GENERALIZED GRIGORCHUK'S OVERGROUPS: GROWTH, CLUSTER POINTS, AND ASSOCIATED RATIONAL MAPS

A Dissertation

by

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ABSTRACT

In this dissertation, we construct and investigate a family $\left\{\widetilde{\mathcal{G}}_\omega \mid \omega \in \{0,1,2\}^{\mathbb{N}}\right\}$ of groups that generalize the famous Grigorchuk's overgroup. Our work is spitted into three parts: (i) study of growth, (ii) study of topological and algebraic properties of the closure of the family $\left\{\widetilde{\mathcal{G}}_\omega\right\}$ in the space $\mathcal{M}_8$ of marked 8-generated groups, and (iii) developing the technical tools of dynamic origin for study the spectral problems associated with the groups $\widetilde{\mathcal{G}}_\omega$.

In the first part, we show, if ω is eventually constant, then $\widetilde{\mathcal{G}}_\omega$ is of polynomial growth, and if ω is not eventually constant, then $\widetilde{\mathcal{G}}_\omega$ is of intermediate growth. In the case of non-eventually constant ω, we give a universal lower bound for the growth rate and an upper bound for homogeneous sequences.

The second part contains the observation that this family is not closed, and the closure is the union of the (countable) set of isolated points and a Cantor set. The cluster points are constructed using branch-type algorithms and are closely related to the Lamplighter groups. Finally, we show that the generalized overgroups that are of intermediate growth are infinitely presented.

The final part is dedicated to studying the Schur complements and multi-dimensional rational maps associated with the generalized overgroups. First, we compute the Schur complements and multi-dimensional rational maps associated with some groups, including the generalized overgroups. These rational maps can be realized as two-dimensional and do belong to a two-parametric family of maps. The two-parametric maps have the integrability property of being semi-conjugate to the Chebyshev map. We show that any random iterations of two-parametric maps, viewed as maps on projective space, are algebraically stable in a rational variety.

DEDICATION

To Sanduni,

whom I share my life with

දුරු රටක හුදකලාව සිටි මා අසලට නොසිතූ විරූ මොහොතක පැමිණ,

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මගේ දයාබර සඳුනිට

To my mom,

who supported me from my birth

“මම” නමැති ප්‍රාණියාට ජීවන හුස්ම පිඹ,

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දහසකුත් එකක් බාදක කම්කටොලු හමුවේ මහමෙරක් මෙන් නොසැලී මා අසලින් හුන්

මගේ දයාබර අම්මාට

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First and foremost I am extremely grateful to my advisor, Professor Rostislav Grigorchuk. His immense knowledge, plentiful experience, and invaluable advice have helped me from the very first day that I started working with him. The extraordinary patience he had towards me has supported me not only in the field of mathematics but also in my personal life.

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Contributors

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The articles [Sam20, Sam22] and a part of [GS21] are reproduced in Chapters 3, 4, and 5, respectively. Parts of above mentioned articles are also used in Chapters 1 and 2. The discussion in Section 5.5 consist of an ongoing project with Nguyen-Bac Dang, Rostislav Grigorchuk, and Mikhail Lyubich.

All other work conducted for the dissertation was completed by the student independently.

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1. INTRODUCTION*

The growth rate of groups is a long studied area [Švа55, Mil68, Gri91] and it was known that growth rates of groups can vary from polynomial growth through intermediate growth to exponential growth. First group of intermediate growth (the growth which is neither polynomial nor exponential), known as the first Grigorchuk's torsion group $\mathcal{G}$, was constructed by Rostislav Grigorchuk in 1980 [Gri80] as a finitely generated infinite torsion group and later [Gri83] it was shown that it has intermediate growth.

At the same time, in [Gri83, Gri84b] (also see [Gri85]) an uncountable family of groups $\left\{\mathcal{G}_\omega \mid \omega \in \Omega = \{0,1,2\}^{\mathbb{N}}\right\}$, known as generalized Grigorchuk's groups were constructed. They consist of groups of intermediate growth when sequence ω is not virtually constant and of polynomial growth when sequence ω is virtually constant [Gri84b].

Since the construction of the first Grigorchuk group, there was an expansion of the area of study and new groups of intermediate growth were introduced [Gri84a, KP13, BE14, BGN15, Nek18]. The group $\widetilde{\mathcal{G}}$ known as the Grigorchuk's overgroup [BG00a] is an infinite finitely generated group of intermediate growth which shares many properties with first Grigorchuk's group [BG02]. In contrast, the Grigorchuk's overgroup has an element of infinite order [BG00a].

Grigorchuk's space $\mathcal{M}_k$ of marked groups with $k(\geqslant 2)$ generators was introduced in 1984 [Gri84b]. It is a totally disconnected, compact metric space with complicated structure of isolated points as shown by Y. de Cornulier, L. Guyot and W. Pitsch [dCGP07] and non-trivial perfect kernel that is homeomorphic to a Cantor set. The space also was studied in [Cha00, CG05, BK20] and other articles.

The space of marked groups was used by Grigorchuck to show the family $\{\mathcal{G}_\omega\}_{\omega\in\Omega}$ consists of

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infinitely presented groups (when ω is not virtually constant). Also, a modification of the construction lead him to show in [Gri84b], that the family is closed and perfect subset of $\mathcal{M}_4$ and hence is homeomorphic to a Cantor set.

The further investigations showed usefulness of spaces $\mathcal{M}_k, k \geqslant 2$ for study of group properties such as (non-elementary) amenability and for constructions in group theory, in particular to study of IRS (invariant random subgroups) on a free group and other groups [Bow15, BGN15].

In 1957, M. Day asked whether all amenable groups are elementary amenable [Day57]. It was answered negatively, by the construction of groups of intermediate growth [Gri84b]. Next examples negating Day's problem came from theory of self-similar groups. One such group is the Basilica group [GZ02], which is amenable but not sub-exponentially amenable [BV05]. Most recent examples of non-elementary amenable groups are topological full groups associated with minimal Cantor system, which were used to construct finitely generated simple non-elementary amenable groups [JM13].

In 1996, Stepin observed that constructions similar to the one in [Gri84b], can lead to new families of non-elementary amenable groups [Ste96]. Namely, if one finds suitable Cantor set of groups containing a countable dense subset of (perhaps elementary) amenable groups and a co-meager set consisting of non-elementary groups, then standard argument based on Baire category insures that there is a co-meager set of non-elementary amenable groups. (See [WW17] for non-constructive proof of existence of non-elementary amenable groups using set theoretic approach.)

The groups $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ belong to an important class of groups called self-similar groups. Self-similar groups were used to solve several outstanding problems in different areas of mathematics. They provide an elegant contribution to the general Burnside problem [Gri80], to the J. Milnor problem on growth [Gri83, Gri84b], to the von Neumann - Day problem on non-elementary amenability [Gri84b, Gri98], to the Atiyah problem in L^2-Betti numbers [GLSZ00], etc. Self-similar groups have applications in many areas of mathematics such as dynamical systems, operator algebras, random walks, spectral theory of groups and graphs, geometry and topology, computer science, and many more (see the surveys [GNS00, BGN03, Gri05, GN07, Gri11, Gri14, GNŠ15, GLN17] and the monograph [Nek05]).

Multi-dimensional rational maps appear in the study of spectral properties of graphs and unitary representations of groups (including representations of Koopman type). The spectral theory of such objects is closely related to the theory of joint spectrum of a pencil of operators in a Hilbert (or more generally in a Banach) space and is implicitly considered in [BG00b] and explicitly outlined in [Yan09].

These multi-dimensional rational maps are *very special* and *quite degenerate* as claimed by N. Sibony and M. Lyubich, respectively. Nevertheless, they are interesting and useful, as, on the one hand, they are responsible for the associated spectral problems, on the other hand, they give a lot of material for people working in dynamics, being quite different from the maps that were considered before.

Some of them demonstrate features of integrability, which means that they semiconjugate to lower-dimensional maps, while the others do not seem to have integrability features and their dynamics (at least on an experimental level) demonstrate the chaotic behavior.

In this dissertation, we construct a family of groups called *generalized overgroups* and explore many properties of them. The construction of these groups, discussed in Section 2.5, closely follows the construction of $\{\mathcal{G}_\omega\}_{\omega\in\Omega}$. Chapter 2 contains some basic preliminaries that are used throughout the dissertation. The study of generalized overgroups are divided into three parts, which are discussed in Chapter 3, 4, and 5. Most of the results discussed here are published in three articles [Sam20], [Sam22] and [GS21].

Chapter 3, extracted from the article [Sam20], discusses the growth of the generalized overgroups. There, we give the description on the growth rate of generalized overgroups (see Theorem 3.1) and give upper and lower bounds for some subclass of groups (see Theorem 3.2 and Proposition 3.4).

Chapter 4 is devoted to study the structure of the set consisting of generalized overgroups as a subset of the space of marked groups of 8-generators. The set is not closed and the closure of it is the union of a Cantor set and the set of isolated points (see Theorem 4.2). The cluster points not in the above set are constructed in Section 4.2.2 and their properties are discussed (see Theorem 4.3). Material of this chapter is published in the article [Sam22].

Chapter 5, the final chapter, consists of the results from the article [GS21], written in collaboration with Rostislav Grigorchuk, and some results obtained under the guidance of Nguyen-Bac Dang, Rostislav Grigorchuk, and Mikhail Lyubich. There, we discuss the method of Schur complements, which can be used to compute spectra of groups. In Section 5.4, the computations of the Schur complements and associated rational maps, for the groups discussed in this text, are given. Integrability properties of these rational maps and related 2-parametric maps are presented in Section 5.5 (see Theorem 5.2 and 5.3).

At the end of the dissertation, there is a short appendix, where some computation are presented. These are used in the preceding chapters, but does not fall in line with the flow of the main text.

2. PRELIMINARIES*

In this chapter, we will introduce some preliminary notions and facts, which will be used in the rest of the text.

2.1 Growth of Groups

Let G be a finitely generated group and let S be a finite *symmetric* (i.e., $s^{-1} \in S$ if $s \in S$) set, not containing the identity of G, that generates G. Now consider the *alphabet* (i.e., a collection of letters) S and let W be a *word* over the alphabet S (by a word over an alphabet, we mean a freely reduced element of the free group generated by the alphabet). The number of letters in W is denoted by $|W|$ and for $s \in S$, the number of occurrences of s in W is denoted by $|W|_s$. For $g \in G$, the *length* of g, denoted by $|g|$, is defined by,

$$|g| = \min\{|W| \colon g = W \text{ in } G\}.$$

Definition 2.1. *Let G be a group generated by a finite symmetric set S. The growth function of G with respect to S (also known as the volume growth function), $\gamma_{G,S} \colon \mathbb{N}_0 \to \mathbb{N}$, is defined by,*

$$\gamma_{G,S}(n) = |B_{G,S}(n)|,$$

where $B_{G,S}(n) = \{g \in G \colon |g| \leqslant n\}$.

Observe that the set $B_{G,S}(n)$ is the ball of radius n and center 1 (the identity in G) in the *Cayley graph*, $\mathrm{Cay}(G, S)$, which is defined in Section 2.2.

There is a partial order relation $\preceq$ for growth functions defined by $f \preceq g$ if and only if there

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are constants A and B such that $f(n) \leqslant Ag(Bn)$ for all n. We define an equivalence relation $\simeq$ by, $f \simeq g$ if and only if $f \preceq g$ and $g \preceq f$. The equivalence class of $\gamma_{G,S}(n)$ is known as the *growth rate* of the group G. The growth rate of a group does not depend on the generating set. So we denote the growth rate of a group G, by $\gamma_G(n)$. Growth rate can be *polynomial*, *exponential*, or *intermediate* if $\gamma_{G,S}(n) \simeq n^d$ for some positive integer d, $\gamma_{G,S}(n) \simeq e^n$, or $n^d \precnsim \gamma_{G,S}(n) \precnsim e^n$ for all positive integers d, respectively. Growth above polynomial is called *super-polynomial* and growth below exponential is called *sub-exponential*.

The *growth exponent* $\lambda_{G,S}$ of a group G generated by S, is given by $\lambda_{G,S} = \lim_n \left(\gamma_{G,S}(n)\right)^{1/n}$, and $\lambda_{G,S} \geqslant 1$ for any finitely generated group G. Note that $1/\lambda_{G,S}$ is the radius of convergence of the generating function of $\{\gamma_{G,S}(n)\}$. An easy exercise shows that, for finitely generated, infinite group G with generating set S,

$$\lim_n \left(\gamma_{G,S}(n)\right)^{1/n} = \lim_n \left(\gamma'_{G,S}(n)\right)^{1/n}, \tag{2.1}$$

by looking at the radii of convergence of generating functions of $\{\gamma_{G,S}(n)\}$ and $\left\{\gamma'_{G,S}(n)\right\}$, where $\gamma'_{G,S}(n) = |B_{G,S}(n) \backslash B_{G,S}(n-1)| = \gamma_{G,S}(n) - \gamma_{G,S}(n-1)$ is the *spherical growth function* of G with respect to the generating set S. For finite indexed subgroup H of G,

$$\gamma_{H,S}(n) \leqslant \gamma_{G,S}(n) \leqslant \gamma_{H,S}(n+N),$$

where $\gamma_{H,S}(n) = |B_{G,S}(n) \cap H|$ and N is the maximum of lengths of *right coset representatives* of H in G. Thus for an infinite group G, we get,

$$\lim_n \left(\gamma_{H,S}(n)\right)^{1/n} = \lim_n \left(\gamma'_{H,S}(n)\right)^{1/n} = \lim_n \left(\gamma_{G,S}(n)\right)^{1/n}. \tag{2.2}$$

Here $\gamma'_{H,S}(n) = |(B_{G,S}(n) \backslash B_{G,S}(n-1)) \cap H|$. It is known that $\lambda_{G,S} > 1 \iff G$ has exponential growth [Gri84b].

2.2 Graphs Associated With Groups

A *graph* is an ordered pair (V, E), consisting a set V of *vertices* and a set E of *edges*, together with two maps $i, t\colon E \to V$. For an edge $e \in E$, the vertex $i(e)$ is called the *initial vertex* and the vertex $t(e)$ is called the *terminal vertex* of e. In the case of $i(e) = t(e)$, we say the edge e is a *loop*. So, our definition of a graph, is called as a directed multi-graph or an oriented multi-graph, in graph theory. Depending on the situation, graph can be *non-oriented* (if the edges are independent of the orientation, i.e., instead of the two maps i, t, the graph has only one map from E to the set of unordered pairs of V) and *labeled* (if edges are colored by elements of a certain alphabet). We only consider *connected locally finite graphs* (the later means that each vertex is incident to a finite number of edges). The *degree* $d(u)$ of the vertex u is the number of edges incident to it (where each edge from or to u contributes 1 to the degree and each loop contributes 2 to the degree). A graph is of *uniformly bounded degree* if there is a constant C such that $d(v) \leqslant C$ for all $v \in V$, and is a *regular graph* if all vertices have the same degree.

There is a rich source of examples of graphs coming from groups, such as the Cayley graphs and the Schreier graphs.

Definition 2.2. *Let G be a group generated by a set S (usually, we assume $|S| < \infty$, which makes G finitely generated). The left Cayley graph, denoted by $\mathrm{Cay}_l(G, S)$, is the graph with the vertex set G and the edge set $\{(g, sg) \mid g \in G \text{ and } s \in S \cup S^{-1}\}$, where g is the initial vertex and sg is the terminal vertex of the edge (g, sg).*

Similarly, one can define the *right Cayley graph*, $\mathrm{Cay}_r(G, S)$. There is a natural *graph isomorphism* (i.e., a bijection between set of vertices, preserving edge adjacencies and directions) between the left and right Cayley graphs. They are *vertex transitive*, i.e., the group $\mathrm{Aut}(\mathrm{Cay}(G, S))$ of automorphisms acts transitively on the set of vertices. This is due to that fact that the right translations by elements of G on the vertex set induce automorphisms of $\mathrm{Cay}_l(G, S)$. When speaking about Cayley graph, we usually keep in mind the left Cayley graph. Depending on the situation, Cayley graphs are considered as labeled (the edge (g, sg) has the label s), or unlabeled (if labels do not play a role). Cayley graphs can also be converted into undirected graphs by identification of pairs $(g, sg), (sg, s^{-1}(sg)) = (sg, g)$ of mutually inverse edges. Examples of Cayley graphs are presented in Figure 2.1. Non-oriented Cayley graph $\mathrm{Cay}(G, S)$ is d-regular with $d = 2|S \backslash S_2| + |S_2|$, where $S_2 \subset S$ is the set of generators whose order is two (involutions).

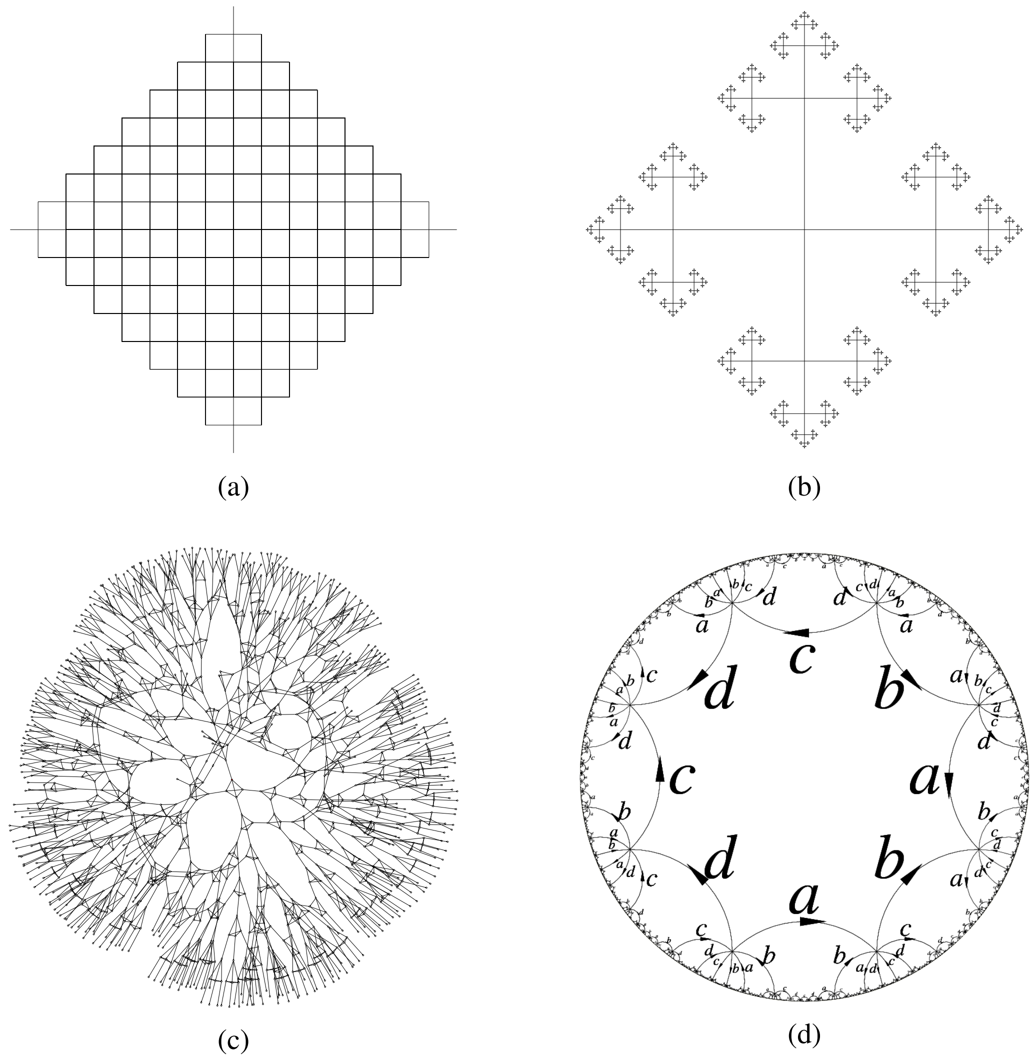


Figure 2.1: Cayley graphs of (**a**) $\mathbb{Z}^2$, (**b**) free group of rank 2, (**c**) group of intermediate growth $\mathcal{G}$, (**d**) surface group of genus 2.

Definition 2.3. *Let G be a group generated by a set S and let H be a subgroup of G. The left Schreier graph (also known as the left Schreier coset graph), denoted by* $\mathrm{Sch}_l(G, H, S)$, *is* *the graph with the vertex set* $G/H = \{gH \mid g \in G\}$*, the set of left cosets, and the edge set* $\{(gH, sgH) \mid g \in G$ *and* $s \in S \cup S^{-1}\}$*, where* gH *is the initial vertex and* sgH *is the terminal vertex of the edge* (gH, sgH).

Again, one can consider a right version of the definition, oriented or non-oriented, labeled or unlabeled versions of the Schreier graph (see [NP20, DDMN10, BDN17] for applications).

Given a set X, on which the group G acts, and a distinguished point $x_0 \in X$, there is an associated graph called the *orbital graph*, in which the vertex set is Gx_0, the orbit of x_0, the edge set is $\{(x, sx) \mid x \in Gx_0$ and $s \in S\}$, where the initial and terminal vertices of the edge (x, sx) are x, sx, respectively. Note that the Schreier graph $\mathrm{Sch}(G, H, S)$ is an orbital graph with respect to the action on G/H by left multiplication. Conversely, every orbital graph of a transitive action (any action can be converted in to a transitive action by restricting the space to a single orbit) can be identified with the Schreier graph $\mathrm{Sch}(G, G_{x_0}, S)$. Therefore, the orbital graphs and Schreier graphs are the same.

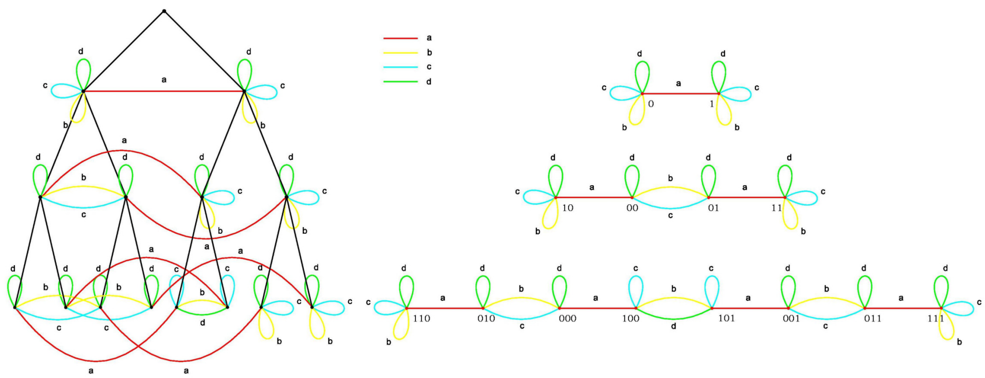
Cayley graph $\mathrm{Cay}(G, S)$ is isomorphic to the Schreier graph $\mathrm{Sch}(G, H, S)$ when the subgroup $H = \{1\}$ is the trivial subgroup. Non-oriented Schreier graphs are also d-regular with d given by the same expression as of Cayley graphs, but in contrast with Cayley graphs, they may have a trivial group of automorphism. Examples of Schreier graphs are presented in the Figure 2.2.

Schreier graphs have much more applications in mathematics being able to provide a geometrical-combinatorial representation of many objects and situations. In particular, they are used to approximate fractals, Julia sets, study the dynamics of groups of iterated monodromy, Hanoi Tower Game on d pegs for $d \geqslant 3$, etc.

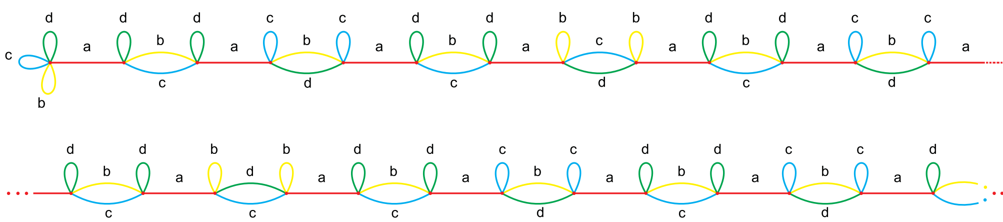
2.3 Groups Acting on Binary Rooted Tree $\mathcal{T}_d$

Let $X = \{x_1, \ldots, x_d\}$ be an alphabet over d symbols $x_1, \ldots, x_d$. We denote the *free monoid* generated by X (i.e., the set of finite words over the alphabet X with *concatenation* operation) by X^*, where the *empty word* is denote by $\emptyset$. Let $X^{\mathbb{N}}$ denote the set of infinite words over X.

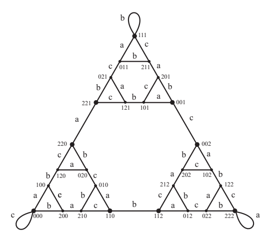
The *d-regular rooted tree* $\mathcal{T}_d$ is the *labeled infinite graph* with vertex set X^*, distinguished vertex $\emptyset$ called the root, and the edge set E, where two vertices u, v are connected by an edge in E



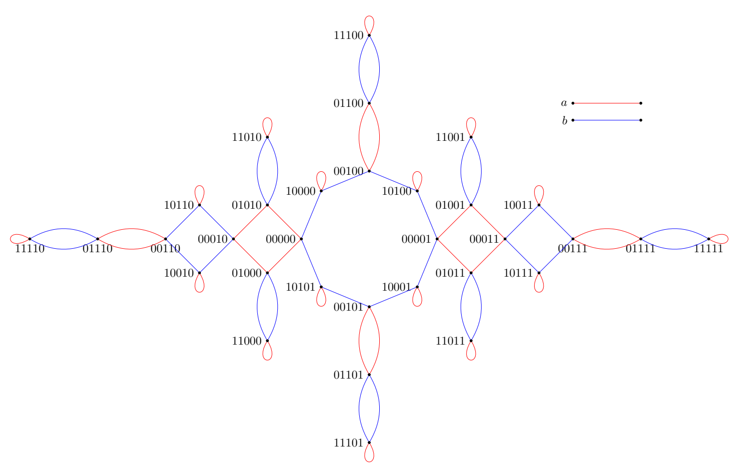
(a)



(b)



(c)



(d)

Figure 2.2: Schreier graphs of (**a**) $\mathcal{G}$ (finite), (**b**) $\mathcal{G}$ (infinite and bi-infinite), (**c**) Hanoi group $H^{(3)}$, (**d**) Basilica.

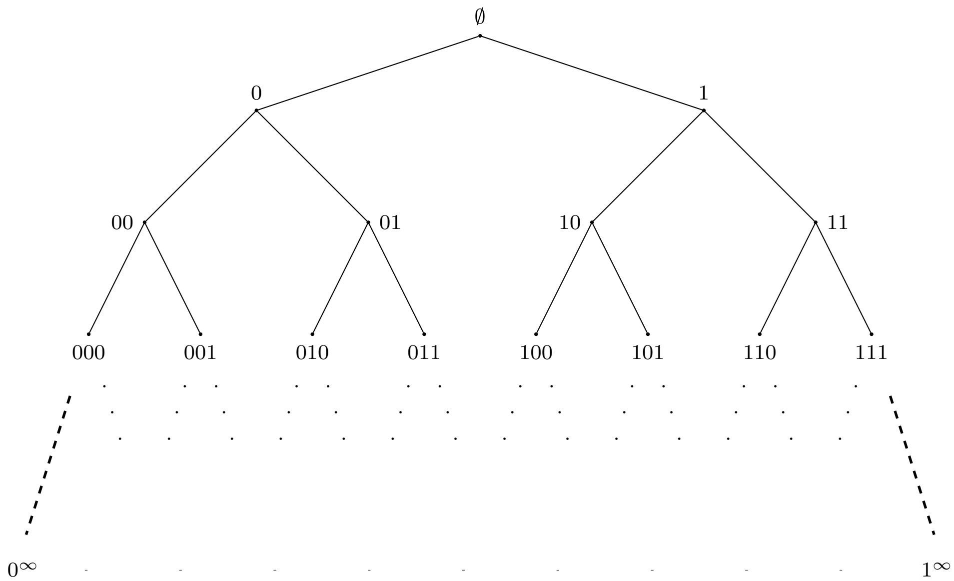


Figure 2.3: Labeled binary rooted tree $\mathcal{T}_2$

if and only if $u = xv$ or $v = xu$ for some $x \in X$. Figure 2.3 represents the *binary rooted tree* $\mathcal{T}_2$, geometrically. We may abuse the notation and write $v \in \mathcal{T}_d$ to indicate a vertex v. For each $n \geqslant 0$, the set of vertices of $\mathcal{T}_d$ whose label has n letters is called the level n of $\mathcal{T}_d$.

The *boundary* of $\mathcal{T}_d$, denoted by $\partial\mathcal{T}_d$, is the set of infinite words $X^{\mathbb{N}}$. It is a topological space under the (Tychonoff) product induced by the discrete space X. Thus, the $\partial\mathcal{T}_d$ is homeomorphic to a Cantor set. The boundary $\partial\mathcal{T}_d$ is a measure space together with the Bernoulli measure μ induced by the distribution on X. In this text, we restrict μ to be the uniform Bernoulli measure.

A bijective map on vertex set of $\mathcal{T}_d$ is said to be an *automorphism* of $\mathcal{T}_d$ if it preserves the tree structure. In other words, g is an automorphism of $\mathcal{T}_d$ if g fixes the root (i.e., $g(\emptyset) = \emptyset$) and preserves the edge adjacencies (i.e., u, v are adjacent in $\mathcal{T}_d$ if and only if $g(u), g(v)$ are adjacent in $\mathcal{T}_d$). Thus, automorphisms preserve each level of $\mathcal{T}_d$ and permute vertices within each level. This is called the *permutation action* of the automorphism on levels of $\mathcal{T}_d$. The collection of automorphisms of $\mathcal{T}_d$, denoted by $\mathrm{Aut}(\mathcal{T}_d)$, is a group under composition operation. Subgroups of $\mathrm{Aut}(\mathcal{T}_d)$ are called the *groups acting on* $\mathcal{T}_d$.

For $g \in \mathrm{Aut}(\mathcal{T}_d), v \in \mathcal{T}_d$, there is a unique element in $\mathrm{Aut}(\mathcal{T}_d)$, denoted by $g|_v$, such that

$g(vu) = g(v)g|_v(u)$, for all $u \in T_2$. The element $g|_v$ is called the *section of g at v*. Some basic properties of sections are given below.

Proposition 2.1. *Let $f, g \in \mathrm{Aut}(\mathcal{T}_d)$ and $v, u \in \mathcal{T}_d$. Then*

1. $(fg)|_v = f|_{g(v)}g|_v$,

2. $g|_{uv} = (g|_u)|_v$.

Proof. Let $f, g \in \mathrm{Aut}(\mathcal{T}_d)$ and $v, u \in \mathcal{T}_d$. First note that $(fg)(vw) = f(g(vw)) = f(g(v)g|_v(w)) = f(g(v))f|_{g(v)}(g|_v(w)) = (fg)(v)(f|_{g(v)}g|_v)(w)$ and $(fg)(vw) = (fg)(v)(fg)|_v(w)$, and therefore $(fg)(v)(f|_{g(v)}g|_v)(w) = (fg)(v)(fg)|_v(w)$ for all $w \in \mathcal{T}_d$. Since $w \in \mathcal{T}_d$ is arbitrary, we obtain the first assertion.

Now note that $g(uvw) = g(uv)g|_{uv}(w) = g(u)g|_u(v)g|_{uv}(w)$ and $g(uvw) = g(u)g|_u(vw) = g(u)g|_u(v)(g|_u)|_v(w)$, and therefore $g(u)g|_u(v)g|_{uv}(w) = g(u)g|_u(v)(g|_u)|_v(w)$ for all $w \in \mathcal{T}_d$. Since $w \in \mathcal{T}_d$ is arbitrary, we obtain the second assertion. □

Let $\sigma\colon \partial\mathcal{T}_d \to \partial\mathcal{T}_d$ be the shift map. Then, σ is a measure preserving transformation. The group of automorphisms $\mathrm{Aut}(\mathcal{T}_d)$ acts on $\partial\mathcal{T}_d$ in a canonical way by, $g \cdot \xi = g(x)g|_x \cdot \sigma\xi$, where x is the first symbol of $\xi \in \partial\mathcal{T}_d$. It can be seen that this action is an action by homeomorphisms. Note that,

$$\sigma g \cdot x\xi = g|_x \cdot \sigma x\xi, \tag{2.3}$$

for $x \in X$ and $\xi \in \partial\mathcal{T}_d$, since $\sigma g \cdot x\xi = \sigma g(x)g|_x \cdot \xi = g|_x \cdot \xi = g|_x \cdot \sigma x\xi$.

Any automorphism is uniquely identified by its sections at vertices of level 1 and the permutation action on level 1. This identification is called the *wreath recursion* and induces an isomorphism of groups given by,

$$\begin{aligned}\mathrm{Aut}(\mathcal{T}_d) &\cong (\mathrm{Aut}(\mathcal{T}_d))^d \rtimes \mathcal{S}_d \\ g &\leftrightarrow ((g|_x)_{x\in X}; \tau_g))\,, \end{aligned} \tag{2.4}$$

where τ_g is the permutation on level 1 of $\mathcal{T}_d$ by g and the action of $\mathcal{S}_d$ on $(\mathrm{Aut}(\mathcal{T}_d))^d$ is by permutation of coordinates.

For $V \subset X^*$, define *stabilizer of* V, denoted by $\mathrm{Stab}(V)$, to be the subgroup of automorphisms that fix all the vertices in V. If V is singleton, say $V = \{v\}$, we denote $\mathrm{Stab}(\{v\})$ by $\mathrm{Stab}(v)$. The *level stabilizer*

$$\mathrm{Stab}(n) = \bigcap_{v \in \text{ level } n \text{ of } \mathcal{T}_d} \mathrm{Stab}(v)$$

contains automorphisms that fix all the vertices in the n-th level. If an automorphism fixes a vertex v, then it fix all the vertices in the *ray* $\emptyset - v$ (by the ray $u - v$, we mean the sequence of distinct vertices starting with u, ending with v, and each consecutive pair of vertices are adjacent). In particular, in the case of $\mathcal{T}_2$, it fixes a vertex of level 1 on ray $\emptyset - v$. Since there are only two vertices on level 1, fixing one vertex forces the other vertex to be fixed. So, any automorphism of $\mathcal{T}_2$ that fix one non-root vertex is in $\mathrm{Stab}(1)$. Also, an automorphism that fixes n-th level fixes all the levels above n.

Since the automorphisms in $\mathrm{Stab}(1)$ fix the vertices of level 1, the wreath recursions (2.4) translates into,

$$\begin{aligned} \psi \colon \mathrm{Stab}(1) &\cong (\mathrm{Aut}(\mathcal{T}_d))^d \\ g &\mapsto (g|_x)_{x \in X}. \end{aligned} \tag{2.5}$$

The map ψ is called the *natural embedding*. If $g \in \mathrm{Stab}(2)$, then $g|_x \in \mathrm{Stab}(1)$ for each $x \in X$. Thus, by applying ψ to $g|_{x_1}, \ldots, g|_{x_d}$, and using Proposition 2.1, we obtain $(g|_{x_1 x})_{x \in X}, \ldots, (g|_{x_d x})_{x \in X}$, respectively. We may abuse the notation and write $\psi^2(g) = \psi \circ \psi(g) = (g|_{xy})_{x,y \in X}$, which is an isomorphism from $\mathrm{Stab}(2)$ to $(\mathrm{Aut}(\mathcal{T}_d))^{d^2}$. Applying the above argument inductively, we obtain

$$\begin{aligned} \psi^n \colon \mathrm{Stab}(n) &\cong (\mathrm{Aut}(\mathcal{T}_d))^{d^n} \\ g &\mapsto (g|_{i_1 i_2 \ldots i_n})_{i_1, i_2, \ldots, i_n \in X}. \end{aligned} \tag{2.6}$$

Here the d^n-tuple $(g|_{i_1 i_2 \ldots i_n})_{i_1, i_2, \ldots, i_n \in X}$ is called the *decomposition* of g into the depth n. We may omit ψ, ψ^n and write $g = (g|_x)_{x \in X}$ and $g = (g|_{i_1 i_2 \ldots i_n})_{i_1, i_2, \ldots, i_n \in X}$, respectively, if there are no ambiguity.

Now consider the case where $d = 2$. It is the convention to use the alphabet $\{0, 1\}$ (i.e., $x_1 = 0$ and $x_2 = 1$). Let $V_{1^\infty} = \{1^n \colon n \in \mathbb{N}\}$ and let $g \in \mathrm{Stab}(V_{1^\infty})$. Note that $1^{n+m} = g(1^{n+m}) = g(1^n 1^m) = g(1^n) g|_{1^n}(1^m) = 1^n g|_{1^n}(1^m)$ and so $g|_{1^n}(1^m) = 1^m$, for each $n, m \in \mathbb{N}$. Therefore $g|_{1^n} \in \mathrm{Stab}(V_{1^\infty})$ for each $n \in \mathbb{N}$. By applying ψ, we obtain $g = (g|_0, g|_1) = (g|_0, (g|_{10}, g|_{11})) = \ldots$. This induces an isomorphism

$$\begin{aligned} \mathrm{Stab}(V_{1^\infty}) &\cong (\mathrm{Aut}(\mathcal{T}_d))^{\mathbb{N}} \\ g &\mapsto \{g|_{1^n 0}\}_{n \in \mathbb{N}} \,. \end{aligned} \tag{2.7}$$

We write $g = \{g|_{1^n 0}\}_{n \in \mathbb{N}}$ to indicate the above isomorphism. In this case, since $g|_{1^k} \in \mathrm{Stab}(V_{1^\infty})$, we have

$$\begin{aligned} g|_{1^k} &= \{(g|_{1^k})\,|_{1^n 0}\}_{n \in \mathbb{N}} \\ &= \{g|_{1^{k+n} 0}\}_{n \in \mathbb{N}} \,, \end{aligned} \tag{2.8}$$

using Proposition 2.1.

Now let us define an important class of groups, called *self-similar groups*.

Definition 2.4. *A group G acting on the d-regular rooted tree $\mathcal{T}_d$ is said to be self-similar if for all $g \in G$ and $x \in X$ the section $g|_x$ coming from wreath recursion* (2.4) *belongs to G.*

An alternative way to define self-similar groups is via *Mealy automata* (also known as the *transducers* or the *sequential machines*. See [BGN03] for more on automata).

Examples of self-similar groups that appear in this text are the first Grigorchuk group $\mathcal{G}$ and the Grigorchuk's overgroup $\widetilde{\mathcal{G}}$ (see Section 2.5).

Definition 2.5. *Let G be a self-similar group.*

1. *G is said to be contracting if there is a finite subset N of G such that $g|_v \in N$, for all $g \in G$ and for all sufficiently large v.*

2. *For a contracting group G, the smallest such set N is called the nucleus of G.*

3. *Suppose G is contracting with the nucleus N. Let $n \in \mathbb{N}$ and $g \in G$ be such that $g \in \mathrm{Stab}(n)$ and $g|_v \in N$ for all vertices of level n. Then the collection of sections of g at level n is called the level n nucleus of g.*

The families of groups $\{\mathcal{G}_\omega\}_{\omega\in\Omega}$ and $\left\{\widetilde{\mathcal{G}}_\omega\right\}_{\omega\in\Omega}$, that are of main focus in this text (see Section 2.5), are not necessarily self-similar (in fact, they are almost surely non-self-similar under the uniform Bernoulli measure on Ω). But they have self-similar type properties, which motivate the next definition.

Definition 2.6. *Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable collection of groups acting on $\mathcal{T}_d$.*

1. *The collection $\{G_n\}_{n\in\mathbb{N}}$ is said to be self-similar if for each $n \in \mathbb{N}$ and for each $g \in G_n$, the sections of g at level k are in G_{n+k}, for each $k \in \mathbb{N}$.*

2. *Self-similar collection $\{G_n\}_{n\in\mathbb{N}}$ is said to be contracting if there is a collection of finite subsets $\{N_n\}_{n\in\mathbb{N}}$ with the same size (i.e., $|N_n|$ is independent of n) satisfying the property that for all $n \in \mathbb{N}$ and for all $g \in G_n$, all the sections of g at level k are in N_{n+k}, for all sufficiently large k.*

3. *For a contracting collection $\{G_n\}_{n\in\mathbb{N}}$, the smallest such collection $\{N_n\}_{n\in\mathbb{N}}$ is called the nucleus of $\{G_n\}_{n\in\mathbb{N}}$.*

4. *Suppose $\{G_n\}_{n\in\mathbb{N}}$ is contracting with the nucleus $\{N_n\}_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$ and $g \in G_n$. If there is a $k \in \mathbb{N}$ such that $g \in \mathrm{Stab}(k)$ and $g|_v \in N_{n+k}$ for all vertices of level k, then the collection of sections of g at level k is called the level k nucleus of g.*

We will use Definition 2.4, Definition 2.5 when talking about the groups $\mathcal{G}, \widetilde{\mathcal{G}}$ and Definition 2.6 when talking about groups $\mathcal{G}_\omega, \widetilde{\mathcal{G}}_\omega$.

2.4 Space of Marked Groups

The space of marked groups with k generators $\mathcal{M}_k$, introduced in [Gri84b] is the space consisting of tuples (G, S), where G is a $k-$generated group and S is an ordered set of k elements generating it. Two points (G_1, S_1) and (G_2, S_2) are identified if the canonical map $S_1 \to S_2$ preserving order, extends to a group isomorphism $G_1 \to G_2$. In geometrical view point, this means the Cayley graphs $\mathrm{Cay}(G_1, S_1)$ and $\mathrm{Cay}(G_2, S_2)$ are order isomorphic.

The space $\mathcal{M}_k$ is a metric space together with the *Cayley metric* d given by,

$$d((G_1, S_1), (G_2, S_2)) = 2^{-n},$$

where n is the largest integer such that the balls of radius n centered at identity of the Cayley graphs $\mathrm{Cay}(G_1, S_1)$ and $\mathrm{Cay}(G_2, S_2)$ are order isomorphic. It was shown in [Gri84b] that $\mathcal{M}_k$ is a compact totally disconnected metric space.

Let (G, S) be a point of $\mathcal{M}_k$. Any element in G can be attached to the ordered set S, to obtain a point in $\mathcal{M}_{k+1}$. We will use this fact in this text by viewing some 3-generated group as points of $\mathcal{M}_4$ and 4-generated groups as points in $\mathcal{M}_8$. A canonical way to attache an element to the generating set is to attach the identity as the $k+1$-th generator. Thus, every point of $\mathcal{M}_k$ can be thought of as a point of $\mathcal{M}_n$, for all n not less than k. In fact, this is an embedding of $\mathcal{M}_k$ into $\mathcal{M}_n$. Therefore, one may consider the space $\mathcal{M} = \bigcup_{k \in \mathbb{N}} \mathcal{M}_k$, of finitely generated marked groups, on which we are not concentrating on, since it does not play a role in this text.

2.5 Generalized Grigorchuk's Group $\mathcal{G}_\omega$ and Generalized Grigorchuk's Overgroup $\widetilde{\mathcal{G}}_\omega$

Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$, the set of sequences of three symbols $0, 1, 2$, and define $\Omega_0, \Omega_1, \Omega_2$ to be the subsets of Ω, where Ω_0 the set of all sequences with all three symbols occurring infinitely often, Ω_1 the set of all sequences with exactly two symbols occurring infinitely often, and Ω_2 the set of all eventually constant sequences. Let $\sigma \colon \Omega \to \Omega$ be the left shift. i.e., $(\sigma\omega)_n = \omega_{n+1}$. Denote $h^{-1}gh$, the conjugate of g by h, by g^h, and $g^{-1}h^{-1}gh$, the commutator of g, h, by $[g, h]$, for any group elements g, h.

First, let us define two groups Γ and $\widetilde{\Gamma}$. Let $S = \{a, b, c, d\}$, $\widetilde{S} = \left\{a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\right\}$ be two sets of symbols and let $R, \widetilde{R}$ be the collections of relations on $S, \widetilde{S}$, respectively, where

$$R = \begin{cases} s^2 = 1 & \text{for all } s \in S \\ [s,t] = 1 & \text{for all } s, t \in S \backslash \{a\} \\ bcd = 1 & \end{cases}, \qquad \widetilde{R} = \begin{cases} s^2 = 1 & \text{for all } s \in \widetilde{S} \\ [s,t] = 1 & \text{for all } s, t \in \widetilde{S} \backslash \{a\} \\ bcd = 1 & \\ s\widetilde{s} = \widetilde{a} & \text{for all } s \in \{b, c, d\} \end{cases}. \tag{2.9}$$

Now define $\Gamma = \langle S \mid R \rangle$ and $\widetilde{\Gamma} = \left\langle \widetilde{S} \mid \widetilde{R} \right\rangle$. The relations in $R, \widetilde{R}$ are called *simple reductions* of $\Gamma, \widetilde{\Gamma}$, respectively. Note that $\langle S \backslash \{a\} \mid R \rangle \cong \mathbb{Z}_2^2$, $\left\langle \widetilde{S} \backslash \{a\} \mid \widetilde{R} \right\rangle \cong \mathbb{Z}_2^3$, and the element a is not related to any other element. Therefore, $\Gamma \cong \mathbb{Z} *_f \mathbb{Z}_2^2$ and $\widetilde{\Gamma} \cong \mathbb{Z} *_f \mathbb{Z}_2^3$, where the component $\mathbb{Z}$ corresponds to the free group generated by the element a. Here, $*_f$ stands for the free product. Thus, any element in $\Gamma, \widetilde{\Gamma}$ can be written in the reduced form

$$(a) * a * a \ldots a * a * (a), \tag{2.10}$$

using simple reductions (2.9), where first and last a can be omitted and $*$'s represent letters in $S \backslash \{a\}, \widetilde{S} \backslash \{a\}$, respectively.

Now lets consider automorphism group of binary rooted tree. Let 1 be the identity in $\mathrm{Aut}(\mathcal{T}_2)$ and let $P \in \mathrm{Aut}(\mathcal{T}_2)$, such that $P(0u) = 1u$ and $P(1u) = 0u$ for each $u \in X^*$. Thus, P is defined by the wreath recursion $P = (1, 1; \tau)$, where τ is the permutation in $\mathcal{S}_2$.

For $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \Omega$, define $b_\omega, c_\omega, d_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega \in \mathrm{Stab}(V_{1^\infty})$ to be the elements identified with sequences $\{B_n^\omega\}, \{C_n^\omega\}, \{D_n^\omega\}, \left\{\widetilde{B}_n^\omega\right\}, \left\{\widetilde{C}_n^\omega\right\}, \left\{\widetilde{D}_n^\omega\right\}$, respectively, where,

$$B_n^\omega = \begin{cases} P & \omega_n = 0 \text{ or } 1 \\ 1 & \omega_n = 2 \end{cases}, \qquad \widetilde{B}_n^\omega = \begin{cases} 1 & \omega_n = 0 \text{ or } 1 \\ P & \omega_n = 2 \end{cases},$$

$$C_n^\omega = \begin{cases} P & \omega_n = 0 \text{ or } 2 \\ 1 & \omega_n = 1 \end{cases}, \qquad \widetilde{C}_n^\omega = \begin{cases} 1 & \omega_n = 0 \text{ or } 2 \\ P & \omega_n = 1 \end{cases},$$

$$D_n^\omega = \begin{cases} P & \omega_n = 1 \text{ or } 2 \\ 1 & \omega_n = 0 \end{cases}, \qquad \widetilde{D}_n^\omega = \begin{cases} 1 & \omega_n = 1 \text{ or } 2 \\ P & \omega_n = 0 \end{cases}. \tag{2.11}$$

Also define $a_\omega, \widetilde{a}_\omega \in \mathrm{Aut}(\mathcal{T}_2)$ by $a_\omega = P$ and $\widetilde{a}_\omega = \{P\}_{n\in\mathbb{N}}$. Note that $a_\omega, \widetilde{a}_\omega$ does not depend on ω and so we may drop the subscript and write $a, \widetilde{a}$, respectively. Define $S_\omega = \{a_\omega, b_\omega, c_\omega, d_\omega\}$ and $\widetilde{S}_\omega = \left\{a_\omega, b_\omega, c_\omega, d_\omega, \widetilde{a}_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega\right\}$. Now lets look at some properties of these automorphisms.

Proposition 2.2. *The element P is an involution. Furthermore,*

1. $s^2 = 1$ *for all* $s \in \widetilde{S}_\omega$.
2. $b_\omega c_\omega d_\omega = 1$.
3. $[s, t] = 1$ *for all* $s, t \in \widetilde{S}_\omega \backslash \{a_\omega\}$.
4. $s\widetilde{s} = \widetilde{a}$ *for all* $s \in S \backslash \{a_\omega\}$.

Proof. Note that $P^2(0u) = P(P(0u)) = P(1u) = 0u$ and $P^2(1u) = P(P(1u)) = P(0u) = 1u$ for any $u \in X^*$. Thus $P^2 = 1$. Since $P \neq 1$, P is an involution.

To prove the first assertion, let $s \in \widetilde{S}_\omega$. If $s = a_\omega$ we are done as $a_\omega = P$ is an involution. Suppose $s \neq a_\omega$. Then $s = \{s_n\}$, where $s_n \in \{1, P\}$ for all n. Therefore $s^2 = \{s_n\} \times \{s_n\} = \{s_n^2\} = \{1\} = 1$, which completes the proof of assertion one.

Observe that for each n, two of B_n^ω, C_n^ω, and D_n^ω are P's and the other is a 1. Thus $B_n^\omega C_n^\omega D_n^\omega = 1$ for each n. Therefore, $b_\omega c_\omega d_\omega = \{B_n^\omega\} \times \{C_n^\omega\} \times \{D_n^\omega\} = \{B_n^\omega C_n^\omega D_n^\omega\} = \{1\} = 1$, which proves the second assertion.

Now let $s, t \in \widetilde{S}_\omega \backslash \{a_\omega\}$. Then $s = \{s_n\}$ and $t = \{t_n\}$, where $s_n, t_n \in \{1, P\}$ for all n. Since s_n, t_n commute for each n, s and t commute, which proves the third assertion.

Finally, to prove the last assertion, let $s \in S \backslash \{a_\omega\}$. Then $s = \{s_n\}$, where $s_n \in \{1, P\}$ for all n, and so $\widetilde{s} = \{\widetilde{s}_n\}$. Note that if $s_n = P$, then $\widetilde{s}_n = 1$ and if $s_n = 1$, then $\widetilde{s}_n = P$. So, $s_n\widetilde{s}_n = P$. Therefore, $s\widetilde{s} = \{s_n\} \times \{\widetilde{s}_n\} = \{s_n\widetilde{s}_n\} = \{P\}_{n\in\mathbb{N}} = \widetilde{a}$. This completes the proof. □

The above proposition shows that the sets S_ω and $\widetilde{S}_\omega$ satisfy the simple reductions presented in (2.9). The next proposition summarizes the properties of the sections of the elements of $\widetilde{S}_\omega$.

Proposition 2.3. *For each $g \in \mathrm{Aut}(\mathcal{T}_2)$, the wreath recursion of the conjugate g^P of g by P (which is the same as g^a or g^{a_ω}) is given by,*

$$g^P = (g|_1, g|_0; \tau_g),$$

where the wreath recursion of g is $(g|_0, g|_1; \tau_g)$. The natural embedding of the elements in $\widetilde{S}_\omega$ and their conjugates by P are;

$$\begin{aligned}
b_\omega &= (B_0^\omega, b_{\sigma\omega}), & c_\omega &= (C_0^\omega, c_{\sigma\omega}), & d_\omega &= (D_0^\omega, d_{\sigma\omega}), & \widetilde{a}_\omega &= (P, \widetilde{a}_{\sigma\omega}),\\
\widetilde{b}_\omega &= (\widetilde{B}_0^\omega, \widetilde{b}_{\sigma\omega}), & \widetilde{c}_\omega &= (\widetilde{C}_0^\omega, \widetilde{c}_{\sigma\omega}), & \widetilde{d}_\omega &= (\widetilde{D}_0^\omega, \widetilde{d}_{\sigma\omega}), & &\\
b_\omega^{a_\omega} &= (b_{\sigma\omega}, B_0^\omega), & c_\omega^{a_\omega} &= (c_{\sigma\omega}, C_0^\omega), & d_\omega^{a_\omega} &= (d_{\sigma\omega}, D_0^\omega), & \widetilde{a}_\omega^{a_\omega} &= (\widetilde{a}_{\sigma\omega}, P),\\
\widetilde{b}_\omega^{a_\omega} &= (\widetilde{b}_{\sigma\omega}, \widetilde{B}_0^\omega), & \widetilde{c}_\omega^{a_\omega} &= (\widetilde{c}_{\sigma\omega}, \widetilde{C}_0^\omega), & \widetilde{d}_\omega^{a_\omega} &= (\widetilde{d}_{\sigma\omega}, \widetilde{D}_0^\omega). & &
\end{aligned} \tag{2.12}$$

Proof. Let $g = (g|_0, g|_1; \tau_g)$ and consider its conjugate by the involution P. Then, $g^P = P^{-1}gP = PgP = (1, 1; \tau)(g|_0, g|_1; \tau_g)(1, 1; \tau) = (g|_1, g|_0; \tau\tau_g)(1, 1; \tau) = (g|_1, g|_0; \tau\tau_g\tau) = (g|_1, g|_0; \tau_g)$, using the permutation action and the fact that $\mathcal{S}_2$ is abelian.

To prove (2.12), we will show $b_\omega = (B_0^\omega, b_{\sigma\omega})$, which consequently shows $b_\omega^{a_\omega} = (b_{\sigma\omega}, B_0^\omega)$ using the first assertion of the proposition. The rest follow similarly, and so we omit their proofs. Since $b_\omega = \{B_n^\omega\}_{n\in\mathbb{N}}$, its section at 0 is $b_\omega|_0 = B_0^\omega$, and its section at 1 is $b_\omega|_1 = \{B_n^\omega\}_{n=1}^\infty = \{B_n^{\sigma\omega}\}_{n\in\mathbb{N}} = b_{\sigma\omega}$, by (2.8). Hence we get the result. □

Now we are ready to define the groups that are of interest for this text.

Definition 2.7. *The generalized Grigorchuk's group $\mathcal{G}_\omega$ (introduced in [Gri84b]) is the group generated by $S_\omega = \{a_\omega, b_\omega, c_\omega, d_\omega\}$. The group $\widetilde{\mathcal{G}}_\omega$ generated by $\widetilde{S}_\omega = \left\{a_\omega, b_\omega, c_\omega, d_\omega, \widetilde{a}_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega\right\}$, is called the generalized overgroup.*

By looking at the generating sets, we can observe that $\mathcal{G}_\omega \leqslant \widetilde{\mathcal{G}}_\omega$. Using Proposition 2.2, it can be seen that the group $\mathcal{G}_\omega$ is generated by $\{a_\omega, b_\omega, c_\omega\}$, and the group $\widetilde{\mathcal{G}}_\omega$ is generated by $\{a_\omega, b_\omega, c_\omega, \widetilde{a}_\omega\}$. By Proposition 2.2, we observe that the elements in $S_\omega, \widetilde{S}_\omega$ satisfy the simple reductions (2.9), for all ω. Therefore, the canonical maps $S \to S_\omega\colon s \mapsto s_\omega$ and $\widetilde{S} \to \widetilde{S}_\omega\colon s \mapsto s_\omega$ extend to surjective homomorphisms $\pi\colon \Gamma \to \mathcal{G}_\omega$ and $\widetilde{\pi}\colon \widetilde{\Gamma} \to \widetilde{\mathcal{G}}_\omega$, respectively. As a consequence of this, the elements in $\mathcal{G}_\omega$ and $\widetilde{\mathcal{G}}_\omega$ have the reduced form (2.10).

When the sequence $\omega = (012)^\infty$, the generalized Grigorchuk group becomes the *first Grigorchuk group* [Gri80], which will be denoted by $\mathcal{G}$, and the generalized overgroup becomes the *Grigorchuk's overgroup* [BG00a], which we will denote by $\widetilde{\mathcal{G}}$. It is customary to write the generators of these groups without the subscript $(012)^\infty$ and they have the following wreath recursion realization:

$$\begin{aligned} &b = (a, c), && c = (a, d), && d = (1, b), && \widetilde{a} = (a, \widetilde{a}), \\ &\widetilde{b} = (1, \widetilde{c}), && \widetilde{c} = (1, \widetilde{d}), && \widetilde{d} = (a, \widetilde{b}). \end{aligned}$$

For a subgroup G of $\mathrm{Aut}(\mathcal{T}_2)$, denote the n-th level stabilizer of G by $\mathrm{Stab}_G(n)$. So, $\mathrm{Stab}_G(n) = \mathrm{Stab}(n) \cap G$. Let $\widetilde{H}_\omega := \widetilde{H}_\omega^{(1)} := \mathrm{Stab}_{\widetilde{\mathcal{G}}_\omega}(1)$. An element $g \in \widetilde{\mathcal{G}}_\omega$, belongs to the first level stabilizer if τ_g, the permutation action of g on the level 1, is trivial. Write g in the reduced form (2.10). Note that, if the number of a_ω's in the reduced form is even, then τ_g becomes trivial and if the number of a_ω's is odd, then τ_g is non trivial. Therefore, $g \in \widetilde{H}_\omega$ if and only if g has even number of a_ω's in its reduced form.

Now suppose $g \in \widetilde{H}_\omega$. Then, the reduced form of g has even number of a_ω's, so, we can gather each sub-word of the form $a_\omega * a_\omega$ and rewrite as $*^{a_\omega}$. This shows that the subgroup $\widetilde{H}_\omega$ is generated by $\left\{s_\omega, s_\omega^{a_\omega}\colon s_\omega \in \widetilde{S}_\omega \backslash \{a_\omega\}\right\}$, and by (2.12), we observe the natural embedding maps

$\widetilde{H}_\omega$ to $\widetilde{\mathcal{G}}_{\sigma\omega} \times \widetilde{\mathcal{G}}_{\sigma\omega}$. Also note that the elements in $\widetilde{H}_\omega$ can be written in the form,

$$(*^{a_\omega}) * *^{a_\omega} * *^{a_\omega} \ldots * *^{a_\omega} * (*^{a_\omega}), \tag{2.13}$$

where $*$'s represent elements in $\widetilde{S}_\omega \backslash \{a_\omega\}$, and the first and the last $*^{a_\omega}$ may be omitted.

Following the natural embedding $\psi\colon \widetilde{H}_\omega \to \widetilde{\mathcal{G}}_{\sigma\omega} \times \widetilde{\mathcal{G}}_{\sigma\omega}$ described above, we will construct natural substitution rules (which will also be called as the natural embedding) that depends on the sequence ω, denoted by $\widetilde{\psi}_\omega$, on words of Γ and $\widetilde{\Gamma}$ with even number of a's in it. First, let us define $\widetilde{\Theta} \subset \widetilde{\Gamma}$, containing all reduced words $W \in \widetilde{\Gamma}$, with even number of a's in its reduced form. By a simple parity argument, we can see that $\widetilde{\Theta}$ is in fact a subgroup of $\widetilde{\Gamma}$. Similarly, we can define Θ, the subgroup of Γ, containing words with even number of a's in its reduced form. Similarly to (2.13), the elements in $\Theta, \widetilde{\Theta}$ has the form

$$(*^{a}) * *^{a} * \ldots * *^{a} * (*^{a}), \tag{2.14}$$

where $*$'s represent elements in $S\backslash \{a\}, \widetilde{S}\backslash \{a\}$, respectively. Here, first and last $*^a$ may be omitted. Then, Θ and $\widetilde{\Theta}$ are the subgroups generated by the sets $\{s, s^a \colon s \in S\backslash \{a\}\}$ and $\left\{s, s^a \colon s \in \widetilde{S}\backslash \{a\}\right\}$, respectively. First define $\widetilde{\psi}_\omega$ on $\left\{s, s^a \colon s \in \widetilde{S}\backslash \{a\}\right\} \cup \{1\}$, similar to (2.12), by $\widetilde{\psi}_\omega(1) = (1, 1)$ and

$$\begin{aligned}
&\widetilde{\psi}_\omega(b) = (B_0^\omega, b), && \widetilde{\psi}_\omega(c) = (C_0^\omega, c), && \widetilde{\psi}_\omega(d) = (D_0^\omega, d), && \widetilde{\psi}_\omega(\widetilde{a}) = (a, \widetilde{a}),\\
&\widetilde{\psi}_\omega(\widetilde{b}) = (\widetilde{B}_0^\omega, \widetilde{b}), && \widetilde{\psi}_\omega(\widetilde{c}) = (\widetilde{C}_0^\omega, \widetilde{c}), && \widetilde{\psi}_\omega(\widetilde{d}) = (\widetilde{D}_0^\omega, \widetilde{d}), &&\\
&\widetilde{\psi}_\omega(b^a) = (b, B_0^\omega), && \widetilde{\psi}_\omega(c^a) = (c, C_0^\omega), && \widetilde{\psi}_\omega(d^a) = (d, D_0^\omega), && \widetilde{\psi}_\omega(\widetilde{a}^a) = (\widetilde{a}, a),\\
&\widetilde{\psi}_\omega(\widetilde{b}^a) = (\widetilde{b}, \widetilde{B}_0^\omega), && \widetilde{\psi}_\omega(\widetilde{c}^a) = (\widetilde{c}, \widetilde{C}_0^\omega), && \widetilde{\psi}_\omega(\widetilde{d}^a) = (\widetilde{d}, \widetilde{D}_0^\omega), &&
\end{aligned} \tag{2.15}$$

by replacing P's by a's. Now, we extend the definition of $\widetilde{\psi}_\omega$ to a map $\widetilde{\Theta} \to \widetilde{\Gamma} \times \widetilde{\Gamma}$, by rewriting $W \in \widetilde{\Theta}$ in the form (2.14), then applying the substitution rule (2.14), and reducing it. Thus, given $W \in \widetilde{\Theta}$, we obtain $\widetilde{\psi}_\omega(W) = (W_0, W_1)$, where W_0, W_1 are the reductions of $\widehat{W}_0, \widehat{W}_1$, and $\widehat{W}_0, \widehat{W}_1$ are the words obtained by applying (2.14) to W.

We can also extend the map $\widetilde{\psi}_\omega$ to tuples of words by coordinate wise evaluation. That is, $\widetilde{\psi}_\omega(W_1, W_2, \ldots, W_k) = \left(\widetilde{\psi}_\omega(W_1), \widetilde{\psi}_\omega(W_2), \ldots, \widetilde{\psi}_\omega(W_k)\right)$. Now apply $\widetilde{\psi}_{\sigma^{n-1}\omega} \circ \ldots \circ \widetilde{\psi}_{\sigma\omega} \circ \widetilde{\psi}_\omega$ to *decompose* W into 2^n reduced words $\{W_{i_1 \ldots i_n}\}$, if no indeterminacy occurs. We may drop the subscript ω in $\widetilde{\psi}_\omega$ for convenience. $\widetilde{\psi}_{\sigma^{n-1}\omega} \circ \ldots \circ \widetilde{\psi}_{\sigma\omega} \circ \widetilde{\psi}_\omega(W)$ will be called the application of $\widetilde{\psi}$, n times, to the word W. We will omit writing the natural substitution rule and write $W = (W_0, W_1)$, $W = \{W_{i_1 \ldots i_n}\}$ instead of $\widetilde{\psi}_\omega(W) = (W_0, W_1)$, $\widetilde{\psi}_{\sigma^{n-1}\omega} \circ \ldots \circ \widetilde{\psi}_{\sigma\omega} \circ \widetilde{\psi}_\omega(W) = \{W_{i_1 \ldots i_n}\}$, respectively, if there are no ambiguity.

If $W = (W_0, W_1)$ and $W' = (W'_0, W'_1)$, then $W^{W'} = (W_0^{W'_0}, W_1^{W'_1})$, and using (2.14) and (2.15), we get $W^a = (W_1, W_0)$. Therefore, for any $W' \in \widetilde{\Gamma}$ (not necessarily in $\widetilde{\Theta}$),

$$W^{W'} = \begin{cases} (W_0^{W'_0}, W_1^{W'_1}) & \text{if } W' \in \widetilde{\Theta} \text{ and } W' = (W'_0, W'_1) \\ (W_1^{W'_0}, W_0^{W'_1}) & \text{if } W' \notin \widetilde{\Theta} \text{ and } aW' = (W'_0, W'_1) \end{cases}. \tag{2.16}$$

Now, let us examine the relation of lengths of words and their decompositions.

Proposition 2.4. *Let $W \in \widetilde{\Theta}$ and let $\widehat{W}_0, \widehat{W}_1 \in \widetilde{\Gamma}$ be the words (not necessarily reduced) obtained by applying* (2.15) *to W. Then,*

$$\left|\widehat{W}_0\right|, \left|\widehat{W}_1\right| \leqslant \frac{|W|+1}{2} \quad \textit{and} \quad \left|\widehat{W}_0\right| + \left|\widehat{W}_1\right| \leqslant |W| + 1. \tag{2.17}$$

In tha case of W can be decomposed into the depth n, we have,

$$|W_{i_1 \ldots i_n}| \leqslant \frac{|W|}{2^n} + 1 - \frac{1}{2^n}, \tag{2.18}$$

where $W = \{W_{i_1 \ldots i_n}\}$.

Proof. Let $W \in \widetilde{\Theta}$ and rewrite W in the form (2.14). Note that each $*$ and $*^a$ in (2.14) of W, contributes either a letter or no letters (if the corresponding coordinate is 1) to each of $\widehat{W}_0$ and $\widehat{W}_1$. Suppose there are k number of $*$'s in W. Then $\left|\widehat{W}_0\right|, \left|\widehat{W}_1\right| \leqslant k$. If W starts and ends with a $*$, i.e., $W = **^a**^a \ldots **^a*$, then $|W| = 2k-1$. If $W = (*^a)**^a**^a \ldots **^a*$ or $W = **^a**^a \ldots **^a*(*^a)$, then $|W| = 2k$. If $W = (*^a) * *^a * *^a \dots * *^a * (*^a)$, then $|W| = 2k+1$. In either case, $|W|+1 \geqslant 2k$, and therefore we obtain (2.17). Note that, $|W_i| \leqslant \left|\widehat{W_i}\right| \leqslant \frac{|W|+1}{2} = \frac{|W|}{2} + 1 - \frac{1}{2}$, and using this inductively, we obtain (2.18). □

In fact, we can give a better upper bound,

$$\left|\widehat{W_0}\right| + \left|\widehat{W_1}\right| \leqslant |W| + 1 - \alpha, \tag{2.19}$$

where α is the number of letters in W, whose first coordinate of the natural embedding is 1. As a direct corollary of Proposition 2.4, we obtain:

Corollary 2.1. *For $g \in \widetilde{H}_\omega$,*

$$|g|_0|, |g|_1| \leqslant \frac{|g|+1}{2}, \quad \textit{and} \quad |g|_0| + |g|_1| \leqslant |g| + 1. \tag{2.20}$$

3. ON GROWTH OF GENERALIZED GRIGORCHUK'S OVERGROUPS*

This chapter is extracted from the article [Sam20].

3.1 Introduction

The growth rate $\gamma_{\mathcal{G}}(n)$ of the first Grigorchuk group $\mathcal{G}$ was first shown to be bounded below by $e^{\sqrt{n}}$ and bounded above by $e^{n^{\beta}}$, where $\beta = \log_{32} 31 \approx 0.991$ [Gri83, Gri84b]. In 1998, Laurent Bartholdi [Bar98] and in 2001, Roman Muchnik and Igor Pak [MP01] independently refined the upper bound to $\gamma_{\mathcal{G}}(n) \le e^{n^{\alpha}}$, where $\alpha = \log(2)/\log(2/\eta) \approx 0.767$ and η is the real root of the polynomial $x^3 + x^2 + x - 2$. Recent work of Anna Erschler and Tianyi Zheng [EZ20] showed $\gamma_{\mathcal{G}}(n) \ge e^{n^{(\alpha-\epsilon)}}$ for any positive ϵ. The Grigorchuk's overgroup $\widetilde{\mathcal{G}}$ is of intermediate growth [BG02] and as a corollary to Proposition 3.4 and Theorem 3.2$''$, the growth rate $\gamma_{\widetilde{\mathcal{G}}}(n)$ of $\widetilde{\mathcal{G}}$ satisfies, $\exp\left(\frac{n}{\log^{2+\epsilon} n}\right) \le \gamma_{\widetilde{\mathcal{G}}}(n) \le \exp\left(\frac{n\log(\log n)}{\log n}\right)$ for any $\epsilon > 0$.

First introduced technique for getting an upper bound for $\mathcal{G}$ uses the *strong contraction property* [Gri84b] (also known as *sum contraction property*), which says that there is a finite index subgroup H of $\mathcal{G}$ such that any element $g \in H$ can be uniquely decomposed into some elements, whose sum of lengths in not larger than $C|g| + D$, where $0 < C < 1$ and D are constants independent of g [Gri84b]. Later this technique was developed and many variants were introduced [Bar03, Fra20]. In 2004, to get a lower bound for certain class of groups of intermediate growth, Anna Erschler introduced a method for partial description of the Poisson boundary [Ers04]. This idea was used to get the current known best lower bound for the growth of $\mathcal{G}$ [EZ20]. We will be using a version of strong contraction property in this text.

The growth rates of the family $\left\{\widetilde{\mathcal{G}}_{\omega}, \omega \in \Omega\right\}$ of generalized Grigorchuk's overgroups are given by the theorem below.

Theorem 3.1. *Let $\omega \in \Omega$. Then $\widetilde{\mathcal{G}}_{\omega}$ is of polynomial growth if ω is virtually constant and $\widetilde{\mathcal{G}}_{\omega}$ is of*

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intermediate growth if ω is not virtually constant.

Recall that Ω_0, Ω_1 be subsets of Ω, where Ω_0 is the set consisting of all sequences containing $0, 1$ and 2 infinitely often, Ω_1 is the set consisting of sequences containing exactly two symbols infinitely often. Define Ω_0^* to be the subset of Ω_0 containing sequences $\omega = \{\omega_n\}$, such that there is an integer $M = M(\omega)$ with the property that for all $k \geqslant 1$, the set $\{\omega_k, \omega_{k+1}, \ldots, \omega_{k+M-1}\}$ contains all three symbols 0,1 and 2. Similarly, define Ω_1^* to be the subset of Ω_1 containing sequences $\omega = \{\omega_n\}$, such that there is an integer $M = M(\omega)$ with the property that for all $k \geqslant 1$, the set $\{\omega_k, \omega_{k+1}, \ldots, \omega_{k+M-1}\}$ contains at least two symbols. Let $\Omega^* = \Omega_0^* \cup \Omega_1^*$. Sequences in Ω^* are called *homogeneous sequences*.

Theorem 3.2. *Let $\omega \in \Omega^*$. Then*

$$\gamma_{\widetilde{\mathcal{G}}_\omega}(n) \preceq \exp\left(\frac{n \log(\log n)}{\log n}\right).$$

Theorem 3.2 provides an upper bound for growth of $\widetilde{\mathcal{G}}_\omega$ only for homogeneous sequences. In fact, it is impossible to give a unifying upper bound for growth of $\widetilde{\mathcal{G}}_\omega$, for all $\omega \in \Omega_0 \cup \Omega_1$. This follows from Theorem 7.1 of [Gri84b], together with the fact that $\mathcal{G}_\omega \subset \widetilde{\mathcal{G}}_\omega$. However, it is possible to provide a unifying lower bound for the growth of groups $\widetilde{\mathcal{G}}_\omega$ for all $\omega \in \Omega_0 \cup \Omega_1$ by a function of type $\exp\left\{\left(\frac{n}{\log^{2+\epsilon}(n)}\right)\right\}$ for arbitrary $\epsilon > 0$ (see Proposition 3.4).

We prove Theorem 3.1 in Section 3.2 and Theorem 3.2 in Section 3.3.

3.2 Growth of Generalized Overgroups $\widetilde{\mathcal{G}}_\omega$

Proposition 3.1. *$\widetilde{\mathcal{G}}_\omega$ has subexponential growth for each $\omega \in \Omega_1 \cup \Omega_2$.*

Before proceeding to the proof, we start with three lemmas.

Lemma 3.1. *A non-decreasing semi-multiplicative function $\gamma(n)$ with argument a natural number, can be extended to a non-decreasing semi-multiplicative function $\gamma(x)$, with argument a non-negative real number.*

Proof. See Lemma 3.1 of [Gri84b]. □

Lemma 3.2. *For any* $\omega \in \Omega, \widetilde{\lambda}_\omega \leqslant \widetilde{\lambda}_{\sigma\omega}$.

Proof. Denote $\widetilde{B}_\omega(n) = B_{\widetilde{\mathcal{G}}_\omega, \widetilde{S}_\omega}(n)$ and $\widetilde{H}_\omega(n) = \widetilde{H}_\omega \cap \widetilde{B}_\omega(n)$. Any element $g \in \widetilde{B}_\omega(n)$ is either in $\widetilde{H}_\omega$ or is of the form $g = ag'$, where $g' \in \widetilde{H}_\omega$ and $|g'| \leqslant |g| + 1 \leqslant n + 1$. Thus,

$$\widetilde{\gamma}_\omega(n) = |\widetilde{B}_\omega(n)| \leqslant |\widetilde{H}_\omega(n)| + |\widetilde{H}_\omega(n+1)| \leqslant 2|\widetilde{H}_\omega(n+1)|.$$

For each $g \in \widetilde{H}_\omega$, $g|_0, g|_1 \in \widetilde{\mathcal{G}}_{\sigma\omega}$ satisfy (2.20) and so,

$$|\widetilde{H}_\omega(n)| \leqslant |\widetilde{B}_{\sigma\omega}(\frac{n+1}{2})|^2 = \left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+1}{2})\right)^2.$$

Therefore,

$$\widetilde{\gamma}_\omega(n) \leqslant 2\left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^2.$$

Consequently,

$$\begin{aligned} \widetilde{\lambda}_\omega &= \lim_n \left(\widetilde{\gamma}_\omega(n)\right)^{1/n} \\ &\leqslant \lim_n \left(2\left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^2\right)^{1/n} \\ &= \lim_n \left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^{2/n} = \widetilde{\lambda}_{\sigma\omega}. \end{aligned}$$

□

Let $\Omega_{1,2}$ contains all the sequences of Ω having at most two symbols.

Lemma 3.3. *For any* $\omega \in \Omega_{1,2}$, $\widetilde{\mathcal{G}}_\omega = \mathcal{G}_\omega$.

Proof. First note that $\widetilde{a}_\omega \in \mathcal{G}_\omega \implies \widetilde{a}_\omega b_\omega, \widetilde{a}_\omega c_\omega, \widetilde{a}_\omega d_\omega \in \mathcal{G}_\omega \implies \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega \in \mathcal{G}_\omega \implies \widetilde{\mathcal{G}}_\omega \subset \mathcal{G}_\omega \implies \widetilde{\mathcal{G}}_\omega = \mathcal{G}_\omega$. To prove Lemma 3.3, we only need to show that $\widetilde{a}_\omega \in \mathcal{G}_\omega$. For definiteness we may assume ω consists only of symbols $0, 1$. Then by (2.11), $b_\omega = \{P, P, P, ...\} = \widetilde{a}_\omega$. Therefore $\widetilde{a}_\omega \in \mathcal{G}_\omega$ and thus the result is true. □

Proof of Proposition 3.1. Let $\omega \in \Omega_1 \cup \Omega_2$. Then there exists $N \in \mathbb{N}$ such that $\sigma^N \omega \in \Omega_{1,2}$. Then by Lemma 3.3, $\widetilde{\mathcal{G}}_{\sigma^N \omega} = \mathcal{G}_{\sigma^N \omega}$. Therefore $\widetilde{\lambda}_{\sigma^N \omega} = \lambda_{\sigma^N \omega}$. For any ω, $\mathcal{G}_\omega$ is of intermediate growth if $\omega \in \Omega_1$ and of polynomial growth if $\omega \in \Omega_2$ [Gri84b]. Thus $\lambda_{\sigma^N \omega} = 1$. So by Lemma 3.2, $\widetilde{\lambda}_\omega \leqslant \widetilde{\lambda}_{\sigma^N \omega} = 1$. Thus $\widetilde{\mathcal{G}}_\omega$ is of subexponential growth. □

Proposition 3.2. *$\widetilde{\mathcal{G}}_\omega$ has intermediate growth for $\omega \in \Omega_1$.*

Proof. By Proposition 3.1, $\widetilde{\mathcal{G}}_\omega$ is of subexponential growth. Since $\mathcal{G}_\omega \subset \widetilde{\mathcal{G}}_\omega$ and $\mathcal{G}_\omega$ is of super-polynomial growth [Gri84b], $\widetilde{\mathcal{G}}_\omega$ is of super-polynomial growth. Hence $\widetilde{\mathcal{G}}_\omega$ is of intermediate growth. □

Proposition 3.3. *$\widetilde{\mathcal{G}}_\omega$ has polynomial growth for $\omega \in \Omega_2$.*

Proof. Since $\omega \in \Omega_2$, there is a natural number N such that $\omega_n = \omega_N$ for all $n \geqslant N$, where $\omega = \{\omega_n\}$. Then $\widetilde{\mathcal{G}}_{\sigma^{N-1}\omega} = \langle a, \widetilde{a} \rangle \cong \mathbb{D}_\infty$, the infinite Dihedral group. Let $\mathbb{G}$ be the subgroup of $\mathrm{Aut}(\mathcal{T}_2)$ containing elements g such that $g|_v \in \langle a, \widetilde{a} \rangle$ for all v in level $N-1$ of $\mathcal{T}_2$. Then $\widetilde{\mathcal{G}}_\omega \subset \mathbb{G}$. Let $\mathbb{G}_0$ be the subgroup of $\mathbb{G}$ containing automorphisms fixing all vertices in the first $N-1$ levels of $\mathcal{T}_2$. Note that $\mathbb{G}_0 \lhd \mathbb{G}$ and $[\mathbb{G} : \mathbb{G}_0] \leqslant 2^{2^N - 1}$. But $\mathbb{G}_0 \cong \langle a, \widetilde{a} \rangle^{2^{N-1}} \cong \mathbb{D}_\infty^{2^{N-1}}$. Thus $\mathbb{G}_0$ is virtually abelian and of polynomial growth. Since $[\mathbb{G} : \mathbb{G}_0] < \infty, \mathbb{G}$ is of polynomial growth. $\widetilde{\mathcal{G}}_\omega \subset \mathbb{G}$ implies that $\widetilde{\mathcal{G}}_\omega$ is of polynomial growth. □

Theorem 3.3. *$\widetilde{\mathcal{G}}_\omega$ has intermediate growth for $\omega \in \Omega_0$.*

We will, from now on, consider $\widetilde{S}_\omega = \left\{ a_\omega, b_\omega, c_\omega, d_\omega, \widetilde{a}_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega \right\}$ as the generating set of $\widetilde{\mathcal{G}}_\omega$. A reduced word W satisfying $g = W$ in $\widetilde{\mathcal{G}}_\omega$ and $|g| = |W|$ is called a minimal representation of g. For any $\epsilon > 0$ define $\mathcal{F}^\epsilon(n) = \mathcal{F}^\epsilon_\omega(n)$ to be the set of length n elements g in $\widetilde{\mathcal{G}}_\omega$ such that for any minimal representation W of g over alphabet $\widetilde{S}_\omega$,

$$|W|_* > (1/2 - \epsilon)n, \quad \text{for some } * \in \widetilde{S}_\omega \backslash \{a\}. \tag{3.1}$$

So for any minimal representation of elements in $\mathcal{F}^\epsilon(n)$, its reduced form (2.10) has most of $*$s as the same letter. Now define $\mathcal{D}^\epsilon(n) = \mathcal{D}^\epsilon_\omega(n)$ to be the complement of $\mathcal{F}^\epsilon(n)$ in $\widetilde{B}_\omega(n) \backslash \widetilde{B}_\omega(n-1)$, the sphere of radius n. Thus if $g \in \mathcal{D}^{\epsilon}(n)$, then g has a minimal representation W satisfying,

$$|W|_{*} \leqslant (1/2-\epsilon)n, \quad \text{for all } * \in \widetilde{S}_{\omega} \backslash \{a\} . \tag{3.2}$$

For any $\delta > 0$ define $\widetilde{\mathcal{F}}^{\delta}(n')$ to be the set of words W' over the alphabet $\widetilde{S}_{\omega} \backslash \{a\}$ of length n' such that,

$$|W'|_{*} > (1-\delta)n', \quad \text{for some } * \in \widetilde{S}_{\omega} \backslash \{a\} . \tag{3.3}$$

Therefore, each word in $\widetilde{\mathcal{F}}^{\delta}(n')$ has mostly equal letters.

Lemma 3.4. *Let $0 < \epsilon < 1/2$ and let W be a minimal representation of an element in $\mathcal{F}^{\epsilon}(n)$. Let W' be the word obtained by deleting all letters a from W. Then $W' \in \widetilde{\mathcal{F}}^{\delta}(n')$ where*

$$\frac{n-1}{2} \leqslant n' \leqslant \frac{n+1}{2}, \tag{3.4}$$

$$\delta = 2\epsilon + \frac{(1-2\epsilon)}{n-1}. \tag{3.5}$$

Proof. Since W is a reduced word, by (2.10), we observe that, $2|W|_{a} - 1 \leqslant |W| \leqslant 2|W|_{a} + 1$. Thus $\frac{|W|-1}{2} \leqslant |W|_{a} \leqslant \frac{|W|+1}{2}$, and so $\frac{|W|-1}{2} \leqslant |W| - |W|_{a} \leqslant \frac{|W|+1}{2}$. So we get (3.4).

By (3.1), $|W'|_{*} = |W|_{*} > (1/2-\epsilon)n \geqslant (1/2-\epsilon)(2n'-1) = \left(1 - 2\epsilon - \frac{(1-2\epsilon)}{2n'}\right) n' \geqslant \left(1 - 2\epsilon - \frac{(1-2\epsilon)}{n-1}\right) n' = (1-\delta)n'$, from (3.5). $\square$

Lemma 3.5. *If $\delta < 1$, then $\overline{\lim_{k}} |\widetilde{\mathcal{F}}^{\delta}(k)|^{1/k} \leqslant (1-\delta)^{-1}(\delta/6)^{-\delta}$.*

Proof. Any word $W \in \widetilde{\mathcal{F}}^{\delta}(k)$ can be constructed by choosing a letter $*$ out of $\left\{b, c, d, \widetilde{b}, \widetilde{c}, \widetilde{d}, \widetilde{a}\right\}$, which satisfies (3.3). So, W contains the letter $*$ at least $k - \lfloor \delta k \rfloor$ times and possibly t times more, where $0 \leqslant t \leqslant \lfloor \delta k \rfloor$. The rest of the positions of W can be filled by the other six letters with frequencies $i_1, \dots, i_6$, where $\sum i_j = \lfloor \delta k \rfloor - t$. Therefore, we have,

$$\left|\widetilde{\mathcal{F}}^{\delta}(k)\right| \leqslant 7 + 7 \sum_{t=0}^{\lfloor \delta k \rfloor} \sum_{\sum i_j = \lfloor \delta k \rfloor - t} \frac{k!}{(k - \lfloor \delta k \rfloor + t)! i_1! \dots i_6!}.$$

Let $(\delta k-t)_* := 6\left\lfloor\dfrac{\lfloor\delta k-t\rfloor}{6}\right\rfloor$ be the largest integer not greater than $\lfloor\delta k-t\rfloor$, that is divisible by 6. Since $i_1,\dots,i_6$ are non negative integers, we have,

$$i_1!\dots i_6! \geqslant \left\lfloor\frac{\sum i_j}{6}\right\rfloor!^6 = \left\lfloor\frac{\lfloor\delta k\rfloor-t}{6}\right\rfloor!^6 = \left\lfloor\frac{\lfloor\delta k-t\rfloor}{6}\right\rfloor!^6 = \left(\frac{(\delta k-t)_*}{6}\right)!^6.$$

Since the number of ways to choose non negative integers $i_1,\dots,i_6$ such that $\sum i_j = \lfloor\delta k\rfloor - t$ is $\binom{\lfloor\delta k\rfloor-t+5}{5}$, we get,

$$\begin{aligned}
\left|\widetilde{\mathcal{F}}^\delta(k)\right| &\leqslant 7+7\sum_{t=0}^{\lfloor\delta k\rfloor}\binom{\lfloor\delta k\rfloor-t+5}{5}\frac{k!}{(k-\lfloor\delta k\rfloor+t)!\left(\frac{(\delta k-t)_*}{6}\right)!^6}\\
&\leqslant 7+7\binom{\lfloor\delta k\rfloor+5}{5}\sum_{t=0}^{\lfloor\delta k\rfloor}\frac{k!}{(k-\lfloor\delta k\rfloor+t)!\left(\frac{(\delta k-t)_*}{6}\right)!^6}\\
&\leqslant (\lfloor\delta k\rfloor+5)^5\sum_{t=0}^{\lfloor\delta k\rfloor}\frac{k!}{(k-\lfloor\delta k\rfloor+t)!\left(\frac{(\delta k-t)_*}{6}\right)!^6}\\
&\leqslant (\lfloor\delta k\rfloor+5)^5\sum_{t=0}^{\lfloor\delta k\rfloor}\frac{e\sqrt{k}k^k e^{-k}e^{(k-\lfloor\delta k\rfloor+t)}e^{(\delta k-t)_*}}{(k-\lfloor\delta k\rfloor+t)^{(k-\lfloor\delta k\rfloor+t)}\left(\frac{(\delta k-t)_*}{6}\right)^{(\delta k-t)_*}}.
\end{aligned}$$

Here we used the Stirling's formula $\dfrac{n^n}{e^n}\leqslant n! \leqslant e\sqrt{n}\dfrac{n^n}{e^n}$. Since $0\leqslant(\lfloor\delta k\rfloor-t)-(\delta k-t)_*\leqslant 6$,

$$\begin{aligned}
\left|\widetilde{\mathcal{F}}^\delta(k)\right| &\leqslant e(\lfloor\delta k\rfloor+5)^5\sum_{t=0}^{\lfloor\delta k\rfloor}\frac{\sqrt{k}k^{(\lfloor\delta k\rfloor-t)-(\delta k-t)_*}e^{(\delta k-t)_*-(\lfloor\delta k\rfloor-t)}}{\left(1-\frac{\lfloor\delta k\rfloor}{k}+\frac{t}{k}\right)^{(k-\lfloor\delta k\rfloor+t)}\left(\frac{(\delta k-t)_*}{6k}\right)^{(\delta k-t)_*}}\\
&\leqslant e(\lfloor\delta k\rfloor+5)^5\sum_{t=0}^{\lfloor\delta k\rfloor}\frac{\sqrt{k}k^6}{\left(1-\frac{\lfloor\delta k\rfloor}{k}+\frac{t}{k}\right)^{(k-\lfloor\delta k\rfloor+t)}\left(\frac{(\delta k-t)_*}{6k}\right)^{(\delta k-t)_*}}\\
&\leqslant ek^6(\lfloor\delta k\rfloor+5)^5\sqrt{k}(1-\delta)^{-k}\sum_{t=0}^{\lfloor\delta k\rfloor}\left(\frac{(\delta k-t)_*}{6k}\right)^{-(\delta k-t)_*}.
\end{aligned}$$

Note that the real valued function, $\xi \mapsto \xi^{-\xi}$ for $\xi > 0$, is an increasing function on the interval $(0, e^{-1})$. Since $\delta/6 < 1/6 < e^{-1}$, we get,

$$\left(\frac{(\delta k - x)_*}{6k}\right)^{-\left(\frac{(\delta k - x)_*}{6k}\right)} \leqslant \left(\frac{\delta}{6}\right)^{-\left(\frac{\delta}{6}\right)}.$$

Therefore,

$$\left|\widetilde{\mathcal{F}}^{\delta}(k)\right| \leqslant e k^6 (\lfloor \delta k \rfloor + 5)^5 \sqrt{k} (1-\delta)^{-k} (\lfloor \delta k \rfloor + 1) \left(\frac{\delta}{6}\right)^{-\left(\frac{\delta}{6}\right)6k}.$$

Hence,

$$\overline{\lim_{k}} \left|\widetilde{\mathcal{F}}^{\delta}(k)\right|^{1/k} \leqslant (1-\delta)^{-1} (\delta/6)^{-\delta}.$$

□

Corollary 3.1. *Let* $\epsilon < 1/2$. *Then,* $\overline{\lim_{n}} |\mathcal{F}^{\epsilon}(n)|^{1/n} \leqslant (1-2\epsilon)^{-1/2} (\epsilon/3)^{-\epsilon}$.

Proof. If n is even, then minimal representations of at most two elements in $\mathcal{F}^{\epsilon}(n)$ give the same word in $\widetilde{\mathcal{F}}^{\delta}(n/2)$. So,

$$|\mathcal{F}^{\epsilon}(n)| \leqslant 2 |\widetilde{\mathcal{F}}^{\delta}(n/2)|.$$

If n is odd, then for each element in $\mathcal{F}^{\epsilon}(n)$, we can assign a unique word in $\widetilde{\mathcal{F}}^{\delta}((n-1)/2)$ or $\widetilde{\mathcal{F}}^{\delta}((n+1)/2)$, and so,

$$|\mathcal{F}^{\epsilon}(n)| \leqslant |\widetilde{\mathcal{F}}^{\delta}((n-1)/2)| + |\widetilde{\mathcal{F}}^{\delta}((n+1)/2)|.$$

Note that,

$$\overline{\lim_{n}} |\widetilde{\mathcal{F}}^{\delta}(n/2)|^{1/n} \leqslant \lim_{n} \left((1-\delta)^{-1} (\delta/6)^{-\delta}\right)^{1/2},$$

$$\overline{\lim_{n}} |\widetilde{\mathcal{F}}^{\delta}((n-1)/2)|^{1/n} \leqslant \lim_{n} \left((1-\delta)^{-1} (\delta/6)^{-\delta}\right)^{1/2},$$

$$\overline{\lim_{n}} |\widetilde{\mathcal{F}}^{\delta}((n+1)/2)|^{1/n} \leqslant \lim_{n} \left((1-\delta)^{-1} (\delta/6)^{-\delta}\right)^{1/2},$$

and thus,

$$\overline{\lim_n}|\mathcal{F}^\epsilon(n)|^{1/n} \leqslant \lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{1/2}.$$

Since $\delta = 2\epsilon + \dfrac{(1-2\epsilon)}{n-1}$, $\lim_n \delta = 2\epsilon$ and therefore,

$$\lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{-1/2} = (1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon}.$$

Hence we get the desired result. □

For each $s \geqslant 1$, let $\widetilde{H}_\omega^{(s)} := \left\{g \in \widetilde{\mathcal{G}}_\omega \mid g(v) = v \text{ for } v \text{ in level } s\right\}$ and denote the canonical generators of $\widetilde{\mathcal{G}}_{\sigma^s\omega}$ by $a, b_s, c_s, d_s, \widetilde{a}, \widetilde{b}_s, \widetilde{c}_s, \widetilde{d}_s$. We assign above symbols, when $s = 0$, to the generators of $\widetilde{\mathcal{G}}_\omega$. Using the map $\widetilde{\psi}$, we get the following;

$$\begin{aligned}
\omega_s = 0 \implies b_{s-1} &= (a, b_s) \quad c_{s-1} = (a, c_s) \quad d_{s-1} = (1, d_s) \quad \widetilde{a} = (a, \widetilde{a}) \\
\widetilde{b}_{s-1} &= (1, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (1, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (a, \widetilde{d}_s),
\end{aligned}$$

$$\begin{aligned}
\omega_s = 1 \implies b_{s-1} &= (a, b_s) \quad c_{s-1} = (1, c_s) \quad d_{s-1} = (a, d_s) \quad \widetilde{a} = (a, \widetilde{a}) \\
\widetilde{b}_{s-1} &= (1, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (a, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (1, \widetilde{d}_s),
\end{aligned}$$

$$\begin{aligned}
\omega_s = 2 \implies b_{s-1} &= (1, b_s) \quad c_{s-1} = (a, c_s) \quad d_{s-1} = (a, d_s) \quad \widetilde{a} = (a, \widetilde{a}) \\
\widetilde{b}_{s-1} &= (a, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (1, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (1, \widetilde{d}_s).
\end{aligned} \tag{3.6}$$

Let W be a minimal representation of an element in $\widetilde{H}_\omega^{(s)}$. Then there are $\tilde{W}_0, \tilde{W}_1$ such that $W = (\tilde{W}_0, \tilde{W}_1)$ using substitutions in (3.6). Let W_0, W_1 be obtained by doing simple reductions on $\tilde{W}_0, \tilde{W}_1$. Let α_1 denote the number of such simple reductions. So W_0, W_1 are minimal representations of words in $\widetilde{H}_{\sigma^1\omega}^{(s-1)}$ and by (2.17),

$$|W_0| + |W_1| \leqslant \left|\widetilde{W_0}\right| + \left|\widetilde{W_1}\right| - \alpha_1 \leqslant |W| + 1 - \alpha_1. \tag{3.7}$$

Now there are $\tilde{W}_{00}, \tilde{W}_{01}, \tilde{W}_{10}, \tilde{W}_{11}$ such that $W_0 = (\tilde{W}_{00}, \tilde{W}_{01})$, $W_1 = (\tilde{W}_{10}, \tilde{W}_{11})$ using substitutions in (3.6). Let $W_{00}, W_{01}, W_{10}, W_{11}$ be obtained by doing simple reductions on $\tilde{W}_{00}, \tilde{W}_{01}, \tilde{W}_{10}$, $\tilde{W}_{11}$. Let α_2 denote the number of such simple reductions. So $W_{00}, W_{01}, W_{10}, W_{11}$ are minimal representations of elements in $\widetilde{H}^{(s-2)}_{\sigma^2\omega}$ and applying (3.7), we get,

$$\begin{aligned}|W_{00}| + |W_{01}| + |W_{10}| + |W_{11}| &\leqslant |W_0| + 1 + |W_1| + 1 - \alpha_2 \\ &\leqslant |W| + 2^2 - 1 - (\alpha_1 + \alpha_2).\end{aligned}$$

Proceeding this manner we get $\{W_{i_1 i_2 \dots i_s}\}_{i_j \in \{0,1\}}$ minimal representations of elements in $\widetilde{H}^{(s-s)}_{\sigma^s\omega} = \widetilde{\mathcal{G}}_{\sigma^s\omega}$. Denote by α_s the number of simple reductions done to obtain $\{W_{i_1 i_2 \dots i_s}\}_{i_j \in \{0,1\}}$ from $\left\{\tilde{W}_{i_1 i_2 \dots i_s}\right\}_{i_j \in \{0,1\}}$. Then by applying (3.7) repeatedly, we get,

$$\sum_{i_1, i_2, \dots, i_s} |W_{i_1 i_2 \dots i_s}| \leqslant |W| + 2^s - 1 - \sum_{1}^{s-1} \alpha_i. \tag{3.8}$$

Let $X_0 := |W|_{d_0} + |W|_{\tilde{b}_0} + |W|_{\tilde{c}_0}$, $Y_0 := |W|_{c_0} + |W|_{\tilde{b}_0} + |W|_{\tilde{d}_0}$ and $Z_0 := |W|_{b_0} + |W|_{\tilde{c}_0} + |W|_{\tilde{d}_0}$. Also for $j = 1, 2, \dots s$, let

$$\begin{aligned}X_j &= \sum \left(|W_{i_1 i_2 \dots i_j}|_{d_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{b}_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{c}_j} \right), \\ Y_j &= \sum \left(|W_{i_1 i_2 \dots i_j}|_{c_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{b}_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{d}_j} \right), \\ Z_j &= \sum \left(|W_{i_1 i_2 \dots i_j}|_{b_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{c}_j} + |W_{i_1 i_2 \dots i_j}|_{\tilde{d}_j} \right).\end{aligned}$$

Lemma 3.6. *Let $\epsilon > 0, n_\epsilon \in \mathbb{N}$ such that $n_\epsilon \epsilon > 5/2$. Let $n \geqslant n_\epsilon$. Let $s \in \mathbb{N}$ such that ω_s is the first time that the third symbol appears in ω. Let W be a minimal representation of an element in $\mathcal{D}^\epsilon(n) \cap \widetilde{H}^{(s)}_\omega$. Then,*

$$\sum_{i_1, i_2, \dots, i_s} |W_{i_1 i_2 \dots i_s}| \leqslant \left(1 - \frac{\epsilon}{5}\right) n + 2^s - 1.$$

Proof. For definiteness, suppose the sequence ω begins with the symbol 0, first 1 appears in the t-th position, and first 2 appears in the s-th position. That is, $\omega_1 = \dots = \omega_{t-1} = 0, \omega_t = 1, \omega_m \neq 2$

for every $m < s$, and $\omega_s = 2$. First note that each simple reduction decreases Y_i, Z_i by at most 2. Thus,

$$Y_{t-1} \geqslant Y_0 - 2\sum_1^{t-1} \alpha_i \geqslant Y_0 - 2\sum_1^{s-1} \alpha_i \quad \text{and} \quad Z_{s-1} \geqslant Z_0 - 2\sum_1^{s-1} \alpha_i. \tag{3.9}$$

Since $\omega_1 = 0$ there are X_0 of letters in W, with 1 in their first coordinate when written using (3.6). Thus we modify (3.8), as done in (2.19) to be,

$$\sum_{i_1,i_2,\ldots,i_s} |W_{i_1 i_2 \ldots i_s}| \leqslant n + 2^s - 1 - \sum_1^{s-1} \alpha_i - X_0.$$

Similarly, since $\omega_t = 1$ and $\omega_s = 2$, we get,

$$\sum_{i_1,i_2,\ldots,i_s} |W_{i_1 i_2 \ldots i_s}| \leqslant n + 2^s - 1 - \sum_1^{s-1} \alpha_i - X_0 - Y_{t-1} - Z_{s-1}. \tag{3.10}$$

Now let us show that $X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i > n\epsilon/5$. To the contrary assume $X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i \leqslant n\epsilon/5$. Therefore, $\sum_1^{s-1} \alpha_i \leqslant n\epsilon/5$ and by (3.9) and (3.10),we get,

$$\begin{aligned} X_0 + Y_0 + Z_0 &\leqslant X_0 + \left(Y_{t-1} + 2\sum_1^{s-1} \alpha_i\right) + \left(Z_{s-1} + 2\sum_1^{s-1} \alpha_i\right) \\ &\leqslant \left(X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i\right) + 3\left(\sum_1^{s-1} \alpha_i\right) \\ &\leqslant \frac{4}{5} n\epsilon. \end{aligned}$$

But $n = |W| \leqslant |W|_a + |W|_{\tilde{a}} + X_0 + Y_0 + Z_0 \leqslant \frac{n+1}{2} + \frac{n}{2} - n\epsilon + \frac{4}{5}n\epsilon$, since $|W|_{\tilde{a}} \leqslant (1/2 - \epsilon)n$ by (3.2). Thus $n\epsilon \leqslant 5/2$, which is a contradiction. So $X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i > n\epsilon/5$. Therefore,

$$\sum_{i_1,i_2,\ldots,i_s} |W_{i_1 i_2 \ldots i_s}| \leqslant \left(1 - \frac{\epsilon}{5}\right) n + 2^s - 1.$$

□

Proof of Theorem 3.3. Take a fixed $0 < \epsilon < 1/2$. Suppose first that there are positive integers k, s, such that there exists an infinite set $N_0 \subset \mathbb{N}$ where,

$$\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right| \geqslant \left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k\omega}(n)\right|, \tag{3.11}$$

for all $n \in N_0$. By Lemma 3.2 and (2.2),

$$\begin{aligned}
\widetilde{\lambda}_\omega &\leqslant \widetilde{\lambda}_{\sigma^k\omega} \\
&= \lim_n |\widetilde{\gamma}_{\sigma^k\omega}(n)|^{1/n} \\
&= \lim_n |\gamma'_{\widetilde{\mathcal{G}}_{\sigma^k\omega},\widetilde{S}_{\sigma^k\omega}}(n)|^{1/n} \\
&= \lim_{n\in N_0} |\gamma'_{\widetilde{\mathcal{G}}_{\sigma^k\omega},\widetilde{S}_{\sigma^k\omega}}(n)|^{1/n} \\
&= \lim_{n\in N_0} \left(\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right| + \left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k\omega}(n)\right|\right)^{1/n}.
\end{aligned}$$

Using (3.11), we get,

$$\begin{aligned}
\widetilde{\lambda}_\omega &\leqslant \overline{\lim_{n\in N_0}} \left(2\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right|\right)^{1/n} \\
&= \overline{\lim_{n\in N_0}} \left(\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right|\right)^{1/n} \\
&\leqslant \overline{\lim_{n\in N_0}} |\mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)|^{1/n} \\
&\leqslant \overline{\lim_n} |\mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)|^{1/n}.
\end{aligned}$$

Using Corollary 3.1 we get,

$$\widetilde{\lambda}_\omega \leqslant (1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon}. \tag{3.12}$$

Now suppose that for every $k, s \in \mathbb{N}$, there exists an $N(k,s)$ such that for all $n \geqslant N(k,s)$,

$$\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right| < \left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k\omega}(n)\right|. \tag{3.13}$$

As before, let $\widetilde{H}^{(s)}_\omega(n) := \widetilde{B}_\omega(n) \cap \widetilde{H}^{(s)}_\omega$ and $\widetilde{H}^{(s)}_{\sigma^k\omega}(n) := \widetilde{B}_{\sigma^k\omega}(n) \cap \widetilde{H}^{(s)}_{\sigma^k\omega}$. Let ω =

$\omega_1 \ldots \omega_{s_1}\omega_{s_1+1} \ldots \omega_{s_1+s_2}\omega_{s_1+s_2+1} \ldots \omega_{s_1+s_2+s_3} \ldots$ where s_1 is the first time third symbol appears in ω, s_2 is the first time third symbol appears in $\sigma^{s_1}\omega$, and so on.

Since $[\widetilde{\mathcal{G}}_\omega : \widetilde{H}_\omega^{(s_1)}] \leqslant 2^{2^{s_1}-1} =: K_1$, there is a fixed Schreier system of representatives of the right cosets of $\widetilde{\mathcal{G}}_\omega$ modulo $\widetilde{H}_\omega^{(s_1)}$ with Schreier representatives of length less than K_1. So for any $g \in \widetilde{B}_\omega(n)$, there are $h \in \widetilde{H}_\omega^{(s_1)}$, l a Schreier representative such that $g = hl$ and since $|l| \leqslant K_1$, we have $|h| \leqslant n + K_1$. Therefore,

$$\left|\widetilde{B}_\omega(n)\right| \leqslant K_1\left|\widetilde{H}_\omega^{(s_1)}(n+K_1)\right|. \tag{3.14}$$

Let $N_1 = \max\{n_\epsilon, N(0,s_1)\}$, where n_ϵ is defined in Lemma 3.6 and $N(0,s_1)$ is defined in (3.13). Note that,

$$\begin{aligned}\left|\widetilde{H}_\omega^{(s_1)}(n+K_1)\right| &= 1 + \sum_{k=1}^{n+K_1}\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \left(\widetilde{B}_\omega(k)\backslash\widetilde{B}_\omega(k-1)\right)\right| \\ &\leqslant N_1\left|\widetilde{B}_\omega(N_1)\right| + \sum_{k=N_1}^{n+K_1}\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \left(\widetilde{B}_\omega(k)\backslash\widetilde{B}_\omega(k-1)\right)\right|.\end{aligned}$$

From (3.13), for $k \geqslant N_1$,

$$\begin{aligned}&\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \left(\widetilde{B}_\omega(k)\backslash\widetilde{B}_\omega(k-1)\right)\right| \\ =&\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \mathcal{F}^\epsilon(k)\right| + \left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \mathcal{D}^\epsilon(k)\right| \\ \leqslant& 2\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \mathcal{D}^\epsilon(k)\right|.\end{aligned}$$

Therefore,

$$\left|\widetilde{H}_\omega^{(s_1)}(n+K_1)\right| \leqslant N_1\left|\widetilde{B}_\omega(N_1)\right| + 2\sum_{k=N_1}^{n+K_1}\left|\widetilde{H}_\omega^{(s_1)}(n+K_1) \cap \mathcal{D}^\epsilon(k)\right|.$$

Now using Lemma 3.6,

$$\left|\widetilde{H}^{(s_1)}_\omega(n+K_1)\right| \leqslant N_1\left|\widetilde{B}_\omega(N_1)\right| + 2\sum_{j_1,\dots,j_{2^{s_1}}}\left|\widetilde{B}_{\sigma^{s_1}\omega}(j_1)\right|\dots\left|\widetilde{B}_{\sigma^{s_1}\omega}(j_{2^{s_1}})\right|, \tag{3.15}$$

where $\sum_{i=1}^{2^{s_1}} j_i \leqslant \left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1$.

Note that,

$$\widetilde{\lambda}_{\sigma^{s_1}\omega} = \lim_j |\widetilde{B}_{\sigma^{s_1}\omega}(j)|^{1/j},$$

and therefore, for each $\delta > 0$, there exists an $J = J(\delta)$ such that for $j \geqslant J$,

$$|\widetilde{B}_{\sigma^{s_1-1}\omega}(j)| \leqslant (\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^j.$$

Thus for all j

$$|\widetilde{B}_{\sigma^{s_1-1}\omega}(j)| \leqslant |\widetilde{B}_{\sigma^{s_1-1}\omega}(J)|(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^j,$$

which implies,

$$\begin{aligned}\left|\widetilde{B}_{\sigma^{s_1}\omega}(j_1)\right|\dots\left|\widetilde{B}_{\sigma^{s_1}\omega}(j_{2^{s_1}})\right| &\leqslant \left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}}(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\sum_{i=1}^{2^{s_1}} j_i} \\ &\leqslant \left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}}(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1}. \end{aligned} \tag{3.16}$$

The number of summands in the right hand side of (3.15) is,

$$\begin{aligned}\binom{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1+2^{s_1}}{2^{s_1}} &\leqslant \binom{n+K_1+2^{s_1+1}-1}{2^{s_1}} \\ &\leqslant (n+K_1+2^{s_1+1}-1)^{2^{s_1}}. \end{aligned} \tag{3.17}$$

Now by (3.14), (3.15), (3.16) and (3.17) we get,

$$\left|\widetilde{B}_\omega(n)\right| \leqslant K_1N_1\left|\widetilde{B}_\omega(N_1)\right|$$

$$+ \left(2K_1(n+K_1+2^{s_1+1}-1)^{2^{s_1}}\left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}} \right.$$
$$\left. \times(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1}\right).$$

Therefore,

$$\widetilde{\lambda}_\omega = \lim_n \left|\widetilde{B}_\omega(n)\right|^{1/n} \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta\right)^{\left(1-\frac{\epsilon}{5}\right)}.$$

Since δ is arbitrary,

$$\widetilde{\lambda}_\omega \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)}.$$

In the same way, still under the assumption (3.13), and replacing ω by $\omega, \sigma^{s_1}\omega, \sigma^{s_1+s_2}\omega, \sigma^{s_1+s_2+s_3}\omega, \ldots$, we get,

$$\begin{aligned}
\widetilde{\lambda}_\omega &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)} \\
\widetilde{\lambda}_{\sigma^{s_1}\omega} &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+s_2}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)} \\
\widetilde{\lambda}_{\sigma^{s_1+s_2}\omega} &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+s_2+s_3}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)} \\
&\vdots
\end{aligned}$$

Thus for each $k \in \mathbb{N}$,

$$\widetilde{\lambda}_\omega \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+\ldots+s_k}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)^k}. \tag{3.18}$$

But the growth index λ of a group with 8 generators of order 2 cannot exceed 9. Since k may be chosen arbitrarily large, it follows from (3.18) that $\widetilde{\lambda}_\omega = 1$. If there exists an $\epsilon > 0$ satisfying (3.13), then $\widetilde{\lambda}_\omega = 1$. If not, then for all $\epsilon > 0$ we have (3.11). Thus by (3.12) and

$$\lim_{\epsilon\to 0}(1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon} = 1,$$

we get $\widetilde{\lambda}_\omega = 1$ in all cases. Since $\widetilde{\lambda}_\omega = 1$, $\widetilde{\mathcal{G}}_\omega$ has subexponential growth.

We know $\mathcal{G}_\omega \subset \widetilde{\mathcal{G}}_\omega$ and by [Gri84b], $\mathcal{G}_\omega$ is of intermediate growth. Therefore $\widetilde{\mathcal{G}}_\omega$ is of intermediate growth. ◻

Note that the Theorem 3.1 follows directly from Proposition 3.2, Proposition 3.3, and Theorem 3.3.

3.3 Growth bounds for Generalized Overgroups

Proposition 3.4. *Let $\omega \in \Omega_0 \cup \Omega_1$. Then for each $\epsilon > 0$,*

$$\gamma_{\widetilde{\mathcal{G}}_\omega}(n) \geq \exp\left\{\left(\frac{n}{\log^{2+\epsilon}(n)}\right)\right\}.$$

Proof. Let $\omega \in \Omega_0 \cup \Omega_1$. We may assume ω has infinitely many 0's and 2's. Then, by (2.11), b_ω as a sequence of P's and I's contains both symbols infinitely often. By Theorem 2 of [Ers04] the group generated by elements $a, b_\omega, \widetilde{a}$ has growth bounded below by $\exp\left\{\left(\frac{n}{\log^{2+\epsilon}(n)}\right)\right\}$. Since $\widetilde{\mathcal{G}}_\omega$ contains the elements $a, b_\omega, \widetilde{a}$, we get the required result. ◻

Theorem 3.2′. *Let $\omega \in \Omega_1^*$. Then,*

$$\gamma_{\widetilde{\mathcal{G}}_\omega}(n) \preceq \exp\left\{\left(\frac{n\log(\log(n))}{\log(n)}\right)\right\}.$$

Proof. Since $\omega \in \Omega_1^*$, there is an N such that $\sigma^N\omega$ contains exactly two symbols, say i, j. Then by Lemma 3.3, $\widetilde{\mathcal{G}}_{\sigma^N\omega} = \mathcal{G}_{\sigma^N\omega}$. By theorem 3 of [Ers04], we get,

$$\gamma_{\widetilde{\mathcal{G}}_{\sigma^n\omega}}(n) \preceq \exp\left\{\left(\frac{n\log(\log(n))}{\log(n)}\right)\right\},$$

and therefore,

$$\begin{aligned}
\gamma_{\widetilde{\mathcal{G}}_\omega}(n) &\approx \left(\gamma_{\widetilde{\mathcal{G}}_{\sigma^n\omega}}(n)\right)^{2^N} \\
&\preceq \left(\exp\left\{\left(\frac{n\log(\log(n))}{\log(n)}\right)\right\}\right)^{2^N} \\
&\approx \exp\left\{\left(\frac{n\log(\log(n))}{\log(n)}\right)\right\}.
\end{aligned}$$

□

While Theorem 3.3 states that $\widetilde{\mathcal{G}}_\omega$ has intermediate growth for all $\omega \in \Omega_0$, for homogeneous sequences from Ω_0^*, we can actually provide an explicit upper bound on growth.

Theorem 3.2″. *Let $\omega \in \Omega_0^*$. Then,*

$$\gamma_{\widetilde{\mathcal{G}}_\omega}(n) \leqslant \exp\left\{\left(\frac{n\log(\log(n))}{\log(n)}\right)\right\}.$$

Proof. The proof follows similarly as of the proof of Theorem 3 of [Ers04] by replacing Lemma 6.2 (1) of [Ers04] by Lemma 3.6. □

Theorem 3.2′ together with Theorem 3.2″ implies Theorem 3.2.

4. GENERALIZED GRIGORCHUK'S OVERGROUPS IN THE SPACE OF MARKED GROUPS*

This chapter is extracted from the article [Sam22].

4.1 Introduction

Recall that $\Omega_2 \subset \Omega = \{0,1,2\}^{\mathbb{N}}$ is the set consisting of virtually constant sequences. If $\omega \in \Omega \backslash \Omega_2$, then $\mathcal{G}_\omega$ has intermediate growth [Gri84b]. In [Gri84b] it was shown that the closure of the set $\mathcal{Z} = \{\mathcal{G}_\omega \mid \omega \in \Omega \backslash \Omega_2\}$ in $\mathcal{M}_4$, denoted by $\overline{\mathcal{Z}}$, is a closed set without isolated points (hence homeomorphic to a Cantor set) and $\overline{\mathcal{Z}} \backslash \mathcal{Z}$ is a countable set consisting of virtually metabelian groups, with one such group $\mathcal{G}_\omega^\alpha$ (defined using an algorithm α for the word problem) for each $\omega \in \Omega_2$. So,

$$\overline{\mathcal{Z}} = \mathcal{Z} \cup \{\mathcal{G}_\omega^\alpha \mid \omega \in \Omega_2\} = \text{Cantor set}.$$

The Grigorchuk's overgroup $\widetilde{\mathcal{G}}$ is important, in particular, because as is shown by Y. Vorobets (private communication), it constitutes a big part of the topological full group $[[(\Lambda, T)]]$ associated with substitutional dynamical system (Λ, T) generated by Lysënok's substitution $\sigma\colon a \mapsto aca,\ b \mapsto d,\ c \mapsto b,\ d \mapsto c$, which was initially used to describe a presentation of $\mathcal{G}$ [Lys85], where T denotes the shift map in the space $\Lambda = \{a,b,c,d\}^{\mathbb{Z}}$.

In this chapter we describe the structure of the closure of the set $\mathcal{X} = \left\{\widetilde{\mathcal{G}}_\omega \mid \omega \in \Omega\right\}$ in $\mathcal{M}_8$, which happens to be much more complicated than in the case of classical Grigorchuk groups (see Figure 4.1).

Constructions in this chapter are based on algorithms α and β_{ij} for i, j, distinct elements of $\{0,1,2\}$, which will be defined in Section 4.2.1. The algorithm α is a branch type algorithm, similar to the one introduced in [Gri84b]. Algorithms β_{ij} were introduced in order to construct 'new' class of modified overgroups (see Section 4.2.2). We hope that the methods introduced

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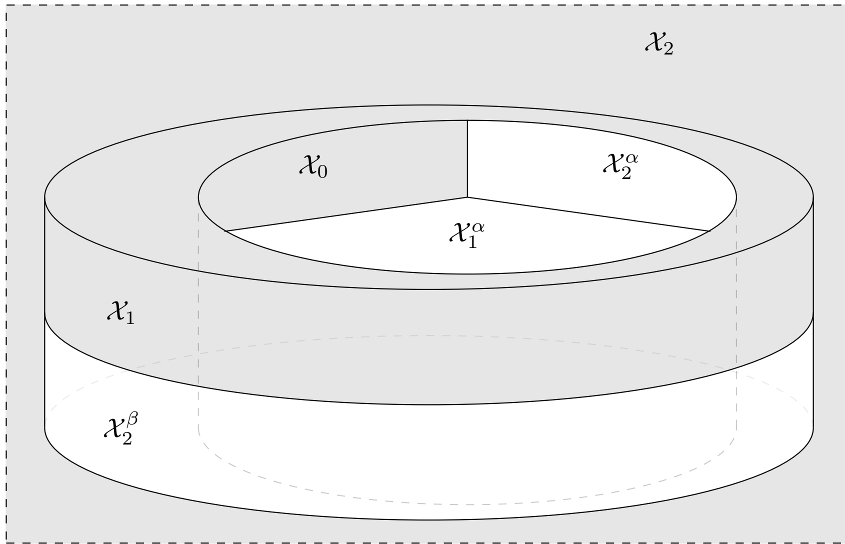

Figure 4.1: Structure of topological closure of $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \mathcal{X}_2$ in $\mathcal{M}_8$

here will contribute to the study in the direction of constructing new example of non-elementary amenable groups.

Recall that $\Omega_0, \Omega_1 \subset \Omega$, where Ω_0 is the set of all sequences with all three symbols occurring infinitely often and $\Omega_1 = \Omega \backslash (\Omega_0 \cup \Omega_2)$ is the set of all sequences with exactly two symbols occurring infinitely often. We use the word "oracle" to represent a sequence in Ω.

Using algorithms α and β_{ij} for $i, j \in \{0, 1, 2\}$, we define modified overgroups $\widetilde{\mathcal{G}}_\omega^\alpha$ and $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ (see Section 4.2.2) as those for which the word problem is decidable by the corresponding algorithm, assuming that the oracle ω is known. We define the following subsets of $\mathcal{M}_8$:

$$
\begin{aligned}
\mathcal{X} &= \left\{(\widetilde{\mathcal{G}}_\omega, \widetilde{S}_\omega)\right\}_{\omega \in \Omega} \text{ ; union of all shaded regions in Figure 4.1,} \\
\mathcal{X}_i &= \left\{(\widetilde{\mathcal{G}}_\omega, \widetilde{S}_\omega)\right\}_{\omega \in \Omega_i} \text{ ; for } i = 0, 1, 2, \\
\mathcal{X}_i^\alpha &= \left\{(\widetilde{\mathcal{G}}_\omega^\alpha, \widetilde{S}_\omega^\alpha)\right\}_{\omega \in \Omega_i} \text{ ; for } i = 1, 2, \\
\mathcal{X}_2^\beta &= \left\{(\widetilde{\mathcal{G}}_\omega^\beta, \widetilde{S}_\omega^\beta) \mid \beta \in \{\beta_{01}, \beta_{12}, \beta_{20}\}\ \right\}_{\omega \in \Omega_2}, \\
\mathcal{Y} &= \mathcal{X}_0 \cup \mathcal{X}_1^\alpha \cup \mathcal{X}_2^\alpha \text{ ; middle cylinder in Figure 4.1,}
\end{aligned}
\tag{4.1}
$$

where $\widetilde{S}_\omega, \widetilde{S}^\alpha_\omega$, and $\widetilde{S}^\beta_\omega$ are the natural generating sets for corresponding groups. In the following text, the topological closure and the set of limit points (or cluster points) of a set V will be denoted by $\overline{V}, V_\sharp$, respectively.

Theorem 4.1. *The sets $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_1^\alpha, \mathcal{X}_2^\alpha$, and $\mathcal{X}_2^\beta$ are mutually disjoint subsets of $\mathcal{M}_8$. In any set other than $\mathcal{X}_2^\beta$, different corresponding oracles ω give rise to different groups. In $\mathcal{X}_2^\beta$, there are two different groups for each corresponding oracle ω.*

Theorem 4.2.

1. *$\overline{\mathcal{X}} = \mathcal{X}_\sharp \sqcup \mathcal{X}_2$, where the set $\mathcal{X}_2$ consists of the set of isolated points of $\mathcal{X}$.*

2. *$\mathcal{X}_\sharp, \mathcal{Y}$ are homeomorphic to a Cantor set.*

3. *Furthermore, we have following relations:*

   (a) $\mathcal{Y} = (\mathcal{X}_0)_\sharp = (\mathcal{X}_1^\alpha)_\sharp = (\mathcal{X}_2^\alpha)_\sharp$.

   (b) $\mathcal{X}_\sharp = \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^\beta = (\mathcal{X}_1)_\sharp = (\mathcal{X}_2^\beta)_\sharp = (\mathcal{X}_2)_\sharp$.

It is worth to mention that the limit groups that appear in [Gri84b] are of the lamplighter type and one of them (building block) is a 2-extension of the lamplighter group $\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$ [BG14]. In our situation the lamplighter group also plays an important role and the building blocks constitute the group $\mathcal{L}$ as well as $\mathcal{L}_2 := \mathbb{Z}_2^2 \wr \mathbb{Z}$ and their direct products.

Theorem 4.3. *Let $\{i, j, k\} = \{0, 1, 2\}$.*

1. *Let $\omega \in \Omega_2$ and let N be the smallest index such that only i appears in ω after N. Then $\widetilde{\mathcal{G}}^\alpha_\omega$ is commensurable to $(\widetilde{\mathcal{G}}^\alpha_{i^\infty})^{2^N}$, which is virtually $(\mathcal{L}_2)^{2^N}$. Therefore $\widetilde{\mathcal{G}}^\alpha_\omega$ is elementary amenable and of exponential growth.*

2. *Let $\omega \in \Omega_2$ and let N be the smallest index such that only i appears in ω after N. Then $\widetilde{\mathcal{G}}^{\beta_{ij}}_\omega$ is commensurable to $(\widetilde{\mathcal{G}}^{\beta_{ij}}_{i^\infty})^{2^N}$, which is virtually $(\mathcal{L})^{2^N}$. Therefore $\widetilde{\mathcal{G}}^{\beta_{ij}}_\omega$ is elementary amenable and of exponential growth.*

3. *Let $\omega \in \Omega_1$ and let N be the smallest index such that no k appears in ω after N. Then $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$ is commensurable to $(\widetilde{\mathcal{G}}^{\alpha}_{\sigma^N\omega})^{2^N}$. $\widetilde{\mathcal{G}}^{\alpha}_{\sigma^N\omega}$ contains $\mathcal{L}$ as a subgroup and is an extension of a non elementary amenable group by an abelian group. Therefore $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$ is non elementary amenable and of exponential growth.*

It is known that the groups in $\mathcal{X}_2$ have polynomial growth and the groups in $\mathcal{X}_0$ and $\mathcal{X}_1$ have intermediate growth (see Chapter 3). As a consequence of Theorem 4.3, we have;

Corollary 4.1. *Groups in the set $\mathcal{X}_0 \cup \mathcal{X}_1$ are of intermediate growth, groups in the set $\mathcal{X}_2$ are of polynomial growth, and groups in $\mathcal{X}^{\alpha}_1 \cup \mathcal{X}^{\alpha}_2 \cup \mathcal{X}^{\beta}_2$ are of exponential growth.*

If G is a finitely presented group in $\mathcal{M}_k$ with finite set of relations R, such that $G_n \to G$ for some sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{M}_k$, then G maps onto G_n for sufficiently large n. This can be obtained by considering the ball of radius n centered at identity of the Cayley graph of G, where $n/2$ is larger than the maximum of lengths of relations in R. In particular, the growth rate of G is not less than the growth growth rate of G_n. By Theorem 4.2 3, for ω non virtually constant, $\widetilde{\mathcal{G}}_{\omega}$ is a limit point of $\mathcal{X}^{\beta}_2$ and so there is a sequence $\{G_n\}$ of groups of exponential growth (by Corollary 4.1) in $\mathcal{X}^{\beta}_2$ converging to $\widetilde{\mathcal{G}}_{\omega}$. Therefore, by the contra-positive of above argument, we get following corollary:

Corollary 4.2. *$\widetilde{\mathcal{G}}_{\omega}$ is infinitely presented for $\omega \in \Omega \backslash \Omega_2$.*

The Cantor-Bendixson rank is an invariant of topological spaces. It is the least ordinal at which the removal of isolated points makes no change to the space. If the topological space is Polish (complete, metrizable and separable), then the Cantor-Bendixson rank is countable [Kec95]. As a consequence of Theorem 4.2, the Cantor-Bendixson rank of $\overline{\mathcal{X}}$ is one.

4.2 Modified Overgroups

4.2.1 Algorithms for the Word Problem

First we define inductively the algorithm α, which solves the word problem for $\widetilde{\mathcal{G}}_{\omega}$, when $\omega \in \Omega_0$. Given any reduced word $W \in \Gamma$, if it has even number of 'a's (i.e. $W \in \Theta$), use $\widetilde{\psi}$

to get two reduced words W_0, W_1. If $W \notin \Theta$, terminate the process. Now suppose we have 2^n reduced words $\{W_{i_1 \dots i_n}\} \subset \Gamma$. If at least one of them is not in Θ, terminate the process. If all the words are in Θ, use $\widetilde{\psi}$ to obtain 2^{n+1} reduced words $\{W_{i_1 \dots i_{n+1}}\}$. Follow this process N times, where $N = \lceil \log_2 |W| \rceil$, to obtain 2^N reduced words $\{W_{i_1 i_2 \dots i_N}\}$. Then by (2.18), we get $|W_{i_1 i_2 \dots i_N}| \leqslant \frac{|W|}{2^N} + 1 - \frac{1}{2^N} \leqslant 1$, and thus the level N nucleus is achieved. The algorithm α gives positive result if all words $W_{i_1 i_2 \dots i_N}$ are the empty word. That is the level N nucleus of W consists of empty words.

Let $\{i, j, k\} = \{0, 1, 2\}$ (we will use this notation of i, j, k throughout rest of the text). Inductively define algorithm β_{ij} which solves the word problem for $\widetilde{\mathcal{G}}_\omega$, when $\omega \in \Omega_1$ and i, j occur in ω infinitely often. Let N_0 be the largest index such that $\omega_{N_0} = k$. Given any reduced word $W \in \Gamma$, similarly to above, if $W \notin \Theta$, end the process. And if $W \in \Theta$, use $\widetilde{\psi}$ to get two reduced words W_0, W_1. Follow this process N times, where $N = \max\{N_0, \lceil \log_2 |W| \rceil\}$, to obtain 2^N reduced words $\{W_{i_1 i_2 \dots i_N}\}$, if such words exist. Note that $N \geqslant N_0$ guarantees that $\sigma^N \omega$ does not contain symbol k in it. By (2.18), $|W_{i_1 i_2 \dots i_N}| \leqslant \frac{|W|}{2^N} + 1 - \frac{1}{2^N} \leqslant 1$ and so the level N nucleus is achieved. The algorithm gives positive result if all words $W_{i_1 i_2 \dots i_N}$ are either empty word or e_{ij}, where $e_{01} = \widetilde{b}, e_{12} = \widetilde{d}$ and $e_{20} = \widetilde{c}$. That is the level N nucleus of W consists of empty words and 'e_{ij}'s.

4.2.2 Modified Overgroups

Here we will introduce new collection of groups using the algorithms described above, named modified overgroups, similar to modified Grigorchuk groups $\mathcal{G}_\omega^\alpha$ introduced in [Gri84b]. (The notation used in [Gri84b] is $\widetilde{\mathcal{G}}_\omega$, which is already taken to overgroups in this text.)

Let $\omega \in \Omega$. Define N_ω^α to be the subgroup of Γ consisting of all the words of Γ that yield a positive result when the algorithm α is applied. Since any conjugate of the empty word is the empty word, using (2.16) multiple times we obtain that N_ω^α is normal in Γ. Define modified overgroup $\widetilde{\mathcal{G}}_\omega^\alpha = \Gamma / N_\omega^\alpha$. Let $\pi^\alpha \colon \Gamma \to \widetilde{\mathcal{G}}_\omega^\alpha$ be the canonical epimorphism. We denote the generating set of $\widetilde{\mathcal{G}}_\omega^\alpha$ by $\widetilde{S}_\omega^\alpha = \left\{a_\omega^\alpha, b_\omega^\alpha, c_\omega^\alpha, d_\omega^\alpha, \widetilde{a}_\omega^\alpha, \widetilde{b}_\omega^\alpha, \widetilde{c}_\omega^\alpha, \widetilde{d}_\omega^\alpha\right\}$.

Now let $\omega \in \Omega_1 \cup \Omega_2$ with at most finitely many 'k's. Define $N_\omega^{\beta_{ij}}$ to be the subgroup of Γ consisting of all the words of Γ that yield a positive result when the algorithm β_{ij} is applied. Note that by choosing $W = e_{ij}$ in (2.16), we obtain that any conjugate of e_{ij} has nuclei consisting of only the empty words and e_{ij}'s, at sufficiently large level. This together with (2.16) yield, $N_\omega^{\beta_{ij}}$ is normal in Γ. Define modified overgroup $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}} = \Gamma/N_\omega^{\beta_{ij}}$. Let $\pi^{\beta_{ij}}\colon \Gamma \to \widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ be the canonical epimorphism. We denote the generating set of $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ by $\widetilde{S}_\omega^{\beta_{ij}} = \left\{a_\omega^{\beta_{ij}}, b_\omega^{\beta_{ij}}, c_\omega^{\beta_{ij}}, d_\omega^{\beta_{ij}}, \widetilde{a}_\omega^{\beta_{ij}}, \widetilde{b}_\omega^{\beta_{ij}}, \widetilde{c}_\omega^{\beta_{ij}}, \widetilde{d}_\omega^{\beta_{ij}}\right\}$.

Proposition 4.1.

1. *If $\omega \in \Omega_0$, then $\widetilde{\mathcal{G}}_\omega^\alpha = \widetilde{\mathcal{G}}_\omega$ and if $\omega \in \Omega_1 \cup \Omega_2$, then $\widetilde{\mathcal{G}}_\omega^\alpha$ surjects onto $\widetilde{\mathcal{G}}_\omega$ with non trivial kernel.*

2. *If $\omega \in \Omega_1$, then $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}} = \widetilde{\mathcal{G}}_\omega$ and if $\omega \in \Omega_2$, then $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ surjects onto $\widetilde{\mathcal{G}}_\omega$ with non trivial kernel.*

Proof. 1. Consider surjections $\pi\colon \Gamma \to \widetilde{\mathcal{G}}_\omega$ and $\pi^\alpha\colon \Gamma \to \widetilde{\mathcal{G}}_\omega^\alpha$. By definition of $\widetilde{\mathcal{G}}_\omega^\alpha$, we obtain that $\ker(\pi^\alpha) \subset \ker(\pi)$. Thus $\widetilde{\mathcal{G}}_\omega^\alpha$ surjects onto $\widetilde{\mathcal{G}}_\omega$.

Let $\omega \in \Omega_0$. Then for any n, each element in $\widetilde{\mathcal{G}}_{\sigma^n\omega}$ of length one will never be the identity. Therefore, $\ker(\pi^\alpha) = \ker(\pi)$, and so the modified overgroup $\widetilde{\mathcal{G}}_\omega^\alpha$ is isomorphic to the generalized overgroup $\widetilde{\mathcal{G}}_\omega$.

Now let $\omega \in \Omega_1 \cup \Omega_2$. Then for some N, $\sigma^N\omega$ contains at most two symbols. Say $\sigma^N\omega$ does not contain 2. Thus $\widetilde{b}_{\sigma^n\omega} = 1$ in $\widetilde{\mathcal{G}}_{\sigma^n\omega}$ for $n \geqslant N$. Note that, $W(01) \in \Gamma$ constructed in (4.6) is in $\ker(\pi)$, but not in $\ker(\pi^\alpha)$, since level n nucleus of $W(01)$ consists of '1's and '$\widetilde{b}$'s, for sufficiently large n. Therefore, $\widetilde{\mathcal{G}}_\omega^\alpha$ surjects onto $\widetilde{\mathcal{G}}_\omega$ with non trivial kernel.

2. Now consider surjections, π and $\pi^{\beta_{ij}}$. By the definition of $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$, we get $\ker(\pi^{\beta_{ij}}) \subset \ker(\pi)$, and thus $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ surjects onto $\widetilde{\mathcal{G}}_\omega$.

Let $\omega \in \Omega_1$ with finitely many 'k's. Then each element in $\widetilde{\mathcal{G}}_{\sigma^n\omega}$ of length one will never be the identity, unless it is e_{ij}. Therefore, $\ker(\pi^{\beta_{ij}}) = \ker(\pi)$, and so $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ is isomorphic to $\widetilde{\mathcal{G}}_\omega$.

Now let $\omega \in \Omega_2$. Without loss of generality, suppose $i = 0, j = 1$. Then for some N, $\sigma^N\omega$ contains only one symbol. Say $\sigma^N\omega$ contain only 0's. Thus $\widetilde{c}_{\sigma^n\omega} = 1$ in $\widetilde{\mathcal{G}}_{\sigma^n\omega}$ for $n \geqslant N$. Note that, $W(02) \in \Gamma$ constructed in (4.6), has only '1's and '$\widetilde{c}$'s in its level n nucleus, for sufficiently large n. So $W(02) \in \ker(\pi)$. Recall that $e_{ij} = e_{01} = \widetilde{b}$, and therefore $W(02) \notin \ker(\pi^{\beta_{ij}})$. Hence $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$ surjects onto $\widetilde{\mathcal{G}}_\omega$ with non trivial kernel. □

The following proposition is useful in comparing two groups.

Proposition 4.2. *Let $r \in \mathbb{N}$ and let $\omega, \eta \in \Omega$ such that $\omega_t = \eta_t$ for each $t \leqslant N$, where $N > \log_2(2r)$.*

1. *If ω, η have all three symbols after the N-th position, then the balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega, \widetilde{\mathcal{G}}_\eta$ are identical.*
2. *If ω has all three symbols after the N-th position, then the balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega, \widetilde{\mathcal{G}}_\eta^\alpha$ are identical.*
3. *The balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega^\alpha, \widetilde{\mathcal{G}}_\eta^\alpha$ are identical.*
4. *If ω, η have exactly the same two symbols, say $\{i, j\}$, after the N-th position, then the balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega, \widetilde{\mathcal{G}}_\eta$ are identical.*
5. *If ω has only i, j and η has no k, after the N-th position, then the balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega, \widetilde{\mathcal{G}}_\eta^{\beta_{ij}}$ are identical.*
6. *If ω, η has no k, after the N-th position, then the balls of radius r of Cayley graphs of $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}, \widetilde{\mathcal{G}}_\eta^{\beta_{ij}}$ are identical.*

Proof. 1. We will say two words W, X over alphabets of generators of $\widetilde{\mathcal{G}}_\omega, \widetilde{\mathcal{G}}_\eta$, are equal if their corresponding letters match. Let W, X be equal words of length at most $2r$. Suppose $W = 1$ in $\widetilde{\mathcal{G}}_\omega$. Thus we can decompose W into two words $\{W_0, W_1\}$, four words $\{W_{00}, W_{01}, W_{10}, W_{11}\}$, ..., 2^N words $\{W_{i_1 i_2 \dots i_N}\}$, where all these words represents identity in corresponding groups. By (2.18), $|W_{i_1 i_2 \dots i_N}| \leqslant \frac{|W|}{2^N} + 1 - \frac{1}{2^N} < 2$. This, together with the fact that ω has all three symbols, implies $W_{i_1 i_2 \dots i_N} = 1$ as a word. Also note that all the words $W_{i_1 i_2 \dots i_N}$ are described by first N symbols of ω. Since first N symbols of ω and η are equal, $X = 1$ in $\widetilde{\mathcal{G}}_\eta$. Therefore we proved 1. The same argument works for 2 and 3.

4. Since ω, η only have i, j after N-th position, the only length one element which represents identity is e_{ij}. Therefore the proof in 1 with a slight modification works. The argument of 4 works for 5 and 6. □

The modified overgroups behave nicely under limits.

Corollary 4.3. *Let $\{\omega^{(n)}\}$ be a sequence in Ω that converges to $\omega \in \Omega$. Then $\widetilde{\mathcal{G}}^{\alpha}_{\omega^{(n)}}$ converges to $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$. Additionally, if there is an N such that no k appears after the N-th position of each of $\{\omega^{(n)}\}$, then $\widetilde{\mathcal{G}}^{\beta_{ij}}_{\omega^{(n)}}$ converges to $\widetilde{\mathcal{G}}^{\beta_{ij}}_{\omega}$.*

Proof. Since $\omega^{(n)} \to \omega$, for sufficiently large n, $\omega, \omega^{(n)}$ satisfy the hypothesis of Proposition 4.2. By Proposition 4.2 3, balls of arbitrary radius k of Cayley graphs of $\widetilde{\mathcal{G}}^{\alpha}_{\omega^{(n)}}$ and $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$, are identical for sufficiently large n. Therefore, $\widetilde{\mathcal{G}}^{\alpha}_{\omega^{(n)}} \to \widetilde{\mathcal{G}}^{\alpha}_{\omega}$.

Now suppose there is an N such that no k appears after the N-th position of each of $\{\omega^{(n)}\}$. Then by a similar argument, using Proposition 4.2 6, we get $\widetilde{\mathcal{G}}^{\beta_{ij}}_{\omega^{(n)}} \to \widetilde{\mathcal{G}}^{\beta_{ij}}_{\omega}$. □

4.2.3 Modified Overgroups for Some $\omega \in \Omega$

Now we will look at the modified overgroups and see what their structures are. In fact we will prove Theorem 4.3 using propositions that are provided in this section. First we will introduce some words in Γ and substitution rules on words in Γ which will be used throughout this section.

Let $\omega \in \Omega$ be a sequence with at most two symbols. Let $y \in S \backslash \{a\}$ be such that for each $n \in \mathbb{N}$, the decomposition of y into depth n using (2.15), has nucleus

$$(1, 1, \ldots, 1, y). \tag{4.2}$$

Since ω has at most two symbols, such y exists. For $n \in \mathbb{Z}$, define $v_n(y) = v_n$ by

$$v_n = \begin{cases} y^{(a\tilde{a})^n} & ; n \geqslant 0 \\ y^{(a\tilde{a})^{-n-1}a} & ; n < 0 \end{cases}. \tag{4.3}$$

For any $W \in \Gamma$, $\left(y^W\right)^2 = (y^2)^W = 1$ since y is an involution. So, $v_n^2 = 1$ for all $n \in \mathbb{Z}$. Note that $v_n^a = v_{-n-1}$, since if $n \geqslant 0$, $v_{-n-1} = y^{(a\tilde{a})^n a} = \left(y^{(a\tilde{a})^n}\right)^a = v_n^a$ and if $n < 0$, $v_n^a = \left(y^{(a\tilde{a})^{-n-1}a}\right)^a = y^{(a\tilde{a})^{-n-1}} = v_{-n-1}$. A direct calculation shows that $v_n^{(a\tilde{a})} = v_{n+1}$.

Let $n \geqslant 0$. Then $v_{2n} = y^{(a\tilde{a})^{2n}} = y^{(\tilde{a}^a\tilde{a})^n}$. Thus, $\widetilde{\psi}(v_{2n}) = (1^{(\tilde{a}a)^n}, y^{(a\tilde{a})^n}) = (1, v_n)$. Now let $n < 0$. Then $v_{2n} = y^{(a\tilde{a})^{-2n-1}a} = y^{(\tilde{a}^a\tilde{a})^{-n-1}\tilde{a}^a}$, and therefore $\widetilde{\psi}(v_{2n}) = (1^{(\tilde{a}a)^{-n-1}\tilde{a}}, y^{(a\tilde{a})^{-n-1}a}) = (1, v_n)$. Whence for even n we have $v_n = (1, v_{n/2})$ via $\widetilde{\psi}$. A similar calculation shows that $v_n = (v_{-(n+1)/2}, 1)$ when n is odd. We summarize the above discussion as the next proposition.

Proposition 4.3.

1. $v_n^2 = 1$ *for* $n \in \mathbb{Z}$.
2. $v_n^a = v_{-n-1}$ *for* $n \in \mathbb{Z}$.
3. $v_n = (1, v_{n/2})$ *for even* n.
4. $v_n = (v_{-(n+1)/2}, 1)$ *for odd* n.
5. $a\tilde{a}$ *acts on* v_n *by conjugation and* $v_n^{(a\tilde{a})} = v_{n+1}$.

Note that by applying $\widetilde{\psi}$ to v_n, we obtain v_m in one coordinate, for some $m \in \mathbb{Z}$ such that $|m| < |n|$, if $n \neq 0, 1$. This fact will be used in next proposition, which has more properties of $\{v_n\}$.

Proposition 4.4. *Let* n, m *be distinct integers. Then,*

1. v_n *achieves a nucleus at some level. Furthermore, each nucleus of* v_n *has all coordinates equal to 1, except for one coordinate, which is equal to* y*. Therefore,* $v_n \neq 1$ *and* $v_n v_m = v_m v_n$ *in* $\widetilde{\mathcal{G}}_\omega^\alpha$.
2. *For each level, nuclei of* v_n *and* v_m*, if exist, are different. So,* $v_n \neq v_m$ *in* $\widetilde{\mathcal{G}}_\omega^\alpha$.

Proof. 1. We use induction on $|n|$. Note that $v_0 = y = (1, y) = (1, 1, 1, y), v_{-1} = (v_0, 1) = (y, 1) = (1, y, 1, 1)$ and $v_1 = (v_{-1}, 1) = (y, 1, 1, 1)$, which proves the base cases. Let $|n| > 1$. Suppose the statement in 1 is true for $|i| < |n|$. By Proposition 4.3 3 and 4, $v_n = (1, v_{n/2})$ or $v_n = (v_{-(n+1)/2}, 1)$. Since $|n| > 1$, we have $|n/2|, |-(n+1)/2| < |n|$. Thus by the induction hypothesis, we obtain the desired result. Since $y, 1$ commute, we get $v_n v_m = v_m v_n$ in $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$. Having a non trivial nucleus guarantees $v_n \neq 1$ in $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$.

2. First note that, v_0, v_1, and v_{-1} have distinct nuclei in each level. We will use induction on $|n| + |m|$. Let $|n| + |m| > 1$. Suppose the statement is true for i, j if $|i| + |j| < |n| + |m|$. If n, m are of different parity, it is clear from Proposition 4.3 3 and 4, that nuclei of v_n, v_m are different. If they are of same parity, apply Proposition 4.3 3 and 4. Then we obtain v_i, v_j from which the induction hypothesis can be applied. Thus by induction we get the desired result. Having different nuclei of same level guarantees $v_n \neq v_m$ in $\widetilde{\mathcal{G}}^{\alpha}_{\omega}$. □

Given $y, y' \in S\backslash\{a\}$ of the form (4.2), following the proof of Proposition 4.4 1 together with the fact that y, y' commutes with each other gives the following corollary.

Corollary 4.4. *Let $y, y' \in S\backslash\{a\}$ of the form* (4.2)*. Then for each $n, m \in \mathbb{Z}$ we have the equality $v_n(y)v_m(y') = v_m(y')v_n(y)$.*

Now we will introduce two substitution rules ξ_0, ξ_1:

$$\xi_0 = \begin{cases} a \mapsto \widetilde{a} \\ \widetilde{a} \mapsto a\widetilde{a}a \\ y \mapsto aya \end{cases} \qquad \xi_1 = \begin{cases} a \mapsto a\widetilde{a}a \\ \widetilde{a} \mapsto \widetilde{a} \\ y \mapsto y \end{cases} \tag{4.4}$$

Note that $\xi_1((a\widetilde{a})^n) = (a\widetilde{a})^{2n}, \xi_1((a\widetilde{a})^n a) = (a\widetilde{a})^{2n+1}a, \xi_0((a\widetilde{a})^n) = a(a\widetilde{a})^{2n}a$ and $\xi_0((a\widetilde{a})^n a) = a(a\widetilde{a})^{2n+1}$. Then $\xi_1(v_n) = v_{2n} = (1, v_n)$ and $\xi_0(v_n) = v_{-2n-1} = (v_n, 1)$. Now we will recursively construct words $V(y)_{i_1 i_2 \dots i_n} = V_{i_1 i_2 \dots i_n}$, for $i_1, i_2, \dots, i_n \in \{0, 1\}$, by,

$$\begin{aligned} V_{\varnothing} &= v_0 \\ V_{i_1 i_2 \dots i_n} &= \xi_{i_1}(V_{i_2 \dots i_n}). \end{aligned} \tag{4.5}$$

It is easy to see that $V_{i_1 i_2 \dots i_n} = v_r$ for some $r \in \mathbb{Z}$ and has a nucleus of depth n with y in $i_1 i_2 \dots i_n$-th coordinate and empty word in other coordinates (see Figure 4.2). Now we will

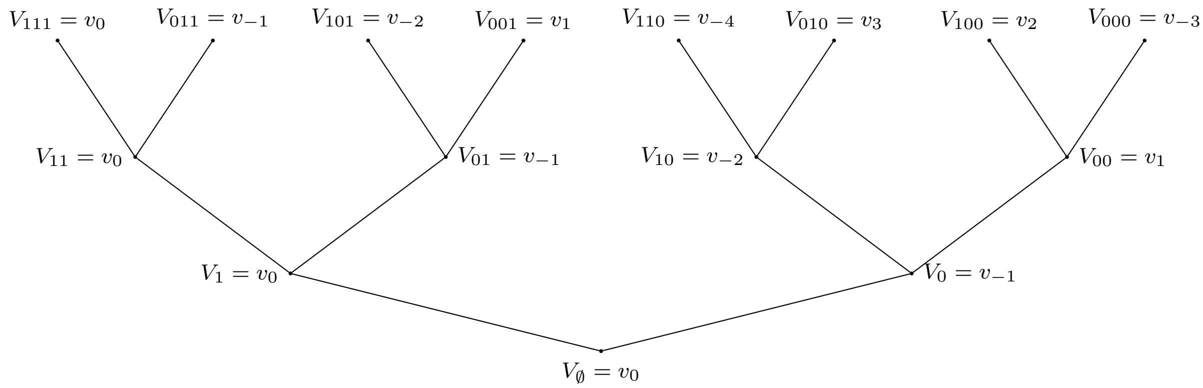

Figure 4.2: $V_{i_1 i_2 \ldots i_n}$ values of first 3 levels

introduce some propositions, which describe the group structure of modified groups for $\omega = 0^\infty$ and $\omega \in \{0,1\}^{\mathbb{N}}$.

Proposition 4.5. *$\widetilde{\mathcal{G}}^{\alpha}_{0^\infty}$ is virtually $\mathcal{L}_2$ of index 2.*

Proof. Let $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}^{\alpha}_{0^\infty}$ and let $G := \widetilde{\mathcal{G}}_{0^\infty}$. We will drop the subscript 0^∞ and superscript α, of each generator, for the convenience. Note that in G we have $b = c = \widetilde{d} = \widetilde{a}$ and $d = \widetilde{b} = \widetilde{c} = 1$. Therefore G is isomorphic to the infinite dihedral group D_∞ generated by a and b. Also note that $d, \widetilde{b}, \widetilde{c}$ have nuclei of the form (4.2). Let ϕ be the surjection from $\widetilde{\mathcal{G}}$ to G described in Proposition 4.1 1.

Lemma 4.1. *$Ker(\phi) = \left\langle\left\langle d, \widetilde{b}, \widetilde{c} \right\rangle\right\rangle = \left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^2$. Here $\langle\langle\cdot\rangle\rangle$ denotes the normal closure.*

Proof. The inclusion $\left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle \leqslant \left\langle\left\langle d, \widetilde{b}, \widetilde{c} \right\rangle\right\rangle \leqslant Ker(\phi)$ is trivial since $d = \widetilde{b} = \widetilde{c} = 1$ in G. To show the other inclusion, let $g \in Ker(\phi)$ and let W be a reduced word representing g in $\widetilde{\mathcal{G}}$. Since $g \in Ker(\phi)$, $W = 1$ in G. But a word is the identity in G if and only if its nucleus of some level contains only $1, d, \widetilde{b}, \widetilde{c}$. Say, W has a nucleus of level n with only $1, d, \widetilde{b}, \widetilde{c}$. We can construct a word W' using $V(d)_{i_1 i_2 \ldots i_n}, V(\widetilde{b})_{i_1 i_2 \ldots i_n}$ and $V(\widetilde{c})_{i_1 i_2 \ldots i_n}$ so that the level n nucleus of W' is the same as the level n nucleus of W. Thus $g = W = W'$ in $\widetilde{\mathcal{G}}$

and since W' represents a group element in $\left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle$, we obtain $Ker(\phi) \leqslant \left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle$. Therefore we get the equality of three groups.

Note that $d = \widetilde{b}\widetilde{c}$ and $v_n(d) = v_n(\widetilde{b})v_n(\widetilde{c})$ for each n. Thus, $\left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle = \left\langle v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle$. Since $\widetilde{b}, \widetilde{c}$ commute and are distinct, $\left\{ v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\}$ consists of mutually commutative distinct elements, by Corollary 4.4.

Now we will show that there are no linear dependencies in $\left\{ v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\}$. To the contrary, suppose there is a relation involving $v_{n_i}(\widetilde{b}), i = 1, 2, \dots, r$ and $v_{m_j}(\widetilde{c}), j = 1, 2, \dots, s$. By commutativity, using the fact that all these elements are involutions, we can assume that this relation has a form

$$W = \prod_{i=1}^{r} v_{n_i}(\widetilde{b}) \prod_{j=1}^{s} v_{m_j}(\widetilde{c}).$$

Let N be the level where all of the involved elements are decomposed to their nuclei. Then by Proposition 4.41 the nucleus of each of $v_{n_i}(\widetilde{b})$ will have exactly one position holding $\widetilde{b}$, and the nuclei of all other $v_{n_{i'}}(\widetilde{b})$ for $i' \neq i$ must have empty word at that position (otherwise, since there is only one position equal to $\widetilde{b}$ in the nucleus, we would obtain that $v_{n_i}(\widetilde{b}) = v_{n_{i'}}(\widetilde{b})$, contradicting to Proposition 4.4 1). Similar argument can be made for elements $v_{m_j}(\widetilde{c})$. Therefore, the decomposition of W at level N will contain a nontrivial coordinate holding one of $\widetilde{b}, \widetilde{c}$, or $\widetilde{b}\widetilde{c} = d$ for each $v_{n_i}(\widetilde{b})$ and $v_{m_j}(\widetilde{c})$ in W and, hence, W cannot represent the trivial element in $\widetilde{\mathcal{G}}$. This contradicts the assumption of having linear dependency. Thus $\left\langle v_n(d), v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle = \left\langle v_n(\widetilde{b}), v_n(\widetilde{c}) \mid n \in \mathbb{Z} \right\rangle \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^2$. This completes the proof of lemma. □

Note that the generator of $\langle a\widetilde{a} \rangle$ acts on $Ker(\phi)$ by shifting its generators. Also note that $Ker(\phi)$ and $\langle a\widetilde{a} \rangle$ intersects trivially, since $a\widetilde{a}$ is of infinite order and all elements of $Ker(\phi)$ are involutions. So, $Ker(\phi) \rtimes \langle a\widetilde{a} \rangle$ is isomorphic to $\mathcal{L}_2 = \mathbb{Z}_2^2 \wr \mathbb{Z}$.

Conjugating the generators of $Ker(\phi) \rtimes \langle a\widetilde{a} \rangle$ by generators of $\widetilde{\mathcal{G}}$, we see that $Ker(\phi) \rtimes \langle a\widetilde{a} \rangle$ is normal in $\widetilde{\mathcal{G}}$. The quotient $\widetilde{\mathcal{G}}/Ker(\phi) \cong D_\infty$ maps onto the quotient $\widetilde{\mathcal{G}}/(Ker(\phi) \rtimes \langle a\widetilde{a} \rangle)$. The kernel of the homomorphism from $\widetilde{\mathcal{G}}/Ker(\phi)$ to $\widetilde{\mathcal{G}}/(Ker(\phi) \rtimes \langle a\widetilde{a} \rangle)$ is generated by the image of $a\widetilde{a}$ in $\widetilde{\mathcal{G}}/Ker(\phi)$. So $Ker(\phi) \rtimes \langle a\widetilde{a} \rangle$ has index two in $\widetilde{\mathcal{G}}$, and therefore $\widetilde{\mathcal{G}}$ is virtually

$Ker(\phi) \rtimes \langle a\widetilde{a} \rangle \cong \mathcal{L}_2$ with index two. $\square$

Proposition 4.6. *$\widetilde{\mathcal{G}}_{i\infty}^{\beta_{ij}}$ is virtually $\mathcal{L}$ with index two.*

Proof. For simplicity, we will prove this for $i = 0, j = 1$ and $\omega = 0^\infty$. We will show $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}} \cong \mathcal{G}_\omega^\alpha$. Here $\mathcal{G}_\omega^\alpha$ is the group defined in Section 6 of [Gri84b], which is denoted by $\widetilde{\mathcal{G}}$ in [Gri84b]. So, $\mathcal{G}_\omega^\alpha = \Gamma'/N'$ where Γ' is the subgroup of Γ generated by $\{a, b, c, d\}$, and N' is the normal subgroup of Γ' consisting all the words that yield positive result when the algorithm α is applied. Let $\pi' \colon \Gamma' \to \mathcal{G}_\omega^\alpha$ be the canonical epimorphism.

Note that in $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$, $\widetilde{b} = 1$ and so $\widetilde{a} = b, \widetilde{c} = d, \widetilde{d} = c$. Now define $f \colon \Gamma \to \Gamma'$ by,

$$f : \begin{cases} s & \mapsto & s & \text{for } s \in \{a, b, c, d\} \\ \widetilde{b} & \mapsto & 1 \\ \widetilde{a} & \mapsto & b \\ \widetilde{c} & \mapsto & d \\ \widetilde{d} & \mapsto & c \end{cases}.$$

Then f is a surjective homomorphism. Since $\widetilde{\psi}$ agrees on ordered sets $\widetilde{S}_\omega^{\beta_{ij}}$ and $\{a, b, c, d, b, 1, d, c\}$, $W \in \ker(\pi^{\beta_{ij}})$ if and only if $f(W) \in \ker(\pi')$. Thus $f \colon \Gamma \to \Gamma'$ induces a well defined monomorphism $\hat{f} \colon \widetilde{\mathcal{G}}_\omega^{\beta_{ij}} \to \mathcal{G}_\omega^\alpha$. f being a surjection implies that $\hat{f}$ is a surjection, and therefore $\hat{f}$ is an isomorphism. This completes the proof, since $\mathcal{G}_\omega^\alpha$ is virtually $\mathcal{L}$ with index two by Theorem 2 of [BG14]. $\square$

Proposition 4.7. *Let $\omega \in \{0,1\}^{\mathbb{N}}$. Then $\widetilde{\mathcal{G}}_\omega^\alpha$ contains $\mathcal{L}$ as a subgroup and is an extension of $\widetilde{\mathcal{G}}_\omega$ by $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2$.*

Proof. Let $\omega \in \{0,1\}^{\mathbb{N}}$. Let $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}_\omega^\alpha = \left\langle a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \right\rangle$ and let $G := \widetilde{\mathcal{G}}_\omega$. We will drop the subscript ω and superscript α, of generators for the convenience. Note that in G we have $b = \widetilde{a}$ and $\widetilde{b} = 1$ and therefore $\widetilde{b}$ has nuclei of the form (4.2). Let ϕ be the surjection from $\widetilde{\mathcal{G}}$ to G described in Proposition 4.1 1. Then by a similar argument as in the proof of Lemma 4.1,

$Ker(\phi) = \left\langle\left\langle\widetilde{b}\right\rangle\right\rangle = \left\langle v_n(\widetilde{b}) \mid n \in \mathbb{Z}\right\rangle \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$. Hence $\widetilde{\mathcal{G}}$ is an extension of G by $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2$. Also since $Ker(\phi) \cap \langle a\widetilde{a}\rangle = \langle 1\rangle$ and $a\widetilde{a}$ acts on $Ker(\phi)$ by shifting (by Proposition 4.3 5), $Ker(\phi) \rtimes \langle a\widetilde{a}\rangle \cong \mathcal{L}$ is a subgroups of $\widetilde{\mathcal{G}}$. □

Proof of Theorem 4.3. Note that for any $\omega \in \Omega$, $\widetilde{\mathcal{G}}_\omega$ is commensurable to $(\widetilde{\mathcal{G}}_{\sigma^N\omega})^{2^N}$. This, together with Proposition 4.5, 4.6 and 4.7, proves the result. □

4.3 Closure and Cluster Points of $\widetilde{\mathcal{G}}_\omega$ in $\mathcal{M}_8$

Recall the notation introduced in (4.1).

$$\begin{aligned}
\mathcal{X} &= \left\{(\widetilde{\mathcal{G}}_\omega, \widetilde{S}_\omega)\right\}_{\omega\in\Omega} \\
\mathcal{X}_i &= \left\{(\widetilde{\mathcal{G}}_\omega, \widetilde{S}_\omega)\right\}_{\omega\in\Omega_i} \quad \text{; for } i = 0,1,2 \\
\mathcal{X}_i^\alpha &= \left\{(\widetilde{\mathcal{G}}_\omega^\alpha, \widetilde{S}_\omega^\alpha)\right\}_{\omega\in\Omega_i} \quad \text{; for } i = 1,2 \\
\mathcal{X}_2^\beta &= \left\{(\widetilde{\mathcal{G}}_\omega^\beta, \widetilde{S}_\omega^\beta) \mid \beta \in \{\beta_{01}, \beta_{12}, \beta_{20}\}\right\}_{\omega\in\Omega_2} \\
\mathcal{Y} &= \mathcal{X}_0 \cup \mathcal{X}_1^\alpha \cup \mathcal{X}_2^\alpha
\end{aligned}$$

Then $\mathcal{X}$ is the disjoint union of $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2$. In order to prove the Theorem 4.1 we use the following propositions.

Proposition 4.8. *Generalized overgroups and modified overgroups corresponding to different oracles ω, are different in $\mathcal{M}_8$.*

Proof. Recall that two points $(G_1, S_1), (G_2, S_2) \in \mathcal{M}_8$ are equal if and only if the canonical map $S_1 \to S_2$ that preserves the order, extends to an isomorphism $G_1 \to G_2$. Thus by restricting S_1, S_2 to ordered sets of r elements S_1', S_2', respectively, give rise to equal points $(\langle S_1'\rangle, S_1'), (\langle S_2'\rangle, S_2') \in \mathcal{M}_r$.

Note that the classical Grigorchuk's groups and their modifications can be obtained by restricting corresponding generating sets of generalized overgroups and modified overgroups. By

[Gri84b], different oracles ω give rise to different classical Grigorchuk's groups and their modifications in $\mathcal{M}_4$. Therefore by above argument, we get the result. $\square$

Form the above proposition we can see that the sets $\mathcal{X}_0, (\mathcal{X}_1 \cup \mathcal{X}_1^{\alpha}), (\mathcal{X}_2 \cup \mathcal{X}_2^{\alpha} \cup \mathcal{X}_2^{\beta})$ are disjoint. This, together with Corollary 4.1, yields,

Corollary 4.5. $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_1^{\alpha}, (\mathcal{X}_2^{\alpha} \cup \mathcal{X}_2^{\beta})$ *are disjoint.*

Now let us prove $\mathcal{X}_2^{\alpha}, \mathcal{X}_2^{\beta}$ are disjoint.

Proposition 4.9. $\mathcal{X}_2^{\alpha}, \mathcal{X}_2^{\beta}$ *are disjoint. In fact, for* $\omega \in \Omega_2$ *with infinitely many* i*'s, the groups* $\widetilde{\mathcal{G}}_{\omega}^{\alpha}, \widetilde{\mathcal{G}}_{\omega}^{\beta_{ij}}$ *and* $\widetilde{\mathcal{G}}_{\omega}^{\beta_{ik}}$ *are different.*

Let ω contain finitely many 'k's. We will construct a word $W(ij)$ such that its nucleus consists only of '1's and 'e_{ij}'s, with not all '1's. For ease of writing let us assume ω contains finitely many '2's. We will construct the word $W(01)$. Recall that $e_{01} = \widetilde{b}$. Let $\omega = \omega_1\omega_2 \dots \omega_n 2^t \eta$, where $\omega_n \neq 2$ and $\eta \in \{0,1\}^{\mathbb{N}}$. Now for $r = 0, 1, \dots, n$, define

$$X_r = \begin{cases} b & ;\omega_r \neq 2 \\ \widetilde{b} & ;\omega_r = 2 \end{cases},$$

$$Y_r = X_n^{X_{n-1}^{\dots X_r^a}},$$

$$Z_r = (\widetilde{b} Y_r)^2,$$

$$W(01) = (Z_1)^{2^t}. \tag{4.6}$$

The decomposed diagram of $W(01)$ of depth $n + t$ is given in the Figure 4.3 and thus its level $n + t$ nucleus consists of only $1, \widetilde{b}$. Using similar constructions, we can construct words $W(02), W(12)$.

Proof of Proposition 4.9. Suppose $\omega = \omega_1\omega_2 \dots \omega_n 2^t \eta$, where $\omega_n \neq 2$ and $\eta \in \{0,1\}^{\mathbb{N}}$. Let $W = W(01)$ defined as above. Then W represents the identity element in $\widetilde{\mathcal{G}}_{\omega}^{\beta_{01}}$ but not the identity in $\widetilde{\mathcal{G}}_{\omega}^{\alpha}$ and $\widetilde{\mathcal{G}}_{\omega}^{\beta_{02}}$. Similarly using the word $W(02)$, we can show $\widetilde{\mathcal{G}}_{\omega}^{\alpha} \neq \widetilde{\mathcal{G}}_{\omega}^{\beta_{02}}$. $\square$

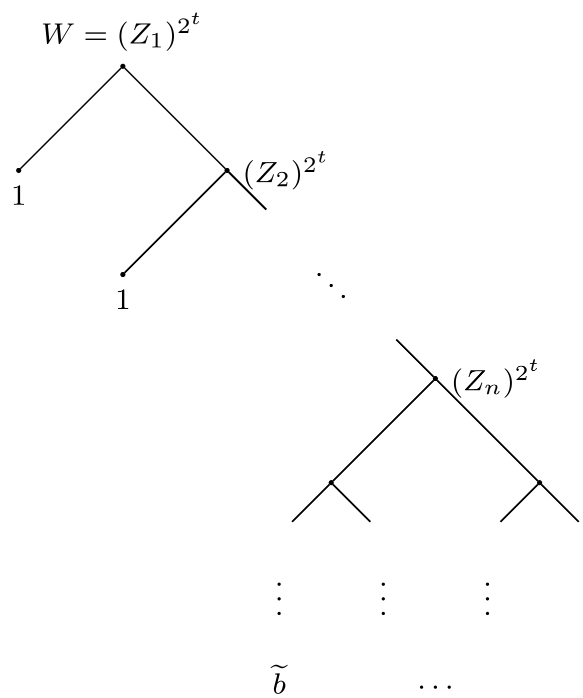

Figure 4.3: Decomposition of $W(01)$ in to the depth $n+t$

Proof of Theorem 4.1. Directly from Proposition 4.8, 4.9 and Corollary 4.5. □

Now we will prove Theorem 4.2. We will use few lemmas in order to do this.

Lemma 4.2. *Let $\omega, \omega^{(n)} \in \Omega$ for all $n \in \mathbb{N}$ and $\omega^{(n)} \to \omega$. Suppose $G = \lim \widetilde{\mathcal{G}}_{\omega^{(n)}}$ exists and $G \neq \widetilde{\mathcal{G}}_{\omega^{(n)}}$, for all n. Then $G = \widetilde{\mathcal{G}}_{\omega}, \widetilde{\mathcal{G}}_{\omega}^{\alpha}$ or $\widetilde{\mathcal{G}}_{\omega}^{\beta_{ij}}$. Moreover $G \in \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta}$ and so $G \notin \mathcal{X}_2$.*

Proof. First let $\omega \in \Omega_0$. Let $r \in \mathbb{N}$ and let $N > \log_2(2r)$. Since $\omega^{(n)} \to \omega$ and $\omega \in \Omega_0$, for sufficiently large n, we may assume $\omega^{(n)}$ has all three symbols after the N-th position and $\omega^{(n)}, \omega$ agrees till the N-th position. Using Proposition 4.2 1, and letting $r \to \infty$, we get $G = \widetilde{\mathcal{G}}_{\omega}$.

Now let $\omega \in \Omega_1$. Let N_0 be the smallest index such that only two symbols appear after N_0-th position. Suppose for each $N \geqslant N_0$, there are infinitely many 'n's such that $\omega^{(n)}$ contains all three symbols after N-th position. Then by Proposition 4.2 2, there is a subsequence $\left\{\omega^{(n_t)}\right\}_{t \geqslant 1}$ of $\left\{\omega^{(n)}\right\}$, such that $\widetilde{\mathcal{G}}_{\omega^{(n_t)}} \to \widetilde{\mathcal{G}}_{\omega}^{\alpha}$ as $t \to \infty$. Since the subsequential limits and limit of the sequence agree, we get $G = \widetilde{\mathcal{G}}_{\omega}^{\alpha}$. Now suppose there is $N \geqslant N_0$ such that for all but finitely many n,

$\omega^{(n)}$ contains at most two symbols after the N-th position. Since $\omega^{(n)} \to \omega$, we may assume $\omega^{(n)}$ contains exactly the same two symbols as of ω, for sufficiently large n. Then by Proposition 4.2 4, we obtain $G = \widetilde{\mathcal{G}}_\omega$.

Finally let $\omega \in \Omega_2$. Let N_0 be the smallest index such that only one symbol, say i, appear after the N_0-th position. Suppose for each $N \geqslant N_0$, there are infinitely many 'n's such that $\omega^{(n)}$ contains all three symbols after the N-th position. Then by Proposition 4.2 2, there is a subsequence of $\{\omega^{(n)}\}$, which converges to $\widetilde{\mathcal{G}}_\omega^\alpha$. Thus, $G = \widetilde{\mathcal{G}}_\omega^\alpha$. Now suppose for each $N \geqslant N_0$, there are infinitely many 'n's such that $\omega^{(n)}$ contains exactly two symbols, say i, j, after the N-th position. Then by Proposition 4.2 5, there is a subsequence of $\{\omega^{(n)}\}$, which converges to $\widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$. Thus, $G = \widetilde{\mathcal{G}}_\omega^{\beta_{ij}}$. If neither of above is true, then there is $N \geqslant N_0$ such that for all but finitely many n, $\omega^{(n)}$ contains exactly one symbol. Since $\omega^{(n)} \to \omega$, that symbol has to be i. Thus for sufficiently large n, $\omega^{(n)} = \omega$. This impossible since $G \neq \widetilde{\mathcal{G}}_{\omega^{(n)}}$.

From above, we can conclude that $G \in \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^\beta$ and $G \notin \mathcal{X}_2$. □

Proof of Theorem 4.2 1. To the contrary, suppose there is an $\eta \in \Omega_2$ such that $\widetilde{\mathcal{G}}_\eta \in \mathcal{X}_2$ is a limit point. Then there exists a sequence $\{\mathcal{G}_{\omega^{(n)}}\}$ converging to $\widetilde{\mathcal{G}}_\eta$. Since Ω is compact, by passing to a subsequence, if necessary, we may assume $\omega^{(n)} \to \omega$, for some $\omega \in \Omega$. By Lemma 4.2, $\widetilde{\mathcal{G}}_\eta = \lim \widetilde{\mathcal{G}}_{\omega^{(n)}} \notin \mathcal{X}_2$, which is a contradiction. □

Proof of Theorem 4.2 3 (a). Let $G \in \mathcal{Y}_\sharp = (\mathcal{X}_0 \cup \mathcal{X}_1^\alpha \cup \mathcal{X}_2^\alpha)_\sharp$. By Proposition 4.1 1, $\widetilde{\mathcal{G}}_\omega = \widetilde{\mathcal{G}}_\omega^\alpha$. Then there exists $\{\omega^{(n)}\} \subset \Omega$ such that $\widetilde{\mathcal{G}}_{\omega^{(n)}}^\alpha \to G$. By compactness of Ω we may assume $\omega^{(n)} \to \omega$ for some $\omega \in \Omega$. Then $G = \widetilde{\mathcal{G}}_\omega^\alpha$ by Corollary 4.3. This together with Corollary 4.3 implies that

$$\omega^{(n)} \to \omega \iff \left(\widetilde{\mathcal{G}}_{\omega^{(n)}}^\alpha \to \widetilde{\mathcal{G}}_\omega^\alpha\right).$$

Therefore $\mathcal{Y} \cong \Omega$ and $\mathcal{X}_0 \cong \Omega_0, \mathcal{X}_1^\alpha \cong \Omega_1, \mathcal{X}_2^\alpha \cong \Omega_2$. Thus, $\mathcal{Y}$ is homeomorphic to a Cantor set and $\mathcal{Y} = (\mathcal{X}_0)_\sharp = (\mathcal{X}_1^\alpha)_\sharp = (\mathcal{X}_2^\alpha)_\sharp$. □

Proof of Theorem 4.2 3 (b). First we will show $\mathcal{X}_\sharp \subset \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^\beta$. Let G be a limit point of $\mathcal{X}$. Thus there exists a sequence $\{\mathcal{G}_{\omega^{(n)}}\}$ converging to G. Since Ω is compact, we may assume

$\omega^{(n)} \to \omega$, for some $\omega \in \Omega$. Then by Lemma 4.2, $G \in \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta}$. Therefore $\mathcal{X}_\sharp \subset \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta}$.

Now we will show $\mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta} \subset (\mathcal{X}_1)_\sharp$. Let $\omega \in \Omega$ and choose $\omega^{(n)} = \omega_1\omega_2 \ldots \omega_n(012)(ij)^{\infty}$, for each n. Then using Proposition 4.2 2, we get $\widetilde{\mathcal{G}}_{\omega^{(n)}} \to \widetilde{\mathcal{G}}_{\omega}^{\alpha}$. So $\mathcal{Y} \subset (\mathcal{X}_1)_\sharp$. Now let $\omega \in \Omega_1 \cup \Omega_2$ with finitely many k's. Choose $\omega^{(n)} = \omega_1\omega_2 \ldots \omega_n(ij)^{\infty}$, for each n. Using Proposition 4.2 5, we get $\widetilde{\mathcal{G}}_{\omega^{(n)}} \to \widetilde{\mathcal{G}}_{\omega}^{\beta_{ij}}$. So $\mathcal{X}_1 \cup \mathcal{X}_2^{\beta} \subset (\mathcal{X}_1)_\sharp$. Therefore $\mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta} \subset (\mathcal{X}_1)_\sharp$.

Using a similar argument by choosing $\omega^{(n)} = \omega_1\omega_2 \ldots \omega_n(012)(i)^{\infty}$ and again choosing $\omega^{(n)} = \omega_1\omega_2 \ldots \omega_n(ij)(i)^{\infty}$, we can show $\mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta} \subset (\mathcal{X}_2)_\sharp$.

Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are subsets of $\mathcal{X}$, we get $\mathcal{X}_\sharp = (\mathcal{X}_1)_\sharp = (\mathcal{X}_2)_\sharp = \mathcal{Y} \cup \mathcal{X}_1 \cup \mathcal{X}_2^{\beta}$. Corollary 4.3 together with Proposition 4.1 implies that $(\mathcal{X}_1)_\sharp = (\mathcal{X}_2^{\beta})_\sharp$ and so we get the desired result. □

Now we will complete the proof of Theorem 4.2.

Proof of Theorem 4.2 2. We already proved $\mathcal{Y}$ is homeomorphic to a Cantor set. Now let us prove that $\mathcal{X}_\sharp$ is also homeomorphic to a Cantor set. Note that the set $\mathcal{X}_\sharp$ is a perfect set. (That is a closed set with all its point being limit points). The space $\mathcal{M}_8$ is a totally disconnected compact metric space. Let us recall that any non empty, totally disconnected, compact, perfect metric space is homeomorphic to the Cantor set. Therefore, $\mathcal{X}_\sharp$ is homeomorphic to the Cantor set. □

5. SCHUR COMPLEMENT METHOD AND ASSOCIATED RATIONAL MAPS*

This chapter consists of some results from the article [GS21] and some results obtained under the guidance of Nguyen-Bac Dang, Rostislav Grigorchuk, and Mikhail Lyubich.

5.1 Introduction

The study of spectra of graphs and groups has applications in graph theory, quantum chemistry, signal processing, ect. The spectrum of a group is defined to be the spectrum of the *Markov operator* operator associated with the Cayley graph of the group. The Markov operator M of a d-regular non-oriented graph (V, E) acts on the Hilbert space $\ell^2(V)$ and is defined by

$$(Mf)(x) = \frac{1}{d} \sum_{y \sim x} f(y),$$

for $f \in \ell^2(V)$, where $x \sim y$ is the adjacency relation. In the case of the Cayley graph of the group G, the Markov operator is given by,

$$(Mf)(g) = \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} f(sg),$$

for $f \in \ell^2(G)$ and $g \in G$.

The operator $L = I - M$ where I is the identity operator is called the *discrete Laplace operator*. Operators M and L can be defined also for non-regular graphs as it is done for instance in [MW89, Chu97]. The Markov operator M is a self-adjoint operator with the norm $\|M\| \leqslant 1$ and its spectrum is contained in $[-1, 1]$. The name "Markov" comes from the fact that M is the Markov operator associated with the random walk on the graph (V, E) in which a transition $u \to v$ occurs with probability $1/d$, if u and v are adjacent vertices.

A more general concept called *weighted Markov operator* is used when the graph is *weighted*,

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in the sense that there is a weight function on the set of edges. Given a symmetric probability distribution on the generators of a group, the weighted Markov operator is associated with the random walk on the (left) Cayley graph. This give rise to the concept called *joint spectrum of pencil of operators* of contracting self-similar groups (see [BG00b, Yan09] for more on this).

The Schur complement method, discussed in Section 5.3, is a useful tool in linear algebra, networks, differential operators, applied mathematics [Cot74]. In particular, it can be used to compute the spectra and joint spectra of some self-similar groups, as seen in [GN07]. Schur complements can be used to construct multi-dimensional maps called *Schur transformations* (also known as *Schur renormalization transformations*), which happen to be rational maps, in some situations. The dynamical properties of these maps are closely related to the spectral problem of corresponding groups [DGL21].

In Section 5.4, we calculate Schur complements, Schur transformations, and associated 2-dimensional rational maps for the first Grigorchuk group $\mathcal{G}$, the overgroup $\widetilde{\mathcal{G}}$, the generalized Grigorchuk groups $\mathcal{G}_\omega$, and generalized overgroups $\widetilde{\mathcal{G}}_\omega$. The 2-dimensional rational maps for $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are given in (5.18) and (5.26), respectively.

For generalized groups $\mathcal{G}_\omega$ and $\widetilde{\mathcal{G}}_\omega$, we obtain 2-dimensional rational maps F_0, F_1, F_2 given in (5.32), associated with $\mathcal{G}_\omega$, and $\widetilde{F}_0, \widetilde{F}_1, \widetilde{F}_2$ given in (5.36), associated with $\widetilde{\mathcal{G}}_\omega$. Note that these maps depend on three parameters in the case of $\mathcal{G}_\omega$ and on seven parameters in the case of $\widetilde{\mathcal{G}}_\omega$. We are particularly interested in studying random dynamics of F_0, F_1, F_2 and $\widetilde{F}_0, \widetilde{F}_1, \widetilde{F}_2$. Dynamical pictures representing a Julia set basin of attraction for random iteration of these maps are shown in Figure 5.1.

There is a 2-parametric family of maps $\{F_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{C} \text{ and } \alpha \neq 0\}$, where $F_{\alpha,\beta}\colon \mathbb{C}^2 \to \mathbb{C}^2$ is given by,

$$F_{\alpha,\beta}(x, v) = \left(\frac{\alpha x^2}{(v+\beta)^2 - \alpha^2}, v - \frac{(v+\beta)x^2}{(v+\beta)^2 - \alpha^2}\right). \tag{5.1}$$

The condition $\alpha \neq 0$, enables $F_{\alpha,\beta}$ to be a *dominant map* (i.e., the image of the map is not contained in an algebraic curve). The 2-parametric maps conjugates to the maps in (5.18), (5.26) (see Proposition 5.2), and semi-conjugate to a lower-dimensional map as seen in the next theorem:

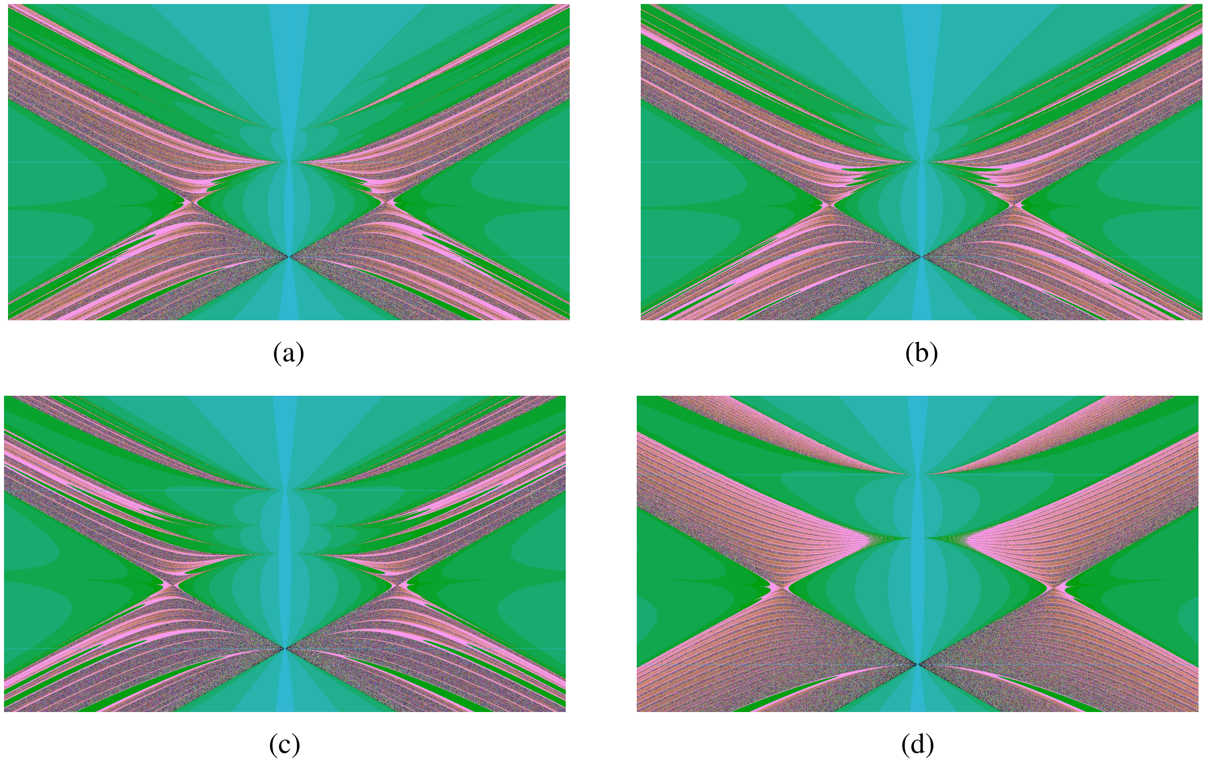


Figure 5.1: Dynamical pictures of $F_{\omega_{n-1}} \circ \ldots \circ F_{\omega_0}$ for (**a**) $\omega = (012)^\infty$ and $(y, z, u) = (1, 2, 3)$, (**b**) $\omega = (01)^\infty$ and $(y, z, u) = (1, 2, 3)$, (**c**) a random ω and $(y, z, u) = (1, 2, 3)$, and (**d**) a random ω and $(y, z, u) = (1, 3, 3)$.

Theorem 5.1. *For any $\alpha \neq 0$ and β, the 2-parametric map $F_{\alpha,\beta}$, given by* (5.1)*, is semi-conjugate to the map $t\colon z \mapsto 2z^2 - 1$ (i.e., there is a rational map $f_s\colon \mathbb{C}^2 \dashrightarrow \mathbb{C}$ satisfying $f_s \circ F_{\alpha,\beta} = t \circ f_s$). The map t is called the Chebyshev map or the Ulam - von Neumann map.*

A map f on a *rational variety* (see Appendix B.2) X is said to be *algebraically stable* if no algebraic curve is contracted via iterates of f to an indeterminacy point of f. That is, for each algebraic curve C on X and for each $n \in \mathbb{N}$, if $f^n(C) := f \circ \ldots \circ f(C)$ is a point, then that point is not an indeterminacy point of f. This concept can be extended to a sequence of maps $\{f_k\}$. We say $\{f_k\}$ is *algebraically stable* if no algebraic curve is contracted to an indeterminacy point via the ordered iterates of $\{f_k\}$ (i.e., $f_n \circ \ldots \circ f_1(C)$ is not an indeterminacy point, for $n \in \mathbb{N}$ and for any algebraic curve C). We have an algebraic stability condition on any sequence of 2-parametric maps.

Theorem 5.2. *There is a rational variety X, obtained by blowing up two points of $\mathbb{P}^2$, such that for each sequence $\{f_n\}$ of two-parametric maps, where $f_n = F_{\alpha_n,\beta_n}$ is of the form* (5.1)*, the sequence $\left\{\widehat{f}_n\right\}$ of lifted maps to X, is algebraically stable.*

As a direct corollary of Theorem 5.2, we obtain the following algebraic stability condition for iterated rational maps on generalized groups.

Theorem 5.3. *Let $\omega \in \Omega = \{0,1,2\}^{\mathbb{N}}$ be arbitrary and let X be the rational variety as in Theorem 5.2. Then,*

1. *The sequence $\left\{\widehat{F}_{\omega_n}\right\}$ of lifted maps, which corresponds to the group $\mathcal{G}_\omega$, is algebraically stable, if $y+z, y+u, z+u$ are non-zero,*
2. *The sequence $\left\{\widehat{\widehat{F}}_{\omega_n}\right\}$ of lifted maps, which corresponds to the group $\mathcal{G}_\omega$, is algebraically stable, if $y+z+q+t, y+u+q+s, z+u+q+r$ are non-zero,*

We will prove Theorem 5.2 and Theorem 5.3 in Section 5.5.

5.2 Self-similar Representations and Matrix Recursions

In order to define a self-similar representation, we will need a few preliminary definitions.

Definition 5.1. *Let H be an infinite dimensional Hilbert space. A map*

$$\psi\colon H \to H^d = H \oplus \ldots \oplus H$$

is called a d-fold similarity (or simply, a d-similarity) if it is an isomorphism of Hilbert spaces.

Definition 5.2. *The Cuntz algebra $\mathcal{O}_d$ is the universal C^*-algebra given by the presentation*

$$\mathcal{O}_d \cong \langle a_1, \ldots, a_d \mid a_1a_1^* + \ldots + a_da_d^* = 1, a_i^*a_i = 1, i = 1, \ldots, d\rangle . \tag{5.2}$$

Note that multiplying the relation $\sum_j a_ja_j^* = 1$ by a_i^* on the left and by a_i on the right, we get, $\sum_{j\neq i}(a_j^*a_i)^*(a_j^*a_i) = 0$. This is a sum of positive elements and so $a_j^*a_i = 0$ if $j \neq i$. Therefore, the Cuntz algebra $\mathcal{O}_d$ is equipped with the set of relations,

$$\left\{ \sum_j a_j a_j^* = 1, a_i^* a_i = 1, a_j^* a_i = 0, \text{ for } 1 \leqslant i, j \leqslant d \text{ and } i \neq j \right\}, \tag{5.3}$$

which we call the *Cuntz relations*.

There is a one to one correspondence between the collection of the $*$-representations of the Cuntz algebra $\mathcal{O}_d$ to $\mathcal{B}(H)$ and the collection of the d-similarities on H, where $\mathcal{B}(H)$ denotes the space of bounded linear operators on H, as seen by the next theorem.

Theorem 5.4 (Proposition 3.1 of [GN07])**.** *Let H be an infinite dimensional separable Hilbert space. Then, there is a bijective correspondence between $*$-representations $\rho\colon \mathcal{O}_d \to \mathcal{B}(H)$ and d-similarities $\psi\colon H \to H^d$.*

Given a $*$-representation $\rho\colon \mathcal{O}_d \to \mathcal{B}(H)$, the corresponding d-similarity $\psi_\rho\colon H \to H^d$ is given by, $\psi_\rho(h) = (\rho(a_1^*)(h), \ldots, \rho(a_d^*)(h))$, where $a_1, \ldots, a_d$ are generators of $\mathcal{O}_d$.

Conversely, given a d-similarity $\psi\colon H \to H^d$, the corresponding $*$-representation $\rho_\psi\colon \mathcal{O}_d \to \mathcal{B}(H)$ can be described by,

$$\rho_\psi(a_i)(h) = \psi^{-1}(0, \ldots, 0, h, 0, \ldots, 0), \tag{5.4}$$

for $h \in H$, where h in the right hand side is at the i-th coordinate of H^d.

The main example that we consider is the Hilbert space $L^2(\partial\mathcal{T}_d, \mu)$ of square integrable functions on the boundary of $\mathcal{T}_d$, with respect to uniform Bernoulli measure μ. Then, there is a natural d-similarity indexed by the d symbols of the alphabet X, given by

$$\psi\colon L^2(\partial\mathcal{T}_d, \mu) \to \bigoplus_{x \in X} L^2(\partial\mathcal{T}_d, \mu),$$
$$(\psi f)_x(\xi) = \frac{1}{\sqrt{d}} f(x\xi),$$

for $f \in L^2(\partial\mathcal{T}_d, \mu)$, $\xi \in \partial\mathcal{T}_d$, and $x \in X$. This arise from the self-similarity property of $\partial\mathcal{T}_d$.

By Theorem 5.4, we obtain the corresponding $*$-representation to the above d-similarity, $\rho\colon \mathcal{O}_d \to \mathcal{B}(L^2(\partial\mathcal{T}_d, \mu))$ given by,

$$(\rho(a_x)f)(\xi) = \begin{cases} \sqrt{d}f(\sigma\xi) & \text{if } \xi = x\sigma\xi \\ 0 & \text{if } \xi \neq x\sigma\xi \end{cases}, \tag{5.5}$$

where, σ is the shift operator on $\partial\mathcal{T}_d$.

Now, we are ready to define self-similar representations.

Definition 5.3. *Let G be a self-similar group acting on the d-regular rooted tree $\mathcal{T}_d$. A unitary representation $\pi\colon G \to \mathcal{B}(L^2(\partial\mathcal{T}_d, \mu))$ is said to be self-similar if*

$$\pi(g) \circ \rho(a_x) = \rho(a_{gx}) \circ \pi(g|_x), \tag{5.6}$$

for $g \in G$ and $x \in X$. Here, ρ is the representation given in (5.5).

Let $\kappa\colon G \to \mathcal{B}(L^2(\partial\mathcal{T}_d, \mu))$ be the *Koopman representation* given by

$$(\kappa(g)f)(\xi) = f(g^{-1}\xi). \tag{5.7}$$

Then κ is a unitary representation. Note that,

$$\begin{aligned}(\kappa(g) \circ \rho(a_x)f)(\xi) &= (\rho(a_x)f)(g^{-1}\xi) \\ &= \begin{cases} \sqrt{d}f(\sigma g^{-1}\xi) & ;\text{if } g^{-1}\xi = x\sigma g^{-1}\xi \\ 0 & ;\text{otherwise} \end{cases} \\ &= \begin{cases} \sqrt{d}f(\sigma g^{-1}\xi) & ;\text{if } \xi = g(x)g|_x\sigma g^{-1}\xi \\ 0 & ;\text{otherwise} \end{cases}\end{aligned}$$

$$= \begin{cases} \sqrt{d} f(g|_x^{-1}\eta) & ;\text{if } \xi = g(x)\eta \\ 0 & ;\text{otherwise} \end{cases}$$

$$= \begin{cases} \sqrt{d}(\kappa(g|_x)f)(\eta) & ;\text{if } \xi = g(x)\eta \\ 0 & ;\text{otherwise} \end{cases}$$

$$= (\rho(a_{gx}) \circ \kappa(g|_x)f)(\xi),$$

by using (2.3) and Proposition 2.1, and therefore Koopman representation is self-similar.

Remark 5.1. The right hand side of (5.7) is usually written with a normalizing factor $\sqrt{\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}}$, square root of the Radon-Nikodym derivative of the pullback measure $g_*\mu$. The pullback measure is given by $g_*\mu(A) = \mu(g^{-1}A)$ for $A \subset \partial\mathcal{T}_d$. But in our case, this normalizing factor is 1 since the action of $\mathrm{Aut}(\mathcal{T}_d)$ on $\mathcal{T}_d$ is uniform measure preserving.

Now let us define the matrix recursions.

Definition 5.4. *Let A be an algebra. A matrix recursion on A is a homomorphism*

$$\varphi \colon A \to M_d(A),$$

where $M_d(A)$ is the algebra of $d \times d$ matrices over A.

Given a d-similarity $\psi \colon H \to H^d$, there is a natural matrix recursion φ on the algebra of bounded operators $\mathcal{B}(H)$. Let $M \in \mathcal{B}(H)$. Then, $\psi \circ M \circ \psi^{-1} \in \mathcal{B}(H^d)$, and so it is associated with the matrix, denoted by $\varphi(M)$, whose columns are obtained by the transposes of $\psi \circ M \circ \psi^{-1}$ images under basic elements of H^d. Note that,

$$\begin{aligned} c_j^T(h) &= (\psi \circ M \circ \psi^{-1})(0, \dots, 0, h, 0, \dots, 0) \\ &= (\psi \circ M)(\rho_\psi(a_j)h) && ;\text{by (5.4)} \\ &= \psi(M \circ \rho_\psi(a_j)h) \end{aligned}$$

$$= (\rho_\psi(a_1^*) \circ M \circ \rho_\psi(a_j)h, \ldots, \rho_\psi(a_d^*) \circ M \circ \rho_\psi(a_j)h), \qquad \text{;by Theorem } 5.4$$

where c_j^T is the transpose of the j-th column of the matrix $\varphi(M)$. Here, the h in the top line appears in the j-th position. Therefore, the matrix recursion of M is,

$$\varphi(M) = \left(\rho_\psi(a_i^*) \circ M \circ \rho_\psi(a_j)\right)_{i,j}. \tag{5.8}$$

We will write M instead of $\varphi(M)$ if there are no ambiguities.

Any self-similar representation of a self-similar group acting on d-regular tree, naturally leads to a matrix recursion on the group algebra, using the idea discussed above. Let G be a self-similar group acting on $\mathcal{T}_d$ and let $\pi\colon G \to \mathcal{B}(L^2(\partial\mathcal{T}_d, \mu))$ be a self-similar representation. Consider the natural d-similarity $\psi\colon L^2(\partial\mathcal{T}_d, \mu) \to \bigoplus_{x\in X} L^2(\partial\mathcal{T}_d, \mu)$. Let $g \in G$. Then,

$$\begin{aligned}\rho(a_y^*) \circ \pi(g) \circ \rho(a_x) &= \rho(a_y^*) \circ \rho(a_{g(x)}) \circ \pi(g|_x) \\ &= \begin{cases} \pi(g|_x) & \text{if } g(x) = y \\ 0 & \text{otherwise} \end{cases},\end{aligned}$$

for $x, y \in X$, using (5.6) and (5.3). Here, ρ is the representation of Cuntz algebra corresponding to the d-similarity ψ. Therefore, using (5.8) we obtain the matrix recursion $\varphi(g)$ of g given by,

$$\varphi(g)_{y,x} = \begin{cases} \pi(g|_x) & \text{if } g(x) = y \\ 0 & \text{otherwise} \end{cases}, \tag{5.9}$$

where $\varphi(g)_{y,x}$ is the entry in y-th row and x-th column of the matrix $\varphi(g)$.

In the case of Koopman representation (i.e., $\pi = \kappa$), by identifying $\kappa(g)$ with g, we define a matrix recursion using (5.9), given by

$$\varphi(g) = (g_{y,x})_{y,x\in X}, \text{ where}$$

$$g_{y,x} = \begin{cases} g|_x & \text{if } g(x) = y \\ 0 & \text{otherwise} \end{cases}. \tag{5.10}$$

We define a matrix recursion φ on the group algebra $\mathbb{C}[G]$, by extending (5.10) to $\mathbb{C}[G]$ linearly. We may write g in place of $\varphi(g)$ if there will be no ambiguity.

Now consider the case of $d = 2$. Then, by (5.10), we obtain the matrix recursions on elements of $\mathrm{Aut}(\mathcal{T}_2)$, introduced in Section 2.5 as follows:

$$\begin{aligned}
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & a &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \widetilde{a} &= \begin{pmatrix} a & 0 \\ 0 & \widetilde{a} \end{pmatrix}, \\
b &= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, & c &= \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, & d &= \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \\
\widetilde{b} &= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{c} \end{pmatrix}, & \widetilde{c} &= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{d} \end{pmatrix}, & \widetilde{d} &= \begin{pmatrix} a & 0 \\ 0 & \widetilde{b} \end{pmatrix}, \\
b_\omega &= \begin{pmatrix} B_0^\omega & 0 \\ 0 & b_{\sigma\omega} \end{pmatrix}, & c_\omega &= \begin{pmatrix} C_0^\omega & 0 \\ 0 & c_{\sigma\omega} \end{pmatrix}, & d_\omega &= \begin{pmatrix} D_0^\omega & 0 \\ 0 & d_{\sigma\omega} \end{pmatrix}, \\
\widetilde{b}_\omega &= \begin{pmatrix} \widetilde{B}_0^\omega & 0 \\ 0 & \widetilde{b}_{\sigma\omega} \end{pmatrix}, & \widetilde{c}_\omega &= \begin{pmatrix} \widetilde{C}_0^\omega & 0 \\ 0 & \widetilde{c}_{\sigma\omega} \end{pmatrix}, & \widetilde{d}_\omega &= \begin{pmatrix} \widetilde{D}_0^\omega & 0 \\ 0 & \widetilde{d}_{\sigma\omega} \end{pmatrix},
\end{aligned} \tag{5.11}$$

Here, $\omega \in \Omega$, and $B_0^\omega, C_0^\omega, D_0^\omega, \widetilde{B}_0^\omega, \widetilde{C}_0^\omega, \widetilde{D}_0^\omega$ are defined in (2.11).

5.3 Schur Complements

Let H be a Hilbert space that can be decomposed into a direct sum of two non-trivial Hilbert spaces H_1 and H_2. That is, $H = H_1 \oplus H_2$, where $H_i \neq \{0\}$ for $i = 1, 2$. Let $M \in \mathcal{B}(H)$ be a bounded operator. Then M has the metrix representation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.12}$$

that arise from the above decomposition, where

$$A\colon H_1 \to H_1, \qquad B\colon H_2 \to H_1, \qquad C\colon H_1 \to H_2, \qquad D\colon H_2 \to H_2,$$

are bounded operators.

First and second *Schur complements*, denoted by S_1 and S_2, are partially defined maps given by,

$$\begin{aligned} S_1\colon \mathcal{B}(H) &\to \mathcal{B}(H_1) & \qquad S_2\colon \mathcal{B}(H) &\to \mathcal{B}(H_2) \\ M &\mapsto A - BD^{-1}C, & M &\mapsto D - CA^{-1}B, \end{aligned}$$

for any $M \in \mathcal{B}(H)$. Here, A, B, C, and D are operators given by the matrix representation (5.12) of M. Note that $S_1(M)$ is defined when D is invertible, and $S_2(M)$ is defined when A is invertible. Invertibility of M is closely related with the invertibility of Schur complements, as can be seen by the next proposition.

Proposition 5.1 ([GN07]). *Let M be a bounded operator with matrix representation given by* (5.12). *If D is invertible, then M is invertible if and only if $S_1(M)$ is invertible and*

$$M^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}BD^{-1} \\ -D^{-1}CS_1^{-1} & D^{-1}CS_1^{-1}BD^{-1} + D^{-1} \end{pmatrix},$$

where $S_1 = S_1(M)$.

A similar statement holds for $S_2(M)$ when A is invertible. The above expression for M^{-1} is called the *Frobenius formula*. In the case $\dim H < \infty$, the determinant $|M|$ of matrix M satisfies

$$|M| = |S_1(M)||D|,$$

which is known as the *Schur formula*.

There is nothing special in decomposition of H into a direct sum of two subspaces. If $H = H_1 \oplus \ldots \oplus H_d$ and

$$M = \begin{pmatrix} M_{11} & \ldots & M_{1,d} \\ \vdots & \ddots & \vdots \\ M_{d1} & \ldots & M_{dd} \end{pmatrix}$$

for $M_{ij}\colon H_i \to H_j$ and $H = H_1 \oplus H_1^{\perp}$, where $H_1^{\perp} = H_2 \oplus \ldots \oplus H_d$, then we are back in the case $d = 2$. By change of the order of the summands (putting H_i on the first place) one can define the i-th Schur complement $S_i(M)$, for each $i = 1, \ldots, d$.

If $\dim H = \infty$ and $\psi\colon H \to H^d$ is a d-similarity, then $S_i(M) = (\rho_\psi(a_i^*)M^{-1}\rho_\psi(a_i))^{-1}$, where ρ_ψ is the representation of Cuntz algebra that corresponds to the d-similarity ψ (see Proposition 5.4 in [GN07]). Therefore, for each $d \geqslant 2$, one can define $\mathcal{S}_H^*$ the semigroup generated by the Schur complements S_i, $1 \leqslant i \leqslant d$ with the operation of composition. We will call $\mathcal{S}_H^*$ the *Schur semigroup*. For a general element of this semigroup, we get the following expression,

$$S_{i_1} \circ \ldots \circ S_{i_k}(M) = \big(\rho_\psi(a_{i_k} \ldots a_{i_1})^* M^{-1} \rho_\psi(a_{i_1} \ldots a_{i_k})\big)^{-1}$$

(see Corollary 5.5 in [GN07]).

The Schur semigroup $\mathcal{S}_H^*$ consists of partially defined transformations on the infinite dimensional space $\mathcal{B}(H)$. Let $L \subset \mathcal{B}(H)$ be a finite dimensional subspace which is invariant with respect to $\mathcal{S}_H^*$. The restriction of each Schur complement gives rise to a $\mathbb{C}^{\dim(L)} \to \mathbb{C}^{\dim(L)}$ map called *Schur map* or *Schur transformation*. The semigroup generated by Schur transformations is denoted by $\mathcal{S}_L^*$. We are particularly interested in the case where the Schur transformations are rational maps. We will examine such examples in Section 5.4.

5.4 Computation of Schur Complements, Schur Transformations, and Associated Rational Maps

In this section, we will compute Schur complements, Schur transformations and rational maps associated with the Grigorchuk group $\mathcal{G}$, the overgroup $\widetilde{\mathcal{G}}$, generalized Grigorchuk groups $\mathcal{G}_\omega$, and generalized overgroups $\widetilde{\mathcal{G}}_\omega$. For $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, we will consider the finite dimensional subspaces generated by the natural generators of the group together with the identity. We will see that these subspaces are invariant with respect to the Schur semigroups. In contrast, for the groups $\mathcal{G}_\omega$ and $\widetilde{\mathcal{G}}_\omega$, these corresponding subspaces are not invariant. But there is a natural way to define Schur transformations, which can be seen in Section 5.4.3.

5.4.1 For the Grigorchuk Group $\mathcal{G}$

Recall that the Grigorchuk group $\mathcal{G}$ is generated by a, b, c, d. Let $M = xa + yb + zc + ud + v1$ be an element of the group algebra $\mathbb{C}[\mathcal{G}]$. Using the matrix recursions (5.11), we identify,

$$M = \begin{pmatrix} (y+z)a + (u+v)1 & x \\ x & ub + yc + zd + v1 \end{pmatrix}. \tag{5.13}$$

First, we will calculate the first Schur complement $S_1(M)$, which is defined when $D = v1 + ub + yc + zd$ is invertible. Since the group generated by $\{1, b, c, d\}$ is isomorphic to $\mathbb{Z}_2^2$ (via the identification $1, b, c, d$ with $(0,0), (1,0), (0,1), (1,1)$, respectively), by (A.6) and (A.2), we obtain that D is invertible if and only if

$$(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z) \neq 0, \tag{5.14}$$

and if the condition in (5.14) is satisfied, then by (A.7),

$$\begin{aligned} D^{-1} =& \frac{1}{4}\left(\frac{1}{v+u+y+z} + \frac{1}{v-u+y-z} + \frac{1}{v+u-y-z} + \frac{1}{v-u-y+z}\right)1 \\ &+ \frac{1}{4}\left(\frac{1}{(v+u+y+z)} - \frac{1}{v-u+y-z} + \frac{1}{v+u-y-z} - \frac{1}{v-u-y+z}\right)b \end{aligned}$$

$$+\frac{1}{4}\left(\frac{1}{(v+u+y+z)}+\frac{1}{v-u+y-z}-\frac{1}{v+u-y-z}-\frac{1}{v-u-y+z}\right)c$$
$$+\frac{1}{4}\left(\frac{1}{(v+u+y+z)}-\frac{1}{v-u+y-z}-\frac{1}{v+u-y-z}+\frac{1}{v-u-y+z}\right)d.$$

Therefore, the first Schur complement

$$\begin{aligned}
S_1(M) =& A - BD^{-1}C \\
=& (y+z)a + (v+u)1 - x^2D^{-1} \\
=& (y+z)a \\
&+ \left(v+u-x^2\frac{2uyz - v(-v^2+u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\right)1 \\
&- x^2\frac{2vyz - u(v^2-u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}b, \\
&- x^2\frac{2vuz - y(v^2+u^2-y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}c, \\
&- x^2\frac{2vuy - z(v^2+u^2+y^2-z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}d.
\end{aligned}$$

This leads to the Schur transformation $S_1^{\mathcal{G}} \colon \mathbb{C}^5 \to \mathbb{C}^5$ given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ -x^2\frac{2vyz - u(v^2-u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)} \\ -x^2\frac{2vuz - y(v^2+u^2-y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)} \\ -x^2\frac{2vuy - z(v^2+u^2+y^2-z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)} \\ v+u-x^2\frac{2uyz - v(-v^2+u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)} \end{pmatrix}. \tag{5.15}$$

Now, we will calculate the second Schur complement $S_2(M)$ which is defined when $A = (y+z)a + (u+v)1$ is invertible. Since the group generated by $\{1, a\}$ is isomorphic to $\mathbb{Z}_2$ (via the identification $1, a$ with $0, 1$, respectively), by (A.4) and (A.2), we obtain that A is invertible if and only if

$$(v+u+y+z)(v+u-y-z) \neq 0, \tag{5.16}$$

and if the condition in (5.16) is satisfied, then A^{-1} is given by,

$$\begin{aligned} A^{-1} &= \frac{1}{2}\left(\frac{1}{v+u+y+z}+\frac{1}{v+u-y-z}\right)1+\frac{1}{2}\left(\frac{1}{v+u+y+z}-\frac{1}{v+u-y-z}\right)a \\ &= \frac{v+u}{(v+u+y+z)(v+u-y-z)}1-\frac{y+z}{(v+u+y+z)(v+u-y-z)}a. \end{aligned}$$

Therefore, the second Schur complement

$$\begin{aligned} S_2(M) =& v1+ub+yc+zd-x^2A^{-1} \\ =& \frac{x^2(y+z)}{(v+u+y+z)(v+u-y-z)}a+ub+yc+zd \\ &+\left(v-\frac{x^2(v+u)}{(v+u+y+z)(v+u-y-z)}\right)1. \end{aligned}$$

This leads to the Schur transformation $S_2^{\mathcal{G}}\colon \mathbb{C}^5 \to \mathbb{C}^5$ given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{x^2(y+z)}{(v+u+y+z)(v+u-y-z)} \\ u \\ y \\ z \\ v-\dfrac{x^2(v+u)}{(v+u+y+z)(v+u-y-z)} \end{pmatrix}. \tag{5.17}$$

The map $S_2^{\mathcal{G}}$ fixes second, third and fourth coordinates when $y=z=u=1$, and so we may restrict the map to the first and the fifth coordinates. Therefore, we get the $\mathbb{C}^2 \to \mathbb{C}^2$ map given by

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2x^2}{(v+3)(v-1)} \\ v-\dfrac{x^2(v+1)}{(v+3)(v-1)} \end{pmatrix}.$$

By the change of coordinates $(x,v) \to (-x,-1-y)$, we obtain the $\mathbb{C}^2 \to \mathbb{C}^2$ map

$$F\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2x^2}{4-y^2} \\ y + \dfrac{x^2 y}{4-y^2} \end{pmatrix}. \tag{5.18}$$

When $y = z = u = 1$, the second, third and fourth coordinates of the map $S_1^{\mathcal{G}}$ are equal and the common value is $\dfrac{x^2}{(v+3)(v-1)}$. By re-normalization (i.e., multiplying by $\frac{(v+3)(v-1)}{x^2}$) we obtain a map which fixes second, third and fourth coordinates. So we may restrict it to the first and the fifth coordinates and get $\mathbb{C}^2 \to \mathbb{C}^2$ map

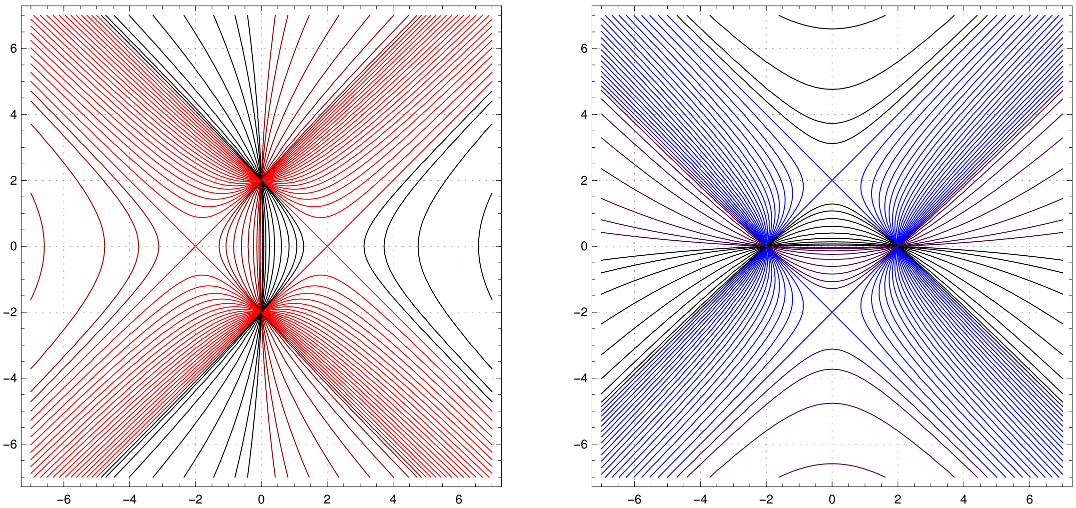
$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2(v+3)(v-1)}{x^2} \\ -2 - v + (v+1)\dfrac{(v+3)(v-1)}{x^2} \end{pmatrix}.$$

By the change of coordinates $(x,v) \to (-x,-1-y)$, we obtain $\mathbb{C}^2 \to \mathbb{C}^2$ map

$$G\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2(4-y^2)}{x^2} \\ -y - \dfrac{y(4-y^2)}{x^2} \end{pmatrix}. \tag{5.19}$$

The map F demonstrates features of an integrable map as it has two almost transversal families of *horizontal hyperbolas* $\mathcal{F}_\theta = \{(x,y)\colon 4 + x^2 - y^2 - 4\theta x = 0\}$ and *vertical hyperbolas* $\mathcal{H}_\eta = \{(x,y)\colon 4 - x^2 + y^2 - 4\eta y = 0\}$, shown in Figure 5.2. The first family $\{\mathcal{F}_\theta\}$ is invariant as a family and $F^{-1}(\mathcal{F}_\theta) = \mathcal{F}_{\theta_1} \sqcup \mathcal{F}_{\theta_2}$, where θ_1, θ_2 are preimages of θ under the Chebyshev map $t\colon z \mapsto 2z^2 - 1$, and the family $\{\mathcal{H}_\eta\}$ consists of invariant curves.

The set $\mathcal{K}$ shown in Figure 5.3a (we will call this set the "cross") is of special interest for us as it represents the joint spectrum of several families of operators associated with the element $m(x,y) = -xa + b + c + d - (y+1)1$ of the group algebra $\mathbb{R}[\mathcal{G}]$ [BG00b, GN07, DG17]. It can be foliated by the hyperbolas $\mathcal{F}_\theta$, $-1 \leqslant \theta \leqslant 1$ as shown in Figure 5.3b (or by hyperbolas $\mathcal{H}_\eta$, $-1 \leqslant \eta \leqslant 1$ shown in Figure 5.3c). The F-preimages of the border line $x + y = 2$ constitutes a dense



(a) (b)

Figure 5.2: Foliation of $\mathbb{R}^2$ by (**a**) horizontal hyperbolas $\mathcal{F}_\theta$ where, maroon, red and black corresponds to $\theta < -1, \theta \in [-1, 1]$ and $\theta > 1$, respectively, and (**b**) vertical hyperbola $\mathcal{H}_\eta$ where, purple, blue and black corresponds to $\eta < -1, \eta \in [-1, 1]$ and $\eta > 1$, respectively.

family of curves for $\mathcal{K}$ (the same is true for G-preimages) and $\mathcal{K}$ is completely invariant set for F or G (i.e., $F^{-1}(\mathcal{K}) \subset \mathcal{K}$ and $F(\mathcal{K}) \subset \mathcal{K}$, so $F(\mathcal{K}) = \mathcal{K}$).

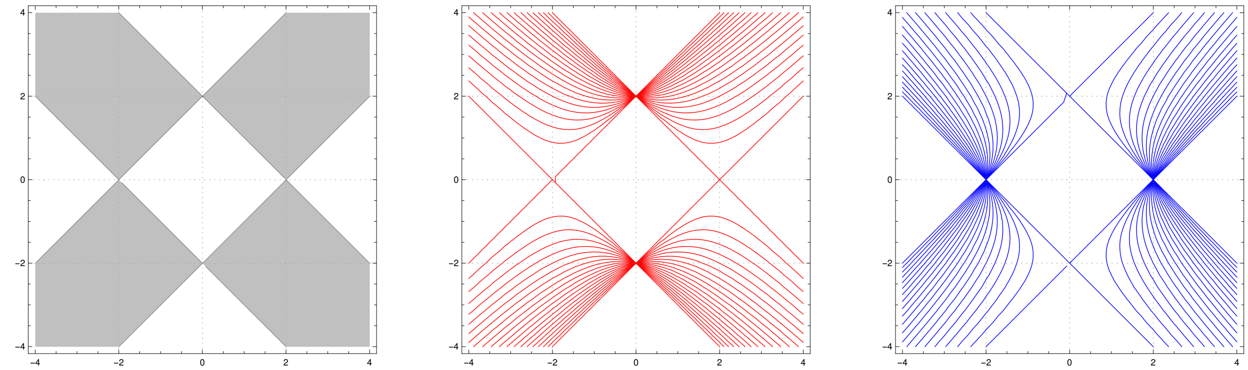
The map F is comprehensively investigated in [DGL21] (its close relative is studied in [GY17] and [GY20] from a different point of view) and serves as a basis for the integrability theory developed there. The map G happens to be more complicated and its study is ongoing.

5.4.2 For the Overgroup $\widetilde{\mathcal{G}}$

Recall that the overgroup $\widetilde{\mathcal{G}}$ is generated by the elements $a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$. Let $M = xa + yb + zc + ud + q\widetilde{a} + r\widetilde{b} + s\widetilde{c} + t\widetilde{d} + v1$ be an element of the group algebra $\mathbb{C}[\widetilde{\mathcal{G}}]$. Using the matrix recursions (5.11), we identify,

$$M = \begin{pmatrix} (y+z+q+t)a + (u+r+s+v)1 & x \\ x & ub + yc + zd + q\widetilde{a} + t\widetilde{b} + r\widetilde{c} + s\widetilde{d} + v1 \end{pmatrix}. \tag{5.20}$$

Now, let us calculate $S_1(M)$, which is defined for invertible $D = ub + yc + zd + q\widetilde{a} + t\widetilde{b} +$



(a) (b) (c)

Figure 5.3: (**a**) The "cross" $\mathcal{K}$, (**b**) foliation of the cross by real slices of horizontal hyperbolas $\mathcal{F}_\theta$ ($\theta \in [-1,1]$), and (**c**) foliation of the cross by real slices of vertical hyperbolas $\mathcal{H}_\eta$ ($\eta \in [-1,1]$).

$r\widetilde{c} + s\widetilde{d} + v1$. The group generated by $\{1, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\}$ is isomorphic to $\mathbb{Z}_2^3$ (via the identification $1, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ with $(0,0,0), (1,0,0), (0,1,0), (1,1,0), (1,1,1), (0,1,1), (1,0,1), (0,0,1)$, respectively). Define

$$
\begin{aligned}
\hat{D}_{000} &= v + u + y + s + z + r + t + q, \\
\hat{D}_{100} &= v - u + y + s - z - r + t - q, \\
\hat{D}_{010} &= v + u - y + s - z + r - t - q, \\
\hat{D}_{001} &= v + u + y - s + z - r - t - q, \\
\hat{D}_{110} &= v - u - y + s + z - r - t + q, \\
\hat{D}_{101} &= v - u + y - s - z + r - t + q, \\
\hat{D}_{011} &= v + u - y - s - z - r + t + q, \\
\hat{D}_{111} &= v - u - y - s + z + r + t - q. \qquad (5.21)
\end{aligned}
$$

By (A.8) and (A.2), we obtain that D is invertible if and only if

$$\prod_{i,j,k \in \{0,1\}} \hat{D}_{ijk} \neq 0, \qquad (5.22)$$

and by (A.9),

$$\begin{aligned}
D^{-1} =&\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)1\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)b\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)c\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)\widetilde{d}\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)d\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\widetilde{c}\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\widetilde{b}\\
&+\frac{1}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)\widetilde{a}. \qquad (5.23)
\end{aligned}$$

Therefore, the first Schur complement

$$\begin{aligned}
S_1(M) =&A-BD^{-1}C\\
=&(y+z+q+t)a+(u+r+s+v)1-x^2D^{-1}\\
=&(y+z+q+t)a\\
&+\Bigg((u+r+s+v)\\
&\quad-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\Bigg)1\\
&-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)b\\
&-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)c\\
&-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)\widetilde{d}\\
&-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)d
\end{aligned}$$

$$
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\widetilde{c}
$$
$$
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\widetilde{b}
$$
$$
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right)\widetilde{a}.
$$

This gives the Schur transformation $S_1^{\tilde{\mathcal{G}}}\colon \mathbb{C}^9 \to \mathbb{C}^9$ given by

$$
\begin{pmatrix} x \\ y \\ z \\ u \\ q \\ r \\ s \\ t \\ v \end{pmatrix} \mapsto
\begin{pmatrix}
y+z+q+t \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}-\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}-\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right) \\
-\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}-\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}-\frac{1}{\hat{D}_{101}}-\frac{1}{\hat{D}_{011}}-\frac{1}{\hat{D}_{111}}\right) \\
\Bigg((u+r+s+v) \\
\quad -\frac{x^2}{8}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{100}}+\frac{1}{\hat{D}_{010}}+\frac{1}{\hat{D}_{001}}+\frac{1}{\hat{D}_{110}}+\frac{1}{\hat{D}_{101}}+\frac{1}{\hat{D}_{011}}+\frac{1}{\hat{D}_{111}}\right)\Bigg)
\end{pmatrix}.
$$

Finally, we will calculate $S_2(M)$ when $A=(y+z+q+t)a+(u+r+s+v)1$ is invertible. Since the group generated by elements $1, a$ is isomorphic to $\mathbb{Z}_2$ (via the identification $1, a$ with $0, 1$, respectively), by (A.4) and (A.2), we obtain, A is invertible if and only if

$$
(v+u+r+s+y+z+q+t)(v+u+r+s-y-z-q-t)\neq 0, \tag{5.24}
$$

and if the condition in (5.24) is satisfied, then A^{-1} is given by,

$$
A^{-1}=\frac{1}{2}\left(\frac{1}{\hat{D}_{000}}+\frac{1}{\hat{D}_{010}}\right)1+\frac{1}{2}\left(\frac{1}{\hat{D}_{000}}-\frac{1}{\hat{D}_{010}}\right)a
$$

using the notation from (5.21). Therefore, the second Schur complement

$$\begin{aligned} S_2(M) =& D - CA^{-1}B \\ =& ub + yc + zd + q\widetilde{a} + t\widetilde{b} + r\widetilde{c} + s\widetilde{d} + v1 - x^2A^{-1} \\ =& -\frac{x^2}{2}\left(\frac{1}{\hat{D}_{000}} - \frac{1}{\hat{D}_{010}}\right) a + ub + yc + zd + q\widetilde{a} + t\widetilde{b} + r\widetilde{c} + s\widetilde{d} \\ & \qquad + \left(v - \frac{x^2}{2}\left(\frac{1}{\hat{D}_{000}} + \frac{1}{\hat{D}_{010}}\right)\right) 1. \end{aligned}$$

Then by substituting from (5.21), we obtain the Schur transformation $S_2^{\tilde{\mathcal{G}}} : \mathbb{C}^9 \to \mathbb{C}^9$ given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ q \\ r \\ s \\ t \\ v \end{pmatrix} \mapsto \begin{pmatrix} \frac{x^2(y+z+q+t)}{(v+u+r+s+y+z+q+t)(v+u+r+s-y-z-q-t)} \\ u \\ y \\ z \\ q \\ t \\ r \\ s \\ v - \frac{x^2(v+u+r+s)}{(v+u+r+s+y+z+q+t)(v+u+r+s-y-z-q-t)} \end{pmatrix}. \tag{5.25}$$

Note that, choosing $y = z = u$ and $r = s = t$ converts $S_2^{\tilde{\mathcal{G}}}$ to a 2-dimensional map. For simplicity, we choose all the variables, except the first and the last, to be 1. Then we get the $\mathbb{C}^2 \to \mathbb{C}^2$ map

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \frac{4x^2}{(v+7)(v-1)} \\ v - \frac{x^2(v+3)}{(v+7)(v-1)} \end{pmatrix}.$$

By the change of coordinates $(x, v) \to (-x, -3 - y)$, we obtain the $\mathbb{C}^2 \to \mathbb{C}^2$ map

$$\widetilde{F}\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2x^2}{16 - y^2} \\ y + \dfrac{x^2 y}{16 - y^2} \end{pmatrix}. \tag{5.26}$$

5.4.3 For the Generalized Grigorchuk Groups and Generalized Overgroups

Recall that the generalized Grigorchuk group $\mathcal{G}_\omega$ is generated by $a_\omega, b_\omega, c_\omega, d_\omega$. Let $M = xa_\omega + yb_\omega + zc_\omega + ud_\omega + v1$ be an element of the group algebra $\mathbb{C}[\mathcal{G}_\omega]$. Using the matrix recursions (5.11) and (2.11), we identify,

$$M = \begin{pmatrix} (p_\omega + q_\omega)a_{\sigma\omega} + (r_\omega + v)1 & x \\ x & yb_{\sigma\omega} + zc_{\sigma\omega} + ud_{\sigma\omega} + v1 \end{pmatrix}, \tag{5.27}$$

where

$$(p_\omega, q_\omega, r_\omega) = \begin{cases} (y, z, u) & ; \omega_0 = 0 \\ (u, y, z) & ; \omega_0 = 1 \\ (z, u, y) & ; \omega_0 = 2 \end{cases} \cdot \tag{5.28}$$

Here ω_0 is the first symbol of the sequence ω. Note that $(p_\omega, q_\omega, r_\omega)$ is determined by ω_0 and so we may write $(p_{\omega_0}, q_{\omega_0}, r_{\omega_0})$ in place of $(p_\omega, q_\omega, r_\omega)$.

First, we will calculate the first Schur complement $S_1(M)$, which is defined when $D = v1 + yb_{\sigma\omega} + zc_{\sigma\omega} + ud_{\sigma\omega}$ is invertible. Note that D is invertible if and only if the condition (5.14) is satisfied, in which case we obtain,

$$\begin{aligned} D^{-1} =& \frac{1}{4}\left(\frac{1}{v+u+y+z} + \frac{1}{v-y+z-u} + \frac{1}{v+y-z-u} + \frac{1}{v-y-z+u}\right)1 \\ &+ \frac{1}{4}\left(\frac{1}{v+u+y+z} - \frac{1}{v-y+z-u} + \frac{1}{v+y-z-u} - \frac{1}{v-y-z+u}\right)b_{\sigma\omega} \\ &+ \frac{1}{4}\left(\frac{1}{v+u+y+z} + \frac{1}{v-y+z-u} - \frac{1}{v+y-z-u} - \frac{1}{v-y-z+u}\right)c_{\sigma\omega} \end{aligned}$$

$$+\frac{1}{4}\left(\frac{1}{v+u+y+z}-\frac{1}{v-y+z-u}-\frac{1}{v+y-z-u}+\frac{1}{v-y-z+u}\right)d_{\sigma\omega}.$$

Therefore, the first Schur complement

$$\begin{aligned}
S_1(M) =&(p_\omega+q_\omega)a_{\sigma\omega}\\
&+\left(v+r_\omega-x^2\frac{2uyz-v(-v^2+u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\right)1\\
&-x^2\frac{2vzu-y(v^2-y^2+u^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}b_{\sigma\omega},\\
&-x^2\frac{2vyu-z(v^2+u^2+y^2-z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}c_{\sigma\omega},\\
&-x^2\frac{2vyz-u(v^2-u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}d_{\sigma\omega}.
\end{aligned}$$

Note that the Schur complement can be viewed as a map from the linear span of $\{a_\omega, b_\omega, c_\omega, d_\omega, 1\}$ to the linear span of $\{a_{\sigma\omega}, b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega}, 1\}$. So, we can define the first Schur transformation $S_1^{\mathcal{G}_\omega}: \mathbb{C}^5 \to \mathbb{C}^5$ given by

$$\begin{pmatrix} x\\ y\\ z\\ u\\ v\end{pmatrix} \mapsto \begin{pmatrix} p_\omega+q_\omega\\ -x^2\frac{2vzu-y(v^2-y^2+u^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\\ -x^2\frac{2vyu-z(v^2+u^2+y^2-z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\\ -x^2\frac{2vyz-u(v^2-u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\\ v+r_\omega-x^2\frac{2uyz-v(-v^2+u^2+y^2+z^2)}{(v+u+y+z)(v-u+y-z)(v+u-y-z)(v-u-y+z)}\end{pmatrix}. \tag{5.29}$$

Now, we will calculate the second Schur complement $S_2(M)$ which is defined when $A = (p_\omega+q_\omega)a_{\sigma\omega}+(r_\omega+v)1$ is invertible. By a similar calculation, we obtain that A is invertible if and only if

$$(v+r_\omega+p_\omega+q_\omega)(v+r_\omega-p_\omega-q_\omega)\neq 0, \tag{5.30}$$

and if the condition in (5.30) is satisfied, then A^{-1} is given by,

$$A^{-1} = \frac{v + r_\omega}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} 1 - \frac{p_\omega + q_\omega}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} a_{\sigma\omega}.$$

Therefore, the second Schur complement

$$\begin{aligned} S_2(M) = & \frac{x^2(p_\omega + q_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} a_{\sigma\omega} + y b_{\sigma\omega} + z c_{\sigma\omega} + u d_{\sigma\omega} \\ & + \left(v - \frac{x^2(v + r_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} \right) 1. \end{aligned}$$

This leads to the Schur transformation $S_2^{\mathcal{G}_\omega} : \mathbb{C}^5 \to \mathbb{C}^5$ given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{x^2(p_\omega + q_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} \\ y \\ z \\ u \\ v - \dfrac{x^2(v + r_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} \end{pmatrix}. \tag{5.31}$$

Observe that $S_2^{\mathcal{G}_\omega}$ fixes the second, third, and fourth coordinates. Thus, by restricting to first and fifth coordinates, we obtain a $\mathbb{C}^2 \to \mathbb{C}^2$ map

$$F_{\omega_0} : \begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{x^2(p_\omega + q_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} \\ v - \dfrac{x^2(v + r_\omega)}{(v + r_\omega + p_\omega + q_\omega)(v + r_\omega - p_\omega - q_\omega)} \end{pmatrix}, \tag{5.32}$$

where

$$(\alpha_\omega, \beta_\omega) = (p_\omega + q_\omega, r_\omega) = \begin{cases} (y + z, u) & ; \omega_0 = 0 \\ (y + u, z) & ; \omega_0 = 1 \\ (z + u, y) & ; \omega_0 = 2 \end{cases} \cdot \tag{5.33}$$

Now let us consider the generalized overgroup $\widetilde{\mathcal{G}}_\omega$, generated by $a_\omega, b_\omega, c_\omega, d_\omega, \widetilde{a}_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega$. Let $M = xa_\omega + yb_\omega + zc_\omega + ud_\omega + q\widetilde{a}_\omega + r\widetilde{b}_\omega + s\widetilde{c}_\omega + t\widetilde{d}_\omega + v1$ be an element of the group algebra $\mathbb{C}[\widetilde{\mathcal{G}}_\omega]$. By a similar calculation as of above, we obtain the second Schur transformation $S_2^{\widetilde{\mathcal{G}}_\omega} : \mathbb{C}^9 \to \mathbb{C}^9$ given by,

$$\begin{pmatrix} x \\ y \\ z \\ u \\ q \\ r \\ s \\ t \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{x^2\alpha_\omega}{(v+\beta_\omega+\alpha_\omega)(v+\beta_\omega-\alpha_\omega)} \\ y \\ z \\ u \\ q \\ r \\ s \\ t \\ v - \dfrac{x^2(v+\beta_\omega)}{(v+\beta_\omega+\alpha_\omega)(v+\beta_\omega-\alpha_\omega)} \end{pmatrix}, \tag{5.34}$$

where

$$(\alpha_\omega, \beta_\omega) = \begin{cases} (y+z+q+t, u+r+s) & ;\omega_0 = 0 \\ (y+u+q+s, z+r+t) & ;\omega_0 = 1 \\ (z+u+q+r, y+r+s) & ;\omega_0 = 2 \end{cases} \cdot \tag{5.35}$$

By restricting to the first and last coordinates, we obtain a $\mathbb{C}^2 \to \mathbb{C}^2$ map given by

$$\widetilde{F}_{\omega_0} : \begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{x^2\alpha_\omega}{(v+\beta_\omega+\alpha_\omega)(v+\beta_\omega-\alpha_\omega)} \\ v - \dfrac{x^2(v+\beta_\omega)}{(v+\beta_\omega+\alpha_\omega)(v+\beta_\omega-\alpha_\omega)} \end{pmatrix}. \tag{5.36}$$

We omit the calculation of the first Schur transformation as it is more complicated to be written down.

5.5 Two-Parametric Maps and Rational Maps Associated with $\mathcal{G}_\omega, \widetilde{\mathcal{G}}_\omega$

We have calculated the rational maps associated to $\mathcal{G}_\omega$ and $\widetilde{\mathcal{G}}_\omega$ to be (5.32) and (5.36), respectively, in Section 5.4.3. If their corresponding $\alpha_\omega \neq 0$, then they are of the form (5.1). Let $f = F_{\alpha,\beta}$, where $F_{\alpha,\beta}$ is given by (5.1). Thus, $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$. Our first observation is, the map f and the maps $F, \widetilde{F}$ given in (5.18), (5.26) are closely related.

Proposition 5.2. *The map f is conjugate to the map*

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{\gamma x^2}{\gamma^2 - v^2} \\ v + \dfrac{x^2 v}{\gamma^2 - v^2} \end{pmatrix},$$

for any non-zero γ. In particular, f is conjugate to F and $\widetilde{F}$.

Proof. First, consider the map $h\colon (x, v) \mapsto (-x, -v - \beta)$. Then, h is an involution and therefore is invertible. By conjugating f by h, we obtain

$$\begin{aligned} f^h(x, v) &= h^{-1} \circ f \circ h(x, v) \\ &= h \circ f \circ h(x, v) \\ &= h \circ f(-x, -v - \beta) \\ &= h\left(\frac{\alpha x^2}{v^2 - \alpha^2}, -v - \beta + \frac{x^2 v}{v^2 - \alpha^2}\right) \\ &= \left(-\frac{\alpha x^2}{v^2 - \alpha^2}, v + \beta - \frac{x^2 v}{v^2 - \alpha^2} - \beta\right) \\ &= \left(\frac{\alpha x^2}{\alpha^2 - v^2}, v + \frac{x^2 v}{\alpha^2 - v^2}\right). \end{aligned}$$

Now let g be the multiplication by α/γ map, i.e., $g\colon (x, v) \mapsto \left(\dfrac{\alpha}{\gamma} x, \dfrac{\alpha}{\gamma} v\right)$. Then, g is invertible and the inverse is the multiplication by γ/α map. Therefore,

$$f^{h \circ g}(x, v) = g^{-1} \circ f^h \circ g(x, v)$$

$$
\begin{aligned}
&= g^{-1} \circ f^h\left(\frac{\alpha}{\gamma}x, \frac{\alpha}{\gamma}v\right) \\
&= g^{-1}\left(\frac{\alpha x^2}{\gamma^2 - v^2}, \frac{\alpha}{\gamma}v + \frac{\alpha x^2 v}{\gamma(\gamma^2 - v^2)}\right) \\
&= \left(\frac{\gamma x^2}{\gamma^2 - v^2}, v + \frac{x^2 v}{\gamma^2 - v^2}\right),
\end{aligned}
$$

which proves the result. Choosing $\gamma = 2$ and $\gamma = 4$, we obtain that f is conjugate to F and $\widetilde{F}$, respectively. $\square$

Proof of Theorem 5.1. We know that the map f is conjugate to F, using Proposition 5.2. By Theorem 5.1.(i) of [DGL21], F is semi-conjugate to t, the Chebyshev map, via the map $(x, v) \mapsto \frac{4 - v^2 + x^2}{4x}$. Thus, f is semi-conjugate to the Chebyshev map. $\square$

Now let us view f as a map on $\mathbb{P}^2$. So, in homogeneous coordinates, the map f becomes

$$
f = \left[\alpha x^2 w : v((v + \beta w)^2 - (\alpha w)^2) - (v + \beta w)x^2 : ((v + \beta w)^2 - (\alpha w)^2)w\right]. \tag{5.37}
$$

We will denote the three polynomials in the coordinates of f as f_0, f_1, f_2. So $f = [f_0 : f_1 : f_2]$. First we will look at the *indeterminacy points* (the points for which the function is not defined, i.e., f_0, f_1, f_2 are all simultaneously zero) and fixed points of f.

Proposition 5.3. *The map f is of algebraic degree 3 and topological degree 2.*

1. *It has five indeterminacy points: Two points $P = [0 : -(\beta + \alpha) : 1]$, $Q = [0 : -(\beta - \alpha) : 1]$ on vertical line and three points $I_0 = [1 : 0 : 0]$, $I_1 = [1 : 1 : 0]$, $I_2 = [-1 : 1 : 0]$ at infinity.*

2. *The point (except indeterminacy points) on the vertical line $\{x = 0\}$ and the point $[-\alpha : -\beta : 1]$ are all the fixed points for f.*

Proof. By observation, we see f is of algebraic degree 3 and topological degree 2.

First, let us calculate indeterminacy points, i.e., points of which all three of f_0, f_1, f_2 are zero. Letting $f_0 = 0$, we obtain $x = 0$ or $w = 0$. That is, all the indeterminacies lie on the vertical line $\{x = 0\}$ or on the line at infinity $\{w = 0\}$.

To find the indeterminacies of vertical line, let $x = 0$. Then, the points making $f_1 = f_2 = 0$ satisfy $(v + \beta w)^2 - (\alpha w)^2 = 0$ and therefore $v = -(\beta + \alpha)w$ or $v = -(\beta - \alpha)w$. Thus, we obtain the points $P = [0 : -(\beta + \alpha) : 1]$ and $Q = [0 : -(\beta - \alpha) : 1]$.

To find the indeterminacies at infinity, let $w = 0$. Then, $f_2 = 0$ and $f_1 = v(v^2 - x^2)$. By making $f_1 = 0$, we obtain $v = 0$, $v = x$, or $v = -x$. Thus, we obtain the points $I_0 = [1 : 0 : 0]$, $I_1 = [1 : 1 : 0]$, and $I_2 = [-1 : 1 : 0]$. This completes the proof of assertion 1.

Now, let us calculate the fixed points. Suppose $f = [f_0 : f_1 : f_2] = \lambda\,[x : v : w]$, for some $\lambda \in \mathbb{C}$. First, note that if $w = 0$, then $x = 0$ and $v = 1$, which is the point at infinity on vertical line. Suppose $w \neq 0$. By $f_2 = \lambda w$, we get $\lambda = (v + \beta w)^2 - (\alpha w)^2$. Using $f_1 = \lambda v$, we obtain $(v + \beta w)x^2 = 0$. Thus, $x = 0$ or $v = -\beta w$.

It is clear from (5.37) that $\{x = 0\}$ is an invariant line of fixed points. So, suppose $x \neq 0$. By $v = -\beta w$, we get $\lambda = -(\alpha w)^2$. Finally, using $f_0 = \lambda x$, we obtain $\alpha x^2 w = -\alpha^2 x w^2$. Since we have $\alpha \neq 0$, $x \neq 0$, and $w \neq 0$, we conclude $x = -\alpha w$, giving the fixed point $[-\alpha : -\beta : 1]$. This completes the proof. □

The map has following properties, which we will use to study the dynamics of f.

Proposition 5.4.

1. *The point* $I_0 = [1 : 0 : 0]$ *is not in the image of* f.

2. *The only points that map to the vertical line are the points on the vertical line and the points on the line at infinity. Moreover, the line at infinity maps to the point* $[0 : 1 : 0]$.

Proof. Suppose a point $[x : v : w]$ is mapped to a point at infinity. Then $f_2\,[x : v : w] = 0$. Thus, $w = 0$, in which case $f\,[x : v : w] = [0 : 1 : 0]$, or $((v + \beta w)^2 - (\alpha w)^2) = 0$, in which case $f_1 = \pm f_0$, that consequently makes $f\,[x : v : w] = [\pm 1 : 1 : 0]$. Therefore, no point is mapped to the point I_0.

To show the second assertion, suppose a point $[x : v : w]$ is mapped to the vertical line. Then, $f_0\,[x : v : w] = 0$ and so we get $x = 0$ or $w = 0$. Therefore, the point $[x : v : w]$ is either on the vertical line, or on the line at infinity. In the case of $w = 0$, we have $f_2[x : v : w] = 0$ and so the image is $[0 : 1 : 0]$. This completes the proof. $\square$

Next step is to study the *contracting curves* (curves that are collapsed to a point via the map) of the map f. To do it, let us look at the jacobian $j(f)$ and its determinant $|j(f)|$. The jacobian is given by

$$j(f) = \begin{pmatrix} 2\alpha x w & 0 & \alpha x^2 \\ -2x(v+\beta w) & (v+\beta w)(3v+\beta w) - \alpha^2 w^2 - x^2 & 2(\beta(v+\beta w) - \alpha^2 w)v - \beta x^2 \\ 0 & 2w(v+\beta w) & 3(\beta(v+\beta w) - \alpha^2 w)w + (v+\beta w)v \end{pmatrix}, \tag{5.38}$$

and therefore the determinant is,

$$|j(f)| = 6\alpha x w(v + (\beta - \alpha)w)(v + (\beta + \alpha)w)\left((v+\beta w)^2 - \alpha^2 w^2 - x^2\right). \tag{5.39}$$

Equating the determinant of the jacobian to zero, we obtain the curves; the vertical line $\{x = 0\}$, the line at infinity $\{w = 0\}$, the line $L' = \{v + (\beta + \alpha)w = 0\}$ passing through I_0 and P, the line $L = \{v + (\beta - \alpha)w = 0\}$ passing through I_0 and Q, and the conic $C = \{(v + \beta w)^2 - \alpha^2 w^2 - x^2 = 0\}$ passing through points I_1, I_2, P, and Q. The Figure 5.4 represents the fixed points, the indeterminacy points, and contracting curves of f, graphically. By Proposition 5.3 assertion 2, we observe that the vertical line $\{x = 0\}$ is not a contracting curve. The dynamics of the above contracting curves can be summarize as follows:

Proposition 5.5. *The map f collapses;*

1. *The line at infinity $\{w = 0\} \setminus \{I_0, I_1, I_2\}$ to the fixed point $[0 : 1 : 0]$.*

2. *The line $L' \setminus \{I_0, P\}$ to the indeterminacy point I_1.*

3. *The line $L \setminus \{I_0, Q\}$ to the indeterminacy point I_2.*

4. *The conic $C \setminus \{I_1, I_2, P, Q\}$ to the point $[\alpha : -\beta : 1]$.*

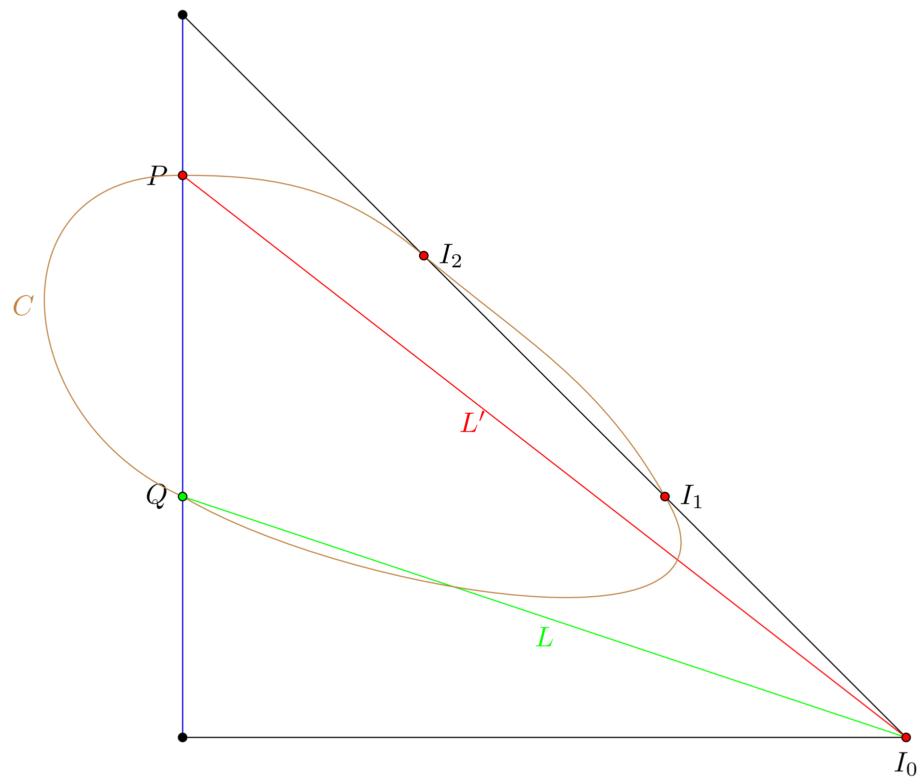


Figure 5.4: Curves L', L, and C that are contracting to a point via f.

Proof. The first assertion is directly obtained from the second assertion of Proposition 5.4. Consider a point $[x : v : w]$ on L'. Then, $v + \beta w = -\alpha w$. Thus, $f_1 = \alpha x^2 w = f_0$ and $f_2 = 0$. Therefore, $f\,[x : v : w] = [1 : 1 : 0] = I_1$. This proves the second assertion. The third assertion follows similarly.

To prove the last assertion, take a point $[x : v : w]$ on C. So, $(v + \beta w)^2 - \alpha^2 w^2 = x^2$. Using it, we obtain, $f_1 = -\beta x^2 w$, and $f_2 = x^2 w$. Therefore, $f\,[x : v : w] = [\alpha x^2 w : -\beta x^2 w : x^2 w] = [\alpha : -\beta : 1]$. This completes the proof. $\square$

Proposition 5.5 shows that there are algebraic curves that collapse to points of indeterminacies. In order to avoid this complication, let us *blow-up* $\mathbb{P}^2$ at the indeterminacy points I_1, I_2 (see Appendix B.2). Let this space, $\mathrm{BL}_{I_1,I_2}(\mathbb{P}^2)$, be denoted by X, and let π_X be the *blow-down map*. Denote the *lift* of f to X, by $\widehat{f}$. We will examine the dynamics of $\widehat{f}$ on E_1, E_2, the *exceptional* *divisors* at I_1, I_2, respectively (see figure 5.5).

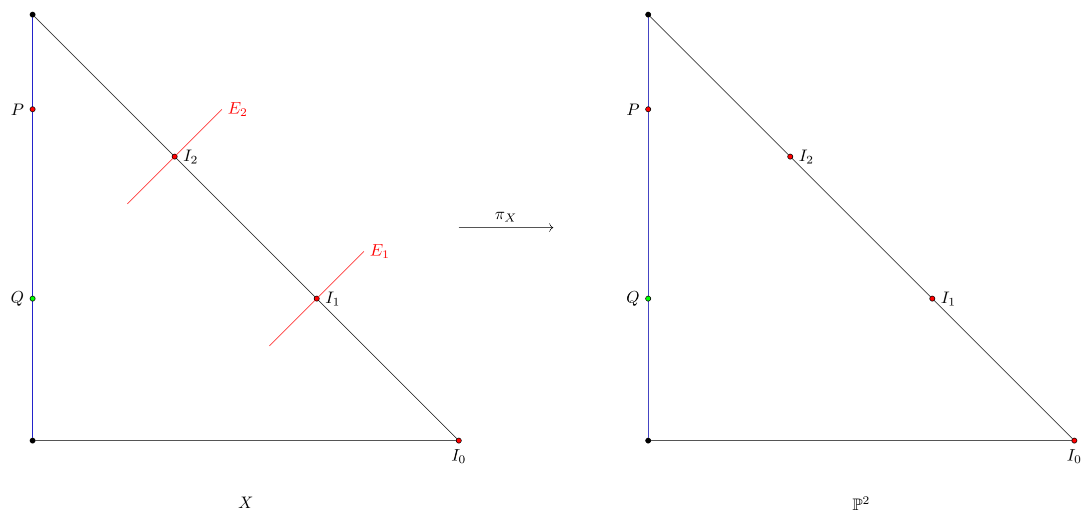


Figure 5.5: Blow up X of $\mathbb{P}^2$ at indeterminacy points I_1 and I_2.

Proposition 5.6.

1. *The lifted map $\widehat{f}$ is regular on E_1, and its image $\widehat{f}(E_1)$ is the strict transform of the line $\{x = \alpha, w = 1\} \cup \{[0 : 1 : 0]\}$.*

2. *The strict transform of $L' \setminus \{I_0, P\}$ is mapped to E_1, which avoids indeterminacies.*

3. *The lifted map $\widehat{f}$ is regular on E_2, and its image $\widehat{f}(E_2)$ is the strict transform of the line $\{x = \alpha, w = 1\} \cup \{[0 : 1 : 0]\}$.*

4. *The strict transform of $L \setminus \{I_0, Q\}$ is mapped to E_2, which avoids indeterminacies.*

Proof. First, consider the point I_1 and the exceptional divisor E_1. Let $R = [x : v : w]$ be an arbitrary point in $\mathbb{P}^2$ not in the vertical line $\{x = 0\}$. So, $x \neq 0$. There are two ways to choose a local coordinate system (e, l) such that the equation of the exceptional divisor E_1 is $\{e = 0\}$:

1. $(e, l) = \left(\dfrac{w}{x}, \dfrac{v - x}{w}\right)$, assuming $w \neq 0$, in which case $\pi_X(e, l) = [1 : 1 + le : e]$.

2. $(e, l) = \left(\frac{v-x}{x}, \frac{w}{v-x}\right)$, assuming $x \neq v$, in which case $\pi_X(e, l) = [1 : 1+e : le]$.

Suppose $w \neq 0$ and so we can choose the fist option, $(e, l) = \left(\frac{w}{x}, \frac{v-x}{w}\right)$. Then, using the fact that $\pi_X(e, l) = [1 : 1+le : e]$, we obtain,

$$f \circ \pi_X \colon (e, l) \mapsto \big[\alpha \colon 2l + \beta + e(3l^2 + 4\beta l + \beta^2 - \alpha^2) + le^2((l+\beta)^2 - \alpha^2 \\ : \ (1 + \beta e + le)^2 - (\alpha e)^2\big]. \quad (5.40)$$

On the exceptional divisor E_1 (i.e., when $e = 0$), the image is $[\alpha : \beta + 2l : 1]$, which parameterize the line $\{x = \alpha, w = 1\}$, and therefore the lift map $\widehat{f}$ is regular on $E_1 \backslash \{l = \infty\}$. In order to take care of $l = \infty$, which corresponds to $w = 0$, let us consider the second coordinate chart $(e, l) = \left(\frac{v-x}{x}, \frac{w}{v-x}\right)$. Then, $\pi_X(e, l) = [1 : 1+e : le]$ and

$$f \circ \pi_X \colon (e, l) \mapsto \big[\alpha l \colon 2 + \beta l + ((1+\beta l)^2 - (\alpha l)^2)e^2 + (3 + 4\beta l + (\beta^2 - \alpha^2)l^2)e \\ : \ l((1 + e + \beta le)^2 - (\alpha le)^2\big].$$

To obtain E_1, we make $e = 0$, and obtain $[\alpha l : \beta l + 2 : l]$. We are concerned with the case of $w = 0$, which corresponds to $l = 0$, and thus we get the point $[0 : 1 : 0]$. This completes the proof of regularity of $\widehat{f}$ on E_1. The $\widehat{f}$ image of E_1 is the strict transform of $\{x = \alpha, w = 1\} \cup \{[0 : 1 : 0]\}$, which proves the first assertion.

Using (5.40) and the first coordinate chart,$(e, l) = \left(\frac{w}{x}, \frac{v-x}{w}\right)$, the lifted map $\widehat{f}$ is given by,

$$\widehat{f} \colon (e, l) \mapsto \left(\frac{(1 + \beta e + le)^2 - (\alpha e)^2}{\alpha}, \frac{(2 + le)l + (1 + le)(\beta - \alpha)}{1 + e(l + \beta - \alpha)}\right), \quad (5.41)$$

for (e, l) such that the point $[x : v : w] \in \mathbb{P}^2$ corresponding to (e, l) does not lie on the vertical line $\{x = 0\}$ nor on the line at infinity, and the image of $[x : v : w]$ does not lie on the vertical line (that is $f_0[x : v : w] \neq 0$).

Let $[x : v : w]$ be on the line L'. Then, $v = -(\alpha + \beta)$, $w = 1$, and therefore the corresponding point in X is $(e,l) = \left(\frac{1}{x}, -(\alpha+\beta+x)\right)$. Using (5.41), we obtain the image $\widehat{f}(e,l) = \left(0, \frac{x^2 - 2\alpha(\alpha+\beta)}{2\alpha}\right)$, which is a point of E_1. Therefor, the image of the strict transform of L', does not hit indeterminacies, and hence we are done with the second assertion.

Now, consider the point I_2 and the exceptional divisor E_2. Similar to above, let $R = [x : v : w]$, where $x \neq 0$. The two ways to pick the coordinate chart are;

1. $(e,l) = \left(\frac{w}{x}, \frac{v+x}{w}\right)$, assuming $w \neq 0$, in which case $\pi_X(e,l) = [1 : le - 1 : e]$.
2. $(e,l) = \left(\frac{v+x}{x}, \frac{w}{v+x}\right)$, assuming $x + v \neq 0$, in which case $\pi_X(e,l) = [1 : e - 1 : le]$.

Suppose $w \neq 0$ and so we can choose the first option, $(e,l) = \left(\frac{w}{x}, \frac{v+x}{w}\right)$. Then, using the fact that $\pi_X(e,l) = [1 : le - 1 : e]$, we obtain,

$$f \circ \pi_X \colon (e,l) \mapsto \big[\alpha \colon 2l + \beta - e(3l^2 + 4\beta l + \beta^2 - \alpha^2) + le^2((l+\beta)^2 - \alpha^2 \\ : (1 - \beta e - le)^2 - (\alpha e)^2\big]. \quad (5.42)$$

On the exceptional divisor E_2 (i.e., when $e = 0$), the image is $[\alpha : \beta + 2l : 1]$, which parameterize the line $\{x = \alpha, w = 1\}$, and therefore the lift map $\widehat{f}$ is regular on $E_2 \backslash \{l = \infty\}$. In order to take care of $l = \infty$, which corresponds to $w = 0$, let us consider the second coordinate chart $(e,l) = \left(\frac{v+x}{x}, \frac{w}{v+x}\right)$. Then, $\pi_X(e,l) = [1 : e - 1 : le]$ and

$$f \circ \pi_X \colon (e,l) \mapsto \big[\alpha l \colon 2 + \beta l + ((1+\beta l)^2 - (\alpha l)^2)e^2 - (3 + 4\beta l + (\beta^2 - \alpha^2)l^2)e \\ : \; l((1 - e - \beta le)^2 - (\alpha le)^2\big].$$

To obtain E_2, we make $e = 0$, and obtain $[\alpha l : \beta l + 2 : l]$. We are concerned with the case of $w = 0$, which corresponds to $l = 0$, and thus we get the point $[0 : 1 : 0]$. Thus $\widehat{f}$ is regular on E_2. The $\widehat{f}$ image of E_2 is the strict transform of $\{x = \alpha, w = 1\} \cup \{[0 : 1 : 0]\}$, which proves the third assertion.

Using (5.42) and the first coordinate chart, $(e,l) = \left(\frac{w}{x}, \frac{v+x}{w}\right)$, the lifted map $\widehat{f}$ is given by,

$$\widehat{f}\colon (e,l) \mapsto \left(\frac{(1-\beta e - le)^2 - (\alpha e)^2}{\alpha}, \frac{(2-le)l + (1-le)(\beta+\alpha)}{1 - e(l+\beta+\alpha)}\right), \tag{5.43}$$

for (e,l) such that the point $[x : v : w] \in \mathbb{P}^2$ corresponding to (e,l) does not lie on the vertical line $\{x = 0\}$ nor on the line at infinity, and the image of $[x : v : w]$ does not lie on the vertical line (that is $f_0\,[x : v : w] \neq 0$).

Now let $[x : v : w]$ be on the line L. Then, $v = -(\beta - \alpha)$, $w = 1$, and therefore the corresponding point in X is $(e,l) = \left(\frac{1}{x}, -(\beta - x - \alpha)\right)$. Using (5.43), we obtain the image $\widehat{f}(e,l) = \left(0, \frac{-x^2 + 2\alpha(\alpha-\beta)}{2\alpha}\right)$, which is a point of E_2. Therefor, the image of the strict transform of L, does not hit indeterminacies. This completes the last assertion. □

Now let us examine the images of the map $\widehat{f}$. If (e,l) is not in $E_1 \cup E_2$, then the image $\widehat{f}(e,l)$ is the strict transform of the point $f\,[x : v : w]$, where $[x : v : w]$ is the strict transform of (e,l). Note that the preimage of the line at infinity is the union of L', L, and the line at infinity, and by Proposition 5.4 and Proposition 5.6 we obtain that the image $\widehat{f}(e,l)$ does not hit the indeterminacies on the line at infinity. Similarly, we can show $\widehat{f}(e,l)$ does not hit indeterminacies on vertical line. If $(e,l) \in E_1 \cup E_2$, then by above proposition, the $\widehat{f}$ image does not hit the strict transforms of I_0 and the vertical line (excluding the point $[0 : 1 : 0]$). So, we obtain the next corollary:

Corollary 5.1. *The map $\widehat{f}$ avoids the strict transform of the indeterminacy point I_0, and no points not in the strict transform of the vertical line $\{x = 0, w = 1\}$ are mapped to the strict transform of the vertical line $\{x = 0, w = 1\}$.*

Now we are ready to prove the algebraic stability.

Proof of Theorem 5.2. Let $X = \mathrm{BL}_{I_1,I_2}(\mathbb{P}^2)$ be the blow up of $\mathbb{P}^2$ at points I_1, I_2. We will show the sequence $\left\{\widehat{f}_n\right\}$ of functions on X is algebraically stable. To prove it, we need to show that no algebraic curve collapse to an indeterminacy point. Let us denote the lines L', L corresponding to map f_n, by L'_n, L_n, respectively.

First, note that the points I_0, I_1, I_2 are common indeterminacy points for all f_n, and the other indeterminacies occur on the vertical line $\{x = 0\}$. By Corollary 5.1, the strict transform of the point I_0 is not in the image of any map $\widehat{f}_n$, and therefore no curve will ever collapse to the strict transform of I_0, under any iteration.

Points not in the strict transform of the vertical line $\{x = 0, w = 1\}$ will never map to the strict transform of $\{x = 0, w = 1\}$, by Corollary 5.1. Therefore, no curve will collapse to a point on $\{x = 0, w = 1\}$. In particular, no algebraic curve will collapse to an indeterminacy point on the vertical line.

Therefore, $\left\{\widehat{f}_n\right\}$ avoids indeterminacies on the vertical line and on I_0. Note that the images of points not on lines L'_n and L_n do not collapse to I_1 and I_2, and by Proposition 5.6, the strict transforms of L'_n and L_n hit no indeterminacies. Therefore, the sequence of maps $\left\{\widehat{f}_n\right\}$ is algebraically stable. □

Proof of Theorem 5.3. The rational maps (5.32), (5.36) associated to $\mathcal{G}_\omega$, $\widetilde{\mathcal{G}}_\omega$, are of the form (5.1) if their corresponding $\alpha_\omega \neq 0$. The condition $\alpha_\omega \neq 0$ becomes $y + z, y + u, z + u$ are non-zero, in the case of $\mathcal{G}_\omega$, by (5.33) and $y + z + q + t, y + u + q + s, z + u + q + r$ are non-zero, in the case of $\widetilde{\mathcal{G}}_\omega$, by (5.35). Now the result follows directly from Theorem 5.2. □

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APPENDIX A

CALCULATING INVERSES IN GROUP ALGEBRA

A.1 For Finite Abelian Groups

Let G be an abelian group. Then all irreducible representations of G are one dimensional. Let $\hat{G}$ denote the complete set of all irreducible representations of G. It is known that the map

$$\mathbb{C}[G] \to \bigoplus_{\rho \in \hat{G}} \mathbb{C}$$

$$\phi = \sum_{g \in G} \phi_g g \mapsto \hat{\phi} = \left(\hat{\phi}_\rho\right)_{\rho \in \hat{G}},$$

where $\hat{\phi}_\rho = \sum_{g \in G} \phi_g \rho(g)$, is an isomorphism of algebras. In order to calculate ϕ^{-1}, suppose $\phi\psi = 1$. Then applying the above map, we get $\hat{\phi}_\rho \hat{\psi}_\rho = 1$ for all $\rho \in \hat{G}$. Thus for all $\rho \in \hat{G}$,

$$\hat{\psi}_\rho = 1/\hat{\phi}_\rho. \tag{A.1}$$

This shows that the necessary and sufficient condition for ϕ to be invertible is $\hat{\phi}_\rho \neq 0$ for all $\rho \in \hat{G}$. In other words,

$$\prod_{\rho \in \hat{G}} \hat{\phi}_\rho \neq 0. \tag{A.2}$$

Now we will restrict our calculations to the situations where $G = \mathbb{Z}_2^n$ for $n \in \mathbb{N}$. Note that each irreducible representation of $\mathbb{Z}_2^n$ is of the form $\rho_{i_1 i_2 \ldots i_n}$. Here $\rho_{i_1 i_2 \ldots i_n}$ is defined by

$$\rho_{i_1 i_2 \ldots i_n}(e_j) = (-1)^{i_j}, \tag{A.3}$$

where e_j is the n-tuple in G with all but j-th entry are 0. In this case we denote the coefficient of $(i_1, i_2, \ldots, i_n)$ in ϕ by $\phi_{i_1 i_2 \ldots i_n}$ and $\hat{\phi}_{\rho_{i_1 i_2 \ldots i_n}}$ by $\hat{\phi}_{i_1 i_2 \ldots i_n}$.

A.2 The Group $\mathbb{Z}_2$ of Order Two

First consider $n = 1$. That is, the group $\mathbb{Z}_2$. Let $\phi = \sum_{g \in \mathbb{Z}_2} \phi_g g \in \mathbb{C}[\mathbb{Z}_2]$. Using (A.3) we get,

$$\begin{aligned} \hat{\phi}_0 &= \phi_0 + \phi_1, \\ \hat{\phi}_1 &= \phi_0 - \phi_1. \end{aligned} \tag{A.4}$$

Suppose $\psi = \sum_{g \in \mathbb{Z}_2} \psi_g g$ is the inverse of ϕ. Then, by (A.3) and (A.1), we obtain

$$\begin{aligned} \psi_0 + \psi_1 &= 1/\hat{\phi}_0, \\ \psi_0 - \psi_1 &= 1/\hat{\phi}_1, \end{aligned}$$

and solving these equations gives,

$$\begin{aligned} \psi_0 &= \frac{1}{2}\left(1/\hat{\phi}_0 + 1/\hat{\phi}_1\right), \\ \psi_1 &= \frac{1}{2}\left(1/\hat{\phi}_0 - 1/\hat{\phi}_1\right). \end{aligned} \tag{A.5}$$

A.3 The Klein Group $\mathbb{Z}_2^2$

Now consider $n = 2$. That is, the group $\mathbb{Z}_2^2$. Let $\phi = \sum_{g \in \mathbb{Z}_2^2} \phi_g g \in \mathbb{C}[\mathbb{Z}_2^2]$. Using (A.3) we get,

$$\begin{aligned} \hat{\phi}_{00} &= \phi_{00} + \phi_{10} + \phi_{01} + \phi_{11}, \\ \hat{\phi}_{10} &= \phi_{00} - \phi_{10} + \phi_{01} - \phi_{11}, \\ \hat{\phi}_{01} &= \phi_{00} + \phi_{10} - \phi_{01} - \phi_{11}, \\ \hat{\phi}_{11} &= \phi_{00} - \phi_{10} - \phi_{01} + \phi_{11}. \end{aligned} \tag{A.6}$$

Suppose $\psi = \sum_{g \in \mathbb{Z}_2^2} \psi_g g$ is the inverse of ϕ. Then, by (A.3) and (A.1), we obtain

$$\psi_{00} = \frac{1}{4}\left(1/\hat{\phi}_{00} + 1/\hat{\phi}_{10} + 1/\hat{\phi}_{01} + 1/\hat{\phi}_{11}\right),$$

$$
\begin{aligned}
\psi_{10} &= \frac{1}{4}\left(1/\hat{\phi}_{00} - 1/\hat{\phi}_{10} + 1/\hat{\phi}_{01} - 1/\hat{\phi}_{11}\right),\\
\psi_{01} &= \frac{1}{4}\left(1/\hat{\phi}_{00} + 1/\hat{\phi}_{10} - 1/\hat{\phi}_{01} - 1/\hat{\phi}_{11}\right),\\
\psi_{11} &= \frac{1}{4}\left(1/\hat{\phi}_{00} - 1/\hat{\phi}_{10} - 1/\hat{\phi}_{01} + 1/\hat{\phi}_{11}\right).
\end{aligned} \tag{A.7}
$$

A.4 The Group $\mathbb{Z}_2^3$

Finally consider $n = 3$. That is, the group $\mathbb{Z}_2^3$. Let $\phi = \sum_{g\in\mathbb{Z}_2^3} \phi_g g \in \mathbb{C}[\mathbb{Z}_2^3]$. Using (A.3) we get,

$$
\begin{aligned}
\hat{\phi}_{000} &= \phi_{000} + \phi_{100} + \phi_{010} + \phi_{001} + \phi_{110} + \phi_{101} + \phi_{011} + \phi_{111},\\
\hat{\phi}_{100} &= \phi_{000} - \phi_{100} + \phi_{010} + \phi_{001} - \phi_{110} - \phi_{101} + \phi_{011} - \phi_{111},\\
\hat{\phi}_{010} &= \phi_{000} + \phi_{100} - \phi_{010} + \phi_{001} - \phi_{110} + \phi_{101} - \phi_{011} - \phi_{111},\\
\hat{\phi}_{001} &= \phi_{000} + \phi_{100} + \phi_{010} - \phi_{001} + \phi_{110} - \phi_{101} - \phi_{011} - \phi_{111},\\
\hat{\phi}_{110} &= \phi_{000} - \phi_{100} - \phi_{010} + \phi_{001} + \phi_{110} - \phi_{101} - \phi_{011} + \phi_{111},\\
\hat{\phi}_{101} &= \phi_{000} - \phi_{100} + \phi_{010} - \phi_{001} - \phi_{110} + \phi_{101} - \phi_{011} + \phi_{111},\\
\hat{\phi}_{011} &= \phi_{000} + \phi_{100} - \phi_{010} - \phi_{001} - \phi_{110} - \phi_{101} + \phi_{011} + \phi_{111},\\
\hat{\phi}_{111} &= \phi_{000} - \phi_{100} - \phi_{010} - \phi_{001} + \phi_{110} + \phi_{101} + \phi_{011} - \phi_{111}.
\end{aligned} \tag{A.8}
$$

Suppose $\psi = \sum_{g\in\mathbb{Z}_2^3} \psi_g g$ is the inverse of ϕ. Then, by (A.3) and (A.1), we obtain

$$
\begin{aligned}
\psi_{000} &= \frac{1}{8}\left(1/\hat{\phi}_{000} + 1/\hat{\phi}_{100} + 1/\hat{\phi}_{010} + 1/\hat{\phi}_{001} + 1/\hat{\phi}_{110} + 1/\hat{\phi}_{101} + 1/\hat{\phi}_{011} + 1/\hat{\phi}_{111}\right),\\
\psi_{100} &= \frac{1}{8}\left(1/\hat{\phi}_{000} - 1/\hat{\phi}_{100} + 1/\hat{\phi}_{010} + 1/\hat{\phi}_{001} - 1/\hat{\phi}_{110} - 1/\hat{\phi}_{101} + 1/\hat{\phi}_{011} - 1/\hat{\phi}_{111}\right),\\
\psi_{010} &= \frac{1}{8}\left(1/\hat{\phi}_{000} + 1/\hat{\phi}_{100} - 1/\hat{\phi}_{010} + 1/\hat{\phi}_{001} - 1/\hat{\phi}_{110} + 1/\hat{\phi}_{101} - 1/\hat{\phi}_{011} - 1/\hat{\phi}_{111}\right),\\
\psi_{001} &= \frac{1}{8}\left(1/\hat{\phi}_{000} + 1/\hat{\phi}_{100} + 1/\hat{\phi}_{010} - 1/\hat{\phi}_{001} + 1/\hat{\phi}_{110} - 1/\hat{\phi}_{101} - 1/\hat{\phi}_{011} - 1/\hat{\phi}_{111}\right),\\
\psi_{110} &= \frac{1}{8}\left(1/\hat{\phi}_{000} - 1/\hat{\phi}_{100} - 1/\hat{\phi}_{010} + 1/\hat{\phi}_{001} + 1/\hat{\phi}_{110} - 1/\hat{\phi}_{101} - 1/\hat{\phi}_{011} + 1/\hat{\phi}_{111}\right),\\
\psi_{101} &= \frac{1}{8}\left(1/\hat{\phi}_{000} - 1/\hat{\phi}_{100} + 1/\hat{\phi}_{010} - 1/\hat{\phi}_{001} - 1/\hat{\phi}_{110} + 1/\hat{\phi}_{101} - 1/\hat{\phi}_{011} + 1/\hat{\phi}_{111}\right),
\end{aligned}
$$

$$\psi_{011} = \frac{1}{8}\left(1/\hat{\phi}_{000} + 1/\hat{\phi}_{100} - 1/\hat{\phi}_{010} - 1/\hat{\phi}_{001} - 1/\hat{\phi}_{110} - 1/\hat{\phi}_{101} + 1/\hat{\phi}_{011} + 1/\hat{\phi}_{111}\right),$$
$$\psi_{111} = \frac{1}{8}\left(1/\hat{\phi}_{000} - 1/\hat{\phi}_{100} - 1/\hat{\phi}_{010} - 1/\hat{\phi}_{001} + 1/\hat{\phi}_{110} + 1/\hat{\phi}_{101} + 1/\hat{\phi}_{011} - 1/\hat{\phi}_{111}\right). \quad \text{(A.9)}$$

APPENDIX B

COMPLEX DYNAMICS

Let $\mathbb{P}^2$ be the 2-dimensional complex projective space and let $[x : v : w]$ be a generic point on it. Thus, $[0 : 0 : 0]$ is an undefined point and $[\lambda x : \lambda v : \lambda w] = [x : v : w]$, for any $\lambda \in \mathbb{C} \setminus \{0\}$. For a self map f on $\mathbb{P}^2$, denote the coordinate functions of f by f_0, f_1, and f_2. That is, $f = [f_0 : f_1 : f_2]$.

B.1 Rational Maps

Consider a self map f on $\mathbb{P}^2$ given by polynomial functions f_0, f_1, and f_2. The points which are not in the domain of f are called the *indeterminacy points*. Thus, the indeterminacy points are the points for which f_0, f_1, and f_2 are simultaneously zero. Observe that the set of indeterminacies is a *Zariski closed* set (i.e., an algebraic subset of the ambient space). This idea can be generalized to any projective surface as below.

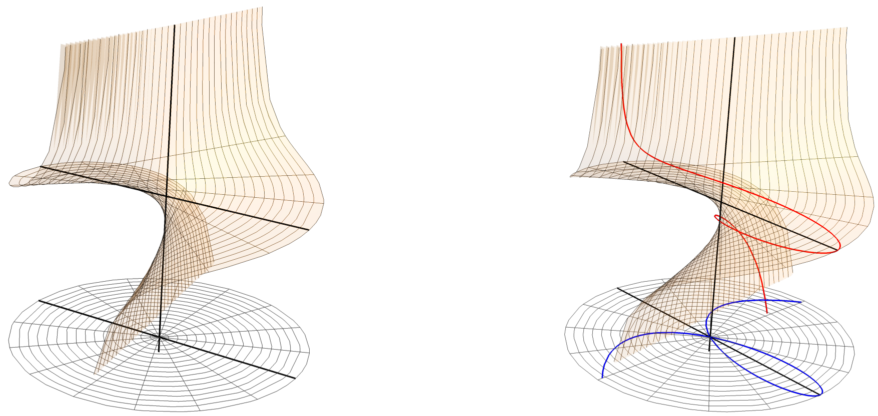
Definition B.1. *Let X, Y be two smooth projective surfaces and let U, V be Zariski open subsets of them,respectively. A map $f : U \to V$ is said to be a rational map if it is given by polynomials in some coordinate system. We denote this by $f : X \dashrightarrow Y$.*

In the case of $U = X$, i.e., there are no indeterminacies in X, then the rational map is said to be *regular*. A non-regular rational map can be restricted to a subsurface to obtain a regular rational map of the subspace.

B.2 Blow-ups

There are situation where a map has an indeterminacy and it is useful to remove (or get rid of) this indeterminacy, or there is a curve with a singularity that we wish to remove. The technique of blow-up comes handy in these situations. We will define blow-ups for $\mathbb{C}^2$ and then it naturally extends to $\mathbb{P}^2$.

Let $p = (x_0, v_0) \in \mathbb{C}^2$. The blow-up of $\mathbb{C}^2$ at point p, denoted by $\mathrm{BL}_p(\mathbb{C}^2)$, is the space obtained by attaching a projective line to $\mathbb{C}^2$ at the point p, which represents the tangent direction



(a) Blow-up of a point.

(b) Strict transform of a curve.

Figure B.1: Blow-up

at p. Thus, $\mathrm{BL}_p(\mathbb{C}^2) = \{((x,v),[\lambda:\mu]) \in \mathbb{C}^2 \times \mathbb{P} \mid \lambda(x-x_0) = \mu(y-y_0)\}$. The projective line that is attached to the surface is called the *exceptional divisor* and the space $\mathrm{BL}_p(\mathbb{C}^2)$ is called the *rational variety*. The point $((x,v),[\lambda:\mu])$ in $\mathrm{BL}_p(\mathbb{C}^2)$ is identified with the point (x,v) in $\mathbb{C}^2$. This identification, π, is called the *blow-down map*, where it collapses the exceptional divisor to the point p. The Figure B.1 (a) represents the blow-up graphically.

For a curve C in $\mathbb{C}^2$, the blow-up of C is called the *strict transform* of C. Thus, the strict transform of C is given by $\overline{\pi^{-1}(C \setminus \{p\})}$. The Figure B.1 (b) represents a curve (in blue color) with a singularity at the point that is blown-up and its strict transform (in red color). It shows how blow-up can be used to deal with singularities, graphically. See the appendix of [DGL21] and the book [GH78] for more on blow-ups.