}^j g (B_{k(\gamma)}^j)^{-1} \sigma_j^{-1}) f(\mathcal{H}\gamma) \right. \tag{3.3}$$

$$\left. - \sum_j \sum_i \sum_{\substack{\delta\in\Delta \\ \gamma:=\delta(x)}} \mathbb{1}_{p^{-1}(B_i^j)}(\gamma) h(\sigma_j B_{k(\gamma)}^j g (B_{k(\gamma)}^j)^{-1} \sigma_j^{-1}) f(\mathcal{H}\gamma) \right|$$

$$+\left|\sum_j \sum_i \sum_{\substack{\delta\in\Sigma\setminus\Delta\\ \gamma:=\delta(x)}} \mathbb{1}_{p^{-1}(B_i^j)}(\gamma) h(\sigma_j B^j_{k(\gamma)} g (B^j_{k(\gamma)})^{-1}\sigma_j^{-1}) f(\mathcal{H}\gamma)\right| \tag{3.4}$$

$$+\left|\sum_j \sum_i \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x}} \mathbb{1}_{p^{-1}(B_i^j)}(\gamma) h(\sigma_j B^j_{k(\gamma)} g (B^j_{k(\gamma)})^{-1}\sigma_j^{-1}) f(\mathcal{H}\gamma)\right. \tag{3.5}$$

$$\left. - \sum_i \sum_j h(\sigma_j B_i^j g (B_i^j)^{-1}\sigma_j^{-1}) \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x\\ k(\gamma)=i}} f(\mathcal{H}\gamma)\right|$$

$$+\left|\sum_i \sum_j h(\sigma_j B_i^j g (B_i^j)^{-1}\sigma_j^{-1}) \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x\\ k(\gamma)=i}} f(\mathcal{H}\gamma) - \sum_i \sum_j h(\sigma_j B_i^j g (B_i^j)^{-1}\sigma_j^{-1}) \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x\\ n(\gamma)=j}} f(\mathcal{H}\gamma)\right| \tag{3.6}$$

$$+\left|\sum_i \sum_j \big(h((\rho_i^j)^{-1}\eta_i B_i^j g (B_i^j)^{-1}\eta_i^{-1}\rho_i^j) - h(\eta_i B_i^j g (B_i^j)^{-1}\eta_i^{-1})\big) \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x\\ n(\gamma)=j}} f(\mathcal{H}\gamma)\right| \tag{3.7}$$

$$+\left|\sum_i \sum_j h(\eta_i B_i^j g (B_i^j)^{-1}\eta_i^{-1}) \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x\\ n(\gamma)=j}} f(\mathcal{H}\gamma) - \sum_{\substack{\mathcal{H}\gamma\in\mathcal{H}\setminus\mathcal{G}\\ s(\gamma)=x}} h(\sigma_{n(\gamma)} g \sigma_{n(\gamma)}^{-1}) f(\mathcal{H}\gamma)\right| \tag{3.8}$$

We consider each line separately to get an upper bound. Not every line is bounded universally with respect to g and $x = s(g)$ but, by integrating each line we get an upper bound for $\int_x \sum_{s(g)=x} F(g,x)$.

We start with the first line and integrate to get

$$\int_x \sum_{s(g)=x} (eq.3.1) \leq \sum_{\delta\in\Delta} \int_x \sum_{s(g)=x} \left| h(\eta_{k(\delta x)} g \eta_{k(\delta x)}^{-1}) - \sum_j h(\eta_{k(\delta x)} B^j_{k(\delta x)} g (B^j_{k(\delta x)})^{-1} \eta_{k(\delta x)}^{-1}) \right|$$

$$\leq \sum_{\delta\in\Delta} \frac{\varepsilon}{100|\Delta|} \leq \frac{\varepsilon}{100}$$

Now we once more integrate to see $\int_x \sum_{s(g)=x}(eq.3.2)$

$$\leq \sum_{\delta\in\Delta}\sum_{j}\int_x \sum_{s(g)=x}\left|h(\rho^j_{k(\delta x)}\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1}(\rho^j_{k(\delta x)})^{-1}) - h(\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1})\right|$$
$$\leq \sum_{\delta\in\Delta}\sum_{j\in J^\tau_\delta}\int_x \sum_{s(g)=x}\left|h(\rho^j_{k(\delta x)}\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1}(\rho^j_{k(\delta x)})^{-1}) - h(\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1})\right|$$
$$+\sum_{\delta\in\Delta}\sum_{j\in\mathbb{N}\setminus J^\tau_\delta}\int_x \sum_{s(g)=x}\left|h(\rho^j_{k(\delta x)}\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1}(\rho^j_{k(\delta x)})^{-1}) - h(\sigma_j B^j_{k(\delta x)} g(B^j_{k(\delta x)})^{-1}\sigma_j^{-1})\right|$$
$$\leq \sum_{\delta\in\Delta}\sum_{j\in J^\tau_\delta}\frac{\varepsilon}{100|\Delta||J^\tau_\delta|} + \sum_{\delta\in\Delta}\frac{2\varepsilon}{100|\Delta|} \leq \frac{3\varepsilon}{100}$$

For a fixed γ, there is a unique i such that the characteristic function $\mathbb{1}_{p^{-1}(B^j_i)}(\gamma)$ is nonzero which means (eq. 3.3)$=0$. We get $\int_x \sum_{s(g)=x}$(eq. 3.4) $\leq \frac{\varepsilon}{100}$ by our choice of Δ and that $\sum_{s(g)=x} h(g) = 1$. The multiplication $g(B^j_{k(\gamma)})^{-1}$ doesn't exist if $k(\gamma)\neq i$, so (eq. 3.5)$=0$.

For (eq. 3.6), notice that since $s(g)=x$, then $h(\sigma_j B^j_i g(B^j_i)^{-1}\sigma_j^{-1}) = 0$ if $x \not\in B^j_i$. So we can restrict our considerations so that $s(\gamma)\in B^j_i$. Now $k(\gamma)=i$ means $\gamma\in\mathcal{H}\eta_i B^j_i = \mathcal{H}\sigma_j B^j_i$. This gives us that $k(\gamma)=i$ if and only if $n(\gamma)=j$ and so (eq.3.6)$=0$.

We first simplify (eq. 3.7) a little bit $\int_x \sum_{s(g)=x}$(eq. 3.7)

$$\leq \sum_{\delta\in\Sigma}\int_x\sum_{s(g)=x}\left|\sum_i\sum_j \mathbb{1}_{p^{-1}(B^j_i)}(\delta x)\Big(h((\rho_i^{n(\gamma)})^{-1}\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1}\rho_i^{n(\gamma)})\right.$$
$$\left.- h(\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1})\Big)\right| f(\mathcal{H}\delta x)$$
$$\leq \sum_{\delta\in\Delta}\int_x\sum_{s(g)=x}\left|\sum_i\Big(h((\rho_i^{n(\gamma)})^{-1}\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1}\rho_i^{n(\gamma)}) - h(\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1})\Big)\right|$$
$$+\sum_{\delta\in\Delta\setminus\Sigma} 2\|h\|_1 f(\mathcal{H}\delta x)$$
$$\leq \sum_{\delta\in\Delta}\sum_{i\in I^\tau_\delta}\int_x\sum_{s(g)=x}\left|h((\rho_i^{n(\gamma)})^{-1}\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1}\rho_i^{n(\gamma)}) - h(\eta_i B_i^{n(\gamma)} g(B_i^{n(\gamma)})^{-1}\eta_i^{-1})\right|$$

$$+\sum_{\delta\in\Delta}\sum_{i\in\mathbb{N}\setminus I_\delta^\tau}\int_x\sum_{s(g)=x}\Big|h((\rho_i^{n(\gamma)})^{-1}\eta_i B_i^{n(\gamma)} g (B_i^{n(\gamma)})^{-1}\eta_i^{-1}\rho_i^{n(\gamma)})$$

$$- h(\eta_i B_i^{n(\gamma)} g (B_i^{n(\gamma)})^{-1}\eta_i^{-1})\Big| + \frac{2\varepsilon}{100}$$

$$\leq \sum_{\delta\in\Delta}\sum_{i\in I_\delta^\tau}\frac{\varepsilon}{100|\Delta||I_\delta^\tau|} + \sum_{\delta\in\Delta}\frac{\varepsilon}{100|\Delta|} + \frac{2\varepsilon}{100} \leq \frac{4\varepsilon}{100}$$

The last step works mutatis mutandi like 3.1 through 3.6, to get

$$\int_x \sum_{s(g)=x}(eq.3.8) \leq \frac{5\varepsilon}{100}$$

giving us

$$\int_x \sum_{s(g)=x} F(g,x) \leq \frac{14\varepsilon}{100}$$

and we have proven the claim. □

Using the claim, we found a measurable function $\xi : \mathcal{G} \to [0,1]$ that satisfies conditions 1,2, and 3. These conditions correspond to finding a nontrivial diffuse conjugation-invariant net of measurable functions. Hence, $\mathcal{G}$ is inner amenable by remark 3.3.4. □

4. ORBIT EQUIVALENCE OF WREATH PRODUCTS

4.1 Introduction

Two groups are said to be orbit equivalent if and only if they admit free probability measure preserving (p.m.p.) ergodic actions on a standard probability space that generate the same orbit equivalence relation. It is easy to observe that finite groups of a given order form an orbit equivalence class. A much more remarkable result due to Ornstein and Weiss is that all infinite amenable groups are orbit equivalent[39]. Some additional work shows that these too form their own orbit equivalence class. In the last 20 years, there's been significant progress in the development of various rigidity results, one of the most celebrated of which is Popa's cocycle superrigidity theorem which implies that any action that is orbit equivalent to a Bernoulli shift of a wide variety of groups(including icc groups either with property(T) or which are nonamenable direct products of infinite groups) must, in fact, be isomorphic to the aforementioned Bernoulli shift[44][45]. At the other end of the spectrum, results showing algebraically dissimilar groups are orbit equivalent appear to be less common with the prominent exception of Ornstein and Weiss. We prove some antirigidity results in the nonamenable setting.

Theorem 4.1.1. *Let Γ be a countable group that contains an infinite amenable group as a free factor. The groups $A \wr \Gamma$ and $B \wr \Gamma$ are orbit equivalent for all finite groups A, B.*

In particular, this implies $C_2 \wr \mathbb{F}_2$ is orbit equivalent to $C_3 \wr \mathbb{F}_2$. This was previously unknown although a consequence of work of Lewis Bowen implies that the group von Neumann algebras $L(C_2 \wr \mathbb{F}_2) \cong L(C_i \wr \mathbb{F}_2)$ are isomorphic[4]. We do this by showing the canonical wreath product actions are orbit equivalent. By contrast, we also show the following rigidity. This rigidity should be viewed as complimentary to the rigidity found in the compact setting by Chifan, Popa, and Sizemore[6].

Theorem 4.1.2. *If Γ is a sofic Bernoulli superrigid group with no nontrivial finite normal subgroups and $|A| \neq |B|$, then the wreath product actions of $A \wr \Gamma$ and $B \wr \Gamma$ are not stably orbit* *equivalent.*

To study these things, we introduce the notion of cofinitely equivariant maps between shift spaces. Cofinitely equivariant maps have appeared implicitly in the study of finitary isomorphisms between shifts[34]. For $x, y \in S^\Gamma$, we use $x \sim y$ to denote that x and y differ in only finitely many coordinates. A map between shift spaces $\phi : (A^\Gamma, \mu^\Gamma) \to (B^\Gamma, \nu^\Gamma)$ is said to be cofinitely equivariant if it is a measure isomorphism such that for every $x \sim y \in A^\Gamma$ and for every $\gamma \in \Gamma$, equivariance holds modulo finite errors $\gamma \cdot \phi(x) \sim \phi(\gamma \cdot y)$. In fact, for certain groups we are able to give a complete classification of when two shift spaces admit a cofinitely equivariant map between them.

4.2 Cofinitely equivariant maps

In the sequel, let Γ be a discrete countable group. Given an ergodic quasi-probability measure preserving (quasi-pmp) action $\Lambda \curvearrowright (X, \mu)$, define an associated quasi-pmp action of the wreath product $\Lambda \wr \Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$ where $(\lambda, \gamma) \cdot f(\omega) \mapsto \lambda(\gamma^{-1}\omega) \cdot f(\gamma^{-1}\omega)$.

Proposition 4.2.1. *Let $\Lambda_1 \curvearrowright (X, \mu)$ and $\Lambda_2 \curvearrowright (Y, \nu)$ be orbit equivalent free ergodic actions. The associated actions of the wreath products, $\Lambda_1 \wr \Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$ and $\Lambda_2 \wr \Gamma \curvearrowright (Y^\Gamma, \nu^\Gamma)$ are orbit equivalent.*

Proof. Let $\phi : (X, \mu) \to (Y, \nu)$ be a measure isomorphism that witnesses the orbit equivalence between $\Lambda_1 \curvearrowright (X, \mu)$ and $\Lambda_2 \curvearrowright (Y, \nu)$. The map $\phi^\Gamma : (X^\Gamma, \mu^\Gamma) \to (Y^\Gamma, \nu^\Gamma)$ witnesses the orbit equivalence between the wreath product actions. □

Type is an invariant of orbit equivalence of actions first defined by Krieger[33] for $\mathbb{Z}$-actions although the definition works perfectly well for actions of countable discrete groups. Krieger showed that type completely classifies orbit equivalence of free ergodic actions of $\mathbb{Z}$.

Theorem 4.2.2 (Dye, Krieger, Ornstein-Weiss)**.** *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be free ergodic actions of amenable groups. Then these actions are orbit equivalent if and only if they have the same type and $|\Gamma| = |\Lambda|$.*

When A is an amenable group, this means we have a canonical (up to orbit equivalence) action of the wreath product $A \wr \Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$ that only depends on the cardinality of A and the type of the wreath product action.

For $x, y \in S^\Gamma$ and a fixed action $\Lambda \curvearrowright S$, we use $x \sim y$ to denote that x and y differ in only finitely many coordinates and for each coordinate i in which they differ, there exists $\lambda \in \Lambda$ such that $\lambda x_i = y_i$.

Definition 4.2.3. A map between shift spaces $\phi : (X^\Gamma, \mu^\Gamma) \to (Y^\Gamma, \nu^\Gamma)$ is said to be **cofinitely equivariant** if it is a measure isomorphism such that for every $x \sim y \in X^\Gamma$ and for every $\gamma \in \Gamma$ we have $\gamma \cdot \phi(x) \sim \phi(\gamma \cdot y)$.

It should be noted that an equivariant map is, in general, not necessarily cofinitely equivariant in our sense. We say two shift spaces are **cofinitely equivariant** if there exists a cofinitely equivariant map between them.

Proposition 4.2.4. *Let A, B be amenable groups with actions $A \curvearrowright X$ and $B \curvearrowright Y$. If (X^Γ, μ^Γ) and (Y^Γ, ν^Γ) are cofinitely equivariant then the wreath product actions $A \wr \Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$ and $B \wr \Gamma \curvearrowright (Y^\Gamma, \mu^\Gamma)$ are orbit equivalent.*

Proof. Let $\phi : (X^\Gamma, \mu^\Gamma) \to (Y^\Gamma, \nu^\Gamma)$ be a cofinitely equivariant map. We claim this is an orbit equivalence. All that needs to be checked is that orbits are bijectively mapped to orbits. This follows from the fact that the $\sim$ equivalence relation is the orbit equivalence relation for the action of $\bigoplus_\Gamma A$ and for $\bigoplus_\Gamma B$. $\square$

Associate to each shift space over a finite alphabet a quantity called the **type** of that shift space. As was mentioned earlier, each shift space has an associated quasi-pmp action. We define the type of the shift space as the type of the associated wreath product action. The fact that this is invariant for cofinite equivariance follows from type of an action being an invariant of orbit equivalence.

For shift spaces, by a minor calculation, there is the following characterization of type. For a probability measure μ supported on finitely many points X, define

$$tp(\mu) := \{r \in \mathbb{R} \mid \text{ there exist } x, y \in A \text{ with } \frac{\mu(x)}{\mu(y)} = r\}$$

In general, if $A \curvearrowright (X, \mu)$ is a free ergodic action of a countable group, we let $\rho : A \times X \to \mathbb{R}^+$ be the unique cocycle such that for every $a \in A$ and Borel $C \subseteq X$, we have $\mu(a \cdot C) = \int_C \rho(a, x) d\mu(x)$. Now, for convenience we denote the statement $P_r :=$ (for every $\varepsilon > 0$ there exists a positive measure set $C \subseteq X$ and $a \in A$ with $|\rho(a, x) - r| < \varepsilon$ for $x \in C$) and define

$$tp(\mu) := \{r \in \mathbb{R} \mid P_r\}.$$

We now close this in the standard topology to get the type

$$TYPE(\mu) := \overline{< tp(\mu) >}^{\mathbb{R}}.$$

There are three possible outcomes when calculating the type of a shift space. We list them

- II_1 where $TYPE(\mu) = \{1\}$
- III_1 where $TYPE(\mu) = (0, \infty)$
- III_λ where $TYPE(\mu) = \{\lambda^n \mid n \in \mathbb{Z}\}$

with the names corresponding to the classical setting.

In particular, type II_1 corresponds to the associated action being measure preserving and to case of the measure on the shift space being uniform.

Type turns out to be a complete invariant for cofinite equivariance of shift spaces of Γ where Γ is amenable. The proof of this reduces to the results of Golodets-Sinel'shchikov and Feldman-Sutherland-Zimmer which we list here. However, we first need a couple of definitions to understand the statements of these theorems.

Definition 4.2.5. The **full group of an equivalence relation** $\mathcal{R}$ is defined

$[\mathcal{R}] := \{T \in \text{ Aut}(X, \mu) | (x, Ty) \in \mathcal{R} \text{ for } \mu - \text{almost every } (x, y)\}/\mu.$

Definition 4.2.6. The **automorphism group of the equivalence relation** is

$N[\mathcal{R}] := \{T \in \text{ Aut}(X, \mu) | (Tx, Ty) \in \mathcal{R} \text{ for } \mu - \text{almost every } (x, y)\}/\mu.$

The group $N[\mathcal{R}]$ is also referred to as the normalizer of the full group since it consists exactly of the elements of Aut(X, μ) that conjugate the full group to itself. In particular, $[\mathcal{R}]$ is a normal subgroup of $N[\mathcal{R}]$.

Definition 4.2.7. Let $\mathcal{R}$ be an equivalence relation and Γ a countable discrete. A **cocycle** of $\mathcal{R}$ into Γ is a function $\alpha : \mathcal{R} \to \Gamma$ such that $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$.

Definition 4.2.8. Let $\mathcal{R}$ be an equivalence relation on (X, μ). Two cocycles $\alpha, \beta : \mathcal{R} \to \Gamma$ are said to be **weakly conjugate** if there exists $\theta \in N[\mathcal{R}]$ and $f : X \to \Gamma$ such that

$$f(y)\alpha(y, x)f(x)^{-1} = \beta(\theta y, \theta x).$$

For the rest of the section, the equivalence relation $\mathcal{R}$ is assumed to be hyperfinite and not necessarily pmp.

Theorem 4.2.9 (Golodets-Sinel'shchikov[22])**.** *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation and let $\alpha, \beta : \mathcal{R} \to \Gamma$ be two surjective cocycles into a countable discrete group G. Then α and β are weakly conjugate.*

Definition 4.2.10. Let $\mathcal{R} \subseteq \mathcal{S}$ be an inclusion of equivalence relations. Assume $\mathcal{R}$ is ergodic and let $\epsilon : N[\mathcal{R}] \to N[\mathcal{R}]/[\mathcal{R}]$ be the quotient map. We say $\mathcal{R}$ is **normal** in $\mathcal{S}$ if there exists a countable group Q and map $\phi : Q \to N[\mathcal{R}]$ such that $\epsilon \circ \phi$ is a faithful homomorphism and $\mathcal{S} = \{(x, y) \in X \times X | (x, \phi(q)(y)) \in \mathcal{R} \text{ for some } q \in Q\}$.

The group Q is actually uniquely determined (up to isomorphism) by $\mathcal{R} \subseteq \mathcal{S}$ and we say $\mathcal{S}/\mathcal{R} \cong Q$.

Theorem 4.2.11 (Feldman-Sutherland-Zimmer[14])**.** *Let $\mathcal{S}$ be a hyperfinite equivalence relation. Let $\mathcal{R}, \mathcal{R}'$ be normal ergodic subrelations of $\mathcal{S}$ such that $\mathcal{S}/\mathcal{R} \cong \mathcal{S}/\mathcal{R}' \cong Q$ for some countable group Q. If $\mathcal{S}$ is type II_1 or $\mathcal{R}$ is type III, then $\mathcal{R} = \theta(\mathcal{R}')$ where θ is some automorphism of $\mathcal{S}$.*

The proof of this theorem relies on Golodets-Sinel'shchikov.

Now, we apply these theorems to get a complete classification of shift spaces of amenable groups.

Theorem 4.2.12. *Let Γ be amenable and A, B be two amenable groups acting on X, Y respectively. Let (X^Γ, μ^Γ) and (Y^Γ, ν^Γ) be two shift spaces. There exists a cofinitely equivariant map between them if and only if $type(\mu) = type(\nu)$.*

Proof. Assume $type(\mu) = type(\nu)$. Consider the amenable groups $G := A \wr \Gamma = \bigoplus_\Gamma A \rtimes \Gamma$ and $H := \bigoplus_\Gamma B \rtimes \Gamma$ along with the normal subgroups $\bigoplus_\Gamma A$ and $\bigoplus_\Gamma B$. The wreath products are amenable and the shift spaces have the same type so the wreath product actions generate hyperfinite equivalence relations of the same type. By Ornstein-Weiss, we may assume they both generate the hyperfinite equivalence relation $\mathcal{S}$. Now, consider the orbit equivalence relations $\mathcal{R}_A$ and $\mathcal{R}_B$ generated by the actions of $\bigoplus_\Gamma C_A$ and $\bigoplus_\Gamma C_B$. These are normal ergodic equivalence relations such that $\mathcal{S}/\mathcal{R}_A \cong \mathcal{S}/R_B \cong \Gamma$. In this situation, $type(\mathcal{R}_A) = type(\mathcal{S}) = type(\mathcal{R}_B)$.

Now, applying Feldman-Sutherland-Zimmer gives us an automorphism of $\mathcal{S}$ sending $\mathcal{R}_A$ to $\mathcal{R}_B$. But $\mathcal{R}_A$ and $\mathcal{R}_B$ are just the $\sim$ equivalence relations. By letting Γ act on the quotient in the natural way, we get that this automorphism is a cofinitely equivariant map.

For the other direction, a cofinite equivariance of shift spaces gives an orbit equivalence of the associated actions, but type is an OE invariant by Krieger[33]. $\square$

Theorem 4.2.13. *Let $\phi : (X^\Lambda, \mu^\Lambda) \to (Y^\Lambda, \nu^\Lambda)$ be an cofinitely equivariant map. If Λ is a free factor of the group Γ, then there is an cofinitely equivariant map $\phi' : (X^\Gamma, \mu^\Gamma) \to (Y^\Gamma, \nu^\Gamma)$.*

Proof. First, $\Gamma = \Lambda * \Delta$ for some group Δ, so every element $\gamma \in \Gamma$ can be written uniquely as a reduced product of elements from Λ and Δ, i.e. $\gamma = \delta_1\lambda_2 \cdots \delta_{k-1}\lambda_k$ where $\delta_i \neq e \neq \lambda_i$ if

$1 < i < k$. We will often abuse notation and refer to the identity coset of $\Lambda, \Delta \subseteq \Gamma$ as Λ, Δ respectively.

Consider the action of Γ on itself by left translation. Define $\pi : \Gamma \to \Lambda$ by $\pi(\gamma) = \lambda_k$. Observe that $\pi|_{g\Lambda}$ is a bijection for every coset $g\Lambda$.

Throughout the rest of the proof a will be an element of Λ and b will be an element of Δ.

For $\gamma \notin \Lambda$, the following occurs because $\delta_{k-1} \neq e$,

$$\pi(a\gamma) = \pi(a\delta_1\lambda_2 \cdots \delta_{k-1}\lambda_k) = \lambda_k = \pi(a\delta_1\lambda_2 \cdots \delta_{k-1}\lambda_k) = \pi(\gamma) \tag{4.1}$$

and, for every $\gamma \in \Gamma$,

$$\pi(b\gamma) = \pi((b\delta_1)\lambda_2 \cdots \delta_{k-1}\lambda_k) = \lambda_k = \pi(\delta_1\lambda_2 \cdots \delta_{k-1}\lambda_k) = \pi(\gamma) \tag{4.2}$$

We will also define an auxiliary map $\psi : X^\Gamma \times \Gamma \to X^\Lambda$ in the following manner. Fix $x \in X^\Gamma$ and an element $\gamma \in \Gamma$. Since π is a bijection on each coset, we will write π_γ^{-1} for the inverse of $\pi|_{\gamma\Lambda}$. Define $\psi(x,\gamma) \in X^\Lambda$ by $\psi(x,\gamma)(\lambda) := x(\pi_\gamma^{-1}(\lambda))$.

By definition of π, we get the equality $g^{-1}[\pi_{g\gamma}^{-1}(\lambda)] = \pi_\gamma^{-1}(\lambda)$. Following from (4.1) and (4.2), we get $b^{-1}\pi_{b\gamma}^{-1}(b\gamma) = \pi_\gamma^{-1}(\gamma)$ and for $\gamma \notin e\Lambda = \langle a \rangle$, that $a^{-1}\pi_{a\gamma}^{-1}(a\gamma) = \pi_\gamma^{-1}(\gamma)$. And hence we get the following equalities

$$\psi(b \cdot x, b\gamma) = (b \cdot x)(\pi_{b\gamma}^{-1}(b\gamma)) = x(\pi_\gamma^{-1}(\gamma)) = \psi(x,\gamma)$$

for every $\gamma \in \Gamma$ and similarly

$$\psi(a \cdot x, a\gamma) = \psi(x,\gamma)$$

when $\gamma \notin e\Lambda$.

For $\gamma \in \Lambda$, however, $a \cdot \psi(x,\gamma) = \psi(a \cdot x, \gamma)$.

We now define $\phi'(x)$ by $\phi'(x)(\gamma) := \phi(\psi(x,\gamma))(\pi(\gamma))$. In essence, we just apply ϕ to x on each coset independently. First, we check this ϕ' is cofinitely equivariant; it is enough to check for

$g \in \Lambda \cup \Delta$. Let $x \sim y \in X^\Gamma$. We want to show that $g \cdot \phi'(x)$ and $\phi'(g \cdot y)$ for $g \in \Lambda \cup \Delta$ differ in only finitely many coordinates. Again, a will be an element of Λ and b and element of Δ.

Since x and y differ in at most finitely many coordinates, they only differ on at most finitely many cosets of Λ. Call the collection of cosets on which they differ S.

Note that for every coset $\gamma'\Lambda \notin aS \cup \{e\mathbb{Z}\}$ and element $\gamma \in \gamma'\Lambda$

$$a \cdot \phi'(x)(\gamma) = \phi(\psi(x, a^{-1}\gamma))(\pi(a^{-1}\gamma)) = \phi(\psi(y, a^{-1}\gamma))(\pi(\gamma)) = \phi(\psi(a \cdot y, \gamma))(\pi(\gamma)) = \phi'(a \cdot y)(\gamma).$$

Similarly, for every coset $\gamma'\Lambda \notin bS$ and $\gamma \in \gamma'\Lambda$,

$$b \cdot \phi'(x)(\gamma) = \phi'(b \cdot y)(\gamma).$$

As for one of the finitely many cosets $\gamma\Lambda \in aS \setminus \{e\Lambda\}$, we have $\psi(a \cdot x, \gamma) \sim \psi(a \cdot y, \gamma)$. So, by assumption $\phi(\psi(a \cdot x, \gamma)) \sim \phi(\psi(a \cdot y, \gamma))$. Following the outline above,

$$a \cdot \phi'(x)(\gamma) = \phi(\psi(x, a^{-1}\gamma))(\pi(\gamma)) \sim \phi(\psi(y, a^{-1}\gamma))(\pi(\gamma)) = \phi'(a \cdot y)(\gamma)$$

and similarly for $\gamma\Lambda \in bS$, giving $b \cdot \phi'(x)(\gamma) \sim \phi'(b \cdot x)(\gamma)$

The last situation is $\gamma\Lambda = \Lambda$ and comparing $a \cdot \phi'(x)(\gamma)$ and $\phi'(a \cdot y)(\gamma)$. We have that $\psi(x, \gamma) \sim \psi(y, \gamma)$. So calculating

$$a \cdot \phi'(x)(\gamma) = a \cdot \phi(\psi(x, \gamma))(\pi(\gamma)) \sim \phi(a \cdot \psi(y, \gamma))(\pi(\gamma)) = \phi(\psi(a \cdot y, \gamma))(\pi(\gamma)) = \phi'(a \cdot y)(\gamma)$$

gives us $g \cdot \phi'(x) \sim \phi'(g \cdot y)$ for $g \in \Gamma$.

All that's left is to show that ϕ' is a measure isomorphism. It is clearly a bijection, so we only need to show it pushes forward μ^Γ to ν^Γ. Pick a cylinder set $C \subset X^\Gamma$. This set is characterized by a finite set $F_C \subseteq \Gamma$ and a function $F_C \to X$. Decompose $F_C = \bigsqcup_S F_C^s$ where $F_C^s \subseteq s\Lambda$. Now, we have cylinder sets $C_s \subseteq X^\Gamma$ characterized by the finite sets F_C^s and the restriction of $F_C \to A$ and

$C = \bigcap_S C_s$. The sets C_s are all independent and so we only need to show ϕ_Γ^S is sends μ^Γ to ν^Γ on cylinder sets C where $F_C \subseteq s\Lambda$.

Consider the σ-algebra Σ generated by cylinder sets C where $F_C \subseteq s\Lambda$. The algebra $(X^\Gamma, \Sigma, \mu^\Gamma)$ is measure isomorphic to (X^Λ, μ^Λ) in the natural way and ϕ' acts as ϕ on Σ. Hence, it pushes forward μ^Λ to ν^Λ. But, $\mu^\Gamma|_\Sigma = \mu^\Lambda|_\Sigma$ and $\nu^\Gamma|_\Sigma = \nu^\Lambda|_\Sigma$. Thus, ϕ' is a measure isomorphism. □

Theorem 4.2.14. *Assume $\Lambda \leq \Gamma$ is a finite index subgroup. If there is a cofinitely equivariant $\phi_\Lambda : (X^\Lambda, \mu^\Lambda) \to (Y^\Lambda, \nu^\Lambda)$, then there is a cofinitely equivariant map $\phi_\Gamma : (X^\Gamma, \mu^\Gamma) \to (Y^\Gamma, \nu^\Gamma)$.*

Proof. We proceed similarly to the previous theorem. For a set of coset representatives S so that $\Gamma = \bigsqcup_{s\in S} s\Lambda$, we can define maps $s : \Gamma \to S, \pi_S : \Gamma \to \Lambda$ by $\gamma = s(\gamma)\pi_S(\gamma)$. Since π_S is a bijection on each coset, write $\pi_{S,\gamma}^{-1} := (\pi_S|_{\gamma\Lambda})^{-1}$. Now we can define $\psi_S : X^\Gamma \times \Gamma \to X^\Lambda$ by $\psi_S(x,\gamma)(\lambda) := x(\pi_{S,\gamma}^{-1}(\lambda))$.

Before continuing, we make a couple of observations. For a fixed δ, we can define a new set of coset representatives $T := \delta^{-1}S$. Now, $\delta t(\gamma) = s(\delta\gamma)$ and $\pi_S(\delta\gamma) = \pi_T(\gamma)$. Calculating, we get

$$\pi_{S,\delta\gamma}^{-1}(\lambda) = s(\delta\gamma)\lambda = \delta t(\gamma)\lambda = \delta(\pi_{T,\gamma}^{-1}(\lambda))$$

and hence, $\psi_S(\delta x, \delta\gamma) = \psi_T(x,\gamma)$.

We can define $\phi_\Gamma^S(x)(\gamma) := \phi_\Lambda\left(\psi_S(x,\gamma)\right)(\pi_S(\gamma))$. Note that if $T = \delta^{-1}S$, then a simple calculation gives $\phi_\Gamma^T(x)(\gamma) = \phi_\Gamma^S(\delta x)(\delta\gamma)$. Let $x \sim y$. We now observe that $\phi_\Gamma^T(y) \sim \phi_\Gamma^S(x)$. Indeed, there are only finitely many cosets, so we only need to check $\phi_\Gamma^T(y)|_{\gamma\Lambda} \sim \phi_\Gamma^S(x)|_{\gamma\Lambda}$ for a fixed γ. Before calculating, note that $s(\gamma\lambda)$ and $t(\gamma\lambda)$ are constant as λ varies and so is $\kappa := s(\gamma\lambda)^{-1}t(\gamma\lambda) \in \Lambda$. Hence, $\pi_T(\gamma\lambda) = \kappa\pi_S(\gamma\lambda)$ and $\pi_{T,\gamma\lambda}^{-1} = \pi_{S,\gamma\lambda}^{-1} \circ \kappa^{-1}$ giving $\kappa \cdot \psi_S(y,\gamma\lambda) = \psi_T(y,\gamma\lambda)$. Calculating we get

$$\phi_\Gamma^T(y)(\gamma\lambda) = \phi_\Lambda(\psi_T(y,\gamma\lambda))(\pi_T(\gamma\lambda)) = \phi_\Lambda(\kappa \cdot \psi_S(y,\gamma\lambda))(\pi_T(\gamma\lambda))$$

and

$$\phi_\Gamma^S(x)(\gamma\lambda) = \phi_\Lambda(\psi_S(x,\gamma\lambda))(\pi_S(\gamma\lambda)) = \kappa \cdot \phi_\Lambda(\psi_S(x,\gamma\lambda))(\pi_T(\gamma\lambda)).$$

Since $\psi_S(x,\gamma\lambda) \sim \psi_S(y,\gamma\lambda)$ and by assumption on ϕ_Λ, we get $\phi_\Gamma^T(y) \sim \phi_\Gamma^S(x)$. One last calculation

$$\delta^{-1} \cdot \phi_\Gamma^S(x)(\gamma) = \phi_\Gamma^S(x)(\delta\gamma) = \phi_\Gamma^T(\delta^{-1}x)(\gamma) \sim \phi_\Gamma^S(\delta^{-1}y)(\gamma)$$

tells us that $\delta \cdot \phi_\Gamma^S(x) \sim \phi_\Gamma^S(\delta \cdot y)$.

The proof that ϕ_Γ^S is a measure isomorphism is identical to the one given in the proof of theorem 4.2.13. $\square$

Let $\mathcal{C}$ be the collection of groups that contains amenable groups and is closed under finite index inclusions and taking free products with countable groups.

Corollary 4.2.15. *Let $\Gamma \in \mathcal{C}$. Type completely classifies cofinite equivariance of Γ shift spaces.*

Corollary 4.2.16. *Let $\Gamma \in \mathcal{C}$ and A, B be amenable groups. Then the natural pmp wreath product actions $A \wr \Gamma \curvearrowright X^\Gamma$ and $B \wr \Gamma \curvearrowright Y^\Gamma$ are orbit equivalent.*

Corollary 4.2.17. *Let Γ be orbit equivalent to a group in $\mathcal{C}$ and A, B be amenable groups. Then $A \wr \Gamma$ is orbit equivalent to $B \wr \Gamma$.*

Proof. This is a consequence of corollary 4.2.16 and [9, Corollary 7.4]. $\square$

4.3 Rigidity

As opposed to when $\Gamma \in \mathcal{C}$, there exist Γ where the groups $C_k \wr \Gamma$ and $C_l \wr \Gamma$ are not orbit equivalent via the primitive wreath product action.

Bowen and Tucker-Drob give the name **Bernoulli superrigid** to groups that satisfy the conclusion of Popa's cocycle superrigidity theorem and showed that Bernoulli superrigidity is an invariant of measure equivalence of groups[5].

Corollary 4.3.1 (Popa)**.** *Let $\Gamma \curvearrowright A^\Gamma$ be a Bernoulli shift of a Bernoulli superrigid group. Then for every cocycle $\omega : \Gamma \times A^\Gamma \to L$ taking values in a discrete countable group L, there exists a homomorphism $\rho : \Gamma \to L$ and measurable map $\theta : A^\Gamma \to L$ such that $c(\gamma, x) = \theta(\gamma x)\rho(\gamma)\theta(x)^{-1}$.*

We give some simple examples of Bernoulli superrigid groups and more can be found in the literature.

Example 4.3.2 (Popa[44][45]). Γ is Bernoulli superrigid if there is an infinite normal subgroup $N \triangleleft \Gamma$ such that one of the following holds

- (Γ, N) has relative property (T)
- $N = H \times K$ where H is infinite and K is nonamenable.

We now prove some lemmas that relate Bernoulli superrigidity and cocycles into a group ring.

Lemma 4.3.3. *Bernoulli superrigid groups have 1 end.*

Proof of Claim. This is because Bernoulli superrigid groups have vanishing first ℓ^2-Betti number as can be seen in [41, Corollary 3.3]. Bernoulli superrigid groups are necessarily nonamenable and hence, not 2-ended.

Assume for contradiction that G is Bernoulli superrigid and not 1-ended. By Stallings theorem[46] in the finitely generated case and Dicks-Dunwoody[10, Theorem IV.6.10] in general, if G is not 1-ended, then it must be infinitely ended and therefore satisfy one of the following

1. $G = C *_D E$ where $[C : D] > 2$ and $D \neq E$ and $|E| \leq C$ and $|D| < \infty$
2. $G = C*_D$ where $C \neq D$ and $|D| < \infty$
3. G is locally finite

For case 1, $\beta_1^{(2)}(D) = 0$ since D is finite,

$$\beta_1^{(2)}(C *_D E) \geq \beta_1^{(2)}(C) + \beta_1^{(2)}(E) - \frac{1}{|C|} - \frac{1}{|E|} + \frac{1}{|D|} > 0$$

since $\frac{2}{|E|} \leq \frac{1}{|D|}$ and $\frac{3}{|C|} \leq \frac{1}{|D|}$.

For case 2, $\beta_1^{(2)}(D) = 0$ since D is finite,

$$\beta_1^{(2)}(C*_D) \geq \beta_1^{(2)}(C) - \frac{1}{|C|} + \frac{1}{|D|} > 0$$

since $\frac{2}{|C|} \leq \frac{1}{|D|}$. Proofs of these well-known inequalities can be found in [42].

Case 3 can't occur since G is nonamenable. Hence, we arrive at a contradiction and Bernoulli superrigid groups must be 1-ended. $\square$

Lemma 4.3.4. *Let* Γ *be 1-ended and* B *abelian. Every additive cocycle of* Γ *into the group ring* $\bigoplus_\Gamma B$ *is a coboundary.*

Proof. This is the content of Dicks-Dunwoody[10, Theorem IV.6.10]. $\square$

Several parts of the following proof come directly from the previous work of Furman about SOE-superrigidity[16].

Theorem 4.3.5. *Let* Γ *be a sofic Bernoulli superrigid group with no nontrivial finite normal subgroups and let* A, B *be amenable groups with* $|A| \neq |B|$. *The pmp wreath product actions of* $A \wr \Gamma$ *and* $B \wr \Gamma$ *are not stably orbit equivalent.*

Proof. We prove the contrapositive. Assume without loss of generality that A, B are abelian and the actions $A \curvearrowright X$ and $B \curvearrowright Y$ are compact since the cardinality of the groups determines the orbit equivalence class of the pmp wreath product action. The existence of a stable orbit equivalence gives us two sets $X' \subseteq X^\Gamma, Y' \subseteq Y^\Gamma$ and a map $\phi' : (X', \mu') \to (Y', \nu')$ between the restricted measure spaces that sends the restricted orbits to restricted orbits. Enumerate Γ and by ergodicity we can define $\pi : X^\Gamma \to X'$ by $\pi(x) = \gamma x$ where γ is the least such that $\gamma x \in X'$. Now we get a map $\phi = \phi' \circ \pi : X^\Gamma \to Y'$ with $\phi_* \mu^\Gamma \sim \nu'$ which defines a cocycle $c : \Gamma \times X^\Gamma \to B \wr \Gamma$ by $c(\gamma, x)\phi(x) = \phi(\gamma x)$. By Bernoulli superrigidity, there exist a group homomorphism $\rho : \Gamma \to B \wr \Gamma$ and measurable map $\theta : X^\Gamma \to B \wr \Gamma$ such that $c(\gamma, x) = \theta(\gamma x)^{-1} \rho(\gamma) \theta(x)$.

Defining $\psi : X^\Gamma \to Y^\Gamma$ by $\psi(x) = \theta(x)\phi(x)$, we get

$$\begin{aligned}\psi(\gamma x) &= \theta(\gamma x)\phi(\gamma x) = \rho(\gamma)\theta(x)\phi(x) \\ &= \rho(\gamma)\theta(x)\phi(x) = \rho(\gamma)\psi(x)\end{aligned}$$

and $\psi_* \mu^\Gamma$ is absolutely continuous with respect to ν^Γ. Let

$$f(y) = \frac{d\psi_* \mu^\Gamma}{d\nu^\Gamma}(y), Y_1 = \{y \in Y | f(y > 0)\}, F = f \circ \psi.$$

The function $F : X^\Gamma \to \mathbb{R}$ is now a Γ-equivariant measurable function and, by ergodicity, $F(x) = \nu^\Gamma(Y_1)$ almost everywhere. So we can assume ψ lands in Y_1 and $\psi(\gamma x) = \rho(\gamma)\psi(x)$. Hence, Y_1 is invariant under the action of $\rho(\Gamma)$. We will later show that $\rho(\Gamma)$ acts on Y^Γ ergodically and so Y_1 will end up being the whole space.

We will now show ρ is injective. By countability, there exists $\lambda \in B \wr \Gamma$ such that $W := \theta^{-1}(\lambda) \cap X'$ has positive measure. Take $e \neq \gamma \in \ker \rho$. For $x \in W \cap \gamma^{-1} W$,

$$\lambda\phi(\gamma x) = \psi(\gamma x) = \psi(x) = \lambda\phi(x).$$

By freeness and since $x, \gamma x \in X'$, the measure $\mu^\Gamma(W \cap \gamma^{-1} W) = 0$ is null. Now, the collection of equal measure sets $\{\gamma^{-1} W\}_{\gamma \in \ker \rho}$ is pairwise disjoint. Hence, $\ker \rho \subseteq \Gamma$ is a finite normal subgroup and, by our assumption, trivial.

We show the map ψ is a measure isomorphism between invariant conull subsets of X^Γ and Y_1. Take a conull set $\hat{X}$ of $x \in X^\Gamma$ on which $\psi(\gamma x) = \rho(\gamma)\psi(x)$ holds for all $\gamma \in \Gamma$ and let $X_1 = \bigcap_\gamma \gamma \hat{X}$ be a Γ-invariant conull set. Since ρ is injective and $\psi(\gamma x) = \rho(\gamma)\psi(x)$, the map ψ is injective on individual orbits in X_1. Therefore ψ is injective on X_1. The map $\psi|_{\theta^{-1}(\{\lambda\})}$ is equal to the map $x \mapsto \lambda\phi(x)|_{\theta^{-1}(\{\lambda\})}$, which a Borel isomorphism and pushes μ^Γ forward to $\nu(Y_1)\nu$. Therefore, we see that ψ is a measure isomorphism from X_1 to $\psi(Y_1)$.

The homomorphism ρ decomposes as $\rho = (\rho_1, \rho_2)$ where $\rho_2 : \Gamma \to \Gamma$ is a homomorphism and $\rho_1 : \Gamma \to \bigoplus_\Gamma B$ is an additive 1-cocycle with respect to the Γ-action on $\bigoplus_\Gamma B$ given by $\gamma \cdot f(\lambda) \mapsto f(\rho_2(\gamma)^{-1}\lambda)$.

We now show the homomorphism ρ_2 is injective. Assume for contradiction ρ_2 is not injective. Hence, $\Gamma' := \ker \rho_2 = \{\gamma | \rho_2(\gamma) = e\}$ is nontrivial and therefore, infinite. The action $\Gamma' \curvearrowright (A^\Gamma, \mu^\Gamma)$ is a Bernoulli action, but $\rho(\Gamma') \leq \bigoplus_\Gamma B$ so the action of Γ' on (B^Γ, ν^Γ) implemented through ρ is compact. Here, ψ defines an isomorphism from a Bernoulli action to a compact action of the infinite group Γ', which is a contradiction.

Now consider the cocycle ρ_1. We aim to show that ρ_1 is a 1-coboundary. The Γ-module $\bigoplus_\Gamma B$ is isomorphic to $\bigoplus_{[\Gamma:\rho_2(\Gamma)]} \bigoplus_{\rho_2(\Gamma)} B$ as Γ-modules. The cocycle decomposes as $\rho_1 = \bigoplus_{[\Gamma:\rho_2(\Gamma)]} \rho_\gamma$ where $\rho_\gamma : \rho_2(\Gamma) \to \bigoplus_{\rho_2(\Gamma)} B$. By lemma 4.3.3 and injectivity of ρ_2, the group $\rho_2(\Gamma)$ is 1-ended. Hence, lemma 4.3.4 tells us that the 1-cocycles $\rho_\gamma : \rho_2(\Gamma) \to \bigoplus_{\rho_2(\Gamma)} B$ are all 1-coboundaries, i.e. there exist $\xi_\gamma \in \bigoplus_{\rho_2(\Gamma)} B$ such that $\rho_\gamma(\delta) = \delta\xi_\gamma - \xi_\gamma$. Define $\xi := \bigoplus_{[\Gamma:\rho_2(\Gamma)]} \xi_\gamma$.

In order to show ρ_1 is a 1-coboundary it remains to show that ξ is finitely supported. Since our group is Bernoulli superrigid, it is not amenable and, in particular, it is not locally finite. Hence, there is a finitely generated infinite subgroup $\Delta = \langle S \rangle \leq \rho_2(\Gamma)$. For each of the finitely many generators $s \in S$, for only finitely many γ the quantity $\rho_\gamma(s)$ is nontrivial. Call this set of $T_s \subseteq [\Gamma : \rho_2(\Gamma)]$. Hence, by the cocycle identity, there exists a finite subset $T = \bigcup_S T_s \subseteq [\Gamma : \rho_2(\Gamma)]$ such that for every $\delta \in \Delta$, the function $\rho_\gamma(\delta)$ is nontrivial only if $\gamma \in T$. Now take $\gamma \notin T$. For every $\delta \in \Delta$,

$$e = \rho_\gamma(\delta) = \delta\xi_\gamma - \xi_\gamma,$$

i.e. $\xi_\gamma = \delta\xi_\gamma$ for every $\delta \in \Delta$. But ξ_γ are finitely supported, hence must be trivial. Therefore, ξ_γ is nontrivial only if $\gamma \in T$ and the function ξ is actually finitely supported on $\bigcup_{\gamma \in T} \operatorname{supp}(\xi_\gamma)$. We conclude that $\xi \in \bigoplus_\Gamma B$ and $\rho_1(\gamma) = \gamma\xi - \xi$ showing that ρ_1 is a 1-coboundary.

We conjugate by (ξ, e) to get

$$\begin{aligned}(\xi, e)\rho(\gamma)(\xi, e)^{-1} &= (\xi, e)(\rho_2(\gamma) \cdot \xi - \xi, \rho_2(\gamma))(-\xi, e) \\ &= (\xi, e)(-\xi, \rho_2(\gamma)) = (e, \rho_2(\gamma)).\end{aligned}$$

Hence, without loss of generality, we can replace $\theta(x)$ by $(\xi, e)\theta(x)$ to assume that ρ is an embedding sending Γ to a subgroup of $\{e\} \times \Gamma$. Now, we notice that $\rho(\Gamma)$ acts on Y^Γ by the Bernoulli shift. This means that the positive measure invariant set Y_1 must be the entirety of Y^Γ.

But, we calculated earlier that $\psi(\gamma x) = \rho(\gamma)\psi(x)$ so ψ is a measure preserving equivariant map between the original $\Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$ and the action $\Gamma \curvearrowright (Y^\Gamma, \nu^\Gamma)$ by $\gamma \cdot f(\lambda) \mapsto f(\rho(\gamma)^{-1}\lambda)$. This second action is isomorphic to a Bernoulli shift with base space $Y^{[\Gamma:\rho_2(\Gamma)]}$. Note that for finite groups $|A| = |X|$ and $|B| = |X|$ and for infinite A, the space X^Γ is infinite. By [3] and [30] and soficity, we get that $|A| \geq |B|$. This argument is symmetric in A and B, so we can repeat it to get $|A| = |B|$. $\square$

5. SUMMARY

Orbit equivalence gives us a rich environment in which to study various measure-theoretic and representation-theoretic properties of groups. The orbit equivalence relation explores connections between many areas of mathematics, most notably connecting the study of algebraic and measurable properties of groups. We study various invariants and examples in this context.

In the setting of groupoids, we primarily study the inner amenable objects. We show these objects have cost 1 and are closed under passing to a coamenable supergroupoid. In relation to this, we prove the equivalence of several definitions of amenable actions of discrete measured groupoids. Much of the work here involves finding alternate proofs of well-known lemmas in the setting of groups and then generalizing these arguments to groupoids.

In the setting of groups, we primarily focused on wreath product groups. Here, we provide new examples of both orbit equivalence antirigidity and rigidity results. In doing so, we introduce the notion of cofinite equivariance which naturally relates to orbit equivalence of wreath products.

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