ON GEOMETRY OF 2-NONDEGENERATE, HYPERSURFACE-TYPE CAUCHY–RIEMANN MANIFOLDS ENCODED IN DYNAMICAL LEGENDRIAN CONTACT STRUCTURES

A Dissertation

by

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ABSTRACT

We construct canonical absolute parallelisms over real-analytic manifolds equipped with 2-nondegenerate, hypersurface-type CR structures of arbitrary odd dimension not less than 7 whose Levi kernel has constant rank belonging to a broad subclass of CR structures that we label as *recoverable*. For this we develop a new approach based on a reduction to a special flag structure, called the *dynamical Legendrian contact structure*, on the leaf space of the CR structure's associated Levi foliation. This extends antecedent results of Curtis Porter and Igor Zelenko, for which they developed a kind of bigraded Tanaka prolongation, from the case of regular CR symbols constituting a discrete set in the set of all CR symbols to the case of the arbitrary CR symbols for which the original CR structure can be uniquely recovered from its corresponding dynamical Legendrian contact structure. We find an explicit criterion for this recoverability. The method developed here clarifies the relationship between the bigraded Tanaka prolongation of regular symbols and their usual Tanaka prolongation, providing a geometric interpretation of conditions under which these two constructions coincide.

Motivated by the search for homogeneous models with given non-regular symbols, we describe a process of reduction of an initial natural frame bundle, which is needed to treat structures with non-regular CR symbols. We demonstrate this reduction procedure for examples whose underlying manifolds have dimensions 7 and 9. We prove that for every $n \geq 3$ the sharp upper bound for the dimension of the symmetry groups of homogeneous, 2-nondegenerate, $(2n+1)$-dimensional CR manifolds of hypersurface type with a 1-dimensional Levi kernel is equal to n^2+7. This supports Beloshapka's conjecture stating that hypersurface models with a maximal finite dimensional group of symmetries for a given dimension of the underlying manifold are Levi nondegenerate. Essential to the calculation of this upper bound is a classification of the CR symbols, which we also derive. Lastly, we classify (up to local equivalence) the 7-dimensional maximally symmetric (among structures with a given CR symbol) homogeneous, 2-nondegenerate, 7-dimensional CR manifolds, of which there are eight, and give a similar partial classification of the 9-dimensional models.

DEDICATION

To my parents and sister for their constant support

from Harvey's heyday at the end of Wimberly Drive

to the present

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NOMENCLATURE

B^T	The transpose of a matrix B
B^*	The conjugate transpose of a matrix B
$B_{i,j}$	The (i,j) entry of a matrix B
$B_{(i,j)}$	The (i,j) block of a matrix B that has been assigned a partition into blocks
$\delta_{i,j}$	The Kronecker δ, equal to 1 if $i = j$ and equal to zero otherwise
E_x	The fiber of a fiber bundle E over the base point x
$\Gamma(E)$	The space of smooth sections of a fiber bundle E
$\mathfrak{g}^0$	A CR symbol
$\mathfrak{g}^{0,\mathrm{mod}}$	A modified CR symbol
$\mathfrak{g}^{0,\mathrm{red}}$	A reduced modified CR symbol
H	A nondegenerate Hermitian matrix in Chapter 3 and a CR structure in all other chapters
$J_{\lambda,m}$	The $m \times m$ Jordan matrix with a 1-dimensional eigenspace and eigenvalue λ
K	A CR structure's Levi kernel
$\mathcal{L}$	A CR structure's Levi form
ℓ	A CR structure's reduced Levi form
$\mathrm{LG}(V)$	The Lagrangian Grassmannian of a symplectic vector space V
$M_{\lambda,m}$	A fundamental block in the canonical form for antilinear operators (see (3.1.4))
$N_{\lambda,m}$	A fundamental block in the canonical form for Hermitian matrices (see (3.1.5))
S_m	The $m \times m$ $(0,1)$-matrix satisfying $(S_m)_{i,j} = \delta_{i+j,m+1}$
T_m	The $m \times m$ $(0,1)$-matrix equal to $J_{0,m}$

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1. INTRODUCTION

This dissertation contributes to the program of solving local equivalence problems in Cauchy–Riemann (or just CR) geometry initiated by the celebrated classical results of É. Cartan [7], Tanaka [40], and Chern–Moser [8]. Motivation for this program stems from an observation by Poincaré that real hypersurfaces embedded into complex vector spaces inherit some geometric structure from the complex structure on the ambient space into which they are embedded, and that this inherited structure (which in modern terminology is called a CR structure) has local invariants. A revelatory early example of this phenomenon appeared from contrasting open subsets in $\mathbb{C}^2$, namely the open unit ball $B := \{z \in \mathbb{C}^2 : |z| < 1\}$ and the open unit polydisc $P := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \text{ and } |z_2| < 1\}$. Following Riemmann's Mapping Theorem, there was keen interest in whether or not there exist biholomorphisms between open sets in $\mathbb{C}^2$ such as B and P, and Poincaré had the insight to address this question by contrasting the local geometry of these two sets' respective boundaries [31]. The boundaries themselves are real hypersurfaces in $\mathbb{C}^2$ (i.e., real 3-dimensional embedded submanifolds).

For a real hypersurface M in $\mathbb{C}^2$, the maximal complex sub-bundle in its tangent bundle TM encodes the geometric structure, called a CR structure, that M inherits from its embedding into $\mathbb{C}^2$. The (local) equivalence problem for these structures is to determine when two such hypersurfaces (locally) admit a diffeomorphism that preserves their respective CR structures. Poincaré discovered that the boundaries of B and P are not locally equivalent at any points, and he raised the question of what invariants distinguish the local geometry of hypersurfaces in $\mathbb{C}^2$.[1] Later, Segre observed in [34] that Poincaré's problem is equivalent to classifying (up to analytic point transformations) second order ordinary differential equations of the form $y'' = f(x, y, y')$. Shortly thereafter, applying what he referred to as his *general method of equivalence*, in [7] É. Cartan

[1]After referring to his observations as *résultats partiels et fragmentaires* (partial and fragmentary results) toward a research program of clarifying the link between conformal geometry and the theory of functions of two variables, Poincaré concluded the article [31] with the sentiment that he hopes this program, when complete, will shed light on the theory of functions of multiple variables.

identified local invariants that completely distinguish hypersurfaces in $\mathbb{C}^2$, essentially classifying CR structures that appear on these hypersurfaces. It is rumored that Cartan obtained similar results for hypersurfaces in $\mathbb{C}^3$ (i.e., for CR structures on 5-dimensional manifolds), but the work was not recorded. Cartan noted that motivation for his work on Poincaré's problem includes its significance for differential equations shown by Segre in [34]. Segre's observations linking hypersurfaces in $\mathbb{C}^2$ to differential equations were generalized in [37, 29], where the geometry of hypersurfaces in $\mathbb{C}^n$ for $n \geq 2$ is shown to be fundamentally linked to a more general family of differential equations. Also, meeting and exceeding Poincaré's early hopes that the geometry of hypersurfaces in $\mathbb{C}^2$ would have application to the theory of functions of 2 complex variables, deep relationships have since been found between the geometry of hypersurfaces in $\mathbb{C}^n$ and the theory of functions of several complex variables, such as the characterization of domains of holomorphy in terms of the geometry of their boundaries (e.g., see [27, Theorem 5.1.3]), which further motivates the study of CR geometry on manifolds of higher dimension.

Extensions of Cartan's solution for treating CR manifolds of higher dimension have been achieved by specializing to treat only the CR structures whose local invariants satisfy certain conditions. The first broad generalizations of Cartan's work for CR manifolds of higher dimension are specialized to the class of structures known as Levi-nondegenerate structures [8, 40]. The solution to the local equivalence problem for Levi-nondegenerate CR structures of hypersurface type is well understood in the general framework of parabolic geometries [5, 6, 43].

This dissertation is focussed around extending those local equivalence results via a construction of canonical absolute parallelisms (developed in Chapter 2) for CR structures that are uniformly (i.e., at every point) Levi-degenerate while still satisfying further nondegeneracy conditions called 2-nondegeneracy (defined below). For such structures on 5-dimensional manifolds – that is, the lowest dimension in which 2-nondegeneracy occurs – the structure of absolute parallelism was constructed only recently and independently in the following three papers (preceded by the work [13] for a more restricted class of structures): Isaev and Zaitsev [24], Medori and Spiro [28], and Merker and Pocchiola [30]. For treating higher dimensions the complexity of calculations and number of branching cases needed to be analyzed in order to implement Cartan's method of equivalence drastically increases, whereas the algebraic method used by Tanaka and the general framework of parabolic geometries that worked for treating the Levi-nondegenerate case is not directly applicable. Hence, new methods are required to treat this problem in higher dimensions. A method of bigraded Tanaka prolongation was developed in [33] that is able to treat the local equivalence problem for a specialized class of 2-nondegenerate structures defined by the property of having a *regular CR symbol* (defined in Section 2.2) on manifolds of arbitrary dimension, the most general previous result in this direction. Yet still the local equivalence problem remains intractable to previously known methods for a preponderance 2-nondegenerate structures. In Chapter 2, we develop a new method for constructing canonical absolute parallelisms – solving the local equivalence problem by reducing it to the classically solved equivalence problem for absolute parallelisms (e.g., [36, Chapter VII, Theorem 4.1]) – for the expansive class of 2-nondegenerate, hypersurface-type CR structures that can be encoded in a special associated flag structure that we call a *dynamical Legendrian contact structure* (Definition 2.1.3).

Another classical problem setting in differential geometry is to find homogeneous structures with the symmetry group of maximal dimension among all geometric structures of a certain class. In CR geometry this problem is classically solved for the class of Levi-nondegenerate CR structures of hypersurface type of arbitrary dimension ([40, 8]). In Chapter 4, we apply the general theory developed in Chapter 2 to solve this problem for 2-nondegenerate CR structures of hypersurface type with a rank 1 Levi kernel. Previously the solution to this problem was given only in the 5-dimensional case [24, 28, 30], which, again, is the case of the smallest possible dimension in which 2-nondegenrate structures exist. We give the solution for arbitrary dimension (which a priori is odd) greater than 5, extending a previous result of [33] that applies only under the aforementioned additional restriction that the considered structures' CR symbols are regular. This result supports Beloshapka's conjecture [24, Conjecture 5.6] stating in particular that the homogeneous hypersurface model with maximal finite dimensional group of symmetries is Levi nondegenerate.

To discuss this in more detail, let us now set some working definitions. A *Cauchy–Riemann (or* *just CR) structure of hypersurface type* on a $(2n+1)$-dimensional real manifold M is an integrable, totally real, complex rank n distribution H contained in the complexified tangent bundle $\mathbb{C}TM$, that is,

$$[H,H] \subset H \quad \text{ and } \quad H \cap \overline{H} = 0. \tag{1.0.1}$$

Here $\mathbb{C}TM$ denotes the tensor product $TM \otimes \mathbb{C}$ of the real tangent bundle TM and the trivial complex line bundle on M. Note that this is merely an abstract definition of the very structure that a hypersurface in a complex space inherits from the complex structure on the ambient space into which it is embedded. Indeed, if we regard M as a hypersurface in $\mathbb{C}^{2n+1}$, let $J : T\mathbb{C}^{2n+1} \to T\mathbb{C}^{2n+1}$ denote the standard complex structure on $\mathbb{C}^{2n+1}$ (i.e., the fiberwise linear bundle map given in standard coordinates as multiplication by i), and let D denote the maximal sub-bundle of TM invariant under J, then the pair $(D, J|_D)$ encodes a structure that M inherits from the complex structure J, and, with this notation, a hypersurface-type CR structure H (in particular, satisfying (1.0.1)) can be defined in terms of the pair $(D, J|_D)$ as the i-eigenspace of the linear extension of J to the complexified distribution $\mathbb{C}D$. Conversely, if the CR structure H is obtained from such a pair $(D, J|_D)$ in this way then we can describe $(D, J|_D)$ in terms of H by setting $D = \big(H \oplus \overline{H}\big) \cap TM$ and taking $J|_D$ to be the restriction of the operator on $H \oplus \overline{H}$ whose i-eigenspace and $(-i)$-eigenspace is H and $\overline{H}$ respectively.

Fixing some notation, if E is a fiber bundle over a base space B then we let $\Gamma(E)$ denote the space of smooth sections of E, and, for $p \in B$, we let E_p denote the fiber of E over p. For a given CR structure H, a Hermitian form $\mathcal{L}$, called the *Levi form* of the CR structure H on M, is defined on fibers of H by the formula

$$\mathcal{L}(X_p, Y_p) := \frac{i}{2}\left[X, \overline{Y}\right]_p \pmod{H_p \oplus \overline{H}_p} \qquad \forall\, X, Y \in \Gamma(H) \text{ and } p \in M, \tag{1.0.2}$$

where $\mathcal{L}$ takes values in the quotient spaces $\mathbb{C}T_pM/\left(H_p \oplus \overline{H}_p\right)$. Note that the coset represented by $\frac{i}{2}[X,\overline{Y}]_p$ in $\mathbb{C}T_pM/\left(H_p \oplus \overline{H}_p\right)$ depends only on the values of X and Y at p rather than the values of X and Y in a neighborhood of p. We let K denote the kernel of the Levi form. A CR structure with $K_p = 0$ is called *Levi-nondegenerate* at p, and it is called Levi-degenerate at p otherwise.

Furthermore, assuming that K is a distribution, that is, $\dim K_p$ is independent of $p \in M$, 2-*nondegeneracy* can be defined as follows: we say that the CR structure H on M is 2-*nondegenerate at a point* $p \in M$ if $K_p \neq 0$ and, for any $Y \in \Gamma(K)$ with $Y_p \neq 0$, there exists $X \in \Gamma(H)$ such that $\left[X, \overline{Y}\right]_p \notin K_p \oplus \overline{K}_p \oplus \overline{H}_p$. The structure H is 2-*nondegenerate* if it is 2-nondegenerate at every point in M. Equivalently, if for $v \in K_p$ and $y \in \overline{H}_p/\overline{K}_p$, we take $V \in \Gamma(K)$ and $Y \in \Gamma(\overline{H})$ such that $V(p) = v$ and $Y(p) \equiv y \pmod{\overline{K}}$, and define a linear map $\mathrm{ad}_v : \overline{H}_p/\overline{K}_p \to H_p/K_p$ by

$$\mathrm{ad}_v(y) := [V, Y]|_p \pmod{K_p \oplus \overline{H}_p}, \tag{1.0.3}$$

and similarly define a linear map $\mathrm{ad}_v : H_p/K_p \to \overline{H}_p/\overline{K}_p$ for $v \in \overline{K}_p$, then a Levi-degenerate CR structure is 2-nondegenerate at p if and only if there is no nonzero $v \in K_p$ (equivalently, no nonzero $v \in \overline{K}_p$) such that $\mathrm{ad}_v = 0$.

The generalization of this definition to arbitrary $k \geq 1$ via the *Freeman sequence* under analogous constant rank assumptions was given in [14]. A more general definition, without the assumption that K is a distribution and for arbitrary $k \geq 1$ can be found in the monograph [2, chapter XI]. Note that our definition of 2-nondegeneracy (the Freeman definition of k-nondegeneracy) and the definition in [2] under the assumption that K is a distribution (respectively, of constancy of ranks in the Freeman sequence) are equivalent (see [26, Appendix]). Freeman's concept of k-nondegeneracy organizes CR structures into a heirarchy of nondegeneracies, and the class of 2-nondegenerate structures can be seen as the next class in this hierarchy following the first class, which is that of Levi nondegenerate CR structures.

As a direct consequence of the Jacobi identity for every $v \in \overline{K}_p$ the antilinear operator $\overline{\mathrm{ad}_v}$: $H_p/K_p \to H_p/K_p$, defned by $\overline{\mathrm{ad}_v}(x) := \overline{\mathrm{ad}_v(x)}$, is a self-adjoint antilinear operator with respect to the Hermitian form ℓ induced on H_p/K_p by the Levi form $\mathcal{L}$ on H_p, that is,

$$\ell(\overline{\mathrm{ad}_v} x, y) = \ell(\overline{\mathrm{ad}_v} y, x), \quad \forall x, y \in H_p/K_p.$$

We call ℓ the *reduced Levi form* of H.

If H is a hypersurface-type CR structure, $n = \operatorname{rank} H$, and $r = \operatorname{rank} K$, then the assumption of 2-nondegeneracy implies that

$$\binom{n-r+1}{2} \geq r, \tag{1.0.4}$$

as the left side is exactly the dimension of the space of self-adjoint antilinear operators on a fiber of H/K (equal to the dimension of $(n-r+1)\times(n-r+1)$ symmetric matrices), which cannot be less than $\operatorname{rank} K$ because the mapping $v \mapsto \mathrm{ad}_v$ is injective on each fiber K_p of K. This implies in particular that, as noted before, among hypersurface-type CR manifolds, the lowest dimension in which 2-nondegeneracy can occur is $\dim M = 5$ (i.e., with $n = 2$ and $r = 1$).

Referred to briefly above, the most general previous results on constructing canonical absolute parallelisms for 2-nondegenerate, hypersurface-type CR structures of dimension higher than 5 (and without an assumption of semisimplicity of the symmetry group of homogeneous models as in [17, 18] and implicitly in [32]) were obtained in [33] for a specialized class of structures, where under specific algebraic conditions a bigraded (i.e., $\mathbb{Z}\times\mathbb{Z}$-graded) analogue of Tanaka's prolongation procedure was developed to construct a canonical absolute parallelism for these CR structures in arbitrary (odd) dimension with Levi kernel of arbitrary admissible dimension. The starting point of these constructions was the introduction of the notion of a bigraded Tanaka symbol of a CR structure at a point (called the *CR symbol*; see Definition 2.2), playing the role of the Tanaka symbol in the standard Tanaka theory [42, 46], which is not applicable here. As with the usual Tanaka symbol, the bigraded Tanaka symbol contains the information about Lie brackets of sections of the (complexified) tangent bundle adapted to a filtration (determined by the CR structure) that remains after a passage from the filtered structure to a natural bigraded structure at a point (see Section 2.2 for more detail), but in contrast to the standard theory the bigraded symbol is not a Lie algebra in general, but rather a bigraded vector space.

The algebraic assumption of [33] under which the bigraded Tanaka prolongation approach works is that the CR symbol they start with is a Lie algebra. Such a CR symbol is called regular. Yet, for fixed $n = \operatorname{rank} H$ and $r = \operatorname{rank} K$ satisfying (1.0.4), apart from the case in which the equality in (1.0.4) holds, that is, when $r = \binom{l+1}{2}$ and $n = \frac{l(l+3)}{2}$ for some positive integer l, which was treated in [17, 18], the non-regular symbols constitute a generic subset in the set of all symbols (see Lemma 2.6.5 for the proof), and a goal motivating the work in this dissertation is to treat structures exhibiting non-regular CR symbols because these are precisely the structures for which, in general, previously known methods for approaching the local equivalence problem, such as the methods in [33], are not applicable. For this, in the real-analytic category, we develop an alternative approach based on a (local) reduction of the original CR structure to a sort of flag structure in the spirit of [10] (see Definition 2.1.4 below) or, equivalently, to families of Legendrian contact structures (following the terminology of [11], see Remark 2.1.2 below) on the space of leaves of the Levi foliation (or shortly the Levi leaf space) of the complexified manifold. We call these flag structures *dynamical Legendrian contact structures*. Specifically, the Levi leaf space is endowed with a contact distribution, and within the Lagrangian Grassmannian of each fiber of this contact distribution (defined with respect to its canonical conformal symplectic form) the original CR structure induces a pair of submanifolds with complex dimension equal to the rank of the Levi kernel (see Section 2.1 for more detail). In particular, if the Levi kernel is one-dimensional, then in each fiber of the contact distribution one has a pair of curves of Lagrangian subspaces.

We leverage this correspondence between CR structures and dynamical Legendrian contact structures to obtain a canonical construction of absolute parallelisms for the CR structures, which is the main result developed in Chapter 2. These parallelisms can be applied to obtain new local equivalence results specifically for *recoverable* 2-nondegenerate, hypersurface-type CR structures, which are those that are uniquely determined by their corresponding dynamical Legendrian contact structure. The set of recoverable structures is a broad family that, at least in the case where the Levi kernel has rank 1, includes the aforementioned non-regular structures. Chapters 4 and 5 are dedicated to applications of the underlying theory developed in Chapter 2, whereas Chapter 3 develops a linear algebra result that is essential for the analysis in Chapters 4 and 5.

In particular, in Chapter 4 we show that the symmetry group of a homogeneous, 2-nondegenerate, hypersurface-type, $(2n+1)$-dimensional CR Manifold is not greater than n^2+7 in any case where

$n \geq 3$, an upper bound that, moreover, can be attained for all $n \geq 3$. The analagous upper bound was previously known for the other case, which is where $n = 2$ [24, 28, 30], but it was not known for any of the cases treated here, that is for each $n \geq 3$.

In Chapter 5, we classify the CR symbols associated with homogeneous CR models on manifolds of dimension 7, and partially classify these symbols for structures on manifolds of dimension 9 with a rank 1 Levi kernel. Moreover, for 7-dimensional manifolds we classify the associated (maximal) *reduced modified CR symbols*, an object introduced in Chapter 2 that encodes a larger set of local invariants than the aforementioned CR symbols do. The importance of this classification is that from each reduced modified CR symbol in the classification, we can construct a distinct homogeneous, 2-nondegenerate, hypersurface-type CR Manifold, contributing to the larger program of classifying homogeneous CR manifolds. The fact that all regular symbols correspond to homogeneous models was shown in [33], wherein they show that each regular symbol is associated with a (locally) unique maximally symmetric homogeneous model. The classification in Chapter 5 extends these results in low dimensions by finding homogeneous models associated with regular symbols that have smaller dimensional symmetry groups than the models found in [33] and by finding homogeneous models associated with non-regular symbols as well. An early conjecture that all homogeneous models have a regular symbol was in fact the original motivation for the work in this dissertation. In Chapter 5, we identify two 7-dimensional homogeneous models and nine 9-dimensional homogenous models with non-regular CR symbols.

2. UNDERLYING THEORY: A CONSTRUCTION OF CANONICAL ABSOLUTE PARALLELISMS

Inspired by the theory developed in [9] for geometry of a single submanifold in flag varieties, we apply a description of the local differential geometry of pairs of submanifolds in Lagrangian Grassmannians to assign to our original CR structure a kind of Tanaka structure, in general of nonconstant type and with the symbol at every point different from the original CR symbol of [33] (correspondingly called the *modified CR symbol*), for which both a Tanaka-like prolongation procedure for the construction of canonical moving frames and upper bounds for the dimension of the pseudogroup of local symmetries can be established, which is this chapter's most fundamental result, Theorem 2.3.5. A nonstandard aspect in this theorem is that the modified symbol of our structure is not necessarily a Lie algebra and it varies from point to point. So to prove Theorem 2.3.5 (see Section 2.8) we make certain modifications to the standard Tanaka prolongation in the spirit of [46], obtaining a microlocal version of the standard construction.

In Section 2.1, we give criteria (Proposition 2.1.6) for when by passing from a CR structure to its corresponding dynamical Legendrian contact structure we do not lose any information, that is, the former can be uniquely recovered from the latter. In particular, we show that in the case of $\operatorname{rank} K = 1$ the CR structure is recoverable if and only if, for a generator v of K, the operator ad_v has rank greater than 1 and, consequently, in this case every CR structure with non-regular CR symbol is recoverable. Moreover, for fixed $\operatorname{rank} H > 1$ and given signature of the reduced Levi form ℓ (i.e., the Hermitian form induced on H/K from the Levi form), among all regular CR symbols, if ℓ is sign-indefinite (Figure 2.1) then there are exactly two symbols for which the operator ad_v has rank 1 and are consequently non-recoverable, whereas if ℓ is sign-definite then there is exactly one such symbol. The two non-recoverable symbols arising in the sign-indefinite case are distinguished by whether the antilinear operator $\overline{\operatorname{ad}_v}$ is nilpotent or not. The former is not possible in the sign-definite case.

In Section 2.4, we prove that structures with non-regular CR symbols cannot have constant

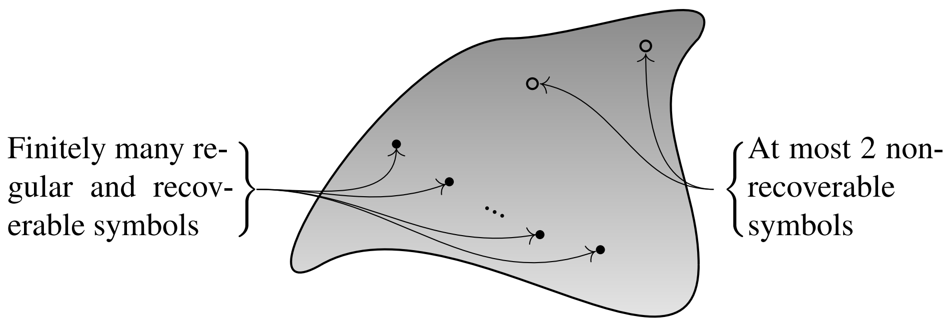

Figure 2.1: Moduli space of CR symbols for $\operatorname{rank} K = 1$, fixed $\dim M > 5$, and fixed signature of $\mathcal{L}$

modified symbols (Theorem 2.4.2), a notion introduced in Section 2.3, which motivates the reduction procedure of Section 2.5. In particular, Section 2.5 introduces another theorem on absolute parallelisms (Theorem 2.5.2), which gives more precise upper bounds for the dimension of algebras of infinitesimal symmetries than Theorem 2.3.5 does in certain cases. As a consequence, we obtain that if the CR symbol $\mathfrak{g}^0$ is regular and recoverable then its usual Tanaka prolongation and the bigraded Tanaka prolongation defined in [33, section 3] coincide.

Although to every regular CR symbol one can assign a homogeneous model, the existence of homogeneous models exhibiting a given non-regular CR symbol turned out to be a subtle question. In Section 2.6, we show (Theorem 2.6.1) that for any fixed rank $r > 1$, in the set of all CR symbols associated with 2-nondegenerate, hypersurface-type CR manifolds of odd dimension greater than $4r + 1$ with rank r Levi kernel, the CR symbols not associated with any homogeneous model are generic, and for $r = 1$ the same result holds if the reduced Levi form is sign-definite, that is, when the CR structure is pseudoconvex.

Despite these non-existence results for generic symbols, such homogeneous models do exist for specific non-regular symbols. In Section 2.7, we demonstrate our constructions with three examples. All three examples are actually homogeneous CR manifolds exhibiting the maximally symmetric structures described in Theorem 2.5.2, and they illustrate novel applications of this chapter's main results. Example 2.7.1 has a non-regular CR symbol, and, as such, the method developed in this chapter is the only known way to build an absolute parallelism over such CR manifolds such that the parallelism's automorphisms are all induced by its underlying CR manifold's symmetries. The other two examples have the same regular CR symbol in the sense of [33, Definition 2.2] but different modified CR symbols, and, as such, while the construction of an absolute parallelism given in [33] is the same for both examples, the construction given here varies, resulting in parallelisms of different dimensions for each example whose dimension matches that of the underlying CR manifold's symmetry group. The classification of non-regular CR symbols that admit homogeneous models is a nontrivial problem even for small dimensions and will be treated partially in Chapter 5.

2.1 The Levi leaf space and its flag structure

From now on we assume that K is a distribution of rank r, that is, $\dim K_p = r$ for all $p \in M$. Note that directly from the definition (1.0.2) it follows that K is an involutive distribution. In this section we introduce an important geometric object, the space of leaves of the foliation by integral submanifolds of the distribution $K \oplus \overline{K}$, called the Levi leaf space. Since K and $\overline{K}$ are subbundles in the complexified tangent bundle $\mathbb{C}TM$, to define such leaves we must "complexify" the manifold M, at least locally. For this to work we have to assume that all considered objects, namely the manifold M and the CR-structure given by H, are real-analytic. Under the real-analytic assumption, locally (i.e., in some neighborhood of any point in M) we can consider a complex manifold $\mathbb{C}M$, a *complexification of* M, by extending the transition maps between charts, which are real-analytic by definition, to analytic functions of complex variables. We can then extend locally the real-analytic distributions H and K to the holomorphic distributions on $\mathbb{C}M$ which, for simplicity, will be denoted by the same letters H and K. The conjugation in local charts of $\mathbb{C}M$ defines an involution τ on $\mathbb{C}M$ such that M is the set of its fixed points. Using this involution we can extend $\overline{H}$ and $\overline{K}$ by the formulas

$$\overline{H} := \tau_*(H) \quad \text{and} \quad \overline{K} := \tau_*(K), \tag{2.1.1}$$

so $\overline{H}$ and $\overline{K}$ are antiholomorphic extensions of the corresponding distributions from M to $\mathbb{C}M$.

Furthermore, distributions $H \oplus \overline{H}$ and $K \oplus \overline{K}$ in $\mathbb{C}M$ are holomorphic as they are holomorphic extensions of the real parts of the corresponding distributions on M. Also note that the constructed extended distribution K on $\mathbb{C}M$ is involutive as is $K \oplus \overline{K}$, so $\mathbb{C}M$ is foliated by the maximal integral (complex) submanifolds of $K \oplus \overline{K}$. This foliation is called the *Levi foliation* and will be denoted by $\mathrm{Fol}(K \oplus \overline{K})$ and, after an appropriate shrinking of $\mathbb{C}M$, which always can be done as our considerations are local, we can assume that the space of leaves of this foliation

$$\mathcal{N} = \mathbb{C}M/\mathrm{Fol}(K \oplus \overline{K})$$

has a natural structure of a (complex) manifold. The manifold $\mathcal{N}$ is called the *Levi leaf space* of the original CR structure.

Let $\pi : \mathbb{C}M \to \mathcal{N}$ be the natural projection, sending a point $p \in \mathbb{C}M$ to the leaf of the Levi foliation passing through p. Since, by construction, for every vector field $X \in \Gamma(K \oplus \overline{K})$, we have

$$[X, H \oplus \overline{H}] \subset \Gamma(H \oplus \overline{H}). \tag{2.1.2}$$

The set

$$\mathcal{D} := \pi_* \left(H \oplus \overline{H}\right) \tag{2.1.3}$$

is a well defined (complex) corank 1 distribution on $\mathcal{N}$.

Moreover, since $X \in \Gamma(H \oplus \overline{H})$ satisfies (2.1.2) if and only if $X \in \Gamma(K \oplus \overline{K})$, the distribution $\mathcal{D}$ is contact, that is, if α is a 1-form on $\mathcal{N}$ annihilating $\mathcal{D}$, then $d\alpha|_{D_\gamma}$ is nondegenrate at every point $\gamma \in \mathcal{N}$. The form

$$\omega_\gamma := d\alpha|_{D_\gamma}$$

is, up to a multiplication by a nonzero constant, a well defined symplectic form on $\mathcal{D}_\gamma$, that is, it defines a *canonical conformal symplectic structure* on $\mathcal{D}_\gamma$.

For every $\gamma \in \mathcal{N}$ and every $p \in \pi^{-1}(\gamma)$, considered as the leaf of the foliation $\mathrm{Fol}(K \oplus \overline{K})$ in $\mathbb{C}M$, set

$$\widehat{J}_\gamma^-(p) := \pi_* H_p \quad \text{and} \quad \widehat{J}_\gamma^+(p) := \pi_* \overline{H}_p.$$

From the involutivity of the distributions H and $\overline{H}$, it follows that $J_\gamma^-(p)$ and $J_\gamma^+(p)$ are Lagrangian subspaces of $\mathcal{D}_\gamma$ with respect to the symplectic form ω_γ, that is, they are elements of the Lagrangian Grassmannian $\mathrm{LG}\big(\mathcal{D}_\gamma\big)$.

Finally, the distributions K and $\overline{K}$ are involutive and define foliations $\mathrm{Fol}(K)$ and $\mathrm{Fol}(\overline{K})$, respectively. Obviously the leaves of $\mathrm{Fol}(K)$ (and of $\mathrm{Fol}(\overline{K})$) foliate the leaves of $\mathrm{Fol}(K \oplus \overline{K})$. Since $[K, H] \subset H$, the space $J_\gamma^-(p)$ is the same for every p in the same leaf of $\mathrm{Fol}(K)$ in $\pi^{-1}(\gamma)$ for $\gamma \in \mathcal{N}$. Hence, we can define the map

$$J_\gamma^- : \pi^{-1}(\gamma)/\mathrm{Fol}(K) \to \mathrm{LG}\big(\mathcal{D}_\gamma\big) \tag{2.1.4}$$

such that, given $p \in \pi^{-1}(\gamma)/\mathrm{Fol}(K)$, we have $\widehat{J}_\gamma^-(p) := \widehat{J}_\gamma^-(\hat{p})$ for some (and therefore any) $\widehat{p} \in \mathbb{C}M$ lying on the leaf containing p of the foliation $\mathrm{Fol}(K)$.

Remark 2.1.1. *Recall that the tangent space* $T_\Lambda \mathrm{LG}(\mathcal{D}_\gamma)$ *to the Lagrangian Grassmannian* $\mathrm{LG}(\mathcal{D}_\gamma)$ *at the point* Λ *is identified with an appropriate subspace in* $\mathrm{Hom}(\Lambda, \mathcal{D}_\gamma/\Lambda)$. *Also, for every* $p \in \pi^{-1}(\gamma)$, *the map* $(\pi_*)_p$ *identifies* H_p/K_p *with* $J_\gamma^-(p)$ *and* $\overline{H}_p/\overline{K}_p$ *with* $J_\gamma^+(p)$. *Using these identifications and basic properties of Lie derivatives, for every* $v \in K_p$ *(or* $v \in \overline{K}_p$*) we can identify the operator* ad_v *defined by* (1.0.3) *with the operator* $\big(J_\gamma^+\big)_* v$ *(or respectively* $\big(J_\gamma^-\big)_* v$*).*

By the identification of the previous remark, the 2-nondegeneracy condition implies that, after an appropriate shrinking of $\mathbb{C}M$, the map J_γ^- from (2.1.4) is a well defined injective immersion, that is, its image is a submanifold of $\mathrm{LG}(\mathcal{D}_\gamma)$ of complex dimension equal to $\operatorname{rank} K$. Similarly, the map

$$J_\gamma^+ : \pi^{-1}(\gamma)/\mathrm{Fol}(\overline{K}) \to \mathrm{LG}\big(\mathcal{D}_\gamma\big). \tag{2.1.5}$$

is a well defined injective immersion and its image is a submanifold of $\mathrm{LG}(\mathcal{D}_\gamma)$ of complex dimension equal to $\operatorname{rank} K$ as well. In the sequel, by J_γ^- and J_γ^+ we will denote the images of the maps in (2.1.4) and (2.1.5), respectively, rather than the maps themselves.

Remark 2.1.2. *Note that, by construction, if $\Lambda^- \in J_\gamma^-$ and $\Lambda^+ \in J_\gamma^+$ then Λ^- and Λ^+ are transversal as subspaces of $\mathcal{D}_\gamma$, that is, $\mathcal{D}_\gamma = \Lambda^- \oplus \Lambda^+$. Recall that a Legendrian contact structure on an odd dimensional distribution is a contact distribution Δ together with the fixed splitting of each fiber Δ_x by a pair of transversal Lagrangian subspaces smoothly depending on x (see [11]). Any section s of the bundle $\pi : \mathbb{C}M \to \mathcal{N}$, defines the Legendrian contact structure on $\mathcal{N}$ given by the distribution $\mathcal{D}$ and the splitting of $\mathcal{D}_\gamma$, given by $J_\gamma^-\big(s(\gamma)\big)$ and $J_\gamma^+\big(s(\gamma)\big)$.*

Motivated by the previous constructions and Remark 2.1.2 we introduce the following definition.

Definition 2.1.3. *A dynamical Legendrian contact structure (with involution) on an odd-dimensional complex manifold $\mathcal{M}$ is a contact distribution Δ together with an involution σ on $\mathcal{M}$ and a fixed pair of k-dimensional submanifods Λ_x^- and Λ_x^+ in the Lagrangian Grassmannian $\mathrm{LG}(\Delta_x)$ of each fiber such that the following conditions hold:*

1. *the submanifolds Λ_x^- and Λ_x^+ are smoothly dependent on x and any point of Λ_x^-, considered as a Lagrangian subspace of Δ_x, is transversal to any point of Λ_x^-, considered as a Lagrangian subspace of Δ_x.*

2. *$\Lambda_x^- = \sigma_* \Lambda_\sigma(x)^+$.*

Such dynamical Legendrian contact structures with involution will be denoted just by the triple $(\Delta, \Lambda^-, \Lambda^+)$ when the involution is determined by context or by the triple $(\Delta, \Lambda^-, \tau)$.

Definition 2.1.4. *Letting H be a 2-nondegenerate hypersurface-type CR structure on the manifold M, the dynamical Legendrian contact structure $\{\mathcal{D}, J^-, J^+\}$ on the Levi leaf space $\mathcal{N}$, with $\mathcal{D}$, J^-, J^+, and involution τ defined by* (2.1.3)*,* (2.1.4)*,* (2.1.5)*, and the sentence before* (2.1.1)*, respectively, is called* the dynamical Legendrian contact structure associated with the germ (at some point in M) of the CR structure H.

The reason that, in Definition 2.1.4, we say that dynamical Legendrian contact structures are associated with germs of CR structures rather than with the whole CR manifold, is that the construction of $\{\mathcal{D}, J^-, J^+\}$ is well defined only after an appropriate shrinking of M (i.e., after possibly replacing M by a neighborhood of any given point in M).

Remark 2.1.5. *If $\{\mathcal{D}, J^-, J^+\}$ is the dynamical Legendrian contact structure associated with the germ of the 2-nondegenerate CR structure H of hypersurface type on the manifold M then $\mathbb{C}M$ is locally canonically diffeomorphic to the bundle $\Pi : J^- \times J^+ \to \mathcal{N}$ with the fiber over the point $\gamma \in \mathcal{N}$ equal to $J^-_\gamma \times J^+_\gamma$, where here $J^- \times J^+ := \bigcup_{\gamma\in\mathcal{N}} \left(J^-_\gamma \times J^+_\gamma\right)$. This canonical diffeomorphism, denoted by F, is given by*

$$F(p) := (J^-_\gamma(p), J^+_\gamma(p)).$$

Moreover, each fiber $J^-_\gamma \times J^+_\gamma$ is foliated by two foliations with leaves $\{J^-_\gamma \times J^+_\gamma(p)\}_{p\in\pi^{-1}(\gamma)}$ and $\{J^-_\gamma(p) \times J^+_\gamma\}_{p\in\pi^{-1}(\gamma)}$, respectively. Denote by V_1 and V_2 the distribution tangent to this foliation. Then it is clear that

$$\begin{aligned} &V_1 = F_*K, \quad V_2 = F_*\overline{K}, \\ &F_*(H_p + \overline{K}_p) = \left\{ y \in T_{F(p)}(J^- \times J^+) \,\middle|\, \Pi_* y \in J^-_{\pi(p)}(p) \right\}, \quad \textit{and} \\ &F_*(\overline{H}_p + K_p) = \left\{ y \in T_{F(p)}(J^- \times J^+) \,\middle|\, \Pi_* y \in J^+_{\pi(p)}(p) \right\}. \end{aligned} \tag{2.1.6}$$

Finally the distribution H is an involutive subdistribution of $H + \overline{K}$.

The main idea of the present chapter is to study the local equivalence problem for the dynamical Legendrian contact structures associated with CR structures instead of the CR structures themselves. Before doing this, we have to understand the conditions under which passing from the CR structure to the corresponding dynamical Legendrian contact structure does not lose any information, that is, under which the former can be uniquely reconstructed from the latter. Such CR structures will be called *recoverable*.

To describe the conditions for recoverability, recall ([36]) that, given two vector spaces V and W and a subpsace Z in $\mathrm{Hom}(V, W)$, the *anti-symmetrization* (Spencer) operator $\partial : \mathrm{Hom}(V, Z) \to$

$\mathrm{Hom}(V \wedge V, W)$ is defined by

$$\partial(f)(v_1, v_2) = f(v_1)v_2 - f(v_2)v_1, \quad v_1, v_2 \in V,\ f \in \mathrm{Hom}(V, Z). \tag{2.1.7}$$

The kernel of the operator ∂ is called the *first prolongation* of the subspace $Z \subset \mathrm{Hom}(V, W)$ and is denoted by $Z_{(1)}$.

Now for $v \in \overline{K}_p$, take $\mathrm{ad}_v : H_p/K_p \to \overline{H}_p/\overline{K}_p$ to be as in the sentence after (1.0.3). From the assumption of 2-nondegeneracy it follows that the map $v \mapsto \mathrm{ad}_v$ identifies $\overline{K}_p$ with a subspace in $\mathrm{Hom}(H_p/K_p, \overline{H}_p/\overline{K}_p)$, which is denoted by $\mathrm{ad}\,\overline{K}_p$.

Proposition 2.1.6. *A 2-nondegenerate, hypersurface-type CR structure H is recoverable in a neighborhood of a point p if and only if the first prolongation $\left(\mathrm{ad}\overline{K}_p\right)_{(1)}$ of the space $\mathrm{ad}\overline{K}_p$ vanishes.*

Proof. From Remark 2.1.5 it follows that a CR structure H is recoverable if and only if H is the unique involutive subdistribution of $H + \overline{K}$ of rank $n = \mathrm{rank}\, H$, transversal to $\overline{K}$ and containing K, because the reconstruction can be obtained using formulas (2.1.6). Let H' be another complex complex rank n involutive subdistrubution of $H + \overline{K}$ that is transversal to $\overline{K}$ and containing K. Fix the point $p \in \mathbb{C}M$. For each $y \in \mathbb{C}M$, there exists a linear map $f_y : H_y/K_y \to \overline{K}_y$ such that H' is the graph of f_y characterized by

$$H'_y = \{v + f_y(v) \mid v \in H_y\}.$$

In the sequel by f we mean the field of linear maps $\{f_y\}_{y \in \mathbb{C}M}$. For two vectors $y_1, y_2 \in H_p$, let $Y_1, Y_2 \in \Gamma(H)$ be such that $Y_i(p) = y_i$ for $i \in \{1, 2\}$, and let Y_1' and Y_2' be the associated vector fields in $\Gamma(H')$ such that

$$Y_i' = Y_i + f(Y_i).$$

We have

$$[Y_1', Y_2']_p = [Y_1, Y_2]_p + [Y_1, f(Y_2)]_p + [f(Y_1), Y_2] + [f(Y_1), f(Y_2)]_p \tag{2.1.8}$$

$$\equiv [Y_1, f(Y_2)]_p + [f(Y_1), Y_2]_p \pmod{H_p + \overline{K}_p} \tag{2.1.9}$$

because $[Y_1, Y_2]$ and $[f(Y_1), f(Y_2)]$ both belong to $H_p + \overline{K}_p$ due to the involutivity of H and $\overline{K}$. Since H' is involutive, it follows that the left side of (2.1.8) belongs to H_p'. Hence, using (1.0.3), (2.1.9) can be written as

$$\mathrm{ad}_{f_p(y_1)} y_2 - \mathrm{ad}_{f_p(y_2)} y_1 = 0. \tag{2.1.10}$$

Since y_1 and y_2 are arbitrary elements of H_p/K_p, by (2.1.7) and (2.1.10), we get that $f_p \in \left(\mathrm{ad}\overline{K}_p\right)_{(1)}$, so the vanishing of $\left(\mathrm{ad}\overline{K}_p\right)_{(1)}$ for generic p is equivalent to $H' = H$. $\square$

Based on this proposition we obtain the following sufficient condition for the recoverability of the CR structure.

Proposition 2.1.7. *If for a generic point p there is no nonzero subspace L of $\overline{K}_p$ satisfying,*

$$\dim\left(\bigcap_{v \in L} \ker \mathrm{ad}_v\right) \geq \operatorname{rank} H - \operatorname{rank} K - \dim L, \tag{2.1.11}$$

then the original CR structure given by H is recoverable.

Proof. By Proposition 2.1.6 it is sufficient to prove that if (2.1.11) cannot be satisfied for some L then $\left(\mathrm{ad}\overline{K}_p\right)_{(1)} = 0$ for generic p. To prove the contrapositive of this statement, assume that for a generic p there is a nonzero $f \in \left(\mathrm{ad}\overline{K}_p\right)_{(1)}$. Set $L = \mathrm{Im} f$. Then

$$\dim \ker f = \operatorname{rank} H - \operatorname{rank} K - \dim L. \tag{2.1.12}$$

On the other hand, if $y \in \ker f$, then from the definition of the first prolongation for every $z \in H_p/K_p$

$$\mathrm{ad}_{f(z)} y = \mathrm{ad}_{f(z)} y - \mathrm{ad}_{f(y)} z = 0,$$

because $f(y) = 0$, which means that $y \in \ker \mathrm{ad}_v$ for every $v \in L$, that is,

$$\ker f \subset \left(\bigcap_{v \in L_p} \ker \mathrm{ad}_v \right).$$

This and (2.1.12) implies (2.1.11), that is if $\left(\mathrm{ad}\overline{K}_p\right)_{(1)} \neq 0$ then there exists a space L satisfying (2.1.11), which is the contrapositive of what we needed to prove. □

Corollary 2.1.8. *If* $\operatorname{rank} K = 1$ *then the original CR structure given by* H *is recoverable if and only if* $\mathrm{ad}\overline{K}_p$ *is generated by an operator of rank greater than* 1 *for a generic point* p.

Proof. First note that for $\operatorname{rank} K = 1$ the only nonzero subspace of $\overline{K}_p$ is $\overline{K}_p$ itself and the inequality (2.1.11) is equivalent to the condition that $\mathrm{ad} K_p$ is generated by an operator of rank 1. This and Proposition 2.1.7 imply the "if" part of the corollary.

To prove the "only if" part assume by contradiction that $\mathrm{ad}\overline{K}_p$ is generated by rank 1 operator ad_v, $v \in \overline{K}_p$ for generic p. Accordingly, $f \in \mathrm{Hom}(H_p/K_p), \mathrm{ad}\overline{K}_p)$ such that $\ker f = \ker \mathrm{ad}_v$ will be a nonzero element of $\mathrm{ad}\left(\mathrm{ad}\overline{K}_p\right)_{(1)} = 0$. Hence, by Proposition 2.1.6 the CR structure is not uniquely recoverable, which leads to the contradiction. □

Remark 2.1.9. *Propositions 2.1.6, 2.1.7, and Corollary 2.1.8 can be reformulated in an obvious way to define CR structures in terms of their dynamical Legendrian contact structures, specifically, using the identifications in Remark 2.1.1 and the formulas in* (2.1.6). *In all such reformulations we have to replace "unique" recovery by "at most one," because in general a dynamical Legendrian contact structure need not be associated with any CR structure as the distribution in the right side of the second line of* (2.1.6) *may not contain any involutive subdistribution of rank equal to the dimension of* $J^-_\gamma(p)$.

2.2 Symbols of 2-nondegenerate CR structures

Let us assume that $(\mathcal{D}, J^-, J^+)$ is a dynamical Legendrian contact structure with involution τ on $\mathcal{N}$ associated with a germ of some 2-nondegenerate, hypersurface-type CR structure H on a manifold M.

For $\gamma \in \mathcal{N}$ and $p \in \pi^{-1}(\gamma)$, note that there is an identification $J_\gamma^+(p) \cong \big(J_\gamma^-(p)\big)^*$ determined by the symplectic form ω_γ). Using this identification, the tangent space $T_{J_\gamma^-(p)}J_\gamma^-$ can be canonically identified with a subspace of $\mathrm{Sym}^2(J_\gamma^-(p))^*\big) \subset \mathrm{Hom}\big(J_\gamma^-(p), J_\gamma^+(p)\big)$, which will be denoted by $\delta^-(p)$.

Note that any element y of $\mathrm{Sym}^2\big((J_\gamma^-(p))^*\big)$, considered as a self-adjoint operator from $J_\gamma^-(p)$ to $(J_\gamma^-(p))^* \cong J_\gamma^+(p)$ can be extended to an element $\tilde{y}$ of the conformal symplectic algebra $\mathfrak{csp}(\mathcal{D}_\gamma)$ by setting $\tilde{y}|_{J_\gamma^-(p)} = y$ and $\tilde{y}|_{J_\gamma^+(p)} = 0$. In the sequel we will regard $\delta^-(p)$ as an element of $\mathfrak{csp}(\mathcal{D}_\gamma)$.

The tangent space $T_{J_\gamma^+(p)}J_\gamma^+$ can be canonically identified with a subspace of $\mathrm{Hom}\big(J_\gamma^+(p), J_\gamma^-(p)\big)$ similarly, which will be denoted by $\delta^+(p)$ and will be also considered as a subspace of $\mathfrak{csp}(\mathcal{D}_\gamma)$. Note that $\delta^+(p)$ is obtained from $\delta^-(p)$ by the involution of $\mathfrak{csp}(\mathcal{D}_\gamma)$ induced by the differential of the involution τ whenever p belongs to the fixed point set of τ.

Definition 2.2.1. *The symbol of the 2-nondegenerate, hypersurface-type CR structure H on a manifold M at a point p in $\mathbb{C}M$ is the orbit of the pair of subspaces $\big(\delta^-(p), \delta^+(p)\big)$ under the action of the conformal symplectic group $CSp(\mathcal{D}_\gamma)$ on $\mathfrak{csp}(\mathcal{D}_\gamma) \times \mathfrak{csp}(\mathcal{D}_\gamma)$ induced by the* Ad*-action of $CSp(\mathcal{D}_\gamma)$ on $\mathfrak{csp}(\mathcal{D}_\gamma)$. The 2-nondegenerate, hypersurface-type CR structure H is said to have constant symbol (or be of constant type) if its symbols at every two points are isomorphic via a conformal symplectic transformation between the corresponding fibers of $\mathcal{D}$ at these points. If p belongs to M then τ induces an involution on the CR symbol, and, in this case, we refer to the symbol at p as the CR symbol with involution.*

In the sequel for simplicity we will work with 2-nondegenerate, hypersurface-type CR structures with constant symbol. Our constructions can in principle be extended to structures of non-constant type, in the spirit of the proof of Theorem 2.3.5 given in Section 2.8, using an appropriate identifying space in the very beginning.

Definition 2.2.1 is formally different from the notion of CR symbol as a certain bigraded (i.e., $(\mathbb{Z} \times \mathbb{Z})$-graded) vector space introduced in [33, Definition 2.2], but it contains equivalent information. To relate these definitions in the subsequent paragraph we use a certain bigrading to denote different spaces. The actual meaning of this bigrading is explained in detail in [33], but it is not crucial here.

The symbol in Definition 2.2.1 of the 2-nondegenerate, hypersurface-type CR structure H with Levi kernel K on a $(2n+1)$-dimensional manifold at a point p in M is encoded by

1. a $2n$-dimensional conformal symplectic space $\mathfrak{g}_{-1}$ (over $\mathbb{C}$) with antilinear involution σ,

2. a splitting $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$, where $\mathfrak{g}_{-1,\pm 1}$ are Lagrangian with $\mathfrak{g}_{-1,1} = \sigma(\mathfrak{g}_{-1,-1})$, and

3. two $(\operatorname{rank} K)$-dimensional subspaces $\mathfrak{g}_{0,2}$ and $\mathfrak{g}_{0,-2}$ belonging to $\operatorname{Sym}(\mathfrak{g}^*_{-1,-1}) \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ and $\operatorname{Sym}(\mathfrak{g}^*_{-1,1}) \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ respectively, such that one is obtained from the other via the map induced by σ.

Specifically, we can identify σ with the antilinear involution induced by the complex conjugation τ defined on $\mathbb{C}M$, and make the identifications

$$\mathfrak{g}_{-1} \cong \mathcal{D}_{\pi(p)}, \quad \mathfrak{g}_{-1,1} \cong J^+_{\pi(p)}(p), \quad \mathfrak{g}_{-1,-1} \cong J^-_{\pi(p)}(p), \quad \text{and} \quad \mathfrak{g}_{0,\pm 2} \cong \delta^{\pm}(p). \tag{2.2.1}$$

Remark 2.2.2. *The symbol in Definition 2.2.1 of the 2-nondegenerate, hypersurface-type CR structure at a point in $\mathbb{C}M$ not in M is also encoded by the objects in the 3-item list above but excluding their properties referencing an involution.*

Further we can consider the $(2n+1)$-dimensional Heisenberg algebra $\mathfrak{g}_- := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ characterized as being a central extension of $\mathfrak{g}_{-1}$ whose center is $\mathfrak{g}_{-2}$ and such that for any representative ω of the conformal symplectic structure on $\mathfrak{g}_{-1}$ there exists a nonzero $z \in g_{-2}$ for which $[x, y] := \omega(x, y)z$ for every $x, y \in \mathfrak{g}_{-1}$.

Finally, let $\mathfrak{g}_{0,0}$ be the maximal subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ such that

$$[\mathfrak{g}_{0,0}, \mathfrak{g}_{-1,\pm 1}] \subset \mathfrak{g}_{-1,\pm 1} \quad \text{and} \quad [\mathfrak{g}_{0,0}, \mathfrak{g}_{0,\pm 2}] \subset \mathfrak{g}_{0,\pm 2}, \tag{2.2.2}$$

where the brackets in the first line are just the action of elements $\mathfrak{g}_{0,0} \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ on $\mathfrak{g}_{-1}$ and the brackets in the second line are as in $\mathfrak{csp}(\mathfrak{g}_{-1})$.

The subspace

$$\mathfrak{g}^0 = \mathfrak{g}_- \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,2} \oplus \mathfrak{g}_{0,0} \subset \mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1}) \tag{2.2.3}$$

together with the involution induced on it by the involution σ on $\mathfrak{g}_{-1}$ is the symbol introduced in [33, Definition 2.2]. So there is a bijective correspondence between the notion of CR symbols (at points in M) introduced here and in [33].

Here in the semidirect sum we mean the natural action of $\mathfrak{csp}(\mathfrak{g}_{-1})$ on $\mathfrak{g}_-$ induced from the standard action on $\mathfrak{g}_{-1}$. This induced action actually identifies $\mathfrak{csp}(\mathfrak{g}_{-1})$ with the algebra of derivations of $\mathfrak{g}_-$, so in the sequel we freely regard elements of each space $\mathfrak{g}_{0,i}$ as endomorphisms of both $\mathfrak{g}_{-1}$ and $\mathfrak{g}_-$, letting context dictate which interpretation is being applied.

Definition 2.2.3. *The CR symbol is called* regular *if* $[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}] \subset \mathfrak{g}_{0,0}$, *where these brackets are as in* $\mathfrak{csp}(\mathfrak{g}_{-1})$ *or, equivalently,* $\mathfrak{g}^0$ *is a Lie subalgebra of* $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$.

Remark 2.2.4. *By construction a CR symbol is regular if and only if* $[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}] \subset \mathfrak{g}_{0,0}$ *or equivalently*

$$[\mathfrak{g}_{0,2}, [\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}]] \subset \mathfrak{g}_{0,2} \quad \textit{and} \quad [\mathfrak{g}_{0,-2}, [\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}]] \subset \mathfrak{g}_{0,-2}. \tag{2.2.4}$$

A canonical absolute parallelism for CR structures with the regular symbols was constructed in [33, Theorem 3.2] using bigraded Tanaka prolongations. The set of regular CR symbols, at least for the case of $\operatorname{rank} K = 1$, is a rather small discrete subset of the set of all symbols. To describe this in more detail, we need the following observation.

Remark 2.2.5. *A CR symbol can be also encoded by*

1. *a real line of Hermitian forms spanned by a form* $\ell : \mathfrak{g}_{-1,1} \times \mathfrak{g}_{-1,1} \to \mathbb{C}$ *defined by* $\ell(x, y) = i\omega\left(x, \sigma(y)\right)$, *for some representative* ω *of the conformal symplectic form on* $\mathfrak{g}_{-1}$, *and*

2. *a vector space of* ℓ*-self-adjoint antilinear operators on* $\mathfrak{g}_{-1,1}$ *defined by* $\{x \mapsto [v, \sigma(x)] \mid v \in \mathfrak{g}_{0,2}\}$

because the symbol's Heisenberg algebra structure on $\mathfrak{g}_-$ *can be recovered from the line of Hermitian forms spanned by* ℓ, *and the subspaces* $\mathfrak{g}_{0,\pm 2}$ *can be recovered from the vector space of*

ℓ-self-adjoint antilinear operators. In particular, if $\operatorname{rank} K = 1$*, then the vector space in item (2) is generated by an ℓ-self-adjoint antilinear operator A, so the CR symbol is encoded in the pair* $(\mathbb{R}\ell, \mathbb{C}A)$*. Canonical forms for these pairs were obtained in [38].*

This allows us to state the following proposition.

Proposition 2.2.6 ([33, section 4], see also Remark 2.4.3 for generalization to an arbitrary $\operatorname{rank} K$)**.** *If* $\operatorname{rank} K = 1$*, $\mathbb{R}\ell$ is the real line of Hermitian forms, and $\mathbb{C}A$ is the complex line of ℓ-selfadjoint antilinear operators spanned by a representative $A : \mathfrak{g}_{-1,1} \to \mathfrak{g}_{-1,1}$ such that the CR structure's symbol is encoded in* $(\mathbb{R}\ell, \mathbb{C}A)$ *as described above, then the CR symbol is regular if and only if*

$$A^3 \in \mathbb{C}A. \tag{2.2.5}$$

In particular, if $\operatorname{rank} K = 1$ and $\mathrm{ad}\overline{K}$ is generated by a rank 1 operator, then the corresponding antilinear operator A is of rank 1 and so it satisfies (2.2.5). Therefore the CR symbol in this case is always regular. This together with Corollary 2.1.8 immediately implies the following corollary.

Corollary 2.2.7. *If the symbol of the CR structure H at a point p is not regular and* $\operatorname{rank} K = 1$ *then the CR structure H is recoverable in some neighborhood of p, that is, H is uniquely determined by the dynamical Legendrian contact structure $(\mathcal{D}, J^+, \tau)$ associated with the germ of H at p.*

Moreover, based on the classification of regular symbols from [33, section 4], for fixed $\operatorname{rank} H > 1$ and given signature of ℓ, among all regular CR symbols, there are exactly two symbols for which the operator A has rank 1 and is consequently non-recoverable if the reduced Levi form is sign-indefinite, and exactly one if the reduced Levi form is sign-definite. These two symbols in the sign-definite case are distinguished by whether the corresponding antilinear operator A is nilpotent or not. The former is not possible in the sign-definite case.

Finally note that the recoverability criteria of Proposition 2.1.6 can be reformulated in terms of the symbol of the CR structure as follows.

Proposition 2.2.8. *A 2-nondegenerate, hypersurface-type CR structure with symbol $\mathfrak{g}^0(p)$ at a point p is recoverable in a neighborhood of p if and only if the first prolongation $(\mathfrak{g}_{0,2})_{(1)}$ of the space $\mathfrak{g}_{0,2}$ considered as the subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ vanishes.*

The CR symbols satisfying the condition of Proposition (2.2.8) will be called recoverable.

2.3 Modified CR symbols and the construction of absolute parallelisms

Assume that H is a 2-nondegenerate hypersurface-type CR structure with constant symbol $\mathfrak{g}^0$ (having the involution σ) as in (2.2.3). Here $\mathfrak{g}^0$, which we will call the "model space," is a fixed representative in the equivalence class of CR symbols. On the other hand, at every point $p \in \mathbb{C}M$ we have the representative $\mathfrak{g}^0(p)$ of the equivalence class of CR symbols, obtained by the identifications (2.2.1) and the corresponding bigraded components of it will be denoted by $\mathfrak{g}_{i,j}(p)$, including $\mathfrak{g}_{0,0}(p)$ defined accordingly as in (2.2.2).

Note that $\mathfrak{g}_-$ and each $\mathfrak{g}_{i,j}$ is determined by the symbol of the CR structure at a point p, so we will write $\mathfrak{g}_-(p)$ or $\mathfrak{g}_{i,j}(p)$ instead of $\mathfrak{g}_-$ or $\mathfrak{g}_{i,j}$ whenever we need to emphasize the dependence of $\mathfrak{g}_-$ or $\mathfrak{g}_{i,j}$ on p.

We now define two bundles, one over $\mathbb{C}M$ and another one over $\mathcal{N}$ with the same total space P^0. For the first bundle $\mathrm{pr} : P^0 \to \mathbb{C}M$, its fiber $\mathrm{pr}^{-1}(p)$ over $p \in \mathbb{C}M$ is comprised of all *adapted frames*, or bigraded Lie algebra isomorphisms, that is,

$$\mathrm{pr}^{-1}(p) = \left\{ \varphi : \mathfrak{g}_- \to \mathfrak{g}_-(p) \;\middle|\; \begin{array}{l} \varphi(\mathfrak{g}_{i,j}) = \mathfrak{g}_{i,j}(p) \quad \forall\, (i,j) \in \{(-1,\pm 1),(-2,0)\} \\ \varphi^{-1} \circ \mathfrak{g}_{0,\pm 2}(p) \circ \varphi = \mathfrak{g}_{0,\pm 2} \\ \varphi([y_1,y_2]) = [\varphi(y_1),\varphi(y_2)] \quad \forall\, y_1,y_2 \in \mathfrak{g}_- \end{array} \right\}.$$

Here $\mathfrak{g}_{-2,0} := \mathfrak{g}_{-2}$ and $\mathfrak{g}_{-2,0}(p) := \mathfrak{g}_{-2}(p)$. The second bundle is $\pi \circ \mathrm{pr} : P^0 \to \mathcal{N}$. Furthermore, we define a bundle $\mathrm{pr} : \Re P^0 \to M$ by

$$\Re P^0 := \{\varphi \in P^0 \,|\, \mathrm{pr}(\varphi) \in M \text{ and } \tau_* \circ \varphi = \varphi \circ \sigma\}, \tag{2.3.1}$$

where σ denotes the fixed involution on the model space $\mathfrak{g}^0$ and τ_* denotes the involution on

$J^+ \times J^-$ induced by the map τ introduced just before (2.1.1). Note that the set-inclusion conditions defining the set in (2.3.1) have some redundancy, but we have stated them as such for clarity.

For any $\psi \in P^0$ with $\gamma = \pi \circ \mathrm{pr}(\psi)$, the tangent space of the fiber $(P^0)_\gamma = (\pi \circ \mathrm{pr})^{-1}(\gamma)$ of the second bundle at ψ can be identified with a subspace of $\mathfrak{csp}(\mathfrak{g}_-)$ by the map $\theta_0 : T_\psi(P^0)_\gamma \to \mathfrak{csp}(\mathfrak{g}_-)$ given by

$$\theta_0\big(\psi'(0)\big) := \big(\psi(0)\big)^{-1}\psi'(0) \tag{2.3.2}$$

where $\psi : (-\epsilon, \epsilon) \to (P^0)_\gamma$ denotes an arbitrary curve in $(P^0)_\gamma$ with $\psi(0) = \psi$. Let

$$\mathfrak{g}_0^{\mathrm{mod}}(\psi) := \theta_0(T_\psi(P^0)_\gamma).$$

Here the superscript mod stands for modified in order to distinguish it from the space $\mathfrak{g}_0 := \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$, defined in [33], which is in general another subspace of $\mathfrak{csp}(\mathfrak{g}_-)$.

Definition 2.3.1. *The space $\mathfrak{g}^{0,\mathrm{mod}}(\psi) := \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi)$ is called the modified CR symbol of the CR structure at the point $\psi \in P^0$.*

The bigrading $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$ of $\mathfrak{g}_{-1}$ confers a bigrading on $\mathfrak{csp}(\mathfrak{g}_{-1})$ with weighted components given by

$$\big(\mathfrak{csp}(\mathfrak{g}_{-1})\big)_{0,i} = \big\{\varphi \in \mathfrak{csp}(\mathfrak{g}_{-1}) \,|\, \varphi(\mathfrak{g}_{-1,j}) \subset \mathfrak{g}_{-1,i+j}\, \forall i \in \{-1, 1\}\big\},$$

yielding the decomposition

$$\mathfrak{csp}(\mathfrak{g}_{-1}) = \big(\mathfrak{csp}(\mathfrak{g}_{-1})\big)_{0,-2} \oplus \big(\mathfrak{csp}(\mathfrak{g}_{-1})\big)_{0,0} \oplus \big(\mathfrak{csp}(\mathfrak{g}_{-1})\big)_{0,2}. \tag{2.3.3}$$

For an element $\varphi \in \mathfrak{csp}(\mathfrak{g}_{-1})$ (or subset $S \subset \mathfrak{csp}(\mathfrak{g}_{-1})$), we write $\varphi_{0,i}$ (or $S_{0,i}$) to denote the natural projection of φ (or S) into $\big(\mathfrak{csp}(\mathfrak{g}_{-1})\big)_{0,i}$ with respect to the decomposition in (2.3.3).

By construction

$$\left(\theta_0\left(\mathrm{pr}_*^{-1}(K_p)\right)\right)_{0,2} = \mathfrak{g}_{0,2} \qquad \forall\, p \in M, \tag{2.3.4}$$

and

$$\left(\theta_0\left(\mathrm{pr}_*^{-1}\left(\overline{K}_p\right)\right)\right)_{0,-2} = \mathfrak{g}_{0,-2} \qquad \forall\, p \in M. \tag{2.3.5}$$

The bundle $\mathrm{pr} : P^0 \to \mathbb{C}M$ is a principal bundle whose structure group, which we label $G_{0,0}$, has the Lie algebra $\mathfrak{g}_{0,0}$, and θ_0 is the principal connection on this bundle, which means, in particular, that

$$\theta_0\left(\mathrm{pr}_*^{-1}(0)\right) = \mathfrak{g}_{0,0}. \tag{2.3.6}$$

Since

$$T_\psi(P^0)_\gamma \cong \mathrm{pr}_*^{-1}(0) \oplus K_{\mathrm{pr}(\psi)} \oplus \overline{K}_{\mathrm{pr}(\psi)}$$

and the subspace $\theta_0\left(\mathrm{pr}_*^{-1}(0)\right) \subset \theta_0\left(T_\psi(P^0)_\gamma\right)$ belongs to the kernel of the projections $\varphi \mapsto \varphi_{0,\pm 2}$, it follows from (2.3.4), (2.3.5), and (2.3.6) that

$$\dim \theta_0\left(T_\psi(P^0)_\gamma\right) = \dim \theta_0\left(\mathrm{pr}_*^{-1}(0)\right) + \dim\left(\theta_0\left(\mathrm{pr}_*^{-1}(K_{\mathrm{pr}(\psi)})\right)\right)_{0,2} + \dim\left(\theta_0\left(\mathrm{pr}_*^{-1}\left(\overline{K}_{\mathrm{pr}(\psi)}\right)\right)\right)_{0,-2}$$

$$= \dim \mathfrak{g}_{0,0} + \dim K_{\mathrm{pr}(\psi)} + \dim \overline{K}_{\mathrm{pr}(\psi)} = \dim T_\psi(P^0)_\gamma,$$

and hence θ_0 is injective on each tangent space $T_\psi(P^0)_\gamma$. Accordingly,

$$\dim \mathfrak{g}^0\big(\mathrm{pr}(\psi)\big) = \dim \mathfrak{g}^{0,\mathrm{mod}}(\psi) \qquad \forall\, \psi \in P^0.$$

Now we are going to define the universal Tanaka prolongation of the modified symbol $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$

by analogy with the standard Tanaka theory [42, 46]. A nonstandard feature here is that the modified CR symbol is not necessarily a Lie algebra. In the standard Tanaka theory the universal Tanaka prolongation of a graded Lie algebra $\mathfrak{g}^0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ can be shortly defined as the largest $\mathbb{Z}$-graded subalgebra containing $\mathfrak{g}^0$ as its nonnegative part satisfying the additional condition that nonnegatively graded elements have nontrivial brackets with $\mathfrak{g}_{-1}$ or, equivalently, one can define each positively graded component of this resulting algebra recursively. Contrastingly, since $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ is not necessarily a Lie algebra, the recursive definition is the only available one to define this prolongation of $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ because the resulting universal prolongation is also not a Lie algebra. In more detail, we recursively define the subspaces $\mathfrak{g}_k^{\mathrm{mod}}(\psi)$ by

$$\mathfrak{g}_1^{\mathrm{mod}}(\psi) := \left\{ f \;\middle|\; \begin{array}{l} f \in \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) + \mathrm{Hom}\big(\mathfrak{g}_{-1}, \mathfrak{g}_0(\psi)\big) \\ f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)]\, \forall\, v_1, v_2 \in \mathfrak{g}_- \end{array} \right\} \tag{2.3.7}$$

and

$$\mathfrak{g}_k^{\mathrm{mod}}(\psi) := \left\{ f \in \bigoplus_{i<0} \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_{i+k}(\psi)) \;\middle|\; \begin{array}{l} f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)] \\ \forall\, v_1, v_2 \in \mathfrak{g}_- \end{array} \right\} \qquad \forall\, k > 1, \tag{2.3.8}$$

and the universal Tanaka prolongation of $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ is the space

$$\mathfrak{u}(\psi) := \mathfrak{g}_- \oplus \bigoplus_{k \geq 0} \mathfrak{g}_k^{\mathrm{mod}}(\psi).$$

Note that the antilinear involution on $\mathfrak{g}_-$ naturally induces an involution on $\mathfrak{csp}(\mathfrak{g}_-)$, which in turn induces a natural involution on each modified symbol $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ whenever $\psi \in \Re P^0$.

To state our main result we introduce the following definitions.

Definition 2.3.2. *Fix a CR symbol* $\mathfrak{g}^0 = \mathfrak{g}_- \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$ *of the form in* (2.2.3). *A subspace* $\mathfrak{g}^{0,\mathrm{mod}} = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}$ *of* $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ *is called an abstract modified CR symbol (with involution) of type* $\mathfrak{g}^0$ *if the following properties hold*

1. $\dim \mathfrak{g}_0^{\mathrm{mod}} = \dim \mathfrak{g}_0$*;*

2. $\mathfrak{g}_0^{\mathrm{mod}} \cap (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} = \mathfrak{g}_{0,0}$;

3. $\left(\mathfrak{g}_0^{\mathrm{mod}}\right)_{0,\pm 2} = \mathfrak{g}_{0,\pm 2}$, *where* $\left(\mathfrak{g}_0^{\mathrm{mod}}\right)_{0,\pm 2}$ *stands for the image of the projection to the subspace* $\mathfrak{csp}(\mathfrak{g}_{-1})$ *with respect to the splitting* (2.3.3) *of* $\mathfrak{csp}(\mathfrak{g}_{-1})$*;*

4. *The subspace* $\mathfrak{g}_0^{\mathrm{mod}}$ *is invariant with respect to the involution on* $\mathfrak{g}_{-1} \oplus \mathfrak{csp}(\mathfrak{g}_{-1})$.

Definition 2.3.3. *A point* $\psi_0 \in P^0$ *is* regular *(with respect to the Tanaka prolongation) if the maps* $\psi \mapsto \dim \mathfrak{g}_k^{\mathrm{mod}}$ *are constant in a neighborhood of* ψ_0 *for all* $k \geq 0$.

If we assume that there exists an integer $l \geq 0$ such that the set of points $\psi \in P^0$ with $\mathfrak{g}_l^{\mathrm{mod}}(\psi) = 0$ is generic, then the regularity property in Definition 2.3.3 is also generic.

Remark 2.3.4. *For a 2-nondegenerate CR structure whose Levi form has a rank one kernel such that* $\mathfrak{g}_{0,2}$ *is generated by an element of rank greater than one, the regularity property in Definition 2.3.3 is generic. This result will be shown in a forthcoming paper.*

Now we formulate the main result of this chapter.

Theorem 2.3.5. *Fix a CR symbol* $\mathfrak{g}^0$ *and a corresponding modified CR symbol* $\mathfrak{g}^{0,\mathrm{mod}}$ *so that its universal Tanaka prolongation* $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$ *is finite dimensional.*

1. *Given a* 2*-nondegenerate, hypersurface-type CR structure with symbol* $\mathfrak{g}^0$ *such that there exists a regular point* ψ_0 *(w.r.t. the Tanaka prolongation) in the bundle* $\Re P^0$ *of this structure with* $\mathfrak{g}^{0,\mathrm{mod}}(\psi_0) = \mathfrak{g}^{0,\mathrm{mod}}$*, there exists a bundle over a neighborhood of* ψ_0 *in* $\Re P^0$ *of dimension equal to* $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$ *that admits a canonical absolute parallelism.*

2. *The real dimension of the algebra of infinitesimal symmetries of a* 2*-nondegenerate, hypersurface-type CR structure of the previous item is not greater than* $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$.

The proof of this theorem is obtained via a modification of the approach of [46] to the original Tanaka prolongation procedure for the construction of absolute parallelisms developed in [42]. A sketch of the proof with emphasis of the main modifications is given in Section 2.8.

Let us clarify the meaning of the term canonical absolute parallelism in the formulation of item (1) of Theorem 2.3.5. Similar to the standard Tanaka theory in [42], wherein a sequence of affine fiber bundles

$$\mathbb{C}M \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \cdots$$

are constructed, we will recursively construct a sequence of bundles $\{P^i \to \mathcal{O}^{i-1}\}_{1\leq i\leq l+\mu}$ such that each base space $\mathcal{O}^i$ is a neighborhood in P^i fitting into the sequence of fiber bundles

$$\mathbb{C}M \leftarrow \mathcal{O}^0 \leftarrow \mathcal{O}^1 \leftarrow \mathcal{O}^2 \leftarrow \cdots . \tag{2.3.9}$$

Moreover, each of the spaces in (2.3.9) have an involution defined on them and by restricting the maps in (2.3.9) to their respective fixed point sets one obtains another sequence of fiber bundles

$$M \leftarrow \Re\mathcal{O}^0 \leftarrow \Re\mathcal{O}^1 \leftarrow \Re\mathcal{O}^2 \leftarrow \cdots .$$

The fibers of each P^i are affine spaces with modeling vector space of dimension equal to $\dim \mathfrak{g}_i^{\mathrm{mod}}(\psi_0)$ from (2.3.8). If the positively graded part of the universal Tanaka prolongation $\mathfrak{u}(\psi_0)$ of $\mathfrak{g}^{0,\mathrm{mod}}(\psi_0)$ consists of l nonzero graded components, then all $\mathcal{O}^i$ with $i \geq l$ are identified with each other by the bundle projections, which are diffeomorphisms in those cases. The bundle $\mathcal{O}^{l+2}$ is an e-structure over $\mathcal{O}^{l+1}$, which determines a canonical absolute parallelism on $\mathcal{O}^l$ via aforementioned identification between $\mathcal{O}^{l+1}$ and $\mathcal{O}^l$. It is important to note that for any $0 < i \leq l$ the recursive construction of the bundle $\mathcal{O}^{i+1}$ over $\mathcal{O}^i$ depends on a choice of normalization conditions, as in the standard Tanaka prolongation theory, and also on a choice of identifying spaces, which is a new feature required for the prolongation procedure to work in the presence of nonconstant (modified) symbols. Algebraically, "normalization condition" refers to a choice of vector space complement to the image of a certain Lie algebra cohomology differential along with identifying spaces that are the choices of complementary subspaces to $\mathfrak{g}_k^{\mathrm{mod}}$ in the ith algebraic Tanaka prolongation of the algebra $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$. Therefore, the word "canonical" in item (1) of Theorem 2.3.5 means that for any CR structure of the type referred to in item (1), the same fixed normalization conditions are applied in each step of the construction of the sequence (2.3.9). This also guaranties the preservation the constructed bundles under the action of the group of symmetries of the underlying structure which essentially implies item (2) of the theorem.

2.4 CR structures with constant modified symbols

In this section we study the CR structures with constant modified CR symbol $\mathfrak{g}^{0,\mathrm{mod}}$, meaning that $\mathfrak{g}^{0,\mathrm{mod}}(\psi) = \mathfrak{g}^{0,\mathrm{mod}}$ for every ψ in P^0. We prove that in this case the modified CR symbol must be a Lie algebra (Proposition 2.4.1) and that the CR structure has a regular symbol at every point in the sense of Definition 2.2.3 (Theorem 2.4.2).

Proposition 2.4.1. *If a 2-nondegenerate CR structure of hypersurface type has a constant modified symbol $\mathfrak{g}^{0,\mathrm{mod}}$ then the degree zero component $\mathfrak{g}_0^{\mathrm{mod}}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_-)$.*

Proof. For a point $p \in M$, let $\gamma = \pi(p)$ and let $\varphi_0 \in (P^0)_\lambda$ be fixed. The differential of the projection $\pi : \mathbb{C}M \to \mathcal{N}$ induces an isomorphism

$$\pi_* : \mathfrak{g}_{-1}(p) \oplus \mathfrak{g}_{-2}(p) \to \mathcal{D}_\gamma \oplus T_\gamma\mathcal{N}/\mathcal{D}_\gamma.$$

The fiber $(P^0)_\lambda$ of P^0 can be identified with the submanifold

$$G := \left\{ (\pi_* \circ \varphi_0)^{-1} \circ \pi_* \circ \psi \;\middle|\; \psi \in (P^0)_\lambda \right\}$$

in the Lie group $CSp(\mathfrak{g}_-)$. If we apply the Maurer–Cartan form $\Omega : T\,CSp(\mathfrak{g}_-) \to \mathfrak{csp}(\mathfrak{g}_-)$ of $CSp(\mathfrak{g}_-)$ to a tangent space of G then its image will be a subspace of $\mathfrak{csp}(\mathfrak{g}_-)$, and moreover we can actually show that

$$\Omega(T_gG) = \mathfrak{g}_0^{\mathrm{mod}} \qquad \forall\, g \in G. \tag{2.4.1}$$

Indeed, let us compute $\Omega(T_gG)$. For a point $g = (\pi_* \circ \varphi_0)^{-1} \circ \pi_* \circ \psi \in G$ and a vector $v \in T_gG$, let $\psi : (-\epsilon, \epsilon) \to (P^0)_\lambda$ be a curve with $\psi(0) = \psi$ such that

$$v = (\pi_* \circ \varphi_0)^{-1} \circ \pi_* \left(\psi'(0)\right).$$

The value of the Maurer–Cartan form Ω at v is given by

$$\Omega(v) = \left((\pi_* \circ \varphi_0)^{-1} \circ \pi_* \circ \psi\right)^{-1} \circ (\pi_* \circ \varphi_0)^{-1} \circ \pi_* \left(\psi'(0)\right) = \psi^{-1}\psi'(0) = \theta_0\big(\psi'(0)\big),$$

so indeed $\Omega(v)$ belongs to $\mathfrak{g}_0^{\mathrm{mod}}$. Equation (2.4.1) now follows because $\dim G = \dim P^0 = \dim \mathfrak{g}_0^{\mathrm{mod}}$.

Let us now show that $\mathfrak{g}_0^{\mathrm{mod}}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_-)$. Fix two vectors $v_1, v_2 \in \mathfrak{g}_0^{\mathrm{mod}}$, and set $V_i = \Omega^{-1}(v_i)$; that is, V_i is the left-invariant vector field on $CSp(\mathfrak{g}_{-1})$ whose value at the identity is v_i. Since V_1 and V_2 are both tangent to G at each point in G, the left-invariant vector field $[V_1, V_2]$ is also tangent to G at every point in G. In particular, letting e denote the identity element in $CSp(\mathfrak{g}_-)$, we have

$$[v_1, v_2] = [V_1, V_2]_e \in T_eG = \mathfrak{g}_0^{\mathrm{mod}},$$

which shows that $\mathfrak{g}_0^{\mathrm{mod}}$ is closed under Lie brackets. □

For the remainder of Section 2.4 our goal is to prove the following theorem.

Theorem 2.4.2. *If a 2-nondegenerate CR structure of hypersurface type has a constant modified symbol $\mathfrak{g}^{0,\mathrm{mod}}$ then the CR structure has a regular CR symbol in the sense of Definition 2.2.3 and the modified symbol equals the CR symbol as described in* (2.2.3).

Before proving this theorem we introduce a matrix representation of $\mathfrak{g}_0^{\mathrm{mod}}$ and describe the conditions for $\mathfrak{g}_0^{\mathrm{mod}}$ to be a Lie subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ in terms of this matrix representation. These conditions will be also used for proving the results on non-existence of certain homogeneous CR structures in Section 2.5.

Let ℓ be the Hermitian form on $\mathfrak{g}_{-1,1}$ as in item (1) of Remark 2.2.5. Still setting $r = \mathrm{rank} K$, set $m = \mathrm{rank} H/K = n - r$, and let $\{e_i\}_{i=1}^m$ be a basis of $\mathfrak{g}_{-1,1}$. Let $\{\overline{e_i}\}_{i=1}^m$ be the vectors obtained via the antilinear involution on $\mathfrak{g}_{-1}$. Identify $\mathfrak{g}_{-1,1}$ with $\mathbb{C}^m$ by identifying $(e_1,\ldots,e_m)$ with the standard basis of $\mathbb{C}^m$, and let H_ℓ be the Hermitian matrix representing ℓ with respect to this identification, that is, $\ell(e_i,e_j) = e_j^* H_\ell e_i$, where $(\cdot)^*$ denotes taking the conjugate transpose. We define the basis $(a_1,\ldots,a_{2m})$ of $\mathfrak{g}_{-1}$ by the rule

$$a_i = \begin{cases} e_i & : i \in \{1,\ldots,m\} \\ \overline{e_{i-m+1}} & : i \in \{m+1,\ldots,2m\}. \end{cases} \tag{2.4.2}$$

Let ω be the symplectic form on $\mathfrak{g}_{-1}$ represented by the matrix

$$J_\ell := i\begin{pmatrix} 0 & H_\ell \\ -H_\ell^T & 0 \end{pmatrix};$$

that is, identifying $\mathfrak{g}_{-1}$ with $\mathbb{C}^{2m}$ by identifying $(a_1,\ldots,a_{2m})$ with the standard basis of $\mathbb{C}^{2m}$, we have $\omega(a_i,a_j) = a_j^T J_\ell a_i$.

This symplectic form ω represents the conformal symplectic form previously defined on $\mathfrak{g}_{-1}$. By representing operators with respect to the same basis (i.e., as given in (2.4.2)), we identify the conformal symplectic Lie algebra with a matrix Lie algebra given by

$$\mathfrak{csp}(\mathfrak{g}_{-1}) = \left\{ \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} + cI \;\middle|\; \begin{array}{l} X_{2,2} = -H_\ell^{-1} X_{1,1}^T H_\ell,\ X_{2,1} = H_\ell^{-1} X_{2,1}^T \overline{H_\ell}, \\ X_{1,2} = \overline{H_\ell}^{-1} X_{1,2}^T H_\ell, \text{ and } c \in \mathbb{C} \end{array} \right\}. \tag{2.4.3}$$

Let $C_1,\ldots,C_r$ be matrices such that the spaces $\mathfrak{g}_{0,2}$ and $\mathfrak{g}_{0,-2}$ are spanned by matrices of the form

$$\begin{pmatrix} 0 & C_i \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ \overline{C_i} & 0 \end{pmatrix} \qquad \forall\, i \in \{1,\ldots,r\} \tag{2.4.4}$$

respectively. In what follows, we consider the Lie algebras of square matrices α satisfying

$$\alpha C_i H_\ell^{-1} + C_i H_\ell^{-1} \alpha^T \in \operatorname{span}\{C_j H_\ell^{-1}\}_{j=1}^r \quad \forall\, i \in \{1,\ldots,\mathrm{r}\} \tag{2.4.5}$$

and respectively

$$\alpha^T H_\ell \overline{C_i} + H_\ell \overline{C_i} \alpha \in \operatorname{span}\{H_\ell \overline{C_j}\}_{j=1}^r \quad \forall\, i \in \{1, \ldots, r\}, \tag{2.4.6}$$

and we define the algebra $\mathscr{A}$ to be their intersection, that is,

$$\mathscr{A} := \{\alpha \mid \alpha \text{ satisfies (2.4.5) and (2.4.6)}\}\,. \tag{2.4.7}$$

Remark 2.4.3. *Using* (2.2.4)*, it can be shown by direct computations that the CR symbol* $\mathfrak{g}^0$ *is regular if and only if*

$$C_i \overline{C}_j C_k + C_k \overline{C}_j C_i \in \operatorname{span}_{\mathbb{C}} \{C_s\}_{s=1}^r, \quad \forall i, j, k \in \{1, \ldots, r\}, \tag{2.4.8}$$

where the matrices are as in (2.4.4).

The four properties in Definition 2.3.2 imply that under the identification in (2.4.3), the space $\mathfrak{g}_0^{\mathrm{mod}}$ has a decomposition $\mathfrak{g}_0^{\mathrm{mod}} = \mathfrak{X}_{0,2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{X}_{0,-2}$ such that, for $i \in \{1, \ldots, r\}$ there exist $m \times m$ matrices Ω_i for which $\mathfrak{X}_{0,2}$ and $\mathfrak{X}_{0,-2}$ are spanned by the matrices

$$\begin{pmatrix} \Omega_i & C_i \\ 0 & -H_\ell^{-1}\Omega_i^T H_\ell \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\overline{H_\ell}^{-1}\Omega_i^* \overline{H_\ell} & 0 \\ \overline{C_i} & \overline{\Omega_i} \end{pmatrix} \qquad \forall\, i \in \{1, \ldots, r\} \tag{2.4.9}$$

respectively, and, moreover, $\mathfrak{g}_{0,0}$ consists of block diagonal matrices in terms of the block decomposition given in (2.4.3). By Proposition 2.4.1, $\mathfrak{g}_0^{\mathrm{mod}}$ is a Lie subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$, and hence $\mathfrak{X}_0$ is a matrix Lie algebra. In particular,

$$[\mathfrak{g}_{0,0}, \mathfrak{X}_{0,\pm 2}] \subset \mathfrak{X}_{0,\pm 2} \oplus \mathfrak{g}_{0,0}, \tag{2.4.10}$$

and

$$[\mathfrak{X}_{0,-2}, \mathfrak{X}_{0,2}] \subset \mathfrak{X}_{0,-2} \oplus \mathfrak{X}_{0,2} \oplus \mathfrak{g}_{0,0}. \tag{2.4.11}$$

The following proposition is obtained by straightforward calculation using the identification in (2.4.3) and applying the commutator relations in (2.4.10) and (2.4.11).

Proposition 2.4.4. *The modified CR symbol $\mathfrak{g}^{0,\mathrm{mod}}$ is a Lie subalgebra of $\mathfrak{g}_{-}\rtimes\mathfrak{csp}(\mathfrak{g}_{-1})$ if and only if there exist coefficients $\eta_{\alpha,i}^{s}\in\mathbb{C}$ and $\mu_{i,j}^{s}\in\mathbb{C}$ indexed by $\alpha\in\mathscr{A}$ and $i,j,s\in\{1,\dots,\operatorname{rank}K\}$ such that the system of relations*

$$\left.\begin{array}{ll}
\textit{(i)} & \alpha C_iH_\ell^{-1}+C_iH_\ell^{-1}\alpha^T=\sum_{s=1}^{r}\eta_{\alpha,i}^{s}C_sH_\ell^{-1}\\
\textit{(ii)} & [\alpha,\Omega_i]-\sum_{s=1}^{r}\eta_{\alpha,i}^{s}\Omega_s\in\mathscr{A}\\
\textit{(iii)} & \Omega_j^TH_\ell\overline{C_i}+H_\ell\overline{C_i}\Omega_j=\sum_{s=1}^{r}\mu_{i,j}^{s}H_\ell\overline{C_s}\\
\textit{(iv)} & \left[\overline{H_\ell^{-1}\Omega_i^TH_\ell},\Omega_j\right]+C_j\overline{C_i}-\sum_{s=1}^{r}\left(\overline{\mu_{i,j}^{s}}\Omega_s+\mu_{j,i}^{s}\overline{H_\ell^{-1}\Omega_s^TH_\ell}\right)\in\mathscr{A}
\end{array}\right\}\tag{2.4.12}$$

holds for all $\alpha\in\mathscr{A}$ and $i,j\in\{1,\dots,\operatorname{rank}K\}$. Note that condition (i) on its own is satisfied automatically by the definition of $\mathscr{A}$, but satisfying (i) and (ii) simultaneously with the same coefficients $\eta_{\alpha,i}^{s}$ is not automatic.

Remark 2.4.5. *Under the identification in* (2.4.3)*,*

$$\mathfrak{g}_{0,0}=\mathit{span}_{\mathbb{C}}\left\{\left(\begin{array}{cc}\alpha & 0\\ 0 & -H_\ell^{-1}\alpha^TH_\ell\end{array}\right)+cI\;\middle|\;\alpha\in\mathscr{A}\text{ and }c\in\mathbb{C}\right\}.\tag{2.4.13}$$

Now we are ready to prove Theorem 2.4.2. Since I belongs to $\mathscr{A}$, by setting $\alpha=\frac{1}{2}I$, we get $\eta_{\alpha,i}^{s}=\delta_{i,s}$ from item (i) of (2.4.12), where $\delta_{i,s}$ denotes the Kronecker symbol. Substituting this α and the corresponding $\eta_{\alpha,i}^{s}$ into the equation in item (ii) of (2.4.12), we get that $\Omega_i\in\mathscr{A}$ for all $i\in\{1,\dots,r\}$. Then subtracting the matrix of the form appearing in (2.4.13) with $\alpha=\Omega_i$ from the matrices appearing in (2.4.9) as the generators of $\mathfrak{X}_{0,2}$ and using (2.4.4), we get that the space $\mathfrak{g}_{0,2}$ belongs to $\mathfrak{g}_0^{\mathrm{mod}}$. A similar argument implies that $\mathfrak{g}_{0,-2}$ belongs to $\mathfrak{g}_0^{\mathrm{mod}}$, and hence $\mathfrak{g}_0=\mathfrak{g}_0^{\mathrm{mod}}$. Accordingly, by Proposition 2.4.1, if a CR structure satisfies the hypothesis of Theorem 2.4.2 then

$\mathfrak{g}_0$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ and therefore $\mathfrak{g}^0$ is a regular symbol. This completes the proof of Theorem 2.4.2.

2.5 Reduction to level sets of modified symbols

According to Theorem 2.4.2, a CR structure with a non-regular symbol cannot have a constant modified symbol on P^0. Consequently, for such structures the upper bound for the algebra of infinitesimal symmetries given in Theorem 2.3.5 is far from being sharp in the case of non-regular $\mathfrak{g}^0$ and can be improved under appropriate natural assumptions. The standard way to deal with structures with nonconstant invariants (e.g., the modified CR symbol in our case) is to make a reduction to the level set of these invariants.

In more detail, given an abstract modified CR symbol $\mathfrak{g}^{0,\mathrm{mod}}$ of type $\mathfrak{g}^0$ the set $P^0(\mathfrak{g}^{0,\mathrm{mod}})$ consisting of all $\psi \in P^0$ such that $\mathfrak{g}^{0,\mathrm{mod}}(\psi) = \mathfrak{g}^{0,\mathrm{mod}}$ is called *the level set of* $\mathfrak{g}^{0,\mathrm{mod}}$ in P^0. Assume that $P^0(\mathfrak{g}^{0,\mathrm{mod}})$ is a smooth submanifold of P^0 such that

$$\mathbb{C}M = \mathrm{pr}\left(P^0(\mathfrak{g}^{0,\mathrm{mod}})\right). \tag{2.5.1}$$

The condition in (2.5.1) is motivated by the study of homogeneous CR manifolds, that is, CR manifolds whose groups of symmetries act transitively. If $\theta_0\left(T_\psi P^0(\mathfrak{g}^{0,\mathrm{mod}})\right)$ is the same subspace $\widetilde{\mathfrak{g}^{0,\mathrm{mod}}}$ for all $\psi \in P^0(\mathfrak{g}^{0,\mathrm{mod}})$, then we say that $P^0(\mathfrak{g}^{0,\mathrm{mod}})$ is a *reduction of* P^0 *with constant reduced modified symbol* $\widetilde{\mathfrak{g}^{0,\mathrm{mod}}}$.

If, on the other hand, $\theta_0\left(T_\psi P^0(\mathfrak{g}^{0,\mathrm{mod}})\right)$ is not constant on $P^0(\mathfrak{g}^{0,\mathrm{mod}})$, then we can repeat the process of restriction to a level set. If the chosen level set projects onto a set containing $\mathbb{C}M$ and the image of the tangent spaces to it under θ_0 is a fixed subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ independently of a point of the level set, then we again obtain the reduction of P^0 with constant reduced modified symbol (after two steps of reduction), and if not then we can repeat the process again. In this way, at least in the homogeneous case, after a finite number of steps we will arrive to a submanifold $P^{0,\mathrm{red}}$ of P^0 that projects onto $\mathbb{C}M$, and such that the tangent spaces to it are mapped under θ_0 is a fixed subspace $\mathfrak{g}^{0,\mathrm{red}}$ of $\mathfrak{csp}(\mathfrak{g}_{-1})$. Also, at least in the homogeneous case, in every step of this reduction procedure the level set can be chosen so that it has a nonempty intersection with $\Re P^0$ and hence $\mathfrak{g}^{0,\mathrm{red}}$ will inherit an involution from the involution defined on $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ for any $\psi \in \Re P^0 \cap P^{0,\mathrm{red}}$. In this case we will say that the bundle P^0 associated with the CR structure admits a reduction with constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$.

Note that the subspace $\mathfrak{g}^{0,\mathrm{red}}$ must be a Lie subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ by literally the same arguments as in the proof of Proposition 2.4.1. These constructions motivate the following definition, which generalizes Definition 2.3.2.

Definition 2.5.1. *Fix a CR symbol* $\mathfrak{g}^0 = \mathfrak{g}_- \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$ *of the form in* (2.2.3). *A Lie subalgebra* $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{red}}$ *of* $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ *is called an abstract reduced modified CR symbol (with involution) of type* $\mathfrak{g}^0$ *if the following properties hold:*

1. $\dim \mathfrak{g}_0^{\mathrm{red}} = \dim\left(\mathfrak{g}_0^{\mathrm{red}} \cap \mathfrak{g}_{0,0}\right) + 2\dim \mathfrak{g}_{0,2} \leq \dim \mathfrak{g}_0$*;*

2. $\mathfrak{g}_0^{\mathrm{red}} \cap \left(\mathfrak{csp}(\mathfrak{g}_{-1})\right)_{0,0} \subset \mathfrak{g}_{0,0}$*;*

3. $\left(\mathfrak{g}_0^{\mathrm{red}}\right)_{0,\pm 2} = \mathfrak{g}_{0,\pm 2}$*, where* $\left(\mathfrak{g}_0^{\mathrm{red}}\right)_{0,\pm 2}$ *stands for the image under the projection to the subspace* $\left(\mathfrak{csp}(\mathfrak{g}_{-1})\right)_{0,\pm 2}$ *with respect to the splitting* (2.3.3) *of* $\mathfrak{csp}(\mathfrak{g}_{-1})$*;*

4. *The subspace* $\mathfrak{g}_0^{\mathrm{red}}$ *is invariant with respect to the involution on* $\mathfrak{g}_{-1} \oplus \mathfrak{csp}(\mathfrak{g}_{-1})$.

To any abstract reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, we construct corresponding special homogeneous CR structures as follows. Set $\mathfrak{g}_{0,0}^{\mathrm{red}} := \mathfrak{g}_0^{\mathrm{red}} \cap \left(\mathfrak{csp}(\mathfrak{g}_{-1})\right)_{0,0}$. Denote by $G^{0,\mathrm{red}}$ and $G_{0,0}^{\mathrm{red}}$ connected Lie groups with Lie algebras $\mathfrak{g}^{0,\mathrm{red}}$ and $\mathfrak{g}_{0,0}^{\mathrm{red}}$, respectively, such that $G_{0,0}^{\mathrm{red}} \subset G^{0,\mathrm{red}}$, and denote by $\Re G^{0,\mathrm{red}}$ and $\Re G_{0,0}^{\mathrm{red}}$ the corresponding real parts with respect to the involution on $\mathfrak{g}^{0,\mathrm{red}}$, meaning that $\Re G^{0,\mathrm{red}}$ and $\Re G_{0,0}^{\mathrm{red}}$ are the maximal subgroups of $G^{0,\mathrm{red}}$ and $G_{0,0}^{\mathrm{red}}$ whose tangent spaces belong to the left translations of the fixed point set of the involution on $\mathfrak{g}^{0,\mathrm{red}}$ and on $\mathfrak{g}_{0,0}^{\mathrm{red}}$ respectively.

Let $M_0^{\mathbb{C}} = G^{0,\mathrm{red}}/G_{0,0}^{\mathrm{red}}$ and $M_0 = \Re G^{0,\mathrm{red}}/\Re G_{0,0}^{\mathrm{red}}$. In both cases here we use left cosets. For every pair (i,j) with $i < 0$, let $\widehat{D}_{i,j}^{\mathrm{flat}}$ be the left-invariant distribution on $G^{0,\mathrm{red}}$ such that it is equal to $\mathfrak{g}_{i,j}$ at the identity. Also, for $j=\pm 2$, let $\widehat{D}^{\text{flat}}_{0,\pm 2}$ be the left-invariant distributions equal to

$$\mathfrak{g}_0^{\text{red}} \cap \Big((\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} \oplus (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,j} \Big)$$

at the identity. Since all $\mathfrak{g}_{i,j}$ are invariant under the adjoint action of $G^{\text{red}}_{0,0}$, the push-forward of each $\widehat{D}^{\text{flat}}_{i,j}$ to $M_0^{\mathbb{C}}$ is a well defined distribution, which we denote by $D^0_{i,j}$. Let D^0_{-1} be the distribution which is the sum of $D^{\text{flat}}_{i,j}$ with $i=-1$. We restrict all of these distributions to M_0, considering them as subbundles of the complexified tangent bundle of M_0. The distribution $H^{\text{flat}} := D^{\text{flat}}_{-1,1} \oplus D^{\text{flat}}_{0,2}$ defines a CR structure of hypersurface type, and, by construction, the corresponding bundle P^0 associated with this CR structure admits a reduction with constant reduced modified symbol $\mathfrak{g}^{0,\text{red}}$. The structure H^{flat} on the constructed homogeneous model M_0 is called the *flat CR structure with constant reduced modified symbol* $\mathfrak{g}^{0,\text{red}}$.

Theorem 2.5.2. *Assume that for a given a CR symbol $\mathfrak{g}^0$ there exists a reduced modified symbol $\mathfrak{g}^{0,\text{red}}$ of type $\mathfrak{g}^0$ with finite dimensional Tanaka prolongation. Then the following three statements hold.*

1. *Given a 2-nondegenerate, hypersurface-type CR structure such that the corresponding bundle P^0 contains a subbundle $P^{0,\text{red}}$ with the reduced modified symbol $\mathfrak{g}^{0,\text{red}}$, there exists a bundle over $\Re P^{0,\text{red}} := P^{0,\text{red}} \cap \Re P^0$ of dimension equal to $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\text{red}})$ that admits a canonical absolute parallelism;*

2. *The dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate, hypersurface-type CR structure of item (1) is not greater than $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\text{red}})$. Moreover, if we assume that the CR symbol $\mathfrak{g}^0$ is recoverable then the algebra of infinitesimal symmetries of the flat CR structure with constant reduced modified symbol $\mathfrak{g}^{0,\text{red}}$ is isomorphic to the real part $\Re\mathfrak{u}(\mathfrak{g}^{0,\text{red}})$ of $\mathfrak{u}(\mathfrak{g}^{0,\text{red}})$;*

3. *If the CR symbol $\mathfrak{g}^0$ is recoverable then any CR structure with the constant reduced modified symbol $\mathfrak{g}^{0,\text{red}}$ whose algebra of infinitesimal symmetries has real dimension equal to*

$\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ *is locally to equivalent to the flat CR structure with reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$.

Items (1) and (2) of this theorem follow from the standard Tanaka theory for constant symbols [42] applied, after the reduction of the bundle P^0, to the bundle $P^{0,\mathrm{red}}$. Item (3) follows from the fact that $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ always contains the grading element E; that is, the element such that $[E, x] = ix$, for $x \in \mathfrak{g}_i$ with $i \in \{-1, -2\}$, which also implies that $[E, x] = 0$ for $x \in \mathfrak{g}_0^{\mathrm{red}}$. This grading element is the generator of the natural family of dilations on the fibers of P^0. Explicitly, given a point $\psi_0 \in P^0$, the grading element in $\mathfrak{g}^{0,\mathrm{red}}$ is the velocity at $t = 0$ of the curve $\psi(t)$ such that $\psi(t)|_{\mathfrak{g}_{-2}} = I$ and $\psi(t)|_{\mathfrak{g}_{-1}} = e^t\, \psi_0|_{\mathfrak{g}_{-1}}$. The fact that the grading element is tangent to the level sets after reduction follows from the fact that this orbit $\psi(t)$ belongs to the same level set of modified CR symbols. Further, the algebra of infinitesimal symmetries of a CR structure has a natural filtration such that, in the case where the dimension of this algebra is equal to $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$, the associated graded algebra is isomorphic to $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$. The existence of this grading element in the reduced symbol implies that the filtered algebra of infinitesimal symmetries is isomorphic to its associated graded algebra (considered as filtered algebras) (see [12, Lemma 3]), that is, to $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$, which implies that the CR structure is locally equivalent to the flat one.

We conclude this section with the following corollary, which relates our Theorem 2.5.2 with the main theorem of [33, Theorem 3.2].

Corollary 2.5.3. *If the CR symbol* $\mathfrak{g}^0$ *is regular and recoverable then its usual Tanaka prolongation and the bigraded Tanaka prolongation defined in [33, section 3] coincide.*

Proof. In the case of regular $\mathfrak{g}^0$ there is a flat CR structure with the constant modified symbol equal to $\mathfrak{g}^0$ so that there is no reduction of the bundle P^0, and, from the assumption of recoverability, item (2) of Theorem 2.5.2 gives the same algebra of infinitesimal symmetry as item (2) of [33, Theorem 3.2]. The former algebra is the usual Tanaka prolongation of $\mathfrak{g}^0$ whereas the latter is the bigraded one. □

Note that without the assumption of recoverability the statement of the previous corollary is wrong. For example, for $\operatorname{rank} K = 1$, if $\operatorname{ad}K$ is generated by a rank 1 operator, then the usual Tanaka prolongation is infinite dimensional and the bigraded Tanaka prolongation is not.

2.6 Generic CR symbols and nonexistence of homogeneneous models

In this section we prove the following theorem.

Theorem 2.6.1. *For any fixed rank $r > 1$, in the set of all CR symbols associated with 2-nondegenerate, hypersurface-type CR manifolds of odd dimension greater than $4r + 1$ with rank r Levi kernel and with reduced Levi form of arbitrary signature, the CR symbols not associated with any homogeneous model are generic. For $r = 1$, the same statement holds if the reduced Levi form is sign-definite, that is, when the CR structure is pseudoconvex.*

Remark 2.6.2. *We believe that the pseudoconvexity assumption in the case of $r = 1$ is not essential and can be omitted through more subtle analysis of the corresponding modification of system* (2.4.12) *than the analysis we apply for the pseudoconvex case (see Remark 2.6.6 below for more detail). For the discussion on sharpness of the lower bounds for the dimension of the ambient manifold see Remark 2.6.4 below. The goal here is to exhibit the phenomena of non-existence of homogeneous models for generic basic data such as the CR symbol rather than to get the most general results in this direction.*

Proof. The proof consists of a series of lemmas:

The following lemma is about the structure of the algebra $\mathscr{A}$, defined in (2.4.7), for generic CR symbols. Note that the inclusion

$$\{sI \mid s \in \mathbb{C}\} \subset \mathscr{A} \tag{2.6.1}$$

always holds.

Lemma 2.6.3. *For any fixed rank r, in the set of all CR symbols associated with 2-nondegenerate, hypersurface-type CR manifolds of odd dimension greater than $4r + 1$ with rank r Levi kernel, the subset of CR symbols such that the algebra*

$$\mathscr{A} = \{sI \mid s \in \mathbb{C}\} \tag{2.6.2}$$

is generic.

Proof. Fix a CR symbol $\mathfrak{g}^0$, and, still using $m = n - r$, let H_ℓ and $\{C_j\}_{j=1}^r$ be the $m \times m$ matrices associated with $\mathfrak{g}^0$ as in (2.4.4), where H_ℓ represents the reduced Levi form. Since the system for the algebra $\mathscr{A}$ given by (2.4.5)-(2.4.6) is overdetermined and linear (in α) and the inclusion (2.6.1) always holds, to prove our lemma it is enough to prove that for fixed signature of the reduced Levi form ℓ (or equivalently, signature of the Hermitian matrix H_ℓ) there exists at least one tuple of matrices $\{C_j\}_{j=1}^r$ for which (2.6.2) holds.

Assume that the matrices C_j are nonsingular for all $1 \leq j \leq r$. If

$$A_i = \alpha C_i H_\ell^{-1}, \tag{2.6.3}$$

with α satisfying (2.4.5), then we have that $A_i + A_i^T \in \operatorname{span}\{C_j H_\ell^{-1}\}_{j=1}^r$. Recalling that the set of solutions of the matrix equation $A + A^T = S$ (with respect to A for a fixed symmetric matrix S) is $\frac{1}{2}S + \mathfrak{so}(m)$ and that $\alpha = A H_\ell C_i^{-1}$ from (2.6.3), we have that α satisfies system (2.4.5) if and only if

$$A_i \in \operatorname{span}\{C_j H_\ell^{-1}\}_{j=1}^r + \mathfrak{so}(m) \qquad \forall\, i \in \{1, \ldots, m\},$$

which is equivalent to

$$\alpha \in \bigcap_{i=1}^r \left(\operatorname{span}\{C_j C_i^{-1}\}_{j=1}^r + \mathfrak{so}(m) H_\ell C_i^{-1}\right). \tag{2.6.4}$$

Similar analysis of (2.4.6) implies that α satisfies system (2.4.6) if and only if

$$\alpha \in \bigcap_{i=1}^r \left(\operatorname{span}\left\{\overline{C}_i^{-1}\overline{C}_j\right\}_{j=1}^r + \overline{C}_i^{-1} H^{-1}\mathfrak{so}(m)\right).$$

Hence, the algebra $\mathscr{A}$ satisfies

$$\mathscr{A} = \bigcap_{i=1}^r \left(\left(\operatorname{span}\left\{C_j C_i^{-1}\right\}_{j=1}^r + \mathfrak{so}(m) H_\ell C_i^{-1}\right) \bigcap \left(\operatorname{span}\left\{\overline{C}_i^{-1}\overline{C}_j\right\}_{j=1}^r + \overline{C}_i^{-1} H^{-1}\mathfrak{so}(m)\right)\right). \tag{2.6.5}$$

Choose a basis in $\mathfrak{g}_{-1,1}$ such that

$$H_\ell = \operatorname{diag}(\underbrace{1,\ldots,1}_{q\,\text{entries}},-1,\ldots,-1). \tag{2.6.6}$$

We consider the splitting of $\mathfrak{gl}(m)$ into the space $\mathfrak{gl}^{\text{diag}}(m)$ of diagonal $m \times m$ matrices and the space $\mathfrak{gl}^{\text{holl}}(m)$ of $m \times m$ matrices with all zeros on the diagonal, sometimes called hollow matrices,

$$\mathfrak{gl}(m) = \mathfrak{gl}^{\text{diag}}(m) \oplus \mathfrak{gl}^{\text{holl}}(m). \tag{2.6.7}$$

Since, by our assumptions, $m > r$, we can take a special tuple $\{C_j\}_{j=1}^r$ such that every matrix C_j is nonsingular and diagonal, that is,

$$C_j = \operatorname{diag}(\lambda_{j,1},\ldots\lambda_{j,m}). \tag{2.6.8}$$

Accordingly, for every $i \in \{1,\ldots r\}$, we have the following inclusions:

1. The spaces $\operatorname{span}\left\{C_j C_i^{-1}\right\}_{j=1}^r$ and $\operatorname{span}\left\{\overline{C}_i^{-1}\overline{C}_j\right\}_{j=1}^r$ belong to $\mathfrak{gl}^{\text{diag}}(m)$;
2. The spaces $\mathfrak{so}(m)H_\ell C_i^{-1}$ and $\overline{C}_i^{-1}H^{-1}\mathfrak{so}(m)$ belong to $\mathfrak{gl}^{\text{holl}}(m)$.

Based on the splitting in (2.6.7), for $\alpha \in \mathfrak{gl}(m)$, we let $\alpha^{\text{diag}} \in \mathfrak{gl}^{\text{diag}}(m)$ and $\alpha^{\text{holl}} \in \mathfrak{gl}^{\text{holl}}(m)$ denote the matrices for which

$$\alpha = \alpha^{\text{diag}} + \alpha^{\text{holl}}.$$

From (2.6.5) and the splitting in (2.6.7) it follows that

$$\alpha^{\text{holl}} \in \bigcap_{i=1}^r \left(\left(\mathfrak{so}(m)H_\ell C_i^{-1}\right) \bigcap \left(\overline{C}_i^{-1}H^{-1}\mathfrak{so}(m)\right)\right) \qquad \forall\, \alpha \in \mathscr{A} \tag{2.6.9}$$

and

$$\alpha^{\text{diag}} \in \bigcap_{i=1}^r \left(\left(\operatorname{span}\left\{C_j C_i^{-1}\right\}_{j=1}^r\right) \bigcap \left(\operatorname{span}\left\{\overline{C}_i^{-1}\overline{C}_j\right\}_{j=1}^r\right)\right) \qquad \forall\, \alpha \in \mathscr{A}. \tag{2.6.10}$$

In particular, for a fixed matrix $\alpha \in \mathscr{A}$, by (2.6.4) and (2.6.9), there exists B and $\widetilde{B}$ in $\mathfrak{so}(m)$ such that

$$\alpha^{\text{holl}} = BH_\ell C_1^{-1} = \overline{C}_1^{-1} H^{-1} \widetilde{B}. \tag{2.6.11}$$

Let

$$\varepsilon_i^q = \begin{cases} 1, & 1 \leq i \leq q, \\ -1, & q+1 \leq i \leq m, \end{cases}$$

and in the following calculation we denote the (i,j) entry of a matrix X by $X_{i,j}$. Using (2.6.6) and (2.6.8), we get that

$$(BH_\ell C_1^{-1})_{i,j} = \frac{\varepsilon_j^q B_{i,j}}{\lambda_{1,j}} \quad \text{and} \quad \left(\overline{C}_1^{-1} H^{-1} B\right)_{i,j} = \frac{\varepsilon_i^q \widetilde{B}_{i,j}}{\overline{\lambda_{1,i}}}.$$

From this and (2.6.11) we have

$$B_{i,j} = \frac{\varepsilon_i^q \varepsilon_j^q \lambda_{1,j}}{\overline{\lambda_{1,i}}} \widetilde{B}_{i,j},$$

and hence, since B and $\widetilde{B}$ are skew symmetric,

$$\frac{\varepsilon_i^q \varepsilon_j^q \lambda_{1,j}}{\overline{\lambda_{1,i}}} \widetilde{B}_{i,j} = B_{i,j} = -B_{j,i} = -\frac{\varepsilon_i^q \varepsilon_j^q \lambda_{1,i}}{\overline{\lambda_{1,j}}} \widetilde{B}_{j,i} = \frac{\varepsilon_i^q \varepsilon_j^q \lambda_{1,i}}{\overline{\lambda_{1,j}}} \widetilde{B}_{i,j}. \tag{2.6.12}$$

If we now assume that

$$|\lambda_{1,i}| \neq |\lambda_{1,j}| \qquad \forall\, i \neq j \tag{2.6.13}$$

then it follows from (2.6.12) that $B = 0$, and hence $\alpha^{\text{holl}} = BH_\ell C_1^{-1} = 0$, that is, $\alpha = \alpha^{\text{diag}}$.

In other words, (2.6.13) implies that (2.6.5) can be simplified to

$$\mathscr{A} = \bigcap_{i=1}^{r} \left(\left(\operatorname{span} \{C_j C_i^{-1}\}_{j=1}^{r} \right) \bigcap \left(\operatorname{span} \left\{ \overline{C}_i^{-1} \overline{C}_j \right\}_{j=1}^{r} \right) \right). \tag{2.6.14}$$

If $r = 1$ then (2.6.14) is equivalent to (2.6.2), which is what we wanted to show, so let us now assume that $r > 1$.

Now use that if $\alpha \in \mathcal{A}$ then there exist $\{\nu_{i,j}\}_{i,j=1}^r$ such that

$$\alpha = \sum_{j=1}^{r} \nu_{i,j} C_j C_i^{-1} = \nu_{i,i} I + \sum_{j \neq i} \nu_{i,j} C_j C_i^{-1}, \tag{2.6.15}$$

which implies by comparing the diagonal entries that for every $i > 1$

$$\sum_{j=1}^{r} \frac{\lambda_{j,s}}{\lambda_{1,s}} \nu_{1,j} = \sum_{j=1}^{r} \frac{\lambda_{j,s}}{\lambda_{i,s}} \nu_{i,j}, \quad \forall s = \{1, \dots, m\}$$

or, equivalently,

$$\sum_{j>1} \lambda_{j,s} \lambda_{i,s} \nu_{1,j} - \sum_{1 \le j \le r, j \neq i} \lambda_{j,s} \lambda_{1,s} \nu_{i,j} + \lambda_{1,s} \lambda_{i,s} (\nu_{1,1} - \nu_{i,i}) = 0, \quad \forall 1 \le s \le m, 2 \le i \le r, \tag{2.6.16}$$

which is a system of $(r-1)m$ linear homogeneous equations with respect to $r^2 - 1$ unknowns $\{\nu_{i,j} | 1 \le i, j \le r, i \neq j\}$ and $\{\nu_{1,1} - \nu_{i,i} | 2 \le i \le r\}$.

It remains to note that $(r-1)m \ge r^2 - 1$ if and only if $m > r$ and for generic tuples $\{\lambda_{i,s} | 1 \le i \le r, 1 \le s \le m\}$ the rank of the matrix in the system (2.6.16) is $r^2 - 1$. Indeed, this matrix (after appropriate rearrangement of columns) has a block-triangular form with $r - 1$ diagonal blocks.

The maximal size diagonal block has size $m \times (2r-2)$ and it is obtained from columns that use variables appearing in the equations in system (2.6.16) with $i = 2$ (i.e., $\{\nu_{1,j} | j > 1\}$, $\{\nu_{2,j} | j > 1\}$) and $\{\nu_{1,1} - \nu_{2,2} | 2 \le i \le r\}$ and its sth row consists of evaluations of quadratic monomials $x_i x_j$, $i \le j$ such that at least one i or j takes values in $\{1, 2\}$ at the point $(x_1, \dots x_r) = (\lambda_{1,s}, \dots, \lambda_{r,s})$. This diagonal block has maximal rank at generic points; the determinant of each of its maximal minors is a nonzero polynomial in the λ's, because when calculating this minor as the alternating (according to the signature of permutations) sum of the corresponding products of the entries of the submatrix corresponding to this minor there cannot be cancellation, as each term of this alternating sum gives a unique monomial. The latter follows from the fact that distinct rows of this diagonal block depend on disjoint sets of variables and that the monomials at different columns are different.

The other $r-2$ diagonal blocks are of size $m \times r$ and they are parameterized by $i > 2$. The block corresponding to a given $i > 2$ is obtained from columns of the matrix of the system (2.6.16) that correspond to the variables $\{\nu_{i,j} | 1 \leq j \leq r, j \neq i\}$ and $\eta_{1,1} - \eta_{i,i}$. Similar to the maximal size diagonal block, the sth row of the diagonal block under consideration consists of evaluations of quadratic monomials $x_1 x_j$, $1 \leq j \leq r$ such that at the point $(x_1, \ldots x_r) = (\lambda_{1,s}, \ldots, \lambda_{r,s})$. This diagonal block has maximal rank at generic points by the same reason as in the previous paragraph.

Therefore, the matrix corresponding to System (2.6.16) has maximal rank, which implies that $\nu_{i,j} = 0$ for all $i \neq j$ and $\nu_{i,i} = \nu_{1,1}$ for all i. So, by (2.6.15) the matrix α must be a multiple of identity, which proves (2.6.2). $\square$

Remark 2.6.4. *The lower bound $4r + 3$ for the dimension of manifold in Lemma 2.6.3 is sharp for $r = 1$, as for 5-dimensional manifold there is only one CR symbol and it is regular and does not satisfy* (2.6.2). *However, for $r > 1$ this bound is strictly greater than the minimal dimension for which non-regular CR symbols exist, so we expect that this bound is not sharp, but our method of proof using diagonal C's, cannot improve it.*

Lemma 2.6.5. *For fixed $n = \operatorname{rank} H$ and $r = \operatorname{rank} K$ such that the strict inequality in* (1.0.4) *holds, the non-regular symbols constitute a generic subset in the set of all CR symbols.*

Proof. We prove this using the same principle that was applied for the proof of Lemma 2.6.3; that is, we will characterize non-regularity as nonsolvability of a certain overdetermined algebraic system and then find one example of a CR symbol for which this system has no solution.

Given a CR symbol $\mathfrak{g}_0$ represented by the matrices $\{C_i\}_{i=1}^r$ as in (2.4.4), the condition for regularity of $\mathfrak{g}_0$ in Remark 2.4.3 is given by the system of equations (2.4.8).

Let $m = n - r$ as before. First consider the case when $n > 2r$ or $m > r$. Working with respect to a basis of $\mathfrak{g}_{-1,1}$ such that H_ℓ is as in (2.6.6), choose $\{C_i\}_{i=1}^r$ and $\{\lambda_{i,s}\}_{1 \leq i \leq r, 1 \leq s \leq m}^m$ satisfying

(2.6.8). The system (2.4.8) can be rewritten as

$$\sum_{l=1}^{r} \nu_{i,j,k}^{l} \lambda_{l,s} = 2\lambda_{i,s}\overline{\lambda_{j,s}}\lambda_{k,s} \qquad \forall\, i,j,k \in \{1,\dots,r\}, s \in 1,\dots,m \tag{2.6.17}$$

for some unknowns $\{\nu_{i,j,k}^{l}\}$. Fix the triple i,j,k and consider the $m \times (r+1)$ matrix such that its sth row consists of evaluations of monomials x_l with $1 \leq l \leq r$ and $2x_i\overline{x}_j x_k$ at the point $(x_1, \dots x_r) = (\lambda_{1,s}, \dots, \lambda_{r,s})$. By assumption $m > r$, so the solvability of the linear system (2.6.17) with respect to $\{\nu_{i,j,k}^{l}\}_{l=1}^{m}$ is equivalent to the fact that this matrix, which is exactly the augmented matrix of this nonhomogeneous system, has rank not greater than r. On the other hand, by the same arguments applied at the end of the proof of Lemma 2.6.3 this matrix has rank $r+1$ for a generic tuple of diagonal matrices $\{C_i\}_{i=1}^{m}$, so the system is not solvable generically.

Now consider the case when $m \leq r$ but the strict equality in (1.0.4) holds. Note that this implies $m > 1$. Choose the tuple $\{C_i\}_{i=1}^{r}$ such that the first $m-1$ elements in it are diagonal as in (2.6.8) and the rest have zero on the diagonal and consider the matrix equation (2.4.8) for example for $i = j = k = 1$. By construction, using the splitting (2.6.7), we get that

$$C_1\overline{C_1}C_1 \in \operatorname{span}\{C_i\}_{i=1}^{m-1},$$

which yields exactly the same relation as in the case $r = m-1$, and we can repeat the argument of the case $m > r$. □

Now we are ready to prove our theorem. We will show that as a desired generic set of CR symbols in the theorem one can take the set of non-regular symbols with the algebra $\mathscr{A}$ satisfying (2.6.2) and maybe some additional generic conditions.

As is done in the proof of Lemma 2.6.3, after fixing a symbol $\mathfrak{g}^0$ with these generic properties, we let H_ℓ and $\{C_j\}_{j=1}^{r}$ be a set of $m \times m$ matrices associated with $\mathfrak{g}^0$. There exists a reduced modified CR symbol of type $\mathfrak{g}^0$ if and only if there exist $m \times m$ matrices $\{\Omega_j\}_{j=1}^{r}$ such that the system of relation (2.4.12) can be satisfied after replacing the algebra $\mathscr{A}$ with some subalgebra

$\mathscr{A}_0 \subset \mathscr{A}$. So, to produce a contradiction, let us assume that there exists a reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ of type $\mathfrak{g}^0$, and fix such corresponding $\{\Omega_j\}_{j=1}^r$ and $\mathscr{A}_0$. In particular, under the identification in (2.4.3), $\mathfrak{g}^{0,\mathrm{red}}$ is spanned by matrices of the form in (2.4.9) along with block diagonal matrices of the form in (2.4.13) but with $\mathscr{A}$ replaced by $\mathscr{A}_0$.

If I belongs to the subalgebra $\mathscr{A}_0$ then, for each i, the first two conditions in (2.4.12) imply that Ω_i belongs to $\mathscr{A}_0$, but this implies that $\mathfrak{g}^0$ is regular, contradicting our assumptions. So I does not belong to $\mathscr{A}_0$, and hence, by (2.6.2),

$$\mathscr{A}_0 = 0. \tag{2.6.18}$$

Similar to arguments of Lemma 2.6.3, since the system of relations (iii) and (iv) from (2.4.12) with $\mathscr{A}_0 = 0$ is overdetermined and algebraic (with respect to the unknown matrices Ω_i), then by the classical elimination theory in order to prove generic nonexistence of solutions of this system it is enough to prove that for fixed signature of the reduced Levi form ℓ (or, equivalently, signature of the Hermitian matrix H_ℓ) there exists at least one tuple of matrices $\{C_j\}_{j=1}^r$ for which this system of equation is incompatible.

First consider the case $r > 1$, which is simpler. For each i, condition (iii) of (2.4.12) means that $\alpha = \Omega_i$ satisfies (2.4.6), or equivalently, $\alpha = \overline{H_\ell}^{-1}\Omega_i^*\overline{H_\ell}$ satisfies (2.4.5) and we can repeat the arguments of the proof of Lemma 2.6.3 after formula (2.6.14) to show that for generic tuple of diagonal matrices $\{C_j\}_{j=1}^r$ the matrix Ω_i^{diag} is a multiple of the identity matrix, noticing that in that part of the proof of Lemma 2.6.3 we only used that α satisfies (2.4.5).

Now we apply an argument similar to the one in the proof of Lemma 2.6.3 between (2.6.11) and (2.6.14) to conclude that $\Omega_i^{\mathrm{holl}} = 0$. In more detail, by analogy with (2.6.9) , taking into account that $\alpha = \Omega_i$ satisfies (2.4.6) only, we have that

$$\Omega_i^{\mathrm{holl}} \in \bigcap_{j=1}^{r} \left(\overline{C}_j^{-1} H_\ell^{-1}\mathfrak{so}(m)\right)$$

and therefore by analogy with with (2.6.11), there exist matrices B and $\widetilde{B}$ in $\mathfrak{so}(m)$ such that

$$\Omega_i^{\mathrm{holl}} = \overline{C}_1^{-1} H_\ell^{-1} B = \overline{C}_2^{-1} H_\ell^{-1} \widetilde{B}. \tag{2.6.19}$$

Comparing entries in (2.6.19) and using skew-symmetricity of B and $\widetilde{B}$, we get that in order to guarantee that $\Omega^{\mathrm{holl}} = 0$ we should replace the condition in (2.6.13) by the condition that

$$\begin{vmatrix} \lambda_{1,i} & \lambda_{1,j} \\ \lambda_{2,i} & \lambda_{2,j} \end{vmatrix} \neq 0, \quad \forall i \neq j.$$

Therefore, for a generic tuple of diagonal matrices $\{C_j\}_{j=1}^r$, we get that $\Omega_i = sI \in \mathscr{A}$ and so the symbol is regular, contradicting our assumptions.

Now consider the remaining case, which is where $r = 1$ and H_ℓ is positive definite. Working with respect to a basis of $\mathfrak{g}_{-1,1}$ such that $H_\ell = I$, by (2.6.18) and condition (iv) in (2.4.12), we have

$$\sum_{s=1}^{r} \left(\overline{\mu_{1,1}^s} \Omega_s + \mu_{1,1}^s \Omega_s^* \right) = [\Omega_1^*, \Omega_1] + C_1 \overline{C_1}. \tag{2.6.20}$$

By analogy with (2.6.10) with $r = 1$, taking into account that $\alpha = \Omega_i$ satisfies (2.4.6) only, we have that

$$\Omega_1^{\mathrm{diag}} \in \mathrm{span}\{I\}.$$

Note also that (2.6.19) holds with $r = 1$ and $H_\ell = I$. Fix $\mu \in \mathbb{C}$ and $B \in \mathfrak{so}(m)$ such that

$$\Omega_1^{\mathrm{diag}} = \mu I \quad \text{and} \quad \Omega_1^{\mathrm{holl}} = \overline{C_1}^{-1} B.$$

Using the notation set in (2.6.8), the (i, j) element of $[\Omega_1^*, \Omega_1]$ satisfies

$$([\Omega_1^*, \Omega_1])_{i,j} = \left(\left[\overline{B} C_1^{T^{-1}}, \overline{C_1}^{-1} B \right] \right)_{i,j} = \sum_{k=1}^{m} \left(\frac{1}{|\lambda_{1,k}|^2} - \frac{1}{\overline{\lambda_{1,i}} \lambda_{1,j}} \right) B_{k,i} \overline{B_{k,j}},$$

and, for $1 \le i \le m$, by equating the (i,i) elements of the matrices on each side of (2.6.20) we get

$$2\Re\mu = |\lambda_{1,i}|^2 + \sum_{k=1}^{m} \left(\frac{1}{|\lambda_{1,k}|^2} - \frac{1}{|\lambda_{1,i}|^2} \right) |B_{k,i}|^2 \qquad \forall\, i \in \{1, \ldots, m\}. \tag{2.6.21}$$

Let i_0 be the index such that $|\lambda_{1,i_0}| = \max\{|\lambda_{1,1}|, \ldots, |\lambda_{1,m}|\}$. Accordingly, every term on the right side of (2.6.21) is nonnegative, and hence

$$|\lambda_{1,i_0}|^2 \le 2\Re\mu. \tag{2.6.22}$$

On the other hand, taking the trace of both sides of (2.6.20) yields

$$2m\Re\mu = \sum_{i=1}^{m} |\lambda_{1,i}|^2 < m\, |\lambda_{1,i_0}|^2\,, \tag{2.6.23}$$

where the strict inequality is obtained by imposing the assumption in (2.6.13). Clearly (2.6.22) and (2.6.23) are incompatible, which means that, for the chosen C_1, no choice of Ω_1 satisfies (2.4.12). $\square$

Remark 2.6.6. *The last arguments of the previous proof do not work in the case with* $r = 1$ *and sign-indefinite* H_ℓ*. There, additional analysis of the equations obtained by comparing off-diagonal entries in the matrix equation given by condition (iv) of* (2.4.12) *is needed. Specifically, in general, for each* n *and each* $(n-1)\times(n-1)$ *matrix* H_ℓ *representing the Hermitian form* ℓ*, one needs to find a single matrix* C *representing a nonzero* ℓ*-self-adjoint antilinear operator such that the system* (4.2.4) *is inconsistent, from which it then follows that (repeating the arguments used above), for this given* H_ℓ*, the set of* ℓ*-self-adjoint antilinear operators represented by a matrix* C *for which* (4.2.4) *cannot be satisfied is a nontrivial Zariski open subset in the vector space of all* ℓ*-self-adjoint antilinear operators. In fact this need only be done for each signature of* H_ℓ*, because for a given signature of* H_ℓ *one can always work in a basis with respect to which* H_ℓ *has a cononical form. In Chapter* (5)*, for each* H_ℓ *corresponding to* $n = 3$ *and* $n = 4$ *we obtain examples of* C *for which the* (4.2.4) *cannot be satisfied, and consequently generalize the generic nonexistence result* *of this chapter to include sign-indefinite H_ℓ in low dimensions.*

2.7 Examples

Although Theorem 2.6.1 shows that for non-regular CR symbols homogeneous models appear rarely, they still exist. In this section we describe three examples of CR structures to which Theorems 2.3.5 and 2.5.2 apply. All three examples are actually homogeneous CR manifolds, which means that they exhibit the maximally symmetric structures described in Theorem 2.5.2, and they illustrate novel applications of this chapter's main results.

The first example is non-regular in the sense of Definition (2.2.3). This section's second and third examples have the same CR symbol in the sense of [33, Definition 2.2] but different modified CR symbols, and, as is mentioned in the introduction, while the construction of an absolute parallelism given in [33] is the same for both examples, the construction given here varies, resulting in parallelisms of different dimensions for each example whose dimension matches that of the underlying CR manifold's symmetry group.

Each example here is described in terms of a reduced modified CR symbol, and since the examples are all homogeneous, Theorem 2.5.2 implies that we can indeed describe them up to local equivalence by giving one of their reduced modified symbols as defined in (2.5.1). From a given reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, one can construct globally the homogeneous model (M_0, H^{flat}) exhibiting the flat CR structure with constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ as described in Section 2.5.

Example 2.7.1. Let $\mathfrak{g}_-$ be the five dimensional Heisenberg algebra with a basis $(e_0, \ldots, e_4)$ whose nonzero brackets are given by

$$[e_1, e_4] = [e_2, e_3] = e_0.$$

The basis $(e_1, \ldots, e_4)$ spans $\mathfrak{g}_{-1}$, and we define

$$\mathfrak{g}_{-1,-1} := \mathrm{span}_{\mathbb{C}}\{e_1, e_2\} \quad \text{and} \quad \mathfrak{g}_{-1,1} := \mathrm{span}_{\mathbb{C}}\{e_3, e_4\}.$$

Representing elements of $\mathfrak{csp}(\mathfrak{g}_{-1})$ as matrices with respect to $(e_1, \dots, e_4)$, we define $\mathfrak{g}_0^{\mathrm{red}}$ to be the subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ spanned by the three matrices

$$\left(\begin{array}{cc:cc} 0 & \frac{i}{\sqrt{2}} & 0 & i \\ \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc:cc} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ \hdashline 0 & -i & 0 & \frac{-i}{\sqrt{2}} \\ 1 & 0 & \frac{1}{\sqrt{2}} & 0 \end{array}\right) \tag{2.7.1}$$

and the 4×4 identity matrix. With these definitions set, $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \rtimes \mathfrak{g}_0^{\mathrm{red}}$ is a Lie algebra and it is an abstract reduced modified symbol of type $\mathfrak{g}^0$ in the sense of Definition 2.5.1, where $\mathfrak{g}^0$ is the CR symbol with component $\mathfrak{g}_{0,2}$ generated by the matrix obtained from the first matrix in (2.7.1) by setting diagonal 2×2 blocks equal to zero. In other words, as the matrix C_1 in (2.4.4), we can take

$$C_1 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}. \tag{2.7.2}$$

We also need to describe the antilinear involution $\sigma : \mathfrak{g}^{0,\mathrm{red}} \to \mathfrak{g}^{0,\mathrm{red}}$ associated with this model's CR structure. On $\mathfrak{g}_-$, the map σ is defined by

$$\sigma(e_0) = e_0, \quad \sigma(e_1) = e_3, \quad \sigma(e_2) = e_4,$$

which uniquely defines an antilinear operator on $\mathfrak{g}_-$. We extend σ to an antilinear involution on $\mathfrak{csp}(\mathfrak{g}_{-1})$ by the rule

$$\sigma(\psi)(x) = \sigma\big(\psi \circ \sigma(x)\big).$$

By Remark 2.4.3 the CR symbol $\mathfrak{g}^0$ is not regular, as $C_1 \overline{C}_1 C_1 \notin \mathbb{C} C_1$.

Consider the flat CR structure H^{flat} defined on M_0 with constant reduced modified symbol $\mathfrak{g}_0^{\mathrm{red}}$ as described in section 2.5. Let us now explicitly describe P^0, the modified CR symbols, and level sets of the mapping $\psi \mapsto \theta_0(T_\psi P^0)$ associated with this CR structure.

Let $G^{0,\mathrm{red}}$ be as described in Section 2.5 and let $q : G^{0,\mathrm{red}} \to M_0^{\mathbb{C}}$ be the natural projection. Let us first show that $G^{0,\mathrm{red}}$ can be naturally embedded into P^0. The reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ is, by construction, canonically identified with the tangent space $T_eG^{0,\mathrm{red}}$ of $G^{0,\mathrm{red}}$ at the identity, and, more generally, via the differential of the left translation L_h by an element $h \in G^{0,\mathrm{red}}$, we canonically identify $\mathfrak{g}^{0,\mathrm{red}}$ with the tangent space of $G^{0,\mathrm{red}}$ at h. In particular, for each $g \in G^{0,\mathrm{red}}$, these identifications can be restricted to give canonical isomorphisms $\psi_h : \mathfrak{g}_- \to D_-^{\mathrm{flat}}\big(q(h)\big)$, where $\psi_h = q_* \circ (L_h)_*|_{\mathfrak{g}_-}$ and

$$D_-^{\mathrm{flat}}\big(q(h)\big) := \bigoplus_{(i,j),\, i<0} D_{i,j}^{\mathrm{flat}}\big(q(h)\big).$$

The map $h \mapsto \psi_h$ defines the required embedding of $G^{0,\mathrm{red}}$ into P^0. In the sequel, we identify $G^{0,\mathrm{red}}$ with its image under this embedding.

Using (2.7.2) together with $H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the identification in (2.4.3), it is easy to show that $\mathfrak{g}_{0,0}$ is a 2-dimensional subspace of $\mathfrak{csp}(\mathfrak{g}_-)$ spanned by

$$x_1 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{and} \quad x_2 = I_4 \tag{2.7.3}$$

where I_k denotes the $k \times k$ identity matrix. The element x_2 in (2.7.3) belongs to $\mathfrak{g}^{0,\mathrm{red}}$, and is actually the grading element referred to in the paragraph immediately following Theorem 2.5.2. Counting dimensions, $G^{0,\mathrm{red}}$ has codimension 1 in P^0. Recall that P^0 is a $G_{0,0}$-bundle over M_0. Using the identification in (2.4.3), we consider the one-parametric subgroup $\exp(cx_1) \subset G_{0,0}$, so then P^0 can described by

$$P^0 = \{\psi \circ \exp(cx_1) \,|\, \psi \in G^{0,\mathrm{red}},\, c \in \mathbb{C}\}. \tag{2.7.4}$$

For a given $\psi \in P^0$ such that

$$\psi = \psi_0 \circ \exp(cx_1), \quad \psi_0 \in G^{0,\mathrm{red}} \tag{2.7.5}$$

the degree zero component of the modified symbol $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ is spanned by the two matrices in (2.7.3) together with the two matrices

$$\begin{pmatrix} 0 & \frac{i}{\sqrt{2}}e^{2c} & 0 & i \\ \frac{1}{\sqrt{2}}e^{2c} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-i}{\sqrt{2}}e^{2c} \\ 0 & 0 & \frac{-1}{\sqrt{2}}e^{2c} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{i}{\sqrt{2}}e^{-2c} & 0 & 0 \\ \frac{-1}{\sqrt{2}}e^{-2c} & 0 & 0 & 0 \\ 0 & -i & 0 & \frac{-i}{\sqrt{2}}e^{-2c} \\ 1 & 0 & \frac{1}{\sqrt{2}}e^{-2c} & 0 \end{pmatrix}. \tag{2.7.6}$$

Clearly, the level sets of the mapping $\psi \mapsto \theta_0(T_\psi P^0)$ are parameterized by the the value e^{2c} appearing in (2.7.6). The image of each tangent space to one of these level sets under the soldering form θ_0 is the space spanned by the matrices in (2.7.5) together with the matrix x_2 in (2.7.3), which is the reduced modified symbol corresponding to that level set. The space $G^{0,\mathrm{red}}$ is a connected component of the level set corresponding to $e^{2c} = 1$, which has two connected components. So, we started with an abstract reduced modified symbol $g^{0,\mathrm{red}}$ and we have shown that it is indeed the reduced modified symbol of the level set $P^{0,\mathrm{red}}$ corresponding to $e^{2c} = 1$. Consequently, Theorem 2.5.2 can be applied to the CR structure H^{flat} on M_0 to obtain that this homogeneous model's symmetry group has dimension equal to $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}) = 8$, where this formula follows from a direct calculation that $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}) = \mathfrak{g}^{0,\mathrm{red}}$. By construction, in fact, $G^{0,\mathrm{red}}$ is the connected component of the symmetry group containing the identity.

Note that in the reduction of P^0 we can also use other level sets of the mapping $\psi \mapsto \theta_0(T_\psi P^0)$ to obtain a different reduced modified symbol isomorphic to $\mathfrak{g}^{0,\mathrm{red}}$ from which we could build this same homogeneous model, but we have to make sure that the chosen level set has a nonempty intersection with the real part $\Re P^0$ of the bundle P^0, which happens if and only if the space $\theta_0(T_\psi P^0)$ is invariant under involution on $\mathfrak{csp}(\mathfrak{g}_{-1})$. The latter holds if and only if $\Re c = 0$, and hence $\Re P^0$ belongs to the subset of $\mathrm{pr}^{-1}(M_0) \subset P^0$ containing points at which the modified symbol is characterized by (2.7.6) with $\Re c = 0$.

Example 2.7.2. Let $\mathfrak{g}_-$ be the 7-dimensional Heisenberg algebra with a basis $(e_0, \ldots, e_6)$ whose nonzero brackets are given by

$$[e_1, e_6] = [e_2, e_5] = [e_3, e_4] = e_0.$$

The basis $(e_1, \ldots, e_6)$ spans $\mathfrak{g}_{-1}$, and we define

$$\mathfrak{g}_{-1,-1} := \operatorname{span}_{\mathbb{C}}\{e_1, e_2, e_3\} \quad \text{and} \quad \mathfrak{g}_{-1,1} := \operatorname{span}_{\mathbb{C}}\{e_4, e_5, e_6\}.$$

Representing elements of $\mathfrak{csp}(\mathfrak{g}_{-1})$ as matrices with respect to $(e_1, \ldots, e_6)$, we define $\mathfrak{g}_0^{\text{red}}$ to be the subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ spanned by

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{2.7.7}$$

together with

$$\mathfrak{g}_{0,0}^{\text{red}} = \left\{ \left(\begin{array}{cccccc} c_1 + c_2 & 0 & c_4 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 + c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 - c_2 & 0 & -c_4 \\ 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 - c_3 \end{array} \right) \middle| \; c_i \in \mathbb{C} \right\}. \tag{2.7.8}$$

For this example, we again consider the flat CR structure H^{flat} defined on M_0 with constant reduced modified symbol $\mathfrak{g}_0^{\text{red}}$ and associated Lie group $G^{0,\text{red}}$ as described in section 2.5. Similar to the calculations for Example 2.7.1, we calculate $\mathfrak{g}_{0,0}$ explicitly using (2.4.7) with

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_\ell = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

to obtain that $\mathfrak{g}_{0,0}$ is spanned by matrices of the form in (2.7.8) together with

$$x_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.7.9}$$

Note that, by Remark 2.4.3, the CR symbol of H^{flat} is regular at every point because $C_1\overline{C_1}C_1 = 0$. Now by using (2.7.9) instead of (2.7.3) the formulas in (2.7.4) and (2.7.5) apply for our present example. In particular, for a point $\psi \in P^0$ satisfying (2.7.5), the degree zero component of the modified symbol $\mathfrak{g}_0^{\text{mod}}(\psi)$ is spanned by the matrices in (2.7.7) and (2.7.9) together with the two matrices

$$\begin{pmatrix} 0 & e^{2c} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{2c} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-2c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & e^{-2c} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.7.10}$$

As in Example 2.7.1, the level sets of P^0 are parameterized by the values e^{2c} appearing in (2.7.10), and the level sets having nontrivial intersection with $\Re P^0$ are those for which the corresponding parameter e^{2c} satisfies $\Re(c) = 0$. The Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{0,\text{red}})$ is 14-dimensional with a 1-dimensional positively graded component.

Example 2.7.3. This third example should be contrasted with Example 2.7.2 and compared to the constructions in [33]. For this example, let $\mathfrak{g}^0$ be the CR symbol of the structure in Example 2.7.2 as characterized in (2.2.3), and set $\mathfrak{g}^{0,\text{red}} = \mathfrak{g}^0$. Consider the flat structure H^{flat} on the homogeneous model M_0 constructed from $\mathfrak{g}^{0,\text{red}}$ in Section 2.5. Note that since the CR symbol in Example 2.7.2 is regular, $\mathfrak{g}_0^{\text{red}}$ defined this way is indeed a Lie algebra. In contrast to Example 2.7.2, the model (M_0, H^{flat}) has constant modified symbol, and, again in contrast to Example 2.7.2, the construction of the absolute parallelism for (M_0, H^{flat}) given in this text is equivalent to the construction given in [33]. The Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{0,\text{red}})$ for this example is 16-dimensional with a 2-dimensional positively graded component. The prolongation $\mathfrak{u}(\mathfrak{g}^{0,\text{red}})$ turns out to be equivalent to the bigraded prolongation introduced in [33, Definition 2.2] whose dimension is given in [33, Theorem 5.3]. This equivalence is not incidental, but rather a consequence of recoverability. That is, a CR structure is recoverable if and only if its associated bigraded prolongation is equivalent to the corresponding usual Tanaka prolongation of its CR symbol. As a consequence, just as happens with this example, the methods for constructing the absolute parallelism given here and in [33] are equivalent for any recoverable structure having a constant modified symbol (which, by Theorem 2.4.2, implies that the modified symbol equals its CR symbol and its CR symbol is regular).

Remark 2.7.1. *It turns out that, up to local equivalence, there is exactly one additional homogeneous 2-nondegenerate, hypersurface-type CR manifold that can be described as the flat model of a constant reduced modified CR symbol (as described in Section 2.5) having the same CR symbol that the models in Examples 2.7.2 and 2.7.3 have, and its symmetry group is 11-dimensional.*

2.8 Proof of Theorem 2.3.5

In this section, we modify methods from the theory of Noboru Tanaka's prolongation procedure described in [46] in order to prove Theorem 2.3.5. In particular, we describe the modifications necessary to obtain the bundles $\{P^i\}_{i=1}^{\infty}$ corresponding to (2.3.9), which are required due to the non-constancy of $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$. For structures with constant modified CR symbols, however, the standard Tanaka prolongation procedure can be applied directly without modification. The key modifications appear in the constructions of P^1 and P^2, and we construct these explicitly. Each higher degree prolongation P^k is obtained from P^{k-1} in the same way that P^2 is obtained from P^1.

2.8.1 Constructing the first geometric prolongation

Let $\Pi_0 : P^0 \to M$ denote the natural projection. The contact structure $\mathcal{D}$ on $\mathcal{N}$ lifts to a filtration $D_0^0 \subset D_0^{-1} \subset D_0^{-2}$ of TP^0 given by

$$D_0^0 = (\Pi_0)_*^{-1}(0), \quad D_0^{-1} = (\Pi_0)_*^{-1}(\mathcal{D}), \quad \text{and} \quad D_0^{-2} = TP^0,$$

and we use the notation

$$D_0^i(\psi) := D_0^i \cap T_\psi P^0.$$

Informally, our first goal in this section is to define *structure functions* on P^0, which are elements in

$$\mathcal{S}_0 = \mathrm{Hom}\left(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}, \mathfrak{g}_{-2}\right) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}, \mathfrak{g}_{-1}\right)$$

associated with horizontal subspaces in TP^0, and we will use this as a tool for finding a family of objects similar to Ehresmann connections on P^0 that is naturally associated with the underlying CR structure in a certain sense. But to be more precise, rather than associating horizontal subspaces with a structure function – as is done in the theory of G-structures – the structure functions we introduce will be associated with graded horizontal subspaces, namely pairs of subspaces of the form $\mathcal{H}_\psi = (H_\psi^{-2}, H_\psi^{-1})$ where ψ is a point in P^0,

$$H_\psi^{-2} \subset D_0^{-2}(\psi)/D_0^0(\psi), \quad H_\psi^{-1} \subset D_0^{-1}(\psi), \tag{2.8.1}$$

$$D_0^{-2}(\psi)/D_0^0(\psi) = H_2 \oplus D_0^{-1}(\psi)/D_0^0(\psi), \quad \text{and} \quad D_0^{-1}(\psi) = H_\psi^{-1} \oplus D_0^0(\psi). \tag{2.8.2}$$

Notice that D_0^0 is the domain of the map $\theta_0 : D_0^0 \to \mathfrak{csp}(\mathfrak{g}_{-1})$ introduced in (2.3.2). We call θ_0 the degree zero soldering form on P^0 and introduce additional soldering forms $\theta_{-1} : D_0^{-1} \to \mathfrak{g}_{-1}$ and $\theta_{-2} : D_0^{-2} \to \mathfrak{g}_{-2}$ as follows. The projection Π_0 naturally induces a linear map from $D_0^{-2}(\psi)/D_0^{-1}(\psi) \oplus D_0^{-1}(\psi)/D_0^0(\psi) \oplus D_0^0(\psi) \to \mathcal{D}_{\Pi_0(\psi)} \oplus T_{\Pi_0(\psi)}\mathcal{N}/\mathcal{D}_{\Pi_0(\psi)}$. Using these induced maps and letting

$$\pi_0^{-2} : D_0^{-2} \to D_0^{-2}/D_0^{-1} \quad \text{and} \quad \pi_0^{-1} : D_0^{-1} \to D_0^{-1}/D_0^0$$

denote the natural projections, for a point ψ in P_0, we define

$$\theta_{-1}(v) := \psi^{-1} \circ (\Pi_0)_* \circ \pi_0^{-1}(v) \qquad \forall\, v \in D_0^{-1}(\psi),$$

and

$$\theta_{-2}(v) := \psi^{-1} \circ (\Pi_0)_* \circ \pi_0^{-2}(v) \qquad \forall\, v \in D_0^{-2}(\psi).$$

We have the associated maps $\overline{\theta}_{-2} : D_0^{-2}/D_0^{-1} \to \mathfrak{g}_{-2}$, $\overline{\theta}_{-1} : D_0^{-1}/D_0^{0} \to \mathfrak{g}_{-1}$, and $\overline{\theta}_0 : D_0^0 \to \mathfrak{csp}(\mathfrak{g}_{-1})$ given by

$$\overline{\theta}_i\big(\pi_0^i(v)\big) := \theta_i(v)$$

where $\pi_0^0 : D_0^0 \to D_0^0$ is taken to be the identity map.

Remark 2.8.1. *These soldering forms $\{\theta_i\}$ are similar to the soldering forms introduced in [46, Section 3], but a subtle difference is that the space $\theta_0\left(D_0^0(\psi)\right)$ depends on ψ. For further reference, the forms θ_{-2} and θ_{-1} are close analogues of the forms labeled as $\theta_{-2}^{(0)}$ and $\theta_{-1}^{(0)}$ in [41].*

For a graded horizontal space $\mathcal{H}_\psi$ of the form satisfying (2.8.1) and (2.8.2), we define $\mathrm{pr}_{-2}^{\mathcal{H}_\psi} : D_0^{-2}(\psi)/D_0^0(\psi) \to D_0^{-1}(\psi)/D_0^0(\psi)$ and $\mathrm{pr}_{-1}^{\mathcal{H}_\psi} : D_0^{-1}(\psi) \to D_0^0(\psi)$ to be the projections parallel to the subspaces H_ψ^{-2} and H_ψ^{-1} respectively, and define the map

$$\phi^{\mathcal{H}_\psi} \in \mathrm{Hom}\left(\mathfrak{g}_{-2}, D_0^{-2}(\psi)/D_0^0(\psi)\right) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1}, D_0^{-1}(\psi)\right) \oplus \mathrm{Hom}\left(\theta_0\left(D_0^0(\psi)\right), D_0^0(\psi)\right)$$

by

$$\phi^{\mathcal{H}_\psi}(v) := \begin{cases} \left((\Pi_0)_* \circ \pi_0^{-2}\big|_{H_\psi^{-2}}\right)^{-1} \circ \psi(v) & \text{if } v \in \mathfrak{g}_{-2} \\ \left((\Pi_0)_* \circ \pi_0^{-1}\big|_{H_\psi^{-1}}\right)^{-1} \circ \psi(v) & \text{if } v \in \mathfrak{g}_{-1} \\ \left(\theta_0\big|_{D_0^0(\psi)}\right)^{-1}(v) & \text{if } v \in \theta_0\left(D_0^0(\psi)\right). \end{cases}$$

We can now define the structure function $S_{\mathcal{H}_\psi} \in \mathcal{S}_0$ associated with $\mathcal{H}_\psi$ by the formula

$$S_{\mathcal{H}_\psi}(v_1, v_2) := \begin{cases} \overline{\theta}_{-2}\left([Y_1, Y_2](\psi) + D_0^{-1}(\psi)\right) & \text{if } v_2 \in \mathfrak{g}_{-2} \\ \overline{\theta}_{-1}\left(\mathrm{pr}_{-2}^{\mathcal{H}_\psi}\left([Y_1, Y_2](\psi) + D_0^0(\psi)\right)\right) & \text{if } v_2 \in \mathfrak{g}_{-1} \end{cases} \tag{2.8.3}$$

where Y_1 and Y_2 are vector fields defined on a neighborhood of ψ in P^0 such that, supposing $v_2 \in \mathfrak{g}_i$ for $i \in \{-1, -2\}$,

$$Y_1 \in \Gamma(D_0^{-1}), \quad Y_2 \in \Gamma(D_0^i), \quad \theta_{-1}(Y_1) = v_1, \quad \theta_i(Y_2) = v_2, \quad (Y_1)_\psi = \phi^{\mathcal{H}_\psi}(v_1), \tag{2.8.4}$$

and either

$$\begin{cases} \text{(a)} \quad i = -1 \text{ and } (Y_2)_\psi = \phi^{\mathcal{H}_\psi}(v_2), \text{ or} \\ \text{(b)} \quad i = -2 \text{ and } (Y_2)_\psi \equiv \phi^{\mathcal{H}_\psi}(v_2) \pmod{D_0^0(\psi)}. \end{cases} \tag{2.8.5}$$

The definition of $S_{\mathcal{H}_\psi}$ given in (2.8.3), (2.8.4), and (2.8.5) coincides with the definition in [46, (3.5)], wherein the following lemma is proven.

Lemma 2.8.2 (proven in [46, Section 3])**.** *The definition of $S_{\mathcal{H}_\psi}$ given in* (2.8.3) *does not depend on the choice of vector fields Y_1 and Y_2 satisfying* (2.8.4) *and* (2.8.5).

Considering another graded horizontal space $\widetilde{\mathcal{H}}_\psi$, let us describe the difference between the structure functions $S_{\mathcal{H}_\psi}$ and $S_{\widetilde{\mathcal{H}}_\psi}$. For this we introduce the function

$$f_{\mathcal{H}_\psi \widetilde{\mathcal{H}}_\psi} \in \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1}, \theta_0\left(D_0^0(\psi)\right)\right)$$

defined by

$$f_{\mathcal{H}_\psi \widetilde{\mathcal{H}}_\psi}(v) = \overline{\theta}_{i+1}\left(\phi^{\mathcal{H}_\psi}(v) - \phi^{\widetilde{\mathcal{H}}_\psi}(v)\right) \qquad \forall\, v \in \mathfrak{g}_i \text{ and } i \in \{-1, -2\},$$

and introduce the *anti-symmetrization (or generalized Spencer) operator*

$$\partial^0_\psi : \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1}, \theta_0\left(D^0_0(\psi)\right)\right) \to \mathcal{S}_0$$

defined by

$$\partial^0_\psi f(v_1, v_2) := [f(v_1), v_2] + [v_1, f(v_2)] - f([v_1, v_2]),$$

where the brackets $[\cdot,\cdot]$ are defined by the Lie algebra structure on $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$. It is shown in [46, Proposition 3.1] that

$$S_{\mathcal{H}_\psi} = S_{\widetilde{\mathcal{H}}_\psi} + \partial^0_\psi f_{\mathcal{H}_\psi \widetilde{\mathcal{H}}_\psi}. \tag{2.8.6}$$

In standard Tanaka theory, one defines the so-called first geometric prolongation of P^0, which is a certain fiber bundle P^1 defined over the base space P^0, but here instead we define an analogous first prolongation as a bundle P^1 over a neighborhood $\mathcal{O}^0$ in P^0. For defining this, let $\psi_0 \in \Re P^0$ be as in the item (1) of Theorem 2.3.5. By regularity of ψ_0 there exists an open neighborhood $\mathcal{O}^0 \subset P^0$ of ψ_0 such that there exists a subspace $N_0 \subset \mathcal{S}_0$ for which

$$\mathcal{S}_0 = N_0 \oplus \mathrm{Im}\partial^0_\psi \qquad \forall\, \psi \in \mathcal{O}^0. \tag{2.8.7}$$

Moreover, the natural involutions on each previously defined $\mathfrak{g}_i$ induce the natural involution on the space $\mathcal{S}_0$ and also $\mathrm{Im}\partial^0_\psi$ is invariant under this involution for ψ belong to $\Re\mathcal{O}^0 := \Re P^0 \cap \mathcal{O}^0$, based on the rule that the involution of the tensor product of two elements is the tensor product of the involution of these elements. So, we can take N_0 to be invariant with respect to the involution.

The subspace N_0 is called the *normalization condition of the structure function* for the first prolongation and the choice of N_0 defines the bundle P^1 via the formula

$$P^1 := \left\{ \mathcal{H}_\psi \;\middle|\; \begin{array}{l} \psi \in \mathcal{O}^0 \text{ and } \mathcal{H}_\psi \text{ is a pair of horizontal} \\ \text{spaces in } T_\psi\mathcal{O}^0 \text{ as described in (2.8.1)} \\ \text{and (2.8.2) such that } S_{\mathcal{H}_\psi} \in N_0 \end{array} \right\}, \tag{2.8.8}$$

or, equivalently,

$$P^1 := \left\{ \phi^{\mathcal{H}_\psi} \;\middle|\; \begin{array}{l} \psi \in \mathcal{O}^0 \text{ and } \mathcal{H}_\psi \text{ is a pair of horizontal} \\ \text{spaces in } T_\psi \mathcal{O}^0 \text{ as described in (2.8.1)} \\ \text{and (2.8.2) such that } S_{\mathcal{H}_\psi} \in N_0 \end{array} \right\}. \tag{2.8.9}$$

Since N_0 is invariant with respect to the involution, we have that if $\phi^{\mathcal{H}_\psi} \in P_1$ then $\overline{\phi^{\mathcal{H}_\psi}} \in P_1$ for all $\psi \in \Re\mathcal{O}^0$, where

$$\overline{\phi^{\mathcal{H}_\psi}}(v) := \overline{\phi^{\mathcal{H}_\psi}(\overline{v})}$$

according to the rule of commuting the involution with tensor products. Hence a natural involution is defined on the fibers of P^1 over $\Re O^0$, and the fixed point set of this induced involution is a subspace in P^1 that we denote by $\Re P^1$ and call the real part of P^1.

By (2.8.6), if $\mathcal{H}_\psi$ and $\widetilde{\mathcal{H}}_\psi$ are two elements of P^1 belonging to the fiber $(P^1)_\psi$ of P^1 over the point $\psi \in \mathcal{O}^0$, then $\partial^0_\psi f_{H_\psi \widetilde{\mathcal{H}}_\psi} = 0$, and hence

$$f_{H_\psi \widetilde{\mathcal{H}}_\psi} \in \ker \partial^0_\psi = \mathfrak{g}_1^{\mathrm{mod}}(\psi).$$

Conversely, if $f \in \ker \partial^0_\psi$ and $\mathcal{H}_\psi$ is in the fiber $(P^1)_\psi$ of P^1 over ψ then the graded horizontal space

$$\{(v,w) + f(v,w) \,|\, (v,w) \in \mathcal{H}_\psi\}$$

also belongs to $(P^1)_\psi$. In other words, by the regularity of ψ_0, P^1 is an affine bundle modeled on $\mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$, and each tangent space $T_{\mathcal{H}_\psi} P^1$ is naturally identified with $\mathfrak{g}_1^{\mathrm{mod}}(\psi)$ by the map $\theta_1^{(1)} : T_{\mathcal{H}_\psi}(P^1)_\psi \to \mathfrak{g}_1^{\mathrm{mod}}(\psi)$ defined by the formula

$$X = \frac{d}{dt}\bigg|_{t=0} \left\{ (v,w) + t\theta_1^{(1)}(X)(v,w) \;\middle|\; (v,w) \in \mathcal{H}_\psi \right\}. \tag{2.8.10}$$

Similarly, $\Re P^1$ is an affine bundle over $\Re\mathcal{O}^0$ modeled on $\Re\mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$.

The difference between P^1 defined here and the first geometric prolongation defined in the standard Tanaka theory as in [46] is that the maps $\phi^{\mathcal{H}_\psi}$ appearing in (2.8.9) have different domains because $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ is non-constant. If $\mathfrak{g}_1^{\mathrm{mod}}(\psi_0) \neq 0$ then in order to continue the prolongation procedure, constructing higher degree geometric prolongations, we need to somehow identify the domains of each $\phi^{\mathcal{H}_\psi}$. Moreover, independent of $\mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$, ultimately we still will need to fix an identification of all $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ in order to construct canonical absolute parallelisms. All of this motivates the following introduction of what we call the *identification space* $\mathcal{I}_0$. By regularity of ψ_0, we can fix a subspace $\mathcal{I}_0 \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ invariant under the induced involution on $\mathfrak{csp}(\mathfrak{g}_{-1})$, such that, after possibly shrinking the neighborhood $\mathcal{O}^0$, in addition to (2.8.7), we have

$$\mathfrak{csp}(\mathfrak{g}_{-1}) = \mathcal{I}_0 \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi) \qquad \forall\, \psi \in \mathcal{O}^0.$$

For all ψ in $\mathcal{O}^0$, each $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ is identified with $\mathfrak{g}_0^{\mathrm{mod}}(\psi_0)$ via the projection to the latter that is parallel to $\mathcal{I}_0$. We let

$$\mathrm{pr}^{\mathcal{I}_0} : \mathfrak{g}_- \oplus \mathfrak{csp}(\mathfrak{g}_{-1}) \to \mathfrak{g}^{0,\mathrm{mod}}(\psi_0)$$

denote the map that is equal to the identity on $\mathfrak{g}_-$ and equal to the projection parallel to $\mathcal{I}_0$ on $\mathfrak{csp}(\mathfrak{g}_{-1})$.

2.8.2 Constructing the second geometric prolongation

We define the second geometric prolongation as a bundle over a neighborhood $\mathcal{O}^1$ in P^1 just as we defined P^1 as a bundle over the neighborhood $\mathcal{O}^0$ in P^0. For this we now introduce structure functions associated with graded horizontal spaces in $T\mathcal{O}^1$ and define P^2 to be the bundle of these graded horizontal spaces whose structure functions satisfy a certain normalization condition.

The filtration $D_0^0 \subset D_0^{-1} \subset D_0^{-2}$ of TP^0 lifts to a filtration $D_1^1 \subset D_1^0 \subset D_1^{-1} \subset D_1^{-2}$ of TP^1, where, for $i \in \{0,-1,-2\}$, $D_1^i = (\Pi_1)_*^{-1} D_0^i$, and $D_1^1 = (\Pi_1)_*^{-1}(0)$. We set $D_1^i(\mathcal{H}_\psi) = D_1^i \cap T_{\mathcal{H}_\psi} P^1$. Using the definition of P^1 given in (2.8.8), for each $\mathcal{H}_\psi \in P^1$ and $i \in \{-2,-1,0,1\}$, we also have soldering forms

$$\theta_i^{(1)} : D_1^i(\mathcal{H}_\psi) \to \mathfrak{g}^{0,\mathrm{mod}}(\psi) \oplus \mathfrak{g}_1^{\mathrm{mod}}(\psi)$$

where $\theta_1^{(1)}$ is as given in (2.8.10), and, for $i < 1$, $\theta_i^{(1)}(v) = \theta_i \circ (\Pi_1)_*(v)$. Each $\theta_i^{(1)}$ has the corresponding map $\overline{\theta}_i^{(1)}$ with domain $D_1^i(\mathcal{H}_\psi)/D_1^{i+1}(\mathcal{H}_\psi)$ defined by $\overline{\theta}_1^{(1)} = \theta_1^{(1)}$ and $\overline{\theta}_i^{(1)} = \theta_i \circ (\Pi_1)_*$.

Similar to the definition of graded horizonal subspaces in TP^0, for $p \in P^1$, we define graded horizontal subspaces in T_pP^1 as tuples of subspaces $\mathcal{H}_p = (H_p^0, H_p^{-1}, H_p^{-2})$ such that $H_p^i \subset D_1^i(p)$, $H_p^0 \oplus D_1^1(p) = D_1^0(p)$, $H_p^{-1} \oplus D_1^0(p) = D_1^{-1}(p)$, and $H_p^{-2}/D_1^1(p) \oplus D_1^{-1}(p)/D_1^1(p) = D_1^{-2}(p)/D_1^1(p)$. For each of these graded horizontal subspaces $\mathcal{H}_p$ in T_pP^1, we define $\mathrm{pr}_{-2}^{\mathcal{H}_p} : D_1^{-2}(p)/D_1^1(p) \to D_1^0(p)/D_1^1(p)$ and $\mathrm{pr}_{-1}^{\mathcal{H}_p} : D_1^{-1}(p) \to D_1^1(p)$ to be the projections parallel to the subspaces H_p^{-2} and H_p^{-1} respectively. Analogous to the map defined in (2.8.8), each graded horizontal subspace $\mathcal{H}_p$ in T_pP^1 uniquely determines an isomorphism

$$\phi^{\mathcal{H}_p} : \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}\big(\Pi_1(p)\big) \oplus \mathfrak{g}_1^{\mathrm{mod}}\big(\Pi_1(p)\big) \to H_p^{-2} \oplus H_p^{-1} \oplus H_p^0 \oplus D_1^1(p)$$

such that, for $i \in \{-1, -2\}$, $\phi^{\mathcal{H}_p}(\mathfrak{g}_i) = H_p^i$, $\phi^{\mathcal{H}_p}\big(\mathfrak{g}_0^{\mathrm{mod}}(p)\big) = H_p^0$, and $\phi^{\mathcal{H}_p}\big(\mathfrak{g}_1^{\mathrm{mod}}(p)\big) = D_1^1(p)$. For a graded horizontal subspace $\mathcal{H}_p \subset T_pP^1$ we define its structure function $S_{\mathcal{H}_p}$ to be the element of

$$\mathcal{S}_1 := \mathrm{Hom}\left(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}, \mathfrak{g}_{-1}\right) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}, \mathfrak{g}_0^{\mathrm{mod}}(\psi_0)\right) \oplus \mathrm{Hom}\left(\mathfrak{g}_{-1} \otimes \mathfrak{g}_0^{\mathrm{mod}}(\psi_0), \mathfrak{g}_0^{\mathrm{mod}}(\psi_0)\right)$$

defined by

$$S_{\mathcal{H}_p}(v_1, v_2) := \begin{cases} \overline{\theta}_{-1}^{(1)}\left(\mathrm{pr}_{-2}^{\mathcal{H}_p}[Y_1, Y_2](p) + D_1^0(p)\right) & \text{if } v_2 \in \mathfrak{g}_{-2} \\ \mathrm{pr}^{\mathcal{I}_0} \circ \overline{\theta}_0^{(1)}\left(\mathrm{pr}_{-1}^{\mathcal{H}_p}\left([Y_1, Y_2](p) + D_1^1(p)\right)\right) & \text{if } v_2 \in \mathfrak{g}_{-1} \\ \mathrm{pr}^{\mathcal{I}_0} \circ \overline{\theta}_0^{(1)}\left([Y_1, Y_2](p)\right) & \text{if } v_2 \in \mathfrak{g}_0^{\mathrm{mod}}(\psi_0) \end{cases} \tag{2.8.11}$$

where Y_1 and Y_2 are vector fields defined on a neighborhood of p in P^1 such that, supposing

$v_2 \in \mathfrak{g}_i(\psi_0)$ for $i \in \{0, -1, -2\}$,

$$Y_1 \in \Gamma(D_1^{-1}), \quad Y_2 \in \Gamma(D_1^i), \quad \theta_{-1}^{(1)}(Y_1) = v_1, \quad \theta_i^{(1)}(Y_2) = \left(\mathrm{pr}^{\mathcal{I}_0}\big|_{\mathfrak{g}^{0,\mathrm{mod}}(\Pi_1(p))}\right)^{-1}(v_2),$$

$$(Y_1)_p = \phi^{\mathcal{H}_p}(v_1),$$

and either

$$\begin{cases} \text{(a)} \quad i = 0 \text{ and } (Y_2)_p = \phi^{\mathcal{H}_p} \circ \left(\mathrm{pr}^{\mathcal{I}_0}\big|_{\mathfrak{g}^{0,\mathrm{mod}}(\Pi_1(p))}\right)^{-1}(v_2), \\ \text{(b)} \quad i = -1 \text{ and } (Y_2)_p \equiv \phi^{\mathcal{H}_p}(v_2) \pmod{D_1^1(\psi)}, \text{ or} \\ \text{(c)} \quad i = -2 \text{ and } (Y_2)_p \equiv \phi^{\mathcal{H}_p}(v_2) \pmod{D_1^0(\psi)}. \end{cases} \tag{2.8.12}$$

Comparing this formula for $\mathcal{S}_{\mathcal{H}_p}$ to the structure functions defined (for geometric prolongations of arbitrary degree) in [46, Formula (4.16)], the only difference is that we include the projection $\mathrm{pr}^{\mathcal{I}_0}$ in multiple places, and this modification is necessary because the symbols $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ are non-constant. Notice that if the structure's modified CR symbols are constant on $\mathcal{O}^0$ then the formulas in (2.8.11) and (2.8.12) would be unaffected by the removal of $\mathrm{pr}^{\mathcal{I}_0}$.

For a point $p \in P^1$, we introduce another anti-symmetrization operator

$$\partial_p^1 : \mathrm{Hom}\big(\mathfrak{g}_{-2}, \mathfrak{g}_0^{\mathrm{mod}}(\Pi_1(p))\big) \oplus \mathrm{Hom}\big(\mathfrak{g}_{-1}, \mathfrak{g}_1^{\mathrm{mod}}(\Pi_1(p))\big) \to \mathcal{S}_1$$

defined by

$$\partial_p^1 f(v_1, v_2) = \mathrm{pr}^{\mathcal{I}_0}\left([f \circ \iota(v_1), v_2] + [v_1, f \circ \iota(v_2)] - f \circ \iota([v_1, v_2])\right) \tag{2.8.13}$$

using the identification $\iota = \left(\mathrm{pr}^{\mathcal{I}_0}\big|_{\mathfrak{g}^{0,\mathrm{mod}}(\Pi_1(p))}\right)^{-1}$ for brevity. Note that this definition of ∂_p^1 is similar to the generalized Spencer operator defined for the second geometric prolongation in [46], and the key difference is that our definition of ∂_p^1 here includes intertwining with the projection $\mathrm{pr}^{\mathcal{I}_0}$.

Similar to the construction of the first geometric prolongation, by regularity of ψ_0 there exists an open neighborhood $\mathcal{O}^1 \subset P^1$ with $\psi_0 \in \Pi_1(\mathcal{O}^1)$ such that there exists a subspace $N_1 \subset \mathcal{S}_1$ for which

$$\mathcal{S}_1 = N_1 \oplus \mathrm{Im}\partial_\psi^0 \qquad \forall\, \psi \in \mathcal{O}^0. \tag{2.8.14}$$

We can take it to be invariant with respect to natural involution induced on $\mathcal{S}_1$. We call N_1 the *normalization condition of the structure function* for the first prolongation and the choice of N_1 defines a second geometric prolongation P^2 via the formula

$$P^2 := \left\{ \mathcal{H}_p \;\middle|\; \begin{array}{l} p \in \mathcal{O}^1 \text{ and } \mathcal{H}_p \text{ is a pair of horizontal} \\ \text{spaces in } T_p\mathcal{O}^1 \text{ as described in (2.8.1)} \\ \text{and (2.8.2) such that } S_{\mathcal{H}_\psi} \in N_1 \end{array} \right\}.$$

Just as P^1 has the structure of an affine bundle modeled on the vector space $\mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$, the bundle P^2 has the structure of an affine bundle over $\mathcal{O}^1$ modeled on $\mathfrak{g}_2^{\mathrm{mod}}(\psi_0)$.

Finally, by complete analogy with the first prolongation, we can define the real part $\Re P^2$ of P^2 as an affine bundle over $\Re\mathcal{O}^1 := \mathcal{O}^1 \cap \Re P^1$ modeled on the space $\Re\mathfrak{g}_2^{\mathrm{mod}}(\psi_0)$.

2.8.3 Higher geometric prolongations

To summarize how the above constructions of P^1 and P^2 differ from the geometric prolongations in [46], each bundle P^i is defined over a neighborhood in P^{i-1} and the maps defined in (2.8.11) and (2.8.13) differ from their analogs in [46] in that they are intertwined with the projection $\mathrm{pr}^{\mathcal{I}_0}$. This exact pattern continues for the construction of each higher geometric prolongation P^i with $i > 2$. In particular, for example, letting $\mathfrak{u}_1$ denote the standard first Tanaka prolongation of the graded Lie algebra $\mathfrak{g}_- \oplus \mathfrak{csp}(\mathfrak{g}_{-1})$ (defined using the same formula given in (2.3.7) with $\mathfrak{g}_0^{\mathrm{mod}}(\psi)$ replaced by $\mathfrak{csp}(\mathfrak{g}_{-1})$) and again using the regularity of ψ_0, we can shrink the neighborhood $\mathcal{O}^1$ so that, in addition to (2.8.14), there exists a subspace $\mathcal{I}_1$ in $\mathfrak{u}_1$ that is invariant under the induced involution and satisfies

$$\mathfrak{u}_1 = \mathcal{I}_1 \oplus \mathfrak{g}_1^{\mathrm{mod}}(\psi) \qquad \forall\, \psi \in \mathcal{O}^0.$$

Each $\mathfrak{g}^{0,\mathrm{mod}}(\psi) \oplus \mathfrak{g}_1^{\mathrm{mod}}(\psi)$ is identified with $\mathfrak{g}^{0,\mathrm{mod}}(\psi_0) \oplus \mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$ via the projection $\mathrm{pr}^{\mathcal{I}_1}$ to the latter that is defined on $\mathfrak{g}^{0,\mathrm{mod}}(\psi)$ as $\mathrm{pr}^{\mathcal{I}_0}$ and is defined on $\mathfrak{u}_1$ as the projection onto $\mathfrak{g}_1^{\mathrm{mod}}(\psi_0)$ parallel to $\mathcal{I}_1$. Next, the third geometric prolongation P^3 can be constructed over a neighborhood $\mathcal{O}^2$ in P^2 using the construction in [46] with modification that the structure functions and generalized Spencer operators must be intertwined with the projection $\mathrm{pr}^{\mathcal{I}_1}$ just as the maps in (2.8.11) and (2.8.13) are intertwined with $\mathrm{pr}^{\mathcal{I}_0}$. Repeating the process with these modifications give a microlocal version of the Tanaka prolongation procedure, as it were.

The properties of the geometric prolongations defined in classical Tanaka theory that we are interested in remain unaffected by the modifications to the prolongation procedure made above. In particular,

(a) for each $i > 0$, the space P^i has the structure of an affine bundle over $\mathcal{O}^{i-1}$ modeled on the vector space $\mathfrak{g}_i^{\mathrm{mod}}(\psi_0)$,

(b) for each $i > 0$, P^i and $\mathcal{O}^{i-1}$ has a natural induced involution defined on it, and by restricting to the fixed point sets of these involutions one obtains the space $\Re P^i$ defined as a fiber bundle over $\Re\mathcal{O}^{i-1}$ modeled on the vector space $\Re\mathfrak{g}_i^{\mathrm{mod}}(\psi_0)$,

(c) if $l+1$ is the smallest number for which $\mathfrak{g}_{l+1}^{\mathrm{mod}}(\psi_0) = 0$ then the $l+2$ normalization conditions $N_0, \ldots, N_{l+1}$ chosen in the first $l+2$ steps of the prolongation procedure determine a canonical absolute parallelism both on $\mathcal{O}^l$ and $\Re\mathcal{O}^l$,

(d) the pseudogroup of local symmetries of the underlying CR manifold has a naturally induced partial action on each $\mathcal{O}^i$ and $\Re\mathcal{O}^i$ under which the parallelism mentioned in the last item is invariant.

Item (c) completes the proof of item (1) in Theorem 2.3.5. Since the canonical frame on $\Re\mathcal{O}^l$

referred to in item (c) is invariant under the action of the pseudogroup of local symmetries, each symmetry is uniquely determined locally near a point by its value at that point, and therefore the dimension of this pseudogroup is not greater than the real dimension of the bundle $\Re\mathcal{O}^l$, which establishes item (2) in Theorem 2.3.5. By items (a) and (b) above, this dimension is equal to $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$ as desired.

3. CLASSIFICATION OF CR SYMBOLS OF STRUCTURES WITH A RANK 1 LEVI KERNEL*

The content of the present chapter, although developed for this dissertation, was published previously jointly with Igor Zelenko in [38]. The exposition here is modified from that in [38] slightly and only as needed for coordination with the other chapters. Note that throughout Chapter (3) we use H to denote a Hermitian matrix, whereas in all other chapters we use H to denote a CR structure and H_ℓ to denote a Hermitian matrix.

In the present chapter we find canonical forms for pairs consisting of a nondegenerate Hermitian form ℓ on a complex n-dimensional vector space W and an antilinear operator $A : W \to W$, which is self-adjoint with respect to the form ℓ. By a canonical form, as usual, we mean a specified choice of matrices representing elements of any such pair, chosen from among matrix representations in all possible bases of W. Recall that, in light of Remark 2.2.5, CR symbols of structures whose Levi kernel has rank 1 are given exactly by a pair of the algebraic objects under consideration, and hence the assignment of canonical forms in this chapter's main result, Theorem 3.1.2, indeed yields a classification of CR symbols of structures with a rank 1 Levi kernel. Recall also that a map $A : W \to W$ is an antilinear operator if

$$A(\lambda v + w) = \overline{\lambda} A(v) + A(w) \qquad \forall v, w \in W, \lambda \in \mathbb{C},$$

and an antilinear operator A is self-adjoint with respect to the form ℓ or, shortly, ℓ-self-adjoint if

$$\ell(Av, w) = \ell(Aw, v) \qquad \forall v, w \in W. \tag{3.0.1}$$

This chapter's main result also gives canonical forms for pairs consisting of a nondegenerate

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Hermitian form and a symmetric bilinear form because the set of these pairs is in bijective correspondence with the one we originally considered. Indeed, to the pair (ℓ, A) we can assign the pair (ℓ, ℓ'), where

$$\ell'(v, w) := \ell(w, Av)$$

is a symmetric bilinear form by (3.0.1). From the nondegeneracy of ℓ it follows that the assignment of (ℓ, A) to (ℓ, ℓ') defines the bijection between the two sets of pairs under consideration.

Surprisingly, when we encountered the necessity of finding the canonical forms for pairs (ℓ, A) in the course of our study in CR geometry, we were not able to find the desired results in the literature. The only results in this direction that we found are those addressing the problem of simultaneous diagonalization [4, 23] and those giving canonical forms for a single antilinear operator [19, 20, 22] and, more generally, for a single semi-linear operator [1, 25] or for a square matrix under φ-equivalence [21]. In [4, Theorem 7], it shown that ℓ and A can be simultaneously diagonalized if ℓ is positive definite, and, in [23, Theorem 2.1], the pairs (ℓ, A) admitting a simultaneous diagonalization are classified. Perhaps, the main difficulty here is that the matrix representations for a Hermitian form and an antilinear operator transform differently under a change of the basis (see formulas (3.1.2) below). It also cannot be reduced to the study of canonical forms of pairs of other objects, wherein the matrix representations of each component of the new pairs transforms in the same way under a basis change. An example of the latter reduction is the set of pairs consisting of a nondegenerate Hermitian form ℓ and an ℓ-self-adjoint linear operator that was treated in [16, Theorem 5.1.1] where a canonical form for such pairs is given, which we will refer to as the Gohberg–Lancaster–Rodman form. Although the matrix representations of each component in such pairs transform differently under a basis change, using a process similar to the one in the previous paragraph, we can obtain a bijective correspondence between the set of such pairs and the set of pairs of Hermitian forms (i.e., a pair of the same type of objects), one of which is nondegenerate. In our case, however, such a reduction is not possible and the problem of finding canonical forms cannot be totally reduced to the study of certain classes of matrix pencils, as was classically done using Weierstrass–Kronecker normal forms for matrix pencils (see, for example, [15] and [44]).

To prove Theorem 3.1.2, we develop in section 3.2 a geometric version of the construction of the canonical form for a single antilinear operator of [20] (which was formulated in [22, Theorem 3.1], proved in [20], and stated for completeness in Remark 3.1.3 below) and combine it with a simultaneous normalization of the Hermitian form, which is comparable in certain respects to the method of [16, subsection 5.3] for obtaining the Gohberg–Lancaster–Rodman form, mentioned in the previous paragraph. By a geometric version we are referring to the study of flags of subspaces analogous to the generalized eigenspaces in the standard theory of linear operators as opposed to the algebraic version in [1, 19, 25] based on the theory of invariant factors and manipulations with matrices as in [20, 21]. Our Theorem 3.1.2 is related to the Hong–Horn canonical form of [22, Theorem 3.1] for a single antilinear operator in the same way that the Gohberg–Lancaster–Rodman form in [16, Theorem 5.1.1] is related to the classical Jordan normal form for linear operators.

In section 3.3, for completeness we sketch an alternative approach to the considered problem that leads to an equivalent canonical form, Theorem 3.3.1. This approach was in fact our original one before we found the more natural and apparently more simple approach leading to Theorem 3.1.2. The idea in this alternative approach is as follows: Since A^2 is an ℓ-self-adjoint linear operator whenever A is an ℓ-self-adjoint antilinear operator, one can first bring the pair (ℓ, A^2) to the Gohberg–Lancaster–Rodman form and then find a canonical form for A with minimal changes in the form of ℓ. This requires solving a certain nonlinear matrix equation, which turned out to be feasible.

3.1 A canonical form for pairs consisting of a Hermitian form and a self-adjoint antilinear operator

As in the introduction to this chapter, ℓ denotes a nondegenerate Hermitian form and A denotes an antilinear operator on an n-dimensional complex space W. Unless otherwise stated, throughout this chapter A is assumed to be ℓ-self-adjoint (i.e., (3.0.1) holds).

Choosing a basis $\{e_1, \ldots, e_n\}$ of W, one can represent the form ℓ and the antilinear operator A by $n \times n$ matrices $H = (H_{i,j})$ and $C = (C_{i,j})$ via a standard construction, requiring, for all

$i, j \in \{1, \ldots, n\}$, that

$$H_{i,j} = \ell(e_j, e_i) \quad \text{and} \quad A(e_i) = \sum_{k=1}^{n} C_{k,i} e_k.$$

The conditions that ℓ is a nondegenerate Hermitian form and A is an ℓ-self-adjoint antilinear operator are equivalent to

$$H^* = H, \quad \det H \neq 0, \quad \text{and} \quad (HC)^T = HC, \tag{3.1.1}$$

respectively.

If one chooses another basis $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$, letting $\widetilde{H}$ and $\widetilde{C}$ be the matrices representing the form ℓ and the operator A in this new basis and letting $M = (M_{i,j})$ be the transition matrix from the new basis to the old one, (i.e., $e_j = \sum_{i=1}^{n} M_{i,j}\tilde{e}_i$,) then

$$\widetilde{H} = (M^{-1})^* H M^{-1} \quad \text{and} \quad \widetilde{C} = MC\overline{M}^{-1}. \tag{3.1.2}$$

Our goal is to find a basis in which the matrix representation of the form ℓ and operator A has a particularly simple form. In other words, if we define an action of the matrix group $GL_n(\mathbb{C})$ on the pairs (H, C) of $n \times n$ matrices satisfying (3.1.1) by the mapping

$$\big(M, (H, C)\big) \mapsto \Big(\big(M^{-1}\big)^* HM^{-1}, MC\overline{M}^{-1}\Big), \quad M \in GL_n(\mathbb{C}),$$

then our goal is to choose a representative in each orbit of this action in a canonical way. This canonical representative is usually called the *canonical* or *normal* form of the pair (ℓ, A).

We let T_k be the $k \times k$ matrix whose (i, j) entry is 1 if $j - i = 1$ and zero otherwise, let S_k be the $k \times k$ matrix whose (i, j) entry is 1 if $j + i = k + 1$ and zero otherwise, let I_k be the rank k identity matrix, and let $J_{\lambda,k} = \lambda I_k + T_k$ be the standard $k \times k$ Jordan block corresponding to the eigenvalue λ, that is,

$$J_{\lambda,k} := \left.\overbrace{\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}}^{k \text{ columns}}\right\} k \text{ rows} \quad \text{and} \quad S_k = \left.\overbrace{\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}^{k \text{ columns}}\right\} k \text{ rows}. \tag{3.1.3}$$

To succinctly define new matrices constructed from others, we write

$$M_1 \oplus M_2 \oplus \cdots \oplus M_k = \bigoplus_{i=1}^{k} M_i$$

to denote the block diagonal matrix whose diagonal entries are the matrices $M_1, \ldots, M_k$. For $\lambda \in \mathbb{C}$, we define the $k \times k$ or $2k \times 2k$ matrix $M_{\lambda,k}$ by

$$M_{\lambda,k} := \begin{cases} J_{\lambda,k} & \text{if } \lambda \in \mathbb{R} \\ \begin{pmatrix} 0 & J_{\lambda^2,k} \\ I_k & 0 \end{pmatrix} & \text{otherwise,} \end{cases} \tag{3.1.4}$$

where 0 denotes a matrix of appropriate size with zero in all entries. We define corresponding matrices $N_{\lambda,k}$ by

$$N_{\lambda,k} := \begin{cases} S_k & \text{if } \lambda \in \mathbb{R} \\ S_{2k} & \text{otherwise.} \end{cases} \tag{3.1.5}$$

For a nonnegative integer k, we define

$$W_{\lambda}^{(k)} := \operatorname{span}_{\mathbb{C}} \left\{ v \in W \ : \ (A^2 - \lambda^2 I)^k v = 0 \text{ or } \left(A^2 - \overline{\lambda^2} I\right)^k v = 0 \right\}. \tag{3.1.6}$$

Since A^2 is linear, we can enumerate its eigenvalues, letting $\lambda_1^2, \ldots, \lambda_\mu^2$ be the real eigenvalues of A^2 and $\lambda_{\mu+1}^2, \ldots, \lambda_\gamma^2$ be the distinct eigenvalues of A^2 with positive imaginary part. In the canonical forms below, we assume that each λ_i is the principle square root of λ_i^2. Since the linear operator A^2 is ℓ-self-adjoint, it is easy to show (see, for example, [16, Theorem 4.2.4]) that the space W can be decomposed into pairwise-ℓ-orthogonal A^2-invariant subspaces

$$W = W_{\lambda_1}^{(n)} \oplus W_{\lambda_2}^{(n)} \oplus \cdots \oplus W_{\lambda_\gamma}^{(n)}. \tag{3.1.7}$$

Remark 3.1.1. *In [16], the authors refine this decomposition of W, obtaining a canonical form for (ℓ, A^2). For an ℓ-self-adjoint linear operator B, their theorem, [16, Theorem 5.1.1], states that the domain of B can be decomposed into B-invariant, pairwise ℓ-orthogonal subspaces such that there exists a basis with respect to which the restrictions of ℓ and B to the decomposition's component subspaces are represented by matrices of the form $\pm S_k$ and $J_{\eta,k}$ if $\eta \in \mathbb{R}$ or $\pm S_{2k}$ and $J_{\eta,k} \oplus J_{\overline{\eta},k}$ if $\eta \notin \mathbb{R}$ (this gives a canonical form for (ℓ, A^2) by letting $B = A^2$).*

Note that $W_{\lambda_i}^{(n)}$ is also A-invariant. Indeed , if $v \in W_{\lambda_i}^{(n)}$ and $(A^2 - \lambda^2 I)^k v = 0$, then

$$\left(A^2 - \overline{\lambda_i^2} I\right)^n (Av) = A(A^2 - \lambda_i^2 I)^n v = 0,$$

which shows that $Av \in W_{\lambda_i}^{(n)}$. Similarly, if $v \in W_{\lambda_i}^{(n)}$ and $(A^2 - \overline{\lambda^2} I)^k v = 0$, then

$$\left(A^2 - \lambda_i^2 I\right)^n (Av) = A(A^2 - \overline{\lambda_i^2} I)^n v = 0,$$

which shows that $Av \in W_{\lambda_i}^{(n)}$. This completes the proof of A-invariancy of $W_{\lambda_i}^{(n)}$.

Accordingly, we can normalize ℓ and A on the spaces W_{λ_i} separately to obtain a general canonical form.

Theorem 3.1.2. *The domain of an ℓ-self-adjoint antilinear operator A can be decomposed into A-invariant, pairwise ℓ-orthogonal subspaces such that there exists a basis with respect to which the restrictions of ℓ and A to the decomposition's component subspaces are represented by matrices of the form $\pm N_{\lambda,k}$ and $M_{\lambda,k}$ where $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_\gamma\}$ and $k \in \mathbb{N}$. The corresponding block diagonal matrices representing ℓ and A are unique up to a permutation of the blocks on the* *diagonal.*

Proof. Since the decomposition in (3.1.7) is pairwise ℓ-orthogonal and A-invariant, the result is a corollary of Propositions 3.2.10, 3.2.11, 3.2.12, and 3.2.13. $\square$

Remark 3.1.3. *In [22, Theorem 3.1], the authors show that an antilinear operator A can be represented by a matrix in the form of the matrix given in Theorem 3.1.2 representing the antilinear operator, that is, the domain of A can be decomposed into A-invariant subspaces on which A is represented by $M_{\lambda,k}$ where $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_\gamma\}$ and $k \in \mathbb{N}$ (note, this is achieved without the assumption that A is ℓ-self-adjoint for some Hermitian form ℓ).*

A canonical form for a nonsingular antilinear operator is fully determined by the Jordan matrix representing its square, and we have a similar relationship between Theorem 3.1.2 and the Gohberg–Lancaster–Rodman form, recorded in the following lemma.

Lemma 3.1.4. *If A is nonsingular then the canonical form for (ℓ, A) given in Theorem 3.1.2 is determined by the Gohberg–Lancaster–Rodman form for (ℓ, A^2).*

3.2 Canonical forms for restrictions to generalized eigenspaces

In this section we obtain a canonical form for the restrictions of ℓ and A to the spaces $W_\lambda^{(n)}$, and these results can be taken together to obtain the canonical form in Theorem 3.1.2. The approach we employ varies depending on the eigenvalue λ^2 of A^2, so this section is structured with subsections, each dedicated to a case where λ^2 belongs to a different family. We repeatedly use the following lemma, which is completely analogous to a standard property of linear self-adjoint operators.

Lemma 3.2.1. *If $V \subset \mathbb{C}^n$ is an A-invariant subspace on which ℓ is nondegenerate then the ℓ-orthogonal complement $V^{\perp_\ell}$ of V is also A-invariant.*

Proof. Since V is A-invariant, for any $w \in V$, we have that $Aw \in V$, which implies that, for $v \in V^{\perp_\ell}$, we have $\ell(Aw, v) = 0$. Therefore, since A is ℓ-self-adjoint, for $v \in V^{\perp_\ell}$ and $w \in V$, we have $\ell(Av, w) = \ell(Aw, v) = 0$, which implies that $Av \in V^{\perp_\ell}$. $\square$

3.2.1 Treating generalized eigenspaces with positive eigenvalues

Throughout this subsection we assume $\lambda^2 > 0$, and this subsection's main result is Proposition 3.2.10.

For this special case with $\lambda^2 > 0$, we define three additional filtrations of $W_\lambda^{(n)}$. Namely,

$$W_\lambda^{(k)\pm} := \left\{x \in W \,:\, (A \mp \lambda I)(A^2 - \lambda^2 I)^{k-1}x = 0\right\}, \tag{3.2.1}$$

and

$$\widetilde{W}_\lambda^{(k)} := \left\{x \in W : (A - \lambda I)^k x = 0\right\}. \tag{3.2.2}$$

The following two lemmas address the relationship between the filtrations $\left\{W_\lambda^{(k)}\right\}$, $\left\{W_\lambda^{(k)\pm}\right\}$, and $\left\{\widetilde{W}_\lambda^{(k)}\right\}$, defined by (3.1.6), (3.2.1), and (3.2.2), respectively. Note that, for each k, $W_\lambda^{(k)\pm}$ and $\widetilde{W}_\lambda^{(k)}$ are vector spaces over $\mathbb{R}$ but not over $\mathbb{C}$. In principle, these lemmas can be deduced from the Hong–Horn canonical forms for antilinear operators from [22, Theorem 3.1] (see also Remark 3.1.3 above), but we prefer to give an independent geometric proof of these Lemmas, first, in order to make the presentation self-contained (as the source [20], where [22, Theorem 3.1] is proved, is not easily available), second, because our proofs of these Lemmas are the main ingredient in the new geometric proof of Hong and Horn's result (outlined in section 3.2.5), and, third, because this proof seems to be interesting by itself.

Lemma 3.2.2. *For all positive integers k, we have $W_\lambda^{(k)}/W_\lambda^{(k-1)} = W_\lambda^{(k)+}/W_\lambda^{(k-1)} \oplus W_\lambda^{(k)-}/W_\lambda^{(k-1)}$. Moreover, $W_\lambda^{(k)}/W_\lambda^{(k-1)} = span_{\mathbb{C}}\left(W_\lambda^{(k)+}/W_\lambda^{(k-1)}\right)$.*

Proof. If $x \in W_\lambda^{(k)}$ then $\lambda x \pm Ax \in W_\lambda^{(k)\pm}$ because

$$(A \mp \lambda I)(A^2 - \lambda^2 I)^{k-1}(\lambda x \pm Ax) = (A^2 - \lambda^2 I)^k x = 0,$$

which shows

$$\left\{\lambda x \pm Ax \,:\, x \in W_\lambda^{(k)}\right\}/W_\lambda^{(k-1)} \subset W_\lambda^{(k)\pm}/W_\lambda^{(k-1)}.$$

Accordingly,

$$\begin{aligned}
W_\lambda^{(k)}/W_\lambda^{(k-1)} &\overset{*}{=} \left\{\lambda x - Ax \,:\, x \in W_\lambda^{(k)}\right\}/W_\lambda^{(k-1)} \oplus \left\{\lambda x + Ax \,:\, x \in W_\lambda^{(k)}\right\}/W_\lambda^{(k-1)} \\
&\subset W_\lambda^{(k)-}/W_\lambda^{(k-1)} \oplus W_\lambda^{(k)+}/W_\lambda^{(k-1)} \\
&\overset{**}{\subset} W_\lambda^{(k)}/W_\lambda^{(k-1)}, \qquad (3.2.3)
\end{aligned}$$

where ** holds because $A^2 - \lambda^2 I = (A + \lambda I)(A - \lambda I)$ and * holds for the following reason. Both $\left\{\lambda x - Ax \,:\, x \in W_\lambda^{(k)}\right\}/W_\lambda^{(k-1)}$ and $\left\{\lambda x + Ax \,:\, x \in W_\lambda^{(k)}\right\}/W_\lambda^{(k-1)}$ are disjoint subsets of $W_\lambda^{(k)}/W_\lambda^{(k-1)}$ because they belong to the kernel of $A + \lambda I : W_\lambda^{(k)}/W_\lambda^{(k-1)} \to W_\lambda^{(k)}/W_\lambda^{(k-1)}$ and $A - \lambda I : W_\lambda^{(k)}/W_\lambda^{(k-1)} \to W_\lambda^{(k)}/W_\lambda^{(k-1)}$ respectively, and these kernels are disjoint because if v is in both kernels then $\lambda v \equiv -\lambda v \pmod{W_\lambda^{(k-1)}}$. This shows that the direct sum on the right side of * is naturally a subset of $W_\lambda^{(k)}/W_\lambda^{(k-1)}$. On the other hand, for any $v \in W_\lambda^{(k)}/W_\lambda^{(k-1)}$, we have

$$v = \left(\lambda\left(\frac{v}{\lambda}\right) - A\left(\frac{v}{\lambda}\right)\right) + \left(\lambda\left(\frac{v}{\lambda}\right) + A\left(\frac{v}{\lambda}\right)\right)$$

which shows that $W_\lambda^{(k)}/W_\lambda^{(k-1)}$ is contained in the direct sum on the right side of *.

By (3.2.3), $W_\lambda^{(k)}/W_\lambda^{(k-1)} = \operatorname{span}_{\mathbb{C}}\left(W_\lambda^{(k)+}/W_\lambda^{(k-1)}\right)$ because $W_\lambda^{(k)+}/W_\lambda^{(k-1)} = iW_\lambda^{(k)-}/W_\lambda^{(k-1)}$. □

Remark 3.2.3. *Notice, we have already used the special condition $\lambda^2 > 0$ of 3.2.1, because Lemma 3.2.2 relies on the fact that $A^2 - \lambda^2 I = (A + \lambda I)(A - \lambda I)$.*

Lemma 3.2.4. *Any basis of the real vector space $\widetilde{W}_\lambda^{(k)}$ is also a basis of the complex vector space $W_\lambda^{(k)}$.*

Proof. When $k = 1$, the statement follows from Lemma 3.2.2 because $\widetilde{W}_\lambda^{(1)} = W_\lambda^{(1)+}$ and $W_\lambda^{(0)} = 0$. Proceeding by induction, let us assume any basis of the real vector space $\widetilde{W}_\lambda^{(k-1)}$ is also a basis of the complex vector space $W_\lambda^{(k-1)}$. Suppose $\dim \widetilde{W}_\lambda^{(k-1)} = l$ and $\dim \widetilde{W}_\lambda^{(k)}/\widetilde{W}_\lambda^{(k-1)} = m$, and let $\{e_1, \ldots, e_{l+m}\}$ be a basis of $\widetilde{W}_\lambda^{(k)}$. Without loss of generality, we can assume $\{e_1, \ldots, e_l\} \subset \widetilde{W}_\lambda^{(k-1)}$ because this assumption does not change the real or complex span of $\{e_1, \ldots, e_{l+m}\}$.

First, we show that the vectors $e_{l+1}, \ldots, e_{l+m}$ are linearly independent over $\mathbb{C}$ modulo $W_\lambda^{(k-1)}$. For this, consider a vector $v \in \mathrm{span}_{\mathbb{C}}\{e_{l+1}, \ldots, e_{l+m}\}$ with coefficients $\alpha_{l+1}, \ldots, \alpha_{l+m} \in \mathbb{R}$ and $\beta_{l+1}, \ldots, \beta_{l+m} \in \mathbb{R}$ such that

$$v := \sum_{j=l+1}^{l+m} (\alpha_j + i\beta_j)e_i \in W_\lambda^{(k-1)}.$$

Set

$$v_+ := \sum_{j=l+1}^{l+m} \alpha_j e_i \quad \text{and} \quad v_- := \sum_{j=l+1}^{l+m} i\beta_j e_i.$$

Since $\widetilde{W}_\lambda^k$ is a real vector space, $v_+ \in \widetilde{W}_\lambda^k$, and hence, by (3.2.1),

$$A(A - \lambda I)^{k-1} v_+ = \lambda(A - \lambda I)^{k-1} v_+.$$

Therefore

$$(A + \lambda I)(A - \lambda I)^{k-1} v_+ = 2\lambda(A - \lambda I)^{k-1} v_+ \tag{3.2.4}$$

Notice $(A+\lambda I)^k v_- = 0$ because, for all $l < j \leq l+m$, $(A+\lambda I)^k(i\beta_i e_j) = -i\beta_j(A-\lambda I)^k e_j = 0$. Since $v \in W_\lambda^{(k-1)}$ and $(A + \lambda I)^k v_- = 0$,

$$\begin{aligned} 0 = (A + \lambda I)(A^2 - \lambda^2 I)^{k-1} v &= (A + \lambda I)(A^2 - \lambda^2 I)^{k-1} v_+ + (A - \lambda I)^{k-1}(A + \lambda I)^k v_- \\ &= (A + \lambda I)(A^2 - \lambda^2 I)^{k-1} v_+, \end{aligned}$$

and hence

$$(A^2 - \lambda^2 I)^{k-1} v_+ \in \ker(A + \lambda I). \tag{3.2.5}$$

Furthermore, $(A-\lambda I)^k v_+ = 0$ because, for all $l < j \le l+m$, $(A-\lambda I)^k \alpha_j e_j = 0$, so

$$(A^2 - \lambda^2 I)^{k-1} v_+ \in \ker(A - \lambda I). \tag{3.2.6}$$

Yet, $\ker(A-\lambda I) \cap \ker(A+\lambda I) = 0$, (3.2.5) and (3.2.6) imply

$$v_+ \in \ker(A^2 - \lambda^2 I)^{k-1}, \tag{3.2.7}$$

and (3.2.4) implies

$$(A^2 - \lambda^2 I)^{k-1} v_+ = (A+\lambda I)^{k-1}(A-\lambda I)^{k-1} v_+ = (2\lambda)^{k-1}(A-\lambda I)^{k-1} v_+. \tag{3.2.8}$$

Together, (3.2.7) and (3.2.8) imply that

$$v_+ \in \ker(A-\lambda I)^{k-1} = \widetilde{W}_\lambda^{(k-1)} = \text{span}_{\mathbb{R}}\{e_1, \ldots, e_l\}$$

and hence $v_+ = 0$ because $\text{span}_{\mathbb{R}}\{e_1, \ldots, e_l\} \cap \text{span}_{\mathbb{R}}\{e_{l+1}, \ldots, e_{l+m}\} = 0$. Note that $v_+ = 0$ implies $\alpha_{l+1} = \cdots = \alpha_{l+m} = 0$ because $e_{l+1}, \ldots, e_{l+m}$ are linearly independent over $\mathbb{R}$. Repeating the same argument with v replaced by iv yields $v_- = 0$ and $\beta_{l+1} = \cdots = \beta_{l+m} = 0$ as well. Hence $v = 0$, which shows that

$$\text{span}_{\mathbb{C}}\{e_{l+1}, \ldots, e_{l+m}\} \cap W_\lambda^{(k-1)} = \text{span}_{\mathbb{C}}\{e_{l+1}, \ldots, e_{l+m}\} \cap \text{span}_{\mathbb{C}}\{e_1, \ldots, e_l\} = 0. \tag{3.2.9}$$

Let us now establish the vector space isomorphism $\widetilde{W}_\lambda^{(k)}/\widetilde{W}_\lambda^{(k-1)} \cong W_\lambda^{(k)+}/W_\lambda^{(k-1)}$. The cosets

$$e_{l+1} + W_\lambda^{(k-1)}, \ldots, e_{l+m} + W_\lambda^{(k-1)}$$

are linearly independent vectors (over $\mathbb{R}$) in the space $W_\lambda^{(k)+}/W_\lambda^{(k-1)}$. If we take an arbitrary vector

$w + W_\lambda^{(k-1)} \in W_\lambda^{(k)+}/W_\lambda^{(k-1)}$ then $(A-\lambda I)w \in W_\lambda^{(k-1)}$, so

$$(A+\lambda I)w \equiv 2\lambda w \pmod{W_\lambda^{(k-1)}}.$$

Hence,

$$(2\lambda)^{1-k}(A+\lambda I)^{k-1}w \equiv w \pmod{W_\lambda^{(k-1)}}. \tag{3.2.10}$$

Now observe that $(A+\lambda I)^{k-1}w \in \widetilde{W}_\lambda^{(k)}$. Indeed, from the definitions (3.1.6) and (3.2.1) and the fact that $w \in W_\lambda^{(k)+}$, it follows that

$$(A-\lambda I)^k (A+\lambda I)^{k-1} w = (A-\lambda I)(A^2-\lambda^2 I)^{k-1} w = 0.$$

Hence, by (3.2.10), $w \in \widetilde{W}_\lambda^{(k)}$. Therefore, there exist real coefficients $a_{l+1}, \ldots, a_{l+m}$ such that

$$w \equiv (2\lambda)^{1-k}(A+\lambda I)^{k-1}w \equiv \sum_{i=l+1}^{l+m} a_i e_i \pmod{W_\lambda^{(k)}}.$$

This shows that the cosets

$$e_{l+1} + W_\lambda^{(k-1)}, \ldots, e_{l+m} + W_\lambda^{(k-1)}$$

form a basis of $W_\lambda^{(k)+}/W_\lambda^{(k-1)}$. On the other hand, the cosets

$$e_{l+1} + \widetilde{W}_\lambda^{(k-1)}, \ldots, e_{l+m} + \widetilde{W}_\lambda^{(k-1)}$$

form a basis of $\widetilde{W}_\lambda^{(k)}/\widetilde{W}_\lambda^{(k-1)}$, so the real vector spaces $\widetilde{W}_\lambda^{(k)}/\widetilde{W}_\lambda^{(k-1)}$ and $W_\lambda^{(k)+}/W_\lambda^{(k-1)}$ are isomorphic.

Applying Lemma 3.2.2, we get

$$\dim_{\mathbb{C}} W_\lambda^{(k)}/W_\lambda^{(k-1)} = \dim_{\mathbb{R}} W_\lambda^{(k)+}/W_\lambda^{(k-1)} = \dim_{\mathbb{R}} W_\lambda^{(k)+}/W_\lambda^{(k-1)} = m,$$

which implies

$$\dim_{\mathbb{C}} W_\lambda^{(k)} = m + \dim W_\lambda^{(k-1)} = l + m.$$

We have shown that $v \in W_\lambda^{(k-1)}$ implies $\alpha_{l+1} = \cdots = \alpha_m = \beta_{l+1} = \cdots = \beta_{l+m} = 0$. In particular, we have shown that $v = 0$ implies $\alpha_{l+1} = \cdots = \alpha_{l+m} = \beta_{l+1} = \cdots = \beta_{l+m} = 0$. Therefore, $e_{l+1}, \ldots, e_{l+m}$ are linearly independent over $\mathbb{C}$, that is,

$$\dim_{\mathbb{C}} \text{span}_{\mathbb{C}}\{e_{l+1}, \ldots, e_{l+m}\} = m,$$

so, by the induction hypothesis and (3.2.9),

$$\dim_{\mathbb{C}} \text{span}_{\mathbb{C}}\{e_1, \ldots, e_{l+m}\} = l + m.$$

Therefore, $W_\lambda^{(k)} = \text{span}_{\mathbb{C}}\{e_1, \ldots, e_{l+m}\}$ because $e_1, \ldots, e_{l+m}$ are $l + m$ linearly independent vectors (over $\mathbb{C}$) in $W_\lambda^{(k)}$. $\square$

Corollary 3.2.5. *If $v \in W_\lambda^{(k)}$ then there exist unique vectors $v_+, v_- \in \widetilde{W}_\lambda^{(k)}$ such that $v = v_+ + iv_-$.*

Define s_1 to be the minimal natural number such that $W_\lambda^{(s_1)} = W_\lambda^{(n)}$. We would like to find a vector $v \in \widetilde{W}_\lambda^{(s_1)}$ such that the space

$$V = \text{span}_{\mathbb{C}} \left\{v, (A - \lambda I)v, \ldots, (A - \lambda I)^{s_1 - 1} v\right\} \tag{3.2.11}$$

is an s_1-dimensional A-invariant space on which ℓ is nondegenerate because we can then normalize A and ℓ on the space V and on the ℓ-orthogonal complement of V separately. Proceeding throughout this subsection, for $v \in W_\lambda^{(n)}$, we adopt the notation of letting $v_+, v_- \in \widetilde{W}_\lambda^{(n)}$ be the unique vectors such that $v = v_+ + iv_-$, as given in Corollary 3.2.5.

Lemma 3.2.6. *If H is a Hermitian $k \times k$ matrix, $\lambda > 0$, and $HJ_{\lambda,k}$ is symmetric, then H is a*

Hankel matrix satisfying

$$H_{i,j} = 0 \qquad \forall i + j \leq k.$$

Proof. Let H' be the upper left $(k-1) \times (k-1)$ block of H. Symmetry of $HJ_{\lambda,k}$ implies that $H'J_{\lambda,k-1}$ is symmetric. Since the Lemma is vacuously true for $k = 1$, we can proceed by induction, and assume H' is a Hankel matrix satisfying

$$H'_{i,j} = 0 \qquad \forall i + j \leq k - 1.$$

Computing the $(1,k)$ and $(k,1)$ entries of $HJ_{\lambda,k}$ yields

$$\left(HJ_{\lambda,k}\right)_{1,k} = \lambda H_{1,k} + H_{1,k-1} \quad \text{and} \quad \left(HJ_{\lambda,k}\right)_{k,1} = \lambda \overline{H_{1,k}}.$$

Symmetry of $HJ_{\lambda,k}$ allows us to equate the terms, so

$$H_{1,k-1} = \lambda \left(\overline{H_{1,k}} - H_{1,k}\right) \in \{iz \mid z \in \mathbb{R}\}.$$

Yet, since H' is both Hankel and Hermitian, its entries are all real numbers. In particular, $H_{1,k-1} \in \mathbb{R}$, so

$$H_{1,k-1} = 0 \quad \text{and} \quad H_{1,k} = H_{k,1} \in \mathbb{R}.$$

Equating $\left(HJ_{\lambda,k}\right)_{2,k}$ with $\left(HJ_{\lambda,k}\right)_{k,2}$ yields

$$H_{1,k} - H_{2,k-1} = \lambda \left(H_{2,k} - \overline{H_{2,k}}\right) \in \{iz \mid z \in \mathbb{R}\}$$

which implies $H_{k,1} = H_{2,k-1}$ because, by the induction hypothesis, $H_{2,k-1} \in \mathbb{R}$. Accordingly,

$$H_{i,j} = H_{1,k} \qquad \forall i + j = k + 1$$

because H' is Hankel.

We conclude this proof with induction. Supposing, for some $1 < m \leq k$, we have $H_{k,j} = H_{j,k}$ and $H_{k,j} = H_{j+1,k-1}$ for all $j < m$, let us establish that $H_{k,m} = H_{m,k}$ and $H_{k,m} = H_{m+1,k-1}$, where we interpret $H_{k,m} = H_{m+1,k-1}$ as vacuously true for $m = k$ (i.e., since $H_{i,j}$ is only defined for $\max\{i,j\} \leq k$, we can extend the definition of $H_{i,j}$ for $\max\{i,j\} > k$ in a way that satisfies the equations $H_{k,k+j} = H_{k+j,k}$ and $H_{k,k+j} = H_{k+j+1,k-1}$ for all j by construction, and, of course, this extension's definition has no relevance to the normalization of H). Symmetry of $HJ_{\lambda,k}$ implies

$$H_{m,k-1} + \lambda H_{m,k} = \left(HJ_{\lambda,k}\right)_{m,k} = \left(HJ_{\lambda,k}\right)_{k,m} = H_{k,m-1} + \lambda H_{k,m},$$

and hence

$$H_{m,k} - H_{k,m} = \lambda^{-1}(H_{k,m-1} - H_{m,k-1}) = 0$$

because, by the induction hypothesis, $H_{k,m-1} = H_{m,k-1}$. If $m = k$ then there is nothing more to check, that is, $H_{k,m} = H_{m+1,k-1}$ is vacuously true. Similarly, if $m = k-1$ then we have already shown $H_{k,m} = H_{m+1,k-1}$. For $m < k-1$, we have

$$H_{m+1,k-1} + \lambda H_{m+1,k} = \left(HJ_{\lambda,k}\right)_{m+1,k} = \left(HJ_{\lambda,k}\right)_{k,m+1} = H_{k,m} + \lambda H_{k,m+1},$$

and hence

$$H_{k,m} - H_{m+1,k-1} = \lambda(H_{m+1,k} - H_{k,m+1}) = \lambda(H_{m+1,k} - \overline{H_{m+1,k}}) \in \{z \,|\, iz \in \mathbb{R}\},$$

which implies $H_{k,m} = H_{m+1,k-1}$ because, since H' is a real matrix, $H_{m+1,k-1} \in \mathbb{R}$. This completes the proof by induction. $\square$

Lemma 3.2.7. *If a nondegenerate Hermitian form ℓ and antilinear operator A are represented respectively by the $k \times k$ matrices H and $J_{\lambda,k}$, where H is a Hankel matrix satisfying*

$$H_{i,j} = 0 \qquad \forall\, i + j \leq k,$$

then there is a basis with respect to which ℓ and A are represented by $\pm S_k$ and $J_{\lambda,k}$ respectively.

Proof. Every transformation of the matrices representing ℓ and A given by the rule (3.1.2) can be induced by a change of basis, so it will suffice to find M such that

$$M^* H M = S_k \quad \text{and} \quad M^{-1} J_{\lambda,k} \overline{M} = J_{\lambda,k}. \tag{3.2.12}$$

To satisfy $M^{-1} J_{\lambda,k} \overline{M} = J_{\lambda,k}$, let us suppose M is a real upper-triangular Toeplitz matrix, and define $h_0, \ldots, h_{k-1} \in \mathbb{R}$ and $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{R}$ to be the coefficients for which

$$H = S_k \left(\sum_{i=1}^{k} h_{i-1} T_k^{i-1} \right) \quad \text{and} \quad M = \sum_{i=1}^{k} \alpha_{i-1} T_k^{i-1}.$$

Note, $h_1, \ldots, h_k$ must be real because H is Hermitian and Hankel. For our particular choice of M, we have $M^* = S_k M S_k$, so

$$M^* H M = S_k \left(\sum_{i=1}^{k} \alpha_{i-1} T_k^{i-1} \right) \left(\sum_{i=1}^{k} h_{i-1} T_k^{i-1} \right) \left(\sum_{i=1}^{k} \alpha_{i-1} T_k^{i-1} \right) = S_k \left(\sum_{i=0}^{k-1} \sum_{r+s+t=i} \alpha_r \alpha_s h_t T_k^{i} \right).$$

Therefore, we need to solve the equation

$$\sum_{i=0}^{k-1} \sum_{r+s+t=i} \alpha_r \alpha_s h_t T_k^{i} = \pm I_k, \tag{3.2.13}$$

that is, we need to choose α_i such that (3.2.13) holds. Comparing entries of the main diagonal in (3.2.13), we find that $\alpha_0 = h_0^{-1/2}$, so let us choose $\alpha_0 = |h_0|^{-1/2}$. Note, $h_0 \neq 0$ because ℓ is nondegenerate, and hence this choice of α_0 is well defined. Having fixed α_0, comparing entries in the first super-diagonal of (3.2.13) shows that we can choose α_1 as the solution to a linear equation with real coefficients so that entries in the first super-diagonal of (3.2.13) match. Proceeding similarly, for $1 < j < k$, after choosing $\alpha_0, \ldots, \alpha_{j-1}$ so that entries in the main diagonal and the first $j-1$ super-diagonals of (3.2.13) match, comparing entries in the j super-diagonal of (3.2.13) shows that we can choose α_j as the solution to a linear equation with real coefficients so that entries in the j super-diagonal of (3.2.13) match; moreover, the variable α_j does not appear in the first $j-1$ super-diagonals of (3.2.13), so, by choosing α_j in this way, we ensure that entries the first j super-diagonals of (3.2.13) match. By choosing $\alpha_0, \ldots, \alpha_k$ in this way we obtain (3.2.12) by construction. □

Lemma 3.2.8. *There exists a vector $v \in W_\lambda^{(n)}$ such that the space in* (3.2.11) *is an s_1-dimensional A-invariant space on which ℓ is nondegenerate.*

Proof. It can be seen from the Gohberg–Lancaster–Rodman canonical form for ℓ and A^2 (given in [16, Theorem 5.1.1] and summarized in Remark 3.1.1) that there exists a vector $v' \in W_\lambda^{(s_1)}$ for which

$$\ell\left(v', (A^2 - \lambda^2 I)^{(s_1-1)} v'\right) \neq 0. \tag{3.2.14}$$

Using the decomposition of Corollary 3.2.5, define the coefficients

$$a_0 := \ell\big(v'_+, (A-\lambda I)^{(s_1-1)} v'_+\big) + \ell\big(v'_-, (A-\lambda I)^{(s_1-1)} v'_-\big),$$

$$a_1 := \ell\big(v'_+, (A-\lambda I)^{(s_1-1)} v'_+\big) - \ell\big(v'_-, (A-\lambda I)^{(s_1-1)} v'_-\big),$$

and

$$b_1 := \ell\big(v'_-, (A-\lambda I)^{(s_1-1)} v'_+\big) - \ell\big(v'_+, (A-\lambda I)^{(s_1-1)} v'_-\big).$$

By direct computation, we obtain the finite Fourier series

$$2(2\lambda)^{1-s_1} \ell\left(\left(e^{i\theta} v'\right)_+, (A-\lambda I)^{s_1-1}\left(e^{i\theta} v'\right)_+\right) = a_0 + a_1 \cos(2\theta) + b_1 \sin(2\theta) \tag{3.2.15}$$

Also, since $v'_+, v'_- \in \widetilde{W}_\lambda^{(s_1)}$, $A(A-\lambda I)^{s_1-1} v'_+ = \lambda (A-\lambda I)^{s_1-1} v'_+$ and $A(A+\lambda I)^{s_1-1} i v'_- =$

$-\lambda(A+\lambda I)^{s_1-1}iv'_-$, and hence

$$(A^2-\lambda^2 I)^{(s_1-1)}v' = (A+\lambda I)^{(s_1-1)}(A-\lambda I)^{(s_1-1)}v'_+ + (A-\lambda I)^{(s_1-1)}(A+\lambda I)^{(s_1-1)}iv'_-$$
$$= (2\lambda)^{s_1-1}(A-\lambda I)^{(s_1-1)}v'_+ + i(2\lambda)^{s_1-1}(A-\lambda I)^{(s_1-1)}v'_-.$$

So, by (3.2.14),

$$0 \neq (2\lambda)^{1-s_1}\ell\left(v', (A^2-\lambda^2 I)^{(s_1-1)}v'\right) = a_0 + ib_1.$$

If the left side of (3.2.15) is zero for all $\theta \in \mathbb{R}$ then $a_0 = a_1 = b_1 = 0$, so, by (3.2.14), there exists $\theta \in \mathbb{R}$ such that

$$\ell\left(\left(e^{i\theta}v'\right)_+, (A-\lambda I)^{s_1-1}\left(e^{i\theta}v'\right)_+\right) \neq 0. \tag{3.2.16}$$

Fixing $\theta \in \mathbb{R}$ so that (3.2.16) holds, define

$$v := \left(e^{i\theta}v'\right)_+, \tag{3.2.17}$$

so, by (3.2.16),

$$\ell\big(v, (A-\lambda I)^{s_1-1}v\big) \neq 0. \tag{3.2.18}$$

Proceeding, let V be as in (3.2.11) with v as in (3.2.17). Define basis vectors

$$e_i = (A-\lambda I)^{i-1}v \qquad (i = 1, \ldots, s_1).$$

The matrix representing the restriction $A|_V$ of A to V with respect to the basis $\{e_i\}_{1\leq i\leq s_1}$ is J_{λ,s_1}. Let H be the matrix representing the restriction of ℓ to V with respect to the basis $\{e_i\}_{1\leq i\leq s_1}$. Since A is ℓ-self-adjoint, HJ_{λ,s_1} is symmetric. Therefore, applying Lemma 3.2.6, H is a Hankel matrix satisfying

$$H_{i,j} = 0 \qquad \forall i + j \leq k,$$

and hence, by (3.2.18),

$$\det H = \sqrt{2} \sin\left(\frac{2k\pi + \pi}{4}\right) H_{1,k}^k = \sqrt{2} \sin\left(\frac{2k\pi + \pi}{4}\right) \Big(\ell\left(v, (A - \lambda I)^{s_1 - 1} v\right)\Big)^k \neq 0.$$

That is, ℓ is nondegenerate on V, as was needed. $\square$

Corollary 3.2.9. *There is an s_1-dimensional A-invariant space V on which ℓ is nondegenerate, and there is a basis of V with respect to which the restrictions $\ell|_V$ and $A|_V$ of ℓ and A to V are represented by the matrices $\pm N_{\lambda,s_1}$ and $M_{|\lambda|,s_1}$ respectively.*

Proof. By Lemma 3.2.8, there exists an s_1-dimensional A-invariant space V on which ℓ is nondegenerate and there exists a basis of V with respect to which the restrictions $\ell|_V$ and $A|_V$ of ℓ and A to V are represented by the matrices H and J_{λ,s_1}, where H is a Hankel matrix satisfying

$$H_{i,j} = 0 \qquad \forall\, i + j \leq s_1.$$

Therefore, by Lemma 3.2.7, there is a basis $\{e_1, \ldots, e_{s_1}\}$ of V with respect to which $\ell|_V$ and $A|_V$ are represented by S_{s_1} and J_{λ,s_1} respectively. If $\lambda > 0$ then this completes the proof because $J_{\lambda,s_1} = M_{|\lambda|,s_1}$. If, on the other hand, $\lambda < 0$ then we observe $\ell|_V$ and $A|_V$ are represented by S_{s_1} and $J_{-\lambda,s_1} = M_{|\lambda|,s_1}$ with respect to the basis $\{ie_1, \ldots, ie_{s_1}\}$. So, in either case, we can find a basis with respect to which $\ell|_V$ and $A|_V$ are represented by $N_{\lambda,s_1} = S_{s_1}$ and $M_{|\lambda|,s_1}$. $\square$

For the following proposition, let $r_1, \ldots, r_{n_\lambda}$ and $s_1, \ldots, s_{n_\lambda}$ be the positive integers satisfying $s_i > s_{i+1}$ such that the restriction of A^2 to $W_\lambda^{(n)}$ has a Jordan canonical form with r_i Jordan blocks of size $s_i \times s_i$. Note, this definition is consistent with the previous definition of s_1, and

$$W_\lambda^{(n)} \cong \mathbb{C}^\mu \quad \text{where} \quad \mu = \sum_{i=1}^{n_\lambda} r_i s_i.$$

Proposition 3.2.10. *There is a basis of $W_\lambda^{(n)}$ with respect to which the restrictions of ℓ and A to $W_\lambda^{(n)}$ are represented by the matrices*

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}N_{\lambda,s_i}\right) \quad \textit{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}M_{|\lambda|,s_i}\right) \quad \textit{where } \epsilon_{i,j}=\pm 1$$

respectively.

Proof. By Corollary 3.2.9, there is a space $V \subset W_\lambda^{(n)}$ that is A-invariant and ℓ-nondegenerate on which ℓ and A can be represented by matrices of the desired form. By Lemma 3.2.1, we can normalize ℓ and A on V and the ℓ-orthogonal complement $V^{\perp_\ell}$ of V separately, so we can repeat this process, applying Corollary 3.2.9 to $V^{\perp_\ell}$ rather than $W_\lambda^{(n)}$. Iterating the process $\sum_{i=1}^{n_\lambda} r_i$ times completes the normalization. □

3.2.2 Treating generalized eigenspaces with eigenvalue zero

In this subsection we construct a canonical form for the restrictions of ℓ and A to the space $W_0^{(n)}$. Our approach is the same as in the proof of Theorem 4.5 in [33].

Proposition 3.2.11. *The space $W_0^{(n)}$ can be decomposed into A-invariant, pairwise ℓ-orthogonal subspaces such that there exists a basis with respect to which the restrictions of ℓ and A to the decomposition's component subspaces are represented by matrices of the form $M_{0,k}$ and $N_{0,k}$.*

Proof. Let

$$k=\min\left\{m\in\mathbb{N}\,:\,\left(A|_{W_\lambda^{(n)}}\right)^m\equiv 0\right\}.$$

Fix a basis, and let H and C be matrices representing ℓ and A with respect to this basis. If k is odd, then A^{k-1} is ℓ-self-adjoint linear, which implies HC^{k-1} is Hermitian, and hence there is a basis with respect to which the mapping

$$v\mapsto H\left(A^{k-1}v\right) \tag{3.2.19}$$

is represented by a nonzero diagonal matrix. If, on the other hand, k is even, then A^{k-1} is ℓ-self-adjoint antilinear, which implies HC^{k-1} is symmetric. By Takagi's theorem in [39, Theorem 2], for every symmetric matrix S, there exists an invertible matrix U such that $US\overline{U}^{-1}$ is diagonal, and, since the map in (3.2.19) is antilinear whenever k is even, Takagi's theorem implies that there is a basis with respect to which the mapping in (3.2.19) is represented by a nonzero diagonal matrix.

For either parity of k, these observations imply that there exists a vector $a_1 \neq 0$ such that

$$H\left(A^{k-1}a_1\right) = \gamma e^{i\theta} a_1 \qquad \text{for some } \theta \in \mathbb{R}, \gamma > 0. \tag{3.2.20}$$

Furthermore, if k is odd then $\theta \equiv 0 \pmod{\pi}$ because (3.2.19) is a linear operator represented by a Hermitian matrix. Accordingly, for $z \in \mathbb{C}$,

$$\ell(A^{k-1}za_1, za_1) = (HA^{k-1}za_1, za_1) = \begin{cases} \pm\gamma |z|^2 \|a_1\|^2 & \text{if } k \text{ is odd} \\ \gamma e^{i\theta} \overline{z}^2 \|a_1\|^2 & \text{if } k \text{ is even.} \end{cases}$$

Therefore

$$\ell\left(A^{k-1}\frac{1}{\sqrt{\gamma\|a_1\|^2 e^{i\theta/2}}}a_1, \frac{1}{\sqrt{\gamma\|a_1\|^2 e^{i\theta/2}}}a_1\right) = \pm 1.$$

Define

$$\tilde{e}_i = A^{i-1}\frac{1}{\sqrt{\gamma\|a_1\|^2 e^{i\theta/2}}}a_1,$$

and define

$$e_1 = \tilde{e}_1 + \alpha_2\tilde{e}_2 + \ldots + \alpha_k\tilde{e}_k \quad \text{and} \quad e_i = A^{i-1}e_1,$$

where the coefficients $\alpha_2, \ldots, \alpha_k$ are chosen below. For all $i + j > k + 1$ we have

$$\ell(e_i, e_j) = \ell(A^{i-1}e_1, A^{j-1}e_1) = \ell(A^{i+j-2}e_1, e_1) = \ell(0, e_1) = 0.$$

Fix the coefficients $\alpha_2, \ldots, \alpha_k$ such that for all $j < k$ we have

$$\ell(e_1, e_j) = 0.$$

Since A is ℓ-self-adjoint, we have

$$\ell(e_i, e_{i+j}) = \ell(e_{i+j}, e_i),$$

so our choices of $\alpha_2, \ldots, \alpha_k$ ensure

$$\ell(e_i, e_j) = 0 \qquad \forall i + j < k + 1.$$

By construction,

$$\ell(e_1, e_k) = \ell(e_j, e_{k+j-1}) = 1,$$

so the restrictions of ℓ and A to the subspace $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ are represented by S_k and $J_{0,k}$ respectively with respect to the basis $\{e_k, \ldots, e_1\}$.

By Lemma 3.2.1, we can normalize ℓ and A on $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ and the orthogonal complement of $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ separately, so this normalization proceedure can be repeated on the orthoganal complement of $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ until $W_0^{(n)}$ is exhausted. □

3.2.3 Treating generalized eigenspaces with negative eigenvalues

Throughout this subsection we assume $\lambda^2 < 0$ and that the restriction of A^2 to $W_\lambda^{(n)}$ has a Jordan canonical form with $2r_i$ Jordan blocks of size $s_i \times s_i$, where $r_1, \ldots, r_{n_\lambda}$ and $s_1, \ldots, s_{n_\lambda}$ are positive integers satisfying $s_i > s_{i+1}$.

Proposition 3.2.12. *There is a basis of $W_\lambda^{(n)}$ with respect to which the restrictions of ℓ and A to*

$W_\lambda^{(n)}$ *are represented by the matrices*

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}N_{\lambda,s_i}\right) \quad \textit{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}M_{\lambda,s_i}\right) \quad \textit{where } \epsilon_{i,j}=\pm 1$$

respectively.

Proof. Given the Gohberg–Lancaster–Rodman canonical form for ℓ and A^2 summarized in Remark 3.1.1, there exists a vector $a_1 \in W_\lambda^{(n)}$ such that the restrictions of ℓ and A^2 to the s_1-dimensional vector space $\mathrm{span}_{\mathbb{C}}\{a_1, (A^2-\lambda^2 I)a_1, \ldots, (A^2-\lambda^2 I)^{s_1-1}a_1\}$ are represented respectively by S_{s_1} and J_{λ^2,s_1} with respect to the basis $\{(A^2-\lambda^2 I)^{s_1-1}a_1, (A^2-\lambda^2 I)^{s_1-2}a_1, \ldots, a_1\}$.

Defining

$$a_{k+1} = (A^2-\lambda^2 I)a_k \quad \text{and} \quad b_k = Aa_k,$$

and letting $e_1 = \alpha a_1 + \beta b_1$, we have

$$\begin{aligned}\ell\left(e_1, A(A^2-\lambda^2 I)^{s_1-1}e_1\right) &= \alpha^2\ell(a_1,b_{s_1}) + \alpha\beta\big(\ell(a_1,Ab_{s_1}) + \ell(b_1,b_{s_1})\big) + \beta^2\ell(b_1,Ab_{s_1}\\ &= \alpha^2\ell(a_1,b_{s_1}) + \alpha\beta\big(\lambda^2\ell(a_1,a_{s_1}) + \lambda^2\ell(a_{s_1},a_1)\big) + \lambda^2\beta^2\ell(b_1,a_{s_1})\\ &= \alpha^2\ell(a_1,b_{s_1}) \pm 2\lambda^2\alpha\beta + \lambda^2\beta^2\ell(b_1,a_{s_1}).\end{aligned}$$

Clearly, either $\ell(a_1,b_{s_1})=0$ or we can choose $\alpha,\beta\in\mathbb{C}$ such that $(\alpha,\beta)\neq(0,0)$ and

$$\ell\left(e_1, A(A^2-\lambda^2 I)^{s_1-1}e_1\right) = 0.$$

Accordingly, we can assume, by possibly replacing a_1 with e_1 as defined above, that

$$\ell(a_1,b_{s_1}) = 0.$$

With this assumption made, we proceed with e_1 defined as above, and will determine the coefficients α and β later. Note, this assumption implies also that $\ell(b_1,a_{s_1})=0$ because $(A^2-\lambda^2 I)$ is an ℓ-self-adjoint linear operator, and hence

$$\ell(e_1, A(A^2 - \lambda^2 I)^{s_1-1} e_1) = \pm 2\lambda^2 \alpha\beta.$$

Define

$$e_k := (A^2 - \lambda^2 I)^{k-1} e_1 \quad \text{and} \quad e_{s_1+k} := Ae_k \qquad \forall 1 \leq k \leq s_1,$$

and, on the span of $\{e_i\}$, let ℓ and A be represented with respect to the basis $\{e_{s_1}, \ldots, e_1, e_{2s_1}, \ldots, e_{s_1+1}\}$ by the matrices

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & J_{\lambda^2, s_1} \\ I_{s_1} & 0 \end{pmatrix}$$

where the each $H_{i,j}$ is an $s_1 \times s_1$ matrix. The matrices $H_{i,j}$ are Hankel because $A^2 - \lambda^2 I$ is H-self-adjoint. That is, $H_{1,1}$ is Hankel because

$$\begin{aligned}\ell(e_i, e_j) = \ell\left((A^2 - \lambda^2)^{i-1} e_1, (A^2 - \lambda^2)^{j-1} e_1\right) &= \ell\left(e_1, (A^2 - \lambda^2)^{i+j-2} e_1\right) \\ &= \ell(e_1, e_{i+j-1}) \qquad \forall i + j \leq s_1 + 1\end{aligned}$$

and

$$\begin{aligned}\ell(e_i, e_j) = \ell\left((A^2 - \lambda^2)^{i-1} e_1, (A^2 - \lambda^2)^{j-1} e_1\right) &= \ell\left(e_1, (A^2 - \lambda^2)^{i+j-2} e_1\right) \\ &= \ell\left(e_1, 0\right) \qquad \forall i, j \leq s_1 \text{ with } i + j > s_1 + 1.\end{aligned}$$

Similarly, using the identity $((A^2 - \lambda^2 I)v, w) = (v, (A^2 - \lambda^2 I)w)$, we can show $H_{1,2}$, $H_{2,1}$, and $H_{2,2}$ are Hankel.

Since A is ℓ-self-adjoint, HC is symmetric, which, as in Lemma 3.2.6, implies that the (i, j) entry of $H_{2,1}$ is 0 for all $i + j < s_1 + 1$. On the other hand, if $s_1 + 1 < i + j$ then still the (i, j) entry of $H_{2,1}$ is 0 because

$$\ell\left((A^2 - \lambda^2)^{i-1} e_1, (A^2 - \lambda^2)^{j-1} Ae_1\right) = \ell\left((A^2 - \lambda^2)^{i+j-2} e_1, Ae_1\right) = \ell\left(0, Ae_1\right) = 0.$$

Therefore,

$$H_{1,2} = H_{2,1} = \ell\left(e_1, e_{2s_1}\right) S_{s_1}.$$

The same analysis shows that the lower left and upper right $s_1 \times s_1$ blocks of the matrix representing ℓ with respect to the basis $\{a_{s_1}, \ldots, a_1, b_{s_1}, \ldots, b_1\}$ are also multiples of S_{s_1}, that is,

$$\ell(a_i, b_j) = \ell(a_1, b_{s_1})\delta_{i+j,s_1+1}.$$

Direct computation also shows that $H_{1,1}$ is a multiple of S_{s_1}, that is,

$$H_{1,1} = \ell(e_1, e_{s_1})S_{s_1},$$

where

$$\begin{aligned}
\ell(e_1, e_{s_1}) &= |\alpha|^2\ell(a_1, a_{s_1}) + \alpha\overline{\beta}\ell(a_1, b_{s_1}) + \beta\overline{\alpha}\ell(b_1, a_{s_1}) + |\beta|^2\ell(b_1, b_{s_1}) \\
&= \left(|\alpha|^2 + \lambda^2|\beta|^2\right)\ell(a_1, a_{s_1}) + \alpha\overline{\beta}\ell(a_1, b_{s_1}) + \overline{\alpha\overline{\beta}\ell(a_1, b_{s_1})} \\
&= \pm\left(|\alpha|^2 + \lambda^2|\beta|^2\right) + \alpha\overline{\beta}\ell(a_1, b_{s_1}) + \overline{\alpha\overline{\beta}\ell(a_1, b_{s_1})} \\
&= \pm\left(|\alpha|^2 + \lambda^2|\beta|^2\right).
\end{aligned}$$

Since HC is symmetric, it follows that

$$H_{2,2} = \left(H_{1,1}J_{\lambda^2,s_1}\right)^T = \ell(e_1, e_{s_1})S_{s_1}J_{\lambda^2,s_1}.$$

Lastly, fixing $\alpha = 1$ and $\beta = 0$, the matrices

$$H = \pm\begin{pmatrix} S_{s_1} & 0 \\ 0 & S_{s_1}J_{\lambda^2,s_1} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & J_{\lambda^2,s_1} \\ I_{s_1} & 0 \end{pmatrix}$$

represent the restrictions of ℓ and A on $\text{span}_{\mathbb{C}}\{e_i\}_{1\leq i\leq 2s_1}$ with respect to a permutation of the basis $\{e_i\}_{1\leq i\leq 2s_1}$. Since H is nonsingular, By Lemma 3.2.1, we can repeat this construction on the ℓ-

orthogonal complement of $\operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_{2s_1}\}$, and hence there exists a basis of $W_\lambda^{(n)}$ with respect to which ℓ and A are represented by the matrices

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}\begin{pmatrix} S_{s_i} & 0 \\ 0 & S_{s_i}J_{\lambda^2,s_i}\end{pmatrix}\right) \quad \text{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\begin{pmatrix} 0 & J_{\lambda^2,s_i} \\ I_{s_i} & 0\end{pmatrix}\right) \tag{3.2.21}$$

where $\epsilon_{i,j} = \pm 1$. In particular, we have shown that if there is a basis with respect to which ℓ and A are represented by

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}S_{2s_i}\right) \quad \text{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\begin{pmatrix} 0 & J_{\lambda^2,s_i} \\ I_{s_i} & 0\end{pmatrix}\right)$$

then there is a basis with respect to which ℓ and A are represented by the matrices in (3.2.21), and hence, noting (3.1.2), there exist a matrix T such that

$$(T^{-1})^*\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}\begin{pmatrix} S_{s_i} & 0 \\ 0 & S_{s_i}J_{\lambda^2,s_i}\end{pmatrix}\right)T^{-1} = \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}N_{\lambda,s_i}\right)$$

and

$$T\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\begin{pmatrix} 0 & J_{\lambda^2,s_i} \\ I_{s_i} & 0\end{pmatrix}\right)\overline{T}^{-1} = \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}M_{\lambda,s_i}\right),$$

which completes the proof. $\square$

3.2.4 Treating generalized eigenspaces with nonreal eigenvalues

Throughout this subsection we assume $\lambda^2 \notin \mathbb{R}$ and that the restriction of A^2 to $W_\lambda^{(n)}$ has a Jordan canonical form with $2r_i$ Jordan blocks of size $s_i \times s_i$, where $r_1, \ldots, 2r_{n_\lambda}$ and $s_1, \ldots, s_{n_\lambda}$ are positive integers satisfying $s_i > s_{i+1}$.

Proposition 3.2.13. *There is a basis of $W_\lambda^{(n)}$ with respect to which the restrictions of ℓ and A to*

$W_\lambda^{(n)}$ *are represented by the matrices*

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}N_{\lambda,s_i}\right) \quad \textit{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}M_{\lambda,s_i}\right) \quad \textit{where } \epsilon_{i,j}=\pm 1$$

respectively.

Proof. Given the Gohberg–Lancaster–Rodman canonical form for ℓ and A^2 summarized in Remark 3.1.1, there exist vectors $a_1, a_1' \in W_\lambda^{(n)}$ such that the restrictions of ℓ and A^2 to the $2s_1$-dimensional vector space $\operatorname{span}_{\mathbb{C}}\{a_1, (A^2-\lambda^2 I)a_1, \ldots, (A^2-\lambda^2 I)^{s_1-1}a_1, a_1', \ldots, (A^2-\overline{\lambda}^2 I)^{s_1-1}a_1'\}$ are represented respectively by S_{2s_1} and $J_{\lambda^2,s_1}\oplus J_{\overline{\lambda}^2,s_1}$ with respect to the basis

$$\{(A^2-\lambda^2 I)^{s_1-1}a_1, \ldots, a_1, (A^2-\overline{\lambda}^2 I)^{s_1-1}a_1', \ldots, a_1'\}.$$

Define

$$a_{k+1}=(A^2-\lambda^2 I)a_k \quad \text{and} \quad a_{k+1}'=\left(A^2-\overline{\lambda}^2 I\right)a_k'.$$

Our goal is to show that there exists a choice of vector a_1 such that $\operatorname{span}_{\mathbb{C}}\{a_1, (A^2-\lambda^2 I)a_1, \ldots, (A^2-\lambda^2 I)^{s_1-1}a_1, a_1', \ldots, (A^2-\overline{\lambda}^2 I)^{s_1-1}a_1'\}$ is A-invariant, so let us proceed assuming otherwise and find a new choice for a_1 that satisfies this property.

Define

$$b_k := Aa_k' \quad \text{and} \quad b_k' := Aa_k.$$

For $1\le i,j\le s_1$,

$$\ell(b_i,b_j)=\ell\left(\overline{\lambda}^2 a_j'+a_{j+1}', a_i'\right)=0 \quad \text{and} \quad \ell(b_i',b_j')=\ell(\lambda^2 a_j+a_{j+1},a_i)=0,$$

and

$$\ell(b_i,b_j')=\ell\left(\lambda^2 a_j+a_{j+1},a_i'\right) \quad \text{and} \quad \ell(b_i',b_j)=\ell\left(\overline{\lambda}^2 a_j'+a_{j+1}',a_i\right).$$

Therefore, the restrictions of ℓ and A^2 to the $2s_1$-dimensional vector space $\operatorname{span}_{\mathbb{C}}\{b_1,\ldots,b_{s_1},b_1',\ldots,b_{s_1}'\}$

are represented respectively by

$$\begin{pmatrix} 0 & J_{\lambda^2,s_1}S_{s_1} \\ J_{\overline{\lambda}^2,s_1}S_{s_1} & 0 \end{pmatrix} \quad \text{and} \quad J_{\lambda^2,s_1} \oplus J_{\overline{\lambda}^2,s_1}$$

with respect to the basis $\operatorname{span}_{\mathbb{C}}\{b_1,\ldots,b_{s_1},b'_1,\ldots,b'_{s_1}\}$.

Letting $e_1 = \alpha a_1 + \beta b_1$, we have

$$\begin{aligned} \ell\left(e_1, A(A^2-\lambda^2 I)^{s_1-1}e_1\right) &= \alpha^2\ell(a_1,b'_{s_1}) + \alpha\beta\big(\ell(a_1,A^2a'_{s_1}) + \ell(b_1,b'_{s_1})\big) + \beta^2\ell(b_1,A^2a'_{s_1}) \\ &= \alpha^2\ell(a_1,b'_{s_1}) + \alpha\beta\big(\lambda^2\ell(a_1,a'_{s_1}) + \overline{\lambda}^2\ell(a'_{s_1},a_1)\big) + \lambda^2\beta^2\ell(b_1,a'_{s_1}) \\ &= \alpha^2\ell(a_1,b'_{s_1}) + \alpha\beta\left(\lambda^2+\overline{\lambda}^2\right) + \lambda^2\beta^2\ell(b_1,a'_{s_1}). \end{aligned} \tag{3.2.22}$$

Define

$$e_k = (A^2-\lambda^2)^{k-1}e_1 \quad \text{and} \quad e_{s_1+k} = Ae_k \qquad \forall 1 \le k \le s_1$$

and, on the span of $\{e_i\}$, let ℓ and A be represented with respect to the basis $\{e_i\}$ by the matrix

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & J_{\lambda^2,s_1} \\ I_{s_1} & 0 \end{pmatrix}$$

where the matrices $H_{i,j}$ are each $s_1 \times s_1$. Direct computation yields $H_{1,1} = H_{2,2} = 0$. Furthermore, $H_{1,2}$ is symmetric because HC is symmetric, and hence $H_{2,1}$ is symmetric as well. If $s_1+1 < i+j$ then the (i,j)th entry of $H_{1,2}$ is zero because

$$\ell\left((A^2-\lambda^2)^{i-1}e_1, (A^2-\overline{\lambda}^2)^{j-1}Ae_1\right) = \ell\left((A^2-\lambda^2)^{i+j-2}e_1, Ae_1\right) = \ell\left(0, Ae_1\right) = 0.$$

Accordingly,

$$\det(H) = \pm\big(|\ell(e_1,e_{2s_1})|\big)^{2s_1},$$

which, by (3.2.22), can be made nonzero for an adequate choice of α and β. Since HC is symmetric, evaluating the lower right $s_1 \times s_1$ block of HC yields, as in Lemma 3.2.6, that

$$H_{2,1} = \overline{H_{1,2}} = \ell(e_1, e_{2s_1})S_{s_1}.$$

By replacing e_1 with $\sqrt{\frac{1}{\ell(e_1,e_{2s_1})}}e_1$, we can assume $\ell(e_1, e_{2s_1}) = \pm 1$, so the restrictions of ℓ and A to $\mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_{2s_1}\}$ are represented by the matrices

$$\pm S_{2s_1} \quad \text{and} \quad \begin{pmatrix} 0 & J_{\lambda^2,s_1} \\ I_{s_1} & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, \dots, e_{2s_1}\}$. By Lemma 3.2.1, we can repeat this normalization proceedure on the ℓ-orthogonal complement of $\mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_{2s_1}\}$, and hence there is a basis of $W_\lambda^{(n)}$ with respect to which ℓ and A are represented by the matrices

$$\bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\epsilon_{i,j}S_{2s_i}\right) \quad \text{and} \quad \bigoplus_{i=1}^{n_\lambda}\left(\bigoplus_{j=1}^{r_i}\begin{pmatrix} 0 & J_{\lambda^2,s_i} \\ I_{s_i} & 0 \end{pmatrix}\right) \qquad \text{where } \epsilon_{i,j} = \pm 1.$$

$\square$

3.2.5 A canonical form for antilinear operators

It is worth noting that methods applied above can be used to obtain the canonical form for antilinear operators (without considering Hermitian forms) given in [22, Theorem 3.1], referred to in Remark 3.1.3, so here we briefly outline how this is done.

On a generalized eigenspace $W_\lambda^{(n)}$ for which $\lambda \notin \mathbb{R}$, in subsections 3.2.3 and 3.2.4 we normalize the restriction $A|_V$ of A to a subspace V, where V is defined to be the space spanned by some Jordan chain of A^2 and the image of A applied to this Jordan chain, and achieve the normalization by first choosing a basis with respect to which A^2 has the Jordan normal form and then transforming this basis to a new one with respect to which A has the form in Theorem 3.1.2, all the while taking care to simultaneously normalize ℓ. The very same procedure can be applied to normalize $A|_V$ without the additional steps needed to normalize ℓ, that is, one can normalize $A|_V$ by reading through the proofs of propositions (3.2.12) and (3.2.13) while disregarding all mention of ℓ (e.g., using the Jordan normal form rather than the Gohberg–Lancaster–Rodman form). Next, letting U denote the A-invariant space on which we have already normalized A, we repeat this normalization on any A-invariant subspace of $W_\lambda^{(n)} \setminus U$ containing a maximal Jordan chain of A^2 rather than applying Lemma 3.2.1 to choose a specific A-invariant complement of U. To find such a subspace, we choose any maximal length Jordan chain of A^2 in $W_\lambda^{(n)} \setminus U$ and consider the subspace spanned by this chain and the image of A applied to this chain.

On the generalized eigenspace $W_0^{(n)}$, we may normalize the restriction $A|_V$ of A to a subspace V, where V is a maximal subspace of $W_0^{(n)}$ that has a basis obtained by applying powers of A to a single vector, by using the procedure in the proof of Proposition 3.2.11, again disregarding all mention of ℓ, that is, rather than choosing $a_1 \in \{v \mid A^{k-1}v \neq 0\} \cap W_0^{(n)}$ such that (3.2.20) holds we simply choose a_1 to be an arbitrary vector in $\{v \mid A^{k-1}v \neq 0\} \cap W_0^{(n)}$. We repeat this normalization on any maximal A-invariant subspace of $W_0^{(n)} \setminus U$ (where U denotes the A-invariant space on which we have already normalized A) that has a basis obtained by applying powers of A to a single vector. To find such a subspace, we choose any vector $v \in W_0^{(n)} \setminus U$ for which the subspace spanned by $\{v, Av, \ldots, A^n v\}$ has maximal dimension.

Lastly, on a generalized eigenspace $W_\lambda^{(n)}$ for which $\lambda^2 > 0$, we apply Lemma 3.2.8 to normalize the restriction $A|_V$ of A to a subspace V, where V is the span of a Jordan chain of A^2 given by Lemma 3.2.8. Note, the proof of Lemma 3.2.8 does not use the assumption that A is ℓ-self-adjoint for some Hermitian form ℓ. And as in the previous two cases, we repeat this normalization on any A-invariant subspace of $W_\lambda^{(n)} \setminus U$ (where, again, U denotes the A-invariant space on which we have already normalized A) containing a maximal length Jordan chain of A^2.

Given that every antilinear operator can be represented by a matrix representing the antilinear operator of a pair in the canonical form of Theorem 3.1.2, we have the following lemma.

Lemma 3.2.14. *Every antilinear operator on $\mathbb{C}^n$ is ℓ-self-adjoint with respect to some nondegenerate Hermitian form ℓ.*

3.3 Alternative canonical forms

We conclude this chapter with a few remarks regarding an alternative approach to deriving a canonical form for the pair (ℓ, A), and we record an alternative canonical form, Theorem 3.3.1, that naturally arises from this approach. The form in Theorem 3.1.2 has some advantages. Its matrices have a minimal number of nonzero entries, for example. The form in Theorem 3.3.1 is, however, better suited for certain applications. Namely, analysis involving antilinear operators often includes consideration of the operators' squares, making use of the squares' linearity and well developed theory for linear operators. The alternative canonical forms of Theorems 3.3.1 and 3.3.2 below are ideal for studying A and A^2 simultaneously because A^2 is represented by a Jordan matrix whenever A is represented by the canonical form of Theorem 3.3.2.

When searching for a canonical form for (ℓ, A), after noticing that a linear operator A^2 is ℓ-self-adjoint whenever the antilinear operator A is ℓ-self-adjoint, it becomes natural to apply the Gohberg–Lancaster–Rodman form to the pair (ℓ, A^2). Specifically, one may try to normalize (ℓ, A) by bringing (ℓ, A^2) to the Gohberg–Lancaster–Rodman form and then changing the basis to normalize A while tracking the changes induced in the matrix representing ℓ (ideally, one would like to achieve this without changing the matrix representing ℓ at all). Indeed, we use this approach in subsections 3.2.3 and 3.2.4, and, from this perspective, noting Lemma 3.1.4, one must wonder why we do not use this approach in section 3.2.1 as well. It turns out to be absolutely viable for the normalization carried out in section 3.2.1, but the method presented in section 3.2.1 is simply more efficient. Applying this alternative approach to carry out the normalization has its own merit, however, because it naturally leads one to discover the canonical form given in Theorem 3.3.1 below.

To explore this further, let us consider the special case wherein $A^2 : \mathbb{C}^n \to \mathbb{C}^n$ has a single eigenvalue λ^2, its only eigenspace is 1-dimensional, and $\lambda^2 > 0$ (note, applying Lemmas 3.2.1 and 3.2.8, one can always reduce to this special case for the normalization carried out in section 3.2.1). Applying the Gohberg–Lancaster–Rodman form to the pair (ℓ, A^2), we can choose a basis of $\mathbb{C}^n$

with respect to which ℓ and A are represented by matrices S_n and C respectively such that

$$C\overline{C} = J_{\lambda^2,n}. \tag{3.3.1}$$

We attempt to normalize A by changing the basis with transformations that preserve the matrix representations of ℓ and A^2. Hence we consider the transformations represented by matrices in the group

$$G := \left\{ M \in M_{n\times n}(\mathbb{C}) \,|\, M^* S_n M = S_n \text{ and } MC\overline{C} = C\overline{C}M \right\}$$

acting on the subspace

$$\mathcal{C} := \{ C \in M_{n\times n}(\mathbb{C}) \,|\, C\overline{C} = J_{\lambda^2,n} \}$$

of $GL_n(\mathbb{C})$, via the action $(M, C) \mapsto MC\overline{M}^{-1}$. It turns out that we can solve (3.3.1), that is, we can completely describe the general form of a matrix C satisfying (3.3.1), and G acts transitively on $\mathcal{C}$.[1] Matrices in (3.3.1) turn out to be upper-triangular and Toeplitz, and, for a matrix $C \in \mathcal{C}$, one can explicitly construct a matrix $M \in G$ such that $MC\overline{M}^{-1} \in GL_n(\mathbb{R})$ and the eigenvalue of $MC\overline{M}^{-1}$ is $|\lambda|$. Choosing M to satisfy these conditions, it turns out that $MC\overline{M}^{-1}$ equals the matrix $M_{|\lambda|,n}$ defined below, which confirms that G acts transitively on $\mathcal{C}$. Of course, we have omitted details of the calculations summarized here, but the summary provides an outline of how one can apply the aforementioned alternative approach to the normalization carried out in section 3.2.1. Furthermore, this summary illustrates how, from one perspective, the alternative canonical form given in Theorem 3.3.1 below arises naturally.

This alternative form features the sequence

$$c_0(\lambda) := \lambda, \quad c_1(\lambda) := \frac{1}{2\lambda}, \quad \text{and} \quad c_i(\lambda) := \frac{-1}{2\lambda}\sum_{j=1}^{i-1} c_j(\lambda)c_{i-j}(\lambda), \tag{3.3.2}$$

[1]The space $\mathcal{C}$ turns out to be homeomorphic to the Cartesian product $S^1 \times \mathbb{R}^{n-1}$ of a circle and Euclidean space with the product topology.

which arises if we try to solve the matrix equation

$$C^2 = J_{\lambda^2,k} \qquad \lambda \neq 0 \tag{3.3.3}$$

by supposing C has the form

$$C = \sum_{i=1}^{k} c_{i-1}(\lambda) T_k^{i-1} \tag{3.3.4}$$

and comparing coefficients, interpreting each side of the equation as a degree $k-1$ polynomial in T_k.[2] An interesting observation is that the sequence

$$|c_1(1/2)| = 1, |c_2(1/2)| = 1, |c_3(1/2)| = 2, \ldots$$

is known as the Catalan numbers, $|c_i(1/2)| = \frac{1}{i+1}\binom{2i}{i}$, which play an important role in combinatorics. The identity

$$c_i(\lambda) = (-1)^i (2\lambda)^{1-2i} |c_i(1/2)| = \frac{(-1)^{i+1}(2\lambda)^{1-2i}}{i+1}\binom{2i}{i},$$

valid for all positive integers i, further illuminates the relationship between $\{c_i(\lambda)\}_{i=1}^{\infty}$ and the Catalan numbers.

[2]If C satisfies (3.3.3) then C satisfies (3.3.4) for some choice of coefficients $c_i(\lambda)$. Nevertheless, proving this fact is not necessary for understanding the provenance of (3.3.2), so we introduce (3.3.4) as though it is not a consequence of (3.3.3).

For $\lambda \in \mathbb{C}$, we define the $k \times k$ or $2k \times 2k$ matrix $\widetilde{M}_{\lambda,k}$ by

$$\widetilde{M}_{\lambda,k} := \begin{cases} \sum_{i=1}^{k} c_{i-1}(\lambda) T_k^{i-1} & \text{if } \lambda \in \mathbb{R} \setminus \{0\} \\ \frac{1}{2}\begin{pmatrix} J_{1,\frac{k}{2}} & -J_{-1,\frac{k}{2}} \\ J_{-1,\frac{k}{2}} & -J_{1,\frac{k}{2}} \end{pmatrix} & \text{if } \lambda = 0 \text{ and } k \text{ is even} \\ \left(\begin{array}{c|c|c} 0 & 0 & I_{\frac{k-1}{2}} \\ \hline I_{\frac{k-1}{2}} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) & \text{if } \lambda = 0 \text{ and } k \text{ is odd} \\ \begin{pmatrix} 0 & \sum_{i=1}^{k} c_{i-1}(\lambda) T_k^{i-1} \\ \sum_{i=1}^{k} c_{i-1}(\overline{\lambda}) T_k^{i-1} & 0 \end{pmatrix} & \text{otherwise,} \end{cases}$$

where 0 denotes a matrix of appropriate size with zero in all entries and, for odd k, $\widetilde{M}_{0,k}$ is a $k \times k$ matrix. We define corresponding matrices $\widetilde{N}_{\lambda,k}$ by

$$\widetilde{N}_{\lambda,k} := \begin{cases} S_k & \text{if } \lambda \in \mathbb{R} \setminus \{0\} \\ S_{\frac{k}{2}} \oplus \left(-S_{\frac{k}{2}}\right) & \text{if } \lambda = 0 \text{ and } k \text{ is even} \\ S_{\lfloor \frac{k}{2} \rfloor} \oplus S_{\lceil \frac{k}{2} \rceil} & \text{if } \lambda = 0 \text{ and } k \text{ is odd} \\ S_k \oplus (-S_k) & \text{if } \lambda^2 < 0 \\ S_{2k} & \text{otherwise,} \end{cases}$$

where $\lceil a \rceil$ denotes the smallest integer not less than a and $\lfloor a \rfloor$ denotes the largest integer not larger than a. For the following theorem, we let $\{\lambda_1, \lambda_2, \ldots, \lambda_\gamma\}$ denote the subset of principle square roots of eigenvalues of A^2 enumerated in section 3.1.

Theorem 3.3.1. *The domain of an ℓ-self-adjoint antilinear operator A can be decomposed into A-invariant, pairwise ℓ-orthogonal subspaces such that there exists a basis with respect to which the restrictions of ℓ and A to the decomposition's component subspaces are represented by matrices of the form $\pm\widetilde{N}_{\lambda,k}$ and $\widetilde{M}_{\lambda,k}$ where $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_\gamma\}$ and $k \in \mathbb{N}$. The corresponding block diagonal matrices representing ℓ and A are unique up to a permutation of the blocks on the diagonal.*

A canonical form for antilinear operators, described in Remark 3.1.3 and section 3.2.5, is given by Hong and Horn in [22, Theorem 3.1]. Since, as is noted in Lemma 3.2.14, every antilinear operator is ℓ-self-adjoint with respect to some nondegenerate Hermitian form ℓ, by applying Theorem 3.3.1 to the pair ℓ and A to get another matrix representation for A, we obtain the following alternative canonical form for antilinear operators.

Theorem 3.3.2. *The domain of an antilinear operator A can be decomposed into A-invariant subspaces such that there exists a basis with respect to which the restriction of A to the decomposition's component subspaces are represented by matrices of the form $\widetilde{M}_{\lambda,k}$ where $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_\gamma\}$ and $k \in \mathbb{N}$. The corresponding block diagonal matrix representing A is unique up to a permutation of the blocks on the diagonal.*

Remark 3.3.3. *In a basis with respect to which A is represented by a matrix with the above canonical form, A^2 is represented by a Jordan matrix. Similarly, if ℓ and A are represented by matrices in the canonical form of Theorem 3.3.1 then the pair (ℓ, A^2) is represented by matrices in the Gohberg–Lancaster–Rodman form. Noting this connection together with Lemma 3.1.4, one can readily show that if A is nonsingular then Theorems 3.1.2 and 3.3.1 are indeed equivalent. To show that each of these theorems is a consequence of the other in the more general case where A is singular, it is not too difficult to explicitly construct a basis change of the maximal subspace on which A is nilpotent transforming the canonical form in Theorem 3.1.2 to the form in Theorem 3.3.1 (and vice versa); for example, considering a change of basis transformation represented by*

$$T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

we have $(T^{-1})^* N_{0,6} T^{-1} = \widetilde{N}_{0,6}$ and $T M_{0,6} \overline{T}^{-1} = \widetilde{M}_{0,6}$, that is, this change of basis transforms a certain matrix representation given by Theorem 3.1.2 to a matrix representation given by Theorem 3.3.1.

4. FIRST APPLICATION: MAXIMALLY SYMMETRIC HOMOGENEOUS MODELS WITH A RANK 1 LEVI KERNEL

As mentioned in the introduction, a classical problem setting in differential geometry is to find homogeneous structures with the symmetry group of maximal dimension among all geometric structures of a certain class. In CR geometry this problem is classically solved for the class of Levi nondegenerate CR structures of hypersurface type of arbitrary dimension ([40, 8]). In this chapter 4 we solve this problem for 2-nondegenerate CR structures of hypersurface type with a rank 1 Levi kernel. Previously the solution to this problem was given only in the 5-dimensional case [24, 28, 30], which is the case of the smallest possible dimension in which 2-nondegenrate structures exist. We give the solution for arbitrary dimension (which a priori is odd) greater than 5 extending the previous result of [33] that work under additional the restrictions the a local invariant called the *CR symbol* is *regular*. This result supports Beloshapka's conjecture [24, Conjecture 5.6] stating in particular that the homogeneous hypersurface model with maximal finite dimensional group of symmetries is Levi nondegenerate.

For a Levi-nondegenerate structure, if the Levi form has signature (p, q) with $p + q = n$ then a maximally symmetric model can be obtained as a real hypersurface in the complex projective space $\mathbb{CP}^{n+1}$, obtained by the complex projectivization of the cone of nonzero vectors in $\mathbb{C}^{n+2}$ that are isotropic with respect to a Hermitian form of signature $(p + 1, q + 1)$, and the algebra of infinitesimal symmetries of this model is isomorphic to $\mathfrak{su}(p + 1, q + 1)$, having dimension $(n + 2)^2 - 1$.

Throughout this chapter we let M denote a $(2n{+}1)$-dimensional homogeneous, 2-nondegenerate, hypersurface-type CR manifold with CR structure H, and we assume that the fiber K_x of the Levi kernel is 1-dimensional at every point $x \in M$, that is, K is a rank 1 distribution. The present chapter is dedicated to finding an upper bound for the dimension of the Lie group Aut(M, H) of symmetries of (M, H). The bound that we obtain is in fact sharp in the sense that it can obtained for some choice of (M, H). As shown in [24, 28, 30] for the lowest dimensional case (i.e., when $\dim M = 5$) this sharp upper bound (even without the homogeneity assumption) is equal to 10 and for the maximally symmetric model the algebra of infinitesimal symmetries is equal to $\mathfrak{so}(3,2)$. The main result here, see Theorem 4.1.2 below, gives this sharp upper bound expressed as a function of $\dim M$ for $\dim M \geq 7$ (equivalently, $n = \frac{1}{2}(\dim M - 1) \geq 3$), namely

$$\dim \mathbf{Aut}(M, H) \leq \frac{1}{4}(\dim M - 1)^2 + 7 = n^2 + 7. \tag{4.0.1}$$

We also show that symmetries of (M, H) are all determined by their third weighted jet. By the *weighted jet* we mean that the derivatives in various directions are calculated according to the filtration

$$(K \oplus \overline{K}) \cap TM \subset (H \oplus \overline{H}) \cap TM \subset TM$$

of TM so that each derivative in a direction in $(K \oplus \overline{K}) \cap TM$ is assigned weight zero, each derivative in a direction in $\Big((H \oplus \overline{H}) \setminus (K \oplus \overline{K})\Big) \cap TM$ is assigned weight 1, and each derivative in a direction in $TM \setminus H \oplus \overline{H}$ is assigned weight 2. These results (even without the assumption of homogeneity) were previously obtained in [33] for the special class of CR structures whose symbols are regular, wherein it was shown by example that the upper bound in (4.0.1) is achieved.

The essential technical bulk of this chapter consists of showing that the dimension of Aut(M, H) for homogeneous structures with non-regular symbol is strictly less than the the right side of (4.0.1) (in fact it is shown in Theorem 4.1.4 below that it is strictly less than $(n-1)^2 + 7$) and that in the non-regular case symmetries of (M, H) are all determined by their first weighted jet.

In the proof of the bound (4.0.1) we apply two results from earlier chapters: the classification of CR symbols for structures with rank-1 Levi kernels developed in chapter 3 and the description of the upper bound for the dimension of symmetry groups in terms of a Tanaka prolongation of the symbol or its reduced version developed in Chapter 2 (Theorem 2.5.2). The proof also features significant application of the formulas derived in Appendix A. In the sequel, we calculate these prolongations and their dimensions for each reduced modified symbol corresponding to a non-regular CR symbol. In particular, we show (Theorem 4.1.4) that the first Tanaka prolongation of each reduced modified symbol corresponding to a non-regular CR symbol is equal to zero and we find the upper bound for the dimension of its (entire) Tanaka prolongaiton.. Analogous analysis for regular CR symbols was previously obtained in [33] with the help of the theory of biagraded Tanaka prolongation. The result on the jth-jet determinacy follows from its equivalence to the vanishing of the jth Tanaka prolongation. In Theorem 4.2.3 for each reduced modified symbol corresponding to a non-regular CR symbol we give more precise upper bound for the dimension of its (entire) Tanaka prolongaiton in terms of the parameters of this nonregular symbol.

Note that at this moment for structures with non-regular symbols (and therefore in the general case) we are not able to remove completely the homogeneity assumption in our results, as this assumption implies that the modified (and reduced modified) symbols introduced in Chapter 2 are constant and therefore are Lie algebras, and we strongly use the latter fact. So, the assumption of homogeneity can be relaxed to the assumption that the structures under consideration admit a constant reduced modified symbol in the sense of Chapter 2, but the question of whether or not there exist CR structures from the considered class with nontransitive symmetry group of dimension higher than the bound in (4.0.1) is still open, although the positive answer to this question is highly unlikely.

4.1 Symmetry bounds and jet determinacy theorems

As a consequence of [33], see Theorems 3.2, 5.1, 5.3 and the last paragraph of section 5 there, one gets the following theorem.

Theorem 4.1.1 (Porter and Zelenko [33]). *If (M, H) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant regular symbol, then*

1. *the dimension of the algebra of infinitesimal symmetries of (M, H) is not greater than $\frac{1}{4}(\dim M - 1)^2 + 7$;*

2. *these symmetries are determined by their third weighted jet;*

3. *the dimension of the algebra of infinitesimal symmetries of (M, H) is equal to $\frac{1}{4}(\dim M -$*

$1)^2 + 7$ *if and only if* (M, H) *is locally equivalent to the flat structure with CR symbol such that the corresponding line of antilinear operators consists of nilpotent ones of rank 1.*

A natural question is whether or not the assumption of regularity of symbol can be removed in the previous theorem. Addressing this question, the main result of the present chapter is the following.

Theorem 4.1.2. *If* (M, H) *is a 2-nondegenerate homogeneous CR structure of hypersurface type with a 1-dimensional Levi kernel and constant symbol (not necessarily regular), then*

1. *statements (1) and (3) of Theorem 4.1.1 are valid;*
2. *if the symbol is non-regular then the (infinitesimal) symmetries of* (M, H) *are determined by their first weighted jet.*

The proof of this theorem is given in sections 4.2 through 4.3 and the appendix. Also, a generalization of this theorem is described in Remark 4.1.5 below. In the remainder of this section, we outline the scheme of the proof of this theorem, based on the constructions and results of chapter 2, namely a slight modification of Proposition 2.4.4 (see Lemma 4.2.1) and the combined implications of Corollary 2.1.8 and Theorem 2.5.2 (see Theorem 4.1.3 below). Theorem 4.1.2 will be essentially reduced to Theorem 4.1.4. The latter theorem is proved in section 4.3 with the help of Appendix A. In this proof we also use the classification of symbols from Chapter 3.

Our analysis in this chapter branches depending on properties of the CR structure's local invariants, namely those encoded in its CR symbol and its reduced modified CR symbols. And indeed the CR structure is associated with reduced modified CR symbols by applying the reduction procedure to P^0 described in Section 2.5 to obtain a reduction $P^{0,\mathrm{red}}$ of P^0 with a constant reduced modified modified symbol $\mathfrak{g}^{0,\mathrm{red}}$. Proceeding, let us fix one such constant reduced modified modified symbol $\mathfrak{g}^{0,\mathrm{red}}$.

Accordingly, $\mathfrak{g}^{0,\mathrm{red}}$ satisfyies the axioms in the definition of abstract reduced modified symbols (Definition 2.5.1). Additionally, as noted in section 2.5, $\mathfrak{g}^{0,\mathrm{red}}$ is a subalgebra of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ because it was obtained from a reduction of P^0 with constant reduced modified CR symbol.

For convenient reference, we now restate some results from Chapter 2.

Theorem 4.1.3 (follows immediately from Corollary 2.1.8 and Theorem 2.5.2)**.** *If (M, H) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, then the dimension of the algebra of infinitesimal symmetries of (M, H) is not greater than* $\dim \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$.

Hence, if we can explicitly calculate $\dim \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ for non-regular CR symbols, then we can obtain an upper bound for the algebra of infinitesimal symmetries of (M, H). This motivates the following theorem, proved in section 4.3.

Theorem 4.1.4. *If a reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ corresponds to a non-regular CR symbol then the following statements hold:*

1. *The first Tanaka prolongation $\mathfrak{g}_1^{\mathrm{red}}$ of $\mathfrak{g}^{0,\mathrm{red}}$ vanishes or, equivalently, the universal Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ of $\mathfrak{g}^{0,\mathrm{red}}$ is equal to $\mathfrak{g}^{0,\mathrm{red}}$.*

2. $\dim \mathfrak{g}^{0,\mathrm{red}}$ *and therefore the dimension of the algebra of infinitesimal symmetries of a homogeneous, 2-nondegenerate, $(2n+1)$-dimensional, CR structure of hypersurface type with rank 1 Levi kernel and non-regular CR symbol is strictly less than* $(n-1)^2 + 7$.

Theorem 4.1.4 is proved in Section 4.3 with the help of the Appendix A. Based on the well-known fact [42, Section 6] that an infinitesimal symmetry of a filtered structure is determined by the jth weighted jet, where j is the minimal nonnegative integer for which the jth Tanaka prolongation is equal to zero, this theorem immediately implies item (2) of Theorem 4.1.2. Item (1) will follow from combining the last theorem with Theorem 4.1.3. In Theorem 4.2.3, for each reduced modified symbol corresponding to a non-regular CR symbol, we give more precise upper bounds (than the ones in item (2) of Theorem 4.1.4) for the dimension of its (entire) Tanaka prolongation in terms of the parameters of this non-regular symbol.

Remark 4.1.5. *Finally note that the arguments of the previous paragraph imply that homogeneity assumption in Theorem 4.1.4 and our main Theorem 4.1.2 can be relaxed to the assumption that structures under consideration admit a constant reduced modified symbol.*

4.2 A matrix representation of the reduced modified CR symbol

Here we restate some of the formulas from Chapter 2 with simplifications specific to the present $\operatorname{rank} K = 1$ case. Without these simplifications, notation in the ensuing analysis would be too dense.

Let ℓ be the reduced Levi form of (M,H) and let A be an ℓ self-adjoint antilinear operator such that $(\mathbb{R}\ell, \mathbb{C}A)$ determines the CR symbol of (M,H) as described in Remark 2.2.5. By construction $\mathfrak{g}^{0,\text{red}}$ is a subset of some modified CR symbol $\mathfrak{g}^{0,\text{mod}}$, which has a matrix representation of the form described in Section 2.4. In particular, we can choose the decomposition $\mathfrak{g}_0^{\text{mod}} = \mathfrak{X}_{0,2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{X}_{0,-2}$ referred to above (2.4.9) such that $\mathfrak{X}_{0,2}$ and $\mathfrak{X}_{0,-2}$ belong to $\mathfrak{g}^{0,\text{red}}$. To use more intuitive notation, let us set $\mathfrak{g}_{0,-}^{\text{red}} := \mathfrak{X}_{0,-2}$ and $\mathfrak{g}_{0,+}^{\text{red}} := \mathfrak{X}_{0,-2}$. So we now have the splitting

$$\mathfrak{g}_0^{\text{red}} = \mathfrak{g}_{0,0}^{\text{red}} \oplus \mathfrak{g}_{0,-}^{\text{red}} \oplus \mathfrak{g}_{0,+}^{\text{red}}.$$

Accordingly, letting H_ℓ and C be matrices representing ℓ and A respectively in some basis of $\mathfrak{g}_{-1}$, (2.4.7) is equivalent to

$$\mathscr{A} := \left\{ \alpha \,\middle|\, \begin{array}{l} \alpha C H_\ell^{-1} + C H_\ell^{-1}\alpha^T = \eta C H_\ell^{-1} \text{ and} \\ \alpha^T H_\ell \overline{C} + H_\ell \overline{C}\alpha = \eta' H_\ell \overline{C} \text{ for some } \eta, \eta' \in \mathbb{C} \end{array} \right\}. \tag{4.2.1}$$

whereas (2.4.9) implies that $\mathfrak{g}_{0,+}^{\text{red}}$ and $\mathfrak{g}_{0,-}^{\text{red}}$ are spanned by

$$\operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} \Omega & C \\ 0 & -H_\ell^{-1}\Omega^T H_\ell \end{pmatrix} \right\} \quad \text{and} \quad \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} -\overline{H_\ell}^{-1}\Omega^* \overline{H_\ell} & 0 \\ \overline{C} & \overline{\Omega} \end{pmatrix} \right\} \tag{4.2.2}$$

respectively. Since $\mathfrak{g}_0^{\text{red}}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$, essentially the same reasoning that justifies Lemma 2.4.4 implies that (2.4.12) holds after replacing the algebra $\mathscr{A}$ with some subalgebra $\mathscr{A}_0$ of $\mathscr{A}$. In other words, we have the following Lemma.

Lemma 4.2.1. *There exists a subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ invariant under the transformation $\alpha \mapsto$*

$\overline{H_\ell}^{-1}\alpha^*\overline{H_\ell}$ such that

$$\mathfrak{g}_{0,0}^{\mathrm{red}} = \left\{ \left(\begin{array}{cc} \alpha & 0 \\ 0 & -H_\ell^{-1}\alpha^T H_\ell \end{array} \right) + cI \,\middle|\, \alpha \in \mathscr{A}_0, \text{ and } c \in \mathbb{C} \right\}, \tag{4.2.3}$$

and there exist coefficients $\{\eta_\alpha\}_{\alpha\in\mathscr{A}_0} \subset \mathbb{C}$ and $\mu \in \mathbb{C}$ such that the system of relations

$$\left.\begin{array}{ll} \text{i)} & \alpha C H_\ell^{-1} + C H_\ell^{-1}\alpha^T = \eta_\alpha C H_\ell^{-1} \\ \text{ii)} & [\alpha, \Omega] - \eta_\alpha \Omega \in \mathscr{A}_0 \\ \text{iii)} & \Omega^T H_\ell \overline{C} + H_\ell \overline{C}\Omega = \mu H_\ell \overline{C} \\ \text{iv)} & \left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}, \Omega\right] + C\overline{C} - \overline{\mu}\Omega - \mu \overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 \end{array}\right\} \tag{4.2.4}$$

holds for all $\alpha \in \mathscr{A}_0$.

Note that the condition regarding invariance under $\alpha \mapsto \overline{H_\ell}^{-1}\alpha^*\overline{H_\ell}$ corresponds to axiom 4 in the definition of abstract reduced modified CR symbols (Definition 2.5.1).

We label the following pair of results from Chapter 2.

Lemma 4.2.2. *The following are equivalent.*

1. *$\mathfrak{g}^0$ is regular.*

2. *$C\overline{C}C$ is a scalar multiple of C.*

Moreover, if Ω is in $\mathscr{A}$ then $\mathfrak{g}^0$ is regular.

Proof. Equivalence of (1) and (2) was established in [33, section 4] and is also given in Proposition (2.2.6). The latter statement is also shown in Chapter 2, although it is not given a numbered result there. We prove it more directly here anyway for clarity because it plays an essential role in this chapter's analysis. For this, let v_+ and v_- be elements in $\mathfrak{g}_{0,2}^{\mathrm{red}}$ and $\mathfrak{g}_{0,-2}^{\mathrm{red}}$ respectively. Note that if Ω is in $\mathscr{A}$ then there exist vectors $w_+, w_- \in \mathfrak{g}_{0,0}$ such that $v_\pm + w_\pm$ belongs to $\mathfrak{g}_{0,\pm2}$. Accordingly,

$$[v_+ + w_+, v_- + w_-] = [w_-, w_+] + [v_+ + w_+, w_-] + [w_+, v_- + w_-] + [v_+, v_-].$$

Since

$$[\mathfrak{g}_{0,0}, \mathfrak{g}_0] \subset \mathfrak{g}_0 \tag{4.2.5}$$

by the definition of $\mathfrak{g}_{0,0}$, the first three terms in the right side of this last equation belong to $\mathfrak{g}_0$. Since $\mathfrak{g}_0^{\text{red}}$ is closed under Lie brackets, $[v_+, v_-]$ belongs to $\mathfrak{g}_0^{\text{red}}$. Hence if Ω is in $\mathscr{A}$ then $[\mathfrak{g}_{0,2}, \mathfrak{g}_{0,-2}] \subset \mathfrak{g}_0 + \mathfrak{g}_0^{\text{red}}$. On the other hand if Ω is in $\mathscr{A}$ then $\mathfrak{g}_0^{\text{red}} \subset \mathfrak{g}_0$. Therefore, if Ω is in $\mathscr{A}$ then $[\mathfrak{g}_{0,2}, \mathfrak{g}_{0,-2}] \subset \mathfrak{g}_0$. Noting (4.2.5), it follows that if Ω is in $\mathscr{A}$ then $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$, that is, $\mathfrak{g}^0$ is regular. $\square$

Now, for completeness, given a non-regular CR symbol $\mathfrak{g}^0$ encoded by the pair (ℓ, A), represented by the pair of matrices (H_ℓ, C) in the canonical basis as as in Theorem 3.1.2 we will give more precise (i.e., in terms of integers $m_1, \dots m_\gamma$ and numbers $\lambda_1, \dots, \lambda_\gamma$) upper bound for the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n+1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to CR symbol $\mathfrak{g}^0$. For this, for every $1 \leq i, j \leq \gamma$, let

$$d(i,j) = \begin{cases} 0, & (\lambda_i \neq \lambda_j) \text{ or } (i = j \text{ and } \lambda_i^2 \text{ is not a nonpositive real number}) \\ \min\{m_i, m_j\} & (i \neq j \text{ and } \lambda_i = \lambda_j > 0) \text{ or } (i = j \text{ and } \lambda_i^2 < 0) \\ 2\min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } (\lambda_i^2 \notin \mathbb{R} \text{ or } \lambda_i = 0) \\ 4\min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } \lambda_i^2 < 0 \\ \lceil \frac{m_i}{2} \rceil & i = j \text{ and } \lambda_i = 0 \end{cases}$$

where $\lceil \frac{m}{2} \rceil$ denotes the ceiling function, i.e. the smallest integer not less than $\frac{m_i}{2}$.

Let

$$d_{\text{total}} := \sum_{i \leq j} d(i,j).$$

Then the following theorem is the direct consequence of item (1) of Theorem 4.1.4 and Lemmas A.0.2, A.0.4, Corollary A.0.5, and Lemma A.0.8:

Theorem 4.2.3. *Given a nonregular CR symbol $\mathfrak{g}^0$ encoded by the pair (ℓ, A) represented by the pair of matrices (H_ℓ, C) in the canonical basis as in Theorem 3.1.2, the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n+1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to CR symbol $\mathfrak{g}^0$ is not greater than $d_{\text{total}}+2n+3$, if at least one λ_i is not zero, and it is not greater than $d_{\text{total}}+2n+4$, if all λ_i are zero.*

Note that the mentioned Lemmas and Corollaries from Appendix A together with (4.2.3) imply that $\dim \mathfrak{g}_{0,0}^{\text{red}}$ is either not greater than $d_{\text{total}}+2$ or $d_{\text{total}}+3$ depending whether or not C is nilpotent. The estimate for $\mathfrak{u}(\mathfrak{g}^{0,\text{red}}) = \mathfrak{g}^{0,\text{red}}$ in Theorem 4.2.3 follows from this and the fact that $\dim(\mathfrak{g}_- + \mathfrak{g}_{0,-2} + \mathfrak{g}_{0,2}) = 2n+1$.

4.3 Proof of Theorem 4.1.4

Let $\sigma : \mathfrak{g}^{0,\text{red}} \to \mathfrak{g}^{0,\text{red}}$ denote the antilinear involution induced by the natural complex conjugation of $\mathbb{C}TM$. We introduce this σ notation to avoid confusion because while working with matrix representations in coordinates we will use the overline notation to denote the standard complex conjugation of coordinates, which is a different involution. Let

$$(e_1, \ldots, e_{2n-2})$$

be a basis of $\mathfrak{g}_{-1}$ with respect to which we get the matrix representation of $\mathfrak{g}_0^{\text{red}}$ given by (4.2.2) and (4.2.3). Notice in particular that $(e_1, \ldots, e_{n-1})$ spans $\mathfrak{g}_{-1,1}$ and

$$\sigma(e_i) := e_{n+i-1} \qquad \forall\, 1 \le i \le n-1.$$

Note that σ extends to an involution defined of $\mathfrak{g}_1^{\text{red}}$ by same formula that we use to extend the natural conjugation from $\mathfrak{g}_-$ to be defined on $\mathfrak{csp}(\mathfrak{g}_{-1})$, that is

$$\sigma(\varphi)(v) := \sigma \circ \varphi \circ \sigma(v) \qquad \forall\, v \in \mathfrak{g}^{0,\text{red}}\, \varphi \in \mathfrak{g}_1^{\text{red}} \tag{4.3.1}$$

defines an involution of $\mathfrak{g}_1^{\mathrm{red}}$.

Recall the definition of the Tanaka prolongation of $\mathfrak{g}_k^{\mathrm{red}}$. Starting with $k = 1$, we recursively define the kth prolongation

$$\mathfrak{g}_k^{\mathrm{red}} := \left\{ \varphi \in \bigoplus_{i=-2}^{-1} \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_{i+k}) \,\middle|\, \begin{array}{l} \varphi([v_1, v_2]) = [\varphi(v_1), v_2] + [v_1, \varphi(v_2)] \\ \forall\, v_1, v_2 \in \mathfrak{g}_- \end{array} \right\} \forall\, k \geq 1.$$

An element φ in $\mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \oplus \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0^{\mathrm{red}})$ belongs to $\mathfrak{g}_1^{\mathrm{red}}$ if and only if

$$\varphi([e_i, e_j]) = \big(\varphi(e_i)\big)(e_j) - \big(\varphi(e_j)\big)(e_i) \quad \forall\, i, j \in \{1, \ldots, 2n-2\}. \tag{4.3.2}$$

Note, here $\varphi(e_i) \in \mathfrak{g}_0^{\mathrm{red}} \subset \mathfrak{csp}(\mathfrak{g}_{-1})$.

Given any element $v \in \mathfrak{g}_{-1}$ let v_- and v_+ be the canonical projections of v to $\mathfrak{g}_{-1,-1}$ and $\mathfrak{g}_{-1,1}$, respectively, with respect to the splitting $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$.

As a direct consequence of (4.3.2) and (4.2.2), if $n \leq j \leq 2n-2$ and $1 \leq i \leq n-1$, then

$$\begin{aligned} \big(\big(\varphi(e_j)\big)e_i\big)_+ &\in \mathrm{span}\{Ae_{j-n+1}\} - \big(\varphi([e_i, e_j])\big)_+ \subset \mathrm{span}\{Ae_{j-n+1}, \big(\varphi(1)\big)_+\}, \\ \big(\big(\varphi(e_i)\big)e_j\big)_- &\in \mathrm{span}\,\big\{\sigma\big(Ae_i\big)\big\} - \big(\varphi([e_i, e_j])\big)_- \subset \mathrm{span}\{Ae_i, \big(\varphi(1)\big)_-\} \end{aligned} \tag{4.3.3}$$

In particular, the upper left $(n-1) \times (n-1)$ block in the matrix $\varphi(e_j)$ and the lower right $(n-1) \times (n-1)$ block in the matrix $\varphi(e_i)$ both have rank at most 2.

Also from (4.3.2) and the fact that $[e_i, e_j] = 0$ for $n \leq i, j \leq 2n-2$, we immediately have that

$$\varphi(e_i)e_j = \varphi(e_j)e_i, \quad n \leq i, j \leq 2n-2. \tag{4.3.4}$$

Lemma 4.3.1. *If the antilinear operator A (or, equivalently the matrix C) has rank greater than 1 and $i \geq n$ then $\varphi(e_i) \in \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,-}^{\mathrm{red}}$, or, equivalently,*

$$\varphi(e_i) = \begin{pmatrix} \alpha_i & 0 \\ c\overline{C} & -H_\ell^{-1}\alpha_i^T H_\ell \end{pmatrix} \qquad \textit{for some } c \in \mathbb{C} \textit{ and } \alpha_i \in \mathscr{A}_0 + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}). \tag{4.3.5}$$

Proof. By (4.2.2), there exists $c \in \mathbb{C}$ such that for every $n \leq j \leq 2n-2$

$$\big((\varphi(e_i))e_j\big)_+ = cAe_{j-n+1} \quad \text{and} \quad \big((\varphi(e_j))e_i\big)_+ \in \operatorname{span}\{Ae_{i-n+1}\}.$$

By (4.3.4), for all $n \leq j \leq 2n-2$,

$$cAe_{j-n+1} \in \operatorname{span}\{Ae_{i-n+1}\}.$$

This implies that $c = 0$, because otherwise $\operatorname{rank} A \leq 1$, contradicting our assumption. Therefore, $\big(\varphi(e_i)v\big)_+ = 0$ for all $v \in \mathfrak{g}_{-1,-1}$, which is equivalent to the statement of the Lemma. □

Similarly, we have the following Lemma.

Lemma 4.3.2. *If the antilinear operator A (or, equivalently the matrix C) has rank greater than 1 and $i < n$ then $\varphi(e_i) \in \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}}$ or, equivalently,*

$$\varphi(e_i) = \begin{pmatrix} \alpha_i & cC \\ 0 & -H_\ell^{-1}\alpha_i^T H_\ell \end{pmatrix} \qquad \textit{for some } c \in \mathbb{C} \textit{ and } \alpha_i \in \mathscr{A}_0 + \mathbb{C}\Omega. \tag{4.3.6}$$

Lemma 4.3.3. *If C has rank greater than 1 and α_i is the matrix defined by* (4.3.5) *and* (4.3.6) *then, for $i < n$, we have*

$$\big(H_\ell \overline{C}\alpha_i\big)^T + H_\ell \overline{C}\alpha_i = \eta H_\ell \overline{C} \textit{ for some } \eta \in \mathbb{C} \tag{4.3.7}$$

and, for $n \leq i$, we have

$$\alpha_i C H_\ell^{-1} + \big(\alpha_i C H_\ell^{-1}\big)^T = \eta C H_\ell^{-1} \textit{ for some } \eta \in \mathbb{C}. \tag{4.3.8}$$

Proof. If α_i is as in (4.3.6) then $\alpha_i \in \mathscr{A} + \mathbb{C}\Omega$, so the definition of $\mathscr{A}$ and item (iii) of (4.2.4) imply (4.3.7). If, on the other hand, α_i is as in (4.3.5) then $\alpha_i \in \mathscr{A} + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell})$, so the definition of $\mathscr{A}$ and item (iii) of (4.2.4) imply (4.3.8). □

Corollary 4.3.4. *If the CR symbol is not regular and the matrix α_i given in* (4.3.5) *or* (4.3.6) *is zero, then $\varphi(e_i) = 0$.*

Proof. Suppose $\alpha_i = 0$. By (4.2.2), (4.2.3), and Lemmas 4.3.1 and 4.3.2, if $\varphi(e_i) \neq 0$ then either $\Omega \in \mathscr{A}$ or $\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}$. The conditions $\Omega \in \mathscr{A}$ and $\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}$ are, however, equivalent, so either $\varphi(e_i) \neq 0$ or $\Omega \in \mathscr{A}$. If the CR symbol is not regular then, by Lemma 4.2.2, $\Omega \notin \mathscr{A}$, and hence $\varphi(e_i) = 0$. $\square$

Lemma 4.3.5. *If an element φ in $\mathfrak{g}_1^{\mathrm{red}}$ satisfies $\varphi(1) = 0$ and*

$$\varphi(e_i) = 0 \qquad \forall\, i \geq n \tag{4.3.9}$$

then

$$\varphi(e_i) = 0 \qquad \forall\, i < n, \tag{4.3.10}$$

and so $\varphi = 0$.

Proof. Since $\varphi(1) = 0$, the left side of (4.3.2) is zero for all i and j. Accordingly, for any $i \in \{1, \ldots, n-1\}$ and $j \in \{n, \ldots, 2n-2\}$, (4.3.2) and (4.3.9) imply that the j column of $\varphi(e_i)$ is zero. Hence, for all $i \in \{1, \ldots, n-1\}$, the latter $n-1$ columns of $\varphi(e_i)$ are all zero. From this and Lemma 4.3.2 (and specifically (4.3.6)), it follows that $H_\ell^{-1}\alpha_i^T H_\ell = 0$. Hence $\alpha_i = 0$ and therefore by (4.3.6) again (4.3.10) holds. $\square$

The *general strategy* of our proof of item (1) of Theorem 4.1.4 is, for a given arbitrary $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$, first to prove that $\varphi(1) = 0$ and then to prove (4.3.9).

We will also need the following equations and notation. In the sequel every $(n-1) \times (n-1)$ matrix X will be also be regarded as an operator having the matrix representation X with respect to the basis $(e_1, \ldots, e_{n-1})$. Let $\{\varphi_i\}_{i=1}^{2n-2} \subset \mathbb{C}$ denote the coefficients satisfying

$$\varphi(1) = \sum_{i=1}^{2n-2} \varphi_i e_i.$$

By (4.3.5), it follows that

$$\big(\varphi(e_i))e_j\big)_- = -\left(H_\ell^{-1}\alpha_i^T H_\ell\right) e_{j-n+1}, \quad \forall\, n \le i, j \le 2n-2.$$

This together with (4.3.4) yields

$$\left(H_\ell^{-1}\alpha_i^T H_\ell\right) e_{j-n+1} = \left(H_\ell^{-1}\alpha_j^T H_\ell\right) e_{i-n+1}, \quad \forall\, n \le i, j \le 2n-2. \tag{4.3.11}$$

Condition (4.3.11) is crucial in the subsequent analysis, namely in the proof of Lemmas 4.3.6 and 4.3.11. Therefore, we need to describe the matrix $H_\ell^{-1}\alpha_j^T H_\ell$, which we begin by first describing the matrix α_j. By (4.3.5), it follows that, for $n \le j \le 2n-2$ and $1 \le i \le n-1$,

$$\big(\varphi(e_j)e_i\big)_+ = \alpha_j e_i.$$

From this and (4.3.3), taking into account that the matrix C represents the antilinear operator A, we have that there exists the unique tuple $(a_i)_{i=1}^{n-1}$ such that

$$\alpha_j e_i = a_i C e_{j-n+1} - (H_\ell)_{i,j-n+1}\big(\varphi(1)\big)_+ \tag{4.3.12}$$

for all $1 \le i \le n-1$ and $n \le j \le 2n-2$. The uniqueness of $(a_i)_{i=1}^{n-1}$ follows from the assumption that $C \neq 0$ and that a_i in (4.3.12) is independent of j.

4.3.1 The first special case

In this subsection, 4.3.1, we consider the special case wherein, for some integer m satisfying $2 \le m \le n-1$, we have

$$H_\ell = S_m \oplus H_\ell' \tag{4.3.13}$$

where H'_ℓ is an arbitrary nondegenerate Hermitian matrix, and

$$C = J_{\lambda,m} \oplus C' \qquad \text{for some } \lambda \geq 0, \tag{4.3.14}$$

where C' is such that (ℓ, A) is represented by (H_ℓ, C). Moreover, we assume that (H_ℓ, C) is in the canonical form of Theorem 3.1.2. In particular,

$$Ce_1 = \lambda e_1, \quad Ce_i = \lambda e_i + e_{i-1} \qquad \forall\, 2 \leq i \leq m, \tag{4.3.15}$$

and

$$H_\ell e_i = e_{m+1-i} \qquad \forall\, 1 \leq i \leq m. \tag{4.3.16}$$

Using (4.3.13) and (4.3.15) we obtain

$$\alpha_n e_i = a_i \lambda e_1 - \delta_{i,m}(\varphi(1))_+ \quad \forall\, i \in \{1, \ldots, n-1\}, \tag{4.3.17}$$

and, for $0 < p < m$,

$$\alpha_{n+p} e_i = a_i e_p + a_i \lambda e_{p+1} - \delta_{i,m-p}(\varphi(1))_+ \quad \forall\, i \in \{1, \ldots, n-1\}. \tag{4.3.18}$$

Now from (4.3.17), we get

$$\alpha_n^T e_1 = \sum_{j=1}^{n-1} a_j e_j - \varphi_1 e_m \quad \text{and} \quad \alpha_n^T e_i = -\varphi_i e_m \qquad \forall\, 2 \leq i \leq n-1.$$

Using this together with (4.3.16) we can get

$$(H_\ell^{-1} \alpha_n^T H_\ell) e_i = -\varphi_{m+1-i} e_1 \qquad \forall\, i \in \{1, \ldots, m-1\}, \tag{4.3.19}$$

$$(H_\ell^{-1}\alpha_n^T H_\ell)e_m \equiv -\varphi_1 e_1 + \lambda \sum_{j=1}^{m} a_{m+1-j} e_j \quad (\text{mod span}\{e_{m+1}, e_{m+2}, \ldots, e_{n-1}\}), \quad (4.3.20)$$

and

$$(H_\ell^{-1}\alpha_n^T H_\ell)e_i = -\left(\sum_{j=m+1}^{n-1} (H_\ell)_{j,i}\varphi_j\right) e_1 = -\left(\sum_{j=1}^{n-1-m} (H'_\ell)_{j,i-m}\varphi_{j+m}\right) e_1 \quad \forall\, i > m, (4.3.21)$$

where H_ℓ is as in (4.3.13).

Similarly, for $0 < p < m$, from (4.3.18) we have

$$\alpha_{n+p}^T e_i = \begin{cases} -\varphi_i e_{m-p}, & i \in \{1, \ldots, n-1\} \setminus \{p, p+1\} \\ -\varphi_p e_{m-p} + \sum_{j=1}^{n-1} a_j e_j, & i = p \\ -\varphi_{p+1} e_{m-p} + \lambda \sum_{j=1}^{n-1} a_j e_j & i = p+1, \end{cases}$$

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_i = -\varphi_{m+1-i} e_{p+1} \qquad \forall\, i \in \{1, \ldots, m\} \setminus \{m-p, m-p+1\}, \quad (4.3.22)$$

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_{m-p} \equiv -\varphi_{p+1} e_{p+1} + \lambda \sum_{j=1}^{m} a_{m+1-j} e_j \quad (\text{mod span}\{e_{m+1}, \ldots, e_{n-1}\}), (4.3.23)$$

and

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_{m-p+1} \equiv -\varphi_p e_{p+1} + \sum_{j=1}^{m} a_{m+1-j} e_j \quad (\text{mod span}\{e_{m+1}, \ldots, e_{n-1}\}).$$

For $p \geq m$,

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_i \in \text{span}\{e_{m+1}, \ldots, e_{n-1}\}. \quad (4.3.24)$$

Lemma 4.3.6. *In the special case of 4.3.1 wherein* (4.3.13) *and* (4.3.14) *hold, if rank*$(C) > 1$ *then*

$$\varphi(1) = 0. \tag{4.3.25}$$

Proof. We will begin by showing that

$$(\varphi(1))_+ = 0. \tag{4.3.26}$$

The proof consists of analysis of equation (4.3.11) in three cases:

1. *Equation* (4.3.11) *for* $i = n$ *and* $j = n + p$ *with* $0 \le p < m - 1$. By (4.3.19)

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = \varphi_{m-p} e_1 \qquad \forall\ 0 \le p < m - 1, \tag{4.3.27}$$

and, by (4.3.22),

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right) e_1 = \varphi_m e_{p+1} \qquad \forall\ 0 \le p < m - 1. \tag{4.3.28}$$

Applying (4.3.27) and (4.3.28) to (4.3.11) with $i = n$ and $j = n + p$ we get

$$\varphi_{m-p} e_1 = \varphi_m e_{p+1} \qquad \forall\ 0 \le p < m - 1.$$

Therefore, using the last equation for $1 \le p < m - 1$ (as for p=0 this equation is a tautology), we get

$$\varphi_2 = \cdots = \varphi_{m-1} = 0,$$

and also that $\varphi_m = 0$ for $m > 2$ (we will give another way to prove the latter identity including the case $m = 2$ in item 3 of the proof below).

2. *Equation* (4.3.11) *for* $i = n$ *and* $j = n + p$ *with* $p \geq m$. By (4.3.21) we get that

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = \left(\sum_{j=1}^{n-1-m} (H'_\ell)_{j,p+1-m}\varphi_{j+m}\right) e_1. \tag{4.3.29}$$

Using (4.3.11), from (4.3.29) and (4.3.24) it follows that $\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = 0$ or, equivalently,

$$\sum_{j=1}^{n-1-m} (H'_\ell)_{j,i}\varphi_{j+m} = 0, \quad 1 \leq i \leq n-1-m.$$

Since the matrix H'_ℓ is nonsigular, this yields

$$\varphi_{m+1} = \cdots = \varphi_{n-1} = 0.$$

3. *Equation* (4.3.11) *for* $i = n$ *and* $j = n + m - 1$. If $v = \lambda \sum_{j=1}^m a_{m+1-j}e_j$, then, by (4.3.20),

$$(H_\ell^{-1}\alpha_n^T H_\ell)e_m \equiv -\varphi_1 e_1 + v \pmod{\operatorname{span}\{e_i\}_{i=m+1}^{n-1}}, \tag{4.3.30}$$

and, by (4.3.23),

$$(H_\ell^{-1}\alpha_{n+m-1}^T H_\ell)e_1 \equiv -\varphi_m e_m + v \pmod{\operatorname{span}\{e_i\}_{i=m+1}^{n-1}}. \tag{4.3.31}$$

Using (4.3.11) again and the fact that $m \geq 2$, from (4.3.30) and (4.3.31) it follows that $\varphi_1 = 0$ and $\varphi_m = 0$. This completes the proof of (4.3.26).

Since (4.3.1) defines an involution of $\mathfrak{g}_1^{\mathrm{red}}$, $\sigma(\varphi)$ also belongs to $\mathfrak{g}_1^{\mathrm{red}}$, so, since φ was an arbitrary element in $\mathfrak{g}_1^{\mathrm{red}}$, the exact same arguments applied above show that $\left(\sigma(\varphi)(1)\right)_+ = 0$. Since $\sigma(1) = 1$,

$$\sigma\left(\left(\varphi(1)\right)_-\right) = \left(\sigma \circ \varphi(1)\right)_+ = \left(\sigma(\varphi)(1)\right)_+ = 0,$$

and hence $\left(\varphi(1)\right)_- = 0$, which, together with (4.3.26) implies (4.3.25). $\square$

Lemma 4.3.7. *In the special case of 4.3.1 wherein* (4.3.13) *and* (4.3.14) *hold, if rank*$(C) > 2$ *then*

$(a_1, \ldots, a_n)C = 0$.

Proof. Consider now the equation in (4.3.8) with $i = n$. The matrix on the right side of (4.3.8) is either zero or it has rank equal to rank(C), which is at least 3 under this lemma's hypothesis. On the other hand, applying (4.3.15), (4.3.16), (4.3.17) and Lemma 4.3.6, we get

$$(\alpha_n C H_\ell^{-1})e_i \in \operatorname{span}\{e_1\} \qquad \forall\, i \in \{1, \ldots, m-1\}, \tag{4.3.32}$$

and, applying (4.3.18) additionally, if $\lambda = 0$ then

$$(\alpha_{n+1} C H_\ell^{-1})e_i \in \operatorname{span}\{e_1\} \qquad \forall\, i \in \{1, \ldots, m-1\}. \tag{4.3.33}$$

Hence, by (4.3.32),

$$\operatorname{rank}\left(\alpha_n C H_\ell^{-1}\right) \leq 1 \tag{4.3.34}$$

and $\operatorname{rank}\left(\alpha_n C H_\ell^{-1} + \left(\alpha_n C H_\ell^{-1}\right)^T\right) \leq 2$ because $\alpha_n C H_\ell^{-1}$ has at most one nonzero row. Similarly, if $\lambda = 0$ then (4.3.33)

$$\operatorname{rank}\left(\alpha_{n+1} C H_\ell^{-1}\right) \leq 1 \tag{4.3.35}$$

and $\operatorname{rank}\left(\alpha_{n+1} C H_\ell^{-1} + \left(\alpha_{n+1} C H_\ell^{-1}\right)^T\right) \leq 2$. Since the matrix on the left side of (4.3.8) has rank at most 2 whenever $i = n$ or $(\lambda, i) = (0, n+1)$, the matrix on the right side of (4.3.8) is zero whenever $i = n$ or $(\lambda, i) = (0, n+1)$. Thus by (4.3.8) the matrix $\alpha_n C H_\ell^{-1}$ is skew symmetric, and the matrix $\alpha_{n+1} C H_\ell^{-1}$ is skew symmetric whenever $\lambda = 0$. This together with (4.3.34) implies that

$$\alpha_n C H_\ell^{-1} = 0, \tag{4.3.36}$$

whereas applying (4.3.35) yields

$$\alpha_{n+1} C H_\ell^{-1} = 0, \tag{4.3.37}$$

whenever $\lambda = 0$. By (4.3.36) and (4.3.17) for $\lambda \neq 0$, or by (4.3.37) and (4.3.18) for $\lambda = 0$, we get that the vector $(a_1, \dots, a_n)CH_\ell^{-1} = 0$, which completes this proof. $\square$

Lemma 4.3.8. *In the special case of 4.3.1 wherein* (4.3.13) *and* (4.3.14) *hold, if rank*$(C) > 2$ *and* $(\lambda, m) \notin \{(0,2),(0,3)\}$ *then* $\mathfrak{g}_1^{\mathrm{red}} = 0$.

Proof. Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (4.3.12). It will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$. Indeed, first plugging this condition and the conclusion (4.3.25) of Lemma 4.3.6 into relation (4.3.12) we obtain that $\alpha_j = 0$ for all $n \leq j \leq 2n-2$. This and Corollary 4.3.4 imply (4.3.9). Thus, the conclusion of the present lemma will follow from (4.3.25) and Lemma 4.3.5.

Notice that since $(a_1, \dots, a_n)C = 0$, we have that $a_i = 0$ for $1 \leq i \leq m$ if $\lambda \neq 0$, and $a_i = 0$ for $1 \leq i \leq m-1$ if $\lambda = 0$. In particular, as $m \geq 2$ we have $a_1 = a_2 = 0$ always, and, since it is assumed that $m > 3$ when $\lambda = 0$, if $\lambda = 0$ then $a_3 = 0$ as well.

To produce a contradiction, assume that there exists an index r such that $a_r \neq 0$ and let r be the minimal such index. By (4.3.17),

$$\alpha_n e_i = \delta_{i,r} a_i \lambda e_1 \quad \forall\, i \leq r, \tag{4.3.38}$$

and, by (4.3.18), for $0 < p < m$,

$$\alpha_{n+p} e_i = \delta_{i,r}(a_i e_p + a_i \lambda e_{p+1}) \quad \forall\, i \leq r. \tag{4.3.39}$$

Note that, by Lemma 4.3.1, $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^*\overline{H}_\ell)$. Since $\mathscr{A}$ is a subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^*\overline{H}_\ell)$ of codimension at most 1, the subspaces $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have a nontrivial intersection. That is, there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0,0)$ and

$$b_1\alpha_n + b_2\alpha_{n+1} \in \mathscr{A}. \tag{4.3.40}$$

By (4.3.38) and (4.3.39) again the first $r-1$ columns of the matrix $b_1\alpha_n + b_2\alpha_n$ vanish and

$$(b_1\alpha_n + b_2\alpha_{n+1})e_r = a_r\Big((\lambda b_1 + b_2)e_1 + \lambda b_2 e_2\Big) \tag{4.3.41}$$

By applying formulas from the appendix (i.e., Section A), we can derive a contradiction from the assumption $\lambda \neq 0$ as follows. Let $b_1\alpha_n + b_2\alpha_{n+1}$ be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of C (referring to the block diagonal partition of C given in Theorem 3.1.2).

By (4.3.40), if $\lambda > 0$ then each (i,j) block of $b_1\alpha_n + b_2\alpha_{n+1}$ is either characterized by Lemma A.0.1 or Corollary A.0.5 and identically zero or it is characterized by Corollary A.0.3 and more specifically characterized by (A.0.10). In particular, if the $(1,j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is nonzero (and therefore characterized by (A.0.10)) and contains part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$, then (A.0.10) implies that the $(j,1)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is nonzero and contained in the first $r-1$ columns of $b_1\alpha_n + b_2\alpha_{n+1}$, which contradicts our definition of r. Accordingly, if $\lambda > 0$ then the $(1,j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ containing part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$ is identically zero, which implies $\lambda b_1 + b_2 = 0$ and $\lambda b_2 = 0$ by (4.3.41). So, if $\lambda > 0$, then we obtain the contradiction $(b_1, b_2) = (0,0)$.

On the other hand, if $\lambda = 0$ then, by Lemma 4.3.1, $\operatorname{span}\{\alpha_{n+2}, \alpha_{n+3}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^*\overline{H}_\ell)$. Similarly to the previous case, $\mathscr{A}$ and $\operatorname{span}\{\alpha_{n+2}, \alpha_{n+3}\}$ have a nontrivial intersection, that is, there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0,0)$ and

$$b_1\alpha_{n+2} + b_2\alpha_{n+3} \in \mathscr{A}. \tag{4.3.42}$$

Note that we are now redefining b_1 and b_2 because the previous definition is no longer needed, and that the b_is in (4.3.42) are not related to the b_is in (4.3.40). By (4.3.38) and (4.3.39) the first $r-1$ columns of the matrix $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ vanish and

$$(b_1\alpha_{n+2} + b_2\alpha_{n+3})e_r = a_r\Big(b_1 e_2 + b_2 e_3\Big). \tag{4.3.43}$$

By applying formulas from the appendix again, we can derive a contradiction now from the assumption $\lambda = 0$. For this, let $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ in (4.3.42) be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of C. By (4.3.42), if $\lambda = 0$ then each (i, j) block of $b_1\alpha_n + b_2\alpha_{n+1}$ is either characterized by Lemma A.0.1 and identically zero or it is characterized by Lemmas A.0.4 and A.0.8 and Corollary A.0.5 and more specifically characterized by (A.0.15), (A.0.16), (A.0.17), and (A.0.23). In particular, if $\lambda = 0$ and the $(1, j)$ block of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ contains part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$, and, furthermore, we assume that the $(1, j)$ block is not identically zero, then this $(1, j)$ block is either characterized by (A.0.17) and (A.0.23) or by (A.0.15) and (A.0.16).

Considering the first possibility where the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is characterized by (A.0.17) and (A.0.23) (i.e., $j = 1$), by (4.3.43), the first m entries of $b_1e_2 + b_2e_3$ form the r column of the $(1, 1)$ block of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$. Since we are assuming that this $(1, 1)$ block is a linear combination of matrices (A.0.17) and (A.0.23) with the latter being a diagonal matrix, noting that $r > 3$, it follows that the first entry in the $r - 1$ column of this $(1, 1)$ block is $-b_1$ and the second entry in the $r - 1$ column of this $(1, 1)$ block is $-b_2$. Yet the $r - 1$ column of the $(1, 1)$ block of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is zero by the definition of r, so we have obtained the contradiction that $(b_1, b_2) = (0, 0)$.

Considering the remaining possibility, which is where the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is characterized by (A.0.15) or (A.0.16), if this $(1, j)$ block is nonzero then (A.0.15) and (A.0.16) imply that the $(j, 1)$ block is nonzero and contained in the first $r - 1$ columns of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$, which contradicts the definition of r.

Hence, the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ must be identically zero because all other possibilities yield contradictions, and yet, by (4.3.43), setting this $(1, j)$ block equal to zero again implies the contradiction $(b_1, b_2) = (0, 0)$. Therefore, there is no index r such that $a_r \neq 0$. □

Lemma 4.3.9. *In the special case of 4.3.1 wherein* (4.3.13) *and* (4.3.14) *hold, if there is a basis* *with respect to which A is represented by the matrix*

$$C = J_{0,3} \oplus J_{c,1} \oplus C'' \quad \textit{for some } c > 0 \tag{4.3.44}$$

or

$$C = J_{0,2} \oplus J_{c,1} \oplus J_{c',1} \oplus C'' \quad \textit{for some } c, c' > 0 \tag{4.3.45}$$

then $\mathfrak{g}_1^{\mathrm{red}} = 0$.

Proof. Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (4.3.12). By the same arguments as in the beginning of the proof of Lemma 4.3.8 , it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$. Note that, by Lemma 4.3.6, in the considered cases $\varphi(1) = 0$. It is more convenient to work with matrices

$$\widetilde{C} = J_{1,c} \oplus J_{0,3} \oplus C'' \tag{4.3.46}$$

or

$$\widetilde{C} = J_{1,c} \oplus J_{1,c'} \oplus J_{0,2} \oplus C'' \tag{4.3.47}$$

instead of C in (4.3.44) and (4.3.45), respectively. This can be done by an obvious permutation of the basis. Also, in the considered cases the rank assumptions of Lemma 4.3.7 with C replaced by $\widetilde{C}$ holds. Therefore, using (4.3.12) with C replaced by $\widetilde{C}$ we get

$$a_1 = a_2 = a_3 = 0. \tag{4.3.48}$$

Note that if we would not replace C by $\widetilde{C}$ we could conclude that $a_1 = a_2 = a_4 = 0$ in the case of (4.3.44) and that $a_1 = a_3 = a_4 = 0$ in the case of (4.3.45), so that is why we make this permutation of the blocks.

Assume for a proof by contradiction that there exists r such that $a_r \neq 0$ and moreover that this is the minimal such index, that is, $a_i = 0$ for all $i < r$. By (4.3.48), $r > 3$. From (4.3.12) with C replaced by $\widetilde{C}$ it follows that in both cases the first $r-1$ columns of the matrices α_i with $n \leq i \leq n+3$ vanish,

$$\alpha_n e_r = a_r c e_1, \quad \text{and} \quad \alpha_{n+3} e_r = a_r e_3. \tag{4.3.49}$$

Further,

$$\alpha_{n+2} e_r = a_r e_2 \tag{4.3.50}$$

if $\tilde{C}$ satisfies (4.3.46) and

$$\alpha_{n+1} e_r = a_r c' e_2 \tag{4.3.51}$$

if $\tilde{C}$ satisfies (4.3.47). Note that, by Lemma 4.3.1, each α_i in these equations belongs to $\mathscr{A} + \mathbb{C}\left(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}\right)$.

Hence, using similar arguments as in the proof of Lemma 4.3.8 we get that the 3-dimensional subspace $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ in the first case and $\operatorname{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$ in the second case has at least a two dimensional intersection with $\mathscr{A}$. Notice further that in either case, the rth columns of matrices in these intersections must have a two-dimensional span because the natural projection from the space $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ (or $\operatorname{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$) to the r column of matrices in this space is injective. Yet, formulas from the appendix (i.e., Section A) can be applied to show that these intersections are at most one-dimensional as follows.

Let us now first assume that $\tilde{C}$ satisfies (4.3.46). Let $B^{(1)}$ and $B^{(2)}$ be matrices belonging to the intersection of $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ and $\mathscr{A}$ such that the r column of $B^{(1)}$ is linearly independent of the r column of $B^{(2)}$. For an $(n-1) \times (n-1)$ matrix B, let $(B_{(i,j)})$ be a partition of B into a block matrix whose diagonal blocks have the same size as the diagonal blocks of C. Let j be the index such that $B_{(1,j)}$ contains part of the r column of B. By Lemma A.0.1, since $c \neq 0$ there exists $i \in \{1,2\}$ such that $B_{(i,j)} = 0$ for all $B \in \mathscr{A}$, because otherwise Lemma A.0.1 implies that the $(1,1)$ and $(2,2)$ blocks of $C\overline{C}$ have the same eigenvalues. In particular, at most one of the $(1,j)$ and $(2,j)$ blocks of any linear combination of $B^{(1)}$ and $B^{(2)}$ is nonzero. It follows that, for each $k \in \{1,2\}$, $B^{(k)}_{(1,j)} = 0$ and $B^{(k)}_{(2,j)} \neq 0$ because otherwise the r column of each $B^{(k)}$ belongs to $\mathrm{span}\{e_1\}$, which contradicts our choice of $B^{(1)}$ and $B^{(2)}$. Moreover, by (4.3.49) and (4.3.50), the first nonzero column of each block $B^{(k)}_{(2,j)}$ has zero in all but its first two entries.

Each $B^{(k)}_{(2,j)}$ is either characterized by Lemma A.0.1 and is identically zero or characterized by Lemma A.0.4 and Corollary A.0.5 and more specifically characterized by (A.0.15), (A.0.16), or (A.0.17) (with $\lambda_i = 0$). If $B^{(k)}_{(2,j)}$ is characterized by (A.0.17) then $j = 2$ and, by (A.0.17), the second entry of the first nonzero column of $B^{(k)}_{(2,2)}$ is zero. If, on the other hand, $B^{(k)}_{(2,j)}$ is characterized by (A.0.15) (or (A.0.16)) and the second entry of the first nonzero column of $B^{(k)}_{(2,j)}$ is nonzero, then, by (A.0.16) (or respectively (A.0.15)), the $B^{(k)}_{(j,2)}$ block of $B^{(k)}$ is nonzero and contained in the first $r-1$ columns of $B^{(k)}$, which contradicts our choice of r. Therefore if $B^{(k)}_{(2,j)}$ is nonzero then the second entry of the first nonzero column of $B^{(k)}_{(2,j)}$ is zero. Yet this contradicts our choice of $B^{(1)}$ and $B^{(2)}$ because it means that the only nonzero entry in the r column of $B^{(1)}$ and $B^{(2)}$ is the second entry.

Let us now address the remaining case, that is, assume that $\tilde{C}$ satisfies (4.3.47). Again, let j be the index such that $B_{(1,j)}$ contains part of the r column a given $(n-1)\times(n-1)$ matrix B. Let $B^{(1)}$ and $B^{(2)}$ be matrices belonging to the intersection of $\mathrm{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$ and $\mathscr{A}$ such that the r column of $B^{(1)}$ is linearly independent from the r column of $B^{(2)}$. From this independence condition and the fact that nonzero entries of these respective rth columns of $B^{(1)}$ and of $B^{(2)}$ appear within their first three entries (the latter is a consequence of (4.3.49) and (4.3.51)), it follows that there exists a matrix B in $\mathrm{span}\{B^{(1)}, B^{(2)}\}$ such that there exists $i \in \{1,2\}$ with $B_{(i,j)} \neq 0$ (because otherwise, the third entry is the only nonzero entry of rth columns of $B^{(1)}$ and $B^{(2)}$, which contradicts the independence of these columns). Since $r > 3$ it follows that $j > 2$. Thus, it follows from Lemma A.0.1 and Corollary A.0.3 that this nonzero $B_{(i,j)}$ with $i \in \{1,2\}$ is characterized by (A.0.10). Yet (A.0.10) implies that the $B_{(j,i)}$ is a nonzero block contained in the first $r-1$ rows of

B, which contradicts our choice of r. $\square$

Lemma 4.3.10. *In the special case of 4.3.1 wherein* (4.3.13) *and* (4.3.14) *hold, if*

$$C = \overbrace{J_{0,2} \oplus \cdots \oplus J_{0,2}}^{k \text{ copies}} \oplus J_{c,1} \oplus J_{0,1} \oplus \cdots \oplus J_{0,1},$$

for some integer k and some $c > 0$ then $\alpha_i = 0$ for all $i \geq n$.

Proof. Let $\varphi \in \mathfrak{g}_1^{\text{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (4.3.12). By the same arguments as in the beginning of the proof of Lemma 4.3.8 , it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$. We work with (H_ℓ, C) in the canonical form of Theorem 3.1.2, so H_ℓ is as in Theorem 3.1.2, that is

$$H_\ell = \epsilon_1 N_{0,2} \oplus \cdots \oplus \epsilon_k N_{0,2} \oplus \epsilon_{k+1} N_{c,1} \oplus \cdots \oplus \epsilon_\gamma N_{0,1}$$

for some coefficients $\epsilon_i = \pm 1$.

For a matrix B in $\mathscr{A}$, let $(B_{(i,j)})$ be a partition of B into a block matrix whose diagonal blocks have the same size as the diagonal blocks of C. By Lemma A.0.4 and Corollary A.0.5 (in the appendix below), we have

$$B_{(i,j)} = \epsilon_i \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \quad \text{and} \quad B_{(j,i)} = -\epsilon_j \begin{pmatrix} b & e \\ 0 & d \end{pmatrix} \qquad \forall\, i, j \leq k$$

and

$$B_{(i,j)} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad \text{and} \quad B_{(j,i)} = \begin{pmatrix} 0 & b \end{pmatrix} \qquad \forall\, i \leq k < j$$

for some $b, c, d, e \in \mathbb{C}$ that depend on (i, j). By Corollary A.0.5 and Lemma A.0.8 (in the appendix below),

$$B_{1,1} = B_{2,2} = \cdots = B_{2k+1,2k+1}, \tag{4.3.52}$$

where here $B_{i,j}$ denotes the (i,j) entry of B rather than the (i,j) block $B_{(i,j)}$. By Lemma A.0.1 and Corollary A.0.5 (in the appendix below),

$$B_{(i,k+1)} = 0 \quad \text{and} \quad B_{(k+1,i)} = 0 \qquad \forall\, i \neq k. \tag{4.3.53}$$

Since, by Lemma 4.3.7, $(a_1, \ldots, a_{n-1})C = 0$, we have

$$a_i = 0 \qquad \text{whenever } i \text{ is odd and } i \leq 2k+1. \tag{4.3.54}$$

From (4.3.12) and Lemma 4.3.6 it follows that, for $0 \leq p \leq n-1$, the i column of the matrix α_{n+p} is equal to a_i times the $p+1$ column of C. In particular, the (i,j) entry of α_{n+2k} is

$$(\alpha_{n+2k})_{i,j} = a_j c \delta_{i,2k+1}. \tag{4.3.55}$$

Since, by Lemma 4.3.1, each α_{n+p} belongs to $\mathscr{A}_0 + \mathbb{C}\left(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}\right)$ and α_{n+2k} does not belong to $\mathscr{A}_0 \setminus \{0\}$, which can be seen by contrasting (4.3.53) and (4.3.55), it follows that

$$\text{either} \quad \alpha_{n+2k} = 0 \quad \text{or} \quad \overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 + \operatorname{span}_{\mathbb{C}}\{\alpha_{n+2k}\}.$$

But $\alpha_{n+2k} = 0$ if and only if $a_1 = \cdots = a_{n-1} = 0$, which is equivalent to what we want to show, so let us proceed assuming

$$\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 + \operatorname{span}_{\mathbb{C}}\{\alpha_{n+2k}\}$$

in order to produce a contradiction. Accordingly, let $\Omega_0 \in \mathscr{A}_0$ and $s \in \mathbb{C}$ be such that

$$\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} = \overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell} + s\alpha_{n+2k}, \tag{4.3.56}$$

or, equivalently,

$$\Omega = \Omega_0 + \overline{s}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}. \tag{4.3.57}$$

Here we will apply another result from the appendix (below), namely Corollary A.0.9, which states that for $B \in \mathscr{A}$, since C is not nilpotent, if $\left(H_\ell\overline{C}B\right)^T + H_\ell\overline{C}B = \mu H_\ell\overline{C}$ then $BCH_\ell^{-1} + CH_\ell^{-1}B^T = \mu CH_\ell^{-1}$. Noting that, by (4.3.54) and (4.3.55), $\overline{CH_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell} = 0$, item (iii) in (4.2.4) and (4.3.57) imply that

$$\left(H_\ell\overline{C}\Omega_0\right)^T + H_\ell\overline{C}\Omega_0 = \mu H_\ell\overline{C}, \tag{4.3.58}$$

and hence Corollary A.0.9 implies that

$$\eta_{\Omega_0} = \mu,$$

where this notation η_{Ω_0} refers to the coefficient with that label in items (i) and (ii) or (4.2.4).

Since the matrix equation $\left(H_\ell\overline{C}X\right)^T + H_\ell\overline{C}X = \mu H_\ell\overline{C}$ is equivalent to

$$\left(\overline{H}_\ell^{-1}X^*\overline{H}_\ell\right)CH_\ell^{-1} + CH_\ell^{-1}\left(\overline{H}_\ell^{-1}X^*\overline{H}_\ell\right)^T = \overline{\mu}CH_\ell^{-1},$$

(4.3.58) implies

$$\eta_{\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell}} = \overline{\mu}. \tag{4.3.59}$$

By (4.3.59), items (i) and (ii) in (4.2.4) imply

$$\left[\Omega, \overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell}\right] + \overline{\mu}\Omega \in \mathscr{A}_0, \tag{4.3.60}$$

and applying the transformation $X \mapsto \overline{H_\ell}^{-1} X^* \overline{H_\ell}$ to the matrix in (4.3.59) yields

$$\left[\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell}, \Omega_0\right] - \mu \overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell} \in \mathscr{A}_0. \tag{4.3.61}$$

Now we analyze item (iv) of (4.2.4). Using (4.3.56), (4.3.57), and lastly (4.3.60), we have

$$\begin{aligned}\left[\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell}, \Omega\right] &= \left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \Omega\right] + [s\alpha_{n+2k}, \Omega_0] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] \\ &\equiv \overline{\mu}\Omega + [s\alpha_{n+2k}, \Omega_0] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] \pmod{\mathscr{A}_0}.\end{aligned}$$

Substituting the last equation into item (iv) of (4.2.4) we get, after the obvious cancellation, that

$$[s\alpha_{n+2k}, \Omega_0] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + C\overline{C} - \mu \overline{H_\ell}^{-1} \Omega^* \overline{H_\ell} \in \mathscr{A}_0. \tag{4.3.62}$$

Similarly, (4.3.56), (4.3.57), and then (4.3.61) yields

$$\begin{aligned}\left[\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell}, \Omega\right] &= \left[\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell}, \Omega_0\right] + \left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \overline{s}\overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] \\ &\equiv \mu \overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell} + \left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \overline{s}\overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right],\end{aligned}$$

where the equivalence is modulo $\mathscr{A}_0$. Substituting the last equation into item (iv) of (4.2.4) we get

$$\left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \overline{s}\overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + C\overline{C} - \overline{\mu}\Omega \in \mathscr{A}_0. \tag{4.3.63}$$

On the other hand, again from (4.3.56) , (4.3.57), and using that $\left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \Omega_0\right] \in \mathscr{A}_0$, we can write

$$\left[\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell}, \Omega\right] \equiv [s\alpha_{n+2k}, \Omega_0] + \left[\overline{H_\ell}^{-1} \Omega_0^* \overline{H_\ell}, \overline{s}\overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right] + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha_{n+2k}^* \overline{H_\ell}\right],$$

where here again the equivalence is modulo $\mathscr{A}_0$. By subtracting the matrix in item (iv) of (4.2.4) from the sum of the matrices in (4.3.62) and (4.3.63) and using the last relation, we get

$$C\overline{C} + |s|^2 \left[\alpha_{n+2k}, \overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell}\right] \in \mathscr{A}_0,$$

or, equivalently,

$$\left(C\overline{C} + |s|^2 \alpha_{n+2k} \overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell}\right) - |s|^2 \overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell} \alpha_{n+2k} \in \mathscr{A}_0. \tag{4.3.64}$$

Notice that the first two terms in (4.3.64), grouped together by parentheses, are matrices whose only potentially nonzero entry is the $(2k+1, 2k+1)$ entry, whereas the other term has the same value in the first $2k+1$ entries of its main diagonal. By (4.3.52), each matrix in $\mathscr{A}_0$ also has the same values in the first $2k+1$ entries of its main diagonal. Moreover, the $(2k+1, 2k+1)$ entry of $C\overline{C}$ is nonzero. Therefore, by (4.3.64),

$$C\overline{C} = -|s|^2 \alpha_{n+2k} \overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell}. \tag{4.3.65}$$

Defining

$$\alpha := |s|^2 \overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell} \alpha_{n+2k},$$

(4.3.64) and (4.3.65) imply that α is in $\mathscr{A}_0$.

It is straightforward to check that, with this definition for α, $\eta_\alpha = 0$ in the notation of item (i) of (4.2.4) (by calculating, for example, the $(1,1)$ entries of the terms in item (i)), and hence items (i) and (ii) of (4.2.4) yield $[\Omega, \alpha] \in \mathscr{A}_0$. Or, equivalently, by (4.3.57), noting that $[\Omega_0, \alpha] \in \mathscr{A}_0$,

$$\overline{s} \left[\overline{H_\ell}^{-1} \alpha^*_{n+2k} \overline{H_\ell}, \alpha\right] \in \mathscr{A}_0. \tag{4.3.66}$$

Notice that $\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\alpha = 0$ because $\left(\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right)^2 = 0$, and hence (4.3.66) implies

$$\overline{s}|s|^2\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right) \in \mathscr{A}_0. \tag{4.3.67}$$

Applying (4.3.65), we get

$$\begin{aligned} -\frac{\overline{s}|s|^2}{|c|^2}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right) &= \frac{\overline{s}}{|c|^2}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(C\overline{C}\right) \\ &= \overline{s}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}, \end{aligned} \tag{4.3.68}$$

where this last equality follows easily from (4.3.55).

By (4.3.57), (4.3.67), and (4.3.68), we get that Ω is in $\mathscr{A}_0$, but this contradicts Lemma 4.2.2. Therefore, the assumption that $\alpha_{n+2k} \neq 0$ must be false, which in turn implies that $a_1 = \cdots = a_{n-1} = 0$, completing this proof. $\square$

4.3.2 The second special case

In this subsection, 4.3.2, we consider the special case where we have some integer $1 \leq m \leq n-1$ such that

$$H_\ell = \left(\begin{array}{c|c} S_{2m} & 0 \\ \hline 0 & H'_\ell \end{array}\right), \tag{4.3.69}$$

where H'_ℓ is an arbitrary nondegenerate Hermitian matrix, and

$$C = \left(\begin{array}{c|c} \overbrace{\begin{array}{c|c} 0 & J_{m,\lambda} \\ \hline I & 0 \end{array}}^{2m \text{ columns}} & 0 \\ \hline 0 & C' \end{array}\right) \left.\vphantom{\begin{array}{c} 0 \\ I \end{array}}\right\} {\scriptstyle 2m \text{ rows}}, \qquad \text{for some } \lambda \in \mathbb{C} \setminus \{x \in \mathbb{R} \,|\, x \geq 0\}, \tag{4.3.70}$$

where C' is a matrix such that (ℓ, A) is represented by (H, C). The analysis in 4.3.2 is similar to that of 4.3.1, but some formulas differ.

By Lemma 4.3.3 there exist coefficients $a_1, \dots, a_{n-1}$, given in (4.3.12), such that, first,

$$\alpha_{n+m} e_i = -\delta_{i,m-1}\big(\varphi(1)\big)_+ + \lambda a_i e_1,$$

second, for any nonnegative integer $p < m$,

$$\alpha_{n+p} e_i = -\delta_{i,2m-p}\big(\varphi(1)\big)_+ + \lambda a_i e_{m+p},$$

and, third, if $0 < p < m$ then

$$\alpha_{n+m+p} e_i = -\delta_{i,2m-p}\big(\varphi(1)\big)_+ + a_i e_p + \lambda a_i e_{p+1},$$

which we use to obtain the following formulas. For $0 \le p < m$, we have

$$(\alpha_{n+p} C H_\ell^{-1}) e_i = (a_{m-i}\lambda - a_{m+1-i}\lambda^2) e_{m+p+1} \qquad \forall\, i \in \{1, \dots, m-1\}, \tag{4.3.71}$$

$$(\alpha_{n+p} C H_\ell^{-1}) e_m = a_1 \lambda^2 e_{m+p+1}, \tag{4.3.72}$$

$$(\alpha_{n+p} C H_\ell^{-1}) e_i = a_{3m+1-i}\lambda e_{m+p+1} - \delta_{i,m+p+1}\big(\varphi(1)\big)_+ \qquad \forall i \in \{m+1, \dots, 2m\}, \tag{4.3.73}$$

and

$$(\alpha_{n+p} C H_\ell^{-1}) e_i \in \operatorname{span}\{e_{m+p+1}\} \qquad \forall i > 2m. \tag{4.3.74}$$

For any nonnegative integer $p < m$ and $1 \le i \le 2m$

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right) e_i \equiv -\varphi_{2m+1-i} e_{p+1} + \delta_{i,m-p} \sum_{j=1}^{2m} a_{2m+1-j}\lambda e_j \pmod{\operatorname{span}\{e_k\}_{k=2m+1}^{n-1}} \tag{4.3.75}$$

and, moreover, this equivalence modulo $\operatorname{span}\{e_k\}_{k=2m+1}^{n-1}$ can be replaced with ordinary strict equivalence whenever $\delta_{i,m-p} = 0$. Also, for $1 \leq i \leq 2m$,

$$\left(H_\ell^{-1}\alpha_{n+m}^T H_\ell\right) e_i \equiv -\varphi_{2m+1-i} e_{m+1} + \delta_{i,2m} \sum_{j=1}^{2m} a_{2m+1-j} \lambda e_j \quad (\operatorname{mod} \operatorname{span}\{e_k\}_{k=2m+1}^{n-1}), \tag{4.3.76}$$

where equivalence modulo $\operatorname{span}\{e_k\}_{k=2m+1}^{n-1}$ can be replaced with ordinary strict equivalence whenever $\delta_{i,2m-1} = 0$. For any $0 < p < m$ and $0 < i < 2m+1$,

$$\begin{aligned}\left(H_\ell^{-1}\alpha_{n+m+p}^T H_\ell\right) e_i = -\varphi_{2m+1-i} e_{m+p+1} &+ \delta_{i,2m-p} \left(\sum_{j=1}^{2m} a_{2m+1-j} \lambda e_j + \sum_{k=2m+1}^{n-1} a_k \lambda e_k\right) \\ &+ \delta_{i,2m-p+1} \left(\sum_{j=1}^{2m} a_{2m+1-j} e_j + \sum_{k=2m+1}^{n-1} a_k e_k\right),\end{aligned} \tag{4.3.77}$$

and for any $0 < p < m$ and $2m < i < n$

$$\left(H_\ell^{-1}\alpha_{n+m+p}^T H_\ell\right) e_i = -\varphi_i e_{m+p+1}.$$

Lastly, for all $i \geq 2m$

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell) e_i = -\left(\sum_{j=2m+1}^{n-1} (H_\ell)_{j,i} \varphi_j\right) e_{p+1} \qquad \forall\, 0 \leq p < m \tag{4.3.78}$$

and

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right) e_i \subset \operatorname{span}\{e_{2m+1}, \ldots, e_{n-1}\} \qquad \forall\, 2m \leq p. \tag{4.3.79}$$

Lemma 4.3.11. *In the special case of 4.3.2 wherein* (4.3.69) *and* (4.3.70) *hold*

$$\varphi(1) = 0.$$

Proof. By the same argument applied at the end of the proof of Lemma 4.3.6, it will suffice to show that $\left(\varphi(1)\right)_+ = 0$. Similar to the proof of Lemma 4.3.6, this proof consists of analysis of equation (4.3.11) in four cases:

1. *Equation* (4.3.11) *for* $i = n$ *and* $j = n + p$ *with* $0 \leq p < m$ *and* $m \neq 1$. By (4.3.75) replacing p with 0 and replacing i with $p + 1$,

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = -\varphi_{2m-p} e_1 \qquad \forall\ 0 \leq p < m - 1, \tag{4.3.80}$$

and, by (4.3.75) with $i = 1$,

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right) e_1 = -\varphi_{2m} e_{p+1} \qquad \forall\ 0 \leq p < m - 1. \tag{4.3.81}$$

Applying (4.3.11), (4.3.80), and (4.3.81) we get $\varphi_{m+2} = \varphi_{m+3} = \cdots = \varphi_{2m} = 0$. Furthermore, by (4.3.75) with $p = 0$ and $i = m$,

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_m \equiv -\varphi_{m+1} e_1 + \sum_{j=1}^{2m} a_{2m+1-j}\lambda e_j \quad (\mathrm{mod\ span}\{e_k\}_{k=2m+1}^{n-1}) \tag{4.3.82}$$

whereas, by (4.3.75) with $p = m - 1$ and $i = 1$,

$$\left(H_\ell^{-1}\alpha_{n+m-1}^T H_\ell\right) e_1 \equiv -\varphi_{2m} e_m + \sum_{j=1}^{2m} a_{2m+1-j}\lambda e_j \quad (\mathrm{mod\ span}\{e_k\}_{k=2m+1}^{n-1}). \tag{4.3.83}$$

Applying (4.3.11), (4.3.83), and (4.3.82) yields $\varphi_{m+1} = 0$ so, altogether, we have shown

$$\varphi_{m+1} = \cdots = \varphi_{2m} = 0. \tag{4.3.84}$$

2. *Equation* (4.3.11) *for* $i = n + m$ *and* $j = n + m + p$ *with* $0 \leq p < m$ *and* $m \neq 1$. By (4.3.77) replacing p with 0 and replacing i with $m + p + 1$,

$$\left(H_\ell^{-1}\alpha_{n+m}^T H_\ell\right) e_{m+p+1} = -\varphi_{m-p} e_{m+1} \qquad \forall\ 0 < p < m - 1, \tag{4.3.85}$$

and, by (4.3.77) with $i = m + 1$,

$$\left(H_\ell^{-1}\alpha_{n+m+p}^T H_\ell\right) e_{m+1} = -\varphi_m e_{m+p+1} \qquad \forall\ 0 < p < m - 1. \tag{4.3.86}$$

Applying (4.3.11), (4.3.85), and (4.3.86) we get $\varphi_2 = \varphi_3 = \cdots = \varphi_m = 0$. Furthermore, by (4.3.77) with $p = 0$ and $i = 2m$,

$$\left(H_\ell^{-1}\alpha_{n+m}^T H_\ell\right) e_{2m} = -\varphi_1 e_{m+1} + \left(\sum_{j=1}^{2m} a_{2m+1-j}\lambda e_j + \sum_{k=2m+1}^{n-1} a_k \lambda e_k\right) \tag{4.3.87}$$

and, by (4.3.75) with $p = m - 1$ and $i = m + 1$,

$$\left(H_\ell^{-1}\alpha_{n+2m-1}^T H_\ell\right) e_{m+1} = -\varphi_m e_{2m} + \left(\sum_{j=1}^{2m} a_{2m+1-j}\lambda e_j + \sum_{k=2m+1}^{n-1} a_k \lambda e_k\right). \tag{4.3.88}$$

Applying (4.3.11), (4.3.87), and (4.3.88) yields $\varphi_1 = \varphi_m = 0$ so, altogether, noting (4.3.84), we have shown

$$\varphi_1 = \cdots = \varphi_{2m} = 0 \quad \text{ if } m > 1. \tag{4.3.89}$$

3. *Equation* (4.3.11) *for $i = n$ and $j = n + m$.* By (4.3.75) and (4.3.76)

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{m+1} = -\varphi_m e_1 \quad \text{ and } \quad \left(H_\ell^{-1}\alpha_{n+m}^T H_\ell\right) e_1 = -\varphi_{2m} e_{m+1}. \tag{4.3.90}$$

By (4.3.11), $\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{m+1} = \left(H_\ell^{-1}\alpha_{n+m}^T H_\ell\right) e_1$, and hence (4.3.90) implies $\varphi_m = \varphi_{2m}$. This is true in particular when $m = 1$, which together with (4.3.89) yields the general result

$$\varphi_1 = \cdots = \varphi_{2m} = 0. \tag{4.3.91}$$

4. *Equation* (4.3.11) *for* $i = n$ *and* $j = n + p$ *with* $p \geq 2m$. By (4.3.78) we get that

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = \left(\sum_{j=2m+1}^{n-1} (H'_\ell)_{j,i}\varphi_j\right) e_1. \tag{4.3.92}$$

Using (4.3.11) again, from (4.3.79) and (4.3.92) it follows that $\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = 0$ or, equivalently,

$$\sum_{j=1}^{n-1-m} (H'_\ell)_{j,i}\varphi_j = 0, \quad \forall\, 1 \leq i \leq n-1-2m. \tag{4.3.93}$$

Since the matrix H'_ℓ is nonsingular, (4.3.93) implies $\varphi_{2m+1} = \cdots = \varphi_{n-1} = 0$, which together with (4.3.91) yields $\varphi_1 = \cdots = \varphi_{n-1} = 0$, that is, $\big(\varphi(1)\big)_+ = 0$. $\square$

Lemma 4.3.12. *In the special case of 4.3.2 wherein* (4.3.69) *and* (4.3.70) *hold,* $(a_1, \ldots, a_{n-1})C = 0$.

Proof. First we want to show that $\alpha_n C H_\ell^{-1}$ is skew symmetric, and we do so by considering two separate cases.

First, consider the case where $m = 1$. By (4.3.72), the $(1,1)$ entry of $\alpha_n C H_\ell^{-1}$ is zero. But the $(1,1)$ entry of $C H_\ell^{-1}$ is nonzero, so (4.3.8) implies that $\alpha_n C H_\ell^{-1}$ is skew symmetric.

Now let us consider the second case, which is where $m > 1$. The right side of (4.3.8) is either zero or its right side has rank equal to rank(C) (which is at least 4 because $m > 1$). On the other hand, using formulas (4.3.71), (4.3.72), (4.3.73), and (4.3.74) for the matrix $\alpha_n C H_\ell^{-1}$ together with Lemma 4.3.11, we can see that the matrix $\alpha_n C H_\ell^{-1}$ has rank at most 1. Therefore the matrix on the left side of (4.3.8) (when setting $i = n$) has rank at most 2, and hence the matrix $\alpha_n C H_\ell^{-1}$ appearing in (4.3.71) must be skew symmetric if $m > 1$.

So, for all values of m, we have shown that $\alpha_n C H_\ell^{-1}$ is skew symmetric and of rank at most 1. Thus it is identically zero, which implies that the rows of α_n are in the left kernel of $C H_\ell^{-1}$. In particular, $(a_1, \ldots, a_n) C H_\ell^{-1} = 0$, which completes this proof because H'_ℓ is nonsingular. $\square$

Lemma 4.3.13. *In the special case of 4.3.2 wherein* (4.3.69) *and* (4.3.70) *hold, if C corresponds to a non-regular CR structure then* $\mathfrak{g}_1^{\mathrm{red}}$.

Proof. Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (4.3.12). By the same arguments as in the beginning of the proof of Lemma 4.3.8 , it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$.

To produce a contradiction, let us assume there exists an index i such that $a_i \neq 0$, and let r be the smallest such index. Since, by Lemma 4.3.12, $(a_1, \ldots, a_n)C = 0$, we have $a_1 = a_2 = 0$, and hence $2 < r$. Also,

$$\alpha_{n+m} e_i = \delta_{i,r} a_r \lambda e_1 \quad \text{and} \quad \alpha_{n+m+1} e_i = \delta_{i,r}(a_r e_1 + a_r \lambda e_2) \qquad \forall\, i \leq r. \tag{4.3.94}$$

By Lemma 4.3.1, $\operatorname{span}\{\alpha_{n+m}, \alpha_{n+m+1}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^* \overline{H}_\ell)$. Since $\mathscr{A}$ is a subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^* \overline{H}_\ell)$ of codimension at most 1 it has a nontrivial intersection with $\operatorname{span}\{\alpha_{n+m}, \alpha_{n+m+1}\}$, and hence there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0, 0)$ and

$$b_1 \alpha_{n+m} + b_2 \alpha_{n+m+1} \in \mathscr{A}. \tag{4.3.95}$$

By (4.3.94) the first $r-1$ columns of the matrix $b_1\alpha_n + b_2\alpha_n$ vanish and

$$(b_1\alpha_n + b_2\alpha_{n+1})e_r = a_r\Big((\lambda b_1 + b_2)e_1 + \lambda b_2 e_2\Big). \tag{4.3.96}$$

Using results from the appendix (Section A below), we can now derive a contradiction as follows. Let $b_1\alpha_n + b_2\alpha_{n+1}$ be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of C. By (4.3.95), each (i, j) block of $b_1\alpha_n + b_2\alpha_{n+1}$ is either characterized by Lemma A.0.1 and identically zero or it is characterized by Corollaries A.0.3 and A.0.5 and more specifically characterized by (A.0.11), (A.0.12), (A.0.13), (A.0.14), and (A.0.17). Notice that if this $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is characterized by (A.0.17) then $j = 1$, and clearly no matrix of the form in (A.0.17) can have nonzero values in either of the first two entries of its first nonzero column, which shows that this $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ containing part of the r

column of $b_1\alpha_n + b_2\alpha_{n+1}$ must be zero if it is characterized by (A.0.17).

If, on the other hand, the $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is characterized by (A.0.11) or (A.0.12) (respectively (A.0.13) or (A.0.14)), is nonzero, and contains part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$, then (A.0.11) and (A.0.12) (respectively (A.0.13) and (A.0.14)) imply that the $(j, 1)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is nonzero and contained in the first $r - 1$ columns of $b_1\alpha_n + b_2\alpha_{n+1}$, which contradicts our definition of r. Therefore, the $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ containing part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$ is identically zero, which, by (4.3.96), implies that $\lambda b_1 + b_2 = 0$ and $\lambda b_2 = 0$. Yet this yields the contradiction $(b_1, b_2) = (0, 0)$. $\square$

4.3.3 The third special case

In this subsection, 4.3.3, we consider the special case where (H_ℓ, C) corresponds to a non-regular CR structure and C is diagonal. Working in the normal form of Theorem 3.1.2, H_ℓ is diagonal too. Since C corresponds to a non-regular CR structure, the matrix $C\overline{C}$ has at least two distinct nonzero eigenvalues, so we can assume without loss of generality that there are numbers $\lambda_1, \ldots, \lambda_{n-1}, \in \mathbb{C}$ and $\epsilon_1, \ldots, \epsilon_{n-1} \in \{1, -1\}$ such that $|\lambda_1| \neq |\lambda_2|$, $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and

$$C = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}) \quad \text{and} \quad H_\ell = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_{n-1}).$$

Accordingly, by (4.3.12),

$$\alpha_{n+p} e_i = a_i \lambda_{p+1} e_{p+1} - \delta_{i,p+1}\varphi(1) \qquad \forall\, 0 \leq p < n, \tag{4.3.97}$$

$$\alpha_n C H_\ell^{-1} e_i = \lambda_i \varepsilon_i a_i \big(\lambda_1 e_1 - \delta_{i,1}\varphi(1)\big), \tag{4.3.98}$$

$$H^{-1}\alpha_{n+p}^T H e_1 = \pm\varphi_1 e_{p+1} \qquad \forall 0 \leq p < n, \tag{4.3.99}$$

and

$$H^{-1}\alpha_n^T He_{p+1} = \pm\varphi_{p+1}e_1 \qquad \forall 0 < p < n. \tag{4.3.100}$$

By (4.3.11), we can equate $H^{-1}\alpha_n^T He_{p+1}$, and hence (4.3.99) and (4.3.100) yields

$$\varphi_1 = \varphi_2 = \cdots = \varphi_{n-1} = 0. \tag{4.3.101}$$

Formula in (4.3.98) now simplifies giving that $\alpha_n CH_\ell^{-1}$ is a matrix with at most 1 nonzero row, and hence the left side of (4.3.8) (when setting $i = n$) cannot be a diagonal matrix of rank greater than one. Yet the right side of (4.3.8) is a diagonal matrix that is either zero or of rank greater than 1, so the right side of (4.3.8) must be zero for the equation to hold. Since the left side of (4.3.8) is zero, (4.3.98) and (4.3.101) imply that

$$\lambda_1 a_1 = \lambda_2 a_2 = \cdots = \lambda_{n-1}a_{n-1} = 0$$

because $\lambda_1 \neq 0$. In particular,

$$a_1 = a_2 = 0 \tag{4.3.102}$$

because λ_1 and λ_2 are both nonzero.

Lemma 4.3.14. *If (H_ℓ, C) corresponds to a non-regular CR structure and C is diagonal then $\mathfrak{g}_1^{\mathrm{red}} = 0$.*

Proof. Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (4.3.12). Recall that $\big(\varphi(1)\big)_+ = 0$ implies $\varphi(1) = 0$, by the same argument applied at the end of the proof of Lemma 4.3.6, and hence $\varphi(1) = 0$ by (4.3.101). Accordingly, by the same arguments as in the beginning of the proof of Lemma 4.3.8, it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$.

To produce a contradiction, assume that there exists r such that $\kappa_r \neq 0$ and r is the minimal index with this property. By (4.3.102) we have that $r > 2$. Noting (4.3.97), by Lemma 4.3.1, $\kappa_r \neq 0$ implies $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H}_\ell)$. Accordingly, $\kappa_r \neq 0$ yields that $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have at least a 1-dimensional intersection. By (4.3.102) and (4.3.97), nonzero entries in the matrices in $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ can only appear in their first two rows and moreover they do not appear in their first two columns. Yet, in the appendix (Section A below), we describe the matrices in $\mathscr{A}$ explicitly. In particular, given that H_ℓ and C are diagonal, the description of $\mathscr{A}$ in the appendix implies that every matrix in $\mathscr{A}$ with nonzero entries in its first two rows also has nonzero entries in its first two columns, which implies that $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have a trivial intersection, a clear contradiction. $\square$

By combining the results of Lemmas 4.3.8, 4.3.9, 4.3.10, 4.3.13 , and 4.3.14, we finish the proof of Theorem 4.1.4, because these Lemmas account for all non-regular symbols.

To prove item (2) of Theorem 4.1.4 note that by (4.2.3) and Lemma A.0.10 for the reduced modified CR symbol corresponding to a non-regular symbol

$$\dim\, \mathfrak{g}_{0,0}^{\mathrm{red}} = \dim\, \mathscr{A} + 1 < n^2 - 4n + 7$$

Therefore, from item (1) of the theorem under consideration and the fact that $\dim\, \mathfrak{g}_0^{\mathrm{red}} = \dim\, \mathfrak{g}_{0,0}^{\mathrm{red}} + 2$ and $\dim\, \mathfrak{g}_- = 2n-1$, it follows that

$$\dim\, \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}) = \dim\, \mathfrak{g}^{0,\mathrm{red}} < (2n-1) + (n^2 - 4n + 7) + 2 = (n-1)^2 + 7,$$

which together with Theorem 4.1.3 completes the proof of item (2) of Theorem 4.1.4.

5. SECOND APPLICATION: LOCAL INVARIANTS OF HOMOGENEOUS MODELS IN LOW DIMENSIONS

The regular CR symbols of CR structures with a rank 1 Levi kernel were classified in [33], wherein they show that all such regular CR symbols are exhibited by homogeneous CR manifolds. In particular, in the terminology of the present text, if $\mathfrak{g}^0$ is one such regular CR symbol, then we can consider the abstract reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}^0$ and consider the flat CR structure (M_0, H^{flat}) with constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ defined in Section 2.5. For this choice of $\mathfrak{g}^{0,\mathrm{red}}$, the homogeneous model (M_0, H^{flat}) is equivalent to the maximally symmetric homogeneous model with CR symbol $\mathfrak{g}^0$ studied in [33].

As is mentioned in Chapter 2 (and shown in Example 2.7.1), despite the non-existence results for generic symbols associated with homogeneous models derived in Section 2.6, homogeneous models do exist for specific non-regular symbols. The theory developed in chapter 2 prepares a programatic approach to finding such examples, which we implement here. In fact, the original motivation for developing the theory in this dissertation was to prove or disprove the conjecture that *all homogeneous, 2-nondegenerate, hypersurface-type CR manifolds have regular symbols*, which we now know is false (as shown in Example 2.7.1).

In this chapter we classify the CR symbols associated with homogeneous models of dimension 7 and give a partial classification of these symbols for dimension 9. For each CR symbol $\mathfrak{g}^0$ in this classification, we find a corresponding abstract reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ whose associated flat model has the CR symbol $\mathfrak{g}^0$. We furthermore classify all such 7-dimensional flat models (up to local equivalence). We start with the dimension 7 case because it is the lowest dimension in which we find nontrivial local invariants encoded in the CR symbol. Indeed, there is only one CR symbol associated with a 2-nondegenerate, hypersurface-type CR structure of dimension 5, the lowest dimension for which 2-nondegeneracy can occur. Our method centers around analysis of the system of relations in Lemma 2.4.4 with the same modification of replacing $\mathscr{A}$ by $\mathscr{A}_0$ that was applied in Chapter 4 to obtain (4.2.4).

In more detail, throughout this chapter, we let $\mathscr{A}_0$ denote a subalgebra of the algebra $\mathscr{A}$ defined in (2.4.7) invariant under the transformation $\alpha \mapsto \overline{H_\ell}^{-1}\alpha^*\overline{H_\ell}$. This invariance property that we assume corresponds to axiom 4 in the definition of abstract reduced modified CR symbols (Definition 2.5.1). We are concerned only with the abstract reduced modified CR symbols that are *associated with homogeneous models and maximal*, meaning that each of these reduced modified CR symbols can be obtained as the constant reduced modified CR symbol of some reduction of the frame bundle P^0 via the reduction procedure described in Section 2.5. In particular, the reduction procedure ensures that any constant reduced modified CR symbol that it produces is a maximal subspace of its corresponding modified CR symbol that is also a Lie subalgebra of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$, which motivates the following definition.

Definition 5.0.1. *A subspace* $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{red}}$ *of* $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ *for some Heisenberg algebra* $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ *(with its grading as given throughout this text, e.g., as introduced in the paragraph following Remark 2.2.2) is a* reduced modified symbol associated with a homogeneous model *with CR symbol* $\mathfrak{g}^0$ *if*

1. $\mathfrak{g}^{0,\mathrm{red}}$ *is an abstract reduced modified CR symbol of type* $\mathfrak{g}^0$ *in the sense of Definition 2.5.1.*

2. $\mathfrak{g}^{0,\mathrm{red}}$ *is a Lie algebra.*

Furthermore, we say that a reduced modified symbol associated with a homogeneous model is maximal *if*

3. $\mathfrak{g}^{0,\mathrm{red}}$ *is the maximal subspace of* $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ *with the previous two properties.*

Remark 5.0.2. *Finding (or classifying all) maximal reduced modified CR symbols associated with homogeneous* 2*-nondegenerate, hypersurface-type CR manifolds, is equivalent to finding (or classifying all) tuples*

$$(H_\ell, \{C_1, \ldots, C_{n-r}\}, \{\Omega_1, \ldots, \Omega_{n-r}\}, \mathscr{A}_0) \tag{5.0.1}$$

for which the system of relations (2.4.12) *is consistent after replacing $\mathscr{A}$ with $\mathscr{A}_0$, with the added condition that $\mathscr{A}_0$ is a maximal subalgebra of $\mathscr{A}$ for which* (5.0.1) *satisfies* (2.4.12).

Indeed, each solution to (2.4.12) (of the form in (5.0.1)) determines the reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ that, in terms of its matrix representations, can be described as the symbol for which there is a splitting $\mathfrak{g}_0^{\mathrm{red}} = \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,-}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}}$ such that $\mathfrak{g}_{0,0}$ is as given in (4.2.3), $\mathfrak{g}_{0,-}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}}$ is spanned by the matrices in (2.4.9), and the symbol's corresponding reduced Levi form (which determines the algebra's structure equations on the negatively graded part of $\mathfrak{g}^{0,\mathrm{red}}$) is represented by H_ℓ.

For each $\mathfrak{g}^{0,\mathrm{red}}$ that we find, the flat model with constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ is uniquely determined by the construction in Section 2.5, so we sort the maximal reduced modified symbols associated with homogeneous models into equivalence classes such that two symbols are equivalent if and only if their corresponding flat models are. In other words we consider the follows equivalence relation.

Definition 5.0.3 (reduced modified CR symbol equivalence)**.** *Letting $\Re G_{0,0}$ denote the maximal subgroup of the linear group $CSp(\mathfrak{g}_{-1})$ whose tangent bundle is contained in the left-invariant distribution generated by*

$$\mathfrak{g}_{0,0} \cap \{v \in \mathfrak{csp}(\mathfrak{g}_{-1}) \,|\, \sigma(v) = v\},$$

we say that two maximal reduced modified CR symbols associated with homogeneous models $\mathfrak{g}^{0,\mathrm{red}}$ and $\widehat{\mathfrak{g}}^{0,\mathrm{red}}$ are equivalent *or* of the same type *if there exists an element g in $\Re G_{0,0}$ such that* $\mathrm{Ad}_g(\mathfrak{g}_0^{\mathrm{red}}) = \widehat{\mathfrak{g}}_0^{\mathrm{red}}$*, where, here,* Ad_g *denotes the standard adjoint action of $CSp(\mathfrak{g}_{-1})$ on its Lie algebra $\mathfrak{csp}(\mathfrak{g}_{-1})$.*

This equivalence relation comes from the observation that the bundle P^0 (referenced in Section 2.5) is a principal bundle over the complexified flat model $M_0^{\mathbb{C}}$ (also in Section 2.5) with a structure group that we label $G_{0,0}$. Given a maximal reduction $P^{0,\mathrm{red}}$ of P^0 with constant reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$, the action of $G_{0,0}$ on P^0 yields transformations of $P^{0,\mathrm{red}}$, where each group element in $G_{0,0}$ transforms $P^{0,\mathrm{red}}$ into a different reduction of P^0 with its own constant reduced modified CR symbol, and the new symbol obtained in this way is related to the old symbol via the adjoint action of $G_{0,0}$ on $\mathfrak{csp}(\mathfrak{g}_{-1})$. We consider only the orbits of $\Re G_{0,0}$ in Definition (5.0.3) rather than the larger orbits of $G_{0,0}$, because we are searching only for the reduced modified CR symbols that are invariant under the natural involution $\mathfrak{csp}(\mathfrak{g}_{-1})$ coming from the complex structure on P^0, in other words, satisfying axiom 4 in the definition of abstract reduced modified CR symbols (Definition 2.5.1).

For $\dim M = 7$, we classify the equivalence classes determined by Definition (5.0.3), obtaining a total of nine distinct classes, each corresponding to a different homogeneous model. Of these nine models, six where found in [33], and the other three are new. One of these new models is given in Example 2.7.1. Another exhibits the non-regular CR symbol encoded by (H_ℓ, C), where C can be any Jordan matrix consisting of a single nonsingular Jordan block. The third new model corresponds to a regular CR symbol and its symmetry group is strictly smaller than that of the maximally symmetric model having the same CR symbol found in [33].

For $\dim M = 9$, we find nine non-regular CR symbols exhibited by homogeneous models whose Levi kernel has rank 1 and, in particular, obtain reduced modified CR symbols whose flat models indeed exhibit these CR symbols. Additionally there are 11 regular CR symbols exhibited by homogeneous models whose Levi kernel has rank 1, which were classified in [33]. Hence, there are at least twenty locally distinguished 9-dimensional homogenous 2-nondegenerate, hypersurface-type CR manifolds with a rank-1 Levi kernel (there are in fact more because among these CR symbols, some are exhibited by multiple homogeneous models, as is illustrated by Examples 2.7.2 and 2.7.3). This is, however, only summarizing a partial classification. Recall that the matrices H_ℓ and C encoding each such structure's CR symbol represent respectively a Hermitian form ℓ and an antilinear operator A whose domain is 3-dimensional. The partial classification obtained here, is a complete classification for the symbols whose corresponding linear operator A^2 has at most 2 eigenvalues (not counting multiplicity). It remains to classify the symbols associated with 9-dimensional homogenous models having a rank-1 Levi kernel for which A^2 has 3 distinct eigenvalues. Although this latter classification is beyond the scope of this chapter, show that such symbols are generically not associated with homogeneous models, extending the results of Section 2.6 that established this generic nonexistance only for pseudoconvex structures. Specifically, for each possible signature of H_ℓ we find a pair (H_ℓ, C) encoding a CR symbol that cannot be exhibited by a homogeneous model, which, as is described in Remark 2.6.6, implies the generic nonexistance result.

5.1 7-dimensional models

Here we classify the (equivalence classes of) maximal reduced modified CR symbols associated with homogeneous models on 7-dimensional manifolds. We still set $\dim(M) = 2n+1$ and $\operatorname{rank} K = r$, so in this section we have $n = 3$. Note that by (1.0.4) with $n = 3$, we get that $r = 1$, which corresponds to the fact that all 2-nondegenerate 7-dimensional CR manifolds have a rank 1 Levi kernel. Accordingly, each of reduced modified symbols that we are classifying in this section is determined by a tuple $\{H_\ell, C_1, \Omega_1, \mathscr{A}_0\}$. Here we have writen C_1 and Ω_1 to match the notation of (2.4.9), but for convenience let us drop the subscript since they are unnecessary in this case with $r = 1$. Accordingly, we will analyze the simplified system in (4.2.4) rather than the more general system in (2.4.12) (using $\mathscr{A}_0$ in place of $\mathscr{A}$).

Since $n - r = 2$, H_ℓ and C are 2×2 matrices, and, by Theorem 3.1.2, we can assume (after possibly rescaling ℓ and C by different real coefficients) that they have one of the forms

$$H_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for some } \epsilon = \pm 1, \tag{5.1.1}$$

$$H_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = I \quad \text{for some } \epsilon = \pm 1, \tag{5.1.2}$$

$$H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.1.3}$$

$$H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.1.4}$$

$$H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \tag{5.1.5}$$

$$H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & e^{i\theta} \\ 1 & 0 \end{pmatrix} \quad \text{for some } \theta \in \left(0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right), \tag{5.1.6}$$

$$H_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{for some } \epsilon = \pm 1,\ \lambda > 1, \tag{5.1.7}$$

or

$$H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.1.8}$$

These possible forms for the pair (H_ℓ, C) are ordered above to highlight a few key patterns. By Lemma 4.2.2, the CR symbols corresponding to (5.1.2), (5.1.1), (5.1.3), and (5.1.4) are regular and thus of the type classified in [33]. Therefore, as mentioned in the introduction, (5.1.2), (5.1.1), (5.1.3), and (5.1.4) are all associated with homogeneous models, and it remains for us to determine which reduced modified CR symbols exist corresponding to these four cases. We will establish the following result.

Theorem 5.1.1. *Up to local equivalence, there are nine 7-dimensional flat 2-nondegenerate, hypersurface-type CR manifolds (referring to the concept of* flat models *introduced in Section 2.5). In particular:*

1. *There exist three equivalence classes of reduced modified CR symbols corresponding to* (5.1.1)*, one for $\epsilon = 1$ and two for $\epsilon = -1$.*

2. *There exist two equivalence classes of reduced modified CR symbols corresponding to* (5.1.2)*, one for each parameter setting of ϵ.*

3. *There exists one equivalence class of reduced modified CR symbols corresponding to each of the four cases* (5.1.3)*,* (5.1.4)*,* (5.1.5)*, and* (5.1.8)*.*

4. No reduced modified CR symbols correspond to any of the cases in (5.1.6) *and* (5.1.7).

Each of these equivalence classes of reduced modified CR symbols corresponds to one of the aforementioned nine models.

Corollary 5.1.2. *There exist homogeneous 2-nondegenerate, hypersurface-type CR manifolds whose CR symbol is nonregular. Specifically, there are two such maximally symmetric 7-dimensional models and there CR symbols correspond to* (5.1.5) *and* (5.1.8). *Applying Theorem 2.5.2 and Theorem 4.1.4 to the corresponding reduced modified symbols that we obtain in this chapter, one gets that each of these two models has an 8-dimensional symmetry group.*

Moreover, in this section, we will explicitly describe the equivalence classes of reduced modified CR symbols referred to in Theorem 5.1.1. Since our classification goal in this section reduces to describing the tuples $(H_\ell, C, \Omega, \mathscr{A}_0)$ for which the system (4.2.4) is consistent, let us suppose that Ω and $\mathscr{A}_0$ are fixed such that (4.2.4) is satisfied. We also let $\mathfrak{g}^0$ be the CR symbol encoded by the pair (H_ℓ, C), as described in Remark 2.2.5. Depending on the value of (H_ℓ, C) we will either describe this pair $(\Omega, \mathscr{A}_0)$ in more detail or derive a contradiction from our assumption that such a pair exists.

5.1.1 Symbols corresponding to formulas (5.1.1) through (5.1.4)

Suppose that H_ℓ and C are as in (5.1.1), (5.1.2), (5.1.3), or (5.1.4). Since, by Lemma 4.2.2, $\mathfrak{g}^0$ is regular, the algebra $\mathfrak{g}^0$ is itself a reduced modified CR symbol corresponding to the pair (H_ℓ, C). This reduced modified symbol is described by taking $\mathscr{A}_0 = \mathscr{A}$ and taking Ω to be any matrix in $\mathscr{A}$. Hence, all that remains for us to do is determine whether or not there exist reduced modified symbols for which Ω is not in $\mathscr{A}$.

If (H_ℓ, C) is as in (5.1.1), then there turns out to be exactly one equivalence class of solutions with Ω not in $\mathscr{A}$ provided that $\epsilon = -1$, whereas there is no such solution if $\epsilon = 1$. So we record this as a lemma.

Lemma 5.1.3. *Suppose (H_ℓ, C) is as in* (5.1.1). *If $\epsilon = -1$ then there exists exactly one type of reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ (in the sense of Definition 5.0.3) such that the matrix Ω is not in $\mathscr{A}$,* *and if $\epsilon = 1$ then there is no such reduced modified symbol. In the former case, this equivalence class of reduced modified symbols is represented by any one of the symbols described by* (5.1.1) *and*

$$\Omega = e^{i\theta} \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{3}{4}} & 0 \end{pmatrix} \quad \textit{and} \quad \mathscr{A}_0 = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right\} \quad \textit{for some } \theta \in \mathbb{R}. \tag{5.1.9}$$

Proof. Notice that $\mathscr{A}$ is the space of all 2×2 diagonal matrices, and item (iii) in (4.2.4) implies $\Omega_{1,2} = 0$. With $(\Omega_1)_{1,2} = 0$, we get

$$\left[\overline{H_\ell^{-1}\Omega_i^T H_\ell}, \Omega_j\right] = \begin{pmatrix} \epsilon|(\Omega_1)_{2,1}|^2 & \epsilon\overline{(\Omega_1)_{2,1}}((\Omega_1)_{2,2} - (\Omega_1)_{1,1}) \\ \Omega\Omega_{2,1}\overline{(\Omega_{2,2} - \Omega_{1,1})} & -\epsilon|\Omega_{2,1}|^2 \end{pmatrix}. \tag{5.1.10}$$

The coefficient μ in item (iii) of (4.2.4) is equal to $2\Omega_{1,1}$, and hence, by (5.1.10), labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α, we have

$$\alpha = \begin{pmatrix} \epsilon|\Omega_{2,1}|^2 - 4|\Omega_{1,1})|^2 + 1 & \epsilon\overline{\Omega_{2,1}}(\Omega_{2,2} - 3\Omega_{1,1}) \\ \Omega_{2,1}\overline{(\Omega_{2,2} - 3\Omega_{1,1})} & a \end{pmatrix}, \tag{5.1.11}$$

where

$$a = -\epsilon|\Omega_{2,1}|^2 - 2\left(\overline{\Omega_{1,1}}\Omega_{2,2} + \Omega_{1,1}\overline{\Omega_{2,2}}\right).$$

By item (iv) of (4.2.4), α belongs to $\mathscr{A}$, and is therefore diagonal. Since we are searching for a solution with Ω not in $\mathscr{A}$, we can assume that $\Omega_{2,1} \neq 0$, and hence setting the off-diagonal entries in (5.1.11) equal to zero yields,

$$\Omega_{2,2} = 3\Omega_{1,1}.$$

Accordingly

$$\alpha = \begin{pmatrix} \epsilon|\Omega_{2,1}|^2 - 4|\Omega_{1,1})|^2 + 1 & 0 \\ 0 & -\epsilon|\Omega_{2,1}|^2 - 12|\Omega_{1,1})|^2 \end{pmatrix}. \tag{5.1.12}$$

Evaluating item (i) in (4.2.4) with α given by (5.1.12), we obtain $\eta^1_{\alpha,1} = 2\epsilon|\Omega_{2,1}|^2 - 8|\Omega_{1,1})|^2 + 2$, and since, noting $\Omega_{1,2} = 0$,

$$[\alpha, \Omega] = \begin{pmatrix} 0 & 0 \\ \left(-2\epsilon|\Omega_{2,1}|^2 - 8|\Omega_{1,1})|^2 - 1\right)\Omega_{2,1} & 0 \end{pmatrix},$$

$(2,1)$ entry of $[\alpha, \Omega] - \eta^1_{\alpha,1}\Omega$ is equal to

$$\left([\alpha, \Omega] - \eta^1_{\alpha,1}\Omega\right)_{2,1} = -\left(4\epsilon|\Omega_{2,1}|^2 + 3\right)\Omega_{2,1}. \tag{5.1.13}$$

By item (ii) in (4.2.4), $[\alpha, \Omega] - \eta^1_{\alpha,1}\Omega$ belongs to $\mathscr{A}$, and hence $\left([\alpha, \Omega] - \eta^1_{\alpha,1}\Omega\right)_{2,1} = 0$. If $\epsilon = 1$ then we have obtained a contradiction because then the value in (5.1.13) is nonzero. Accordingly, if $\Omega \notin \mathscr{A}$ then $\epsilon = -1$. Accordingly, setting (5.1.13) equal to zero with $\epsilon = -1$, we get

$$|\Omega_{2,1}|^2 = \frac{3}{4}. \tag{5.1.14}$$

By (5.1.12) and (5.1.14)

$$\alpha = \begin{pmatrix} -4|\Omega_{1,1}|^2 - \frac{1}{4} & 0 \\ 0 & 3\left(-4|\Omega_{1,1}|^2 - \frac{1}{4}\right) \end{pmatrix},$$

and hence

$$\operatorname{span}\left\{\frac{\alpha}{-4|\Omega_{1,1}|^2 - \frac{1}{4}}\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right\} \subset \mathscr{A}_0. \tag{5.1.15}$$

Notice that I is not in $\mathscr{A}_0$ because then items (i) and (ii) of (4.2.4) would imply that Ω is in $\mathscr{A}_0$, so equality actually holds in (5.1.15), which together with (5.1.14) implies that (5.1.1) and (5.1.9) indeed give a solution to the system (4.2.4).

Lastly, we need to show that changing the parameter θ in (5.1.9) does not change the equivalence class represented by the corresponding reduced modified CR symbol. To see this last observation, consider the 1-parameter subgroup

$$\left\{ \left(\begin{array}{cc} e^{it}I & 0 \\ 0 & e^{-it}I \end{array} \right) \middle| \, t \in \mathbb{R} \right\} \tag{5.1.16}$$

of $CSp(\mathfrak{g}_{-1})$. This subgroup belongs to the group $\Re G_{0,0}$ (introduced in Definition 5.0.3) and it acts transitively (via the natural adjoint action) on the set of reduced modified symbols parameterized by θ described by (5.1.1) and (5.1.9). □

Lemma 5.1.4. *If (H_ℓ, C) is as in* (5.1.2)*,* (5.1.3) *or* (5.1.4) *then the only corresponding reduced modified symbol is the one described by taking $\mathscr{A}_0 = \mathscr{A}$ and taking Ω to be any matrix in $\mathscr{A}$.*

Proof. If (H_ℓ, C) is as in (5.1.2), it is easily checked that item (iii) in (4.2.4) implies that both of the set inclusion conditions in (4.2.1) are satisfied by setting $\alpha = \Omega$. In other words, if (H_ℓ, C) is as in (5.1.2) then Ω is in $\mathscr{A}$.

This also happens if (H_ℓ, C) is as in (5.1.4) instead by exactly the same calculation, which is clear because if (H_ℓ, C) is as in (5.1.4) then CH_ℓ^{-1} and $H_\ell\overline{C}$ are the same in this case as they are in when (5.1.2) holds.

Similarly, if (H_ℓ, C) is as in (5.1.3) then $\mathscr{A}$ is the space of 2×2 upper-triangular matrices, and item (iii) in (4.2.4) implies $\Omega_{2,1} = 0$. In other words, if (H_ℓ, C) is as in (5.1.3) then again we get that Ω is in $\mathscr{A}$. □

5.1.2 Symbols corresponding to formulas (5.1.5) through (5.1.8)

In each of these cases (i.e., in (5.1.5) through (5.1.8)), $\mathscr{A}$ is spanned by the identity matrix I. If I is in $\mathscr{A}_0$ then items (i) and (ii) of (4.2.4) imply that Ω is in $\mathscr{A}_0$, which contradicts Lemma 4.2.2, so

$$\mathscr{A}_0 = 0. \tag{5.1.17}$$

Proceeding, suppose first that H_ℓ and C are as in (5.1.5) and (5.1.6). To treat both cases with common formulas, for the case where H_ℓ and C are as in (5.1.5), we set $\theta = \frac{\pi}{2}$ so that C is described by the same formula as in (5.1.6). Item (iii) in (4.2.4) implies

$$\Omega_{1,1} = \Omega_{2,2}, \quad \Omega_{1,2} = -e^{-i\theta}\Omega_{2,1}, \quad \text{and} \quad \mu = 2\Omega_{1,1}. \tag{5.1.18}$$

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.1.18) to simplify α, we obtain

$$\alpha_{1,1} = e^{i\theta} - 4|\Omega_{1,1}|^2 + (e^{-i\theta} - e^{i\theta})|\Omega_{2,1}|^2. \tag{5.1.19}$$

Since item (iv) of (4.2.4) gives that α belongs to $\mathscr{A}_0$, by (5.1.17), the value in (5.1.19) is equal to zero, and hence

$$\Omega_{1,1} = 0 \quad \text{and} \quad |\Omega_{2,1}|^2 = \frac{e^{i\theta}}{e^{i\theta} - e^{-i\theta}}. \tag{5.1.20}$$

Since $0 < \theta < \pi$ and $0 \leq |\Omega_{2,1}|$, (5.1.20) implies that $\theta = \frac{\pi}{2}$, and hence the system (4.2.4) is inconsistent if (H_ℓ, C) is as in (5.1.6), which yields the following result.

Lemma 5.1.5. *There are no reduced modified CR symbols corresponding to any of the cases in* (5.1.6)*.*

Lemma 5.1.6. *There exists exactly one equivalence class of reduced modified symbols $\mathfrak{g}^{0,\mathrm{red}}$ (in the sense of Definition 5.0.3) corresponding to the case where (H_ℓ, C) is as in* (5.1.5)*. This equivalence class of reduced modified symbols is represented by any one of the symbols described by* (5.1.5) *and*

$$\Omega = e^{i\theta}\begin{pmatrix} 0 & i\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{pmatrix} \quad \textit{and} \quad \mathscr{A}_0 = 0 \quad \textit{for some } \theta \in \mathbb{R}. \tag{5.1.21}$$

Proof. By (5.1.17), (5.1.18), and (5.1.20), if $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies (4.2.4) with (H_ℓ, C) as in (5.1.5) then, indeed (5.1.21) holds. Conversely, if $(H_\ell, C, \Omega, \mathscr{A}_0)$ is as in (5.1.5) and (5.1.21) then it is straightforward to check that the system (4.2.4) is consistent.

We finish this proof using the same conclusion as in the proof of Lemma 5.1.3. That is, the 1-parameter subgroup of $CSp(\mathfrak{g}_{-1})$ given in (5.1.16) acts transitively on the set of reduced modified symbols parameterized by θ described by (5.1.5) and (5.1.21). In other words, as θ varies in (5.1.21) the corresponding reduced modified CR symbols belong to the same equivalence class (in the sense of Definition 5.0.3). $\square$

The flat model corresponding to the class of reduced modified CR symbols of Lemma 5.1.6 is described in detail in Example 2.7.1.

The following lemmas address the cases in (5.1.7) and (5.1.8).

Lemma 5.1.7. *There are no reduced modified CR symbols corresponding to either of the cases (i.e., $\epsilon = 1$ and $\epsilon = -1$) in* (5.1.7).

Proof. Item (iii) in (4.2.4) implies

$$\Omega_{1,1} = \Omega_{2,2}, \quad \Omega_{1,2} = -\epsilon\lambda\Omega_{2,1}, \quad \text{and} \quad \mu = 2\Omega_{1,1}. \tag{5.1.22}$$

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.1.22) to simplify α, we obtain

$$\alpha_{1,2} = 2\epsilon(\lambda\overline{\Omega_{1,1}}\Omega_{2,1} - \Omega_{1,1}\overline{\Omega_{2,1}}) \quad \text{and} \quad \alpha_{2,1} = -2(\overline{\Omega_{1,1}}\Omega_{2,1} - \lambda\Omega_{1,1}\overline{\Omega_{2,1}}) \tag{5.1.23}$$

and

$$\alpha_{1,1} = 1 - 4|\Omega_{1,1}|^2 + \epsilon(1-\lambda^2)|\Omega_{2,1}|^2 \text{ and } \alpha_{2,2} = \lambda^2 - 4|\Omega_{1,1}|^2 - \epsilon(1-\lambda^2)|\Omega_{2,1}|^2 \tag{5.1.24}$$

Since item (iv) of (4.2.4) gives that α belongs to $\mathscr{A}_0$, by (5.1.17), the values in (5.1.23) and (5.1.24) are equal to zero. Accordingly,

$$(\lambda^2 - \epsilon)\Omega_{1,1}\overline{\Omega_{2,1}} = \frac{\alpha_{1,2} + \lambda\alpha_{2,1}}{2} = 0,$$

which implies that either $\Omega_{1,1} = 0$ or $\Omega_{2,1} = 0$ because $\lambda^2 \neq 1$. If $\Omega_{2,1} = 0$ then Ω a multiple of the identity, which implies that $\Omega \in \mathscr{A}$, contradicting Lemma 4.2.2. Therefore, $\Omega_{1,1} = 0$. Yet if $\Omega_{1,1} = 0$, since $\alpha_{1,1} = \alpha_{2,2} = 0$, the two equations in (5.1.24) respectively imply

$$1 = -\epsilon(1-\lambda^2)|\Omega_{2,1}|^2 \quad \text{and} \quad \lambda^2 = \epsilon(1-\lambda^2)|\Omega_{2,1}|^2$$

which contradicts the fact that λ is real. $\square$

Lemma 5.1.8. *There exists exactly one equivalence class of reduced modified symbols* $\mathfrak{g}^{0,\mathrm{red}}$ *(in the sense of Definition 5.0.3) corresponding to the case where* (H_ℓ, C) *is as in* (5.1.8)*. This equivalence class of reduced modified symbols is represented by any one of the symbols described by* (5.1.8) *and*

$$\Omega = e^{i\theta}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathscr{A}_0 = 0 \quad \textit{for some } \theta \in \mathbb{R}. \tag{5.1.25}$$

Proof. Item (iii) in (4.2.4) implies

$$\Omega_{1,1} = 2\Omega_{1,2} + \Omega_{2,2}, \quad \Omega_{2,1} = 0, \quad \text{and} \quad \mu = 2\left(\Omega_{1,2} + \Omega_{2,2}\right). \tag{5.1.26}$$

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.1.26) to simplify α, we obtain

$$\alpha_{1,2} = -2\left(\Omega_{1,2}\left(\overline{\Omega_{1,2} + \Omega_{2,2}}\right) + \overline{\Omega_{1,2}}\left(3\Omega_{1,2} + \Omega_{2,2}\right) - 1\right) \tag{5.1.27}$$

and

$$\begin{aligned} \alpha_{1,1} &= 1 - 2\overline{\Omega_{2,2}}\left(\Omega_{1,2} + \Omega_{2,2}\right) - 2\left(\overline{\Omega_{1,2} + \Omega_{2,2}}\right)\left(2\Omega_{1,2} + \Omega_{2,2}\right) \\ &= \left(1 - 4\left|\Omega_{1,2}\right|^2\right) - 2\left(\overline{\Omega_{2,2}}\Omega_{1,2} + \Omega_{2,2}\overline{\Omega_{1,2}}\right) - 2\left|\Omega_{2,2}\right|^2 - 4\overline{\Omega_{2,2}}\Omega_{1,2}. \end{aligned} \tag{5.1.28}$$

Since item (iv) of (4.2.4) gives that α belongs to $\mathscr{A}_0$, by (5.1.17), $\alpha = 0$. Setting the value in

(5.1.27) equal to zero is equivalent to

$$\Omega_{2,2}\overline{\Omega_{1,2}} + \overline{\Omega_{2,2}}\Omega_{1,2} = 1 - 4\left|\Omega_{1,2}\right|^2. \tag{5.1.29}$$

Setting $\alpha_{1,1} = 0$ and applying (5.1.29) to simplify (5.1.28), we obtain

$$\begin{aligned} 0 &= \left(1 - 4\left|\Omega_{1,2}\right|^2\right) - 2\left(\overline{\Omega_{2,2}}\Omega_{1,2} + \Omega_{2,2}\overline{\Omega_{1,2}}\right) - 2\left|\Omega_{2,2}\right|^2 - 4\overline{\Omega_{2,2}}\Omega_{1,2} \\ &= -\left(1 - 4\left|\Omega_{1,2}\right|^2\right) - 2\left|\Omega_{2,2}\right|^2 - 4\overline{\Omega_{2,2}}\Omega_{1,2}. \end{aligned} \tag{5.1.30}$$

Therefore, $\overline{\Omega_{2,2}}\Omega_{1,2}$ is a real number and (5.1.29) implies

$$\Omega_{2,2} = \frac{1 - 4\left|\Omega_{1,2}\right|^2}{2\overline{\Omega_{1,2}}}. \tag{5.1.31}$$

Together (5.1.30) and (5.1.31) imply

$$\frac{\left(1 - 4\left|\Omega_{1,2}\right|^2\right)^2}{2\left|\Omega_{1,2}\right|^2} = 2\left|\Omega_{2,2}\right|^2 = -\left(1 - 4\left|\Omega_{1,2}\right|^2\right) - 4\overline{\Omega_{2,2}}\Omega_{1,2} = -3\left(1 - 4\left|\Omega_{1,2}\right|^2\right),$$

which is equivalent to

$$0 = \left(1 - 4\left|\Omega_{1,2}\right|^2\right)^2 + 6\left(1 - 4\left|\Omega_{1,2}\right|^2\right)\left|\Omega_{1,2}\right|^2 = (1 - 4\left|\Omega_{1,2}\right|^2)(2\left|\Omega_{1,2}\right|^2 + 1). \tag{5.1.32}$$

By (5.1.31) and (5.1.32),

$$\left|\Omega_{1,2}\right|^2 = \frac{1}{4} \quad \text{and} \quad \Omega_{2,2} = 0. \tag{5.1.33}$$

Therefore, noting (5.1.27) and (5.1.33), if $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies (4.2.4) with (H_ℓ, C) as in (5.1.8) then (5.1.25) holds. Conversely, if $(H_\ell, C, \Omega, \mathscr{A}_0)$ is as in (5.1.8) and (5.1.25) then it is straightforward to check that the system (4.2.4) is consistent.

We finish this proof using the same conclusion as in the proof of Lemma 5.1.3. That is, the 1-

parameter subgroup of $CSp(\mathfrak{g}_{-1})$ given in (5.1.16) acts transitively on the set of reduced modified symbols parameterized by θ described by (5.1.8) and (5.1.21). In other words, as θ varies in (5.1.25) the corresponding reduced modified CR symbols belong to the same equivalence class (in the sense of Definition 5.0.3). $\square$

5.2 9-dimensional models with a rank 1 Levi kernel

Here we partially classify the CR symbols exhibited by homogeneous models on 9-dimensional manifolds with a rank 1 Levi kernel. Specifically, for a CR symbol encoded by the pair (H_ℓ, C) representing the Hermitian form ℓ and ℓ-self-adjoing antilinear operator A obtained from a CR structure on a 9-dimensional manifold with a rank 1 Levi kernel, we classify all such symbols that can exhibited by homogeneous model, with the the additional constraint that linear operator A^2 has at most 2 eigenvalues. We

For each CR symbol $\mathfrak{g}^0$ in this partial classification we also find at least one abstract reduced modified CR symbol whose corresponding flat model (defined in Section 2.5) has $\mathfrak{g}^0$ as its CR symbol. This classification is formally less ambitious than the one carried out in section 5.1 in the sense that we are not attempting to classify all reduced modified CR symbols here. Accordingly, for simplicity, we will not aim to find abstract reduced modified CR symbols that satisfy the maximality condition (i.e., condition 3) in Definition 5.0.1. Indeed, imposing this maximality condition is useful primarily for classifying the flat models associated with reduced modified symbols. For example, the maximality condition insures that the classes of symbols referred to in Theorem 5.1.1 indeed each correspond to a different homogeneous model, whereas an analogous classification of abstract symbols satisfying Definition 5.0.1 without the maximality condition would yield several different equivalence classes of abstract reduced modified CR symbols sharing a common corresponding flat model.

Our approach will be similar to that in Section 5.1. Specifically, for a given pair (H_ℓ, C), if a CR symbol encoded by the pair $(\mathbb{R}H_\ell, \mathbb{C}C)$, as described in Remark 2.2.5, is exhibited by a homogeneous model then there exists Ω and $\mathscr{A}_0$ such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies the system of relations in (4.2.4) because the reduction procedure described in Section 2.5 can be applied to this homogeneous model to obtain an associated constant reduced modified CR symbol encoded in a tuple of the form $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfying (4.2.4). In other words, our classification goal in this section reduces to finding all pairs (H_ℓ, C) such that there exists a tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfying (4.2.4). We will need the following lemma.

Lemma 5.2.1. *If the tuple* $(H_\ell, C, \Omega, \mathscr{A}_0)$ *satisfies* (4.2.4) *then so does* $(H_\ell, C, e^{i\theta}\Omega, \mathscr{A}_0)$ *for all* $\theta \in \mathbb{R}$.

Proof. This is essentially immediate after replacing Ω in (4.2.4) with $e^{i\theta}\Omega$. Specifically, the parameter μ in item (iii) and (iv) of (4.2.4) is consequently replaced with $e^{i\theta}\mu$; so item (iii) clearly still holds after this replacement, whereas item (iv) remains unchanged after this replacement (and some obvious simplification). The matrix $[\alpha, \Omega] - \eta_\alpha \Omega$ in item (ii) of (4.2.4) is rescaled by $e^{i\theta}$ after this replacement, so item (ii) of (4.2.4) clearly still holds after this replacement. Lastly, item (i) of (4.2.4) is independent of Ω, so it too is unaffected by this replacement. $\square$

Since $\dim(M) = 9$ and $\operatorname{rank} K = 1$, H_ℓ and C are 3×3 matrices. It is shown in [33] that all regular symbols are exhibited by homogeneous models, so what remains to for us to do is find the non-regular symbols exhibited by homogeneous models. The regular symbols for our present special case were classified in [33], and taking H_ℓ and C to be in the canonical form of Theorem 3.1.2, it follows from Lemma 4.2.2 that the (H_ℓ, C) encodes a regular CR symbol if and only if either C is nilpotent or C is diagonal with exactly one nonzero eigenvale (and possibly an eigenvalue of zero as well). Accordingly, by Theorem 3.1.2, if (H_ℓ, C) encodes a non-regular CR symbol then we can assume (after possibly rescaling ℓ and C by different real coefficients) that they have one of the forms

$$H_\ell = \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{H'_ℓ}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & \epsilon \end{array}\right) \quad \text{and} \quad C = \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{C'}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right) \quad \text{for some } \epsilon = \pm 1, \tag{5.2.1}$$

where (H'_ℓ, C') is a pair of 2×2 matrices having one of the forms appearing for the pair (H_ℓ, C)

in (5.1.1) through (5.1.8),

$$H_\ell = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \begin{array}{l} \text{for some } \epsilon = \pm 1 \\ \text{and } 0 < \lambda, \end{array} \tag{5.2.2}$$

$$H_\ell = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & e^{i\theta} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \begin{array}{l} \text{for some } \epsilon = \pm 1,\, 0 < \lambda, \\ \text{and } \theta \in (0, \pi], \end{array} \tag{5.2.3}$$

$$H_\ell = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon' \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \begin{array}{l} \text{for some } \epsilon, \epsilon' = \pm 1 \\ \text{and } 1 < \lambda, \end{array} \tag{5.2.4}$$

$$H_\ell = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.5}$$

and

$$H_\ell = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } \epsilon = \pm 1, \tag{5.2.6}$$

or, alternatively, C is diagonal and has three distinct eigenvalues.

5.2.1 Symbols corresponding to formula (5.2.1)

In Section 5.1 we classified the maximal reduced modified CR symbols (and therefore all CR symbols) associated with 7-dimensional homogeneous models. Each of the reduced modified CR symbols in that classification can be used to construct a 9-dimensional homogeneous model in accord with the following lemma.

Lemma 5.2.2. *If the tuple $\{H'_\ell, C', \Omega', \mathscr{A}'_0\}$ encodes a reduced modified CR symbol associated with a 7-dimensional homogeneous model (in the sense of Remark 5.0.2 and the paragraph following* *it) then the tuple* $\{H_\ell, C, \Omega, \mathscr{A}_0\}$ *defined by* (5.2.1) *and*

$$\Omega = \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{Ω'}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right) \quad \textit{and} \quad \mathscr{A}_0 = \left\{ \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{α}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right) \,\middle|\, \alpha \in \mathscr{A}_0' \right\}$$

encodes a reduced modified CR symbol associated with a 9*-dimensional homogeneous model (i.e., it satisfies system* (4.2.4)*).*

Proof. This follows almost immediately from the observation that the system (4.2.4) is consistent if we replace $\{H_\ell, C, \Omega, \mathscr{A}_0\}$ by $\{H_\ell', C', \Omega', \mathscr{A}_0'\}$. □

Lemma 5.2.2 does not necessarily, however, describe all CR symbols of the form in (5.2.1) that are associated with homogeneous models. Working toward a description of all such symbols, we prove the following lemma.

Lemma 5.2.3. *Each CR symbol encoded by a pair* (H_ℓ, C) *of the form in* (5.2.1) *where* (H_ℓ', C') *has the form of the pair of matrices in* (5.1.6) *is not associated with a homogeneous model.*

Proof. To produce a contradiction, suppose that Ω and $\mathscr{A}_0$ are such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies system (4.2.4). In this case, the algebra $\mathscr{A}$ in (2.4.7) is given by

$$\mathscr{A} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}, \tag{5.2.7}$$

which can be verified directly from (2.4.7) and is derived explicitly in Appendix A. Let Ω' denote the upper left 2×2 block of Ω, and let $\mathscr{A}_0'$ denote the matrix algebra of 2×2 matrices spanned by the identity matrix. Notice that by Lemma 5.1.5, the system in (4.2.4) is not consistent if we replace $(H_\ell, C, \Omega, \mathscr{A}_0)$ by $(H_\ell', C', \Omega', \mathscr{A}_0')$ because, by Lemma 5.1.5, the CR symbol encoded by (H_ℓ', C') does not correspond to a homogeneous model.

Item (iii) in (4.2.4) implies

$$\Omega_{1,1} = \Omega_{2,2}, \quad \Omega_{1,3} = \Omega_{2,3} = 0, \quad \Omega_{1,2} = -e^{i\theta}\Omega_{2,1}, \quad \text{and} \quad \mu = 2\Omega_{1,1}. \tag{5.2.8}$$

It follows that

$$(\Omega_{3,1}, \Omega_{3,2}) \neq (0,0) \tag{5.2.9}$$

because otherwise every matrix in the system in (4.2.4) is block diagonal, from which our assumption that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies (4.2.4) implies that (4.2.4) is consistent after replacing $(H_\ell, C, \Omega, \mathscr{A}_0)$ by $(H'_\ell, C', \Omega', \mathscr{A}'_0)$, a contradiction.

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.2.8) to simplify α, we obtain

$$\alpha_{1,3} = \epsilon\left(e^{i\theta}\overline{\Omega_{3,1}}\Omega_{2,1} + \overline{\Omega_{3,2}}\left(\Omega_{3,3} - 3\Omega_{1,1}\right)\right), \tag{5.2.10}$$

$$\alpha_{2,3} = \epsilon\left(-\overline{\Omega_{3,2}}\Omega_{2,1} + \overline{\Omega_{3,1}}\left(\Omega_{3,3} - 3\Omega_{1,1}\right)\right), \tag{5.2.11}$$

$$\alpha_{1,1} = e^{i\theta} - 4\left|\Omega_{1,1}\right|^2 + \epsilon\overline{\Omega_{3,2}}\Omega_{3,1}, \quad \text{and} \quad \alpha_{2,2} = e^{i\theta} - 4\left|\Omega_{1,1}\right|^2 + \epsilon\overline{\Omega_{3,1}}\Omega_{3,2}. \tag{5.2.12}$$

By item (iv) of (4.2.4) and (5.2.7) $\alpha_{1,1} = \alpha_{2,2}$, and hence (5.2.12) implies

$$\overline{\Omega_{3,2}}\Omega_{3,1} = \overline{\Omega_{3,1}}\Omega_{3,2}. \tag{5.2.13}$$

Also by item (iv) of (4.2.4) and (5.2.7) $\alpha_{1,3} = \alpha_{2,3} = 0$, and hence (5.2.10) and (5.2.11) imply

$$\overline{\Omega_{3,1}}\Omega_{2,1} = -e^{-i\theta}\overline{\Omega_{3,2}}\left(\Omega_{3,3} - 3\Omega_{1,1}\right) \quad \text{and} \quad \overline{\Omega_{3,2}}\Omega_{2,1} = \overline{\Omega_{3,1}}\left(\Omega_{3,3} - 3\Omega_{1,1}\right). \tag{5.2.14}$$

Multiplying terms of the equations in (5.2.14) by either $\Omega_{3,2}$ or $\Omega_{3,1}$ yields

$$\overline{\Omega_{3,1}}\Omega_{3,2}\Omega_{2,1} = -e^{-i\theta}\left|\Omega_{3,2}\right|^2\left(\Omega_{3,3} - 3\Omega_{1,1}\right) \quad \text{and} \quad \overline{\Omega_{3,2}}\Omega_{3,1}\Omega_{2,1} = \left|\Omega_{3,1}\right|^2\left(\Omega_{3,3} - 3\Omega_{1,1}\right),$$

and hence, by (5.2.13),

$$-e^{-i\theta}\left|\Omega_{3,2}\right|^2\left(\Omega_{3,3}-3\Omega_{1,1}\right)=\left|\Omega_{3,1}\right|^2\left(\Omega_{3,3}-3\Omega_{1,1}\right),$$

which leaves two possibilities, namely either $-e^{-i\theta}\left|\Omega_{3,2}\right|^2=\left|\Omega_{3,1}\right|^2$ or

$$\Omega_{3,3}=3\Omega_{1,1}. \tag{5.2.15}$$

Note that, since $\theta\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$, $-e^{-i\theta}\left|\Omega_{3,2}\right|^2=\left|\Omega_{3,1}\right|^2$ would imply $\Omega_{3,1}=\Omega_{3,2}=0$, contradicting (5.2.9). Therefore (5.2.17) indeed holds.

By (5.2.9), (5.2.14), and (5.2.17), we get $\Omega_{2,1}=0$, so, noting (5.2.8),

$$\Omega_{1,2}=\Omega_{2,1}=\Omega_{1,3}=\Omega_{2,3}=0. \tag{5.2.16}$$

Applying (5.2.17) to simplify α, we get

$$\alpha_{1,2}=\epsilon\left|\Omega_{3,1}\right|^2 \quad \text{and} \quad \alpha_{2,1}=\epsilon\left|\Omega_{3,2}\right|^2. \tag{5.2.17}$$

Since, by item (iv) in (4.2.4), $\alpha\in\mathscr{A}$, (5.2.7) implies $\alpha_{1,2}=\alpha_{2,1}=0$, and hence (5.2.17) implies $\Omega_{3,1}=\Omega_{3,2}=0$, contradicting (5.2.12). $\square$

Corollary 5.2.4. *If a CR symbol encoded by a pair* (H_ℓ,C) *of the form in* (5.2.1) *is associated with a homogeneous model then either the pair* (H'_ℓ,C') *encodes a CR symbol associated with some* 7*-dimensional homogeneous model (and is therefore of the form classified in Section 5.1) or* (H'_ℓ,C') *has the form of the pair in* (5.1.7)*.*

Proof. Suppose that a CR symbol encoded by a pair (H_ℓ,C) of the form in (5.2.1) is associated with a homogeneous model. By Theorem 5.1.1, the pairs of matrices in each formula among (5.1.1) through (5.1.5) as well as (5.1.8) encode a CR symbol associated with some 7-dimensional homogeneous model, so, by Lemma (5.2.2), (H'_ℓ,C') could be of one of the form in (5.1.1) through

(5.1.5) as well as (5.1.8). By Lemma 5.2.5, (H'_ℓ, C') cannot be of the form in (5.1.6), so the remaining possibility is that (H'_ℓ, C') is of the form in (5.1.7). $\square$

5.2.2 Symbols corresponding to formula (5.2.2)

There are exactly two CR symbols encoded by a pair (H_ℓ, C) of the form in (5.2.2) that are associated with a homogeneous model. In particular, the system in (4.2.4) is satisfied by the tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.2) with $\lambda = 1$ and

$$\Omega = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathscr{A}_0 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & -\epsilon a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}. \tag{5.2.18}$$

Note that if (H_ℓ, C) is of the form in (5.2.2) with $\lambda = 1$ then $\mathscr{A}_0$ as defined in (5.2.63) is indeed a subalgebra of $\mathscr{A}$ as is required; specifically, in this case $\mathscr{A}$ is the algebra spanned by $\mathscr{A}_0$ and the identity matrix, which can be verified directly from (2.4.7) and is also derived explicitly in Appendix A.

Hence this tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.2) with $\lambda = 1$ and (5.2.18) encodes a reduced modified CR symbol whose flat model exhibits the CR symbol encoded by (H_ℓ, C). There are two choices for the parameter ϵ in (5.2.2), so the construction works for each of the two corresponding CR symbols.

All other CR symbols encoded by a pair (H_ℓ, C) of the form in (5.2.2) are not associated with a homogeneous model, which is the content of the following lemma.

Lemma 5.2.5. *Each CR symbol encoded by a pair (H_ℓ, C) of the form in* (5.2.2) *with $\lambda \neq 1$ is not associated with a homogeneous model.*

Proof. Let (H_ℓ, C) be of the form in (5.2.4) with $\lambda \neq 1$, and suppose that Ω and $\mathscr{A}_0$ are such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies system (4.2.4).

Item (iii) in (4.2.4) implies

$$\Omega_{2,1} = 0, \quad \Omega_{2,2} = \Omega_{1,1} - 2\Omega_{1,2}, \quad \Omega_{1,3} = \epsilon\lambda \left(\Omega_{3,1} - \Omega_{3,2}\right), \tag{5.2.19}$$

$$\Omega_{2,3} = -\epsilon\lambda\Omega_{3,1}, \quad \Omega_{3,3} = \Omega_{1,1} - \Omega_{1,2}, \quad \text{and} \quad \mu = 2\left(\Omega_{1,1} - \Omega_{1,2}\right). \tag{5.2.20}$$

Since $\lambda \neq 0$, the algebra $\mathscr{A}$ in (2.4.7) is given by

$$\mathscr{A} = \operatorname{span}_{\mathbb{C}}\left\{I\right\}, \tag{5.2.21}$$

which can be verified directly from (2.4.7) and is also derived explicitly in Appendix A.

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.2.19) and (5.2.20) to simplify α, we obtain

$$\alpha_{2,1} = \epsilon\left(1 - \lambda^2\right)\left|\Omega_{3,1}\right|^2. \tag{5.2.22}$$

By item (iv) is (4.2.4), $\alpha \in \mathscr{A}$, and hence, by (5.2.21), $\alpha_{2,1} = 0$, which, by (5.2.22), implies

$$\Omega_{3,1} = 0. \tag{5.2.23}$$

Applying (5.2.19), (5.2.20), and (5.2.23) to simplify α, we obtain

$$\alpha_{1,1} = 1 + 2\overline{\Omega_{1,2}}\left(3\Omega_{1,1} - 2\Omega_{1,2}\right) - 2\overline{\Omega_{1,1}}\left(2\Omega_{1,1} - \Omega_{1,2}\right), \tag{5.2.24}$$

$$\alpha_{2,2} = 1 + 2\overline{\Omega_{1,2}}\left(\Omega_{1,1} - 2\Omega_{1,2}\right) - 2\overline{\Omega_{1,1}}\left(2\Omega_{1,1} - 3\Omega_{1,2}\right), \tag{5.2.25}$$

and

$$\alpha_{3,3} = \lambda^2 + 4\overline{\Omega_{1,2}}\left(\Omega_{1,1} - \Omega_{1,2}\right) - 4\overline{\Omega_{1,1}}\left(\Omega_{1,1} - \Omega_{1,2}\right). \tag{5.2.26}$$

Since $\alpha \in \mathscr{A}$, (5.2.21) implies that $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$. Accordingly, applying (5.2.24), (5.2.25), and (5.2.26) yields

$$0=\alpha_{1,1}-\alpha_{2,2}=4\left(\overline{\Omega_{1,2}}\Omega_{1,1}-\overline{\Omega_{1,1}}\Omega_{1,2}\right) \tag{5.2.27}$$

and

$$0=\alpha_{1,1}-\alpha_{3,3}=1-\lambda^2+2\left(\overline{\Omega_{1,2}}\Omega_{1,1}-\overline{\Omega_{1,1}}\Omega_{1,2}\right). \tag{5.2.28}$$

Yet, this is a contradiction because together (5.2.27) and (5.2.28) imply $\lambda^2=1$, whereas this lemmas hypothesis supposes that λ is a positive number not equal to 1. $\square$

5.2.3 Symbols corresponding to formula (5.2.3)

The CR symbol encoded by the pair (H_ℓ, C) of the form in (5.2.3) with $\theta=\pi$ and $\lambda=\sqrt{3}$ is associated with a homogeneous model. In particular, the system in (4.2.4) is satisfied by the tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.5) and

$$\Omega=\frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1}{2} & -\epsilon & -\sqrt{3}e^{i\phi} \\ -\epsilon & \frac{1}{2} & \sqrt{3}e^{i\phi} \\ \epsilon e^{i\phi} & e^{i\phi} & \frac{1}{2}\end{pmatrix} \quad \text{and} \quad \mathscr{A}_0=0 \qquad \text{for some } \phi\in\mathbb{R}, \tag{5.2.29}$$

where ϵ is the parameter defining H_ℓ in (5.2.3). Hence this tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.3) and (5.2.29) encodes a reduced modified CR symbol whose flat model exhibits the CR symbol encoded by (H_ℓ, C) as in (5.2.3) with $\theta=\pi$, $\lambda=\sqrt{3}$, and $\epsilon=1$.

We will show that there are no other CR symbols encoded by a pair (H_ℓ, C) of the form in (5.2.3) associated with homogeneous models. For this, let (H_ℓ, C) be of the form in (5.2.3), and suppose that Ω and $\mathscr{A}_0$ are such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies system (4.2.4).

Item (iii) in (4.2.4) implies

$$\Omega_{1,1}=\Omega_{2,2}=\Omega_{3,3}, \quad \Omega_{1,2}=-e^{-i\theta}\Omega_{2,1}, \quad \Omega_{1,3}=-\epsilon\lambda\Omega_{3,1}, \quad \text{and} \quad \Omega_{2,3}=-e^{i\theta}\epsilon\lambda\Omega_{3,2}$$

that is

$$\Omega = \begin{pmatrix} \Omega_{1,1} & -e^{-i\theta}\Omega_{2,1} & -\epsilon\lambda\Omega_{3,1} \\ \Omega_{2,1} & \Omega_{1,1} & -e^{i\theta}\epsilon\lambda\Omega_{3,2} \\ \Omega_{3,1} & \Omega_{3,2} & \Omega_{1,1} \end{pmatrix}. \tag{5.2.30}$$

Applying (5.2.30) to simplify item (iii) in (4.2.4) yields

$$\mu = 2\Omega_{1,1}. \tag{5.2.31}$$

Since $\lambda \neq 0$, the algebra $\mathscr{A}$ in (2.4.7) is spanned by I, which can be verified directly from (2.4.7) and is also derived explicitly in Appendix A. If I is in $\mathscr{A}_0$ then items (i) and (ii) of (4.2.4) imply that Ω is in $\mathscr{A}_0$, which contradicts Lemma 4.2.2, so

$$\mathscr{A}_0 = 0. \tag{5.2.32}$$

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.2.30) and (5.2.31) to simplify α, we obtain

$$\operatorname{trace}(\alpha) = e^{i\theta} + e^{-i\theta} + \lambda^2 - 12\left|\Omega_{1,1}\right|^2. \tag{5.2.33}$$

By item (iv) of (4.2.4), $\alpha \in \mathscr{A}_0$, and hence, by (5.2.32), $\alpha = 0$. Setting the value in (5.2.33) equal to zero yields

$$\left|\Omega_{1,1}\right|^2 = \frac{e^{i\theta} + e^{-i\theta} + \lambda^2}{12}. \tag{5.2.34}$$

and

$$\lambda^2 \geq -e^{i\theta} - e^{-i\theta} = -2\cos(\theta). \tag{5.2.35}$$

By Lemma 5.2.1, we can assume without loss of generality that $0 \leq \Omega_{1,1}$. With this assumption, (5.2.34) implies

$$\Omega_{1,1} = \frac{\sqrt{e^{i\theta} + e^{-i\theta} + \lambda^2}}{2\sqrt{3}}. \tag{5.2.36}$$

Now using (5.2.30), (5.2.31), and (5.2.36) to simplify α, we obtain

$$\alpha_{1,1} = -\frac{1}{3}\left(e^{-i\theta} - 2e^{i\theta} + \lambda^2 + 3(e^{i\theta} - e^{-i\theta})\left|\Omega_{2,1}\right|^2 - 3\epsilon\left(1 - e^{-i\theta}\lambda^2\right)\overline{\Omega_{3,2}}\Omega_{3,1}\right). \tag{5.2.37}$$

Since $\alpha_{1,1} = 0$ and $1 - e^{-i\theta}\lambda^2 \neq 0$, (5.2.37) implies

$$\overline{\Omega_{3,2}}\Omega_{3,1} = \epsilon\frac{e^{-i\theta} - 2e^{i\theta} + \lambda^2 + 3(e^{i\theta} - e^{-i\theta})\left|\Omega_{2,1}\right|^2}{-3\left(1 - e^{-i\theta}\lambda^2\right)} \tag{5.2.38}$$

Accordingly, there are two possibilities; either

$$\overline{\Omega_{3,2}}\Omega_{3,1} = 0$$

or

$$\Omega_{3,1} = \epsilon\frac{e^{-i\theta} - 2e^{i\theta} + \lambda^2 + 3(e^{i\theta} - e^{-i\theta})\left|\Omega_{2,1}\right|^2}{-3\left(1 - e^{-i\theta}\lambda^2\right)\overline{\Omega_{3,2}}}.$$

Lemma 5.2.6. *Each CR symbol encoded by a pair (H_ℓ, C) of the form in* (5.2.3) *with $\theta = \pi$ and $\lambda \neq \sqrt{3}$ is not associated with a homogeneous model.*

Proof. Suppose that Ω and $\mathscr{A}_0$ are such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies system (4.2.4) with (H_ℓ, C) of the form in (5.2.3) with $\theta = \pi$ and $\lambda \neq \sqrt{3}$. Applying $\theta = \pi$ to simplify (5.2.38), we obtain

$$\Omega_{3,1} = \epsilon\left(3\overline{\Omega_{3,2}}\right)^{-1}. \tag{5.2.39}$$

With $\theta = \pi$, applying (5.2.30), (5.2.31), (5.2.36), and (5.2.39) to simplify α, we obtain

$$\alpha_{1,2} = -\epsilon \frac{\lambda^2}{9\left|\Omega_{3,2}\right|^2} + \epsilon \left|\Omega_{3,2}\right|^2 - \frac{\sqrt{\lambda^2 - 2}\left(\Omega_{2,1} + \overline{\Omega_{2,1}}\right)}{\sqrt{3}} \tag{5.2.40}$$

and

$$\alpha_{2,1} = \epsilon \frac{1}{9\left|\Omega_{3,2}\right|^2} - \epsilon\lambda^2 \left|\Omega_{3,2}\right|^2 - \frac{\sqrt{\lambda^2 - 2}\left(\Omega_{2,1} + \overline{\Omega_{2,1}}\right)}{\sqrt{3}}. \tag{5.2.41}$$

Since α is in $\mathscr{A}_0$ (and is therefore zero, by (5.2.32)), (5.2.40) and (5.2.41), we have

$$0 = \alpha_{1,2} - \alpha_{2,1} = \epsilon \frac{(1+\lambda^2)(9\left|\Omega_{3,2}\right|^4 - 1)}{9\left|\Omega_{3,2}\right|^2},$$

and hence $\left|\Omega_{3,2}\right| = \frac{1}{\sqrt{3}}$, or equivalently,

$$\Omega_{3,2} = \frac{1}{\sqrt{3}} e^{i\phi} \qquad \text{for some } \phi \in \mathbb{R}. \tag{5.2.42}$$

Applying (5.2.42) to simplify (5.2.41) and using $\alpha_{1,2} = 0$, we get

$$0 = \alpha_{1,2} = \epsilon \frac{1}{3}\left(1 - \lambda^2 - \epsilon 2\sqrt{3}\sqrt{\lambda^2 - 2}\left(\Omega_{2,1} + \overline{\Omega_{2,1}}\right)\right),$$

which implies

$$\Omega_{2,1} = \epsilon \frac{1 - \lambda^2}{2\sqrt{3}\sqrt{\lambda^2 - 2}} + i\Im(\Omega_{2,1}), \tag{5.2.43}$$

where $\Im(\Omega_{2,1})$ denotes the imaginary part of $\Omega_{2,1}$.

Now applying (5.2.42) and (5.2.43) in addition to (5.2.30), (5.2.31), (5.2.36), and (5.2.39) with $\theta = \pi$, to simplify α, we obtain

$$\alpha_{1,3} = \frac{e^{-i\phi}\left(3 - \lambda^2 - \epsilon 2i\sqrt{3}\sqrt{\lambda^2 - 2}\Im(\Omega_{2,1})\right) - \epsilon\lambda e^{i\phi}\left(3 - \lambda^2 + \epsilon 2i\sqrt{3}\sqrt{\lambda^2 - 2}\Im(\Omega_{2,1})\right)}{6\epsilon\sqrt{\lambda^2 - 2}}.$$

Since $\alpha_{1,3} = 0$,

$$e^{-i\phi}\left(3 - \lambda^2 - \epsilon 2i\sqrt{3}\sqrt{\lambda^2 - 2}\Im(\Omega_{2,1})\right) = \epsilon\lambda e^{i\phi}\left(3 - \lambda^2 + \epsilon 2i\sqrt{3}\sqrt{\lambda^2 - 2}\Im(\Omega_{2,1})\right) \tag{5.2.44}$$

By comparing the norms of the values on each side of (5.2.44) we see that if $\lambda \neq 1$ then each side of (5.2.44) must be zero. Accordingly, by (5.2.44), either $\lambda = 1$ or

$$3 - \lambda^2 - \epsilon 2i\sqrt{3}\sqrt{\lambda^2 - 2}\Im(\Omega_{2,1}), \tag{5.2.45}$$

Yet, by (5.2.35) with $\theta = \pi$, $\lambda > \sqrt{2}$, so (5.2.46) holds, which is equivalent to

$$\lambda = \sqrt{3} \quad \text{and} \quad \Im(\Omega_{2,1}) = 0. \tag{5.2.46}$$

This completes the proof of the lemma. □

5.2.4 Symbols corresponding to formula (5.2.4)

Lemma 5.2.7. *There is no CR symbol encoded by a pair (H_ℓ, C) of the form in* (5.2.4) *associated with a homogeneous model.*

Proof. Let (H_ℓ, C) be of the form in (5.2.4), and suppose that Ω and $\mathscr{A}_0$ are such that $(H_\ell, C, \Omega, \mathscr{A}_0)$ satisfies system (4.2.4).

Item (iii) in (4.2.4) implies

$$\Omega_{1,1} = \Omega_{2,2} = \Omega_{3,3}, \quad \Omega_{1,2} = -\epsilon\Omega_{2,1}, \quad \Omega_{1,3} = -\epsilon'\lambda\Omega_{3,1}, \quad \text{and} \quad \Omega_{2,3} = -\epsilon\epsilon'\lambda\Omega_{3,2}$$

that is

$$\Omega = \begin{pmatrix} \Omega_{1,1} & -\epsilon\Omega_{2,1} & -\epsilon'\lambda\Omega_{3,1} \\ \Omega_{2,1} & \Omega_{1,1} & -\epsilon\epsilon'\lambda\Omega_{3,2} \\ \Omega_{3,1} & \Omega_{3,2} & \Omega_{1,1} \end{pmatrix}, \tag{5.2.47}$$

and item (iii) in (4.2.4) yields

$$\mu = 2\Omega_{1,1}. \tag{5.2.48}$$

The algebra $\mathscr{A}$ in (2.4.7) is given by

$$\mathscr{A} = \left\{ \left(\begin{array}{ccc} a & b & 0 \\ -\epsilon b & a & 0 \\ 0 & 0 & a \end{array} \right) \middle| \, a \in \mathbb{C} \right\} \tag{5.2.49}$$

which can be verified directly from (2.4.7) and is also derived explicitly in Appendix A.

Labeling the matrix $\left[\overline{H_\ell^{-1}\Omega^T H_\ell}, \Omega\right] + C\overline{C} - \left(\overline{\mu}\Omega + \mu\overline{H_\ell^{-1}\Omega^T H_\ell}\right)$ in item (iv) of (4.2.4) α and applying (5.2.47) and (5.2.48) to simplify α, we obtain

$$\alpha_{1,1} = 1 - 4\left|\Omega_{1,1}\right|^2 + \epsilon'\left(1-\lambda^2\right)\left|\Omega_{3,1}\right|^2, \tag{5.2.50}$$

$$\alpha_{2,2} = 1 - 4\left|\Omega_{1,1}\right|^2 + \epsilon\epsilon'\left(1-\lambda^2\right)\left|\Omega_{3,2}\right|^2, \tag{5.2.51}$$

$$\alpha_{3,3} = \lambda^2 - 4\left|\Omega_{1,1}\right|^2 - \epsilon'\left(1-\lambda^2\right)\left|\Omega_{3,1}\right|^2 - \epsilon\epsilon'\left(1-\lambda^2\right)\left|\Omega_{3,2}\right|^2,$$

$$\alpha_{1,2} = \alpha_{2,1} = -2\epsilon\overline{\Omega_{2,1}}\Omega_{1,1} + 2\epsilon\overline{\Omega_{1,1}}\Omega_{2,1} - \epsilon'\lambda^2\overline{\Omega_{3,2}}\Omega_{3,1} + \epsilon'\overline{\Omega_{3,1}}\Omega_{3,2},$$

and

$$\alpha_{2,1} = 2\overline{\Omega_{2,1}}\Omega_{1,1} - 2\overline{\Omega_{1,1}}\Omega_{2,1} + \epsilon\epsilon'\overline{\Omega_{3,2}}\Omega_{3,1} - \epsilon\epsilon'\lambda^2\overline{\Omega_{3,1}}\Omega_{3,2}. \tag{5.2.52}$$

Since, by item (iv) in (4.2.4), $\alpha \in \mathscr{A}$, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and hence (5.2.50), (5.2.51), and ch3

9d case f item 4 d imply

$$0=\epsilon'\frac{\alpha_{2,2}-\alpha_{1,1}}{(\lambda^2-1)}=\epsilon\left|\Omega_{3,2}\right|^2-\left|\Omega_{3,1}\right|^2=\frac{3\left|\Omega_{3,1}\right|^2+\epsilon'}{2}-\epsilon'\frac{\alpha_{3,3}-\alpha_{2,2}}{2(\lambda^2-1)}=\frac{3\left|\Omega_{3,1}\right|^2+\epsilon'}{2} \quad (5.2.53)$$

and

$$\begin{aligned}0=\epsilon'\frac{\alpha_{2,2}-\alpha_{1,1}}{(\lambda^2-1)}=\epsilon\left|\Omega_{3,2}\right|^2-\left|\Omega_{3,1}\right|^2=\epsilon'\frac{\alpha_{3,3}-\alpha_{2,2}}{2(\lambda^2-1)}-\frac{\epsilon 3\left|\Omega_{3,2}\right|^2+\epsilon'}{2}\\=-\frac{\epsilon 3\left|\Omega_{3,2}\right|^2+\epsilon'}{2}.\end{aligned} \quad (5.2.54)$$

By (5.2.53),

$$\epsilon'=-1 \quad \text{and} \quad \left|\Omega_{3,1}\right|^2=\frac{1}{3} \quad (5.2.55)$$

which together with (5.2.54) implies that

$$\epsilon=1 \quad \text{and} \quad \left|\Omega_{3,2}\right|^2=\frac{1}{3}. \quad (5.2.56)$$

Note that $\alpha_{1,2}=-\epsilon\alpha_{2,1}$, by (5.2.49), because $\alpha\in\mathscr{A}$, and hence, by (5.2.52), (5.2.52), (5.2.55), and (5.2.56),

$$0=\frac{\alpha_{1,2}+\alpha_{2,1}}{\lambda^2-1}=\overline{\Omega_{3,2}}\Omega_{3,1}+\overline{\Omega_{3,1}}\Omega_{3,2},$$

which together with (5.2.55) and (5.2.56) implies

$$\Omega_{3,2}=\pm i\Omega_{3,1}. \quad (5.2.57)$$

By Lemma 5.2.1, we can assume without loss of generality that $\Omega_{1,1}$ is real. Accordingly, applying (5.2.47), (5.2.48), (5.2.55), (5.2.56), and (5.2.57) to simplify α, we obtain

$$\alpha_{1,3} = 2\Omega_{1,1}\left(\overline{\Omega_{3,1}} - \lambda\Omega_{3,1}\right) \pm \left(i\overline{\Omega_{3,1}}\Omega_{2,1} - \lambda i\overline{\Omega_{2,1}}\Omega_{3,1}\right), \tag{5.2.58}$$

so, since (5.2.49) implies that $\alpha_{3,1} = 0$, by (5.2.59), we have

$$2\Omega_{1,1}\overline{\Omega_{3,1}} \pm i\overline{\Omega_{3,1}}\Omega_{2,1} = \lambda\left(\overline{2\Omega_{1,1}\overline{\Omega_{3,1}} \pm i\overline{\Omega_{3,1}}\Omega_{2,1}}\right)$$

because $\Omega_{1,1}$ is real. Since $\lambda > 1$, it follows that

$$2\Omega_{1,1}\overline{\Omega_{3,1}} \pm i\overline{\Omega_{3,1}}\Omega_{2,1} = 0,$$

which together with (5.2.55) yields

$$\Omega_{1,2} = \pm 2i\Omega_{1,1}. \tag{5.2.59}$$

Finally, applying (5.2.47), (5.2.48), (5.2.55), (5.2.56), (5.2.57), and ch3 9d case f omega simplified k with the constraint that $\Omega_{1,1}$ is real to simplify α, we obtain

$$\alpha = \frac{1}{3}\begin{pmatrix} 2+\lambda^2-12\Omega_{1,1}^2 & -i(2+\lambda^2-24\Omega_{1,1}^2) & 0 \\ i(2+\lambda^2-24\Omega_{1,1}^2) & 2+\lambda^2-12\Omega_{1,1}^2 & 0 \\ 0 & 0 & 2+\lambda^2-12\Omega_{1,1}^2 \end{pmatrix} \tag{5.2.60}$$

and

$$\Omega = \frac{1}{3}\begin{pmatrix} \Omega_{1,1} & -\left(\pm 2i\Omega_{1,1}\right) & \lambda\Omega_{3,1} \\ \pm 2i\Omega_{1,1} & \Omega_{1,1} & \pm i\lambda\Omega_{3,1} \\ \Omega_{3,1} & \pm i\Omega_{3,1} & \Omega_{1,1} \end{pmatrix} \tag{5.2.61}$$

where the $\pm$ signs are all consistent with the sign in (5.2.57). Using (5.2.60) to solve for the coefficient η_α in item (i) of (4.2.4) we get $\eta_\alpha = \frac{2}{3}\left(2+\lambda^2-12\Omega_{1,1}^2\right)$. Accordingly, using (5.2.60) and (5.2.61) to calculate $[\alpha,\Omega] - \eta_\alpha\Omega$, we obtain a contradiction directly; specifically, the $(1,3)$ entry in $[\alpha,\Omega] - \eta_\alpha\Omega$ turns out to equal $-\frac{1}{3}\lambda(3+\lambda^2)\Omega_{3,1}$, which is nonzero by (5.2.55), and yet

(5.2.49) and item (ii) of (4.2.4) imply that the $(1,3)$ entry in $[\alpha, \Omega] - \eta_\alpha \Omega$ is zero.

$\square$

5.2.5 Symbols corresponding to formulas (5.2.5) and (5.2.6)

The CR symbol encoded by the pair (H_ℓ, C) of the form in (5.2.5) is associated with a homogeneous model. Indeed, the system in (4.2.4) is satisfied by the tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.5) and

$$\Omega = \begin{pmatrix} -\frac{3}{2} & -1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathscr{A}_0 = 0. \tag{5.2.62}$$

Each CR symbol encoded by the pair (H_ℓ, C) of the form in (5.2.6) – of which there are two, distinguished by the parameter ϵ in (5.2.6) – is associated with a homogeneous model. For example, the system in (4.2.4) is satisfied by the tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by (5.2.6) and

$$\Omega = \begin{pmatrix} -\frac{3}{2} & -1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathscr{A}_0 = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{C} \right\}. \tag{5.2.63}$$

Note that if (H_ℓ, C) is of the form in (5.2.6) then $\mathscr{A}_0$ as defined in (5.2.63) is indeed a subalgebra of $\mathscr{A}$ as is required; specifically, in this case $\mathscr{A}$ is the algebra spanned by $\mathscr{A}_0$ and the identity matrix, which can be verified directly from (2.4.7) and is also derived explicitly in Appendix A.

Hence this tuple $(H_\ell, C, \Omega, \mathscr{A}_0)$ defined by either (5.2.5) and (5.2.62) or (5.2.6) and (5.2.63) encodes a reduced modified CR symbol whose flat model exhibits the CR symbol encoded by (H_ℓ, C).

6. SUMMARY AND DISCUSSION

This work is inspired by the antecedent theory developed for CR geometry by Curtis Porter and Igor Zelenko in [33] wherein they define and study the local invariants of 2-nondegenerate, hypersurface-type CR structures encoded in their corresponding CR symbols. In particular they develop a construction of canonical absolute parallelisms for the structures whose Levi kernels have constant rank and whose CR symbols are *regular*, solving the local equivalence problem for maximally symmetric CR manifolds whose CR symbols are regular, and furthermore obtaining sharp upper bounds for the dimension of these structures' symmetry groups expressed in terms of their underlying manifolds' dimension. A natural question, and starting point for this dissertation, concerns the extent to which these results be extended by relaxing the assumption that the structures' CR symbols are regular.

To treat 2-nondengenerate structures without imposing the regularity assumption on their CR symbols we introduce (in Section 2.1) and study a correspondence between these CR structures and the geometry of their associated dynamical Legendrian contact structures. The latter geometries are naturally amenable to Tanaka prolongation and the general theory developed by Noboru Tanaka for analysis of filtered structures, whereas this general theory is not straightforwardly applicable to the analysis of 2-nondegenerate structures. A broad class of CR structures that we call *recoverable* are uniquely determined by their associated dynamical Legendrian contact structure, and our study of dynamical Legendrian contact structures directly transmutes into analysis of these recoverable CR structures' geometry. Necessary and sufficient conditions for a CR structure to be recoverable are given in Propositions 2.1.6 and 2.2.8.

The fundamental theory developed in Chapter 2, essential for the conclusions in Chapters 4 and 5, is a construction of canonical absolute parallelisms for 2-nondengenerate, hypersurface-type CR structures whose Levi kernel has constant rank (Theorems 2.3.5 and 2.5.2). This construction supplies effective machinery for the study of recoverable CR structures, solving the local equivalence problem for these structures by reducing it to the local equivalence problem for $\{e\}$-structures.

From this construction we also obtain bounds for the dimension of these structures' symmetry groups and show that structures that attain these bounds are locally unique (Theorem 2.5.2).

Along with this construction, we introduce modified CR symbols and reduced modified CR symbols, which are subspaces of a certain Lie algebra encoding more local invariants than their antecedent CR symbols encode. We show (in Sections 2.4 and 2.5) that homogeneous 2-nondegenerate, hypersurface-type CR structures admit reduced modified CR symbols that have the structure of a Lie algebra. This is a strong algebraic constraint on the local invariants of these homogeneous structures, which we use to show that – excluding a few exceptional parameter settings for the dimension of the CR manifold and the rank of its Levi kernel – generic CR symbols are not found in homogeneous structures. Specifically, we show that, for any fixed rank $r > 1$, in the set of all CR symbols associated with 2-nondegenerate, hypersurface-type CR manifolds of odd dimension greater than $4r + 1$ with a rank r Levi kernel, the CR symbols not associated with any homogeneous model are generic, and, for $r = 1$, the same result holds if the CR structure is pseudoconvex. Furthermore, for $r = 1$, the analysis in Chapter 5 together with the general arguments in Chapter 2 establish this generic nonexistence result without the pseudoconvexity assumption for manifolds of dimension at most 9 (Remark 2.6.6). Later chapters (Chapters 3, 4, and 5) are geared toward deeper analysis of the 2-nondegenerate structures whose Levi kernels have rank 1.

In Chapter 3 we classify the CR symbols of 2-nondegenerate structures whose Levi kernel has rank 1 (Theorem 3.1.2). This classification is equivalent (see Remark 2.2.5) to the linear algebra problem of obtaining a canonical form for pairs (ℓ, A) consisting of a Hermitian form ℓ and an ℓ-self-adjoint antilinear operator A. It is also equivalent to obtaining canonical forms for pairs consisting of a nondegenerate Hermitian form and a symmetric bilinear form. It was somewhat surprising that these pairs were not previously classified given the fundamental role that Hermitian and bilinear forms perform in so many areas of linear algebra and theoretical physics and the extensive lineage of similar classifications [1, 4, 16, 15, 19, 20, 21, 22, 23, 25, 44, 45]. In any case, the canonical forms obtained in Theorem 3.1.2 complement a large body of literature on canonical forms in linear algebra.

In Chapter 4 we obtain a sharp upper bound for the dimensions of the symmetry groups of homogeneous 2-nondegenerate, hypersurface-type CR manifolds whose Levi kernel has rank 1 expressed in terms of the manifolds' dimensions, namely the dimension of the symmetry group of such a CR manifold M does not exceed $\frac{1}{4}(\dim M-1)^2+7$ (Theorem 4.1.2). The technical work in Chapter 4 is actually a calculation of this upper bound for the structures whose CR symbols are non-regular, and the general upper bound follows by combining this upper bound for non-regular CR symbols with the upper bound for regular CR symbols previously obtained in [33]. It turns out that the maximally symmetric models have regular CR symbols, and hence the upper bound obtained in [33] for regular CR symbols is also the general upper bound.

In the very recent paper [3] it was shown that for $\dim M = 7$, without the homogeneity assumption, the upper bound for the dimension of the group of symmetries of 2-nondegenerate, hypersurface-type CR structures with a rank 1 Levi kernel is 17. The sharp bound (4.0.1) for the homogeneous case is 16 and an example of the structure from the considered class with a 17-dimensional symmetry group is unknown. The result of Chapter 4 (communicated in a private correspondence) was in fact used in [3] to reduce the bound from 18, obtained initially by the methods of normal forms, to 17 (see Proposition 16 there).

Within the symmetry group of each homogeneous 2-nondegenerate, hypersurface-type CR manifold whose Levi kernel has rank 1, the isotropy subgoups' Lie algebras each contain a special subalgebra tangent to the subgroup of symmetries determined by their first weighted jet (i.e., a subalgebra of the algebra of infinitesimal symmetries whose constituent vector fields are all zero at a given point and are all fully determined by their Lie derivatives with vector fields contained in the distribution $H\oplus\overline{H}\cap TM$, where H denotes the CR structure on M). In Appendix A, we derive an explicit formula for a matrix representation of this special subalgebra, and this formula is essential for the analysis carried out in Chapter 4. Prior to obtaining the general formulas derived in Appendix A, we developed an alternative method for calculating the dimension of this special subalgebra, which is recorded in Appendix B. Specifically, Appendix B outlines a way to apply techniques of [15] to calculate the dimension of the transformation group preserving two symmetric forms of which at least one is nondegenerate. The result of Appendix B is not necessary for obtaining the other results in this dissertation because, wherever it would otherwise be applied the results of Appendix A suffice. Appendix B is included rather solely because it is a result of independent interest generalizing a formula obtained in [35] and complementing the calculations in Appendix A.

In Chapter 5, we classify the CR symbols associated with homogeneous CR models on manifolds of dimension 7, of which there are eight altogether, and partially classify these symbols for structures on manifolds of dimension 9 with a rank 1 Levi kernel. Moreover, we classify the 7-dimension flat models associated with given reduced modified CR symbols, of which there are nine altogether. The fact that all regular symbols correspond to homogeneous models was shown in [33], wherein they show that each regular symbol is associated with a (locally) unique maximally symmetric homogeneous model. The classification in Chapter 5 extends these results in low dimensions by finding homogeneous models associated with regular symbols that have smaller dimensional symmetry groups than the models found in [33] and by finding homogeneous models associated with non-regular symbols as well as. An early conjecture that all homogeneous models have a regular symbol was in fact the original motivation for the work in this dissertation. In Chapter 5, we identify two 7-dimensional homogeneous models and nine 9-dimensional homogenous models with non-regular CR symbols.

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APPENDIX A

ON THE LIE ALGEBRA OF THE ISOTROPY SUBGROUP OF SYMMETRIES DETERMINED BY THEIR FIRST WEIGHTED JET

In this appendix we give a general formula for matrices in the algebra $\mathscr{A}$ defined in (4.2.1) together with an outline for how the formula can be verified. Naturally, it is easier to verify the formula than to derive it, and, since the formula is ancillary to this paper's topic, we omit the analysis used to derive it. The formula depends on the matrices H_ℓ and C representing the pair (ℓ, A).

In the sequel we assume that H_ℓ and C are in the canonical form prescribed by Theorem 3.1.2 We will also use the notation of Section 4.2, and, in particular, we let $\lambda_1, \ldots, \lambda_\gamma$, $m_1, \ldots, m_\gamma$, $\epsilon_1, \ldots, \epsilon_\gamma$, M_{λ_i, m_i} and N_{λ_i, m_i} be as in Theorem 3.1.2. Recall that, in particular, this means the real and imaginary parts of each λ_i are both nonnegative.

Define the *bi-orthogonal subalgebra* of $\mathscr{A}$ to be

$$\mathscr{A}^o := \{B \in \mathscr{A} \mid BCH_\ell^{-1} + CH_\ell^{-1}B^T = B^T H_\ell \overline{C} + H_\ell \overline{C} B = 0\},$$

where this name is reflecting the observation that $\mathscr{A}^o$ is analogous to an intersection of two orthogonal algebras. In this appendix, we first obtain a formula describing the elements in $\mathscr{A}^o$ and then obtain a formula for a subspace $\mathscr{A}^s \subset \mathscr{A}$ complimentary to $\mathscr{A}^o$, that is, such that

$$\mathscr{A} = \mathscr{A}^o \oplus \mathscr{A}^s. \tag{A.0.1}$$

Such a space $\mathscr{A}^s$ is spanned by elements that we call *conformal scaling* elements of $\mathscr{A}$, referring to the observation that these are analogous to non-orthogonal elements in an intersection of two conformally orthogonal algebras.

To begin, let B be an $(n-1) \times (n-1)$ matrix in $\mathscr{A}^o$ and partition B into blocks $\{B_{(i,j)}\}_{i,j=1}^{\gamma}$

where the number of rows in $B_{(i,j)}$ is the same as in the matrix M_{λ_i,m_i} and the number of columns in $B_{(i,j)}$ is the same as in the matrix M_{λ_j,m_j}. Similarly, we partition $H_\ell\overline{C}B$ and BCH_ℓ^{-1} into blocks $\{(H_\ell\overline{C}B)_{(i,j)}\}_{i,j=1}^{\gamma}$ and $\{(BCH_\ell^{-1})_{(i,j)}\}_{i,j=1}^{\gamma}$ whose sizes are the same as in the partition of B.

Let us now derive a relationship between the blocks $B_{(i,j)}$ and $B_{(j,i)}$. To simplify formulas, we assume $\epsilon_i = \epsilon_j$. To treat the more general case where possibly $\epsilon_i \neq \epsilon_j$, one can simply replace N_{λ_i,m_i} (or N_{λ_j,m_j}) with $\epsilon_i N_{\lambda_i,m_i}$ (or $\epsilon_j N_{\lambda_j,m_j}$) in all of the subsequent formulas.

We have

$$(BCH_\ell^{-1})_{(i,j)} = B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j} \quad \text{and} \quad (H_\ell\overline{C}B)_{(i,j)} = N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)},$$

so, since $B \in \mathscr{A}$,

$$\left(M_{\lambda_i,m_i}N_{\lambda_i,m_i}\right)^T B_{(j,i)}^T = -B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j}$$

and

$$B_{(j,i)}^T\left(N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}}\right)^T = -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}.$$

Since A is ℓ-self-adjoint, each matrix $N_{\lambda_k,m_k}\overline{M_{\lambda_k,m_k}}$ and $M_{\lambda_k,m_k}N_{\lambda_k,m_k}$ is symmetric (one can also verify this by directly using the canonical form), and hence

$$M_{\lambda_i,m_i}N_{\lambda_i,m_i}B_{(j,i)}^T = -B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j}, \tag{A.0.2}$$

and

$$B_{(j,i)}^T N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}} = -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}. \tag{A.0.3}$$

Multiplying both sides of (A.0.3) by $M_{\lambda_j,m_j}N_{\lambda_j,m_j}$ from the right and then applying (A.0.2) yields

$$\begin{aligned} B_{(j,i)}^T N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j}N_{\lambda_j,m_j} &= -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j} \qquad \text{(A.0.4)} \\ &= N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}N_{\lambda_i,m_i}B_{(j,i)}^T. \end{aligned}$$

Multiplying (A.0.4) by N_{λ_i,m_i} from the left and by N_{λ_j,m_i} from the right yields

$$\left(N_{\lambda_i,m_i}B_{(i,j)}^T N_{\lambda_j,m_j}\right)\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j} = \overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}\left(N_{\lambda_i,m_i}B_{(i,j)}^T N_{\lambda_j,m_j}\right). \tag{A.0.5}$$

Notice that (A.0.2) is also equivalent to

$$N_{\lambda_i,m_i}M_{\lambda_i,m_i}\left(N_{\lambda_j,m_j}B_{(j,i)}N_{\lambda_i,m_i}\right)^T = -\left(N_{\lambda_i,m_i}B_{(i,j)}N_{\lambda_j,m_j}\right)N_{\lambda_j,m_j}M_{\lambda_j,m_j}. \tag{A.0.6}$$

Equation (A.0.5) gives us all restrictions on the general form of $B_{(i,j)}$ that are not coming from the relationship between $B_{(i,j)}$ and other blocks in the matrix B. Equation (A.0.6), on the other hand, gives us the restrictions on the general form of $B_{(i,j)}$ coming from its relationship with $B_{(j,i)}$. Moreover, if (A.0.5) and (A.0.6) are satisfied for i and j then B is in $\mathscr{A}^o$ because (A.0.2) and (A.0.3) hold. In other words, our present goal is to solve the system of matrix equations in (A.0.5) and (A.0.6), and whenever $(\lambda_i, \lambda_j) \neq (0,0)$, this exercise is equivalent to first solving the matrix equation

$$X\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j} = \overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}X, \tag{A.0.7}$$

and then, for the case where $i = j$, solving the system of equations consisting of (A.0.7) and

$$N_{\lambda_i,m_i}M_{\lambda_i,m_i}X^T = -XN_{\lambda_i,m_i}M_{\lambda_i,m_i}.$$

The case where $\lambda_i = \lambda_j = 0$ requires special treatment because, in this case, contrary to the case where $(\lambda_i, \lambda_j) \neq (0,0)$, even if $i \neq j$ solutions for $B_{(i,j)}$ in (A.0.5) need not satisfy (A.0.6) for any matrix $B_{(j,i)}$.

Equation (A.0.7) is of the form analyzed in [15, Chapter 8]. In fact, an explicit solution to (A.0.7) is given in [15, Chapter 8], but the solution is expressed in terms of a basis with respect to which $\overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}$ and $\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j}$ have their Jordan normal forms. On the other hand, the transition matrix from the initially considered basis to a basis of the Jordan normal form is block-diagonal with the blocks corresponding to the Jordan blocks. Hence, the following lemma can be obtained from the solution in [15, Chapter 8].

Lemma A.0.1. *If $\lambda_i \neq \lambda_j$ then $B_{(i,j)} = 0$.*

Proof. Since the real and imaginary parts of λ_i and λ_j are all nonnegative, if $\lambda_i \neq \lambda_j$ then the eigenvalues of $\overline{M_{\lambda_i,m_i}} M_{\lambda_i,m_i}$ all differ from the eigenvalues of $\overline{M_{\lambda_j,m_j}} M_{\lambda_j,m_j}$. Accordingly, by [15, Chapter 8, Theorem 1 and Equation (11)], the matrix X in (A.0.7) is zero. □

Given Lemma A.0.1, all that remains is to find the general formula for $B_{(i,j)}$ when $\lambda_i = \lambda_j$. We will say that a Toplitz $p \times q$ matrix is an *upper-triangular Toeplitz matrix*, if the only nonzero entries appear on or above the main diagonal in their right-most $p \times p$ block if $p \leq q$, and the top-most $q \times q$ block if $p \geq q$ (in the terminology of [15, Chapter 8] they are called regular upper-triangular, but we avoid this terminology because the term “regular" is already assigned in the present paper to another concept).

Lemma A.0.2. *Suppose $\lambda_i = \lambda_j$ and $m_i \leq m_j$. The dimension of the space of solutions of* (A.0.7) *is equal to*

1. *m_i if $\lambda_i > 0$;*

2. *$2m_i$ if $\lambda_i^2 \notin \mathbb{R}$;*

3. *$4m_i$ if $\lambda_i^2 < 0$.*

Proof. We use [15, Chapter 8, Theorem 1] again for each of the cases.

Suppose first that $\lambda_i > 0$. If $\lambda > 0$ then $\overline{M_{\lambda,m}} M_{\lambda,m}$ is similar to the Jordan matrix $J_{\lambda^2,m}$. Let U_i and U_j be invertible matrices such that $U_j \overline{M_{\lambda_j,m_j}} M_{\lambda_j,m_j} U_j^{-1} = J_{\lambda_j^2,m_j}$ and $U_i \overline{M_{\lambda_i,m_i}} M_{\lambda_i,m_i} U_i^{-1} = J_{\lambda_i^2,m_i}$. For a matrix X satisfying (A.0.7), set $\widetilde{X} = U_j^{-1} X U_i$ so that, by (A.0.7),

$$\widetilde{X} J_{\lambda_j^2,m_j} = J_{\lambda_i^2,m_i} \widetilde{X}. \tag{A.0.8}$$

It is shown in [15, Chapter 8, Theorem 1] that the space of solutions of (A.0.8) consists of upper-triangular Toeplitz matrices. Therefore, the space of solutions of (A.0.8) has dimension m_i, which shows item (1) because $X \mapsto U_j^{-1} X U_i$ gives an isomorphism between the space of solutions of (A.0.8) and the space of solutions of (A.0.7).

Let us now suppose $\lambda_i^2 \notin \mathbb{R}$ or $\lambda_i^2 < 0$. If $\lambda^2 \notin \mathbb{R}$ or $\lambda^2 < 0$ then

$$\overline{M_{\lambda,m}} M_{\lambda,m} = J_{\lambda^2,m} \oplus J_{\overline{\lambda}^2,m}. \tag{A.0.9}$$

For a matrix X satisfying (A.0.7), consider the 2×2 block matrix partition $(X_{(r,s)})_{r,s\in\{1,2\}}$ of X whose blocks are all $m_i \times m_j$ matrices. It is shown in [15, Chapter 8, Theorem 1] that the space of solutions of (A.0.7) with $\overline{M_{\lambda_i,m_i}} M_{\lambda_i,m_i}$ and $\overline{M_{\lambda_j,m_j}} M_{\lambda_j,m_j}$ of the form in (A.0.9) consists of matrices $(X_{(r,s)})_{r,s\in\{1,2\}}$ for which each $X_{(r,s)}$ is an upper-triangular Toeplitz matrix, where, moreover, if $\lambda_i^2 \neq \overline{\lambda_i}^2$ then $X_{(1,2)} = X_{(2,1)} = 0$. Accordingly, if $\lambda_i^2 \notin \mathbb{R}$ (respectively $\lambda_i^2 < 0$) then solutions to (A.0.7) are determined by two (respectively four) upper-triangular Toeplitz $m_i \times m_i$ matrices. Items (2) and (3) follow because each upper-triangular Toeplitz $m_i \times m_i$ is determined by m_i variables. $\square$

Corollary A.0.3. *If $m_i \leq m_j$, $\lambda_i = \lambda_j = \lambda$ and $\lambda \neq 0$ then the matrices $B_{(i,j)}$ and $B_{(j,i)}$ are described by one of three formulas, where the correct formula depends on λ.*

1. *If $\lambda > 0$ then $B_{(i,j)}$ and $B_{(j,i)}$ respectively equal*

$$\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{}^{\substack{m_j - m_i \\ \textit{columns}}}} & \\ 0 & \cdots & 0 & \\ \vdots & & \vdots & \sum_{k=0}^{m_i-1} b_k T_{m_i}^k \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \end{array}\right) \quad \textit{and} \quad -\epsilon_i\epsilon_j \left(\begin{array}{ccc} \multicolumn{3}{c}{\sum_{k=0}^{m_i-1} b_k T_{m_i}^k} \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right) \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \substack{m_j \; - \; m_i \\ \textit{rows,}} \tag{A.0.10}$$

for some coefficients $\{b_k\}$.

2. *If $\lambda^2 \notin \mathbb{R}$ then*

$$B_{(i,j)} = \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\qquad}^{\substack{m_j - m_i \\ \textit{columns}}} & \qquad\qquad\qquad & \overbrace{\qquad\qquad}^{\substack{m_j - m_i \\ \textit{columns}}} & \qquad\qquad\qquad \end{array} \\ \left(\begin{array}{ccc|c|ccc|c} 0 & \cdots & 0 & & 0 & \cdots & 0 & \\ \vdots & & \vdots & \sum\limits_{k=0}^{m_i-1} a_k T_{m_i} & \vdots & & \vdots & 0 \\ \cline{4-4}\cline{8-8} \vdots & & \vdots & 0 & \vdots & & \vdots & \sum\limits_{k=0}^{m_i} b_k T_{m_i} \\ 0 & \cdots & 0 & & 0 & \cdots & 0 & \end{array}\right) \end{array}, \qquad \text{(A.0.11)}$$

and

$$B_{(j,i)} = -\epsilon_i \epsilon_j \left(\begin{array}{c|c} \sum\limits_{k=0}^{m_i} a_k T_{m_i} & 0 \\ \hline \multicolumn{2}{c}{\begin{matrix} 0 & \cdots\cdots\cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots\cdots\cdots & 0 \end{matrix}} \\ \hline 0 & \sum\limits_{k=0}^{m_i} b_k T_{m_i} \\ \hline \multicolumn{2}{c}{\begin{matrix} 0 & \cdots\cdots\cdots\cdots & 0 \\ \vdots & & \vdots \end{matrix}} \end{array}\right) \begin{array}{l} \vphantom{\sum\limits_{k=0}^{m_i}} \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \substack{m_j - m_i \\ \textit{rows}} \\ \vphantom{\sum\limits_{k=0}^{m_i}} \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \end{matrix}}\right\} \substack{m_j - m_i \\ \textit{rows,}} \end{array} \qquad \text{(A.0.12)}$$

for some coefficients $\{a_k, b_k\}$.

3. *If $\lambda^2 < 0$ then*

$$B_{(i,j)} = \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\qquad}^{\substack{m_j - m_i \\ \textit{columns}}} & \qquad\qquad\qquad\qquad\qquad & \overbrace{\qquad\qquad}^{\substack{m_j - m_i \\ \textit{columns}}} & \qquad\qquad\qquad \end{array} \\ \left(\begin{array}{ccc|c|ccc|c} 0 & \cdots & 0 & & 0 & \cdots & 0 & \\ \vdots & & \vdots & \sum\limits_{k=0}^{m_i-1} c_k T_{m_i}^k & \vdots & & \vdots & \sum\limits_{k=0}^{m_i} d_k T_{m_i}^k \\ \cline{4-4}\cline{8-8} \vdots & & \vdots & \sum\limits_{k=0}^{m_i-1} \left(\sum_{r=0}^{k} e_r\right) T_{m_i}^k & \vdots & & \vdots & \sum\limits_{k=0}^{m_i} f_k T_{m_i}^k \\ 0 & \cdots & 0 & & 0 & \cdots & 0 & \end{array}\right) \end{array}, \qquad \text{(A.0.13)}$$

and

$$B_{(j,i)} = \epsilon_i \epsilon_j \left(\begin{array}{c|c} -\sum\limits_{k=0}^{m_i} c_k T_{m_i}^k & \sum\limits_{k=0}^{m_i} e_k T_{m_i}^k \\ \hline \multicolumn{2}{c}{0 \quad \cdots\cdots\cdots\cdots \quad 0} \\ \multicolumn{2}{c}{\vdots \qquad\qquad\qquad\qquad \vdots} \\ \multicolumn{2}{c}{0 \quad \cdots\cdots\cdots\cdots \quad 0} \\ \hline \sum\limits_{k=0}^{m_i} \left(\sum_{r=0}^{k} d_r \right) T_{m_i}^k & -\sum\limits_{k=0}^{m_i} f_k T_{m_i}^k \\ \hline \multicolumn{2}{c}{0 \quad \cdots\cdots\cdots\cdots \quad 0} \\ \multicolumn{2}{c}{\vdots \qquad\qquad\qquad\qquad \vdots} \end{array} \right) \begin{array}{l} \\ \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \substack{m_j - m_i \\ \textit{rows}} \\ \\ \left. \begin{array}{l} \\ \\ \end{array} \right\} \substack{m_j - m_i \\ \textit{rows,}} \end{array} \tag{A.0.14}$$

for some coefficients $\{a_k, b_k, c_k, d_k, e_k, f_k\}$.

Proof. Using the formula for $B_{(i,j)}$ given in (A.0.10), (A.0.11), and (A.0.13), it is straightforward to check that (A.0.7) holds with $X = B_{(i,j)}$. Moreover, this formula for $B_{(i,j)}$ is the most general formula with this property because, by Lemma A.0.2, it has the maximum number of parameters possible. Lastly, the formula for $B_{(j,i)}$ given in (A.0.10), (A.0.12), and (A.0.14) is obtained through another straightforward calculation by applying (A.0.6) directly to the formula for $B_{(i,j)}$. □

To simplify notation in the following lemma, for an integer q, we let $[q]_2$ denote the residue of q modulo 2, that is, $[q]_2 = 0$ if q is even and $[q]_2 = 1$ if q is odd.

Lemma A.0.4. *If* $m_i \leq m_j$ *and* $\lambda_i = \lambda_j = 0$ *then*

$$B_{(i,j)} = \begin{array}{l} \quad \overbrace{\hphantom{0 \quad \cdots \quad 0}}^{\substack{m_j - m_i \\ \textit{columns}}} \\ \left(\begin{array}{ccc|cccccc} 0 & \cdots & 0 & c_1^1 & c_2^1 & \cdots & & \cdots & c_{m_i}^1 \\ \vdots & & \vdots & 0 & c_1^0 & c_2^0 & \cdots & \cdots & c_{m_i-1}^0 \\ \vdots & & \vdots & 0 & 0 & c_1^1 & c_2^1 & \cdots & c_{m_i-2}^1 \\ \vdots & & \vdots & \vdots & & \ddots & c_1^0 & \cdots & c_{m_i-3}^0 \\ \vdots & & \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & & \cdots & 0 & c_1^{[m_i]_2} \end{array} \right) \end{array}, \tag{A.0.15}$$

and

$$B_{(j,i)} = -\epsilon_i\epsilon_j \left(\begin{array}{cccccc} c_1^{[m_i+1]_2} & c_2^{[m_i+2]_2} & \cdots & & \cdots & c_{m_i}^{[2m_i]_2} \\ 0 & c_1^{[m_i+2]_2} & c_2^{[m_i+3]_2} & \cdots & \cdots & c_{m_i-1}^{[2m_i]_2} \\ 0 & 0 & c_1^{[m_i+3]_2} & c_2^{[m_i+4]_2} & \cdots & c_{m_i-2}^{[2m_i]_2} \\ \vdots & & \ddots & c_1^{[m_i+4]_2} & \cdots & c_{m_i-3}^{[2m_i]_2} \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & c_1^{[2m_i]_2} \\ \hline \multicolumn{6}{l}{\left. \begin{matrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{matrix} \right\} m_j - m_i \textit{ rows}} \end{array} \right) \tag{A.0.16}$$

for some coefficients $\{a_k, b_k, c_k^1, c_k^0\}$.

Proof. Let us refer to the main diagonal of the upper right $m_i \times m_i$ block in each matrix $B_{(i,j)}$ and $B_{(j,i)}$ as that matrix's reference diagonal.

Notice that equations (A.0.2) and (A.0.3) hold in the present context with $\lambda_i = \lambda_j = 0$ and $\epsilon_i = \epsilon_j$. Let us assume $\epsilon_i = \epsilon_j$, noting that for the other case, where $\epsilon_i \neq \epsilon_j$, we would first change the sign of the right side of (A.0.2) and (A.0.3) and then proceed with exactly the same calculations.

Applying (A.0.2), we find that the last row of $B_{(i,j)}$ contains only zeros below the reference diagonal, and, applying (A.0.3), we find that the first column of $B_{(i,j)}$ contains only zeros to the left of the reference diagonal. Similarly, by (A.0.2) and (A.0.3), the first column and last row of $B_{(j,i)}$ contain zeros in their entries that are below or to the left of the reference diagonal. After substituting 0 in for those entries, applying (A.0.2) again, we now find that the second to last row of $B_{(i,j)}$ (or of $B_{(j,i)}$) contains only zeros below (or to the left of) the reference diagonal, whereas, by applying (A.0.3) again, we find that the second column of $B_{(i,j)}$ (or of $B_{(j,i)}$) contains only zeros to the left of (or below) the reference diagonal. Repeating this analysis, we eventually find that all entries in $B_{(i,j)}$ and $B_{(j,i)}$ that are below or to the left of the reference diagonal are zero.

Let us now calculate the restrictions that (A.0.2) and (A.0.3) impose on the remaining nonzero entries in $B_{(i,j)}$ and $B_{(j,i)}$. For the next observations, we use the term *secondary transpose* to refer to the transformation of square matrices described by reflecting their entries over the secondary diagonal, that is, sending the (i, j) entry of an $m \times m$ matrix to the $(m+1-j, m+1-i)$ entry. Applying (A.0.2), we see that upper left $(m_i - 1) \times (m_i - 1)$ block of the upper right $m_i \times m_i$ block of $B_{(i,j)}$ is equal to -1 (or $-\epsilon_i\epsilon_j$ in the general case) times the secondary transpose of the upper left $(m_i - 1) \times (m_i - 1)$ block of $B_{(j,i)}$. Similarly, applying (A.0.3), we see that lower right $(m_i - 1) \times (m_i - 1)$ block of the upper right $m_i \times m_i$ block of $B_{(i,j)}$ is equal to -1 (or $-\epsilon_i\epsilon_j$ in the general case) times the secondary transpose of the lower right $(m_i - 1) \times (m_i - 1)$ block of $B_{(j,i)}$. These last two observations, taken together, complete this proof. □

Corollary A.0.5. *For all $i \in \{1, \dots, \gamma\}$,*

$$B_{(i,i)} = \begin{cases} \left(\sum_{k=1}^{\lceil m_i/2 \rceil} a_k T_{m_i}^{m_i-2k+1}\right) I_{\text{alt},m_i} & \textit{if } \lambda_i = 0 \\ \begin{pmatrix} 0 & \sum\limits_{k=0}^{m_i-1} a_k T_{m_i}^k \\ \sum\limits_{k=0}^{m_i-1} \left(\sum_{r=0}^{k} a_r\right) T_{m_i}^k & 0 \end{pmatrix} & \textit{if } \lambda_i^2 < 0 \\ 0 & \textit{otherwise,} \end{cases} \tag{A.0.17}$$

where $I_{\text{alt},m}$ denotes the $m \times m$ diagonal matrix with a 1 in its odd columns and a -1 in its even columns.

Proof. This follows immediately from the formulas in Corollary A.0.3 and Lemma A.0.4 with $i = j$. □

The previous results provide a general formula for matrices in $\mathscr{A}^o$. We now focus on obtaining a general formula of a subspace $\mathscr{A}^s$ satisfying (A.0.1).

Lemma A.0.6. *Either $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 1$ or $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 2$, and the latter case occurs if and only if there exists a matrix X in $\mathscr{A}$ satisfying*

$$XCH_\ell^{-1} + CH_\ell^{-1}X^T = 2CH_\ell^{-1} \Leftrightarrow (X - I)^T H_\ell C^{-1} + H_\ell C^{-1}(X - I) = 0, \tag{A.0.18}$$
$$X^T H_\ell \overline{C} + H_\ell \overline{C} X = 0.$$

Proof. Define

$$\mathscr{A}_1^o := \left\{X \;\middle|\; XCH_\ell^{-1} + CH_\ell^{-1}X^T = 0\right\} \quad \text{and} \quad \mathscr{A}_2^o := \left\{X \;\middle|\; X^T H_\ell \overline{C} + H_\ell \overline{C} X = 0\right\}.$$

Since $\mathscr{A}^o = \mathscr{A}_1^o \cap \mathscr{A}_2^o$,

$$\dim(\mathscr{A}^o) + \dim(\mathscr{A}_1^o + \mathscr{A}_2^o) = \dim(\mathscr{A}_1^o) + \dim(\mathscr{A}_2^o), \tag{A.0.19}$$

and, letting $\mathbb{C}I$ denote $\operatorname{span}\{I\}$, since $\mathscr{A} = (\mathscr{A}_1^o + \mathbb{C}I) \cap (\mathscr{A}_2^o + \mathbb{C}I)$,

$$\begin{aligned}
\dim(\mathscr{A}) + \dim\left(\mathscr{A}_1^o + \mathscr{A}_2^o + \mathbb{C}I\right) &= \dim\left(\mathscr{A}_1^o + \mathbb{C}I\right) + \dim\left(\mathscr{A}_2^o + \mathbb{C}I\right) \\
&= \dim(\mathscr{A}_1^o) + \dim(\mathscr{A}_2^o) + 2 \\
&= \dim(\mathscr{A}^o) + \dim(\mathscr{A}_1^o + \mathscr{A}_2^o) + 2,
\end{aligned}$$

where this last equation holds by (A.0.19). Therefore,

$$\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = \dim(\mathscr{A}_1^o + \mathscr{A}_2^o) - \dim\left(\mathscr{A}_1^o + \mathscr{A}_2^o + \mathbb{C}I\right) + 2,$$

and hence

$$\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = \begin{cases} 1 & \text{if } I \notin \mathscr{A}_1^o + \mathscr{A}_2^o \\ 2 & \text{if } I \in \mathscr{A}_1^o + \mathscr{A}_2^o. \end{cases}$$

In particular, $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 2$ if and only if there exists $X \in \mathscr{A}_2^o$ such that $(I - X) \in \mathscr{A}_1^o$, which is equivalent to (A.0.18). $\square$

Lemma A.0.7. *If* $C = M_{m,\lambda}$ *and* $\lambda \neq 0$ *then* $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 1$.

Proof. We assume that (H_ℓ, C) is in the canonical form of Theorem 3.1.2, so $H_\ell = S_m$, where S_m is defined in (3.1.3). Fix a subspace $\mathscr{A}^s$ of $\mathscr{A}$ satisfying (A.0.1). To produce a contradiction, let us assume that $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) \neq 1$. By Lemma A.0.6, we can assume that there exists a matrix X in $\mathscr{A}^s$ satisfying (A.0.18). Since $H_\ell C^{-1}$ and $H_\ell \overline{C}$ are symmetric, condition (A.0.18) is fundamentally related to the two symmetric forms Q_1 and Q_2 defined by

$$Q_1(v,w) := w^T H_\ell C^{-1} v \quad \text{and} \quad Q_2(v,w) := w^T H_\ell \overline{C} v.$$

Note that

$$Q_2(v,w) = Q_1\left(C\overline{C}v, w\right) = Q_1\left(A^2 v, w\right),$$

where A is, again, the antilinear operator represented by C.

Let us now work instead with respect to a basis that is orthonormal with respect to Q_1, that is, letting L denote the matrix representing the linear operator A^2 in this basis, we have

$$Q_1(v,w) = w^T v \quad \text{and} \quad Q_2 = w^T L v$$

in this new basis. By [15, Chapter 11.3, Corollary 2], we can assume without loss of generality that

$$L = \begin{cases} \frac{1}{2}(I + iS_m)J_{\lambda,m}(I - iS_m) & \text{if } \lambda^2 > 1 \\ \frac{1}{2}(I + iS_m)J_{\lambda,m}(I - iS_m) \oplus \frac{1}{2}(I + iS_m)J_{\lambda,m}(I - iS_m) & \text{otherwise.} \end{cases}$$

The second equation in (A.0.18) implies that X is in the Lie algebra of the transformation group that preserves Q_2, whereas the first equation of (A.0.18) implies that $X - I$ is in the Lie algebra of the transformation group that preserves Q_1. That is, with respect to the new basis, $(X - I) = -(X - I)^T$ and $X^T L + LX = 0$,which is equivalent to

$$(X - I) = -(X - I)^T \quad \text{and} \quad [X, L] = 0. \tag{A.0.20}$$

Defining the pair of matrices (S, J) by

$$(S, J) = \begin{cases} (I + iS_m, J_{\lambda,m}) & \text{if } \lambda^2 > 1 \\ ((I + iS_m) \oplus (I + iS_m), J_{\lambda,m} \oplus J_{\lambda,m}) & \text{otherwise,} \end{cases}$$

the condition $[X, L] = 0$ is equivalent to

$$\left[S^{-1}XS, J\right] = 0. \tag{A.0.21}$$

Solving for the matrix X in $[X, L] = 0$ is a classical problem of Frobenious whose general solution is given in [15, Chapter 8]. In [15, Chapter 8], a formula is given for matrices that commute with a Jordan matrix such as J, so we have rewritten $[X, L] = 0$ as in (A.0.21), in order to apply the solution of [15, Chapter 8] directly. The formula in [15, Chapter 8] gives that, after partitioning the matrix $S^{-1}XS$ into size $m \times m$ blocks, each block of $S^{-1}XS$ in this partition is an upper-triangular Toeplitz matrix. If X is a Toeplitz matrix then $(I + iS_m)X(I - iS_m)$ is symmetric because S_mX and XS_m are both symmetric whereas $X^T = S_mXS_m$. Accordingly, letting X' denote the upper left $m \times m$ block of X, since $(I - iS_m)(X' - I)(I + iS_m)$ is Toeplitz,

$$\begin{aligned} X' - I &= \frac{1}{4}(I + iS_m)\big[(I - iS_m)(X' - I)(I + iS_m)\big](I - iS_m) \\ &= \left(\frac{1}{4}(I + iS_m)\big[(I - iS_m)(X' - I)(I + iS_m)\big](I - iS_m)\right)^T = (X' - I)^T. \end{aligned} \tag{A.0.22}$$

By (A.0.20) and (A.0.22), $X' = I$, which contradicts the upper left $m \times m$ block of the second matrix equation in (A.0.18). □

With Lemmas A.0.6 and A.0.7 established we now give a general formula for a subspace $\mathscr{A}^s$ of $\mathscr{A}$ satisfying (A.0.1).

Lemma A.0.8. *For a subspace $\mathscr{A}^s$ of $\mathscr{A}$ satisfying* (A.0.1)*,* $\dim(\mathscr{A}^s) = 2$ *if and only if C is* *nilpotent. In particular, if*

$$C = J_{0,m_1} \oplus \ldots \oplus J_{0,m_\gamma}$$

then, to satisfy (A.0.1)*, we can take the subspace* $\mathscr{A}^s$ *of* $\mathscr{A}$ *spanned by the identity matrix and the matrix*

$$\bigoplus_{i=1}^{\gamma} D_{m_i},$$

where, for an integer m*,* D_m *denotes the* $m \times m$ *diagonal matrix defined by*

$$D_m := \mathrm{Diag}\left(\frac{m}{2}, \frac{m}{2} - 1, \ldots, \frac{m}{2} - m + 1\right). \tag{A.0.23}$$

Proof. Suppose that (H_ℓ, C) is in the canonical form of Theorem 3.1.2, specifically such that

$$C = J_{\lambda_1,m_1} \oplus \cdots \oplus J_{\lambda_\gamma,m_\gamma},$$

and suppose that $\dim(\mathscr{A}^s) = 2$. As is shown in the proof of Lemma A.0.6, we can assume without loss of generality that there exists a matrix X in $\mathscr{A}^s$ satisfying (A.0.18). In particular, partitioning X into a block matrix whose diagonal blocks $X_{(i,i)}$ are size $m_i \times m_i$, the blocks $X_{(i,i)}$ satisfy

$$X_{(i,i)} M_{m_i,\lambda_i} N_{m_i,\lambda_i} + M_{m_i,\lambda_i} N_{m_i,\lambda_i} X_{(i,i)}^T = 2 M_{m_i,\lambda_i} N_{m_i,\lambda_i} \tag{A.0.24}$$

and

$$X_{(i,i)}^T N_{m_i,\lambda_i} \overline{M_{m_i,\lambda_i}} + N_{m_i,\lambda_i} \overline{M_{m_i,\lambda_i}} X_{(i,i)} = 0. \tag{A.0.25}$$

Lemma A.0.7 implies that (A.0.24) and (A.0.24) are consistent if and only if $\lambda_i = 0$, and hence if $\mathscr{A}^s = 2$ then C is nilpotent.

Conversely, if C is nilpotent then $\lambda_1 = \cdots = \lambda_\gamma = 0$. Hence, by (3.1.4) and (3.1.5) the relations (A.0.24) and (A.0.25) can be rewritten as

$$X_{(i,i)} J_{0,m_i} S_{m_i} + J_{0,m_i} S_{m_i} X_{(i,i)}^T = 2 J_{0,m_i} S_{m_i} \quad \text{and} \quad X_{(i,i)}^T S_{m_i} J_{0,m_i} + S_{m_i} J_{0,m_i} X_{(i,i)} = 0 \tag{A.0.26}$$

for each i individually. Assuming that $B_{(i,i)} = \mathrm{Diag}\left(x_1^i, \ldots x_{m_i}^i\right)$, by comparing the entries of (A.0.26) with the help of the expressions for matrices J_{0,m_1} and S_{m_i} from (3.1.3), one gets that (A.0.26) is equivalent to

$$\begin{aligned} x_j^i + x_{m_i - j}^i &= 2 \qquad \forall\, 1 \leq j \leq m-1, \\ x_j^i + x_{m-j+2}^i &= 0 \qquad \forall\, 2 \leq j \leq m. \end{aligned} \tag{A.0.27}$$

Finally, it is clear that taking $X_{(i,i)} = D_{m_i}$, where D_{m_i} is as in (A.0.23), satisfies (A.0.27) which completes the proof. $\square$

As a direct consequence of the previous Lemma, since for nilpotent C we have $\mathscr{A} = \mathscr{A}^o + \mathbb{C}I$, one gets immediately the following

Corollary A.0.9. *If C is not nilpotent then in* (4.2.1) *one can take $\eta' = \eta$.*

Now we prove the final result of this section.

Lemma A.0.10. *If H_ℓ and C are in the canonical form prescribed by Theorem 3.1.2 and $C \neq 0$ then*

$$\dim(\mathscr{A}) \leq n^2 - 4n + 6. \tag{A.0.28}$$

Moreover, this bound is attained if and only if (ℓ, A) can be represented by the pair (H_ℓ, C) in the canonical form of Theorem 3.1.2 with

$$C = J_{0,2} \oplus \overbrace{J_{0,1} \oplus \cdots \oplus J_{0,1}}^{n-3 \text{ copies}}. \tag{A.0.29}$$

Proof. Assume that

$$\dim(\mathscr{A}) \geq n^2 - 4n + 6, \tag{A.0.30}$$

and that (H_ℓ, C) are in the canonical form of Theorem 3.1.2. We will still use the notation of Theorem 3.1.2 as well, in particular referring to the sequence $(\lambda_1, \ldots, \lambda_\gamma)$.

Suppose that the λ_is are not all the same. Without loss of generality, we can assume that $(\lambda_1, \ldots, \lambda_\gamma)$ is enumerated so that there exists an integer k such that

$$\lambda_1 = \ldots = \lambda_k \quad \text{and} \quad \lambda_j \neq \lambda_1 \qquad \forall j > k. \tag{A.0.31}$$

Define

$$s = \sum_{i=1}^{k} [\text{number of rows in } M_{\lambda_i, m_i}]$$

where k is as in (A.0.31). By Lemma A.0.1, for every matrix B in $\dim(\mathscr{A}^o + \operatorname{span}\{I\})$, the upper right $(s) \times (n-1-s)$ block and the lower left $(n-1-s) \times (s)$ block of B is zero. Moreover, since the λ_is are not all zero, there is at least one index i such that $B_{(i,i)}$ has zeros on its main diagonal. Accordingly, if the λ_is are not all the same, then

$$\dim(\mathscr{A}^o) + 1 = \dim(\mathscr{A}^o + \operatorname{span}\{I\}) \leq (n-1)^2 - 2s(n-1-s).$$

Since

$$2n - 4 \leq 2j(n-1-j) \qquad \forall\, 1 \leq j < n-1,$$

it follows that

$$\dim(\mathscr{A}) = \dim(\mathscr{A}^o) + 1 \leq (n-1)^2 - 2s(n-1-s) \leq (n-1)^2 - 2n + 4 = n^2 - 4n + 5,$$

where the identity $\dim(\mathscr{A}) = \dim(\mathscr{A}^o) + 1$ follows from Lemma A.0.8 and the assumption that the λ_is are not all the same. Clearly, this contradicts (A.0.30), so if (A.0.30) holds then there exists a value $\lambda \in \mathbb{C}$ such that

$$\lambda = \lambda_i \qquad \forall\, i. \tag{A.0.32}$$

If (A.0.32) holds with $\lambda \neq 0$ then Corollaries A.0.3 and A.0.5 imply that each matrix B in $\mathscr{A}^o$ is fully determined by its entries above the main diagonal, and hence, applying Lemma A.0.8,

$$\dim(\mathscr{A}) \leq \frac{(n-1)(n-2)}{2} + 1 < n^2 - 4n + 6 \qquad \forall\, n \geq 2.$$

Therefore, if (A.0.32) holds with $\lambda \neq 0$ then our assumption (A.0.30) fails.

In other words, – assuming for a moment that (A.0.30) can be satisfied, which we will prove below by giving an explicit example – if $\dim(\mathscr{A})$ is maximized then we can assume without loss of generality that

$$C = J_{0,m_1} \oplus \cdots \oplus J_{0,m_\gamma} \quad \text{with} \quad m_1 \geq \cdots \geq m_\gamma. \tag{A.0.33}$$

For B in $\mathscr{A}^o$, let us partition B as is done in Lemma A.0.4. By Lemma A.0.4, for $i < j$ the $B_{(i,j)}$ and $B_{(j,i)}$ blocks are together determined by $2m_j$ parameters, whereas, by Corollary A.0.5, the $B_{(i,i)}$ block is determined by $\lceil \frac{m_i}{2} \rceil$ parameters, where $\lceil \frac{m_i}{2} \rceil$ denotes the ceiling function, that is, the smallest integer not less than $\frac{m_i}{2}$. Hence, by counting the number of parameters determining B, Lemma A.0.4 and Corollary A.0.5 imply that if (A.0.33) holds then

$$\dim(\mathscr{A}^o) = \sum_{k=1}^{\gamma} \left(\left\lceil \frac{m_k}{2} \right\rceil + 2(k-1)m_k \right). \tag{A.0.34}$$

Let $r \in \{1, \ldots, \gamma\}$ be an integer such that

$$m_i = 1 \qquad \forall\, i > r,$$

and to compare with C, let us also consider the matrix

$$C' = J_{0,m_1} \oplus \cdots \oplus J_{0,m_{r-1}} \oplus J_{0,1} \oplus \cdots \oplus J_{0,1}.$$

In other words, C' is obtained from C by replacing the last nonzero block on the diagonal of C with zeros. We will compute the dimension of $\mathscr{A}^o$ corresponding to the case where $C = C'$, but, since are going to compare this to the sum in (A.0.34), for clarity let $\mathscr{A}'$ denote the algebra that we would otherwise denote by $\mathscr{A}^o$ corresponding to this case where $C = C'$, and let $\mathscr{A}^o$ still denote the algebra refered to in (A.0.34).

Notice that the kth summand in (A.0.34) counts the number of parameters determining the blocks $B_{(i,j)}$ of a matrix B in $\mathscr{A}^o$ for which $\max\{i, j\} = k$. If we compare the general formula for a matrix B in $\mathscr{A}^o$ to that of a matrix B' in $\mathscr{A}'$, the only difference appears in the blocks $B_{(i,j)}$ of B for which $\max\{i, j\} = r$, and hence a formula for $\dim(\mathscr{A}')$ should match the formula in (A.0.34), except that the rth summand will change. Using Lemma A.0.4 and Corollary A.0.5, it is however straightforward to work out exactly how this rth summand of (A.0.34).

Specifically, in replacing the formula for B with the formula for B', the $B_{(r,r)}$ block is replaced with the $m_r \times m_r$ matrix having m_r^2 independent parameters, whereas, for all $i < r$, $B_{(i,r)}$ (respectively $B_{(r,i)}$) is replaced with a matrix having m_r independent parameters in its first row (respectively column) and zeros elsewhere. Accordingly,

$$\dim(\mathscr{A}') = \dim(\mathscr{A}^o) - \left(\left\lceil \frac{m_r}{2} \right\rceil + 2(r-1)m_r\right) + m_r^2 + 2(r-1)m_r \geq \dim(\mathscr{A}^o). \tag{A.0.35}$$

Since equality holds in (A.0.35) if and only if $m_r = 1$, the dimension of $\mathscr{A}^o$ is maximized with C as in (A.0.33) if and only if

$$C = J_{0,m_1} \oplus \overbrace{J_{0,1} \oplus \cdots \oplus J_{0,1}}^{n-1-m_1 \text{ copies}}, \tag{A.0.36}$$

in which case, by (A.0.34),

$$\dim \mathscr{A}^o = \left\lceil \frac{m_1}{2} \right\rceil + \sum_{k=2}^{n-m_1} (2k-1) = \left\lceil \frac{m_1}{2} \right\rceil + (n-m_1)^2 - 1. \tag{A.0.37}$$

Since $C \neq 0$, this last sum is maximized with C as in (A.0.36) if and only if C is as in (A.0.29), in which case applying (A.0.37) with $m_1 = 2$ yields (A.0.28) because, by Lemma A.0.8, if C is as in (A.0.36) then $\dim \mathscr{A} = \dim \mathscr{A}^o + 2$. □

APPENDIX B

DIMENSION OF THE TRANSFORMATION GROUP PRESERVING TWO SYMMETRIC FORMS OF WHICH AT LEAST ONE IS NONDEGENERATE

The main result of Appendix B is Lemma B.0.1, which gives a formula describing the dimension of the group of transformations preserving two symmetric forms of which at least one is nondegenerate. In particular, this formula gives the dimension of the algebra $\mathscr{A}$ (defined in (4.2.1)) studied in Appendix A for the case where C is nonsingular. Of course, this dimension can alternatively be calculated using the formulas in Appendix A.

For this special case wherein C is nonsingular, we can alternatively characterize $\mathscr{A}$ as an intersection of two certain indefinite orthogonal algebras. Specifically, still letting H_ℓ and C be matrices representing a nondegenerate Hermitian form ℓ and an ℓ-self-adjoint antilinear operator A, if C is nonsingular, then we define bilinear forms Q_1 and Q_2 by

$$Q_1(v,w) = \langle H_\ell C^{-1} v, w\rangle \quad \text{and} \quad Q_2(v,w) = \langle H_\ell \overline{C} v, w\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^{n-1}$. Note,

$$Q_1(v,w) = Q_2(C\overline{C}v, w). \tag{B.0.1}$$

The form Q_1 is symmetric because

$$Q_1(v,w) = \langle H_\ell C^{-1}(C\tilde{v}), C\tilde{w}\rangle \overset{*}{=} \langle H_\ell \tilde{w}, CC^{-1}(C\tilde{v})\rangle = \langle H_\ell C^{-1}(C\tilde{w}), C\tilde{v}\rangle = Q_1(w,v),$$

where $\tilde{v} = C^{-1}v$ and $\tilde{w} = C^{-1}w$, and $*$ holds because A is ℓ-selfadjoint. Similarly, Q_2 is symmetric because $H_\ell\overline{C}$ is a symmetric matrix. For a symmetric form Q, let us denote by $CO(Q)$ the group of matrices that preserve the quadratic form associated with Q up to a scalar multiple, and let $\mathfrak{co}(Q)$ denote the Lie algebra of $CO(Q)$. With this notation, (4.2.1) equivalent to

$$\mathscr{A} = \mathfrak{co}(Q_1) \cap \mathfrak{co}(Q_2). \tag{B.0.2}$$

Of course, the description of $\mathscr{A}$ in (B.0.2) does not apply to the general case where C may be singular because, although the form Q_2 is well defined, Q_1 might not be.

Notice that the algebra in (B.0.2) is indeed the Lie algebra of the group of transformations (conformally) preserving two symmetric forms of which at least one is nondegenerate. Our formula given in Lemma B.0.1 generalizes the formula given [35], wherein the authors consider only groups that preserve a pair of nondegenerate symmetric forms Q and Q' for which $Q'(x, y) = Q(Lx, y)$ for some diagonalizable matrix L, whereas Lemma B.0.1 allows for that matrix L to be arbitrary, which is important in light of (B.0.1) because we want to treat the case where $L = C\overline{C}$.

For the remainder of Appendix B, let Q and Q' denote symmetric forms on a finite dimensional vector space V. We assume Q is nondegenerate, so there exists a linear operator L such that

$$Q'(x, y) = Q(Lx, y).$$

Define

$$\mathcal{A} := \mathcal{A}(Q, Q') := \left\{ B \in \mathrm{Aut}(V) \;\middle|\; \begin{array}{l} Q'(Bx, y) + Q'(x, By) = 0 \text{ and} \\ Q(Bx, y) + Q(x, By) = 0 \quad \forall x, y \in V \end{array} \right\},$$

that is, $\mathcal{A}$ is the intersection of the Lie algebras of the groups of operators that preserve the forms Q and Q'.

We would like to describe the algebra $\mathcal{A}$. For this first let us adopt the notation

$$E_\lambda^{(i)} := \{v \in V \;:\; (L - \lambda I)^i v = 0\} \quad \text{and} \quad E_\lambda := \bigoplus_{i \in \mathbb{N}} E_\lambda^{(i)}.$$

For each $\lambda \in \mathrm{Spec}(L)$, define

$$n_\lambda = \dim\left(\ker\left(L - \lambda I\right)\right)$$

Let us work in a basis in which L has a Jordan normal form,

$$L = \bigoplus_{\lambda \in \mathrm{Spec(L)}} \bigoplus_{i=1}^{n_\lambda} J_{\lambda, m_{\lambda,i}}, \tag{B.0.3}$$

where the matrices $J_{\lambda,m_{\lambda,i}}$ denote the standard Jordan blocks as in (3.1.3) and $m_{\lambda,i} \geq m_{\lambda,j}$ for $i > j$.

Further, we call a $p \times q$ matrix B an upper-triangular Toeplitz matrix (a regular upper-triangular in the terminology of [15]) if it has the form

$$\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{}^{q-p\text{ columns}}} & \\ 0 & \cdots & 0 & \\ \vdots & & \vdots & \sum_{k=0}^{p-1} b_k T_p^k \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \end{array}\right) \text{ if } p \leq q, \quad \text{and} \quad \begin{array}{l} \left(\begin{array}{ccc} \multicolumn{3}{c}{\sum_{k=0}^{q-1} b_k T_q^k} \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right) \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} p-q \text{ rows} \end{array} \quad \text{if } q < p$$

for some sequence of coefficients (b_i). Here again the T_p and T_q refers to the notation given in the beginning of Section 3.

Theorem B.0.1. *There exists a basis in which L has the Jordan canonical form* (B.0.3)*, and the algebra $\mathcal{A}$ consists of the block diagonal matrices*

$$B = \bigoplus_{\lambda \in \mathrm{Spec}(L)} B_\lambda,$$

where each diagonal block B_λ has size $n_\lambda \times n_\lambda$ and is partitioned into blocks $\{B^\lambda_{(k,j)}\}_{k,j=1}^{n_\lambda}$ where $B^\lambda_{(j,k)}$ is a matrix of size $m_{\lambda,j} \times m_{\lambda,k}$ with the following properties:

1. *$B^\lambda_{(k,j)}$ are upper-triangular Toeplitz matrices;*

2. *If $m_{\lambda,j} \leq m_{\lambda,k}$ the right $m_{\lambda,j} \times m_{\lambda,j}$ block of $B^\lambda_{(k,j)}$ equals the upper $m_{\lambda,j} \times m_{\lambda,j}$ block of*

$-B^\lambda_{(j,k)}$, so, in particular, $B^\lambda_{(j,k)}$ is determined by $B^\lambda_{(k,j)}$, and $B^\lambda_{(j,j)} = 0$.

Moreover,

$$\dim \mathcal{A} = \sum_{\lambda \in \mathrm{Spec(L)}} \sum_{k=1}^{n_\lambda - 1} k m_{\lambda,k+1} = \sum_{\lambda \in \mathrm{Spec(L)}} \sum_{k \in \mathbb{N}} \binom{d_{\lambda,k}}{2}, \tag{B.0.4}$$

where $d_{\lambda,i} = \dim\left(E^{(i)}_\lambda / E^{(i-1)}_\lambda\right)$.

Proof. Since Q is nondegenerate, we can also work in a basis orthonormal with respect to Q. The matrices of the operators with respect to this basis will be denoted by the same letters as the matrices of the same operators with respect to the basis in which L has a Jordan normal form, but with the circumflex ˆ above the letter. In this basis

$$\mathcal{A} = \left\{ \hat{B} \;\middle|\; \hat{B} = -\hat{B}^T \text{ and } \left[\hat{L}, \hat{B}\right] = 0 \right\}.$$

Note that $\left[\hat{L}, \hat{B}\right] = 0$ implies that $[L, B] = 0$. In [15, Chapter 8, Section 2] an explicit description is given for matrices that commute with a matrix in a Jordan normal form. From this description it follows that a matrix B in $\mathcal{A}$ (w.r.t. the basis pertaining to (B.0.3)) must satisfy condition (1) of Theorem B.0.1. It remains to show that the additional condition

$$\hat{B} = -\hat{B}^T \tag{B.0.5}$$

implies condition (2).

According to [15, Chapter 11.3, Corollary 2] there exists a basis orthonormal with respect to Q such that in this basis

$$\hat{L} = \bigoplus_{\lambda \in \mathrm{Spec}(L)} \bigoplus_{i=1}^{n_\lambda} J_\lambda^{(m_{\lambda,i})}$$

where $J_\lambda^{(m_{\lambda,j})}$ denotes the $m_{\lambda,j} \times m_{\lambda,j}$ symmetric matrix defined by

$$J_\lambda^{(m_{\lambda,j})} := \frac{1}{2}(I + iS_{m_{\lambda,j}})J_{\lambda,m_{\lambda,j}}(I - iS_{m_{\lambda,j}}),$$

with $S_{m_{\lambda,i}}$ and $J_{\lambda,m_{\lambda,i}}$ again being as in (3.1.3).

By construction

$$\hat{B} = SBS^{-1},$$

where $S = \bigoplus_{j=1}^{n_\lambda}(I + iS_{m_{\lambda,j}})$. Note that $S^2_{m_{\lambda,j}} = I$. Therefore condition (B.0.5) is equivalent to

$$\begin{aligned}(I + iS_{m_{\lambda,k}})B^\lambda_{(k,j)}(I - iS_{m_{\lambda,j}}) &= 2\hat{B}^\lambda_{(k,j)} = -2\left(\hat{B}^\lambda_{(j,k)}\right)^T \\ &= -\left((I + iS_{m_{\lambda,j}})B^\lambda_{(j,k)}(I - iS_{m_{\lambda,k}})\right)^T = (iS_{m_{\lambda,k}} - I)\left(\hat{B}^\lambda_{(j,k)}\right)^T(I + iS_{m_{\lambda,j}}),\end{aligned}$$

which implies that

$$B^\lambda_{(k,j)} = -\frac{1}{4}(iS_{m_{\lambda,k}} - I)^2\left(B^\lambda_{(j,k)}\right)^T(I + iS_{m_{\lambda,j}})^2 = -S_{m_{\lambda,k}}\left(B^\lambda_{(j,k)}\right)^T S_{m_{\lambda,j}}.$$

Noting that $\left(B^\lambda_{(j,k)}\right)^T$ is upper-triangular Toeplitz, this yields condition (2) of Theorem B.0.1.

Further, the block $B^\lambda_{(j,k)}$ with $j < k$ is determined by $m_{\lambda,k}$ variables. Therefore, counting the number of variables that determine B, we obtain the first equality in the chain of the equalities (B.0.4). It is an elementary exercise to get the second equality in this chain. □

The significance of Theorem B.0.1 for our present broader study of CR geometry is that, regarding the algebra $\mathscr{A}$ in (B.0.2), by setting $Q_1 = Q'$, $Q_2 = Q$, and $L = C\overline{C}$, we get $\mathscr{A} = \mathcal{A} \oplus \mathrm{span}_{\mathbb{C}}(I)$ (an identity that indeed holds when C is nonsingular).

For example, we conclude this appendix with an application of Lemma B.0.1.

Lemma B.0.2. *If $C\overline{C}$ is invertible and its eigenspaces are all 1-dimensional then*

$$\mathscr{A} = \{ sI \mid s \in \mathbb{C} \}. \tag{B.0.6}$$

Proof. Setting $Q' = Q_1$, $Q = Q_2$, and $T = C\overline{C}$, applying Lemma B.0.1 yields $\dim(\mathscr{A}) = 1$ because $\mathscr{A} = \mathcal{A} \oplus \text{span}_{\mathbb{C}}(I)$. Since $\mathscr{A}$ contains multiples of I, (B.0.6) is correct. $\square$

Lemma B.0.2 is, of course, alternatively immediately obtained from the formulas derived in Appendix A. The method for obtaining it described Appendix B, regarding $\mathscr{A}$ as an intersection of two familiar Lie algebras is, however, more conceptual.