MATHEMATICAL ANALYSIS OF THE PRIMITIVE EQUATIONS WITH ROTATION

A Dissertation

by

QUYUAN LIN

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Edriss S. Titi
Committee Members,	Jean-Luc Guermond
	Peter Howard
	R. Lee Panetta
Head of Department,	Sarah Witherspoon

May 2021

Major Subject: Mathematics

ABSTRACT

Large planetary scale dynamics of the oceans and the atmosphere is governed by the primitive equations (PEs). It is well-known that the $3D$ viscous PEs is globally (in time) well-posed in Sobolev spaces. On the other hand, the inviscid primitive equations (IPEs) without rotation is known to be ill-posed in all Sobolev spaces, and some of its smooth solutions can form singularities in finite time. In this thesis, the above results are extended in the presence of rotation (Coriolis force). More specifically, certain finite-time blowup solutions to the IPEs with rotation are constructed, and it is established that the IPEs with rotation is ill-posed in the sense that the perturbation around a certain steady state background flow is both linearly and nonlinearly ill-posed in all Sobolev spaces, and is linearly ill-posed in Gevrey class of order $s > 1$.

Although the IPEs is ill-posed in Sobolev spaces and Gevrey class of order $s > 1$, it is shown in this thesis that the $3D$ IPEs is locally (in time) well-posed in the space of analytic functions, i.e., the Gevrey class of order $s = 1$, for a short interval of time that is independent of the rotation rate. By the comparison between the $3D$ IPEs and the $2D$ Euler equations, one can establish the long-time existene of solutions to the $3D$ IPEs provided the analytic norm of the initial baroclinic mode is small enough, while the initial barotropic mode can be large. Moreover, one can show that, in the case of "well-prepared" analytic initial data (only the Sobolev norm of the baroclinic mode is small depending on the rotation rate, while the analytic norm can be large), the regularizing effect of the Coriolis force by providing a lower bound for the life-span of the solutions which grows toward infinity with the rotation rate. The latter is achieved by a delicate analysis of a simple limit resonant system whose solution approximates the corresponding solution of the $3D$ IPEs with the same initial data.

The PEs with only vertical viscosity (also called the hydrostatic Navier-Stokes equations) is believed to be ill-posed in Sobolev spaces. To overcome the potential ill-posedness, some weak dissipations are introduced in the horizontal directions, which are the linear (Rayleigh-like friction) damping terms. With these damping terms, it is established that this system is locally well-posed with general Sobolev initial data and globally well-posed with small Sobolev initial data. In order to study the possible finite-time blow-up and to give a reliable numerical regularization, it is proposed to study the Voigt α-regularization of this model, which is an inviscid regularization. One is able to establish the global well-posedness of the regularized model for arbitrary Sobolev initial data. In addition, it is shown that the solutions of the regularized model converge to those of the original model on the interval of the existence of the latter, as $\alpha \to 0$. Based on this convergence result, a blowup criterion of the original model is established.

DEDICATION

To my mother, my father, and my loved one.

ACKNOWLEDGMENTS

My sincere and deep gratitude goes first and foremost to my advisor, Dr. Edriss S. Titi, who has been a tremendous mentor and has given me a lot of patient guidance. His advice on both mathematical research and career has been invaluable.

I would like to thank Dr. Chongsheng Cao, Dr. Tej-eddine Ghoul, and Dr. Slim Ibrahim for their generous help, time, and many inspiring discussions in my research. Working with them gives me an enjoyable experience on collaborating with other mathematicians.

My gratitude also goes to Dr. Jean-Luc Guermond, Dr. Peter Howard and Dr. R. Lee Panetta for serving on committee and giving several important courses during my graduate study.

I convey special thank to Dr. Peter Howard and Dr. Peter Kuchment, who have been always patient and helpful when I ask some general questions in mathematics and writing.

I would also like to thank Mr. Hussain Ibdah, Dr. Xin Liu, and Dr. Cecilia Mondaini for their kind help in some discussions about mathematics.

I am very grateful to Dr. Jianxin Zhou, who contributes a lot to the exchange program between my undergraduate school, Beihang University, and Texas A&M University. This program led me to the graduate study in Texas A&M University.

I appreciate all the help from the Department of Mathematics at Texas A&M University. It would be impossible for me to pursue a research career without the support of the department. I would especially like to thank former Associate Head for Graduate Programs Dr. Peter Howard, current Associate Head for Graduate Programs Dr. Andrea Bonito, and Mrs. Monique Stewart in Graduate Programs for their support, guidance, and help on my graduate study.

I want to thank the Texas A&M University Office of Graduate and Professional Studies to allow me to construct this LaTeX thesis template. Special thanks to JaeCee Crawford, Amy Motquin, Ashley Schmitt, Rachel Krolczyk, and Roberta Caton for carefully reviewing this material.

Last but not least, I want to express my deepest gratitude to my family and friends. My work would not have been possible without their warm love, continued patience, and endless support.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Professor Edriss S. Titi of the Department of Mathematics. All other work conducted for the dissertation was completed by the student independently.

Funding Sources

Graduate study was supported by the Department of Mathematics from Texas A&M University.

NOMENCLATURE

PEs	Primitive equations
IPEs	Inviscid primitive equations

TABLE OF CONTENTS

Page

1. INTRODUCTION AND LITERATURE REVIEW

1.1 The Primitive Equations of Oceanic and Atmospheric Dynamics

In the atmospheric and oceanic dynamics, the Boussinesq approximation model is accepted as the fundamental model that governs their motion. In the case of incompressible flows, the dimensionless version of this system reads as

$$\begin{cases} \partial_t \mathcal{U} + \mathcal{U}\cdot\nabla\mathcal{U} - \nu_h^B \Delta_h \mathcal{U} - \nu_z^B \partial_{zz}\mathcal{U} + \Omega e_3 \times \mathcal{U} + \nabla p + T e_3 = 0, \\ \nabla\cdot\mathcal{U} = 0, \\ \partial_t T + \mathcal{U}\cdot\nabla T - \kappa_h^B \Delta_h T - \kappa_z^B \partial_{zz} T = 0 \end{cases} \tag{1.1}$$

in the domain $\mathcal{D} \subseteq \mathbb{R}^3$, where the velocity field $\mathcal{U} = (\mathcal{V}, w)$ with horizontal velocity $\mathcal{V} = (u, v)$ and vertical velocity w, the pressure p, and the temperature T are the unknown quantities which are functions of the independent time and space variables (t, x, y, z). The $3D$ gradient is denoted by $\nabla = (\partial_x, \partial_y, \partial_z)$, and the $2D$ horizontal gradient and Laplacian are denoted by $\nabla_h = (\partial_x, \partial_y)$ and $\Delta_h = \partial_{xx} + \partial_{yy}$, respectively. The nonnegative constants $\nu_h^B, \nu_z^B, \kappa_h^B$ and κ_z^B are the horizontal viscosity, the vertical viscosity, the horizontal diffusivity and the vertical diffusivity coefficients, respectively. The parameter $\Omega \in \mathbb{R}$ stands for the speed of rotation in the Coriolis force, and $e_3 = (0, 0, 1)$ is the unit vector in the z direction. All the quantities mentioned above are dimensionless.

For planetary scale oceanic and atmospheric dynamics the vertical scale (a few kilometers for the ocean, 10-20 kilometers for the atmosphere) is much smaller than the horizontal scales (several thousands of kilometers). By virtue of this thinness (the ratio of the vertical height or depth to horizontal width is small) of the ocean and atmosphere, the above Boussinesq equations (1.1) is considered in a thin domain $\mathcal{D}_\epsilon := \{(x, y, z) : 0 \le z \le \epsilon, (x, y) \in \mathbb{R}^2\}$ for ϵ a small positive parameter. Following Azérad-Guillén [3] and Li-Titi [69], it is assumed that $(\nu_h^B, \nu_z^B, \kappa_h^B, \kappa_z^B) = (\nu_h, \epsilon^2\nu_z, \kappa_h, \epsilon^2\kappa_z)$, where $(\nu_h, \nu_z, \kappa_h, \kappa_z) = \mathcal{O}(1)$. By rescaling, the following new unknowns are introduced

$$\begin{cases} \mathcal{V}_\epsilon(t,x,y,z) = \mathcal{V}(t,x,y,\epsilon z), \quad w_\epsilon(t,x,y,z) = \frac{1}{\epsilon} w(t,x,y,\epsilon z), \\ p_\epsilon(t,x,y,z) = p(t,x,y,\epsilon z), \quad T_\epsilon(t,x,y,z) = \epsilon T(t,x,y,\epsilon z), \end{cases}$$

for $(x,y,z) \in \mathcal{D} := \{(x,y,z) : 0 \le z \le 1, (x,y) \in \mathbb{R}^2\}$. Boussinesq system (1.1), defined on the ϵ-dependent domain $\mathcal{D}_\epsilon$, can be transformed to the following scaled Boussinesq equations

$$\begin{cases} \partial_t \mathcal{V}_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h \mathcal{V}_\epsilon + w_\epsilon \partial_z \mathcal{V}_\epsilon - \nu_h \Delta_h \mathcal{V}_\epsilon - \nu_z \partial_{zz} \mathcal{V}_\epsilon + \Omega \mathcal{V}_\epsilon^\perp + \nabla_h p_\epsilon = 0, \\ \epsilon^2 (\partial_t w_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h w_\epsilon + w_\epsilon \partial_z w_\epsilon - \nu_h \Delta_h w_\epsilon - \nu_z \partial_{zz} w_\epsilon) + \partial_z p_\epsilon + T_\epsilon = 0, \\ \nabla_h \cdot \mathcal{V}_\epsilon + \partial_z w_\epsilon = 0, \\ \partial_t T_\epsilon + \mathcal{V}_\epsilon \cdot \nabla_h T_\epsilon + w_\epsilon \partial_z T_\epsilon - \kappa_h \Delta_h T_\epsilon - \kappa_z \partial_{zz} T_\epsilon = 0 \end{cases} \tag{1.2}$$

in the fixed domain $\mathcal{D}$. Here the notation $\mathcal{V}_\epsilon^\perp = (-v_\epsilon, u_\epsilon)$ is used.

By taking the formal limit $\epsilon \to 0^+$, and assuming that $(\mathcal{V}_\epsilon, w_\epsilon, p_\epsilon, T_\epsilon)$ converges to $(\mathcal{V}, w, p, T)$ in a suitable sense, the vertical momentum equation in (1.2) degenerates to the following hydrostatic balance

$$\partial_z p + T = 0.$$

As a result, one obtains the following dimensionless system, known as the primitive equations (PEs)

$$\begin{cases} \partial_t \mathcal{V} + \mathcal{V} \cdot \nabla_h \mathcal{V} + w \partial_z \mathcal{V} - \nu_h \Delta_h \mathcal{V} - \nu_z \partial_{zz} \mathcal{V} + \Omega \mathcal{V}^\perp + \nabla_h p = 0, \\ \partial_z p + T = 0, \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0, \\ \partial_t T + \mathcal{V} \cdot \nabla_h T + w \partial_z T - \kappa_h \Delta_h T - \kappa_z \partial_{zz} T = 0 \end{cases} \tag{1.3}$$

in the fixed domain $\mathcal{D}$. Here the notation $\mathcal{V}^\perp = (-v, u)$ is used.

The above small aspect ratio limit, from system (1.2) to system (1.3) with $\epsilon \to 0^+$, can be rigorously justified. The weak convergence of such limit was proved in Azérad-Guillén [3], while the strong convergence was established by Li-Titi [69] with error estimates in terms of the small aspect ratio ϵ.

Investigating the well-posedness of the PEs is a crucial step for understanding, from both the physical and mathematical points of view, the validity and limitation of their derivation. By well-posedness for a initial value problem, it means:

1. given initial data of a chosen space, there exists a time $\mathcal{T} > 0$ such that a solution exists in this space for all time $t \in [0, \mathcal{T}]$;
2. the solution is unique;
3. the solution map is continuous with respect to initial data.

This notion of well-posedness is introduced by Hadamard [47]. If any one of these three conditions is violated, then the problem is ill-posed. If the first statement is true for any positive time $\mathcal{T} > 0$, then one says this problem is globally (in time) well-posed. If the solution leaves the space at a finite time, i.e., the corresponding norm becomes infinity at a finite time, then one says the solution blows up in finite time, and this problem is only locally (in time) well-posed.

1.2 History and Introduction

The PEs form a fundamental block in models of the oceanic and atmospheric dynamics, and have been a standard framework for studying geostrophic adjustment of frontal anomalies in a rotating continuously stratified fluid of strictly rectilinear fronts and jets, see, e.g., Blumen [11], Gill [42, 43], Haltiner–Williams [50], Hermann–Owens [49], Holton [52], Kuo–Polvani [63], Lewandowski [65], Majda [75], Pedlosky [81], Plougonven–Zeitlin [82], Rossby [84], Vallis [89], Washington–Parkinson [90], Zeng [92], and references therein.

After the PEs were formally derived, it was commonly believed that their mathematical analysis is much harder than the original Navier-Stokes equations (or Boussinesq equations). The main difficulty in the study of the PEs is on the loss of one horizontal derivative in the vertical velocity

w. To be more specific, w does not have an evolution equation, and can only be derived as

$$w(t, x, y, z) = -\int_0^z \nabla_h \cdot \mathcal{V}(t, x, y, s) ds \tag{1.4}$$

through the divergence free condition (third equation in system (1.3)) and the boundary conditions on w ($w = 0$ at $z = 0$ and $z = 1$). The expression of w in (1.4) clearly shows that there is one loss of horizontal derivative. However, as one will see below, as long as there is horizontal viscosity ($\nu_h > 0$), the global regularity of the $3D$ PEs can be achieved. Remarkably, the global regularity of the $3D$ Navier-Stokes equations is one of the most famous and challenging mathematical problems.

1.2.1 Viscous Primitive Equations

The mathematical studies of the PEs started by Lions–Temam–Wang [71–73] in the 1990s. They considered the PEs with both full viscosity and full diffusivity ($\nu_h, \nu_z, \kappa_h, \kappa_z > 0$) and established the global existence of weak solutions. The uniqueness of weak solutions to the $3D$ viscous PEs is still an open problem, while the weak solutions to $2D$ viscous PEs (independent of spatial variable y) turn out to be unique, see Bresch et al. [14].

In the context of strong solutions, for the $2D$ case, the local well-posedness was established by Guillén-González et al. [45], and the global existence of strong solutions was proved by Bresch et al. [15] and Temam–Ziane [88]. For the $3D$ case, Cao–Titi [25] separated the velocity field $\mathcal{V}$ into the barotropic mode $\overline{\mathcal{V}}$ (the average in z variable) and baroclinic mode $\widetilde{\mathcal{V}}$ (the fluctuation, i.e., $\widetilde{\mathcal{V}} = \mathcal{V} - \overline{\mathcal{V}}$). By taking advantage of the fact that effectively the unknown pressure is a function of only two spatial horizontal variables x and y, the pressure term does not appear in the evolution equation of the baroclinic mode, and thus the control of L^6 norm can be achieved for the baroclinic mode. Based on this, Cao–Titi [25] firstly established that the PEs with both full viscosity and full diffusivity are globally well-posed with the relevant physical boundary conditions. Such result is also achieved later on by Kobelkov [55], see also the subsequent articles of Kukavica–Ziane [61, 62] for Dirichlet boundary conditions, as well as Hieber–Kashiwabara [51] for some progress towards relaxing the smoothness on the initial data by using the semigroup method.

1.2.2 Inviscid Primitive Equations

When $\nu_h = \nu_z = 0$, the inviscid primitive equations (IPEs) without coupling with the temperature is also called the hydrostatic Euler equations. In the absence of rotation ($\Omega = 0$), the linearized IPEs near certain shear-flows has been shown to be ill-posed in Sobolev spaces by Renardy [83]. Later on, the nonlinear ill-posedness of the IPEs without rotation was established by Han-Kwan and Nguyen in [48], where they built an abstract framework to show that the IPEs is ill-posed in any Sobolev space. These results on the ill-posedness of non-rotating IPEs were extended to the case when the rotation is present ($\Omega \neq 0$) in the work by Ibrahim–Lin–Titi [53].

The linear ill-posedness of the IPEs mentioned above shows that the linearized $2D$ IPEs (as well as the $3D$ case, see Section 3.1), around a special steady state background flow, has unstable solutions of the form $u(t,x,z) = e^{2\pi ikx}e^{\sigma_k t}u_k(z)$, where $\Re\sigma_k = \lambda k$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Such Kelvin-Helmholtz type instability, which is similar to the one appears in the context of vortex sheets (see, e.g., Caflisch–Orellana [16], and the survey paper by Bardos–Titi [9] and reference therein), precludes the construction of solutions in Sobolev spaces for general initial data. To overcome this strong instability, one should consider initial data u_0 that are strongly localized in Fourier, typically for which $|\hat{u}_0(k,z)| \lesssim e^{-\delta|k|^{1/s}}$ with $\delta > 0$ and $s \geq 1$. Such localization condition corresponds to Gevrey class of order s in the x variable. Kelvin-Helmholtz type instability forces us to choose $s = 1$ for the well-posedness result, which is the space of analytic functions. This is consistent with positive results reported in the work of Kukavica–Temam–Vicol–Ziane [59] and in our work Ghoul–Ibrahim–Lin–Titi [41]. Notably, for the Prandtl equations, which have some similarities in the structure with the PEs, is shown by Gérard-Varet and Dormy [39] that its linearization around a special background flow has unstable solutions of similar form, but with $\Re\sigma_k \sim \lambda\sqrt{k}$ for $k \gg 1$ arbitrarily large and some positive $\lambda \in \mathbb{R}_+$. This implies that the optimal Gevrey class order s for Prandtl equation is $s = 2$, which is consistent with the positive results reported by the work of Dietert and Gérard-Varet [33] and the work of Li–Masmoudi–Yang [70]. This shows that the linear instability of the IPEs is “worse" than that of the Prandtl equations.

Due to the ill-posedness discussed above, in order to show the well-posedness of the IPEs, one needs to assume either some special structures (local Rayleigh condition) on the initial data or real analyticity for general initial data, see, e.g., Brenier [12, 13], Grenier [44], Kukavica–Masmoudi–Vicol–Wong [58], Kukavica–Temam–Vicol–Ziane [59], Masmoudi–Wong [76], and our work Ghoul–Ibrahim–Lin–Titi [41]. In particular, the authors in [59] established the local well-posedness of the $3D$ IPEs in the space of analytic functions, but the time of existence they obtained shrinks to zero as the rate of rotation $|\Omega|$ increases toward infinity. This is contrary to the cases of the $3D$ fast rotating Euler, Navier–Stokes and Boussinesq equations, where the limit of fast rotation leads to either strong "dispersion" or averaging mechanism that weakens the nonlinear effects and hence allows for establishing the global regularity result in the Navier-Stokes case, and prolongs the life-span of the solutions in the Euler case, by Babin–Mahalov–Nicolaenko [5–8] (see also the work of Chemin–Desjardines–Gallagher–Grenier [29], Dutrifoy [34], Embid–Majda [36], Ibrahim–Yoneda [54], Koh–Lee–Takada [56], and references therein). In addition, the readers are referred to Babin–Ilyin–Titi [4], Guo–Simon–Titi [46], Kostianko–Titi–Zelik [57], and Liu–Tamdor [74] for simple examples demonstrating the above mechanism. This suggests that one should be able to show that the fast rotation prolongs the life-span of the solution to the $3D$ IPEs. Indeed, it is shown in [41] that the IPEs is locally well-posed in the space of analytic functions with a time interval that is independent of Ω, and the life-span of solutions to the IPEs can be prolonged with fast rotation and "well-prepared" initial data.

By virtue of local well-posedness and long-time existence of solutions to the $3D$ IPEs, the next question is, whether one can show that the solutions exist globally or form singularities in finite time? In the case of non-rotating IPEs, it was proven that smooth solutions to the IPEs can develop singularities in finite time, see Cao–Ibrahim–Nakanishi–Titi [17] and Wong [91]. This result is extended to the case when $\Omega \neq 0$ in [53]. By virtue of the finite-time blowup results, one can conclude that there is no hope to show the global well-posedness of the $3D$ IPEs, even with fast rotation. The optimal result one can achieve is that fast rotation prolongs the life-span of solutions to the $3D$ IPEs.

In summary, for the IPEs with rotation, the following results will be reported in this dissertation:

1. The perturbed rotating IPEs, around a certain steady state background flow depending on the rotation rate Ω, is both linearly and nonlinearly ill-posed in Sobolev spaces, and is linearly ill-posed in Gevrey class of order $s > 1$.

2. The $3D$ rotating IPEs is locally well-posed in the space of analytic functions for a short interval of time that is independent of Ω. This improves the result reported in [59].

3. Finite-time blowup solutions to the rotating IPEs with initial data depending on Ω is constructed, and the explicit upper bound of the time for the blowup is provided.

4. Independently of $|\Omega|$, the life-span of the analytic solutions to the rotating IPEs tends to infinity as the analytic norm of the initial baroclinic mode goes to zero. As a corollary, one can show in this case that the analytic solutions of the $3D$ IPEs converge to the global analytic solutions of the limit system, which is governed by the $2D$ Euler equations.

5. The life-span of the solutions goes toward infinity, with $|\Omega| \to \infty$. This is established for “well-prepared" initial data, namely, when only the Sobolev norm (but not the analytic norm) of the baroclinic mode is small enough, depending on $|\Omega|$. Furthermore, for large $|\Omega|$ and “well-prepared" initial data, one can show that the solution to the $3D$ IPEs is approximated by the solution to a simple limit resonant system with the same initial data.

1.2.3 The Primitive Equations With Weak Dissipation

The global regularity results of the $3D$ PEs mentioned in Section 1.2.1 are in the case of full viscosity and full diffusivity. Motivated by physical and mathematical considerations, it is of great interest to investigate the PEs with partial viscosity and/or partial diffusivity. There have been several mathematical studies of these models. The global existence and uniqueness of strong solutions for the $3D$ PEs with full viscosity ($\nu_h, \nu_z > 0$) and with either only horizontal diffusivity or only vertical diffusivity ($\kappa_h > 0, \kappa_z = 0$ or $\kappa_h = 0, \kappa_z > 0$) have been established by Cao–Titi [26] and Cao–Li–Titi [18, 19]. Concerning partial viscosity, global well-posedness of the $3D$ PEs with only horizontal viscosity ($\nu_h > 0, \nu_z = 0$) and with either only horizontal diffusivity or only vertical diffusivity was established by Cao–Li–Titi in [20–22]. See also the survey paper by Li–Titi [68].

On the other hand, for the $3D$ PEs with only vertical viscosity ($\nu_h = 0, \nu_z > 0$), which without coupling with the temperature is also called the hydrostatic Navier-Stokes equations, there is no results concerning the well-posedness in Sobolev spaces. Indeed, Renardy [83] has indicated, without providing details, that one should be able to show the linear ill-posedness of the PEs with only vertical viscosity, in any Sobolev space, by using matched asymptotics. The reason of ill-posedness is the loss of one horizontal derivative in w as indicated in (1.4), and the loss of horizontal dissipation since $\nu_h = 0$. Therefore, in order to establish well-posedness in Sobolev spaces for general initial data, in addition to vertical viscosity, some additional horizontal dissipative terms are necessary. For this reason, one considers the following reduced $3D$ PEs with weak dissipation

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \epsilon_1 u - \Omega v + \partial_x p - \nu_z \partial_{zz} u = 0, \\ \partial_t v + u\partial_x v + w\partial_z v + \epsilon_1 v + \Omega u - \nu_z \partial_{zz} v = 0, \\ \epsilon_2 w + \partial_z p + T = 0, \\ \partial_x u + \partial_z w = 0, \\ \partial_t T - \kappa_h \partial_{xx} T - \kappa_z \partial_{zz} T + u\partial_x T + w\partial_z T = 0 \end{cases} \tag{1.5}$$

in the thin domain $\{(x,z) : 0 \le z \le 1, x \in \mathbb{R}\}$. The term reduced model means that the relevant physical quantities depend only on two spatial variables x and z. Here $\epsilon_1 > 0$ and $\epsilon_2 > 0$ represent the linear (Rayleigh-like friction) damping coefficients. The consideration of $\epsilon_2 > 0$ is inspired by Samelson–Vallis [85] and Salmon [86, p. 150]. The linear damping term $\epsilon_2 w$ is the key to show well-posedness, since from $\partial_x u = -\partial_z w$, one obtains the horizontal dissipation and therefore is able to overcome the ill-posedness. Accordingly, when $\epsilon_2 > 0$, one can view the term $\epsilon_2 w$ as having a “regularizing" effect, since it annihilates the ill-posedness indicated by Renardy [83] when $\epsilon_2 = 0$. This also indicates that the damping term $\epsilon_2 w$ has a non-negligible effect on the dynamics and leads to a reliable numerical regularization. In terms of physical motivation, the damping terms $\epsilon_1 u$, $\epsilon_1 v$, and $\epsilon_2 w$ can be interpreted as the Rayleigh friction with the bottom of ocean (continental shelf). Such linear damping terms are also considered in the $3D$ Salmon's planetary geostrophic oceanic dynamics model by Cao–Titi [27], where they were able to show global regularity of this model. The consideration $\epsilon_2 > 0$ is crucial in their work since it is well known that when $\epsilon_2 = 0$ the planetary geostrophic model of ocean circulation is ill-posed (see, e.g., [27] and reference therein). This in turn motivated Salmon to introduce the friction term $\epsilon_2 w$, with $\epsilon_2 > 0$, in the planetary geostrophic model to overcome this problem. Consequently, this provides an additional motivation for taking $\epsilon_2 > 0$ in our system. It is shown in the work by Cao–Lin–Titi [23] that system (1.5) is locally well-posed with general Sobolev initial data and globally well-posed with small Sobolev initial data.

From a mathematical perspective, system (1.5) with $\Omega = 0, v \equiv 0, T \equiv 0$ is reminiscent of the famous Prandtl system in the upper half space. Ill-posedness of Prandtl system in Sobolev spaces was established by Gérard-Varet and Dormy [39] (see more details in Section 1.2.2), and by Gérard-Varet and Nguyen [40]. The existence of finite-time blow-up for Prandtl system was shown by E–Enquist [35]. On the other hand, well-posedness results of the Prandtl system have been obtained by assuming either real analyticity or some special structures of the initial data, see, e.g., Kukavica–Masmoudi–Vicol–Wong [58], Kukavica–Vicol [60], Masmoudi–Wong [77], and Oleinik [79].

In order to study the possible finite-time blow-up of system (1.5), and to give a reliable numerical regularization, it is proposed to study the Voigt α-regularization with respect to z variable of

(1.5). More specifically, consider the following system

$$\begin{cases} \partial_t(u - \alpha^2\partial_{zz}u) + u\partial_x u + w\partial_z u + \epsilon_1 u - \Omega v + \partial_x p - \nu_z\partial_{zz}u = 0, \\ \partial_t(v - \alpha^2\partial_{zz}v) + u\partial_x v + w\partial_z v + \epsilon_1 v + \Omega u - \nu_z\partial_{zz}v = 0, \\ \epsilon_2 w + \partial_z p + T = 0, \\ \partial_x u + \partial_z w = 0, \\ \partial_t T - \kappa_h\partial_{xx}T - \kappa_z\partial_{zz}T + u\partial_x T + w\partial_z T = 0, \end{cases} \tag{1.6}$$

where $\alpha > 0$. Voigt α-regularization is also used in the study of the $3D$ Euler equations, see, e.g., Cao–Lunasin–Titi [28], Larios–Petersen–Titi–Wingate [66], and Larios–Titi [67]. It is established in [23] that system (1.6) is globally well-posed for general Sobolev initial data. In addition, by taking $\Omega = 0, v \equiv 0$, and $T \equiv 0$ in system (1.5) and system (1.6) (these considerations are just for mathematical simplicity), one can prove the convergence of the strong solutions of system (1.6) to system (1.5) as $\alpha \to 0$. At the end, based on the convergence, a blowup criterion of system (1.5) with $\Omega = 0, v \equiv 0$, and $T \equiv 0$ is established.

In summary, for the PEs with only vertical viscosity and linear damping terms, the following results will be reported in this dissertation:

1. System (1.5) is locally well-posed with general Sobolev initial data and globally well-posed with small Sobolev initial data.

2. System (1.6) is globally well-posed for general Sobolev initial data.

3. When $\Omega = 0, v \equiv 0$, and $T \equiv 0$ in system (1.5) and system (1.6), the strong solutions of system (1.6) converge to those of system (1.5) on the time interval of the existence of system (1.5) as $\alpha \to 0$. The considerations of $\Omega = 0, v \equiv 0$, and $T \equiv 0$ are just for mathematical simplicity.

4. Based on the result of convergence, a blowup criterion of system (1.5) with $\Omega = 0, v \equiv 0$, and $T \equiv 0$ is given.

1.3 Outline of The Dissertation

Chapter 2 consists of preliminary background materials. The notation that will be used in this work are introduced, including functional settings and several projections. Later, lemmas that will be used in this dissertation are listed, together with the proofs for some of them. The proofs of Lemma 2.2.11–2.2.17 will be presented in Appendix A.

Chapter 3 is dedicated for the detailed mathematical analysis of the IPEs. In Section 3.1, one first recall previous results about the ill-posedness of the IPEs in the absence of rotation, then extend these results to the case when rotation is present. In Section 3.2, it is shown that the IPEs are locally well-posed in the space of analytic functions. By using the projections introduced in Chapter 2, one can first reformulate the problem. Later, using the standard Galerkin method and the energy estimate, the existence of the solutions is established. The uniqueness of solutions and continuous dependence on the initial data are also proved. In Section 3.3, following two different methods in [17] and [91], one is able to to extend the blowup results to the case when the rotation is present. An explicit example is also provided to discuss the intuition for the long-time existence results. In Section 3.4, it is established that the life-span of the solutions to the $3D$ IPEs can be prolonged to infinity as long as the analytic norm of the initial baroclinic mode goes to zero. Moreover, the solutions to the $3D$ IPEs converge to the solution to the $2D$ Euler equations in this situation. Finally, Section 3.5 focus on the study of the effect of rotation on the life-span of the solutions. By using more projections to reformulate the problem furthermore, one can derive a limit resonant system as the rotation rate $|\Omega| \to \infty$. As the limit resonant system is globally well-posed, one can investigate the perturbed system and establish technical energy estimates to show that the life-span of the solutions to the $3D$ IPEs goes to infinity as long as the rotation rate $|\Omega| \to \infty$ and the Sobolev norm of the initial baroclinic mode converges to zero depending on $|\Omega|$. Moreover, in this situation, the solutions to the $3D$ IPEs can be approximated by the solutions to the limit resonant system with the same initial data. This section ends up with some remarks and discussions.

Chapter 4 consists of the study of the PEs with only vertical viscosity and weak horizontal dissipation that is the linear (Rayleigh-like friction) damping. Section 4.1 studies the local well-posedness of this system in certain Sobolev space with general initial data. This section starts with the reformulation of the problem and formal *a priori* energy estimates. Later, by constructing the Galerkin approximation system of our problem, the existence of the solutions is shown by rigorous Galerkin procedure. Next, the uniqueness of the solutions and continuous dependence on the initial data are established. At the end, one can consider a special case, i.e., setting $\Omega = 0, v \equiv 0$ and $T \equiv 0$, and prove similar result with less requirement on the regularity of initial data. Section 4.2 is dedicated for the study of the global well-posedness of the system provided that the initial data is small enough depending on the viscosity, diffusivity, and the linear damping coefficients. In Section 4.3, it is proposed to study the Voigt α-regularization of our original model, and one can establish the global well-posedness in Sobolev spaces for arbitrary initial data, i.e., without smallness assumption. For the special case when $\Omega = 0, v \equiv 0$ and $T \equiv 0$, similar result is established with less requirement on the regularity of initial data. In Section 4.4, it is shown that the solutions of the Voigt α-regularization model converge to the solution of the original model on the interval of the existence of the latter, as $\alpha \to 0$, in the case when $\Omega = 0, v \equiv 0$ and $T \equiv 0$. Such requirement is just for mathematical simplification. In Section 4.5, a blowup criterion of the original model is given based on the convergence result.

Chapter 5 contains the conclusion and summary of this dissertation.

Appendix A is dedicated for the detailed proofs of Lemma 2.2.11–2.2.17 that are associated with the energy estimate of nonlinear terms in the space of analytic functions. These results can be used for other study on the PEs in the space of analytic functions.

2. NOTATION AND PRELIMINARIES

In this chapter, we introduce the notation and collect some preliminary results that will be used in this dissertation.

2.1 Notation

The universal constant C appears in this dissertation may change from step to step. When we use subscript for C, e.g., C_r, it means that the constant depends only on r. We use $f \lesssim g$ and $f \gtrsim g$ to mean $f \le Cg$ and $f \ge Cg$, respectively.

We use the notation $\boldsymbol{x} := (\boldsymbol{x}', z)$ to represent the spatial variables, where $\boldsymbol{x}'$ and z represent the horizontal and vertical variables, respectively. In the $2D$ case $\boldsymbol{x}' = x$, while in the $3D$ case $\boldsymbol{x}' = (x, y)$.

For domain $U \subset \mathbb{R}^d$, where $d = 2$ or 3, we denote by $L^p(U)$, for $p \ge 1$, the Lebesgue space of real valued functions $f(\boldsymbol{x})$ satisfying $\int_U |f(\boldsymbol{x})|^p d\boldsymbol{x} < \infty$, and denote the corresponding norm by

$$\|f\|_{L^p} := \|f\|_{L^p(U)} = \Big(\int_U |f(\boldsymbol{x})|^p d\boldsymbol{x} \Big)^{\frac{1}{p}}. \tag{2.1}$$

When the function f is a vector field in $\mathbb{R}^m$, by abuse of notation, we still use $f \in L^p(U)$ instead of $f \in \Big(L^P(U)\Big)^m$ when there is no confusion. When $p = 2$, we use the notation

$$\|f\| := \|f\|_{L^2(U)} \tag{2.2}$$

for simplicity, and denote the inner product in $L^2(U)$ by

$$\langle f, g \rangle := \int_U f(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} \tag{2.3}$$

for functions $f, g \in L^2(U)$.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$ denote mult-indices. The notation

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d}, \qquad |\alpha| = \sum_{j=1}^{d} \alpha_j, \qquad \alpha! = \prod_{j=1}^{d} \alpha_j! \tag{2.4}$$

will be used throughout. For $r \geq 0$ an integer, we denote by $H^r(U) = W^{r,2}(U)$ the Sobolev space of real valued functions f satisfying $\sum\limits_{|\alpha| \leq r} \|D^\alpha f\|_{L^2}^2 < \infty$, and denote the corresponding norm by

$$\|f\|_{H^r} := \Big(\sum_{|\alpha| \leq r} \|D^\alpha f\|_{L^2}^2 \Big)^{\frac{1}{2}}. \tag{2.5}$$

For $s > 0$, a function $f \in C^\infty(U)$ is said to be in Gevrey class of order s, denoted by $f \in G^s(U)$, if there exist constants $\rho > 0$ and $M > 0$ such that for every $\boldsymbol{x} \in U$ and $\alpha \in \mathbb{N}^d$, one has

$$|\partial^\alpha f(\boldsymbol{x})| \leq M \Big(\frac{\alpha!}{\rho^{|\alpha|}} \Big)^s. \tag{2.6}$$

When $U = \mathbb{T}^d \subset \mathbb{R}^d$, where $\mathbb{T}^d$ is the d-dimensional torus with unit length, we use $\hat{f}_{\boldsymbol{k}}$ to denote the Fourier coefficient of function $f \in L^2(\mathbb{T}^d)$, so that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \hat{f}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k} \cdot \boldsymbol{x}}, \qquad \hat{f}_{\boldsymbol{k}} = \int_{\mathbb{T}^d} e^{-2\pi i \boldsymbol{k} \cdot \boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.7}$$

In this case, the H^r norm can also be defined in the following way:

$$\|f\|_{H^r} := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^d} (1 + |\boldsymbol{k}|^{2r}) |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}. \tag{2.8}$$

Notice that here $r \geq 0$ is not necessary an integer. The Sobolev space $H^r(\mathbb{T}^d)$ is the set of all $L^2(\mathbb{T}^d)$ functions for which (2.8) is finite. We also denote the corresponding H^r semi-norm by

$$\|f\|_{\dot{H}^r} := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\boldsymbol{k}|^{2r} |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}. \tag{2.9}$$

For more details about Sobolev spaces, see Adams [2].

Denote by $A = \sqrt{-\Delta}$, subject to periodic boundary condition, where Δ is the $3D$ Laplacian. For each $s > 0$ and $r \geq 0$, we define a family, parameterized by $\tau \geq 0$, of normed spaces

$$\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) := \{f \in H^r(\mathbb{T}^3) : \|e^{\tau A^{1/s}} f\|_{H^r} < \infty\}, \tag{2.10}$$

where the norm is defined by

$$\|e^{\tau A^{1/s}} f\|_{H^r} := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^3} (1 + |\boldsymbol{k}|^{2r} e^{2\tau |\boldsymbol{k}|^{1/s}}) |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}. \tag{2.11}$$

Denote the semi-norm by

$$\|A^r e^{\tau A^{1/s}} f\| := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^3} |\boldsymbol{k}|^{2r} e^{2\tau |\boldsymbol{k}|^{1/s}} |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}, \tag{2.12}$$

then it is easy to see that

$$\|e^{\tau A^{1/s}} f\|_{H^r}^2 = \|A^r e^{\tau A^{1/s}} f\|^2 + \|f\|^2. \tag{2.13}$$

As we will see later in Lemma 2.2.1, the following relationship holds:

$$G^s(\mathbb{T}^3) = \bigcup_{\tau > 0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)). \tag{2.14}$$

For more details about Gevrey class, we refer the readers to Ferrari–Titi [37], Foias–Temam [38], and Levermore–Oliver [64].

Given time $\mathcal{T} > 0$, denote by $L^p(0, \mathcal{T}; X)$ the space of functions $f : [0, T] \to X$ satisfying $\int_0^{\mathcal{T}} \|f(t)\|_X^p dt < \infty$, where X is a Banach space and $\|\cdot\|_X$ represents its norm. Similarly, denote by $C([0, \mathcal{T}]; X)$ the space of continuous functions $f : [0, T] \to X$.

Next, we define several projections that will be used in this dissertation. For $\varphi \in L^2(\mathbb{T}^3)$, denote by

$$P_0\varphi := \overline{\varphi} = \int_0^1 \varphi(\boldsymbol{x}', z)dz. \tag{2.15}$$

We call

$$\overline{\varphi} := P_0\varphi \tag{2.16}$$

the barotropic mode of φ, and

$$\widetilde{\varphi} := \varphi - \overline{\varphi} = (I - P_0)\varphi \tag{2.17}$$

the baroclinic mode of φ.

For $\varphi(\boldsymbol{x}') \in L^2(\mathbb{T}^2)$, denote the $2D$ horizontal Leray projection by

$$\mathbb{P}_h\varphi := \varphi - \nabla_h\Delta_h^{-1}\nabla_h \cdot \varphi. \tag{2.18}$$

Here, we denote by $\phi = \Delta_h^{-1}\varphi$ when $\Delta_h\phi = \varphi$ and $\int_{\mathbb{T}^2} \phi(\boldsymbol{x}')d\boldsymbol{x}' = \int_{\mathbb{T}^2} \varphi(\boldsymbol{x}')d\boldsymbol{x}' = 0$.

Next, for $\varphi \in L^2(\mathbb{T}^3)$, define projections $P_\pm$ as

$$P_\pm\varphi := \frac{1}{2}(\widetilde{\varphi} - i\widetilde{\varphi}^\perp). \tag{2.19}$$

The projections P_0 and $P_\pm$ play an important role in the analysis of the $3D$ IPEs. We will see later in section 3.5 the details of the derivations of these projections and why they are crucial.

2.2 Preliminaries

In this section, we list lemmas that will be used in this dissertation, together with the proofs for some of them. We start with the following lemma that comes from Levermore–Oliver [64] and addresses the relation between Gevrey class $G^s(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$.

Lemma 2.2.1. *For any $s > 0$ and $r \geq 0$, we have*

$$G^s(\mathbb{T}^3) = \bigcup_{\tau>0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)). \tag{2.20}$$

Althought our definition of the norm $\|e^{\tau A^{1/s}} f\|_{H^r}$ is slightly different from [64], the proof of this lemma is almost the same, and we refer the readers to [64].

The next lemma also comes from [64] (see also [37]), addressing an important property of the space $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$.

Lemma 2.2.2. *If $s \geq 1$, $\tau \geq 0$, and $r > 3/2$, then $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ is a Banach algebra, and for any $f, g \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have*

$$\|e^{\tau A^{1/s}}(fg)\|_{H^r} \leq C_{r,s}\|e^{\tau A^{1/s}} f\|_{H^r}\|e^{\tau A^{1/s}} g\|_{H^r}. \tag{2.21}$$

For the semi-norm, we also have a similar estimate

$$\|A^r e^{\tau A^{1/s}}(fg)\| \leq C_{r,s}\Big(|\hat{f}_0| + \|A^r e^{\tau A^{1/s}} f\|\Big)\Big(|\hat{g}_0| + \|A^r e^{\tau A^{1/s}} g\|\Big). \tag{2.22}$$

For the proof, we refer the readers to [37] for the case when $s = 1$, and to [80] for the case when $s > 1$.

The following lemma addresses that we can use P_0 and $P_\pm$ to decompose any vector filed φ into three parts that are orthogonal to each other.

Lemma 2.2.3. *For any $\varphi \in L^2(\mathbb{T}^3)$, we have the following decomposition:*

$$\varphi = P_0\varphi + P_+\varphi + P_-\varphi. \tag{2.23}$$

Moreover, we have the following properties:

$$P_\pm P_\pm \varphi = P_\pm \varphi, \qquad P_0 P_0 \varphi = P_0 \varphi, \qquad P_\pm P_\mp \varphi = P_0 P_\pm \varphi = P_\pm P_0 \varphi = 0. \tag{2.24}$$

Proof. The proof is straightforward from the definition of P_0 and $P_\pm$, and that $\overline{\widetilde{\varphi}} = \widetilde{\overline{\varphi}} = 0$. □

For projections $P_0, P_\pm$, we have the following properties.

Lemma 2.2.4. *For $f, g \in L^2(\mathbb{T}^3)$, we have*

$$\langle P_0 f, g \rangle = \langle f, P_0 g \rangle = \langle P_0 f, P_0 g \rangle, \tag{2.25}$$

and

$$\langle P_\pm f, g \rangle = \langle f, P_\mp g \rangle. \tag{2.26}$$

If $f \in H^r(\mathbb{T}^3)$ with $r \geq 0$, then for $|\alpha| \leq r$, we have

$$\partial^\alpha P_0 f = P_0 \partial^\alpha f \quad \textit{and} \quad \partial^\alpha P_\pm f = P_\pm \partial^\alpha f. \tag{2.27}$$

Moreover, if $f \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \geq 0$, we have

$$A^r e^{\tau A^{1/s}} P_0 f = P_0 A^r e^{\tau A^{1/s}} f. \tag{2.28}$$

Proof. For (2.25), we compute

$$\begin{aligned}
\langle P_0 f, g \rangle &= \int_{\mathbb{T}^3} \Big(\int_0^1 f(\boldsymbol{x}', z) dz \Big) g(\boldsymbol{x}', z) d\boldsymbol{x}' dz \\
&= \int_{\mathbb{T}^2} \Big(\int_0^1 f(\boldsymbol{x}', z) dz \Big) \Big(\int_0^1 g(\boldsymbol{x}', z) dz \Big) d\boldsymbol{x}' = \langle P_0 f, P_0 g \rangle \\
&= \int_{\mathbb{T}^3} f(\boldsymbol{x}', z) \Big(\int_0^1 g(\boldsymbol{x}', z) dz \Big) d\boldsymbol{x}' dz = \langle f, P_0 g \rangle.
\end{aligned} \tag{2.29}$$

For (2.26), one has

$$\begin{aligned}
\langle P_\pm f, g \rangle &= \frac{1}{2} \int_{\mathbb{T}^3} (\widetilde{f} \pm i \widetilde{f}^\perp)(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} \\
&= \frac{1}{2} \int_{\mathbb{T}^3} \Big((fg - \overline{f} g) \pm i (f^\perp g - \overline{f}^\perp g) \Big)(\boldsymbol{x}) d\boldsymbol{x} \\
&= \frac{1}{2} \int_{\mathbb{T}^3} \Big((fg - f\overline{g}) \mp i (f g^\perp - f \overline{g}^\perp) \Big)(\boldsymbol{x}) d\boldsymbol{x}
\end{aligned}$$

$$= \frac{1}{2}\int_{\mathbb{T}^3} f(\boldsymbol{x})(\widetilde{g} \mp i\widetilde{g}^{\perp})(\boldsymbol{x})d\boldsymbol{x} = \langle f, P_{\mp} g\rangle. \tag{2.30}$$

For (2.27), if $\alpha_3 = 0$, we have

$$\partial^\alpha P_0 f = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\int_0^1 f(\boldsymbol{x}', z)dz = \int_0^1 \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(\boldsymbol{x}', z)dz = P_0\partial^\alpha f, \tag{2.31}$$

and

$$\begin{aligned}\partial^\alpha P_{\pm} f &= \frac{1}{2}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\Big[(f - P_0 f) \pm i(f - P_0 f)^{\perp}\Big] \\ &= \frac{1}{2}\Big[(\partial^\alpha f - P_0\partial^\alpha f) \pm i(\partial^\alpha f - P_0\partial^\alpha f)^{\perp}\Big] = P_{\pm}\partial^\alpha f.\end{aligned} \tag{2.32}$$

If $\alpha_3 > 0$, thanks to periodic boundary condition, we have

$$\partial^\alpha P_0 f = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3}\int_0^1 f(\boldsymbol{x}', z)dz = 0 = \int_0^1 \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3} f(\boldsymbol{x}', z)dz = P_0\partial^\alpha f, \tag{2.33}$$

and

$$\begin{aligned}\partial^\alpha P_{\pm} f &= \frac{1}{2}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3}\Big[(f - P_0 f) \pm i(f - P_0 f)^{\perp}\Big] = \frac{1}{2}(\partial^\alpha f \pm i\partial^\alpha f^{\perp}) \\ &= \frac{1}{2}\Big[(\partial^\alpha f - P_0\partial^\alpha f) \pm i(\partial^\alpha f - P_0\partial^\alpha f)^{\perp}\Big] = P_{\pm}\partial^\alpha f.\end{aligned} \tag{2.34}$$

Therefore, for any $|\alpha| \le r$, (2.27) holds. The proof of (2.28) is straightforward, so we omit it. □

For Leray projection $\mathbb{P}_h$, we have the following properties.

Lemma 2.2.5. *For $f, g \in L^2(\mathbb{T}^3)$, we have*

$$\langle \mathbb{P}_h f, g\rangle = \langle f, \mathbb{P}_h g\rangle, \tag{2.35}$$

and

$$\mathbb{P}_h P_0 f = P_0 \mathbb{P}_h f. \tag{2.36}$$

If $f \in H^r(\mathbb{T}^3)$ with $r \geq 0$, then for $|\alpha| \leq r$, we have

$$\partial^\alpha \mathbb{P}_h f = \mathbb{P}_h \partial^\alpha f. \tag{2.37}$$

Moreover, if $f \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \geq 0$, we have

$$A^r e^{\tau A^{1/s}} \mathbb{P}_h f = \mathbb{P}_h A^r e^{\tau A^{1/s}} f. \tag{2.38}$$

Proof. For the proof of (2.35) and (2.37), see [32]. For (2.36), we compute

$$\mathbb{P}_h P_0 f = P_0 f - \nabla_h \Delta_h^{-1} \nabla_h \cdot (P_0 f) = P_0 f - P_0(\nabla_h \Delta_h^{-1} \nabla_h \cdot f) = P_0 \mathbb{P}_h f. \tag{2.39}$$

For (2.38), one has

$$\begin{aligned} A^r e^{\tau A^{1/s}} \mathbb{P}_h f &= A^r e^{\tau A^{1/s}} \Big[\sum_{\boldsymbol{k} \neq \boldsymbol{0}} (\hat{f}_{\boldsymbol{k}} - \frac{\boldsymbol{k} \cdot \hat{f}_{\boldsymbol{k}}}{|\boldsymbol{k}|^2} \boldsymbol{k}) e^{2\pi i \boldsymbol{k} \cdot \boldsymbol{x}} + \hat{f}_{\boldsymbol{0}} \Big] \\ &= \sum_{\boldsymbol{k} \neq \boldsymbol{0}} |\boldsymbol{k}|^r e^{\tau |\boldsymbol{k}|^{1/s}} (\hat{f}_{\boldsymbol{k}} - \frac{\boldsymbol{k} \cdot \hat{f}_{\boldsymbol{k}}}{|\boldsymbol{k}|^2} \boldsymbol{k}) e^{2\pi i \boldsymbol{k} \cdot \boldsymbol{x}} = \mathbb{P}_h A^r e^{\tau A^{1/s}} f. \end{aligned} \tag{2.40}$$

□

For the relation between the norm of $\mathcal{V}$ and the norms of $\overline{\mathcal{V}}, \widetilde{\mathcal{V}}$ in $L^2(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have the following lemma.

Lemma 2.2.6. *Let $\mathcal{V} = P_0 \mathcal{V} + (I - P_0)\mathcal{V} = \overline{\mathcal{V}} + \widetilde{\mathcal{V}}$. Suppose that $r \geq 0$, $s > 0$, and $\tau \geq 0$, we have*

$$\|\mathcal{V}\|^2 = \|\overline{\mathcal{V}}\|^2 + \|\widetilde{\mathcal{V}}\|^2, \tag{2.41}$$

and

$$\|e^{\tau A^{1/s}}\mathcal{V}\|_{H^r}^2 = \|e^{\tau A^{1/s}}\overline{\mathcal{V}}\|_{H^r}^2 + \|e^{\tau A^{1/s}}\widetilde{\mathcal{V}}\|_{H^r}^2. \tag{2.42}$$

Proof. Using Fourier representation of $\mathcal{V}, \overline{\mathcal{V}}, \widetilde{\mathcal{V}}$, one has

$$\mathcal{V} = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \hat{\mathcal{V}}_{\boldsymbol{k}} e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \overline{\mathcal{V}} = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}} \hat{\mathcal{V}}_{\boldsymbol{k}} e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \widetilde{\mathcal{V}} = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} \hat{\mathcal{V}}_{\boldsymbol{k}} e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.43}$$

Then we have

$$\|\mathcal{V}\|^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} |\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}} |\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 + \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 = \|\overline{\mathcal{V}}\|^2 + \|\widetilde{\mathcal{V}}\|^2, \tag{2.44}$$

and

$$\begin{aligned}
&\|e^{\tau A^{1/s}}\mathcal{V}\|_{H^r}^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} (1 + |\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 \\
&= \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}} (1 + |\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 + \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} (1 + |\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{\mathcal{V}}_{\boldsymbol{k}}|^2 \\
&= \|e^{\tau A^{1/s}}\overline{\mathcal{V}}\|_{H^r}^2 + \|e^{\tau A^{1/s}}\widetilde{\mathcal{V}}\|_{H^r}^2.
\end{aligned} \tag{2.45}$$

□

Denote by $u_\pm = \frac{1}{2}e^{\mp i\Omega t}(\widetilde{\mathcal{V}} \pm i\widetilde{\mathcal{V}}^\perp)$. For the relation between the norm of $\widetilde{\mathcal{V}}$ and the norms of $u_\pm$ in $L^2(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have the following Lemma.

Lemma 2.2.7. *Let $u_\pm = \frac{1}{2}e^{\mp i\Omega t}(\widetilde{\mathcal{V}} \pm i\widetilde{\mathcal{V}}^\perp)$. Suppose that $r \geq 0$, $s > 0$, and $\tau \geq 0$, we have*

$$\|u_+\|^2 = \|u_-\|^2 = \frac{1}{2}\|\widetilde{\mathcal{V}}\|^2, \tag{2.46}$$

and

$$\|e^{\tau A^{1/s}}u_+\|_{H^r}^2 = \|e^{\tau A^{1/s}}u_-\|_{H^r}^2 = \frac{1}{2}\|e^{\tau A^{1/s}}\widetilde{\mathcal{V}}\|_{H^r}^2. \tag{2.47}$$

Proof. For (2.46), we have

$$\|u_+\|^2 = \|u_-\|^2 = \langle u_+, u_- \rangle = \frac{1}{4}\langle \widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}^\perp, \widetilde{\mathcal{V}} - i\widetilde{\mathcal{V}}^\perp \rangle = \frac{1}{2}\|\widetilde{\mathcal{V}}\|^2. \tag{2.48}$$

For (2.47), notice that

$$\begin{aligned} \|A^r e^{\tau A^{1/s}} u_+\|^2 &= \|A^r e^{\tau A^{1/s}} u_-\|^2 = \langle A^r e^{\tau A^{1/s}} u_+, A^r e^{\tau A^{1/s}} u_- \rangle \\ &= \frac{1}{4}\langle A^r e^{\tau A^{1/s}} (\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}^\perp), A^r e^{\tau A^{1/s}} (\widetilde{\mathcal{V}} - i\widetilde{\mathcal{V}}^\perp) \rangle = \frac{1}{2}\|A^r e^{\tau A^{1/s}} \widetilde{\mathcal{V}}\|^2. \end{aligned} \tag{2.49}$$

Thanks to (2.13), we know (2.47) holds. □

The following anisotropic estimate in $\mathbb{T}^2$ is similar to the one in Cao–Wu [24].

Lemma 2.2.8. *Assume that $f, g, h, g_z, h_x \in L^2(\mathbb{T}^2)$. Then*

$$\int_{\mathbb{T}^2} |fgh| dxdz \le C\|f\|\|g\|^{\frac{1}{2}}(\|g\|^{\frac{1}{2}} + \|g_z\|^{\frac{1}{2}})\|h\|^{\frac{1}{2}}(\|h\|^{\frac{1}{2}} + \|h_x\|^{\frac{1}{2}}).$$

Proof. First, recall that by one-dimensional Agmon's inequality (or Gagliardo–Nirenberg interpolation inequality), for $\phi \in H^1(0,1)$, one has

$$\|\phi\|_{L^\infty(0,1)} \le C\left(\|\phi\|_{L^2(0,1)} + \|\phi\|^{\frac{1}{2}}_{L^2(0,1)}\|\phi_x\|^{\frac{1}{2}}_{L^2(0,1)}\right). \tag{2.50}$$

Therefore, by Hölder's inequality and Agmon's inequality (2.50),

$$
\begin{aligned}
&\int_{\mathbb{T}^2} |fgh| dxdz \\
\leq & C \int_0^1 \left[\Big(\int_0^1 |f(x,z)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_0^1 |g(x,z)|^2 dx \Big)^{\frac{1}{2}} \Big(\sup_{0\leq x\leq 1} |h(x,z)| \Big) \right] dz \\
\leq & C \int_0^1 \left\{ \Big(\int_0^1 |f(x,z)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_0^1 |g(x,z)|^2 dx \Big)^{\frac{1}{2}} \right. \\
& \left. \left[\Big(\int_0^1 |h(x,z)|^2 dx \Big)^{\frac{1}{4}} \Big(\int_0^1 |h_x(x,z)|^2 dx \Big)^{\frac{1}{4}} + \Big(\int_0^1 |h(x,z)|^2 dx \Big)^{\frac{1}{2}} \right] \right\} dz \\
\leq & C \|f\| \sup_{0\leq z\leq 1} \Big(\int_0^1 |g(x,z)|^2 dx \Big)^{\frac{1}{2}} \|h\|^{\frac{1}{2}} (\|h\|^{\frac{1}{2}} + \|h_x\|^{\frac{1}{2}})
\end{aligned}
\tag{2.51}
$$

By Minkowski's inequality, Agmon's inequality (2.50), and Hölder inequality,

$$
\begin{aligned}
&\sup_{0\leq z\leq 1} \Big(\int_0^1 |g(x,z)|^2 dx \Big)^{\frac{1}{2}} \\
\leq & C \Big(\int_0^1 \sup_{0\leq z\leq 1} |g(x,z)|^2 dx \Big)^{\frac{1}{2}} \\
\leq & C \left(\int_0^1 \left[\Big(\int_0^1 |g(x,z)|^2 dz \Big)^{\frac{1}{2}} \Big(\int_0^1 |g_z(x,z)|^2 dz \Big)^{\frac{1}{2}} + \int_0^1 |g(x,z)|^2 dz \right] dx \right)^{\frac{1}{2}} \\
\leq & C \|g\|^{\frac{1}{2}} (\|g\|^{\frac{1}{2}} + \|g_z\|^{\frac{1}{2}}).
\end{aligned}
\tag{2.52}
$$

Inserting (2.52) to (2.51) yields the desired inequality. □

Next we prove the following result which will be used in Chapter 4.

Lemma 2.2.9. *Assume that $f \in H^1(\mathbb{T}^2)$ and $f_{xz} \in L^2(\mathbb{T}^2)$. Then $f \in L^\infty(\mathbb{T}^2)$. Moreover,*

$$\|f\|_{L^\infty} \leq C \left(\|f\|_{H^1}^2 + \|f_{xz}\|^2 \right)^{\frac{1}{2}}.$$

Proof. Let $\{\hat{f}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}$ be the Fourier coefficients of f. By Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|f\|_{L^\infty} &\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\hat{f}_{\boldsymbol{k}}| = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \frac{|\hat{f}_{\boldsymbol{k}}|(1+k_1^2+k_2^2+k_1^2k_2^2)^{\frac{1}{2}}}{(1+k_1^2+k_2^2+k_1^2k_2^2)^{\frac{1}{2}}} \\ &\leq \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\hat{f}_{\boldsymbol{k}}|^2(1+k_1^2+k_2^2+k_1^2k_2^2)\right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^2} \frac{1}{(1+k_1^2)(1+k_2^2)}\right)^{\frac{1}{2}} \\ &\leq C\Big(\|f\|_{H^1}^2 + \|f_{xz}\|^2\Big)^{\frac{1}{2}} < \infty.\end{aligned} \tag{2.53}$$

Therefore, $f \in L^\infty(\mathbb{T}^2)$. □

We also need the following Aubin-Lions theorem.

Lemma 2.2.10. *(Aubin-Lions Lemma, cf. Simon [87] Corollary 4) Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then it holds that*

1. *If $\mathcal{F}$ is a bounded subset of $L^p(0,T;X)$, where $1 \leq p < \infty$, and $\mathcal{F}_t := \{\frac{\partial f}{\partial t} | f \in \mathcal{F}\}$ is bounded in $L^1(0,T;Y)$, then $\mathcal{F}$ is relative compact in $L^p(0,T;B)$.*

2. *If $\mathcal{F}$ is a bounded subset of $L^\infty(0,T;X)$ and $\mathcal{F}_t$ is bounded in $L^q(0,T;Y)$, where $q > 1$, then $\mathcal{F}$ is relative compact in $C([0,T];B)$.*

The following lemmas concern the estimates of nonlinear terms in the space of analytic functions, and will be used in the study of $3D$ IPEs. We only list them here, and will provide detailed proofs in Appendix A.

First, we estimate nonlinear terms of the form $f \cdot \nabla_h g$.

Lemma 2.2.11. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 2$ and $\tau \geq 0$, one has*

$$\begin{aligned}\Big|\Big\langle A^r e^{\tau A}(f\cdot\nabla_h g), A^r e^{\tau A} h\Big\rangle\Big| \leq C_r\Big[&(\|A^r e^{\tau A} f\| + |\hat{f}_0|)\|A^{r+1/2}e^{\tau A} g\|\|A^{r+1/2}e^{\tau A}h\| \\ &+ \|A^{r+1/2}e^{\tau A} f\|\|A^r e^{\tau A} g\|\|A^r e^{\tau A} h\|\Big].\end{aligned} \tag{2.54}$$

Similarly, we estimate $(\nabla_h \cdot f)g$ in the following:

Lemma 2.2.12. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 2$ and $\tau \geq 0$, one has*

$$\begin{aligned}\left|\left\langle A^r e^{\tau A}\big((\nabla_h \cdot f)g\big), A^r e^{\tau A} h\right\rangle\right| \leq & C_r\Big[(\|A^r e^{\tau A} g\| + |\hat{g}_0|)\|A^{r+1/2}e^{\tau A} f\|\|A^{r+1/2}e^{\tau A}h\| \\ & + \|A^{r+1/2}e^{\tau A}g\|\|A^r e^{\tau A} f\|\|A^r e^{\tau A} h\|\Big].\end{aligned} \tag{2.55}$$

Next, we provide an estimate for $(\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds)\partial_z g$ in the following:

Lemma 2.2.13. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 2$, $\tau \geq 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned}&\left|\left\langle A^r e^{\tau A}\big((\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds)\partial_z g\big), A^r e^{\tau A} h\right\rangle\right| \\ \leq & C_r\Big(\|A^r e^{\tau A} f\|\|A^{r+1/2}e^{\tau A} g\|\|A^{r+1/2}e^{\tau A}h\| \\ & + \|A^r e^{\tau A} g\|\|A^{r+1/2}e^{\tau A} f\|\|A^{r+1/2}e^{\tau A}h\| \\ & + \|A^r e^{\tau A} h\|\|A^{r+1/2}e^{\tau A} f\|\|A^{r+1/2}e^{\tau A}g\|\Big).\end{aligned} \tag{2.56}$$

Lemma 2.2.14–2.2.17 play an essential role in the study of the effect of fast rotation to the $3D$ IPEs. First, let us state the following:

Lemma 2.2.14. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 5/2$ and $\tau \geq 0$, one has*

$$\begin{aligned}&\left|\left\langle A^r e^{\tau A}(f \cdot \nabla_h g), A^r e^{\tau A} h\right\rangle - \left\langle f \cdot \nabla_h A^r e^{\tau A} g, A^r e^{\tau A} h\right\rangle\right| \\ &\leq C_r\|A^r f\|\|A^r g\|\|A^r h\| + C_r\tau\|A^{r+1/2}e^{\tau A} f\|\|A^{r+1/2}e^{\tau A} g\|\|A^{r+1/2}e^{\tau A}h\|.\end{aligned} \tag{2.57}$$

Lemma 2.2.14 can also be used in the study of Euler equations, since it involves nonlinear term similar to that appearing in the Euler equations. The next three lemmas provide the estimates for nonlinear terms which are specific to the structure of the PEs.

Lemma 2.2.15. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 5/2$ and $\tau \geq 0$, one has*

$$\begin{aligned}&\left|\left\langle A^r e^{\tau A}\big((\nabla_h \cdot f)g\big), A^r e^{\tau A} h\right\rangle - \left\langle (\nabla_h \cdot A^r e^{\tau A} f)g, A^r e^{\tau A} h\right\rangle\right| \\ &\leq C_r\|A^r f\|\|A^r g\|\|A^r h\| + C_r\tau\|A^{r+1/2}e^{\tau A} f\|\|A^{r+1/2}e^{\tau A} g\|\|A^{r+1/2}e^{\tau A}h\|.\end{aligned} \tag{2.58}$$

Lemma 2.2.16. *For $f \in \mathcal{D}(e^{\tau A} : H^{r+3/2}(\mathbb{T}^3))$ and $g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 5/2$, $\tau \geq 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds\big)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle \\
&\qquad\qquad\qquad - \Big\langle \big(\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds\big) A^r e^{\tau A}\partial_z g, A^r e^{\tau A} h\Big\rangle\Big| \\
&\leq C_r\|A^{r+1}f\|\|A^r g\|\|A^r h\| + C_r\tau\|A^{r+3/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|. \qquad (2.59)
\end{aligned}$$

Lemma 2.2.17. *For $g \in \mathcal{D}(e^{\tau A} : H^{r+3/2}(\mathbb{T}^3))$ and $f, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2}(\mathbb{T}^3))$, where $r > 5/2$, $\tau \geq 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds\big)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle \\
&\qquad\qquad\qquad - \Big\langle \partial_z g A^r e^{\tau A}\big(\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s)ds\big), A^r e^{\tau A} h\Big\rangle\Big| \\
&\leq C_r\|A^{r+1}g\|\|A^r f\|\|A^r h\| + C_r\tau\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}h\|. \qquad (2.60)
\end{aligned}$$

3. INVISCID PRIMITIVE EQUATIONS

In this chapter, we will provide details of the mathematical analysis of the IPEs

$$\begin{cases} \partial_t \mathcal{V} + \mathcal{V} \cdot \nabla_h \mathcal{V} + w \partial_z \mathcal{V} + \Omega \mathcal{V}^\perp + \nabla_h p = 0, \\ \partial_z p = 0, \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0. \end{cases} \tag{3.1}$$

System (3.1) is derived from system (1.3) by taking $\nu_h = \nu_z = 0$ (inviscid), and under the observation that any smooth solution $(\mathcal{V}, T)$ to system system (1.3) with initial condition $T_0 = 0$ must satisfy $T \equiv 0$. We consider system (3.1) in the domain $\mathcal{D} = \{(x, y, z) : 0 \leq z \leq 1, (x, y) \in \mathbb{R}^2\}$, subject to the boundary condition

$$w|_{z=0,1} = 0, \tag{3.2}$$

and the initial conditions

$$\mathcal{V}|_{t=0} = \mathcal{V}_0. \tag{3.3}$$

Notice that we do not have initial condition for w, since w_0 must satisfy the compatible condition

$$w_0 = -\int_0^z \nabla_h \cdot \mathcal{V}_0(x, y, s) ds. \tag{3.4}$$

We will do the mathematical analysis of system (3.1) in the following order:

Ill-posedness in Sobolev spaces

$\rightarrow$ Local well-posedness in the space of analytic functions

$\rightarrow$ Finite-time blowup of solutions

$\rightarrow$ Long-time existence of solutions.

3.1 Ill-posedness *

In this section, we study the ill-posedness of system (3.1). First, observe that if initially $\mathcal{V}_0 = (u_0, v_0)$ is independent of the y variable, then any smooth solution $\mathcal{V} = (u, v)$ to system (3.1) remains independent of the y variable. Indeed, the above statement is the consequence of the uniqueness of smooth enough solutions to the IPEs, which is established in the space of analytic functions in [41, 59], and also in Section 3.2. Therefore, under these assumptions on the initial data, namely,

$$\mathcal{V}_0(x, y, z) = \mathcal{V}_0(x, z), \tag{3.5}$$

we obtain the following reduced IPEs system from original IPEs system (3.1)

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u - \Omega v + \partial_x p = 0, \\ \partial_t v + u\partial_x v + w\partial_z v + \Omega u = 0, \\ \partial_z p = 0, \\ \partial_x u + \partial_z w = 0. \end{cases} \tag{3.6}$$

We first review the results in [48, 83] about the linear and nonliear ill-posedness of system (3.6) in the absence of rotation.

*Reprinted with permission from “Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation” by Slim Ibrahim, Quyuan Lin, and Edriss S. Titi, 2021. Journal of Differential Equations, Volume 286, Pages 557–577, Copyright [2021] by Elsevier.

3.1.1 Without Rotation

When $\Omega = 0$, i.e., the rotation is absent, system (3.6) can be written as

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + w\partial_z v = 0, \\ \partial_z p = 0, \\ \partial_x u + \partial_z w = 0. \end{cases} \tag{3.7}$$

Observe that if the initial data satisfies $v_0 = 0$, then any smooth solutions (u, v) to system (3.7) must satisfy $v \equiv 0$. Therefore, we obtain a further reduced system that does not involve with v:

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \partial_x p = 0, \\ \partial_z p = 0, \\ \partial_x u + \partial_z w = 0. \end{cases} \tag{3.8}$$

Now observe that any shear flow $(U, W, P) = (U(z), 0, 0)$ is a steady solution to system (3.8). Denote by

$$(u, w, p) = (U + \widetilde{u}, W + \widetilde{w}, P + \widetilde{p}) = (U(z) + \widetilde{u}, \widetilde{w}, \widetilde{p}). \tag{3.9}$$

Here the tilde notation is used to represent the perturbation, not the baroclinic mode. The perturbation $(\widetilde{u}, \widetilde{w}, \widetilde{p})$ around this shear flow satisfies

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u}\partial_x \widetilde{u} + U\partial_x \widetilde{u} + \widetilde{w}\partial_z \widetilde{u} + \widetilde{w}U' + \partial_x \widetilde{p} = 0, \\ \partial_z \widetilde{p} = 0, \\ \partial_x \widetilde{u} + \partial_z \widetilde{w} = 0. \end{cases} \tag{3.10}$$

The linearization of system (3.10) around the zero steady state solution is

$$\begin{cases} \partial_t \widetilde{u} + U\partial_x \widetilde{u} + \widetilde{w}U' + \partial_x \widetilde{p} = 0, \\ \partial_z \widetilde{p} = 0, \\ \partial_x \widetilde{u} + \partial_z \widetilde{w} = 0. \end{cases} \tag{3.11}$$

In the work by Renardy [83], he considered system (3.11) with boundary conditions

$$\widetilde{u} \text{ is periodic in } x \text{ with period } 1, \quad \widetilde{w}|_{z=0,1} = 0. \tag{3.12}$$

We state the following result from [83], with some changes compared to the version in [83].

Theorem 3.1.1. *System* (3.11) *with boundary conditions* (3.12) *is ill-posed in Sobolev spaces and Gevrey class of order* $s > 1$.

Proof. Observe that system (3.11) with boundary conditions (3.12) has solutions of the form

$$\widetilde{u}(x,z,t) = \chi'(z) \exp\Big(2\pi i n(x - ct)\Big), \tag{3.13}$$

where c solves the following equation

$$\int_0^1 \Big(U(z) - c\Big)^{-2} dz = 0, \tag{3.14}$$

and χ is given by

$$\chi(z) = K(U(z) - c) \int_0^z (U(z) - c)^{-2} dz \tag{3.15}$$

for some constant K. Moreover, it was shown in [83] that when $U(z)$ is odd about $z = \frac{1}{2}$ and $U^{-2}(z)$ is integrable over $[0,1]$, there exists purely imaginary root $c = i\beta$ of (3.14), with $\beta \in \mathbb{R}$ and $\beta \neq 0$. $U(z)$ can be chosen to be analytic (smooth). As an example, $U(z) = \tanh(\frac{z-1/2}{d})$ for small $d > 0$, see Chen–Morrison [30].

Now suppose system (3.11) with boundary conditions (3.12) is well-posed in Sobolev spaces and Gevrey class of order $s > 1$ in the sense of Hadamard. Then for the initial data

$$\widetilde{u}_0(x, z) = \chi'(z) \exp\Big(2\pi i n x\Big), \tag{3.16}$$

which is all Sobolev spaces and Gevrey class of order $s > 1$, by uniqueness of solutions, the solution must have the form (3.13). Since c is purely imaginary, this implies Kelvin-Helmholtz type instability. Therefore, System (3.11) with boundary conditions (3.12) is ill-posed in Sobolev spaces and Gevrey class of order $s > 1$. □

In [48], Han-Kwan and Nguyen considered the nonlinear perturbation system (3.10) with boundary conditions (3.12). Based on the ill-posedness of the linear perturbation system (3.11), they established the following result regarding the ill-posedness of system (3.10).

Theorem 3.1.2. *There exists a stationary background shear flow $U(z)$ such that the following holds. For all $s \in \mathbb{N}$, $\alpha \in (0, 1]$, and $k \in \mathbb{N}$, there are families of solutions $(\tilde{u}_\epsilon)_{\epsilon>0}$ to system* (3.10) *with boundary conditions* (3.12)*, and corresponding times $t_\epsilon = \mathcal{O}(\epsilon|\log \epsilon|)$, and $(x_0, z_0) \in \mathbb{T} \times (0, 1)$ such that*

$$\lim_{\epsilon \to 0} \frac{\|\partial_z \tilde{u}_\epsilon\|_{L^2([0,t_\epsilon]\times\Omega_\epsilon)}}{\|\partial_z \tilde{u}_\epsilon|_{t=0}\|^{\alpha}_{H^s(\mathbb{T}\times(0,1))}} = +\infty, \tag{3.17}$$

where $\Omega_\epsilon = B(x_0, \epsilon^k) \times B(z_0, \epsilon^k)$.

Remark 1. Equation (3.17) indicates that system (3.10) does not satisfy the third condition for the well-posedness in the sense of Hadamard. The shear flow $U(z)$ used in Theorem 3.1.2 is the same as the one mentioned in Theorem 3.1.1, and it can be chosen to be analytic.

Remark 2. Theorem 3.1.1 and Theorem 3.1.2 imply the linear and nonlinear ill-posedness of system (3.8) in Sobolev spaces and linear ill-posedness in Gevrey class of order $s > 1$. In the case when $\Omega = 0$, system (3.8) is derived from the original $3D$ system (3.1) with initial data satisfying (3.5) and $v_0 = 0$. Therefore, the $3D$ IPEs (3.1) with $\Omega = 0$ has the same results about ill-posedness.

3.1.2 With Rotation

We will extend Theorem 3.1.1 and Theorem 3.1.2 to the case when $\Omega \neq 0$ for system (3.6). When the rotation is present ($\Omega \neq 0$), $v \equiv 0$ is no longer a solution to system (3.6) unless $u = 0$. Therefore, one needs to consider the evolution of v in system (3.6). We consider system (3.6) in the horizontal channel $\{(x, z) : 0 \leq z \leq 1, x \in \mathbb{R}\}$, subject to the boundary condition (3.2).

Observe that the steady state background flow

$$(U, V, W, P) = (U(z), -\Omega x, 0, -\frac{1}{2}\Omega^2 x^2) \tag{3.18}$$

is a solution to system (3.6) with boundary condition (3.2). Here the x-direction component U is the shear flow used in Theorem 3.1.1 and Theorem 3.1.2, the y-direction component V is a Couette shear flow, depending on Ω, in the x variable. Observe that this background flow has infinite energy. We consider the periodic perturbation around this steady state background flow for system (3.6). Denote by

$$(u, v, w, p) = (U + \widetilde{u}, V + \widetilde{v}, W + \widetilde{w}, P + \widetilde{p}). \tag{3.19}$$

Then the perturbation $(\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{p})$ around this steady state background flow satisfies

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u}\partial_x \widetilde{u} + U\partial_x \widetilde{u} + \widetilde{w}\partial_z \widetilde{u} + \widetilde{w}U' + \partial_x \widetilde{p} - \Omega \widetilde{v} = 0, \\ \partial_t \widetilde{v} + \widetilde{u}\partial_x \widetilde{v} + U\partial_x \widetilde{v} + \widetilde{w}\partial_z \widetilde{v} = 0, \\ \partial_z \widetilde{p} = 0, \\ \partial_x \widetilde{u} + \partial_z \widetilde{w} = 0, \end{cases} \tag{3.20}$$

with boundary conditions

$$\widetilde{u}, \widetilde{v} \text{ are periodic in } x \text{ with period } 1, \quad \widetilde{w}|_{z=0,1} = 0. \tag{3.21}$$

In order to show the ill-posedness of system (3.20)–(3.21), we assume by contradiction that it is well-posed. Then by uniqueness we see that if $\tilde{v}_0 = 0$, then $\tilde{v} \equiv 0$. Therefore, system (3.20) with boundary conditions (3.21) reduces to system (3.10) with boundary conditions (3.12). It follows directly from Theorem 3.1.1 and Theorem 3.1.2 that the perturbed system (3.20)–(3.21) is both linearly and nonlinearly ill-posed in any Sobolev space, and is linearly ill-posed in Gevrey class of order $s > 1$. To be more specific, we have:

Theorem 3.1.3. *System* (3.6) *with boundary condition* (3.2) *is both linearly and nonlinearly ill-posed in Sobolev spaces, and is linearly ill-posed in Gevrey class of order* $s > 1$*, in the sense that the perturbed system* (3.20)–(3.21) *around the certain steady state background flow (3.18) is both linear and nonlinearly ill-posed in Sobolev spaces, and is linearly ill-posed in Gevrey class of order* $s > 1$.

Remark 3. System (3.6) is derived from system (3.1) with initial data satisfying (3.5). Therefore, the $3D$ IPEs (3.1) with arbitrary $\Omega \in \mathbb{R}$ has the same results about the ill-posedness.

3.2 Local Well-posedness

The $3D$ IPEs is ill-posed in Sobolev spaces and Gevrey class of order $s > 1$ since the linearization around certain shear flow exhibits Kelvin-Helmholtz type instability. To overcome this strong instability, one should consider initial data u_0 that are strongly localized in Fourier, typically for which $|\hat{u}_0(\boldsymbol{k}, z)| \lesssim e^{-\delta|\boldsymbol{k}|^{1/s}}$ with $\boldsymbol{k} \in \mathbb{Z}^2$, $\delta > 0$, and $s = 1$. Such localization condition corresponds to Gevrey class of order $s = 1$ in the horizontal variables x and y, which is exactly the space of analytic functions. Therefore, we will work in the space of analytic functions for local well-posedness of the $3D$ IPEs.

In [59], Kukavica–Temam–Vicol–Ziane have shown the local well-posedness of the $3D$ IPEs in the space of analytic functions. However, the time of existence they obtained shrinks to zero as the rate of rotation $|\Omega|$ increases toward infinity. This is contrary to what we expect: fast rotation should prolong the life-span of solutions. We prove this result again using different framework, and improve their result by showing that the time of existence can be independent of Ω.

Before we state the result, we will first do the reformulation of the problem.

3.2.1 Reformulation of The Problem

Instead of the physical domain $\mathcal{D} = \left\{(x, y, z) : 0 \leq z \leq 1, (x, y) \in \mathbb{R}^2\right\}$ and the boundary condition (3.2) discussed at the beginning of this chapter, we consider the domain to be three-dimensional unit torus $\mathbb{T}^3$, and subject to the following boundary conditions and initial condition:

$$\mathcal{V} \text{ is periodic in } \boldsymbol{x} \text{ with period } 1,\ \mathcal{V} \text{ is even in } z \text{ and } w \text{ is odd in } z. \tag{3.22}$$

$$\mathcal{V}|_{t=0} = \mathcal{V}_0. \tag{3.23}$$

Observe that the space of periodic functions with respect to z with the symmetry condition (3.22) is invariant under the dynamics of system (3.1). If the original physical domain is $\mathcal{D} = \left\{(x, y, z) : 0 \leq z \leq H, (x, y) \in \mathbb{T}^2\right\}$ with $H = \frac{1}{2}$, then the solution to system (3.1) in $\mathbb{T}^3$ subject to (3.22) restricted on the original domain will solve the original physical problem. Notably here we should also assume the initial condition $\mathcal{V}_0$ for the original physical problem is even extendable in z variable so that we are able to work in $\mathbb{T}^3$. Working in $\mathbb{T}^3$ allows us to use Fourier analysis, and makes the mathematical presentation simpler and more beautiful.

In this work, we assume that

$$\int_{\mathbb{T}^3} \mathcal{V}_0(\boldsymbol{x}) d\boldsymbol{x} = 0. \tag{3.24}$$

This assumption is made to simplify the mathematical presentation. See Remark 5 for detailed explanation. Integrating the first equation in system (3.1) over $\mathbb{T}^3$, by integration by parts, thanks to the third equation in system (3.1) (incompressible condition) and boundary conditions (3.22), we obtain

$$\partial_t \int_{\mathbb{T}^3} \mathcal{V} d\boldsymbol{x} + \Omega \int_{\mathbb{T}^3} \mathcal{V}^{\perp} d\boldsymbol{x} = 0. \tag{3.25}$$

Therefore, for any time $t \geq 0$, $\mathcal{V}$ has zero mean in $\mathbb{T}^3$:

$$\int_{\mathbb{T}^3} \mathcal{V}(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{T}^3} \mathcal{V}_0(\boldsymbol{x}) d\boldsymbol{x} = 0. \tag{3.26}$$

Denote by

$$\dot{L}^2 := \Big\{ \varphi \in L^2(\mathbb{T}^3, \mathbb{R}^2) : \int_{\mathbb{T}^3} \varphi(\boldsymbol{x}) d\boldsymbol{x} = 0 \Big\}. \tag{3.27}$$

From the third equation in system (3.1) (incompressible condition) and boundary conditions (3.22), we know that

$$\nabla_h \cdot \overline{\mathcal{V}} = \int_0^1 \nabla_h \cdot \mathcal{V}(\boldsymbol{x}', z) dz = - \int_0^1 \partial_z w(\boldsymbol{x}', z) dz = 0. \tag{3.28}$$

Since $\nabla_h \cdot \overline{\mathcal{V}} = 0$, and $\overline{\mathcal{V}}$ has zero mean over $\mathbb{T}^2$ due to (3.26), we know there exists a stream function $\psi(\boldsymbol{x}')$ such that $\overline{\mathcal{V}} = \nabla_h^\perp \psi = (-\partial_y \psi, \partial_x \psi)$. That is, $\mathcal{V} \in \mathcal{S}$, where

$$\mathcal{S} := \Big\{ \varphi \in \dot{L}^2 : \nabla_h \cdot \overline{\varphi} = 0 \Big\} = \Big\{ \varphi \in \dot{L}^2 : \varphi = \nabla_h^\perp \psi(\boldsymbol{x}') + \widetilde{\varphi}(\boldsymbol{x}) \Big\}. \tag{3.29}$$

The time of existence of solutions to the $3D$ IPEs obtained in [59] shrinks to zero as $|\Omega|$ increases toward infinite. The reason behind this is that the pressure term was computed explicitly, which contains Ω. This makes the estimates depend on Ω, and thus forces the time of existence of solutions shrink to zero as $|\Omega|$ increases toward infinite. In order to show the time can be independent of Ω, the idea is to eliminate the pressure term in system (3.1). As in the study of Navier-Stokes equations, one can use Leray projection to eliminate pressure. Notice that although $\nabla_h \cdot \mathcal{V} = -\partial_z w \neq 0$, we have another incompressible condition (3.28). Therefore, the idea is to apply Leray projection on the evolution of the barotropic mode $\overline{\mathcal{V}}$.

For this reason, we apply $\mathbb{P}_h P_0$ and $I - P_0$ to the first equation in system (3.1). Recall that $\overline{\mathcal{V}} = \nabla_h^\perp \psi$. Therefore,

$$\overline{\mathcal{V}}^\perp = -\nabla_h \psi \tag{3.30}$$

can be combined with $\nabla_h p$. By integration by parts, thanks to (3.22) and (3.28), we derive the evolution equations for the barotropic mode $\overline{\mathcal{V}}$ and the baroclinic mode $\widetilde{\mathcal{V}}$:

$$\begin{cases} \partial_t \overline{\mathcal{V}} + \mathbb{P}_h\Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}\Big) + \mathbb{P}_h P_0\Big((\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}\Big) = 0, & (3.31) \\ \partial_t \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} - P_0\Big(\widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}\Big) & \\ \qquad - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}} + \Omega \widetilde{\mathcal{V}}^{\perp} = 0. & (3.32) \end{cases}$$

In summary, we have the following lemma.

Lemma 3.2.1. *For $\mathcal{V} \in \mathcal{S}$, system* (3.1) *is equivalent to system (3.31)–(3.32).*

We will work on system (3.31)–(3.32) in the domain $\mathbb{T}^3$, subject to the following symmetry boundary conditions and initial conditions:

$$\overline{\mathcal{V}}, \widetilde{\mathcal{V}} \text{ are periodic in } \mathbb{T}^3 \text{ and are even in } z; \tag{3.33}$$

$$\overline{\mathcal{V}}|_{t=0} = \overline{\mathcal{V}}_0 = P_0 \mathcal{V}_0, \quad \widetilde{\mathcal{V}}|_{t=0} = \widetilde{\mathcal{V}}_0 = (I - P_0)\mathcal{V}_0, \quad \nabla_h \cdot \overline{\mathcal{V}}_0 = 0. \tag{3.34}$$

3.2.2 Main Results

We have the following theorem concerning the local well-posedness of system (3.31)–(3.34).

Theorem 3.2.2. *Assume $\overline{\mathcal{V}}_0, \widetilde{\mathcal{V}}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Then there exist a time*

$$\mathcal{T} = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2)} > 0, \tag{3.35}$$

and a function

$$\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2), \tag{3.36}$$

both independent of Ω, such that there exists a unique solution

$$(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in \mathcal{S} \cap L^\infty\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))\big) \cap L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3))\big) \tag{3.37}$$

to system (3.31)–(3.34) *on* $[0, \mathcal{T}]$. *Moreover, the unique solution* $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}})$ *depends continuously on the initial data, in the sense of (3.107).*

Thanks to Lemma 2.2.6 and Lemma 3.2.1, we have the following corollary for the original system (3.1) with boundary and initial conditions (3.22)–(3.23).

Corollary 3.2.3. *Assume* $\mathcal{V}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ *with* $r > 5/2$ *and* $\tau_0 > 0$. *Let* $\Omega \in \mathbb{R}$ *be arbitrary and fixed. Then there exist a time* $\mathcal{T}$ *defined in (3.35) and a function* $\tau(t)$ *defined in (3.36), both independent of* Ω, *such that there exists a unique solution*

$$\mathcal{V} \in \mathcal{S} \cap L^\infty\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))\big) \cap L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3))\big) \tag{3.38}$$

to system (3.1) *with* (3.22)–(3.23) *on* $[0, T]$. *Moreover, the unique solution* $\mathcal{V}$ *depends continuously on the initial data.*

For the proof of Theorem 3.2.2, we first work on Galerkin approximating system of (3.31)–(3.34), and provide energy estimates. Then, using Aubin-Lions compactness theorem (Lemma 2.2.10), one can show the existence of solutions to system (3.31)–(3.34). Finally, we establish the uniqueness of solutions and its continuous dependence on the initial data.

3.2.3 Galerkin Approximating System

In this section, we employ the standard Galerkin approximation procedure. For $\boldsymbol{k} \in \mathbb{Z}^3$, let

$$\phi_{\boldsymbol{k}} = \phi_{k_1,k_2,k_3} := \begin{cases} \sqrt{2} e^{2\pi i(k_1 x + k_2 y)} \cos(2\pi k_3 z) & \text{if } k_3 \neq 0 \\ e^{2\pi i(k_1 x + k_2 y)} & \text{if } k_3 = 0, \end{cases} \tag{3.39}$$

and

$$\mathcal{E} := \{\phi \in L^2(\mathbb{T}^3) \mid \phi = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} a_{\boldsymbol{k}} \phi_{\boldsymbol{k}},\ a_{-k_1,-k_2,k_3} = a^*_{k_1,k_2,k_3},\ \sum_{\boldsymbol{k} \in \mathbb{Z}^3} |a_{\boldsymbol{k}}|^2 < \infty\}, \tag{3.40}$$

here a^* means the complex conjugate of a. Notice that $\mathcal{E}$ is a closed subspace of $L^2(\mathbb{T}^3)$, and consists of real valued functions which are even in z variable. For any $m \in \mathbb{N}$, denote by

$$\mathcal{E}_m := \{\phi \in L^2(\mathbb{T}^3) \mid \phi = \sum_{|\boldsymbol{k}|\leq m} a_{\boldsymbol{k}}\phi_{\boldsymbol{k}},\ a_{-k_1,-k_2,k_3} = a^*_{k_1,k_2,k_3}\}, \tag{3.41}$$

the finite-dimensional subspaces of $\mathcal{E}$. For any function $f \in L^2(\mathbb{T}^3)$, denote by

$$f_{\boldsymbol{k}} := \int_{\mathbb{T}^3} f(\boldsymbol{x})\phi^*_{\boldsymbol{k}}(\boldsymbol{x})d\boldsymbol{x}, \tag{3.42}$$

and write

$$\Pi_m f := \sum_{|\boldsymbol{k}|\leq m} f_{\boldsymbol{k}}\phi_{\boldsymbol{k}}. \tag{3.43}$$

Notice that here the definition (3.42) is slightly different from the Fourier coefficient (2.7). Π_m are orthogonal projections from $L^2(\mathbb{T}^3)$ to $\mathcal{E}_m$.

Now let

$$\overline{\mathcal{V}}_m = \sum_{0\neq|\boldsymbol{k}|\leq m, k_3=0} a_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(\boldsymbol{x}'), \quad \widetilde{\mathcal{V}}_m = \sum_{|\boldsymbol{k}|\leq m, k_3\neq 0} b_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(\boldsymbol{x}', z), \tag{3.44}$$

for $m \geq 1$, and

$$\overline{\mathcal{V}}_m|_{m=0} = \widetilde{\mathcal{V}}_m|_{m=0} = 0 \tag{3.45}$$

when $m = 0$. From this definition, we know that $\overline{\mathcal{V}}_m = P_0(\overline{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m)$ and $\widetilde{\mathcal{V}}_m = (I - P_0)(\overline{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m)$. Moreover, for each $m \in \mathbb{N}$, we have

$$\int_{\mathbb{T}^3} \overline{\mathcal{V}}_m d\boldsymbol{x} = \int_{\mathbb{T}^3} \widetilde{\mathcal{V}}_m d\boldsymbol{x} = 0. \tag{3.46}$$

For each $m \geq 1$, consider the following Galerkin approximation system for our model (3.31)–

(3.32):

$$\begin{cases} \partial_t\overline{\mathcal{V}}_m + \Pi_m\mathbb{P}_h\Big(\overline{\mathcal{V}}_m\cdot\nabla_h\overline{\mathcal{V}}_m\Big) + \Pi_m\mathbb{P}_hP_0\Big((\nabla_h\cdot\widetilde{\mathcal{V}}_m)\widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m\Big) = 0, & (3.47)\\ \partial_t\widetilde{\mathcal{V}}_m + \Pi_m\Big[\widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m\cdot\nabla_h\overline{\mathcal{V}}_m + \overline{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m \\ \qquad -P_0\Big(\widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m + (\nabla_h\cdot\widetilde{\mathcal{V}}_m)\widetilde{\mathcal{V}}_m\Big) \\ \qquad -\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}_m\Big] + \Omega\widetilde{\mathcal{V}}_m^{\perp} = 0, & (3.48) \end{cases}$$

subject to the following initial conditions:

$$\overline{\mathcal{V}}_m|_{t=0} = \Pi_m\overline{\mathcal{V}}_0, \quad \widetilde{\mathcal{V}}_m|_{t=0} = \Pi_m\widetilde{\mathcal{V}}_0. \tag{3.49}$$

For each $m \geq 1$, the Galerkin approximation, system (3.47)–(3.49), corresponds to a first order system of ordinary differential equations, in the coefficients $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$, for $1 \leq |\boldsymbol{k}| \leq m$, with quadratic nonlinearity. Therefore, by the theory of ordinary differential equations, there exists some $t_m > 0$ such that system (3.47)–(3.49) admits a unique solution $(\overline{\mathcal{V}}_m, \widetilde{\mathcal{V}}_m)$ on the interval $[0, t_m]$. Observe that from (3.49), we have $a_{\boldsymbol{k}}(0)$ and $b_{\boldsymbol{k}}(0) \in \mathbb{C}$ satisfying $a_{-k_1,-k_2,k_3}(0) = a^*_{k_1,k_2,k_3}(0)$ and $b_{-k_1,-k_2,k_3}(0) = b^*_{k_1,k_2,k_3}(0)$. Thanks to the uniqueness of the solutions of the ODE system, we conclude that $a_{-k_1,-k_2,k_3}(t) = a^*_{k_1,k_2,k_3}(t)$ and $b_{-k_1,-k_2,k_3}(t) = b^*_{k_1,k_2,k_3}(t)$, for $t \in [0, t_m]$. Therefore, $\overline{\mathcal{V}}_m, \widetilde{\mathcal{V}}_m \in \mathcal{E}_m$. Thanks to (3.49), we know that $\nabla_h\cdot\overline{\mathcal{V}}_m(t=0) = 0$. Applying $2D$ divergence on (3.47), we have $\partial_t(\nabla_h\cdot\overline{\mathcal{V}}_m) = 0$. Therefore, we know $\nabla_h\cdot\overline{\mathcal{V}}_m = 0$.

In next section, we provide the energy estimates for the Galerkin approximation system.

3.2.4 Energy Estimates

In this section, we establish the energy estimates for the Galerkin approximation system (3.47)–(3.49). By virtue of Lemma 2.2.4 and Lemma 2.2.5, and since $\nabla_h\cdot\overline{\mathcal{V}}_m = 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\overline{\mathcal{V}}_m\|^2 + \|\widetilde{\mathcal{V}}_m\|^2) = 0. \tag{3.50}$$

Integrating in time yields

$$\|\overline{\mathcal{V}}_m(t)\|^2 + \|\widetilde{\mathcal{V}}_m(t)\|^2 = \|\overline{\mathcal{V}}_m(0)\|^2 + \|\widetilde{\mathcal{V}}_m(0)\|^2 \le \|\overline{\mathcal{V}}_0\|^2 + \|\widetilde{\mathcal{V}}_0\|^2. \tag{3.51}$$

Therefore, (3.51) implies that the solution $\overline{\mathcal{V}}_m$ and $\widetilde{\mathcal{V}}_m$ exist global in time.

Next, employing Lemma 2.2.4 and Lemma 2.2.5, we derive the estimate for the analytic norm, that is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\overline{\mathcal{V}}_m\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 - \Big\langle A^r e^{\tau A}\Big(\overline{\mathcal{V}}_m\cdot\nabla_h\overline{\mathcal{V}}_m\Big), A^r e^{\tau A}\overline{\mathcal{V}}_m\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot\widetilde{\mathcal{V}}_m)\widetilde{\mathcal{V}}_m\Big), A^r e^{\tau A}\overline{\mathcal{V}}_m\Big\rangle - \Big\langle A^r e^{\tau A}\Big(\widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m\Big), A^r e^{\tau A}\overline{\mathcal{V}}_m\Big\rangle, \end{aligned} \tag{3.52}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{\mathcal{V}}_m\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2 - \Big\langle A^r e^{\tau A}\Big(\widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m\Big), A^r e^{\tau A}\widetilde{\mathcal{V}}_m\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}\Big(\widetilde{\mathcal{V}}_m\cdot\nabla_h\overline{\mathcal{V}}_m\Big), A^r e^{\tau A}\widetilde{\mathcal{V}}_m\Big\rangle - \Big\langle A^r e^{\tau A}\Big(\overline{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m\Big), A^r e^{\tau A}\widetilde{\mathcal{V}}_m\Big\rangle \\ &\quad +\Big\langle A^r e^{\tau A}\Big(\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}_m\Big), A^r e^{\tau A}\widetilde{\mathcal{V}}_m\Big\rangle. \end{aligned} \tag{3.53}$$

Add estimates (3.52)–(3.53) together, and add $\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2$ to both sides. By Lemma 2.2.11–2.2.13, since $\overline{\mathcal{V}}_m$ and $\widetilde{\mathcal{V}}_m$ have zero mean, thanks to Young's inequality we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|A^r e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big) + \Big(\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big) \\ &\le \Big(\dot{\tau} + C_r(\|A^r e^{\tau A}\overline{\mathcal{V}}_m\| + \|A^r e^{\tau A}\widetilde{\mathcal{V}}_m\|) + 1\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big) \\ &\le \Big(\dot{\tau} + C_r(1 + \|e^{\tau A}\overline{\mathcal{V}}_m\|^2_{H^r} + \|e^{\tau A}\widetilde{\mathcal{V}}_m\|^2_{H^r})\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big). \end{aligned} \tag{3.54}$$

Remark 4. Here we add the term $\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2$ to both sides so that one can obtain the regularity in $L^2\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A}:H^{r+1/2}(\mathbb{T}^3))\big)$.

Let τ satisfy

$$\dot{\tau} + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2) = 0, \tag{3.55}$$

for which we can solve out that

$$\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2). \tag{3.56}$$

Denote by

$$\mathcal{T} = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2)} > 0, \tag{3.57}$$

we know that

$$\tau(t) \geq \tau(\mathcal{T}) = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2)} > 0 \tag{3.58}$$

on $t \in [0, \mathcal{T}]$. Here we require C_r to be large enough such that

$$C_r \geq 2(\widetilde{C}_r + C_{r-\frac{1}{2}}), \tag{3.59}$$

where $\widetilde{C}_r$ appears in (3.103) and $C_{r-\frac{1}{2}}$ appears in (3.104). By the continuity of τ, $\|e^{\tau A}\overline{\mathcal{V}}_m\|_{H^r}^2$, and $\|e^{\tau A}\widetilde{\mathcal{V}}_m\|_{H^r}^2$, there exists a maximal time $\mathcal{T}_1 \in (0, \mathcal{T}]$ such that

$$\|e^{\tau(t)A}\overline{\mathcal{V}}_m(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{\mathcal{V}}_m(t)\|_{H^r}^2 \leq 2(\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2) + 1 \tag{3.60}$$

on $t \in [0, \mathcal{T}_1]$. We claim that $\mathcal{T}_1 = \mathcal{T}$. On $[0, \mathcal{T}_1]$, from (3.60), we know

$$\dot{\tau} + C_r(1 + \|e^{\tau A}\overline{\mathcal{V}}_m\|_{H^r}^2 + \|e^{\tau A}\widetilde{\mathcal{V}}_m\|_{H^r}^2) \leq \dot{\tau} + 2C_r(1 + \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2) = 0. \tag{3.61}$$

From (3.54), on $t \in [0, \mathcal{T}_1]$, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|A^r e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big) + \Big(\|A^{r+1/2}e^{\tau A}\overline{\mathcal{V}}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}_m\|^2\Big) \le 0, \quad (3.62)$$

and thus

$$\|A^r e^{\tau(\mathcal{T}_1)A}\overline{\mathcal{V}}_m(\mathcal{T}_1)\|^2 + \|A^r e^{\tau(\mathcal{T}_1)A}\widetilde{\mathcal{V}}_m(\mathcal{T}_1)\|^2 \le \|A^r e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2. \quad (3.63)$$

This together with (3.51) give us

$$\|e^{\tau(\mathcal{T}_1)A}\overline{\mathcal{V}}_m(\mathcal{T}_1)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_1)A}\widetilde{\mathcal{V}}_m(\mathcal{T}_1)\|_{H^r}^2 \le \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2. \quad (3.64)$$

Therefore, if $\mathcal{T}_1 < \mathcal{T}$, then by continuity, there exists some $\mathcal{T}_2$ such that $\mathcal{T}_1 < \mathcal{T}_2 < \mathcal{T}$ and

$$\begin{aligned}\|e^{\tau(\mathcal{T}_2)A}\overline{\mathcal{V}}_m(\mathcal{T}_2)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_2)A}\widetilde{\mathcal{V}}_m(\mathcal{T}_2)\|_{H^r}^2 &\le \|e^{\tau(\mathcal{T}_1)A}\overline{\mathcal{V}}_m(\mathcal{T}_1)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_1)A}\widetilde{\mathcal{V}}_m(\mathcal{T}_1)\|_{H^r}^2 + 1 \\ &\le 2(\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2) + 1, \quad (3.65)\end{aligned}$$

which contradicts to the maximum assumption on $\mathcal{T}_1$. Therefore, $\mathcal{T}_1 = \mathcal{T}$. Thus, (3.60)–(3.62) are satisfied on $[0, \mathcal{T}]$. Therefore, (3.61) holds on $[0, \mathcal{T}]$, and we obtain

$$\|e^{\tau(t)A}\overline{\mathcal{V}}_m(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{\mathcal{V}}_m(t)\|_{H^r}^2 \le \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2. \quad (3.66)$$

For arbitrary fixed $\mathcal{T}^* \in [0, \mathcal{T}]$, from (3.56), we know that $\min\limits_{t\in[0,\mathcal{T}^*]} \tau(t) = \tau(\mathcal{T}^*)$. For $t \in [0, \mathcal{T}^*]$, integrating (3.54) from 0 to t in time, we obtain

$$\begin{aligned}&\|A^r e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}_m(t)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}_m(t)\|^2 \\ &\quad + 2\int_0^t \Big(\|A^{r+1/2}e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}_m(s)\|^2 + \|A^{r+1/2}e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}_m(s)\|^2\Big) ds \\ &\le \|A^r e^{\tau(t)A}\overline{\mathcal{V}}_m(t)\|^2 + \|A^r e^{\tau(t)A}\widetilde{\mathcal{V}}_m(t)\|^2\end{aligned}$$

$$+2\int_0^t \Big(\|A^{r+1/2}e^{\tau(s)A}\overline{\mathcal{V}}_m(s)\|^2 + \|A^{r+1/2}e^{\tau(s)A}\widetilde{\mathcal{V}}_m(s)\|^2\Big)ds$$
$$\leq \|A^r e^{\tau_0 A}\overline{\mathcal{V}}_m(0)\|^2 + \|A^r e^{\tau_0 A}\widetilde{\mathcal{V}}_m(0)\|^2 \leq \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}^2. \tag{3.67}$$

The estimates (3.51) and (3.67) together imply that

$$\begin{aligned}&\overline{\mathcal{V}}_m, \widetilde{\mathcal{V}}_m \text{ are uniformly bounded in}\\ &L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(t)A}:H^r)\big)\cap L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(t)A}:H^{r+1/2})\big),\end{aligned} \tag{3.68}$$

and

$$\begin{aligned}&\overline{\mathcal{V}}_m, \widetilde{\mathcal{V}}_m \text{ are uniformly bounded in}\\ &L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^r)\big)\cap L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r+1/2})\big).\end{aligned} \tag{3.69}$$

By Banach–Alaoglu theorem, there exist a subsequence, denoted also by $\overline{\mathcal{V}}_m, \widetilde{\mathcal{V}}_m$, and corresponding limits, $\overline{\mathcal{V}}, \widetilde{\mathcal{V}}$, respectively, such that

$$\begin{aligned}&\overline{\mathcal{V}}_m \to \overline{\mathcal{V}}, \;\; \widetilde{\mathcal{V}}_m \to \widetilde{\mathcal{V}} \text{ weakly* in } L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^r)\big)\\ &\text{and weakly in } L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r+1/2})\big).\end{aligned} \tag{3.70}$$

Moreover, $\overline{\mathcal{V}}$ and $\widetilde{\mathcal{V}}$ also satisfy the bound in (3.67). By virtue of $P_0\overline{\mathcal{V}}_m = \overline{\mathcal{V}}_m$ and $P_0\widetilde{\mathcal{V}}_m = 0$ for any $m \in \mathbb{N}$, thanks to the convergence in (3.70), one has $P_0\overline{\mathcal{V}} = \overline{\mathcal{V}}$ and $P_0\widetilde{\mathcal{V}} = 0$, which clarifies the notation.

In order to apply Aubin-Lions compactness theorem (Lemma 2.2.10), we need some estimates on $\partial_t\overline{\mathcal{V}}_m$ and $\partial_t\widetilde{\mathcal{V}}_m$. By taking L^2 inner product of (3.47) and (3.48) with arbitrary $\phi \in L^2(\mathbb{T}^3)$, thanks to Lemma 2.2.4 and 2.2.5, we have

$$\Big\langle \partial_t\overline{\mathcal{V}}_m, \phi\Big\rangle + \Big\langle \overline{\mathcal{V}}_m\cdot\nabla_h\overline{\mathcal{V}}_m, \mathbb{P}_h\Pi_m\phi\Big\rangle + \Big\langle (\nabla_h\cdot\widetilde{\mathcal{V}}_m)\widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m\cdot\nabla_h\widetilde{\mathcal{V}}_m, P_0\mathbb{P}_h\Pi_m\phi\Big\rangle = 0, \tag{3.71}$$

and

$$
\begin{aligned}
\Big\langle \partial_t \widetilde{\mathcal{V}}_m, \phi \Big\rangle + \Big\langle \widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m + \overline{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m \\
- P_0 \Big(\widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + (\nabla_h \cdot \widetilde{\mathcal{V}}_m) \widetilde{\mathcal{V}}_m \Big) \\
- \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_m(\boldsymbol{x}', s) ds \Big) \partial_z \widetilde{\mathcal{V}}_m + \Omega \widetilde{\mathcal{V}}_m^{\perp}, \Pi_m \phi \Big\rangle = 0.
\end{aligned}
\tag{3.72}
$$

By Hölder inequality and Sobolev inequality, thanks to $\|\Pi_m \phi\| \leq \|\phi\|$, $\|\mathbb{P}_h \phi\| \leq \|\phi\|$, and $\|P_0 \phi\| \leq \|\phi\|$ for any $\phi \in L^2(\mathbb{T}^3)$, since $r > 5/2$, we have

$$
\Big| \langle \partial_t \overline{\mathcal{V}}_m, \phi \rangle \Big| \leq C_r (\|\overline{\mathcal{V}}_m\|_{H^r}^2 + \|\widetilde{\mathcal{V}}_m\|_{H^r}^2) \|\phi\|, \tag{3.73}
$$

$$
\Big| \langle \partial_t \widetilde{\mathcal{V}}_m, \phi \rangle \Big| \leq C_r (\|\overline{\mathcal{V}}_m\|_{H^r}^2 + \|\widetilde{\mathcal{V}}_m\|_{H^r}^2 + |\Omega| \|\widetilde{\mathcal{V}}_m\|) \|\phi\|. \tag{3.74}
$$

Next, applying $A^{r-1/2} e^{\tau(\mathcal{T}^*)A}$ to (3.47) and (3.48), and taking L^2 inner product of (3.47) and (3.48) with arbitrary $\phi \in L^2(\mathbb{T}^3)$, thanks to Lemma 2.2.4 and 2.2.5, we have

$$
\begin{aligned}
\Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} \partial_t \overline{\mathcal{V}}_m, \phi \Big\rangle + \Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} (\overline{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m), \mathbb{P}_h \Pi_m \phi \Big\rangle \\
+ \Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} \Big((\nabla_h \cdot \widetilde{\mathcal{V}}_m) \widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m \Big), P_0 \mathbb{P}_h \Pi_m \phi \Big\rangle = 0,
\end{aligned}
\tag{3.75}
$$

and

$$
\begin{aligned}
\Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} \partial_t \widetilde{\mathcal{V}}_m, \phi \Big\rangle + \Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} \Big[\widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m + \overline{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m \\
- P_0 \Big(\widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + (\nabla_h \cdot \widetilde{\mathcal{V}}_m) \widetilde{\mathcal{V}}_m \Big) \\
- \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_m(\boldsymbol{x}', s) ds \Big) \partial_z \widetilde{\mathcal{V}}_m + \Omega \widetilde{\mathcal{V}}_m^{\perp} \Big], \Pi_m \phi \Big\rangle = 0.
\end{aligned}
\tag{3.76}
$$

By Cauchy–Schwarz inequality and Lemma 2.2.2, since $r > 5/2$, we have

$$
\begin{aligned}
& \Big| \Big\langle A^{r-1/2} e^{\tau(\mathcal{T}^*)A} \partial_t \overline{\mathcal{V}}_m, \phi \Big\rangle \Big| \\
\leq & C_r \Big(\|e^{\tau(\mathcal{T}^*)A} \overline{\mathcal{V}}_m\|_{H^r} \|e^{\tau(\mathcal{T}^*)A} \overline{\mathcal{V}}_m\|_{H^{r+1/2}} + \|e^{\tau(\mathcal{T}^*)A} \widetilde{\mathcal{V}}_m\|_{H^r} \|e^{\tau(\mathcal{T}^*)A} \widetilde{\mathcal{V}}_m\|_{H^{r+1/2}} \Big) \|\phi\|,
\end{aligned}
\tag{3.77}
$$

and

$$
\begin{aligned}
&\left|\left\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\partial_t\widetilde{\mathcal{V}}_m,\phi\right\rangle\right|\\
\leq &C_r\Big(\|e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}_m\|^2_{H^{r+1/2}}+\|e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}_m\|^2_{H^{r+1/2}}+|\Omega|\|A^re^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}_m\|\Big)\|\phi\|.
\end{aligned}
\tag{3.78}
$$

By virtue of the bound (3.69), from (3.73)–(3.74) and (3.77)–(3.78), we have

$$
\begin{aligned}
&\partial_t\overline{\mathcal{V}}_m \text{ are uniformly bounded in } L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big),\\
&\partial_t\widetilde{\mathcal{V}}_m \text{ are uniformly bounded in } L^1\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big).
\end{aligned}
\tag{3.79}
$$

By Banach–Alaoglu theorem, we have

$$
\begin{aligned}
&\partial_t\overline{\mathcal{V}}_m\to\partial_t\overline{\mathcal{V}} \text{ weakly in } L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big), \text{ weakly* in } L^\infty\big(0,\mathcal{T}^*;L^2\big),\\
&\partial_t\widetilde{\mathcal{V}}_m\to\partial_t\widetilde{\mathcal{V}} \text{ weakly* in } L^1\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big).
\end{aligned}
\tag{3.80}
$$

From (3.69) and (3.79), since $\mathcal{D}(e^{\tau A}:H^{r_1})\hookrightarrow\hookrightarrow\mathcal{D}(e^{\tau A}:H^{r_2})$ when $r_1>r_2$, by Lemma 2.2.10, for a subsequence and $0<\epsilon<1/2$, the following strong convergence holds:

$$
\begin{aligned}
&\overline{\mathcal{V}}_m\to\overline{\mathcal{V}},\ \ \widetilde{\mathcal{V}}_m\to\widetilde{\mathcal{V}} \text{ strongly in}\\
&C\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-\epsilon})\big)\cap L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r+1/2-\epsilon})\big).
\end{aligned}
\tag{3.81}
$$

3.2.5 Existence of Solutions

In this section, we establish the local in time existence of solutions to system (3.31)–(3.34). More specifically, we show the limit functions $\overline{\mathcal{V}}$ and $\widetilde{\mathcal{V}}$ we get from previous section satisfy (3.31)–(3.32) and (3.37). First, since $\nabla_h\cdot\overline{\mathcal{V}}_m=0$ for any $m\in\mathbb{N}$ and thanks to the convergence (3.81), one has $\nabla_h\cdot\overline{\mathcal{V}}=0$. By virtue of (3.46) and (3.81), we know

$$
\int_{\mathbb{T}^3}\overline{\mathcal{V}}d\boldsymbol{x}=\int_{\mathbb{T}^3}\widetilde{\mathcal{V}}d\boldsymbol{x}=0. \tag{3.82}
$$

Therefore, $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in \mathcal{S}$.

Next, by taking inner product of equation (3.47) and (3.48) with test function $\phi \in L^2(0, \mathcal{T}^*; L^2)$ in $L^2((0, \mathcal{T}^*) \times \mathbb{T}^3)$, we have

$$\Big\langle \partial_t \overline{\mathcal{V}}_m + \Pi_m \mathbb{P}_h \Big(\overline{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m \Big) + \Pi_m \mathbb{P}_h P_0 \Big((\nabla_h \cdot \widetilde{\mathcal{V}}_m) \widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m \Big), \phi \Big\rangle = 0, \tag{3.83}$$

and

$$\begin{aligned}\Big\langle \partial_t \widetilde{\mathcal{V}}_m + \Pi_m \Big[& \widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + \widetilde{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m + \overline{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m \\ & - P_0 \Big(\widetilde{\mathcal{V}}_m \cdot \nabla_h \widetilde{\mathcal{V}}_m + (\nabla_h \cdot \widetilde{\mathcal{V}}_m) \widetilde{\mathcal{V}}_m \Big) \\ & - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_m(\boldsymbol{x}', s) ds \Big) \partial_z \widetilde{\mathcal{V}}_m \Big] + \Omega \widetilde{\mathcal{V}}_m^{\perp}, \phi \Big\rangle = 0.\end{aligned} \tag{3.84}$$

From (3.70) and (3.80), we know that

$$\langle \Omega \widetilde{\mathcal{V}}_m^{\perp}, \phi \rangle \to \langle \Omega \widetilde{\mathcal{V}}^{\perp}, \phi \rangle, \quad \langle \partial_t \overline{\mathcal{V}}_m, \phi \rangle \to \langle \partial_t \overline{\mathcal{V}}, \phi \rangle \quad \langle \partial_t \widetilde{\mathcal{V}}_m, \phi \rangle \to \langle \partial_t \widetilde{\mathcal{V}}, \phi \rangle. \tag{3.85}$$

For nonlinear terms, we consider, for example,

$$\begin{aligned} & \Big| \Big\langle \Pi_m \mathbb{P}_h \Big(\overline{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m \Big), \phi \Big\rangle - \Big\langle \mathbb{P}_h \Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}} \Big), \phi \Big\rangle \Big| \\ = & \Big| \Big\langle \overline{\mathcal{V}}_m \cdot \nabla_h \overline{\mathcal{V}}_m, \Pi_m \mathbb{P}_h \phi \Big\rangle - \Big\langle \overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}, \mathbb{P}_h \phi \Big\rangle \Big| \\ \leq & \Big| \Big\langle \overline{\mathcal{V}}_m \cdot \nabla_h (\overline{\mathcal{V}}_m - \overline{\mathcal{V}}), \Pi_m \mathbb{P}_h \phi \Big\rangle \Big| + \Big| \Big\langle (\overline{\mathcal{V}}_m - \overline{\mathcal{V}}) \cdot \nabla_h \overline{\mathcal{V}}, \Pi_m \mathbb{P}_h \phi \Big\rangle \Big| \\ & + \Big| \Big\langle \overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}, (\Pi_m \mathbb{P}_h \phi - \mathbb{P}_h \phi) \Big\rangle \Big| \\ \leq & C_r \Big(\| \overline{\mathcal{V}}_m \|_{L^\infty(0, \mathcal{T}^*; H^r)} + \| \overline{\mathcal{V}} \|_{L^\infty(0, \mathcal{T}^*; H^r)} \Big) \| \overline{\mathcal{V}}_m - \overline{\mathcal{V}} \|_{L^2(0, \mathcal{T}^*; H^r)} \| \phi \|_{L^2(0, \mathcal{T}^*; L^2)} \\ & + C_r \| \overline{\mathcal{V}} \|^2_{L^4(0, \mathcal{T}^*; H^r)} \| \Pi_m \phi - \phi \|_{L^2(0, \mathcal{T}^*; L^2)}, \end{aligned} \tag{3.86}$$

and

$$
\begin{aligned}
&\Big|\Big\langle \Pi_m\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}_m,\phi\Big\rangle-\Big\langle\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}},\phi\Big\rangle\Big|\\
\le&\Big|\Big\langle\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}_m(\boldsymbol{x}',s)ds\Big)\partial_z(\widetilde{\mathcal{V}}_m-\widetilde{\mathcal{V}}),\Pi_m\phi\Big\rangle\Big|\\
&+\Big|\Big\langle\Big(\int_0^z \nabla_h\cdot(\widetilde{\mathcal{V}}_m-\widetilde{\mathcal{V}})(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}},\Pi_m\phi\Big\rangle\Big|\\
&+\Big|\Big\langle\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}},(\Pi_m\phi-\phi)\Big\rangle\Big|\\
\le&C_r\Big(\|\widetilde{\mathcal{V}}_m\|_{L^\infty(0,\mathcal{T}^*;H^r)}+\|\widetilde{\mathcal{V}}\|_{L^\infty(0,\mathcal{T}^*;H^r)}\Big)\|\widetilde{\mathcal{V}}_m-\widetilde{\mathcal{V}}\|_{L^2(0,\mathcal{T}^*;H^r)}\|\phi\|_{L^2(0,\mathcal{T}^*;L^2)}\\
&+C_r\|\widetilde{\mathcal{V}}\|^2_{L^4(0,\mathcal{T}^*;H^r)}\|\Pi_m\phi-\phi\|_{L^2(0,\mathcal{T}^*;L^2)},
\end{aligned}
\tag{3.87}
$$

where we have used Hölder inequality, Sobolev inequality, and $r>5/2$. By virtue of of (3.69), (3.70) and (3.81), the right hand side of (3.86) and (3.87) go to zero as $m\to\infty$.

One can show similarly that all other nonlinear terms converge to corresponding limit terms. Therefore, for arbitrary $\phi\in L^2(0,\mathcal{T}^*;L^2)$, we have

$$
\Big\langle \partial_t\overline{\mathcal{V}}+\mathbb{P}_h\Big(\overline{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}\Big)+\mathbb{P}_hP_0\Big((\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}+\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}\Big),\phi\Big\rangle=0, \tag{3.88}
$$

and

$$
\begin{aligned}
&\Big\langle\partial_t\widetilde{\mathcal{V}}+\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}+\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}+\overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}-P_0\Big(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}_m+(\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}\Big)\\
&\quad-\Big(\int_0^z\nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}+\Omega\widetilde{\mathcal{V}}^\perp,\phi\Big\rangle=0.
\end{aligned}
\tag{3.89}
$$

This implies that (3.31) and (3.32) hold in $L^2(0,\mathcal{T}^*;L^2)$. From (3.80), we conclude that (3.31) holds in $L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big)$ and (3.32) holds in $L^1\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big)$.

Next, due to (3.81), one has, for every $t\in[0,\mathcal{T}^*]$, $\overline{\mathcal{V}}_m(t)\to\overline{\mathcal{V}}(t)$, $\widetilde{\mathcal{V}}_m(t)\to\widetilde{\mathcal{V}}(t)$ in L^2. In particular, $\overline{\mathcal{V}}_m(0)\to\overline{\mathcal{V}}(0)$, $\widetilde{\mathcal{V}}_m(0)\to\widetilde{\mathcal{V}}(0)$ in L^2. On the other hand, by (3.49), we have $\overline{\mathcal{V}}_m(0)\to\overline{\mathcal{V}}_0$, $\widetilde{\mathcal{V}}_m(0)\to\widetilde{\mathcal{V}}_0$ in L^2. As a result, $\overline{\mathcal{V}},\widetilde{\mathcal{V}}$ satisfy the desired initial condition: $\overline{\mathcal{V}}(0)=\overline{\mathcal{V}}_0$

and $\widetilde{\mathcal{V}}(0) = \widetilde{\mathcal{V}}_0$. Recall that the choice of $\mathcal{T}^* \in [0, \mathcal{T}]$ is arbitrary, and in particular we can choose $\mathcal{T}^* = \mathcal{T}$ so that all of the results above hold with $\mathcal{T}^*$ replaced by $\mathcal{T}$.

Finally, we need to show (3.37). First, we have shown that $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in \mathcal{S}$. Recall that for arbitrary $\mathcal{T}^* \in [0, \mathcal{T}]$, by (3.67) and the convergence (3.70), we have

$$\|A^r e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}(t)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}(t)\|^2 \leq \|A^r e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2, \tag{3.90}$$

for $t \in [0, \mathcal{T}^*]$, and especailly for $t = \mathcal{T}^*$. Therefore,

$$\|A^r e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}(\mathcal{T}^*)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}(\mathcal{T}^*)\|^2 \leq \|A^r e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2, \tag{3.91}$$

for any $\mathcal{T}^* \in [0, \mathcal{T}]$. Since the L^2 energy is conversed, we have

$$\|e^{\tau(\mathcal{T}^*)A}\overline{\mathcal{V}}(\mathcal{T}^*)\|^2_{H^r} + \|e^{\tau(\mathcal{T}^*)A}\widetilde{\mathcal{V}}(\mathcal{T}^*)\|^2_{H^r} \leq \|e^{\tau_0 A}\overline{\mathcal{V}}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2_{H^r}. \tag{3.92}$$

Therefore, the solution $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^\infty\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r)\big)$ for $\tau = \tau(t)$ defined in (3.56).

In order to show $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$, define the inner product:

$$\langle f, g\rangle_H := \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \int_0^{\mathcal{T}} (1 + |\boldsymbol{k}|^{2r+1} e^{2\tau(t)|\boldsymbol{k}|})(\hat{f}_{\boldsymbol{k}} \cdot \hat{g}_{\boldsymbol{k}}) dt. \tag{3.93}$$

$L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$ with inner product defined by (3.93) is a Hilbert space. By setting $\mathcal{T}^* = \mathcal{T}$ in (3.68), we know $\{\overline{\mathcal{V}}_m\}$ and $\{\widetilde{\mathcal{V}}_m\}$ are bounded sequence in this Hilbert space, and therefore there exist weak limit $\overline{\mathcal{V}}^*$ and $\widetilde{\mathcal{V}}^*$ such that

$$\int_0^{\mathcal{T}} \|e^{\tau(t)A}\overline{\mathcal{V}}^*(t)\|^2_{H^{r+1/2}} dt \leq \liminf_{m\to\infty} \int_0^{\mathcal{T}} \|e^{\tau(t)A}\overline{\mathcal{V}}_m(t)\|^2_{H^{r+1/2}} dt < \infty, \tag{3.94}$$

$$\int_0^{\mathcal{T}} \|e^{\tau(t)A}\widetilde{\mathcal{V}}^*(t)\|^2_{H^{r+1/2}} dt \leq \liminf_{m\to\infty} \int_0^{\mathcal{T}} \|e^{\tau(t)A}\widetilde{\mathcal{V}}_m(t)\|^2_{H^{r+1/2}} dt < \infty. \tag{3.95}$$

Thus, $(\overline{\mathcal{V}}^*, \widetilde{\mathcal{V}}^*) \in L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$. By uniqueness of weak limit, we know $\overline{\mathcal{V}} =$

$\overline{\mathcal{V}}^*, \widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}^*$, thus, $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$. Therefore, (3.37) holds. The existence of solutions to system (3.31)–(3.34) is proved.

3.2.6 Uniqueness of Solutions and Continuous Dependence on The Initial Data

In this section, we show the uniqueness of solutions and the continuous dependence on the initial data. Let $(\overline{\mathcal{V}}_1, \widetilde{\mathcal{V}}_1)$ and $(\overline{\mathcal{V}}_2, \widetilde{\mathcal{V}}_2)$ be two strong solutions to system (3.31)–(3.34) with initial data $((\overline{\mathcal{V}}_0)_1, (\widetilde{\mathcal{V}}_0)_1)$ and $((\overline{\mathcal{V}}_0)_2, (\widetilde{\mathcal{V}}_0)_2)$, respectively. Assume the radius of analyticity for initial data $((\overline{\mathcal{V}}_0)_1, (\widetilde{\mathcal{V}}_0)_1)$ is τ_{10}, and for $((\overline{\mathcal{V}}_0)_2, (\widetilde{\mathcal{V}}_0)_2)$ is τ_{20}. Let $\tau_0 = \min\{\tau_{10}, \tau_{20}\}$, and

$$M = \max\Big\{\|e^{\tau_{10}A}(\overline{\mathcal{V}}_0)_1\|_{H^r}^2 + \|e^{\tau_{10}A}(\widetilde{\mathcal{V}}_0)_1\|_{H^r}^2, \|e^{\tau_{20}A}(\overline{\mathcal{V}}_0)_2\|_{H^r}^2 + \|e^{\tau_{20}A}(\widetilde{\mathcal{V}}_0)_2\|_{H^r}^2\Big\}. \quad (3.96)$$

Denote by $\overline{\mathcal{V}} = \overline{\mathcal{V}}_1 - \overline{\mathcal{V}}_2$ and $\widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}_1 - \widetilde{\mathcal{V}}_2$. By virtue of (3.56) and (3.57), we define

$$\widetilde{\tau}(t) = \tau_0 - 2tC_r(1+M), \quad \widetilde{\mathcal{T}} = \frac{\tau_0}{1 + 2C_r(1+M)}. \quad (3.97)$$

Here C_r satisfies (3.59).

From previous sections, and by the definition of τ_0 and M, we know

$$(\overline{\mathcal{V}}_i, \widetilde{\mathcal{V}}_i), (\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^\infty\big(0, \widetilde{\mathcal{T}}; \mathcal{D}(e^{\widetilde{\tau}(t)A} : H^r)\big) \cap L^2\big(0, \widetilde{\mathcal{T}}; \mathcal{D}(e^{\widetilde{\tau}(t)A} : H^{r+1/2})\big), \quad (3.98)$$

and

$$\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_i\|_{H^r}^2 + \|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_i\|_{H^r}^2 \le M \quad (3.99)$$

for $i = 1, 2$. From (3.31)–(3.32), it is clear that

$$\begin{cases} \partial_t \overline{\mathcal{V}} + \mathbb{P}_h\Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}}\Big) \\ \qquad + \mathbb{P}_h P_0\Big((\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1 + (\nabla_h \cdot \widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}}\Big) = 0, & (3.100) \\ \partial_t \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} \\ \qquad - P_0\Big((\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1 + (\nabla_h \cdot \widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}}\Big) \\ \qquad - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}}_1 - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_2(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}} + \Omega \widetilde{\mathcal{V}}^\perp = 0. & (3.101) \end{cases}$$

Taking L^2 inner product of (3.100) with $\overline{\mathcal{V}}$ and (3.101) with $\widetilde{\mathcal{V}}$, applying $A^{r-1/2}e^{\widetilde{\tau}A}$ to (3.100) and (3.101) and taking L^2 inner product with $A^{r-1/2}e^{\widetilde{\tau}A}\overline{\mathcal{V}}$ and $A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}$, correspondingly, thanks to Lemma 2.2.4 and Lemma 2.2.5, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|e^{\widetilde{\tau}(t)A}\overline{\mathcal{V}}(t)\|^2_{H^{r-1/2}} + \|e^{\widetilde{\tau}(t)A}\widetilde{\mathcal{V}}(t)\|^2_{H^{r-1/2}}\Big) - \dot{\widetilde{\tau}}\Big(\|A^r e^{\widetilde{\tau}A}\overline{\mathcal{V}}\|^2 + \|A^r e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\|^2\Big) \\ &+ \Big\langle \overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1 + (\nabla_h \cdot \widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}}, \overline{\mathcal{V}}\Big\rangle \\ &+ \Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1 + (\nabla_h \cdot \widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}} \\ &\qquad\qquad + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}}\Big), A^{r-1/2}e^{\widetilde{\tau}A}\overline{\mathcal{V}}\Big\rangle \\ &+ \Big\langle \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} \\ &\qquad - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}}_1 - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_2(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}\Big\rangle \\ &+ \Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big[\widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} \\ &\quad - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}}_1 - \Big(\int_0^z \nabla_h \cdot \widetilde{\mathcal{V}}_2(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{\mathcal{V}}\Big], A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\Big\rangle = 0. \qquad (3.102) \end{aligned}$$

Thanks to Hölder inequality, Young's inequality and Sobolev inequality, since $r > 5/2$, and noticing that $\overline{\mathcal{V}}$ and $\widetilde{\mathcal{V}}$ have zero mean over $\mathbb{T}^3$, we can apply Poincaré inequality to have

$$\begin{aligned} &\Big|\Big\langle \overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1 + (\nabla_h \cdot \widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}}, \overline{\mathcal{V}}\Big\rangle \\ &\quad + \Big\langle \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}_1 + \widetilde{\mathcal{V}}_2 \cdot \nabla_h \overline{\mathcal{V}} + \overline{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}_1 + \overline{\mathcal{V}}_2 \cdot \nabla_h \widetilde{\mathcal{V}} \end{aligned}$$

$$
\begin{aligned}
&-\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}_1-\Big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}},\widetilde{\mathcal{V}}\Big\rangle\Big|\\
&\leq \widetilde{C}_r\Big(\|\overline{\mathcal{V}}_1\|_{H^r}+\|\overline{\mathcal{V}}_2\|_{H^r}+\|\widetilde{\mathcal{V}}_1\|_{H^r}+\|\widetilde{\mathcal{V}}_2\|_{H^r}\Big)\Big(\|\overline{\mathcal{V}}\|^2_{H^{r-1/2}}+\|\widetilde{\mathcal{V}}\|^2_{H^{r-1/2}}\Big)\\
&\leq \widetilde{C}_r\Big(\|\overline{\mathcal{V}}_1\|_{H^r}+\|\overline{\mathcal{V}}_2\|_{H^r}+\|\widetilde{\mathcal{V}}_1\|_{H^r}+\|\widetilde{\mathcal{V}}_2\|_{H^r}\Big)\Big(\|A^r e^{\widetilde{\tau}A}\overline{\mathcal{V}}\|^2+\|A^r e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\|^2\Big),
\end{aligned}
\tag{3.103}
$$

where the last step we apply Poincaré inequality. For higher order part, thanks to Lemma 2.2.11–2.2.13, by Young's inequality, we have

$$
\begin{aligned}
&\Big|\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big(\overline{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}_1+\overline{\mathcal{V}}_2\cdot\nabla_h\overline{\mathcal{V}}+(\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}_1+(\nabla_h\cdot\widetilde{\mathcal{V}}_2)\widetilde{\mathcal{V}}\\
&\qquad\qquad +\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}_1+\widetilde{\mathcal{V}}_2\cdot\nabla_h\widetilde{\mathcal{V}}\Big),A^{r-1/2}e^{\widetilde{\tau}A}\overline{\mathcal{V}}\Big\rangle\\
&+\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big[\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}_1+\widetilde{\mathcal{V}}_2\cdot\nabla_h\widetilde{\mathcal{V}}+\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}_1+\widetilde{\mathcal{V}}_2\cdot\nabla_h\overline{\mathcal{V}}+\overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}_1+\overline{\mathcal{V}}_2\cdot\nabla_h\widetilde{\mathcal{V}}\\
&\quad-\Big(\int_0^z\nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}_1-\Big(\int_0^z\nabla_h\cdot\widetilde{\mathcal{V}}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{\mathcal{V}}\Big],A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\Big\rangle\Big|\\
&\leq C_{r-\frac12}\Big(\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_2\|_{H^r}\Big)\\
&\qquad\qquad\times\Big(\|A^r e^{\widetilde{\tau}A}\overline{\mathcal{V}}\|^2+\|A^r e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\|^2\Big).
\end{aligned}
\tag{3.104}
$$

Combining (3.102)–(3.104), thanks to (3.59), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|e^{\widetilde{\tau}(t)A}\overline{\mathcal{V}}(t)\|^2_{H^{r-1/2}}+\|e^{\widetilde{\tau}(t)A}\widetilde{\mathcal{V}}(t)\|^2_{H^{r-1/2}}\Big)\\
&\leq\Big[\dot{\widetilde{\tau}}+\frac12 C_r\Big(\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_2\|_{H^r}\Big)\Big]\\
&\quad\times\Big(\|A^r e^{\widetilde{\tau}A}\overline{\mathcal{V}}\|^2+\|A^r e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}\|^2\Big).
\end{aligned}
\tag{3.105}
$$

Since $\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_i\|^2_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_i\|^2_{H^r}\leq M$ for $i=1,2$, by Cauchy–Schwarz inequality, we know that

$$
\begin{aligned}
&\dot{\widetilde{\tau}}+\frac12 C_r\Big(\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{\mathcal{V}}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{\mathcal{V}}_2\|_{H^r}\Big)\\
&\leq -2C_r(1+M)+\sqrt{2}C_r\sqrt{M}\leq(\frac{\sqrt{2}}{2}-2)C_r(1+M)<0,
\end{aligned}
\tag{3.106}
$$

for $t \in [0, \widetilde{\mathcal{T}}]$. Therefore, for $t \in [0, \widetilde{\mathcal{T}}]$, we have

$$\|e^{\widetilde{\tau}(t)A}\overline{\mathcal{V}}(t)\|^2_{H^{r-1/2}} + \|e^{\widetilde{\tau}(t)A}\widetilde{\mathcal{V}}(t)\|^2_{H^{r-1/2}} \leq \|e^{\widetilde{\tau}_0 A}\overline{\mathcal{V}}_0\|^2_{H^{r-1/2}} + \|e^{\widetilde{\tau}_0 A}\widetilde{\mathcal{V}}_0\|^2_{H^{r-1/2}}. \tag{3.107}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $\overline{\mathcal{V}}_0 = \widetilde{\mathcal{V}}_0 = 0$ and $\tau_{10} = \tau_{20}$, we have $\overline{\mathcal{V}} = \widetilde{\mathcal{V}} = 0$ for all $t \in [0, \widetilde{\mathcal{T}}]$. Moreover, from (3.57), (3.97), and the definition of M in (3.96), we know $\widetilde{\mathcal{T}} = \mathcal{T}$. Therefore, the solution is unique, and we complete the proof of Theorem 3.2.2.

Remark 5. In case that $\int_{\mathbb{T}^3} \mathcal{V}(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathbb{T}^2} \overline{\mathcal{V}}(\boldsymbol{x}')d\boldsymbol{x}' \neq 0$, the only change in system (3.31)–(3.34) is in (3.31) which will become

$$\partial_t \overline{\mathcal{V}} + \mathbb{P}_h\Big(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}\Big) + \mathbb{P}_h P_0\Big((\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}}\Big) + \Omega \int_{\mathbb{T}^2} \overline{\mathcal{V}}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}' = 0. \tag{3.108}$$

The additional term $\Omega \int_{\mathbb{T}^2} \overline{\mathcal{V}}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'$ appearing in (3.108) does not change the energy estimates. Since

$$\int_{\mathbb{T}^2} \overline{\mathcal{V}}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}' \cdot \int_{\mathbb{T}^2} \overline{\mathcal{V}}(\boldsymbol{x}')d\boldsymbol{x}' = 0, \tag{3.109}$$

the conservation of L^2 norm does not change. Since $\Omega \int_{\mathbb{T}^2} \overline{\mathcal{V}}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'$ is a constant vector in spatial variables, when we apply the operator $A^r e^{\tau A}$ to it, it will disappear. Therefore, this additional term does not affect the higher order energy estimates. Thus, when $\int_{\mathbb{T}^2} \overline{\mathcal{V}}(\boldsymbol{x}')d\boldsymbol{x}' \neq 0$, we still have the same results.

In the next section, we show that the local well-posedness result is sharp in the sense that it cannot be extended to a global result by constructing finite-time blowup smooth solutions.

3.3 Finite-time Blowup †

We have established the local well-posedness of the $3D$ IPEs for a time that is independend of the rotation rate $|\Omega|$. The next question is, whether the solutions can exist globally in time, or blow

†Reprinted with permission from "Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation" by Slim Ibrahim, Quyuan Lin, and Edriss S. Titi, 2021. Journal of Differential Equations, Volume 286, Pages 557–577, Copyright [2021] by Elsevier.

up in finite time? In this section, we will answer this question by constructing finite-time blowup smooth solutions, and therefore the global well-posedness cannot be obtained.

We again assume the initial condition satisfies (3.5), and consider the reduced IPEs system (3.6). In the absense of rotation ($\Omega = 0$), Cao–Ibrahim–Nakanishi–Titi [17] and Wong [91] have constructed smooth solutions that blow up in finite time. In the followings, we will show that their results still hold for any $\Omega \in \mathbb{R}$. Therefore, our results contain and extend their results.

We first reduce (3.6) further more.

3.3.1 Reduced System

We assume that u and v are odd in the x variable, and that w and p are even in the x variable. Observe that such symmetric conditions are invariant under smooth dynamics of system (3.6). From the last equation in system (3.6) (incompressible condition) and boundary condition (3.2), we know

$$\int_0^1 u_x(t,x,z)dz = 0. \tag{3.110}$$

Differentiating the first two equations in system (3.6) with respect to x, we have

$$\begin{cases} u_{xt} + u\,u_{xx} + u_x^2 + w_x u_z + w u_{xz} - \Omega\, v_x + p_{xx} = 0, & (3.111) \\ v_{xt} + u_x\, v_x + u v_{xx} + w_x v_z + w v_{xz} + \Omega\, u_x = 0. & (3.112) \end{cases}$$

Thanks to (3.110), integrating (3.111) with respect to z over the interval $[0,1]$, an integration by parts together with last two equations in system (3.6) and boundary condition (3.2) implies

$$p_{xx} = \int_0^1 \Big[-2(uu_x)_x + \Omega v_x \Big] dz. \tag{3.113}$$

Let

$$W(t,z) = w(t,0,z), \quad V(t,z) = -v_x(t,0,z).$$

Plugging (3.113) back to (3.111), and by virtue of the oddness of u and v and evenness of w and

p, system (3.111)–(3.112) restricts on the line $x = 0$ becomes

$$\begin{cases} W_{tz} - (W_z)^2 + WW_{zz} + 2\int_0^1 W_z^2(t,z)dz - \Omega V + \Omega \int_0^1 V(t,z)dz = 0, & (3.114) \\ V_t - W_z V + WV_z + \Omega W_z = 0. & (3.115) \end{cases}$$

The corresponding initial and boundary conditions are

$$W(0,z) = w_0(0,z) =: W_0(z), \quad V(0,z) = v_0(0,z) =: V_0(z), \tag{3.116}$$

$$W(t,0) = W(t,1) = 0. \tag{3.117}$$

The uniqueness of smooth solutions to system (3.114)–(3.117) is needed for establishing the blowup result. However, in the following proposition we show the local well-posedness of system (3.114)–(3.117) with initial condition satisfying $(W_0, V_0) \in H^2 \times H^1$.

Proposition 3.3.1. *Suppose that* $(W_0, V_0) \in H^2 \times H^1$. *Then there exists a time* $\mathcal{T}$ *such that there exists a unique solution* (W, V) *to system* (3.114)–(3.117) *for* $t \in [0, \mathcal{T}]$, *which depends continuously on the initial data* (W_0, V_0). *Moreover,* (W, V) *satisfy*

$$\begin{aligned} &W \in L^\infty(0, \mathcal{T}; H^2) \cap C([0, \mathcal{T}]; H^1), \\ &V \in L^\infty(0, \mathcal{T}; H^1) \cap C([0, \mathcal{T}]; L^2). \end{aligned} \tag{3.118}$$

Proof. First observe that due to the boundary condition (3.117), the Poincaré inequality implies that $\|W_z\|_{L^2}$ is equivalent to $\|W\|_{H^1}$ and $\|W_{tz}\|_{L^2}$ is equivalent to $\|W_t\|_{H^1}$.

For the existence of solutions, we only provide the formal energy estimates. These estimates can be justified rigorously by deriving them first to the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness lemma (Lemma 2.2.10).

Taking the z derivative to (3.114)–(3.115), we have

$$\begin{cases} W_{tzz} + WW_{zzz} - W_z W_{zz} - \Omega V_z = 0, & (3.119) \\ V_{zt} + WV_{zz} - VW_{zz} + \Omega W_{zz} = 0. & (3.120) \end{cases}$$

By taking the L^2 inner product of (3.114) with W_z, (3.115) with V, (3.119) with W_{zz}, and (3.120) with V_z, by integration by parts and thanks to (3.117), one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|W_z\|_{L^2}^2+\|V\|_{L^2}^2+\|W_{zz}\|_{L^2}^2+\|V_z\|_{L^2}^2\Big)\\=&\frac{3}{2}\int_0^1 W_z\Big(W_z^2+V^2+W_{zz}^2+\frac{1}{3}V_z^2\Big)dz+\int_0^1 VV_zW_{zz}dz.\end{aligned} \tag{3.121}$$

Thanks to Hölder inequality, Sobolev inequality, and Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|W_z\|_{L^2}^2+\|V\|_{L^2}^2+\|W_{zz}\|_{L^2}^2+\|V_z\|_{L^2}^2\Big)\\\leq&C\Big(\|W_z\|_{L^2}^2+\|V\|_{L^2}^2+\|W_{zz}\|_{L^2}^2+\|V_z\|_{L^2}^2\Big)^{3/2}.\end{aligned} \tag{3.122}$$

This implies that there exists some finite time $\mathcal{T}>0$ such that

$$W\in L^\infty(0,\mathcal{T};H^2),\ V\in L^\infty(0,\mathcal{T};H^1). \tag{3.123}$$

In order to apply Aubin-Lions compactness lemma, we also need the energy estimates for $\partial_t W$ and $\partial_t V$. By taking the L^2 inner product of (3.114) with arbitrary $\phi\in L^2(0,1)$, thanks to Hölder inequality and Sobolev inequality, one has

$$\Big|\Big\langle W_{tz},\phi\Big\rangle\Big|\leq C(\|W\|_{H^2}^2+\|V\|_{L^2})\|\phi\|_{L^2}. \tag{3.124}$$

From (3.123), we have $W_{tz}\in L^\infty(0,\mathcal{T};L^2)$, which is equivalent to

$$W_t\in L^\infty(0,\mathcal{T};H^1). \tag{3.125}$$

Similarly, by taking the L^2 inner product of (3.115) with arbitrary $\phi\in L^2(0,1)$, thanks to Hölder inequality, Sobolev inequality, and Young's inequality, one has

$$\Big|\Big\langle V_t,\phi\Big\rangle\Big|\leq C(\|W\|_{H^1}^2+\|W\|_{H^1}+\|V\|_{H^1}^2)\|\phi\|_{L^2}. \tag{3.126}$$

Thanks to (3.123), one obtains

$$V_t \in L^\infty(0,\mathcal{T};L^2). \tag{3.127}$$

By virtue of Aubin-Lions compactness lemma, thanks to (3.123), (3.125), and (3.127), we have

$$W \in C([0,\mathcal{T}];H^1),\ V \in C([0,\mathcal{T}];L^2). \tag{3.128}$$

For the uniqueness and continuous dependence on the initial data, we suppose (W_1,V_1) and (W_2,V_2) are two solutions. Denote by $W = W_1 - W_2$, $\bar{W} = \frac{1}{2}(W_1+W_2)$, $V = V_1 - V_2$, and $\bar{V} = \frac{1}{2}(V_1+V_2)$. Then (3.114)–(3.117) implies that

$$\begin{cases} W_{tz} - 2\bar{W}_z W_z + W\bar{W}_{zz} + \bar{W}W_{zz} + 4\int_0^1 \bar{W}_z W_z dz - \Omega V + \Omega\int_0^1 V dz = 0, & (3.129)\\ V_t - W_z\bar{V} - \bar{W}_z V + W\bar{V}_z + \bar{W}V_z + \Omega W_z = 0, & (3.130)\end{cases}$$

with boundary condition

$$W(t,0) = W(t,1) = 0. \tag{3.131}$$

Multiplying (3.129) by W_z, and (3.130) by V, integrating with respect to z over the interval $[0,1]$, then an integration by parts together with the boundary condition (3.131) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|W_z\|_{L^2}^2 + \|V\|_{L^2}^2\Big)\\ =&\int_0^1 \Big(\frac{5}{2}\bar{W}_z W_z^2 - \bar{W}_{zz}WW_z + W_z\bar{V}V + \frac{3}{2}\bar{W}_z V^2 - W\bar{V}_z V\Big)dz. \end{aligned} \tag{3.132}$$

Using Hölder inequality, Young's inequality, and Sobolev inequality one obtains

$$\frac{d}{dt}\Big(\|W_z\|_{L^2}^2 + \|V\|_{L^2}^2\Big) \le C\Big(\|\bar{W}\|_{H^2} + \|\bar{V}\|_{H^1}\Big)\Big(\|W_z\|_{L^2}^2 + \|V\|_{L^2}^2\Big). \tag{3.133}$$

Thanks to Grönwall inequality, since $\bar{W} \in L^\infty(0,\mathcal{T};H^2)$, $\bar{V} \in L^\infty(0,\mathcal{T};H^1)$, for any $t \in [0,\mathcal{T}]$

we have

$$\begin{aligned}\|W_z(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2 \leq & \Big(\|W_z(0)\|_{L^2}^2 + \|V(0)\|_{L^2}^2\Big) \\ & \times \exp\Big(C\int_0^t \|\bar{W}(s)\|_{H^2} + \|\bar{V}(s)\|_{H^1} ds\Big).\end{aligned} \tag{3.134}$$

This implies continuous dependence on the initial data, and in particular the uniqueness. □

Remark 6. As we have seen in Section 3.1, the original IPEs system (3.6) is ill-posed in all Sobolev spaces. On the other hand, we establish in Proposition 3.3.1 the local well-posedness of the reduced system (3.114)–(3.115) in certain Sobolev space. The main reason behind this discrepancy is that when we restrict (3.111)–(3.112) to the line $x = 0$ to get the reduced system (3.114)–(3.115), the terms $w_x u_z$ and $w_x v_z$ in (3.111)–(3.112) disappear due to symmetry. These very terms, $w_x u_z$ and $w_x v_z$, are those which lose one horizontal derivative that forbids the well-posedness of the original system in Sobolev spaces.

Next, we use two different approaches to construct finite-time blowup solutions. One follows in [17], and another one follows [91].

3.3.2 First Method

In this section we follow the method used in [17]. We first introduce the following proposition from [17] and provide further analysis strengthening its conclusion. Observe that in [31] (see also [78, section 4], and references therein), a similar problem, arising in a different fluid dynamic context, has been investigated.

Proposition 3.3.2. *Consider the following nonlinear nonlocal degenerate elliptic boundary value problem:*

$$\phi' - (\phi')^2 + \phi\phi'' + 2\int_0^1 (\phi'(z))^2 dz = 0, \quad \phi(0) = \phi(1) = 0. \tag{3.135}$$

Then for each $\alpha \in (0,1)$*, the boundary value problem (3.135) has a nontrivial solution* $\phi_\alpha \in C^{2,\alpha}([0,1])$.

Recall that, for an integer k, and $0 < \alpha < 1$ the space $C^{k,\alpha}$ is endowed with the norm

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Proof. For each $m > 0$, the existence of nontrivial solution to the boundary value problem (3.135) satisfying the additional constraint

$$2 \int_0^1 (\phi'(z))^2 dz = m^2 \tag{3.136}$$

has been established in [17]. Let $\alpha \in (0, 1)$ and define

$$m := \sqrt{\Big(\frac{1+\alpha}{2(1-\alpha)}\Big)^2 - \frac{1}{4}} > 0. \tag{3.137}$$

That is

$$\alpha = \frac{\sqrt{m^2 + 1/4} - \frac{1}{2}}{\sqrt{m^2 + 1/4} + \frac{1}{2}}. \tag{3.138}$$

Denoting by $\psi := \phi'$, it was shown in [17] that the nontrivial solution ϕ of problem (3.135) satisfying (3.136) can be written as

$$\phi = C(m)(\psi_+ - \psi)^{\frac{\psi_+}{\psi_+ - \psi_-}} (\psi - \psi_-)^{\frac{-\psi_-}{\psi_+ - \psi_-}}, \tag{3.139}$$

where

$$\psi_\pm(m) := \pm\sqrt{m^2 + 1/4} + 1/2, \tag{3.140}$$

and

$$C(m) = \frac{1}{B(\frac{\psi_+}{\psi_+ - \psi_-}, \frac{-\psi_-}{\psi_+ - \psi_-})}. \tag{3.141}$$

Here $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is the Beta function. Moreover, it was also shown in [17] that

ϕ, ψ and z satisfy

$$(\phi,\psi)(z=0)=(0,\psi_+),\quad (\phi,\psi)(z=1)=(0,\psi_-),\quad \psi_-\le\psi\le\psi_+, \tag{3.142}$$

and

$$\frac{d\psi}{dz}=\frac{-1}{C(m)}(\psi_+-\psi)^{\frac{-\psi_-}{\psi_+-\psi_-}}(\psi-\psi_-)^{\frac{\psi_+}{\psi_+-\psi_-}}. \tag{3.143}$$

Therefore, ψ is a continuous and decreasing function of z, and smooth in $(0,1)$. From (3.142) and (3.143), one has

$$z(\psi)=-C(m)\int_{\psi_+}^{\psi}(\psi_+-\tilde{\psi})^{\frac{\psi_-}{\psi_+-\psi_-}}(\tilde{\psi}-\psi_-)^{\frac{-\psi_+}{\psi_+-\psi_-}}d\tilde{\psi}, \tag{3.144}$$

and

$$z(\psi)-1=-C(m)\int_{\psi_-}^{\psi}(\psi_+-\tilde{\psi})^{\frac{\psi_-}{\psi_+-\psi_-}}(\tilde{\psi}-\psi_-)^{\frac{-\psi_+}{\psi_+-\psi_-}}d\tilde{\psi}. \tag{3.145}$$

Next, we establish that $\phi(z)\in C^{2,\alpha}([0,1])$. From (3.139) and (3.143), we know when ψ is away from ψ_+ and ψ_-, i.e., z is away from 0 and 1, $\phi(z)$ is smooth. Therefore, we only need to consider when ψ is close to ψ_+ and ψ_-. From (3.144), one has

$$z(\psi)=C(m)B(\frac{\psi_+-\psi}{\psi_+-\psi_-};\frac{\psi_+}{\psi_+-\psi_-},\frac{-\psi_-}{\psi_+-\psi_-}), \tag{3.146}$$

where $B(x;a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt$ is the incomplete Beta function ($0\le x\le 1$). Moreover, from (3.141), we know that

$$z(\psi)=\frac{B(\frac{\psi_+-\psi}{\psi_+-\psi_-};\frac{\psi_+}{\psi_+-\psi_-},\frac{-\psi_-}{\psi_+-\psi_-})}{B(\frac{\psi_+}{\psi_+-\psi_-},\frac{-\psi_-}{\psi_+-\psi_-})}=I(\frac{\psi_+-\psi}{\psi_+-\psi_-};\frac{\psi_+}{\psi_+-\psi_-},\frac{-\psi_-}{\psi_+-\psi_-}), \tag{3.147}$$

where $I(x;a,b)=\frac{B(x;a,b)}{B(a,b)}$ is the regularized Beta function. When $x\in(0,1)$, by series expansion (cf. [1, p. 944]), one has

$$I(x;a,b) = \frac{x^a(1-x)^b}{aB(a,b)}\Big\{1 + \sum_{n=0}^{\infty} \frac{B(a+1,n+1)}{B(a+b,n+1)} x^{n+1}\Big\}. \tag{3.148}$$

Therefore, for $\psi_- < \psi < \psi_+$, we can write

$$\begin{aligned} z(\psi) = \frac{C(m)}{\psi_+}(\psi_+ - \psi)^{\frac{\psi_+}{\psi_+ - \psi_-}} (\psi - \psi_-)^{\frac{-\psi_-}{\psi_+ - \psi_-}} \\ \times\Big\{1 + \sum_{n=0}^{\infty} \frac{B(\frac{\psi_+}{\psi_+ - \psi_-} + 1, n+1)}{B(1,n+1)} \Big(\frac{\psi_+ - \psi}{\psi_+ - \psi_-}\Big)^{n+1}\Big\}. \end{aligned} \tag{3.149}$$

Letting

$$h_1(\psi) := \sum_{n=0}^{\infty} \frac{B(\frac{\psi_+}{\psi_+ - \psi_-} + 1, n+1)}{B(1,n+1)} \Big(\frac{\psi_+ - \psi}{\psi_+ - \psi_-}\Big)^{n+1}, \tag{3.150}$$

then $h_1(\psi) \geq 0$ and $h_1(\psi)$ is smooth on $\psi \in (\psi_-, \psi_+]$. Combine (3.143) and (3.149), we find that for $z \in (0,1)$,

$$\frac{d\psi}{dz} = -C(m)^{\frac{\psi_- - \psi_+}{\psi_+}} \Big(\frac{\psi_+}{1 + h_1(\psi(z))}\Big)^{\frac{-\psi_-}{\psi_+}} \Big(\psi(z) - \psi_-\Big)^{\frac{\psi_+ + \psi_-}{\psi_+}} z^{\frac{-\psi_-}{\psi_+}}. \tag{3.151}$$

From this expression and since $h_1(\psi(z)))$ is smooth on $z \in [0,1)$, we conclude that $\frac{d\psi}{dz}$ is continuous on $z \in [0,1)$, and smooth on $z \in (0,1)$. Observe that $\alpha = \frac{-\psi_-}{\psi_+}$, thus we have

$$\begin{aligned} \lim_{z \to 0^+} \frac{|\frac{d\psi}{dz}(z) - \frac{d\psi}{dz}(0)|}{|z - 0|^\alpha} &= \lim_{z \to 0^+} C(m)^{\frac{\psi_- - \psi_+}{\psi_+}} \Big(\frac{\psi_+}{1 + h_1(\psi(z))}\Big)^{\frac{-\psi_-}{\psi_+}} \Big(\psi(z) - \psi_-\Big)^{\frac{\psi_+ + \psi_-}{\psi_+}} \\ &= \lim_{\psi \to \psi_+} C(m)^{\frac{\psi_- - \psi_+}{\psi_+}} \Big(\frac{\psi_+}{1 + h_1(\psi)}\Big)^{\frac{-\psi_-}{\psi_+}} (\psi - \psi_-)^{\frac{\psi_+ + \psi_-}{\psi_+}} \\ &= C(m)^{\frac{\psi_- - \psi_+}{\psi_+}} \psi_+^{\frac{-\psi_-}{\psi_+}} (\psi_+ - \psi_-)^{\frac{\psi_+ + \psi_-}{\psi_+}} < \infty. \end{aligned} \tag{3.152}$$

Therefore, $\psi(z) \in C^{1,\alpha}([0,1))$, and thus $\phi(z) \in C^{2,\alpha}([0,1))$. Similarly, from (3.145), for $\psi_- <$

$\psi < \psi_+$, we can write

$$\begin{aligned}1 - z(\psi) =& I(\frac{\psi - \psi_-}{\psi_+ - \psi_-}; \frac{-\psi_-}{\psi_+ - \psi_-}, \frac{\psi_+}{\psi_+ - \psi_-}) \\ =& \frac{-C(m)}{\psi_-}(\psi - \psi_-)^{\frac{-\psi_-}{\psi_+ - \psi_-}}(\psi_+ - \psi)^{\frac{\psi_+}{\psi_+ - \psi_-}} \\ & \times \Big\{1 + \sum_{n=0}^{\infty} \frac{B(\frac{-\psi_-}{\psi_+ - \psi_-} + 1, n+1)}{B(1, n+1)}(\frac{\psi - \psi_-}{\psi_+ - \psi_-})^{n+1}\Big\}.\end{aligned} \tag{3.153}$$

Letting

$$h_2(\psi) := \sum_{n=0}^{\infty} \frac{B(\frac{-\psi_-}{\psi_+ - \psi_-} + 1, n+1)}{B(1, n+1)}(\frac{\psi - \psi_-}{\psi_+ - \psi_-})^{n+1}, \tag{3.154}$$

then $h_2(\psi) \geq 0$ and $h_2(\psi)$ is smooth on $\psi \in [\psi_-, \psi_+)$. Combine (3.143) and (3.153), we find that

$$\frac{d\psi}{dz} = -C(m)^{\frac{\psi_+ - \psi_-}{\psi_-}} \Big(\frac{-\psi_-}{1 + h_2(\psi(z))}\Big)^{\frac{-\psi_+}{\psi_-}} \Big(\psi_+ - \psi(z)\Big)^{\frac{\psi_+ + \psi_-}{\psi_-}} (1 - z)^{\frac{-\psi_+}{\psi_-}}. \tag{3.155}$$

From this expression and since $h_2(\psi(z)))$ is smooth on $z \in (0, 1]$, observe that $\frac{-\psi_+}{\psi_-} > 1$ and since $\psi(z) \in C^{1,\alpha}([0, 1))$, we know that indeed $\psi(z) \in C^{1,\alpha}([0, 1])$. Therefore, $\phi(z) \in C^{2,\alpha}([0, 1])$. □

Now we state first blowup result.

Theorem 3.3.3. *Let $\phi(z)$ be a nontrivial solution of the boundary value problem (3.135), and let $f(x)$ be a smooth odd periodic function with period 1, satisfying $f'(0) = 1$. Suppose that (u, v, w, p) is a smooth solution to system* (3.6)*, subject to the boundary condition (3.2), with initial condition*

$$u_0(x, z) = -f(x)\phi'(z), \quad v_0(x, z) = -\Omega f(x). \tag{3.156}$$

Then the solution blows up at sometime $\mathcal{T} \in (0, 1]$.

Proof. From Proposition 3.3.2, one can conclude that $\phi(z) \in H^2(0, 1)$. For smooth solution (u, v, w, p), we know (3.114)–(3.115) is the governing system when $x = 0$. From (3.156), we know the initial data for system (3.114)–(3.115) will be

$$W_0(z) = \phi(z), \quad V_0(z) = \Omega. \tag{3.157}$$

Thanks to Proposition 3.3.1, one can observe that

$$W(t,z) = \frac{\phi(z)}{1-t}, \quad V(t,z) \equiv \Omega \tag{3.158}$$

is the unique solution to (3.114)–(3.115) subject to initial condition (3.157) and boundary condition (3.117). Then we see $W(t,z)$ blows up at $t = 1$, and therefore, the solution (u, v, w, p) must blow up at sometime $\mathcal{T} \in (0, 1]$. □

3.3.3 Second Method

In this section we provide another approach that adopts ideas from [91]. This approach requires some additional conditions on the initial data, but avoids technical issue on the function ϕ as in Proposition 3.3.2. Moreover, this approach allows the initial data to be analytic, which guarantees the existence of solutions to the IPEs in the space of analytic functions.

Theorem 3.3.4. *Suppose that (u, v, w, p) is a smooth solution to system* (3.6)*, subject to the boundary condition (3.2), with initial condition (u_0, v_0) satisfying the following conditions:*

$$\begin{cases} u_0(x,z) \textit{ and } v_0(x,z) \textit{ are smooth odd periodic functions in } x \textit{ with period } 1, & (3.159) \\ u_0(x,z) \textit{ satisfies the compatibility condition } \displaystyle\int_0^1 u_0(x,z)dz = 0, & (3.160) \\ \partial_x v_0(0,z) = -\Omega \ \textit{ for all } z \in [0,1], & (3.161) \\ \partial_{xz} u_0(0,0) = 0, \ \partial_{xzz} u_0(0,z) < 0 \ \textit{ for all } z \in [0,1]. & (3.162) \end{cases}$$

Then the solution blows up at sometime $\mathcal{T} \in (0, \frac{-3}{\partial_x u_0(0,1)}]$.

Before proving this theorem, we first state the following lemma, which is similar to Lemma 2.4 in [91]. Since our settings are slightly different from [91], we also provide a detailed proof.

Lemma 3.3.5. *The smooth solution (u, v, w, p) stated in Theorem 3.3.4 satisfies*

$$\partial_{xz} u(t,0,0) = 0, \tag{3.163}$$

and, as long as the solution remains smooth at time t*, we have*

$$\partial_{xzz}u(t,0,z) < 0, \text{ for all } z \in [0,1]. \tag{3.164}$$

In other words, condition (3.162) is invariant in time.

Proof. For arbitrary $y_0 \in \mathbb{R}$ and $z_0 \in [0,1]$, consider the following system of characteristic equations

$$\begin{cases} \frac{dX}{dt} = u(t,X,Z), \\ \frac{dY}{dt} = v(t,X,Z), \\ \frac{dZ}{dt} = w(t,X,Z) \end{cases} \tag{3.165}$$

with the initial data

$$\begin{cases} X(0) = 0, \\ Y(0) = y_0, \\ Z(0) = z_0. \end{cases} \tag{3.166}$$

By virtue of oddness of u and v in the x variable, the solution (X,Y,Z) must satisfies

$$X(t) \equiv 0, \quad Y(t) \equiv y_0. \tag{3.167}$$

It means that particles starting from the line segment

$$L := \Big\{(x,y,z) : x = 0,\ y = y_0,\ z \in [0,1]\Big\} \tag{3.168}$$

can only move along this line segment. Moreover, when $z_0 = 0$ or $z_0 = 1$, thanks to the boundary condition (3.2), the solution must satisfy additionally $Z(t) \equiv 0$ or $Z(t) \equiv 1$, respectively. This means that the particles stationed at $(0, y_0, 0)$ and $(0, y_0, 1)$ do not move.

On the line segment L (3.168), we again consider the reduced system (3.114)–(3.115). By virtue of the last equation in system (3.6), the boundary condition (3.2), and the assumption

(3.161), one obtains that the corresponding initial and boundary conditions are

$$W(0,z) = -\int_0^z \partial_x u_0(0,s)ds, \quad V(0,z) = \Omega, \tag{3.169}$$

$$W(t,0) = W(t,1) = 0. \tag{3.170}$$

From (3.169), thanks to Proposition 3.3.1, we observe that $V \equiv \Omega$ is the unique solution to equation (3.115). Plugging this back into equation (3.114), we obtain

$$W_{tz} - (W_z)^2 + WW_{zz} + 2\int_0^1 W_z^2(t,z)dz = 0. \tag{3.171}$$

By taking one derivative with respect to z of (3.171), we have

$$W_{tzz} - W_z W_{zz} + WW_{zzz} = 0. \tag{3.172}$$

From (3.162), (3.169) and (3.170), we know that $W_{zz}(0,0) = 0$ and $W(t,0) = 0$. These together with the last equation in system (3.6) and (3.172) imply that

$$\partial_{xz} u(t,0,0) = W_{zz}(t,0) = 0. \tag{3.173}$$

By taking two derivatives with respect to z of (3.171), we have

$$W_{tzzz} - W_{zz}^2 + WW_{zzzz} = 0. \tag{3.174}$$

Since the particles on the line segment L only move along this line, therefore, (3.174) implies

$$\begin{aligned} \frac{d}{dt}W_{zzz}(t,Z(t)) &= \frac{d}{dt}w_{zzz}(t,0,Z(t)) \\ &= W_{tzzz}(t,Z(t)) + WW_{zzzz}(t,Z(t)) = W_{zz}^2(t,Z(t)) \geq 0. \end{aligned} \tag{3.175}$$

Let $\mathcal{T} > 0$ such that the solution (u,v,w,p) of system (3.6) remains smooth on $[0,\mathcal{T}]$. Then

(3.175) implies that as long as $W_{zzz}(0, Z(0)) > 0$, we have $W_{zzz}(\mathcal{T}, Z(\mathcal{T})) > 0$. In order to show that $W_{zzz}(\mathcal{T}, z^*) > 0$ for each $z^* \in [0, 1]$, we need to find $z_0 \in [0, 1]$ such that $Z(0) = z_0$ and $Z(\mathcal{T}) = z^*$. For this purpose, we define

$$\tau = \mathcal{T} - t, \quad \tilde{Z}(\tau) = Z(t). \tag{3.176}$$

Then, we have the following ordinary differential equation

$$\frac{d\tilde{Z}(\tau)}{d\tau} = \frac{dZ(t)}{dt}\frac{dt}{d\tau} = -\frac{dZ(t)}{dt} = -W(t, Z(t)) = -W(\mathcal{T} - \tau, \tilde{Z}(\tau)), \tag{3.177}$$

with initial condition

$$\tilde{Z}(0) = Z(\mathcal{T}) = z^*. \tag{3.178}$$

Since W is smooth on $t \in [0, \mathcal{T}]$, we have a unique solution $\tilde{Z}(\tau)$ on $\tau \in [0, \mathcal{T}]$. Define $z_0 := \tilde{Z}(\mathcal{T})$, then we see from (3.176) that $Z(0) = \tilde{Z}(\mathcal{T}) = z_0$ and $Z(\mathcal{T}) = \tilde{Z}(0) = z^*$. From incompressible condition, we know that $\partial_{xzz}u(t, 0, z) = -W_{zzz}(t, z)$, therefore,

$$\partial_{xzz}u(t, 0, z) < 0, \quad \text{for all } z \in [0, 1]. \tag{3.179}$$

□

We also need the following lemma. For details, see Lemma 2.5 in [91].

Lemma 3.3.6. *Let $f : [0, 1] \to \mathbb{R}$ be a C^2 function with the following properties:*

1. *$f'(0) = 0$ and $f''(z) > 0$ for any $z \in [0, 1]$,*

2. *$\int_0^1 f(z)dz = 0$.*

Then $f(1) > 0$ and

$$\int_0^1 f^2(z)dz \le \frac{1}{3}f(1)^2. \tag{3.180}$$

Now let us return to the proof of Theorem 3.3.4.

Proof. (Proof of Theorem 3.3.4) From the proof in Lemma 3.3.5, we know under the assumptions in Theorem 3.3.4, we have

$$W_{tz} - (W_z)^2 + WW_{zz} + 2\int_0^1 W_z^2(t,z)dz = 0. \tag{3.181}$$

From Lemma 3.3.5, we know $W_z(t,\cdot) = -\partial_x u(t,0,\cdot)$ satisfies both conditions in the Lemma 3.3.6. Therefore, we have $W_z(t,1) > 0$ and $\int_0^1 W_z(t,z)^2 dz \leq \frac{1}{3}W_z(t,1)^2$. Using this inequality in (3.181), and restrict at the point $z = 1$, thanks to the boundary condition (3.170), we have

$$W_{zt}(t,1) = W_z(t,1)^2 - 2\int_0^1 W_z(t,z)^2 dz \geq \frac{1}{3}W_z(t,1)^2. \tag{3.182}$$

Since $W_z(0,1) > 0$, it follows that

$$W_z(t,1) \geq \frac{3W_z(0,1)}{3 - W_z(0,1)t}. \tag{3.183}$$

Then we see $W_z(t,1)$ blows up before or at the time $\frac{3}{W_z(0,1)} = \frac{-3}{\partial_x u_0(0,1)}$. Therefore, the solution (u,v,w,p) must blow up at sometime $\mathcal{T} \in (0, \frac{-3}{\partial_x u_0(0,1)}]$. □

Remark 7. The requirements (3.159)–(3.162) allow the initial condition to be real analytic. As an example, consider u_0 and v_0 to be:

$$u_0(x,z) = \lambda(-z^2 + \frac{1}{3})\sin x, \quad v_0(x,z) = -\Omega \sin x, \tag{3.184}$$

with $\lambda > 0$. For analytic initial data, system (3.6) is local well-posed (from Theorem 3.2.2). Therefore, for arbitrary $\Omega \in \mathbb{R}$, we have initial data such that the solution of $3D$ IPEs exists, and also blows up in finite time. For initial data (u_0, v_0), notice that $\int_0^1 u_0(x,z)dz = 0$ and v_0 is independent of the z variable. This implies that the baroclinic mode of the initial data is $(u_0, 0)$, and the barotropic mode of the initial data is $(0, v_0)$. We know from above that the guaranteed blowup time is

$$\frac{-3}{\partial_x u_0(0,1)} = \frac{9}{2\lambda}. \tag{3.185}$$

When $|\Omega| \gg 1$, we have:

- when $\lambda = |\Omega|$, the baroclinic mode satisfies $(u_0, 0) \sim |\Omega|$, and the whole initial data satisfies $(u_0, v_0) \sim |\Omega|$. The guaranteed blowup time in this case satisfies $\mathcal{T} \sim \frac{1}{|\Omega|}$;
- when $\lambda = 1$, the baroclinic mode satisfies $(u_0, 0) \sim 1$, while the whole initial data satisfies $(u_0, v_0) \sim |\Omega|$. The guaranteed blowup time in this case satisfies $\mathcal{T} \sim 1$;
- when $\lambda = \frac{1}{|\Omega|}$, this implies a smallness condition on the baroclinic $(u_0, 0) \sim \frac{1}{|\Omega|}$, while the whole initial data satisfies $(u_0, v_0) \sim |\Omega|$. The guaranteed blowup time in this case satisfies $\mathcal{T} \sim |\Omega|$.

Based on the observations above, one can expect that the lower bound of the life-span of the $3D$ IPEs in the space of analytic functions can be prolonged with fast rotation, and with some smallness conditions on the size of the baroclinic mode. We will investigate this problem in the next section.

Remark 8. It remains interesting to know whether for arbitrary Ω there exists a blowup solution with initial data (u_0, v_0) whose barotropic and baroclinic modes are both of order 1. Moreover, to estimate the corresponding blowup time $\mathcal{T}$ as $|\Omega| \to \infty$. Observe that if the blowup time $\mathcal{T} \sim 1$ as $|\Omega| \to \infty$, this would imply that fast rotation does not prolong the life-span of the solution to the $3D$ IPEs unless, as it has been noted above, a smallness condition on the size of the baroclinic mode is met.

3.4 Long-time Existence

In previous sections, we have shown local well-posedness and constructed smooth solutions that blows up at finite time. In this section, we will show that with certain assumption on the initial data, the life-span of solutions to the $3D$ IPEs can be prolonged to infinity. For this result, we do not take advantage of fast rotation, i.e., this result is independent of Ω.

We start with the following important observation. When $\widetilde{\mathcal{V}}_0 = 0$ in system (3.31)–(3.32), $\widetilde{\mathcal{V}} \equiv 0$ by uniqueness of the solutions. Then, system (3.31)–(3.32) only contains the evolution of the barotropic mode $\overline{\mathcal{V}}$, and it is governed by the $2D$ Euler equations since $\widetilde{\mathcal{V}} \equiv 0$. It is well-known that the $2D$ Euler equations are globally well-posed in Sobolev spaces. Moreover, it has been shown by Levermore–Oliver [64] that the $2D$ Euler equations are also globally well-posed in the space of analytic functions.

Based on this observation, one can expect that the life-span of solutions to the $3D$ IPEs can be prolonged as long as the $\widetilde{\mathcal{V}}_0$ is small. Since we are working in the space of analytic functions, we need the smallness of the analytic norm of $\widetilde{\mathcal{V}}_0$. This is a strong assumption, and we will see later, if the rotation rate is fast enough, we can obtain long-time existence result by putting a much weaker assumption on the initial data.

We first review the $2D$ Euler equations.

3.4.1 $2D$ Euler Equations

Consider the following $2D$ Euler equations in $\mathbb{T}^3$:

$$\begin{cases} \partial_t \overline{V} + \overline{V} \cdot \nabla_h \overline{V} + \nabla_h P = 0, & (3.186) \\ \nabla_h \cdot \overline{V} = 0, & (3.187) \end{cases}$$

with initial condition

$$\overline{V}|_{t=0} = \overline{V}_0. \tag{3.188}$$

Here $\overline{V}$ depends only on two horizontal variables $\boldsymbol{x}'$. The global existence of solutions to system (3.186)–(3.188) in Sobolev spaces H^r with $r \geq 3$ is a classical result, see, e.g., [10]. Moreover, from equation (3.84) in [10], for $r \geq 3$, we have

$$\frac{d}{dt}\|\overline{V}\|_{H^r} \leq C_r \|\overline{V}\|_{H^r}(1 + \ln^+ \|\overline{V}\|_{H^r}). \tag{3.189}$$

Let $\|\overline{V}_0\|_{H^r} \leq M$ for some $M \geq 0$. Since $\ln^+ x + 1 \leq 2\ln(x+e)$, by setting $W(t) = \|\overline{V}(t)\|_{H^r} + e$, from (3.189), we have

$$\frac{d}{dt} W \leq C_r W \ln W. \tag{3.190}$$

Therefore, we get the following bound:

$$\|\overline{V}(t)\|_{H^r} \leq W(t) \leq W(0)^{e^{C_r t}} = (\|\overline{V}_0\|_{H^r} + e)^{e^{C_r t}} \leq (M+e)^{e^{C_r t}} =: \theta_{M,r}(t). \tag{3.191}$$

We need the following lemma for the purpose of this section. For its proof, we refer the reader to [64]. It is also a special case of Lemma 2.2.14.

Lemma 3.4.1. *For $f, g \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$ where $r > 5/2$ and $\tau \geq 0$, one has*

$$\begin{aligned} \Big|\langle A^r e^{\tau A}(f \cdot \nabla_h g), A^r e^{\tau A} g\rangle\Big| &\leq C_r(\|A^r f\|\|A^r g\|^2 + \|\nabla_h \cdot f\|_{L^\infty}\|A^r e^{\tau A} g\|^2) \\ &\quad + C_r \tau \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\|^2. \end{aligned} \tag{3.192}$$

Moreover, if $r > 3$, then $\|A^{r+1/2} e^{\tau A} f\|$ can be replaced by $\|A^r e^{\tau A} f\|$.

Based on Lemma 3.4.1, Levermore–Oliver [64] proved the global existence of solutions to system (3.186)–(3.188) for initial data in the space of analytic functions. For completion, we state it here, with slight difference compared with the original statement in [64].

Proposition 3.4.2. *Assume $\overline{V}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 3$ and $\tau_0 > 0$, and suppose that $\|e^{\tau_0 A}\overline{V}_0\|_{H^r} \leq M$ for some $M \geq 0$. There exists a non-increasing function*

$$\tau(t) = \tau_0 \exp\Big(-C_r \int_0^t h(s)ds\Big), \tag{3.193}$$

where

$$h^2(t) := \|e^{\tau_0 A}\overline{V}_0\|_{H^r}^2 + C_r \int_0^t \theta_{M,r}^3(s)ds, \tag{3.194}$$

and $\theta_{M,r}(t)$ defined in (3.191), such that for any given time $\mathcal{T} > 0$, there exists a unique solution

$$\overline{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))) \tag{3.195}$$

to system (3.186)–(3.188). Moreover, there exist constants $C_M > 1$ and $C_r > 1$ such that

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^r}^2 \leq h^2(t) \leq C_M^{\exp(C_r t)}. \tag{3.196}$$

3.4.2 Long-time Existence of The $3D$ IPEs

The following is the main theorem of this section, which concerns the long-time exsitence of solutions to system (3.31)–(3.34) in the case when the analytic norm of $\widetilde{\mathcal{V}}_0$ is small.

Theorem 3.4.3. *Assume $\overline{\mathcal{V}}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$, $\widetilde{\mathcal{V}}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Let $M \geq 0$ and $\epsilon \geq 0$, and suppose that $\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^{r+1}} \leq M$ and $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r} \leq \epsilon$. Then there are constants $C_M > 1$ and $C_r > 1$, and a function $K(t) = C_M^{\exp(C_r t)}$, such that if $\mathcal{T} = \mathcal{T}(\tau_0, \epsilon, M, r)$ satisfies*

$$\int_0^{\mathcal{T}} e^{K(s)} ds = \frac{\tau_0}{2\epsilon}, \tag{3.197}$$

then the unique solution obtained in Theorem 3.2.2 satisfies $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, with

$$\tau(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \epsilon \int_0^t e^{K(s)} ds). \tag{3.198}$$

In particular, from (3.197), $\mathcal{T} \gtrsim \ln(\ln(\ln(\frac{1}{\epsilon}))) \to \infty$, as $\epsilon \to 0^+$.

Thanks to Lemma 2.2.6 and Lemma 3.2.1, we immediately have the following corollary.

Corollary 3.4.4. *Assume $\mathcal{V}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$, and the conditions of Theorem 3.4.3 hold. Then the unique solution obtained in Corollary 3.2.3 satisfies $\mathcal{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, with $\mathcal{T}$ defined in (3.197) and τ defined in (3.198).*

Remark 9. For the proof of Theorem 3.4.3, we only establish formal energy estimates. However, these formal estimates can be justified rigorously by establishing them first for the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem (Lemma 2.2.10), as we did in the previous section.

Remark 10. The constants C_M and C_r in $K(t)$ may change from step to step in the proof, and always larger than 1. When necessary, we use $K_1(t), K_2(t), ...$ to emphasize the changes in C_M and C_r. Two useful inequalities are

$$\int_0^t K(s)ds \leq K_1(t), \quad \int_0^t e^{K(s)}ds \leq e^{K_1(t)} \tag{3.199}$$

for some new $K_1(t)$. At the end, we choose some suitable and large enough C_M and C_r for the $K(t)$ in Theorem 3.4.3. Similar abuse of notation will also be used in the rest of sections.

Proof. (proof of Theorem 3.4.3.) Let $\overline{V}$ be the unique global solution to the $2D$ Euler equations (3.186)–(3.188) in the space $\mathcal{D}(e^{\tau_1(t)A} : H^{r+1}(\mathbb{T}^3))$, with initial condition $\overline{V}_0 = \overline{\mathcal{V}}_0$ and $\tau_1(t)$ satisfying (3.193). Let $\overline{\phi} = \overline{\mathcal{V}} - \overline{V}$. Applying $\mathbb{P}_h$ to (3.186), taking the difference between (3.31) and (3.186), and writing (3.32) in terms of $\overline{V}$ and $\overline{\phi}$, we have

$$\begin{cases} \partial_t\overline{\phi} + \mathbb{P}_h\Big(\overline{\phi}\cdot\nabla_h\overline{\phi} + \overline{\phi}\cdot\nabla_h\overline{V} + \overline{V}\cdot\nabla_h\overline{\phi}\Big) + \mathbb{P}_h P_0\Big((\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}\Big) = 0, & (3.200) \\ \partial_t\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + \overline{\phi}\cdot\nabla_h\widetilde{\mathcal{V}} + \overline{V}\cdot\nabla_h\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\overline{\phi} + \widetilde{\mathcal{V}}\cdot\nabla_h\overline{V} & \\ \qquad -P_0\Big((\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}\Big) - \big(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(s)ds\big)\partial_z\widetilde{\mathcal{V}} + \Omega\widetilde{\mathcal{V}}^\perp = 0, & (3.201) \end{cases}$$

with initial condition

$$\overline{\phi}(0) = \overline{\mathcal{V}}_0 - \overline{V}_0 = 0, \quad \widetilde{\mathcal{V}}(0) = \widetilde{\mathcal{V}}_0. \tag{3.202}$$

First, by virtue of (3.51), and since the L^2 energy is conserved for $\overline{V}$, thanks to triangle inequality, we have

$$\|\overline{\phi}\|^2 + \|\widetilde{\mathcal{V}}\|^2 \le 2(\|\overline{\mathcal{V}}\|^2 + \|\overline{V}\|^2 + \|\widetilde{\mathcal{V}}\|^2) = 4\|\overline{\mathcal{V}}_0\|^2 + 2\|\widetilde{\mathcal{V}}_0\|^2. \tag{3.203}$$

Next, applying $A^r e^{\tau A}$ to equation (3.200) and (3.201), and taking L^2 inner product with $A^r e^{\tau A}\overline{\phi}$ and $A^r e^{\tau A}\widetilde{\mathcal{V}}$, respectively, thanks to Lemma 2.2.4 and Lemma 2.2.5, since $P_0\widetilde{\mathcal{V}} = 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\overline{\phi}\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 - \Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^r e^{\tau A}((\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}), A^r e^{\tau A}\overline{\phi}\Big\rangle,\end{aligned} \tag{3.204}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{\mathcal{V}}\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}\|^2 - \Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle - \Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle \\ &\quad +\Big\langle A^r e^{\tau A}\big((\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}', s)ds)\partial_z\widetilde{\mathcal{V}}\big), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle.\end{aligned} \tag{3.205}$$

Thanks to Lemma 2.2.11–2.2.13, we have

$$\begin{aligned}&\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\ &\quad +\Big|\Big\langle A^r e^{\tau A}((\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle\Big| \\ &\quad +\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle\Big| \\ &\quad +\Big|\Big\langle A^r e^{\tau A}\big((\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}', s)ds)\partial_z\widetilde{\mathcal{V}}\big), A^r e^{\tau A}\widetilde{\mathcal{V}}\Big\rangle\Big| \\ &\le C_r(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|)(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}\|^2).\end{aligned} \tag{3.206}$$

Here we use the fact that $\widetilde{\mathcal{V}}$ and $\overline{\phi}$ have zero mean value. For $\widetilde{\mathcal{V}}$, since $\overline{\widetilde{\mathcal{V}}} = 0$, so its mean is zero.

For $\overline{\phi}$, since $\overline{V}$ and $\overline{\mathcal{V}}$ both have zero mean value, therefore, $\overline{\phi}$ also has zero mean value.

By virtue of Lemma 3.4.1, since $\nabla_h \cdot \overline{V} = 0$, one obtains

$$\begin{aligned}
&\left|\left\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\right\rangle\right| + \left|\left\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\widetilde{\mathcal{V}}), A^r e^{\tau A}\widetilde{\mathcal{V}}\right\rangle\right| \\
&\le C_r\|A^r\overline{V}\|(\|A^r\overline{\phi}\|^2 + \|A^r\widetilde{\mathcal{V}}\|^2) \\
&\qquad\qquad + C_r\tau\|A^{r+1/2}e^{\tau A}\overline{V}\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}\|^2). \qquad (3.207)
\end{aligned}$$

From Lemma 2.2.2, thanks to Cauchy–Schwarz inequality, and since $\widetilde{\mathcal{V}}$ and $\overline{\phi}$ have zero mean, we have

$$\begin{aligned}
&\left|\left\langle A^r e^{\tau A}(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\widetilde{\mathcal{V}}\right\rangle\right| + \left|\left\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\overline{\phi}\right\rangle\right| \\
&\le C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|^2). \qquad (3.208)
\end{aligned}$$

Combining all of the estimates above, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|^2) \\
&\le \Big(\dot{\tau} + C_r(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|) + C_r\tau\|e^{\tau A}\overline{V}\|_{H^{r+1}}\Big) \\
&\qquad\qquad \times\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{\mathcal{V}}\|^2\Big) \\
&\quad + C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}\Big(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|^2\Big). \qquad (3.209)
\end{aligned}$$

As indicated in Remark 10, we will use $K_0, K_1, K_2, ...$ to indicate the change in $K(t)$ from step to step, and all of them are increasing double exponentially in t. Recall that τ_1 is defined by (3.193). Indeed, there exists a function $K_0(t)$ such that $\tau_1(t) \ge \tau_0 e^{-\int_0^t K_0(s)ds}$. Let $\tau \le \tau_1$. Recall from (3.196), we have

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}} \le \|e^{\tau_1(t)A}\overline{V}(t)\|_{H^{r+1}} \le C_M^{\exp(\widetilde{C_r}t)} =: K_1(t). \qquad (3.210)$$

Denote by

$$F = \|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{\mathcal{V}}\|^2, \quad G = \|A^{r+1/2} e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2} e^{\tau A}\widetilde{\mathcal{V}}\|^2. \tag{3.211}$$

We can rewrite (3.209) as

$$\frac{d}{dt}F \le 2(\dot{\tau} + C_r F^{1/2} + \tau K_2)G + K_2 F. \tag{3.212}$$

Notice that when τ satisfies

$$\dot{\tau} + C_r F^{1/2} + \tau K_2 \le 0, \tag{3.213}$$

we have

$$F(t) \le F(0)e^{\int_0^t K_2(s)ds} \le F(0)e^{K_3(t)}, \tag{3.214}$$

and therefore

$$C_r F(t)^{1/2} \le F(0)^{1/2} e^{K_4(t)}. \tag{3.215}$$

Notice that $F(0) = \|A^r e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2 \le \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|^2_{H^r} \le \epsilon^2$. From (3.213), we require that

$$\frac{d}{dt}(\tau e^{\int_0^t K_2(s)ds}) + \epsilon e^{\int_0^t K_2(s)ds} e^{K_4(t)} \le 0. \tag{3.216}$$

Thanks to (3.199), we have

$$e^{\int_0^t K_2(s)ds} e^{K_4(t)} \le e^{K_5(t)} \tag{3.217}$$

for some new $K_5(t)$. Therefore, instead of (3.216), we require that

$$\frac{d}{dt}(\tau e^{\int_0^t K_2(s)ds}) + \epsilon e^{K_5(t)} \le 0. \tag{3.218}$$

Integrating (3.216) from 0 to t in time, we have

$$\tau(t)e^{\int_0^t K_2(s)ds} \leq \tau_0 - \epsilon \int_0^t e^{K_5(s)}ds. \tag{3.219}$$

Recall that we also need $\tau(t) \leq \tau_1(t)$ and we know that $\tau_1(t) \geq \tau_0 e^{-\int_0^t K_0(s)ds}$. Therefore, for a new and suitable function $K(t)$, we can set

$$\tau(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \epsilon \int_0^t e^{K(s)}ds) \tag{3.220}$$

such that $\tau(t)$ satisfies the condition in (3.213) and also $\tau(t) \leq \tau_1(t)$. One can see $\tau(t) > 0$ on $t \in [0, \mathcal{T}]$ when $\mathcal{T}$ satisfies

$$\int_0^{\mathcal{T}} e^{K(s)}ds = \frac{\tau_0}{2\epsilon}. \tag{3.221}$$

Since $K(t)$ is double exponential in time, and $\int_0^{\mathcal{T}} e^{K(s)}ds \leq \mathcal{T}e^{K(\mathcal{T})} \leq e^{2K(\mathcal{T})}$, we have $\mathcal{T} \gtrsim \ln(\ln(\ln(\frac{1}{\epsilon}))) \to \infty$ as $\epsilon \to 0^+$.

From (3.214), since $\overline{\phi}$ and $\widetilde{\mathcal{V}}$ have zero mean, we can apply Poincaré inequality to obtain

$$\|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{\mathcal{V}}(t)\|_{H^r}^2 \leq \epsilon^2 e^{K(t)} \tag{3.222}$$

when $K(t)$ is chosen suitably, on $t \in [0, \mathcal{T}]$, with $\tau(t)$ defined by (3.220). From (3.196), and since $\tau \leq \tau_1$, we know $\|e^{\tau(t)A}\overline{V}(t)\|_{H^r}$ is also bounded on $t \in [0, \mathcal{T}]$. By triangle inequality, we have

$$\begin{aligned} &\|e^{\tau(t)A}\overline{\mathcal{V}}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{\mathcal{V}}(t)\|_{H^r} \\ &\leq \|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r} + \|e^{\tau(t)A}\overline{V}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{\mathcal{V}}(t)\|_{H^r} < \infty \end{aligned} \tag{3.223}$$

on $t \in [0, \mathcal{T}]$. Therefore, the time of existence of the solution to system (3.31)–(3.34) satisfies (3.197). $\square$

3.4.3 Convergence to The $2D$ Euler Equations

Based on Theorem 3.4.3, we have the following result concerning the convergence of solutions of the $3D$ IPEs (3.31)–(3.34) to solutions of the $2D$ Euler equations (3.186)–(3.188) in the space of analytic functions.

Theorem 3.4.5. *Assume a sequence of initial data $\{\overline{\mathcal{V}}_0^n = \overline{\mathcal{V}}_0\}_{n\in\mathbb{N}} \subset \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$ and $\{\widetilde{\mathcal{V}}_0^n\}_{n\in\mathbb{N}} \subset \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^{r}(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega\in\mathbb{R}$ be arbitrary and fixed. Suppose $\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^{r+1}} \leq M$ for some $M\geq 0$, and $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0^n\|_{H^r} \leq \epsilon_n$ with $\epsilon_n \to 0$, as $n\to\infty$. Then there are constants $C_M > 1$, $C_r > 1$, and a function $K(t) = C_M^{\exp(C_r t)}$, such that for each $n\in\mathbb{N}$, if the function $\tau^n(t)$ and the time $\mathcal{T}_n$ satisfy*

$$\tau^n(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \epsilon_n\int_0^t e^{K(s)}ds), \qquad \int_0^{\mathcal{T}_n} e^{K(s)}ds = \frac{\tau_0}{2\epsilon_n}, \tag{3.224}$$

the solution to system (3.31)–(3.34) with initial data $(\overline{\mathcal{V}}_0^n, \widetilde{\mathcal{V}}_0^n)$ satisfies

$$(\overline{\mathcal{V}}^n, \widetilde{\mathcal{V}}^n) \in \mathcal{S}\cap L^\infty(0,\mathcal{T}_n; \mathcal{D}(e^{\tau^n A} : H^r(\mathbb{T}^3))).$$

Let $\overline{V} \in \mathcal{S}\cap L^\infty(0,\infty;\mathcal{D}(e^{\tau^0(t)A} : H^r(\mathbb{T}^3)))$ be the unique global solution to the $2D$ Euler equations (3.186)–(3.188) with initial data $\overline{V}(0) = \overline{\mathcal{V}}_0$. Then, $(\overline{\mathcal{V}}^n, \widetilde{\mathcal{V}}^n)$ converges to $\overline{V}$ for $t\in[0,\mathcal{T}_0]$, as $n\to\infty$, in the following sense:

$$\|e^{\tau^0(t)A}(\overline{\mathcal{V}}^n + \widetilde{\mathcal{V}}^n - \overline{V})(t)\|_{H^r} \leq \epsilon_n e^{K(t)} \to 0, \quad \text{as} \quad n\to\infty. \tag{3.225}$$

Proof. Denote by $\overline{\phi}^n = \overline{\mathcal{V}}^n - \overline{V}$. By virtue of the proof of Theorem 3.4.3, we just need to prove the estimate (3.225). Since $\tau^0(t)\leq\tau^n(t)$ for any $n\in\mathbb{N}$, from (3.222), one has

$$\begin{aligned}\|e^{\tau^0(t)A}\widetilde{\mathcal{V}}^n(t)\|_{H^r} + \|e^{\tau^0(t)A}\overline{\phi}^n(t)\|_{H^r} &\leq \|e^{\tau^n(t)A}\widetilde{\mathcal{V}}^n(t)\|_{H^r} + \|e^{\tau^n(t)A}\overline{\phi}^n(t)\|_{H^r} \\ &\leq \epsilon_n e^{K(t)}\end{aligned} \tag{3.226}$$

when the function $K(t)$ is chosen suitably. Therefore, we have

$$\|e^{\tau^0(t)A}(\overline{\mathcal{V}}^n + \widetilde{\mathcal{V}}^n - \overline{V})(t)\|_{H^r} \leq \|e^{\tau^0(t)A}\widetilde{\mathcal{V}}^n(t)\|_{H^r} + \|e^{\tau^0(t)A}\overline{\phi}^n(t)\|_{H^r} \leq \epsilon_n e^{K(t)}. \quad (3.227)$$

As $n \to \infty$, we have $\epsilon_n \to 0$, and therefore, $\epsilon_n e^{K(t)} \to 0$. This completes the proof. $\square$

3.5 Effect of Rotation

In this section, we investigate the effect of rotation on the life-span of solutions to the $3D$ IPEs. Our goal is to show that fast rotation, i.e., when $|\Omega|$ is large, the life-span can be prolonged. For this purpose, we investigate the rotation operator, and do further reformulation of the $3D$ IPEs.

3.5.1 Reformulation of The Problem

For $\varphi \in \dot{L}^2$ where $\dot{L}^2$ is defined in (3.27), the rotating opeartor is

$$\mathcal{J}\varphi := \varphi^\perp = (-\varphi_2, \varphi_1). \quad (3.228)$$

Inspired by the $2D$ Leray projection, we define the projection $P_S : \dot{L}^2 \to \mathcal{S}$ as

$$P_S\varphi := \widetilde{\varphi} + \mathbb{P}_h\overline{\varphi}. \quad (3.229)$$

Then, we can define an operator $P : \mathcal{S} \to \mathcal{S}$ as

$$P\varphi := P_S(\mathcal{J}\varphi). \quad (3.230)$$

A direct computation using $\nabla_h \cdot \overline{\varphi} = 0$, we obtain

$$P\varphi = \widetilde{\varphi}^\perp. \quad (3.231)$$

It is easy to see that the kernel of P is

$$\ker P = \Big\{\varphi \in \mathcal{S} : \widetilde{\varphi}^{\perp} = 0\Big\} = \Big\{\varphi \in \mathcal{S} : \varphi = \overline{\varphi}\Big\}. \tag{3.232}$$

Therefore, the projection $P_0 : \mathcal{S} \to \ker P$ indeed is the projection into the kernel of P.

Notice that if we consider $\mathcal{V}_0 \in \ker P$, i.e., consider $\widetilde{\mathcal{V}}_0 = 0$, the $3D$ IPEs reduce to the $2D$ Euler, and this has been discussed in section 3.4. In order to investigate the effect of rotation, we further study the evolution of the baroclinic mode. This can be done by further decomposing the baroclinic mode in order to identify the resonant and non-resonant parts due to the rotation.

Since the rotation matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

corresponding to $P\widetilde{\mathcal{V}} = \mathcal{J}\widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}^{\perp}$ has eigenvalues $\pm i$, with eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$, we can define

$$P_{+}\varphi := \Big\langle (I-P_0)\varphi, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}\Big\rangle_E \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\Big\langle \widetilde{\varphi}, \overline{\begin{pmatrix} 1 \\ i \end{pmatrix}}\Big\rangle_E \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(\widetilde{\varphi} + i\widetilde{\varphi}^{\perp}), \tag{3.233}$$

and

$$P_{-}\varphi := \Big\langle (I-P_0)\varphi, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}\Big\rangle_E \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}\Big\langle \widetilde{\varphi}, \overline{\begin{pmatrix} 1 \\ -i \end{pmatrix}}\Big\rangle_E \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(\widetilde{\varphi} - i\widetilde{\varphi}^{\perp}). \tag{3.234}$$

Here the inner product $\langle \cdot, \cdot \rangle_E$ is the usual Euclidean inner product. These projections P_+ and P_- are also defined in section 2.1, and we give the derivation here. Similar ideas and projections for $3D$ rotating Euler equations can be found in Dutrifoy [34] and Koh–Lee–Takada [56].

In fact, the operator P has three eigenvalues, 0 and $\pm i$. These three projections P_0 and $P_\pm$ project $\mathcal{V}$ into the eigenspaces corresponding to 0 and $\mp i$. From Lemma 2.2.3, we know we can use P_0 and $P_\pm$ to decompose any vector filed $\varphi \in L^2(\mathbb{T}^3)$ into three parts that are orthogonal to each other.

Now observe that we can write $\widetilde{\mathcal{V}}^\perp$ in equation (3.32) as $\widetilde{\mathcal{V}}^\perp = -i(P_+\mathcal{V} - P_-\mathcal{V})$. Hence applying $P_\pm$ to (3.32), we have

$$\begin{aligned}\partial_t P_\pm \mathcal{V} + P_\pm\Big(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}} + \overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} - P_0(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + (\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}})\\ - (\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}\Big) \mp i\Omega P_\pm\mathcal{V} = 0.\end{aligned} \tag{3.235}$$

By setting

$$u_+ = e^{-i\Omega t}P_+\mathcal{V}, \quad u_- = e^{i\Omega t}P_-\mathcal{V}, \tag{3.236}$$

and multiplying $e^{-i\Omega t}$ to the equation for $P_+\mathcal{V}$ and $e^{i\Omega t}$ to the equation for $P_-\mathcal{V}$, we can rewrite (3.235) as

$$\begin{aligned}\partial_t u_\pm + e^{\mp i\Omega t}P_\pm\Big(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}} + \overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} - P_0(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + (\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}})\\ - (\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}\Big) = 0.\end{aligned} \tag{3.237}$$

For the u_+ part, thanks to Lemma 2.2.3 and (3.233), we have

$$\begin{cases}P_+(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}) = \frac{1}{2}(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}^\perp) - \frac{1}{2}P_0\Big(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}^\perp\Big)\\ \qquad = \frac{1}{2}\widetilde{\mathcal{V}}\cdot\nabla_h(\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}^\perp) - \frac{1}{2}P_0\Big(\widetilde{\mathcal{V}}\cdot\nabla_h(\widetilde{\mathcal{V}} + i\widetilde{\mathcal{V}}^\perp)\Big)\\ \qquad = e^{i\Omega t}\Big(\widetilde{\mathcal{V}}\cdot\nabla_h u_+ - P_0(\widetilde{\mathcal{V}}\cdot\nabla_h u_+)\Big), & (3.238)\\ P_+(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}) = \frac{1}{2}(\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}} + i\widetilde{\mathcal{V}}\cdot\nabla_h\overline{\mathcal{V}}^\perp) = \frac{1}{2}\widetilde{\mathcal{V}}\cdot\nabla_h(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp), & (3.239)\\ P_+(\overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}) = \frac{1}{2}(\overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + i\overline{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}}^\perp) = e^{i\Omega t}(\overline{\mathcal{V}}\cdot\nabla_h u_+), & (3.240)\\ P_+P_0\Big(\widetilde{\mathcal{V}}\cdot\nabla_h\widetilde{\mathcal{V}} + (\nabla_h\cdot\widetilde{\mathcal{V}})\widetilde{\mathcal{V}}\Big) = 0. & (3.241)\end{cases}$$

Observe that by integration by parts one has

$$
\begin{aligned}
&P_+\Big((\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}\Big)\\
=&\frac{1}{2}\Big((\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}+i(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}^\perp\Big)\\
&-\frac{1}{2}P_0\Big((\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}+i(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z\widetilde{\mathcal{V}}^\perp\Big)\\
=&e^{i\Omega t}(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z u_+ + e^{i\Omega t}P_0\Big((\nabla_h\cdot\widetilde{\mathcal{V}})u_+\Big).
\end{aligned}
\tag{3.242}
$$

Therefore, u_+ part in (3.237) becomes

$$
\begin{aligned}
\partial_t u_+ = -\Big(&\widetilde{\mathcal{V}}\cdot\nabla_h u_+ + \overline{\mathcal{V}}\cdot\nabla_h u_+ - P_0(\widetilde{\mathcal{V}}\cdot\nabla_h u_+ + (\nabla_h\cdot\widetilde{\mathcal{V}})u_+)\\
&-(\int_0^z \nabla_h\cdot\widetilde{\mathcal{V}}(\boldsymbol{x}',s)ds)\partial_z u_+\Big) - \frac{1}{2}e^{-i\Omega t}(\widetilde{\mathcal{V}}\cdot\nabla_h)(\overline{\mathcal{V}}+i\overline{\mathcal{V}}^\perp).
\end{aligned}
\tag{3.243}
$$

Using $\widetilde{\mathcal{V}} = u_+e^{i\Omega t} + u_-e^{-i\Omega t}$, we can furthermore rewrite (3.243) as

$$
\begin{aligned}
\partial_t u_+ = -e^{i\Omega t}\Big(&u_+\cdot\nabla_h u_+ - P_0(u_+\cdot\nabla_h u_+ + (\nabla_h\cdot u_+)u_+)\\
&-(\int_0^z \nabla_h\cdot u_+(\boldsymbol{x}',s)ds)\partial_z u_+\Big)\\
&-\Big(\overline{\mathcal{V}}\cdot\nabla_h u_+ + \frac{1}{2}(u_+\cdot\nabla_h)(\overline{\mathcal{V}}+i\overline{\mathcal{V}}^\perp)\Big)\\
&-e^{-2i\Omega t}\frac{1}{2}(u_-\cdot\nabla_h)(\overline{\mathcal{V}}+i\overline{\mathcal{V}}^\perp)\\
&-e^{-i\Omega t}\Big(u_-\cdot\nabla_h u_+ - P_0(u_-\cdot\nabla_h u_+ + (\nabla_h\cdot u_-)u_+)\\
&-(\int_0^z \nabla_h\cdot u_-(\boldsymbol{x}',s)ds)\partial_z u_+\Big).
\end{aligned}
\tag{3.244}
$$

From (3.244), one can identify the resonant and non-resonant parts due to the rotation. The resonant part is

$$
\partial_t u_+ + \overline{\mathcal{V}}\cdot\nabla_h u_+ + \frac{1}{2}(u_+\cdot\nabla_h)(\overline{\mathcal{V}}+i\overline{\mathcal{V}}^\perp) = 0.
$$

Notice that u_- is the complex conjugate of u_+. Therefore, by taking complex conjugate of

(3.244), we obtain the evolution equation for u_- as:

$$\begin{aligned}\partial_t u_- = -e^{-i\Omega t}\Big(& u_- \cdot \nabla_h u_- - P_0(u_- \cdot \nabla_h u_- + (\nabla_h \cdot u_-)u_-) \\ & -(\int_0^z \nabla_h \cdot u_-(\boldsymbol{x}', s)ds)\partial_z u_-\Big) \\ & -\Big(\overline{\mathcal{V}} \cdot \nabla_h u_- + \frac{1}{2}(u_- \cdot \nabla_h)(\overline{\mathcal{V}} - i\overline{\mathcal{V}}^\perp)\Big) \\ & -e^{2i\Omega t}\frac{1}{2}(u_+ \cdot \nabla_h)(\overline{\mathcal{V}} - i\overline{\mathcal{V}}^\perp) \\ & -e^{i\Omega t}\Big(u_+ \cdot \nabla_h u_- - P_0(u_+ \cdot \nabla_h u_- + (\nabla_h \cdot u_+)u_-) \\ & -(\int_0^z \nabla_h \cdot u_+(\boldsymbol{x}', s)ds)\partial_z u_-\Big). \end{aligned} \tag{3.245}$$

Here the resonant part is

$$\partial_t u_- + \overline{\mathcal{V}} \cdot \nabla_h u_- + \frac{1}{2}(u_- \cdot \nabla_h)(\overline{\mathcal{V}} - i\overline{\mathcal{V}}^\perp) = 0.$$

Next, we reformulate the evolution of the barotropic mode (3.31). Using $\widetilde{\mathcal{V}} = u_+ e^{i\Omega t} + u_- e^{-i\Omega t}$, we can rewrite (3.31) as:

$$\begin{aligned}&\partial_t \overline{\mathcal{V}} + \mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+ \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_+\Big) \\ &\quad + e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_- \cdot \nabla_h u_- + (\nabla_h \cdot u_-)u_-\Big) \\ &\quad + \mathbb{P}_h P_0\Big(u_+ \cdot \nabla_h u_- + u_- \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_- + (\nabla_h \cdot u_-)u_+\Big) = 0.\end{aligned}$$

Since $u_\pm = e^{\mp i\Omega t} P_\pm \mathcal{V} = \frac{1}{2}e^{\mp i\Omega t}(\widetilde{\mathcal{V}} \pm i\widetilde{\mathcal{V}}^\perp)$, and $\mathbb{P}_h$ commutes with P_0, the last term becomes

$$\begin{aligned}&\mathbb{P}_h P_0\Big(u_+ \cdot \nabla_h u_- + u_- \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_- + (\nabla_h \cdot u_-)u_+\Big) \\ &= P_0\mathbb{P}_h\Big(u_+ \cdot \nabla_h u_- + u_- \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_- + (\nabla_h \cdot u_-)u_+\Big) \\ &= \frac{1}{2}P_0\mathbb{P}_h\Big(\widetilde{\mathcal{V}} \cdot \nabla_h \widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}^\perp \cdot \nabla_h \widetilde{\mathcal{V}}^\perp + (\nabla_h \cdot \widetilde{\mathcal{V}})\widetilde{\mathcal{V}} + (\nabla_h \cdot \widetilde{\mathcal{V}}^\perp)\widetilde{\mathcal{V}}^\perp\Big) \\ &= \frac{1}{2}P_0\mathbb{P}_h(\nabla_h|\widetilde{\mathcal{V}}|^2) = 0.\end{aligned}$$

Therefore, one obtains

$$\partial_t \overline{\mathcal{V}} + \mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+ \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_+\Big)$$
$$+ e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_- \cdot \nabla_h u_- + (\nabla_h \cdot u_-)u_-\Big) = 0. \tag{3.246}$$

Here the resonant part is

$$\partial_t \overline{\mathcal{V}} + \mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}}) = 0,$$

which is the $2D$ Euler equations.

In summary, we have the following lemma.

Lemma 3.5.1. *For $\mathcal{V} \in \mathcal{S}$, system* (3.1) *is equivalent to system (3.244)–(3.246).*

3.5.2 Limit Resonant System

In previous section, we have done the reformulation to see the resonant and non-resonant parts due to rotation. In this section, we derive the formal resonant limit resonant system of the original system (3.1) (or equivalently, system (3.244)–(3.246) by Lemma 3.5.1) as $|\Omega| \to \infty$, and establish some properties of the limit resonant system.

Recall from (3.244), we have

$$\begin{aligned}
\partial_t u_+ = & -e^{i\Omega t}\Big(u_+ \cdot \nabla_h u_+ - P_0(u_+ \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_+) \\
& \qquad - (\int_0^z \nabla_h \cdot u_+(\boldsymbol{x}', s)ds)\partial_z u_+\Big) \\
& - \Big(\overline{\mathcal{V}} \cdot \nabla_h u_+ + \frac{1}{2}(u_+ \cdot \nabla_h)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp)\Big) \\
& - e^{-i\Omega t}\Big(u_- \cdot \nabla_h u_+ - P_0(u_- \cdot \nabla_h u_+ + (\nabla_h \cdot u_-)u_+) \\
& \qquad - (\int_0^z \nabla_h \cdot u_-(\boldsymbol{x}', s)ds)\partial_z u_+\Big) \\
& - e^{-2i\Omega t}\frac{1}{2}(u_- \cdot \nabla_h)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp) \\
=: & -e^{i\Omega t} I_1 - I_0 - e^{-i\Omega t} I_{-1} - e^{-2i\Omega t} I_{-2},
\end{aligned} \tag{3.247}$$

where

$$
\begin{aligned}
&I_1 := u_+ \cdot \nabla_h u_+ - P_0\Big(u_+ \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_+\Big) - \Big(\int_0^z \nabla_h \cdot u_+(\boldsymbol{x}', s)ds\Big)\partial_z u_+, \\
&I_0 := \overline{\mathcal{V}} \cdot \nabla_h u_+ + \frac{1}{2}(u_+ \cdot \nabla_h)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp), \\
&I_{-1} := u_- \cdot \nabla_h u_+ - P_0\Big(u_- \cdot \nabla_h u_+ + (\nabla_h \cdot u_-)u_+\Big) - \Big(\int_0^z \nabla_h \cdot u_-(\boldsymbol{x}', s)ds\Big)\partial_z u_+, \\
&I_{-2} := \frac{1}{2}(u_- \cdot \nabla_h)(\overline{\mathcal{V}} + i\overline{\mathcal{V}}^\perp).
\end{aligned}
\tag{3.248}
$$

Observe that I_0 is a typical resonant term. Unlike the case of the $3D$ Euler equations where there are frequency selection resonances, in this resonance term, I_0, all frequencies resonate.

We can rewrite (3.247) as

$$
\begin{aligned}
&\partial_t\Big[u_+ - \frac{i}{\Omega}\Big(e^{i\Omega t}I_1 - e^{-i\Omega t}I_{-1} - \frac{1}{2}e^{-2i\Omega t}I_{-2}\Big)\Big] \\
&= -\frac{i}{\Omega}\Big(e^{i\Omega t}\partial_t I_1 - e^{-i\Omega t}\partial_t I_{-1} - \frac{1}{2}e^{-2i\Omega t}\partial_t I_{-2}\Big) - I_0.
\end{aligned}
\tag{3.249}
$$

Denote by the formal limits of $u_+, u_-, \overline{\mathcal{V}}$ to be $U_+, U_-, \overline{V}$. By taking limit $|\Omega| \to \infty$ formally, we obtain the limit resonant equation for u_+ is

$$
\partial_t U_+ = -(\overline{V} \cdot \nabla_h)U_+ - \frac{1}{2}(U_+ \cdot \nabla_h)(\overline{V} + i\overline{V}^\perp). \tag{3.250}
$$

By taking the complex conjugate, we obtain the limit resonant equation for u_- is

$$
\partial_t U_- = -(\overline{V} \cdot \nabla_h)U_- - \frac{1}{2}(U_- \cdot \nabla_h)(\overline{V} - i\overline{V}^\perp). \tag{3.251}
$$

For the limit equation for $\overline{\mathcal{V}}$, recall from (3.246) that

$$
\begin{aligned}
\partial_t\overline{\mathcal{V}} + \mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla_h\overline{\mathcal{V}}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+ \cdot \nabla_h u_+ + (\nabla_h \cdot u_+)u_+\Big) \\
+ e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_- \cdot \nabla_h u_- + (\nabla_h \cdot u_-)u_-\Big) = 0.
\end{aligned}
$$

Observe that $\mathbb{P}_h(\overline{\mathcal{V}} \cdot \nabla_h \overline{\mathcal{V}})$ is a typical resonant term. Using the similar method in the derivation of U_+, we can derive the limit resonant equation for $\overline{\mathcal{V}}$ as

$$\partial_t \overline{V} + \mathbb{P}_h(\overline{V} \cdot \nabla_h \overline{V}) = 0. \tag{3.252}$$

Observe that (3.252) is the $2D$ Euler equation.

Consider the initial conditions

$$\Big(\overline{V}_0, (U_+)_0, (U_-)_0\Big) = \Big(\overline{\mathcal{V}}_0, \frac{1}{2}(\widetilde{\mathcal{V}}_0 + i\widetilde{\mathcal{V}}_0^{\perp}), \frac{1}{2}(\widetilde{\mathcal{V}}_0 - i\widetilde{\mathcal{V}}_0^{\perp})\Big) \tag{3.253}$$

for system (3.250)–(3.252). Since $\mathcal{V}_0 \in \mathcal{S}$, we have $\nabla_h \cdot \overline{V} = 0$, $P_0\overline{V} = \overline{V}$, and $P_0 U_\pm = 0$.

Besides the equations for $U_\pm$ and $\overline{V}$, we also want a baroclinic mode $\widetilde{V}$ similar as in the original system. Since initially $U_\pm(0) = \frac{1}{2}(\widetilde{\mathcal{V}}_0 \pm i\widetilde{\mathcal{V}}_0^{\perp})$, we define $\widetilde{V} := U_+ + U_-$ so that $U_\pm = \frac{1}{2}(\widetilde{V} \pm i\widetilde{V}^{\perp})$. From (3.250)–(3.251), we have

$$\partial_t \widetilde{V} + (\overline{V} \cdot \nabla_h)\widetilde{V} + \frac{1}{2}(\widetilde{V} \cdot \nabla_h \overline{V} - \widetilde{V}^{\perp} \cdot \nabla_h \overline{V}^{\perp}) = 0. \tag{3.254}$$

Since $\nabla_h \cdot \overline{V} = 0$, (3.254) is equivelent to

$$\partial_t \widetilde{V} + \overline{V} \cdot \nabla_h \widetilde{V} + \frac{1}{2}\widetilde{V}^{\perp}(\nabla_h^{\perp} \cdot \overline{V}) = 0. \tag{3.255}$$

Since $P_0 U_\pm = 0$, we see $P_0 \widetilde{V} = 0$.

Therefore, we consider the following limit resonant system

$$\begin{cases} \partial_t \overline{V} + \mathbb{P}_h(\overline{V} \cdot \nabla_h \overline{V}) = 0, & (3.256) \\ \partial_t \widetilde{V} + \overline{V} \cdot \nabla_h \widetilde{V} + \frac{1}{2}\widetilde{V}^{\perp}(\nabla_h^{\perp} \cdot \overline{V}) = 0, & (3.257) \\ \overline{V}(0) = \overline{V}_0, \quad \widetilde{V}(0) = \widetilde{V}_0, & (3.258) \end{cases}$$

with $P_0\overline{V} = \overline{V}$ and $P_0\widetilde{V} = 0$. Now notice that (3.256) is the $2D$ Euler equation, and (3.257) is a linear transport equation with an additional stretching term.

Next, we establish the global well-posedness of limit resonant system (3.256)–(3.258) in both Sobolev spaces and the space of analytic functions. Notice that the global well-posedness of (3.256) has been established in Proposition 3.4.2.

Proposition 3.5.2. *Assume* $\overline{V}_0 \in \mathcal{S} \cap H^{r+1}(\mathbb{T}^3)$ *and* $\widetilde{V}_0 \in \mathcal{S} \cap H^r(\mathbb{T}^3)$ *with* $r > 5/2$. *Let* $M \geq 0$, *and suppose that* $\|\overline{V}_0\|_{H^{r+1}} \leq M$. *Then there exist constants* $C_M > 1$ *and* $C_r > 1$, *and a function* $K(t) := C_M^{\exp(C_r t)}$, *such that for any give time* $\mathcal{T} > 0$, *there exists a unique solution* $\overline{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; H^{r+1}(\mathbb{T}^3))$ *and* $\widetilde{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; H^r(\mathbb{T}^3))$ *of system (3.256)–(3.258) on* $[0, \mathcal{T}]$, *and satisfies*

$$\|\overline{V}(t)\|_{H^{r+1}} \leq K(t), \quad \|\widetilde{V}(t)\|_{H^r} \leq \|\widetilde{V}_0\|_{H^r} e^{K(t)}. \tag{3.259}$$

Moreover, assume $\overline{V}_0 \in \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$ *and* $\widetilde{V}_0 \in \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ *with* $r > 5/2$ *and* $\tau_0 > 0$, *and suppose that* $\|e^{\tau_0 A}\overline{V}_0\|_{H^{r+1}} \leq M$. *Then there exists a function*

$$\tau(t) = \tau_0 \exp(-\int_0^t K(s) ds), \tag{3.260}$$

such that for any given time $\mathcal{T} > 0$, *there exists a unique solution* $\overline{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1}(\mathbb{T}^3)))$ *and* $\widetilde{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$ *of system (3.256)–(3.258) on* $[0, \mathcal{T}]$ *such that*

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}} \leq K(t), \quad \|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r} \leq \|e^{\tau_0 A}\widetilde{V}_0\|_{H^r} e^{K(t)}. \tag{3.261}$$

Proof. We will use the notation $K_1, K_2, ...$ as indicated in Remark 10. The global well-posedness of the $2D$ Euler equations in Sobolev spaces and corresponding growth estimate is classical, see [10]. From (3.191), we obtain that $\|\overline{V}\|_{H^{r+1}} \leq K_1(t)$ for some function $K_1(t)$.

For the growth of $\|\widetilde{V}\|_{H^r}$, by standard energy estimate, since $\nabla_h \cdot \overline{V} = 0$ and $r > \frac{5}{2}$, we have

$$\frac{d}{dt}\|\widetilde{V}\|_{H^r}^2 \leq C_r \|\overline{V}\|_{H^{r+1}} \|\widetilde{V}\|_{H^r}^2. \tag{3.262}$$

By Gronwall inequality, and by virture of the growth of $\|\overline{V}\|_{H^{r+1}}$, we obtain

$$\|\widetilde{V}(t)\|_{H^r} \leq \|\widetilde{V}_0\|_{H^r} \exp(\frac{1}{2}C_r \int_0^t K_1(s)ds) \leq \|\widetilde{V}_0\|_{H^r} e^{K(t)} \tag{3.263}$$

for some suitable function $K(t)$, such that $\|\overline{V}(t)\|_{H^{r+1}} \leq K(t)$ also holds. By virtue of these formal energy estimates, the global well-posedness of system (3.256)–(3.258) in Sobolev spaces follows.

The global well-posedness of the $2D$ Euler equations in the space of analytic functions and the corresponding growth estimate are established in Proposition 3.4.2. From Proposition 3.4.2, we can first choose some suitable functions $K_1(t)$ and $K_2(t)$ such that $\tau(t) \leq \tau_0 \exp(-\int_0^t K_1(s)ds)$ and $\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}} \leq K_2(t)$.

For the baroclinic mode $\widetilde{V}$, first, it is easy to see the L^2 energy is conserved. Next, using Lemma 2.2.2 and Lemma 3.4.1, since $r > 5/2$ and $\int_{\mathbb{T}^3} \widetilde{V}(\boldsymbol{x})d\boldsymbol{x} = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{V}\|^2 \\
&= \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{V}\|^2 - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\widetilde{V}), A^r e^{\tau A}\widetilde{V}\Big\rangle - \frac{1}{2}\Big\langle A^r e^{\tau A}(\nabla_h^\perp\cdot\overline{V})\widetilde{V}^\perp, A^r e^{\tau A}\widetilde{V}\Big\rangle \\
&\leq (\dot{\tau} + C_r\tau\|A^{r+1}e^{\tau A}\overline{V}\|)\|A^{r+1/2}e^{\tau A}\widetilde{V}\|^2 + C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}\|A^r e^{\tau A}\widetilde{V}\|^2.
\end{aligned} \tag{3.264}$$

For suitable $K_1(t)$ and $K_2(t)$, we have

$$\dot{\tau} + C_r\tau\|A^{r+1}e^{\tau A}\overline{V}\| \leq \tau(-K_1 + C_r K_2) \leq 0. \tag{3.265}$$

Therefore, by Gronwall inequality, thanks to (3.199), for some suitable function $K(t)$, we have

$$\begin{aligned}
\|A^r e^{\tau(t)A}\widetilde{V}(t)\|^2 &\leq \|A^r e^{\tau_0 A}\widetilde{V}_0\|^2 \exp(\int_0^t C_r\|e^{\tau(s)A}\overline{V}(s)\|_{H^{r+1}}ds) \\
&\leq \|e^{\tau_0 A}\widetilde{V}_0\|_{H^r}^2 e^{K(t)}.
\end{aligned} \tag{3.266}$$

Since L^2 energy is conserved, we have

$$\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r} \leq \|e^{\tau_0 A}\widetilde{V}_0\|_{H^r} e^{K(t)}. \tag{3.267}$$

We can choose $K(t)$ large enough such that $\tau(t) = \tau_0 \exp(-\int_0^t K(s)ds)$ and $\|e^{\tau(t)A}\overline{V}\|_{H^{r+1}} \leq K(t)$. Notice that $\tau(\mathcal{T}) > 0$ for any finite time $\mathcal{T} < \infty$. Therefore, the solution $(\overline{V}, \widetilde{V})$ exists in the space of analytic functions globally in time. □

Remark 11. The use of $K(t)$ above still follows Remark 10. The conclusion is that the growth of $\|\overline{V}(t)\|_{H^{r+1}}$ and $\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}}$ are double exponential in time, while the growth of $\|\widetilde{V}(t)\|_{H^r}$ and $\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r}$ are triple exponential in time.

Remark 12. Since $U_\pm = \frac{1}{2}(\widetilde{V} + i\widetilde{V}^\perp)$, similar as Lemma 2.2.7, for $r \geq 0$ and $\tau \geq 0$, we have

$$\|U_+\|^2 = \|U_-\|^2 = \frac{1}{2}\|\widetilde{V}\|^2, \tag{3.268}$$

and

$$\|e^{\tau A}U_+\|_{H^r}^2 = \|e^{\tau A}U_-\|_{H^r}^2 = \frac{1}{2}\|e^{\tau A}\widetilde{V}\|_{H^r}^2. \tag{3.269}$$

Therefore, the growing bounds of $\|\widetilde{V}\|_{H^r}$ and $\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r}$ also apply to $\|U_\pm(t)\|_{H^r}$ and $\|e^{\tau(t)A}U_\pm(t)\|_{H^r}$.

3.5.3 Main Results

With the help of fast rotation, i.e., when $|\Omega|$ is large, we show that the time of existence of the solution in the space of analytic functions can be prolonged as long as the Sobolev norm $\|\widetilde{\mathcal{V}}_0\|_{H^r}$ is small depending on Ω, while the analytic norm $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^r}$ can be large (of order 1). We call such initial data as "well-prepared" initial data. The following theorem is the main result of this section.

Theorem 3.5.3. *Assume $\overline{\mathcal{V}}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+3}(\mathbb{T}^3))$, $\widetilde{\mathcal{V}}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+2}(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $M \geq 0$ and $\delta > 0$, then there exist constants $C_{\tau_0} > 1$, $C_{M,\tau_0} > 1$, $C_r > 1$,*

$\widetilde{C}_{M,\tau_0} > 1$, $\widetilde{C}_r > 1$, *and functions* $\widetilde{K}(t) := e^{C_{M,\tau_0}^{\exp(C_r t)}}$, $\widetilde{K}_0(t) := e^{\widetilde{C}_{M,\tau_0}^{\exp(\widetilde{C}_r t)}}$, *with* $\widetilde{K}(t) > \widetilde{K}_0(t)$. *Suppose that* $|\Omega_0| \geq C_{\tau_0} e^{\widetilde{K}(1)}$, *and that* $\|e^{\tau_0 A}\overline{\mathcal{V}}_0\|_{H^{r+3}} + \|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \leq M$ *with* $\|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$. *Then there exists a time* $\mathcal{T} = \mathcal{T}(\tau_0, |\Omega_0|, M, r) \geq 1$ *satisfying*

$$C_{\tau_0} e^{\widetilde{K}(\mathcal{T})} = |\Omega_0|, \tag{3.270}$$

such that when $|\Omega| \geq |\Omega_0|$, *the unique solution* $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}})$ *to system* (3.31)–(3.34) *obtained in Theorem 3.2.2 satisfies* $(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, *with*

$$\tau(t) = \Big(\tau_0 - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_0(s)}}} ds - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{|\Omega_0|} ds\Big) e^{-\int_0^t \widetilde{K}_0(s) ds} > 0. \tag{3.271}$$

In particular, from (3.270), $\mathcal{T} \gtrsim \ln(\ln(\ln(\ln|\Omega_0|))) \to \infty$, *as* $|\Omega_0| \to \infty$.

Thanks to Lemma 3.2.1 and Lemma 2.2.6, we immediately have the following corollary.

Corollary 3.5.4. *Suppose* $\mathcal{V}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+3}(\mathbb{T}^3))$, *and the conditions of Theorem 3.5.3 hold. Then the unique solution* $\mathcal{V}$ *obtained in Corollary 3.2.3 satisfies* $\mathcal{V} \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, *when* $|\Omega| \geq |\Omega_0|$, *with* $\mathcal{T}$ *defined in (3.270) and* τ *defined in (3.271).*

In the rest of this section, we focus on system (3.244)–(3.246), which is equivalent to system (3.31)–(3.32) due to Lemma 3.2.1 and Lemma 3.5.1. To prove Theorem 3.5.3, we consider the difference between the original system (3.244)–(3.246) and the limit resonant system (3.250)–(3.252). We call such difference system as perturbed system. Then by the formal energy estimate, we show that the solution to the perturbed system exists for a long time. This together with the global existence of the solution to system (3.250)–(3.252) give us the long-time existence of the solution to system (3.244)–(3.246), and therefore the long-time existence of the solution to system (3.31)–(3.32).

In the following section, we first give a rational behind the smallness of the initial baroclinic mode.

3.5.4 A Rational Behind The Smallness of The Initial Baroclinic Mode

The result of Theorem 3.5.3 is for "well-prepared" initial data, namely, for a given fixed $\delta > 0$, $\|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$. Before we go into the proof of Theorem 3.5.3, we briefly rationalize, below, the reason behind this smallness condition on the baroclinic mode.

Consider the linear IPEs:

$$\begin{cases} \partial_t \mathcal{V} + \Omega \mathcal{V}^\perp + \nabla_h p = 0, & (3.272) \\ \partial_z p = 0, & (3.273) \\ \nabla_h \cdot \mathcal{V} + \partial_z w = 0, & (3.274) \end{cases}$$

whose explicit solution is

$$\mathcal{V}(\boldsymbol{x}, t) = \overline{\mathcal{V}}_0(\boldsymbol{x}') + \mathcal{R}(t)\widetilde{\mathcal{V}}_0(\boldsymbol{x}), \tag{3.275}$$

where

$$\mathcal{R}(t) := \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix}. \tag{3.276}$$

We see there is no "decay" due to rotation in the linear level. This is different from the linearized $3D$ Euler equations with rotation, for which one can obtain certain decay due to dispersion/averaging mechanism, see, e.g., [34, 56].

Now let us look back to our nonlinear IPEs (3.31)–(3.32). The first equation (3.31) is the evolution of the barotropic mode, which is the $2D$ Euler with source terms coming from the baroclinic mode. The second equation (3.32) is the evolution of the baroclinic mode, which is the Burger's equations with rotation and other nonlinear coupling terms. For the Burger's equations with rotation, it is shown in [4, 74] that when the rotation rate $|\Omega|$ is large enough depending on the initial data, the solution exists globally in time because of the absence of resonance between the rotation and nonlinearity, which allows a very strong averaging mechanism that weakens the nonlinearity. In our case, however, the additional coupling nonlinear terms in (3.32) resonate with the rotation term, which does not allow for this simple scenario to take place. However, thanks to the smallness assumption on the initial baroclinic mode, the additional coupling nonlinear terms are initially small, which allows us to push this argument further.

Another reason behind this smallness assumption is indicated in Remark 7. It suggests that the smallness condition on the baroclinic mode is required to guarantee the long-time existence of solutions to the $3D$ IPEs with fast rotation.

Further reasoning for the smallness condition on the initial baroclinic mode will be provided in Remark 14 and Remark 15, below.

3.5.5 The Perturbed System Around $|\Omega| = \infty$

In Section 3.5.2, we see that the limit resonant system (3.250)–(3.252) is globally well-posed. Therefore, the idea to show long-time existence of the solution is to consider the difference between the original system (3.244)–(3.246) and the limit resonant system (3.250)–(3.252).

Denote by

$$\overline{\phi} = \overline{\mathcal{V}} - \overline{V}, \quad \phi_{\pm} = u_{\pm} - U_{\pm}. \tag{3.277}$$

Taking the difference between (3.246) and (3.252), (3.244) and (3.250), (3.245) and (3.251), we obtain

$$\begin{cases} \partial_t\overline{\phi} + \mathbb{P}_h\Big[\overline{\phi}\cdot\nabla_h\overline{V} + \overline{\phi}\cdot\nabla_h\overline{\phi} + \overline{V}\cdot\nabla_h\overline{\phi} + e^{2i\Omega t}P_0\Big(Q_{1,+,+} + Q_{2,+,+}\Big) \\ \qquad\qquad + e^{-2i\Omega t}P_0\Big(Q_{1,-,-} + Q_{2,-,-}\Big)\Big] = 0, & (3.278) \\ \partial_t\phi_+ + \overline{\phi}\cdot\nabla_h U_+ + \overline{\phi}\cdot\nabla_h\phi_+ + \overline{V}\cdot\nabla_h\phi_+ + \frac{1}{2}(\phi_+\cdot\nabla_h)(\overline{V} + i\overline{V}^\perp) \\ \quad + \frac{1}{2}(\phi_+\cdot\nabla_h)(\overline{\phi} + i\overline{\phi}^\perp) + \frac{1}{2}(U_+\cdot\nabla_h)(\overline{\phi} + i\overline{\phi}^\perp) \\ \quad + e^{i\Omega t}\Big(Q_{1,+,+} - P_0Q_{1,+,+} - P_0Q_{2,+,+} - Q_{3,+,+}\Big) \\ \quad + e^{-i\Omega t}\Big(Q_{1,-,+} - P_0Q_{1,-,+} - P_0Q_{2,-,+} - Q_{3,-,+}\Big) + e^{-2i\Omega t}Q_{4,-,+} = 0, & (3.279) \\ \partial_t\phi_- + \overline{\phi}\cdot\nabla_h U_- + \overline{\phi}\cdot\nabla_h\phi_- + \overline{V}\cdot\nabla_h\phi_- + \frac{1}{2}(\phi_-\cdot\nabla_h)(\overline{V} - i\overline{V}^\perp) \\ \quad + \frac{1}{2}(\phi_-\cdot\nabla_h)(\overline{\phi} - i\overline{\phi}^\perp) + \frac{1}{2}(U_-\cdot\nabla_h)(\overline{\phi} - i\overline{\phi}^\perp) \\ \quad + e^{-i\Omega t}\Big(Q_{1,-,-} - P_0Q_{1,-,-} - P_0Q_{2,-,-} - Q_{3,-,-}\Big) \\ \quad + e^{i\Omega t}\Big(Q_{1,+,-} - P_0Q_{1,+,-} - P_0Q_{2,+,-} - Q_{3,+,-}\Big) + e^{2i\Omega t}Q_{4,+,-} = 0, & (3.280) \end{cases}$$

where

$$Q_{1,\pm,\mp} = \phi_\pm\cdot\nabla_h U_\mp + \phi_\pm\cdot\nabla_h\phi_\mp + U_\pm\cdot\nabla_h\phi_\mp + U_\pm\cdot\nabla_h U_\mp, \tag{3.281}$$

$$Q_{2,\pm,\mp} = (\nabla_h\cdot\phi_\pm)U_\mp + (\nabla_h\cdot\phi_\pm)\phi_\mp + (\nabla_h\cdot U_\pm)\phi_\mp + (\nabla_h\cdot U_\pm)U_\mp, \tag{3.282}$$

$$\begin{aligned} Q_{3,\pm,\mp} = &(\int_0^z \nabla_h\cdot\phi_\pm(\boldsymbol{x}',s)ds)\partial_z U_\mp + (\int_0^z \nabla_h\cdot\phi_\pm(\boldsymbol{x}',s)ds)\partial_z\phi_\mp \\ &+(\int_0^z \nabla_h\cdot U_\pm(\boldsymbol{x}',s)ds)\partial_z\phi_\mp + (\int_0^z \nabla_h\cdot U_\pm(\boldsymbol{x}',s)ds)\partial_z U_\mp, \end{aligned} \tag{3.283}$$

$$\begin{aligned} Q_{4,\pm,\mp} = &\frac{1}{2}\Big[(\phi_\pm\cdot\nabla_h)(\overline{V} \mp i\overline{V}^\perp) + (\phi_\pm\cdot\nabla_h)(\overline{\phi} \mp i\overline{\phi}^\perp) \\ &+(U_\pm\cdot\nabla_h)(\overline{\phi} \mp i\overline{\phi}^\perp) + (U_\pm\cdot\nabla_h)(\overline{V} \mp i\overline{V}^\perp)\Big]. \end{aligned} \tag{3.284}$$

Recall that we supplement the initial conditions for the limit resonant system (3.250)–(3.252) as

$$\overline{V}_0 = \overline{\mathcal{V}}_0, \quad (U_\pm)_0 = (u_\pm)_0 = \frac{1}{2}(\widetilde{\mathcal{V}}_0 \pm i\widetilde{\mathcal{V}}_0^\perp). \tag{3.285}$$

Therefore, the initial conditions for the perturbed system is

$$\overline{\phi}_0 = 0, \quad (\phi_\pm)_0 = 0. \tag{3.286}$$

3.5.6 Proof of Theorem 3.5.3

In this subsection, we prove Theorem 3.5.3. From Proposition 3.5.2, let $\overline{V}$ and $U_\pm$ be the global solution in $\mathcal{S} \cap \mathcal{D}(e^{\tau(t)A} : H^{r+3}(\mathbb{T}^3))$ and $\mathcal{S} \cap \mathcal{D}(e^{\tau(t)A} : H^{r+2}(\mathbb{T}^3))$, respectively, to system (3.250)-(3.252), with initial data (3.285) and $\tau(t)$ defined by (3.260).

Next, we provide the energy estimate in the space of analytic functions for system (3.278)–(3.280). Applying $A^r e^{\tau A}$ to (3.278)–(3.280), and taking L^2 inner product of (3.278) with $A^r e^{\tau A}\overline{\phi}$, (3.279) with $2A^r e^{\tau A}\phi_-$, and (3.280) with $2A^r e^{\tau A}\phi_+$, thanks to Lemma 2.2.4 and Lemma 2.2.5, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\overline{\phi}\|^2 =& \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 - \Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &- \Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &- e^{2i\Omega t}\Big\langle A^r e^{\tau A}(Q_{1,+,+} + Q_{2,+,+}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &- e^{-2i\Omega t}\Big\langle A^r e^{\tau A}(Q_{1,-,-} + Q_{2,-,-}), A^r e^{\tau A}\overline{\phi}\Big\rangle,\end{aligned} \tag{3.287}$$

and

$$\begin{aligned}&\frac{d}{dt}(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) = 2\dot{\tau}(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2) \\ &-2\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle \\ &-2\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h \phi_+), A^r e^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h \phi_-), A^r e^{\tau A}\phi_+\Big\rangle \\ &-2\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h \phi_+), A^r e^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h \phi_-), A^r e^{\tau A}\phi_+\Big\rangle \\ &-\Big\langle A^r e^{\tau A}(\phi_+\cdot\nabla_h(\overline{V} + i\overline{V}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}(\phi_-\cdot\nabla_h(\overline{V} - i\overline{V}^\perp)), A^r e^{\tau A}\phi_+\Big\rangle \\ &-\Big\langle A^r e^{\tau A}(\phi_+\cdot\nabla_h(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}(\phi_-\cdot\nabla_h(\overline{\phi} - i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_+\Big\rangle \\ &-\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h(\overline{\phi} - i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_+\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&-2e^{i\Omega t}\Big(\Big\langle A^r e^{\tau A}(Q_{1,+,+}-Q_{3,+,+}), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(Q_{1,+,-}-Q_{3,+,-}), A^r e^{\tau A}\phi_+\Big\rangle\Big)\\
&-2e^{-i\Omega t}\Big(\Big\langle A^r e^{\tau A}(Q_{1,-,+}-Q_{3,-,+}), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(Q_{1,-,-}-Q_{3,-,-}), A^r e^{\tau A}\phi_+\Big\rangle\Big)\\
&-2e^{2i\Omega t}\Big\langle A^r e^{\tau A} Q_{4,+,-}, A^r e^{\tau A}\phi_+\Big\rangle - 2e^{-2i\Omega t}\Big\langle A^r e^{\tau A} Q_{4,-,+}, A^r e^{\tau A}\phi_-\Big\rangle. \qquad (3.288)
\end{aligned}
$$

There are totally 71 different nonlinear terms in (3.287) and (3.288). We separate them into the following four different types. We use V to denote the velocity field of the limit resonant system, i.e., $\overline{V}$ and $U_\pm$, and use ϕ to denote the velocity filed of the perturbed system, i.e., $\overline{\phi}$ and $\phi_\pm$.

- Type 1: terms that are trilinear in ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 2: terms that are bilinear in ϕ with no derivative of ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 3: terms that are linear in ϕ, e.g., $e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 4: terms that are bilinear in ϕ and a derivative of ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.

For type 1 nonlinear terms (19 terms), using Lemma 2.2.11–2.2.13, and for type 2 nonlinear terms (15 terms), using Lemma 2.2.2, since $\overline{\phi}$, $\phi_\pm$, $\overline{V}$ and $U_\pm$ all have zero mean value in $\mathbb{T}^3$, we have

$$
\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\\
&+\Big|e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h U_+ + \phi_+\cdot\nabla_h\phi_+ + (\nabla_h\cdot U_+)\phi_+ + (\nabla_h\cdot\phi_+)\phi_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\\
&+\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h U_- + \phi_-\cdot\nabla_h\phi_- + (\nabla_h\cdot U_-)\phi_- + (\nabla_h\cdot\phi_-)\phi_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\\
&+2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle\Big| + 2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle\Big|\\
&+2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h \phi_+), A^r e^{\tau A}\phi_-\Big\rangle\Big| + 2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla_h \phi_-), A^r e^{\tau A}\phi_+\Big\rangle\Big|\\
&+\Big|\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h(\overline{V}+i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h(\overline{V}-i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big|\\
&+\Big|\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h(\overline{\phi}+i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h(\overline{\phi}-i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big|\\
&+2\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h U_+ + \phi_+\cdot\nabla_h\phi_+ - (\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}', s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+2\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h U_- + \phi_+\cdot\nabla_h\phi_- - (\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+2\Big|e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h U_+ + \phi_-\cdot\nabla_h\phi_+ - (\int_0^z \nabla_h\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&+2\Big|e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h U_- + \phi_-\cdot\nabla_h\phi_- - (\int_0^z \nabla_h\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla_h(\overline{V} - i\overline{V}^\perp) + \phi_+\cdot\nabla_h(\overline{\phi} - i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla_h(\overline{V} + i\overline{V}^\perp) + \phi_-\cdot\nabla_h(\overline{\phi} + i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
\le\ & C_r\Big(\|A^{r+1}e^{\tau A}\overline{V}\| + \|A^{r+1}e^{\tau A}U_+\| + \|A^{r+1}e^{\tau A}U_-\|\Big) \\
&\qquad\qquad \times\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\quad + C_r\Big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_-\|\Big) \\
&\qquad\qquad \times\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big).
\end{aligned}
\tag{3.289}
$$

For type 3 nonlinear terms (14 terms), when $\Omega \neq 0$, we first explain the idea on the sample term $e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle$. Indeed, by differentiation by parts, we have

$$
\begin{aligned}
& e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
=\ & \frac{1}{2i\Omega}\partial_t\Big(e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big) \\
& -\frac{1}{2i\Omega}e^{2i\Omega t}\partial_t\Big(\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big).
\end{aligned}
\tag{3.290}
$$

We leave the first term until integrating in time. For the second term, we have

$$
\begin{aligned}
& -\frac{1}{2i\Omega}e^{2i\Omega t}\partial_t\Big(\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big) \\
\le\ & \frac{1}{|\Omega|}|\dot{\tau}|\Big|\Big\langle A^{r+1}e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \frac{1}{2|\Omega|}\Big|\Big\langle A^r e^{\tau A}\partial_t(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
& +\frac{1}{2|\Omega|}\Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\partial_t\overline{\phi}\Big\rangle\Big| := I_1 + I_2 + I_3.
\end{aligned}
\tag{3.291}
$$

Thanks to Cauchy–Schwarz inequality, Lemma 2.2.2, and Lemma 2.2.5, since $\overline{\phi}$, $\phi_\pm$, $\overline{V}$ and $U_\pm$

all have zero mean value in $\mathbb{T}^3$, and since $r > 5/2$, from (3.250) and (3.278), we have

$$\begin{aligned} I_1 &\le \frac{C_r}{|\Omega|}|\dot{\tau}|\|A^{r+1}e^{\tau A}U_+\|\|A^{r+2}e^{\tau A}U_+\|\|A^r e^{\tau A}\overline{\phi}\| \\ &\le \frac{C_r}{|\Omega|^2}|\dot{\tau}|^2 + C_r\|A^{r+2}e^{\tau A}U_+\|^4\|A^r e^{\tau A}\overline{\phi}\|^2, \end{aligned} \tag{3.292}$$

$$\begin{aligned} I_2 &\le \frac{C}{|\Omega|}\Big(\Big|\Big\langle A^r e^{\tau A}\Big\{\Big(\overline{V}\cdot\nabla_h U_+ + \frac{1}{2}(U_+\cdot\nabla_h)(\overline{V}+i\overline{V}^{\perp})\Big)\cdot\nabla_h U_+\Big\}, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\ &\qquad + \Big|\Big\langle A^r e^{\tau A}\Big\{U_+\cdot\nabla_h\Big(\overline{V}\cdot\nabla_h U_+ + \frac{1}{2}(U_+\cdot\nabla_h)(\overline{V}+i\overline{V}^{\perp})\Big)\Big\}, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\Big) \\ &\le \frac{C_r}{|\Omega|}\|A^{r+2}e^{\tau A}U_+\|^2\|A^{r+2}e^{\tau A}\overline{V}\|\|A^r e^{\tau A}\overline{\phi}\| \\ &\le C_r\|A^{r+2}e^{\tau A}U_+\|^2\|A^{r+2}e^{\tau A}\overline{V}\|^2\|A^r e^{\tau A}\overline{\phi}\|^2 + \frac{C_r}{|\Omega|^2}\|A^{r+2}e^{\tau A}U_+\|^2, \end{aligned} \tag{3.293}$$

and

$$\begin{aligned} I_3 &\le \frac{C}{|\Omega|}\Big|\Big\langle A^r e^{\tau A}\mathbb{P}_h(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\Big\{\overline{\phi}\cdot\nabla_h\overline{V} + \overline{\phi}\cdot\nabla_h\overline{\phi} + \overline{V}\cdot\nabla_h\overline{\phi} \\ &\qquad + e^{2i\Omega t}P_0\Big(Q_{1,+,+}+Q_{2,+,+}\Big) + e^{-2i\Omega t}P_0\Big(Q_{1,-,-}+Q_{2,-,-}\Big)\Big\}\Big\rangle\Big| \\ &\le \frac{C}{|\Omega|}\Big|\Big\langle A^{r+1}e^{\tau A}\mathbb{P}_h(U_+\cdot\nabla_h U_+), A^{r-1}e^{\tau A}\Big\{\overline{\phi}\cdot\nabla_h\overline{V} + \overline{\phi}\cdot\nabla_h\overline{\phi} + \overline{V}\cdot\nabla_h\overline{\phi} \\ &\qquad + e^{2i\Omega t}P_0\Big(Q_{1,+,+}+Q_{2,+,+}\Big) + e^{-2i\Omega t}P_0\Big(Q_{1,-,-}+Q_{2,-,-}\Big)\Big\}\Big\rangle\Big| \\ &\le \frac{C_r}{|\Omega|}\|A^{r+2}e^{\tau A}U_+\|^2\Big[\|A^r e^{\tau A}\overline{V}\|^2 + \|A^r e^{\tau A}U_+\|^2 + \|A^r e^{\tau A}U_-\|^2 \\ &\qquad + \|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big]. \end{aligned} \tag{3.294}$$

Applying differentiation by parts to all the type 3 nonlinear terms (14 terms), one obtains

$$\begin{aligned} &-e^{2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot U_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &\qquad + \Big\langle A^r e^{\tau A}\Big((U_+\cdot\nabla_h)(\overline{V}-i\overline{V}^{\perp})\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big] \\ &-e^{-2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_-), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot U_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ &\qquad + \Big\langle A^r e^{\tau A}\Big((U_-\cdot\nabla_h)(\overline{V}+i\overline{V}^{\perp})\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big] \end{aligned}$$

$$
\begin{aligned}
&-2e^{i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big] \\
&-2e^{-i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}', s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}', s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big] \\
&= -\frac{1}{2i\Omega}\partial_t\Big\{e^{2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot U_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\qquad +\Big\langle A^r e^{\tau A}\Big((U_+\cdot\nabla_h)(\overline{V}-i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\} \\
&\quad +\frac{1}{2i\Omega}\partial_t\Big\{e^{-2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_-), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot U_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\qquad +\Big\langle A^r e^{\tau A}\Big((U_-\cdot\nabla_h)(\overline{V}+i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big]\Big\} \\
&\quad -\frac{2}{i\Omega}\partial_t\Big\{e^{i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\} \\
&\quad +\frac{2}{i\Omega}\partial_t\Big\{e^{-i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_+), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h U_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}', s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}', s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\} \\
&\quad +R =: \partial_t N + R, \qquad (3.295)
\end{aligned}
$$

where R corresponds the remaining terms.

Using the similar estimates for (3.291), thanks to Young's inequality, when $|\Omega| > 1$, we have

$$
\begin{aligned}
|R| \le C_r\Big(&\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) \\
&\times\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi\|^2 + \|A^r e^{\tau A}\phi\|^2\Big)
\end{aligned}
$$

$$+\frac{C_r}{|\Omega|}\Big(|\dot{\tau}|^2+\|A^{r+2}e^{\tau A}\overline{V}\|^4+\|A^{r+2}e^{\tau A}U_+\|^4+\|A^{r+2}e^{\tau A}U_-\|^4+1\Big). \tag{3.296}$$

For $\partial_t N$, since $\overline{\phi}(0)=\phi_+(0)=\phi_-(0)=0$, using Lemma 2.2.2, since $\overline{V}$ and $U_\pm$ have zero mean value in $\mathbb{T}^3$, by Young's inequality, we have

$$\begin{aligned}|\int_0^t \partial_s N(s)ds| = |N(t)| \le \frac{C_r}{|\Omega|}\Big(&\|A^{r+1}e^{\tau A}\overline{V}\|^2+\|A^{r+1}e^{\tau A}U_+\|^2+\|A^{r+1}e^{\tau A}U_-\|^2\Big)\\ &\times\Big(\|A^r e^{\tau A}\overline{\phi}\|+\|A^r e^{\tau A}\phi_+\|+\|A^r e^{\tau A}\phi_+\|\Big).\end{aligned} \tag{3.297}$$

The difficulties are on the estimate of type 4 nonlinear terms (23 terms). Thanks to Lemma 3.4.1, since $\nabla_h\cdot\overline{V}=0$, we have

$$\begin{aligned}&\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\\ &\le C_r\|A^r e^{\tau A}\overline{V}\|\|A^r e^{\tau A}\overline{\phi}\|^2 + C_r\tau\|A^{r+1/2}e^{\tau A}\overline{V}\|\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2.\end{aligned} \tag{3.298}$$

Thanks to Lemma 2.2.14, by integration by parts, we have

$$\begin{aligned}&\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\phi_+), A^r e^{\tau A}\phi_-\Big\rangle+\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\phi_-), A^r e^{\tau A}\phi_+ >\Big|\\ &\le\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\phi_+), A^r e^{\tau A}\phi_-\Big\rangle-\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- >\Big|\\ &\quad+\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla_h\phi_-), A^r e^{\tau A}\phi_+\Big\rangle-\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big|\\ &\quad+\Big|\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle+\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ >\Big|\\ &\le C_r\|A^r e^{\tau A}\overline{V}\|(\|A^r e^{\tau A}\phi_+\|^2+\|A^r e^{\tau A}\phi_-\|^2)\\ &\quad+C_r\tau\|A^{r+1/2}e^{\tau A}\overline{V}\|(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2),\end{aligned} \tag{3.299}$$

where

$$\Big|\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle+\Big\langle \overline{V}\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big|=0 \tag{3.300}$$

by integration by parts and $\nabla_h\cdot\overline{V}=0$.

Thanks to Lemma 2.2.14 and Lemma 2.2.16, since $r > 5/2$, by integration by parts and by Sobolev inequality, we have

$$\begin{aligned}
&\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h\phi_+), A^r e^{\tau A}\phi_-\Big\rangle + e^{i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h\phi_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\quad -e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\quad -e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+ > \Big| \\
\le& \Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h\phi_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- > \Big| \\
&+\Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla_h\phi_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ > \Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_+, A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad -\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle \\
&\quad -\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_+, A^r e^{\tau A}\phi_-\Big\rangle \\
&\quad -\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_-, A^r e^{\tau A}\phi_+ > \Big| \\
\le& C_r\|A^{r+1}e^{\tau A}U_+\|(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad +C_r\tau\|A^{r+3/2}e^{\tau A}U_+\|(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2),
\end{aligned} \tag{3.301}$$

where

$$\begin{aligned}
&\Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle \\
&-\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_+, A^r e^{\tau A}\phi_-\Big\rangle \\
&-\Big\langle(\int_0^z \nabla_h\cdot U_+(\boldsymbol{x}', s)ds)A^r e^{\tau A}\partial_z\phi_-, A^r e^{\tau A}\phi_+ > \Big| = 0
\end{aligned} \tag{3.302}$$

by integration by parts. Similarly, we have

$$\begin{aligned}
&\Big|e^{-i\Omega t}\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h\phi_+), A^r e^{\tau A}\phi_-\Big\rangle + e^{-i\Omega t}\Big\langle A^r e^{\tau A}(U_-\cdot\nabla_h\phi_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\quad -e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}',s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\quad -e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot U_-(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\le C_r\|A^{r+1}e^{\tau A}U_-\|(\|A^r e^{\tau A}\phi_+\|^2+\|A^r e^{\tau A}\phi_-\|^2) \\
&\quad +C_r\tau\|A^{r+3/2}e^{\tau A}U_-\|(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (3.303)
\end{aligned}$$

Next, since $-iU_+ = U_+^\perp$, we have

$$\begin{aligned}
&\Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle(\nabla_h\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad +\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}(\overline{\phi}-i\overline{\phi}^\perp), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\le \Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\overline{\phi}, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\quad +\Big|\Big\langle(\nabla_h\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle U_+^\perp\cdot\nabla_h A^r e^{\tau A}\overline{\phi}^\perp, A^r e^{\tau A}\phi_+ >\Big| \\
&\le \Big|\Big\langle(\nabla_h\cdot U_+)A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}\phi_+\cdot\nabla_h U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&\quad +\Big|\Big\langle U_+^\perp\cdot\nabla_h A^r e^{\tau A}\overline{\phi}^\perp, A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\phi_+\cdot\nabla_h A^r e^{\tau A}\overline{\phi}, U_+\Big\rangle\Big|. \qquad (3.304)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\Big|\Big\langle U_+^\perp\cdot\nabla_h A^r e^{\tau A}\overline{\phi}^\perp, A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\phi_+\cdot\nabla_h A^r e^{\tau A}\overline{\phi}, U_+\Big\rangle\Big| \\
&= \Big|\Big\langle(\nabla_h\cdot A^r e^{\tau A}\overline{\phi})U_+, A^r e^{\tau A}\phi_+\Big\rangle\Big| = 0, \qquad (3.305)
\end{aligned}$$

therefore, by Sobolev inequality and Hölder inequality, and since $r > 5/2$, we have

$$\begin{aligned}
&\Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle(\nabla_h\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad +\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}(\overline{\phi}-i\overline{\phi}^\perp), A^r e^{\tau A}\phi_+\Big\rangle\Big|
\end{aligned}$$

$$\le C_r\|\nabla_h U_+\|_{L^\infty}(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2)$$
$$\le C_r\|A^r e^{\tau A} U_+\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2). \tag{3.306}$$

Based on this, thanks to Lemma 2.2.14 and Lemma 2.2.15, we have

$$\begin{aligned}
&\Big|e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla_h\phi_+ + (\nabla_h\cdot\phi_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad + e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla_h(\overline{\phi} - i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
\le &\Big|\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla_h\phi_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big((\nabla_h\cdot\phi_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle(\nabla_h\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla_h\overline{\phi}\Big), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\overline{\phi}, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla_h\overline{\phi}^\perp\Big), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\overline{\phi}^\perp\Big\rangle, A^r e^{\tau A}\phi_+\Big| \\
&+\Big|\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle(\nabla_h\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad+\Big\langle U_+\cdot\nabla_h A^r e^{\tau A}(\overline{\phi} - i\overline{\phi}^\perp), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
\le &C_r\|A^r e^{\tau A}U_+\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2) \\
&+C_r\tau\|A^{r+1/2}e^{\tau A}U_+\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2).
\end{aligned} \tag{3.307}$$

Similarly, we have

$$\begin{aligned}
&\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_-\cdot\nabla_h\phi_- + (\nabla_h\cdot\phi_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad + e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_-\cdot\nabla_h(\overline{\phi} + i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
\le &C_r\|A^r e^{\tau A}U_-\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&+C_r\tau\|A^{r+1/2}e^{\tau A}U_-\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2).
\end{aligned} \tag{3.308}$$

For the rest parts in type 4, there is no cancellation as above. First, by Hölder inequality, we have

$$
\begin{aligned}
&\left|\left\langle U_+ \cdot \nabla_h A^r e^{\tau A}(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- \right\rangle\right| \\
&\le \left|\left\langle A^{1/2} U_+ \cdot \nabla_h A^{r-1/2} e^{\tau A}(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- \right\rangle\right| \\
&\quad + \left|\left\langle U_+ \cdot \nabla_h A^{r-1/2} e^{\tau A}(\overline{\phi} + i\overline{\phi}^\perp)), A^{r+1/2} e^{\tau A}\phi_- \right\rangle\right| \\
&\le C_r(\|U_+\|_{L^\infty} + \|A^{1/2}U_+\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2).
\end{aligned} \tag{3.309}
$$

Based on this, using Lemma 2.2.14–2.2.17, we have

$$
\begin{aligned}
&\left|\left\langle A^r e^{\tau A}(U_+ \cdot \nabla_h(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- > \right|\right. \\
&\le \left|\left\langle A^r e^{\tau A}(U_+ \cdot \nabla_h(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- \right\rangle - \left\langle U_+ \cdot \nabla_h A^r e^{\tau A}(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- \right\rangle\right| \\
&\quad + \left|\left\langle U_+ \cdot \nabla_h A^r e^{\tau A}(\overline{\phi} + i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- \right\rangle\right| \\
&\le C_r\|A^r e^{\tau A}U_+\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad + C_r(\tau\|A^{r+1/2}e^{\tau A}U_+\| + \|U_+\|_{L^\infty} + \|A^{1/2}U_+\|_{L^\infty}) \\
&\qquad\qquad \times(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2).
\end{aligned} \tag{3.310}
$$

Similarly,

$$
\begin{aligned}
&\left|\left\langle A^r e^{\tau A}(U_- \cdot \nabla_h(\overline{\phi} - i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_+ > \right|\right. \\
&\le C_r\|A^r e^{\tau A}U_-\|(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2) \\
&\quad + C_r(\tau\|A^{r+1/2}e^{\tau A}U_-\| + \|U_-\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty}) \\
&\qquad\qquad \times(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2).
\end{aligned} \tag{3.311}
$$

Next, by Hölder inequality, we have

$$
\left|\left\langle (\partial_z U_+) A^r e^{\tau A}(\int_0^z \nabla_h \cdot \phi_+(\boldsymbol{x}', s)ds), A^r e^{\tau A}\phi_- \right\rangle\right|
$$

$$
\begin{aligned}
&\le \Big|\Big\langle (A^{1/2}\partial_z U_+)A^{r-1/2}e^{\tau A}(\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad + \Big|\Big\langle (\partial_z U_+)A^{r-1/2}e^{\tau A}(\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds), A^{r+1/2} e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r(\|\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (3.312)
\end{aligned}
$$

Based on this, thanks to Lemma 2.2.14 to 2.2.17, we obtain

$$
\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\quad - \Big\langle (\partial_z U_+)A^r e^{\tau A}(\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad + \Big|\Big\langle (\partial_z U_+)A^r e^{\tau A}(\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r\|A^{r+1}e^{\tau A}U_+\|(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad + C_r(\tau\|A^{r+3/2}e^{\tau A}U_+\| + \|\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty}) \\
&\qquad \times(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (3.313)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\le C_r(\|A^{r+1}e^{\tau A}U_+\| + \|A^{r+1}e^{\tau A}U_-\|)(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad + C_r\Big(\tau\|A^{r+3/2}e^{\tau A}U_+\| + \tau\|A^{r+3/2}e^{\tau A}U_-\| + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty} \\
&\qquad + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\Big)\Big(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big). \qquad (3.314)
\end{aligned}
$$

Finally, taking summation of (3.287) and (3.288), and using estimates (3.289)–(3.314) for all the nonlinear terms (71 terms), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\leq \Big[\dot{\tau} + C_r\big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_+\|\big) \\
&\qquad +C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\| + \|A^{r+3/2}e^{\tau A}U_+\| + \|A^{r+3/2}e^{\tau A}U_-\|\big) \\
&\qquad +C_r\big(\|U_+\|_{L^\infty} + \|U_-\|_{L^\infty} + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty} \\
&\qquad\qquad +\|A^{1/2}U_+\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\big)\Big] \\
&\qquad \times\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big) \\
&\quad +C_r\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) \\
&\qquad\qquad \times\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\quad +\frac{C_r}{|\Omega|}\Big(|\dot{\tau}|^2 + \|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) + \partial_t N. \qquad (3.315)
\end{aligned}
$$

Notice that eventually we will set

$$
\begin{aligned}
&\dot{\tau} + C_r\big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_+\|\big) \\
&\quad +C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\| + \|A^{r+3/2}e^{\tau A}U_+\| + \|A^{r+3/2}e^{\tau A}U_-\|\big) \\
&\quad +C_r\big(\|U_+\|_{L^\infty} + \|U_-\|_{L^\infty} + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty} \\
&\qquad +\|A^{1/2}U_+\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\big) = 0. \qquad (3.316)
\end{aligned}
$$

Therefore, by Sobolev inequality, Poincare' inequality, and Young's inequality, since $r > 5/2$, $\tau \leq \tau_0$, and $U_\pm$ have zero mean value, we have

$$
\begin{aligned}
|\dot{\tau}|^2 &\leq C_r\big(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_+\|^2\big) \\
&\quad +C_r(\tau_0^2+1)\big(\|A^{r+1/2}e^{\tau A}\overline{V}\|^2 + \|A^{r+3/2}e^{\tau A}U_+\|^2 + \|A^{r+3/2}e^{\tau A}U_-\|^2\big). \qquad (3.317)
\end{aligned}
$$

By Young's inequality, the term $\frac{|\dot{\tau}|^2}{|\Omega|}$ can be combined with other terms, and we can rewrite (3.315) as

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\leq \Big[\dot{\tau} + C_r\big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_+\|\big) \\
&\qquad + C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\| + \|A^{r+3/2}e^{\tau A}U_+\| + \|A^{r+3/2}e^{\tau A}U_-\|\big) \\
&\qquad + C_r\big(\|U_+\|_{L^\infty} + \|U_-\|_{L^\infty} + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty} \\
&\qquad\qquad + \|A^{1/2}U_+\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\big)\Big] \\
&\quad \times\Big[\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big] \\
&\quad + C_r\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) \\
&\qquad\qquad \times\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\quad + \frac{C_{r,\tau_0}}{|\Omega|}\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) + \partial_t N.
\end{aligned}
\tag{3.318}
$$

We set

$$
F := \frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2, \tag{3.319}
$$

$$
G := \|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2, \tag{3.320}
$$

and denote by

$$
K(t) := C_{M,\tau_0}^{\exp(C_r t)}, \qquad \widetilde{K}(t) := e^{K(t)}, \tag{3.321}
$$

which are double exponential and triple exponential in time. We will follow the rule on the use of notation as indicated in Remark 10. From Proposition 3.5.2 and thanks to Lemma 2.2.7, when $\|e^{\tau_0 A}\overline{V}_0\|_{H^{r+3}} + \|e^{\tau_0 A}\widetilde{V}_0\|_{H^{r+2}} \leq M$, we have

$$
\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+3}} \leq K(t) \leq \widetilde{K}_1(t), \qquad \|e^{\tau(t)A}U_\pm(t)\|_{H^{r+2}} \leq \widetilde{K}_1(t), \tag{3.322}
$$

provided that $\tau(t)$ satisfies (3.260). Observe that in (3.318), $\|U_\pm\|_{L^\infty}$, $\|A^{1/2}U_\pm\|_{L^\infty}$, $\|\partial_z U_\pm\|_{L^\infty}$, and $\|A^{1/2}\partial_z U_\pm\|_{L^\infty}$ are the terms force the smallness assumption on Sobolev norm of the baroclinic mode. For $\delta > 0$, by Proposition 3.5.2 and Lemma 2.2.7, thanks to Sobolev inequality, when $\|\widetilde{V}_0\|_{H^{3+\delta}} = \|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$, we have

$$\|U_\pm\|_{L^\infty} + \|A^{1/2}U_\pm\|_{L^\infty} + \|\partial_z U_\pm\|_{L^\infty} + \|A^{1/2}\partial_z U_\pm\|_{L^\infty} \leq C\|\widetilde{V}\|_{H^{3+\delta}} \leq \frac{C\widetilde{K}_1(t)}{|\Omega_0|}. \quad (3.323)$$

Since $|\Omega| \geq |\Omega_0|$, we can rewrite (3.318) as

$$\frac{dF}{dt} \leq (\dot{\tau} + C_r F^{1/2} + \tau\widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|})G + \widetilde{K}_2 F + \frac{\widetilde{K}_2}{|\Omega_0|} + \partial_t N. \quad (3.324)$$

By setting $\dot{\tau} + C_r F^{1/2} + \tau\widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|} = 0$, we have

$$\frac{dF}{dt} \leq \widetilde{K}_2 F + \frac{\widetilde{K}_2}{|\Omega_0|} + \partial_t N. \quad (3.325)$$

By Grönwall inequality, we have

$$\frac{d}{dt}(Fe^{-\int_0^t \widetilde{K}_2(s)ds}) \leq \frac{\widetilde{K}_2}{|\Omega_0|} + (\partial_t N)e^{-\int_0^t \widetilde{K}_2(s)ds}. \quad (3.326)$$

Integrating from 0 to t, noticing that $F(0) = 0$, we have

$$F(t)e^{-\int_0^t \widetilde{K}_2(s)ds} \leq \frac{1}{|\Omega_0|}\int_0^t \widetilde{K}_2(s)ds + \int_0^t (\partial_s N(s))e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}ds. \quad (3.327)$$

From (3.297), we know $|N(t)| \leq \frac{1}{|\Omega_0|}\widetilde{K}_3(t)F^{1/2}$. Moreover, $\frac{1}{|\Omega_0|}\widetilde{K}_3(t)F^{1/2}$ is increasing in time. By integration by parts in time, thanks to Cauchy–Schwarz inequality, since $N(0) = 0$, we have

$$\begin{aligned}&\int_0^t (\partial_s N(s))e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}ds \leq |N(t)| + \int_0^t |N(s)||\partial_s e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}|ds \\ &\leq \frac{1}{|\Omega_0|}\widetilde{K}_3 F^{1/2} + \frac{t}{|\Omega_0|}\widetilde{K}_3 F^{1/2}\widetilde{K}_2 \leq \frac{1}{|\Omega_0|}\widetilde{K}_4 + \frac{1}{|\Omega_0|}F. \quad (3.328)\end{aligned}$$

Thus, we have

$$F(t) \leq \frac{1}{|\Omega_0|} e^{\widetilde{K}_5(t)} + \frac{1}{|\Omega_0|} e^{\widetilde{K}_5(t)} F(t), \tag{3.329}$$

which is equivalent to

$$F(t) \leq \frac{e^{\widetilde{K}_5(t)}}{|\Omega_0| - e^{\widetilde{K}_5(t)}}. \tag{3.330}$$

Plugging this back to $\dot{\tau} + C_r F^{1/2} + \tau \widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|} = 0$, we can require that

$$\dot{\tau} + \frac{e^{\widetilde{K}_6(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_6(t)}}} + \tau \widetilde{K}_6 + \frac{1}{|\Omega_0|} \widetilde{K}_6 \leq 0. \tag{3.331}$$

By Gronwall inequality, we can require

$$\frac{d}{dt}(\tau e^{\int_0^t \widetilde{K}_6(s) ds}) \leq \frac{-e^{\widetilde{K}_7(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_6(t)}}} - \frac{e^{\widetilde{K}_7(t)}}{|\Omega_0|}. \tag{3.332}$$

Integrating from 0 to t, for some suitable function $\widetilde{K}_0(t)$, we can require

$$\tau(t) = \Big(\tau_0 - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_0(s)}}} ds - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{|\Omega_0|} ds\Big) e^{-\int_0^t \widetilde{K}_0(s) ds}. \tag{3.333}$$

Notice that τ in (3.333) also satisfies (3.260) when $\widetilde{K}_0(t)$ is chosen suitably. In order to have $\tau(t) > 0$, we just need to require that

$$\tau_0 \geq \frac{3e^{\widetilde{K}_8(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_8(t)}}} \text{ and } \tau_0 \geq \frac{3e^{\widetilde{K}_8(t)}}{|\Omega_0|} \tag{3.334}$$

for some suitable function $\widetilde{K}_8(t) > \widetilde{K}_0(t)$.

For some new $\widetilde{K}(t) > \widetilde{K}_8(t)$ and the given Ω_0, let $\mathcal{T}$ satisfy

$$C_{\tau_0} e^{\widetilde{K}(\mathcal{T})} = |\Omega_0|, \tag{3.335}$$

then the two conditions in (3.334) are satisfied on $t \in [0, \mathcal{T}]$. Thus, $\tau(t) > 0$ on $t \in [0, \mathcal{T}]$. From (3.335), we know that $e^{\widetilde{K}(\mathcal{T})} \geq \frac{|\Omega_0|}{2C_{\tau_0}}$, and thus the time $\mathcal{T}$ satisfies

$$\mathcal{T} \gtrsim \ln(\ln(\ln(\ln |\Omega_0|))) \to \infty, \tag{3.336}$$

as $|\Omega_0| \to \infty$.

When $\widetilde{K}(t)$ is chosen suitably, from (3.330), we know

$$\|A^r e^{\tau(t)A} \overline{\phi}(t)\|^2 + \|A^r e^{\tau(t)A} \phi_+(t)\|^2 + \|A^r e^{\tau(t)A} \phi_-(t)\|^2 \leq \frac{e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}} < \infty \tag{3.337}$$

on $t \in [0, \mathcal{T}]$. Since $\overline{\phi}$ and $\phi_\pm$ have zero mean value in $\mathbb{T}^3$, by Poincaré inequality, the L^2 norm can be bounded by the higher order norm. Therefore, we have

$$\|e^{\tau(t)A} \overline{\phi}(t)\|_{H^r}^2 + \|e^{\tau(t)A} \phi_+(t)\|_{H^r}^2 + \|e^{\tau(t)A} \phi_-(t)\|_{H^r}^2 \leq \frac{2e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}} < \infty \tag{3.338}$$

on $t \in [0, \mathcal{T}]$. Since $\tau(t)$ satisfies (3.260), we know that

$$\|e^{\tau(t)A} \overline{V}(t)\|_{H^r}^2 + \|e^{\tau(t)A} U_+(t)\|_{H^r}^2 + \|e^{\tau(t)A} U_-(t)\|_{H^r}^2 < \infty \tag{3.339}$$

on $t \in [0, \mathcal{T}]$. Since $\overline{\mathcal{V}} = \overline{\phi} + \overline{V}$ and $\widetilde{u}_\pm = \widetilde{\phi}_\pm + \widetilde{U}_\pm$, by triangle inequality, thanks to Lemma 2.2.7, we have

$$\|e^{\tau(t)A} \overline{\mathcal{V}}(t)\|_{H^r}^2 + \|e^{\tau(t)A} \widetilde{\mathcal{V}}(t)\|_{H^r}^2 < \infty \tag{3.340}$$

on $t \in [0, \mathcal{T}]$. Therefore, we obtain

$$(\overline{\mathcal{V}}, \widetilde{\mathcal{V}}) \in L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))). \tag{3.341}$$

This completes the proof of Theorem 3.5.3.

3.5.7 Approximation by The Limit Resonant System

As a consequence of the proof of Theorem 3.5.3, the following theorem describes the approximation of the solution to the original system (3.31)–(3.34) by the solution to the limit resonant system (3.256)–(3.258) in the space of analytic functions, for large rotation rate $|\Omega|$ and small initial baroclinic mode in Sobolev norm.

Theorem 3.5.5. *Suppose the conditions in Theorem 3.5.3 hold, and let $(\overline{V}, \widetilde{V})$ be the solution to system (3.256)–(3.258) with initial data $(\overline{\mathcal{V}}_0, \widetilde{\mathcal{V}}_0)$. Denote by $\overline{\phi} = \overline{\mathcal{V}} - \overline{V}$ and $\widetilde{\phi} = \widetilde{\mathcal{V}} - \widetilde{V}$, then, for $|\Omega| \geq |\Omega_0|$, one has*

$$\|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{\phi}(t)\|_{H^r} \lesssim \frac{e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}}, \tag{3.342}$$

for $t \in [0, \mathcal{T}]$ with $\mathcal{T}$ given by (3.270) and $\tau(t)$ given by (3.271).

Proof. The proof is an immediate consequence of (3.338). □

3.5.8 Remarks and Discussions

Remark 13. To emphasize the difference between smallness in analytic norm and in Sobolev norm, for $|\Omega| \gg 1$, consider

$$\widetilde{\mathcal{V}}_0 = c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad k_3 \neq 0, \tag{3.343}$$

with $|\boldsymbol{k}| = \lceil \tau_0^{-1} \ln|\Omega| \rceil$ and $|c_{\boldsymbol{k}}| = (\ln|\Omega|)^{-r-2}|\Omega|^{-1}$. When $0 < \delta < 1$, since $r > 5/2$, we have $\|\widetilde{\mathcal{V}}_0\|_{H^{3+\delta}} \leq \|\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \sim |\Omega|^{-1}$, $\|e^{\tau_0 A}\widetilde{\mathcal{V}}_0\|_{H^{r+2}} \sim 1$. Therefore, one can construct a sequence of initial data

$$\{(\widetilde{\mathcal{V}}_0)_\Omega\} = c_{\boldsymbol{k}(\Omega)} e^{i\boldsymbol{k}(\Omega)\cdot\boldsymbol{x}}, \tag{3.344}$$

where $|\boldsymbol{k}(\Omega)| = \lceil \tau_0^{-1} \ln|\Omega| \rceil$ and $|c_{\boldsymbol{k}(\Omega)}| = (\ln|\Omega|)^{-r-2}|\Omega|^{-1}$. Then as $|\Omega| \to \infty$, the existence time of solutions $\mathcal{T} \to \infty$, with initial condition $\|e^{\tau_0 A}(\widetilde{\mathcal{V}}_0)_\Omega\|_{H^{r+2}} \sim 1$. This result needs fast rotation, and is very different from Theorem 3.4.3.

Remark 14. In estimate (3.310) we have the resonance term

$$(U_+ \cdot \nabla_h)(\overline{\phi} + i\overline{\phi}^{\perp}) = (U_+ \cdot \nabla_h \overline{\phi} - U_+^{\perp} \cdot \nabla_h \overline{\phi}^{\perp}) = U_+^{\perp}(\nabla_h^{\perp} \cdot \overline{\phi}), \tag{3.345}$$

which involves the vorticity $\nabla_h^{\perp} \cdot \overline{\phi}$. Notice that in the limit resonant system (3.256)–(3.257), the evolution of the barotropic mode $\overline{V}$ is independent of the baroclinic mode $\widetilde{V}$, and therefore we can control the vorticity $\nabla_h^{\perp} \cdot \overline{V}$. However, for the original system (3.31)–(3.32) (or the perturbed system (3.278)–(3.280)), the evolution of the barotropic mode $\overline{\mathcal{V}}$ (or $\overline{\phi}$) depends also on the baroclinic mode $\widetilde{\mathcal{V}}$ (or $\phi_{\pm}$). Therefore, we are unable to control (3.345) without the smallness condition on the initial baroclinic mode.

Remark 15. In estimate (3.313), we have the term

$$e^{i\Omega t}(\int_0^z \nabla_h \cdot \phi_+(\boldsymbol{x}', s) ds) \partial_z U_+. \tag{3.346}$$

Despite the oscillation, we are unable apply similar methods as in type 3 due to the loss of derivative on the baroclinic mode. For this term, we do not have cancellation as other terms in type 4. Therefore, we are forced to require the smallness condition on the initial baroclinic mode.

4. PRIMITIVE EQUATIONS WITH WEAK DISSIPATION *

In this chapter, we study the PEs with weak dissipation

$$
\begin{cases}
u_t + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x - \nu u_{zz} = 0, & (4.1)\\
v_t + uv_x + wv_z + \epsilon_1 v + \Omega u - \nu v_{zz} = 0, & (4.2)\\
\epsilon_2 w + p_z + T = 0, & (4.3)\\
u_x + w_z = 0, & (4.4)\\
T_t - \kappa \Delta T + uT_x + wT_z = 0 & (4.5)
\end{cases}
$$

in the horizontal channel $\{(x, z) : 0 \leq z \leq H, x \in \mathbb{R}\}$.

Remark 16. Since system (4.1)–(4.5) is independent of y variable, the notation

$$\nabla = (\partial_x, \partial_z)$$

and

$$\Delta = \partial_{xx} + \partial_{zz}$$

will be used.

We complement this system with the boundary conditions

$$
\begin{aligned}
&\Big(u_z, v_z, w, T\Big)\Big|_{z=0,H} = 0,\\
&u, v, w, T \text{ are periodic in } x \text{ with period } 1. \qquad (4.6)
\end{aligned}
$$

and the initial condition

$$\Big(u, v, T\Big)\Big|_{t=0} = \Big(u_0, v_0, T_0\Big). \tag{4.7}$$

*Reprinted with permission from "On the Well-Posedness of Reduced $3D$ Primitive Geostrophic Adjustment Model with Weak Dissipation" by Chongsheng Cao, Quyuan Lin, and Edriss S. Titi, 2020. Journal of Mathematical Fluid Mechanics, Volume 22:32, Pages 1–34, Copyright [2020] by Springer.

In particular, without loss of generality, we choose $H = \frac{1}{2}$. Similar as in Chapter 3, we first consider system (4.1)–(4.5) in the unit two dimensional torus $\mathbb{T}^2$, subject to the following symmetric boundary and initial conditions:

$$u,\ v,\ w,\ p\ \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \tag{4.8}$$

$$u,\ v,\ p\ \text{ are even in } z,\ \text{ and } w,\ T \text{ are odd in z}; \tag{4.9}$$

$$\Big(u, v, T\Big)\Big|_{t=0} = \Big(u_0, v_0, T_0\Big). \tag{4.10}$$

After solving this problem in $\mathbb{T}^2$ subject to (4.8)–(4.10), the solution restricted on original horizontal channel $\{(x, z) : 0 \leq z \leq \frac{1}{2}, x \in \mathbb{R}\}$ will solve the original physical problem with corresponding boundary conditions (4.6) and initial conditions (4.7). Notably here we should also assume the initial condition (u_0, v_0, T_0) for the original physical problem is even, even, and odd extentable in z variable, respectively, so that we are able to work in $\mathbb{T}^2$.

4.1 Local Well-posedness

We first study the local well-posedness in Sobolev space for system (4.1)–(4.5) subject to boundary and initial conditions (4.8)–(4.10).

4.1.1 Reformulation of The Problem.

First, let us reformulate the system (4.1)–(4.5) by deriving equations for w, p_x and p_z in terms of u, v and T. For the sake of simplicity, we drop the argument t in functions when there is no confusion.

First, from (4.4) and by boundary condition (4.9), i.e., $w(x, 0) = 0$, we have

$$w(x, z) = -\int_0^z u_x(x, s) ds. \tag{4.11}$$

From (4.3) and (4.11), we have

$$p_z(x, z) = -T(x, z) - \epsilon_2 w(x, z) = -T(x, z) + \epsilon_2 \int_0^z u_x(x, s) ds. \tag{4.12}$$

Next, we will derive equation for p_x. Notice that since $w(x,0) = w(x,1) = 0$, from (4.11), one has the compatibility condition

$$\int_0^1 u_x(x,z)dz = 0. \tag{4.13}$$

Let us denote by

$$c(t) := \int_0^1 u(x,z,t)dz, \quad d(x,t) := \int_0^1 v(x,z,t)dz. \tag{4.14}$$

Integrating (4.1) with respect to z over $(0,1)$, using boundary condition (4.8) and (4.9), one has:

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \Big(uu_x(x,z) + wu_z(x,z) + p_x(x,z)\Big)dz = \Omega d(x,t).$$

By integration by parts and using (4.4), (4.8) and (4.9), we get

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \Big((u^2)_x(x,z) + p_x(x,z)\Big)dz = \Omega d(x,t). \tag{4.15}$$

Integrating (4.15) with respect to x over $(0,1)$, using compatibility condition (4.13), we have

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \int_0^1 \Big((u^2)_x(x,z) + p_x(x,z)\Big)dxdz = \Omega \int_0^1 d(x,t)dx.$$

Thanks to (4.8), we have

$$\dot{c}(t) + \epsilon_1 c(t) = \Omega \int_0^1 d(x,t)dx. \tag{4.16}$$

Plugging (4.16) back into (4.15) yields

$$\int_0^1 p_x(x,z)dz = \Omega \int_0^1 v(x,z)dz - \Omega \int_0^1 \int_0^1 v(x,z)dxdz - \int_0^1 2uu_x(x,z)dz. \tag{4.17}$$

Next, from (4.11) and (4.12), we have

$$p(x,z) = p_s(x) + \epsilon_2 \int_0^z \int_0^s u_x(x,\xi) d\xi ds - \int_0^z T(x,s) ds, \tag{4.18}$$

where $p_s(x) = p(x,0)$ is the pressure at $z = 0$. By differentiating (4.18) with respect to x, and integrating respect to z over $(0,1)$, by virtue of (4.17), we have

$$\begin{aligned}(p_s)_x(x) = \int_0^1 \Big[\int_0^{z'} T_x(x,s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi) d\xi ds + \Omega v(x,z') - 2uu_x(x,z') \Big] dz' \\ -\Omega \int_0^1 \int_0^1 v(x',z') dx' dz'.\end{aligned} \tag{4.19}$$

Therefore, by differentiating (4.18) with respect to x, and using (4.19), we have

$$\begin{aligned}p_x(x,z) = \epsilon_2 \int_0^z \int_0^s u_{xx}(x,\xi) d\xi ds - \int_0^z T_x(x,s) ds \\ + \int_0^1 \Big[\int_0^{z'} T_x(x,s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi) d\xi ds + \Omega v(x,z') - 2uu_x(x,z') \Big] dz' \\ -\Omega \int_0^1 \int_0^1 v(x',z') dx' dz'.\end{aligned} \tag{4.20}$$

By virtue of (4.11), (4.12) and (4.20), and since p is determined up to a constant, the unknowns for system (4.1)–(4.5) are only (u, v, T). Therefore, we reformulate system (4.1)–(4.5) to the following system:

$$\begin{cases} u_t - \nu u_{zz} + u\, u_x + w u_z + \epsilon_1 u - \Omega v + p_x = 0, & (4.21) \\ v_t - \nu v_{zz} + u\, v_x + w v_z + \epsilon_1 v + \Omega u = 0, & (4.22) \\ T_t - \kappa \Delta T + u\, T_x + w\, T_z = 0, & (4.23) \end{cases}$$

with w, p_x, p_z defined by

$$\begin{cases} w(x,z) := -\int_0^z u_x(x,s)ds, & (4.24)\\ p_x(x,z) := \epsilon_2 \int_0^z \int_0^s u_{xx}(x,\xi)d\xi ds - \int_0^z T_x(x,s)ds \\ \qquad + \int_0^1 \Big[\int_0^{z'} T_x(x,s)ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi)d\xi ds \\ \qquad\qquad + \Omega v(x,z') - 2uu_x(x,z')\Big]dz' \\ \qquad - \Omega \int_0^1 \int_0^1 v(x',z')dx'dz', & (4.25)\\ p_z(x,z) := -T(x,z) + \epsilon_2 \int_0^z u_x(x,s)ds. & (4.26) \end{cases}$$

In this section, we are interested in system (4.21)–(4.26) in the unit two dimensional torus $\mathbb{T}^2$, subject to the following symmetry boundary conditions and initial conditions:

$$u,\ v \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \tag{4.27}$$

$$u,\ v \text{ are even in } z, \text{ and } T \text{ is odd in z}; \tag{4.28}$$

$$\Big(u, v, T\Big)\Big|_{t=0} = \Big(u_0, v_0, T_0\Big). \tag{4.29}$$

It's worth mentioning again that our system (4.21)–(4.26) satisfies the compatibility condition (4.13).

By virtue of (4.24)–(4.26) and (4.27), (4.28), one obtains that w, p also satisfy the symmetry conditions:

$$w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period } 1; \tag{4.30}$$

$$p \text{ is even in } z, \text{ and } w \text{ is odd in z}. \tag{4.31}$$

From (4.24) and (4.26), and by differentiating (4.24) with respect to z, we have

$$\epsilon_2 w + p_z + T = 0, \tag{4.32}$$

$$u_x + w_z = 0. \tag{4.33}$$

Therefore, we have the following conclusion.

Lemma 4.1.1. *System (4.21)–(4.26) subject to (4.27)–(4.29) is equivalent to original system (4.1)–(4.5) subject to (4.8)–(4.10).*

4.1.2 Main Results

The following is the definition of the strong solutions to system (4.21)–(4.26).

Definition 4.1.2. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.27) and (4.28), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Given time $\mathcal{T} > 0$, we say (u, v, T) is a strong solution to system (4.21)–(4.26), subject to (4.27)–(4.29), on the time interval $[0, \mathcal{T}]$, if*

1. *u, v and T satisfy the symmetry conditions (4.27) and (4.28);*

2. *u, v and T have the regularities*

$$\begin{cases} u, v, T, u_x, v_x, T_x \in L^\infty(0, \mathcal{T}; H^1), \\ u_z, v_z, u_{xz}, v_{xz} \in L^2(0, \mathcal{T}; H^1), \\ T, T_x \in L^2(0, \mathcal{T}; H^2), \\ u, v, T \in L^\infty(0, \mathcal{T}; L^\infty) \cap C([0, \mathcal{T}]; L^2), \\ \nabla u, \nabla v, \nabla T \in L^2(0, \mathcal{T}; L^\infty), \\ \partial_t u, \partial_t v, \partial_t T \in L^2(0, \mathcal{T}; L^2); \end{cases}$$

3. *u, v and T satisfy system (4.21)–(4.23) in the following sense:*

$$\partial_t u - \nu u_{zz} + u u_x + w u_z + \epsilon_1 u - \Omega v + p_x = 0 \;\; in \;\; L^2(0, \mathcal{T}; L^2),$$
$$\partial_t v - \nu v_{zz} + u v_x + w v_z + \epsilon_1 v + \Omega u = 0 \;\; in \;\; L^2(0, \mathcal{T}; L^2),$$

$$\partial_t T - \kappa \Delta T + uT_x + wT_z = 0 \ \ in \ \ L^2(0, \mathcal{T}; L^2),$$

with w, p_x, p_z defined by (4.24)–(4.26), and fulfill the initial condition (4.29).

We have the following result concerning the existence and uniqueness of strong solutions to system (4.21)–(4.26), subject to (4.27)–(4.29), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for some positive time $\mathcal{T}$.

Theorem 4.1.3. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.27) and (4.28), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Then there exists some time $\mathcal{T} > 0$ such that there exists a unique strong solution (u, v, T) of system (4.21)–(4.26), subject to (4.27)–(4.29), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.*

To prove Theorem 4.1.3, we first establish formal *a priori* estimates for the solutions of system (4.21)–(4.26). These estimates can be justified rigorously by deriving them first to the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem (Lemma 2.2.10). Based on these, we show the existence of the strong solutions. At the end, we establish the uniqueness of strong solutions, and the continuous dependence on the initial data.

4.1.3 *A priori* Estimates

In this section, we start by assuming that system (4.21)-(4.26) holds for smooth functions and we establish the following formal *a priori* estimates.

By taking the L^2-inner product of equation (4.21) with $u, -\Delta u, \Delta u_{xx}$, equation (4.22) with $v, -\Delta v, \Delta v_{xx}$, equation (4.32) with $w, -\Delta w, \Delta w_{xx}$ and equation (4.23) with $T, -\Delta T, \Delta T_{xx}$, and by integration by parts, thanks to (4.27) and (4.30), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2 \\
&\qquad\qquad\qquad + \|T\|^2 + \|\nabla T\|^2 + \|\nabla T_x\|^2\Big) \\
&+\nu\Big(\|u_z\|^2 + \|v_z\|^2 + \|\nabla u_z\|^2 + \|\nabla v_z\|^2 + \|\nabla u_{xz}\|^2 + \|\nabla v_{xz}\|^2\Big) \\
&+\epsilon_1\Big(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon_2\Big(\|w\|^2+\|\nabla w\|^2+\|\nabla w_x\|^2\Big)+\kappa\Big(\|\nabla T\|^2+\|\Delta T\|^2+\|\Delta T_x\|^2\Big)\\
&=\int_{\mathbb{T}^2}(uu_x+wu_z-\Omega v+p_x)(-u+\Delta u-\Delta u_{xx})\\
&\qquad+(uv_x+wv_z+\Omega u)(-v+\Delta v-\Delta v_{xx})\\
&\qquad+(p_z+T)(-w+\Delta w-\Delta w_{xx})\\
&\qquad+(uT_x+wT_z)(-T+\Delta T-\Delta T_{xx})\,dxdz. \qquad (4.34)
\end{aligned}
$$

By integration by parts, thanks to (4.27), (4.30) and (4.33), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2}(-\Omega v+p_x)\,(-u+\Delta u-\Delta u_{xx})+\Omega u\,(-v+\Delta v-\Delta v_{xx})\\
&\quad+p_z\,(-w+\Delta w-\Delta w_{xx})+(uu_x+wu_z)\,(-u+u_{zz})\\
&\quad+(uv_x+wv_z)(-v)+(uT_x+wT_z)(-T)\,dxdz=0. \qquad (4.35)
\end{aligned}
$$

Therefore, the right-hand side of (4.34) becomes

$$
\begin{aligned}
&\int_{\mathbb{T}^2}(uu_x+wu_z)\,(u_{xx}-u_{xxxx}-u_{xxzz})+(uv_x+wv_z)\,(\Delta v-v_{xxxx}-v_{xxzz})\\
&\quad+T\,(-w+\Delta w-\Delta w_{xx})+(uT_x+wT_z)\,(\Delta T-\Delta T_{xx})\,dxdz. \qquad (4.36)
\end{aligned}
$$

Denote by

$$
\begin{cases}
Y:=1+\|u\|^2+\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2+\|\nabla u_x\|^2+\|\nabla v_x\|^2\\
\qquad\qquad+\|T\|^2+\|\nabla T\|^2+\|\nabla T_x\|^2 & (4.37)\\
F:=\|u_z\|^2+\|v_z\|^2+\|\nabla u_z\|^2+\|\nabla v_z\|^2+\|\nabla u_{xz}\|^2+\|\nabla v_{xz}\|^2, & (4.38)\\
G:=\|w\|^2+\|\nabla w\|^2+\|\nabla w_x\|^2, & (4.39)\\
K:=\|\nabla T\|^2+\|\Delta T\|^2+\|\Delta T_x\|^2. & (4.40)
\end{cases}
$$

From (4.11), by Hölder inequality and Minkowski inequality, we have

$$
\|w\|=\Big(\int_0^1\int_0^1\Big|\int_0^z u_x(x,s)ds\Big|^2dxdz\Big)^{\frac{1}{2}}
$$

$$\leq \int_0^z \left(\int_0^1 \int_0^1 |u_x(x,s)|^2 dxdz\right)^{\frac{1}{2}} ds$$
$$\leq \int_0^1 \left(\int_0^1 \int_0^1 |u_x(x,s)|^2 dxdz\right)^{\frac{1}{2}} ds$$
$$\leq \left(\int_0^1 \int_0^1 \int_0^1 |u_x(x,s)|^2 dxdzds\right)^{\frac{1}{2}} = \|u_x\|. \tag{4.41}$$

Similarly, one can get

$$\|w_x\| \leq \|u_{xx}\|. \tag{4.42}$$

Let us estimate each term in (4.36). By integration by parts, using Cauchy–Schwarz inequality, Young's inequality and Lemma 2.2.8, thanks to (4.27), (4.30), (4.33), (4.41) and (4.42), we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xx} dxdz\right| \\
&\leq C\|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_z\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xx}\|^{\frac{1}{2}})\|u_{xx}\| \\
&\quad + C\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_z\|^{\frac{1}{2}})\|u_{xx}\| \\
&\leq CY^{\frac{3}{2}} \leq CY^3,
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xxxx}\, dxdz\right| = \left|\int_{\mathbb{T}^2} (3u_x u_{xx} + w_{xx}u_z + 2w_x u_{xz})\, u_{xx}\, dxdz\right| \\
&\leq C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xx}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{3}{2}}(\|u_{xx}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}}) \\
&\quad + \|w_{xx}\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}}) \\
&\quad + \|w_x\|\|u_{xz}\|^{\frac{1}{2}}(\|u_{xz}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}})\Big] \\
&\leq \frac{\epsilon_2}{6}\|w_{xx}\|^2 + \frac{\nu}{6}\|u_{xxz}\|^2 + C_{\epsilon_2,\nu}Y^3 \leq \frac{\nu}{10}F + \frac{\epsilon_2}{6}G + C_{\epsilon_2,\nu}Y^3,
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xxzz}\, dxdz\right| = \left|\int_{\mathbb{T}^2} (u_x u_{xz} + w_x u_{zz})\;\, u_{xz}\, dxdz\right| \\
&\leq C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|u_{xz}\|^{\frac{3}{2}}(\|u_{xz}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}})
\end{aligned}$$

$$
\begin{aligned}
&+\|u_{xz}\|\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|u_{zz}\|^{\frac{1}{2}}(\|u_{zz}\|^{\frac{1}{2}}+\|u_{xzz}\|^{\frac{1}{2}})\\
&\le \frac{\nu}{10}(\|u_{zz}\|^2+\|u_{xxz}\|^2+\|u_{xzz}\|^2)+C_\nu Y^2 \le \frac{\nu}{10}F+C_\nu Y^3,
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(uv_x+wv_z)\ \Delta v\ dxdz\right|\\
&\le C(\|u\|+\|u_z\|)(\|v_x\|+\|v_{xx}\|)(\|v_{xx}\|+\|v_{zz}\|)\\
&\quad +C(\|w\|+\|w_z\|)(\|v_z\|+\|v_{xz}\|)(\|v_{xx}\|+\|v_{zz}\|)\\
&\le \frac{\nu}{10}\|v_{zz}\|^2+C_\nu Y^2 \le \frac{\nu}{10}F+C_\nu Y^3,
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
&|\int_{\mathbb{T}^2}(uv_x+wv_z)\ v_{xxxx}dxdz|\\
&=|\int_{\mathbb{T}^2}(u_{xx}v_x+w_{xx}v_z+2u_xv_{xx}+2w_xv_{xz})\ v_{xx}dxdz|\\
&\le C\Big[\|v_{xx}\|\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}}+\|v_{xx}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&\quad +\|w_{xx}\|\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}}+\|v_{xz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\\
&\quad +\|v_{xx}\|^{\frac{3}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}}+\|u_{xx}\|^{\frac{1}{2}})\\
&\quad +\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|v_{xx}\|\Big]\\
&\le \frac{\epsilon_2}{6}\|w_{xx}\|^2+\frac{\nu}{10}(\|u_{xxz}\|^2+\|v_{xxz}\|^2)+C_{\epsilon_2,\nu}Y^3 \le \frac{\nu}{10}F+\frac{\epsilon_2}{6}G+C_{\epsilon_2,\nu}Y^3,
\end{aligned}
\tag{4.47}
$$

$$
\begin{aligned}
&|\int_{\mathbb{T}^2}(uv_x+wv_z)\ v_{xxzz}dxdz|\\
&=|\int_{\mathbb{T}^2}(u_{xz}v_x+v_{xx}u_z-v_zu_{xx}+w_xv_{xz})\ v_{xz}dxdz|\\
&\le C\Big[\|v_{xz}\|\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}}+\|v_{xz}\|^{\frac{1}{2}})\|u_{xz}\|^{\frac{1}{2}}(\|u_{xz}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&\quad +\|v_{xz}\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}}+\|u_{xz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\\
&\quad +\|v_{xz}\|\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}}+\|v_{xz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&\quad +\|v_{xz}\|\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\Big]
\end{aligned}
$$

$$\leq \frac{\nu}{10}(\|u_{xxz}\|^2 + \|v_{xxz}\|^2) + C_\nu Y^2 \leq \frac{\nu}{10}F + C_\nu Y^3, \tag{4.48}$$

$$\begin{aligned} & |\int_{\mathbb{T}^2} T\,(-w + \Delta w - \Delta w_{xx})\,dxdz| \\ & \leq \|T\|\|w\| + \|\nabla T\|\|\nabla w\| + \|\nabla T_x\|\|\nabla w_x\| \leq \frac{\epsilon_2}{6}G + C_{\epsilon_2}Y, \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} & |\int_{\mathbb{T}^2} (uT_x + wT_z)\,(\Delta T - \Delta T_{xx})dxdz \\ & \leq |\int_{\mathbb{T}^2} (uT_x + wT_z)\,\Delta T dxdz| + |\int_{\mathbb{T}^2} (u_xT_x + uT_{xx} + wT_{xz} + w_xT_z)\,\Delta T_x dxdz| \\ & \leq C\Big[\|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_x\|^{\frac{1}{2}})\|T_x\|^{\frac{1}{2}}(\|T_x\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}}) \\ & \qquad + \|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_z\|^{\frac{1}{2}})\|T_z\|^{\frac{1}{2}}(\|T_z\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}})\Big]\|\Delta T\| \\ & \quad + C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xx}\|^{\frac{1}{2}})\|T_x\|^{\frac{1}{2}}(\|T_x\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}}) \\ & \qquad + \|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_z\|^{\frac{1}{2}})\|T_{xx}\|^{\frac{1}{2}}(\|T_{xx}\|^{\frac{1}{2}} + \|T_{xxx}\|^{\frac{1}{2}}) \\ & \qquad + \|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|T_{xz}\|^{\frac{1}{2}}(\|T_{xz}\|^{\frac{1}{2}} + \|T_{xzz}\|^{\frac{1}{2}}) \\ & \qquad + \|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}} + \|w_{xz}\|^{\frac{1}{2}})\|T_z\|^{\frac{1}{2}}(\|T_z\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}})\Big]\|\Delta T_x\| \\ & \leq \frac{\kappa}{2}(\|\Delta T\|^2 + \|\Delta T_x\|^2) + C_\kappa Y^3 \leq \frac{\kappa}{2}K + C_\kappa Y^3. \end{aligned} \tag{4.50}$$

From the estimates above, (4.34) becomes

$$\frac{dY}{dt} + \nu F + \epsilon_2 G + \kappa K \leq C_{\epsilon_2,\nu,\kappa} Y^3. \tag{4.51}$$

Therefore, we have $\frac{dY}{dt} \leq C_{\epsilon_2,\nu,\kappa}Y^3$, and this implies that

$$Y(t) \leq \sqrt{\frac{Y(0)^2}{1 - Y(0)^2 C_{\epsilon_2,\nu,\kappa} t}}.$$

Choose

$$\mathcal{T} = \frac{3}{4C_{\epsilon_2,\nu,\kappa}Y(0)^2}.$$

From above, we have $Y(t) \leq 2Y(0)$ on $[0,\mathcal{T}]$. Plugging it in (4.51), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G + \kappa K \leq 8C_{\epsilon_2,\nu,\kappa}Y(0)^3, \text{ for } t \in [0,\mathcal{T}]. \tag{4.52}$$

Integrating above from 0 to t for any time $t \in [0,\mathcal{T}]$, we obtain

$$Y(t) + \int_0^t \Big(\nu F(s) + \epsilon_2 G(s) + \kappa K(s)\Big)\, ds \leq Y(0) + 8C_{\epsilon_2,\nu,\kappa}tY(0)^3.$$

Therefore, we have

$$\begin{cases} u, v, T, u_x, v_x, T_x \in L^\infty(0,\mathcal{T};H^1), \\ u_z, v_z, u_{xz}, v_{xz} \in L^2(0,\mathcal{T};H^1), \\ T, T_x \in L^2(0,\mathcal{T};H^2). \end{cases} \tag{4.53}$$

By virtue of (4.53) and (4.41), we have

$$w \in L^\infty(0,\mathcal{T};H^1). \tag{4.54}$$

Thanks to Lemma 2.2.9, from (4.53), we also have

$$u, v, T \in L^\infty(0,\mathcal{T};L^\infty), \quad \nabla u, \nabla v, \nabla T \in L^2(0,\mathcal{T};L^\infty). \tag{4.55}$$

4.1.4 Existence of The Strong Solutions

In this section, we employ the standard Galerkin approximation procedure to show the existence of the strong solutions. Let

$$\phi_{\boldsymbol{k}} = \phi_{k_1,k_2} := \begin{cases} \sqrt{2}\exp\left(2\pi i k_1 x\right)\cos(2\pi k_2 z) & \text{if } k_2 \neq 0 \\ \exp\left(2\pi i k_1 x\right) & \text{if } k_2 = 0, \end{cases} \tag{4.56}$$

$$\psi_{\boldsymbol{k}} = \psi_{k_1,k_2} := \sqrt{2}\exp\left(2\pi i k_1 x\right)\sin(2\pi k_2 z), \tag{4.57}$$

and

$$\begin{aligned} \mathcal{E} &:= \{\phi \in L^2(\mathbb{T}^2) \mid \phi = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} a_{\boldsymbol{k}}\phi_{\boldsymbol{k}},\ a_{-k_1,k_2} = a^*_{k_1,k_2},\ \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |a_{\boldsymbol{k}}|^2 < \infty\}, \\ \mathcal{O} &:= \{\psi \in L^2(\mathbb{T}^2) \mid \psi = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} a_{\boldsymbol{k}}\psi_{\boldsymbol{k}},\ a_{-k_1,k_2} = a^*_{k_1,k_2},\ \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |a_{\boldsymbol{k}}|^2 < \infty\}. \end{aligned}$$

Observe that functions in $\mathcal{E}$ and $\mathcal{O}$ are even and odd with respect to z variable, respectively. Moreover, $\mathcal{E}$ and $\mathcal{O}$ are closed subspace of $L^2(\mathbb{T}^2)$, orthogonal to each other and consist of real valued functions. For any $m \in \mathbb{N}$, denote by

$$\begin{aligned} \mathcal{E}_m &:= \{\phi \in L^2(\mathbb{T}^2) \mid \phi = \sum_{|k|\leq m} a_{\boldsymbol{k}}\phi_{\boldsymbol{k}},\ a_{-k_1,k_2} = a^*_{k_1,k_2}\}, \\ \mathcal{O}_m &:= \{\psi \in L^2(\mathbb{T}^2) \mid \psi = \sum_{|k|\leq m} a_{\boldsymbol{k}}\psi_{\boldsymbol{k}},\ a_{-k_1,k_2} = a^*_{k_1,k_2}\}, \end{aligned}$$

the finite-dimensional subspaces of $\mathcal{E}$ and $\mathcal{O}$, respectively. For any function $f \in L^2(\mathbb{T}^2)$, denote by

$$\bar{f}_{\boldsymbol{k}} := \int_{\mathbb{T}^2} f(x,z)\phi^*_{\boldsymbol{k}}(x,z)dxdz, \quad \tilde{f}_{\boldsymbol{k}} := \int_{\mathbb{T}^2} f(x,z)\psi^*_{\boldsymbol{k}}(x,z)dxdz, \tag{4.58}$$

and write

$$P_m f := \sum_{|k|\leq m} \bar{f}_{\boldsymbol{k}}\phi_{\boldsymbol{k}}, \quad \Pi_m f := \sum_{|k|\leq m} \tilde{f}_{\boldsymbol{k}}\psi_{\boldsymbol{k}}. \tag{4.59}$$

Then P_m and Π_m are the orthogonal projections from $L^2(\mathbb{T}^2)$ to $\mathcal{E}_m$ and $\mathcal{O}_m$, respectively. Now let

$$u_m = \sum_{|\boldsymbol{k}|=0}^{m} a_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(x,z), \quad v_m = \sum_{|\boldsymbol{k}|=0}^{m} b_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(x,z), \quad T_m = \sum_{|\boldsymbol{k}|=0}^{m} c_{\boldsymbol{k}}(t)\psi_{\boldsymbol{k}}(x,z),$$

and consider the Galerkin approximation system for our model (4.21)–(4.26) as following:

$$\begin{cases} \partial_t u_m - \nu\partial_{zz}u_m + P_m[u_m\partial_x u_m + w_m\partial_z u_m] + \epsilon_1 u_m - \Omega v_m + \partial_x p_m = 0, & (4.60)\\ \partial_t v_m - \nu\partial_{zz}v_m + P_m[u_m\partial_x v_m + w_m\partial_z v_m] + \epsilon_1 v_m + \Omega u_m = 0, & (4.61)\\ \partial_t T_m - \kappa\Delta T_m + \Pi_m[u_m\partial_x T_m + w_m\partial_z T_m] = 0, & (4.62) \end{cases}$$

with $w_m, \partial_x p_m, \partial_z p_m$ defined by:

$$\begin{cases} w_m(x,z) := -\int_0^z \partial_x u_m(x,s)ds, & (4.63)\\ \partial_x p_m(x,z) := \epsilon_2\int_0^z\int_0^s \partial_{xx}u_m(x,\xi)d\xi ds - \int_0^z \partial_x T_m(x,s)ds \\ \qquad + \int_0^1 \Big[\int_0^{z'} \partial_x T_m(x,s)ds - \epsilon_2\int_0^{z'}\int_0^s \partial_{xx}u_m(x,\xi)d\xi ds + \Omega v_m(x,z')\Big]dz' \\ \qquad - P_m\int_0^1 2u_m\partial_x u_m(x,z')dz' - \Omega\int_0^1\int_0^1 v_m(x',z')dx'dz', & (4.64)\\ \partial_z p_m(x,z) := -T_m(x,z) + \epsilon_2\int_0^z \partial_x u_m(x,s)ds, & (4.65) \end{cases}$$

subject to the following initial conditions:

$$u_m(0) = P_m u_0, \quad v_m(0) = P_m v_0, \quad T_m(0) = \Pi_m T_0. \tag{4.66}$$

Observe that the definitions of $w_m, \partial_x p_m$ and $\partial_z p_m$ are inspired by (4.24)–(4.26). Moreover, notice that

$$(\partial_x p_m)_z(x,z) = -\partial_x T_m(x,z) + \epsilon_2\int_0^z \partial_{xx}u_m(x,s)ds = (\partial_z p_m)_x(x,z),$$

hence (4.64) and (4.65) are compatible. Moreover, from (4.63) and (4.65), we have

$$\begin{cases} \epsilon_2 w_m + \partial_z p_m + T_m = 0, & (4.67) \\ \partial_x u_m + \partial_z w_m = 0. & (4.68) \end{cases}$$

For each $m \in \mathbb{N}$, the Galerkin approximation, system (4.60)–(4.62), together with (4.63)–(4.65), correspond to a first order system of ordinary differential equations, in the coefficients $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ and $c_{\boldsymbol{k}}$ for $0 \leq |k| \leq m$, with quadratic nonlinearity. Therefore, by the theory of ordinary differential equations, there exists some $t_m > 0$ such that system (4.60)–(4.62) together with (4.63)–(4.65) admit a unique solution (u_m, v_m, T_m) on the interval $[0, t_m]$.

Observe that from (4.66), we have $a_{\boldsymbol{k}}(0), b_{\boldsymbol{k}}(0), c_{\boldsymbol{k}}(0) \in \mathbb{C}$ satisfying $a_{-k_1,k_2}(0) = a^*_{k_1,k_2}(0)$, $b_{-k_1,k_2}(0) = b^*_{k_1,k_2}(0)$, and $c_{-k_1,k_2}(0) = c^*_{k_1,k_2}(0)$. Thanks to the uniqueness of the solutions of the ODE system, we conclude that $a_{-k_1,k_2}(t) = a^*_{k_1,k_2}(t), b_{-k_1,k_2}(t) = b^*_{k_1,k_2}(t)$, and $c_{-k_1,k_2}(t) = c^*_{k_1,k_2}(t)$, for $t \in [0, t_m]$. Therefore, $u_m, v_m \in \mathcal{E}_m$, and $T_m \in \mathcal{O}_m$.

Since (u_m, v_m, T_m) have finitely many modes, they are smooth functions, and therefore if one repeats the arguments concerning the *a priori* estimates for the solution, one obtains the same estimates for Galerkin approximate solution (u_m, v_m, T_m). More specifically, for each fixed $m \in \mathbb{N}$, there exists

$$\mathcal{T}_m := \frac{3}{4C_{\epsilon_2,\nu,\kappa} Y_m(0)^2} \tag{4.69}$$

such that

$$\frac{dY_m}{dt} + \nu F_m + \epsilon_2 G_m + \kappa K_m \leq 8C_{\epsilon_2,\nu,\kappa} Y_m(0)^3 \leq 8C_{\epsilon_2,\nu,\kappa} Y(0)^3, \text{ for } t \in [0, \mathcal{T}_m]. \tag{4.70}$$

Here Y is defined in (4.37), and Y_m, F_m, G_m, K_m are similar to (4.37)–(4.40), but with subscript m for all terms. Moreover,

$$\mathcal{T}_m \geq \frac{3}{4C_{\epsilon_2,\nu,\kappa} Y(0)^2} =: \mathcal{T} \tag{4.71}$$

uniformly in m. Therefore, (4.70) holds for all m for $t \in [0, \mathcal{T}]$. In particular, the $L^2(\mathbb{T}^2)$ norm of (u_m, v_m, T_m) is uniformly bounded for $t \in [0, \mathcal{T}]$. Hence, the solution (u_m, v_m, T_m) exists at least for $t \in [0, \mathcal{T}]$. From (4.70), we have the following uniform bounds for the sequence of the Galerkin approximate solutions (u_m, v_m, T_m) and the corresponding w_m:

$$\begin{cases} u_m, v_m, T_m, \partial_x u_m, \partial_x v_m, \partial_x T_m \ \text{ are uniformly bounded in } \ L^\infty(0, \mathcal{T}; H^1), \\ \partial_z u_m, \partial_z v_m, \partial_{xz} u_m, \partial_{xz} v_m \ \text{ are uniformly bounded in } \ L^2(0, \mathcal{T}; H^1), \\ T_m, \partial_x T_m \ \text{ are uniformly bounded in } \ L^2(0, \mathcal{T}; H^2), \\ w_m \ \text{ are uniformly bounded in } \ L^\infty(0, \mathcal{T}; H^1), \\ u_m, v_m, T_m \ \text{ are uniformly bounded in } \ L^\infty(0, \mathcal{T}; L^\infty), \\ \nabla u_m, \nabla v_m, \nabla T_m \ \text{ are uniformly bounded in } \ L^2(0, \mathcal{T}; L^\infty). \end{cases} \tag{4.72}$$

By Banach–Alaoglu theorem, there exist a subsequence, denoted also by (u_m, v_m, w_m, T_m), and corresponding limits, (u, v, w, T), respectively, such that

$$\begin{cases} u_m \to u, \ \ v_m \to v, \ \ T_m \to T \text{ weakly * in } L^\infty(0, \mathcal{T}; H^1) \text{ and weakly in } L^2(0, \mathcal{T}; H^2), \\ w_m \to w \text{ weakly * in } L^\infty(0, \mathcal{T}; H^1) \text{ and weakly in } L^2(0, \mathcal{T}; H^1), \end{cases} \tag{4.73}$$

and the limits (u, v, w, T) satisfy (4.53)–(4.55). From the closeness of $\mathcal{E}$ and $\mathcal{O}$, (u, v, w, T) satisfy

$$u, v \in \mathcal{E}, \ \ w, T \in \mathcal{O}, \tag{4.74}$$

and therefore satisfy the symmetry conditions (4.27) and (4.28).

Now let us verify that the limit w we get from (4.73) satisfies the definition (4.24). Define the space

$$\mathcal{V} := \text{span}_{\boldsymbol{k} \in \mathbb{Z}^2}\{\phi_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}\}, \tag{4.75}$$

where $\phi_{\boldsymbol{k}}, \psi_{\boldsymbol{k}}$ are defined in (4.56) and (4.57). By taking inner product of (4.68) with test function

$\phi \in \mathcal{V}$ in $L^2(\mathbb{T}^2)$, and using (4.73) to pass $m \to \infty$, we have

$$0 = \langle \partial_x u_m + \partial_z w_m, \phi \rangle \to \langle u_x + w_z, \phi \rangle.$$

Since $\mathcal{V}$ is dense in $L^2(\mathbb{T}^2)$, and by virtue of (4.53)–(4.55), we have $u_x + w_z = 0$ at least in $L^\infty(0, \mathcal{T}; H^1)$. Thanks to (4.74), we know $w = 0$ at $z = 0$, and therefore we can write $w(x,z) = -\int_0^z u_x(x,s)ds$, which is exactly (4.24).

In order to obtain the strong convergence of the approximate solutions, we shall derive uniform bounds for $\partial_t u_m, \partial_t v_m$ and $\partial_t T_m$. Let us first estimate $\partial_t u_m$. By taking inner product of equation (4.60) with test function $\phi \in \mathcal{V}$ in $L^2(\mathbb{T}^2)$, we obtain

$$\begin{aligned}
\Big|\langle \partial_t u_m, \phi \rangle\Big| &= \Big|\langle P_m[u_m \partial_x u_m + w_m \partial_z u_m] + \epsilon_1 u_m - \Omega v_m + \partial_x p_m - \nu \partial_{zz} u_m, \phi \rangle\Big| \\
&\le |\langle u_m \partial_x u_m, P_m \phi \rangle| + |\langle w_m \partial_z u_m, P_m \phi \rangle| + |\langle \epsilon_1 u_m, \phi \rangle| \\
&\qquad + |\langle \Omega v_m, \phi \rangle| + |\langle \nu \partial_{zz} u_m, \phi \rangle| + |\langle \partial_x p_m, \phi \rangle| \\
&=: A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned} \tag{4.76}$$

By Cauchy–Schwarz inequality and Lemma 2.2.8, and using the fact $\|P_m \phi\| \le \|\phi\|$, we have

$$\begin{aligned}
A_1 &= |\langle u_m \partial_x u_m, P_m \phi \rangle| \\
&\le C \|u_m\|^{\frac{1}{2}} (\|u_m\|^{\frac{1}{2}} + \|\partial_x u_m\|^{\frac{1}{2}}) \|\partial_x u_m\|^{\frac{1}{2}} (\|\partial_x u_m\|^{\frac{1}{2}} + \|\partial_{xz} u_m\|^{\frac{1}{2}}) \|\phi\|, && (4.77) \\
A_2 &= |\langle w_m \partial_z u_m, P_m \phi \rangle| \\
&\le C \|w_m\|^{\frac{1}{2}} (\|w_m\|^{\frac{1}{2}} + \|\partial_z w_m\|^{\frac{1}{2}}) \|\partial_z u_m\|^{\frac{1}{2}} (\|\partial_z u_m\|^{\frac{1}{2}} + \|\partial_{xz} u_m\|^{\frac{1}{2}}) \|\phi\|, && (4.78) \\
A_3 + A_4 + A_5 &= |\langle \epsilon_1 u_m, \phi \rangle| + |\langle \Omega v_m, \phi \rangle| + |\langle \nu \partial_{zz} u_m, \phi \rangle| \\
&\le C_{\epsilon_1, \nu, \Omega} (\|u_m\| + \|v_m\| + \|\partial_{zz} u_m\|) \|\phi\|. && (4.79)
\end{aligned}$$

From (4.63) and (4.64), we have

$$
\begin{aligned}
A_6 = \Big|\langle \partial_x p_m, \phi\rangle\Big| &\le \Big|\epsilon_2\langle \int_0^1\int_0^{z'} \partial_x w_m(x,s)dsdz', \phi\rangle\Big| + \Big|\langle \int_0^1\int_0^{z'} \partial_x T_m(x,s)dsdz', \phi\rangle\Big| \\
&+\Big|2\langle \int_0^1 u_m\partial_x u_m(x,z')dz', P_m\phi\rangle\Big| + \Big|\langle \Omega\int_0^1 v_m(x,z')dz', \phi\rangle\Big| \\
&+\Big|\epsilon_2\langle \int_0^z \partial_x w_m(x,\xi)d\xi, \phi\rangle\Big| + \Big|\langle \int_0^z \partial_x T_m(x,s)ds, \phi\rangle\Big| \\
&+\Big|\langle \Omega\int_0^1\int_0^1 v_m(x',z')dx'dz', \phi\rangle\Big| \qquad (4.80)\\
=: B_1+B_2+B_3+B_4+B_5+B_6+B_7. \qquad (4.81)
\end{aligned}
$$

By Hölder inequality, Sobolev inequailty and Lemma 2.2.9, and the fact $\|P_m\phi\| \le \|\phi\|$, we have

$$
\begin{aligned}
B_1+B_5 &= \Big|\epsilon_2\int_0^1\int_0^1\int_0^1\int_0^{z'} \partial_x w_m(x,s)dsdz'\phi(x,z)dxdz\Big| \\
&\quad+\Big|\epsilon_2\int_0^1\int_0^1\int_0^z \partial_x w_m(x,s)ds\phi(x,z)dxdz\Big| \\
&\le \epsilon_2\int_0^1\int_0^1\int_0^1\int_0^1 |\partial_x w_m(x,s)||\phi(x,z)|dsdz'dxdz \\
&\quad+\epsilon_2\int_0^1\int_0^1\int_0^1 |\partial_x w_m(x,s)||\phi(x,z)|dsdxdz \\
&\le \epsilon_2\int_0^1\int_0^1 \|\partial_x w_m(s)\|_{L^2_x}ds\|\phi(z)\|_{L^2_x}dz \\
&\le \epsilon_2\|\partial_x w_m\|\|\phi\|, \qquad (4.82)
\end{aligned}
$$

$$
\begin{aligned}
B_2+B_6 &= \Big|\int_0^1\int_0^1\int_0^1\int_0^{z'} \partial_x T_m(x,s)dsdz'\phi(x,z)dxdz\Big| \\
&\quad+\Big|\int_0^1\int_0^1\int_0^z \partial_x T_m(x,s)ds\phi(x,z)dxdz\Big| \\
&\le \int_0^1\int_0^1\int_0^1\int_0^1 |\partial_x T_m(x,s)||\phi(x,z)|dsdz'dxdz \\
&\quad+\int_0^1\int_0^1\int_0^1 |\partial_x T_m(x,s)||\phi(x,z)|dsdxdz \\
&\le \int_0^1\int_0^1 \|\partial_x T_m(s)\|_{L^2_x}\|\phi(z)\|_{L^2_x}dsdz
\end{aligned}
$$

$$\leq \|\partial_x T_m\|\|\phi\|, \tag{4.83}$$

$$\begin{aligned}
B_3 &= \Big|2\langle \int_0^1 u_m \partial_x u_m(x,z')dz', P_m\phi\rangle\Big| \\
&\leq 2\int_0^1\int_0^1\int_0^1 |u_m\partial_x u_m(x,z')||P_m\phi(x,z)|dz'dxdz \\
&\leq 2\int_0^1\int_0^1 \|u_m\partial_x u_m(z')\|_{L^2_x}\|P_m\phi(z)\|_{L^2_x}dz'dz \\
&\leq 2\|u_m\partial_x u_m\|\|P_m\phi\| \\
&\leq C\|u_m\|_{L^\infty}\|\partial_x u_m\|\|\phi\|,
\end{aligned} \tag{4.84}$$

and

$$\begin{aligned}
B_4 + B_7 &= \Big|\langle \Omega \int_0^1 v_m(x,z')dz', \phi\rangle\Big| + \Big|\langle \Omega \int_0^1\int_0^1 v_m(s,r)dsdr, \phi\rangle\Big| \\
&\leq \Omega\|v_m\|\|\phi\|.
\end{aligned} \tag{4.85}$$

From the above and the estimates for A_1–A_5, using (4.72), since $\mathcal{V}$ is dense in $L^2(\mathbb{T}^2)$, we have

$$\partial_t u_m \text{ are uniformly bounded in } L^2(0,\mathcal{T};L^2). \tag{4.86}$$

By using similar estimates for $\partial_t v_m$, we can get

$$\partial_t v_m \text{ are uniformly bounded in } L^2(0,\mathcal{T};L^2). \tag{4.87}$$

For $\partial_t T_m$, taking inner product of equation (4.62) with some test function $\phi \in \mathcal{V}$ in $L^2(\mathbb{T}^2)$, we obtain

$$|\langle \partial_t T_m, \phi\rangle| \leq |\langle u_m\partial_x T_m + w_m\partial_z T_m, \Pi_m\phi\rangle| + |\kappa\langle \Delta T_m, \phi\rangle| := C_1 + C_2 + C_3.$$

By Cauchy–Schwarz inequality and Lemma 2.2.8, and using the fact $\|\Pi_m\phi\| \le \|\phi\|$, we have

$$
\begin{aligned}
C_1 &= |\langle u_m\partial_x T_m, \Pi_m\phi\rangle| \\
&\le C\|u_m\|^{\frac{1}{2}}(\|u_m\|^{\frac{1}{2}} + \|\partial_x u_m\|^{\frac{1}{2}})\|\partial_x T_m\|^{\frac{1}{2}}(\|\partial_x T_m\|^{\frac{1}{2}} + \|\partial_{xz} T_m\|^{\frac{1}{2}})\|\phi\|, \\
C_2 &= |\langle w_m\partial_z T_m, \Pi_m\phi\rangle| \\
&\le C\|w_m\|^{\frac{1}{2}}(\|w_m\|^{\frac{1}{2}} + \|\partial_z w_m\|^{\frac{1}{2}})\|\partial_z T_m\|^{\frac{1}{2}}(\|\partial_z T_m\|^{\frac{1}{2}} + \|\partial_{xz} T_m\|^{\frac{1}{2}})\|\phi\|, \\
C_3 &= |\kappa\langle \Delta T_m, \phi\rangle| \le C_\kappa\|T_m\|_{H^2}\|\phi\|.
\end{aligned}
$$

From the estimates above, using (4.72), since $\mathcal{V}$ is dense in $L^2(\mathbb{T}^2)$, we have

$$
\partial_t T_m \text{ are uniformly bounded in } L^2(0,\mathcal{T};L^2). \tag{4.88}
$$

Then, we infer from (4.86)–(4.88) that there is a subsequence, also denoted by (u_m, v_m, T_m), such that

$$
\partial_t u_m \to \partial_t u, \ \ \partial_t v_m \to \partial_t v, \ \ \partial_t T_m \to \partial_t T \text{ weakly in } L^2(0,\mathcal{T};L^2), \tag{4.89}
$$

By (4.72), (4.86)–(4.88), and thanks to Lemma 2.2.10, we have, for a subsequence, the following strong convergence holds:

$$
u_m \to u, \ \ v_m \to v, \ \ T_m \to T \ \text{ in } \ L^2(0,\mathcal{T};H^1) \cap C([0,\mathcal{T}],L^2). \tag{4.90}
$$

By virtue of (4.24) and (4.63), using Hölder inequality, we have

$$
\begin{aligned}
\|w_m - w\| &= \|\int_0^z (\partial_x u_m - u_x)(x,s)ds\| \\
&\le \int_0^1\int_0^1\int_0^1 |\partial_x u_m - u_x|(x,s)dsdxdz \le \|\partial_x u_m - u_x\|.
\end{aligned} \tag{4.91}
$$

Therefore, by (4.90) and above, we have

$$w_m \to w \ \text{ in } \ L^2([0,\mathcal{T}],L^2). \tag{4.92}$$

Next, we show the convergence of equations (4.60)–(4.62) to the corresponding limits. Taking inner product of equation (4.60) with test function $\psi \in L^2(0,\mathcal{T};H^1)$ in $L^2((0,\mathcal{T})\times\mathbb{T}^2)$, we have

$$\Big\langle \partial_t u_m - \nu\partial_{zz}u_m + P_m[u_m\partial_x u_m + w_m\partial_z u_m] + \epsilon_1 u_m - \Omega v_m + \partial_x p_m, \psi \Big\rangle = 0. \tag{4.93}$$

First, by virtue of (4.73) and (4.89), we have

$$\begin{cases} \langle -\nu\partial_{zz}u_m, \psi\rangle \to \langle -\nu u_{zz}, \psi\rangle, \\ \langle \epsilon_1 u_m, \psi\rangle \to \langle \epsilon_1 u, \psi\rangle, \\ \langle -\Omega v_m, \psi\rangle \to \langle -\Omega v, \psi\rangle, \\ \langle \partial_t u_m, \psi\rangle \to \langle \partial_t u, \psi\rangle, \end{cases} \tag{4.94}$$

as $m\to\infty$.

For the nonlinear terms, we have

$$\begin{aligned} &\langle P_m[u_m\partial_x u_m + w_m\partial_z u_m] + \partial_x p_m, \psi\rangle - \langle uu_x + wu_z + p_x, \psi\rangle \\ &= \langle u_m\partial_x u_m + w_m\partial_z u_m, P_m\psi\rangle + \langle \partial_x p_m, \psi\rangle - \langle uu_x + wu_z + p_x, \psi\rangle \\ &= \langle (u_m - u)\partial_x u_m, P_m\psi\rangle + \langle u(\partial_x u_m - u_x), P_m\psi\rangle + \langle uu_x, P_m\psi - \psi\rangle \\ &\quad + \langle (w_m - w)\partial_z u_m, P_m\psi\rangle + \langle w(\partial_z u_m - u_z), P_m\psi\rangle \\ &\quad + \langle wu_z, P_m\psi - \psi\rangle + \langle \partial_x p_m - p_x, \psi\rangle \\ &=: D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7. \end{aligned} \tag{4.95}$$

By Hölder inequality, Young's inequality and Sobolev inequality, thanks to Lemma 2.2.8, using

(4.53)–(4.55), (4.72), and (4.90)–(4.92), since

$$|P_m\psi\|_{L^2(0,\mathcal{T};H^1)} \le \|\psi\|_{L^2(0,\mathcal{T};H^1)}, \quad \|\Pi_m\psi\|_{L^2(0,\mathcal{T};H^1)} \le \|\psi\|_{L^2(0,\mathcal{T};H^1)},$$

and $\|P_m\psi - \psi\|_{L^2(0,\mathcal{T};H^1)} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}
|D_1| &= |\langle (u_m - u)\partial_x u_m, P_m\psi\rangle| \\
&\le \|u_m - u\|_{L^4(0,\mathcal{T};L^2)}\|\partial_x u_m\|_{L^4(0,\mathcal{T};L^4)}\|P_m\psi\|_{L^2(0,\mathcal{T};L^4)} \\
&\le C\|u_m - u\|_{L^4(0,\mathcal{T};L^2)}\|\partial_x u_m\|_{L^4(0,\mathcal{T};H^1)}\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \qquad (4.96)\\
|D_2| &= |\langle u(\partial_x u_m - u_x), P_m\psi\rangle| \\
&\le \|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\|\partial_x u_m - u_x\|_{L^2(0,\mathcal{T};L^2)}\|P_m\psi\|_{L^2(0,\mathcal{T};L^2)} \\
&\le C\|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\|\partial_x u_m - u_x\|_{L^2(0,\mathcal{T};L^2)}\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \qquad (4.97)\\
|D_3| &= |\langle uu_x, P_m\psi - \psi\rangle| \\
&\le \|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\|u_x\|_{L^2(0,\mathcal{T};L^2)}\|P_m\psi - \psi\|_{L^2(0,\mathcal{T};L^2)} \\
&\le C\|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\|u_x\|_{L^2(0,\mathcal{T};L^2)}\|P_m\psi - \psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \qquad (4.98)\\
|D_4| &= |\langle (w_m - w)\partial_z u_m, P_m\psi\rangle| \\
&\le C\|w_m - w\|_{L^2(0,\mathcal{T};L^2)}\Big(\|\partial_z u_m\|_{L^\infty(0,\mathcal{T};L^2)} + \|\partial_{xz} u_m\|_{L^\infty(0,\mathcal{T};L^2)}\Big) \\
&\qquad \times\Big(\|P_m\psi\|_{L^2(0,\mathcal{T};L^2)} + \|\Pi_m\psi_z\|_{L^2(0,\mathcal{T};L^2)}\Big) \\
&\le C\|w_m - w\|_{L^2(0,\mathcal{T};L^2)}\Big(\|\partial_z u_m\|_{L^\infty(0,\mathcal{T};L^2)} + \|\partial_{xz} u_m\|_{L^\infty(0,\mathcal{T};L^2)}\Big) \\
&\qquad \times\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \qquad (4.99)\\
|D_5| &= |\langle w(\partial_z u_m - u_z), P_m\psi\rangle| \\
&\le \Big(\|w\|_{L^\infty(0,\mathcal{T};L^2)} + \|w_z\|_{L^\infty(0,\mathcal{T};L^2)}\Big)\|\partial_z u_m - u_z\|_{L^2(0,\mathcal{T};L^2)} \\
&\qquad \times\Big(\|P_m\psi\|_{L^2(0,\mathcal{T};L^2)} + \|P_m\psi_x\|_{L^2(0,\mathcal{T};L^2)}\Big) \\
&\le \Big(\|w\|_{L^\infty(0,\mathcal{T};L^2)} + \|w_z\|_{L^\infty(0,\mathcal{T};L^2)}\Big)\|\partial_z u_m - u_z\|_{L^2(0,\mathcal{T};L^2)} \\
&\qquad \times\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \qquad (4.100)
\end{aligned}$$

$$
\begin{aligned}
|D_6| &= |\langle w u_z, P_m\psi - \psi\rangle| \\
&\quad \le \|w\|_{L^2(0,\mathcal{T};L^4)} \|u_z\|_{L^\infty(0,\mathcal{T};L^2)} \|P_m\psi - \psi\|_{L^2(0,\mathcal{T};L^4)} \\
&\quad \le C\|w\|_{L^2(0,\mathcal{T};H^1)} \|u_z\|_{L^\infty(0,\mathcal{T};L^2)} \|P_m\psi - \psi\|_{L^2(0,\mathcal{T};H^1)} \to 0,
\end{aligned} \tag{4.101}
$$

as $m \to \infty$. For D_7, from (4.25) and (4.64), and using (4.24) and (4.63) we have

$$
\begin{aligned}
|D_7| &= |\langle \partial_x p_m - p_x, \psi\rangle| \\
&\le \Big|\langle \epsilon_2 \int_0^1 \int_0^{z'} (\partial_x w_m - w_x)(x,s) ds dz', \psi\rangle\Big| + \Big|\langle \epsilon_2 \int_0^z (\partial_x w_m - w_x)(x,s) ds, \psi\rangle\Big| \\
&\quad + \Big|\langle \int_0^1 \int_0^{z'} (\partial_x T_m - T_x)(x,s) ds dz', \psi\rangle\Big| + \Big|\langle \int_0^z (\partial_x T_m - T_x)(x,s) ds, \psi\rangle\Big| \\
&\quad + 2\Big|\langle \int_0^1 (u_m \partial_x u_m - u u_x)(x,z') dz', P_m\psi\rangle\Big| + 2\Big|\langle \int_0^1 u u_x(x,z') dz', P_m\psi - \psi\rangle\Big| \\
&\quad + \Omega\Big|\langle \int_0^1 (v_m - v)(x,z') dz', \psi\rangle\Big| + \Omega\Big|\langle \int_0^1 \int_0^1 (v_m - v)(x',z') dx' dz', \psi\rangle\Big| \\
&=: E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8.
\end{aligned} \tag{4.102}
$$

By integration by parts, using Hölder inequality and Sobolev inequality, thanks to (4.53)–(4.55), (4.72), (4.74) and (4.90)–(4.92), we have

$$
\begin{aligned}
E_1 + E_2 &= \Big|\langle \epsilon_2 \int_0^1 \int_0^{z'} (\partial_x w_m - w_x)(x,s) ds dz', \psi\rangle\Big| \\
&\qquad\qquad + \Big|\langle \epsilon_2 \int_0^z (\partial_x w_m - w_x)(x,s) ds, \psi\rangle\Big| \\
&= \Big|\langle \epsilon_2 \int_0^1 \int_0^{z'} (w_m - w)(x,s) ds dz', \psi_x\rangle\Big| + \Big|\langle \epsilon_2 \int_0^z (w_m - w)(x,s) ds, \psi_x\rangle\Big| \\
&\le \epsilon_2 \int_0^{\mathcal{T}} \int_0^1 \int_0^1 \int_0^1 |(w_m - w)(x,s)||\psi_x(x,z)| dx ds dz dt \\
&\le \epsilon_2 \|w_m - w\|_{L^2(0,\mathcal{T};L^2)} \|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0,
\end{aligned} \tag{4.103}
$$

$$
\begin{aligned}
E_3 + E_4 &= \Big|\langle \int_0^1 \int_0^{z'} (\partial_x T_m - T_x)(x,s) ds dz', \psi\rangle\Big| + \Big|\langle \int_0^z (\partial_x T_m - T_x)(x,s) ds, \psi\rangle\Big| \\
&= \Big|\langle \int_0^1 \int_0^{z'} (T_m - T)(x,s) ds dz', \psi_x\rangle\Big| + \Big|\langle \int_0^z (T_m - T)(x,s) ds, \psi_x\rangle\Big| \\
&\le \int_0^{\mathcal{T}} \int_0^1 \int_0^1 \int_0^1 |(T_m - T)(x,s)||\psi_x(x,z)| ds dx dz dt
\end{aligned}
$$

$$\leq \|T_m - T\|_{L^2(0,\mathcal{T};L^2)}\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \tag{4.104}$$

$$E_5 = 2\Big|\langle \int_0^1 (u_m\partial_x u_m - uu_x)(x,z')dz', P_m\psi\rangle\Big|$$
$$= \Big|\langle \int_0^1 (u_m + u)(u_m - u)(x,z')dz', P_m\psi_x\rangle\Big|$$
$$\leq \int_0^{\mathcal{T}}\int_0^1\int_0^1 \|(u_m+u)(z')\|_{L_x^4}\|(u_m-u)(z')\|_{L_x^4}\|(P_m\psi_x(z)\|_{L_x^2}dz'dzdt$$
$$\leq C(\|u_m\|_{L^\infty(0,\mathcal{T};H^1)} + \|u\|_{L^\infty(0,\mathcal{T};H^1)})\|u_m - u\|_{L^2(0,\mathcal{T};H^1)}\|\psi\|_{L^2(0,\mathcal{T};H^1)} \to 0, \tag{4.105}$$

$$E_6 = 2\Big|(\int_0^1 uu_x(x,z')dz', P_m\psi - \psi\rangle\Big|$$
$$\leq \int_0^{\mathcal{T}}\int_0^1\int_0^1\int_0^1 |uu_x(x,z')||(P_m\psi - \psi\rangle(x,z)|dz'dxdzdt$$
$$\leq C\|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\int_0^{\mathcal{T}}\int_0^1\int_0^1 \|u_x(z')\|_{L_x^2}\|(P_m\psi - \psi\rangle(z)\|_{L_x^2}dz'dzdt$$
$$\leq C\|u\|_{L^\infty(0,\mathcal{T};L^\infty)}\|u_x\|_{L^2(0,\mathcal{T};L^2)}\|P_m\psi - \psi\|_{L^2(0,\mathcal{T};L^2)} \to 0, \tag{4.106}$$

$$E_7 + E_8 = \Omega\Big|(\int_0^1 (v_m - v)(x,z')dz', \psi\rangle\Big| + \Omega\Big|(\int_0^1\int_0^1 (v_m - v)(x',z')dx'dz', \psi\rangle\Big|$$
$$\leq \Omega\|v_m - v\|_{L^2(0,\mathcal{T};L^2)}\|\psi\|_{L^2(0,\mathcal{T};L^2)} \to 0, \tag{4.107}$$

as $m \to \infty$. From the estimates above, $D_7 \to 0$ as $m \to \infty$. Consequently, we can pass $m \to \infty$ in (4.93) to get

$$\Big\langle \partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x, \psi\Big\rangle = 0, \tag{4.108}$$

for $\psi \in L^2(0,\mathcal{T};H^1)$. Therefore, we have

$$\partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x = 0 \text{ in } L^2(0,\mathcal{T};H^{-1}). \tag{4.109}$$

By virtue of (4.89), all the terms in (4.109) are actually in $L^2(0,\mathcal{T};L^2)$. Consequently, we have

$$\partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x = 0 \text{ in } L^2(0,\mathcal{T};L^2). \tag{4.110}$$

Using analogous arguments, we can obtain the desired results for v and T,

$$\partial_t v - \nu v_{zz} + u v_x + w v_z + \epsilon_1 v + \Omega u = 0 \ \text{ in } \ L^2(0, \mathcal{T}; L^2), \tag{4.111}$$

and

$$\partial_t T - \kappa \Delta T + u T_x + w T_z = 0 \ \text{ in } \ L^2(0, \mathcal{T}; L^2). \tag{4.112}$$

Finally, due to (4.90), one has, for every $t \in [0, \mathcal{T}]$, $u_m(t) \to u(t)$, $v_m(t) \to v(t)$, $T_m(t) \to T(t)$ in L^2. In particular, $u_m(0) \to u(0)$, $v_m(0) \to v(0)$, $T_m(0) \to T(0)$ in L^2. On the other hand, by (4.66), we have $u_m(0) \to u_0$, $v_m(0) \to v_0$, and $T_m(0) \to T_0$ in L^2. As a result, (u, v, T) satisfies the desired initial condition: $u(0) = u_0$, $v(0) = v_0$ and $T(0) = T_0$.

We obtain the local in time existence of strong solutions to system (4.21)–(4.26), subjects to (4.27)–(4.29), on the interval $[0, \mathcal{T}]$.

4.1.5 Uniqueness of Solutions and Continuous Dependence on The Initial Data

In this section, we will show the continuous dependence on the initial data and the uniqueness of the strong solutions. Let $(u_1, v_1, w_1, p_1, T_1)$ and $(u_2, v_2, w_2, p_2, T_2)$ be two strong solutions of system (4.21)–(4.26), with the initial data $((u_0)_1, (v_0)_1, (T_0)_1)$ and $((u_0)_2, (v_0)_2, (T_0)_2)$, respectively. Denote by $u = u_1 - u_2$, $v = v_1 - v_2$, $w = w_1 - w_2$, $p = p_1 - p_2$, $T = T_1 - T_2$. It is clear that

$$\begin{cases} \partial_t u - \nu u_{zz} + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - \Omega v + p_x = 0, & (4.113) \\ \partial_t v - \nu v_{zz} + u_1 v_x + w_1 v_z + u(v_2)_x + w(v_2)_z + \epsilon_1 v + \Omega u = 0, & (4.114) \\ \epsilon_2 w + p_z + T = 0, & (4.115) \\ u_x + w_z = 0, & (4.116) \\ \partial_t T - \kappa \Delta T + u_1 T_x + w_1 T_z + u(T_2)_x + w(T_2)_z = 0. & (4.117) \end{cases}$$

By taking the inner product of equation (4.113) with u, (4.114) with v, (4.115) with w, and

(4.117) with T, in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.27), (4.33), and (4.116), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d(\|u\|^2+\|v\|^2+\|T\|^2)}{dt}+\epsilon_1(\|u\|^2+\|v\|^2)\\
&\qquad\qquad +\nu(\|u_z\|^2+\|v_z\|^2)+\epsilon_2\|w\|^2+\kappa\,\|\nabla T\|^2\\
&\leq \left|\int_{\mathbb{T}^2}(u(u_2)_x+w(u_2)_z)\;u\;dxdz\right|+\left|\int_{\mathbb{T}^2}(u(v_2)_x+w(v_2)_z)\;v\;dxdz\right|\\
&\quad+\left|\int_{\mathbb{T}^2}wT\;dxdz\right|+\left|\int_{\mathbb{T}^2}(u(T_2)_x+w(T_2)_z)\;T\;dxdz\right| =: I_1+I_2+I_3+I_4. \qquad (4.118)
\end{aligned}$$

By integration by parts, using Hölder inequality and Young's inequality, thanks to (4.27), (4.33), and (4.116), we have

$$\begin{aligned}
I_1&=\left|\int_{\mathbb{T}^2}(u(u_2)_x+w(u_2)_z)\;u\;dxdz\right|\\
&\leq\frac{\epsilon_2}{8}\|w\|^2+C_{\epsilon_2}(\|(u_2)_x\|_{L^\infty}+\|(u_2)_z\|_{L^\infty}^2)\|u\|^2, \qquad (4.119)
\end{aligned}$$

$$\begin{aligned}
I_2&=\left|\int_{\mathbb{T}^2}(u(v_2)_x+w(v_2)_z)\;v\;dxdz\right|\\
&\leq\frac{\epsilon_2}{8}\|w\|^2+C_{\epsilon_2}(\|(v_2)_x\|_{L^\infty}+\|(v_2)_z\|_{L^\infty}^2)(\|u\|^2+\|v\|^2), \qquad (4.120)
\end{aligned}$$

$$I_3=\left|\int_{\mathbb{T}^2}wT\;dxdz\right|\leq\frac{\epsilon_2}{8}\|w\|^2+C_{\epsilon_2}\|T\|^2, \qquad (4.121)$$

and

$$\begin{aligned}
I_4&=\left|\int_{\mathbb{T}^2}(u(T_2)_x+w(T_2)_z)\;T\;dxdz\right|\\
&\leq\frac{\epsilon_2}{8}\|w\|^2+C_{\epsilon_2}(\|(T_2)_x\|_{L^\infty}+\|(T_2)_z\|_{L^\infty}^2)(\|u\|^2+\|T\|^2). \qquad (4.122)
\end{aligned}$$

From the estimates above, we obtain

$$\begin{aligned}&\frac{d(\|u\|^2+\|v\|^2+\|T\|^2)}{dt}+\epsilon_1(\|u\|^2+\|v\|^2)\\&\qquad\qquad+\nu(\|u_z\|^2+\|v_z\|^2)+\epsilon_2\|w\|^2+\kappa\|\nabla T\|^2\\&\le C_{\epsilon_2}K\left(\|u\|^2+\|v\|^2+\|T\|^2\right),\end{aligned}\tag{4.123}$$

where

$$K=1+\|\nabla u_2\|_{L^\infty}^2+\|\nabla v_2\|_{L^\infty}^2+\|\nabla T_2\|_{L^\infty}^2.\tag{4.124}$$

Thanks to (4.55), we obtain $K\in L^1(0,\mathcal{T})$. Therefore, by Gronwall inequality, we obtain

$$\begin{aligned}&\|u(t)\|^2+\|v(t)\|^2+\|T(t)\|^2\\&\le(\|u(t=0)\|^2+\|v(t=0)\|^2+\|T(t=0)\|^2)\exp(C_{\epsilon_2}\int_0^t K(s)ds),\end{aligned}\tag{4.125}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0)=v(t=0)=T(t=0)=0$, we have $u(t)=v(t)=T(t)=0$, for all $t\ge 0$. Therefore, the strong solution is unique.

4.1.6 The Special Case: $\Omega=0, v\equiv 0$ and $T\equiv 0$

In this section, we assume that $\Omega=0$, $v\equiv 0$ and $T\equiv 0$. In this case, system (4.1)–(4.5) will be reduced to

$$\begin{cases}\partial_t u-\nu u_{zz}+uu_x+wu_z+\epsilon_1 u+p_x=0, & (4.126)\\ \epsilon_2 w+p_z=0, & (4.127)\\ u_x+w_z=0. & (4.128)\end{cases}$$

Remark 17. There are two reasons why we consider this special case. Firstly, notice that when $\epsilon_1=\epsilon_2=0$, system (4.126)–(4.128) is exactly the $2D$ hydrostatic Navier-Stokes equations. So we can regard system (4.126)–(4.128) as the hydrostatic Navier-Stokes equations with damping. Secondly, as we will see later, we can show the local regularity of strong solution to system (4.126)–(4.128) for initial conditions with less regularity. The reason why we need to assume more regularity for initial data to system (4.21)–(4.23) is that we need to bound terms which contain v_{xx}. For u_{xx}, we can use incompressible condition $u_{xx} = -w_{xz}$ to avoid such an issue. Therefore, in the case when we do not have the evolution equation for v, we can require less for the initial data.

As before, our domain is $\mathbb{T}^2$, and the boundary and initial condition are

$$\begin{cases} u,\ w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period 1}, & (4.129)\\ u \text{ and } p \text{ are even in } z, \text{ and } w \text{ is odd in } z, & (4.130)\\ u|_{t=0} = u_0. & (4.131)\end{cases}$$

Using an analogue argument to that in section 4.1.1, system (4.126)–(4.128) subject to (4.129)–(4.131) is equivalent to the following:

$$u_t - \nu u_{zz} + u\,u_x + wu_z + \epsilon_1 u + p_x = 0, \tag{4.132}$$

with w, p_x, p_z defined by

$$\begin{cases} w(x,z) := -\int_0^z u_x(x,s)ds, & (4.133)\\ p_x(x,z) := \epsilon_2 \int_0^z \int_0^s u_{xx}(x,\xi)d\xi ds \\ \qquad\qquad + \int_0^1 \Big[-\epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi)d\xi ds - 2uu_x(x,z')\Big] dz', & (4.134)\\ p_z(x,z) := \epsilon_2 \int_0^z u_x(x,s)ds, & (4.135)\end{cases}$$

subject to the following symmetry boundary condition and initial condition

$$\begin{cases} u \text{ is periodic in } x \text{ and } z \text{ with period 1, and is even in } z; & (4.136)\\ u|_{t=0} = u_0. & (4.137)\end{cases}$$

By virtue of (4.133)–(4.135) and (4.136), we obtain that w, p also satisfy the symmetry conditions

$$\begin{cases} w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period } 1; & (4.138) \\ p \text{ is even in } z, \text{ and } w \text{ is odd in } z. & (4.139) \end{cases}$$

By virtue of (4.133) and (4.135), and by differentiating (4.133) with respect to z, we have

$$\epsilon_2 w + p_z = 0, \tag{4.140}$$

$$u_x + w_z = 0. \tag{4.141}$$

In this section, we are interested in system (4.132)–(4.135) in the unit two-dimensional flat torus $\mathbb{T}^2$, subject to (4.136)–(4.137). First, we give the definition of strong solution to system (4.132)–(4.135).

Definition 4.1.4. *Suppose that $u_0 \in H^1(\mathbb{T}^2)$ satisfies the symmetry conditions (4.136), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xz} u_0 \in L^2(\mathbb{T}^2)$. Given time $\mathcal{T} > 0$, we say u is a strong solution to system (4.132)–(4.135), subject to (4.136)–(4.137), on the time interval $[0, \mathcal{T}]$, if*

1. *u satisfies the symmetry condition (4.136);*

2. *u has the regularities*

$$\begin{cases} u \in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2) \cap C([0, \mathcal{T}], L^2) \cap L^\infty(0, \mathcal{T}; L^\infty), \\ u_z \in L^2(0, \mathcal{T}; L^\infty), \\ u_{xz} \in L^\infty(0, \mathcal{T}; L^2), \\ u_{xzz} \in L^2(0, \mathcal{T}; L^2), \\ \partial_t u \in L^2(0, \mathcal{T}; L^2); \end{cases}$$

3. u satisfies system (4.132) in the following sense:

$$\partial_t u - \nu u_{zz} + u u_x + w u_z + \epsilon_1 u + p_x = 0, \quad \textit{in} \;\; L^2(0,\mathcal{T}; L^2),$$

with w, p_x, p_z *defined by (4.133)–(4.135), and fulfill the initial condition (4.137).*

We have the following result concerning the locally existence and uniqueness of strong solutions to system (4.132)–(4.135), subject to (4.136)–(4.137), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for some positive time $\mathcal{T}$.

Theorem 4.1.5. *Suppose that* $u_0 \in H^1(\mathbb{T}^2)$ *satisfies the symmetry conditions (4.136), with the compatibility condition* $\int_0^1 \partial_x u_0 dz = 0$*. Moreover, suppose that* $\partial_{xz} u_0 \in L^2(\mathbb{T}^2)$*. Then there exists some* $\mathcal{T} > 0$ *such that there is a unique strong solution u of system (4.132)–(4.135), subject to (4.136)–(4.137), on the interval* $[0, \mathcal{T}]$*. Moreover, the unique strong solution u depends continuously on the initial data.*

Proof. For sake of simplicity, we will only do *a priori* estimates formally here. By taking the inner product of equation (4.132) with u, $-u_{zz}$, and equation (4.140) with w, $-w_{zz}$, in $L^2(\mathbb{T}^2)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|u_z\|^2\right) + \nu\left(\|u_z\|^2 + \|u_{zz}\|^2\right) + \epsilon_1\left(\|u\|^2 + \|u_z\|^2\right) + \epsilon_2\left(\|w\|^2 + \|w_z\|^2\right) \\ &= -\int_{\mathbb{T}^2} (u u_x + w u_z)\,(u - u_{zz})\, dxdz - \int_{\mathbb{T}^2} \Big(p_x (u - u_{zz}) + p_z (w - w_{zz})\Big)\, dxdz. \end{aligned} \tag{4.142}$$

By integration by parts, thanks to (4.136), (4.138) and (4.141), we have

$$-\int_{\mathbb{T}^2} (u u_x + w u_z)\,(u - u_{zz})\, dxdz - \int_{\mathbb{T}^2} \Big(p_x (u - u_{zz}) + p_z (w - w_{zz})\Big)\, dxdz = 0. \tag{4.143}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} &\|u(t)\|^2 + \|u_z(t)\|^2 + 2\int_0^t \Big[\nu\big(\|u_z(s)\|^2 + \|u_{zz}(s)\|^2\big) + \epsilon_2\big(\|w(s)\|^2 + \|w_z(s)\|^2\big)\Big] ds \\ &\qquad \le \|u(0)\|^2 + \|u_z(0)\|^2. \end{aligned} \tag{4.144}$$

From the estimates above, we obtain

$$\begin{cases} u,\ u_z \text{ bounded in } L^\infty(0,\mathcal{T};L^2), \\ w,\ u_{zz},\ w_z=-u_x \text{ bounded in } L^2(0,\mathcal{T};L^2), \end{cases} \tag{4.145}$$

for arbitrary $\mathcal{T}>0$. By taking the inner product of equation (4.132) with $-u_{xx}, u_{xxzz}$ and equation (4.140) with $-w_{xx}$, w_{xxzz} in $L^2(\mathbb{T}^2)$, integrating by parts, thanks to (4.136) and (4.138) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_x\|^2+\|u_{xz}\|^2)+\nu\left(\|u_{xz}\|^2+\|u_{xzz}\|^2\right) \\ &\qquad\qquad +\epsilon_1(\|u_x\|^2+\|u_{xz}\|^2)+\epsilon_2(\|w_x\|^2+\|w_{xz}\|^2) \\ &=\int_{\mathbb{T}^2}(uu_x+wu_z)\ (u_{xx}-u_{xxzz})\ dxdz \\ &\qquad\qquad +\int_{\mathbb{T}^2}\Big(p_x(u_{xx}-u_{xxzz})+p_z(w_{xx}-w_{xxzz})\Big)dxdz. \end{aligned} \tag{4.146}$$

By integration by parts, thanks to (4.136), (4.138) and (4.141), we have

$$\int_{\mathbb{T}^2}\Big(p_x(u_{xx}-u_{xxzz})+p_z(w_{xx}-w_{xxzz})\Big)dxdz=0. \tag{4.147}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_x\|^2+\|u_{xz}\|^2)+\nu\left(\|u_{xz}\|^2+\|u_{xzz}\|^2\right) \\ &\qquad\qquad +\epsilon_1(\|u_x\|^2+\|u_{xz}\|^2)+\epsilon_2(\|w_x\|^2+\|w_{xz}\|^2) \\ &\le \Big|\int_{\mathbb{T}^2}(uu_x+wu_z)\ (u_{xx}-u_{xxzz})\ dxdz\Big|. \end{aligned} \tag{4.148}$$

Denote by

$$\begin{cases} Y:=1+\|u_x\|^2+\|u_{xz}\|^2, & (4.149) \\ F:=\|u_{xz}\|^2+\|u_{xzz}\|^2, & (4.150) \\ G:=\|w_x\|^2+\|w_{xz}\|^2, & (4.151) \\ K:=1+\|u\|^2+\|u_z\|^2+\|u_{zz}\|^2. & (4.152) \end{cases}$$

By integration by parts and Lemma 2.2.8, using Young's inequality, thanks to (4.41), (4.42), (4.136), (4.138) and (4.141), we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xx} dxdz\right| \\
&\leq C\|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_x\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|w_{xz}\| \\
&\qquad + C\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_z\|^{\frac{1}{2}})\|w_{xz}\| \\
&\leq \frac{\epsilon_2}{4} G + C_{\epsilon_2} K Y^2,
\end{aligned} \tag{4.153}$$

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} (uu_x + wu_z)\ u_{xxzz}\ dxdz\right| = \left|\int_{\mathbb{T}^2} (u_x u_{xz} + w_x u_{zz})\ \ u_{xz}\ dxdz\right| \\
&\leq C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|w_{xz}\|^{\frac{1}{2}})\|u_{xz}\|^{\frac{3}{2}}(\|u_{xz}\|^{\frac{1}{2}} + \|u_{xzz}\|^{\frac{1}{2}}) \\
&\qquad + \|u_{xz}\|\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}} + \|w_{xz}\|^{\frac{1}{2}})\|u_{zz}\|^{\frac{1}{2}}(\|u_{zz}\|^{\frac{1}{2}} + \|u_{xzz}\|^{\frac{1}{2}})\Big] \\
&\leq \frac{\epsilon_2}{4} G + \frac{\nu}{2} F + C_{\epsilon_2,\nu} K Y^2.
\end{aligned} \tag{4.154}$$

From the estimates above and by (4.145), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G \leq C_{\epsilon_2,\nu} K Y^2, \quad \text{with } \ K \in L^1(0,\mathcal{T}) \ \text{ for arbitrary } \ \mathcal{T} > 0. \tag{4.155}$$

Therefore, we have $\frac{dY}{dt} \leq C_{\epsilon_2,\nu} K Y^2$, and this implies that

$$Y(t) \leq \frac{Y(0)}{1 - Y(0) C_{\epsilon_2,\nu} \int_0^t K ds}. \tag{4.156}$$

Let $\mathcal{T}$ be such that

$$\int_0^{\mathcal{T}} K ds = \frac{1}{2Y(0) C_{\epsilon_2,\nu}}. \tag{4.157}$$

From above, we have $Y(t) \leq 2Y(0)$ on $[0,\mathcal{T}]$. Plugging it in (4.155), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G \leq 4 C_{\epsilon_2,\nu} K Y(0)^2, \ \text{ for } t \in [0,\mathcal{T}]. \tag{4.158}$$

Integrating above from 0 to t for any time $t \in [0, \mathcal{T}]$, we obtain

$$Y(t) + \int_0^t (\nu F(s) + \epsilon_2 G(s)) \ ds \le Y(0) + 4C_{\epsilon_2,\nu} Y(0)^2 \int_0^t K(s) ds. \tag{4.159}$$

From the estimates above, by virtue of (4.141), (4.145) and (4.41), we obtain

$$\begin{cases} u \in L^\infty(0,\mathcal{T};H^1) \cap L^2(0,\mathcal{T};H^2), \quad u_{xz} \in L^\infty(0,\mathcal{T};L^2), \quad u_{xzz} \in L^2(0,\mathcal{T};L^2), & (4.160) \\ w, w_z, w_{zz} \in L^\infty(0,\mathcal{T};L^2), \quad w_x, w_{xz} \in L^2(0,\mathcal{T};L^2). & (4.161) \end{cases}$$

Using Galerkin method, one can obtain local existence of strong solution to system (4.132)–(4.135), subject to (4.136)–(4.137). Next, we show the continuous dependence of solutions on the initial data and the uniqueness of the strong solutions. Let (u_1, w_1, p_1) and (u_2, w_2, p_2) be two strong solutions of system (4.132)–(4.135), and initial data $(u_0)_1$ and $(u_0)_2$, respectively. Denote by $u = u_1 - u_2, w = w_1 - w_2, p = p_1 - p_2$. It is clear that

$$\begin{cases} \partial_t u + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - \nu u_{zz} + p_x = 0, & (4.162) \\ \epsilon_2 w + p_z = 0. & (4.163) \end{cases}$$

By taking the inner product of equation (4.162) with u, (4.163) with w in $L^2(\mathbb{T}^2)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d\|u\|^2}{dt} + \epsilon_1\|u\|^2 + \epsilon_2\|w\|^2 + \nu\|u_z\|^2 \\ =& \int_{\mathbb{T}^2} u(u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z) + (p_x u + p_z w) \ dxdz. \end{aligned} \tag{4.164}$$

By integration by parts, thanks to (4.136), (4.138) and (4.141), we have

$$\int_{\mathbb{T}^2} u(u_1 u_x + w_1 u_z) + (p_x u + p_z w) \ dxdz = 0. \tag{4.165}$$

Therefore, we have

$$\frac{1}{2}\frac{d\|u\|^2}{dt} + \epsilon_1\|u\|^2 + \epsilon_2\|w\|^2 + \nu\|u_z\|^2 \le \Big| \int_{\mathbb{T}^2} u\Big(u(u_2)_x + w(u_2)_z\Big) \ dxdz \Big|. \tag{4.166}$$

From (4.160) and (4.161), and by Lemma 2.2.9, we obtain that $w_2, (u_2)_z \in L^2(0, \mathcal{T}; L^\infty)$. Therefore, using Young's inequality and Hölder inequality, we have

$$
\begin{aligned}
&|\int_{\mathbb{T}^2} u^2 (u_2)_x dxdz| = |\int_{\mathbb{T}^2} u^2 (w_2)_z dxdz| = |2\int_{\mathbb{T}^2} u u_z w_2 \, dxdz| \\
&\quad \le \int_{\mathbb{T}^2} \left(\frac{\nu}{2} |u_z|^2 + C|uw_2|^2 \right) dxdz \le C_\nu \|w_2\|_{L^\infty}^2 \|u\|^2 + \frac{\nu}{2}\|u_z\|^2
\end{aligned}
\tag{4.167}
$$

and

$$
\begin{aligned}
|\int_{\mathbb{T}^2} uw(u_2)_z dxdz| &\le \int_{\mathbb{T}^2} \left(\frac{\epsilon_2}{2}|w|^2 + C|u(u_2)_z|^2 \right) dxdz \\
&\le C_{\epsilon_2} \|(u_2)_z\|_{L^\infty}^2 \|u\|^2 + \frac{\epsilon_2}{2}\|w\|^2.
\end{aligned}
\tag{4.168}
$$

From the estimates above, we obtain

$$
\frac{d}{dt}\|u\|^2 + \epsilon_1 \|u\|^2 + \epsilon_2 \|w\|^2 + \nu \|u_z\|^2 \le C_{\epsilon_2, \nu} (\|w_2\|_{L^\infty}^2 + \|(u_2)_z\|_{L^\infty}^2) \|u\|^2. \tag{4.169}
$$

Thanks to Gronwall inequality, we have

$$
\|u(t)\|^2 \le \|u(0)\|^2 \exp\left(C_{\epsilon_2,\nu} \int_0^t \left(\|w_2(s)\|_{L^\infty}^2 + \|(u_2)_z(s)\|_{L^\infty}^2 \right) ds \right). \tag{4.170}
$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0) = 0$, we have $u(t) = 0$, for all $t \in [0, \mathcal{T}]$. Therefore, the strong solution is unique. □

4.2 Global Well-posedness with Small Initial Data

In previous section, we establish the local well-posedness of system (4.1)–(4.5) subject to boundary and initial conditions (4.8)–(4.10). In this section, we will show the following result concerning the global existence and uniqueness of strong solutions to system (4.21)–(4.26), subject to boundary and initial conditions (4.27)–(4.29), provided that the initial data is small enough.

Theorem 4.2.1. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.27) and (4.28), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that*

$$\|u_0\|_{H^1} + \|v_0\|_{H^1} + C_0\|T_0\|_{H^1} + \|\partial_x u_0\|_{H^1} + \|\partial_x v_0\|_{H^1} + C_0\|\partial_x T_0\|_{H^1} << 1$$

is small enough, for some $C_0 > 0$ determined in (4.187). Then for any time $\mathcal{T} > 0$, there exists a unique strong solution (u, v, T) of system (4.21)–(4.26), subject to (4.27)–(4.29), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.

Proof. From Theorem 4.1.3, we know there exists time $\mathcal{T}^* > 0$ such that there is a unique strong solution (u, v, T) of system (4.21)–(4.26), subject to (4.27)–(4.29), on the interval $[0, \mathcal{T}^*]$. Assume the maximal time $\mathcal{T}$ for existence of solution is finite, then it is necessary to have

$$\limsup_{t \to \mathcal{T}^-} (\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|T(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|T_x(t)\|_{H^1}) = \infty.$$

We will prove this is not true for any finite time $\mathcal{T} > 0$, and therefore $\mathcal{T} = \infty$.

First, notice that since T is an odd function with respect to z variable, we have

$$\int_{\mathbb{T}^2} T \, dxdz \equiv 0. \tag{4.171}$$

By taking the L^2-inner product of (4.21) with $u, -\Delta u, \Delta u_{xx}$, (4.22) with $v, -\Delta v, \Delta v_{xx}$, (4.32) with $w, -\Delta w, \Delta w_{xx}$, and (4.23) with $C_0 T, -C_0 \Delta T, C_0 \Delta T_{xx}$, in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (4.27), (4.30) and (4.33), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2 \\
&\qquad\quad + C_0\|T\|^2 + C_0\|\nabla T\|^2 + C_0\|\nabla T_x\|^2\Big) \\
&\quad + \nu\Big(\|u_z\|^2 + \|v_z\|^2 + \|\nabla u_z\|^2 + \|\nabla v_z\|^2 + \|\nabla u_{xz}\|^2 + \|\nabla v_{xz}\|^2\Big) \\
&\quad + \epsilon_1\Big(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2\Big) \\
&\quad + \epsilon_2\Big(\|w\|^2 + \|\nabla w\|^2 + \|\nabla w_x\|^2\Big) + C_0\kappa\Big(\|\nabla T\|^2 + \|\Delta T\|^2 + \|\Delta T_x\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^2} (uu_x + wu_z - \Omega v + p_x)\,(-u + \Delta u - \Delta u_{xx}) + (uv_x + wv_z + \Omega u)\,(-v + \Delta v - \Delta v_{xx}) \\
&\qquad + (p_z + T)\,(-w + \Delta w - \Delta w_{xx}) + C_0\,(uT_x + wT_z)\,(-T + \Delta T - \Delta T_{xx})\,dxdz \\
&= \int_{\mathbb{T}^2} (uu_x + wu_z)\,(u_{xx} - u_{xxxx} - u_{xxzz}) + (uv_x + wv_z)\,(\Delta v - v_{xxxx} - v_{xxzz}) \\
&\qquad + T\,(-w + \Delta w - \Delta w_{xx}) + C_0\,(uT_x + wT_z)\,(\Delta T - \Delta T_{xx})\,dxdz. \qquad (4.172)
\end{aligned}
$$

Denote by

$$
\begin{cases}
Y := \|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2 + C_0\|T\|^2 \\
\qquad\qquad + C_0\|\nabla T\|^2 + C_0\|\nabla T_x\|^2, & (4.173) \\
F := \|u_z\|^2 + \|v_z\|^2 + \|\nabla u_z\|^2 + \|\nabla v_z\|^2 + \|\nabla u_{xz}\|^2 + \|\nabla v_{xz}\|^2, & (4.174) \\
G := \|w\|^2 + \|\nabla w\|^2 + \|\nabla w_x\|^2, & (4.175) \\
H := \|\nabla T\|^2 + \|\Delta T\|^2 + \|\Delta T_x\|^2, & (4.176) \\
K := \|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\nabla u_x\|^2 + \|\nabla v_x\|^2. & (4.177)
\end{cases}
$$

We estimate each term in (4.172). By integration by parts, using Poincaré inequality, Young's inequality and Lemma 2.2.8, thanks to (4.27), (4.30), (4.33), (4.41), (4.42) and (4.171), we have

$$
\begin{aligned}
&\left| \int_{\mathbb{T}^2} (uu_x + wu_z)\, w_{xz} dxdz \right| \\
&\le C\|u\|^{\frac12}(\|u\|^{\frac12} + \|u_z\|^{\frac12})\|u_x\|^{\frac12}(\|u_x\|^{\frac12} + \|u_{xx}\|^{\frac12})\|w_{xz}\| \\
&\qquad + C\|u_z\|^{\frac12}(\|u_z\|^{\frac12} + \|u_{xz}\|^{\frac12})\|w\|^{\frac12}(\|w\|^{\frac12} + \|w_z\|^{\frac12})\|w_{xz}\| \\
&\le C(\|w\|^2 + \|w_z\|^2 + \|w_{xz}\|^2)(\|u\| + \|u_z\| + \|u_{xz}\|) \\
&\le CGY^{1/2}, \qquad (4.178)
\end{aligned}
$$

$$
\begin{aligned}
&\left| \int_{\mathbb{T}^2} (uu_x + wu_z)\; u_{xxxx}\; dxdz \right| \\
&= \left| \int_{\mathbb{T}^2} (3w_z w_{xz} + w_{xx} u_z + 2w_x u_{xz})\; u_{xx}\; dxdz \right| \\
&\le C\Big[\|w_z\|^{\frac12}(\|w_z\|^{\frac12} + \|w_{xz}\|^{\frac12})\|w_{xz}\|\|u_{xx}\|^{\frac12}(\|u_{xx}\|^{\frac12} + \|u_{xxz}\|^{\frac12})
\end{aligned}
$$

$$
\begin{aligned}
&+\|w_{xx}\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}}+\|u_{xz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&+\|w_x\|\|u_{xz}\|^{\frac{1}{2}}(\|u_{xz}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\Big]\\
\le{}& C(\|\nabla w_x\|^2+\|\nabla u\|^2+\|\nabla u_x\|^2+\|u_{xxz}\|^2)(\|\nabla u\|+\|\nabla u_x\|)\\
\le{}& C(F+G+K)Y^{1/2}, \qquad (4.179)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(uu_x+wu_z)\,u_{xxzz}\,dxdz\right|\\
={}&\left|\int_{\mathbb{T}^2}(u_xu_{xz}+w_xu_{zz})\,\;u_{xz}\,dxdz\right|\\
\le{}& C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}}+\|u_{xz}\|^{\frac{1}{2}})\|u_{xz}\|^{\frac{3}{2}}(\|u_{xz}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&+\|u_{xz}\|\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|u_{zz}\|^{\frac{1}{2}}(\|u_{zz}\|^{\frac{1}{2}}+\|u_{xzz}\|^{\frac{1}{2}})\Big]\\
\le{}& C(F+G+K)Y^{1/2}, \qquad (4.180)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(uv_x+wv_z)\,\Delta v\,dxdz\right|\\
\le{}& C(\|u\|+\|u_z\|)(\|v_x\|+\|v_{xx}\|)(\|v_{xx}\|+\|v_{zz}\|)\\
&+C(\|w\|+\|w_z\|)(\|v_z\|+\|v_{xz}\|)(\|v_{xx}\|+\|v_{zz}\|)\\
\le{}& C(K+F)Y^{1/2}, \qquad (4.181)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_{\mathbb{T}^2}(uv_x+wv_z)\,v_{xxxx}dxdz|\\
={}&|\int_{\mathbb{T}^2}(u_{xx}v_x+w_{xx}v_z+2u_xv_{xx}+2w_xv_{xz})\,v_{xx}dxdz|\\
\le{}& C\Big[\|v_{xx}\|\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}}+\|v_{xx}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\\
&+\|w_{xx}\|\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}}+\|v_{xz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\\
&+\|v_{xx}\|^{\frac{3}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}}+\|u_{xx}\|^{\frac{1}{2}})\\
&+\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|v_{xx}\|\Big]\\
\le{}& C(K+F+G)Y^{1/2}, \qquad (4.182)
\end{aligned}
$$

$$
\begin{aligned}
& |\int_{\mathbb{T}^2} (uv_x + wv_z)\ v_{xxzz} dxdz| \\
&= |\int_{\mathbb{T}^2} (u_{xz}v_x + v_{xx}u_z - v_z u_{xx} + w_x v_{xz})\ v_{xz} dxdz| \\
&\le C\Big[\|v_{xz}\|\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|u_{xz}\|^{\frac{1}{2}}(\|u_{xz}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}}) \\
&\quad + \|v_{xz}\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}} + \|v_{xxz}\|^{\frac{1}{2}}) \\
&\quad + \|v_{xz}\|\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}}) \\
&\quad + \|v_{xz}\|\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}} + \|w_{xz}\|^{\frac{1}{2}})\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}} + \|v_{xxz}\|^{\frac{1}{2}})\Big] \\
&\le C(K + F + G)Y^{1/2}, \qquad (4.183)
\end{aligned}
$$

$$
\begin{aligned}
& |\int_{\mathbb{T}^2} T\,(-w + \Delta w - \Delta w_{xx})\,dxdz| \\
&\le \|T\|\|w\| + \|\nabla T\|\|\nabla w\| + \|\nabla T_x\|\|\nabla w_x\| \\
&\le \frac{\epsilon_2}{2}G + \frac{1}{2\epsilon_2}(\|T\|^2 + \|\nabla T\|^2 + \|\nabla T_x\|^2) \\
&\le \frac{\epsilon_2}{2}G + \frac{1}{2\epsilon_2}(C_p\|\nabla T\|^2 + \|\nabla T\|^2 + \|\nabla T_x\|^2) \\
&\le \frac{\epsilon_2}{2}G + \frac{1}{2\epsilon_2}(C_p + 1)H, \qquad (4.184)
\end{aligned}
$$

where, thanks to (4.171), we apply Poincaré inequality to obtain the last inequality. Finally,

$$
\begin{aligned}
& C_0|\int_{\mathbb{T}^2} (uT_x + wT_z)\ (\Delta T - \Delta T_{xx}) dxdz| \\
&\le |\int_{\mathbb{T}^2} (uT_x + wT_z)\ \Delta T dxdz| + |\int_{\mathbb{T}^2} (u_xT_x + uT_{xx} + wT_{xz} + w_xT_z)\ \Delta T_x dxdz| \\
&\le C_0 C\Big[\|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_z\|^{\frac{1}{2}})\|T_x\|^{\frac{1}{2}}(\|T_x\|^{\frac{1}{2}} + \|T_{xx}\|^{\frac{1}{2}}) \\
&\qquad + \|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_z\|^{\frac{1}{2}})\|T_z\|^{\frac{1}{2}}(\|T_z\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}})\Big]\|\Delta T\| \\
&\quad + C_0 C\Big[\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|T_x\|^{\frac{1}{2}}(\|T_x\|^{\frac{1}{2}} + \|T_{xx}\|^{\frac{1}{2}}) \\
&\qquad + \|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_z\|^{\frac{1}{2}})\|T_{xx}\|^{\frac{1}{2}}(\|T_{xx}\|^{\frac{1}{2}} + \|T_{xxx}\|^{\frac{1}{2}}) \\
&\qquad + \|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|T_{xz}\|^{\frac{1}{2}}(\|T_{xz}\|^{\frac{1}{2}} + \|T_{xzz}\|^{\frac{1}{2}}) \\
&\qquad + \|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}} + \|w_{xz}\|^{\frac{1}{2}})\|T_z\|^{\frac{1}{2}}(\|T_z\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}})\Big]\|\Delta T_x\|
\end{aligned}
$$

$$\leq C_0 C H Y^{1/2}. \tag{4.185}$$

From the estimates above, we obtain

$$\begin{aligned}\frac{1}{2}\frac{dY}{dt} + \nu(1 - \frac{C}{\nu}Y^{\frac{1}{2}})F + \epsilon_1(1 - \frac{C}{\epsilon_1}Y^{\frac{1}{2}})K \\ + \epsilon_2(\frac{1}{2} - \frac{C}{\epsilon_2}Y^{\frac{1}{2}})G + C_0\kappa(1 - \frac{C}{\kappa}Y^{\frac{1}{2}} - \frac{C_p+1}{2\epsilon_2\kappa C_0})H \leq 0.\end{aligned} \tag{4.186}$$

Choose

$$C_0 = \frac{C_p + 1}{\epsilon_2 \kappa}. \tag{4.187}$$

Observe that if $Y_0 < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$, there exists $t^* > 0$ such that $\frac{dY}{dt} \leq 0$ on $[0, t^*]$, and hence $Y(t) \leq Y_0$ for $t \in [0, t^*]$, and in particular, $Y(t^*) < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$. Thus we can repeat this procedure to arbitrary time $t > 0$ to get $Y(t) \leq Y_0 < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$ for all time. This implies the required bound for the global in time existence of strong solution. □

4.3 Voigt α-regularization

In order to study the possible finite-time blow-up of system (4.1)–(4.5), and to give a reliable numerical regularization, in this section, we study the Voigt α-regularization of system (4.1)–(4.5) with $\nu = 0$, which is

$$\begin{cases} (u - \alpha^2 u_{zz})_t + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x = 0, & (4.188) \\ (v - \alpha^2 v_{zz})_t + uv_x + wv_z + \epsilon_1 v + \Omega u = 0, & (4.189) \\ \epsilon_2 w + p_z + T = 0, & (4.190) \\ u_x + w_z = 0, & (4.191) \\ T_t - \kappa\Delta T + uT_x + wT_z = 0. & (4.192) \end{cases}$$

Remark 18. We take $\nu = 0$ here. Indeed, when $\nu > 0$, the system has additional dissipation, and thus is easier to study. One can repeat the procedures below and get similar result when $\nu > 0$.

As in the case of system (4.1)–(4.5), our domain is $\mathbb{T}^2$, and the boundary and initial conditions are

$$u,\ v,\ w,\ p \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1, \tag{4.193}$$

$$u,\ v \text{ and } p \text{ are even in } z,\ \text{ and } w,\ T \text{ are odd in } z, \tag{4.194}$$

$$\Big(u, v, T\Big)|_{t=0} = \Big(u_0, v_0, T_0\Big). \tag{4.195}$$

Using analogue argument as for system (4.1)–(4.5), system (4.188)–(4.192) subject to (4.193)–(4.195) is equivalent to the following:

$$\begin{cases} (u - \alpha^2 u_{zz})_t + u\, u_x + w u_z + \epsilon_1 u - \Omega v + p_x = 0, & (4.196)\\ (v - \alpha^2 v_{zz})_t + u\, v_x + w v_z + \epsilon_1 v + \Omega u = 0, & (4.197)\\ T_t - \kappa \Delta T + u\, T_x + w\, T_z = 0, & (4.198) \end{cases}$$

with w, p_x, p_z defined by

$$\begin{cases} w(x,z) := -\displaystyle\int_0^z u_x(x,s)ds, & (4.199)\\ p_x(x,z) := \epsilon_2 \displaystyle\int_0^z \int_0^s u_{xx}(x,\xi)d\xi ds - \int_0^z T_x(x,s)ds & \\ \qquad + \displaystyle\int_0^1 \Big[\int_0^{z'} T_x(x,s)ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi)d\xi ds & \\ \qquad\qquad + \Omega v(x,z') - 2u u_x(x,z') \Big] dz' & \\ \qquad - \Omega \displaystyle\int_0^1 \int_0^1 v(x',z')dx'dz', & (4.200)\\ p_z(x,z) := -T(x,z) + \epsilon_2 \displaystyle\int_0^z u_x(x,s)ds, & (4.201) \end{cases}$$

subject to the symmetry boundary conditions and initial conditions (4.27)–(4.29). We also have (4.32) and (4.33), for which we repeat here:

$$\begin{cases} \epsilon_2 w + p_z + T = 0, & (4.202)\\ u_x + w_z = 0. & (4.203) \end{cases}$$

In this section, we will show the global regularity of strong solution to system (4.196)–(4.201), subject to (4.27)–(4.29), for arbitrary initial data without smallness assumption.

4.3.1 Main Results

First, we give the definition of strong solution to system (4.196)–(4.201), subject to (4.27)–(4.29).

Definition 4.3.1. *Suppose that $u_0, v_0 \in H^2(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.27) and (4.28), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xxz}u_0, \partial_{xxz}v_0 \in L^2(\mathbb{T}^2)$. Given time $\mathcal{T} > 0$, we say (u, v, T) is a strong solution to the system (4.196)–(4.201), subject to (4.27)–(4.29), on the time interval $[0, \mathcal{T}]$, if*

1. *u, v and T satisfy the symmetry conditions (4.27) and (4.28);*
2. *u, v and T have the regularities*

$$
\begin{cases}
u, v \in L^\infty(0,\mathcal{T}; H^2) \cap C([0,\mathcal{T}]; H^1), \\
u_{xxz}, v_{xxz} \in L^\infty(0,\mathcal{T}; L^2), \\
\partial_t u \in L^\infty(0,\mathcal{T}; L^2) \cap L^2(0,\mathcal{T}; H^1), \\
T \in L^2(0,\mathcal{T}; H^2) \cap L^\infty(0,\mathcal{T}; H^1) \cap C([0,\mathcal{T}]; L^2), \\
\partial_t v \in L^\infty(0,\mathcal{T}; H^1) \\
\partial_t T \in L^2(0,\mathcal{T}; L^2);
\end{cases}
$$

3. *u, v and T satisfy system (4.196)–(4.198) in the following sense:*

$$
\partial_t(u - \alpha^2 u_{zz}) + uu_x + wu_z + \epsilon_1 u - \Omega v + p_x = 0 \ \textit{ in } \ L^\infty(0,\mathcal{T}; L^2) \cap L^2(0,\mathcal{T}; H^1),
$$
$$
\partial_t(v - \alpha^2 v_{zz}) + uv_x + wv_z + \epsilon_1 v + \Omega u = 0 \ \textit{ in } \ L^\infty(0,\mathcal{T}; H^1),
$$
$$
\partial_t T - \kappa \Delta T + uT_x + wT_z = 0 \ \textit{ in } \ L^2(0,\mathcal{T}; L^2),
$$

with w, p_x, p_z defined by (4.199)–(4.201), and fulfill the initial condition (4.29).

We have the following result concerning the existence and uniqueness of strong solutions to system (4.196)–(4.201), subject to (4.27)–(4.29), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for any positive time $\mathcal{T}$.

Theorem 4.3.2. *Suppose that $u_0, v_0 \in H^2(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.27) and (4.28), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xxz} u_0, \partial_{xxz} v_0 \in L^2(\mathbb{T}^2)$. Let time $\mathcal{T} > 0$. Then there exists a unique strong solution (u, v, T) of system (4.196)–(4.201), subject to (4.27)–(4.29), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.*

For the proof, we will only do formal *a priori* energy estimates and omit the details of showing existence of solutions. Then we show the uniqueness of solutions and the continuous dependence on the initial data.

4.3.2 *A priori* Estimates

By taking the L^2-inner product of equation (4.196) with u, equation (4.197) with v, equation (4.202) with w and equation (4.198) with T, in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (4.27), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2 \|u_z\|^2 + \alpha^2 \|v_z\|^2\Big) \\
&\qquad\qquad + \epsilon_1 \|u\|^2 + \epsilon_1 \|v\|^2 + \epsilon_2 \|w\|^2 + \kappa \|\nabla T\|^2 \\
&= -\int_{\mathbb{T}^2} (u u_x + w u_z)\, u \; dxdz - \int_{\mathbb{T}^2} (u v_x + w v_z)\, v \; dxdz \\
&\quad - \int_{\mathbb{T}^2} (u p_x + w p_z + wT) \; dxdz - \int_{\mathbb{T}^2} (u T_x + w T_z)\, T \; dxdz. \qquad (4.204)
\end{aligned}
$$

By integration by parts, thanks to (4.27), (4.30) and (4.203), we have

$$
\begin{aligned}
&-\int_{\mathbb{T}^2} (u u_x + w u_z)\, u \; dxdz - \int_{\mathbb{T}^2} (u v_x + w v_z)\, v \; dxdz \\
&\quad - \int_{\mathbb{T}^2} (u p_x + w p_z) \; dxdz - \int_{\mathbb{T}^2} (u T_x + w T_z)\, T \; dxdz = 0. \qquad (4.205)
\end{aligned}
$$

By Cauchy–Schwarz inequality and Young's inequality, we have

$$-\int_{\mathbb{T}^2} wT dxdz \leq \|w\|\|T\| \leq \frac{\epsilon_2}{2}\|w\|^2 + C_{\epsilon_2}\|T\|^2. \tag{4.206}$$

As a result of the above, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2\|u_z\|^2 + \alpha^2\|v_z\|^2\Big) \\
&\qquad\qquad +2\epsilon_1\|u\|^2 + 2\epsilon_1\|v\|^2 + \epsilon_2\|w\|^2 + 2\kappa\|\nabla T\|^2 \\
&\leq C_{\epsilon_2}(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2\|u_z\|^2 + \alpha^2\|v_z\|^2).
\end{aligned} \tag{4.207}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned}
&(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2\|u_z\|^2 + \alpha^2\|v_z\|^2)(t) \\
&\quad + \int_0^t \left(2\epsilon_1\|u(s)\|^2 + 2\epsilon_1\|v(s)\|^2 + \epsilon_2\|w(s)\|^2 + 2\kappa\|\nabla T(s)\|^2\right)\, ds \\
&\leq (\|u_0\|^2 + \|v_0\|^2 + \|T_0\|^2 + \alpha^2\|\partial_z u_0\|^2 + \alpha^2\|\partial_z v_0\|^2)e^{C_{\epsilon_2}t}.
\end{aligned} \tag{4.208}$$

Consequently, we have

$$\begin{cases}
u, v, u_z, v_z \in L^\infty(0,\mathcal{T}; L^2), \\
w \in L^2(0,\mathcal{T}; L^2), \\
T \in L^\infty(0,\mathcal{T}; L^2) \cap L^2(0,\mathcal{T}; H^1)
\end{cases} \tag{4.209}$$

for arbitrary $\mathcal{T} > 0$.

By taking the L^2-inner product of equation (4.196) with $-u_{zz}$ and equation (4.202) with $-w_{zz}$ in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (4.27) and (4.30), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d(\|u_z\|^2 + \alpha^2\|u_{zz}\|^2)}{dt} + \epsilon_1\|u_z\|^2 + \epsilon_2\|w_z\|^2 \\
&= \int_{\mathbb{T}^2} (uu_x + wu_z - \Omega v)u_{zz} + (u_{zz}p_x + w_{zz}p_z + w_{zz}T)\, dxdz.
\end{aligned} \tag{4.210}$$

By integration by parts, thanks to (4.27), (4.30) and (4.203), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} (uu_x + wu_z)u_{zz} dxdz + \int_{\mathbb{T}^2} (u_{zz}p_x + w_{zz}p_z)\, dxdz \\
&= -\int_{\mathbb{T}^2} [u_z u_x + uu_{xz} + w_z u_z + wu_{zz}]u_z dxdz - \int_{\mathbb{T}^2} p(u_x + w_z)_{zz} dxdz \\
&= -\frac{1}{2}\int_{\mathbb{T}^2} u_z^2(u_x + w_z)dxdz - \int_{\mathbb{T}^2} p(u_x + w_z)_{zz} dxdz = 0.
\end{aligned} \tag{4.211}
$$

By integration by parts, using Cauchy–Schwarz inequality and Young's inequality, thanks to (4.27) and (4.30), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} (-\Omega v u_{zz} + w_{zz}T)\, dxdz = \int_{\mathbb{T}^2} (\Omega v_z u_z - w_z T_z)\, dxdz \\
&\leq \Omega \|v_z\|\|u_z\| + \|w_z\|\|T_z\| \\
&\leq \frac{\epsilon_2}{2}\|w_z\|^2 + C_{\epsilon_2}\|T_z\|^2 + C_\Omega \|v_z\|(1 + \|u_z\|^2 + \alpha^2\|u_{zz}\|^2).
\end{aligned} \tag{4.212}
$$

As a result of the above, we have

$$
\begin{aligned}
&\frac{d(1 + \|u_z\|^2 + \alpha^2\|u_{zz}\|^2)}{dt} + \epsilon_1\|u_z\|^2 + \epsilon_2\|w_z\|^2 \\
&\leq C_\Omega\|v_z\|(1 + \|u_z\|^2 + \alpha^2\|u_{zz}\|^2) + C_{\epsilon_2}\|T_z\|^2.
\end{aligned} \tag{4.213}
$$

Thanks to Gronwall inequality, we obtain

$$
\begin{aligned}
&\|u_z(t)\|^2 + \alpha^2\|u_{zz}(t)\|^2 + \int_0^t \left(2\epsilon_1\|u_z(s)\|^2 + \epsilon_2\|w_z(s)\|^2\right)\, ds \\
&\leq C_{\epsilon_2}\left(1 + \int_0^t \|T_z(s)\|^2 ds + \|\partial_z u_0\|^2 + \alpha^2\|\partial_{zz}u_0\|^2\right)\exp\left(C_\Omega\int_0^t \|v_z(s)\|ds\right).
\end{aligned} \tag{4.214}
$$

By virtue of (4.209) and the above, we have

$$
u_{zz} \in L^\infty(0, \mathcal{T}; L^2), \quad w_z = -u_x \in L^2(0, \mathcal{T}; L^2), \tag{4.215}
$$

for arbitrary $\mathcal{T} > 0$.

By taking the L^2-inner product of equation (4.196) with $-u_{xx}$ and equation (4.202) with $-w_{xx}$, in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (4.27) and (4.30), we get

$$\frac{1}{2}\frac{d(\|u_x\|^2 + \alpha^2 \|u_{xz}\|^2)}{dt} + \epsilon_1\|u_x\|^2 + \epsilon_2\|w_x\|^2$$
$$= \int_{\mathbb{T}^2} (uu_x + wu_z - \Omega v)u_{xx} + (u_{xx}p_x + w_{xx}p_z + w_{xx}T)\, dxdz. \tag{4.216}$$

By integration by parts, thanks to (4.27), (4.30) and (4.203), we have

$$\int_{\mathbb{T}^2} (u_{xx}p_x + w_{xx}p_z)\, dxdz = 0. \tag{4.217}$$

By integration by parts, using Cauchy–Schwarz inequality and Young's inequality, thanks to (4.27), (4.30) and (4.203), we have

$$-\int_{\mathbb{T}^2} \Omega v u_{xx} dxdz = \int_{\mathbb{T}^2} \Omega v w_{xz} dxdz = -\int_{\mathbb{T}^2} \Omega v_z w_x dxdz$$
$$\leq C_{\Omega,\epsilon_2}\|v_z\|^2 + \frac{\epsilon_2}{6}\|w_x\|^2, \tag{4.218}$$

and

$$\int_{\mathbb{T}^2} T w_{xx}\, dxdz = -\int_{\mathbb{T}^2} T_x w_x\, dxdz \leq C_{\epsilon_2}\|T_x\|^2 + \frac{\epsilon_2}{6}\|w_x\|^2. \tag{4.219}$$

By integration by parts, using Young's inequality and Lemma 2.2.8, thanks to (4.27), (4.30) and (4.203), we have

$$\int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xx}\, dxdz = -\int_{\mathbb{T}^2} ((u_x)^3 + w_x u_z u_x)\, dxdz$$
$$= -\int_{\mathbb{T}^2} (-w_z (u_x)^2 + w_x u_z u_x)\, dxdz = -\int_{\mathbb{T}^2} (2wu_x u_{xz} + w_x u_z u_x)\, dxdz$$
$$\leq C\big[\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|u_{xz}\|$$
$$+\|w_x\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\|u_x\|^{\frac{1}{2}}(\|u_x\|^{\frac{1}{2}} + \|u_{xz}\|^{\frac{1}{2}})\big]$$

$$\leq CC_{\epsilon_2}\left(1+\|w\|^2+\|u_x\|^2+\|u_z\|^2\right)\left(1+\|u_x\|^2+\alpha^2\|u_{xz}\|^2\right)+\frac{\epsilon_2}{6}\|w_x\|^2. \quad (4.220)$$

From the estimates above, we have

$$\begin{aligned}&\frac{d(1+\|u_x\|^2+\alpha^2\|u_{xz}\|^2)}{dt}+\epsilon_1\|u_x\|^2+\epsilon_2\|w_x\|^2\\ &\leq C_{\epsilon_2}\left(1+\|w\|^2+\|u_x\|^2+\|u_z\|^2\right)\left(1+\|u_x\|^2+\alpha^2\|u_{xz}\|^2\right)\\ &\quad +C_{\epsilon_2,\Omega}(\|v_z\|^2+\|T_x\|^2). \end{aligned} \quad (4.221)$$

By Gronwall inequality, we obtain

$$\begin{aligned}&\|u_x(t)\|^2+\alpha^2\|u_{xz}(t)\|^2+\int_0^t\left(2\epsilon_1\|u_x(s)\|^2+\epsilon_2\|w_x(s)\|^2\right)\,ds\\ &\leq C_{\epsilon_2,\Omega}\left(1+\int_0^t\left(\|v_z(s)\|^2+\|T_x(s)\|^2\right)ds+\|\partial_x u_0\|^2+\alpha^2\|\partial_{xz}u_0\|^2\right)\\ &\quad\times\exp\left(C_{\epsilon_2}\int_0^t\left(1+\|w(s)\|^2+\|u_x(s)\|^2+\|u_z(s)\|^2\right)ds\right). \end{aligned} \quad (4.222)$$

By virtue of (4.209), (4.215) and the above, we have

$$u,u_z\in L^\infty(0,\mathcal{T};H^1),\quad w\in L^2(0,\mathcal{T};H^1), \quad (4.223)$$

for arbitrary $\mathcal{T}>0$.

By virtue of (4.223), (4.41) and (4.42), we have

$$w\in L^\infty(0,\mathcal{T};L^2)\cap L^2(0,\mathcal{T};H^1), \quad (4.224)$$

for arbitrary $\mathcal{T}>0$. By taking the L^2-inner product of equation (4.197) with $-\Delta v$ in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.27), we have

$$\frac{1}{2}\frac{d(\|\nabla v\|^2+\alpha^2\|\nabla v_z\|^2)}{dt}+\epsilon_1\|\nabla v\|^2$$

$$= \int_{\mathbb{T}^2} \Big[(uv_x + wv_z)\ (v_{xx} + v_{zz}) + \Omega u \Delta v \Big] dxdz. \tag{4.225}$$

By integration by parts, using Cauchy–Schwarz inequality and Lemma 2.2.8, thanks to (4.27), (4.30) and (4.203), we have

$$\begin{aligned} &| \int_{\mathbb{T}^2} uv_x v_{xx}\ dxdz| = | \int_{\mathbb{T}^2} \frac{1}{2} u_x v_x^2\ dxdz| = | \int_{\mathbb{T}^2} \frac{1}{2} w_z v_x^2\ dxdz| = | \int_{\mathbb{T}^2} wv_x v_{xz}\ dxdz| \\ &\le C\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|v_{xz}\|, \end{aligned} \tag{4.226}$$

$$| \int_{\mathbb{T}^2} uv_x v_{zz}\ dxdz| \le C\|u\|^{\frac{1}{2}}(\|u\|^{\frac{1}{2}} + \|u_x\|^{\frac{1}{2}})\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|v_{zz}\|, \tag{4.227}$$

$$\begin{aligned} &| \int_{\mathbb{T}^2} wv_z v_{xx}\ dxdz| = | \int_{\mathbb{T}^2} v_x(w_x v_z + wv_{xz})\ dxdz| \\ &\le C\|w_x\|\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}}) \\ &\qquad + C\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}} + \|v_{xz}\|^{\frac{1}{2}})\|v_{xz}\|, \end{aligned} \tag{4.228}$$

$$| \int_{\mathbb{T}^2} wv_z v_{zz}\ dxdz| \le C\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}})\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}} + \|v_{zz}\|^{\frac{1}{2}})\|v_{zz}\|, \tag{4.229}$$

and

$$| \int_{\mathbb{T}^2} \Omega u \Delta v\ dxdz| = | \int_{\mathbb{T}^2} \Omega \nabla u \nabla v\ dxdz| \le C_\Omega \|\nabla u\|\|\nabla v\|. \tag{4.230}$$

As a result of the above and by Young's inequality, we conclude

$$\begin{aligned} &\frac{d(1 + \|\nabla v\|^2 + \alpha^2 \|\nabla v_z\|^2)}{dt} + 2\epsilon_1 \|\nabla v\|^2 \\ &\le C_\Omega \left(1 + \|u\|^2 + \|w\|^2 + \|\nabla u\|^2 + \|w_x\|^2\right) \left(1 + \|\nabla v\|^2 + \alpha^2 \|\nabla v_z\|^2\right). \end{aligned} \tag{4.231}$$

Thanks to Gronwall inequality, we obtain

$$
\begin{aligned}
&\|\nabla v(t)\|^2 + \alpha^2 \|\nabla v_z(t)\|^2 + \int_0^t 2\epsilon_1 \|\nabla v(s)\|^2 ds \\
\leq & \left(1 + \|\nabla v_0\|^2 + \alpha^2 \|\nabla \partial_z v_0\|^2\right) \\
& \times \exp\left(C_\Omega \int_0^t \left(1 + \|u\|^2 + \|w\|^2 + \|\nabla u\|^2 + \|w_x\|^2\right)(s)\, ds \right).
\end{aligned} \tag{4.232}
$$

By virtue of (4.209), (4.215), (4.223) and the above, we have

$$
v, v_z \in L^\infty(0, \mathcal{T}; H^1), \tag{4.233}
$$

for arbitrary $\mathcal{T} > 0$.

By taking the L^2-inner product of equation (4.198) with $-\Delta T$ in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.27), we have

$$
\frac{1}{2}\frac{d\|\nabla T\|^2}{dt} + \kappa \|\Delta T\|^2 = \int_{\mathbb{T}^2} (uT_x + wT_z)\Delta T. \tag{4.234}
$$

By Lemma 2.2.8 and Young's inequality, thanks to (4.203), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} (uT_x + wT_z)\Delta T \\
\leq & C\Big(\|u\|^{\frac{1}{2}} (\|u\|^{\frac{1}{2}} + \|u_z\|^{\frac{1}{2}}) \|T_x\|^{\frac{1}{2}} (\|T_x\|^{\frac{1}{2}} + \|T_{xx}\|^{\frac{1}{2}}) \|\Delta T\| \\
& \quad + \|w\|^{\frac{1}{2}} (\|w\|^{\frac{1}{2}} + \|w_z\|^{\frac{1}{2}}) \|T_z\|^{\frac{1}{2}} (\|T_z\|^{\frac{1}{2}} + \|T_{xz}\|^{\frac{1}{2}}) \|\Delta T\| \Big) \\
\leq & \frac{\kappa}{2} \|\Delta T\|^2 + C_\kappa \left(1 + \|u\|^4 + \|u_z\|^4 + \|w\|^4 + \|w_z\|^4\right) \|\nabla T\|^2, \\
= & \frac{\kappa}{2} \|\Delta T\|^2 + C_\kappa \left(1 + \|u\|^4 + \|u_z\|^4 + \|w\|^4 + \|u_x\|^4\right) \|\nabla T\|^2.
\end{aligned} \tag{4.235}
$$

As a result of the above we conclude

$$
\frac{d\|\nabla T\|^2}{dt} + \kappa \|\Delta T\|^2 \leq C_\kappa \left(1 + \|u\|^4 + \|u_z\|^4 + \|w\|^4 + \|u_x\|^4\right) \|\nabla T\|^2. \tag{4.236}
$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned}
&\|\nabla T(t)\|^2 + \kappa \int_0^t \|\Delta T(s)\|^2 \, ds \\
&\leq \|\nabla T_0\|^2 \exp\left(C_\kappa \int_0^t \left(1 + \|u\|^4 + \|u_z\|^4 + \|w\|^4 + \|u_x\|^4\right)(s) ds \right). \qquad (4.237)
\end{aligned}$$

By virtue of (4.209), (4.215), (4.223), (4.224) and the above, we have

$$T \in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2). \qquad (4.238)$$

By taking the L^2-inner product of equation (4.196) with u_{xxxx}, equation (4.197) with v_{xxxx}, and equation (4.202) with w_{xxxx} in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.27) and (4.30), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d(\|u_{xx}\|^2 + \|v_{xx}\|^2 + \alpha^2 \|u_{xxz}\|^2 + \alpha^2 \|v_{xxz}\|^2)}{dt} \\
&\qquad + \epsilon_1 \|u_{xx}\|^2 + \epsilon_1 \|v_{xx}\|^2 + \epsilon_2 \|w_{xx}\|^2 \\
&= -\int_{\mathbb{T}^2} (uu_x + wu_z - \Omega v)\, u_{xxxx} \, dxdz - \int_{\mathbb{T}^2} (uv_x + wv_z + \Omega u)\, v_{xxxx} \, dxdz \\
&\qquad - \int_{\mathbb{T}^2} (u_{xxxx} p_x + w_{xxxx} p_z + w_{xxxx} T) \, dxdz. \qquad (4.239)
\end{aligned}$$

By integration by parts, thanks to (4.27), (4.30) and (4.203), , we have

$$\int_{\mathbb{T}^2} (-\Omega v u_{xxxx} + \Omega u v_{xxxx}) \, dxdz - \int_{\mathbb{T}^2} (u_{xxxx} p_x + w_{xxxx} p_z) \, dxdz = 0. \qquad (4.240)$$

By integration by parts, using Young's inequality and Lemma 2.2.8, thanks to (4.27), (4.30) and (4.203), we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^2} (uu_x + wu_z)\, u_{xxxx} \, dxdz \right| = \left| \int_{\mathbb{T}^2} (5wu_{xxz} - \frac{1}{2} w_{xx} u_z - 2w_x u_{xz})\, u_{xx} \, dxdz \right| \\
&\leq C \Big[\|w\|^{\frac{1}{2}} (\|w\|^{\frac{1}{2}} + \|w_x\|^{\frac{1}{2}}) \|u_{xx}\|^{\frac{1}{2}} (\|u_{xx}\|^{\frac{1}{2}} + \|u_{xxz}\|^{\frac{1}{2}}) \|u_{xxz}\|
\end{aligned}$$

$$
\begin{aligned}
&+\|w_{xx}\|\|u_z\|^{\frac{1}{2}}(\|u_z\|^{\frac{1}{2}}+\|u_{xz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}}) \\
&+\|w_x\|\|u_{xz}\|^{\frac{1}{2}}(\|u_{xz}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\|u_{xx}\|^{\frac{1}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|u_{xxz}\|^{\frac{1}{2}})\Big] \\
\leq\ & \frac{\epsilon_2}{6}\|w_{xx}\|^2+C_{\epsilon_2}\left(1+\|w\|^2+\|w_x\|^2+\|u_z\|^2+\|u_{xz}\|^2\right) \\
&\qquad \times\left(1+\|u_{xx}\|^2+\alpha^2\|u_{xxz}\|^2\right), \qquad (4.241)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{T}^2}(uv_x+wv_z)\,v_{xxxx}dxdz\Big|=\Big|\int_{\mathbb{T}^2}(u_{xx}v_x+w_{xx}v_z-4wv_{xxz}+2w_xv_{xz})\,v_{xx}dxdz\Big| \\
\leq\ &\Big|\int_{\mathbb{T}^2}(w_xv_{xz}+w_{xx}v_z-4wv_{xxz}+2w_xv_{xz})\,v_{xx}dxdz\Big|+\Big|\int_{\mathbb{T}^2}w_xv_xv_{xxz}\,dxdz\Big| \\
&\leq C\Big[\|w_x\|\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}}) \\
&\quad+\|w_{xx}\|\|v_z\|^{\frac{1}{2}}(\|v_z\|^{\frac{1}{2}}+\|v_{xz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}}) \\
&\quad+\|w\|^{\frac{1}{2}}(\|w\|^{\frac{1}{2}}+\|w_x\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|v_{xxz}\| \\
&\quad+\|w_x\|\|v_{xz}\|^{\frac{1}{2}}(\|v_{xz}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}})\|v_{xx}\|^{\frac{1}{2}}(\|v_{xx}\|^{\frac{1}{2}}+\|v_{xxz}\|^{\frac{1}{2}}) \\
&\quad+\|w_x\|^{\frac{1}{2}}(\|w_x\|^{\frac{1}{2}}+\|w_{xz}\|^{\frac{1}{2}})\|v_x\|^{\frac{1}{2}}(\|v_x\|^{\frac{1}{2}}+\|v_{xx}\|^{\frac{1}{2}})\|v_{xxz}\|\Big] \\
\leq\ &\frac{\epsilon_2}{6}\|w_{xx}\|^2+C_{\epsilon_2}\left(1+\|w\|^2+\|w_x\|^2+\|v_x\|^2+\|v_z\|^2+\|v_{xz}\|^2\right) \\
&\qquad \times\left(1+\|v_{xx}\|^2+\alpha^2\|v_{xxz}\|^2\right). \qquad (4.242)
\end{aligned}
$$

By integration by parts, using Cauchy-Schwarz inequality and Young's inequality, thanks to (4.27) and (4.30), we have

$$
|\int_{\mathbb{T}^2}Tw_{xxxx}dxdz|=|\int_{\mathbb{T}^2}T_{xx}w_{xx}dxdz|\leq\|T_{xx}\|\|w_{xx}\|\leq\frac{\epsilon_2}{6}\|w_{xx}\|^2+C_{\epsilon_2}\|T_{xx}\|^2. \qquad (4.243)
$$

As a result of the above, we conclude

$$
\begin{aligned}
&\frac{d(1+\|u_{xx}\|^2+\|v_{xx}\|^2+\alpha^2\|u_{xxz}\|^2+\alpha^2\|v_{xxz}\|^2)}{dt} \\
&\qquad +2\epsilon_1\|u_{xx}\|^2+2\epsilon_1\|v_{xx}\|^2+\epsilon_2\|w_{xx}\|^2
\end{aligned}
$$

$$\leq C_{\epsilon_2}\left(1+\|w\|^2+\|w_x\|^2+\|u_z\|^2+\|u_{xz}\|^2+\|v_x\|^2+\|v_z\|^2+\|v_{xz}\|^2+\|T_{xx}\|^2\right)$$
$$\times\left(1+\|u_{xx}\|^2+\|v_{xx}\|^2+\alpha^2\|u_{xxz}\|^2+\alpha^2\|v_{xxz}\|^2\right). \quad (4.244)$$

By Gronwall inequality, we obtain

$$\begin{aligned}
&\|u_{xx}(t)\|^2+\|v_{xx}(t)\|^2+\alpha^2\|u_{xxz}(t)\|^2+\alpha^2\|v_{xxz}(t)\|^2\\
&\quad+\int_0^t\left(2\epsilon_1\|u_{xx}(s)\|^2+2\epsilon_1\|v_{xx}(s)\|^2+\epsilon_2\|w_{xx}(s)\|^2\right)\,ds\\
&\leq(1+\|\partial_{xx}u_0\|^2+\|\partial_{xx}v_0\|^2+\alpha^2\|\partial_{xxz}u_0\|^2+\alpha^2\|\partial_{xxz}v_0\|^2)\\
&\qquad\times\exp\Big(C_{\epsilon_2}\int_0^t(1+\|w(s)\|^2+\|w_x(s)\|^2+\|u_z(s)\|^2+\|u_{xz}(s)\|^2\\
&\qquad\qquad+\|\nabla v(s)\|^2+\|v_{xz}(s)\|^2+\|T_{xx}(s)\|^2)ds\Big).
\end{aligned} \quad (4.245)$$

From (4.209), (4.215), (4.223), (4.224), (4.233), (4.238) and the above, we have

$$u,v\in L^\infty(0,\mathcal{T};H^2),\quad u_{xxz},v_{xxz}\in L^\infty(0,\mathcal{T};L^2),\quad w\in L^2(0,\mathcal{T};H^2) \quad (4.246)$$

for arbitrary $\mathcal{T}>0$.

By virtue of (4.246), thanks (4.41) and (4.42), we have

$$w\in L^\infty(0,\mathcal{T};H^1)\cap L^2(0,\mathcal{T};H^2), \quad (4.247)$$

for arbitrary $\mathcal{T}>0$.

By standard Galerkin method, one can establish the existence of solutions. Moreover, the solutions satisfy the desired regularity.

4.3.3 Uniqueness of Solutions and Continuous Dependence on The Initial Data

To finish the proof of Theorem 4.3.2, in this section, we prove the uniqueness of solutions and continuous dependence on the initial data. Let (u_1,v_1,w_1,p_1,T_1) and (u_2,v_2,w_2,p_2,T_2) be two strong solutions of the system (4.196)–(4.201), and initial data $((u_0)_1,(v_0)_1,(T_0)_1)$ and

$((u_0)_2, (v_0)_2, (T_0)_2)$, respectively. Denote by $u = u_1 - u_2, v = v_1 - v_2, w = w_1 - w_2, p = p_1 - p_2, T = T_1 - T_2$. Thanks to (4.202) and (4.203), it is clear that

$$\begin{cases} \partial_t \left(u - \alpha^2 u_{zz}\right) + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - \Omega v + p_x = 0, & (4.248) \\ \partial_t \left(v - \alpha^2 v_{zz}\right) + u_1 v_x + w_1 v_z + u(v_2)_x + w(v_2)_z + \epsilon_1 v + \Omega u = 0, & (4.249) \\ \epsilon_2 w + p_z + T = 0, & (4.250) \\ u_x + w_z = 0, & (4.251) \\ \partial_t T - \kappa \Delta T + u_1 T_x + w_1 T_z + u(T_2)_x + w(T_2)_z = 0. & (4.252) \end{cases}$$

By taking the inner product of equation (4.248) with u, (4.249) with v, (4.250) with w, and (4.252) with T in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.27), (4.203) and (4.251), we get

$$\begin{aligned} &\frac{1}{2}\frac{d(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2 \|u_z\|^2 + \alpha^2 \|v_z\|^2)}{dt} \\ &\qquad + \epsilon_1 \|u\|^2 + \epsilon_1 \|v\|^2 + \epsilon_2 \|w\|^2 + \kappa \|\nabla T\|^2 \\ &\le \left| \int_{\mathbb{T}^2} (u(u_2)_x + w(u_2)_z)\, u \, dxdz \right| + \left| \int_{\mathbb{T}^2} (u(v_2)_x + w(v_2)_z)\, v \, dxdz \right| \\ &\quad + \left| \int_{\mathbb{T}^2} wT \, dxdz \right| + \left| \int_{\mathbb{T}^2} (u(T_2)_x + w(T_2)_z)\, T \, dxdz \right|. \end{aligned} \tag{4.253}$$

By integration by parts, using Hölder inequality and Young's inequality, thanks to (4.27), (4.203) and (4.251) and Lemma 2.2.8,

$$\begin{aligned} &\left| \int_{\mathbb{T}^2} (u(u_2)_x + w(u_2)_z)\, u \, dxdz \right| \\ &\le C \int_{\mathbb{T}^2} |w u_z u_2| + |w(u_2)_z u| \, dxdz, \\ &\le \frac{\epsilon_2}{8} \|w\|^2 + C_{\epsilon_2} \|u_2\|_{L^\infty}^2 \|u_z\|^2 \\ &\qquad + C_{\epsilon_2} \|(u_2)_z\| (\|(u_2)_z\| + \|(u_2)_{xz}\|)(\|u\|^2 + \alpha^2 \|u_z\|^2), \end{aligned} \tag{4.254}$$

$$\left| \int_{\mathbb{T}^2} (u(v_2)_x + w(v_2)_z)\, v \, dxdz \right|$$

$$\leq \frac{\epsilon_2}{8}\|w\|^2 + C_{\epsilon_2}\|(v_2)_x\|_{L^\infty}(\|u\|^2 + \|v\|^2)$$
$$+C_{\epsilon_2}\|(v_2)_z\|(\|(v_2)_z\| + \|(v_2)_{xz}\|)(\|v\|^2 + \alpha^2\|v_z\|^2), \tag{4.255}$$

$$\left|\int_{\mathbb{T}^2} wT\ dxdz\right| \leq \frac{\epsilon_2}{8}\|w\|^2 + C_{\epsilon_2}\|T\|^2, \tag{4.256}$$

and

$$\left|\int_{\mathbb{T}^2} (u(T_2)_x + w(T_2)_z)\ T\ dxdz\right|$$
$$\leq C_{\epsilon_2,\kappa}\|T\|\|(T_2)_x\|^{\frac{1}{2}}(\|(T_2)_x\|^{\frac{1}{2}} + \|(T_2)_{xx}\|^{\frac{1}{2}})(\|u\| + \|u_z\|) + \frac{\epsilon_2}{8}\|w\|^2$$
$$+\frac{\kappa}{2}\|T_z\|^2 + C_{\epsilon_2,\kappa}(1 + \|(T_2)_z\|^2)(\|(T_2)_z\|^2 + \|(T_2)_{xz}\|^2)\|T\|^2$$
$$\leq \frac{\epsilon_2}{8}\|w\|^2 + \frac{\kappa}{2}\|\nabla T\|^2 + C_{\epsilon_2,\kappa}\Big(1 + \|(T_2)_x\| + \|(T_2)_{xx}\|$$
$$+\|(T_2)_z\|^4 + \|(T_2)_z\|^2\|(T_2)_{xz}\|^2\Big)$$
$$\times\Big(\|u\|^2 + \|T\|^2 + \alpha^2\,\|u_z\|^2\Big). \tag{4.257}$$

From the estimates above, we obtain

$$\frac{d(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2\,\|u_z\|^2 + \alpha^2\,\|v_z\|^2)}{dt}$$
$$+\epsilon_1\|u\|^2 + \epsilon_1\|v\|^2 + \epsilon_2\|w\|^2 + \kappa\,\|\nabla T\|^2$$
$$\leq C_{\epsilon_2,\kappa}K\left(\|u\|^2 + \|v\|^2 + \|T\|^2 + \alpha^2\,\|u_z\|^2 + \alpha^2\,\|v_z\|^2\right), \tag{4.258}$$

where

$$K = 1 + \|u_2\|_{L^\infty}^2 + \|(v_2)_x\|_{L^\infty} + \|(u_2)_z\|^2 + \|(u_2)_{xz}\|^2 + \|(v_2)_z\|^2 + \|(v_2)_{xz}\|^2$$
$$+\|(T_2)_x\| + \|(T_2)_{xx}\| + \|(T_2)_z\|^4 + \|(T_2)_z\|^2\|(T_2)_{xz}\|^2. \tag{4.259}$$

Using Lemma 2.2.9, thanks to (4.238) and (4.246), we obtain $K \in L^1(0,\mathcal{T})$. Therefore, by

Gronwall inequality, we obtain

$$\begin{aligned}&\|u(t)\|^2 + \|v(t)\|^2 + \|T(t)\|^2 + \alpha^2 \|u_z(t)\|^2 + \alpha^2 \|v_z(t)\|^2 \\ \leq &\Big(\|u(t=0)\|^2 + \alpha^2\|u_z(t=0)\|^2 + \|v(t=0)\|^2 + \alpha^2\|v_z(t=0)\|^2 \\ &\qquad + \|T(t=0)\|^2\Big) \exp(C_{\epsilon_2,\kappa} \int_0^t K(s)ds). \end{aligned} \tag{4.260}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0) = v(t=0) = T(t=0) = 0$, we have $u(t) = v(t) = T(t) = 0$, for all $t \geq 0$. Therefore, the strong solution is unique.

4.3.4 The Special Case: $\Omega = 0, v \equiv 0$ and $T \equiv 0$

In this section, we consider the special case when $\Omega = 0, v \equiv 0$ and $T \equiv 0$. System (4.188)–(4.192) reduces to the following system

$$\begin{cases} (u - \alpha^2 u_{zz})_t + u\, u_x + w u_z + p_x = 0, & (4.261) \\ \epsilon_2 w + p_z + T = 0, & (4.262) \\ u_x + w_z = 0, & (4.263) \\ T_t - \kappa \Delta T + u\, T_x + w\, T_z = 0. & (4.264) \end{cases}$$

We impose similar boundary and initial conditions for this system:

$u,\ w,\ p$ and T are periodic in x and z with period 1, (4.265)

$u,\ p$ are even in $z,$ and $w,\ T$ are odd in $z,$ (4.266)

$$\Big(u, T\Big)|_{t=0} = \Big(u_0, T_0\Big). \tag{4.267}$$

Using an analogue argument to that in section 4.1.1, system (4.261)–(4.264) subject to (4.265)–(4.267) is equivalent to the following:

$$\begin{cases} (u-\alpha^2 u_{zz})_t + u\,u_x + wu_z + \epsilon_1 u + p_x = 0, & (4.268)\\ T_t - \kappa\Delta T + u\,T_x + w\,T_z = 0, & (4.269)\end{cases}$$

with w, p_x, p_z defined by

$$\begin{cases} w(x,z) := -\int_0^z u_x(x,s)ds, & (4.270)\\ p_x(x,z) := \epsilon_2 \int_0^z \int_0^s u_{xx}(x,\xi)d\xi ds - \int_0^z T_x(x,s)ds \\ \qquad\qquad + \int_0^1 \Big[\int_0^{z'} T_x(x,s)ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x,\xi)d\xi ds - 2uu_x(x,z')\Big] dz', & (4.271)\\ p_z(x,z) := -T(x,z) + \epsilon_2 \int_0^z u_x(x,s)ds. & (4.272)\end{cases}$$

We are interested in the system (4.268)–(4.272) in the unit two dimensional torus $\mathbb{T}^2$, subject to the following symmetry boundary conditions and initial conditions:

u and T are periodic in x and z with period 1; (4.273)

u is even in z, and T is odd in z; (4.274)

$(u,T)|_{t=0} = (u_0, T_0).$ (4.275)

We have the global well-posedness for system (4.268)–(4.272), for initial condition with less regularity. i.e., for $u_0, \partial_z u_0 \in H^1$ and $T_0 \in H^1$. Let us give the definition of strong solution first.

Definition 4.3.3. *Suppose that $u_0 \in H^1(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.273) and (4.274), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_z u_0 \in H^1$. Given time $\mathcal{T} > 0$, we say (u,T) is a strong solution to the system (4.268)–(4.272), subjecto to (4.273)–(4.275), on the time interval $[0,\mathcal{T}]$, if*

1. *u and T satisfy the symmetry conditions (4.273) and (4.274);*

2. *u and T have the regularities*

$$\begin{cases} u \in L^\infty(0,\mathcal{T};H^1) \cap C([0,\mathcal{T}];L^2), \\ u_z \in L^\infty(0,\mathcal{T};H^1), \\ \partial_t u \in L^2(0,\mathcal{T};L^2), \\ T \in L^2(0,\mathcal{T};H^2) \cap L^\infty(0,\mathcal{T};H^1) \cap C([0,\mathcal{T}];L^2), \\ \partial_t T \in L^2(0,\mathcal{T};L^2); \end{cases}$$

3. *u, T satisfy system (4.268)–(4.269) in the following sense:*

$$(u-\alpha^2 u_{zz})_t + u\,u_x + w u_z + \epsilon_1 u + p_x = 0 \;\; \textit{in} \;\; L^2(0,\mathcal{T};L^2);$$
$$T_t - \kappa\Delta T + u\,T_x + w\,T_z = 0 \;\; \textit{in} \;\; L^2(0,\mathcal{T};L^2),$$

with w, p_x, p_z defined by (4.270)–(4.272), and fulfill the initial condition (4.275).

Based on Theorem 4.3.2, we have the following theorem on the existence and uniqueness of strong solutions to system (4.268)–(4.272), subject to (4.273)–(4.275), on $\mathbb{T}^2 \times (0,\mathcal{T})$, for any positive time $\mathcal{T}$. The proof is similar as Theorem 4.3.2, and we omit the details here.

Theorem 4.3.4. *Suppose that $u_0 \in H^1(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.273) and (4.274), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_z u_0 \in H^1$. Given time $\mathcal{T} > 0$. Then there exists a unique strong solution (u, T) of the system (4.268)–(4.272), subject to (4.273)–(4.275), on the interval $[0,\mathcal{T}]$. Moreover, the unique strong solution (u, T) depends continuously on the initial data. Same result holds when $T \equiv 0$.*

Remark 19. The reason why we need to assume more regularity for the initial data to system (4.196)–(4.198) is that we need a bound for $\|(v_2)_x\|_{L^\infty}$ appears in (4.255). If we do not have the evolution equation in v, we can require less on the initial data.

4.4 Convergence as $\alpha \to 0$

In this section, we will prove the convergence of the strong solution of the following system

$$\begin{cases} (u^\alpha - \alpha^2 u^\alpha_{zz})_t - \nu u^\alpha_{zz} + u^\alpha u^\alpha_x + w^\alpha u^\alpha_z + \epsilon_1 u^\alpha + p^\alpha_x = 0, & (4.276)\\ \epsilon_2 w^\alpha + p^\alpha_z = 0, & (4.277)\\ u^\alpha_x + w^\alpha_z = 0 & (4.278)\end{cases}$$

subject to the following symmetric boundary conditions and initial condition

$$u^\alpha,\ w^\alpha \ \text{ and } \ p^\alpha \ \text{ are periodic in } x \text{ and } z \text{ with period 1;} \tag{4.279}$$

$$u^\alpha,\ p^\alpha \ \text{ are even in } z, \ \text{ and } \ w^\alpha \ \text{ is odd in } z; \tag{4.280}$$

$$u^\alpha|_{t=0} = u^\alpha_0, \tag{4.281}$$

to the strong solution of system (4.126)–(4.128) subject to (4.129)–(4.131), as $\alpha \to 0$.

Remark 20. The global well-posedness of system (4.276)–(4.278) subject to (4.279)–(4.281) can be obtained as in section 4.3. Moreover, as indicated in the last part of section 4.3, we only need to assume that $u^\alpha_0, \partial_z u^\alpha_0 \in H^1(\mathbb{T}^2)$ since we do not have the evolution equation in v^α.

Theorem 4.4.1. *Suppose that $u_0, \{u^\alpha_0\}_{0<\alpha\le 1} \subset H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (4.129)–(4.130) and (4.279)–(4.280), with the compatibility conditions $\int_0^1 \partial_x u_0 dz = 0$ and $\int_0^1 \partial_x u^\alpha_0 dz = 0$, for $\forall\ 0 < \alpha \le 1$, and suppose that $\partial_{xz} u_0 \in L^2(\mathbb{T}^2), \{\partial_z u^\alpha_0\}_{0<\alpha\le 1} \subset H^1(\mathbb{T}^2)$. Moreover, suppose there exists some constant $M > 0$ such that the following uniform bound for initial data holds:*

$$\sup_{0<\alpha\le 1} \Big(\|u^\alpha_0\| + \|\partial_z u^\alpha_0\| + \alpha \|\partial_{zz} u^\alpha_0\| \Big) \le M. \tag{4.282}$$

Let $\mathcal{T} > 0$ be such that u is the strong solution of system (4.126)–(4.128) on $[0, \mathcal{T}]$ with initial data u_0. Let u^α be the strong solution to system (4.276)–(4.278) on $[0, \mathcal{T}]$ with initial data u^α_0. If $u^\alpha_0 \to u_0$ in L^2, as $\alpha \to 0$, then $u^\alpha \to u$ in $L^\infty(0, \mathcal{T}; L^2)$, and $u^\alpha_z \to u_z$ in $L^2(0, \mathcal{T}; L^2)$, as $\alpha \to 0$.

Proof. Let us first derive the uniform bounds of some norms of the strong solution u^α. By taking the L^2-inner product of equation (4.276) with $u^\alpha, -u^\alpha_{zz}$, and equation (4.277) with $w^\alpha, -w^\alpha_{zz}$, in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (4.279), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u^\alpha\|^2+(\alpha^2+1)\,\|u^\alpha_z\|^2+\alpha^2\,\|u^\alpha_{zz}\|^2\Big)+\epsilon_1\Big(\|u^\alpha\|^2+\|u^\alpha_z\|^2\Big)\\&\quad+\epsilon_2\Big(\|w^\alpha\|^2+\|w^\alpha_z\|^2\Big)+\nu\Big(\|u^\alpha_z\|^2+\|u^\alpha_{zz}\|^2\Big)\\&=-\int_{\mathbb{T}^2}\left(u^\alpha u^\alpha_x+w^\alpha u^\alpha_z\right)\,\left(u^\alpha-u^\alpha_{zz}\right)\,dxdz\\&\quad-\int_{\mathbb{T}^2}\Big(p^\alpha_x\left(u^\alpha-u^\alpha_{zz}\right)+p^\alpha_z\left(w^\alpha-w^\alpha_{zz}\right)\Big)dxdz. \qquad (4.283)\end{aligned}$$

By integration by parts, thanks to (4.278) and (4.279), we have

$$\begin{aligned}&-\int_{\mathbb{T}^2}\left(u^\alpha u^\alpha_x+w^\alpha u^\alpha_z\right)\left(u^\alpha-u^\alpha_{zz}\right)\,dxdz\\&\quad-\int_{\mathbb{T}^2}\Big(p^\alpha_x\left(u^\alpha-u^\alpha_{zz}\right)+p^\alpha_z\left(w^\alpha-w^\alpha_{zz}\right)\Big)dxdz=0. \qquad (4.284)\end{aligned}$$

As a result of the above, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|u^\alpha\|^2+(\alpha^2+1)\,\|u^\alpha_z\|^2+\alpha^2\,\|u^\alpha_{zz}\|^2\Big)\\&\quad+\nu\Big(\|u^\alpha_z\|^2+\|u^\alpha_{zz}\|^2\Big)+\epsilon_2\Big(\|w^\alpha\|^2+\|w^\alpha_z\|^2\Big)\le 0. \qquad (4.285)\end{aligned}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned}&\|u^\alpha(t)\|^2+\,\|u^\alpha_z(t)\|^2+\int_0^t\Big[\nu\left(\|u^\alpha_z(s)\|^2+\|u^\alpha_{zz}(s)\|^2\right)\\&\qquad\qquad+\epsilon_2\left(\|w^\alpha(s)\|^2+\|w^\alpha_z(s)\|^2\right)\Big]\,ds\\&\le\|u^\alpha_0\|^2+(1+\alpha^2)\|\partial_z u^\alpha_0\|^2+\alpha^2\|\partial_{zz}u^\alpha_0\|^2, \qquad (4.286)\end{aligned}$$

for $t\in[0,\mathcal{T}]$. Thanks to the uniform bound for initial data (4.282), we have

$$\sup_{0<\alpha\le 1}\Big(\|u^\alpha\|_{L^\infty(0,\mathcal{T};L^2)}+\|u^\alpha_z\|_{L^\infty(0,\mathcal{T};L^2)}+\nu\|u^\alpha_{zz}\|_{L^2(0,\mathcal{T};L^2)}$$

$$+\epsilon_2\|w^\alpha\|_{L^2(0,\mathcal{T};L^2)} + \epsilon_2\|w_z^\alpha\|_{L^2(0,\mathcal{T};L^2)}\Big) \le C(M), \tag{4.287}$$

where $C(M)$ is a constant depending on M, but not on α. Now subtracting (4.126)–(4.127) from (4.276)–(4.277), we obtain

$$\begin{cases} \partial_t[(u^\alpha - u) - \alpha^2(u_{zz}^\alpha - u_{zz})] - \nu(u_{zz}^\alpha - u_{zz}) + \epsilon_1(u^\alpha - u) + (p_x^\alpha - p_x) \\ \qquad = (u - u^\alpha)u_x + (u_x - u_x^\alpha)u^\alpha + (w - w^\alpha)u_z + (u_z - u_z^\alpha)w^\alpha - \alpha^2\partial_t u_{zz}, & (4.288) \\ \epsilon_2(w^\alpha - w) + (p_z^\alpha - p_z) = 0. & (4.289) \end{cases}$$

By taking the inner product of equation (4.288) with $u^\alpha - u$ and equation (4.289) with $w^\alpha - w$, by integration by parts, and using (4.128) and (4.278), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u^\alpha - u\|^2 + \alpha^2\|u_z^\alpha - u_z\|^2) + \nu\,\|u_z^\alpha - u_z\|^2 + +\epsilon_1\|u^\alpha - u\|^2 + \epsilon_2\|w^\alpha - w\|^2 \\ =& \int_{\mathbb{T}^2}\Big[(u - u^\alpha)^2 w_z + (u^\alpha - u)(u_x - u_x^\alpha)u^\alpha + (w - w^\alpha)(u^\alpha - u)u_z \\ &+(u_z - u_z^\alpha)(u^\alpha - u)w^\alpha + (p_x - p_x^\alpha)(u^\alpha - u) + (p_z - p_z^\alpha)(w^\alpha - w) \\ &+\alpha^2\partial_t u_{zz}(u - u^\alpha)\Big]dxdz. \end{aligned} \tag{4.290}$$

We estimate each term in (4.290). By integration by parts, using Hölder inequality and Young's inequality, thanks to (4.128), (4.129), (4.278) and (4.279), we have

$$\begin{aligned} &\int_{\mathbb{T}^2}(u - u^\alpha)^2 w_z dxdz = -2\int_{\mathbb{T}^2}(u^\alpha - u)_z(u^\alpha - u)w dxdz \\ \le& \frac{\nu}{2}\|u_z^\alpha - u_z\|^2 + C_\nu\|w\|_\infty^2\|u^\alpha - u\|^2, \end{aligned} \tag{4.291}$$

$$\begin{aligned} &\int_{\mathbb{T}^2}\Big[(u^\alpha - u)(u_x - u_x^\alpha)u^\alpha + (u_z - u_z^\alpha)(u^\alpha - u)w^\alpha\Big]dxdz \\ =& \frac{1}{2}\int_{\mathbb{T}^2}(u^\alpha - u)^2(u_x^\alpha + w_z^\alpha)dxdz = 0, \end{aligned} \tag{4.292}$$

$$\int_{\mathbb{T}^2}(w-w^\alpha)(u^\alpha-u)u_z dxdz \le \frac{\epsilon_2}{2}\|w^\alpha-w\|^2 + C_{\epsilon_2}\|u_z\|_\infty^2\|u^\alpha-u\|^2, \tag{4.293}$$

$$\begin{aligned}&\int_{\mathbb{T}^2}\Big[(p_x-p_x^\alpha)(u^\alpha-u)+(p_z-p_z^\alpha)(w^\alpha-w)\Big]dxdz\\ &=\int_{\mathbb{T}^2}(p-p^\alpha)[(u-u^\alpha)_x+(w-w^\alpha)_z]dxdz=0,\end{aligned} \tag{4.294}$$

and

$$\begin{aligned}\alpha^2\int_{\mathbb{T}^2}\partial_t u_{zz}(u-u^\alpha)dxdz &= \alpha^2\int_{\mathbb{T}^2}u_t(u-u^\alpha)_{zz}dxdz\\ &\le C\alpha^2\|u_t\|(\|u_{zz}^\alpha\|+\|u_{zz}\|).\end{aligned} \tag{4.295}$$

From all the estimates above, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|u^\alpha-u\|^2+\alpha^2\|u_z^\alpha-u_z\|^2)+\nu\|u_z^\alpha-u_z\|^2+\epsilon_1\|u^\alpha-u\|^2+\epsilon_2\|w^\alpha-w\|^2\\ &\le C_{\nu,\epsilon_2}(\|w\|_\infty^2+\|u_z\|_\infty^2)(\|u^\alpha-u\|^2+\alpha^2\|u_z^\alpha-u_z\|^2)+C\alpha^2\|u_t\|(\|u_{zz}^\alpha\|+\|u_{zz}\|).\end{aligned} \tag{4.296}$$

Denote by

$$F:=\|w\|_\infty^2+\|u_z\|_\infty^2,\quad G:=\|u_t\|(\|u_{zz}^\alpha\|+\|u_{zz}\|), \tag{4.297}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}(\|u^\alpha-u\|^2+\alpha^2\|u_z^\alpha-u_z\|^2)+\nu\|u_z^\alpha-u_z\|^2+\epsilon_1\|u^\alpha-u\|^2+\epsilon_2\|w^\alpha-w\|^2\\ &\quad\le C_{\nu,\epsilon_2}F(\|u^\alpha-u\|^2+\alpha^2\|u_z^\alpha-u_z\|^2)+C\alpha^2G.\end{aligned} \tag{4.298}$$

Notice that the constants appear above do not depend on α. Thanks to Gronwall inequality, we obtain

$$\|u^\alpha-u\|^2(t)+\alpha^2\|u_z^\alpha-u_z\|^2(t)$$

$$
\begin{aligned}
&+\int_0^t \Big(\nu\|u_z^\alpha - u_z\|^2(s) + \epsilon_1\|u^\alpha - u\|^2(s) + \epsilon_2\|w^\alpha - w\|^2(s)\Big)ds \\
&\le (\|u_0^\alpha - u_0\|^2 + \alpha^2\|\partial_z u_0^\alpha - \partial_z u_0\|^2)\exp\Big(C_{\nu,\epsilon_2}\int_0^t F(s)ds\Big) \\
&\quad +C\alpha^2\exp\Big(C_{\nu,\epsilon_2}\int_0^t F(s)ds\Big)\int_0^t G(s)ds \\
&= (\|u_0^\alpha - u_0\|^2 + \alpha^2\|\partial_z u_0^\alpha - \partial_z u_0\|^2)\exp\Big(C_{\nu,\epsilon_2}\int_0^t F(s)ds\Big) + C\alpha^2 H(t), \qquad (4.299)
\end{aligned}
$$

where $H(t)$ is defined as

$$
H(t) := \exp\Big(C_{\nu,\epsilon_2}\int_0^t F(s)ds\Big)\int_0^t G(s)ds. \qquad (4.300)
$$

By virture of the regularity of strong solution to system (4.126)–(4.128), and the uniform bound (4.287), using Lemma 2.2.9, we have $F, G \in L^1(0,\mathcal{T})$. By virtue of uniform bound (4.287), we have $\alpha^2 H(t) \to 0$, as $\alpha \to 0$. Since $u_0^\alpha \to u_0$ in L^2, and thanks to (4.282), we have $u^\alpha \to u$ in $L^\infty(0,\mathcal{T};L^2)$, $u_z^\alpha \to u_z$ in $L^2(0,\mathcal{T};L^2)$, and $w^\alpha \to w$ in $L^2(0,\mathcal{T};L^2)$, as $\alpha \to 0$.

□

4.5 Blowup Criterion

In this section we give a blow-up criterion for system (4.126)–(4.128) subjects to (4.129)–(4.131). The following result follows the idea in [66].

Theorem 4.5.1. *With the same assumptions in Theorem 4.4.1, and take $u_0^\alpha = u_0$ for all α. Suppose there exists some time $\mathcal{T}^* < \infty$ such that*

$$
\limsup_{\alpha\to 0^+}\Big(\alpha^2 \sup_{t\in[0,\mathcal{T}^*]}\|u_z^\alpha(t)\|^2\Big) > 0, \qquad (4.301)
$$

then the solution for system (4.126)–(4.128) blows up on $[0,\mathcal{T}^*]$.

Proof. Assume the solution for system (4.126)–(4.128) will not blow up on $[0,\mathcal{T}^*]$, then $u \in L^\infty(0,\mathcal{T}^*;H^1)$ and $\partial_t u \in L^2(0,\mathcal{T}^*;L^2)$. By taking the inner product of equation (4.126) with u and equation (4.127) with w in $L^2(\mathbb{T}^2)$, by integration by parts and thanks to (4.128) and (4.129), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|u_z\|^2 + \epsilon_1\|u\|^2 + \epsilon_2\|w\|^2 = 0. \tag{4.302}$$

Integrating (4.302) from 0 to t for $t \in [0, \mathcal{T}^*]$, we have

$$\|u(t)\|^2 + 2\int_0^t \Big(\nu\|u_z(s)\|^2 + \epsilon_1\|u(s)\|^2 + \epsilon_2\|w(s)\|^2\Big) ds = \|u_0\|^2. \tag{4.303}$$

On the other hand, using analogue argument for system (4.276)–(4.278), we have

$$\begin{aligned}&\alpha^2\|u_z^\alpha(t)\|^2 + \|u^\alpha(t)\|^2 + 2\int_0^t \left(\nu\|u_z^\alpha(s)\|^2 + \epsilon_1\|u^\alpha(s)\|^2 + \epsilon_2\|w^\alpha(s)\|^2\right) ds \\ &= \|u_0^\alpha\|^2 + \alpha^2\|\partial_z u_0^\alpha\|^2 = \|u_0\|^2 + \alpha^2\|\partial_z u_0\|^2.\end{aligned} \tag{4.304}$$

From (4.299) and thanks to the fact that $u_0^\alpha = u_0$, for any $t \in [0, \mathcal{T}^*]$, we have

$$\|u^\alpha(t)\| \geq \|u(t)\| - C\alpha H^{1/2}(t) \geq \|u(t)\| - C\alpha H^{1/2}(\mathcal{T}^*), \tag{4.305}$$

since $H^{1/2}(t)$ is monotonically increasing. By virtue of (4.303), we know $\|u_0\| \geq \|u(t)\| \geq \|u(\mathcal{T}^*)\|$ for any $t \in [0, \mathcal{T}^*]$. Therefore, we can take $\alpha < \frac{\|u(\mathcal{T}^*)\|}{CH^{1/2}(\mathcal{T}^*)}$ to guarantee the right hand side of (4.305) is positive. Take square on (4.305), we obtain

$$\begin{aligned}\|u^\alpha(t)\|^2 &\geq \|u(t)\|^2 - 2\alpha CH^{1/2}(\mathcal{T}^*)\|u(t)\| + C^2\alpha^2 H(\mathcal{T}^*) \\ &\geq \|u(t)\|^2 - 2\alpha CH^{1/2}(\mathcal{T}^*)\|u_0\| + C^2\alpha^2 H(\mathcal{T}^*).\end{aligned} \tag{4.306}$$

Subtracting (4.304) from (4.303), we have

$$\|u(t)\|^2 - \|u^\alpha(t)\|^2$$

$$= \alpha^2\|u_z^\alpha(t)\|^2 - \alpha^2\|\partial_z u_0\|^2 + 2\int_0^t \Big(\nu\|u_z^\alpha(s)\|^2 + \epsilon_1\|u^\alpha(s)\|^2 + \epsilon_2\|w^\alpha(s)\|^2\Big)$$
$$-\Big(\nu\|u_z(s)\|^2 + \epsilon_1\|u(s)\|^2 + \epsilon_2\|w(s)\|^2\Big)ds. \tag{4.307}$$

Combining (4.307) with (4.306), we obtain

$$\begin{aligned}\alpha^2\|u_z^\alpha(t)\|^2 \le \alpha^2\|\partial_z u_0\|^2 &+ 2\alpha C H^{1/2}(\mathcal{T}^*)\|u_0\| - C^2\alpha^2 H(\mathcal{T}^*)\\ &+2\int_0^t \left(\nu\|u_z(s)\|^2 + \epsilon_1\|u(s)\|^2 + \epsilon_2\|w(s)\|^2\right)\\ &- \left(\nu\|u_z^\alpha(s)\|^2 + \epsilon_1\|u^\alpha(s)\|^2 + \epsilon_2\|w^\alpha(s)\|^2\right) ds.\end{aligned} \tag{4.308}$$

By Cauchy–Schwarz inequality and Hölder inequality, thanks to (4.160)–(4.161) and the uniform bound (4.287), we have the estimate for the last term in (4.308):

$$\begin{aligned}&2\int_0^t \left(\nu\|u_z(s)\|^2 + \epsilon_1\|u(s)\|^2 + \epsilon_2\|w(s)\|^2\right)\\ &\qquad - \left(\nu\|u_z^\alpha(s)\|^2 + \epsilon_1\|u^\alpha(s)\|^2 + \epsilon_2\|w^\alpha(s)\|^2\right) ds\\ &= 2\int_0^t \Big[\nu\big(u_z - u_z^\alpha, u_z + u_z^\alpha\big) + \epsilon_1\big(u - u^\alpha, u + u^\alpha\big) + \epsilon_2\big(w - w^\alpha, w + w^\alpha\big)\Big] ds\\ &\le 2\int_0^t \Big[\nu\|u_z - u_z^\alpha\|\|u_z + u_z^\alpha\| + \epsilon_1\|u - u^\alpha\|\|u + u^\alpha\| + \epsilon_2\|w - w^\alpha\|\|w + w^\alpha\|\Big] ds\\ &\le C_{\nu,\epsilon_1,\epsilon_2}\left(\|u_z - u_z^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)} + \|u - u^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)} + \|w - w^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)}\right).\end{aligned} \tag{4.309}$$

Plugging this back into (4.308), we have

$$\begin{aligned}\alpha^2\|u_z^\alpha(t)\|^2 &\le \alpha^2\|\partial_z u_0\|^2 + 2\alpha C H^{1/2}(\mathcal{T}^*)\|u_0\| - C^2\alpha^2 H(\mathcal{T}^*)\\ &+C_{\nu,\epsilon_1,\epsilon_2}\left(\|u_z - u_z^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)} + \|u - u^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)} + \|w - w^\alpha\|_{L^2(0,\mathcal{T}^*;L^2)}\right).\end{aligned} \tag{4.310}$$

By virtue of Theorem 4.4.1, the right hand side of (4.310) is independent of t, and it converges to

0 as $\alpha \to 0$. Therefore, by taking $\limsup_{\alpha\to 0^+} \sup_{t\in[0,\mathcal{T}^*]}$ on both hand sides of (4.310), we obtain

$$\limsup_{\alpha\to 0^+} \left(\alpha^2 \sup_{t\in[0,\mathcal{T}^*]} \|u_z^\alpha(t)\|^2\right) = 0, \tag{4.311}$$

which contradicts to (4.301). □

Remark 21. By considering the convergence for the whole system, i.e., the convergence of the strong solution of system (4.188)–(4.192) to the corresponding solution of system (4.1)–(4.5), we can establish a similar blow-up criterion for system (4.1)–(4.5).

5. CONCLUSION AND SUMMARY

In this dissertation, our discussions mainly focus on two topics. The IPEs and PEs with weak dissipation.

In Chapter 3, we focus on the IPEs. We start with the ill-posedness of the IPEs, from the case without rotation to the case with rotation. These results suggest we should work in the space of analytic functions for the well-posedness. Next, we establish the local well-posedness, as suggested, in the space of analytic functions. Notably, the time of existence we obtain is independent of the rate of rotation. This improves the result in [59]. Given the local well-posedness, the next natural question is whether one can extend it to a global one, or establish the finite-time blowup. Indeed, we construct smooth solutions that blow up in finite time, for both non-rotating and rotating system.

The blowup results indicate that there is no hope to show the global well-posedness of the IPEs. The best result one can expect is the long-time existence of solutions to the IPEs under some assumptions. We start with a simple observation, namely when the baroclinic mode is zero, the $3D$ IPEs reduce to the $2D$ Euler equations that are globally well-posed. Therefore, we put only smallness assumption on the analytic norm of the initial baroclinic mode to obtain the first long-time existence result. By virtue of the effect of fast rotation on the life-span for other models, we propose to investigate the case of the IPEs in the space of analytic functions. By some delicate analysis and using some well-chosen projections, we derive the formal limit resonant system as the rotation rate $|\Omega| \to \infty$, and show that this limit system is globally well-posed. Eventually, we are able to establish that the life-span of solutions to the $3D$ IPEs is prolonged to infinity together with the rotation rate $|\Omega|$, for "well-prepared" initial data. Here the "well-prepared" initial data means that only the Sobolev norm (not the analytic norm) of the initial baroclinic mode is small depending on $|\Omega|$. We end up with some remarks and discussions about why we still need the "well-prepared" initial data instead of arbitrary initial data. This is the optimal result we can achieve so far.

In Chapter 4, we focus on the PEs with weak dissipation. The weak dissipation is the vertical viscosity and linear damping. The consideration of the linear damping terms is due to the ill-posedness in Sobolev spaces of the PEs with only vertical viscosity suggested by Renardy [83]. With the help of these linear damping terms, we are able to show the local well-posedness with arbitrary Sobolev initial data and the global well-posedness with small Sobolev initial data. In order to study the possible finite-time blow-up of the system, and to give a reliable numerical regularization, we propose to study the Voigt α-regularization of our model. For the regularized model, we are able to establish the global well-posedness for arbitrary Sobolev initial data. Moreover, we show the convergence of solutions as $\alpha \to 0$. Based on the convergence result, we derive a blowup criterion of the original model.

REFERENCES

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions,* Dover, 1973, reprint.

[2] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] P. Azérad and F. Guillén, *Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics,* SIAM J. Math. Anal. **33** (2001), 847–859.

[4] A. Babin, A.A. Ilyin, and E.S. Titi, *On the regularization mechanism for the spatially periodic Korteweg–de Vries equation,* Commun. Pure Appl. Math. **64** (2011), 591–648 .

[5] A. Babin, A. Mahalov, and B. Nicolaenko, *Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids,* Asymptot. Anal. **15:2** (1997), 103–150.

[6] A. Babin, A. Mahalov, and B. Nicolaenko, *Global regularity of 3D rotating Navier–Stokes equations for resonant domains,* Indiana Univ. Math. J. **48:3** (1999), 1133–1176.

[7] A. Babin, A. Mahalov, and B. Nicolaenko, *On the regularity of three-dimensional rotating Euler-Boussinesq equations,* Mathematical Models and Methods in Applied Sciences **9:7** (1999), 1089–1121.

[8] A. Babin, A. Mahalov, and B. Nicolaenko, *Fast singular oscillating limits and global regularity for the 3D primitive equations of geophysics,* Math. Model. Numer. Anal. **34** (1999), 201–222.

[9] C. Bardos and E. S. Titi, *Euler equations of incompressible ideal fluids,* Russian Mathematical Surveys **62:3** (2007), 409–451.

[10] A. L. Bertozzi and A. J. Majda, *Vorticity and Incompressible Flow,* Cambridge University Press, Cambridge, 2002.

[11] W. Blumen, *Geostrophic adjustment,* Rev. Geophys. Space Phys. **10** (1972), 485–528.

[12] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles,* Nonlinearity **12:3** (1999), 495–512.

[13] Y. Brenier, *Remarks on the derivation of the hydrostatic Euler equations,* Bull. Sci. Math. **127:7** (2003), 585–595.

[14] D. Bresch, F. Guillén-González, N. Masmoudi and M.A. Rodríguez-Bellido, *On the uniqueness of weak solutions of the two-dimensional primitive equations,* Differential Integral Equations **16** (2003), 77–94.

[15] D. Bresch, A. Kazhikhov and J. Lemoine, *On the two-dimensional hydrostatic Navier-Stokes equations,* SIAM J. Math. Anal. **36** (2004), 796–814.

[16] R. E. Caflisch and O. F. Orellana, *Singular solutions and ill-posedness for the evolution of vortex sheets,* SIAM J. Math. Anal. **20:2** (1989), 293–307.

[17] C. Cao, S. Ibrahim, K. Nakanishi and E. S. Titi, *Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics,* Comm. Math. Phys. **337** (2015), 473–482.

[18] C. Cao, J. Li and E. S. Titi, *Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity,* Arch. Rational Mech. Anal. **214** (2014), 35–76.

[19] C. Cao, J. Li and E. S. Titi, *Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity,* J. Differential Equations **257** (2014), 4108–4132.

[20] C. Cao, J. Li and E. S. Titi, *Global well-posedness of the 3D primitive equations with only horizontal viscosity and diffusivity,* Comm. Pure Appl. Math. **69** (2016), 1492–1531.

[21] C. Cao, J. Li and E. S. Titi, *Strong solutions to the 3D primitive equations with only horizontal dissipation: Near H^1 initial data,* J. Funct. Anal. **272:11** (2017), 4606–4641.

[22] C. Cao, J. Li and E. S. Titi, *Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity,* Physica D **412** (2020), https://doi.org/10.1016/j.physd.2020.132606.

[23] C. Cao, Q. Lin and E. S. Titi, *On the well–posedness of reduced $3D$ primitive geostrophic adjustment model with weak dissipation,* J. Math. Fluid Mech. (2020), https://doi.org/10.1007/s00021-020-00495-6.

[24] C. Cao and J. Wu, *Global regularity for the 2D anisotropic Boussinesq Equations with vertical dissipation,* Arch. Ration. Mech. Anal. **208** (2013), 985–1004.

[25] C. Cao and E. S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics,* Ann. Math. **166** (2007), 245–267.

[26] C. Cao and E. S. Titi, *Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusivity,* Comm. Math. Phys. **310** (2012), 537–568.

[27] C. Cao and E. S. Titi, *Regularity "in Large" for the 3D Salmon's Planetary Geostrophic Model of Ocean Dynamics,* Math. Clim. Weather Forecast. **6:1** (2020), 1–15.

[28] Y. Cao, E. Lunasin and E. S. Titi, *Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models,* Commun. Math. Sci. **4** (2006), 823–848.

[29] J.-Y. Chemin, B. Desjardines, I. Gallagher and E. Grenier, *Mathematical Geophysics. An introduction to rotating fluids and the Navier-Stokes Equations,* Clarendon Press, Oxford, 2006.

[30] X. L. Chen and P. J. Morrison, *A sufficient condition for the ideal instability of shear flow with parallel magnetic field,* Phys. Fluids B Plasma Phys. **3:4** (1991), 863–865.

[31] S. Childress, G.R. Ierley, E.A. Spiegel and W.R. Young, *Blow-up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation-point form,* J. Fluid Mech. **203** (1989), 1–22.

[32] P. Constantin and C. Foias, *Navier-Stokes Equations,* The University of Chicago Press, 1988.

[33] H. Dietert and D. Gérard-Varet, *Well-posedness of the Prandtl equations without any structural assumption,* Ann. PDE **5:8** (2019), https://doi.org/10.1007/s40818-019-0063-6

[34] A. Dutrifoy, *Examples of dispersive effects in non-viscous rotating fluids*, J. Math. Pures Appl. **84:9** (2005), 331–356.

[35] W. E and B. Enquist, *Blow up of solutions to the unsteady Prandtl equation,* Comm. Pure Appl. Math. **50:12** (1997), 1287–1293.

[36] P.F. Embid and A.J. Majda, *Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity,* Comm. Partial Differ. Equ. **21** (1996), 619–658.

[37] A. B. Ferrari and E. S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Differ. Equ. **23** (1998), 1–16.

[38] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier–Stokes equations*, J. Funct. Anal. **87** (1989), 359–369.

[39] D. Gérard-Varet and E. Dormy, *On the ill-posedness of the Prandtl equation,* J. Amer. Math. Soc. **23:2** (2010), 591–609.

[40] D. Gérard-Varet and T. Nguyen, *Remarks on the ill-posedness of the Prandtl equation,* Asymptotic Analysis. **77** (2012), 71–88.

[41] T. Ghoul, S. Ibrahim, Q. Lin and E. S. Titi, *On the effect of rotation on the life-span of analytic solutions to the* $3D$ *inviscid primitive equations,* (2020), preprint, arXiv:2010.01740.

[42] A.E. Gill, *Adjustment under gravity in a rotating channel,* J. Fluid Mech. **103** (1976), 275–295.

[43] A.E. Gill, *Atmosphere–Ocean Dynamics,* Academic Press New York, 1982.

[44] E. Grenier, *On the derivation of homogeneous hydrostatic equations,* M2AN Math. Model. Numer. Anal. **33:5** (1999), 965–970.

[45] F. Guillén-González, N. Masmoudi and M.A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the primitive equations,* Differ. Integral Equ. **14** (2001), 1381–1408.

[46] Y. Guo, K. Simon and E.S. Titi *Global well-posedness of a system of nonlinearly coupled KdV equations of Majda and Biello,* Commun. Math. Sci. **13:5** (2015), 1261–1288.

[47] J. Hadamard, *: Lectures on Cauchy's problem in linear partial differential equations*, Dover Publications, New York, 1953.

[48] D. Han-Kwan and T. Nguyen, *Illposedness of the hydrostatic Euler and singular Vlasov equations*, Arch. Ration. Mech. Anal. **221:3** (2016), 1317–1344.

[49] A.J. Hermann and W.B. Owens, *Energetics of gravitational adjustment in mesoscale chimneys*, J. Phys. Oceanogr. **23** (1993), 346–371.

[50] G. Haltiner and R. Williams, *Numerical Weather Prediction and Dynamic Meteorology*, second ed. Wiley, New York, 1984.

[51] M. Hieber and T. Kashiwabara, *Global well-posedness of the three-dimensional primitive equations in L^p-space*, Arch. Rational Mech. Anal. **221** (2016), 1077–1115.

[52] J.R. Holton, *An Introduction to Dynamic Meteorology,* 4th ed. Elsevier Academic Press, 2004.

[53] S. Ibrahim, Q. Lin and E. S. Titi, *Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation,* J. Differ. Equ. **286** (2021), 557–577, https://doi.org/10.1016/j.jde.2021.03.037.

[54] S. Ibrahim and T. Yoneda, *Long time solvability of the Navier-Stokes-Boussinesq equations with almost periodic initial large data* J. Math. Sci. Univ. Tokyo **20:1** (2013), 1–25.

[55] G. M. Kobelkov, *Existence of a solution in the large for the 3D large-scale ocean dynamics equaitons,* C. R. Math. Acad. Sci. Paris **343** (2006), 283–286.

[56] Y. Koh, S. Lee and R. Takada, *Strichartz estimates for the Euler equations in the rotating framework,* J. Differ. Equ. **256** (2014), 707–744.

[57] A. Kostianko, E. S. Titi and S. Zelik, *Large dispertion, averaging and attractors: three 1D paradigms,* Nonlinearity **31** (2018), 317–350.

[58] I. Kukavica, N. Masmoudi, V. Vicol and T. Wong, *On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions,* SIAM J. Math. Anal. **46:6** (2014), 3865–3890.

[59] I. Kukavica, R. Temam, V. Vicol, and M. Ziane, *Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain,* J. Differ. Equ. **250:3** (2011), 1719–1746.

[60] I. Kukavica and V. Vicol, *On the local existence of analytic solutions to the Prandtl boundary layer equations,* Commun. Math. Sci. **11:1** (2013), 269–292.

[61] I. Kukavica and M. Ziane, *The regularity of solutions of the primitive equations of the ocean in space dimension three,* C. R. Math. Acad. Sci. Paris **345** (2007), 257–260.

[62] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean,* Nonlinearity **20** (2007), 2739–2753.

[63] A.C. Kuo and L.M. Polvani, *Time–dependent fully nonlinear geostrophic adjustment,* J. Phys. Oceanogr. **27** (1997), 1614–1634.

[64] C.D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Differ. Equ. **133** (1997), 321–339.

[65] R. Lewandowski, *Analyse Mathématique et Océanographie*, Masson, Paris, 1997.

[66] A. Larios, M. Petersen, E.S. Titi and B. Wingate, *A computational investigation of the finite-time blow-up of the 3D incompressible Euler equations based on the Voigt regularization,* Theor. Comp. Fluid Dyn. **32:1** (2018), 23–34.

[67] A. Larios and E.S. Titi, *On the higher-order global regularity of the inviscid Voigt-regularization of three-dimensional hydrodynamic models,* Discrete Contin. Dyn. Syst. Ser. B **14:2** (2010), 603–627.

[68] J. Li and E. S. Titi, *Recent advances concerning certain class of geophysical flows,* Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer (2018), 933–971.

[69] J. Li and E.S. Titi, *The primitive equations as the small aspect ratio limit of the Navier–Stokes equations: rigorous justification of the hydrostatic approximation,* J. Math. Pures Appl. **124** (2019), 30–58.

[70] W. Li, N. Masmoudi and T. Yang, *Well-posedness in Gevrey function space for $3D$ Prandtl equations without structural assumption,* (2020), preprint, arXiv:2001.10222.

[71] J. L. Lions, R. Temam and S. Wang, *New formulations of the primitive equations of the atmosphere and appliations,* Nonlinearity **5** (1992), 237–288.

[72] J. L. Lions, R. Temam and S. Wang, *On the equations of the large-scale ocean,* Nonlinearity **5** (1992), 1007–1053.

[73] J. L. Lions, R. Temam and S. Wang, *Mathematical study of the coupled models of atmosphere and ocean (CAO III),* J. Math. Pures Appl. **74** (1995), 105–163.

[74] H. Liu and E. Tamdor, *: Rotation prevents finite-time breakdown,* Phys. D **188** (2004), 262–276.

[75] A. J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean (Courant Lecture Notes in Mathematics vol 9),* Providence, RI: American Mathematical Society, 2003.

[76] N. Masmoudi and T. Wong, *On the H^s theory of hydrostatic Euler equations,* Arch. Ration. Mech. Anal. **204:1** (2012), 231–271.

[77] N. Masmoudi and T. Wong, *Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods,* Comm. Pure Appl. Math. **68:10** (2015), 1683–1741.

[78] H. Okamoto, *Well-posedness of the generalized Proudman-Johnson equation without viscosity,* J. Math. Fluid Mech. **11** (2009), 46–59.

[79] O. Oleinik, *On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid,* J. Appl. Math. Mech. **30** (1966), 951–974.

[80] M. Oliver, *"A Mathematical Investigation of Models of Shallow Water with a Varying Bottom,"* Ph.d. dissertation, University of Arizona, Tucson, Arizona, 1996.

[81] J. Pedlosky, *Geophysical Fluid Dynamics,* 2nd edition, Springer, New York, 1987.

[82] R. Plougonven and V. Zeitlin, *Lagrangian approach to the geostrophic adjustment of frontal anomalies in a stratified fluid,* Geophys. Astrophys. Fluid Dyn. **99** (2005), 101–135.

[83] M. Renardy, *Ill-posedness of the hydrostatic Euler and Navier-Stokes equations,* Arch. Ration. Mech. Anal. **194:3** (2009), 877–886.

[84] C.G. Rossby, *The mutual adjustment of pressure and velocity distribution in certain simple current systems, II,* J. Mar. Res. **1** (1938), 239–263.

[85] R. Samelson and G. Vallis, *A simple friction and diffusion scheme for planetary geostrophic basin models,* J. Physical Oceanography **27** (1997), 186–194.

[86] R. Salmon, *Lectures on Geophysical Fluid Dynamics,* Oxford University Press, New York, Oxford, 1998.

[87] J. Simon, *Compact sets in the space* $L^p(0,T;B)$*,* Ann. Mat. Pure Appl. **146** (1987), 65–96.

[88] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, 2003.

[89] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge Univ. Press, 2006.

[90] W. M. Washington and C. L. Parkinson, *An Introduction to Three Dimensional Climate Modeling*, Oxford University Press, Oxford, 1986.

[91] T. K. Wong, *Blowup of solutions of the hydrostatic Euler equations*, Proc. Amer. Math. Soc. **143:3** (2015) 1119–1125.

[92] Q. C. Zeng, *Mathematical and Physical Foundations of Numerical Weather Prediction*, Science Press, Beijing, 1979.

APPENDIX A

ESTIMATES OF NONLINEAR TERMS IN THE SPACE OF ANALYTIC FUNCTIONS

In this appendix, we provide the proof of Lemma 2.2.11–2.2.17. First, we prove Lemma 2.2.11.

Proof of Lemma 2.2.11. First, notice that

$$\left|\left\langle A^r e^{\tau A}(f \cdot \nabla_h g), A^r e^{\tau A} h\right\rangle\right| = \left|\left\langle f \cdot \nabla_h g, A^r e^{\tau A} H\right\rangle\right|, \tag{A.1}$$

where $H = A^r e^{\tau A} h$. We use Fourier representation of f, g and H, in which we can write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^3} \hat{f}_{\boldsymbol{j}} e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{x}}, \tag{A.2}$$

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \hat{g}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.3}$$

$$A^r e^{\tau A} H(\boldsymbol{x}) = \sum_{\boldsymbol{l}\in\mathbb{Z}^3} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} \hat{H}_{\boldsymbol{l}} e^{2\pi i \boldsymbol{l}\cdot\boldsymbol{x}}. \tag{A.4}$$

Therefore,

$$\left|\left\langle f \cdot \nabla_h g, A^r e^{\tau A} H\right\rangle\right| \leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}||\hat{g}_{\boldsymbol{k}}||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{H}_{\boldsymbol{l}}|. \tag{A.5}$$

From $|\boldsymbol{l}| = |\boldsymbol{j}+\boldsymbol{k}| \leq |\boldsymbol{j}| + |\boldsymbol{k}|$ we have the following inequalities:

$$|\boldsymbol{l}|^r \leq (|\boldsymbol{j}| + |\boldsymbol{k}|)^r \leq C_r(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r), \quad e^{\tau|\boldsymbol{l}|} \leq e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|}. \tag{A.6}$$

Applying these inequalities, we have

$$\left|\left\langle f \cdot \nabla_h g, A^r e^{\tau A} H\right\rangle\right| \leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r) e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}|. \tag{A.7}$$

Since $|\boldsymbol{k}|, |\boldsymbol{j}|, |\boldsymbol{l}|$ are all nonnegative, we have $|\boldsymbol{k}|^{1/2} \le (|\boldsymbol{j}| + |\boldsymbol{l}|)^{1/2} \le |\boldsymbol{j}|^{1/2} + |\boldsymbol{l}|^{1/2}$, therefore,

$$\begin{aligned}
&\left|\left\langle f \cdot \nabla_h g, A^r e^{\tau A} H \right\rangle\right| \\
&\le \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}|^{1/2}(|\boldsymbol{j}|^{1/2} + |\boldsymbol{l}|^{1/2})|\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r) e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} |\hat{h}_{\boldsymbol{l}}| \\
&\le \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r \Big(|\boldsymbol{k}|^{1/2} |\boldsymbol{j}|^{r+1/2} |\boldsymbol{l}|^r + |\boldsymbol{k}|^{r+1/2} |\boldsymbol{j}|^{1/2} |\boldsymbol{l}|^r + |\boldsymbol{k}|^{1/2} |\boldsymbol{j}|^r |\boldsymbol{l}|^{r+1/2} \\
&\qquad\qquad + |\boldsymbol{k}|^{r+1/2} |\boldsymbol{l}|^{r+1/2} \Big) \times e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: A_1 + A_2 + A_3 + A_4. \qquad \text{(A.8)}
\end{aligned}$$

Thanks to Cauchy–Schwarz inequality, since $r > 2$, we have

$$\begin{aligned}
A_1 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\boldsymbol{k}|^{1/2} |\boldsymbol{j}|^{r+1/2} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&= C_r \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{1/2} |\hat{g}_{\boldsymbol{k}}| e^{\tau|\boldsymbol{k}|} \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ \boldsymbol{j}\neq 0,-\boldsymbol{k}}} |\boldsymbol{j}|^{r+1/2} e^{\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}| |\boldsymbol{j}+\boldsymbol{k}|^r e^{\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}| \\
&\le C_r \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{1-2r} \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\quad \times \sup_{\boldsymbol{k}\in\mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ \boldsymbol{j}\neq 0,-\boldsymbol{k}}} |\boldsymbol{j}|^{2r+1} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2 \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ \boldsymbol{j}\neq 0,-\boldsymbol{k}}} |\boldsymbol{j}+\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\le C_r \|A^{r+1/2} e^{\tau A} f\| \|A^r e^{\tau A} g\| \|A^r e^{\tau A} h\|, \qquad \text{(A.9)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
A_2 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\boldsymbol{k}|^{r+1/2} |\boldsymbol{j}|^{1/2} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&\le C_r \|A^r e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\| \|A^r e^{\tau A} h\|, \qquad \text{(A.10)}
\end{aligned}$$

and

$$\begin{aligned}
A_3 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\boldsymbol{k}|^{1/2} |\boldsymbol{j}|^r |\boldsymbol{l}|^{r+1/2} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&\le C_r \|A^r e^{\tau A} f\| \|A^r e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|. \qquad \text{(A.11)}
\end{aligned}$$

For A_4, thanks to Cauchy–Schwarz inequality, since $r > 2$, we have

$$\begin{aligned}
A_4 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\boldsymbol{k}|^{r+1/2} |\boldsymbol{l}|^{r+1/2} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{h}_{\boldsymbol{l}}| \\
&= C_r \sum_{\boldsymbol{j}\in\mathbb{Z}^3} e^{\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}} |\boldsymbol{k}|^{r+1/2} |\hat{g}_{\boldsymbol{k}}| e^{\tau|\boldsymbol{k}|} |\boldsymbol{j}+\boldsymbol{k}|^{r+1/2} e^{\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}| \\
&\leq C_r \Big\{ |\hat{f}_0| + \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{-2r} \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2 \Big)^{1/2} \Big\} \\
&\quad \times \sup_{\boldsymbol{j}\in\mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}} |\boldsymbol{k}|^{2r+1} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}} |\boldsymbol{j}+\boldsymbol{k}|^{2r+1} e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\leq C_r (\|A^r e^{\tau A} f\| + |\hat{f}_0|) \|A^{r+1/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|. \qquad \text{(A.12)}
\end{aligned}$$

Combine the estimates for A_1 to A_4, and since $\|A^r e^{\tau A} g\| \leq \|A^{r+1/2} e^{\tau A} g\|$, $\|A^r e^{\tau A} h\| \leq \|A^{r+1/2} e^{\tau A} h\|$, we achieve the desired inequality. □

The proof of Lemma 2.2.12 is almost the same as Lemma 2.2.11, so we omit it. Next, we prove Lemma 2.2.13.

Proof of Lemma 2.2.13. First, one has

$$\Big| \Big\langle A^r e^{\tau A} \big((\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds) \partial_z g \big), A^r e^{\tau A} h \Big\rangle \Big| = \Big| \Big\langle (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds) \partial_z g, A^r e^{\tau A} H \Big\rangle \Big|. \quad \text{(A.13)}$$

Using Fourier representation of f, and noticing that $\overline{f} = 0$, we have

$$f(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0}} \hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}' + i j_3 z)}, \quad \text{(A.14)}$$

where $\boldsymbol{j}' = (j_1, j_2)$. Then we have

$$\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds = \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0, \boldsymbol{j}'\neq 0}} \frac{1}{j_3} \boldsymbol{j}' \cdot \hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}' + i j_3 z)} - \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0, \boldsymbol{j}'\neq 0}} \frac{1}{j_3} \boldsymbol{j}' \cdot \hat{f}_{\boldsymbol{j}} e^{2\pi i \boldsymbol{j}'\cdot\boldsymbol{x}'}. \quad \text{(A.15)}$$

Therefore, we have

$$\Big|\Big\langle(\int_0^z \nabla_h\cdot f(s)ds)\partial_z g, A^r e^{\tau A}H\Big\rangle\Big| \le \Big|\Big\langle(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{1}{j_3}\boldsymbol{j}'\cdot\hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}'+ij_3 z)})\partial_z g, A^r e^{\tau A}H\Big\rangle\Big|$$
$$+\Big|\Big\langle(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{1}{j_3}\boldsymbol{j}'\cdot\hat{f}_{\boldsymbol{j}} e^{i\boldsymbol{j}'\cdot\boldsymbol{x}'})\partial_z g, A^r e^{\tau A}H\Big\rangle\Big| =: I_1+I_2. \tag{A.16}$$

Let us estimate I_2 first. For $\boldsymbol{l}=(\boldsymbol{l}',l_3)=(-\boldsymbol{j}'-\boldsymbol{k}',-k_3)$, by using the inequalities

$$|\boldsymbol{j}'|^{1/2}\le C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}),\quad |\boldsymbol{k}|^{1/2}\le C(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2}),\quad |\boldsymbol{l}|^r\le C_r(|\boldsymbol{j}'|^r+|\boldsymbol{k}|^r), \tag{A.17}$$

we have

$$\begin{aligned} I_2 &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}'||k_3||\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}'|^r+|\boldsymbol{k}|^r)e^{\tau|\boldsymbol{j}'|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\ &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|+|\boldsymbol{j}'||\boldsymbol{k}|^{r+1})e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\ &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}\Big(|\boldsymbol{k}|^{3/2}|\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^r+|\boldsymbol{k}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^{r+1/2}+|\boldsymbol{j}'|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^r \\ &\qquad +|\boldsymbol{j}'||\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}\Big)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: B_1+B_2+B_3+B_4. \end{aligned} \tag{A.18}$$

When $k_3\neq 0$ and $r>2$, we know that $|\boldsymbol{k}|^{1-r}\le|(\boldsymbol{k}',\pm1)|^{1-r}$ and $\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{2-2r}\le C_r$ is finite. Thanks to Cauchy–Schwarz inequality, we have

$$\begin{aligned} B_1 &= \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{k}|^{3/2}|\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\ &= C_r\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{3/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}'|^{r+1/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r \end{aligned}$$

$$
\begin{aligned}
&\quad \times e^{\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}| \\
&\le C_r \sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{1-r} \sum_{k_3\neq 0} |\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2} \\
&\quad \times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2} \\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2} \\
&\quad \times\Big(\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2} \\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{2-2r}\Big)^{1/2} \\
&\quad \times\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2} \\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|. \qquad \text{(A.19)}
\end{aligned}
$$

The estimate for B_2 is similar as B_1, and we can get

$$
B_2 \le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|.
$$

For B_3, thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$
\begin{aligned}
B_3 &= \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}'|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^re^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&= C_r\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}'|^{3/2}|\hat{f}_{\boldsymbol{j}}|e^{\tau|\boldsymbol{j}|}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{r+1/2}e^{\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r \\
&\quad \times e^{\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}| \\
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r+1}|\hat{f}_{\boldsymbol{j}}|^2e^{2\tau|\boldsymbol{j}|}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\quad \times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}
\end{aligned}
$$

$$\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^r e^{\tau A}h\|. \tag{A.20}$$

The estimate for B_4 is similar as B_3, and we can get

$$B_4 \leq C_r\|A^r e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|.$$

The estimates of B_1 to B_4 indicate that I_2 satisfies the desired inequality.

Now let us estimate on I_1. For $\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0$, by using the inequalities

$$|\boldsymbol{j}|^{1/2} \leq C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}), \quad |\boldsymbol{k}|^{1/2} \leq C(|\boldsymbol{j}|^{1/2}+|\boldsymbol{l}|^{1/2}), \quad |\boldsymbol{l}|^r \leq C_r(|\boldsymbol{j}|^r+|\boldsymbol{k}|^r), \tag{A.21}$$

we have

$$\begin{aligned}
I_1 &\leq \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}'||k_3||\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r+|\boldsymbol{k}|^r)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\
&\leq \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^{r+1}|\boldsymbol{k}|+|\boldsymbol{j}||\boldsymbol{k}|^{r+1})e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\
&\leq \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}\Big(|\boldsymbol{k}|^{3/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r+|\boldsymbol{k}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}+|\boldsymbol{j}|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^r \\
&\qquad\qquad +|\boldsymbol{j}||\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}\Big)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: \widetilde{B}_1+\widetilde{B}_2+\widetilde{B}_3+\widetilde{B}_4.
\end{aligned} \tag{A.22}$$

Thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$\begin{aligned}
\widetilde{B}_1 &= \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{k}|^{3/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&= C_r\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{3/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}|^{r+1/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}||\boldsymbol{j}+\boldsymbol{k}|^r e^{\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}| \\
&\leq C_r\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\sum_{k_3\neq 0}|\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\quad\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{2-2r}\Big)^{1/2}\\
&\quad\times\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|, \qquad\qquad \text{(A.23)}
\end{aligned}
$$

where in the second inequality, we use Fubini theorem to exchange the order of $\sum\limits_{j_3\neq 0}$ and $\sum\limits_{k_3\neq 0}$. The estimate for $\widetilde{B}_2$ is similar to $\widetilde{B}_1$, and we can get $\widetilde{B}_2\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|$.

For $\widetilde{B}_3$, thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$
\begin{aligned}
\widetilde{B}_3 &= \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^re^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}|\\
&= C_r\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}|^{3/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}|\boldsymbol{j}+\boldsymbol{k}|^re^{\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|\\
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r+1}|\hat{f}_{\boldsymbol{j}}|^2e^{2\tau|\boldsymbol{j}|}\Big)^{1/2}\\
&\quad\times\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{j}+\boldsymbol{k}|^{2r}e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|, \qquad\qquad \text{(A.24)}
\end{aligned}
$$

where in the first inequality we use $|\boldsymbol{j}|^{2-2r}\le|\boldsymbol{j}'|^{2-2r}$ due to $r>2$. The estimate for $\widetilde{B}_4$ is similar as $\widetilde{B}_3$, and we can get $\widetilde{B}_4\le C_r\|A^re^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|$. The estimates of $\widetilde{B}_1$ to $\widetilde{B}_4$ indicate that I_1 satisfies the desired inequality. □

The proof of Lemma 2.2.14 is similarly to that of Lemma 8 in [64] since it involves nonlinear term similar to that appearing in the Euler equations. The proof of Lemma 2.2.15 is similarly to that of Lemma 2.2.14. Therefore, they are omitted.

The proof of Lemma 2.2.16 is similar to that of Lemma 2.2.17, so we first focus below on the proof of Lemma 2.2.17, and later we sketch the proof of Lemma 2.2.16 with emphasis on the main differences.

Proof of Lemma 2.2.17. First, denote by $H = A^r e^{\tau A} h$, and let

$$\begin{aligned} I :=& \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle \\ &\qquad - \Big\langle \partial_z g A^r e^{\tau A} (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds), A^r e^{\tau A} h\Big\rangle\Big| \\ =& \Big|\Big\langle (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds)\partial_z g, A^r e^{\tau A} H\Big\rangle - \Big\langle \partial_z g A^r e^{\tau A} (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds), H\Big\rangle\Big|. \end{aligned} \tag{A.25}$$

Similar as in the proof of Lemma 2.2.13, using Fourier representation of f, since $\overline{f} = 0$, we have

$$\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds = \sum_{\substack{\boldsymbol{j} \in \mathbb{Z}^3 \\ j_3 \neq 0}} \frac{1}{j_3} \boldsymbol{j}' \cdot \hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot \boldsymbol{x}' + i j_3 z)} - \sum_{\substack{\boldsymbol{j} \in \mathbb{Z}^3 \\ j_3 \neq 0}} \frac{1}{j_3} \boldsymbol{j}' \cdot \hat{f}_{\boldsymbol{j}} e^{2\pi i \boldsymbol{j}' \cdot \boldsymbol{x}'}, \tag{A.26}$$

where $\boldsymbol{j}' = (j_1, j_2)$. Using Fourier representation of g and H, we have

$$\begin{aligned} I \leq& C \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{H}_{\boldsymbol{l}}| |\boldsymbol{j}'| |\boldsymbol{k}| \Big| |\boldsymbol{l}|^r e^{\tau |\boldsymbol{l}|} - |\boldsymbol{j}|^r e^{\tau |\boldsymbol{j}|} \Big| \\ &+ C \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3 + l_3 = 0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{H}_{\boldsymbol{l}}| |\boldsymbol{j}'| |\boldsymbol{k}| \Big| |\boldsymbol{l}|^r e^{\tau |\boldsymbol{l}|} - |(\boldsymbol{j}', 0)|^r e^{\tau |(\boldsymbol{j}', 0)|} \Big| := I_1 + I_2. \end{aligned} \tag{A.27}$$

We estimate I_2 first. By virtue of the following observation [64]:

For $r \geq 1$ and $\tau \geq 0$, and for all positive $\xi, \eta \in \mathbb{R}$, we have

$$|\xi^r e^{\tau\xi} - \eta^r e^{\tau\eta}| \leq C_r |\xi - \eta| \Big(|\xi - \eta|^{r-1} + \eta^{r-1} + \tau(|\xi - \eta|^r + \eta^r) e^{\tau|\xi - \eta|} e^{\tau\eta} \Big); \tag{A.28}$$

with $\xi = |\boldsymbol{l}|$, $\eta = |(\boldsymbol{j}', 0)| = |\boldsymbol{j}'|$, and $|\xi - \eta| = \Big||\boldsymbol{l}| - |(\boldsymbol{j}', 0)|\Big| \leq \Big| - \boldsymbol{l} - (\boldsymbol{j}', 0)\Big| = |\boldsymbol{k}|$, inequality (A.28) implies

$$I_2 \leq C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 \Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}\Big). \quad \text{(A.29)}$$

By the definition of H, and since $e^x \leq 1 + xe^x$ for any $x \geq 0$, we have

$$|\hat{H}_{\boldsymbol{l}}| = |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} |\hat{h}_{\boldsymbol{l}}| \leq |\boldsymbol{l}|^r (1 + \tau|\boldsymbol{l}| e^{\tau|\boldsymbol{l}|}) |\hat{h}_{\boldsymbol{l}}| \leq |\boldsymbol{l}|^r |\hat{h}_{\boldsymbol{l}}| + \tau(|\boldsymbol{j}'| + |\boldsymbol{k}|)|\hat{H}_{\boldsymbol{l}}|. \quad \text{(A.30)}$$

Therefore, one obtains that

$$\begin{aligned} &|\hat{H}_{\boldsymbol{l}}|\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}\Big) \\ &\leq \Big(|\boldsymbol{l}|^r |\hat{h}_{\boldsymbol{l}}| + \tau(|\boldsymbol{j}'| + |\boldsymbol{k}|)|\hat{H}_{\boldsymbol{l}}|\Big)\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1}\Big) + |\hat{H}_{\boldsymbol{l}}|\Big(\tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}\Big) \\ &\leq |\hat{h}_{\boldsymbol{l}}||\boldsymbol{l}|^r (|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1}) + \tau C_r |\hat{H}_{\boldsymbol{l}}|(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}. \end{aligned} \quad \text{(A.31)}$$

Based on this, one has

$$\begin{aligned} I_2 \leq{}& C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 |\boldsymbol{l}|^r (|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1}) \\ &+ \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 (|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} := I_{21} + I_{22}. \end{aligned} \quad \text{(A.32)}$$

Here

$$\begin{aligned} I_{21} ={}& C_r \Big(\sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+1} |\boldsymbol{l}|^r + \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'|^r |\boldsymbol{k}|^2 |\boldsymbol{l}|^r \Big) \\ :={}& I_{211} + I_{212}. \end{aligned} \quad \text{(A.33)}$$

Thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$
\begin{aligned}
I_{211} &= C_r \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}'||\hat{f}_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{r+1}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r|\hat{g}_{\boldsymbol{k}}||\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}| \\
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\quad \times \sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2} \\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|, \qquad \text{(A.34)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{212} &= C_r \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^2|\hat{g}_{\boldsymbol{k}}| \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}'|^r|\hat{f}_{\boldsymbol{j}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}| \\
&\le C_r \sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{1-r}\sum_{k_3\neq 0}|\boldsymbol{k}|^{r+1}|\hat{g}_{\boldsymbol{k}}|\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2} \\
&\quad \times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2} \\
&\le C_r\|A^r f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\quad \times\Big(\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2} \\
&\le C_r\|A^r f\|\|A^r h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{2-2r}\Big)^{1/2}\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|. \qquad \text{(A.35)}
\end{aligned}
$$

Next, for I_{22}, we have

$$
I_{22} = \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}
$$

$$+\tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|^2 e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|} := I_{221}+I_{222}. \tag{A.36}$$

Noticing that $|\boldsymbol{k}|^{1/2} \leq C(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2})$ and $|\boldsymbol{j}'|^{1/2} \leq C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2})$, thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$\begin{aligned}
I_{221} &= \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+2} e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{l}|^r|\boldsymbol{k}|^{r+3/2}(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2}) e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'|^{3/2}|\boldsymbol{l}|^{r+1/2}|\boldsymbol{k}|^{r+3/2} e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{2r+3}|e^{2\tau|\boldsymbol{k}|}\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\qquad \times \sup_{\boldsymbol{j}\in\mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\leq \tau C_r \|A^{r+1/2}e^{\tau A}f\|\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|,
\end{aligned} \tag{A.37}$$

and

$$\begin{aligned}
I_{222} &= \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|^2 e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{k}|^2|\boldsymbol{l}|^r(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}) e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{k}|^{5/2}|\boldsymbol{l}|^{r+1/2} e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}
\end{aligned}$$

$$
\begin{aligned}
&\le \tau C_r \sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{1-r} \sum_{k_3\neq 0} |\boldsymbol{k}|^{r+3/2} e^{\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}| \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r+1} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{j_3\neq 0} \frac{1}{|j_3|^2} \sum_{\boldsymbol{j}'\in\mathbb{Z}^2} |(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1} e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|} |\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2 \Big)^{1/2} \\
&\le \tau C_r \|A^{r+1/2} e^{\tau A} f\| \sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{1-r} \Big(\sum_{k_3\neq 0} |\boldsymbol{k}|^{2r+3} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{k_3\neq 0} \sum_{\boldsymbol{j}'\in\mathbb{Z}^2} |(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1} e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|} |\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2 \Big)^{1/2} \\
&\le \tau C_r \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} h\| \Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{2-2r} \Big)^{1/2} \\
&\qquad\qquad \times \Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2} \sum_{k_3\neq 0} |\boldsymbol{k}|^{2r+3} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\le \tau C_r \|A^{r+1/2} e^{\tau A} f\| \|A^{r+3/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|. \qquad \text{(A.38)}
\end{aligned}
$$

Therefore, I_2 satisfies the desired estimates.

To estimate I_1, we use (A.28) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{j}|$, and with $|\xi - \eta| = \Big||\boldsymbol{l}| - |\boldsymbol{j}|\Big| \le |-\boldsymbol{l} - \boldsymbol{j}| = |\boldsymbol{k}|$, to obtain

$$
I_1 \le C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{H}_{\boldsymbol{l}}| |\boldsymbol{j}'| |\boldsymbol{k}|^2 \Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}|^{r-1} + \tau (|\boldsymbol{k}|^r + |\boldsymbol{j}|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \Big). \qquad \text{(A.39)}
$$

Thanks to (A.31), one obtains that

$$
\begin{aligned}
I_1 &\le C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{h}_{\boldsymbol{l}}| |\boldsymbol{j}'| |\boldsymbol{k}|^2 |\boldsymbol{l}|^r (|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}|^{r-1}) \\
&\quad + \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{H}_{\boldsymbol{l}}| |\boldsymbol{j}'| |\boldsymbol{k}|^2 (|\boldsymbol{k}|^r + |\boldsymbol{j}|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} := I_{11} + I_{12}. \qquad \text{(A.40)}
\end{aligned}
$$

Here

$$
I_{11} \le C_r \Big(\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{h}_{\boldsymbol{l}}| |\boldsymbol{j}| |\boldsymbol{k}|^{r+1} |\boldsymbol{l}|^r + \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}| |\hat{g}_{\boldsymbol{k}}| |\hat{h}_{\boldsymbol{l}}| |\boldsymbol{j}|^r |\boldsymbol{k}|^2 |\boldsymbol{l}|^r \Big)
$$

$$:= I_{111} + I_{112}. \tag{A.41}$$

Thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$\begin{aligned}
I_{111} &= C_r \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}||\hat{f}_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{r+1}|\boldsymbol{j}+\boldsymbol{k}|^r|\hat{g}_{\boldsymbol{k}}||\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}| \\
&\leq C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\quad\times \sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r}|\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2} \\
&\leq C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|,
\end{aligned} \tag{A.42}$$

and

$$\begin{aligned}
I_{112} &= C_r \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^2|\hat{g}_{\boldsymbol{k}}| \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}|^r|\hat{f}_{\boldsymbol{j}}||\boldsymbol{j}+\boldsymbol{k}|^r|\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}| \\
&\leq C_r\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2} \\
&\quad\times \sup_{\boldsymbol{k}\in\mathbb{Z}^3}\Big(\sum_{\boldsymbol{j}\in\mathbb{Z}^3} |\boldsymbol{j}+\boldsymbol{k}|^{2r}|\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2} \\
&\leq C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|.
\end{aligned} \tag{A.43}$$

Next, for I_{12}, we have

$$\begin{aligned}
I_{12} &\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|} \\
&\quad+\tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1}|\boldsymbol{k}|^2e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|} := I_{121} + I_{122}.
\end{aligned} \tag{A.44}$$

Since $|\boldsymbol{k}|^{1/2} \leq C(|\boldsymbol{j}|^{1/2} + |\boldsymbol{l}|^{1/2})$ and $|\boldsymbol{j}|^{1/2} \leq C(|\boldsymbol{k}|^{1/2} + |\boldsymbol{l}|^{1/2})$, thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$\begin{aligned}
I_{121} &= \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{k}|^{r+2} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{l}|^{r}|\boldsymbol{k}|^{r+3/2}(|\boldsymbol{j}|^{1/2}+|\boldsymbol{l}|^{1/2}) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{l}|} \\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{3/2}|\boldsymbol{l}|^{r+1/2}|\boldsymbol{k}|^{r+3/2} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{l}|} \\
&\leq \tau C_r \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2-2r} \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r+1} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2 \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ k_3\neq 0}} |\boldsymbol{k}|^{2r+3} |e^{2\tau|\boldsymbol{k}|} \hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \\
&\qquad \times \sup_{\boldsymbol{j}\in\mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ k_3\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r+1} e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2 \Big)^{1/2} \\
&\leq \tau C_r \|A^{r+1/2} e^{\tau A} f\| \|A^{r+3/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|, \qquad \text{(A.45)}
\end{aligned}$$

and

$$\begin{aligned}
I_{122} &= \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1}|\boldsymbol{k}|^{2} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{k}|^{2}|\boldsymbol{l}|^{r}(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{l}|} \\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{k}|^{5/2}|\boldsymbol{l}|^{r+1/2} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{l}|} \\
&\leq \tau C_r \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2-2r} \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2r+3} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2 \Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r+1} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2 \Big)^{1/2} \\
&\qquad \times \sup_{\boldsymbol{k}\in\mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r+1} e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2 \Big)^{1/2} \\
&\leq \tau C_r \|A^{r+1/2} e^{\tau A} f\| \|A^{r+3/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|. \qquad \text{(A.46)}
\end{aligned}$$

Therefore, I_1 satisfies the desired estimates. The proof is completed. □

Finally, we sketch the proof of Lemma 2.2.16.

Proof of Lemma 2.2.16. Similar as the proof of Lemma 2.2.17, we have

$$
\begin{aligned}
I := &\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle \\
&\qquad - \Big\langle (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds) A^r e^{\tau A}\partial_z g, A^r e^{\tau A} h\Big\rangle\Big| \\
= &\Big|\Big\langle (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds)\partial_z g, A^r e^{\tau A} H\Big\rangle - \Big\langle (\int_0^z \nabla_h \cdot f(\boldsymbol{x}', s) ds) A^r e^{\tau A}\partial_z g, H\Big\rangle\Big|. \\
\leq &C \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|\Big||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |\boldsymbol{k}|^r e^{\tau|\boldsymbol{k}|}\Big| \\
&+ C \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|\Big||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |\boldsymbol{k}|^r e^{\tau|\boldsymbol{k}|}\Big| := I_1 + I_2. \qquad \text{(A.47)}
\end{aligned}
$$

For I_1, since $\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0$, we use (A.28) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{k}|$ and

$$
|\xi - \eta| = \Big||\boldsymbol{l}| - |\boldsymbol{k}|\Big| \leq \Big| - \boldsymbol{l} - \boldsymbol{k}\Big| = |\boldsymbol{j}|,
$$

to conclude

$$
I_1 \leq C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{j}||\boldsymbol{k}|\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}\Big). \qquad \text{(A.48)}
$$

For I_2, since $(\boldsymbol{j}', 0) + \boldsymbol{k} + \boldsymbol{l} = 0$, we use (A.28) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{k}|$ and

$$
|\xi - \eta| = \Big||\boldsymbol{l}| - |\boldsymbol{k}|\Big| \leq \Big| - \boldsymbol{l} - \boldsymbol{k}\Big| = |\boldsymbol{j}'|,
$$

to obtain

$$
I_2 \leq C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3, k_3, \boldsymbol{j}' \neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^2|\boldsymbol{k}|\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}\Big). \qquad \text{(A.49)}
$$

Observe that the difference between the sums in the right-hand sides of (A.48) and (A.39) is manifested in the factors $|\boldsymbol{j}'||\boldsymbol{j}||\boldsymbol{k}|$ and $|\boldsymbol{j}'||\boldsymbol{k}|^2$, and between (A.49) and (A.29) is manifested in the factors $|\boldsymbol{j}'|^2|\boldsymbol{k}|$ and $|\boldsymbol{j}'||\boldsymbol{k}|^2$. Therefore, one can follow the estimates of I_1 in (A.39) and I_2 in (A.29), and obtain that I_1 in (A.48) and I_2 in (A.49) satisfy the desired bound in Lemma 2.2.16.

□