NONLINEAR EMBEDDINGS INTO BANACH SPACES

A Dissertation

by

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ABSTRACT

The study of nonlinear embeddings of Banach spaces has been an active field of research since the beginning of the 20th century with many applications to theoretical computer science (Sparsest Cut problem, Nearest Neighbor Search, etc), geometry (Gromov's positive scalar curvature conjecture, the Novikov conjecture, etc) and group theory (growth of groups, amenability, etc). In this dissertation, we review some pre-existing theory about isometric, bi-Lipschitz, quasi-isometric, and coarse embeddings of metric spaces into Banach spaces, as well as provide some new results. In Section 3 we give a new derivation of optimal bounds from below for the distortion of ℓ_q into p-uniformly convex Banach spaces. In particular, this allows us to present a new proof of the fact that there exists a doubling subset of ℓ_q that does not admit any bi-Lipschitz embedding into $\mathbb{R}^d$ for any $d \in \mathbb{N}$ and $q > 2$ (this result follows from previous work by V. Lafforgue and A. Naor and independent results by Y. Bartal, L. Gottlieb and O. Neiman). We also study how our new approach can be generalized to obtain embeddability obstructions into non-positively curved spaces. In Section 4 we study equivariant coarse embedding into ℓ_q. Those are special kind of coarse embeddings which come with a representation that is connected to the embedding itself. We show that if a normed vector space, viewed as an abelian group under addition, admits an equivariant coarse embedding into ℓ_p then it also embeds in a bi-Lipschitz way into ℓ_p. We discuss potential applications of this result to open problems about coarse embeddings into ℓ_p.

DEDICATION

To my mother, my grandmother, and my grandfather.

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1. INTRODUCTION

Banach spaces are one of the cornerstones of functional analysis and modern mathematics in general. The development of the linear theory and its applications to differential equations, geometry, complexity theory, and other fields, were among some of the most important advances in mathematics of the 20th century. In this dissertation, however, we focus on the non-linear theory of Banach spaces and of its possible applications.

One of the most important results that started this line of research is that of Ribe who proved in [Rib76] that uniformly homeomorphic Banach spaces have uniformly linearly isomorphic finite-dimensional subspaces. It has inspired Bourgain and Lindenstrauss to propose a program of trying to classify properties of Banach spaces using a purely metric language. Bourgain started what is now called Ribe's program (see [N06] for an overview) by showing in [Bou86] that a Banach space is superreflexive if and only if the family of binary trees of finite height does not equi-bi-Lipschitzly embed into it.

Banach spaces behave in a very different way depending on the category of maps we consider. For example, Mazur and Ulam in [MU32] proved that all surjective isometries between Banach spaces consist of a linear isometry and a translation by a constant vector. In Section 2 we discuss another result in a similar direction, namely that of Heinrich and Mankiewicz who in [HM82] proved that if $f\colon X \to Y$ is a bi-Lipschitz embedding from a separable Banach space X into a space Y with the Radon–Nikodym property, then there exists a point of Gâteaux differentiability of f and the derivative at that point is a linear embedding with distortion bounded by distortion of f. On the other side of the spectrum, there are results like one of Kadets who proved in [Kad67] that all separable infinite-dimensional Banach spaces are homeomorphic.

One particular field of research that produces non-trivial results about Banach spaces is large scale geometry. It studies spaces viewed from far away and is interested in global geometric features rather than the local ones. From this point of view, integers are the same as the real line since they look alike if you zoom out sufficiently. Similarly, a disk (or any other bounded geometric space) is indistinguishable from a single point. This is in stark contrast to a lot of classical notions in analysis that focus on the local properties of maps and spaces. Historically these ideas first appeared in the proof of Mostow's rigidity theorem [Mos68] and its strengthening by Margulis [Mar70]. They have later entered the field of geometric group theory for good with the work of Švarc, Milnor, and Wolf on the growth of groups. A celebrated result by Gromov [GM81], which states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, relied heavily on this machinery. The definition of the quasi-isometric equivalence (i.e. a map that preserves the large scale geometry, the concept that was further generalized into coarse equivalence) was finally formulated by Gromov in [GM93], and used in great success in different areas of mathematics ranging from differential geometry to data analysis. Many concepts have been studied extensively in group theory, for example, the existence of the Banach-Tarski paradox [BT24] that had lead von Neumann to introduce the concept of amenability in [VNJ29], turned out to be a large scale invariant. One can find a good overview of the field in [NY12].

A result by Yu from [Yu00] motivated a lot of research in this field, because it connected large scale geometry to K-theory. Namely, Yu proved that if a discrete metric space X with bounded geometry admits a coarse embedding into Hilbert space, then the coarse Baum-Connes conjecture holds for X, i.e. the coarse index map is an isomorphism. Note that if X happens to be a finitely generated group G whose classifying space BG has the homotopy type of a finite CW complex and the metric structure we consider comes from a word-length metric, one can deduce that the strong Novikov conjecture holds for G, i.e. the index map from $K_*(BG)$ to $K_*(C_r^*G)$ is injective, where C_r^*G is the reduced C^* algebra of G. Note that by the index theory, the strong Novikov conjecture implies the Novikov conjecture i.e. that higher signatures (certain numerical invariants in the Pontryagin classes of smooth manifolds) are homotopy invariant. This result was later generalized by Kasparov and Yu in [KGYG] where they proved that if X is a bounded geometry metric space, which admits a coarse embedding into a uniformly convex Banach space, then the coarse Novikov conjecture holds for X; that is, the coarse index map is injective. Those results inspired people to study not only coarse embeddings into Banach spaces but also various coarse embeddings between said spaces.

This dissertation is organized in the following manner. In Section 2, we recall some relevant notions about topology, linear and non-linear theory of Banach spaces, and geometric group theory. We give an overview of existing results related to our work as well as establish some tools we will use later on.

In Section 3, we revisit the main results from [BGN14, BGN15] and [LN14a] about the impossibility of dimension reduction for doubling subsets of ℓ_q for $q > 2$. We provide an alternative elementary proof of this impossibility result that combines the simplicity of the construction in [BGN14, BGN15] with the generality of the approach in [LN14a] (except for L_1 targets). One advantage of this different approach is that it can be naturally generalized to obtain embeddability obstructions into non-positively curved spaces or asymptotically uniformly convex Banach spaces.

In Section 4, we study the category of equivariant coarse embeddings - which are coarse embeddings that satisfy a certain compatibility condition with a predetermined representation into the isometry group of the target space. We find that the existence of such an embedding of a normed vector space into ℓ_p forces it to be bi-Lipschitz embeddable into ℓ_p. We also discuss a possible way to attack the problem of coarse embeddings into ℓ_p using this method. In particular one could hope it can address the question of the existence of a coarse embedding of L_p into ℓ_p.

2. PRELIMINARIES

2.1 Topology

In this section we recall some basic topological properties of spaces, starting with introducing the concept of topology which is an abstract way to define the family of open sets:

Definition 1. *A topological space consists of a set X together with a family τ of its subsets, such that:*

- *the empty set and X itself belong to τ,*
- *τ is closed under taking arbitrary unions,*
- *τ is closed under taking finite intersections.*

Complements of open sets are called closed sets. Now, let us introduce maps that preserve the above structure.

Definition 2. *A function f between topological spaces is called continuous if the pre-image of every open set is open. If f is a bijection and both f and its inverse f^{-1} are continuous we call such a map a homeomorphism.*

Notice that from a topological point of view homeomorphic spaces are indistinguishable as there is a one to one correspondence between both the points of the spaces as well as their open sets.

Many of the spaces that we will consider will belong to the following family of spaces:

Definition 3. *A topological space (X,τ) is called separable if it contains a countable, dense subset. That means there exists a countable family of elements of X such that every nonempty open subset contains an element of the family.*

Let us introduce the additional structure of a metric on a set, which allows us to measure a distance between a pair of points.

Definition 4. *A metric on a set X is a function $d : X \times X \to [0,\infty)$ such that*

- $d(x,y) = 0 \Leftrightarrow x = y$*,*
- *for every $x,y \in X$ we have $d(x,y) = d(y,x)$,*
- *the triangle inequality holds, i.e. for all $x,y,z \in X$ we have $d(x,y) \leqslant d(x,z) + d(z,y)$.*

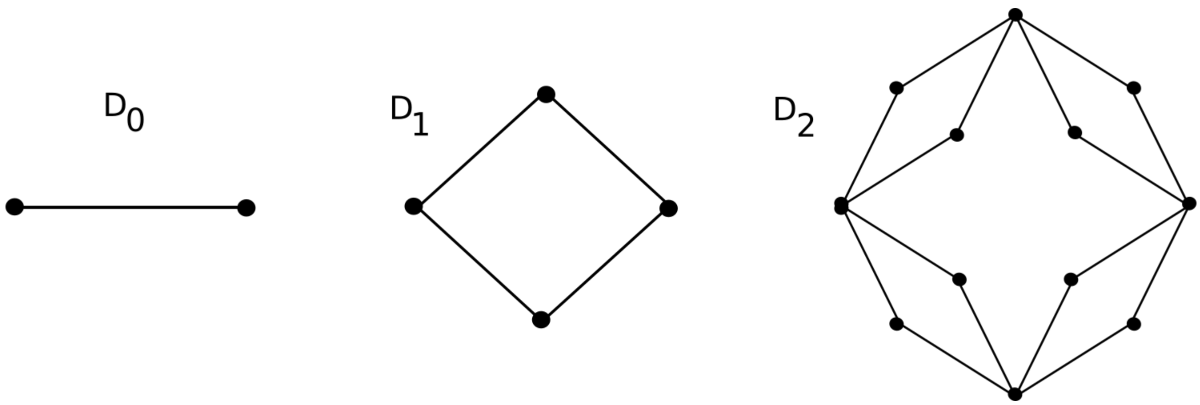
Note that every metric space has a natural topology generated by open balls $B(x,r) = \{y \in X \colon d(x,y) < r\}$. A very basic example of a metric is a graph equipped with the shortest path metric:

Definition 5. *Let (V,E) be a graph with a set of vertices V and a set of edges $E \subset V \times V$. We say that a sequence of vertices $(v_0,\ldots,v_n)$ is a path of length of n between a vertex x and a vertex y if $v_0 = x$, $v_n = y$, and for every $i \in \{0,\ldots n-1\}$ vertices v_i and v_{i+1} are connected by an edge. The shortest path metric on a connected graph (V,E) assigns the distance between the pair of vertices to be equal to the length of a shortest path between them. If the graph is not connected we can assign a fixed distance between its components. We might modify this construction by assigning different weights to the individual edges and then adding it to the length of a path that passes through it. We can also consider directed graphs in which an edge $\{a,b\}$ can be taken only as a path from a to b but not the other way around.*

A lot of graphs we will consider will arise as Cayley graphs of groups, which are discussed in subsection 3. Another type of graphs, that will be important for us later on, are some families that are defined in a recursive way. Let us start by introducing the first family of those, called diamond graphs. The diamond graph of level 0, denoted D_0, consists of two vertices and an edge of length 1 between them. Informally we construct D_n from D_{n-1} in the following way: we replace every edge $\{a,b\}$ of D_{n-1} with a pair of new vertices m_1, m_2 and fours new edges: $\{a,m_1\}$, $\{a,m_2\}$, $\{b,m_1\}$ and $\{b,m_2\}$. See the Figure 2.1 below for a reference.

Note that quite often in order not to change the distance between vertices a,b when performing the recursive step one rescales the length of the new edges by half. And so when using this convention all edges of D_n would have length $\frac{1}{2^n}$.

Figure 2.1: First three diamond graphs D_0, D_1, and D_2.



Now let us make the definition precise. A directed s-t graph $\mathsf{G} = (V,E)$ is a directed graph which has two distinguished vertices $s,t \in V$. To avoid confusion, we will also write sometimes $s(\mathsf{G})$ and $t(\mathsf{G})$. There is a natural way to "compose" directed s-t graphs using the $\oslash$-product defined in [LR10]. Informally, the $\oslash$ operation replaces all the edges of an s-t graph by identical copies of a given s-t-graph. Given two directed s-t graphs H and G, define a new graph $\mathsf{H} \oslash \mathsf{G}$ as follows:

i) $V(\mathsf{H} \oslash \mathsf{G}) \stackrel{\text{def}}{=} V(\mathsf{H}) \cup (E(\mathsf{H}) \times (V(\mathsf{G}) \setminus \{s(\mathsf{G}), t(\mathsf{G})\}))$

ii) For every oriented edge $e = (u,v) \in E(\mathsf{H})$, there are $|E(\mathsf{G})|$ oriented edges,

$$\begin{gathered}\big\{(\{e,v_1\},\{e,v_2\}) \mid (v_1,v_2) \in E(\mathsf{G}) \text{ and } v_1,v_2 \notin \{s(\mathsf{G}),t(\mathsf{G})\}\big\}\\ \cup\big\{(u,\{e,w\}) \mid (s(\mathsf{G}),w) \in E(\mathsf{G})\big\} \cup \big\{(\{e,w\},u) \mid (w,s(\mathsf{G})) \in E(\mathsf{G})\big\}\\ \cup\big\{(\{e,w\},v) \mid (w,t(\mathsf{G})) \in E(\mathsf{G})\big\} \cup \big\{(v,\{e,w\}) \mid (t(\mathsf{G}),w) \in E(\mathsf{G})\big\}\end{gathered}$$

iii) $s(\mathsf{H} \oslash \mathsf{G}) \stackrel{\text{def}}{=} s(\mathsf{H})$ and $t(\mathsf{H} \oslash \mathsf{G}) \stackrel{\text{def}}{=} t(\mathsf{H})$.

It is clear that the $\oslash$-product is associative (in the sense of graph-isomorphism or metric space isometry), and for a directed graph G one can recursively define $\mathsf{G}^{\oslash k}$ for all $k \in \mathbb{N}$ as follows:

- $\mathsf{G}^{\oslash 1} \stackrel{\text{def}}{=} \mathsf{G}$.

- $\mathsf{G}^{\oslash^{k+1}} \stackrel{\text{def}}{=} \mathsf{G}^{\oslash^k} \oslash \mathsf{G}$, for $k \geqslant 1$.

Note that it is sometimes convenient, for some induction purposes, to define $\mathsf{G}^{\oslash^0}$ to be the two-vertex graph with an edge connecting them. Note also that if the base graph G is symmetric the graph $\mathsf{G}^{\oslash^k}$ does not depend on the orientation of the edges.

If one starts with the 4-cycle C_4, the graph $\mathsf{D_k} \stackrel{\text{def}}{=} \mathsf{C}_4^{\oslash^k}$ is the diamond graph of depth k. The countably branching diamond graph of depth k is defined as $\mathsf{D}_k^{\omega} \stackrel{\text{def}}{=} \mathsf{K}_{2,\omega}^{\oslash^k}$, where $\mathsf{K}_{2,\omega}$ is the complete bipartite infinite graph with two vertices on one side, (such that one is $s(\mathsf{K}_{2,\omega})$ and the other $t(\mathsf{K}_{2,\omega})$), and countably many vertices on the other side. Another family of graphs that is closely related to diamond graphs is that of Laakso graphs, $\mathsf{L}_k \stackrel{\text{def}}{=} \mathsf{L}_1^{\oslash^k}$ where the base graph L_1 is the graph depicted below.

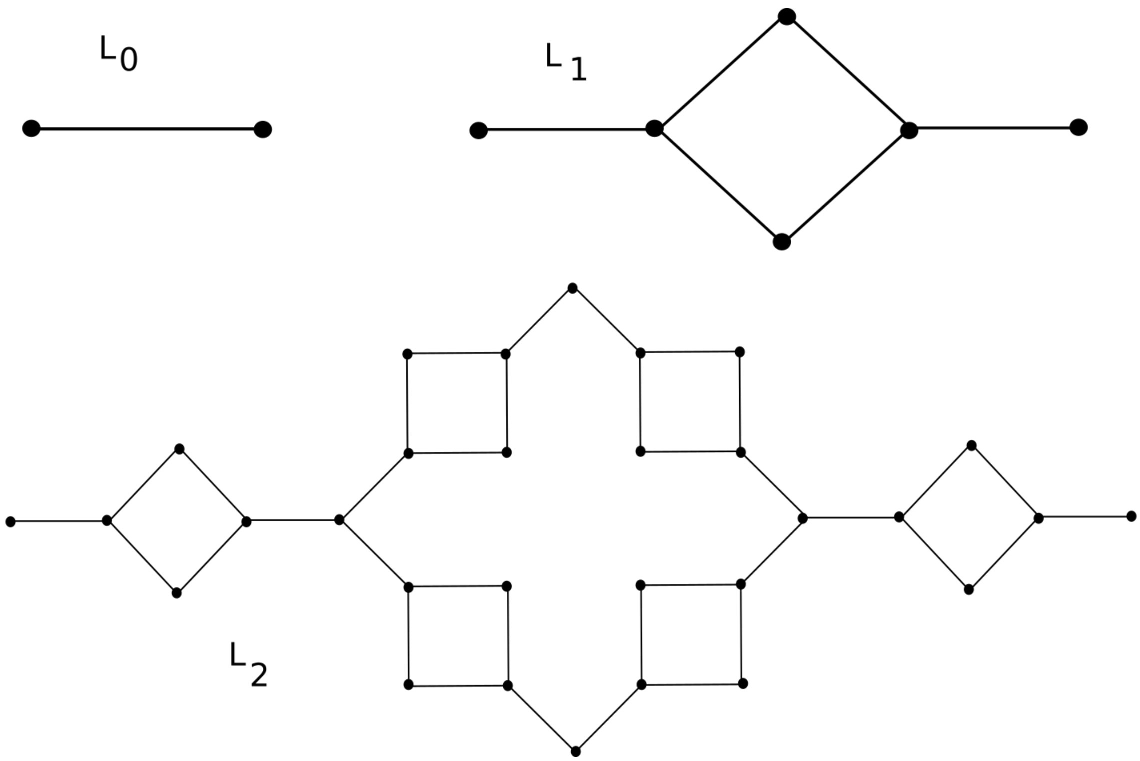
Definition 6. *The Laakso graph of level 0, denoted L_0, consists of two vertices and an edge of length 1 between them. Given L_{n-1} we construct L_n in the following way: we replace every edge $\{s,t\}$ of D_{n-1} with four of new vertices a, b, m_1, m_2 and six new edges: $\{s,a\}, \{b,t\}, \{a,m_1\}$, $\{a,m_2\}$, $\{b,m_1\}$ and $\{b,m_2\}$. See the Figure 2.2 below for a reference.*

As with diamond graphs, one can make sure that performing the recursive step does not change the distance between the original vertices by re-scaling the length of all edges by a factor of four this time. Thus using this convention all edges of L_n would have length $\frac{1}{4^n}$.

The main reason why one decides to work with Laakso graphs, instead of diamond graphs (which share a lot of common properties and are typically easier to handle) is that the former is a doubling metric space.

Definition 7. *A metric space (X,d) is called doubling if there exists a constant $M > 0$ such that for any point $x \in X$ and any radius $r > 0$, it is possible to cover the ball $B(x,2r)$ with the union of at most M balls of radius r.*

Figure 2.2: First three Laakso graphs L_0, L_1, and L_2.



When considering a map between two metric spaces the strongest condition one can ask for is for the function to preserve the distance:

Definition 8. *A map $f\colon (X,d_X) \to (Y,d_Y)$ is an isometry if it preserves distances, i.e. for any $x_1,x_2 \in X$ the following equality holds $d_Y(f(x_1),f(x_2)) = d_X(x_1,x_2)$.*

From the metric point of view spaces that are isometric are the same. A weaker condition that we can ask for is a special type of homeomorphism where additional metric control is imposed over the whole space:

Definition 9. *A bijection $f\colon (X,d_X) \to (Y,d_Y)$ is a uniform homeomorphism if both f and f^{-1} are uniformly continuous, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ so that if $d_X(x_1,x_2) < \delta$ then $d_Y(f(x_1),f(x_2)) < \varepsilon$.*

Since we will be working with sequence spaces let us recall the formal definition of convergence of sequences:

Definition 10. *We say that a sequence* $\{x_i\}_{i=1}^{\infty}$ *of elements in a metric space* (X,d) *converges to* x *if for every* $\varepsilon > 0$ *there exists a natural number* N *such that, for all* $n \geqslant N$ *we have* $d(x, x_n) \leqslant \varepsilon$. *We call* x *the limit of the sequence and denote it by* $\lim_{i\to\infty} x_i$. *Note that this well-defined since limits in metric spaces are unique.*

Also, let us recall a formally weaker condition which of great importance to the field of analysis.

Definition 11. *A Cauchy sequence is a sequence* $\{x_i\}_{i=1}^{\infty}$ *of elements in a metric space* (X,d) *such that for every* $\varepsilon > 0$ *there exists a natural number* N *such that, for all* $n,m \geqslant N$ *we have* $d(x_n, x_m) \leqslant \varepsilon$.

Note that every convergent sequence is a Cauchy sequence, but the converse is not necessarily true. We distinguish the class of spaces where the opposite implication holds.

Definition 12. *A metric space is complete if every Cauchy sequence converges.*

2.2 Banach spaces

In this section we go over the basics of functional analysis, starting with how we define a norm on a vector space, which is an analog of a metric adjusted to the linear setting.

Definition 13. *A norm is a real valued function* $\|\cdot\|$ *defined on a vector space* V *over real or complex numbers, such that:*

- *for every* $v \in V$ $\|v\| \geqslant 0$,
- $\|v\| = 0 \Leftrightarrow v = 0$,
- *for every* $v \in V$ *and every scalar* δ: $\|\delta v\| = |\delta|\|v\|$
- *the triangle inequality holds, i.e. for all* $v,w \in V$ *we have* $\|v+w\| \leqslant \|v\| + \|w\|$.

Note that a norm induces a metric on a vector space by $d(v,w) = \|v-w\|$.

The field of functional analysis focuses on studying normed vector spaces that satisfy an additional metric condition:

Definition 14. *A complete normed vector space is called a Banach space.*

Let us recall the definitions of two basic Banach spaces that we will focus on in this dissertation, the first of which is the sequence space ℓ_p.

Definition 15. *For* $1 \leqslant p < \infty$ *we define* $\ell_p(\mathbb{N})$ *as the vector space of all infinite sequences* $\{x_i\}_1^\infty$ *such that* $\sum_{i=1}^\infty |x_i|^p < \infty$ *equipped with the norm* $\|\{x_i\}_{i=1}^\infty\|_p = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$.

Note that if we replace the index set (natural numbers in the above case) with any other countable infinite set then the resulting spaces are linearly isometric (see below).

The second Banach space that we need to formally define here is L_p which can be viewed as a continuous analog of its predecessor.

Definition 16. *For* $1 \leqslant p < \infty$ *we define* $\mathscr{L}_p(\mathbb{R})$ *to be the space of all measurable functions* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\|f\|_p \equiv \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p} < \infty$. *Furthermore, we denote by* $L_p(\mathbb{R})$ *the quotient space with respect to the kernel of* $||.||_p$*, which defines a complete norm on the said quotient.*

Now we take time to define maps between Banach spaces that have been classically studied:

Definition 17. *A linear map* $T\colon (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ *between normed vector spaces is a linear embedding if there exists a constant* $M > 0$ *s.t. for all* $x \in X$*:* $1/M\|x\|_X \leqslant ||T(x)||_Y \leqslant M\|x\|_X$. *A linear embedding that is bijective is called a linear isomorphism.*

Linear embeddings are also called bounded operators and the infinum of all M is called the norm of T. If two Banach spaces are isomorphic in the above sense they are often considered to be indistinguishable from the Banach space point of view.

We now introduce the very useful concept of dual space, first from a purely algebraic point of view:

Definition 18. *If* V *is a vector space over a field* $\mathbb{K}$ *then let us consider the set of all linear forms* $f\colon V \to \mathbb{K}$. *Together with pointwise addition and scalar multiplication, this set forms a vector space, which is called the algebraic dual space of* V *and denoted by* V^*.

If the vector space V has some additional topological structure we can also put it on V^*. There are different ways to do it, the first of which is called a weak topology.

Definition 19. *If V is a topological vector space over a field $\mathbb{K}$ and V^* its dual space, by weak topology we mean the smallest topology on V^* that makes $f(v)$ a continuous map from V^* to $\mathbb{K}$ for every fixed $v \in V$. More formally: consider the family of all topologies on V^* under which $f(v)$ is a continuous map for all $v \in V$. Intersection of all those topologies is also a topology on V^* so the smallest topology is well defined.*

If V is a normed vector space, we can also put a norm on V^*:

Definition 20. *If V is a vector space with a norm $||.||$ then we can define a norm on V^* by defining $||f|| = \sup_{||x|| \leqslant 1} |f(x)|$ for any $f \in V^*$.*

The topology defined by this norm is called a strong topology since it has more open sets on V^* than the weak topology. Now we give the fundamental example.

Example 1. *Under the strong topology on dual, the following hold:*

- $(\ell_p, \|\cdot\|_p)^*$ *is linearly isometric with* $(\ell_q, \|\cdot\|_q)$ *for* $1 < p < \infty$ *and* q *such that* $1/p + 1/q = 1$,
- $(L_p, \|\cdot\|_p)^*$ *is linearly isometric with* $(L_1, \|\cdot\|_q)$ *for* $1 < p < \infty$ *and* q *such that* $1/p + 1/q = 1$.

We can observe that for the range $1 < p < \infty$ if we take double dual of ℓ_p we recover the same space, that leads us to define the following:

Definition 21. *A Banach space V is called reflexive if the map $J\colon V \to V^{**}$, defined by $J(x)(f) = f(x)$, is a linear isometry.*

As noted above both ℓ_p and L_p spaces are reflexive for $1 < p < \infty$.

We will now cover some classical results which answer the question where ℓ_p or L_p spaces are isomorphic in a linear way. Note that this is a fundamental question since both families of spaces are considered building blocks for many Banach spaces. To begin we mention the very special case of $p = 2$.

Theorem 22. *L_2 and ℓ_2 spaces are linearly isometric as they are both seperable Hilbert spaces.*

In order to distinguish between L_p and ℓ_p spaces for $p \neq 2$ we need to introduce a special kind of bounded operators.

Definition 23. *A bounded linear operator between two Banach spaces X and Y is said to be compact if it maps the closed unit ball of X into a relatively compact subset, that is a set whose closure is compact, in Y.*

One can see compact operators as a natural generalization of finite-rank operators in an infinite-dimensional setting.

We now cover the celebrated theorem by Pitt from the 1930s (see [Pit32] for the original proof, or [AK16] for modern discussion of the result) which classifies bounded operators between ℓ_p and ℓ_q.

Theorem 24. *Every bounded operator $T\colon \ell_p \to \ell_q$ is compact for $1 \leqslant q < p < \infty$.*

Proof. Without a loss of generality we can assume that T has norm one, that is $\|T(x)\|_q \leqslant \|x\|_p$. Since $1 < p$, the dual of ℓ_p is separable. Thus every bounded sequence in ℓ_p has a weakly Cauchy subsequence. In order to prove compactness of T, it is enough to show that it is weak to norm continuous. Let us consider a weakly null sequence (x_i) in ℓ_p. We have to show that $\lim_{i\to\infty} T(x_i) \to 0$. Let assume on the contrary that there exists a weakly null sequence (x_i) with $\|x_i\| = 1$ such that $\|T(x_i)\| \geqslant \delta > 0$. By passing to a subsequence, we may suppose that (x_i) is a basic sequence equivalent to the canonical basis of ℓ_p (see for example Proposition 2.1.3 in [AK16]). But since $\{T(x_i)\}_i$ is also weakly null we can pass to a further subsequence to ensure that $\{T(x_i)\}_i$ is equivalent to a canonical basis of ℓ_q. We have shown that $T\colon \ell_p \to \ell_q$ is just an identity map, but according to our assumption, it is bounded, which leads us to a contradiction. $\square$

Not that the above proof works if we replace ℓ_p with one of its closed subspaces. Also with small modifications the argument can work for $p = \infty$ if as a domain we consider the space c_0 i.e. the space of all sequences converging to 0 equipped with the $||\cdot||_\infty$ norm. Recall that $||\cdot||_\infty$ takes the sup of all possible values of a sequence or a function, and is also used to define ℓ_∞ and L_∞. This allows us to conclude the following linear classification:

Corollary 1. *All the spaces in the family of ℓ_p for $1 \leqslant p < \infty$ and c_0 are mutually not isomorphic.*

The more modern way to prove the above, which also generalizes to L_p spaces, is by using the theory of type and cotype.

Definition 25. *Let $\{\varepsilon_i\}_1^\infty$be a sequence of independent random variables, each with Rademacher distribution, that is $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$, then:*

- *we say that a Banach space X has Rademacher type $p \in [1,2]$ if there exists a positive constant C so that for every finite set of vectors $\{x_i\}_{i=1}^n$ in X:*

$$(\mathbb{E}||\sum_{i=1}^n \varepsilon_i x_i||^p)^{1/p} \leqslant C(\sum_{i=1}^n ||x_i||^p)^{1/p},$$

- *we say that a Banach space X has Rademacher cotype $q \in [2,\infty]$ if there exists a positive constant C so that for every finite set of vectors $\{x_i\}_{i=1}^n$ in X:*

$$(\sum_{i=1}^n ||x_i||^q)^{1/q} \leqslant C(\mathbb{E}||\sum_{i=1}^n \varepsilon_i x_i||^q)^{1/q}.$$

The next result follows from the work of Nordlander in [Nor62] for type and Orlicz in [Orl33a] and [Orl33b] for cotype Note that their results were stated differently because the language of type and cotype did not exist yet.

Theorem 26. *The following type and cotype qualities hold:*

- *for $1 \leqslant p \leqslant 2$, L_p and ℓ_p spaces have type p and cotype 2,*
- *for $2 \leqslant p < \infty$, L_p and ℓ_p spaces have type 2 and cotype p,*

Since type and cotype are isomorphism invariants we are able to deduce from the above majority of the Corollary 1. Additionally this allows us to differentiate between L_p spaces.

Corollary 2. *L_p and L_q spaces not linearly isomorphic for $p \neq q$ and $1 \leqslant p < \infty$.*

We now turn our attention to compare ℓ_p and L_p spaces. One can show (see Proposition 6.4.13 in [AK16]) using a sequence of independent normalized Gaussians on the interval $[0,1]$ that.

Theorem 27. *ℓ_2 isometrically embeds into L_p for $1 \leqslant p < \infty$.*

Recall that by Pitt's theorem all maps from ℓ_2 to ℓ_p are compact for $1 < p < 2$. But the above result tells us that L_p for $1 \leqslant p < 2$ admits a non-compact map from ℓ_2. Thus we can conclude that there is no linear embedding from L_p into ℓ_p for $1 < p < 2$. If L_p and ℓ_p for $p > 2$ were linearly isomorphic, so wuld their dual thus we can conclude the following.

Corollary 3. *L_p and ℓ_p are not linearly isomorphic for $p \neq 2$ in the range $1 \leqslant p < \infty$.*

We conclude this section by summarizing all the linear results into one statement:

Corollary 4. *The family of spaces c_0, ℓ_p, L_q for $1 \leqslant p < q < \infty$ consists of pairwise non-isomorphic spaces.*

2.3 Nonlinear theory

In this section, we focus on different maps between metric spaces, in particular on non linear maps between Banach spaces.

As discussed in Section 1, when considering a map between two metric spaces the strongest condition one can ask for is for the map to be an isometry, i.e. to preserve the distances. The following result by Mazur and Ulam from [MU32], explains the importance of previously discussed linear theory in the study of metric properties of Banach spaces:

Theorem 28. *Let $f\colon X \to Y$ be a (not necessarily linear) isometry between real normed spaces X, Y, that maps 0 to 0 and is surjective. Then, f is a linear isometry.*

The proof of the above statement follows from studying the midpoints of line segments. Based on this result and facts established in the previous section we can deduce the following.

Corollary 5. *The family of spaces* c_0, ℓ_p, L_q *for* $1 \leqslant p < q < \infty$ *consists of pairwise non-isometric spaces.*

The isometry condition is very rigid. Instead of imposing the distance not to change at all between a pair of points, one can instead make sure that we have some kind of control over how much does it change. There are various levels of generality of this idea, but the most basic and oldest condition to have been studied is to ask for a linear control over how much the distance changes. First, recall the definition of Lipschitz map.

Definition 29. *A map* $f\colon (X, d_X) \to (Y, d_Y)$ *is called Lipschitz with constant* $K > 0$ *if for any* $x_1, x_2 \in X$ *the following inequality holds.*

$$d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2).$$

The least constant K *for which the above equation holds is called the Lipschitz constant of* f *and is denoted by* $Lip(f)$.

The map f needs to satisfy additional conditions to be considered a bi-Lipschitz embedding.

Definition 30. *A map* $f\colon (X, d_X) \to (Y, d_Y)$ *is called a bi-Lipschitz embedding if it is one to one and both* f *and* f^{-1} *are Lipschitz. In other words there exist constants* $A, B > 0$ *such that for any* $x_1, x_2 \in X$ *the following inequalities hold.*

$$A d_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant B d_X(x_1, x_2).$$

The distortion of a bi-Lipschitz embedding is defined to be $dist(f) = Lip(f) \cdot Lip(f^{-1})$.

A bi-Lipschitz embedding that is surjective is called a Lipschitz isomorphism.

We now will present some standard results about bi-Lipschitz embeddings between Banach spaces.

Definition 31. *The function f between an open subset X of a Banach spaces V and a Banach space W is called Gâteaux differentiable at a point $x \in X$ if there exists a bounded linear operator $T: X \to Y$ such that for every $u \in X$ the following holds*

$$T(u) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} \tag{2.1}$$

We denote T as $D_f(x)$ and call it a Gâteaux derivative of f at x. Since limits in Banach spaces are unique, it follows that $D_f(x)$ is well defined.

One can impose a stronger condition and require the equality in (2.1) to hold uniformly for all u in a unit sphere. This is in fact equivalent to the notion of Fréchet differentiability.

Definition 32. *The function f between an open subset X of a Banach spaces V and a Banach space W is called Fréchet differentiable at a point $x \in X$ if there exists a bounded linear operator $T: X \to Y$ such that $u \in X$ the following holds*

$$\lim_{\|u\| \to 0} \frac{\|f(x+u) - f(x) - T(u)\|_W}{\|u\|_V} = 0 \tag{2.2}$$

In this case we call T the Fréchet derivative of f at x.

Naturally Fréchet differentiability implies Gâteaux differentiability. The converse is also true if f is Lipschitz and X is a finite dimensional Banach space. This follows from the compactness of the unit sphere of X, something that is clearly not true for the case of $dim(X) = \infty$. There are very few cases when the existence of Fréchet derivatives can be showed when X is infinite dimensional.

Below we give the original definition of the Radon–Nikodym property that played a central role in the study of bi-Lipschitz embeddings between Banach spaces.

Definition 33. *A Banach space X has the Radon–Nikodym property if every operator from L_1 into X is representable, that is for every bounded linear operator $T: L_1[0,1] \to X$ there exists a* *bounded and strongly measurable function* $g\colon [0,1] \to X$ *such that for any* $f \in L_1[0,1]$*:*

$$T(f) = \int_0^1 f(t)g(t)dt.$$

The above property plays an important role in many branches of Banach space theory and thus has many equivalent formulations using different terms such as the extremal structure of closed bounded convex sets or the convergence of vector-valued martingales. The characterization that we need for our purposes is the one that deals with the existence of derivatives of Lipschitz functions.

Proposition 1. *A Banach space* X *has the Radon–Nikodym property if and only if every Lipschitz map from the unit interval* $[0,1]$ *into* X *is differentiable almost everywhere.*

The following result by Dunford and Pettis from [DP40] provides plenty of examples of Banach spaces with Radon–Nikodym property.

Proposition 2. *The following families of spaces have the Radon–Nikodym property.*

- *Reflexive spaces,*
- *Separable dual spaces.*

In particular it follows that ℓ_p spaces for $1 \leqslant p < \infty$ and L_q spaces for $1 < q < \infty$ all have the Radon–Nikodym property, but L_1 does not.

Finally, we are ready to formulate the infinite dimensional version of Rademacher Theorem (see Theorem 14.2.13 in [AK16] for a proof).

Theorem 34. *Let* X *be a separable Banach space and* Y *be a Banach space with the Radon-Nikodym property. Let* $f\colon X \to Y$ *be a Lipschitz map. Then the set of points at which* f *fails to be Gâteaux differentiable is Haar-null.*

The following result by Heinrich and Mankiewicz in [HM82] applies the above differentiability results to obtain linear embeddings from bi-Lipschitz embeddings.

Theorem 35. *Let $f\colon X \to Y$ be a bi-Lipschitz embedding between Banach spaces X and Y. Assume X is separable and that Y has the Radon–Nikodym property. Then X is linearly isomorphic to a subspace of Y.*

Proof. By Theorem (34) we know that there exists a point x of Gâteaux differentiablity of f. Using the fact that f is Lipschitz, for any $t > 0$ and $x, u \in X$ we get

$$A\|tu\| \leqslant \|f(x+tu) - f(x)\| \leqslant B\|tu\|.$$

If we divide all sides by $|t|$ and take the limit as t goes to 0 it follows that

$$A\|u\| \leqslant \|D_f(x)(u)\| \leqslant B\|u\|.$$

By construction $D_f(u)$ is linear, hence we obtained an isomorphic embedding of X into Y. Also note that by our construction, the distortion of f gives us an upper bound on the isomorphism constant of $D_f(x)$ that is the product of its norm and the norm of its inverse. □

The question of whether it is always possible to find a point x so that $D_f(x)$ is surjective has been studied for many years. That would allow us to reduce the Lipschitz isomorphism problem for separable reflexive Banach spaces to the linear isomorphism problem. However, the above result is enough to reduce the embeddability problem from Lipschitz to a linear setting. Thus the results from section 2 about linear embeddings together with Theorem of Heinrich and Mankiewicz allows us to conclude the following.

Corollary 6. *Following can be said about bi-Lipschitz embeddings between some of the classical Banach spaces:*

- *For $1 \leqslant p \neq q < \infty$ neither ℓ_p nor c_0 embed bi-Lipschitzly into ℓ_q.*
- *For $1 \leqslant q < \infty$ there exists no bi-Lipschitz embedding from L_p into L_q unless $p = q$ or $1 \leqslant q \leqslant p \leqslant 2$.*

- *There is no bi-Lipschitz embedding from L_p into ℓ_q unless $p = q = 2$.*

Quasi-isometric embeddings (also known as coarse Lipschitz embeddings in Banach space theory) are a natural extension of bi-Lipschitz embeddings for which the Lipschitz conditions are satisfied only at a certain scale, namely.

Definition 36. *A map $f\colon (X, d_X) \to (Y, d_Y)$ is a quasi-isometric embedding if there exist global constants $0 < A < B$ such that if $x_1, x_2 \in X$ and $d(x, y) > K$ then*

$$Ad_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant Bd_X(x_1, x_2).$$

It is important to emphasize that the requirement for x and y to be at least K apart makes this condition detect only how the map behaves at a large scale. This is in stark contrast to a lot of classical geometric conditions, like continuity, that requires control over local phenomena. Because of this, the field of mathematics that deals with quasi-isometric embeddings and their generalizations is often called large scale geometry.

Below we provide an alternative definition of quasi-isometry that is more common only used in geometric group theory. This is also the version that we will be using later in this dissertation.

Definition 37. *A map $f\colon (X, d_X) \to (Y, d_Y)$ is a quasi-isometric embedding if there exist constants $A, B, C, D > 0$ s.t. for all $x_1, x_2 \in X$,*

$$Ad_X(x_1, x_2) + B \leqslant d_Y(f(x_1), f(x_2)) \leqslant Cd_X(x_1, x_2) + D.$$

The following result follows from the work of Kalton and Randrianarivony in [KR08] where they studied Hamming graphs $G_k(\mathbb{M}) = \{\overline{n} = (n_1, \ldots, n_k)\colon n_i \in \mathbb{M}, n_1 < n_2 < \cdots < n_k\}$ with a metric $d(\overline{n}, \overline{m}) = |\{i : n_i \neq m_i\}|$ for any infinite subset $\mathbb{M}$ of the natural numbers. More precisely they estimate the minimal distortion of any bi-Lipschitz embedding of $G_k(\mathbb{M})$ into ℓ_p-like Banach spaces using Ramsey's Theorem.

Theorem 38. *There is no quasi-isometric embedding of the following spaces into ℓ_p:*

- *ℓ_q space for $1 \leqslant q < p < \infty$*

- *L_q space for any q unless $p = q = 2$.*

Note that the case of quasi-isometric embedding of ℓ_p into L_q for $q > p$ an additional argument that involves midpoints is required.

In the special case of normed vector spaces there exists a direct connection between quasi-isometric embeddings and with previously defined uniform homeomorphisms.

Proposition 3. *Let $f\colon X \to Y$ be a uniform homeomorphism between normed vector spaces. Then f is a surjective quasi isometry.*

It is important to mention here the groundbreaking result by Johnson, Lindenstrauss, and Schechtman.

Theorem 39. *Let X be a Banach space that is uniformly homeomorphic to ℓ_p for some $p \neq 2$ and $1 < p < \infty$. Then X is linearly isomorphic to ℓ_p.*

More recently Mendel and Naor in [NMM08][Theorem 1.10] showed the following by developing a theory of metric cotype, which is an extension of Rademacher cotype to a purely metric setting.

Theorem 40. *A uniform embedding from L_p into L_q exists if and only if $p \leqslant q$ or $q \leqslant p \leqslant 2$.*

Those rigidity results are in stark contrast to what happens if we remove the assumption of the uniformity of the embedding.

Theorem 41. *All separable infinite dimensional Banach spaces are homeomorphic.*

The above result is due to Kadets in [Kad67] and it has been further generalized by Toruńczyk in [Tor81] who showed that all Banach spaces of the same density character are homeomorphic.

Finally, we introduce a concept of coarse embeddings that is one more step in the generalization of isometric maps. Just like in the case of bi-Lipschitz embeddings we required a linear control over how the distance is changing, and for quasi-isometric embeddings we required affine control over it, we now require the existence of any kind of control function - as long as they are proper and non decreasing.

Definition 42. *A map $f\colon (X,d_X) \to (Y,d_Y)$ is a a coarse embedding if there exist non decreasing functions $\rho_-,\rho_+\colon [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and for any $x_1,x_2 \in X$ the following inequalities hold.*

$$\rho_-(d_X(x_1,x_2)) \leqslant d_Y(f(x_1),f(x_2)) \leqslant \rho_+(d_X(x_1,x_2)).$$

In this setting question about embeddability of L_p into ℓ_p has only partial answers. Namely for $1 \leqslant p < 2$ it follows by work of by Mendel and Naor in [NMM04] that L_p coarsly embeds into ℓ_p. This is done by factoring the embedding through Hilbert space, based on the argument by Nowak who proved in [N06] that ℓ_2 coarsely embeds into ℓ_p for $1 \leqslant p < \infty$.

2.4 Group theory

In this section, we recall some basic properties of groups, both the classical algebraic ones and those more geometric in nature. We start by defining the object we are going to study:

Definition 43. *A group is a set G together with a binary operation $*\colon G\times G \to G$ such that :*

- *for any $a,b,c \in G$ $(a*b)*c = a*(b*c)$*
- *there exists an unique element $e \in G$ so that $a = a*e = e*a$ for all $a \in G$*
- *for every element $a \in G$ there exists an unique element $a^{-1} \in G$ so that $e = a*a^{-1} = a^{-1}*a$*

Note that e is called an identity element or neutral element. For the sake of simplifying the notation, we will often skip the group multiplication sign $*$ and we will simply write ab when we mean $a*b$.

If we chose a subset $H \subset G$ that forms a group with respect to operations defined on G we call it a subgroup.

Definition 44. *A subset H is called a subgroup of a group G if $e \in H \subset G$ and H is closed under multiplication and inverse operations defined on H.*

We denote a subgroup by $H \leqslant G$ and by $H \lhd G$ in case if it is a normal subgroup:

Definition 45. *A subgroup N of a group G is called a normal subgroup if it is invariant under conjugation i.e. for every $n \in N$ and $g \in G$ $gng^{-1} \in N$.*

One can gather from the previous section, that objects that we study are not the only thing that matters, but the map we consider between them is an important part of the subject. Now it is time to introduce maps between groups that preserve their algebraic properties.

Definition 46. *A map $f : G \to H$ between two groups is called a homomorphism if $f(g_1 *_G g_2) = f(g_1) *_H f(g_2)$for any $g_1, g_2 \in G$. Homomorphism that is bijective is called a group isomorphism.*

We write $G \cong H$ if two groups are isomorphic and we note that it is equivalent to the existence of two homomorphisms $f\colon G \to H$ and $g\colon H \to G$ such that $f \circ g = id_H$ and $g \circ f = id_G$. From the group-theoretic point of view, two isomorphic groups are indistinguishable from each other.

We now introduce a certain basic finiteness condition for both groups and their elements alike:

Definition 47. *A nontrivial group element $g \in G$ is called torsion if $g^n = e$ for some natural number $n \in \mathbb{N}$. If elements of a group are torsion then the group is called torsion, similarly, if the group has no nontrivial torsion elements then the group is called torsion free.*

We now introduce the very fundamental concept of group action on a set. It is important to remark here that this is the true origin of group theory since groups were first studied as a collection of transformations of certain sets (for example symmetries of geometric objects), rather than abstract algebraic objects.

Definition 48. *We say that a group $(G, *)$ acts on a set X (from the left) if there exists a map $G \times X \to X$ that sends a pair (g, x) to $g \cdot x$ such that :*

- *for any $g, h \in G$ and $x \in X$ $(g * h) \cdot x = g \cdot (h \cdot x)$*
- *$e \cdot x = x$ for all $x \in X$*

One can define the right group action in a similar way. If the group action in question is obvious in the context we will write gx instead of $g \cdot x$.

Notice that every group G is acting on itself by multiplication from the left. If a group acts on a different group, we can use this action to defined a skewed multiplication on their Cartesian product.

Definition 49. *Let a group N act on a group H, then the following operation defines a group multiplication on the set $N \times H$: $(n,g) \times (m,h) \to (n(g \cdot m), gh)$ with (e_N, e_H) being the neutral element. This group is called a semidirect product of N and H and denoted by $N \rtimes H$.*

The above is what is called an outer definition of a semi-direct product. The lemma below shows its connection to the inner definition of semi-direct product, as well as short exact sequences:

Lemma 50. *The following conditions are equivalent:*

- *group G is a semidirect product of N and H i.e. $G = N \rtimes H$*
- *for every $g \in G$ there exists an unique $n \in N$ and $h \in H$ s.t. $g = nh$*
- *for every $g \in G$ there exists $n \in N$ and an unique $h \in H$ s.t. $g = nh$*
- *there exists a homomorphism $p : G \to H$ that is the identity on H and that sends N to the identity element. In other words, there is a split exact sequence*

$$1 \to N \xrightarrow{i} G \xrightarrow{p} H \to 1.$$

Finally, we end with some basic definitions related to group actions:

Definition 51. *Let a group G act on a set X and $g \in G$ and $x \in X$ is, we then define:*

- *the support $supp(g) \subset X$ to consist of all $y \in X$ s.t. $gy \neq y$*
- *the orbit $Gx \subset X$ to consist of all $y \in X$ s.t. $hx = y$ for some $h \in G$*

- *the stabilizer $G_x \leqslant G$ to be a subgroup consisting of all elements $h \in G$ s.t. $hx = x$*
- *we say that a subset $Y \subset X$ is invariant under the group action if $GY = Y$.*

Now we introduce some geometric structures on groups, thus entering the realm of geometric group theory. Even though fundamental concepts here date back to the 19th century and Cayley this field didn't really pick up until the late 20th century. We start by defying how we can measure a length of a group element:

Definition 52. *A length function on a group G is a a function $|\cdot| :\to [0,\infty)$ such that for all $g,h \in G$ the following conditions are satisfied:*

- $|g| = 0$ *if and only if g is the identity element*
- $|g| = |g^{-1}|$
- $|gh| \leqslant |g| + |h|$.

A length function is said to be proper if for any $R > 0$ the set $\{g \in G : |g| \leqslant R\}$ is finite.

In order to define the most natural example of a length function, we need to introduce generating sets.

Definition 53. *A group G is called finitely generated if there exists a finite subset $S \subset G$ such that every element $g \in G$ can be written as a finite word in elements of S i.e. there exist $s_1, \cdots, s_n \in S$ s.t. $g = s_1 * s_2 * \cdots * s_n$.*

We write $G =< S >$ to denote that G is generated by a set S. It is often assumed that the generating set is symmetric $S = S^{-1}$ i.e. if $s \in S$ then $s^{-1} \in S$. Notice that if the group has a finite generating set S then $S \bigcup S^{-1}$ is a finite, symmetric generating set.

Choosing a generating set S for a group G allows us to define a length function on G:

Definition 54. *If G is a group with a finite generating set S we define a word length as $|g|_S = min\{n : g = s_1 * \ldots s_n,$ where $s_i \in S\}$.*

Note that a word length is a proper length function on a group G.

We now explain how to define a metric on a group using a length function:

Definition 55. *Let G be a group equipped with a length function $|\cdot|$ then we can define a word length metric by $d(g,h) = |g^{-1}h|$ for all $g,h \in G$.*

An important property of a metric d defined as above, is the fact that it is invariant under the left action of G on itself that means:

$$d(jg, jh) = |(jg)^{-1}jh| = |g^{-1}j^{-1}jh| = |g^{-1}h| = d(g,h).$$

Given a generating set S, one can see the group generated by it as a graph:

Definition 56. *Let G be a group generated by a finite, symmetric set S. The Cayley graph $Cay(G,S)$ is defined as follows*

- *vertices of $Cay(G,S)$ are the elements of G*
- *two vertices g and h are connected by an edge if and only if $g * h^{-1} \in S$.*

Note that the shortest path metric on $Cay(G,S)$ coincides with the word metric defined by S.With that in mind, Cayley graphs allow us to view discrete groups as geometric objects at their core. However, the geometric structure we define does rely heavily on the generating set S. Fortunately for us that does not bother us too much as large scale properties that we are mostly interested in are preserved under bi-Lipschitz equivalence.

Theorem 57. *Let G be a group equipped with two finite, generating sets S, S' and let d and d' be word length metrics induced by them. Then (G,d) and (G,d') are bi-Lipschitz equivalent.*

Proof. We will show that identity map $id : (G,d) \to (G,d')$ satisfies this condition. First note that since both metrics are left-invariant and id is an isomorphism it is enough to show that for all $g \in G$ and some universal constant D:

$$|g| \leqslant D|g|'$$

$$|g|' \leqslant D|g|$$

where $|\cdot|$ and $|\cdot|'$ are length functions induced by S and S' respectively. Define

$$D = max\{|s'| : s' \in S'\}.$$

Now consider $g \in G$ such that $|g| = n$ and $g = s_1 \dots s_n$ for $s_i \in S$. But all s_i can be written as a product of elements of S', hence:

$$\begin{aligned} |g|' &\leqslant \sum_{i=1}^{n} |s_i|' \\ &\leqslant D|g|. \end{aligned}$$

By interchanging the roles of S and S' we get the other inequality, finishing the proof. $\square$

We now return to the topic of group actions but introducing geometric structure on both the space and the group. To start we define a linear isometry of a normed vector space:

Definition 58. *Let V be a normed vector space. A linear bijection $U\colon V \to V$ is called a linear isometry if it preserves the norm, that is for every $v \in V$ we have $\|U(v)\|_V = \|v\|_V$.*

Note that the set of all linear isometries of a fixed vector space V forms a group under composition that we denote by $Isom(V)$.

We now formalize what does it mean for an arbitrary group to act on a normed vector space by linear isometries.

Definition 59. *We say that a group G acts on a normed vector space V in a linear, isometric way if for every $g \in G$ $v \to g \cdot v$ is a linear isometry from V to V.*

Observe that if G is a group that acts in a linear, isometric way on V if and only if there exists an isometric representation of G i.e. a homomorphism $\pi\colon G \to Isom(V)$ into the group of all linear isometries of V.

Since linear maps can be generalized to affine maps one can follow the same path with group actions:

Definition 60. *An affine isometry of a normed vector space V is a map $A\colon V \to V$ such that for every $v \in V$ $A(v) = U(v) + b$, where U is a linear isometry and b is a fixed vector in V.*

We say that group G acts on V by an affine isometries if for every $g \in G$ there exists an affine isometry $A_g\colon V \to V$ such that $A_{gh} = A_g A_h$. By our previous remark $A_g(v) = \pi_g v + b_g$ where $\pi\colon G \to Isom(V)$ and $b\colon G \to V$. If we rewrite $A_{gh} = A_g A_h$ in this form we reach the identity known as the cocycle condition:

$$b_{gh} = \pi_g b_h + b_g \tag{2.3}$$

We say that a map $f\colon X \to Y$ is metrically proper if a preimage of any bounded set is a bounded set. Note that an affine action of a group G, equipedd with a length function $|\cdot|$, on a normed vector space V is proper if and only if $lim_{|g|\to\infty} \|b_g\|_V = \infty$. The elementary calculation yields the following.

Proposition 4. *Let G be a finitely generated group, which admits a proper, affine, isometric action on a normed vector space V, with a cocycle b. Then b is a coarse embedding.*

Proof. By the cocycle condition:

$$\begin{aligned} b_g - b_h &= b_{h(h^{-1}g)} \\ &= \pi_h b_{h^{-1}g} + b_h - b_h \\ &= \pi_h b_{h^{-1}g}. \end{aligned}$$

Since π is an isometry we get:

$$\|b_g - b_h\| = \|\pi_h b_{h^{-1}g}\| = \|b_{h^{-1}g}\|. \tag{2.4}$$

Thus for any $t \geqslant 0$ we can define ρ_- as

$$\rho_- = inf\{\|b_h\| : |h| \geqslant t\}.$$

For any $g, h \in G$ by the above we have:

$$\|b_g - b_h\| = \|b_{h^{-1}g}\| \geqslant \rho_-(|h^{-1}g|) \geqslant \rho_-(d(h,g))$$

Finally we note that $\lim_{t\to\infty} \rho_-(t) \to \infty$ hence b is indeed a coarse embedding. $\square$

Because of the above admitting a proper, affine, isometric action on V is viewed as a stronger version of a coarse embedding and is also called an equivariant coarse embedding.

3. NO DIMENSION REDUCTION FOR DOUBLING SPACES

In this section we focus on bi-Lipschitz embeddings between ℓ_p spaces, in particular we present a new geometric proof that ℓ_p does not admit a bi-Lipschitz embedding into ℓ_q for $p > q \geqslant 2$. This is done by constructing a family of diamond graph-like objects based on the construction found in [BGN15]. Results in this section come from a joint paper with Florent Baudier and Andrew Swift [BSŚ].

3.1 Overview

The celebrated Johnson-Lindenstrauss [JL84] lemma asserts that any n-point subset of ℓ_2^n admits a bi-Lipschitz embedding with distortion at most $1+\varepsilon$ into ℓ_2^k where $k = O(\frac{\log n}{\varepsilon^2})$. This dimension reduction phenomenon is a fundamental paradigm as it can be used to improve numerous algorithms in theoretical computer science (cf. [Nao18]) both in terms of running time and storage space. Johnson and Lindenstrauss observed that a simple volume argument gives that the dimension must be at least $\Omega(\log\log n)$. Later Alon [Alo03] showed that the bound in the Johnson-Lindenstrauss lemma was tight up to a $\log(1/\varepsilon)$ factor. Recently, Larsen and Nelson [LN17] were able to show the optimality of the dimension bound in the Johnson-Lindenstrauss lemma. A common feature of the subsets exhibiting lower bounds on the dimension is that they have high doubling constants. In [LP01], Lang and Plaut raised the following fundamental question.

Problem 1. *Does a doubling subset of ℓ_2 admit a bi-Lipschitz embedding into a constant dimensional Euclidean space?*

Based on a linear programming argument, Brinkman and Charikar [BC05] proved that there is no dimension reduction in ℓ_1. An enlightening geometric proof was given by Lee and Naor in [LN04]. The subset of ℓ_1 that does not admit dimension reduction is the diamond graph $\mathsf{D_k}$ and has a high doubling constant. However, there does exist a doubling subset[1] of ℓ_1, the Laakso

[1]The results are asymptotic in nature and by doubling we mean that the doubling constant of $\mathsf{L_k}$ is $O(1)$. The

graph L_k, for which existence of a bi-Lipschitz embedding with distortion D into ℓ_1^d implies that $D=\Omega(\sqrt{\log(n)/\log(d)})$, or equivalently there is no bi-Lipschitz embedding of L_k with distortion D in ℓ_p^k if $k=O(n^{1/D^2})$. Therefore, Problem 1 has a negative solution for ℓ_1-targets. That Problem 1 also has a negative solution for ℓ_q-targets for $q>2$ was proved independently by Y. Bartal, L.-A. Gottlieb, and O. Neiman [BGN14, BGN15], and V. Lafforgue and A. Naor [LN14a].

Theorem 1. *For every $q\in(2,\infty)$, there exists a doubling subset of ℓ_q that does not admit any bi-Lipschitz embedding into $\mathbb{R}^d$ for any $d\in\mathbb{N}$.*

In Section 3.2 we give a new proof of Theorem 1. In order to put our contribution into perspective and to highlight the advantages and limits of our alternative proof, we will discuss the two distinct approaches taken in [BGN14, BGN15] and [LN14a], as well as their scopes of application.

The approach undertaken by Lafforgue and Naor is based on classical, albeit subtle, geometric properties of Heisenberg groups. In [LN14a], Lafforgue and Naor construct for every $\varepsilon\in(0,\frac{1}{2}]$ and $q\in[2,\infty)$, an embedding $F_{\varepsilon,q}\colon\mathbb{H}_3(\mathbb{Z})\to L_q(\mathbb{R}^s)$ such that $F_{\varepsilon,q}(\mathbb{H}_3(\mathbb{Z}))$ is 2^{16}-doubling and

$$\forall x,y\in\mathbb{H}_3(\mathbb{Z}),\ \mathsf{d}_W(x,y)^{1-\varepsilon}\leqslant\|F_{\varepsilon,q}(x)-F_{\varepsilon,q}(y)\|\lesssim\frac{\mathsf{d}_W(x,y)^{1-\varepsilon}}{\varepsilon^{1/q}},\tag{3.1}$$

where d_W is the canonical word metric on the discrete 3-dimensional Heisenberg group $\mathbb{H}_3(\mathbb{Z})$, and $\mathbb{R}^s$ is some potentially high-dimensional Euclidean space equipped with the Lebesgue measure. The symbol $\lesssim$ will be conveniently use to hide a universal numerical mulitiplicative constant.

The map $F_{\varepsilon,q}$ is given by a rather elementary formula but showing that it is a bi-Lipschitz embedding of the $(1-\varepsilon)$-snowflaking of $\mathbb{H}_3(\mathbb{Z})$ as in (3.1), and that the image is doubling requires some quite technical analytic computations [2]. By taking $\varepsilon=1/\log n$ in (3.1), the map $F_{1/\log n,q}$ becomes a bi-Lipschitz embedding with distortion $O((\log n)^{1/q})$ of the ball of radius $\sqrt[4]{n}$ into L_q (whose image inherits the doubling property of $F_{\varepsilon,q}(\mathbb{H}_3(\mathbb{Z}))$). Since $\mathbb{H}_3(\mathbb{Z})$ is a finitely generated group of quartic growth, for every $n\geqslant 1$ there exists a n-point subset $X_n\subset\mathbb{H}_3(\mathbb{Z})$ lying in an

classical notation $D=O(f(n))$ (resp. $D=\Omega(f(n))$) means that $D\leqslant\alpha f(n)$ (resp. $D\geqslant\alpha f(n)$) for some constant α and for n large enough. And $D=\Theta(f(n))$ if and only if $[D=O(f(n))]\wedge[D=\Omega(f(n))]$.

[2] Lafforgue and Naor gave an alternate (and of similar difficulty) proof of (3.1) using the Schrödinger representation of Heisenberg groups that we do not discuss here.

annulus enclosed by two balls with radii proportional to $\sqrt[4]{n}$. The image of X_n under $F_{1/\log n,q}$, which will be denoted $\mathscr{H}_n(q)$, is 2^{16}-doubling. A significant advantage of the Heisenberg-based approach of Lafforgue and Naor is that it provides non-embeddability results for the doubling subset $\mathscr{H}_n(q)$ of ℓ_q for a wide class of Banach space targets. It is indeed possible to leverage some deep non-embeddability results available for the subset X_n of $\mathbb{H}_3(\mathbb{Z})$, to derive lower bounds on the distortion of $\mathscr{H}_n(q)$ when embedding $\mathscr{H}_n(q)$ into any p-uniformly convex Banach space for $2 \leqslant p < q$ and even into L_1.

Let $\mathsf{c}_{\mathsf{Y}}(\mathsf{X})$ denote the Y-distortion of X for two metric spaces $(\mathsf{X}, \mathsf{d}_{\mathsf{X}})$ and $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$. The following theorem is a quantitatively explicit version (and updated according to the most recent available bounds) of Theorem 1.2 in [LN14a].

Theorem 2. *For every $q \in (2,\infty)$ and every $n \in \mathbb{N}$, there exists a 2^{16}-doubling n-point subset $\mathscr{H}_n(q)$ of ℓ_q such that*

1. $\mathsf{c}_{\mathfrak{Y}}(\mathscr{H}_n(q)) = \Omega((\log n)^{\frac{1}{p}-\frac{1}{q}})$ *if $\mathfrak{Y}$ is a p-uniformly convex Banach space for $2 \leqslant p < q$*
2. $\mathsf{c}_{L_1}(\mathscr{H}_n(q)) = \Omega((\log n)^{\frac{1}{4}})$.

Moreover, for every $q \in (2,\infty)$, there exists a doubling subset $\mathscr{H}(q)$ of ℓ_q that does not admit a bi-Lipschitz embedding into L_1 or into a p-uniformly convex Banach space for any $p \in (1,q)$.

Assertions (1) and (2) in Theorem 2 follow from the above discussion of the Lafforgue-Naor approach and sharp non-embeddability of Heisenberg balls into p-uniformly convex spaces ([LN14b], which refines earlier results from [ANT13]), and into L_1 (more specifically [NY18], which improves the lower bound in [CKN11]). The moreover part of Theorem 2 follows from a standard argument where $\mathscr{H}(q)$ is a certain disjoint union of the sequence $\{\mathscr{H}_n(q)\}_{n\in\mathbb{N}}$ and which contains an isometric copy of a rescaling of $\mathscr{H}_n(q)$ for every $n \in \mathbb{N}$.

The derivation of Theorem 1 from Theorem 2, which follows from the fact that we can assume without loss of generality that the constant finite-dimensional space is 2-uniformly convex, is standard. Another consequence of assertion (1) in Theorem 2 and classical estimates on the Banach-Mazur distance between finite-dimensional ℓ_r-spaces is the following corollary.

Corollary 7. *For every $q \in (2,\infty)$ and every $n \in \mathbb{N}$, there exists a 2^{16}-doubling n-point subset $\mathscr{H}_n(q)$ of ℓ_q such that*

1. $\mathsf{c}_{\ell_q^d}(\mathscr{H}_n(q)) = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\frac{1}{2}-\frac{1}{q}}\Big)$.

2. $\mathsf{c}_{\ell_p^d}(\mathscr{H}_n(q)) = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\min\left\{\frac{1}{2},\frac{1}{p}\right\}-\frac{1}{q}}\Big)$ *if* $1 < p < q$.

It is worth pointing out that the case $q = 2$ in assertion (1) of Theorem 2 also follows from an important Poincaré-type inequality for the Heisenberg group [ANT13, Theorem 1.4 and Corollary 1.6] which is a precursor of a groundbreaking line of research pertaining to Poincaré-type inequalities in terms of horizontal versus vertical perimeter in Heisenberg groups.

We now turn to the approach of Bartal, Gottlieb, and Neiman.

Theorem 3. [BGN15] *Let $q \in (2,\infty)$, $D \geqslant 1$, and $d \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists a n-point subset $\mathscr{L}_n(p,q,D,d)$ of ℓ_q that is 2^{32}-doubling and such that any bi-Lipschitz embedding of $\mathscr{L}_n(p,q,D,d)$ with distortion D into ℓ_p^d must satisfy*

1. $D = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\frac{1}{2}-\frac{1}{q}}\Big)$ *if* $p = q$

   and

2. $D = \Omega\left(\frac{(\log n)^{\frac{1}{2}-\frac{1}{q}}}{d^{\frac{\max\{p-2,2-p\}}{2p}}}\right)$ *if* $1 \leqslant p < q$.

A conceptual difference between Theorem 3 and Corollary 7 is that in Theorem 3 the finite doubling subsets depend on the distortion, the dimension, and also the host space. Consequently, the sequence $\{\mathscr{L}_n(p,q,D,d)\}_{n\geqslant 1}$ only rules out bi-Lipschitz embeddings for fixed distortion and dimension. Nevertheless, one can still derive Theorem 1 from Theorem 3. This derivation, which was omitted in [BGN14, BGN15], will be recalled at the end of Section 3.2. The doubling subset $\mathscr{L}_n(p,q,D,d)$ of ℓ_q is based on an elementary construction of a $\Theta(6^k)$-point Laakso-like structure in ℓ_q^k that we will recall in Section 3.2 since our new proof of Theorem 1 uses the same construction. The combinatorial proof of Theorem 3 in [BGN15] utilizes a newly introduced method based on potential functions, i.e. functions of the form $\Phi_{p,q}(u,v) = \frac{\|f(u)-f(v)\|_p^p}{\|u-v\|_q^p}$ for some p,q, where $\{u,v\}$

is an “edge" of $\mathscr{L}_n(p,q,D,d)$. The method of potential functions relies heavily on the fact that every map taking values into ℓ_p^d can be decomposed as a sum of d real-valued (coordinate) maps, and this method does not seem to be easily extendable to more general Banach space targets.

In Section 3.2 we present a new proof of Theorem 1. The doubling subsets are identical to the ones of Bartal-Gottlieb-Neiman and they are described in Section 3.2.1. The proof uses a self-improvement argument, which was first employed for metric embedding purposes by Johnson and Schechtman in [JS09], and subsequently in [Klo14], [BZ16], [BCD$^+$17], [Swi18], and [Zha21]; and is carried over in Section 3.2.2. Our proof has several advantages. We prove an analog of Theorem 3 where the n-point doubling subset can be chosen independently of the dimension and improve the estimates in assertion 2. Moreover, the self-improvement approach is rather elementary and yet covers the case of uniformly convex target spaces as in the work of Lafforgue-Naor. However, it does not allow the recovery of the case of an L_1 target as in assertion (2) of Theorem 2. The fact that we will be dealing with abstract metric structures that are not graph metrics requires a significantly more delicate implementation of the self-improvement argument. In Section 3.2.3 we explain how the new proof allows us to derive known tight lower bounds for the distortion of ℓ_p^n into uniformly convex spaces. It is worth mentioning that the lower bounds that can be derived from the Bartal-Gottlieb-Neiman approach and the Lafforgue-Naor approach seem to be often suboptimal. In Section 3.2.4 we extend the technique to cover purely metric targets of non-positive curvature and more generally rounded ball metric spaces. Finally, in Section 3.3 we extend our approach to the asymptotic Banach space setting. For this purpose, we construct countably branching analogs of the structures introduced by Bartal, Gottlieb, and Neiman that provide quantitative obstructions to embeddability into asymptotically midpoint uniformly convex spaces.

3.2 Impossibility of dimension reduction in ℓ_q, $q > 2$

3.2.1 Thin Laakso substructures

Note that the classical Laakso graphs that we introduced in Section 2 do not admit bi-Lipschitz embeddings into any uniformly convex Banach space, in particular into ℓ_p when $p \in (1,\infty)$, and this is due to the fact that there are, at all scales, midpoints that are far apart. The idea of Bartal, Gottlieb, and Neiman was to slightly tweak the Laakso construction by reducing the distance between the midpoints so that these modified metric structures could fit into ℓ_p^k for arbitrarily large dimension k but not into ℓ_p^d for fixed d without incurring a large distortion. It will be convenient to abstract the construction of Bartal, Gottlieb, and Neiman and to that end, we introduce the following definition.

Definition 1 (Thin Laakso substructure)**.** *Let $q \in [1,\infty]$ and $\varepsilon > 0$. For $k \in \mathbb{N}$, we say that a metric space $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ admits a (ε, q)*-thin *k*-Laakso substructure*, if there exists a collection of points $\mathscr{L}_k(\varepsilon, q) \subset \mathsf{X}$ indexed by L_k (and we will identify the points in $\mathscr{L}_k(\varepsilon, q)$ with the corresponding points in $\mathscr{L}_k$) such that for every $1 \leqslant j \leqslant k$ and for all $\{s, a, m_1, m_2, b, t\} \subset \mathscr{L}_k(\varepsilon, q)$ indexed by any copy of the Laakso graph L_1 created at level j, the following interpoint distance equalities hold:*

(c_1) $\mathsf{d}_\mathsf{X}(s,a) = \mathsf{d}_\mathsf{X}(b,t) = \frac{1}{2}\mathsf{d}_\mathsf{X}(a,b) = \frac{1}{4}\mathsf{d}_\mathsf{X}(s,t) > 0$

(c_2) $\mathsf{d}_\mathsf{X}(s,b) = \mathsf{d}_\mathsf{X}(a,t) = \frac{3}{4}\mathsf{d}_\mathsf{X}(t,s)$

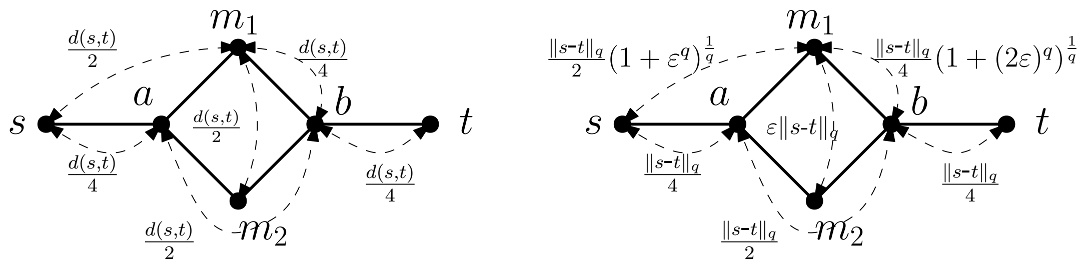
(c_3) $\mathsf{d}_\mathsf{X}(m_1,a) = \mathsf{d}_\mathsf{X}(m_1,b) = \mathsf{d}_\mathsf{X}(m_2,a) = \mathsf{d}_\mathsf{X}(m_2,b) = \frac{1}{4}(1+(2\varepsilon)^q)^{1/q}\mathsf{d}_\mathsf{X}(s,t)$

(c_4) $\mathsf{d}_\mathsf{X}(s,m_1) = \mathsf{d}_\mathsf{X}(m_2,s) = \mathsf{d}_\mathsf{X}(m_1,t) = \mathsf{d}_\mathsf{X}(m_2,t) = \frac{1}{2}(1+\varepsilon^q)^{1/q}\mathsf{d}_\mathsf{X}(s,t)$

(c_5) $\mathsf{d}_\mathsf{X}(m_1,m_2) = \varepsilon \cdot \mathsf{d}_\mathsf{X}(s,t)$ *(midpoint separation).*

The distances in the combinatorial Laakso graph, which is the template for the construction, satisfy $(c_1)-(c_4)$ with $\varepsilon = 0$, and (c_5) with $\varepsilon = \frac{1}{2}$, and the distances for a path graph with 4 points would satisfy $(c_1)-(c_5)$ with $\varepsilon = 0$. The following diagram can help visualize the differences between the distances in the Laakso graph L_1 and the (ε, q)-thin 1-Laakso substructure $\mathscr{L}_1(\varepsilon, q)$ construction.

Figure 3.1: Distances in Laakso graph L_1 and distances in ε-thin Laakso structure $\mathscr{L}_1(\varepsilon,q)$ in ℓ_q^2



The existence of (ε,q)-thin k-Laakso substructures in ℓ_q was proven in [BGN15]. Since we use different notation and a slightly different thinness parameter we will reproduce the proof for the convenience of the reader.

Lemma 1. *Let $q \in [1,\infty]$. For all $k \in \mathbb{N}$ and $\varepsilon > 0$, ℓ_q^{k+1} admits a (ε,q)-thin k-Laakso substructure.*

Proof. Let $\{e_i\}_{i=1}^{k+1}$ be the canonical basis of ℓ_q^{k+1}. The proof is by induction on k. If $k=1$ then $\mathsf{L}_1 = \{s,a,m_1,m_2,b,t\}$ and identifying points in $\mathscr{L}_1(\varepsilon,q)$ with the corresponding points in $\mathscr{L}_1$ we define

$$\begin{gathered} s = -e_1 \quad \textit{and} \quad t = e_1, \\ a = -\frac{1}{2}e_1 \quad \textit{and} \quad b = \frac{1}{2}e_1, \\ m_1 = \varepsilon e_2 \quad \textit{and} \quad m_2 = -\varepsilon e_2. \end{gathered}$$

Observe that the vectors are in ℓ_p^2 and a straightforward verification shows that conditions $(c_1)-(c_5)$ are verified. Assume now that $\mathscr{L}_k(\varepsilon,q)$ has been constructed in ℓ_q^{k+1}. Recall that L_{k+1} is contructed by replacing every edge in L_k with a copy of L_1. For every edge $\{s,t\}$ in L_k we introduce 4 new points as follows:

$$\begin{aligned} a = \frac{3}{4}s + \frac{1}{4}t \quad &\textit{and} \quad b = \frac{1}{4}s + \frac{3}{4}t, \\ m_1 = \frac{s+t}{2} + \frac{\varepsilon}{2}\|s-t\|_q e_{k+2} \quad &\textit{and} \quad m_2 = \frac{s+t}{2} - \frac{\varepsilon}{2}\|s-t\|_q e_{k+2}. \end{aligned}$$

Then

$$\begin{aligned}\|b-m_2\|_q = & \left\|\frac{1}{4}s+\frac{3}{4}t-\frac{s+t}{2}+\frac{\varepsilon}{2}\|s-t\|_q e_{k+2}\right\|_q = \left\|\frac{t-s}{4}+\frac{\varepsilon}{2}\|s-t\|_q e_{k+2}\right\|_q \\ = & \left(\frac{1}{4^q}\|s-t\|_q^q+\frac{\varepsilon^q}{2^q}\|s-t\|_q^q\right)^{1/q} = \frac{\|s-t\|_q}{4}\Big(1+(2\varepsilon)^q\Big)^{1/q},\end{aligned}$$

where in the penultimate equality we used the fact that $\frac{1}{4}(s-t)\in\ell_q^{k+1}$. The other equalities can be checked similarly. □

Remark 1. *It was proved in* [BGN15] *that an* (ε,p)*-thin k-Laakso substructure is* 2^{32}*-doubling whenever* $\varepsilon<\frac{2}{17}$*.*

3.2.2 A proof via a self-improvement argument

In this section we prove Theorem 1 using a self-improvement argument. Recall that a Banach space $\mathfrak{X}$ is *uniformly convex* if for all $t>0$ there exists $\delta(t)>0$ such that for all $x,y\in S_X$, if $\|x-y\|_{\mathfrak{X}}\geqslant t$ then $\|\frac{x+y}{2}\|_{\mathfrak{X}}\leqslant 1-\delta(t)$. The modulus of uniform convexity of $\mathfrak{X}$, denoted $\delta_{\mathfrak{X}}$, is defined by

$$\delta_{\mathfrak{X}}(t)=\inf\left\{1-\left\|\frac{x+y}{2}\right\|_{\mathfrak{X}} : \|x-y\|_{\mathfrak{X}}\geqslant t\right\}. \tag{3.2}$$

Clearly, $\mathfrak{X}$ is uniformly convex if and only if $\delta_{\mathfrak{X}}(t)>0$ for all $t>0$, and we say that $\mathfrak{X}$ is q-uniformly convex (or is uniformly convex of power type q) if $\delta_{\mathfrak{X}}(t)\geqslant ct^q$ for some universal constant $c>0$. A classical result of Pisier [Pis75] states that a uniformly convex Banach space admits a renorming that is q-uniformly convex for some $q\geqslant 2$. The following key lemma is similar to a contraction result for Laakso graphs from [JS09].

Lemma 2. *Let* $p\in(1,\infty)$*. Assume that* $\mathscr{L}_k(\varepsilon,p)$ *is a* (ε,p)*-thin k-Laakso substructure in* $(\mathsf{X},\mathsf{d}_\mathsf{X})$ *and that* $f\colon\mathsf{X}\to(\mathfrak{Y},\|\cdot\|)$ *is a bi-Lipschitz embedding with distortion D. Then for every* $1\leqslant\ell\leqslant k$*, if* $\{s,a,m_1,m_2,b,t\}\subset\mathsf{L}_k(\varepsilon,p)$ *is indexed by a copy of one of the Laakso graphs* L_1 *created at step* ℓ*, we have:*

$$\|f(s)-f(t)\|\leqslant D\mathsf{d}_\mathsf{X}(s,t)(1+\varepsilon^p)^{1/p}\left(1-\delta_{\mathfrak{Y}}\left(\frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}}\right)\right). \tag{3.3}$$

Proof. Assume without loss of generality that for all $x, y \in \mathsf{X}$

$$\mathsf{d}_{\mathsf{X}}(x,y) \leqslant \|f(x)-f(y)\| \leqslant D\mathsf{d}_{\mathsf{X}}(x,y). \tag{3.4}$$

Let $\alpha \stackrel{\mathrm{def}}{=} \frac{\mathsf{d}_{\mathsf{X}}(s,t)}{2}(1+\varepsilon^p)^{1/p}$, and

$$x_1 \stackrel{\mathrm{def}}{=} f(m_1)-f(s), \quad x_2 \stackrel{\mathrm{def}}{=} f(m_2)-f(s), \quad y_1 \stackrel{\mathrm{def}}{=} f(t)-f(m_1), \quad y_2 \stackrel{\mathrm{def}}{=} f(t)-f(m_2).$$

For all $i \in \{1,2\}$, it follows from the upper bound in (3.4) and (c_4) that $\frac{\|x_i\|}{D\alpha} \leqslant 1$ and $\frac{\|y_i\|}{D\alpha} \leqslant 1$. On the other hand, it follows from the lower bound in (3.4) and (c_5) that

$$\frac{\|x_1-x_2\|}{D\alpha} \geqslant \frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}} \quad \text{and} \quad \frac{\|y_1-y_2\|}{D\alpha} \geqslant \frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}}.$$

Therefore

$$\left\|\frac{x_1+x_2}{2D\alpha}\right\| \leqslant 1-\delta_{\mathfrak{Y}}\Big(\frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}}\Big) \quad \text{and} \quad \left\|\frac{y_1+y_2}{2D\alpha}\right\| \leqslant 1-\delta_{\mathfrak{Y}}\Big(\frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}}\Big).$$

Since

$$\begin{aligned} f(t)-f(s) &= (f(t)-f(m_1)+f(m_1)-f(s)+f(t)-f(m_2)+f(m_2)-f(s))/2 \\ &= (y_1+y_2+x_1+x_2)/2, \end{aligned}$$

it follows from the triangle inequality that $\|\frac{f(t)-f(s)}{D\alpha}\| \leqslant 2\Big(1-\delta_{\mathfrak{Y}}\Big(\frac{2\varepsilon}{D(1+\varepsilon^p)^{1/p}}\Big)\Big)$ and the conclusion follows. $\square$

By using the tension between the thinness parameter of a thin Laakso substructure and the power type of the modulus of uniform convexity of the host space we can prove the following distortion lower bound.

Theorem 4. *Let $2 \leqslant p < q$ and assume that $(\mathsf{X}, \mathsf{d_X})$ admits a bi-Lipschitz embedding with distortion D into a p-uniformly convex Banach space $\mathfrak{Y}$. There exists $\varepsilon := \varepsilon(p,q,D,\mathfrak{Y}) > 0$ such that if $(\mathsf{X}, \mathsf{d_X})$ admits a (ε, q)-thin k-Laakso substructure then $D = \Omega(k^{1/p-1/q})$*

Proof. Assume that for all $x, y \in \mathsf{X}$

$$\mathsf{d_X}(x,y) \leqslant \|f(x) - f(y)\| \leqslant D\mathsf{d_X}(x,y), \tag{3.5}$$

and let $\mathscr{L}_k(\varepsilon,q)$ be a (ε,q)-thin k-Laakso substructure with $\varepsilon > 0$ small enough such that $(1+\varepsilon^q)^{1/q} \leqslant 2$. The self-improvement argument uses the self-similar structure of the Laakso graphs. For $1 \leqslant j \leqslant k$ consider the decomposition $\mathsf{L}_{k-j} \oslash \mathsf{L}_j$ of L_k, i.e. L_k is formed by replacing each of the 6^{k-j} edges of L_{k-j} by a copy of L_j. We define D_j to be the smallest constant such that

$$\|f(x) - f(y)\| \leqslant D_j \mathsf{d_X}(x,y), \tag{3.6}$$

for all $4 \times 6^{k-j}$ pairs of points $\{x,y\}$ in $\mathscr{L}_k(\varepsilon, p)$ that are indexed by vertices of a copy of L_j in L_k of the form $\{s, m_i\}$ or $\{m_i, t\}$ for some $i \in \{1,2\}$, where s and t are the farther apart vertices in L_j whose two distinct midpoints are m_1 and m_2.

It is clear that for all $j \in \{1, \dots, k\}$, the inequalities $1 \leqslant D_j \leqslant D$ hold. Assume that $\delta_{\mathfrak{Y}}(t) \geqslant ct^p$ for some constant $c > 0$ (that depends on $\mathfrak{Y}$ only). Fix L_j^0 as one of the 6^{k-j} copies of L_j in the decomposition $\mathsf{L}_{k-j} \oslash \mathsf{L}_j$ of L_k. Observe that $\mathsf{L}_j^0 = \mathsf{L}_1 \oslash \mathsf{L}_{j-1}$ and let $\{s, a, m_1, m_2, b, t\}$ denote the vertices of L_1 in this decomposition of L_j^0. Consider the pair $\{s, m_1\}$ as defined above (the 3 other pairs can be treated similarly) and the two copies of L_{j-1} which contain either s or m_1 and have the vertex a in common. In the proof of Lemma 2 we only used the upper bound in (3.4) for pairs of points of the form described in the definition of D_{j-1}, and because we assumed that $\delta_{\mathfrak{Y}}(t) \geqslant ct^p$ and $(1+\varepsilon^q)^{1/q} \leqslant 2$ we have

$$\|f(s)-f(a)\| \leqslant D_{j-1}\mathsf{d}_{\mathsf{X}}(s,a)(1+\varepsilon^q)^{1/q}\Big(1-\frac{c\varepsilon^p}{D_{j-1}^p}\Big), \tag{3.7}$$

and

$$\|f(a)-f(m_1)\| \leqslant D_{j-1}\mathsf{d}_{\mathsf{X}}(a,m_1)(1+\varepsilon^q)^{1/q}\Big(1-\frac{c\varepsilon^p}{D_{j-1}^p}\Big). \tag{3.8}$$

Then, it follows from the triangle inequality that

$$\|f(s)-f(m_1)\| \leqslant D_{j-1}(\mathsf{d}_{\mathsf{X}}(s,a)+\mathsf{d}_{\mathsf{X}}(a,m_1))(1+\varepsilon^q)^{1/q}\Big(1-\frac{c\varepsilon^p}{D_{j-1}^p}\Big). \tag{3.9}$$

By (c_1) and (c_3) in the construction of the thin Laakso substructures we have

$$\mathsf{d}_{\mathsf{X}}(s,a)+\mathsf{d}_{\mathsf{X}}(a,m_1)=\frac{1}{4}\mathsf{d}_{\mathsf{X}}(s,t)+\frac{1}{4}(1+(2\varepsilon)^q)^{1/q}\mathsf{d}_{\mathsf{X}}(s,t).$$

Since $q\geqslant 1$ we have $(1+(2\varepsilon)^q)^{1/q}\leqslant 1+(2\varepsilon)^q$, and thus

$$\mathsf{d}_{\mathsf{X}}(s,a)+\mathsf{d}_{\mathsf{X}}(a,m_1)\leqslant\frac{1}{4}\mathsf{d}_{\mathsf{X}}(s,t)(2+(2\varepsilon)^q)=\frac{1}{2}\mathsf{d}_{\mathsf{X}}(s,t)(1+2^{q-1}\varepsilon^q)\overset{(c_4)}{=}\frac{1+2^{q-1}\varepsilon^q}{(1+\varepsilon^q)^{1/q}}\mathsf{d}_{\mathsf{X}}(s,m_1).$$

Substituting this last inequality in (3.9) we obtain

$$\|f(s)-f(m_1)\| \leqslant D_{j-1}\mathsf{d}_{\mathsf{X}}(s,m_1)(1+2^{q-1}\varepsilon^q)\Big(1-\frac{c\varepsilon^p}{D_{j-1}^p}\Big).$$

By symmetry of the (ε,q)-thin Laakso substructures, the other pairs of points in the definition of D_j can be treated similarly and hence we have proved that

$$D_j\leqslant D_{j-1}(1+2^{q-1}\varepsilon^q)\Big(1-\frac{c\varepsilon^p}{D_{j-1}^p}\Big).$$

Then,

$$
\begin{aligned}
D_j &\leqslant D_{j-1}(1+2^{q-1}\varepsilon^q) - \frac{(1+2^{q-1}\varepsilon^q)c\varepsilon^p}{D_{j-1}^{p-1}} \\
&\leqslant D_{j-1} + D(2\varepsilon)^q - \frac{c\varepsilon^p}{D^{p-1}}, \qquad (3.10)
\end{aligned}
$$

where in (3.10) we used the fact that $D_{j-1} \leqslant D$ and $1+2^{q-1}\varepsilon^q \geqslant 1$. Rearranging we have

$$
D_{j-1} - D_j \geqslant D\left(\frac{c\varepsilon^p}{D^p} - (2\varepsilon)^q\right), \qquad (3.11)
$$

If we let $\varepsilon = \gamma D^{-\frac{p}{q-p}}$ for some small enough γ to be chosen later (and that depends only on p,q, and c), then

$$
\begin{aligned}
D_{j-1} - D_j &\geqslant D\left(c\gamma^p \frac{D^{-\frac{p^2}{q-p}}}{D^p} - (2\gamma)^q D^{-\frac{pq}{q-p}}\right) \qquad (3.12) \\
&\geqslant D \cdot D^{-\frac{pq}{q-p}}(c\gamma^p - 2^q\gamma^q). \qquad (3.13)
\end{aligned}
$$

If we choose $\gamma \in \left(0, \left(\frac{c}{2^{q+1}}\right)^{1/(q-p)}\right)$, and since $p<q$, we have $c\gamma^p - 2^q\gamma^q \geqslant \frac{c}{2}\gamma^p > 0$. Hence $D_{j-1} - D_j \geqslant \frac{c\gamma^p}{2} D^{1-\frac{pq}{q-p}}$ and summing over $j = 2,\ldots,k$ we get

$$
D \geqslant D_1 - D_k \geqslant \sum_{j=2}^{k} \frac{c\gamma^p}{2} D^{1-\frac{pq}{q-p}} \geqslant \frac{c\gamma^p}{2}(k-1)D^{1-\frac{pq}{q-p}}, \qquad (3.14)
$$

and hence $D \gtrsim k^{1/p-1/q}$. □

Corollary 8 below improves Theorem 3 in several ways. The dependence in the dimension for the thinness parameter is removed. Assertion 1 extends to all p-uniformly convex Banach spaces the bound in assertion 2 of Theorem 3 while improving the bound. Indeed if $\mathfrak{Y} = \ell_p^d$ then $D = \Omega\left(\left(\frac{\log n}{d}\right)^{\min\left\{\frac{1}{2},\frac{1}{p}\right\}-\frac{1}{q}}\right)$.

Corollary 8. *Let $q \in (2,\infty)$, $\mathfrak{Y}$ be a Banach space, and fix $D \geqslant 1$. For every $n \in \mathbb{N}$ there exists* *an n-point subset $\mathscr{L}_n(q,D,\mathfrak{Y})$ of ℓ_q that is 2^{32}-doubling and such that any bi-Lipschitz embedding with distortion D into $\mathfrak{Y}$ must incur*

1. $D = \Omega\Big((\log n)^{\frac{1}{p}-\frac{1}{q}}\Big)$ *if $p \in [2,q)$ and $\mathfrak{Y}$ is a p-uniformly convex Banach space*

   and

2. $D = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\frac{1}{2}-\frac{1}{q}}\Big)$ *if $\mathfrak{Y} = \ell_q^d$*

Proof. Assertion (1) follows immediately from Theorem 4 and Lemma 1. The second assertion follows from the fact that $n = \Theta(6^k)$ and that the Banach-Mazur distance between the 2-uniformly convex spaces ℓ_2^d and ℓ_q^d is at most $d^{1/2-1/q}$. $\square$

Remark 2. *Very recently, Naor and Young* [NY20] *gave the first partial counter-example to the metric Kadec-Pełczyński problem, which asks whether for $1 \leqslant p < r < q < \infty$, a metric space that admits a bi-Lipschitz embedding into L_p and into L_q necessarily admits a bi-Lipschitz embedding into L_r. Naor and Young produced a Heisenberg-type space that does embed into ℓ_1 and into ℓ_q but does not embed into ℓ_r for any $1 < r < 4 \leqslant q$. The fact that what happens for ℓ_r in the range $4 \leqslant r < q$ is not understood seems inherent of the Heisenberg approach. If we could show that the thin Laakso substructures do embed into ℓ_1 then we would have a second counter-example to the metric Kadec-Pełczyński problem which resolves this issue.*

It remains to show how Theorem 1 can be derived from Corollary 8. First observe that for all $q > 2$, $D \geqslant 1$, and every $n \in \mathbb{N}$ the n-point doubling subsets $\mathscr{L}_n(q,D,\ell_2)$ of ℓ_q belong to the unit ball of ℓ_q. Now consider the subset $Z_q \stackrel{\mathrm{def}}{=} \bigcup_{(k,n)\in\mathbb{N}^2} \mathscr{L}_n(q,k,\ell_2) \times \{(4^k,4^n)\} \subset \ell_q \oplus_q \mathbb{R}^2 \equiv \ell_q$. Clearly, Z_q contains an isometric copy of $\mathscr{L}_n(q,k,\ell_2)$ and it can be verified that Z_q is doubling. If $Z_q \subset \ell_q$ admits a bi-Lipschitz embedding with distortion D into ℓ_q^d for some $d \in \mathbb{N}$, then the proof of assertion (2) in Corollary 8 shows that $D = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\frac{1}{2}-\frac{1}{q}}\Big)$ since Z_q contains an isometric copy of $\mathscr{L}_n(q,k,\ell_2)$ for all $n \in \mathbb{N}$, where $k \in \mathbb{N}$ is such that $k \leqslant D < k+1$, and hence D cannot be finite.

3.2.3 Quantitative embeddability of ℓ_q^k into uniformly convex Banach spaces

The alternative proof of Theorem 1 that we proposed has several noteworthy applications. One application concerns the non-embeddability of ℓ_q into L_p and more generally lower bounding the quantitative parameter $\mathsf{c}_{\mathfrak{Y}}(\ell_q^k)$ whenever $\mathfrak{Y}$ is a p-uniformly convex Banach space. It is well known that when $2 \leqslant p < q$, if $\mathfrak{Y}$ has cotype p, and in particular if $\mathfrak{Y}$ is p-uniformly convex, then $\sup_{k\geqslant 1} \mathsf{c}_{\mathfrak{Y}}(\ell_q^k) = \infty$. Quantitatively,

$$\mathsf{c}_{\mathfrak{Y}}(\ell_q^k) = \Omega\Big(k^{\frac{1}{p}-\frac{1}{q}}\Big) \text{ for all } 2 \leqslant p < q. \tag{3.15}$$

and

$$\mathsf{c}_{\mathfrak{Y}}(\ell_q^k) = \Omega\Big(k^{\frac{1}{2}-\frac{1}{q}}\Big) \text{ for all } 1 \leqslant p \leqslant 2 < q. \tag{3.16}$$

The fact that these lower bounds are tight follows from simple estimates of the norm of the formal identity (and its inverse) (e.g. $\|I_{\ell_q^k\to\ell_p^k}\| \cdot \|I^{-1}_{\ell_q^k\to\ell_p^k}\| \leqslant k^{1/p-1/q}$ is a consequence of Hölder's inequality and the monotonicity of the ℓ_r-norms). Since the thin k-Laakso substructure lives in ℓ_q^{k+1} and has $\Theta(6^k)$ points these lower bounds follow directly from the first assertion of Corollary 8. As we point out below the approaches in [LN14a] and [BGN15] seem to only give suboptimal results.

The n-point doubling subset $\mathscr{H}_n(q)$ of Lafforgue and Naor lies in some $L_q(\mathbb{R}^k)$-space and hence in $\ell_q^{n(n-1)/2}$ by a result of [Bal90]. Therefore, if one uses the Heisenberg-type $\sqrt{n}$-point doubling subset of ℓ_q^n one can derive that, for example, $\mathsf{c}_{\ell_2}(\ell_q^n) = \Omega\Big(\log(n)^{\frac{1}{2}-\frac{1}{q}}\Big)$ which is suboptimal. To get the optimal lower bound one would need to be able to show that the doubling subset can actually be embedded into $\ell_q^{\Theta(\log n)}$, which is the best we can hope for due to assertion 1 in Theorem 2. This does not seem to be known and we do not know if this is true.

Following Bartal-Gottlieb-Neiman's approach, one could obtain partial and suboptimal results as follows. Assume that ℓ_q^k admits a bi-Lipschitz embedding into ℓ_p^k. Then one can construct a subset in ℓ_q^k, namely $\mathscr{L}_n(q,D,k-1)$, having $n = \Theta(6^k)$ points and witnessing the fact that D must be large. The estimates in Theorem 3 yield $\mathsf{c}_{\ell_p^k}(\ell_q^k) = \Omega\Big(k^{\frac{1}{p}-\frac{1}{q}}\Big)$ for all $2 \leqslant p < q$. Therefore, the right order of magnitude is captured in the range $2 \leqslant p < q$ but in the (very) restricted case of a finite-dimensional ℓ_p target that has the same dimension as the source space. In the range $1<p<2<q$ one gets $\mathsf{c}_{\ell_p^k}(\ell_q^k)=\Omega\left(k^{1-\frac{1}{p}-\frac{1}{q}}\right)$ which does not capture the right order of magnitude and is clearly suboptimal.

3.2.4 Quantitative embeddability of ℓ_q^k into non-positively curved spaces

Another advantage of the proof via self-improvement is that it can be extended, with a little bit more care, to cover maps taking values into non-positively curved spaces, and more generally to the context of rounded ball metric spaces.

Recall that the η-approximate midpoint set of $x,y\in(\mathsf{X},\mathsf{d_X})$ is defined as

$$\begin{aligned}\mathrm{Mid}(x,y,\eta) &\stackrel{\mathrm{def}}{=} \left\{z\in\mathsf{X}\colon\ \max\left\{\mathsf{d_X}(x,z),\mathsf{d_X}(y,z)\right\}\leqslant\frac{1+\eta}{2}\mathsf{d_X}(x,y)\right\}\\ &= B_{\mathsf{X}}\left(x,\frac{1+\eta}{2}\mathsf{d_X}(x,y)\right)\cap B_{\mathsf{X}}\left(y,\frac{1+\eta}{2}\mathsf{d_X}(x,y)\right)\end{aligned}$$

As usual for an arbitrary set $A\subset\mathsf{X}$, $\mathrm{diam}(A)\stackrel{\mathrm{def}}{=}\sup\left\{\mathsf{d_X}(x,y)\colon\ x,y\in A\right\}$. The following definition is due to T. J. Laakso [Laa02].

A metric space $(\mathsf{X},\mathsf{d_X})$ is a *rounded ball space* if for all $t>0$ there exists $\eta(t)>0$ such that for all $x,y\in\mathsf{X}$

$$\mathrm{diam}\big(\mathrm{Mid}(x,y,\eta(t))\big)<t\cdot\mathsf{d_X}(x,y). \tag{3.17}$$

Remark 3. *Note that for all $x,y\in\mathsf{X}$ and $\eta>0$, $\mathrm{diam}(\mathrm{Mid}(x,y,\eta))\leqslant(1+\eta)\mathsf{d_X}(x,y)$ always holds. Therefore the rounded ball property is non-trivial only for $t\in(0,1]$ and in this case $\eta\in(0,1)$ necessarily.*

Note that a Banach space is a rounded ball space if and only if it is uniformly convex [Laa02, Lemma 5.2]. We can define a *rounded ball modulus* η_{X} as follows

$$\eta_{\mathsf{X}}(t)\stackrel{\mathrm{def}}{=}\sup\{\eta(t)\colon \text{(3.17) holds for all } x,y\in\mathsf{X}\}. \tag{3.18}$$

We will say that $(\mathsf{X},\mathsf{d_X})$ is a rounded space with power type p if $\eta_{\mathsf{X}}(t)\geqslant ct^p$.

The following contraction lemma is an extension, to the purely metric context of rounded ball spaces, of the contraction phenomenon in Lemma 2.

Lemma 3. *Let $(\mathsf{Y},\mathsf{d}_{\mathsf{Y}})$ be a metric space and $\varepsilon>0$ such that $(1+\varepsilon^q)^{1/q}\leqslant 2$. Assume that $\mathscr{L}_k(\varepsilon,q)$ is a (ε,q)-thin k-Laakso substructure in $(\mathsf{X},\mathsf{d}_{\mathsf{X}})$ and that $f\colon \mathsf{X}\to\mathsf{Y}$ satisfies*

$$\frac{1}{A}\mathsf{d}_{\mathsf{X}}(x,y)\leqslant \mathsf{d}_{\mathsf{Y}}(f(x),f(y))\leqslant B\mathsf{d}_{\mathsf{X}}(x,y), \tag{3.19}$$

for some constants $A,B>0$. Then for every $1\leqslant \ell\leqslant k$, if $\{s,a,m_1,m_2,b,t\}\subset \mathsf{L}_k(\varepsilon,q)$ is indexed by a copy of one of the Laakso graphs L_1 created at step ℓ we have:

$$\mathsf{d}_{\mathsf{Y}}(f(s),f(t))\leqslant B\mathsf{d}_{\mathsf{X}}(s,t)(1+\varepsilon^q)^{1/q}\Big(1-\frac{1}{2}\eta_{\mathsf{Y}}(\varepsilon/2AB)\Big). \tag{3.20}$$

Proof. Let $r>0$ be the smallest radius such that $B_{\mathsf{Y}}(f(s),r)\cap B_{\mathsf{Y}}(f(t),r)\supseteq\{f(m_1),f(m_2)\}$. Then

$$r\leqslant \max\{d_{\mathsf{Y}}(f(s),f(m_1)),d_{\mathsf{Y}}(f(s),f(m_2)),d_{\mathsf{Y}}(f(t),f(m_1)),d_{\mathsf{Y}}(f(t),f(m_2))\}$$

and it follows from (3.19) and (c_4) that $r\leqslant \frac{B}{2}(1+\varepsilon^q)^{1/q}\mathsf{d}_{\mathsf{X}}(s,t)$. On the other hand,

$$\operatorname{diam}(B_{\mathsf{Y}}(f(s),r)\cap B_{\mathsf{Y}}(f(t),r))\geqslant d_{\mathsf{Y}}(f(m_1),f(m_2)), \tag{3.21}$$

and thus

$$\begin{aligned}\operatorname{diam}(B_{\mathsf{Y}}(f(s),r)\cap B_{\mathsf{Y}}(f(t),r))&\geqslant \frac{1}{A}\mathsf{d}_{\mathsf{X}}(m_1,m_2)\overset{(c_5)}{=}\frac{1}{A}\varepsilon\mathsf{d}_{\mathsf{X}}(s,t)\\&\overset{(c_4)}{=}\frac{1}{A}\varepsilon\Big(\frac{\mathsf{d}_{\mathsf{X}}(s,m_1)}{(1+\varepsilon^q)^{1/q}}+\frac{\mathsf{d}_{\mathsf{X}}(t,m_1)}{(1+\varepsilon^q)^{1/q}}\Big)\\&\geqslant\frac{\varepsilon}{AB(1+\varepsilon^q)^{1/q}}\Big(\mathsf{d}_{\mathsf{Y}}(f(s),f(m_1))+\mathsf{d}_{\mathsf{Y}}(f(t),f(m_1))\Big)\\&\geqslant\frac{\varepsilon}{2AB}\mathsf{d}_{\mathsf{Y}}(f(s),f(t)),\end{aligned}$$

where in the last inequality we used our assumption on ε and the triangle inequality. Therefore,

$r \geqslant \frac{1+\eta_{\mathsf{Y}}(\varepsilon/(2AB))}{2}\mathsf{d}_{\mathsf{Y}}(f(s),f(t))$ by definition of the rounded ball modulus, and

$$\begin{aligned} d_{\mathsf{Y}}(f(s),f(t)) &\leqslant \frac{2r}{1+\eta_{\mathsf{Y}}(\varepsilon/2AB)} \leqslant \frac{B(1+\varepsilon^q)^{1/q}\mathsf{d}_{\mathsf{X}}(s,t)}{1+\eta_{\mathsf{Y}}(\varepsilon/2AB)} \\ &\leqslant B\mathsf{d}_{\mathsf{X}}(s,t)(1+\varepsilon^q)^{1/q}\Big(1-\frac{1}{2}\eta_{\mathsf{Y}}(\varepsilon/2AB)\Big). \end{aligned}$$

where in the last inequality we used Remark 3. □

A slightly different implementation of the self-improvement argument gives the following extension of Theorem 4 to metric spaces with rounded ball modulus with power type. We only emphasize the few points in the proof that are different.

Theorem 5. *Let* $1 < p < q$ *and let* $(\mathsf{Y},\mathsf{d}_{\mathsf{Y}})$ *be a rounded ball metric space with power type* p*. For all* $D \geqslant 1$ *there exists* $\varepsilon := \varepsilon(D,p,q,\mathsf{Y}) > 0$ *such that if* $(\mathsf{X},\mathsf{d}_{\mathsf{X}})$ *admits a* (ε,q)*-thin* k*-Laakso substructure and embeds bi-Lipschitzly with distortion at most* $D \geqslant 1$ *into* Y *then* $D = \Omega(k^{1/p-1/q})$.

Proof. Assume that for all $x,y \in \mathsf{X}$

$$\frac{1}{A}\mathsf{d}_{\mathsf{X}}(x,y) \leqslant \mathsf{d}_{\mathsf{Y}}(f(x),f(y)) \leqslant B\mathsf{d}_{\mathsf{X}}(x,y), \tag{3.22}$$

with $AB \leqslant D$.

This time we define B_j to be the smallest constant such that

$$\|f(x)-f(y)\| \leqslant B_j\mathsf{d}_{\mathsf{X}}(x,y), \tag{3.23}$$

for all $4\times 6^{k-j}$ pairs of points $\{x,y\}$ in $\mathscr{L}_k(\varepsilon,q)$ that are indexed by vertices of a copy of L_j in L_k of the form $\{s,m_i\}$ or $\{m_i,t\}$ for some $i \in \{1,2\}$, where s and t are the farther apart vertices in L_j whose two distinct midpoints are m_1 and m_2.

It is clear that for all $j \in \{1,\dots,k\}$, the inequalities $1 \leqslant B_j \leqslant B$ hold. Since in the proof of Lemma 3 we have only used the upper bound in (3.22) for pairs of points of the form described in the definition of B_{j-1}, proceeding as in the proof of Theorem 4 we show that

$$AB_{j-1} - AB_j \geqslant AB\left(\frac{c\varepsilon^p}{2^{p+1}(AB)^p} - (2\varepsilon)^q\right).$$

If we let $\varepsilon = \gamma(AB)^{-\frac{p}{q-p}}$ for some small enough γ to be chosen later (and that depends only on p, q, and c), then

$$AB_{j-1} - AB_j \geqslant (AB)^{1-\frac{pq}{q-p}}\left(\frac{c}{2^{p+1}}\gamma^p - 2^q\gamma^q\right).$$

If we choose $0 < \gamma < \left(\frac{c}{2^{p+q+2}}\right)^{1/(q-p)}$ we have $\frac{c}{2^{p+1}}\gamma^p - 2^q\gamma^q \geqslant \frac{c}{2^{p+2}}\gamma^p > 0$. Hence $AB_{j-1} - AB_j \geqslant \frac{c}{2^{p+2}}\gamma^p(AB)^{1-\frac{pq}{q-p}}$ and summing over $j = 2, \ldots, k$ we conclude that $AB \gtrsim k^{1/p-1/q}$. □

We now identify a 4-point inequality that implies the rounded ball property with power type p.

Lemma 4. *Let $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ be a metric space and $p \in (0, \infty)$. If there exists $C \in (0, 2^p]$ such that for all $x_1, x_2, x_3, x_4 \in \mathsf{X}$ we have*

$$\mathsf{d}_\mathsf{X}(x_1,x_3)^p + \mathsf{d}_\mathsf{X}(x_2,x_4)^p \leqslant \frac{C}{4}\Big(\mathsf{d}_\mathsf{X}(x_1,x_2)^p + \mathsf{d}_\mathsf{X}(x_2,x_3)^p + \mathsf{d}_\mathsf{X}(x_3,x_4)^p + \mathsf{d}_\mathsf{X}(x_4,x_1)^p\Big), \tag{3.24}$$

then X is a rounded ball space with $\eta_\mathsf{X}(t) \geqslant t^p/(2^p - 1)$ if $p \geqslant 1$ and with $\eta_\mathsf{X}(t) \geqslant t$ if $p \in (0,1)$.

Proof. Fix $t > 0$ and let $x, y \in \mathsf{X}$ and $\eta \in (0,1)$. If $\mathrm{Mid}(x,y,\eta)$ is empty or reduced to a single point there is nothing to prove. Otherwise, let $w \neq z \in \mathrm{Mid}(x,y,\eta)$. It follows from (3.24) that

$$\mathsf{d}_\mathsf{X}(x,y)^p + \mathsf{d}_\mathsf{X}(w,z)^p \leqslant \frac{C}{4}(\mathsf{d}_\mathsf{X}(x,w)^p + \mathsf{d}_\mathsf{X}(w,y)^p + \mathsf{d}_\mathsf{X}(y,z)^p + \mathsf{d}_\mathsf{X}(z,x)^p),$$

and by the definition of $\mathrm{Mid}(x,y,\eta)$, we have

$$\mathsf{d}_\mathsf{X}(w,z)^p \leqslant \Big(C\frac{(1+\eta)^p}{2^p} - 1\Big)\mathsf{d}_\mathsf{X}(x,y)^p.$$

And since $C \leqslant 2^p$,

$$\mathsf{d}_\mathsf{X}(w,z) \leqslant ((1+\eta)^p - 1)^{\frac{1}{p}}\mathsf{d}_\mathsf{X}(x,y).$$

If $p \geqslant 1$ then $((1+\eta)^p - 1)^{\frac{1}{p}} \leqslant (2^p-1)^{1/p}\eta^{1/p}$, and if $\eta = t^p/(2^p-1)$, then

$$\operatorname{diam}(\operatorname{Mid}(x,y,\eta)) < t\mathsf{d}_{\mathsf{X}}(x,y).$$

If $p \in (0,1)$ then $((1+\eta)^p - 1)^{\frac{1}{p}} \leqslant \eta$, and $\eta = t$ implies that

$$\operatorname{diam}(\operatorname{Mid}(x,y,\eta)) < t\mathsf{d}_{\mathsf{X}}(x,y). \quad \square$$

Inequality (3.24) when $p = 2$ and $C = 4$ is well known under various names: quadrilateral inequality, roundness 2, Enflo type 2 with constant 1. It was proved by Berg and Nikolaev [BN07] (see also [BN08] or [Sat09]) that the quadrilateral inequality characterizes CAT(0)-spaces amongst geodesic metric spaces and that CAT(0)-spaces coincide with Alexandrov spaces of non-positive curvatures; and this provides a rather large class of metric spaces which are rounded ball with power type 2. It is not difficult to show that ultrametric spaces satisfy inequality (3.24) with $p = 1$ and $C = 2$. We give one example of an application of Theorem 6.

Corollary 9. *If $q > 2$ and $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$ is a metric space with roundness 2, in particular an Alexandrov space of non-positive curvature, then*

$$\mathsf{c}_{\mathsf{Y}}(\ell_q^k) = \Omega\left(k^{\frac{1}{2}-\frac{1}{q}}\right).$$

Remark 4. *To the best of our knowledge, the only known proof of Corollary 9 can be found in the work of Eskenazis, Mendel, and Naor in* [EMN19] *where it was shown that Alexandrov spaces of non-positive curvature have metric cotype* 2*. This is a particular case of a much deeper result which says that q-barycentric metric spaces have* sharp *metric cotype q, and whose proof partly relies on a version of Pisier's martingale inequality in the context of nonlinear martingales.*

3.3 Embeddability obstruction via thin $\aleph_0$-branching diamond substructures

Using the self-improvement argument together with the smallness of approximate midpoint sets to prove Theorem 1 has the other advantage of being easily generalizable to the asymptotic setting. It is well-known that the size of a t-approximate metric midpoint set in an asymptotically uniformly convex Banach spaces is “small”. By “small” we mean that the set is included in the (Banach space) sum of a compact set and a ball of small radius. Therefore the techniques from the previous sections can be adequately modified to show that the presence of countably branching versions of the Laakso-type substructure is a bi-Lipschitz embeddability obstruction. A similar fact for countably branching diamond and Laakso graphs was first proved in [BCD+17] and generalized in [Swi18].

The only reason to work with Laakso-type substructures in the previous sections was to produce spaces with the doubling property. In the asymptotic setting, we need to work with substructures whose underlying graphs have vertices with countably many neighbors and fail the doubling property altogether. Therefore, we will only consider simpler diamond-type substructures.

As noted in [BCD+17] it is more convenient to work with the notion of asymptotic midpoint uniform convexity. Let $\mathfrak{X}$ be a Banach space and $t \in (0,1)$. Define

$$\tilde{\delta}_{\mathfrak{X}}(t) \stackrel{\text{def}}{=} \inf_{x\in S_{\mathfrak{X}}} \sup_{Z\in \mathrm{cof}(\mathfrak{X})} \inf_{z\in S_Z} \max\{\|x+tz\|, \|x-tz\|\} - 1.$$

The norm of $\mathfrak{X}$ is said to be asymptotically midpoint uniformly convex if $\tilde{\delta}_{\mathfrak{X}}(t) > 0$ for every $t \in (0,1)$. Being asymptotically midpoint uniformly convexifiable is formally weaker than being asymptotically uniformly convexifiable. However, it is still open whether asymptotic uniform convexity and asymptotic midpoint uniform convexity are equivalent notions up to renorming. We now recall some facts that we will need which can be found in [BCD+17]. A characterization of asymptotic midpoint uniformly convex norms was given in [DKLR+13] in terms of the Kuratowski measure of noncompactness of approximate midpoint sets. Recall that the Kuratowski measure of noncompactness of a subset S of a metric space, denoted by $\alpha(S)$, is defined as the infimum of all $\varepsilon > 0$ such that S can be covered by a finite number of sets of diameter less than ε. Note that it is a property of the metric.

In [DKLR+13] it was shown that a Banach space $\mathfrak{X}$ is asymptotically midpoint uniformly convex if and only if

$$\lim_{t\to 0}\sup_{x\in S_{\mathfrak{X}}}\alpha(\mathrm{Mid}(-x,x,t))=0.$$

To prove the main result of this section we need the following lemma which is a particular case of Lemma 4.3 in [BCD$^+$17].

Lemma 5. *If the norm of a Banach space $\mathfrak{X}$ is asymptotically midpoint uniformly convex, then for every $t\in(0,1)$ and every $x,y\in\mathfrak{X}$ there exists a finite subset S of $\mathfrak{X}$ such that*

$$\mathrm{Mid}(x,y,\tilde{\delta}_{\mathfrak{X}}(t)/4)\subset S+2t\|x-y\|B_{\mathfrak{X}}. \tag{3.25}$$

We define thin diamond substructures that can be used to prove non-embeddability results.

Definition 2 (Thin κ-branching diamond substructure)**.** *Let $p\in[1,\infty)$, $\varepsilon>0$, κ be a cardinal number, and I a set of cardinality κ. For $k\in\mathbb{N}$, we say that a metric space X admits a* (ε,p)-thin κ-branching k-diamond substructure *if there exists a collection $\mathscr{D}_k^\kappa(\varepsilon,p)$ of points indexed by D_k^κ such that for every $1\leqslant\ell\leqslant k$ if $\{s,\{m_i\}_{i\in I},t\}\subset\mathscr{D}_k^\kappa$ is indexed by a copy of one of the diamond created at step ℓ then:*

(d_1) $\mathsf{d}_{\mathsf{X}}(s,m_i)=\mathsf{d}_{\mathsf{X}}(m_i,t)=\frac{1}{2}(1+(2\varepsilon)^p)^{1/p}\mathsf{d}_{\mathsf{X}}(s,t)$, *for all $i\in I$*

(d_2) $\mathsf{d}_{\mathsf{X}}(m_i,m_j)=2^{1-1/p}\varepsilon\cdot\mathsf{d}_{\mathsf{X}}(s,t)$ *for all $i\neq j$.*

In Lemma 6 below, we provide a construction of a (ε,p)-thin $\aleph_0$-branching k-diamond substructure in L_p-spaces, which in turns implies for all $p\in[1,\infty)$, $k\in\mathbb{N}$, and $\varepsilon>0$ the existence of an (ε,p)-thin $\aleph_0$-branching k-diamond substructure.

Lemma 6. *For every $p\in[1,\infty)$, every $\varepsilon>0$, and every $k\in\mathbb{N}$; L_p admits a (ε,p)-thin $\aleph_0$-branching k-diamond substructure.*

Proof. Let $\chi_{i,j,k}$ stand in for the characteristic function $\chi_{\left[k+\frac{i-1}{2^j},k+\frac{i}{2^j}\right]}$. Fix $\varepsilon>0$. The (ε,p)-thin $\aleph_0$-branching k-diamond substructure in L_p with parameter $\varepsilon>0$ is defined recursively as follows. For simplicity, we start the induction with the 0-diamond graph D_0^ω to be a single edge with endpoint s and t, and (again identifying the points in $\mathscr{D}_k^\omega(\varepsilon,p)$ with the vertices of D_k^ω) we define $\mathscr{D}_0^\omega(\varepsilon,p) := \{s,t\}$ by $s \stackrel{\text{def}}{=} \chi_{[0,1]}$ and $t \stackrel{\text{def}}{=} -\chi_{[0,1]}$ and the conditions are vacuously satisfied. Suppose now that $\mathscr{D}_k^\omega$ has already been defined such that $\mathscr{D}_k^\omega \subseteq L_p[0,k+1]$. To construct $\mathscr{D}_{k+1}^\omega$ we introduce for every edge $\{s,t\} \in \mathscr{D}_k^\omega$ and $i \in \mathbb{N}$ a "midpoint" as follows:

$$m_i = \frac{s+t}{2} + \sum_{r=1}^{2^i} (-1)^r \varepsilon \|s-t\|_p \chi_{r,i,k+1}. \tag{3.26}$$

Then,

$$\begin{aligned}\|s-m_i\|_p^p &= \left\|\frac{s-t}{2} - \sum_{r=1}^{2^i}(-1)^r\varepsilon\|s-t\|_p\chi_{r,i,k+1}\right\|_p^p = \left\|\frac{s-t}{2}\right\|_p^p + \left\|\sum_{r=1}^{2^i}(-1)^r\varepsilon\|s-t\|_p\chi_{r,i,k+1}\right\|_p^p \\ &= \left\|\frac{s-t}{2}\right\|_p^p + \varepsilon^p\|s-t\|_p^p = \frac{(1+(2\varepsilon)^p)}{2^p}\|s-t\|_p^p,\end{aligned}$$

wherein the second equality we used the fact that the vectors have disjoint supports (in $[0,k+1]$ and $[k+1,k+2]$, respectively).

For $i<j$, observe that $\chi_{r,i,k+1} = \sum_{\ell=(r-1)2^{j-i}+1}^{r2^{j-i}} \chi_{\ell,j,k+1}$, and so

$$\begin{aligned}\|m_i-m_j\|_p^p &= \left\|\sum_{r=1}^{2^i}(-1)^r\varepsilon\|s-t\|_p\chi_{r,i,k+1} - \sum_{r=1}^{2^j}(-1)^r\varepsilon\|s-t\|_p\chi_{r,j,k+1}\right\|_p^p \\ &= \varepsilon^p\|s-t\|_p^p \left\|\sum_{r=1}^{2^i}\sum_{\ell=(r-1)2^{j-i}+1}^{r2^{j-i}} \left((-1)^r-(-1)^\ell\right)\chi_{\ell,j,k+1}\right\|_p^p \\ &= \varepsilon^p\|s-t\|_p^p \left(\sum_{r=1}^{2^i}\sum_{\ell=(r-1)2^{j-i}+1}^{r2^{j-i}} \int_{k+1+\frac{\ell-1}{2^j}}^{k+1+\frac{\ell}{2^j}} \left|(-1)^r-(-1)^\ell\right|^p dx\right) \\ &= \varepsilon^p\|s-t\|_p^p \left(\sum_{r=1}^{2^i}\frac{2^{j-i}}{2}\cdot 2^{-j}\cdot 2^p\right) \\ &= \varepsilon^p\|s-t\|_p^p\cdot\frac{1}{2}\cdot 2^p \\ &= 2^{p-1}\varepsilon^p\|s-t\|_p^p\end{aligned}$$

□

Next, we prove the contraction principle that is needed in the asymptotic setting.

Lemma 7. *Let $\varepsilon > 0$ such that $(1+(2\varepsilon)^p)^{1/p} \leqslant 2$ and let κ be an infinite cardinality. Assume that $\mathscr{D}_k^\kappa(\varepsilon, p)$ is a (ε, p)-thin κ-branching k-diamond substructure in $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ and that $f\colon \mathsf{X} \to (\mathfrak{Y}, \|\cdot\|)$ is a bi-Lipschitz embedding with distortion D. Then for every $1 \leqslant \ell \leqslant k$, if $\{s, \{m_i\}_{i\in I}, t\} \subset \mathscr{D}_k^\kappa$ is indexed by a copy of one of the diamond graph D_1^κ created at step ℓ, we have:*

$$\|f(s) - f(t)\| \leqslant D\mathsf{d}_\mathsf{X}(s,t)(1+(2\varepsilon)^p)^{1/p}\left(1 - \frac{1}{5}\tilde{\delta}_\mathfrak{Y}\left(\frac{\varepsilon}{16D}\right)\right). \tag{3.27}$$

Proof. Assume that for all $x, y \in \mathsf{X}$

$$\mathsf{d}_\mathsf{X}(x,y) \leqslant \|f(x) - f(y)\| \leqslant D\mathsf{d}_\mathsf{X}(x,y). \tag{3.28}$$

We claim that there exists $j \in \mathbb{N}$ such that

$$f(m_j) \notin \mathrm{Mid}\left(f(s), f(t), \frac{1}{4}\tilde{\delta}_\mathfrak{Y}\left(\frac{\varepsilon}{16D}\right)\right). \tag{3.29}$$

Assuming for a moment that (3.29) holds, then we have either

$$\|f(m_j) - f(t)\| > \frac{1}{2}\left(1 + \frac{1}{4}\tilde{\delta}_\mathfrak{Y}\left(\frac{\varepsilon}{16D}\right)\right)\|f(s) - f(t)\|$$

or

$$\|f(m_j) - f(s)\| > \frac{1}{2}\left(1 + \frac{1}{4}\tilde{\delta}_\mathfrak{Y}\left(\frac{\varepsilon}{16D}\right)\right)\|f(s) - f(t)\|.$$

In both cases it follows from (3.28) and condition (d_1) above that

$$\begin{aligned}\|f(s) - f(t)\| &< D\mathsf{d}_\mathsf{X}(s,t)(1+(2\varepsilon)^p)^{1/p}\left(1 + \frac{1}{4}\tilde{\delta}_X\left(\frac{\varepsilon}{16D}\right)\right)^{-1}\\ &\leqslant D\mathsf{d}_\mathsf{X}(s,t)(1+(2\varepsilon)^p)^{1/p}\left(1 - \frac{1}{5}\tilde{\delta}_X\left(\frac{\varepsilon}{16D}\right)\right).\end{aligned}$$

It remains to prove (3.29). By Lemma 5 there exists a finite subset $S := \{z_1, \ldots, z_n\} \subset \mathfrak{Y}$ such that

$$\mathrm{Mid}\left(f(s),f(t),\frac{1}{4}\tilde{\delta}_{\mathfrak{Y}}\left(\frac{\varepsilon}{16D}\right)\right)\subset S+\frac{\varepsilon}{8D}\|f(s)-f(t)\|B_{\mathfrak{Y}}.$$

If for every $i\in\mathbb{N}$,

$$f(m_i)\in \mathrm{Mid}\left(f(s),f(t),\frac{1}{4}\tilde{\delta}_{\mathfrak{Y}}\left(\frac{\varepsilon}{16D}\right)\right),$$

then $f(m_i)=z_{n_i}+y_i$ with $z_{n_i}\in S$ and $y_i\in\mathfrak{Y}$ so that

$$\|y_i\|\leqslant\frac{\varepsilon}{8D}\|f(s)-f(t)\|.$$

Therefore, for all $i\neq j$,

$$\begin{aligned}
\|z_{n_i}-z_{n_j}\| &\geqslant \|f(m_i)-f(m_j)\|-\|y_i-y_j\| \\
&\geqslant \mathsf{d}_{\mathsf{X}}(m_i,m_j)-\frac{\varepsilon}{4D}\|f(s)-f(t)\| \\
&\geqslant \mathsf{d}_{\mathsf{X}}(m_i,m_j)-\frac{\varepsilon}{4D}(\|f(s)-f(m_i)\|+\|f(m_i)-f(t)\|) \\
&\geqslant 2^{1-1/p}\varepsilon\cdot\mathsf{d}_{\mathsf{X}}(s,t)-\frac{\varepsilon}{4}(1+(2\varepsilon)^p)^{1/p}\mathsf{d}_{\mathsf{X}}(s,t) \\
&\geqslant 2^{1-1/p}\varepsilon\cdot\mathsf{d}_{\mathsf{X}}(s,t)-\frac{\varepsilon}{2}\mathsf{d}_{\mathsf{X}}(s,t) \\
&\geqslant \frac{1}{2}\varepsilon\cdot\mathsf{d}_{\mathsf{X}}(s,t)>0,
\end{aligned}$$

which contradicts the fact that S is finite. $\square$

Since in the proof of Lemma 7 we were careful to only use the upper bound in (3.28) for pairs of points of the form $\{s,m_i\}$ or $\{t,m_i\}$, the derivation of Theorem 6 below from Lemma 7 is by now standard and thus omitted.

Theorem 6. *Let $1\leqslant p<q$ and assume that $(\mathsf{X},\mathsf{d}_{\mathsf{X}})$ admits a bi-Lipschitz embedding with distortion D into a p-asymptotically midpoint uniformly convex Banach space $\mathfrak{Y}$. There exists $\varepsilon:=\varepsilon(p,q,D,\mathfrak{Y})>0$ such that if X admits a (ε,q)-thin $\aleph_0$-branching k-diamond substructure then*

$D = \Omega(k^{1/p-1/q})$.

The following consequence is immediate.

Corollary 10. *$L_q[0,1]$ does not bi-Lipschitzly embed into any p-asymptotically midpoint uniformly convex Banach space if $q > p \geqslant 1$. In particular, $L_q[0,1]$ does not bi-Lipschitzly embed into ℓ_p if $q > p \geqslant 1$.*

Remark 5. *Corollary 10 is not new since it can be shown using classical approximate midpoint techniques (see* [BL00, Chapter 10, Section 2] *or* [KR08] *for instance). The classical approximate midpoint technique provides an obstruction of qualitative nature and relies on some linear arguments but it can handle weaker notions of embeddings. Our proof of Theorem 6, and in turn of Corollary 10, identifies concrete and purely metric structures that provide quantitative obstructions to bi-Lipschitz embeddings.*

4. THERE IS NO EQUIVARIANT COARSE EMBEDDING OF L_p INTO ℓ_p

In this section we focus on investigating the existence of equivariant coarse embeddings of L_p into ℓ_p, this is based on the result proved in [Świ20]. As explained before the problem that we would like to attack is the existence of coarse embedding of L_p into ℓ_p but this provides too many technical obstacles. Because of this, we focus our attention on the equivariant category, and by the end of the section, we provide an idea of how one can use our results to attack the general problem.

It is important to establish that in this section whenever we mention ℓ_p we will mean $\ell_p(\mathbb{Z})$. The choice to make integers the indexing set is made for the readability of some proofs. Note that as stated in section 2 all ℓ_p spaces indexed by a countable, infinite set are isometric.

We begin by introducing the symmetric group of a set.

Definition 61. *For a set X we denote by $Sym(X)$ the group of all bijections of X, with composition as a group multiplication and the identity map as the trivial element.*

In the special case when X is a subset of integers it is common to refer to $Sym(X)$ as a permutation group. If X is of cardinality n we denote $Sym(X)$ by $Sym(n)$. Elements of this group are often represented using the cycle notation, i.e. a group element σ is written as a collection of orbits of elements $x \in X$: $(x, \sigma(x), \sigma(\sigma(x)), \dots)$ with omission of trivial orbits. For example $(123)(45) \in Symm(6)$ denotes a permutation that sends 1 to 2, 2 to 3, 3 to 1, 4 to 5, 5 to 4 and 6 to 6 .

We now generalize the notion of roots of the n-th degree to a group theory setting:

Definition 62. *We say that a group element $g \in G$ is a n-th root of a group element $h \in G$ if $g^n = h$.*

Observe that permutation $(1234)(56)$ and (1234) both give the same group element $(13)(24)$ when applied twice. That illustrates that root elements in $Sym(\mathbb{Z})$ are not unique.

Group homomorphisms preserve the property of being a root of degree n.

Lemma 63. *If h is a n-th root of a group element $g \in G$ and f is a group homomorphism then $f(h)$ is a root of degree n of the element $f(g)$.*

Note that there is a basic connection between the support of an element $g \in G$ and the supports of all its roots, namely.

Lemma 64. *Let a group G act on a set X. Let $g, h \in G$ such that $h^n = g$ for some natural number n. Then $supp(g) \subseteq supp(h)$.*

Proof. Observe that if a group element $h \in G$ fixes $x \in X$ then h^n also fixes x for any natural number n, hence $supp(h^n) \subseteq supp(h)$, but by our assumption was that $h^n = g$ for some $n \in \mathbb{N}$. $\square$

We now focus on studying root elements of $Symm(\mathbb{Z})$ specifically. First let us return to the example of $(1234)(56)$ and (1234) both being square roots of $(13)(24)$, What makes (1234) special is the fact that they have the same support as $(13)(24)$. The following lemma shows that if an element σ of $Sym(\mathbb{Z})$ has a root of nth degree, there always exists a root of the same degree with the same support as σ.

Lemma 65. *Let $Sym(\mathbb{Z})$ denote the group of bijections of integers. If $\sigma \in Sym(\mathbb{Z})$ has a nth root δ. Then there exists a unique nth root γ withe the exactly same support, meaning $supp(\gamma) = supp(\sigma)$. We will denote γ by $\sqrt[n]{\sigma}$.*

Proof. Let $k, l \in supp(\sigma)$ s.t $\sigma(k) = l$ and $m \in supp(\delta) - supp(\sigma)$. Assume that $\delta(k) = m$. Notice that σ and δ commute, since σ belongs to a cyclic subgroup generated by δ. Thus $\sigma(\delta(k)) = \sigma(m) = m$ and $\delta(\sigma(k)) = \delta(l)$ should be equal. But δ is an isomorphism sending k to m, so it can not send l to m as well. Contradiction.

We just showed that for any $k \in supp(\sigma)$, $\delta(k)$ also belongs to $supp(\sigma)$. It mean that all cycles in a cycle decomposition of δ contain either only elements of $supp(\sigma)$ or only those from $supp(\delta) - supp(\sigma)$. After removing later cycles from δ we obtain $supp(\sqrt[n]{\sigma})$. $\square$

Now we're ready to state and prove the key lemma of our theorem. We're gonna study possible homomorphisms from a vector space $(V, +)$, viewed as an abelian group under addition, into $Sym(\mathbb{Z})$. The key observation here is the fact that $Sym(\mathbb{Z})$ has relatively few roots, whereas $(V, +)$ viewed as an abelian group under addition has a lot of them, namely for any $v \in V$ and any natural number n, there exists a root of order n: $\frac{v}{n}$. Since group homomorphism sends n-th root to a n-th root that allows us to reach the following:

Lemma 66. *Let $(V,+)$ be a vector space viewed as an abelian group under addition. Then every homomorphism $\sigma : (V,+) \to Sym(\mathbb{Z})$ is trivial.*

Proof. For any $v \in V$ and natural number n we have $\sigma(\frac{v}{n})^n = \sigma(v)$, hence by Lemma 65 $\sqrt[n]{\sigma(v)}$ always exist. We will show that $\sigma(v) = e$. Consider two cases.

First assume that all elements of $supp(\sigma(v))$ have a finite orbit. Let $k \in supp(\sigma(v))$ and its orbit consist of n integers. Then there exists $\sqrt[n!]{\sigma(v)}$, which sends this n-tuple of integers to itself. Order of all elements of S_n is divides $n!$, so $\sqrt[n!]{\sigma(v)}^{n!} = \sigma(v)$ acts on that n-tuple trivially. Thus $k \notin supp(\sigma(v))$, contradiction.

Now let $k \in supp(\sigma(v))$ have an unbounded orbit. By Lemma 65 $\sqrt{\sigma(v)}$ sends k to some $l \neq k$, which belongs to the orbit of k under $\sigma(v)$. It follows that $\sqrt{\sigma(v)}(l) = \sigma(v)(k)$. We claim that $\sqrt{\sigma(v)}(\sigma(v)^i(l)) = \sigma(v)^{i+1}(k)$ for any integer i. Case $i = 0$ is our basis of induction. Assume $i > 0$ and $\sqrt{\sigma(v)}(\sigma(v)^i(l)) = \sigma(v)^{i+1}(k)$. Then $\sigma(v)(\sqrt{\sigma(v)}(\sigma(v)^i(l))) = \sigma(v)(\sigma(v)^{i+1}(k)) = \sigma(v)^{i+2}(k)$ is equal to $\sqrt{\sigma(v)}(\sigma(v)(\sigma(v)^i(l))) = \sqrt{\sigma(v)}(\sigma(v)^{i+1}(l)))$, proving inductive step. Similarly if $i < 0$ and $\sqrt{\sigma(v)}(\sigma(v)^i(l)) = \sigma(v)^{i+1}(k)$. Then $\sigma(v)(\sqrt{\sigma(v)}(\sigma(v)^{i-1}(l)))$ is equal to $\sqrt{\sigma(v)}(\sigma(v)(\sigma(v)^{i-1}(l))) = \sqrt{\sigma(v)}(\sigma(v)^i(l)))$, finishing the proof of the claim. Since $k = \sigma(v)^i(l)$ for some $i \neq 0$ we have $\sqrt{\sigma(v)}(k) = \sqrt{\sigma(v)}\sigma(v)^i(l) = \sigma(v)^i(k)$, which gives us that i is an integer such that $2i = 1$. Contradiction with the assumption that $\sqrt{\sigma(v)}(k) = l$. Thus $\sigma(v)$ can not be of this form. □

It is time to tie all of the above to Banach spaces now. We recall the classical Banach-Lamperti theorem that characterizes isometries of ℓ_p.

Theorem 67. *Every linear isometry of ℓ_p, for $1 \leqslant p \leqslant \infty, p \neq 2$, is of the form $\{x_i\}_{i\in\mathbb{Z}} \to \{\varepsilon_i x_{\sigma(i)}\}_{i\in\mathbb{Z}}$, where ε_i is 1 or -1 for all i and σ is a permutation of the indexing set $\mathbb{Z}$.*

Notice that the full isometry group is generated by permutation elements (i.e. $Sym(\mathbb{Z})$) together with the change of signs elements (since multiplying by -1 twice gives us 1 that means the group corresponding to a fixed index is $\mathbb{Z}_2$, but since the indexing set is $\mathbb{Z}$ the group generated by those elements is $\mathbb{Z}_2^{\mathbb{Z}}$). Further more observe that $Sym(\mathbb{Z})$ is a normal subgroup and it has a trivial intersection with $\mathbb{Z}_2^{\mathbb{Z}}$ thus we can conclude that $Isom(\ell_p) \cong \mathbb{Z}_2^{\mathbb{Z}} \rtimes Sym(\mathbb{Z})$.

With that in mind we are able to study possible homomorphisms from a vector space V into $Isom(\ell_p)$:

Theorem 68. *All representations $\pi : (V,+) \to Isom(\ell_p)$ are trivial for $p \neq 2$.*

Proof. Recall that the existence of the following short exact sequence follows from the fact that for $p \neq 2$ the isometry group $Isom(\ell_p)$ is ismorphic to $\mathbb{Z}_2^{\mathbb{Z}} \rtimes Sym(\mathbb{Z})$

$$1 \to \mathbb{Z}_2^{\mathbb{Z}} \xrightarrow{i} Isom(\ell_p) \xrightarrow{p} Sym(\mathbb{Z}) \to 1.$$

By Lemma 66 homomorphisms $p \circ \pi : (V,+) \to Sym(\mathbb{Z})$ is trivial, so $\pi(V) \leqslant ker(p) \cong im(i) \cong \mathbb{Z}_2^{\mathbb{Z}}$. But since $\mathbb{Z}_2^{\mathbb{Z}}$ is a torsion group for every $v \in V$ $\pi(v)^2 = e$ thus we conclude that π needs to be trivial. □

Having established that the only representation of $(V,+)$ in $Isom(\ell_p)$ is the trivial one we finally shift our attention to examining the possible proper, affine, isometric actions. The cocycle condition together with some basic calculations yields the following:

Theorem 69. *Every normed vector space $(V,+)$ admitting a proper, affine, isometric action on ℓ_p is bi-Lipschitz embeddable into ℓ_p.*

Proof. By Theorem 68 linear representation of V is trivial. The cocycle condition 2.3 gives us then the existence of an additive, coarse embedding $A : V \to l_p$ i.e. $\rho_-(\|v\|_V) \leqslant \|A(v)\|_{\ell_p} \leqslant \rho_+(\|v\|_V)$. Now let $\alpha \in \mathbb{R}_+$ be such that $\rho_-(\alpha) > 0$.

There exists $n \in \mathbb{N}$ such that $2^n\alpha \leqslant \|v\| \leqslant 2^{n+1}\alpha$. Then $\|A(v)\| = 2^n \|\frac{A(v)}{2^n}\| \geqslant 2^n \rho_-(\alpha) \geqslant \frac{\rho_-(\alpha)}{2\alpha}\|v\|$. Similarly $\|A(v)\| = 2^n\|\frac{A(v)}{2^n}\| \leqslant 2^n\rho_+(2\alpha) \leqslant \frac{\rho_+(2\alpha)}{\alpha}\|v\|$. □

As we have established before L_p does not bi-Lipschitzly embeds into ℓ_p, hence we can conclude the following.

Corollary 70. *There is no proper, affine, isometric action of L_p on ℓ_p for $p \neq 2$.*

In other words, is no equivariant coarse embedding of L_p into ℓ_p. Finally, let us mention a result by Cornulier, Tessera, and Valette from [CTV07] here:

Theorem 71. *Let G be a locally compact, compactly generated, amenable group. If G coarsely embeds into the Hilbert space, then there exists a proper, affine, isometric action of G on the Hilbert space.*

The result states that the existence of a coarse embedding into a Hilbert space guarantees the existence of an equivariant coarse embedding under the conditions stated above. We hope that one can mimic techniques used in the proof to show that the same is true for coarse embeddings from L_p into ℓ_p locally compact, compactly generated topological group, it is abelian which means it is 'as amenable as possible. If one can carry on with research in this direction, the results proved in this section can be used for settling the question of the existence of coarse embeddings from L_p into ℓ_p.

5. SUMMARY

In Section 3 we provided new results about bi-Lipschitz embeddings and their distortions, that relate to the question by Lang-Plaut for sufficient conditions to admit an embedding into a finite dimensional space.

Problem 1. *Does a doubling subset of ℓ_2 admit a bi-Lipschitz embedding into a constant dimensional Euclidean space?*

Instead of studying the subsets of ℓ_2 we focused on different ambient spaces. We presented a new proof of the following result using thin Laakso substructures and a self improvment argument.

Theorem 1. *For every $q \in (2,\infty)$, there exists a doubling subset of ℓ_q that does not admit any bi-Lipschitz embedding into $\mathbb{R}^d$ for any $d \in \mathbb{N}$.*

We were also able to obtain optimal bounds for distortion of an embedding of ℓ_q into a p-uniformly convex Banach space, namely.

Corollary 8. *Let $q \in (2,\infty)$, $\mathfrak{Y}$ be a Banach space, and fix $D \geqslant 1$. For every $n \in \mathbb{N}$ there exists an n-point subset $\mathscr{L}_n(q,D,\mathfrak{Y})$ of ℓ_q that is 2^{32}-doubling and such that any bi-Lipschitz embedding with distortion D into $\mathfrak{Y}$ must incur*

1. $D = \Omega\Big((\log n)^{\frac{1}{p}-\frac{1}{q}}\Big)$ *if $p \in [2,q)$ and $\mathfrak{Y}$ is a p-uniformly convex Banach space*

   and

2. $D = \Omega\Big(\Big(\frac{\log n}{d}\Big)^{\frac{1}{2}-\frac{1}{q}}\Big)$ *if $\mathfrak{Y} = \ell_q^d$*

We also discussed (see Remark 2) possible future applications of our construction to a problem by Kadec and Pełczyński.

Problem 2. *If a metric space X admits a bi-Lipschitz embedding into L_p and into L_q does it also embed into L_r for $1 \leqslant p < r < q < \infty$?*

We were able to adapt our methods to obtain results for distortion into non-positively curved spaces (or more generally spaces with roundness 2).

Corollary 9. *If $q > 2$ and $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ is a metric space with roundness 2, in particular an Alexandrov space of non-positive curvature, then*

$$\mathsf{c}_\mathsf{Y}(\ell_q^k) = \Omega\left(k^{\frac{1}{2}-\frac{1}{q}}\right).$$

Finally we changed our construction and using thin κ-branching diamond substructure, we were able to obtain a new proof a classical result.

Corollary 10. *$L_q[0,1]$ does not bi-Lipschitzly embed into any p-asymptotically midpoint uniformly convex Banach space if $q > p \geqslant 1$. In particular, $L_q[0,1]$ does not bi-Lipschitzly embed into ℓ_p if $q > p \geqslant 1$.*

In Section 4 we focused on coarse embeddings into Banach spaces and their more rigid version called equivariant coarse embeddings. Using the group theoretic properties of the isometry group of ℓ_p, we were able to show the following.

Theorem 69. *Every normed vector space $(V, +)$ admitting a proper, affine, isometric action on ℓ_p is bi-Lipschitz embeddable into ℓ_p.*

Using bi-Lipschitz theory it follows that there is no equivariant coarse embedding of L_p into ℓ_p (see Corollary 70) We remarked that one can hope to use our result together with techniques by Cornulier, Tessera, and Valette from [CTV07] to attack the following open problem.

Problem 3. *Does L_p coarsely embed into ℓ_p for $p \in (2, \infty)$?*

REFERENCES

[AK16] F. Albiac, N. Kalton, Topics in Banach Space Theory 2nd edition, Graduate Texts in Mathematics, Springer 2016

[Alo03] Noga Alon, *Problems and results in extremal combinatorics. I*, 2003, pp. 31–53. EuroComb'01 (Barcelona). MR2025940

[ANT13] Tim Austin, Assaf Naor, and Romain Tessera, *Sharp quantitative nonembeddability of the Heisenberg group into superreflexive Banach spaces*, Groups, Geometry, and Dynamics **7** (2013), no. 3, 497–522.

[Ban31] Stefan Banach, Teoria operacji liniowych, Wydawnictwo Kasy Im. Miankowskiego Instytutu Popoularyzowania Nauki, Warszawa 1931

[Bal90] Keith Ball, *Isometric embedding in l_p-spaces*, European J. Combin. **11** (1990), no. 4, 305–311. MR1067200

[BT24] S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fund. math. **6** (1924), 244–277.

[BSŚ] Florent Baudier, Andrew Swift, and Krzysztof Święcicki, *No dimension reduction for doubling subsets of ℓ_q when $q > 2$ revisited*, accepted to Journal of Mathematical Analysis and Applications, available at `arXiv:2103.05080`.

[BCD$^+$17] Florent Baudier, Ryan Causey, Stephen Dilworth, Denka Kutzarova, Nirina L. Randrianarivony, Thomas Schlumprecht, and Sheng Zhang, *On the geometry of the countably branching diamond graphs*, J. Funct. Anal. **273** (2017), no. 10, 3150–3199. MR3695891

[BZ16] Florent P. Baudier and Sheng Zhang, *(β)-distortion of some infinite graphs*, J. Lond. Math. Soc. (2) **93** (2016), no. 2, 481–501. MR3483124

[BGN14] Yair Bartal, Lee-Ad Gottlieb, and Ofer Neiman, *On the impossibility of dimension reduction for doubling subsets of ℓ_p*, Computational geometry (SoCG'14), 2014, pp. 60–66. MR3382276

[BGN15] ———, *On the impossibility of dimension reduction for doubling subsets of ℓ_p*, SIAM J. Discrete Math. **29** (2015), no. 3, 1207–1222. MR3369992

[BL00] Yoav Benyamini and Joram Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000. MR1727673

[BN07] I. D. Berg and I. G. Nikolaev, *On a distance characterization of A. D. Aleksandrov spaces of nonpositive curvature*, Dokl. Akad. Nauk **414** (2007), no. 1, 10–12. MR2447040

[BN08] ______, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata **133** (2008), 195–218. MR2390077

[Bou86] J. Bourgain, *The metrical interpretation of superreflexivity in Banach spaces*, Israel J. Math. **56** (1986), 222–230.

[BC05] Bo Brinkman and Moses Charikar, *On the impossibility of dimension reduction in l_1*, J. ACM **52** (2005), no. 5, 766–788. MR2176562

[CKN11] Jeff Cheeger, Bruce Kleiner, and Assaf Naor, *Compression bounds for Lipschitz maps from the Heisenberg group to L_1*, Acta Math. **207** (2011), no. 2, 291–373. MR2892612

[CTV07] Y. de Cornulier, R. Tessera, A. Valette, Isometric group actions on Hilbert spaces:growth of cocycles, A. GAFA, Geom. funct. anal. (2007) 17: 770

[DKLR$^+$13] S. J. Dilworth, Denka Kutzarova, N. Lovasoa Randrianarivony, J. P. Revalski, and N. V. Zhivkov, *Compactly uniformly convex spaces and property (β) of Rolewicz*, J. Math. Anal. Appl. **402** (2013), no. 1, 297–307. MR3023259

[DP40] N. Dunford and B.J. Pettis, *Linear operations on summable functions*, Trans. Am. Math. Soc. **47** (1940), 323–392. MR2541760 (2010k:52031)

[EMN19] Alexandros Eskenazis, Manor Mendel, and Assaf Naor, *Nonpositive curvature is not coarsely universal*, Invent. Math. **217** (2019), no. 3, 833–886. MR3989255

[GM93] Gromov M., *Geometric group theory, vol. 2: Asymptotic invariants of infinite groups*, London Math. Soc. Lecture Note Ser., Cambridge University Press **182** (1993).

[GM81] ______, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73.

[HM82] S. Heinrich and P. Mankiewicz, *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math. **73** (1982), no. 3, 225–251. MR675426

[JL84] William B. Johnson and Joram Lindenstrauss, *Extensions of Lipschitz mappings into a Hilbert space*, Conference in modern analysis and probability (New Haven, Conn., 1982), 1984, pp. 189–206. MR737400

[JS09] W. B. Johnson and G. Schechtman, *Diamond graphs and super-reflexivity*, J. Topol. Anal. **1** (2009), no. 2, 177–189. MR2541760 (2010k:52031)

[Kad67] M. I. Kadec, *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funkcional. Anal. i Priložen **1** (1967), 61–70.

[KP61] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1961/62), 161–176. MR152879

[KR08] N. Kalton, N. Randrianarivony, The coarse Lipschitz structure of $\ell_p \bigoplus ell_q$, Math. Ann. 341 (2008), 223–237

[KGYG] Kasparov G. and Yu G., *The coarse geometric Novikov conjecture and uniform convexity, date=2006, journal=Advances in Mathematics, volume=206, pages=1-56,.*

[KR08] Nigel J. Kalton and N. Lovasoa Randrianarivony, *The coarse Lipschitz geometry of $l_p \oplus l_q$*, Math. Ann. **341** (2008), no. 1, 223–237. MR2377476

[Klo14] Benoît R. Kloeckner, *Yet another short proof of Bourgain's distortion estimate for embedding of trees into uniformly convex Banach spaces*, Israel J. Math. **200** (2014), no. 1, 419–422. MR3219585

[Laa02] Tomi J. Laakso, *Plane with A_∞-weighted metric not bi-Lipschitz embeddable to $\mathbb{R}^N$*, Bull. London Math. Soc. **34** (2002), no. 6, 667–676. MR1924353

[LN04] J. R. Lee and A. Naor, *Embedding the diamond graph in L_p and dimension reduction in L_1*, Geom. Funct. Anal. **14** (2004), no. 4, 745–747. MR2084978

[LN14a] Vincent Lafforgue and Assaf Naor, *A doubling subset of L_p for $p > 2$ that is inherently infinite dimensional*, Geom. Dedicata **172** (2014), 387–398. MR3253787

[LN14b] ______, *Vertical versus horizontal Poincar\'e inequalities on the Heisenberg group*, Israel Journal of Mathematics **203** (2014), no. 1, 309 339.

[LN17] Kasper Green Larsen and Jelani Nelson, *Optimality of the Johnson-Lindenstrauss lemma*, 58th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2017, 2017, pp. 633–638. MR3734267

[Nor62] G. Nordlander, *On sign-independent and almost sign-independent convergence in normed linear spaces*, 1962, pp. 287–296.

[Orl33a] W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen I*, 1933, pp. 33–37.

[Orl33b] ______, *Über unbedingte Konvergenz in Funktionenräumen II*, 1933, pp. 41–47.

[LP01] Urs Lang and Conrad Plaut, *Bilipschitz embeddings of metric spaces into space forms*, Geom. Dedicata **87** (2001), no. 1-3, 285–307. MR1866853

[LR10] James R. Lee and Prasad Raghavendra, *Coarse differentiation and multi-flows in planar graphs*, Discrete Comput. Geom. **43** (2010), no. 2, 346–362. MR2579701

[Rib76] M. Ribe, *On uniformly homeomorphic normed spaces*, Ark. Mat. **14** (1976), 237–244.

[Man72] P. Mankiewicz, *On Lipschitz mappings between Fréchet spaces*, Studia Math. **41** (1972), 225–241. MR308724

[Mar70] G. Margulis, *The isometry of closed manifolds of constant negative curvature with the same fundamental group*, Dokl. Akad. Nauk SSSR **192** (1970), 736–737.

[NMM04] M. Mendel and A. Naor, Euclidean quotients of finite metric spaces, Adv. Math. 189 (2004), 451–494.

[Mos68] G. Mostov, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.

[NMM08] A. Naor and Mendel M., *Metric Cotype*, Annals of Mathematics **Volume 168** (2008), Pages 247-298.

[Nao18] Assaf Naor, *Metric dimension reduction: a snapshot of the Ribe program*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, 2018, pp. 759–837. MR3966745

[NY18] Assaf Naor and Robert Young, *Vertical perimeter versus horizontal perimeter*, Ann. of Math. (2) **188** (2018), no. 1, 171–279. MR3815462

[NY20] Assaf Naor and Robert Young, *Foliated corona decompositions*, arXiv e-prints (April 2020), available at `arXiv:2004.12522`.

[N06] P. Nowak, On coarse embeddability into lp-spaces and a conjecture of Dranishnikov, Fund. Math. 189 (2006), no. 2, 111–116.

[NY12] Piotr Nowak, Guoliang Yu, Large Scale Geometry, European Mathematical Society Publishing House, 2012

[MU32] S. Mazur, S. Ulam, Sur les transformations isométriques d'espaces vectoriels normés. C. R. Acad. Sci. Paris 194, 946–948 (1932)

[Pis75] Gilles Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), no. 3-4, 326–350. MR394135

[Pit32] H.R. Pitt, *A note on bilinear forms*, J. Lond. Math. Soc **11** (1932), no. 174–180.

[Sat09] Takashi Sato, *An alternative proof of Berg and Nikolaev's characterization of* $\mathrm{CAT}(0)$*-spaces via quadrilateral inequality*, Arch. Math. (Basel) **93** (2009), no. 5, 487–490. MR2563595

[Swi18] A. Swift, *A coding of bundle graphs and their embeddings into Banach spaces*, Mathematika **64** (2018), no. 3, 847–874. MR3867323

[Świ20] Krzysztof Święcicki, *There is no equivariant coarse embedding of* L_p *into* ℓ_p, arXiv e-prints (2020), available at `arXiv:2012.00097`.

[Tor81] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. **111(3)** (1981), 247–262.

[VNJ29] Von Neumann J., *Zur allgemeinen Theorie des Maßes*, Fundam. Math. **13** (1929), 73–116.

[Yu00] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139(1)** (2000), 201–240.

[Zha21] Sheng Zhang, *A submetric characterization of Rolewicz's property (β)*, arXiv e-prints (January 2021), available at `arXiv:2101.08707`.