# MINIMUM TIME HEADWAY AND STABILIZING CONTROL GAINS FOR VEHICLE PLATOONS WITH TIME DELAY

A Thesis

by

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ABSTRACT

An Adaptive Cruise Control (ACC) system maintains a desired spacing between the vehicles in a platoon through longitudinal control. Maintaining tight longitudinal spacing between vehicles contribute to an increased traffic throughput and road capacity. Most ACC systems adopt a Constant Time Headway Policy (CTHP); a CTHP specifies a desired spacing that is proportional to the speed of the following vehicle with the proportionality constant referred to as the time headway. A smaller time headway leads to enhanced traffic capacity.

Past studies have bounded the minimum time headway which can be stably achieved in the presence of lags. In this study, the minimum limit for time headway achievable with stability guarantees in the presence of bounded time-varying time delays is investigated. Using Hermite-Biehler Theorem for Quasi-Polynomials, the set of all stabilizing control gains of the ACC system is derived as a function of the time headway and the time delay. Similarly, the subset of the above set of control gains preserving string stability is numerically computed.

In this study, it is concluded that for time headway not exceeding twice the upper limit of time delay, there are no control gains that can guarantee individual and string stability. It is observed that for a given time headway, the set of stabilizing control gains during some time delay are stabilizing for any smaller time delay.

DEDICATION

To my parents, brother, family, friends and my teachers

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# NOMENCLATURE

| | |
|---|---|
| CTHP | Constant Time Headway Policy |
| ACC | Adaptive Cruise Control |
| CAV | Connected and Autonomous Vehicles |
| ADAS | Advanced Driver Assistance System |
| V2V | Vehicle to Vehicle |
| IC | Internal Combustion |
| PID | Proportional Integral Derivative |
| CACC | Cooperative Adaptive Cruise Control |
| PD | Proportional Derivative |
| BIBO | Bounded Input Bounded Output |
| LHP | Left Half Plane |
| RHP | Right Half Plane |

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# 1. INTRODUCTION AND LITERATURE REVIEW

This thesis focuses on the effect of time delays on the propagation of errors in a platoon of vehicles equipped with Adaptive Cruise Control (ACC) System and employing a Constant Time Headway Policy (CTHP).

## 1.1 Vehicle Platooning

Vehicle Platoon is a group of vehicles that intercommunicate to move as a tightly spaced group. Forming intelligently structured platoons of autonomously coupled vehicles improves road capacity, traffic throughput and fuel efficiency. Adaptive Cruise Control(ACC) is an Advanced Driver Assistance System (ADAS) focusing on longitudinal control of the vehicle platoon thereby maintaining desired spacing between them. Besides offering partial autonomy and promoting driver convenience, ACC system provides benefits such as:

- Informing the following vehicle about the speed profile of the preceding vehicle so that the amount of braking and accelerating can be decreased. Lesser acceleration and deceleration translate to lesser fuel consumption [1].
- Decreasing the necessary spacing between the vehicles on a highway while travelling at a given speed, ACC system can improve road capacity and traffic throughput [2] [3].
- ACC systems have been claimed to improve traffic safety in congested freeways [4].

Spacing policy governs the longitudinal distance between any two consecutive vehicles in a platoon. Choosing an efficient spacing policy is crucial to the design of an ACC system. The most rudimentary spacing policy is the Constant Spacing Policy where a given constant spacing is maintained irrespective of the vehicle speeds; however, the communication requirements for maintaining the desired spacing are far more stringent when compared with Variable Spacing Policies, where the desired spacing varies as a function of vehicle speed [5]. A Variable Spacing Policy that has been extensively dealt with in literature is a Constant Time Headway Policy (CTHP) [6].

In CTHP, the inter-vehicle spacing is proportional to the vehicle speeds with a proportionality constant namely Time Headway ($h$).

A vehicle platoon may be homogeneous or heterogeneous in terms of the vehicles and in terms of the control applied to various vehicles in the platoon. The discussion in this thesis is limited to a homogeneous platoon involving same type of vehicles grouped with same controller throughout.

In transportation literature [7], the term time headway stands for the time interval between two consecutive cars crossing a point. In a ACC system employing CTHP, the time headway can be viewed as the time taken by two consecutive cars in the platoon to cross a point when the platoon is moving in a steady state. The ACC system can be meritorious in terms of traffic throughput and road capacity only if a stable platoon with a tight spacing thereby a small time headway can be maintained.

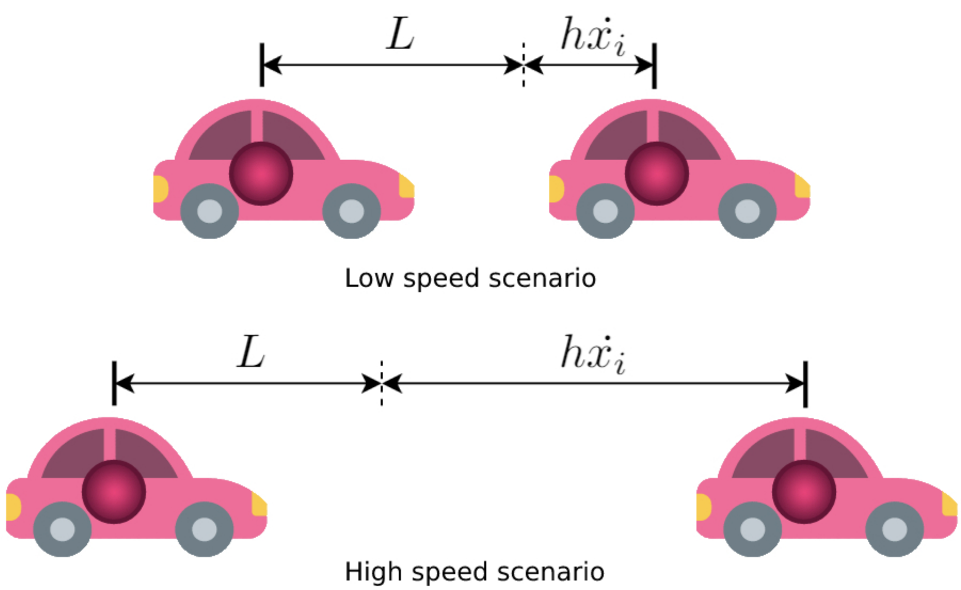


Figure 1.1: Effect of time headway in CTHP

## 1.2 Time Delay

There are several sources of Time Delays in an Autonomous/Connected Vehicle (ACV): communication system delays, backlash in mechanical systems, pressure buildup necessary to overcome pre-loaded springs in pneumatic & hydraulic systems of brake and transmission systems and variable time delays in internal combustion engines. Normally vehicle longitudinal control involves application of pneumatic or hydraulic brakes for deceleration and internal combustion engines for acceleration.

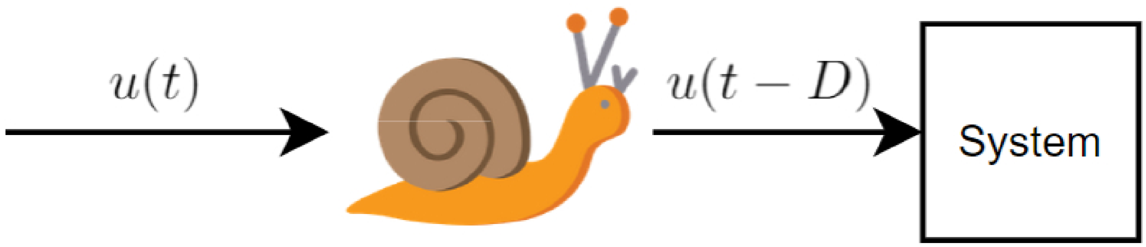


Figure 1.2: Time Delay in systems

- Uncertainties in the wireless channel have been found to cause inevitable time-varying time delays in V2V communication amounting to delays in the order of 100ms [8].
- Time delays occur in the actuation of hydraulic/pneumatic brakes due to the time taken for the operating fluid to build pressure in the brake chamber. Especially, in heavy vehicles commonly employing pneumatic brakes, there is a pronounced variable actuation time delay in the order of 250 ms [9] [10].
- In the throttle response of an IC engine, time delay manifests in various forms such as fuel actuator delay, cycle time delay etc. Usually this delay is in the order of 10ms.

It is therefore important to account for these time delays while analyzing the ACC system. Such time delays alter the behavior of the system drastically and affect the stability of the overall system.

The analysis of a linear time delayed system is itself very complicated owing to the stability being governed by the Hurwitzness of a Quasi-polynomial as opposed to a polynomial in delay-free linear systems. It is important to note that these time delays may not exactly be known, but their bound is.

The composite effect of these time delays results in a slower response from the vehicles. Maintaining a tight spacing requires following vehicles to respond quickly to changes in the preceding vehicle movement. The existence of time delays is adversarial to stably maintaining a small time headway in the platoon. The stability of an ACC system with time delay involving time headway as a parameter will be analysed in the thesis. The objective is to find the minimum possible time headway in the presence of time delay.

### 1.3 Literature Study

In the past, various forms of stability of ACC systems using CTHP with time lags have been analysed. Minimum time headway in the presence of time lags have been discussed for both homogeneous [11] and heterogeneous platoons [12].

The monograph [13] discusses methods to derive the set of all stabilizing PID controls for both delay-free systems and systems with time delays. Hermite-Biehler Theorem for quasi-polynomials due to Pontryagin has been used to derive the set of stabilizing controls of first order linear time delay systems [13]. The above procedure has been extended previously to formulate algorithms that numerically compute the set of stabilizing PID controls of second order time delay systems [14] [15]. By efficiently using the Hermite-Biehler Theorem for quasi-polynomials it is possible to attain analytical solution for stabilizing set of PID controls although the ACC system is a second order time delay system.

The boundary of the set of control gains of a delay-free Cooperative Adaptive Cruise Control (CACC) system that ascertain string stability has been computed numerically without detailing on the effect of the time headway on the stabilizing set [16]. An extension of this methodology can be employed to analyse the string stability of the ACC system for our problem. In [17], string stability of ACC system with time delays has been discussed without reference to a relationship between time headway.

In [2], there are comments on the most desirable $h$ in terms of numerical simulations performed with practical time delay bounds gleaned from empirical studies .

## 1.4 Thesis Outline

The following is a brief description of the structure of the remaining thesis. In chapter 2, the ACC controller architecture considered for the thesis and its component controllers will be discussed. The choice of a suitable vehicle model will be justified. Spacing error transfer function that is crucial for analysing stability will be derived. The notions of stability will be discussed and the minimum time headway problem will be formally defined.

In chapter 3, Hermite Biehler Theorem for Quasi-Polynomials will be defined and the theorem will be employed to derive the region of individual stability. The acquired results will be presented.

In chapter 4, the procedure to find the boundary of string stability region will be deliberated. The procedure will be employed for the ACC system and the results will be presented.

In chapter 5, the conclusions drawn from this study will be stated and possible directions for future work will be identified.

# 2. THE MINIMUM TIME HEADWAY PROBLEM

In this chapter, the minimum time headway problem is posed mathematically. In section (2.1), the controller of the ACC system and the vehicle model employed in designing the controller will be discussed. In the section (2.2), the spacing error transfer function of the system which is crucial for the problem is derived. In the section (2.3), the notions of stability to be considered for qualifying the controller are explained. In the section (2.4), the statement of the Minimum Time Headway problem is presented.

## 2.1 The Controller and Vehicle Model

The aim of the ACC system controller is to maintain desired longitudinal distances between the vehicles of the platoon. The longitudinal control architecture usually breaks this task into two levels (Figure 2.1).

- The Higher Level Controller specifies the desired acceleration based on the error in spacing between the vehicles.

- The Lower Level Controller determines the brake/throttle inputs to track the desired acceleration specified by the Higher Level Controller.

We limit our scope to analysing the design space of the Higher Level Controller which is central to the ACC system. Designing the Lower Level Controller involves a vehicle model which establishes a relationship between brake/throttle inputs and the corresponding effect on acceleration. Generally, accurate models for both the engine and brake systems are complex and nonlinear. By restricting our analysis to maneuvers that do not require the engine or brake to reach their limits, it is possible to discard these nonlinearities from our design procedure. By using Feedback Linearization approach, establishing a linear and homogeneous relationship between the desired acceleration specified by the Higher Level Controller and the actual acceleration of the vehicle is possible [12]. In the past [18], researchers have attained satisfactory results by using a simple

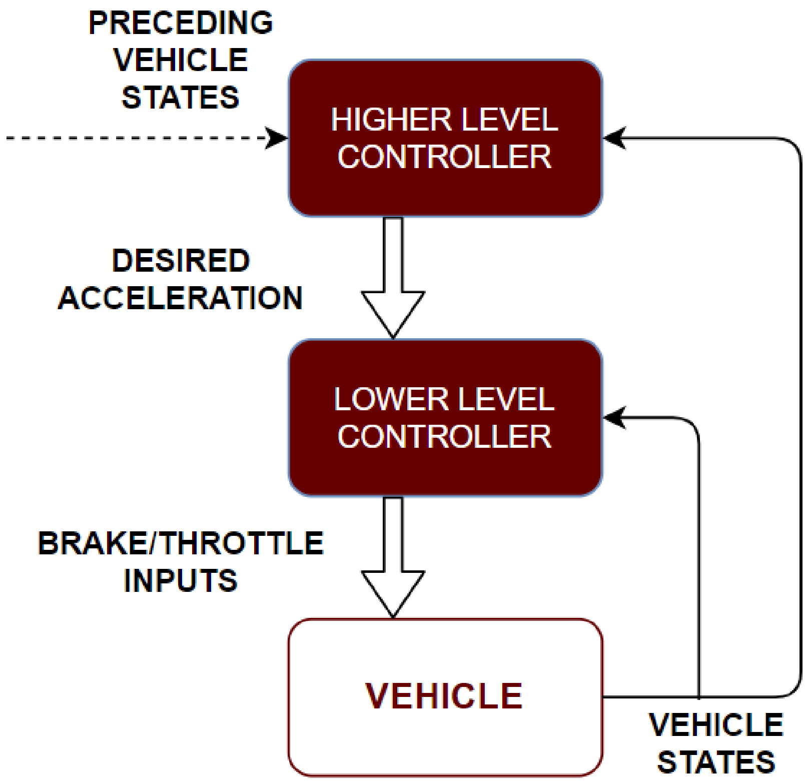


Figure 2.1: The ACC control architecture

point mass vehicle model. For the scope of our study, it is prudent to use the following point mass vehicle model. Let $\ddot{x}_i(t)$ be the acceleration of the $i^{th}$ vehicle and let $u(t)$ be the input desired accleration.

$$\ddot{x}_i(t) = u(t) \tag{2.1}$$

To study the effect of time delay on the design of the controller, it is important to include the effect of time delays in the model. Though delay manifests itself in multiple forms as described earlier, we lump all the delay's together. This lumped delay $D$ represents the composite, effective time delay observed due to various delays. In presence of a pure time delay $D$, the input (desired acceleration) at time $t - D$ drives the actual acceleration at time $t$. On incorporating this effect, the vehicle model becomes,

$$\ddot{x}_i(t) = u(t - D) \tag{2.2}$$

The time delay $D$ is uncertain and time-varying is nature. It is impossible to quantify $D$ at any given moment. However, an upper bound $D_0$ can be established such that $0 \leq D \leq D_0$, by factoring extreme delays from all possible sources.

As discussed earlier, a spacing policy is central to the ACC system and we employ CTHP. The desired spacing in our CTHP involves:

- Stand Still Distance $L$, a constant spacing maintained irrespective of the vehicle speed
- A spacing that scales up with vehicle speed as $h\dot{x}_i$, where $\dot{x}_i$ is the speed of the $i^{th}$ vehicle and the proportionality constant $h$ is called the Time Headway.

The desired position of the $i^{th}$ vehicle $x_{i,des}$ based on CTHP,

$$x_{i,des} = x_{i-1} - L - h\dot{x}_i \tag{2.3}$$

where $x_{i-1}$ is the position of the preceding $(i-1)^{th}$ vehicle.

The error in the desired position of the $i^{th}$ vehicle $e_i$ is:

$$e_{i,p} = x_i - x_{i,des} \tag{2.4}$$

At a steady state, it is desirable for the vehicles to travel at the same longitudinal speed. Let the velocity difference between two consecutive vehicles, say $(i-1)^{th}$ and $i^{th}$ vehicles be the velocity error,

$$e_{i,v} = \dot{x}_i - \dot{x}_{i-1} \tag{2.5}$$

Both the spacing error and velocity error are used to drive the controller. The structure of the proposed controller is similar to a Proportional Derivative (PD) controller. Since the velocity error is not directly the time derivative of the spacing error, the controller is not exactly PD. Let the gain analogous to the Proportional gain scaling the spacing error be $K_p$. Let the gain analogous to the Derivative gain scaling the velocity error be $K_v$. The control law of the ACC system is:

$$u_i = -K_p e_{i,p} - K_v e_{i,v} = -K_p[x_i - x_{i-1} + L + h\dot{x_i}] - K_v[\dot{x}_i - \dot{x_{i-1}}] \tag{2.6}$$

## 2.2 Spacing Error Transfer Function

Having defined the ACC system, we need to derive the spacing error transfer function to discuss the stability of the system. In the frequency domain, the spacing error transfer function can be defined as:

$$H(s) = \frac{E_{i,p}(s)}{E_{i-1,p}(s)} \tag{2.7}$$

where $E_{i,p}(s)$ and $E_{i-1,p}(s)$ are Laplace transforms of $e_{i,p}(t)$ and $e_{i-1,p}(t)$ respectively.
Applying Laplace transform to the vehicle model equation (2.2) gives,

$$s^2 X_i(s) = U(s)e^{-sD} \tag{2.8}$$

where $X_i(s)$ and $U(s)$ are Laplace transforms of $x_i(t)$ and $u(t)$ respectively.
Let the initial conditions be $x_i(0) = l_i$, $\dot{x}_i(0) = v$ and $\ddot{x}_i = 0$. Define $y_i(t)$ such that,

$$y_i(t) = x_i(t) - l_i - vt \tag{2.9}$$

$$\implies \dot{y}_i(t) = \dot{x}_i - v \tag{2.10}$$

$$\implies \ddot{y}_i(t) = \ddot{x}_i \tag{2.11}$$

When the Laplace transform of $y_i(t)$ is $Y_i(s)$, from (2.8) and (2.11),

$$s^2 Y(s) = U(s)e^{-sD} \tag{2.12}$$

Assuming the initial conditions satisfy the following equation,

$$l_i = l_{i-1} - L - hv$$

$$\implies e_{i,p} = y_i - y_{i-1} + h\dot{y}_i$$

Replacing the $x_i$ terms with $y_i$ terms in the control law equation (2.6) we get,

$$u_i = -K_p(y_i - y_{i-1} + h\dot{y}_i) - K_v(\dot{y}_i - \dot{y}_{i-1}) \tag{2.13}$$

Taking Laplace transform of equation (2.13) and combining with equation (2.12) gives,

$$s^2 e^{sD} Y(s) = -K_p(Y_i(s) - Y_{i-1}(s) + hsY_i(s)) - K_v s(Y_i(s) - Y_{i-1}(s))$$

Let the transfer function $G(s) = \frac{Y_i(s)}{Y_{i-1}(s)}$,

$$\implies G(s) = \frac{Y_i(s)}{Y_{i-1}(s)} = \frac{K_p + K_v s}{s^2 e^{sD} + (K_v + K_p h)s + K_p} \tag{2.14}$$

Applying Laplace transform to $e_{i,p}$,

$$E_{i,p}(s) = Y_i(s) - Y_{i-1}(s) + hsY_i(s) = (1 + hs)Y_i(s) - Y_{i-1}(s) \tag{2.15}$$

Substituting $Y_i(s)$ in terms of $Y_{i-1}(s)$ in equation (2.15),

$$\frac{E_{i,p}(s)}{Y_{i-1}(s)} = (1 + hs)G(s) - 1 \tag{2.16}$$

$$\frac{E_{i-1,p}(s)}{Y_{i-2}(s)} = (1 + hs)G(s) - 1$$

$$\implies \frac{E_{i,p}(s)}{E_{i-1,p}(s)} = \frac{Y_{i-1}(s)}{Y_{i-2}(s)} = G(s)$$

So the spacing error transfer function named $H(s) = G(s)$ is given by,

$$H(s) = \frac{E_{i,p}(s)}{E_{i-1,p}(s)} = \frac{K_p + K_v s}{s^2 e^{sD} + (K_v + K_p h)s + K_p} \tag{2.17}$$

## 2.3 Stability

A thorough understanding of the stability of the system is inevitable for ensuring safety. The most basic prerequisite of our controller is to track the longitudinal spacing error between vehicles while meeting the following stability objectives [6].

1. **Individual Stability:** We need to ensure that an individual coupling between two vehicles remains stable. Individual stability can be defined as the ability of a vehicle in the platoon to track bounded acceleration and velocity profiles of the preceding vehicle with a bounded spacing and velocity error »reference. This translates to BIBO stability which can be guaranteed by ensuring that the characteristic equation of the system is Hurwitz (i.e. has all its zeros in the LHP). Let the characteristic equation of our system be $\delta(s)$ given by,

   $$\delta(s) = \text{den}(H(s)) = s^2 e^{sD} + (K_v + K_p h)s + K_p \tag{2.18}$$

   An individual coupling in the platoon is stable if $\delta(s)$ is Hurwitz.

2. **String Stability:** It is important to ensure string stability of the ACC system to maintain a stable platoon. A vehicle platoon is string stable if the spacing errors are constrained from diverging when propagating upstream through the platoon [5] »reference. Various conditions for string stability have been detailed in »reference. We restrict our analysis to the following condition. For a platoon to be considered string stable the magnitude of the spacing error transfer function must not be greater than 1 at any frequency.

   $$\|H(j\omega)\|_\infty \leq 1 \quad \forall \quad \omega \tag{2.19}$$

The string stability of a platoon is relevant only if individual stability of all component pairs of vehicles can be established.

## 2.4 Problem Statement

The minimum time headway problem is approached in three steps.

1. For a given $h$ and $D$, find the set of all gains $(K_p, K_v)$ for which the individual coupling between one pair of vehicles is stable.

   Find $A_{(h,D)}$ where,

$$A_{(h,D)} := \{(K_p, K_v) \quad | \quad \delta(s) \text{is a Hurwitz polynomial}\} \tag{2.20}$$

2. For a given $h$ and $D$, find the set of all gains $(K_p, K_v)$ which satisfy the transfer function magnitude constraint for string stability. Finding $(K_p, K_v)$ that satisfy both individual stability and the magnitude constraint for string stability is more relevant.

   Find $B_{(h,D)}$ where,

$$B_{(h,D)} := \{(K_p, K_v) \quad | \quad \|H(j\omega)\|_\infty \leq 1 \quad \forall \quad \omega\} \tag{2.21}$$

   It is sufficient to find $A_{(h,D)} \cap B_{(h,D)}$.

3. For a given upper limit on time delay $D_0$, find the minimum $h$ for which there exists a pair of gains $(K_p, K_v)$ that guarantee both individual stability and string stability.

   For a given $D_0$,

$$\min \quad h \quad \text{s.t.} \tag{2.22}$$

$$A_{(h,D)} \cap B_{(h,D)} \neq \phi \quad \forall \quad D \leq D_0$$

   Alternately, the problem can be understood as finding the maximum allowable parasitic time delay in an ACC system for maintaining a stable platoon at a desired time headway.

# 3. STABILITY OF ONE COUPLED PAIR OF VEHICLES

In this chapter, the procedure to determine the set of gains $(K_p, K_v)$ ensuring the coupling between a single pair of vehicles in the platoon is stable will be discussed. In the section (3.1), we introduce the Hermite-Biehler theorem which can be employed to study stability in systems with a polynomial characteristic equation. In the section (3.2), we discuss the Hermite-Biehler Theorem for Quasi-Polynomials which is useful in computing the stabilizing gains in systems with delays. In the section (3.3), some conditions given by the Hermite-Biehler Theorem for Quasi-Polynomials are utilised to arrive at primitive bounds for the stabilizing region in the $K_pK_v$-plane. In the section (3.4), the exact boundary of the stabilizing region in the $K_pK_v$-plane is derived in terms of the Time Delay $(D)$ and Time Headway $(h)$.

## 3.1 Hermite-Biehler Theorem

Hermite-Biehler Theorem is based on the monotonic phase increase property of a Hurwitz polynomial. For any Hurwitz polynomial $\delta(s)$ of degree $n$,

- $\delta(j\omega)$ increases strictly and continuously with increasing $\omega$ in $(-\infty, \infty)$
- The plot of $\delta(j\omega)$ on the complex plane moves strictly in the counterclockwise direction and moves through $n$ quadrants i.e., there is a monotonic increase in the phase of $\delta(j\omega)$ with increasing $\omega$ in $(-\infty, \infty)$.

In case of a simple real polynomial $\delta(s)$ of degree $n$, such that:

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n$$

Alternatively $\delta(s)$ written as:

$$\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$$

where $\delta_e(s^2)$ and $s\delta_o(s^2)$ are polynomials formed by the even and odd powered terms in $\delta(s)$. This form is useful as when we consider $\delta(j\omega) = p(\omega) + jq(\omega)$ for $\omega \in \mathbb{R}$, $p(\omega) = \delta_e(-\omega^2)$ and $q(\omega) = \omega\delta_o(-\omega^2)$. So the even and odd powered terms correspond to the real and imaginary parts when the polynomial is written in its parametric form.

Let the non-negative real zeros of $\delta_e(-\omega^2)$ be $\omega_{e_1}, \omega_{e_2}, \omega_{e_3}, \ldots$ and the non-negative real zeros of $\delta_o(-\omega^2)$ be $\omega_{o_1}, \omega_{o_2}, \omega_{o_3}, \ldots$

**Theorem 3.1 (Hermite-Biehler Theorem)** *Let $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n$ be a given real polynomial of degree $n$. Then $\delta(s)$ is Hurwitz stable if and only if all the zeros of $\delta_e(-\omega^2)$, $\delta_o(-\omega^2)$ are real and distinct, $\delta_n$ and $\delta_{n-1}$ are of the same sign, and the non-negative real zeros satisfy the following interlacing property*

$$0 < \omega_{e_1} < \omega_{o_1} < \omega_{e_2} < \omega_{o_2} < \ldots \tag{3.1}$$

## 3.2 Hermite-Biehler Theorem for Quasi-Polynomials

The above version of the Hermite-Biehler theorem cannot be directly applied to the characteristic polynomial of systems with time delay. In presence of time delays, the characteristic equations will have exponential $e^s$ terms resulting in infinite number of roots. So, we focus on a generalization of the Hermite-Biehler Theorem due to Pontryagin which is better suited for testing the Hurwitz property of characteristic equations involving time delay.

In the presence of time delays, the characteristic equations of the systems take the form:

$$\delta(s) = f(s, e^s)$$

where $f(s, t)$ is a polynomial in two variables which is called as a quasi-polynomial. It is defined as:

$$f(s, t) = \sum_{h=0}^{M} \sum_{k=0}^{N} a_{hk} s^h t^k$$

In a characteristic equation of the form $f(s, e^s)$ the presence of the highest power term called Principal Term is crucial for stability. Formally, the Principal Term can be defined as:

**Principal Term:** *$f(s,t)$ is said to have a principal term if there exists a nonzero coefficient $a_{hk}$ where both indices have maximal values. Without loss of generality, we will denote the principal term as $a_{MN}s^M t^N$. This means that for each other term $a_{hk}s^h t^k$, for $a_{hk} \neq 0$; we have either $M > h, N > k$; or $M = h, N > k$; or $M > h, N = k$.*

The following theorem summarizes the importance of the principal term in a characteristic equation:

**Theorem 3.2** *If the polynomial $f(s,t)$ does not have a principal term, then the function $F(s) = f(s, e^s)$ has an infinite number of zeros with arbitrarily large positive real parts.*

Because of Theorem 3.2, it is crucial to ensure the presence of the principal term in characteristic equation of the ACC system before we proceed to analyse stability.

The characteristic equation of the ACC system is,

$$\delta(s) = s^2 e^{sD} + (K_v + K_p h)s + K_p \tag{3.2}$$

Clearly, $s^2 e^{sD}$ is the principal term as it has the highest powers of both $s$ and $e^s$. So our characteristic equation will not contain infinite number of zeros in the RHP.

The following is the Hermite-Biehler Theorem for Quasi-Polynomials due to Pontryagin which can be applied to derive the conditions for our characteristic equation to be Hurwitz.

**Theorem 3.3 (Hermite-Biehler Theorem for Quasi-Polynomials)** *Let $F(s) = f(s, e^s)$, where $f(s,t)$ is a polynomial with a principal term, and write*

$$F(j\omega) = F_r(\omega) + jF_i(\omega)$$

*where $F_r(\omega)$ and $F_i(\omega)$ represent, respectively, the real and imaginary parts of $F(j\omega)$. If all the roots of $F(s)$ lie in the open LHP, then the roots of $F_r(\omega)$ and $F_i(\omega)$ are real, simple, interlacing,* *and*

$$F_i'(\omega)F_r(\omega) - F_i(\omega)F_r'(\omega) > 0 \tag{3.3}$$

*for each* $\omega$ *in* $(^-\infty, \infty)$*, where* $F_r'(\omega)$ *and* $F_i'(\omega)$ *denote the first derivative with respect to u of* $F_r(\omega)$ *and* $F_i(\omega)$*, respectively. Moreover, in order that all the roots of* $F(s)$ *lie in the open LHP, it is sufficient that one of the following conditions be satisfied:*

1. *All the roots of* $F_r(\omega)$ *and* $F_i(\omega)$ *are real, simple, and interlacing and*

$$F_i'(\omega)F_r(\omega) - F_i(\omega)F_r'(\omega) > 0$$

2. *All the roots of* $F_r(\omega)$ *are real and for each root* $\omega = \omega_r$*,*

$$F_i(\omega)F_r'(\omega) < 0 \tag{3.4}$$

3. *All the roots of* $F_i(\omega)$ *are real and for each root* $\omega = \omega_i$*,*

$$F_r(\omega)F_i'(\omega) > 0 \tag{3.5}$$

### 3.3 Primitive Bounds

Before studying the conditions for stability of the ACC system in the presence of time delays, it is necessary to obtain the conditions for the delay-free system to be stable. The characteristic equation of the delay-free system is simply equation (3.2) when $D = 0$,

$$\delta(s) = s^2 + (K_v + K_p h)s + K_p \tag{3.6}$$

For both the roots of (3.6) to lie in the LHP, the conditions are:

The product of the roots must be positive and the sum of the roots must be negative, which gives:

$$K_p > 0 \tag{3.7}$$

$$K_v + K_p h < 0 \tag{3.8}$$

Now conditions for stability of ACC can be derived by applying the Hermite-Biehler Theorem. Replacing $s$ by $j\omega$ in the characteristic equation (3.2),

$$\delta(j\omega) = -\omega^2 e^{j\omega D} + j\omega(K_v + K_p h) + K_p \tag{3.9}$$

Decomposing $\delta(j\omega)$ into the real part $\delta_r(\omega)$ and the imaginary part $F_i(\omega)$, we get:

$$\delta_r(\omega) = K_p - \omega^2 \cos(\omega D) \tag{3.10}$$

$$\delta_i(\omega) = \omega(K_v + K_p h - \omega \sin(\omega D) \tag{3.11}$$

as $e^{j\omega D} = \cos(\omega D) + j\sin(\omega D)$.

An important condition provided by the Theorem 3.3 is that all the roots of $\delta_r(\omega)$ and $\delta_i(\omega)$ must be real. The following theorem enables us to check if all the roots are real.

**Theorem 3.4** *Let $M$ and $N$ denote the highest powers of $s$ and $e^s$, respectively, in $\delta(s)$. Let $\eta$ be an appropriate constant such that the coefficients of terms of highest degree in $\delta_r(\omega)$ and $\delta_i(\omega)$ do not vanish at $\omega = \eta$. Then for the equations $\delta_r(\omega) = 0$ or $\delta_i(\omega) = 0$ to have only real roots, it is necessary and sufficient that in each of the intervals*

$$-2l\pi + \eta < \omega < 2l\pi + \eta, l = l_0, l_0 + 1, l_0 + 2, \ldots$$

*$\delta_r(\omega)$ or $\delta_i(\omega)$ have exactly $4lN + M$ real roots for a sufficiently large $l_0$.*

For the characteristic equation (3.2), $M = 2$ and $N = 1$. $\eta = \frac{\pi}{4}$ can be chosen as the highest degree term doesn't vanish at $\frac{\pi}{4}$. Choosing $l_0 = 1$, we have to check if the number of roots of $\delta_r(\omega)$ and $\delta_i(\omega)$ have exactly $4lN + M = 6$ roots in the interval $z \in \left[-2\pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}\right] = \left[-\frac{7\pi}{4}, \frac{9\pi}{4}\right]$.

Let $z = \omega D$. Applying this variable change to (3.10) and (3.11) and simplifying to find roots we get:

$$\delta_r(z) = D^2 K_p - z^2 \cos(z) = 0 \tag{3.12}$$

$$\implies D^2 K_p = z^2 \cos(z)$$

$$\delta_i(z) = z((K_v + K_p h)D - z \sin(z)) = 0 \tag{3.13}$$

$$\implies z = 0 \quad \text{or} \quad D(K_v + K_p h) = z \sin(z)$$

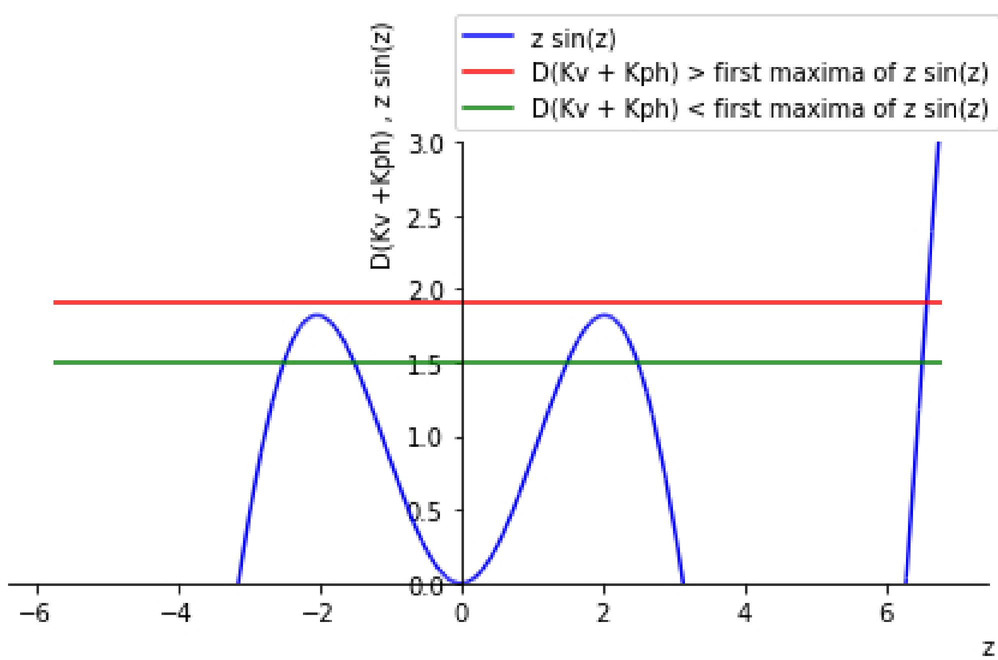


Figure 3.1: $z$ vs $z\sin(z)$ showing real roots of $\delta_i(z)$ for different values of $D(K_v + K_p h)$

For $\delta_i(z)$, $z = 0$ is a solution irrespective of $K_p, K_v, D, h$. The Figure 3.1 shows the plot $z$ vs $z\sin(z)$ for the interval $z \in \left[-\frac{7\pi}{4}, \frac{9\pi}{4}\right]$, from which we can see that the number of roots is 5 if

$D(K_v + K_p h) <$ first maxima$(z\sin(z))$. Including $z = 0$, $\delta_i(z) = 0$ has 6 roots in the interval. If $D(K_v + K_p h) >$ first maxima$(z\sin(z))$, $\delta_i(z) = 0$ has only 2 roots including $z = 0$ in the interval. On choosing any $l \geq 1$, we observe in the interval $\left[-2l\pi + \frac{\pi}{4}, 2l\pi + \frac{\pi}{4}\right]$ there are exactly $4lN + M$ roots of $\delta_i(z)$ if $D(K_v + K_p h) <$ first maxima$(z\sin(z))$; And $4lN + M - 4$ roots if $D(K_v + K_p h) >$ first maxima$(z\sin(z))$. So, for $\delta_i(z)$ to have only real roots,

$$D(K_v + K_p h) < \text{first maxima}(z\sin(z)) \simeq 1.819$$

$$K_v + K_p h < \frac{1.819}{D} \tag{3.14}$$

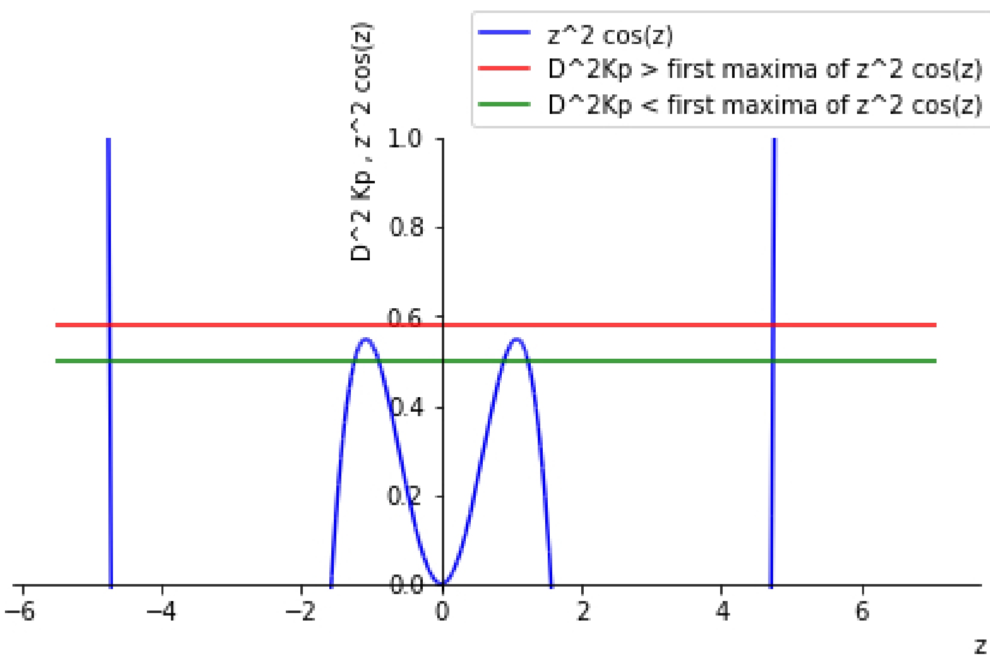


Figure 3.2: $z$ vs $z^2\cos(z)$ showing real roots of $\delta_r(z)$ for different values of $D^2K_p$

Similarly, Figure 3.2 shows $z$ vs $z^2\cos(z)$ plot in the interval $z \in \left[-\frac{7\pi}{4}, \frac{9\pi}{4}\right]$. It can be observed that $\delta_r(z)$ has exactly 6 roots in the interval if $D^2K_p <$ first maxima$(z^2\cos(s))$ but only 2 roots otherwise. In the general case, for any $l \geq 1$, there are exactly $4lN + M$ roots of $\delta_r(z)$ in the interval $\left[-2l\pi + \frac{\pi}{4}, 2l\pi + \frac{\pi}{4}\right]$ if and only if the condition $D^2K_p <$ first maxima$(z^2\cos(s))$ holds. For $\delta_r(z)$ to have only real roots,

$$D^2K_p < \text{first maxima}(z^2\cos(z)) \simeq 0.5498$$

$$K_p < \frac{0.5498}{D^2} \tag{3.15}$$

The equations (3.7), (3.8), (3.14) and (3.15) constitute a set of primitive bounds. For a given $h$ and $D$, the primitive bounds enclose a parallelogram region on the $K_pK_v$-plane as shown by the Figure 3.3. The $K_p$ dimension is proportional to $\frac{1}{D^2}$ and the $K_v$ dimension is proportional to $\frac{1}{D}$. The slope of two lines constituted by (3.8) and (3.14) is $-h$.

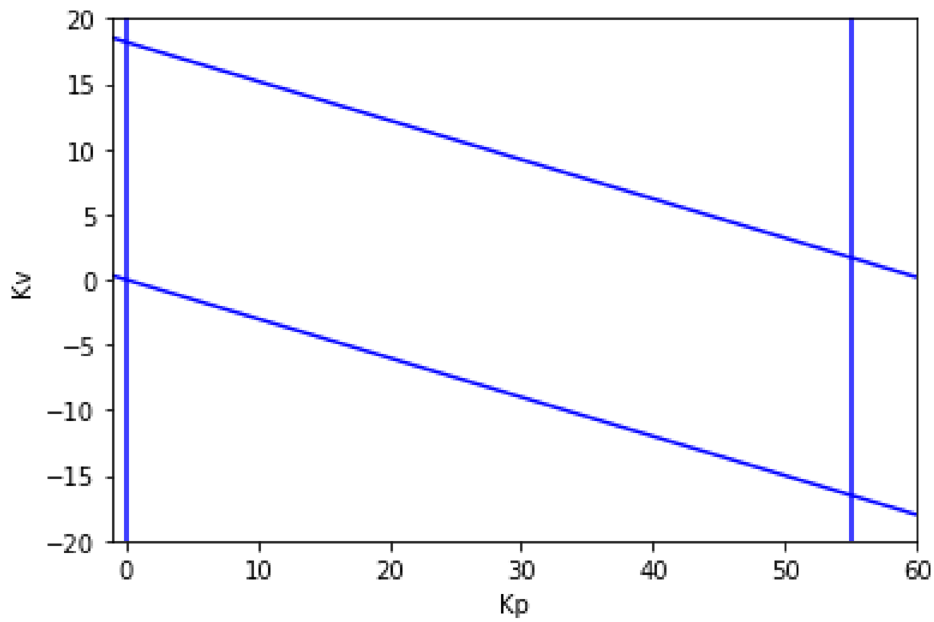


Figure 3.3: Primitive bounds (at D = 0.1, h = 0.3): $K_p > 0$, $K_v + K_ph > 0$, $K_p < \frac{0.5498}{D^2}$, $K_v + K_ph < \frac{1.819}{D}$

### 3.4 Exact boundary of the stabilizing region

The primitive bounds provide conditions for stability of the delay-free system and conditions for $\delta_r(\omega)$, $\delta_i(\omega)$ to have only real roots. Clearly, the stabilizing region in the $K_pK_v$-plane is a subset of the region enclosed by the primitive bounds. One of the conditions stated previously (inequality 3.4) as an equivalent of the monotonic phase increase condition for Quasi-Polynomials can be applied to derive the exact boundary of the stability region. The condition is:

- *All the roots of $F_r(\omega)$ are real and for each root $\omega = \omega_r$,*

$$F_i(\omega)F_r'(\omega) < 0$$

The primitive bounds ensure that all the roots of $\delta_r(\omega)$ are real. At some root $\omega = \omega_r$ of $\delta_r(\omega)$,

$$\delta_r(\omega_r) = K_p - {\omega_r}^2 \cos(\omega_r D) = 0$$

$$\implies K_p = {\omega_r}^2 \cos(\omega_r D)$$

For all values of $K_p$ lying within the primitive bounds, all roots of $\delta_r(\omega)$ are real and there exists infinite number of such $\omega_r$. At some random $\omega_r$ and associated $K_p$, the inequality (3.4) must hold for $\delta(s)$ to be Hurwitz.

$$\delta_r{}'(\omega_r) = -\frac{\partial}{\partial\omega}(\omega^2 \cos(\omega D))\Big|_{\omega=\omega_r} \tag{3.16}$$

At some $\omega = \omega_r$, where $K_p = {\omega_r}^2 \cos(\omega_r D)$ the inequality (3.4) becomes,

$$\delta_i(\omega_r)\delta_r{}'(\omega_r) < 0$$

$$\omega_r(K_v + K_p h - \omega_r \sin(\omega_r D))\frac{\partial}{\partial\omega}(\omega^2 \cos(\omega D))\Big|_{\omega=\omega_r} > 0$$

Overall the expression is even in $\omega$. So we can consider only positive $\omega_r$ and cancel $\omega_r$ from the inequality. Also substituting $K_p$ in terms of $\omega_r$,

$$(K_v - \omega_r \sin(\omega_r D) + h{\omega_r}^2 \cos(\omega_r D))\frac{\partial}{\partial \omega}(\omega^2 \cos(\omega D))\bigg|_{\omega=\omega_r} > 0$$

The condition can be broken into two inequalities based on whether $\omega^2 \cos(\omega D)$ is increasing or decreasing. As when it is increasing $\frac{\partial}{\partial \omega}(\omega^2 \cos(\omega D)) > 0$ and when decreasing $\frac{\partial}{\partial \omega}(\omega^2 \cos(\omega D)) < 0$. So the condition becomes:

$$\begin{aligned} K_v &> \omega_r \sin(\omega_r D) - h{\omega_r}^2 \cos(\omega_r D) \quad \text{if} \quad \omega^2 \cos(\omega D) \text{ is increasing.} \\ K_v &< \omega_r \sin(\omega_r D) - h{\omega_r}^2 \cos(\omega_r D) \quad \text{if} \quad \omega^2 \cos(\omega D) \text{ is decreasing.} \end{aligned} \tag{3.17}$$

As $\omega_r$ can take any value based on the choice of $K_p$, we get upper bounds and lower bounds for $K_v$ for each choice of $K_p$. The entire boundary can be formed by plotting the parametric curve $K_p = {\omega_r}^2 \cos(\omega_r D)$ vs $K_v = \omega_r \sin(\omega_r D) - h{\omega_r}^2 \cos(\omega_r D)$ with $\omega_r$ as a free variable (as shown in Figure 3.4). It is possible to demarcate between the portions of the parametric curve which form lower and upper bounds with ease as an increase or decrease along $K_p = {\omega_r}^2 \cos(\omega_r D)$ is easily observable.

Extracting the tightest bounds i.e minimum upper bounds and maximum lower bounds and combining with the primitive bounds (Figure 3.5), we can complete the exact boundary of the stabilizing region. The exact boundary of the region in the $K_pK_v$-plane which stabilizes the ACC system for a single coupled pair of vehicles is formed by:
The curve:

$$K_p = \omega^2 \cos(\omega D), \quad K_v = \omega \sin(\omega D) - h\omega^2 \cos(\omega D) \quad \text{where} \quad \omega \in \left[0, \frac{\pi}{2}\right] \tag{3.18}$$

and the line:

$$K_p > 0$$

The dimension of the stabilizing region along $K_p$ scales as $\frac{1}{D^2}$. The dimension along $K_v$ scales

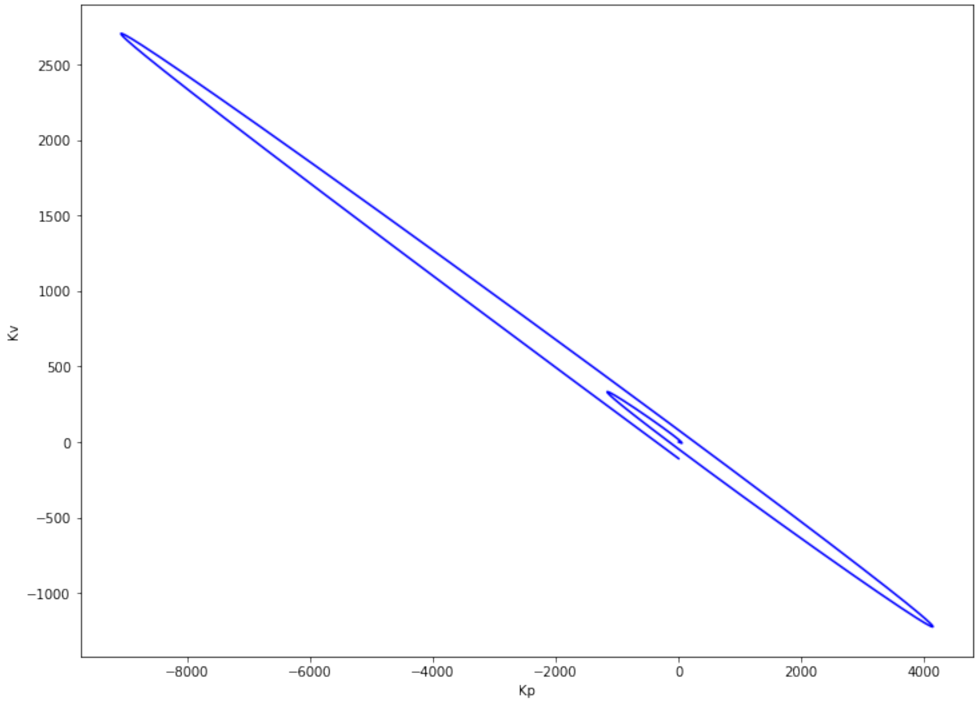


Figure 3.4: Parametric curve: $K_p = {\omega_r}^2 \cos(\omega_r D)$ vs $K_v = \omega_r \sin(\omega_r D) - h{\omega_r}^2 \cos(\omega_r D)$ where $\omega_r$ is a free variable.

as $\frac{1}{D}$. With an increase in $h$, the region gets skewed along negative $K_v$ (Figure 3.6).

Most importantly, for two time delays, $D_1$ and $D_2$, such that $D_1 > D_2$, the stability region when $D = D_1$ lies completely inside the stability region when $D = D_2$ for a given $h$ as shown in Figure 3.7. This property is crucial because for a given $h$, if a pair of gains $(K_p, K_v)$ stabilizes the system for a given delay $D_0$, the system remains stable for all $0 \leq D \leq D_0$.

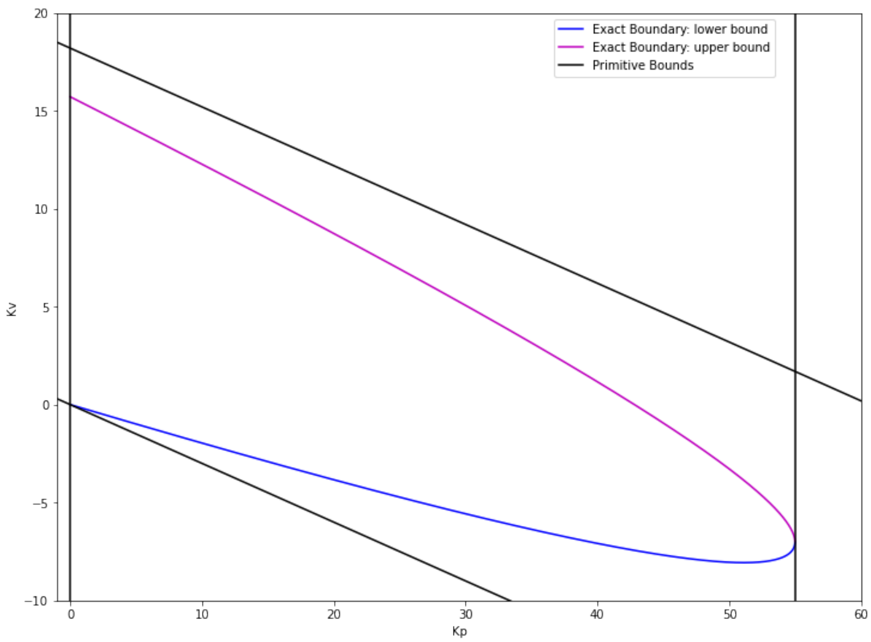


Figure 3.5: Exact boundary given by the tightest bounds along with primitive bounds (D = 0.1, h = 0.3)

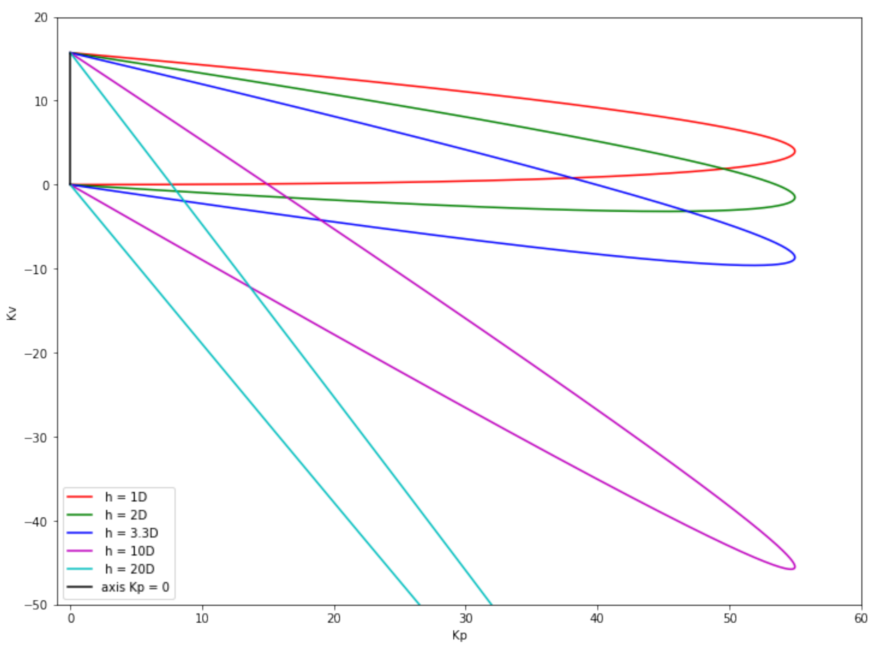


Figure 3.6: Variation of the stabilizing region with h (D = 0.1)

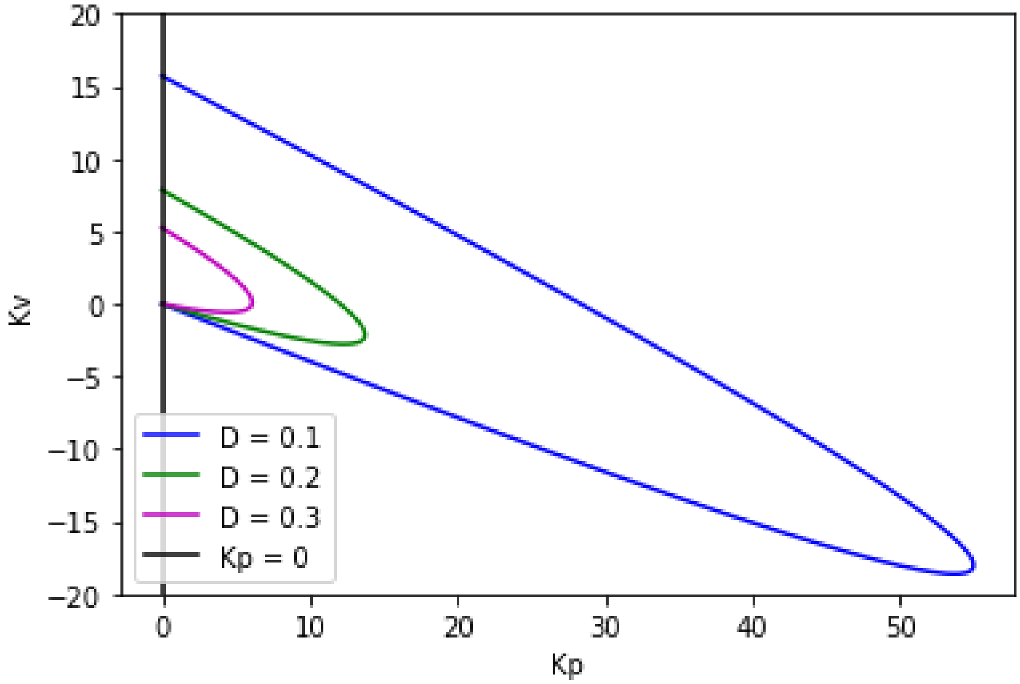


Figure 3.7: Variation of the stabilizing region with D (h = 0.5)

# 4. STRING STABILITY OF THE VEHICLE PLATOON

In this chapter, the procedure to determine the set of all gains $(K_p, K_v)$ for a given $h$,$D$ that ensure string stability of the vehicle platoon is explained. In the section (4.1), the conditions for string stability of a vehicle platoon are defined mathematically and the procedure to find the string stable region on the $K_pK_v$-plane is described briefly. In the section (4.2), the point condition is applied to the ACC system to derive a portion of the boundary of the string stable region. In the section (4.3), the tangent condition is applied to the ACC system to derive the complete boundary of the string stable region. In the section (4.4), the string stable region is compared with the region of stability of one pair of vehicles discussed in chapter 3. The variation of the string stable region with $h$ and $D$ is studied.

## 4.1 String Stability

While individual stability focuses on a single coupled pair of vehicles, string stability is a group property that deals with error propagation along a vehicle platoon. A vehicle platoon is string stable if the spacing error is constrained from amplifying when propagating from the start to end of the platoon. Mathematically, the string stability of the platoon employing our ACC system can be evaluated using the following condition.

Let $H(s)$ be the transfer function between spacing errors of two consecutive vehicles in the platoon. Then the vehicle platoon is considered string stable if:

$$\|H(j\omega)\|_\infty \leq 1 \quad \forall \quad \omega \tag{4.1}$$

This condition ensures that at any frequency, the spacing errors between the vehicles do not amplify.

$$\|H(j\omega)\| = \frac{\|\text{num}(H(j\omega))\|}{\|\text{den}(H(j\omega))\|}$$

So $\|H(j\omega)\| = 1 \implies \|H(j\omega)\|^2 = 1$, can be equivalently written as $\|\text{num}(H(j\omega))\|^2 - \|\text{den}(H(j\omega))\|^2 = 0$, as the denominator remains positive in the individual stability region . Although this condition is not equivalent, it works in the scope of our problem. For convenience defining $\xi(\omega, K_p, K_v)$ as,

$$\xi(\omega, K_p, K_v) = \|\text{den}(H(j\omega))\|^2 - \|\text{num}(H(j\omega))\|^2 \tag{4.2}$$

**Equivalence:**

$$(a) \quad 1 - \|H(j\omega)\|^2 = 0 = \frac{\|\text{den}(H(j\omega))\|^2 - \|\text{num}(H(j\omega))\|^2}{\|\text{den}(H(j\omega))\|^2} = \frac{\xi}{Y}$$

Let $\|\text{den}(H(j\omega))\|^2 = Y$. $Y$ is positive in individual stability boundary.

$$(b) \quad \frac{\partial}{\partial \omega}(1 - \|H(j\omega)\|^2) = \frac{\xi' Y - Y' \xi}{Y^2} = 0$$

System: $(a) = 0$ and $(b) = 0$ becomes $\xi = 0$ and $\xi' = 0$.

Similarly,

$$\frac{\partial^2}{\partial \omega^2}(1 - \|H(j\omega)\|^2) = \frac{\xi'' Y - \xi Y'' - 2(\xi' Y - Y' \xi) Y Y'}{Y^4}$$

At $\omega = 0$, irrespective of $(K_p, K_v)$, $\xi(0) = 0$, $\xi'(0) = 0$.

$$\frac{\partial^2}{\partial \omega^2}(1 - \|H(j\omega)\|^2) = 0 = \frac{\xi''}{Y^3}$$

$\implies \xi'' = 0$.

To derive the boundary of the string stable region it is sufficient to evaluate the following conditions.

1. **Point condition:** In the extremes of $\omega$, the boundary must meet the inequality (4.2). Mathematically this translates to:

   - At $\omega \to 0^+$, $\quad \xi(\omega, K_p, K_v) = 0$. More precisely, $\omega = 0$ must be a local maxima and not a local minima. To obtain the boundary, the first non-zero partial derivative with respect $\omega$ at $\omega = 0$ must be set to $0$.

$$\frac{\partial^{(k)}}{\partial\omega^{(k)}}\xi(\omega, K_p, K_v)\bigg|_{\omega=0} = 0 \tag{4.3}$$

where $k$ is the least natural number such that,

$$\frac{\partial^{(k)}}{\partial\omega^{(k)}}\xi(\omega, K_p, K_v)\bigg|_{\omega=0} \neq 0 \quad \text{for any} \quad (K_p, K_v)$$

- At $\omega \to \infty, \quad \|H(j\omega)\| \to 0$.

2. **Tangent condition:** For a given $h$,$D$ on the $K_pK_v$-plane the boundary is characterised by points where the maxima of $\|H(j\omega)\|$ over all $\omega$ just touches 1. At the peak, $\|H(j\omega)\| = 1$. This can be equivalently stated as,

$$\xi(\omega, K_p, K_v) = 0 \tag{4.4}$$

$$\frac{\partial}{\partial\omega}\xi(\omega, K_p, K_v) = 0 \tag{4.5}$$

At each $\omega$ by finding $K_p, K_v$ for which both the equations (4.4) and (4.5) are satisfied, it is possible to map the entire boundary of the string stable region.

## 4.2 Point condition

For our ACC system, the transfer function between spacing errors $e_{i-1,p}$ and $e_{i,p}$ of two consecutive vehicles is $\delta(s)$. So for string stability of the vehicle platoon,

$$\delta(j\omega) = \left\| \frac{K_v j\omega + K_p}{-\omega^2 e^{j\omega D} + (K_v + K_p h)j\omega + K_p} \right\|_\infty \leq 1 \quad \forall \quad \omega \tag{4.6}$$

$$\implies \xi(\omega, K_p, K_v) = \omega^4 - 2K_p\omega^2\cos\omega D + {K_p}^2h^2\omega^2 + 2K_pK_vh\omega^2 - 2\omega^3\sin\omega D(K_ph + K_v) \tag{4.7}$$

Applying (4.3),

$$\frac{\partial}{\partial \omega}\xi(\omega, K_p, K_v) = 4\omega^3 - 2K_p(2\omega\cos(\omega D) - \omega^2 D\sin(\omega D)) + 2\omega {K_p}^2 h^2 \\ + 4\omega K_p K_v h - (K_p h + K_v)(6\omega^2\sin(\omega D) + 2\omega^3 D\cos(\omega D) \tag{4.8}$$

Clearly $\frac{\partial}{\partial \omega}\xi(\omega, K_p, K_v)\Big|_{\omega=0} = 0$ irrespective of $K_p, K_v$. Calculating the second derivative,

$$\frac{\partial^2}{\partial \omega^2}\xi(\omega, K_p, K_v) = 12\omega^2 - 2K_p(2\cos(\omega D) - 4\omega D\sin(\omega D) - \omega^2 D^2\cos(\omega D)) \\ + 2{K_p}^2 h^2 + 4K_p K_v h \\ - (K_p h + K_v)(12\omega\sin(\omega D) + 12\omega^2 D\cos(\omega D) - 2\omega^3 D^2\sin(\omega D)) \tag{4.9}$$

At $\omega = 0$,

$$\frac{\partial^2}{\partial \omega^2}\xi(\omega, K_p, K_v)\Big|_{\omega=0} = -4K_p + 2{K_p}^2 h^2 + 4K_p K_v h \tag{4.10}$$

Since, $\frac{\partial^2}{\partial \omega^2}\xi(\omega, K_p, K_v)\Big|_{\omega=0} \neq 0$ irrespective of $K_p, K_v$, $k = 2$. The condition for the string stable region boundary is $\frac{\partial^2}{\partial \omega^2}\xi(\omega, K_p, K_v)\Big|_{\omega=0} = 0$.

$$-4K_p + 2{K_p}^2 h^2 + 4K_p K_v h = 0$$

It is sufficient to check for string stability in the region where individual stability has already been established. It is reasonable to consider only $K_p > 0$. The boundary simplifies into,

$$2K_v + K_p h = \frac{2}{h} \tag{4.11}$$

At $\omega \to \infty$, $\|H(j\omega)\| \to 0$ irrespective of $K_p, K_v$ as the denominator is higher in degree than the numerator. So the point condition at $\omega \to \infty$ always holds for our system. For a given $h$ and $D$, the point condition gives the line equation (4.11) which forms a portion of the boundary of the string stable region.

### 4.3 Tangent condition

At $\omega = 0$, the magnitude of the spacing error transfer function is invariably $1$. Also the slope of the magnitude is $0$ at any $K_p, K_v$. Moreover, the magnitude of the transfer function is even in $\omega$. So, it is sufficient to perform the tangent condition analysis for $\omega > 0$.

Cancelling $\omega^2$ throughout from $\xi(\omega, K_p, K_v) = 0$ (refer equation (4.7)),

$$\omega^2 - 2K_p \cos(\omega D) + {K_p}^2 h^2 + 2K_p K_v h - 2\omega \sin(\omega D)(K_v + K_p h) = 0 \tag{4.12}$$

Similarly after cancelling $2\omega$ from $\frac{\partial}{\partial\omega}\xi(\omega, K_p, K_v) = 0$ (refer equation (4.8)),

$$\begin{aligned} 2\omega^2 - K_p(2\cos(\omega D) - \omega D \sin(\omega D)) + {K_p}^2 h^2 + 2K_p K_v h \\ -(K_p h + K_v)(3\omega \sin(\omega D) + \omega^2 D \cos(\omega D)) = 0 \end{aligned} \tag{4.13}$$

Now subtracting equation (4.12) from equation (4.13) and cancelling $\omega$ throughout, the following line equation in the $K_p K_v$-plane is formed.

$$\omega + K_p(D\sin(\omega D) - h\sin(\omega D) - h\omega D\cos(\omega D)) - K_v(\sin(\omega D) + \omega D\cos(\omega D)) = 0 \tag{4.14}$$

Rewriting equation (4.14) to express $K_v$ in terms of $K_p, \omega, h, D$,

$$K_v = K_p \left( \frac{D\sin(\omega D)}{\sin(\omega D) + \omega D\cos(\omega D)} - h \right) + \frac{\omega}{\sin(\omega D) + \omega D\cos(\omega D)} \tag{4.15}$$

Let $P = \frac{D\sin(\omega D)}{\sin(\omega D)+\omega D\cos(\omega D)}$ and $Q = \frac{\omega}{\sin(\omega D)+\omega D\cos(\omega D)}$. The equation (4.15) becomes,

$$K_v = K_p(P - h) + Q \tag{4.16}$$

Substituting $K_p(P-h)+Q$ in place of $K_v$ in equation (4.12) the following quadratic equation in $K_p$ is formed.

$$K_p{}^2(2Ph - h^2) + K_p(2Qh - 2\cos(\omega D) - 2P\omega\sin(\omega D)) + \omega^2 - 2Q\omega\sin(\omega D) = 0 \quad (4.17)$$

Although the solution of the equation cannot be simplified into an elegant expression analytically,

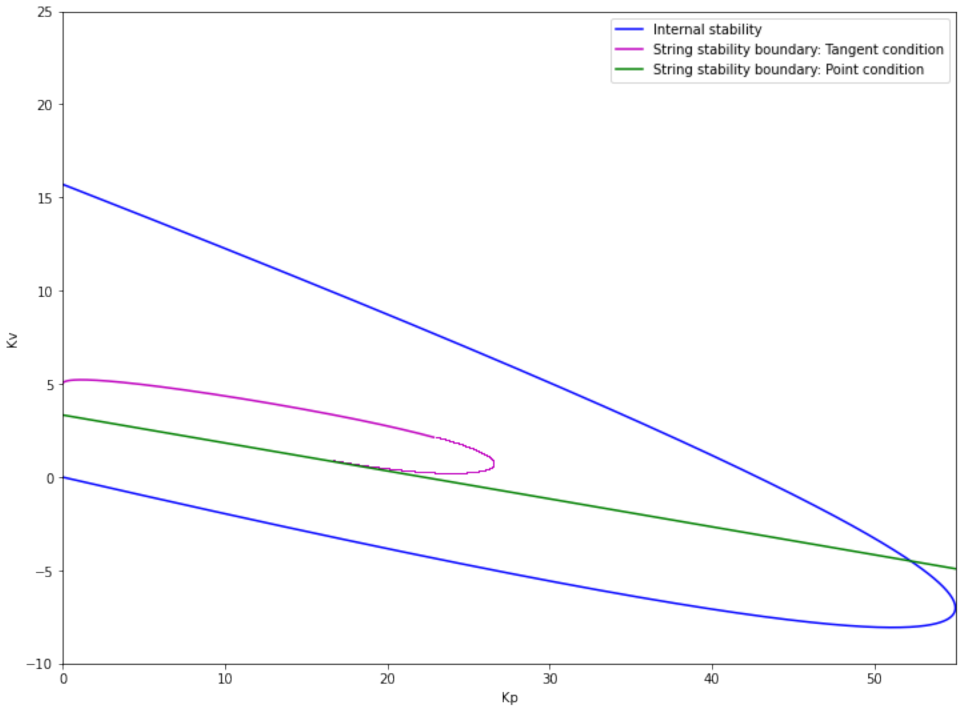


Figure 4.1: String stable boundary in comparison with individual stability boundary at D = 0.1s, h = 0.3s

it is possible to obtain a numerical solution. Moreover, we are only concerned about the boundary points lying inside the string stable region. At a given $h$ and $D$, the boundary points can be obtained by solving equation (4.17) at each $\omega$ numerically. Figure 4.1 shows the string stable boundary acquired from the point and tangent condition inside the individual stability boundary.

### 4.4 Observations on the string stability region

The combined boundary made from boundaries given by point and tangent condition that fall inside the region of individual stability, along with $K_p = 0$ axis, form a closed and bounded string stable region . As shown in Figure 4.2, the string stability inside the region can be verified by sampling points on the $K_pK_v$-plane around the region boundary and observing the magnitude $\|H(j\omega)\|$ varying over $\omega$. From the sampling exercise it is evident that the closed, bounded boundary encloses the string stable region. Moreover, points outside the boundary close to the line derived from the point condition, violate string stability by forming a minima at $\omega = 0$ (Figure 4.3) . Similarly, in case of a point just outside the region boundary acquired from tangent condition, the magnitude forms a peak at some $\omega$ that exceeds $1$ (Figure 4.4).

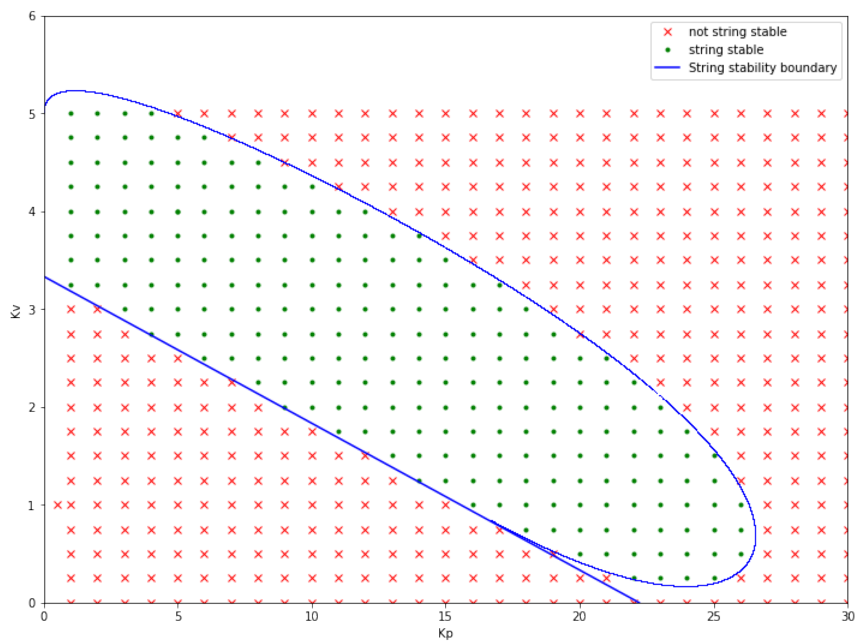


Figure 4.2: Sampled points around the string stability boundary at $D = 0.1s$, $h = 0.3s$

For a given delay $D$, the size of the string stable region expands with an increase in $h$ as shown in Figure 4.5. So at a given delay, a larger time headway would cause the size of the region to

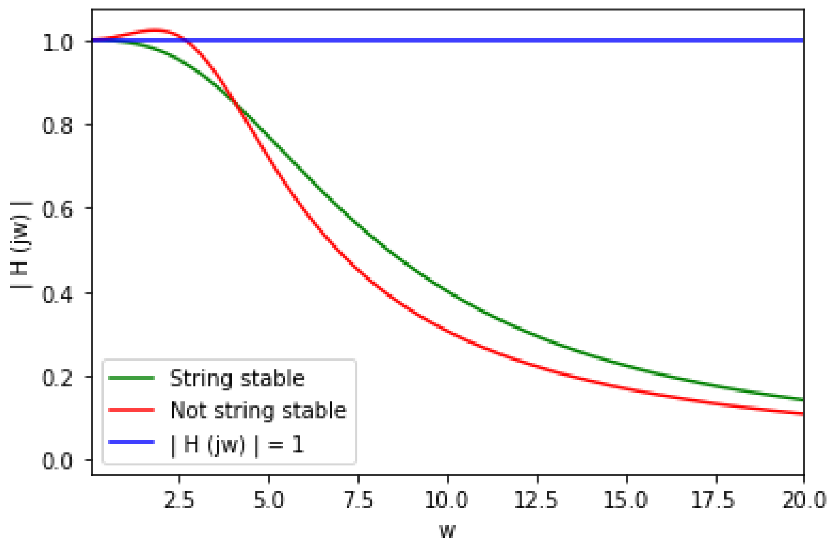


Figure 4.3: $(K_p, K_v)$ samples close to the point condition boundary: The variation of $\|H(j\omega)\|$ with $\omega$ at $D = 0.1s$, $h = 0.3s$ at sampled $(K_p, K_v)$. Stable at $(K_p, K_v) = (8, 2.25)$; Unstable at $(K_p, K_v) = (8, 1.75)$.

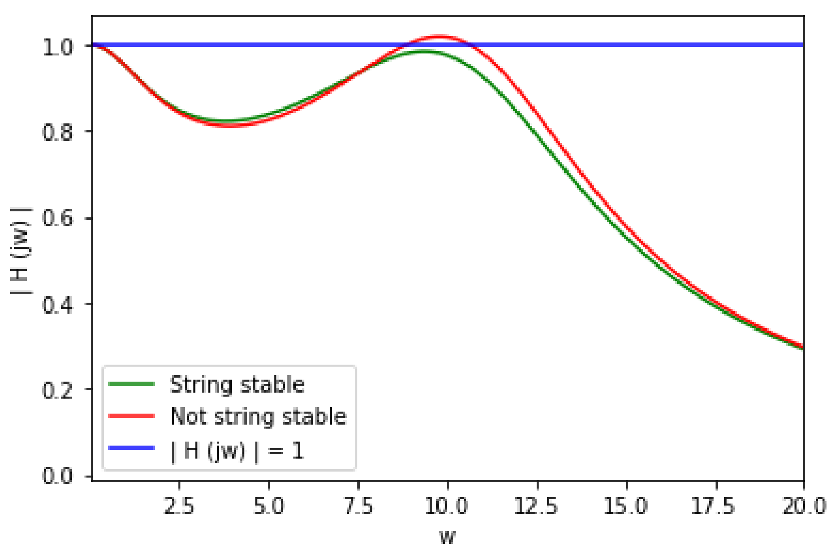


Figure 4.4: $(K_p, K_v)$ samples close to the tangent condition boundary: The variation of $\|H(j\omega)\|$ with $\omega$ at $D = 0.1s$, $h = 0.3s$ at sampled $(K_p, K_v)$. Stable at $(K_p, K_v) = (12, 4)$; Unstable at $(K_p, K_v) = (13, 4)$.

expand. Also with an increase in $h$, the region expands towards the negative $K_v$ axis. The point where the boundary meets $K_p = 0$, remains the same as it depends only on $D$.

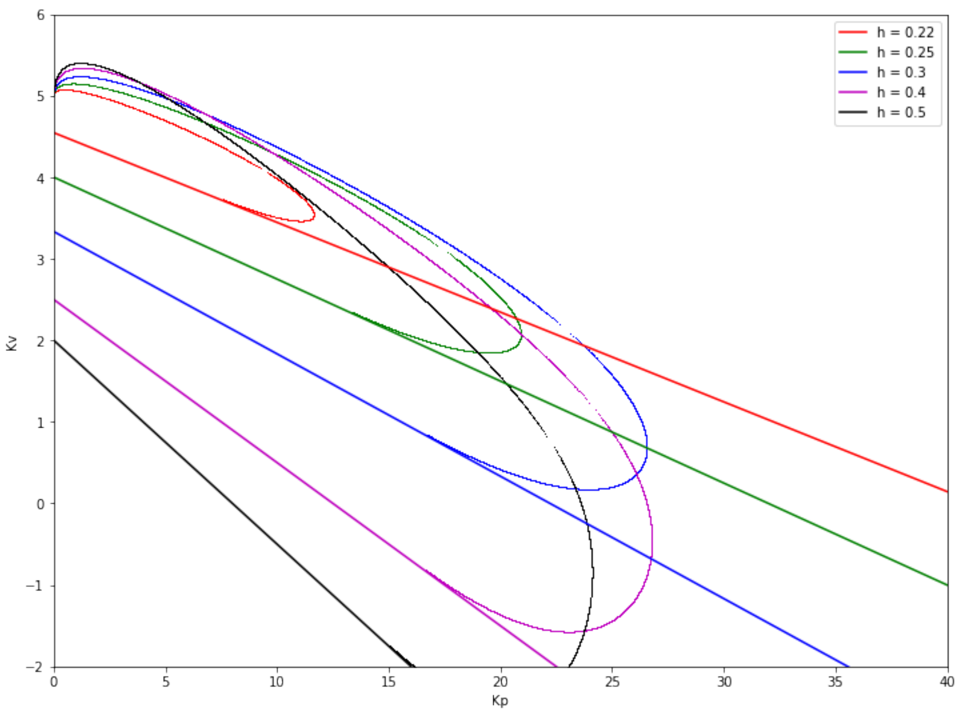


Figure 4.5: The variation of the string stable boundary with $h$ for a given time delay $D = 0.1$

For a given time headway $h$, the size of the region shrinks as the time delay $D$ increases as shown in Figure 4.6. The boundary derived by point condition does not change with a variation in $D$ as it depends only on $h$. So when $D$ increases the size of the region shrinks along the point condition line and $K_v$ axis. Most importantly, For a given $h$, for two delays $D1$ and $D2$, such that $D1 < D2$, the string stable region during $D2$ lies completely inside the string stability region during $D1$. So if the gains $K_p, K_v$ are picked from within the string stability region at the upper limit of delays $D_0$, it ensures that string stability prevails for all the other time delays $0 \leq D \leq D_0$.

However small the time delay, for a large enough $h$ the string stability boundary expands but never intersects the individual stability boundary (Figure 4.7).

Based on the numerical results, for a given $D$ the string stable region exists if and only if $h > 2D$. The observation as shown in Figure 4.7 shows that irrespective of the $D$ as the $h$ value gets closer to $2D$, the string stable region begins to vanish. And for $h \leq 2D$, the string stability

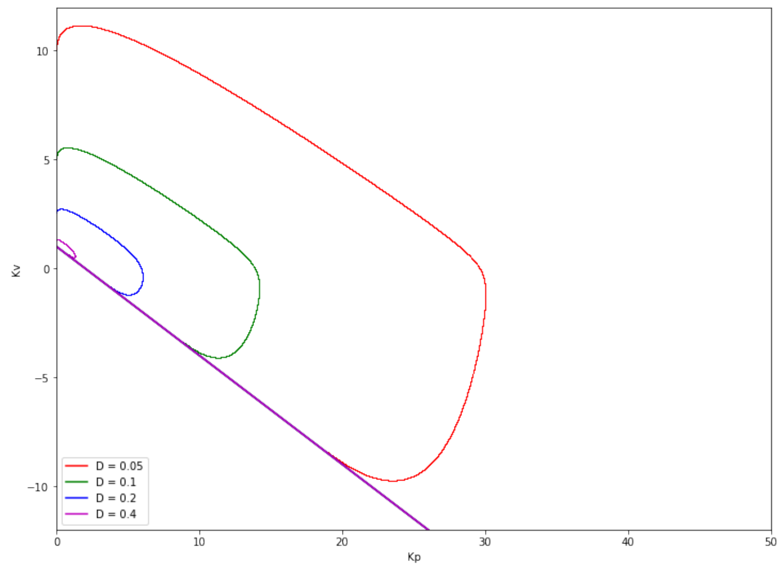


Figure 4.6: The variation of the string stable boundary with $D$ for a given time delay $h = 1$

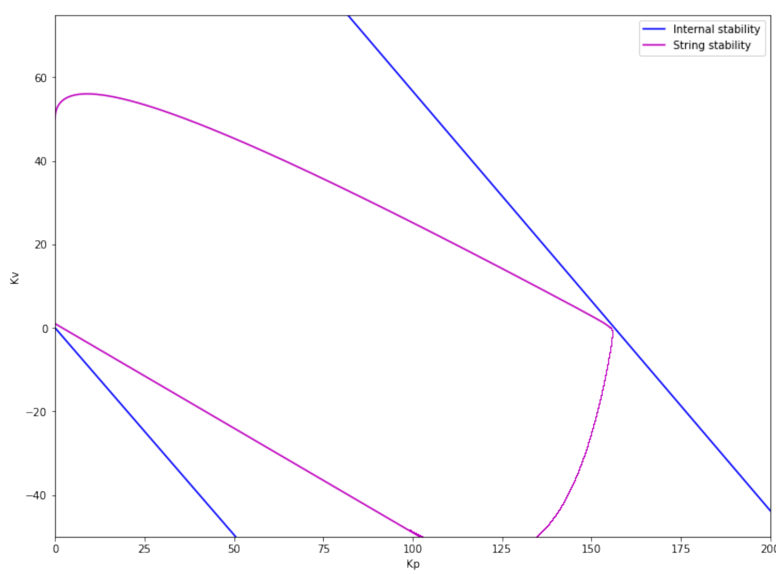


Figure 4.7: The comparison between string stable boundary and individual stability boundary at a small time delay i.e high $\frac{h}{D}$ ratio. $D = 0.1$, $h = 1$.

tangent condition does not produce any boundary inside the individual stability region. As shown in Figure 4.8 even after sampling through the portion of individual stability region that satisfies the string stability due to point condition, there are no string stable points.

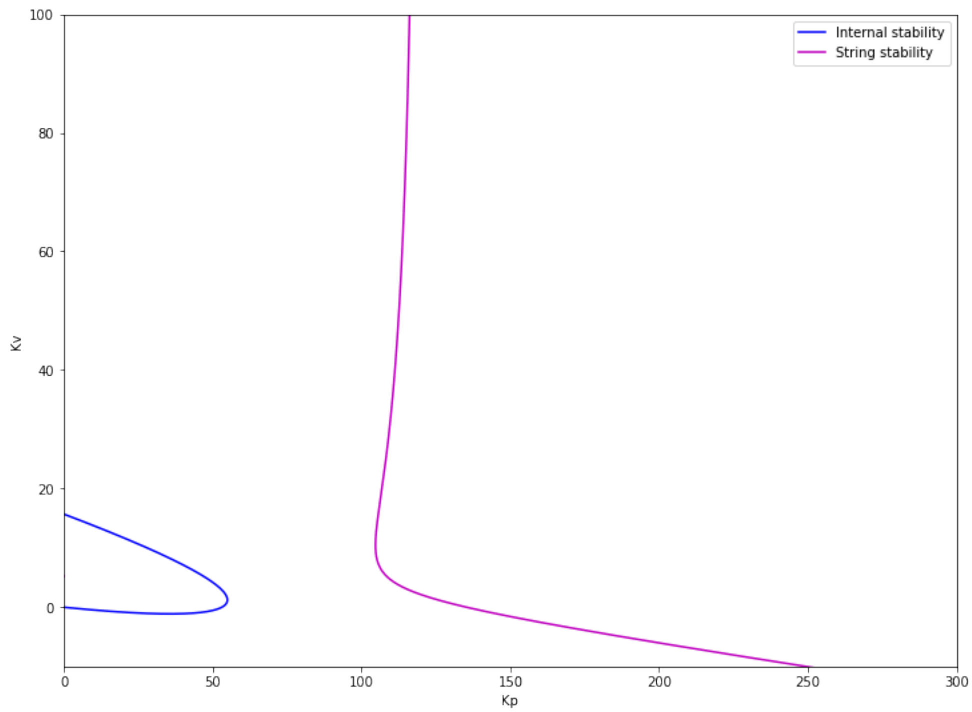


Figure 4.8: The string stability boundary due to tangent condition lies entire outside the individual stability region. $D = 0.1$, $h = 0.15$.

So if the delay bound is $D_0$, the minimum time headway $h$ that can be chosen must be greater than $2D_0$. This will ensure that a string stable region exists for any $D$, such that $0 \leq D \leq D_0$. Alternately, to choose a time headway $h$, it is essential to ensure the composite time delay $D$ of the system is lesser than $\frac{h}{2}$.

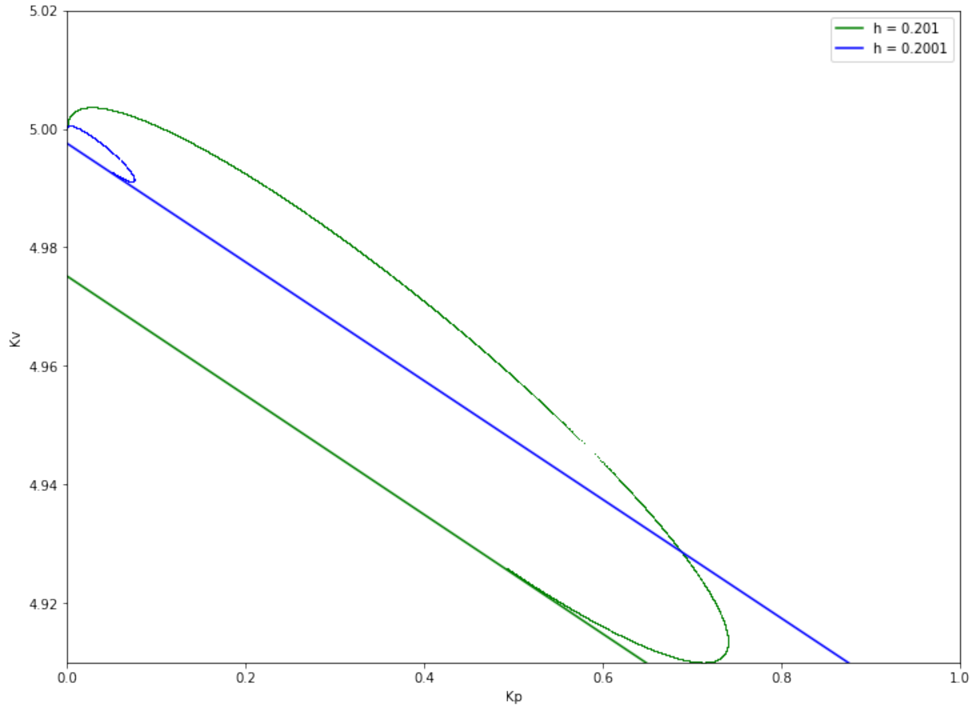


Figure 4.9: The string stable region vanishing as $h$ gets closer to $2D$ at $D = 0.1s$

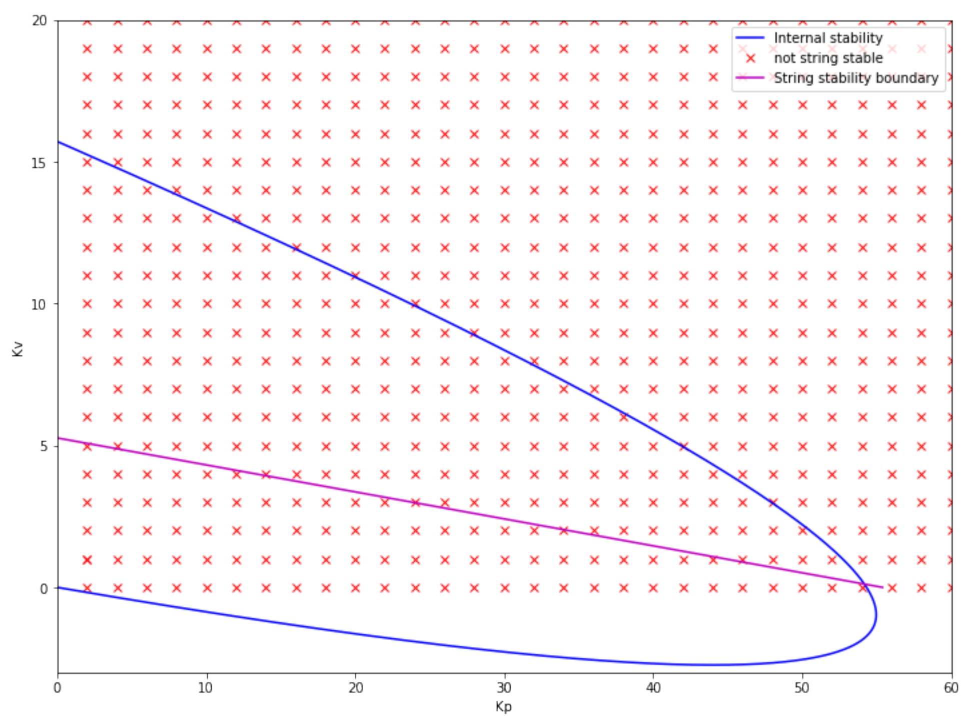


Figure 4.10: Sampling showing complete failure of string stability at $h < 2D$. $h = 0.19$, $D = 0.1$

# 5. SUMMARY AND CONCLUSIONS

For a given $h$ and $D$, the individual stability region on the $K_pK_v$-plane is found to be a connected bounded region. For any two delays $D_1$ and $D_2$ such that $D_1 < D_2$, the individual stability region due to $D_2$ is a subset of the individual stability region due to $D_1$ for a fixed $h$.

Similarly, the string stability region on the $K_pK_v$-plane by definition is a connected bounded subset of the individual stability region. For any two delays $D_1$ and $D_2$ such that $D_1 < D_2$, the string stability region due to $D_2$ is a subset of the string stability region due to $D_1$ for a fixed $h$. For a given $D$ the size of the string stability region increases with an increase in $h$.

In the presence of time-varying time delay bounded by $D_0$, for a string stable region to exist, the time headway must be greater than twice the upper limit of time delay i.e. $h > 2D_0$. Conversely, the maximum allowable time delay $D_0$ for a given time headway $h$ for a string stable region to exist is $D_0 < \frac{h}{2}$.

In future, this work can be extended as follows:

- The string stable region is numerically computed and the minimum time headway was concluded based on numerical results. In future, these results can be corroborated by deriving analytical results
- The system can be generalised to include both lags and time delays
- Time Headway bounds for heterogeneous vehicle platoons with different time delays may be explored. Effect of time delays on stability of a heterogeneous vehicle platoon can be studied

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