FREE DECOMPOSITIONS OF R-DIAGONAL RANDOM VARIABLES

A Dissertation

by

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ABSTRACT

Basic notions for $*$-noncommutative probability spaces and B-valued $*$-noncommutative probability spaces, including Voiculescu's free independence and Speicher's cumulants, are recalled. In both the scalar and more generally, the algebra-valued setting, R-diagonal random variables are defined and we recall some results regarding their decompositions into products of a Haar unitary and a self adjoint element that are $*$-free from one another. Various classes of B-valued Haar unitaries are compared and contrasted with several distinguishing examples. Decompositions of the particular case of $\mathbb{C}^2$-valued circular elements are investigated more thoroughly with computational methods, resulting in a proof that every tracial $\mathbb{C}^2$-valued circular having a free decomposition with a Haar unitary must have a free even decomposition with a normalizing Haar unitary.

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1. INTRODUCTION

Dan Voiculescu started the field of free probability in the 1980s with an effort to solve the free group factors isomorphism problem, which remains an important unsolved problem in operator algebras. Since its introduction, connections have been found between free probability and several other fields, including random matrix theory, combinatorics, and quantum information theory.

Non-commutative random variables are elements of an algebra A over the complex numbers. A noncommutative probability space may have extra structure; for example, it may be a $*$-algebra, a C^*-algebra, or a von Neumann algebra. In the scalar-valued setting, this algebra will be equipped with an expectation functional $\varphi : A \to \mathbb{C}$. One can extend this to a more general algebra-valued setting, where A now contains a unital algebra B and the expectation functional is replaced by a conditional expectation $E : A \to B$. In both situations there is a relationship between random variables known as free independence, which is a natural analogue to the classical notion of independence and has a connection to the algebraic free product.

The free cumulants, introduced by Roland Speicher, give combinatorial tools that have become very important in the theory of free probability. These cumulants can be used to easily define R-diagonal random variables, and in particular, the circular random variables. These random variables are a class of (generally) non-normal random variables that we are most interested in studying. Speicher showed in the scalar-valued setting that a noncommutative random variable a in a tracial C^*-noncommutative probability space is R-diagonal if and only if it has the same $*$-distribution as a product up, where u is a Haar unitary, p is positive, and u and p are $*$-free. In other words, a is tracial R-diagonal if and only if it has a free polar decomposition.

Research into analogues of this result in the B-valued setting have found (see [1]) that if a is a special type of R-diagonal element, then it has the same B-valued $*$-distribution as a product us, where s is even (i.e., self adjoint with vanishing odd moments), u is a Haar unitary, and u and s are $*$-free over B. Moreover, in this case, the Haar unitary u is found to normalize B; i.e., both $uBu^* \subseteq B$ and $u^*Bu \subseteq B$. Thus we have a class of R-diagonal elements that admit a free even decomposition with a normalizing Haar unitary. This work by March Boedihardjo and Ken Dykema also includes an example of a tracial B-valued circular random variable that does not have a free polar decomposition, so the scalar result does not extend to the B-valued situation.

We pick up where this work left off in an effort to answer the question: Are there any tracial B-valued circular random variables having a free even decomposition where the Haar unitary u does not normalize B? We investigate some theoretical aspects of this problem and use computational methods to explore a special case for the algebra B.

Chapter 2 is an overview of the scalar setting. In Section 2.1, the basic notions are defined with some detailed examples of noncommutative $*$-probability spaces, $*$-freeness, and cumulants. Section 2.2 focuses on the scalar R-diagonal random variables. These elements are introduced as a generalization of the Haar unitaries, and we ultimately describe the theorem that every tracial R-diagonal has a free polar decomposition. This theorem motivates the remainder of the dissertation.

Chapter 3 mirrors the style of Chapter 2 for the B-valued setting. Section 3.1 introduces the basic definitions, and in Section 3.2, we give the definitions and some basic properties for Haar unitary, R-diagonal, and circular elements in B-valued noncommutative $*$-probabiltiy spaces. We also include the example by Boediharjo and Dykema showing that some tracial R-diagonals may not have a free polar decomposition outside the scalar setting.

Chapter 4 begins with an investigation into the nuances that B-valued Haar unitaries possess compared to their scalar counterparts. We define in Section 4.1 several different classes of B-valued Haar unitaries and study the relationship between these classes. Section 4.2 recalls the theorem from [1] regarding free even decompositions with normalizing Haar unitaries of B-valued R-diagonal elements whose cumulants satisfy an automorphism condition. We include an example showing that, despite some extra assumptions, we still cannot have any extensions to a free polar decomposition. Finally, we end with Section 4.3 which dives into the particular case $B = \mathbb{C}^2$. Using computational methods, we show that every $\mathbb{C}^2$-valued circular element with a free decomposition containing a Haar unitary must have a free even decomposition with a normalizing Haar unitary.

2. $*$-NONCOMMUTATIVE PROBABILITY SPACES

2.1 $*$-noncommutative random variables, $*$-moments, freeness, and $*$-cumulants

Definition 2.1.1. A *$*$-noncommutative probability space* is a pair (A, φ), where A is a unital $*$-algebra over the complex numbers and φ is a unital positive linear functional on A.

Furthermore, if A is a C*-algebra (resp. von Neumann algebra), then we say (A, φ) is a *C*-noncommutative probability space* (resp. *W*-noncommutative probability space*). If φ is a trace, then (A, φ) is said to be a *tracial*.

The mapping φ is called an *expectation* and elements of A are called *random variables*.

We explore some examples of $*$-noncommutative probability spaces.

Example 2.1.2.

(1) Let $(X, \mathcal{M}, P)$ be a classical probability space; that is, X is a set, $\mathcal{M}$ is a σ-algebra of measurable subsets of X, and $P : \mathcal{M} \to [0, 1]$ is a probability measure. Fix $A = L^\infty(X, P)$ to be the set of all essentially bounded measurable complex-valued functions on X. Define the adjoint $* : A \to A$ by $f^*(x) = \overline{f(x)}$, and define the expectation $\varphi : A \to \mathbb{C}$ by

$$\varphi(f) = \int_X f\, dP.$$

Then (A, φ) is a $*$-noncommutative probability space. Despite this terminology, the algebra A is commutative.

(2) Let G be a group. Define the group algebra $\mathbb{C}G$ to be the complex vector space with basis G. This algebra can be viewed as the collection of all linear combinations

$$\sum_{g \in G} \alpha_g g,$$

where $\alpha_g \in \mathbb{C}$ for each $g \in G$ and $\alpha_g = 0$ for all but finitely many $g \in G$. We can define multiplication and the $*$-operation on $\mathbb{C}G$ by

$$\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right)=\sum_{g,h}\alpha_g\beta_h(gh)=\sum_{k\in G}\left(\sum_{g,h\colon gh=k}\alpha_g\beta_h\right)k$$

and

$$\left(\sum_{g\in G}\alpha_g g\right)^*=\sum_{g\in G}\overline{\alpha_g}g^{-1}.$$

Define the expectation $\tau_G:\mathbb{C}G\to\mathbb{C}$ by

$$\tau_G\left(\sum_{g\in G}\alpha_g g\right)=\alpha_e,$$

where e is the group identity of G. The map τ_G is called the *canonical trace* on $\mathbb{C}G$. It is indeed a trace, due to

$$\begin{aligned}\tau_G\left(\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right)\right)&=\sum_{g,h\colon gh=e}\alpha_g\beta_h\\&=\sum_{h,g\colon hg=e}\beta_h\alpha_g\\&=\tau_G\left(\left(\sum_{h\in G}\beta_h h\right)\left(\sum_{g\in G}\alpha_g g\right)\right).\end{aligned}$$

It is also faithful because

$$0=\tau_G\left(\left(\sum_{g\in G}\alpha_g g\right)^*\left(\sum_{h\in G}\alpha_h h\right)\right)=\sum_{g,h\colon g^{-1}h=e}\overline{\alpha_g}\alpha_h=\sum_{g\in G}|\alpha_g|^2$$

if and only if $\alpha_g=0$ for all $g\in G$. Thus $(\mathbb{C}G,\tau_G)$ is a tracial $*$-noncommutative probability space with a faithful expectation.

(3) Let (A,φ) be a $*$-noncommutative probability space, and define $M_n(A)$ to be the set of $n\times n$

matrices with entries in A. Define $\Phi : M_n(A) \to \mathbb{C}$ by

$$\Phi\left([a_{i,j}]_{1\leq i,j\leq n}\right) = \frac{1}{n}\sum_{i=1}^{n}\varphi(a_{ii}).$$

Since

$$\Phi([a_{i,j}]_{i,j}^*[a_{i,j}]_{i,j}) = \Phi([a_{j,i}^*]_{i,j}[a_{i,j}]_{i,j}) = \Phi\left(\left[\sum_{k=1}^{n} a_{k,i}^* a_{k,j}\right]_{i,j}\right) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\varphi(|a_{k,i}|^2),$$

the positivity of φ implies that of Φ. Therefore $(M_n(A), \Phi)$ is a $*$-noncommutative probability space. Moreover, the above computation shows that if φ is faithful, then Φ is too. If φ is a trace, a similar calcuation

$$\Phi([a_{i,j}]_{i,j}[b_{i,j}]_{i,j}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\varphi(a_{i,k}b_{k,i}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\varphi(b_{i,k}a_{k,i}) = \Phi([b_{i,j}]_{i,j}[a_{i,j}]_{i,j})$$

reveals that Φ is also tracial.

For the rest of the section, we fix a $*$-noncommutative probability space (A, φ).

Definition 2.1.3. A *joint moment* or *moment* of a family $(a_i)_{i\in I}$ of random variables is a number of the form

$$\varphi(a_{\varepsilon(1)}a_{\varepsilon(2)}\cdots a_{\varepsilon(n)}),$$

where $n \in \mathbb{N}$ and $\varepsilon \in I^n$. A $*$-*moment* of a single random variable a is a joint moment of the pair (a, a^*).

Let $\mathbb{C}\langle (X_i)_{i\in I}\rangle$ denote the unital algebra over $\mathbb{C}$ freely generated by the noncommuting indeterminates $(X_i)_{i\in I}$. The *joint distribution* or *distribution* of a family $(a_i)_{i\in I}$ of random variables is the mapping $\Theta : \mathbb{C}\langle (X_i)_{i\in I}\rangle \to \mathbb{C}$ defined by extending

$$\Theta(X_{\varepsilon(1)}\cdots X_{\varepsilon(n)}) = \varphi(a_{\varepsilon(1)}\cdots a_{\varepsilon(n)})$$

linearly. The $*$-*distribution* of a single random variable a is the joint distribution of the pair (a, a^*).

The $*$-distribution of a random variable is an organized collection of all its $*$-moments. Informally we think of the $*$-distribution as an object containing all of the probabilistic information about the random variable. This idea is reinforced by the following theorem, which is a variation of Theorem 4.10 from [4].

Theorem 2.1.4 ([4], Theorem 4.10)**.** *Let (A, φ) and (B, ψ) be $*$-noncommutative probability spaces such that φ and ψ are faithful, and suppose $a \in A$ and $b \in B$ such that $A = \mathrm{Alg}(a)$ (i.e., A is the unital $*$-algebra generated by a) and $B = \mathrm{Alg}(b)$. Then a and b have the same $*$-distribution if and only if there is a unital $*$-isomorphism $\Phi : A \to B$ uniquely determined by $\Phi(a) = b$ that is also a $*$-isomorphism of (A, φ) and (B, ψ); namely, $\psi \circ \Phi = \varphi$.*

Proof. The converse follows from linearity and

$$\varphi(a^{\varepsilon(1)} \cdots a^{\varepsilon(n)}) = \psi(\Phi(a^{\varepsilon(1)} \cdots a^{\varepsilon(n)})) = \psi(\Phi(a)^{\varepsilon(1)} \cdots \Phi(a)^{\varepsilon(n)}) = \psi(b^{\varepsilon(1)} \cdots b^{\varepsilon(n)}),$$

which holds for every $n \in \mathbb{N}$ and $\varepsilon \in \{1, *\}^n$.

Suppose a and b have the same $*$-distribution. Given $P, Q \in \mathbb{C}\langle X, X^* \rangle$, we use the faithfulness of φ and ψ to obtain

$$\begin{aligned} P(a) = Q(a) &\iff \varphi((P(a) - Q(a))^*(P(a) - Q(a))) \\ &\iff \psi((P(b) - Q(b))^*(P(b) - Q(b))) \\ &\iff P(b) = Q(b). \end{aligned}$$

Since $A = \mathrm{Alg}(a) = \{P(a) \mid P \in \mathbb{C}\langle X, X^* \rangle\}$, we can define the map $\Phi : A \to B$ by $\Phi(P(a)) = P(b)$ for every $P \in \mathbb{C}\langle X, X^* \rangle$. By the equivalences above, Φ is well-defined and injective. It is surjective because $B = \mathrm{Alg}(b) = \{P(b) \mid P \in \mathbb{C}\langle X, X^* \rangle\}$. The definition of Φ immediately makes it a unital $*$-homomorphism satisfying $\Phi(a) = b$. Finally $\psi \circ \Phi = \varphi$ because a and b have the same $*$-distribution. Indeed, given $P \in \mathbb{C}\langle X, X^* \rangle$, we have

$$\psi(\Phi(P(a))) = \psi(P(b)) = \varphi(P(a)). \qquad \square$$

Analogous theorems in the contexts of C*-noncommutative probability spaces (see [4], Theorem 4.11) and W*-noncommutative probability spaces also hold.

Those that have studied classical probability will recall that a fundamental property of the field is the notion of independence. We state Voiculescu's definition of free independence, which is an analogue of classical independence in our noncommutative setting.

Definition 2.1.5. Let $(A_i)_{i \in I}$ be a family of unital $*$-subalgebras of A. This family of subalgebras is said to be *$*$-freely independent with respect to φ*, or simply *$*$-free*, if

$$\varphi(a_1 \cdots a_n) = 0$$

whenever the following properties are satisfied:

(a) n is a positive integer;

(b) $\varepsilon \in I^n$ so that $a_i \in A_{\varepsilon(i)}$ for each $1 \leq i \leq n$;

(c) Each a_i is *centered*; namely, $\varphi(a_i) = 0$ for all $1 \leq i \leq n$;

(d) Neighboring elements are from different subalgebras; i.e., $\varepsilon(1) \neq \varepsilon(2), \ldots, \varepsilon(n-1) \neq \varepsilon(n)$.

A family of subsets $(X_i)_{i \in I}$ is *$*$-free* if the corresponding family of unital $*$-subalgebras $(\mathrm{Alg}(X_i))_{i \in I}$ generated by $(X_i)_{i \in I}$is $*$-free.

Since we have chosen to only consider algebras with the $*$-operation, we have chosen to only define $*$-freeness. One could easily define usual free independence by not requiring that the family of unital subalgebras all be $*$-subalgebras. We won't need this version of freeness, but we mention it anyway to justify the appearance of the $*$ in the terminology. Any mention of freeness in the rest of this chapter refers to $*$-freeness.

We will frequently discuss the $*$-freeness between random variables or random variables and sets by replacing each random variable by its singleton set. For example, we'll say that a random variable a and a subset X of A are $*$-free if $\{a\}$ and X are $*$-free.

The definition of $*$-freeness looks a bit artificial and hard to utilize. Using ideas from Example 2.1.7 below, it's not hard to see that if $(A_i)_{i\in I}$ is a family of $*$-free $*$-subalgebras of (A, φ), then the map φ on the $*$-subalgebra generated by the family $(A_i)_{i\in I}$ is uniquely determined by its behavior on the individual $*$-subalgebras. See Proposition 3.1.6 below for a proof of this fact in the more general algebra-valued setting.

Moreover, $*$-freeness has a natural connection with the algebraic free product. Given a family $((A_i, \varphi_i))_{i\in I}$ of $*$-noncommutative probability spaces, there is a positive functional $\varphi = *_{i\in I}\varphi_i$ on the algebraic free product $A = *_{i\in I}A_i$. This construction makes (A, φ) into a $*$-noncommutative probability space containing the $*$-free family $(A_i)_{i\in I}$ of $*$-subalgebras. We discuss (a generalization of) this free product construction more precisely in the algebra-valued section (see Theorem 3.1.8 below).

One part of the definition of $*$-freeness that seems very restrictive is that all of the random variables must be centered. The following definition will be very important to alleviate this problem.

Definition 2.1.6. Given a random variable a, define its *centering* $\mathring{a} = a^\circ$ by $a - \varphi(a)1_A$.

Every random variable can be written as the sum of its centering and a scalar: $a = \mathring{a} + \varphi(a)1_A$. This is a trick that we'll use repeatedly.

Example 2.1.7. In this example we will provide some concrete calculations using $*$-freeness to decompose mixed moments into moments of the individual variables. Fix random variables a, b that are $*$-free.

(1) We write $ab = (\mathring{a} + \varphi(a)1_A)(\mathring{b} + \varphi(b)1_A)$. By freeness and the fact $\mathring{a}, \mathring{b} \in \ker(\varphi)$, we have

$$\varphi(ab) = \varphi(\mathring{a}\mathring{b}) + \varphi(\mathring{a})\varphi(b) + \varphi(a)\varphi(\mathring{b}) + \varphi(a)\varphi(b) = \varphi(a)\varphi(b).$$

(2) We use the same method to write aba^* as a sum of eight terms. We deduce

$$\varphi(aba^*) = \varphi(b)\varphi(\mathring{a}\mathring{a}^*) + \varphi(a)\varphi(b)\varphi(a^*).$$

Unpacking $\mathring{a}\mathring{a}^* = (a - \varphi(a)1_A)(a^* - \varphi(a^*)1_A)$ yields

$$\begin{aligned}\varphi(aba^*) &= \varphi(b)\varphi(\mathring{a}\mathring{a}^*) + \varphi(a)\varphi(b)\varphi(a^*)\\ &= \varphi(b)\varphi(aa^*).\end{aligned}$$

(3) These two examples may mislead one into thinking that $*$-freeness implies some multipicative property for the expectation. We will see this is not the case. Consider a^*bab^*, which can be written as a sum of sixteen terms by employing the method using centerings. Most of these will vanish by one application of $*$-freeness and the definition of centerings, giving

$$\begin{aligned}\varphi(a^*bab^*) = \varphi(a)\varphi(\mathring{a}^*\mathring{b}\mathring{b}^*) + \varphi(b)\varphi(\mathring{a}^*\mathring{a}\mathring{b}^*) + \varphi(b)\varphi(b^*)\varphi(\mathring{a}^*\mathring{a})\\ + \varphi(a^*)\varphi(a)\varphi(\mathring{b}\mathring{b}^*) + \varphi(a^*)\varphi(b)\varphi(a)\varphi(b^*).\end{aligned}$$

By applying centering again, we get $\varphi(\mathring{a}^*\mathring{b}\mathring{b}^*) = \varphi(\mathring{a}^*(\mathring{b}\mathring{b}^*)^\circ) + \varphi(\mathring{a}^*)\varphi(\mathring{b}\mathring{b}^*) = 0$ and similarly $\varphi(\mathring{a}^*\mathring{a}\mathring{b}^*) = 0$. After unpacking $\mathring{a}^*\mathring{a}$ and $\mathring{b}\mathring{b}^*$, we obtain

$$\begin{aligned}\varphi(a^*bab^*) &= \varphi(a)\varphi(\mathring{a}^*\mathring{b}\mathring{b}^*) + \varphi(b)\varphi(\mathring{a}^*\mathring{a}\mathring{b}^*) + \varphi(b)\varphi(b^*)\varphi(\mathring{a}^*\mathring{a})\\ &\quad + \varphi(a^*)\varphi(a)\varphi(\mathring{b}\mathring{b}^*) + \varphi(a^*)\varphi(b)\varphi(a)\varphi(b^*)\\ &= \varphi(b)\varphi(b^*)\varphi(a^*a) + \varphi(a^*)\varphi(a)\varphi(bb^*) - \varphi(a^*)\varphi(b)\varphi(a)\varphi(b^*).\end{aligned}$$

We aim to define the free cumulants of a $*$-noncommutative probability space (A, φ). Before we can do this, we must deviate momentarily to define the noncrossing partitions.

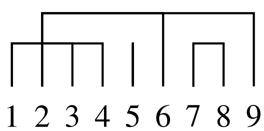
Definition 2.1.8. A partition $\pi = \{V_1, \ldots, V_k\}$ of $\{1, \ldots, n\}$ is said to be *crossing* if there are two distinct blocks V_j and V_k having elements $p_1, p_2 \in V_j$ and $q_1, q_2 \in V_k$ such that

$$p_1 < q_1 < p_2 < q_2.$$

If the partition is not crossing, then we say it is *noncrossing*.

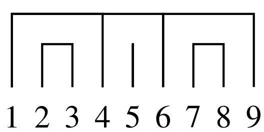
Denote by $NC(n)$ the set of all noncrossing partitions on $\{1, \ldots, n\}$, and let NC be the union of $NC(n)$ over all $n \in \mathbb{N}$. For each n, let 1_n denote the noncrossing partition $\{\{1, \ldots, n\}\}$ consisting of a single block.

We draw graphical representations of partitions on $\{1, \ldots, n\}$ by viewing the integers on a numberline, drawing a vertical line above each one, and joining the vertical lines of the elements in the same block with a hozrizontal line. For example, $\pi = \{\{1, 3, 4\}, \{2, 6, 9\}, \{5\}, \{7, 8\}\}$ and $\sigma = \{\{1, 4, 6\}, \{2, 3\}, \{5\}, \{6, 9\}, \{7, 8\}\}$ can be represented by



$$= \qquad \{\{1, 3, 4\}, \{2, 6, 9\}, \{5\}, \{7, 8\}\}$$

and



$$= \qquad \{\{1, 4, 6\}, \{2, 3\}, \{5\}, \{6, 9\}, \{7, 8\}\}.$$

The juxtaposition of the crossing partition π and the noncrossing partition σ with this graphical representation makes it clear why the terminology "noncrossing" is used. Indeed, only the noncrossing partitions can be drawn in this manner such that the lines from different blocks do not intersect. We will use this notation inline and in subscripts without drawing the corresponding integers.

We present a basic fact about noncrossing partitions that we will need later.

Lemma 2.1.9. *Every noncrossing partition contains an interval block, which is a block consisting only of consecutive integers.*

Proof. We use induction on n, the size of the set $\{1, \ldots, n\}$ that is partitioned.

When $n = 1$, the only partition is noncrossing and it consists of a single interval block. In the general case, let $\pi \in \mathrm{NC}(n)$. If n is in an interval block of π, then π obviously has an interval block. Otherwise, the block V in π which contains n must a gap. Fix $i, k \in \mathbb{N}$ such that $i, i+k+1 \in V$ and $\{i+1, \ldots, i+k\} \cap V = \emptyset$; i.e., $I = \{i+1, \ldots, i+k\}$ is a gap in V of length k. Since π is noncrossing, no integer in I can be paired with an integer outside I. Thus, restricting π to I (and renumbering to preserve order) results in a noncrossing partition of length $k < n$. By the induction hypothesis, this restriction must have an interval block. Hence π has an interval block. $\square$

Now we may introduce Speicher's free cumulants, which provide a different, more combinatorial viewpoint into the probabilistic information of a $*$-noncommutative probability space.

Definition 2.1.10. Given $n \in \mathbb{N}$, the multilinear functional $\kappa_n : A^n \to \mathbb{C}$ is defined recursively by the *moment-cumulant formula*

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{NC}(n)} \kappa_\pi[a_1, \ldots, a_n], \tag{2.1}$$

where each $\kappa_\pi : A^n \to \mathbb{C}$ is a multilinear functional that can be factorized into a product determined by the block structure of π. More precisely, $\kappa_\pi : A^n \to \mathbb{C}$ is the multilinear map defined as the product

$$\kappa_\pi[a_1, \ldots, a_n] = \prod_{V \in \pi} \kappa_\pi(V)[a_1, \ldots, a_n],$$

where, given a block $V = \{i_1 < \cdots < i_m\}$ in π, we define $\kappa_\pi(V)[a_1, \ldots, a_n]$ by

$$\kappa_\pi(V)[a_1, \ldots, a_n] = \kappa_m(a_{i_1}, \ldots, a_{i_m}).$$

The sequence $(\kappa_n)_n$ is called the *free cumulants* or *cumulants* of the $*$-noncommutative probability space (A, φ).

The *cumulants* of a family $(a_i)_{i \in I}$ of random variables are the cumulant maps restricted to only take arguments from the set $\{a_i \mid i \in I\}$. The $*$-*cumulants* of a random variable a are the cumulants of the pair (a, a^*).

Note the use of square brackets for the functions indexed by noncrossing partitions in contrast to the parentheses used for the sequence of cumulants.

The method of obtaining the cumulant maps from the moment-cumulant formula is formally known as Möbius inversion. A detailed treatment of Möbius inversion, and its use in defining these cumulants, can be found in Lectures 10 and 11 from [4]. We show a computation of the first several cumulants to illustrate this process.

Example 2.1.11. We use the moment-cumulant formula (2.1) to write $\kappa_n(a_1, \ldots, a_n)$ in terms of moments for the first few values of n.

(1) When $n = 1$, we immediately get $\varphi(a_1) = \kappa_{|}[a_1] = \kappa_1(a_1)$.

(2) When $n = 2$, we have

$$\varphi(a_1 a_2) = \kappa_{||}[a_1, a_2] + \kappa_{\sqcap}[a_1, a_2] = \kappa_1(a_1)\kappa_1(a_2) + \kappa_2(a_1, a_2),$$

so that $\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$.

(3) When $n = 3$, the moment-cumulant formula yields

$$\begin{aligned}
\varphi(a_1 a_2 a_3) &= \kappa_{|||}[a_1, a_2, a_3] + \kappa_{|\sqcap}[a_1, a_2, a_3] + \kappa_{\overline{\sqcap}}[a_1, a_2, a_3] \\
&\quad + \kappa_{\sqcap|}[a_1, a_2, a_3] + \kappa_{\sqcap\!\sqcap}[a_1, a_2, a_3] \\
&= \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2(a_2 a_3) + \kappa_2(a_1 a_3)\kappa_1(a_2) \\
&\quad + \kappa_2(a_1 a_2)\kappa_1(a_3) + \kappa_3(a_1 a_2 a_3),
\end{aligned}$$

which implies

$$\begin{aligned}\kappa_3(a_1a_2a_3) &= \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2)\varphi(a_3) - \varphi(a_1)[\varphi(a_2a_3) - \varphi(a_2)\varphi(a_3)] \\ &\quad - [\varphi(a_1a_3) - \varphi(a_1)\varphi(a_3)]\varphi(a_2) - [\varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)]\varphi(a_3) \\ &= \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_1a_3)\varphi(a_2) - \varphi(a_1a_2)\varphi(a_3) \\ &\quad + 2\varphi(a_1)\varphi(a_2)\varphi(a_3).\end{aligned}$$

One will notice that the computations will quickly become too unwieldy. Indeed, when $n = 4$, there are 14 noncrossing partitions. Almost all of the these 14 terms will need to use the discovered formulas for κ_2 and κ_3, which will grow the number of total terms even further.

One convenient property about the free cumulants is that that they provide an easier way to use freeness. We will not need to directly use the following theorem, but its statement is included here to illustrate the usefulness of cumulants.

Theorem 2.1.12 ([4], Theorem 11.16)**.** *Let $(\kappa_n)_n$ be the cumulants for a $*$-noncommutative probability space (A, φ). The family $(A_i)_{i\in I}$ of unital $*$-subalgebras of A is $*$-free if and only if*

$$\kappa_n(a_1, \ldots, a_n) = 0$$

whenever $n \geq 2$, $a_i \in A_{\varepsilon(i)}$ with $\varepsilon \in I^n$, and there exists $1 \leq i, j \leq n$ with $\varepsilon(i) \neq \varepsilon(j)$. In other words, the family $(A_i)_{i\in I}$ is $*$-free if and only if every mixed cumulant vanishes.

The benefit of this characterization of freeness over the definition is that the variables are not required to be centered and from different subalgebras than their neighbors.

2.2 R-diagonal random variables and their free polar decompositions

Let (A, φ) be a $*$-noncommutative probability space.

Definition 2.2.1. A random variable $u \in A$ is called a *Haar unitary* if it is unitary (i.e., $u^{-1} = u^*$) and $\varphi(u^k) = 0$ for every integer $k \geq 1$.

Since φ is $*$-preserving, the $*$-distribution of a Haar unitary u is completely determined by

$$\varphi(u^n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

The following example shows how Haar unitaries may appear in the $*$-noncommutative probability spaces from Example 2.1.2.

Example 2.2.2.

(1) Let $A = L^\infty(\mathbb{T}, \mu)$, where μ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Defining $\varphi : A \to \mathbb{C}$ in the usual way

$$\varphi(f) = \int_{\mathbb{T}} f(z)\, d\mu(z),$$

we have a $*$-noncommutative probability space (A, φ). The identity function $f(z) = z$ on $\mathbb{T}$ is a Haar unitary. It is unitary because $zz^* = 1 = z^*z$ for every $z \in \mathbb{T}$. Also, f has the same $*$-distribution as a Haar unitary:

$$\varphi(f^n) = \int_{\mathbb{T}} z^n\, d\mu(z) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The normalized Lebesgue measure on the circle is called the Haar measure, so this example actually explains the terminology we use for Haar unitaries.

(2) Haar unitaries also frequently appear in the group algebra framework. Let G be a group with an element g of infinite order, and consider the $*$-noncommutative probability space $(\mathbb{C}G, \tau_G)$ defined in Example 2.1.2. We prove the element $g := 1 \cdot g$, viewed as an element of $\mathbb{C}G$, is a Haar unitary.

It is a unitary because, if e denotes the identity of G, then

$$gg^* = (1 \cdot g)(1 \cdot g^{-1}) = 1 \cdot e = 1_{\mathbb{C}G}$$

and similarly $g^*g = 1_{\mathbb{C}G}$. Since g has infinite order, the power g^k in G is not the identity whenever k is a positive integer. Therefore $\tau_G(g^k) = 0$ for every integer $k \geq 1$, so g is a Haar unitary.

(3) Suppose (A, φ) is a $*$-noncommutative probability space with a Haar unitary v. Consider the $*$-noncommutative probability space $(M_n(A), \Phi)$ defined in Example 2.1.2. Identifying $M_n(A)$ as $M_n(\mathbb{C}) \otimes A$, we have $\Phi = \operatorname{tr}_n \otimes \varphi$, where $\operatorname{tr}_n : M_n(\mathbb{C}) \to \mathbb{C}$ is the normalized (faithful) trace $\operatorname{tr}_n([\alpha_{i,j}]_{i,j}) = \frac{1}{n}\sum_{i=1}^n \alpha_{i,i}$. Given any unitary matrix $\alpha = [\alpha_{i,j}]$ in $M_n(\mathbb{C})$, we argue that the random variable $u = \alpha \otimes v$ is a Haar unitary.

The unitary property is verified by

$$uu^* = (\alpha \otimes v)(\alpha^* \otimes v^*) = \alpha\alpha^* \otimes vv^* = 1_{M_n(\mathbb{C})} \otimes 1_A = 1_{M_n(A)}$$

and $u^*u = 1_{M_n(A)}$, which holds similarly. Given an integer $k \geq 1$, we have $u^k = \alpha^k \otimes v^k$ so that

$$\Phi(u^k) = \operatorname{tr}_n(\alpha^k)\varphi(v^k) = 0.$$

Therefore u is a Haar unitary.

Fix a Haar unitary u. Since we know the complete $*$-distribution of u, we should be able to determine the $*$-cumulants of u; i.e., cumulants of the form $\kappa_n(u^{\varepsilon(1)}, \ldots, u^{\varepsilon(n)})$ where $\varepsilon \in \{1, *\}^n$.

We can use induction on the length of the cumulant to prove that if the number of u's is different from the number of u^*'s, then the cumulant vanishes. This statement is true whenever $n = 1$ because $\kappa_1(u) = \varphi(u) = 0$ and similarly $\kappa_1(u^*) = 0$. Supposing there is an integer $k \geq 2$ for which the statement holds for all $n < k$, we consider a $*$-cumulant of $\kappa_k(u^{\varepsilon(1)}, \ldots, u^{\varepsilon(k)})$ of u with an unequal number of u's and u^*'s. The moment-cumulant formula implies

$$\sum_{\pi \in \mathrm{NC}(k)} \kappa_\pi[u^{\varepsilon(1)}, \ldots, u^{\varepsilon(k)}] = \varphi(u^{\varepsilon(1)} \cdots u^{\varepsilon(k)}) = 0,$$

where the second equality is due to the fact that u is a Haar unitary. For each $\pi \in \mathrm{NC}(k) \setminus \{1_k\}$, some block V of π must contain indices that correspond to an unequal number of u's and u^*'s. Hence $\pi(V)[u_1, \ldots, u_k] = 0$ by the induction hypothesis, so also $\kappa_\pi[u_1, \ldots, u_k] = 0$. Therefore the displayed equation above becomes $\kappa_{1_k}[u_1, \ldots, u_k] = 0$, which means $\kappa_k(u_1, \ldots, u_k) = 0$. This completes the proof that all $*$-cumulants of u corresponding to an unequal number of u's and u^*'s vanish.

In fact, something stronger is true.

Proposition 2.2.3 ([4], Proposition 15.1)**.** *The only nonvanishing $*$-cumulants of a Haar unitary u are those of the form $\kappa_n(u, u^*, \ldots, u, u^*)$ or $\kappa_n(u^*, u, \ldots, u^*, u)$. Moreover*

$$\kappa_{2n}(u, u^*, \ldots, u, u^*) = \kappa_{2n}(u^*, u, \ldots, u^*, u) = (-1)^{n-1} C_{n-1},$$

where C_n denotes the nth Catalan number.

This leads us to the definition of R-diagonal elements, which were first studied by Alexandru Nica and Roland Speicher (see [5]).

Definition 2.2.4. A random variable a is said to be *R-diagonal* if its only nonvanishing $*$-cumulants are alternating; i.e., are of the form $\kappa_{2n}(a, a^*, \ldots, a, a^*)$ or $\kappa_{2n}(a^*, a, \ldots, a^*, a)$.

Denoting $\beta_n^{(1)} = \kappa_{2n}(a, a^*, \ldots, a, a^*)$ and $\beta_n^{(2)} = \kappa_{2n}(a^*, a, \ldots, a^*, a)$ for each n, the sequences $(\beta_n^{(1)})_n$ and $(\beta_n^{(2)})_n$ are called the *determining sequences* of a.

If the expectation φ is a trace on the $*$-algebra generated by a, then a is called a *tracial R-diagonal* element.

The determining sequences are named as such because they hold all of the information about the $*$-cumulants of an R-diagonal element a, and these $*$-cumulants uniquely determine the element's $*$-distribution by the moment-cumulant formula (2.1). Since each of these $*$-cumulants depend on moments of the form $\varphi((aa^*)^n)$ and $\varphi((a^*a)^n)$, we realize that an R-diagonal element's $*$-distribution is uniquely determined by the distributions of the self adjoint elements aa^* and a^*a.

One especially useful fact is that the R-diagonal property is preserved by the multiplication with a $*$-free random variable.

Proposition 2.2.5 ([4], Proposition 15.8)**.** *If a is R-diagonal and b is a random variable $*$-free from a, then ab and ba are R-diagonal.*

The $*$-distribution of an R-diagonal element is invariant under the multiplication by a $*$-free Haar unitary. In fact, this is characterization.

Theorem 2.2.6 ([4], Theorem 15.10)**.** *Let a and u be random variables in a $*$-noncommutative probability space (A, φ) such that u is a Haar unitary and a, u are $*$-free. Then a is R-diagonal if and only if a and ua have the same $*$-distribution.*

Proof. Since Haar unitaries are R-diagonal, Proposition 2.2.5 implies ua is R-diagonal. Hence, if a and ua have the same $*$-distribution, then a is also R-diagonal.

Conversely suppose a is R-diagonal. To show that a and ua have the same $*$-distribution, it suffices to show that a^*a and aa^* have the same distributions as $(ua)^*(ua)$ and $(ua)(ua)^*$, respectively. The first of these is automatic because $(ua)^*(ua) = a^*u^*ua = a^*a$. For the other pair, we have

$$\varphi(((ua)(ua)^*)^n) = \varphi((uaa^*u^*)^n) = \varphi(u(aa^*)^n u^*) = \varphi(u\varphi(aa^*)u^*) = \varphi(aa^*)$$

for every positive integer n, where the third equality is due to the $*$-freeness of a and u. □

If a is a tracial R-diagonal element, then its two determining sequences coincide. Thus the $*$-distribution of a is uniquely determined by the single sequence $(\beta_n^{(1)})_n$. A special class of self adjoint elements has a similar property.

Definition 2.2.7. A random variable x is said to be *even* if all of its odd $*$-moments vanish; that is, if for all $n \in \mathbb{N}$ and $\varepsilon \in \{1, *\}^{2n+1}$ we have

$$\varphi(x^{\varepsilon(1)} \cdots x^{\varepsilon(2n+1)}) = 0.$$

An even element could equivalently be defined to have all of its odd $*$-cumulants vanish. This can be proved with a simple induction argument on the moment-cumulant formula

$$\varphi(x^{\varepsilon(1)} \cdots x^{\varepsilon(2n+1)}) = \sum_{\pi \in \mathrm{NC}(2n+1)} \kappa_\pi[x^{\varepsilon(1)}, \ldots, x^{\varepsilon(2n+1)}].$$

Therefore, given an even self adjoint element x, the entire $*$-distribution is uniquely determined by its *determining sequence* $(\beta_n^x)_n$ defined by $\beta_n^x = \kappa_{2n}(x, \ldots, x)$. Furthermore, the $*$-distribution of x is determined by the distribution of x^2. This is analogous to the situation with tracial R-diagonal elements. In fact, every tracial R-diagonal can be decomposed into a product of a Haar unitary and a $*$-free even self adjoint such that the determining sequence of the tracial R-diagonal matches that of the even self adjoint.

Theorem 2.2.8 ([4], Proposition 15.12)**.** *Let a be a random variable in a $*$-noncommutative probability space (A, φ). Then a is tracial R-diagonal if and only if there exists a (possibly different) $*$-noncommutative probability space having $*$-free elements u and x such that u is a Haar unitary, x is an even self adjoint, and the $*$-distributions of a and ux coincide. Moreover, the determining sequence of a is the same as the determining sequence of x.*

Proof. Let a be a tracial R-diagonal element. We construct an even self adjoint y in the $*$-

noncommutative probability space $(M_2(A), \Phi)$ from Example 2.1.2, where $\Phi = \mathrm{tr}_2 \otimes \varphi$. Define

$$y = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.$$

This is clearly self adjoint, and the powers of y are given by

$$y^{2n} = \begin{pmatrix} (aa^*)^n & 0 \\ 0 & (a^*a)^n \end{pmatrix}$$

and

$$y^{2n+1} = \begin{pmatrix} 0 & a(a^*a)^n \\ a^*(aa^*)^n & 0 \end{pmatrix}.$$

Thus y is even and $\Phi([y^2]^n) = \varphi([a^*a]^n) = \varphi([aa^*]^n)$ for each $n \in \mathbb{N}$. By invoking the free product construction for $*$-noncommutative probability spaces (see Theorem 6.13 of [4] or Theorem 3.1.8 below with $B = \mathbb{C}$), we may obtain a $*$-noncommutative space (A', φ') containing a Haar unitary u and an even self adjoint x such that u and x are $*$-free and x has the same distribution as y. Then uy is R-diagonal by Proposition 2.2.5,

$$\varphi'([(ux)^*(ux)]^n) = \varphi'(x^{2n}) = \Phi(y^{2n}) = \varphi([a^*a]^n),$$

and

$$\varphi'([(ux)(ux)^*]^n) = \varphi'(ux^{2n}u^*) = \varphi'(x^{2n}) = \Phi(y^{2n}) = \varphi([aa^*]^n),$$

where we used the $*$-freeness of u and x in the second equality. Therefore a and ux have the same $*$-distribution.

Conversely, we suppose there is a $*$-noncommutative probability space (A', φ') containing u and x such that u is Haar unitary, x is even self adjoint, u, x are $*$-free, and a has the same $*$-distribution as ux. Then ux, and thus also a, is R-diagonal by Proposition 2.2.5. Poposition 5.19

of [4] says that if the expectation is a trace on a family of free subalgebras, then it is a trace on the subalgebra generated by these subalgebras. Since u and x is $*$-free, it must be the case that φ' is a trace on the $*$-algebra generated by $\{u, x\}$. Hence ux is a tracial, so a must also be tracial.

To see that a and x have the same determining sequence, it suffices to prove $\varphi([a^*a]^n) = \varphi'([x^2]^n)$ holds for every $n \in \mathbb{N}$. This follows immediately from the facts that $[(ux)^*(ux)]^n = x^{2n}$ and a and ux have the same $*$-distribution. □

By considering the polar decomposition in a C*-noncommutative probability space, a more canonical variation of Theorem 2.2.8 holds.

Theorem 2.2.9 ([4], Proposition 15.13)**.** *Let a be a random variable in a C*-noncommutative probability space (A, φ). Then a is tracial R-diagonal if and only if there are elements u, p in some C*-noncommutative probability space such that*

(a) u is a Haar unitary,

(b) p is positive,

(c) u and p are $*$-free, and

(d) a has the same $*$-distribution as up.

Moreover, the distribution of p is the same as the distribution of $|a|$.

We are motivated by this theorem, so we give its conclusion the name *free polar decomposition*.

Definition 2.2.10. A random variable a has a *free polar decomposition* if there are random variables u and p in some $*$-noncommutative probability space satisfying conditions (a)-(d) in the Theorem 2.2.9

Therefore Theorem 2.2.9 says that a random variable in a C*-noncommutative probability space has a free polar decomposition if and only if it is tracial R-diagonal.

3. B-VALUED $*$-NONCOMMUTATIVE PROBABILITY SPACES

3.1 B-valued random variables, freeness, and $*$-cumulants

Definition 3.1.1. Let B be a unital $*$-algebra over the complex numbers. A *B-valued $*$-noncommutative probability space* is a pair (A, E), where A is a unital $*$-algebra containing a unitally embedded copy of B and $E : A \to B$ is a positive, idempotent linear function that restricts to the identity on B and satisfies $E(b_1 a b_2) = b_1 E(a) b_2$ for every $a \in A$ and $b_1, b_2 \in B$. If B and A are additionally C*-algebras (resp. von Neumann algebras), then we say (A, E) is a *B-valued C*-noncommutative probability space* (resp. *B-valued W*-noncommutative probability space*).

The mapping E is called a *conditional expectation*. Elements of A are called *B-valued random variables*, or simply *random variables*.

In the case $B = \mathbb{C}$, we recover the the definition of a $*$-noncommutative probability space. Henceforth we fix a B-valued $*$-noncommutative probability space (A, E) over some unital $*$-algebra B.

The B-valued probabilistic information about a random variable is captured by its B-valued $*$-moments and B-valued $*$-distribution.

Definition 3.1.2. A *B-valued moment* of a family $(a_i)_{i \in I}$ of random variables is an element of B of the form

$$E(a_{\varepsilon(1)} b_1 \cdots a_{\varepsilon(n-1)} b_{n-1} a_{\varepsilon(n)}),$$

where $n \in \mathbb{N}$, $b_1, \ldots, b_{n-1} \in B$, and $\varepsilon \in I^n$. We typically consider a *B-valued $*$-moment* of a single random variable a, which is a B-valued moment of the pair (a, a^*).

Let $B\langle (X_i)_{i \in I} \rangle$ be the universal algebra over $\mathbb{C}$ generated by B and the indeterminates $(X_i)_{i \in I}$. The *B-valued distribution* of a family $(a_i)_{i \in I}$ is the mapping $\Theta : B\langle (X_i)_{i \in I} \rangle \to B$ defined by extending

$$\Theta(b_0 X_{\varepsilon(1)} b_1 \cdots X_{\varepsilon(n)} b_n) = E(b_0 a_{\varepsilon(1)} b_1 \cdots a_{\varepsilon(n)} b_n)$$

linearly. When we refer to the *B-valued $*$-distribution* of a single random variable a, we mean the B-valued distribution of the pair (a, a^*).

One can mimic the proof Theorem 2.1.4 to prove an analogous property for B-valued $*$-distributions. We could not find a reference to this well-known fact.

Theorem 3.1.3. *Let (A_1, E_1) and (A_2, E_2) be B-valued $*$-noncommutative probability spaces such that E_1 and E_2 are faithful, and suppose $a_1 \in A_1$ and $a_2 \in A_2$ such that $A_1 = \mathrm{Alg}(B \cup \{a_1\})$ and $A_2 = \mathrm{Alg}(B \cup \{a_2\})$. Then a_1 and a_2 have the same B-valued $*$-distribution if and only if there is $*$-isomorphism $\Phi : A_1 \to A_2$ such that*

(a) *Φ acts identically on B,*

(b) *$\Phi(a_1) = a_2$, and*

(c) *$E_2 \circ \Phi = E_1$.*

Proof. Replace all instances of $\mathbb{C}\langle X, X^*\rangle$ by $B\langle X, X^*\rangle$ in the proof of Theorem 2.1.4. □

Dan Voiculescu extended his notion of free indpendence to this more general setting.

Definition 3.1.4. Let $(A_i)_{i\in I}$ be a family of $*$-subalgebras of A that each contains a unitally embedded copy of B. This family of subalgebras is *$*$-free over B* if

$$E(a_1 \cdots a_n) = 0$$

whenever:

(a) n is a positive integer;

(b) $\varepsilon \in I^n$ so that $a_i \in A_{\varepsilon(i)}$ for each $1 \leq i \leq n$;

(c) Each a_i is centered; i.e., $E(a_i) = 0$ for all $1 \leq i \leq n$;

(d) Neighboring elements are from different subalgebras; meaning, $\varepsilon(1) \neq \varepsilon(2), \ldots, \varepsilon(n-1) \neq \varepsilon(n)$.

A family of subsets $(X_i)_{i\in I}$ is $*$-*free over* B if the corresponding family of unital $*$-subalgebras $(\text{Alg}(B, X_i))_{i\in I}$ is $*$-free.

If $(A_i)_{i\in I}$ is $*$-free over B and $a_1, \ldots, a_n$ satisfy the conditions above, then we have

$$E(a_1b_1 \cdots a_{n-1}b_{n-1}a_n) = 0$$

for any $b_1, \ldots, b_{n-1} \in B$ because we can let each b_i be absorbed into one of its neighboring a_i's. We will use this property when applying $*$-freeness because B-valued moments are of the form $E(a_1b_1 \cdots a_{n-1}b_{n-1}a_n)$.

Of course, the definition of $*$-freeness depends on the conditional expectation E. If there is any ambiguity in which conditional expectation our discussion of $*$-freeness over B concerns, we'll replace "over B" with the terminology "with respect to E". If $B = \mathbb{C}$, then $*$-freeness over B is the same as $*$-freeness from Definition 2.1.5.

We recall the *centering* of a random variable a to be the random variable $\mathring{a} = a^\circ$, defined by $\mathring{a} = a - E(a)$. Then $a = \mathring{a} + E(a)$. Thus every B-valued random variable is the sum of a centered random variable and an element of B. We use this idea to show the first of many analogues of the scalar setting. Compare the following example with Example 2.1.7.

Example 3.1.5. We use $*$-freeness over B to decompose mixed B-valued $*$-moments into B-valued $*$-moments of the individual variables. Fix random variables a_1, a_2 that are $*$-free over B, and let $b_1, b_2, b_3 \in B$.

(1) We write $a_1b_1a_2 = (\mathring{a}_1 + E(a_1))b_1(\mathring{a}_2 + E(a_2))$. By $*$-freeness over B we have

$$E(a_1b_1a_2) = E(\mathring{a}_1b_1\mathring{a}_2) + E(\mathring{a}_1)b_1E(a_2) + E(a_1)b_1E(\mathring{a}_2) + E(a_1)b_1E(a_2) = E(a_1)b_1E(a_2).$$

(2) The same technique applied to a_2 allows us to deduce

$$E(a_1b_1a_2b_2a_1^*) = E(a_1b_1\mathring{a}_2b_2a_1^*) + E(a_1b_1E(a_2)b_2a_1^*).$$

By centering a_1 and a_1^*, we can split the first of these terms into four new terms. Each of these will vanish, so we conclude

$$E(a_1 b_1 a_2 b_2 a_1^*) = E(a_1 b_1 E(a_2) b_2 a_1^*).$$

(3) We center a_2 and a_2^* to express $E(a_1^* b_1 a_2 b_2 a_1 b_3 a_2^*)$ as a sum of four terms. Two of these will vanish, so we have

$$E(a_1^* b_1 a_2 b_2 a_1 b_3 a_2^*) = E(a_1^* b_1 \mathring{a}_2 b_2 a_1 b_3 \mathring{a}_2^*) + E(a_1^* b_1 E(a_2) b_2 a_1) E(b_3 a_2^*).$$

We will simplify the first of these two terms. If a_1 is replaced by $\mathring{a}_1$, then it will be zero whenever a_1^* is replaced by either $\mathring{a}_1^*$ or $E(a_1^*)$. Thus we obtain

$$E(a_1^* b_1 \mathring{a}_2 b_2 a_1 b_3 \mathring{a}_2^*) = E(a_1^* b_1 \mathring{a}_2 b_2 E(a_1) b_3 \mathring{a}_2^*) = E(a_1^* b_1 E(\mathring{a}_2 b_2 E(a_1) b_3 \mathring{a}_2^*))$$

after centering $\mathring{a}_2 b_2 E(a_1) b_3 \mathring{a}_2^*$. Unpacking $\mathring{a}_2$ and $\mathring{a}_2^*$ yields

$$E(\mathring{a}_2 b_2 E(a_1) b_3 \mathring{a}_2^*) = E(a_2 b_2 E(a_1) b_3 a_2^*) - E(a_2) b_2 E(a_1) b_3 E(a_2^*),$$

which implies

$$\begin{aligned} E(a_1^* b_1 a_2 b_2 a_1 b_3 a_2^*) = {} & E(a_1^* b_1 E(a_2 b_2 E(a_1) b_3 a_2^*)) - E(a_1^*) b_1 E(a_2) b_2 E(a_1) b_3 E(a_2^*) \\ & + E(a_1^* b_1 E(a_2) b_2 a_1) E(b_3 a_2^*). \end{aligned}$$

In fact, $*$-freeness over B of a family of $*$-subalgebras is enough to determine the conditional expectation on their generated $*$-algebra.

Proposition 3.1.6 ([10], Proposition 1.3)**.** *Suppose $(A_i)_{i \in I}$ is a family of $*$-subalgebras of a B-valued $*$-noncommutative probability space (A, E) that is $*$-free over B. Let A' denote the $*$-*

algebra generated by $\bigcup_{i\in I} A_i$. Then $E\mid_{A'}$ is uniquely determined by the family of maps $(E\mid_{A_i})_{i\in I}$.

Proof. Due to linearity, it suffices to show

$$E(a_1 b_1 \cdots a_{n-1} b_{n-1} a_n) \tag{3.1}$$

is determined by $(E\mid_{A_i})_{i\in I}$ for every $n\in\mathbb{N}$, $b_1,\dots,b_{n-1}\in B$, $\varepsilon\in I^n$, and $a_j\in A_{\varepsilon(j)}$ for each $1\le j\le n$. We may further assume that neighboring a_j's come from different subalgebras, for if a_j, a_{j+1} are in the same subalgebra, then we could combine $a_j b_j a_{j+1}$ to obtain a single element of that algebra. Thus $\varepsilon(1)\neq\varepsilon(2),\dots,\varepsilon(n-1)\neq\varepsilon(n)$.

We proceed by induction on n. When $n=1$, this is clear. In the general case, we center the random variables to get

$$\begin{aligned} E(a_1 b_1 \cdots a_{n-1} b_{n-1} a_n) &= E((\mathring{a}_1 + E(a_1))b_1 \cdots (\mathring{a}_{n-1} + E(a_{n-1}))b_{n-1}(\mathring{a}_n + E(a_n))) \\ &= E(\mathring{a}_1 b_1 \cdots \mathring{a}_{n-1} b_{n-1} \mathring{a}_n) + \text{other terms}, \end{aligned}$$

where the other terms are of the same form as the moment (3.1) but with a length less than n. The term $E(\mathring{a}_1 b_1 \cdots \mathring{a}_{n-1} b_{n-1} \mathring{a}_n)$ vanishes by $*$-freeness, and the other terms can all be written using the maps $E|_{A_i}$, $i\in I$, by the induction hypothesis. This completes the proof. ☐

Now we present the free product construction of $*$-noncommutative probability spaces.

Definition 3.1.7. Let $(A_i)_{i\in I}$ be a family of $*$-algebras that contain a unitally embedded copy of the $*$-algebra B. The *algebraic free product $*$-algebra with amalgamation over B*, denoted by $\underset{i\in I}{*_B} A_i$, is the free $*$-algebra generated by $\bigcup_{i\in I} A_i$ modulo the relations within A_i for each $i\in I$ and the relations identifying the $*$-subalgebra B within all A_i, $i\in I$.

Each $*$-subalgebra A_i is viewed as its image under the canonical embedding $A_i \to \underset{i\in I}{*_B} A_i$.

Theorem 3.1.8 ([8])**.** *Let B be a C*-algebra and let $((A_i, E_i))_{i\in I}$ be a family of B-valued $*$-noncommutative probability spaces. There is a conditional expectation $E = \underset{i\in I}{*_B} E_i$ on $A = \underset{i\in I}{*_B} A_i$ such that* *(a)* (A, E) *is a* B*-valued* $*$*-noncommutative probability space,*

(b) $E|_{A_i} = E_i$ *for each* $i \in I$*, and*

(c) $(A_i)_{i\in I}$ *is* $*$*-free over* B *with respect to* E*.*

Similar constructions exist for families of C*-noncommutative probability spaces and W*-noncommutative probability spaces. These free product constructions are incredibly useful to construct a space with desired families of $*$-free random variables. For example, if we have B-valued random variables a_1 and a_2 that are in different B-valued $*$-noncommutative probability spaces, then the free product construction with amalgamation over B produces a space having copies of a_1 and a_2 that are $*$-free over B.

We'll end the section with a formulation from [3] of Speicher's B-valued cumulant maps.

Definition 3.1.9. Suppose $(a_i)_{i\in I}$ is a family of B-valued random variables. Denote $J = \bigcup_{n\in\mathbb{N}} I^n$. Given $n \in \mathbb{N}$ and $j \in I^n$, the corresponding cumulant map $\alpha_j : B^{n-1} \to B$ is defined recursively by the *B-valued moment-cumulant formula*

$$\mathcal{E}(a_{j(1)}b_1a_{j(2)}\cdots b_{n-1}a_{j(n)}) = \sum_{\pi\in\mathrm{NC}(n)} \hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}], \tag{3.2}$$

where $\mathrm{NC}(n)$ is the set of all noncrossing partitions of $\{1, \ldots, n\}$ and, given $\pi \in \mathrm{NC}(n)$, $\hat{\alpha}_j(\pi)$ is a multilinear map defined in terms of $\alpha_{j'}$ for each j' obtained by restricting j to a block of π. More precisely, if $\pi = 1_n$, then

$$\hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}] = \alpha_j(b_1, \ldots, b_{n-1}),$$

and for $\pi \neq 1_n$, we choose an interval block $\{p, p+1, \ldots, p+q-1\} \in \pi$ with $p \geq 1$ and $q \geq 1$, and let $\pi' \in \mathrm{NC}(n-q)$ be obtained by restricting π to $\{1, \ldots, p-1\} \cup \{p+q, \ldots, n\}$ and then renumbering to preserve order. Then, with $j' = (j(1), \ldots, j(p-1), j(p+q), \ldots, j(n)) \in I^{n-q}$

and $j'' = (j(p), \ldots, j(p+q-1)) \in I^q$, we have

$$\hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}]$$

$$= \begin{cases} \hat{\alpha}_{j'}(\pi')[b_1, \ldots, b_{p-2}, \\ \qquad b_{p-1}\alpha_{j''}(b_p, \ldots, b_{p+q-2})b_{p+q-1}, \\ \qquad b_{p+q}, \ldots, b_{n-1}], & p \geq 2,\ p+q-1 < n, \\ \hat{\alpha}_{j'}(\pi')[b_1, \ldots, b_{p-2}]b_{p-1}\alpha_{j''}(b_p, \ldots, b_{n-1}), & p \geq 2,\ p+q-1 = n, \\ \alpha_{j''}(b_1, \ldots, b_{q-1})b_q\hat{\alpha}_{j'}(\pi')[b_{q+1}, \ldots, b_{n-1}], & p = 1,\ q < n. \end{cases}$$

The collection of maps $(\alpha_j)_{j\in J}$ are called the B-*valued cumulants* of $(a_i)_{i\in I}$. The B-*valued* $*$-*cumulants* of a random variable a are the B-valued cumulants corresponding to the pair (a, a^*).

Recall that an inverval block can always be found within a noncrossing partition by Lemma 2.1.9. It also does not matter in which order the interval blocks are chosen. The following example illustrates the process of defining $\hat{\alpha}_j(\pi)$ in terms of $\alpha_{j'}$'s of smaller size.

Example 3.1.10. Fix $a_1, a_2, \ldots, a_8 \in A$ and $b_1, b_2, \ldots, a_7 \in B$. First, we compute the map $\hat{\alpha}_{(1,2,3,4,5,6,7,8)}(\sqcap\ \sqcap\!\!\!\sqcap\ \sqcap\!\!\sqcap\!\!\sqcap)$. At each step of the recursion, we choose an interval block to introduce a factor of $\alpha_{j'}$. If this interval block is on the left or right end, then this factor appears on the left or right side, respectively. Otherwise, the factor is nested within one of the arguments. This process results in the computation

$$\begin{aligned} &\hat{\alpha}_{(1,2,3,4,5,6,7,8)}(\sqcap\ \sqcap\!\!\!\sqcap\ \sqcap\!\!\sqcap\!\!\sqcap)[b_1, b_2, b_3, b_4, b_5, b_6, b_7] \\ &\qquad = \alpha_{(1,2)}(b_1)b_2\hat{\alpha}_{(3,4,5,6,7,8)}(\sqcap\!\!\!\sqcap\ \sqcap\!\!\sqcap\!\!\sqcap)[b_3, b_4, b_5, b_6, b_7] \\ &\qquad = \alpha_{(1,2)}(b_1)b_2\hat{\alpha}_{(3,4,5)}(\sqcap\!\!\!\sqcap)[b_3, b_4]b_5\alpha_{(6,7,8)}(b_6, b_7) \\ &\qquad = \alpha_{(1,2)}(b_1)b_2\hat{\alpha}_{(3,5)}(\sqcap)[b_3\alpha_{(4)}b_4]b_5\alpha_{(6,7,8)}(b_6, b_7) \\ &\qquad = \alpha_{(1,2)}(b_1)b_2\alpha_{(3,5)}(b_3\alpha_{(4)}b_4)b_5\alpha_{(6,7,8)}(b_6, b_7). \end{aligned}$$

Note how this is closely related to using the block structure of  to parenthesize the a_i's in the word

$$a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6a_7b_7a_8$$

into

$$(a_1b_1a_2)b_2(a_3b_3(a_4)b_4a_5)b_5(a_6b_6a_7b_7a_8),$$

and then replacing each pair of parentheses with the cumulant map whose index is determined by the a_i's appearing within the given pair of parentheses and whose arguments consist of the terms between these a_i's.

We use this method on another example. Given the noncrossing partition  in $\mathrm{NC}(8)$, we parenthesize the word

$$a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6a_7b_7a_8$$

into

$$(a_1b_1a_2b_2(a_3)b_3a_4b_4(a_5b_5(a_6)b_6a_7)b_7a_8).$$

Therefore

$\hat{\alpha}_{(1,2,3,4,5,6,7,8)}($  $)[b_1, b_2, b_3, b_4, b_5, b_6, b_7] = \alpha_{(1,2,4,8)}(b_1, b_2\alpha_{(3)}b_3, b_4\alpha_{(5,7)}(b_5\alpha_{(6)}b_6)b_7).$

Now that the connection between $\hat{\alpha}_j$'s and α_j's is more clear, we compute some cumulants. Compare the following example to Example 2.1.11.

Example 3.1.11. Using the B-valued moment cumulant formula (3.2), we write the first few cumulants in terms of moments. Fix $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in A$, and $b_1, b_2, \ldots, b_{n-1} \in B$.

(1) When $n = 1$, we immediately get $E(a_1) = \hat{\alpha}_{(1)}(\,|\,) = \alpha_{(1)}$.

(2) When $n = 2$, we have

$$E(a_1b_1a_2) = \hat{\alpha}_{(1,2)}(\,|\,|\,)[b_1] + \hat{\alpha}_{(1,2)}(\sqcap)[b_1] = \alpha_{(1)}b_1\alpha_{(2)} + \alpha_{(1,2)}(b_1),$$

so that $\alpha_{(1,2)}(b_1) = E(a_1b_1a_2) - E(a_1)b_1E(a_2)$.

(3) When $n = 3$, the moment-cumulant formula yields

$$
\begin{aligned}
E(a_1b_1a_2b_2a_3) &= \hat{\alpha}_{(1,2,3)}(\,|\,|\,|\,)[b_1,b_2] + \hat{\alpha}_{(1,2,3)}(\,|\sqcap\,)[b_1,b_2] + \hat{\alpha}_{(1,2,3)}(\,\overline{|\,\text{ı}\,|}\,)[b_1,b_2] \\
&\quad + \hat{\alpha}_{(1,2,3)}(\,\sqcap\,|\,)[b_1,b_2] + \hat{\alpha}_{(1,2,3)}(\,\sqcap\!\!\sqcap\,)[b_1,b_2] \\
&= \alpha_{(1)}b_1\alpha_{(2)}b_2\alpha_{(3)} + \alpha_{(1)}b_1\alpha_{(2,3)}(b_2) + \alpha_{(1,3)}(b_1\alpha_{(2)}b_2) \\
&\quad + \alpha_{(1,2)}(b_1)b_2\alpha_{(3)} + \alpha_{(1,2,3)}(b_1,b_2),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\alpha_{(1,2,3)}(b_1,b_2) &= E(a_1b_1a_2b_2a_3) - E(a_1)b_1E(a_2)b_2E(a_3) \\
&\quad - E(a_1)b_1[E(a_2b_2a_3) - E(a_2)b_2E(a_3)] \\
&\quad - [E(a_1b_1E(a_2)b_2a_3) - E(a_1)b_1E(a_2)b_2E(a_3)] \\
&\quad - [E(a_1b_1a_2) - E(a_1)b_1E(a_2)]b_2E(a_3) \\
&= E(a_1b_1a_2b_2a_3) - E(a_1)b_1E(a_2b_2a_3) - E(a_1b_1E(a_2)b_2a_3) \\
&\quad - E(a_1b_1a_2)b_2E(a_3) + 2E(a_1)b_1E(a_2)b_2E(a_3).
\end{aligned}
$$

As was the case in the scalar setting, we will use the B-valued cumulants to study a special class of random variables: the B-valued R-diagonal random variables.

3.2 B-valued R-diagonal and circular random variables

Let B be a unital $*$-algebra and (A, E) be a B-valued $*$-noncommutative probability space. We begin by introducing the B-valued analogue of R-diagonal random variables. Unlike the scalar case, we will start with definition involving moments before seeing the natural generalization in terms of cumulants.

Definition 3.2.1. Given $n \in \mathbb{N}$ and $\varepsilon \in \{1, *\}^n$, the *maximal alternating interval partition* of $\sigma(\varepsilon)$ is the interval partition of $\{1, \ldots, n\}$ whose blocks V are the maximal interval subsets of $\{1, \ldots, n\}$ such that if $j \in V$ and $j+1 \in V$, then $\varepsilon(j) \neq \varepsilon(j+1)$.

For example, if $\varepsilon = (1, 1, *, *, *, 1, *)$, then $\sigma(\varepsilon) = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}\}$.

Definition 3.2.2. A B-valued random variable a is said to be *B-valued R-diagonal* if for every $k \geq 0$ and $b_1, \ldots, b_{2k} \in B$ we have

$$E(ab_1a^*b_2ab_3a^* \cdots b_{2k-2}ab_{2k-1}a^*b_{2k}a) = 0,$$

(i.e., odd alternating moments vanish) and, for $n \geq 1$, $\varepsilon \in \{1, *\}^n$, and $b_1, b_2, \ldots, b_n \in B$ we have

$$E\left(\prod_{V \in \sigma(\varepsilon)} \left(\left(\prod_{j \in V} a^{\varepsilon(j)} b_j\right) - E\left(\prod_{j \in V} a^{\varepsilon(j)} b_j\right)\right)\right) = 0,$$

where the terms in each of the three products are taken in the order of increasing indices.

Before listing some important characterizations of R-diagonal random variables, we'll need some notation.

Definition 3.2.3. A unitary $u \in A$ is called a *Haar unitary* if $E(u^k) = 0$ for all $k \geq 1$.

In the scalar situation, this condition is enough to determine the entire $*$-distribution of Haar unitaries. In fact, the definition of R-diagonal was motivated as a generalization of Haar unitaries. However, due to the nature of B-valued $*$-distributions, a Haar unitary in the B-valued setting may not be R-diagonal. Some of these nuances are explored in Section 4.1.

Definition 3.2.4. The B-valued $*$-noncommutative probability space (A', E') is said to be an *enlargement* of (A, E) if there is an embedding $\theta : A \to A'$ so that $\theta(b) = b$ for each $b \in B$ and $E'(\theta(a)) = E(a)$ for each $a \in A$.

The following theorem, which appears in [1] and parts of which appear in [7], gives some useful characterizations of B-valued R-diagonal elements that include a generalization of Theorem 2.2.6 to the B-valued setting.

Theorem 3.2.5 ([1], Theorem 3.1)**.** *Let $a \in A$. The following are equivalent:*

(a) *a is B-valued R-diagonal.*

(b) *The only non-vanishing B-valued $*$-cumulants of a (namely, cumulants of the pair $(a_1, a_2) = (a, a^*)$) are those that are alternating and of even length. That is, $\alpha_j = 0$ unless j is of the form $(1, 2, \ldots, 1, 2)$ or $(2, 1, \ldots, 2, 1)$.*

(c) *There is an enlargement (A', E') of (A, E) and a unitary $u \in A'$ such that*

  (i) *u commutes with B,*

  (ii) *u is a Haar unitary,*

  (iii) *u and a are $*$-free with respect to E', and*

  (iv) *a and ua have the same B-valued $*$-distribution.*

(d) *If (A', E') is an enlargement of (A, E) and $u \in A'$ is a unitary such that*

  (i) *u commutes with B and*

  (ii) *u and a are $*$-free with respect to E',*

  then a and ua have the same B-valued $*$-distribution.

Condition (b) of the above theorem is the familiar property that defines scalar R-diagonal random variables (see Definition 2.2.4). It will be referred to as the *cumulant property of R-diagonal elements*, whereas Definition 3.2.2 will be called the *moment property of R-diagonal elements*.

In light of the cumulant property for a B-valued R-diagonal a, we can simplify the notation for the B-valued $*$-cumulants of a. Instead of considering the cumulants $(\alpha_j)_{j\in J}$, indexed by $J = \bigcup_{n\in\mathbb{N}}\{1,2\}^n$, we will work with the two sequences $(\beta_k^{(1)})_k$ and $(\beta_k^{(2)})_k$ defined by $\beta_k^{(1)} = \alpha_{(1,2,\ldots,1,2)}$ and $\beta_k^{(2)} = \alpha_{(2,1,\ldots,2,1)}$.

Definition 3.2.6. A *polar decomposition* of a random variable a in a $*$-noncommutative probability space is a pair (v,p) in some B-valued $*$-noncommutative probability space such that v is a partial isometry, p is positive, and the B-valued $*$-distribution of a and vp coincide.

Furthermore, if v and p are $*$-free over B, then the pair is said to be a *free polar decomposition*.

In the scalar setting, every R-diagonal element in a tracial C*-noncommutative probability space has a free polar decomposition Theorem 2.2.9. The analogous statement does not hold in the B-valued setting. In order to present an example showing this phenomenon, it will be convenient to decribe traciality in terms of the cumulants of an R-diagonal random variable. The following proposition, due to Boedihardjo and Dykema, does precisely this.

Proposition 3.2.7 ([1], Proposition 5.1)**.** *Let $a \in A$ be a B-valued R-diagonal element and τ be a trace on B. Then $\tau \circ E$ restricted to $\mathrm{Alg}(B \cup \{a, a^*\})$ is tracial if and only if, for all $k \geq 1$ and all $b_1, \ldots, b_{2k} \in B$, we have*

$$\tau(\beta_k^{(1)}(b_1, \ldots, b_{2k-1})b_{2k}) = \tau(b_1\beta_k^{(2)}(b_2, \ldots, b_{2k})).$$

Since we hope to build an example using cumulants, it will be helpful to consider a very particular type of R-diagonal which has even fewer nonvanishing cumulants.

Definition 3.2.8. A B-valued random variable a is *B-valued circular* if $\alpha_j = 0$ whenever $j \in J \setminus \{(1,2),(2,1)\}$, where $J := \bigcup_{n\geq 1}\{1,2\}^n$, and $(\alpha_j)_{j\in J}$ are the cumulant maps associated to the pair $(a_1, a_2) := (a, a^*)$.

Clearly a B-valued circular element is also B-valued R-diagonal with cumulants $\beta_k^{(1)} = \beta_k^{(2)} = 0$ unless $k = 1$. Nonetheless, we will favor the notation $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ for the nonvanishing B-valued $*$-cumulants of a B-valued circular element.

The next lemma is nothing more than a restatement of the moment-cumulant formula (3.2) for B-valued circular elements. It will prove to be useful for computations appearing later in this section and in Section 4.3.

Lemma 3.2.9. *A B-valued random variable a is B-valued circular if and only if*

$$E(a_{j_1}b_1\cdots a_{j_{n-1}}b_{n-1}a_{j_n}) \\ = \sum_{\{p|j_p\neq j_1\}} \alpha_{(j_1,j_p)}(b_1E(a_{j_2}b_2\cdots a_{j_{p-1}}b_{p-1}))b_pE(a_{j_{p+1}}b_{p+1}\cdots a_{j_{n-1}}b_{n-1}a_{j_n}) \tag{3.3}$$

holds for every $n\geq 1$, $b_1,\dots,b_{n-1}\in B$, and $j=(j_1,\dots,j_n)\in J:=\bigcup_{n\geq 1}\{1,2\}^n$.

Proof. Given any $n\geq 1$ and tuple $j\in\{1,2\}^n$, consider the set T of all triples (k,π',π''), where $2\leq k\leq n$ such that $j_1\neq j_p$, $\pi'\in \mathrm{NC}(p-2)$, and $\pi''\in\mathrm{NC}(n-p)$. We describe a bijection between T and the set N of all noncrossing partitions $\pi\in\mathrm{NC}(n)$ such that the block of 1 is $\{1,p\}$, where $j_1\neq j_p$. Given a partition $\pi\in N$, we define π' and π'' by restricting π to the sets $\{2,\dots,p-1\}$ and $\{p+1,\dots,n\}$, respectively, and renumbering to preserve order. Note that this gives a bijection because a triple (p,π',π'') can uniquely determine a partition $\pi\in N$ such that $\{1,p\}\in\pi$ and both π' and π'' are the appropriate restrictions of π. With this correspondence, we have

$$\hat{\alpha}_j(\pi)[b_1,\dots,b_{n-1}] \\ = \alpha_{(j_1,j_p)}(b_1\hat{\alpha}_{(j_2,\dots,j_{p-1})}(\pi')[b_2,\dots,b_{p-2}]b_{p-1})b_p\hat{\alpha}_{(j_{p+1},\dots,j_n)}(\pi'')[b_{p+1},\dots,b_{n-1}]$$

for every $b_1,\dots,b_{n-1}\in B$. Keeping this correspondence in mind, we use the moment-cumulant formula to write

$$E(a_{j_1}b_1\cdots a_{j_{n-1}}b_{n-1}a_{j_n}) = \sum_{\pi\in\mathrm{NC}(n)}\hat{\alpha}_j(\pi)[b_1,\dots,b_{n-1}] \\ = \sum_{\pi\in N}\hat{\alpha}_j(\pi)[b_1,\dots,b_{n-1}] + \sum_{\pi\in\mathrm{NC}(n)\setminus N}\hat{\alpha}_j(\pi)[b_1,\dots,b_{n-1}].$$

We claim that the second sum over $\mathrm{NC}(n)\setminus N$ will vanish for all $n \geq 1$, $b_1, \ldots, b_{n-1} \in B$, and $j \in \{1,2\}^n$ if and only if a is B-valued circular. In this case, we can combine the two displayed equations above to obtain (3.3); namely,

$$\begin{aligned}
&E(a_{j_1} b_1 \cdots a_{j_{n-1}} b_{n-1} a_{j_n}) \\
&\quad = \sum_{\pi \in N} \hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}] \\
&\quad = \sum_{(p,\pi',\pi'') \in T} \alpha_{(j_1,j_p)}(b_1 \hat{\alpha}_{(j_2,\ldots,j_{p-1})}(\pi')[b_2, \ldots, b_{p-2}] b_{p-1}) b_p \hat{\alpha}_{(j_{p+1},\ldots,j_n)}(\pi'')[b_{p+1}, \ldots, b_{n-1}] \\
&\quad = \sum_{\{p | j_p \neq j_1\}} \alpha_{(j_1,j_p)}(b_1 \sum_{\pi' \in \mathrm{NC}(p-2)} \hat{\alpha}_{(j_2,\ldots,j_{p-1})}(\pi')[b_2, \ldots, b_{p-2}] b_{p-1}) \\
&\qquad\quad b_p \sum_{\pi'' \in \mathrm{NC}(n-p)} \hat{\alpha}_{(j_{p+1},\ldots,j_n)}(\pi'')[b_{p+1}, \ldots, b_{n-1}] \\
&\quad = \sum_{\{p | j_p \neq j_1\}} \alpha_{(j_1,j_p)}(b_1 E(a_{j_2} b_2 \cdots a_{j_{p-1}} b_{p-1})) b_p E(a_{j_{p+1}} b_{p+1} \cdots a_{j_{n-1}} b_{n-1} a_{j_n}),
\end{aligned}$$

where the last equality follows from two more applications of the moment-cumulant formula.

It remains to show that

$$\sum_{\pi \in \mathrm{NC}(n)\setminus N} \hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}] = 0$$

for all $n \geq 1$, $b_1, \ldots, b_{n-1} \in B$, and $j \in \{1,2\}^n$ if and only if a is B-valued circular. Given $\pi \in \mathrm{NC}(n)\setminus N$, let $\{1 < k_2 < \cdots < k_s\}$ be the block of π containing 1. Then the expression $\hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}]$ can be written as a product involving the map $\alpha_{(j_1,j_{k_2},\ldots,j_{k_s})}$. This map is not $\alpha_{(1,2)}$ or $\alpha_{(2,1)}$ because $\pi \notin N$, so if a is B-valued circular, then this map is zero and consequently $\hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}] = 0$. Conversely, suppose a is not B-valued circular, and let $j \notin \{(1,2),(2,1)\}$ be of minimal length such that $\alpha_j(b_1, \ldots, b_{n-1})$ doesn't vanish for some $b_1, \ldots, b_{n-1} \in B$. Then, by minimality, the map $\alpha_{(j_1,j_{k_2},\ldots,j_{k_s})}$ will vanish unless $j = (j_1, j_{k_2}, \ldots, j_{k_s})$, which implies

$$\sum_{\pi \in \mathrm{NC}(n)\setminus N} \hat{\alpha}_j(\pi)[b_1, \ldots, b_{n-1}] = \hat{\alpha}_j(1_n)[b_1, \ldots, b_{n-1}] = \alpha_j(b_1, \ldots, b_{n-1}) \neq 0. \qquad \square$$

One advantage of using B-valued circular elements is that they can be built from a semicircular family.

Definition 3.2.10. A family $(x_i)_{i\in I}$ of self adjoint B-valued random variables is *centered B-valued semicircular* if $\gamma_j = 0$ for every $j \in J := \bigcup_{n\geq 1} I^n$ of length not equal to 2, where $(\gamma_j)_{j\in J}$ are the cumulant maps of the family $(x_i)_{i\in I}$.

We essentially repeat Lemma 3.2.9, except now for a centered B-valued semicircular family. Note the difference in the index of the sum.

Lemma 3.2.11. *A family $(x_i)_{i\in I}$ of B-valued random variables is centered B-valued semicircular if and only if for every $n \geq 1$, $b_1, \ldots, b_{n-1} \in B$, and $j = (j_1, \ldots, j_n) \in J := \bigcup_{n\geq 1} I^n$,*

$$\begin{aligned}&E(x_{j_1} b_1 \cdots x_{j_{n-1}} b_{n-1} x_{j_n})\\ &\qquad = \sum_{p\geq 2} \gamma_{(j_1,j_p)}(b_1 E(x_{j_2} b_2 \cdots x_{j_{p-1}} b_{p-1})) b_p E(x_{j_{p+1}} b_{p+1} \cdots x_{j_{n-1}} b_{n-1} x_{j_n}).\end{aligned}$$

Proof. Similar to the proof of Lemma 3.2.9. □

The following proposition formalizes the connection between B-valued circular elements and centered B-valued semicircular pairs. It appears as Proposition 6.1 in [1]. We include a detailed proof.

Proposition 3.2.12 ([1], Proposition 6.1)**.** *Suppose a is an element of a B-valued $*$-noncommutative probability space and let*

$$x_1 = \operatorname{Re} a = \frac{a + a^*}{2} \qquad \textit{and} \qquad x_2 = \operatorname{Im} a = \frac{a - a^*}{2i}.$$

Let $J = \bigcup_{n\geq 1}\{1,2\}^n$, $(\alpha_j)_{j\in J}$ be the cumulant maps for $(a_1, a_2) = (a, a^*)$, and $(\gamma_j)_{j\in J}$ be the cumulant maps for (x_1, x_2). Then a is B-valued circular if and only if (x_1, x_2) is centered B-valued semicircular, $\gamma_{(1,1)} = \gamma_{(2,2)}$, and $\gamma_{(1,2)} = -\gamma_{(2,1)}$. Under these conditions we have

$$\gamma_{(1,1)} = \frac{1}{4}(\alpha_{(1,2)} + \alpha_{(2,1)}) \qquad \textit{and} \qquad \gamma_{(1,2)} = \frac{i}{4}(\alpha_{(1,2)} - \alpha_{(2,1)}).$$

Proof. Suppose a is B-valued circular. Using the moment-cumulant formula we discover

$$E(a) = E(a^*) = E(aba) = E(a^*ba^*) = 0,$$
$$E(aba^*) = \alpha_{(1,2)}(b), \text{ and}$$
$$E(a^*ba) = \alpha_{(2,1)}(b)$$

for all $b \in B$. Therefore $\gamma_{(1)} = E(x_1) = \frac{1}{2}(E(a) + E(a^*)) = 0$ and

$$\begin{aligned}\gamma_{(1,1)}(b) &= E(x_1bx_1) - \gamma_{(1)}b\gamma_{(1)} \\ &= \frac{1}{4}(E(aba) + E(aba^*) + E(a^*ba) + E(a^*ba^*)) \\ &= \frac{1}{4}(\alpha_{(1,2)}(b) + \alpha_{(2,1)}(b)).\end{aligned}$$

The relations $\gamma_{(1,1)} = \gamma_{(2,2)}$ and $\gamma_{(1,2)} = -\gamma_{(2,1)} = \frac{i}{4}(\alpha_{(1,2)} - \alpha_{(2,1)})$ follow similarly.

To prove (x_1, x_2) is centered B-valued semicircular, we will show the condition in Lemma 3.2.11. Fix $n \geq 1$, $b_1, \ldots, b_{n-1} \in B$, and $j_1, \ldots, j_n \in \{1, 2\}$. We begin by using Lemma 3.2.9 to obtain

$$\begin{aligned}&E(x_{j_1}b_1 \cdots x_{j_{n-1}}b_{n-1}x_{j_n}) \\ &\quad= \frac{1}{2^n}(-i)^{|\{s|j_s=2\}|} \sum_{k_1,\ldots,k_n \in \{1,2\}} (-1)^{|\{s|j_s=k_s=2\}|} E(a_{k_1}b_1 \cdots a_{k_{n-1}}b_{n-1}a_{k_n}) \\ &\quad= \frac{1}{2^n}(-i)^{|\{s|j_s=2\}|} \sum_{k_1,\ldots,k_n \in \{1,2\}} (-1)^{|\{s|j_s=k_s=2\}|} \\ &\qquad \sum_{\{p|k_p \neq k_1\}} \alpha_{(k_1,k_p)}(b_1E(a_{k_2}b_2 \cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1} \cdots a_{k_{n-1}}b_{n-1}a_{k_n})\end{aligned}$$

$$= \frac{1}{2^n}(-i)^{|\{s|j_s=2\}|} \sum_{k_2,\ldots,k_n\in\{1,2\}} (-1)^{|\{s\geq 2|j_s=k_s=2\}|}$$

$$\Bigg[\sum_{\{p|k_p=2\}} \alpha_{(1,k_p)}(b_1E(a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n})$$

$$+ \sum_{\{p|k_p=1\}} (-1)^{\delta_{j_1,2}}\alpha_{(2,k_p)}(b_1E(a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n})\Bigg].$$

For each choice of $k_2,\ldots,k_n \in \{1,2\}$ and each $p \in \{2,\ldots,n\}$, we have a term involving either $\alpha_{(1,k_p)}$ or $\alpha_{(2,k_p)}$, depending on whether $k_p = 2$ or $k_p = 1$. Thus we can instead iterate over $p \in \{2,\ldots,n\}$ and $k_2,\ldots,k_{p-1},k_{p+1},\ldots,k_n \in \{1,2\}$ and include the terms for both $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$. This yields

$$E(x_{j_1}b_1\cdots x_{j_{n-1}}b_{n-1}x_{j_n})$$

$$= \frac{1}{2^n}(-i)^{|\{s|j_s=2\}|}\sum_{p\geq 2}\sum_{\substack{k_s\in\{1,2\}\\ s\geq 2,\, s\neq p}} (-1)^{|\{s\geq 2,\, s\neq p|j_s=k_s=2\}|}$$

$$[(-1)^{\delta_{j_p,2}}\alpha_{(1,2)}(b_1E(a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n})$$

$$+ (-1)^{\delta_{j_1,2}}\alpha_{(2,1)}(b_1E(a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n})]$$

$$= \frac{1}{2^n}(-i)^{|\{s|j_s=2\}|}\sum_{p\geq 2}\sum_{\substack{k_s\in\{1,2\}\\ s\geq 2,\, s\neq p}} (-1)^{|\{s\geq 2,\, s\neq p|j_s=k_s=2\}|}$$

$$[(-1)^{\delta_{j_p,2}}\alpha_{(1,2)} + (-1)^{\delta_{j_1,2}}\alpha_{(2,1)}](b_1E(a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}))b_pE(a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n}).$$

After rearranging, we end the computation with two reversals of our first step to get

$$E(x_{j_1}b_1\cdots x_{j_{n-1}}b_{n-1}x_{j_n})$$

$$= \sum_{p\geq 2}\frac{1}{4}(-i)^{\delta_{j_1,2}+\delta_{j_p,2}}[(-1)^{\delta_{j_p,2}}\alpha_{(1,2)} + (-1)^{\delta_{j_1,2}}\alpha_{(2,1)}]$$

$$\left(b_1E\left(\frac{1}{2^{p-2}}(-i)^{\{2\leq s\leq p-1|j_s=2\}} \sum_{k_2,\ldots,k_{p-1}\in\{1,2\}} (-1)^{\{2\leq s\leq p-1|j_s=k_s=2\}} a_{k_2}b_2\cdots a_{k_{p-1}}b_{p-1}\right)\right)$$

$$b_pE\left(\frac{1}{2^{n-p}}(-i)^{\{p+1\leq s\leq n|j_s=2\}} \sum_{k_{p+1},\ldots,k_n\in\{1,2\}} (-1)^{\{p+1\leq s\leq n|j_s=k_s=2\}} a_{k_{p+1}}b_{p+1}\cdots a_{k_{n-1}}b_{n-1}a_{k_n}\right)$$

$$= \sum_{p\geq 2} \frac{1}{4}(-i)^{\delta_{j_1,2}+\delta_{j_p,2}}[(-1)^{\delta_{j_p,2}}\alpha_{(1,2)} + (-1)^{\delta_{j_1,2}}\alpha_{(2,1)}]$$

$$(b_1 E(x_{k_2} b_2 \cdots x_{k_{p-1}} b_{p-1})) b_p E(x_{k_{p+1}} b_{p+1} \cdots x_{k_{n-1}} b_{n-1} x_{k_n}).$$

It only remains to prove that

$$\frac{1}{4}(-i)^{\delta_{j_1,2}+\delta_{j_p,2}}[(-1)^{\delta_{j_p,2}}\alpha_{(1,2)} + (-1)^{\delta_{j_1,2}}\alpha_{(2,1)}] = \gamma_{(j_1,j_p)}.$$

Denoting the left-hand side of this equation by $L(j_1, j_p)$, we consider each of the four cases.

$$\begin{aligned}
j_1 = j_p = 1 &\quad\Longrightarrow\quad L(1,1) = \frac{1}{4}(\alpha_{(1,2)} + \alpha_{(2,1)}) = \gamma_{(1,1)} \\
j_1 = 1,\, j_p = 2 &\quad\Longrightarrow\quad L(1,2) = \frac{i}{4}(\alpha_{(1,2)} - \alpha_{(2,1)}) = \gamma_{(1,2)} \\
j_1 = 2,\, j_p = 1 &\quad\Longrightarrow\quad L(2,1) = \frac{-i}{4}(\alpha_{(1,2)} - \alpha_{(2,1)}) = \gamma_{(2,1)} \\
j_1 = j_p = 2 &\quad\Longrightarrow\quad L(2,2) = \frac{1}{4}(\alpha_{(1,2)} + \alpha_{(2,1)}) = \gamma_{(2,2)}
\end{aligned}$$

Conversely, the moment-cumulant formula implies $\alpha_{(1,1)} = \alpha_{(2,2)} = 0$ and

$$\alpha_{(1,2)} = 2(\gamma_{(1,1)} - i\gamma_{(1,2)}) \qquad \text{and} \qquad \alpha_{(2,1)} = 2(\gamma_{(1,1)} + i\gamma_{(1,2)}).$$

The rest of the proof follows similarly to the forward direction. □

We will use this proposition to build a B-valued circular element with specified cumulant maps by first appealing to Shlyaktenko's construction [6] to obtain a B-valued centered semicircular pair. Starting with completely positive linear functionals $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ on B, the maps $\gamma_{(i,j)}$, for $i, j = 1, 2$, form a covariance matrix. Shlayktenko's construction produces a semicircular system (x_1, x_2) having cumulant maps $(\gamma_{(i,j)})_{i,j=1,2}$ within a B-valued C*-noncommutative probability space. Thus $a = x_1 + ix_2$ is a B-valued circular element with cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$. Therefore, given completely positive cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$, we can obtain a B-valued circular element in a B-valued C*-noncommutative probability space having these cumulant maps.

Example 3.2.13 (Tracial $\mathbb{C}^2$-circular without free polar decomposition, [1], Example 6.8)**.** Set $B = \mathbb{C}^2$ and let $\tau_B(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_1 + \lambda_2)$ define a trace on B. Define linear functionals $\alpha_{(1,2)}, \alpha_{(2,1)}$ on B by

$$\alpha_{(1,2)}(\lambda_1, \lambda_2) = \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2} + \lambda_2\right)$$
$$\alpha_{(2,1)}(\lambda_1, \lambda_2) = \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_2\right).$$

The maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ are completely positive, so there is a C*-noncommutative probability space (A, E) containing a B-valued circular element a having these as its cumulant maps. We may assume A is the C*-algebra generated by $B \cup \{a\}$, and that the GNS representation of B is faithful. Due to Proposition 3.2.7, $\tau_B \circ E$ is a faithful tracial state on A.

However, a does not have a free polar decomposition. To prove this by way of contradiction, we suppose there exists u and p in some B-valued C*-noncommutative probability space such that u is a Haar unitary, p is positive, u and p are $*$-free over B, and a and up have the same B-valued $*$-distribution. Then we have

$$E(p^2) = E(a^* 1_B a) = \alpha_{(2,1)}(1, 1) = (1, 1) = 1_B,$$

which leads to the contradiction

$$\left(\frac{1}{2}, \frac{3}{2}\right) = \alpha_{(1,2)}(1, 1) = E(a 1_B a^*) = E(up^2 u^*) = E(uE(p^2)u^*) = E(u 1_B u^*) = (1, 1).$$

Therefore a cannot have a free polar decomposition.

4. DECOMPOSITIONS IN B-VALUED SPACES

4.1 Classes of B-valued Haar unitaries

Let B be a unital $*$-algebra and (A, E) be a B-valued $*$-noncommutative probability space.

Definition 4.1.1. Given an element $a \in A$, a $*$-moment

$$E(a^{\varepsilon(1)} b_1 a^{\varepsilon(2)} b_2 \cdots a^{\varepsilon(n-1)} b_{n-1} a^{\varepsilon(n)}),$$

with $n \in \mathbb{N}$, $b_1, \ldots, b_{n-1} \in B$, and $\varepsilon \in \{1, *\}^n$, is said to be *unbalanced* if the number of $*$'s differ from the number of non-$*$'s; i.e.,

$$\#\{j \mid \varepsilon(j) = 1\} \neq \frac{n}{2}.$$

An element $a \in A$ is called *balanced* if all of its unbalanced $*$-moments vanish. An element that is not balanced is called *unbalanced*.

It's an easy consequence of the moment-cumulant formula and the cumulant property of R-diagonal elements (see Theorem 3.2.5(b)) that every B-valued R-diagonal element is balanced. It's also clear that if a unitary is balanced, then it is a *Haar unitary*; i.e., $E(u^k) = 0$ for all integers $k \geq 1$.

Definition 4.1.2. A unitary $u \in A$ is said to *normalize* B if $ubu^*, u^*bu \in B$ for every $b \in B$.

Normalizing Haar unitaries will play an important role in the next section. It follows from the moment property of R-diagonal elements (see Definition 3.2.2) that every normalizing Haar unitary is R-diagonal. Indeed, when u normalizes B, for each $b \in B$ there is $b', b'' \in B$ satisfying $ub = b'u$ and $u^*b = b''u^*$. Consequently, odd alternating moments vanish because, given any $k \geq 1$ and $b_1, \ldots, b_{2k} \in B$, there is $b_0 \in B$ such that $ub_1u^*b_2 \cdots b_{2k-2}ub_{2k-1}u^*b_{2k}u = b_0u$. Fix

$n \geq 1$, $\varepsilon \in \{1, *\}^n$, and $b_1, \ldots, b_n \in B$, and consider the expression

$$E\left(\prod_{V\in\sigma(\varepsilon)}\left(\left(\prod_{j\in V} u^{\varepsilon(j)}b_j\right) - E\left(\prod_{j\in V} u^{\varepsilon(j)}b_j\right)\right)\right). \tag{4.1}$$

If V_0 is an alternating interval block in $\sigma(\varepsilon)$ of even length, then $\prod_{j\in V_0} u^{\varepsilon(j)}b_j \in B$ and consequently the term

$$\left(\prod_{j\in V_0} u^{\varepsilon(j)}b_j\right) - E\left(\prod_{j\in V_0} u^{\varepsilon(j)}b_j\right)$$

vanishes. Thus (4.1) vanishes. If there is no block of even length, then every block in $\sigma(\varepsilon)$ has odd length. Thus

$$E\left(\prod_{j\in V} a^{\varepsilon(j)}b_j\right) = 0$$

for each $V \in \sigma(\varepsilon)$, so expression (4.1) becomes

$$\begin{aligned} E\left(\prod_{V\in\sigma(\varepsilon)}\left(\left(\prod_{j\in V} u^{\varepsilon(j)}b_j\right) - E\left(\prod_{j\in V} u^{\varepsilon(j)}b_j\right)\right)\right) &= E(u^{\varepsilon(1)}b_1 \cdots u^{\varepsilon(n)}b_n) \\ &= b_0 E(u^k) \end{aligned}$$

for some $b_0 \in B$ and $k \in \mathbb{Z}$. Since there are no even length blocks in the maximal alternating interval partition, there must an unqual number of 1's and $*$'s. Thus $k \neq 0$, so (4.1) vanishes by the Haar property. This completes the proof that u is R-diagonal.

The following is a summary of these basic facts.

Normalizing Haar unitaries $\subseteq$ R-diagonal unitaries $\subseteq$ Balanced unitaries $\subseteq$ Haar unitaries

In the case $B = \mathbb{C}$, a Haar unitary is trivially R-diagonal and thus also balanced. The notion of normalizing is also redundant because every element normalizes the scalars. Thus all of these inclusions are actually equalities. This is not the case more generally. The next three examples will show that each inclusion is strict outside the scalar setting.

Example 4.1.3 (Unbalanced Haar unitary)**.** Let $A = M_2(C(\mathbb{T}))$ be the algebra of 2×2 matrices with entries from the continuous functions on the circle. We identify $B = \mathbb{C}^2$ with the subspace of A consisting of diagonal matrices whose entries are constant functions. Let $E : A \to B$ be the conditional expectation given by

$$E\left(\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\right) = \begin{pmatrix} \tau(f_{11}) & 0 \\ 0 & \tau(f_{22}) \end{pmatrix},$$

where τ is the trace on $C(\mathbb{T})$ given by integration with respect to the Haar measure on $\mathbb{T}$. This gives the B-valued C*-noncommutative probability space (A, E).

Let $v \in C(\mathbb{T})$ be the identity function; namely, $z \mapsto z$. Then v is a Haar unitary in $C(\mathbb{T})$ with respect to τ. By identifying A with $M_2(\mathbb{C}) \otimes C(\mathbb{T})$ in the usual way, we define the unitary

$$u = p \otimes v + (1 - p) \otimes v^* \in A,$$

where the projection $p \in M_2(\mathbb{C})$ is given by

$$p = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $u^k = p \otimes v^k + (1 - p) \otimes (v^*)^k$, we have

$$E(u^k) = \frac{1}{2}\begin{pmatrix} \tau(v^k) + \tau((v^*)^k) & 0 \\ 0 & \tau(v^k) + \tau((v^*)^k) \end{pmatrix}.$$

Hence u is a Haar unitary.

To see that u is unbalanced, we consider the matrix unit $e_{11} \in M_2(\mathbb{C})$ as an element of B and compute

$$\begin{aligned}
E(ue_{11}u) &= E\left(\frac{1}{4}\begin{pmatrix} v+v^* & 0 \\ v-v^* & 0 \end{pmatrix}\begin{pmatrix} v+v^* & v-v^* \\ v-v^* & v+v^* \end{pmatrix}\right) \\
&= E\left(\frac{1}{4}\begin{pmatrix} v^2+(v^*)^2+2 & v^2-(v^*)^2 \\ v^2-(v^*)^2 & v^2+(v^*)^2-2 \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

We conclude that u is an unbalanced Haar unitary.

The following example follows a similar pattern.

Example 4.1.4 (Balanced unitary that is not R-diagonal)**.** Let $v = 1 \cdot (1,0)$ and $w = 1 \cdot (0,1)$ be two commuting Haar unitaries in the group C*-algebra C*$(\mathbb{Z} \times \mathbb{Z})$ with respect to the canonical trace τ, which is defined by

$$\tau\left(\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} c_{(n,m)} \cdot (n,m)\right) = c_{(0,0)}.$$

Note that $v^n w^m = 1 \cdot (n,m)$, so

$$\tau(v^n w^m) = 0 \text{ for all } (n,m) \in \mathbb{Z}\times\mathbb{Z} \setminus \{(0,0)\}. \tag{4.2}$$

Let $A = M_2(C^*(\mathbb{Z}\times\mathbb{Z}))$, and identify $B = \mathbb{C}^2$ with the set of diagonal matrices whose entries are only supported on the identity element $(0,0)$ of $\mathbb{Z}\times\mathbb{Z}$. By defining the conditional expectation $E : A \to B$ by

$$E\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right) = \begin{pmatrix} \tau(g_{11}) & 0 \\ 0 & \tau(g_{22}) \end{pmatrix},$$

we have the B-valued C*-noncommutative probability space (A, E).

Taking p as in Example 4.1.3, we define the unitary

$$u = p \otimes v + (1 - p) \otimes w.$$

To show this is a balanced Haar unitary, we fix an unbalanced B-valued $*$-moment

$$x = u^{\varepsilon(1)} b_1 \cdots u^{\varepsilon(n-1)} b_{n-1} u^{\varepsilon(n)},$$

where $n \in \mathbb{N}$, $b_1, \ldots, b_{n-1} \in B$, $\varepsilon \in \{1, *\}^n$, and $m := \#\{j \mid \varepsilon(j) = 1\}$ is not equal to $n/2$. Observe that x is a 2×2 matrix (a_{ij}), where each entry is of the form

$$a_{ij} = \sum_{k=0}^{m} \sum_{\ell=0}^{n-m} \lambda_{k,\ell}^{ij} u^k (u^*)^\ell v^{m-k} (v^*)^{n-m-\ell}.$$

Since $m \neq n/2$, we cannot have both $k = \ell$ and $m - k = n - m - \ell$ occuring simultaneously. We deduce from property (4.2) that each term $u^k (u^*)^\ell v^{m-k} (v^*)^{n-m-\ell}$ inside the sum has a trace of zero. Consequently $\tau(a_{ii}) = 0$ for each i so that $E(x) = 0$. Therefore u is a balanced unitary.

We show u is not B-valued R-diagonal by showing that it violates the moment condition Definition 3.2.2. More precisely, we aim to prove

$$E([u^* b_1 u - E(u^* b_1 u)] b_2 [u b_3 u^* - E(u b_3 u^*)]) \neq 0$$

for some $b_1, b_2, b_3 \in B$. To this end, we compute

$$\begin{aligned} u^* b_1 u &= \frac{1}{4} \begin{pmatrix} b_1^{(1)}(v^* + w^*) & b_1^{(2)}(v^* - w^*) \\ b_1^{(1)}(v^* - w^*) & b_1^{(2)}(v^* + w^*) \end{pmatrix} \begin{pmatrix} v + w & v - w \\ v - w & v + w \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2(b_1^{(1)} + b_1^{(2)}) + (b_1^{(1)} - b_1^{(2)})(vw^* + v^*w) & (b_1^{(1)} - b_1^{(2)})(vw^* - v^*w) \\ -(b_1^{(1)} - b_1^{(2)})(vw^* - v^*w) & 2(b_1^{(1)} + b_1^{(2)}) - (b_1^{(1)} - b_1^{(2)})(vw^* + v^*w) \end{pmatrix}, \end{aligned}$$

which gives

$$E(u^*b_1u) = \frac{1}{2}\begin{pmatrix} b_1^{(1)} + b_1^{(2)} & 0 \\ 0 & b_1^{(1)} + b_1^{(2)} \end{pmatrix}$$

and consequently

$$u^*b_1u - E(u^*b_1u) = \frac{1}{4}\begin{pmatrix} (b_1^{(1)} - b_1^{(2)})(vw^* + v^*w) & (b_1^{(1)} - b_1^{(2)})(vw^* - v^*w) \\ -(b_1^{(1)} - b_1^{(2)})(vw^* - v^*w) & -(b_1^{(1)} - b_1^{(2)})(vw^* + v^*w) \end{pmatrix}.$$

Similar computations yield

$$ub_3u^* - E(ub_3u^*) = \frac{1}{4}\begin{pmatrix} (b_3^{(1)} - b_3^{(2)})(vw^* + v^*w) & -(b_3^{(1)} - b_3^{(2)})(vw^* - v^*w) \\ (b_3^{(1)} - b_3^{(2)})(vw^* - v^*w) & -(b_3^{(1)} - b_3^{(2)})(vw^* + v^*w) \end{pmatrix}.$$

Using these relations and the identities

$$\begin{aligned} (vw^* + v^*w)^2 &= 2 + v^2(w^*)^2 + (v^*)^2w^2 \\ (vw^* + v^*w)(vw^* - v^*w) &= v^2(w^*)^2 - (v^*)^2w^2 \\ (vw^* - v^*w)^2 &= -2 + v^2(w^*)^2 + (v^*)^2w^2, \end{aligned}$$

we obtain

$$\begin{aligned} &E([u^*b_1u - E(u^*b_1u)]b_2[ub_3u^* - E(ub_3u^*)]) \\ &\qquad = \frac{1}{8}\begin{pmatrix} (b_1^{(1)} - b_1^{(2)})(b_2^{(1)} - b_2^{(2)})(b_3^{(1)} - b_3^{(2)}) & 0 \\ 0 & -(b_1^{(1)} - b_1^{(2)})(b_2^{(1)} - b_2^{(2)})(b_3^{(1)} - b_3^{(2)}) \end{pmatrix}. \end{aligned}$$

Taking b_1, b_2, b_3 to all be the matrix unit e_{11} ensures the above expectation is nonzero. Therefore the balanced unitary u is not B-valued R-diagonal.

Before presenting the third example, we remark on a fifth class of unitaries that will be closely connected with the class of R-diagonal unitaries. Every R-diagonal unitary u trivially appears as the polar part of the polar decomposition $u1_A$ of a B-valued R-diagonal element. This leads to the following question: If a unitary appears in the polar decomposition of a B-valued R-diagonal element, does the unitary have to also be B-valued R-diagonal? The following proposition, which is a generalization of Proposition 5.2 from [1], answers this question in the affirmative.

Proposition 4.1.5. *Let B be a von Neumann algebra and (A, E) be a B-valued W*-noncommutative probability space. Suppose $a \in A$ is B-valued R-diagonal and $a = v|a|$ is the polar decomposition of a. Then v is B-valued R-diagonal.*

Proof. Due to the amalgamated free product construction for von Neumann algebras, we may assume without loss of generality that there is a Haar unitary $u \in A$ that commutes with B and is $*$-free from a over B. Thus u and v are $*$-free over B. Using Theorem 3.2.5(c), we realize that a and ua have the same B-valued $*$-distribution. Therefore, since we are in a W*-noncommutative probability space and the polar decomposition of ua is $uv|a|$, the elements v and uv must have the same B-valued $*$-distribution. Now we may employ Theorem 3.2.5(d) to get that v is B-valued R-diagonal. □

We'll also need the following lemma for our third example.

Lemma 4.1.6. *Let B be a unital C*-algebra with a faithful trace τ and let (A, E) be a C*-noncommutative probability space such that $\tau \circ E$ is a trace. If u normalizes B with the automorphism θ on B defined by $\theta(b) = u^*bu$, then*

(a) u *normalizes* $\ker(E)$,

(b) $E(u^*xu) = \theta(E(x))$ *for all* $x \in A$, *and*

(c) $E(uxu^*) = \theta^{-1}(E(x))$ *for all* $x \in A$.

Proof. Assuming (a) holds and writing $x \in A$ as $x = \mathring{x} + E(x)$, we have $u^*\mathring{x}u, u\mathring{x}u^* \in \ker(E)$ so that

$$E(u^*xu) = E(u^*\mathring{x}u) + E(u^*E(x)u) = \theta(E(x))$$

and

$$E(uxu^*) = E(u\mathring{x}u^*) + E(uE(x)u^*) = \theta^{-1}(E(x)).$$

Hence it suffices to prove (a).

Let $x \in \ker(E)$. By the traciality of $\tau \circ E$, we have

$$\begin{aligned}
\tau(E(u^*xu)^*E(u^*xu)) &= (\tau \circ E)(u^*x^*uE(u^*xu)) \\
&= (\tau \circ E)(x^*uE(u^*xu)u^*) \\
&= (\tau \circ E)(x^*\theta^{-1}(E(u^*xu))) \\
&= \tau(E(x^*)\theta^{-1}(E(u^*xu))) \\
&= 0.
\end{aligned}$$

The faithfulness of τ gives $E(u^*xu) = 0$. Similarly $E(uxu^*) = 0$, which completes the proof. □

Now we can go forward with our final example of the section.

Example 4.1.7 (R-diagonal unitary that is not normalizing)**.** Let $B = \mathbb{C}^2$. Recall the example (see Example 3.2.13) of the B-valued circular a that does not have a free polar decomposition. By considering the von Neumann generated by the algebra in that example, we can consider the polar decomposition $u|a|$ of a. Proposition A.1 of [1] shows that u is a unitary, and so by Proposition 4.1.5, u is B-valued R-diagonal. We will use the cumulants of a

$$\begin{aligned}
\alpha_{(1,2)}(\lambda_1, \lambda_2) &= \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2} + \lambda_2\right) \\
\alpha_{(2,1)}(\lambda_1, \lambda_2) &= \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_2\right)
\end{aligned}$$

to show that u fails to normalize B.

Assume u normalizes B with the automorphism θ on B defined by $\theta(b) = u^*bu$. We hope to find a contradiction. The example has B equipped with a faithful trace τ such that $\tau \circ E$ is a trace, so we can apply part (c) of Lemma 4.1.6 to obtain the relation

$$E(aa^*baa^*) = E(u|a|^2u^*bu|a|^2u^*) = \theta^{-1}(E(|a|^2\theta(b)|a|^2)) = \theta^{-1}(E(a^*a\theta(b)a^*a)) \tag{4.3}$$

for all $b \in B$.

Set $b = (1, -1)$. With the use of the moment cumulant formula, we verify that the leftmost expression of (4.3) is equal to

$$E(aa^*baa^*) = \alpha_{1,2}(1_B)b\alpha_{1,2}(1_B) + \alpha_{1,2}(\alpha_{2,1}(b)) = \left(\frac{1}{4}, -\frac{13}{4}\right).$$

Since there are only two automorphisms on $B = \mathbb{C}^2$, we can easily compute the rightmost expression of (4.3) for either case. If θ is the identity on B, then the rightmost expression is

$$\theta^{-1}(E(a^*a\theta(b)a^*a)) = E(a^*aba^*a) = \alpha_{2,1}(1_B)b\alpha_{2,1}(1_B) + \alpha_{2,1}(\alpha_{1,2}(b)) = \left(1, -\frac{3}{2}\right).$$

If θ is the flip on B (i.e., $\theta((\lambda_1, \lambda_2)) = (\lambda_2, \lambda_1)$), then we have

$$\theta^{-1}(E(a^*a\theta(b)a^*a)) = \theta^{-1}(\alpha_{2,1}(1_B)\theta(b)\alpha_{2,1}(1_B) + \alpha_{2,1}(\alpha_{1,2}(\theta(b)))) = \left(\frac{3}{2}, -1\right).$$

In either case we have verified that equation (4.3) is violated. Therefore the R-diagonal unitary u does not normalize B.

4.2 Even decompositions

Definition 4.2.1. A B-valued random variable a in a B-valued $*$-noncommutative probability space is said to be *even* if all of its odd B-valued $*$-moments vanish; that is, if for all $n \in \mathbb{N}$, $\varepsilon \in \{1, *\}^{2n+1}$, and $b_1, \ldots, b_{2n} \in B$, we have

$$E(a^{\varepsilon(1)} b_1 \cdots a^{\varepsilon(2n)} b_{2n} a^{\varepsilon(2n+1)}) = 0.$$

The only previously known result regarding free decompositions of general operator-valued R-diagonal elements is due to the work of Boedihardjo and Dykema. Their work includes in the following theorem, which appears as part of Proposition 5.5 in [1].

Theorem 4.2.2 ([1]). *Suppose B is a unital $*$-algebra and (A, E) is a B-valued $*$-noncommutative probability space. Fix a random variable $a \in A$ and an automorphism θ of B. Then a is B-valued R-diagonal and the kth order B-valued cumulant maps $\beta_k^{(1)}$ and $\beta_k^{(2)}$ satisfy*

$$\beta_k^{(2)}(b_1, \theta(b_2), b_3, \ldots, \theta(b_{2k-2}), b_{2k-1}) = \theta(\beta_k^{(1)}(\theta(b_1), b_2, \theta(b_3), \ldots, b_{2k-2}, \theta(b_{2k-1}))) \tag{4.4}$$

if and only if there exists a B-valued $*$-noncommutative probability space (A', E') containing elements $s, u \in A'$ satisfying the following properties.

(a) s is an even self adjoint element.

(b) u is a B-normalizing Haar unitary with $u^* b u = \theta(b)$ for all $b \in B$.

(c) u and s are $*$-free over B with respect to E'.

(d) a and us have the same B-valued $*$-distribution.

One of the limitations of this theorem, compared to the situation in the scalar setting, is that it concerns a free decomposition with an even self adjoint instead of a free polar decomposition. This cannot be circumvented. The following example demonstrates that a B-valued R-diagonal element satisfying the cumulant condition (4.4) need not admit a free polar decomposition with a normalizing unitary.

Example 4.2.3. Let a be a circular element in a tracial, scalar-valued C*-noncommutative probability space (A_0, τ_0) with $\tau_0(a^*a) = 1$. Suppose a has polar decomposition $a = v|a|$. By Proposition 2.6 from [9], the partial isometry v is a Haar unitary.

Let B be a unital C*-algebra different from $\mathbb{C}$ with a faithful tracial state τ_B. We construct $(A, \tau) = (A_0, \tau_0) * (B, \tau_B)$ using the free product, and let $E : A \to B$ be the τ-preserving conditional expectation. Then (A, E) is a B-valued C*-noncommutative probability space and $\tau \circ E = \tau$. Theorem 12 of [7] gives a way to express the B-valued cumulants of a in terms of its scalar cumulants and the trace τ_B. In particular, this implies a is a B-valued circular element whose cumulant maps $\beta_k^{(1)}$ and $\beta_k^{(2)}$ satisfy

$$\beta_1^{(1)}(b) = \beta_1^{(2)}(b) = \tau_B(b)1_B$$

for all $b \in B$ and $\beta_k^{(1)} = \beta_k^{(2)} = 0$ for all $k \geq 2$. Thus a satisfies the cumulant condition (4.4) from the above theorem whenever θ is τ_B-preserving; i.e., $\tau_B \circ \theta = \tau_B$.

Fix a τ-preserving automorphism θ. To reach a contradiction, we suppose a has the same B-valued $*$-distribution as the product $u|a|$, where u is a normalizing Haar with $u^*bu = \theta(b)$ for all $b \in B$. For a polar decomposition, the B-valued $*$-distribution of u is determined by that of a. Since we already have $a = v|a|$, it must be the case that u and v have the same $*$-distribution. However, this is impossible because v is $*$-free from B with respect to τ. Indeed, fixing a nonzero $b \in \ker(\tau) \cap B$ (which is possible because $B \neq \mathbb{C}$), we observe that $*$-freeness yields

$$\begin{aligned}
&\tau([E(v^*bv) - \theta(b)]^*[E(v^*bv) - \theta(b)]) \\
&\qquad = (\tau \circ E)(v^*b^*vE(v^*bv)) - (\tau \circ E)(v^*b^*v\theta(b)) - (\tau \circ E)(\theta(b^*)v^*b^*v) + (\tau \circ \theta)(b^*b) \\
&\qquad = \tau(v^*b^*vE(v^*bv)) - \tau(v^*b^*v\theta(b)) - \tau(\theta(b^*)v^*b^*v) + (\theta \circ \tau)(b^*b) \\
&\qquad = 0 - 0 - 0 + \tau(b^*b) = \tau(b^*b).
\end{aligned}$$

The faithfulness of τ forces this to be nonzero and indicates $E(v^*bv) \neq \theta(b)$. Hence v and u do not have the same B-valued $*$-distribution, so a cannot have such a polar decomposition.

Theorem 4.2.2 and the previous example motivate the study of even decompositions.

Definition 4.2.4. Let B be a unital $*$-algebra and (A, E) be a B-valued $*$-noncommutative probability space. An *even decomposition* of an element $a \in A$ is a pair (u, s) in some B-valued $*$-noncommutative probability space such that

(a) u is a partial isometry,

(b) s is an even self adjoint element, and

(c) a and us have the same B-valued $*$-distribution.

If additionally u and s are $*$-free over B, then we call (u, s) a *free even decomposition* of a.

We can use the existence of polar decompositions in von Neuamnn algebras to prove that an even decomposition always exists.

Proposition 4.2.5. *Suppose B is a C*-algebra and (A, E) is a B-valued C*-noncommutative probability space. Then every $a \in A$ has an even decomposition.*

Proof. Fix $a \in A$. We may assume without loss of generality that B is a W*-algebra and (A, E) is a B-valued W*-noncommutative probability space. Then a has a polar decomposition $a = v|a|$. Let $A' = A^2$, which contains contains the copy $\{(b, b) \mid b \in B\}$ of B. Define $E' : A' \to B$ by $E'(a_1, a_2) = \frac{1}{2}E(a_1 + a_2)$. Then (A', E') is a B-valued $*$-noncommutative probability space. Letting $s = (|a|, -|a|)$ and $u = (v, -v)$ in A', we have that a and us have the same B-valued $*$-distribution. $\square$

Even decompositions are not unique. This is clear upon revisiting Example 4.2.3, which is an example of a B-valued circular a whose cumulants satisfy (4.4) for every trace preserving automorphism θ. Therefore Theorem 4.2.2 says that, for each of these automorphisms θ, there exists a free even decomposition (u_θ, s_θ) of a, where u is a normalizing unitary satisfying $u_\theta^* b u_\theta = \theta(b)$ for all $b \in B$.

In particular, by taking $B = \mathbb{C}^2$ and $\tau(x, y) = \frac{x+y}{2}$, both automorphisms on B are τ-preserving. Consequently Example 4.2.3 gives rise two even decompositions, where one of the Haar unitaries commutes with B and the other Haar unitary normalizes B and implements the flip automorphism $(x, y) \mapsto (y, x)$. Clearly these two Haar unitaries have different B-valued $*$-distributions.

This leads to the following question: Are there any R-diagonal random variables having a free even decomposition that are not of the type seen in Theorem 4.2.2? In other words, can we find an R-diagonal random variable that has a free even decomposition, but not one with a normalizing unitary? Of course, this is equivalent to asking whether there is an R-diagonal random variable having a free even decomposition whose cumulants do not satisfy equation (4.4) for any automorphism θ. The next section is devoted to answering this question for tracial $\mathbb{C}^2$-valued circular elements.

4.3 $\mathbb{C}^2$-valued circular elements

Recall that a B-valued random variable a is called *B-valued circular* if $\alpha_j = 0$ whenever $j \in J \setminus \{(1,2),(2,1)\}$, where $J = \bigcup_{n\geq 1}\{1,2\}^n$, and $(\alpha_j)_{j\in J}$ are the cumulant maps associated to the pair $(a_1, a_2) = (a, a^*)$.

A simple application of Proposition 3.2.7 implies that if a is a B-valued circular element with associated cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$, and if τ is a trace on B, then $\tau \circ E$ is a trace on $\mathrm{Alg}(B \cup \{a, a^*\})$ if and only if

$$\tau(\alpha_{(1,2)}(b_1)b_2) = \tau(b_1\alpha_{(2,1)}(b_2)) \qquad \text{for all } b_1, b_2 \in B. \tag{4.5}$$

We are interested in studying the free even decompositions of $\mathbb{C}^2$-valued circular elements with the tracial property (4.5). Let (A, E) be a $\mathbb{C}^2$-valued $*$-noncommutative probability space, and fix a $\mathbb{C}^2$-valued circular $a \in A$. Then $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ are given by

$$\alpha_{(1,2)}(x,y) = (r_{11}x + r_{12}y, r_{21}x + r_{22}y)$$
$$\alpha_{(2,1)}(x,y) = (s_{11}x + s_{12}y, s_{21}x + s_{22}y),$$

for some parameters r_{ij}, s_{ij}. Since the conditional expectation is positive and we have both $\alpha_{(1,2)}(x,y) = E(a(x,y)a^*)$ and $\alpha_{(2,1)}(x,y) = E(a^*(x,y)a)$, we must have that $r_{ij} \geq 0$ and $s_{ij} \geq 0$ for each $i,j \in \{1,2\}$.

We find additional restrictions on the parameters so that for some trace $\tau(x,y) = qx + (1-q)y$, where $q \in (0,1)$, the corresponding functional $\tau \circ E$ is tracial. Given elements $b_1 = (x_1, y_1), b_2 = (x_2, y_2) \in \mathbb{C}^2$, we have

$$\tau(\alpha_{(1,2)}(b_1)b_2) = \tau(b_1\alpha_{(2,1)}(b_2))$$
$$\iff \tau(\alpha_{(1,2)}(x_1,y_1)(x_2,y_2)) = \tau((x_1,y_1)\alpha_{(2,1)}(x_2,y_2))$$
$$\iff \tau((r_{11}x_1 + r_{12}y_1)x_2, (r_{21}x_1 + r_{22}y_1)y_2) = \tau(x_1(s_{11}x_2 + s_{12}y_2), y_1(s_{21}x_2 + s_{22}y_2))$$

$$\iff qr_{11}x_1x_2 + qr_{12}y_1x_2 + (1-q)r_{21}x_1y_2 + (1-q)r_{22}y_1y_2$$
$$= qs_{11}x_1x_2 + qs_{12}x_1y_2 + (1-q)s_{21}y_1x_2 + (1-q)s_{22}y_1y_2.$$

This holds for every $b_1, b_2 \in \mathbb{C}^2$ iff $r_{11} = s_{11}$, $r_{22} = s_{22}$, $qr_{12} = (1-q)s_{21}$, and $(1-q)r_{21} = qs_{12}$, which means the cumulants satisfy

$$\begin{aligned} \alpha_{(1,2)}(x,y) &= (r_{11}x + r_{12}y, r_{21}x + r_{22}y) \\ \alpha_{(2,1)}(x,y) &= (r_{11}x + \frac{1-q}{q}r_{21}y, \frac{q}{1-q}r_{12}x + r_{22}y). \end{aligned} \tag{4.6}$$

Our goal is to find conditions under which a has the same $\mathbb{C}^2$-valued $*$-distribution as a product of the form us, where u is a Haar unitary, s is self adjoint and even, and u and s are $*$-free over $\mathbb{C}^2$. That is, we want to determine when a has a free even decomposition with a Haar unitary. We know from Theorem 4.2.2 that if there is an automorphism θ of $\mathbb{C}^2$ such that $\alpha_{(2,1)} = \theta \circ \alpha_{(1,2)} \circ \theta$, then a automatically has a free even decomposition with a normalizing Haar unitary. In fact, this will turn out to be an equivalence because of the following theorem.

Theorem 4.3.1. *Let a be a $\mathbb{C}^2$-valued circular random variable that is tracial in the sense of (4.5) and has the same $\mathbb{C}^2$-valued $*$-distribution as a product up, where u is a Haar unitary, p is self adjoint, and u and p are $*$-free over $\mathbb{C}^2$. Then*

$$\alpha_{(2,1)} = \theta \circ \alpha_{(1,2)} \circ \theta \tag{4.7}$$

for some automorphism θ on $\mathbb{C}^2$.

In particular, this theorem says that if a $\mathbb{C}^2$-valued circular element has a free even decomposition or a free polar decomposition with a Haar unitary, then its cumulants satisfy the automorphism condition (4.7). Combining this theorem with Theorem 4.2.2 implies that all $\mathbb{C}^2$-valued circular elements with a free even decomposition consisting of a Haar unitary must have a free even decomposition with a normalizing Haar unitary.

Corollary 4.3.2. *Let a be a $\mathbb{C}^2$-valued circular random variable that is tracial in the sense of (4.5). Then a has a free even decomposition with a Haar unitary if and only if it has a free even decomposition with a normalizing Haar unitary.*

The rest of the section is devoted to proving Theorem 4.3.1. There are only two automorphisms on $\mathbb{C}^2$, so it's easy to determine what the automorphism condition (4.7) means in terms of the parameters $q, r_{11}, r_{12}, r_{21}, r_{22}$. If θ is the identity, then (4.7) is equivalent to $qr_{12} = (1-q)r_{21}$. If θ is the flip automorphism defined by $\theta((x,y)) = (y,x)$, then (4.7) is equivalent to

$$(r_{11}x + \frac{1-q}{q}r_{21}y, \frac{q}{1-q}r_{12}x + r_{22}y) = (r_{21}y + r_{22}x, r_{11}y + r_{12}x), \quad x, y \in \mathbb{C},$$

which holds iff $r_{11} = r_{22}$ and $q = 1/2$. Hence there exists some automorphism θ on $\mathbb{C}^2$ satisfying (4.7) if and only if the parameters satisfy

$$qr_{12} = (1-q)r_{21} \quad \text{or} \quad (r_{11} = r_{22} \quad \text{and} \quad q = 1/2). \tag{4.8}$$

Suppose a satisfies (4.5) and has the same B-valued distribution as up, where u is a Haar unitary, p is self adjoint, and u and p are $*$-free over $\mathbb{C}^2$. To prove Theorem 4.3.1, we aim to get a contradiction by also assuming that (4.7) does not hold for either automorphism on $\mathbb{C}^2$. That is, we assume the negation of the automorphism condition (4.8) to get

$$qr_{12} \neq (1-q)r_{21} \quad \text{and} \quad (r_{11} \neq r_{22} \quad \text{or} \quad q \neq 1/2). \tag{4.9}$$

This will lead to a contradiction.

In order to find further conditions among the r_{ij}'s that are implied by the existence of a free even decomposition, we use $*$-freeness to find some relations on the $*$-moments of a.

Lemma 4.3.3. *If x is a B-valued circular random variable with the same B-valued $*$-distribution as a product up, where u is Haar unitary, p is self adjoint, and u and p are $*$-free over B, then for* *all integers* $n, m, k \geq 0$ *we have*

$$E((xx^*)^n) = E(uE(p^{2n})u^*) = E(uE((x^*x)^n)u^*)$$

and

$$\begin{aligned} E((xx^*)^n(x^*x)^m(xx^*)^k) &= E(uE(p^{2n}E(u^*E(p^{2m})u)p^{2k})u^*) \\ &\quad - E(uE(p^{2n})E(u^*E(p^{2m})u)E(p^{2k})u^*) \\ &\quad + E(uE(p^{2n})u^*E(p^{2m})uE(p^{2k})u^*) \end{aligned}$$

Proof. If $x = up$, then $xx^* = up^2u^*$ so that

$$E((xx^*)^n) = E((up^2u^*)^n) = E(up^{2n}u^*) = E(uE(p^{2n})u^*) = E(uE((x^*x)^n)u^*),$$

where the penultimate equality is due to freeness.

Given a B-valued random variable y, we recall the notation

$$\mathring{y} = y^\circ := y - E(y),$$

which gives $y = \mathring{y} + E(y)$ and $E(\mathring{y}) = 0$ for every y. Now

$$\begin{aligned} E((xx^*)^n(x^*x)^m(xx^*)^k) &= E(up^{2n}u^*p^{2m}up^{2k}u^*) \\ &= E(up^{2n}u^*(p^{2m})^\circ up^{2k}u^*) + E(up^{2n}u^*E(p^{2m})up^{2k}u^*). \qquad (4.10) \end{aligned}$$

We consider each of these terms separately. The first one is zero because

$$\begin{aligned} E(up^{2n}u^*(p^{2m})^\circ up^{2k}u^*) &= E(u(p^{2n})^\circ u^*(p^{2m})^\circ u(p^{2k})^\circ u^*) + E(u(p^{2n})^\circ u^*(p^{2m})^\circ uE(p^{2k})u^*) \\ &\quad + E(uE(p^{2n})u^*(p^{2m})^\circ u(p^{2k})^\circ u^*) + E(uE(p^{2n})u^*(p^{2m})^\circ uE(p^{2k})u^*), \end{aligned}$$

and $*$-freeness of u and p ensures that each of these terms vanish. We rewrite the second term on the right-hand side of (4.10) to obtain

$$E(up^{2n}u^*E(p^{2m})up^{2k}u^*) = E(up^{2n}[u^*E(p^{2m})u]^\circ p^{2k}u^*) + E(up^{2n}E[u^*E(p^{2m})u]p^{2k}u^*). \quad (4.11)$$

The first term from (4.11) expands to

$$\begin{aligned} E(up^{2n}[u^*E(p^{2m})u]^\circ p^{2k}u^*) &= E(u(p^{2n})^\circ[u^*E(p^{2m})u]^\circ(p^{2k})^\circ u^*) \\ &\quad + E(u(p^{2n})^\circ[u^*E(p^{2m})u]^\circ E(p^{2k})u^*) \\ &\quad + E(uE(p^{2n})[u^*E(p^{2m})u]^\circ(p^{2k})^\circ u^*) \\ &\quad + E(uE(p^{2n})[u^*E(p^{2m})u]^\circ E(p^{2k})u^*), \end{aligned}$$

and $*$-freeness of u and p implies that only the fourth term remains. Hence the first term on the right-hand side of (4.11) satisfies

$$\begin{aligned} E(up^{2n}[u^*E(p^{2m})u]^\circ p^{2k}u^*) &= E(uE(p^{2n})[u^*E(p^{2m})u]^\circ E(p^{2k})u^*) \\ &= E(uE(p^{2n})u^*E(p^{2m})uE(p^{2k})u^*) \\ &\quad - E(uE(p^{2n})E(u^*E(p^{2m})u)E(p^{2k})u^*). \end{aligned}$$

The other term from (4.11) can be rewritten as

$$\begin{aligned} &E(up^{2n}E[u^*E(p^{2m})u]p^{2k}u^*) \\ &\qquad = E(u[p^{2n}E[u^*E(p^{2m})u]p^{2k}]^\circ u^*) + E(uE[p^{2n}E[u^*E(p^{2m})u]p^{2k}]u^*), \end{aligned}$$

and freeness forces the first of these two terms to be zero. This shows that the second term on the right-hand side of (4.11) satisfies

$$E(up^{2n}E[u^*E(p^{2m})u]p^{2k}u^*) = E(uE(p^{2n}E(u^*E(p^{2m})u)p^{2k})u^*).$$

Putting everything together, we conclude

$$\begin{aligned}E((xx^*)^n(x^*x)^m(xx^*)^k) &= E(uE(p^{2n})u^*E(p^{2m})uE(p^{2k})u^*)\\ &- E(uE(p^{2n})E(u^*E(p^{2m})u)E(p^{2k})u^*)\\ &+ E(uE(p^{2n}E(u^*E(p^{2m})u)p^{2k})u^*). \qquad \square\end{aligned}$$

To simplify notation, we define functions $g_1, g_2 : \mathbb{N}_0 \to B$ by

$$\begin{aligned}g_1(n) &= E((xx^*)^n)\\ g_2(n) &= E((x^*x)^n).\end{aligned}$$

Then the relations in Lemma 4.3.3 become

$$g_1(n) = E(ug_2(n)u^*). \tag{4.12}$$

and

$$\begin{aligned}E((xx^*)^n(x^*x)^m(xx^*)^k) &= E(uE(p^{2n}E(u^*g_2(m)u)p^{2k})u^*)\\ &- E(ug_2(n)E(u^*g_2(m)u)g_2(k)u^*)\\ &+ E(ug_2(n)u^*g_2(m)ug_2(k)u^*). \qquad (4.13)\end{aligned}$$

We can use (4.12) in the case of $x = a$ to find that $1_{\mathbb{C}^2} = (1,1) = g_2(0) \in \mathbb{C}^2$ and $g_2(1) = E(a^*a)$ are linearly independent. Indeed, if $\alpha_{(2,1)}(1) = g_2(1) = (c,c)$ for some scalar c, then we have by (4.12) that

$$\alpha_{(1,2)}(1) = g_1(1) = E(ug_2(1)u^*) = E(u(c,c)u^*) = (c,c) = \alpha_{(2,1)}(1).$$

Comparing the first components of $\alpha_{(1,2)}(1)$ and $\alpha_{(2,1)}(1)$ from (4.6) yields the parameter equation

$r_{11}+r_{12} = r_{11}+\frac{1-q}{q}r_{21}$, and thus $qr_{12} = (1-q)r_{21}$. This contradicts (4.9). Therefore $g_2(0) = 1_{\mathbb{C}^2}$ and $g_2(1)$ are linearly independent, so we may write every $b \in \mathbb{C}^2$ as a linear combination of $g_2(0)$ and $g_2(1)$.

Since it will be important to work with these linear combinations, we aim to find the coordinate functionals $P_1, P_2 : \mathbb{C}^2 \to \mathbb{C}$ for the basis $\{g_2(0), g_2(1)\}$ of $\mathbb{C}^2$; i.e., the linear functions P_1, P_2 satisfying $b = P_1(b)g_2(0) + P_2(b)g_2(1)$ for every $b \in \mathbb{C}^2$. Taking $b = (x, y)$ and denoting $g_2(1) = (g_2(1)_1, g_2(1)_2)$, we solve the linear system

$$x = P_1((x,y)) + P_2((x,y))g_2(1)_1$$
$$y = P_1((x,y)) + P_2((x,y))g_2(1)_2$$

to obtain $P_1((x,y)) = x - P_2((x,y))g_2(1)_1$ and $P_2((x,y)) = (x-y)/(g_2(1)_1 - g_2(1)_2)$.

We can further exploit (4.12) to show that the assumption $0 = \sum_{k=0}^{n} c_k g_2(k)$ for some scalars c_i implies

$$\sum_{k=0}^{n} c_k g_1(k) = \sum_{k=0}^{n} c_k E(ug_2(k)u^*) = E\left(u \sum_{k=0}^{k} c_k g_2(k) u^*\right) = E(u0u^*) = 0.$$

That is, any vanishing linear combination of $g_2(k)$'s yields the same vanishing linear combination of $g_1(k)$'s. In particular $g_2(n) = P_1(g_2(n))g_2(0) + P_2(g_2(n))g_2(1)$ so that

$$g_1(n) = P_1(g_2(n))g_1(0) + P_2(g_2(n))g_1(1), \qquad \text{for all } n \in \mathbb{N}_0. \tag{4.14}$$

The first major step in the proof of Theorem 4.3.1 will be to determine conditions on the parameters $q, r_{11}, r_{12}, r_{21}, r_{22}$ that must hold as a consequence of (4.14). Before we can do this, we must learn how to compute the maps g_1 and g_2.

Proposition 4.3.4. *If x is a B-valued circular random variable with cumulants $\beta_{(1,2)}$ and $\beta_{(2,1)}$ associated to $(x_1, x_2) = (x, x^*)$, then the maps g_1 and g_2 can be computed recursively by $g_1(0) =$*

$g_2(0) = 1,$

$$g_1(n) = \sum_{i=1}^{n} \beta_{(1,2)}(g_2(i-1))g_1(n-i), \quad \textit{and} \quad g_2(n) = \sum_{i=1}^{n} \beta_{(2,1)}(g_1(i-1))g_2(n-i).$$

Proof. Clearly $g_1(0) = g_2(0) = 1$. The recursive relation is a consequence of the moment-cumulant formula. Since x is circular, the only noncrossing partitions that need to be considered are noncrossing pair partitions for which each set in the partition pairs an x with an x^*. By considering the n possibilities for which x^* in the word $(xx^*)^n$ is paired with the leftmost x in a given noncrossing pair partition, we obtain

$$g_1(n) = E((xx^*)^n) = \sum_{i=1}^{n} \beta_{(1,2)}(E((x^*x)^{i-1}))E((xx^*)^{n-i}) = \sum_{i=1}^{n} \beta_{(1,2)}(g_2(i-1))g_1(n-i)$$

and, similarly by considering which x is paired with the leftmost x^*,

$$g_2(n) = E((x^*x)^n) = \sum_{i=1}^{n} \beta_{(2,1)}(E((xx^*)^{i-1}))E((x^*x)^{n-i}) = \sum_{i=1}^{n} \beta_{(2,1)}(g_1(i-1))g_2(n-i). \quad \square$$

Applying the preceeding proposition to the case $x = a$, for a fixed n we can express both components of the pairs $g_1(n)$ and $g_2(n)$ as rational functions in the parameters $q, r_{11}, r_{12}, r_{21}, r_{22}$. From here, we use Mathematica's [11] reduce function (see [2] for the full Mathematica notebook) to find that (4.14), applied for $n = 2$ and $n = 3$, together with the assumptions from (4.9), implies that the parameters satisfy one of the following conditions.

Case I: $r_{11} = r_{21}$ and $r_{12} \neq r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$

Case II: $r_{11} \neq r_{21}$ and $r_{12} = r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$

Case III: $r_{11} = r_{21}$ and $r_{12} = r_{22}$

To proceed, we define the multilinear map $M : (\mathbb{C}^2)^{\otimes 3} \to \mathbb{C}^2$ by

$$M(b_1, b_2, b_3) = E(ub_1u^*b_2ub_3u^*)$$

and the map $M_0 : \mathbb{N}_0^3 \to \mathbb{C}^2$ by

$$M_0(n, m, k) = E(ug_2(n)u^*g_2(m)ug_2(k)u^*).$$

Clearly

$$M_0(n, m, k) = M(g_2(n), g_2(m), g_2(k)) \qquad \text{for all } n, m, k \in \mathbb{N}_0. \tag{4.15}$$

In fact, since $g_2(0)$ and $g_2(1)$ form a basis for $\mathbb{C}^2$, M can be obtained from the values of M_0 on the triples in $\{0, 1\}^3$. More specifically, writing $c_i := P_1(b_i)$ and $d_i := P_2(b_i)$ for $i = 1, 2, 3$ gives

$$\begin{aligned} M(b_1, b_2, b_3) &= M(c_1g_2(0) + d_1g_2(1), c_2g_2(0) + d_2g_2(1), c_3g_2(0) + d_3g_2(1)) \\ &= c_1c_2c_3M(g_2(0), g_2(0), g_2(0)) + c_1c_2d_3M(g_2(0), g_2(0), g_2(1)) \\ &\quad + c_1d_2c_3M(g_2(0), g_2(1), g_2(0)) + d_1c_2c_3M(g_2(1), g_2(0), g_2(0)) \\ &\quad + c_1d_2d_3M(g_2(0), g_2(1), g_2(1)) + d_1c_2d_3M(g_2(1), g_2(0), g_2(1)) \\ &\quad + d_1d_2c_3M(g_2(1), g_2(1), g_2(0)) + d_1d_2d_3M(g_2(1), g_2(1), g_2(1)) \\ &= c_1c_2c_3M_0(0, 0, 0) + c_1c_2d_3M_0(0, 0, 1) + c_1d_2c_3M_0(0, 1, 0) \\ &\quad + d_1c_2c_3M_0(1, 0, 0) + c_1d_2d_3M_0(0, 1, 1) + d_1c_2d_3M_0(1, 0, 1) \\ &\quad + d_1d_2c_3M_0(1, 1, 0) + d_1d_2d_3M_0(1, 1, 1). \end{aligned} \tag{4.16}$$

Therefore if we can compute M_0 in terms of the parameters, then we can also compute M in terms of the parameters. This will allow us to test the relation (4.15) against triples in $\mathbb{N}_0^3 \setminus \{0, 1\}^3$, which will ultimately lead to a contradiction in each of our three cases.

Proposition 4.3.5. *The map M_0 can be computed by*

$$M_0(n, m, k) = G(n, g_2(m), k) - N_1(H(n, N_2(g_2(m)), k)) + N_1(g_2(n)N_2(g_2(m))g_2(k)), \tag{4.17}$$

where

$$
\begin{aligned}
N_1 &: \mathbb{C}^2 \to \mathbb{C}^2; && b \mapsto E(ubu^*) \\
N_2 &: \mathbb{C}^2 \to \mathbb{C}^2; && b \mapsto E(u^*bu) \\
H &: \mathbb{N}_0 \times \mathbb{C}^2 \times \mathbb{N}_0 \to \mathbb{C}^2; && (n,b,k) \mapsto E((a^*a)^n b (a^*a)^k) \\
G &: \mathbb{N}_0 \times \mathbb{C}^2 \times \mathbb{N}_0 \to \mathbb{C}^2; && (n,b,k) \mapsto E((aa^*)^n b (aa^*)^k).
\end{aligned}
$$

The maps N_1, H, G can be written in terms of the parameters $q, r_{11}, r_{12}, r_{21}, r_{22}$. Moreover, in Case I and Case II, N_2 and thus M_0 can be written in terms of these parameters.

Proof. To prove (4.17), we employ relation (4.13) from Lemma 4.3.3 with $x = a$ to get

$$
\begin{aligned}
M_0(n,m,k) &= E(ug_2(n)u^*g_2(m)ug_2(k)u^*) \\
&= E((aa^*)^n(a^*a)^m(aa^*)^k) \\
&\quad - E(uE(p^{2n}E(u^*g_2(m)u)p^{2k})u^*) \\
&\quad + E(ug_2(n)E(u^*g_2(m)u)g_2(k)u^*) \\
&= E((aa^*)^n E((a^*a)^m)(aa^*)^k) \\
&\quad - N_1(H(n, N_2(g_2(m)), k)) \\
&\quad + N_1(g_2(n)N_2(g_2(m))g_2(k)) \\
&= G(n, g_2(m), k) - N_1(H(n, N_2(g_2(m)), k)) + N_1(g_2(n)N_2(g_2(m))g_2(k)),
\end{aligned}
\tag{4.18}
$$

wherein the equality $E((aa^*)^n(a^*a)^m(aa^*)^k) = E((aa^*)^n E((a^*a)^m)(aa^*)^k)$ can be seen from the moment-cumulant formula.

We can express both H and G by a recursive formula involving the cumulants $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$.

The moment-cumulant formula gives

$$\begin{aligned}
H(n,b,k) &= E((a^*a)^n b (a^*a)^k) \\
&= \sum_{i=1}^{n} \alpha_{(2,1)}(g_1(i-1)) E((a^*a)^{n-i} b (a^*a)^k) \\
&\quad + \sum_{j=1}^{k} \alpha_{(2,1)}(E((aa^*)^{n-1} aba^* (aa^*)^{j-1})) g_2(k-j) \\
&= \sum_{i=1}^{n} \alpha_{(2,1)}(g_1(i-1)) H(n-i,b,k) + \sum_{j=1}^{k} \alpha_{(2,1)}(H'(n-1,b,j-1)) g_2(k-j),
\end{aligned}$$

where the intermediate map $H' : \mathbb{N}_0 \times \mathbb{C}^2 \times \mathbb{N}_0 \to \mathbb{C}^2$, defined by

$$H'(n,b,k) = E((aa^*)^n aba^* (aa^*)^k),$$

satisfies

$$\begin{aligned}
H'(n,b,k) &= E((aa^*)^n aba^* (aa^*)^k) \\
&= \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1)) E((aa^*)^{n-i} aba^* (aa^*)^k) + \alpha_{(1,2)}(E((a^*a)^n b)) g_1(k) \\
&\quad + \sum_{j=1}^{k} \alpha_{(1,2)}(E((a^*a)^n b (a^*a)^j)) g_1(k-j) \\
&= \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1)) E((aa^*)^{n-i} aba^* (aa^*)^k) \\
&\quad + \sum_{j=0}^{k} \alpha_{(1,2)}(E((a^*a)^n b (a^*a)^j)) g_1(k-j) \\
&= \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1)) H'(n-i,b,k) + \sum_{j=0}^{k} \alpha_{(1,2)}(H(n,b,j)) g_1(k-j).
\end{aligned}$$

Therefore, by Proposition 4.3.4, H can be computed from the cumulants via the recursive relations

$$
\begin{aligned}
H(0,b,k) &= bg_2(k),\\
H'(0,b,k) &= \sum_{j=0}^{k} \alpha_{(1,2)}(bg_2(j))g_1(k-j),\\
H(n,b,k) &= \sum_{i=1}^{n} \alpha_{(2,1)}(g_1(i-1))H(n-i,b,k) + \sum_{j=1}^{k} \alpha_{(2,1)}(H'(n-1,b,j-1))g_2(k-j),\\
H'(n,b,k) &= \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1))H'(n-i,b,k) + \sum_{j=0}^{k} \alpha_{(1,2)}(H(n,b,j))g_1(k-j).
\end{aligned}
$$

Similarly,

$$
G(n,b,k) = \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1))G(n-i,b,k) + \sum_{j=1}^{k} \alpha_{(1,2)}(G'(n-1,b,j-1))g_1(k-j),
$$

where $G' : \mathbb{N}_0 \times \mathbb{C}^2 \times \mathbb{N}_0 \to \mathbb{C}^2$, defined by $G'(n,b,k) = E((a^*a)^n a^*ba(a^*a)^k)$, satisfies

$$
G'(n,b,k) = \sum_{i=1}^{n} \alpha_{(2,1)}(g_1(i-1))G'(n-i,b,k) + \sum_{j=0}^{k} \alpha_{(2,1)}(G(n,b,j))g_2(k-j).
$$

Thus G can be computed recursively by

$$
\begin{aligned}
G(0,b,k) &= bg_1(k),\\
G'(0,b,k) &= \sum_{j=0}^{k} \alpha_{(2,1)}(bg_1(j))g_2(k-j),\\
G(n,b,k) &= \sum_{i=1}^{n} \alpha_{(1,2)}(g_2(i-1))G(n-i,b,k) + \sum_{j=1}^{k} \alpha_{(1,2)}(G'(n-1,b,j-1))g_1(k-j),\\
G'(n,b,k) &= \sum_{i=1}^{n} \alpha_{(2,1)}(g_1(i-1))G'(n-i,b,k) + \sum_{j=0}^{k} \alpha_{(2,1)}(G(n,b,j))g_2(k-j).
\end{aligned}
$$

N_1 can be computed from g_1 and g_2, and thus from the cumulants by Proposition 4.3.4, by writing $b = P_1(b)g_2(0) + P_2(b)g_2(1)$ and using relation (4.12) to get

$$N_1(b) = P_1(b)E(ug_2(0)u^*) + P_2(b)E(ug_2(1)u^*) = P_1(b)g_1(0) + P_2(b)g_1(1).$$

To compute N_2, we apply the $*$-freeness of u and p over $\mathbb{C}^2$ to obtain

$$\begin{aligned}
\alpha_{(1,2)}(N_2(b)) &= E(aE(u^*bu)a^*) \\
&= E(upE(u^*bu)pu^*) \\
&= E(uE(pE(u^*bu)p)u^*) \\
&= E(uE(pu^*bup)u^*) \\
&= E(uE(a^*ba)u^*) \\
&= N_1(\alpha_{(2,1)}(b)), \qquad \text{for all } b \in \mathbb{C}^2, \qquad (4.19)
\end{aligned}$$

which yields the relation $\alpha_{(1,2)} \circ N_2 = N_1 \circ \alpha_{(2,1)}$. In Case I and Case II, since $0 < q < 1$, both r_{11} and r_{22} must be nonzero. Thus $r_{11}r_{22} \neq r_{12}r_{21}$, so the cumulant map $\alpha_{(1,2)}$ is invertible. Therefore (4.19) implies $N_2 = \alpha_{(1,2)}^{-1} \circ N_1 \circ \alpha_{(2,1)}$. □

Unfortunately, N_2 and M_0 cannot be completely described in terms of our parameters in Case III. The technique in the above proof no longer works because $\alpha_{(1,2)}$ is not invertible in this case. Since N_2 is linear and positive, there are parameters $m_{11}, m_{12}, m_{21}, m_{22} \geq 0$ such that

$$N_2((x, y)) = (m_{11}x + m_{12}y, m_{21}x + m_{22}y).$$

Therefore in Case III, we will express M_0 by both the original parameters $q, r_{11}, r_{12}, r_{21}, r_{22}$ and new parameters $m_{11}, m_{12}, m_{21}, m_{22}$.

We can now conclude the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. So far we have shown that if a is a tracial $\mathbb{C}^2$-valued cirulcar element having the same $\mathbb{C}^2$-valued $*$-distribution as a product of a Haar unitary and a $*$-free self adjoint, and if a satisfies (4.9), then one of the following cases must hold.

Case I: $r_{11} = r_{21}$ and $r_{12} \neq r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$

Case II: $r_{11} \neq r_{21}$ and $r_{12} = r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$

Case III: $r_{11} = r_{21}$ and $r_{12} = r_{22}$

We have also described how to compute M in terms of M_0 on $\{0,1\}^3$ by (4.16), and M_0 by Proposition 4.3.5. Using these we will test relation (4.15) against various values of n, m, and k using Mathematica's reduce function. The following tests have been run within the Mathematica notebook [2].

Suppose we are in Case I: $r_{11} = r_{21}$ and $r_{12} \neq r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$. Since $0 < q < 1$ and either $q \neq 1/2$ or $r_{11} \neq r_{22}$, we must have $r_{11} \neq 0$ and $r_{22} \neq r_{11}$. We may assume $r_{11} = 1$. This causes no loss of generality because the $\mathbb{C}^2$-valued circular random variable $d := r_{11}^{-1/2} a$ has cumulant $\gamma_{(1,2)}$ associated with $(d_1, d_2) = (d, d^*)$ satisfying

$$\begin{aligned}\gamma_{(1,2)}((x,y)) &= E(d(x,y)d^*) = r_{11}^{-1} E(a(x,y)a^*) = r_{11}^{-1} \alpha_{(1,2)}(x,y)\\ &= (1 \cdot x + r_{12} r_{11}^{-1} y, 1 \cdot x + r_{22} r_{11}^{-1} y),\end{aligned}$$

and a has a free decomposition up iff d has a free decomposition $u(r_{11}^{-1/2} p)$. With the assumptions $r_{11} = r_{21} = 1$, $r_{12} \neq r_{22}$, $q = r_{11}/(r_{11} + r_{22})$, and $r_{22} \neq 1$, Mathematica's reduce function says that equation (4.15) with $n = 3$, $m = 1$, $k = 3$ implies that r_{22} is negative or non-real. This contradicts the fact that r_{22} is positive.

Now let us consider Case II: $r_{11} \neq r_{21}$ and $r_{12} = r_{22}$ and $q = r_{11}/(r_{11} + r_{22})$. In a similar manner to how we imposed extra assumptions in Case I, we may assume $r_{22} = 1$ and $r_{11} \neq r_{22}$. By testing equation (4.15) again with $n = 3$, $m = 1$, $k = 3$, we deduce from the output of Mathematica's reduce function that r_{11} is negative or non-real. Since r_{11} is positive, this is a contradiction.

Finally, we consider Case III: $r_{11} = r_{21}$ and $r_{12} = r_{22}$. This case has two complications. The first is that we can no longer find the map N_2 in terms of the initial parameters. Instead, as we have already seen, we realize N_2 in terms of new parameters $m_{11}, m_{12}, m_{21}, m_{22}$. This effectively only introduces two more unknowns because

$$1_{\mathbb{C}^2} = N_2(1) = (m_{11} + m_{12}, m_{21} + m_{22})$$

implies $m_{12} = 1 - m_{11}$ and $m_{22} = 1 - m_{21}$. However, the involvement of these parameters still means that we will have to test (4.15) against multiple triples (n, m, k).

The second complication is that Case III doesn't impose any relations involving q, and therefore we need to consider extra assumptions. We accomplish this by splitting into three distinct subcases:

Subcase I: $r_{11} = r_{21} = r_{12} = r_{22}$

Subcase II: $0 = r_{11} = r_{21} \neq r_{12} = r_{22}$

Subcase III: $0 \neq r_{11} = r_{21} \neq r_{12} = r_{22}$

Suppose we are in Subcase I. We cannot have all parameters equal to 0, and we cannot have $q = 1/2$, for either of these would contradict (4.9). Thus we may assume $r_{11} = r_{12} = r_{21} = r_{22} = 1$ and $q \neq 1/2$. The test of (4.15) with $(n, m, k) = (2, 1, 1)$ and the assumptions $m_{12} = 1 - m_{11}$ and $m_{22} = 1 - m_{21}$ yields $m_{12} = m_{22}$. After adding this assumption, and now testing (4.15) with $(n, m, k) = (2, 1, 3)$, we obtain that either q is non-real, negative, greater than 1, or equal to $1 - m_{22}$. Since only the latter is possible, we include that in our test, except this time with $(n, m, k) = (3, 1, 3)$, to determine that m_{22} is either $1/2$, non-real, negative, or greater than 1. Of course, none of these are possible because $0 < q < 1$, $q \neq 1/2$, and $q = 1 - m_{22}$. Therefore we have reached a contradiction.

In Subcase II, we begin with the assumption $r_{12} = r_{22} = 1$, as well as $m_{12} = 1 - m_{11}$ and $m_{22} = 1 - m_{21}$ This subcase only requires two tests of (4.15). The first uses $(n, m, k) = (2, 1, 1)$ to find that $q = m_{21}$. After adding this assumption, the second test uses $(n, m, k) = (3, 1, 1)$ and yields $q = 0$, which is a contradiction.

Lastly, we investigate Subcase III. Without loss of generality, we set $r_{11} = r_{21} = 1$. Now the condition $qr_{12} \neq (1-q)r_{21}$ from (4.9) implies $q \neq 1/(1+r_{22})$. We also continue to assume $m_{12} = 1 - m_{11}$ and $m_{22} = 1 - m_{21}$. Using Mathematica's reduce function to test (4.15) with $(n, m, k) = (2, 1, 1)$ yields that $m_{12} = 1 - q - m_{21}r_{22}^2 + qr_{22}^2$. By adding in this constraint and now testing against $(n, m, k) = (1, 1, 3)$, we obtain that if $r_{22} \neq 0$, then $m_{21} = q$. This motivates the consideration of two sub-subcases: $r_{22} = 0$ and $r_{22} \neq 0$. The former leads to the contradiction $q < 0$ immediately upon testing (4.15) with $(n, m, k) = (3, 1, 3)$. The same triple tested alongside the assumption $r_{22} \neq 0$ and $m_{21} = q$ leads to the discovery that q is negative, greater than 1, or non-real. This gives us our final contradiction. □

5. CONCLUSIONS

In addition to giving some interesting distinctions between classes of Haar unitaries in the B-valued setting, the work in Section 4.1 motivates the desire to find free decompositions involving normalizing Haar unitaries. This motivates the work in Section 4.3, and continues to motivate research in this area.

The main theorem from Section 4.3 is limited in two ways. The first of these limitations is that it only concerns $\mathbb{C}^2$-valued circular elements instead of general B-valued R-diagonal elements. Recalling the discussion preceeding Example 3.2.13, it's clear why we prefer to work with circular elements. Similar techniques may be able to partially alleviate the limitation $B = \mathbb{C}^2$, however the computation would become increasingly complex as the dimension of B and the number of automorphisms on B grows. Also, any similiar result using the same techniques would need to choose a particular B instead of working more generally.

The second limitation is in the conclusion of the theorem, which does not claim that the unitary u normalizes $\mathbb{C}^2$. Instead, we obtain the automorphism condition (4.7), which implies that there is an even decomposition involving some (possibly different) unitary that normalizes $\mathbb{C}^2$. This limitation is in place because of Example 4.2.3 and the discussion at the end of Section 4.2, which show that it's possible for u to not normalize $\mathbb{C}^2$ despite the validity of the (4.7).

However, this example is very special. In particular, the $\mathbb{C}^2$-valued $*$-cumulant maps are scalar-valued and equal to one another. In the notation of Section 4.3, this example has

$$\operatorname{span}\{g_1(k), g_2(k) \mid k \in \mathbb{N}_0\} = \mathbb{C}1_{\mathbb{C}^2},$$

which is in contrast to the situation $\operatorname{span}\{g_2(0), g_2(1)\} = B$ obtained as a consequence of relation (4.12). It might be the case that we only need a small assumption on the $\mathbb{C}^2$-valued circular (or its cumulants) to ensure that u normalizes $\mathbb{C}^2$.

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