THE RELATIVE KÜNNETH THEOREM

A Dissertation

by

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Submitted to the Graduate and Professional School of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2021

Major Subject: Mathematics

ABSTRACT

Let A be a unital ring and B a unital subring. In this dissertation we study the relative homological algebra arising from the pair (A, B). We introduce relative analogues of free, projective, and flat modules, and we show in which sense they generalize their absolute analogues. We systematically characterize these modules in terms of relative free modules, which play a key role in this exposition.

We introduce a section of the connecting homomorphism in the associated long exact sequence to a short exact sequence. We prove that if our original short exact sequence splits, then the associated long exact sequence also splits. We use this to prove that the expected long exact sequences of relative Tor are split. Finally, we use the splitting long exact sequences of Tor to prove a relative version of the Künneth Theorem, where the resultant short exact sequences are split.

DEDICATION

To the perseverant.

> Ever tried. Ever failed. No matter. Try again. Fail again. Fail better.
>
> — Samuel Beckett

ACKNOWLEDGMENTS

I thank my advisor, Sarah J. Witherspoon, for encouraging and supporting me over the years. I also thank Cris Negron, Julia Plavnik, and Chelsea Walton for their mathematical mentoring.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was completed by the student independently.

Funding Sources

The author was partially supported by a Fulbright scholarship. The author was partially supported by NSF grant 1665286, P.I. Sarah Witherspoon, and by NSF grant 2001163, P.I. Sarah Witherspoon. The author was partially supported by NSF grant 1440140, P.I. David Eisenbud and Co-P.I. Helene Barcelo, while in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester.

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1. INTRODUCTION

Among the celebrated unifying theories of mathematics we find homology, arising when Eilenberg and Mac Lane realized that they were doing the same computations in algebra and topology. *Homological algebra* is the product of their search for a streamlined approach to the similarities they found in their respective areas. The insight of homological algebra resided in the realization that one does not need to know much about a specific object, let that be a topological space, a group, or other, to understand it. Instead, information can be obtained through related structures where the object of interest acts. There are several ways of exploiting this idea, a powerful one being the computation and study of the structure of *(co)homology* groups. The construction of these (co)homology groups rely on notions measuring how well behaved an action is, some of the most important ones being *free*, *projective*, *flat*, or *injective*.

The introduction of substructures to the homological constructions yielded what is known as relative homology. While this was implemented fairly straightforwardly and is widely used in topology, it only has sporadic applications in algebra, arguably the most notable of these being Brauer's results concerning representations of finite groups as showcased by Alperin [1]. The formal treatment of *relative homological algebra* has a categorical viewpoint embodied in the foundational work of Eilenberg and Moore [9], whose approach allows them to construct a relative setup through a pair of adjoint functors. This work was facilitated by Buchsbaum's [6] generalizing of the absolute setup to what we now call abelian categories (Buchsbaum's calls them exact categories), as well as Heller's [21] theory of proper morphisms in an additive category. Of course, the framework of derived categories by Gabriel [13] and contributions by Sklyarenko [34] also played an important role in the development of the area. Recently the seminal work of Enochs and Jenda [10, 11] applies this language to commutative algebra, obtaining classical results over certain Cohen-Macaulay rings which include Gorenstein local rings, and extends it to some non-commutative settings. However, the exposition often remains either quite general or very specific, and finding a practical approach for the working mathematician in the literature remains elusive.

The inception of the relative setup by Hochschild [24] and of the clarity of Mac Lane's [27] contextualization in relative abelian categories are not as general as the aforementioned constructions. Nonetheless, since they present a much friendlier perspective of the topic, that will be the one used in this dissertation. In addition, this viewpoint has the advantage of yielding results stronger than in the more general sense, which will be crucial to our interests. These techniques native to the relative setup have been used abundantly, albeit mostly targeted to deal with specific difficulties in an argument concerning the absolute setup. Examples of such uses are approaches to understand the deformations of homomorphisms of Lie groups and algebras [30], some studies of modules over an Artin algebra [2], the establishment of the finite-generation of some cohomology rings of classical Lie superalgebras [3, 4, 29], and a description of the changes of the Hochschild (co)homology of a bound quiver algebra in terms of addition and deletion of arrows to the quiver [7, 8]. Among the notable exceptions that dealt with the formalism itself of the relative setup are the description of the relative Ext groups as n-fold extensions (mimicking the result for the absolute Ext groups) [6], the construction of an isomorphism that explicitly describes the correspondence between split long exact sequences and cocycles in relative Hochschild cohomology [5], and under mild hypotheses the construction of long exact sequences relating the relative Hochschild (co)homology of a pair of algebras with the Hochschild (co)homology of each of the algebras [25].

The main motivation of this dissertation is to construct the necessary tools to further the understanding of Hochschild cohomology [23], with a view towards extending the deformation theory of algebras [14, 15, 16, 17, 17, 18, 19, 20] to the relative setup. There is a well known interplay between the absolute and the relative setup in the cohomology of associative algebras, see for example Weibel's book [35], and several of the works above would benefit from a systematic treatment of the background material. In particular, we are interested in a number of equivalent constructions of the ring structure of relative Hochschild cohomology [31].

1.1 Overview of the contents

In this dissertation we aim to provide a clear and concise exposition to the theory of relative homological algebra for categories of modules over a ring with unit relative to a subring with unit, including formalism and tools that are currently missing in the literature. We do not attempt to prove the most general versions of the results that we exhibit, but the version that can be of most practical use for our goals. This has the twofold advantage that we can reason in categorical terms when these illuminate the path, whereas we can occasionally allow ourselves to reason in more specific terms when convenient.

This being said, many of the references used and given deal with some of the broader setups, and indeed the reader may notice that often even the new results presented here seem to hold in a more general context. When appropriate, we will be stating in which other settings the results are expected to hold, in which settings a proof is in progress, and when known a reference will be provided. However, we warn the reader to curb their enthusiasm since the main results use specificities of the categories of modules over a ring, and hence the tools presented here may be unfit to generalize them.

In Chapter 2 we present several usual constructions in homological algebra, emphasizing properties of split exact sequences because of the crucial role they will play throughout this dissertation. For example, we prove in excruciating detail that a chain complex of modules over a ring is split exact if and only if the identity chain map is null homotopic (see Proposition 2.16). This will be referenced constantly to obtain information about the sequences of modules arising in the relative setup. We also present free, projective, injective, and flat modules, as well as many of their properties and characterizations. Many proofs are included because of the role they will play in the upcoming relative setup. In particular, the characterization of free modules in terms of a universal property (see Proposition 2.23) justifies much of our posterior nomenclature, and showcases how the absolute setup benefits from the relative one. We also include complete proofs of several known benefits of working with flat modules (see Proposition 2.32 and Proposition 2.43), since the techniques provide a taste of what will come.

Additionally, we include the construction of the derived functors Ext and Tor. We also include the Comparison Theorem 2.36 and use it to conclude that not only Ext and Tor are well defined, but that they are functorial. The included proofs will be used in the dissertation to conclude analogous statements about their relative counterparts. This exposition ends with the statement of all four long exact sequences for Ext and Tor as well as the Künneth Theorem. Since we do not use their proofs, we instead refer the reader to the existing literature [22].

In Chapter 3 we present both the relative setup of interest as well as the main results of this dissertation. Proposition 3.2 is a most useful folklore result always stated, never proved, and constantly abused. By providing a complete proof we showcase some of the techniques that will be repeated and referenced throughout the remainder of the dissertation. Lemma 3.3 is a crucial translation of a well known in the absolute case to the relative setup. To prove this result we heavily rely on working over abelian categories, and this is the first instance where our less general context enables stronger conclusions.

We then proceed to introduce the novel concept of a free module in the relative setup in Definition 3.4. This nomenclature is justified in the following pages, culminating in the characterization given in Proposition 3.7. This extremely practical characterization explains the usefulness of free modules as well as the central role they play in our approach. While some of the properties of these modules were known and have been used in the literature, our contextualization as the relative analogue of a free module is new. The concepts of relative projective and injective modules are then presented. Several noteworthy features of interest should be mentioned. First, not only do we prove the expected result of relatively free modules being projective, but in contrast to the absolute case, our proof does not rely on the Axiom of Choice. Second, in Proposition 3.16 we prove the expected characterization of relative projective modules in terms of relative exact sequences, relative free modules, and more. Moreover, we added comprehensive proofs of the facts that there are enough relative projective and relative injective modules. We then introduce a new and improved definition of relative flat module, and include a systematic study of their properties. This includes the essential Remarks 3.22 and 3.23, showing a case where the techniques native to the relative setup are required. The expected relations between flat modules and projective and free modules are proven, and a wealth of examples are computed in complete detail. Our notion of relative flat module is *a priori* not only different from the one presented in [35], but also much stronger. It is remarkable that we are able to prove that they are equivalent. This is a key contribution to the subject, permitting the use of a wealth of techniques that yield much stronger results.

Finally, we present the constructions of relative Tor, the main object of interest, and Ext. All the results obtained rely solely on methods from relative homological algebra, which is an essential point for our perspective on the subject. We provide a complete proof of the Relative Comparison Theorem, that should be compared with the proof in the usual case. We also provide complete proofs of the facts that relative Tor and Ext are well defined and functorial in the appropriate sense. Additionally, we prove the Relative Horseshoe Lemma, which is surprisingly simple compared with the absolute case. Among the main results is a new splitting long exact sequence in homology of Theorem 3.37, arising as a consequence of the existence of a section of the connecting homomorphism Proposition 3.36. The insight of this splitting in homology yields a new splitting in another main result, the relative long exact sequence for Tor in Theorem 3.39, and its counterpart. We conclude with the main new result of this dissertation, and the reason for this work: the Relative Künneth Theorem. Again, the key observation is that we can carry on the splitting of our initial sequences to homology, enabling us to measure the exactness of a total complex in the relative setup.

1.2 Notation and conventions

For the rest of this dissertation, unless otherwise specified, we let A be a ring and k a field. We say that A is an *associative algebra* over k whenever it is a k-module and there are k-morphisms called *multiplication* $\mu_A : A \otimes_k A \to A$ and *unit* $\eta_A : k \to A$ such that the following diagrams commute.

$$\begin{tikzcd}
A \otimes_k A \otimes_k A \arrow[r, "1_A \otimes \mu_A"] \arrow[d, "\mu_A \otimes 1_A"'] & A \otimes_k A \arrow[d, "\mu_A"] \\
A \otimes_k A \arrow[r, "\mu_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes_k A \arrow[r, "\eta_A \otimes 1_A"] \arrow[d, "\cong"] & A \otimes_k A \arrow[d, "\mu_A"] & A \otimes_k k \arrow[l, "1_A \otimes \eta_A"'] \arrow[d, "\cong"] \\
A \arrow[r, "1_A"] & A & A \arrow[l, "1_A"']
\end{tikzcd}$$

Notice that this definition also applies whenever k is a commutative ring with unit 1_k. For $i \in \mathbb{N}$ we will denote $A^{\otimes_k i} = A \otimes_k \cdots \otimes_k A$ the tensor product over k of i copies of A.

Remark 1.1. We will use the Koszul sign convention [26] when dealing with tensor product of morphisms between chain complexes. Whenever $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$, $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$, $(\boldsymbol{R_\bullet}, \boldsymbol{r_\bullet})$, $(\boldsymbol{S_\bullet}, \boldsymbol{s_\bullet})$ are chain complexes of (left or right, as necessary) A-modules and $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q}[\boldsymbol{i}]_\bullet$, $g_\bullet : \boldsymbol{R_\bullet} \to \boldsymbol{S_\bullet}[j]$ are chain morphisms of A-modules for some fixed $i, j \in \mathbb{Z}$, we define $f \otimes g : \boldsymbol{P_\bullet} \otimes_A \boldsymbol{R_\bullet} \to \boldsymbol{Q_\bullet} \otimes_A \boldsymbol{S_\bullet}$ by

$$(f \otimes g)(x \otimes y) := (-1)^{|j||x|} f(x) \otimes g(y)$$

for all $x \in P_{|x|}$ and $y \in R_{|y|}$, where $|x|, |y| \in \mathbb{Z}$. We can see any morphism of A-modules $h : M \to N$ as a chain morphism between chain complexes concentrated in degree zero, and thus this convention also applies.

Given a map $f : M \to N$ between two algebraic structures, we denote its restriction to a substructure $L \subseteq M$ as $f_L : L \to N$. If the map is an isomorphism identifying M and N as equivalent algebraic structures, we denote this by $M \cong N$. A quotient of algebraic structures will be denoted by M/N. The elements in this quotient will be denoted by $\overline{m}$ for $m \in M$, which otherwise said are the equivalence classes of elements represented by m. Given $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q_\bullet}$ and $g_\bullet : \boldsymbol{R_\bullet} \to \boldsymbol{S_\bullet}$ chain morphisms of A-modules that are chain homotopic, we denote this by $f_\bullet \simeq g_\bullet$. Most of the algebraic structures considered in this dissertation are modules over a ring A. Because of this we will rarely use the term "abelian groups", we instead favor $\mathbb{Z}$-modules for consistency with the rest of the dissertation.

2. PRELIMINARIES OF HOMOLOGICAL ALGEBRA

In this chapter we will briefly recall some definitions, concepts, results, and ideas from homological algebra over categories of modules of rings that will be useful throughout this dissertation. Most, if not all, of this section can be found in [32, 35, 28]. We will exploit some explicit constructions that this setting allows and that may not be available in abelian categories. An account sufficient for most of our purposes can be found in [36, Chapter 2 and Appendix A].

Throughout the chapter we let A be an associative unitary ring. The modules over A will be left modules unless otherwise specified.

2.1 Sequences of modules

We now recall the basic concepts of homological algebra: concatenations of morphisms such that their composition is zero, and some of their behaviors.

Definition 2.1. Let $\{M_i\}_{i\in\mathbb{Z}}$ be a family of A-modules and $\{d_i : M_i \to M_{i-1}\}_{i\in\mathbb{Z}}$ be a family of A-morphisms. The composition

$$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is called a *sequence* of A-modules, and is denoted $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$. A sequence is called a *complex* of A-modules whenever $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. A sequence is called *exact* whenever $\operatorname{im}(d_i) = \ker(d_{i-1})$ for all $i \in \mathbb{Z}$. A sequence is *truncated* at M_i for a fixed $i \in \mathbb{Z}$ by replacing $d_{i+1} = 0$ and $M_j = 0$, $d_j = 0$ for all $j \leq i$, $j \in \mathbb{Z}$ in $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$, obtaining

$$\cdots \xrightarrow{d_{i+3}} M_{i+2} \xrightarrow{d_{i+2}} M_{i+1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

Definition 2.2. Let $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ be a sequence of A-modules. Fix $j \in \mathbb{Z}$, for each $i \in \mathbb{Z}$ we set $M[j] = M_{i+j}$ and $d[j] = d_{i+j}$. We call $(\boldsymbol{M[j]_\bullet}, \boldsymbol{d[j]_\bullet})$ a *shifted sequence* of $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$.

Some complexes permit bookkeeping particularly well and thus deserve special attention.

Definition 2.3. Let M, N be A-modules and $i \in \mathbb{N}$. An exact complex of A-modules $(\boldsymbol{L_\bullet}, \boldsymbol{d_\bullet})$ such that $L_{-1} = M$, $L_i = N$, and $L_j = 0$ for $j \notin \{-1, \ldots, i\}$ is called an *i-extension* of M by N.

$$0 \longrightarrow N \xrightarrow{d_i} L_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

A way of measuring the exactness of a complex is its homology.

Definition 2.4. Let $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ be a complex of A-modules. For each $i \in \mathbb{Z}$ we call M_i the *i-chains*, $Z_i(\boldsymbol{M_\bullet}) = \ker(d_i)$ the *i-cycles*, $B_i(\boldsymbol{M_\bullet}) = \operatorname{im}(d_{i+1})$ the *i-boundaries*, $H_i(\boldsymbol{M_\bullet}) = Z_i(\boldsymbol{M_\bullet})/B_i(\boldsymbol{M_\bullet})$ the *i-th homology*.

When it is clear by the context, we may omit the i or the complex $\boldsymbol{M_\bullet}$ when talking about chains, cycles, boundaries, and homology. Notice how the i-cycles $Z_i(\boldsymbol{M_\bullet})$, the i-boundaries $B_i(\boldsymbol{M_\bullet})$, and the i-th homology $H_i(\boldsymbol{M_\bullet})$ are all A-modules for all $i \in \mathbb{Z}$. Whenever we have a complex where the morphisms are going in the opposite direction, we will use $H^i(?)$ to denote its homology.

Proposition 2.5. *A complex of A-modules $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is an exact sequence if and only if $H_i(\boldsymbol{M_\bullet}) = 0$ for all $i \in \mathbb{Z}$.*

Proof. Notice that in a complex of A-modules $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ we have $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$, and thus $B_i(\boldsymbol{M_\bullet}) = \operatorname{im}(d_{i+1}) \subseteq \ker(d_i) = Z_i(\boldsymbol{M_\bullet})$. Hence $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is an exact sequence if and only if $Z_i(\boldsymbol{M_\bullet}) = \ker(d_i) \subseteq \operatorname{im}(d_{i+1}) = B_i(\boldsymbol{M_\bullet})$ if and only if $Z_i(\boldsymbol{M_\bullet}) = B_i(\boldsymbol{M_\bullet})$ if and only if $H_i(\boldsymbol{M_\bullet}) = 0$ for all $i \in \mathbb{Z}$. $\square$

Definition 2.6. Let $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$, $(\boldsymbol{N_\bullet}, \boldsymbol{e_\bullet})$ be complexes of A-modules. A family of A-morphisms $\{f_i : M_i \to N_i\}_{i \in \mathbb{Z}}$ making the following diagram commute is called a *chain map*, and is denoted $f_\bullet : (\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet}) \to (\boldsymbol{N_\bullet}, \boldsymbol{e_\bullet})$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & M_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & M_i \arrow[r, "d_i"] \arrow[d, "f_i"] & M_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r, "e_{i+2}"] & N_{i+1} \arrow[r, "e_{i+1}"] & N_i \arrow[r, "e_i"] & N_{i-1} \arrow[r, "e_{i-1}"] & \cdots
\end{tikzcd}$$

When our complexes are particularly well behaved, we will want chain maps to also behave well.

Definition 2.7. Let M, N be A-modules, $i \in \mathbb{N}$, and $(\boldsymbol{K}_\bullet, \boldsymbol{d}_\bullet)$, $(\boldsymbol{L}_\bullet, \boldsymbol{e}_\bullet)$ be i-extensions of M by N. A chain map $f_\bullet : (\boldsymbol{K}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{L}_\bullet, \boldsymbol{e}_\bullet)$ with $f_{-1} = 1_M$ and $f_i = 1_N$ is called a *map of i-extensions* from $\boldsymbol{K}_\bullet$ to $\boldsymbol{L}_\bullet$.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \xrightarrow{d_i} & K_{i-1} & \xrightarrow{d_{i-1}} & \cdots & \xrightarrow{d_1} & K_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_N} & & \downarrow{\scriptstyle f_{i-1}} & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_M} & & \\
0 & \longrightarrow & N & \xrightarrow{e_i} & L_{i-1} & \xrightarrow{e_{i-1}} & \cdots & \xrightarrow{e_1} & L_0 & \xrightarrow{e_0} & M & \longrightarrow & 0
\end{array}$$

Remark 2.8. Since maps of i-extensions are a binary relation, they generate an equivalence relation. That is, let $\boldsymbol{K}_\bullet$ and $\boldsymbol{L}_\bullet$ be i-extensions of M by N. We say that they are equivalent whenever there exists some $r \in \mathbb{N}$ and $\boldsymbol{X}_\bullet^1, \ldots, \boldsymbol{X}_\bullet^r$ that are i-extensions of M by N such that $\boldsymbol{X}_\bullet^1 = \boldsymbol{K}_\bullet$, $\boldsymbol{X}_\bullet^r = \boldsymbol{L}_\bullet$, and for every $1 \leq j \leq r-1$ there exists a map of i-extensions either from $\boldsymbol{X}_\bullet^i$ to $\boldsymbol{X}_\bullet^{i+1}$ or from $\boldsymbol{X}_\bullet^{i+1}$ to $\boldsymbol{X}_\bullet^i$. We denote this by $\boldsymbol{K}_\bullet \sim \boldsymbol{L}_\bullet$, and it is indeed an equivalence relation.

Notice that chain maps induce well defined A-morphisms in homology. Namely given a chain map $f_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{N}_\bullet, \boldsymbol{e}_\bullet)$ we have a family of A-morphisms $\{f_{*_i} : H_i(\boldsymbol{M}_\bullet) \to H_i(\boldsymbol{N}_\bullet)\}_{i \in \mathbb{Z}}$ such that $f_{*_i}(\overline{m}) = \overline{f_i(m)}$ for all $m \in M_i$ for all $i \in \mathbb{Z}$.

Definition 2.9. Let $f_\bullet, g_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{N}_\bullet, \boldsymbol{e}_\bullet)$ be chain maps. They are called *chain homotopic* whenever there is a family of A-morphisms $\{s_i : M_i \to N_{i+1}\}_{i \in \mathbb{Z}}$ such that $f_i - g_i = s_{i-1}d_i + e_{i+1}s_i$ for all $i \in \mathbb{Z}$, and is denoted $f_\bullet \simeq g_\bullet$. The family $s_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{N}_{\bullet+1}, \boldsymbol{e}_{\bullet+1})$ is called a *homotopy* for $f_\bullet - g_\bullet$. This is illustrated in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & M_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & M_i \arrow[r, "d_i"] \arrow[dl, "s_i"] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & M_{i-1} \arrow[r, "d_{i-1}"] \arrow[dl, "s_{i-1}"] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & \cdots \\
\cdots \arrow[r, "e_{i+2}"] & N_{i+1} \arrow[r, "e_{i+1}"] & N_i \arrow[r, "e_i"] & N_{i-1} \arrow[r, "e_{i-1}"] & \cdots
\end{tikzcd}$$

A homotopy $s_\bullet$ is called a *chain contraction* of $f_\bullet$ whenever $g_\bullet$ is the zero map. A chain contraction of the identity map $1_{\boldsymbol{M}_\bullet}$ is called a *contracting homotopy*.

Proposition 2.10. *Let $f_\bullet, g_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{N}_\bullet, \boldsymbol{e}_\bullet)$ be chain homotopic, then $f_{*_i} : H_i(\boldsymbol{M}_\bullet) \to H_i(\boldsymbol{N}_\bullet)$ and $g_{*_i} : H_i(\boldsymbol{M}_\bullet) \to H_i(\boldsymbol{N}_\bullet)$ coincide for all $i \in \mathbb{Z}$.*

Proof. Let $s_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet) \to (\boldsymbol{N}_{\bullet+1}, \boldsymbol{e}_{\bullet+1})$ be a homotopy for $f_\bullet - g_\bullet$. Given $i \in \mathbb{Z}$ and $m \in \ker(d_i)$

$$f_{*_i}(\overline{m}) = \overline{f_i(m)} = \overline{s_{i-1}d_i(m)} + \overline{e_{i+1}s_i(m)} + \overline{g_i(m)} = g_{*_i}(\overline{m}).$$

□

Proposition 2.11. *Let $f_\bullet : (\boldsymbol{L}_\bullet, \boldsymbol{l}_\bullet) \to (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ and $g_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet) \to (\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet)$ be chain maps, then $g_{*_i} f_{*_i} : H_i(\boldsymbol{L}_\bullet) \to H_i(\boldsymbol{N}_\bullet)$ and $(gf)_{*_i} : H_i(\boldsymbol{L}_\bullet) \to H_i(\boldsymbol{N}_\bullet)$ coincide for all $i \in \mathbb{Z}$.*

Proof. Given $i \in \mathbb{Z}$ and $z \in \ker(l_i)$

$$(gf)_{*_i}(\overline{z}) = \overline{(gf)_i(z)} = \overline{g_i f_i(z)} = g_{*_i}(\overline{f_i(z)}) = g_{*_i}(f_{*_i}(\overline{z})) = (g_{*_i} f_{*_i})(\overline{z}).$$

□

Notice how the equalities in Proposition 2.10 and Proposition 2.11 are of A-morphisms.

Definition 2.12. A sequence $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ of A-modules is called *split* if there is a sequence

$$\cdots \xleftarrow[s_{i+1}]{} M_{i+1} \xleftarrow[s_i]{} M_i \xleftarrow[s_{i-1}]{} \cdots$$

of A-morphisms such that $d_i s_{i-1} d_i = d_i$ for all $i \in \mathbb{Z}$. A sequence $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ is called *split exact* whenever it is split and exact.

We may have sequences that are exact but not split exact.

Example 2.13. Let $A = \mathbb{Z}$, the sequence

$$\cdots \xrightarrow{2\cdot} \mathbb{Z}/(4) \xrightarrow{2\cdot} \mathbb{Z}/(4) \xrightarrow{2\cdot} \mathbb{Z}/(4) \xrightarrow{2\cdot} \cdots$$

is exact but it does not split, all A-morphisms $s_i : \mathbb{Z}/(4) \to \mathbb{Z}/(4)$ yield $(2\cdot)s_i(2\cdot) = 0 \neq (2\cdot)$.

It is important to notice that not all split sequences are exact. Consider this simple example.

Example 2.14. Let M be a non-zero A-module. Consider the sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & M \oplus M \arrow[r, "\pi_1"] \arrow[l, bend left] & M \arrow[r] \arrow[l, bend left, "\iota_1"] & 0 \arrow[r] \arrow[l, bend left] & M \oplus M \arrow[r, "\pi_1"] \arrow[l, bend left] & M \arrow[r] \arrow[l, bend left, "\iota_1"] & 0 \arrow[r] \arrow[l, bend left] & \cdots
\end{tikzcd}
$$

where π_1 is the projection on the first coordinate, and ι_1 is the inclusion on the first coordinate. This is not an exact sequence because $\ker(\pi_1) = 0 \oplus M \neq 0$, but it splits since $\pi_1\iota_1\pi_1(M \oplus M) = \pi_1\iota_1(M) = \pi_1(M \oplus 0) = M = \pi_1(M \oplus M)$.

Remark 2.15. A short exact sequence of A-modules $0 \to L \xrightarrow{g} M \xrightarrow{f} N \to 0$ splits if and only if there exists an A-morphism $s : N \to M$ with $fs = 1_N$. This happens if and only if there exists an A-morphism $t : M \to L$ with $tg = 1_L$.

The following result can be found in [35, Exercise 1.4.3], but it was known long before. Since we will use it later, we include a full proof with it.

Proposition 2.16. *Let $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ be a chain complex of A-modules. It is split exact if and only if the identity map $1_{\boldsymbol{M_\bullet}}$ is null homotopic.*

Proof. Suppose that $1_{\boldsymbol{M_\bullet}}$ is null homotopic. Notice first that by Proposition 2.10 the induced map in homology is zero, so Proposition 2.5 yields that $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is exact. Now by definition of null homotopic, there exist A-morphisms $s_i : M_i \to M_{i+1}$ such that $d_{i+1}s_i + s_{i-1}d_i = 1_{M_i}$ for all $i \in \mathbb{Z}$. Composing d_i with the above and using that $d_i d_{i+1} = 0$ we obtain

$$
d_i = d_i 1_{M_i} = d_i(d_{i+1}s_i + s_{i-1}d_i) = d_i d_{i+1} s_i + d_i s_{i-1} d_i = d_i s_{i-1} d_i
$$

and thus $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is split.

Suppose that $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is split exact. We then have $\operatorname{im}(d_i) = \ker(d_{i-1})$ and A-morphisms $s_i : M_i \to M_{i+1}$ such that $d_i s_{i-1} d_i = d_i$ for all $i \in \mathbb{Z}$. We thus have the short exact sequence

$$
0 \longrightarrow \operatorname{im}(d_{i+1}) \hookrightarrow M_i \xrightarrow{d_i} \operatorname{im}(d_i) \longrightarrow 0
$$

where the map $s_{i-1}|_{\operatorname{im}(d_i)} : \operatorname{im}(d_i) \to M_i$ satisfies that $d_i s_{i-1}|_{\operatorname{im}(d_i)} d_i = d_i = 1_{M_{i-1}} d_i$, so by surjectivity of d_i we have $d_i s_{i-1}|_{\operatorname{im}(d_i)} = 1_{M_{i-1}}$. Hence the above short exact sequence splits so $M_i \cong \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1})$, where we can assume equality without loss of generality. Then $d_i : \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) \to \mathrm{im}(d_{i-1}) \oplus \mathrm{im}(d_i)$, it is just sending $\mathrm{im}(d_i)$ to itself and $\mathrm{im}(d_{i+1})$ to zero. Set $s_i : \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) \to \mathrm{im}(d_{i+1}) \oplus \mathrm{im}(d_{i+2})$ the A-morphism that sends $\mathrm{im}(d_{i+1})$ to itself and $\mathrm{im}(d_i)$ to zero. Now $d_{i+1}s_i(\mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1})) = 0 \oplus \mathrm{im}(d_{i+1})$ and $s_{i-1}d_i(\mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1})) = \mathrm{im}(d_i) \oplus 0$, so $d_{i+1}s_i + s_{i-1}d_i = 1_{\mathrm{im}(d_i)\oplus\mathrm{im}(d_{i+1})}$ for all $i \in \mathbb{Z}$, so $1_{\boldsymbol{M_\bullet}}$ is null homotopic as desired. $\square$

Remark 2.17.

1. Given a split short exact sequence of A-modules $0 \to L \to M \to N \to 0$, the proof of Proposition 2.16 implies that $M \cong N \oplus L$.

2. The sequence $0 \to L \xrightarrow{\iota_1} L \oplus N \xrightarrow{\pi_2} N \to 0$ where $\iota_1 : L \to L \oplus N$ is the inclusion on the first component and $\pi_2 : L \oplus N \to N$ is the projection on the second component, is exact. Since the identity map is null homotopic (via the sections $\pi_1 : L \oplus N \to L$ and $\iota_2 : N \to L \oplus N$), by Proposition 2.16 the sequence is also split. This is the 1-extension of N by L called the *trivial extension*.

Definition 2.18. Let $\{M_{i,j}\}_{i,j\in\mathbb{Z}}$ be a family of A-modules, $\{d^h_{i,j} : M_{i,j} \to M_{i,j-1}\}_{i,j\in\mathbb{Z}}$ and $\{d^v_{i,j} : M_{i,j} \to M_{i-1,j}\}_{i,j\in\mathbb{Z}}$ be families of A-morphisms such that $d^h_{i,j}d^h_{i,j+1} = 0$, $d^v_{i,j}d^v_{i+1,j} = 0$, and $d^h_{i,j}d^v_{i,j+1} + d^h_{i,j}d^h_{i+1,j} = 0$ for all $i, j \in \mathbb{Z}$. The diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & M_{i+1,j-1} \arrow[l] \arrow[d, "d^v_{i+1,j-1}"] & M_{i+1,j} \arrow[l, "d^h_{i+1,j}"'] \arrow[d, "d^v_{i,j}"] & M_{i+1,j+1} \arrow[l, "d^v_{i+1,j+1}"'] \arrow[d, "d^v_{i,j}"] & \cdots \arrow[l] \\
\cdots & M_{i,j-1} \arrow[l] \arrow[d, "d^v_{i,j-1}"] & M_{i,j} \arrow[l, "d^h_{i,j}"'] \arrow[d, "d^v_{i,j}"] & M_{i,j+1} \arrow[l, "d^v_{i,j+1}"'] \arrow[d, "d^v_{i,j+1}"] & \cdots \arrow[l] \\
\cdots & M_{i-1,j-1} \arrow[l] \arrow[d] & M_{i-1,j} \arrow[l, "d^v_{i-1,j}"'] \arrow[d] & M_{i-1,j+1} \arrow[l, "d^v_{i-1,j+1}"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

is called a *bicomplex* of A-modules, and is denoted $(\boldsymbol{M_{\bullet,\bullet}}, \boldsymbol{d^h_{\bullet,\bullet}}, \boldsymbol{d^v_{\bullet,\bullet}})$. The morphisms $\boldsymbol{d^h_{\bullet,\bullet}}$ and $\boldsymbol{d^v_{\bullet,\bullet}}$ are called the *horizontal* and *vertical* differentials, respectively.

We will be mainly using bicomplexes when we tensor two complexes.

Example 2.19. Let $(\boldsymbol{L_\bullet}, \boldsymbol{l_\bullet})$ and $(\boldsymbol{N_\bullet}, \boldsymbol{n_\bullet})$ be complexes of left and right A-modules, respectively. For all $i, j \in \mathbb{Z}$ define $M_{i,j} = L_i \otimes_A N_j$, and the A-morphisms $m^h_{i,j} : M_{i,j} \to M_{i-1,j}$ and $m^v_{i,j} : M_{i,j} \to M_{i,j-1}$ via $m^h_{i,j}(x \otimes y) = l_i(x) \otimes y$ and $m^v_{i,j}(x \otimes y) = (-1)^i x \otimes n_j(y)$ for all $x \in M_i$ and $y \in N_j$. Then $(\boldsymbol{M_{\bullet,\bullet}}, \boldsymbol{m^h_{\bullet,\bullet}}, \boldsymbol{m^v_{\bullet,\bullet}})$ is a bicomplex of $\mathbb{Z}$-modules. We often denote this bicomplex by $(\boldsymbol{L_\bullet} \otimes_A \boldsymbol{N_\bullet}, \boldsymbol{l_\bullet} \otimes \boldsymbol{n_\bullet})$.

We will mainly be interested in the new complexes arising from this construction.

Definition 2.20. Let $(\boldsymbol{M_{\bullet,\bullet}}, \boldsymbol{d^h_{\bullet,\bullet}}, \boldsymbol{d^v_{\bullet,\bullet}})$ be a bicomplex of A-modules. The family of A-modules $\{\mathrm{Tot}_i(\boldsymbol{M_{\bullet,\bullet}}) = \bigoplus_{r+s=i} M_{r,s}\}_{i\in\mathbb{Z}}$ together with the differentials $\{d_i = \bigoplus_{r+s=i} d^h_{r,s} + d^v_{r,s}\}_{i\in\mathbb{Z}}$ is called the *total complex* of $\boldsymbol{M_{\bullet,\bullet}}$, and is denoted $(\mathrm{Tot}_\bullet(\boldsymbol{M_{\bullet,\bullet}}), \boldsymbol{d_\bullet})$.

Example 2.21. Let $(\boldsymbol{L_\bullet}, \boldsymbol{l_\bullet})$ and $(\boldsymbol{N_\bullet}, \boldsymbol{n_\bullet})$ be complexes of left and right A-modules, respectively, and let $(\boldsymbol{L_\bullet} \otimes_A \boldsymbol{N_\bullet}, \boldsymbol{l_\bullet} \otimes \boldsymbol{n_\bullet})$ be their tensor product. Then $(\mathrm{Tot}_\bullet(\boldsymbol{L_\bullet} \otimes_A \boldsymbol{N_\bullet}), \mathrm{Tot}_\bullet(\boldsymbol{l_\bullet} \otimes \boldsymbol{n_\bullet}))$, which we will often denote by $((\boldsymbol{L_\bullet} \otimes_A \boldsymbol{N_\bullet})_\bullet, (\boldsymbol{l_\bullet} \otimes \boldsymbol{n_\bullet})_\bullet)$, is the following complex.

$$\cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} L_j \otimes_A N_{i-j} \xrightarrow{\bigoplus_{j\in\mathbb{Z}} (l_j \otimes 1_{N_{i-j}} + 1_{L_j} \otimes n_{i-j})} \bigoplus_{j\in\mathbb{Z}} L_{j-1} \otimes_A N_{i-j} \longrightarrow \cdots$$

The main motivation behind Section 3.3 is to understand the homology of the total complex of the tensor product of two complexes. This is well known in the absolute setup as Theorem2.44, and we extend it to the relative setup.

2.2 Modules and their properties

We now recall the properties that make free, projective, and injective modules special.

Definition 2.22. An A-module U is said to be *free* if it is isomorphic to a direct sum of copies of A, namely there is a (not necessarily finite) set X such that $U \cong \bigoplus_{x\in X} A_x$ with $A_x = \langle x \rangle \cong A$ for all $x \in X$. We say that X is a *basis* of U.

A sequence of A-modules $(\boldsymbol{U_\bullet}, \boldsymbol{d_\bullet})$ is said to be a *free resolution* of an A-module M when it is an exact sequence bounded on the right by M and U_i is free for all $i \in \mathbb{N}$

$$\cdots \xrightarrow{d_{i+1}} U_i \xrightarrow{d_i} U_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} U_0 \xrightarrow{d_0} M \longrightarrow 0.$$

The following characterization is analogous to [1, Section 5, Proposition 1]. The key difference is that we are not assuming A to be a k-algebra, so our proof requires a slight modification.

Proposition 2.23. *An A-module U is free if and only if U has a subset X such that for every A-module M and every function of sets $g : X \to M$ there is a unique A-morphism $h : U \to M$ with $h|_X = g$ as functions of sets.*

$$\begin{tikzcd} U \arrow[dr, dashed, "h_A"] & \\ X \arrow[u, hook] \arrow[r, "g"] & M \end{tikzcd}$$

Proof. ($\Rightarrow$) [32, Proposition 2.34] Let U be free with basis X, we have X is a subset of U. Every $u \in U \cong \bigoplus_{x \in X} A_x$ can be written uniquely as $u = \sum_{x \in X} a_x x$ where $a_x \in A$ for all $x \in X$, and all but a finite number of them are zero. Let M be an A-module and $g : X \to M$ a function. The function $h : U \to M$ given by $h(u) = \sum_{x \in X} a_x g(u_x)$ is the desired unique A-morphism satisfying $h|_X = g$ as functions of sets.

($\Leftarrow$) Let $X \subseteq U$ be such that for every A-module M and every function $g : X \to M$ there is a unique A-morphism $h : U \to M$ with $h|_X = g$ as functions of sets. Consider the function $g : X \to U/\langle X \rangle$ given by $g(x) = 0$ for all $x \in X$. The canonical projection $h_1 : U \to U/\langle X \rangle$ and the map $h_2 : U \to U/\langle X \rangle$ as $h_2(u) = 0$ for all $u \in U$ both satisfy that $h_1|_X = g = h_2|_X$ as functions of sets. Since this morphism must be unique, we have the canonical projection must be the zero map, and thus $U = \langle X \rangle$. By definition this makes $f : \bigoplus_{x \in X} A_x \to U$ with $f((a_x)_{x \in X}) = \sum_{x \in X} a_x x$ an A-epimorphism. To prove that it is an A-monomorphism suppose that we have $\sum_{i=1}^n a_i x_i = 0$ for some $a_i \in A$, $x_i \in X$, $i = 1, \ldots, n$, and a fixed $n \in \mathbb{N}$. For $j = 1, \ldots, n$ consider the functions $\delta_j : X \to A$ as $\delta_j(x) = 1$ if $x = x_j$ and $\delta_j(x) = 0$ if $x \neq x_j$, they extend to A-morphisms

$D_j : U \to A$. For $j = 1, \ldots, n$ we now have

$$0 = D_j(0) = D_j\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i D_j(x_i) = a_i.$$

Hence $f : \bigoplus_{x \in X} A_x \to U$ is an isomorphism of A-modules, making U a free module.

□

Definition 2.24. An A-module P is said to be *projective* if for every exact sequence $M \xrightarrow{g} N \to 0$ and every A-morphism $h : P \to N$ there is an A-morphism $h' : P \to M$ with $gh' = h$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'_A"'] \arrow[d, "h_A"] & \\ M \arrow[r, "g_A"] & N \arrow[r] & 0 \end{tikzcd}$$

A sequence of A-modules $(\boldsymbol{P_\bullet}, \boldsymbol{d_\bullet})$ is said to be a *projective resolution* of an A-module M when it is an exact sequence bounded on the right by M and P_i is projective for all $i \in \mathbb{N}$

$$\cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Remark 2.25. If we allow ourselves to use the axiom of choice, free modules are projective.

Although we will be almost exclusively interested in the above, for completeness we include the object arising when we reverse all the arrows.

Definition 2.26. An A-module I is said to be *injective* if for every exact sequence $0 \to M \xrightarrow{g} N$ and every A-morphism $h : M \to I$ there is an A-morphism $h' : N \to I$ with $h'g = h$.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & M \arrow[u, "h_A"] \arrow[r, "g_A"] & N \arrow[ul, dashed, "h'_A"'] \end{tikzcd}$$

A sequence of A-modules $(\boldsymbol{I_\bullet}, \boldsymbol{d_\bullet})$ is said to be an *injective resolution* of an A-module M when it is an exact sequence bounded on the left by M and I_i is injective for all $i \in \mathbb{N}$

$$0 \longrightarrow M \xrightarrow{d_{-1}} I_0 \xrightarrow{d_0} \cdots \xrightarrow{d_{i-2}} I_{i-1} \xrightarrow{d_{i-1}} I_i \xrightarrow{d_i} \cdots.$$

We can now give the familiar characterizations of these objects. Some of these can be rephrased with slightly more sophisticated nomenclature. We add the following proof since it illustrates the elementary techniques we will be using throughout the text.

Proposition 2.27. *For P an A-module, the following are equivalent:*

1. *P is projective.*

2. *Every short exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} P \to 0$ splits.*

3. *P is a direct summand of a free A-module.*

4. *$\mathrm{Hom}_A(P, ?)$ is an exact functor.*

Proof. (1.⇒2.) Given a short exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} P \to 0$ we can fit the identity map $1_P : P \to P$ in the diagram

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "h"'] \arrow[d, "1_P"] & \\ 0 \arrow[r] & M \arrow[r, "g"] & N \arrow[r, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

and since P is projective there is an A-morphism $h : P \to N$ with $fh = 1_P$. Thus by Remark 2.15 the short exact sequence splits.

(2.⇒3.) The A-morphism $\pi : \bigoplus_{p\in P} A \to P$ given by $\pi((0, \cdots, 1_p, \cdots, 0)) = p$ for all $p \in P$ is surjective, so the following is a short exact sequence, which by hypothesis must be split.

$$0 \longrightarrow \ker(\pi) \hookrightarrow \bigoplus_{p\in P} A \xrightarrow{\pi} P \longrightarrow 0$$

By Remark 2.17 we have $P \oplus \ker(\pi) \cong \bigoplus_{p\in P} A$. Hence P is a direct summand of the free A-module $\oplus_{p\in P} A$.

(3.⇒4.) Suppose we have a short exact sequence $0 \to L \xrightarrow{g} M \xrightarrow{f} N \to 0$, applying $\mathrm{Hom}_A(P, ?)$ yields the exact sequence $0 \to \mathrm{Hom}_A(P, L) \xrightarrow{g\circ} \mathrm{Hom}_A(P, M) \xrightarrow{f\circ} \mathrm{Hom}_A(P, N)$ because it is always left exact. To see that it is right exact it is enough to prove that $f\circ : \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$ is surjective.

Let P be a direct summand of an A-free module U, let $\pi : U \to P$, $\iota : P \to U$ be the canonical projection and inclusion respectively. Given an A-morphism $l : P \to N$ we can fit it in the following diagram.

$$\begin{tikzcd}
 & U \arrow[d, shift left, "\pi"] \arrow[ddl, dashed, "h'"'] & \\
 & P \arrow[u, shift left, "\iota"] \arrow[d, "l"] & \\
M \arrow[r, "f"] & N \arrow[r] & 0
\end{tikzcd}$$

By Remark 2.25 we have that U is projective and thus there exists an A-morphism $h' : U \to M$ with $fh' = l\pi$. Now $h = h'\iota$ is an A-morphism $h : P \to M$ with $fh = fh'\iota = l\pi\iota = l1_P = l$, and thus $f\circ$ is surjective, and thus $\mathrm{Hom}_A(P, ?)$ is right exact.

(4.$\Rightarrow$1.) Suppose we have an exact sequence $M \xrightarrow{f} N \to 0$ and an A-morphism $h : P \to N$. Since $\mathrm{Hom}_A(P, ?)$ is right exact, applying it to the short exact sequence $0 \to \ker(f) \hookrightarrow M \xrightarrow{f} N \to 0$ yields that $\mathrm{Hom}_A(P, M) \xrightarrow{f\circ} \mathrm{Hom}_A(P, N) \to 0$ is an exact sequence, that is $f\circ$ is surjective. Hence there exists an A-morphism $h' : P \to M$ with $fh' = h$, meaning that P is projective.

□

For completeness, the following can be found between [33, Section 8.4, Theorem 8.4.9] and [32, Section 3.2, Proposition 3.25].

Proposition 2.28. *For I an A-module, the following are equivalent:*

1. *I is injective.*
2. *Every short exact sequence $0 \to I \xrightarrow{i} M \xrightarrow{p} N \to 0$ splits.*
3. *I is a direct summand of a cofree A-module.*
4. *$\mathrm{Hom}_A(?, I)$ is an exact functor.*

Another interesting phenomenon arises when we require that the tensor product with a module is an exact functor.

Definition 2.29. An A-module F is said to be *flat* if for every exact sequence of right A-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

then

$$0 \longrightarrow L \otimes_A F \xrightarrow{f\otimes 1_F} M \otimes_A F \xrightarrow{g\otimes 1_F} N \otimes_A F \longrightarrow 0$$

is an exact sequence of abelian groups. Otherwise said, the functor $? \otimes_A F$ is exact.

A sequence of A-modules $(\boldsymbol{F_\bullet}, \boldsymbol{d_\bullet})$ is said to be a *flat resolution* of an A-module M when it is an exact sequence bounded on the right by M and F_i is projective for all $i \in \mathbb{N}$

$$\cdots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Remark 2.30. The functor $? \otimes_A F$ is right exact for every A-module F, namely for every exact sequence $0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0$ then $L \otimes_A F \overset{f\otimes 1_F}{\to} M \otimes_A F \overset{g\otimes 1_F}{\to} N \otimes_A F \to 0$ is an exact sequence of abelian groups. Hence an A-module F is flat if and only if for every injection $f : L \to M$ then $f \otimes 1_F : L \otimes_A F \to M \otimes_A F$ is also an injection. However, this characterization can be improved, since it turns out that an A-module F is flat if and only if for every right ideal I of A with inclusion $\iota : I \to A$ then $\iota \otimes 1_F : I \otimes_A F \to A \otimes_A F$ is an injection.

An analogous definition of flatness follows for right A-modules by requiring that the functor $F \otimes_A ?$ is exact. The following can be found in [32, Section 3.3, Proposition 3.46].

Proposition 2.31. *1. The ring A is flat as an A-module.*

2. Let $\{M_x\}_{x\in X}$ be a family of A-modules. Then $\bigoplus_{x\in X} M_x$ is flat if and only if each M_x is flat.

3. Let P be a projective A-module. Then P is a flat A-module.

The following illustrates the usefulness of working with flat modules: the kernel of a tensor product of A-morphisms can be understood through the direct sum of their respective kernels.

Proposition 2.32. *Let $f : M \to N$ and $g : P \to Q$ be left and right A-morphisms respectively such that M and P are flat. Then* $\ker(f \otimes_A g) \cong (\ker(f) \otimes_A P) \oplus (M \otimes_A \ker(g))$ *as $\mathbb{Z}$-modules.*

Proof. Consider the short exact sequences

$$0 \longrightarrow \ker(f) \xrightarrow{\iota} M \xrightarrow{f} \mathrm{im}(f) \longrightarrow 0,$$
$$0 \longrightarrow \ker(g) \xrightarrow{\kappa} P \xrightarrow{g} \mathrm{im}(g) \longrightarrow 0.$$

Since applying $? \otimes_A P$, $? \otimes_A \mathrm{im}(g)$, $\mathrm{im}(f)\otimes_A ?$, and $M \otimes_A ?$ is right exact, then

$$\ker(f) \otimes_A P \xrightarrow{\iota\otimes 1_P} M \otimes_A P \xrightarrow{f\otimes 1_P} \mathrm{im}(f) \otimes_A P \longrightarrow 0,$$
$$\ker(f) \otimes_A \mathrm{im}(g) \xrightarrow{\iota\otimes 1_{\mathrm{im}(g)}} M \otimes_A \mathrm{im}(g) \xrightarrow{f\otimes 1_{\mathrm{im}(g)}} \mathrm{im}(f) \otimes_A \mathrm{im}(g) \longrightarrow 0,$$
$$\mathrm{im}(f) \otimes_A \ker(g) \xrightarrow{1_{\mathrm{im}(f)}\otimes \kappa} \mathrm{im}(f) \otimes_A P \xrightarrow{1_{\mathrm{im}(f)}\otimes g} \mathrm{im}(f) \otimes_A \mathrm{im}(g) \longrightarrow 0,$$
$$M \otimes_A \ker(g) \xrightarrow{1_M\otimes \kappa} M \otimes_A P \xrightarrow{1_M\otimes g} M \otimes_A \mathrm{im}(g) \longrightarrow 0,$$

are exact sequences, and the following diagram commutes.

$$\begin{tikzcd}
\ker(f) \otimes_A P \arrow[r, "\iota\otimes 1_P"] \arrow[d, "1_{\ker(f)}\otimes g"] & M \otimes_A P \arrow[r, two heads, "f\otimes 1_P"] \arrow[d, "1_M\otimes g"] & \mathrm{im}(f) \otimes_A P \arrow[d, "1_{\mathrm{im}(f)}\otimes g"] \\
\ker(f) \otimes_A \mathrm{im}(g) \arrow[r, "\iota\otimes 1_{\mathrm{im}(g)}"] & M \otimes_A \mathrm{im}(g) \arrow[r, two heads, "f\otimes 1_{\mathrm{im}(g)}"] & \mathrm{im}(f) \otimes_A \mathrm{im}(g)
\end{tikzcd}$$

Then $(f \otimes g) : M \otimes_A P \to \mathrm{im}(f) \otimes_A \mathrm{im}(g)$ given by $(f \otimes g) = (1_{\mathrm{im}(f)} \otimes g)(f \otimes 1_P)$ is a surjective A-morphism because it is a composition of surjective A-morphisms. Now

$$\ker(f \otimes g) = \{z \in M \otimes_A P : (f \otimes 1_P)(z) \in \ker(1_{\mathrm{im}(f)} \otimes g)\},$$

the preimage of $\ker(1_{\mathrm{im}(f)} \otimes g)$ via $f \otimes 1_P$. Given $x, y \in \ker(1_{\mathrm{im}(f)} \otimes g)$ with $(f \otimes 1_P)(x) = (f \otimes 1_P)(y)$ means that $(f \otimes 1_P)(x - y) = 0$ so $x - y \in \ker(f \otimes 1_P)$ and thus $x = y + z$ for some $z \in \ker(f \otimes 1_P)$. Hence to obtain $(f \otimes 1_P)^{-1}(\ker(1_{\mathrm{im}(f)} \otimes g))$ it suffices to find for each $x \in \ker(1_{\mathrm{im}(f)} \otimes g)$ a single element in its preimage via $f \otimes 1_P$, and then to direct sum with $\ker(f \otimes 1_P)$. We proceed to find A-modules satisfying these conditions. First, the exactness of the above sequences yields $\ker(f \otimes 1_P) = \mathrm{im}(\iota \otimes 1_P)$. Second, we claim that every element in $\ker(1_{\mathrm{im}(f)} \otimes g)$ has at least one preimage via $f \otimes 1_P$ in $\ker(1_M \otimes g)$. This follows from diagram chasing in the above commutative diagram. For this, let $x \in \ker(1_{\mathrm{im}(f)} \otimes g) \subseteq \mathrm{im}(f) \otimes_A P$, by the surjectivity of $f\otimes 1_P$ there is $z\in M\otimes_A P$ with $(f\otimes 1_P)(z)=x$. Since $(f\otimes 1_{\mathrm{im}(g)})(1_M\otimes g)(z)=(1_{\mathrm{im}(f)}\otimes g)(f\otimes 1_P)(z)=(1_{\mathrm{im}(f)}\otimes g)(x)=0$ then $(1_M\otimes g)(z)\in\ker(f\otimes 1_{\mathrm{im}(g)})=\mathrm{im}(\iota\otimes 1_{\mathrm{im}(g)})$ by exactness of the bottom row, so there is $u\in\ker(f)\otimes_A\mathrm{im}(g)$ with $(\iota\otimes 1_{\mathrm{im}(g)})(u)=(1_M\otimes g)(z)$. Since $1_{\ker(f)}\otimes g$ is surjective, there is $w\in\ker(f)\otimes_A P$ with $(1_{\ker(f)}\otimes g)(w)=u$, whence $(1_M\otimes g)(\iota\otimes 1_P)(w)=(\iota\otimes 1_{\mathrm{im}(g)})(1_{\ker(f)}\otimes g)(w)=(\iota\otimes 1_{\mathrm{im}(g)})(u)=(1_M\otimes g)(z)$. In particular $(1_M\otimes g)(z-(\iota\otimes 1_P)(w))=0$ and $(f\otimes 1_P)(z-(\iota\otimes 1_P)(w))=(f\otimes 1_P)(z)-(f\otimes 1_P)(\iota\otimes 1_P)(w)=x$ because $(f\otimes 1_P)(\iota\otimes 1_P)=0$ by exactness of the top row, so $z-(\iota\otimes 1_P)(w)\in\ker(f\otimes 1_M)\subseteq M\otimes_A P$ is the claimed preimage of x via $f\otimes 1_P$. In fact, for every $z\in\ker(1_M\otimes g)$ we have $(f\otimes 1_P)(z)\in\ker(1_{\mathrm{im}(f)}\otimes g)$ because $(1_{\mathrm{im}(f)}\otimes g)(f\otimes 1_P)(z)=(f\otimes 1_{\mathrm{im}(g)})(1_M\otimes g)(z)=0$. Hence, up to elements in $\ker(f\otimes 1_P)$, we can take as the representatives of preimages of elements in $\ker(1_{\mathrm{im}(f)}\otimes g)$ via $f\otimes 1_P$ the module $\ker(1_M\otimes g)=\mathrm{im}(1_M\otimes\kappa)$, by exactness of the above sequences. Then $\ker(f\otimes g)=\mathrm{im}(\iota\otimes 1_P)\oplus\mathrm{im}(1_M\otimes\kappa)=\mathrm{im}(\iota\otimes 1_P+1_M\otimes\kappa)$, making

$$(\ker(f)\otimes_A P)\oplus(M\otimes_A\ker(g))\xrightarrow{\iota\otimes 1_P+1_M\otimes\kappa}M\otimes_A P\xrightarrow{f\otimes g}\mathrm{im}(f)\otimes\mathrm{im}(g)\longrightarrow 0$$

an exact sequence. This holds in complete generality, and will also be used in Proposition 2.43.

Finally, since M and P are flat, we in fact have the short exact sequences

$$\begin{array}{l}0\longrightarrow\ker(f)\otimes_A P\xhookrightarrow{\iota\otimes 1_P}M\otimes_A P\xrightarrow{f\otimes 1_P}\mathrm{im}(f)\otimes_A P\longrightarrow 0,\\ 0\longrightarrow M\otimes_A\ker(g)\xhookrightarrow{1_M\otimes\kappa}M\otimes_A P\xrightarrow{1_M\otimes g}M\otimes_A\mathrm{im}(g)\longrightarrow 0,\end{array}$$

whence $\mathrm{im}(\iota\otimes 1_P)\cong\ker(f)\otimes_A P$ and $\mathrm{im}(1_M\otimes\kappa)\cong M\otimes_A\ker(g)$, so $\ker(f\otimes g)\cong(\ker(f)\otimes_A P)\oplus(M\otimes_A\ker(g))$ as $\mathbb{Z}$-modules. □

In fact, there are a number of similar results depending on which of the modules are flat, such as Proposition 2.32. However, to fully understand their proof, we will need the additional technology of Tor groups.

2.3 Ext and Tor

We now recall the working definitions of the derived functors Ext and Tor, as well as their balance.

Definition 2.33. Let M, N be A-modules, consider $(\boldsymbol{P}_\bullet,\boldsymbol{p}_\bullet)$ a projective resolution of M

$$\cdots \xrightarrow{p_{i+1}} P_i \xrightarrow{p_i} P_{i-1} \xrightarrow{p_{i-1}} \cdots \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0,$$

applying the functor $\mathrm{Hom}_A(?, N)$ we obtain the complex

$$\cdots \xleftarrow{p_{i+1}^*} \mathrm{Hom}_A(P_i, N) \xleftarrow{p_i^*} \mathrm{Hom}_A(P_{i-1}, N) \xleftarrow{p_{i-1}^*} \cdots \xleftarrow{p_0^*} \mathrm{Hom}_A(P_0, N) \longleftarrow 0,$$

and taking its homology yields the *Ext groups* of M with coefficients in N, namely for all $i \in \mathbb{N}$

$$\begin{aligned}
\mathrm{Ext}_A^i(M, N) &= H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)) = \ker(p_{i+1}^*)/\operatorname{im}(p_i^*) \\
\mathrm{Ext}_A^\bullet(M, N) &= \bigoplus_{i\in\mathbb{N}} \mathrm{Ext}_A^i(M, N).
\end{aligned}$$

Again, for completeness, we include the following definition.

Definition 2.34. Let M be a right A-module and N be a left A-module, consider $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ a projective resolution of M

$$\cdots \xrightarrow{p_{i+1}} P_i \xrightarrow{p_i} P_{i-1} \xrightarrow{p_{i-1}} \cdots \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0,$$

applying the functor $? \otimes_A N$ we obtain the complex

$$\cdots \xrightarrow{p_{i+1}\otimes 1} P_i \otimes_A N \xrightarrow{p_i \otimes 1} P_{i-1} \otimes_A N \xrightarrow{p_{i-1}\otimes 1} \cdots \xrightarrow{p_1 \otimes 1} P_0 \otimes_A N \longrightarrow 0,$$

and taking its homology yields the *Tor groups* of M with coefficients in N, namely for all $i \in \mathbb{N}$

$$\begin{aligned}
\mathrm{Tor}_i^A(M, N) &= H_i(\boldsymbol{P_\bullet} \otimes_A N) = \ker(p_i \otimes 1)/\operatorname{im}(p_{i+1} \otimes 1) \\
\mathrm{Tor}_\bullet^A(M, N) &= \bigoplus_{i\in\mathbb{N}} \mathrm{Tor}_i^A(M, N).
\end{aligned}$$

Recall also the characterization of flat modules in terms of vanishing Tor.

Theorem 2.35. *For F a right A-module, the following are equivalent:*

1. *F is flat.*

2. $\mathrm{Tor}_i^A(F, M) = 0$ *for all A-modules M and for all positive $i \in \mathbb{N}$.*

3. $\operatorname{Tor}_1^A(F, M) = 0$ *for all A-modules M.*

As expected, these definitions are independent of the choice of projective resolution. For this result, we first need the so called Comparison Theorem.

Theorem 2.36 (Comparison Theorem)**.** *Let M, N be A-modules, $f : M \to N$ an A-morphism, $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ a (not necessarily exact) sequence of projective modules bounded on the right by M, $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ an exact sequence of (not necessarily projective) modules bounded on the right by N. Then there exists a chain map $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q_\bullet}$ making the completed diagram commute. This chain map is unique up to homotopy.*

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] \arrow[d, dashed, "f_2"] & P_1 \arrow[r, "p_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "p_0"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "q_3"] & Q_2 \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & N \arrow[r] & 0
\end{tikzcd}$$

Proof. The existence of $f_i : P_i \to Q_i$ for all $i \in \mathbb{N}$ follows by induction. For $i = 0$ consider

$$\begin{tikzcd}
P_0 \arrow[r, "p_0"] \arrow[d, dashed, "f_0"'] & M \arrow[d, "f"] & \\
Q_0 \arrow[r, "q_0"] & N \arrow[r] & 0,
\end{tikzcd}$$

since P_0 is projective the diagram guarantees the existence of an A-morphism $f_0 : P_0 \to Q_0$ with $q_0 f_0 = f p_0$. Suppose now for induction that we have the following commutative square.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "p_{i+1}"] & P_i \arrow[r, "p_i"] \arrow[d, "f_i"] & P_{i-1} \arrow[d, "f_{i-1}"] \\
Q_{i+1} \arrow[r, "q_{i+1}"] & Q_i \arrow[r, "q_i"] & Q_{i-1}
\end{tikzcd}$$

Exactness of the bottom sequence gives $\operatorname{im}(q_{i+1}) = \ker(q_i)$ and hence $q_i f_i p_{i+1} = f_{i-1} p_i p_{i+1} = 0$ yields $\operatorname{im}(f_i p_{i+1}) \subseteq \operatorname{im}(q_{i+1})$. Consider the following diagram.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, dashed, "f_{i+1}"'] & P_i \arrow[d, "f_i"] & \\
Q_{i+1} \arrow[r, "q_{i+1}"] & \operatorname{im}(q_{i+1}) \arrow[r] & 0
\end{tikzcd}$$

Indeed, since P_{i+1} is projective we obtain the existence of an A-morphism $f_{i+1} : P_{i+1} \to Q_{i+1}$ with $q_{i+1} f_{i+1} = f_i p_i$.

Suppose that $g_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q_\bullet}$ is another chain map making the completed diagram commute, the uniqueness up to a homotopy $r_i : P_i \to Q_{i+1}$ follows from an explicit construction of the homotopy by induction on $i \in \mathbb{N} \cup \{-1\}$. For $i = -1$, set $P_{-1} = M$, $P_{-2} = 0$, $Q_{-1} = N$, $r_{-1} : M \to Q_0$, $f_{-1} = g_{-1} = f$, and $r_{-1} = r_{-2} = 0$. The commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "f_1"] & P_0 \arrow[r, "p_0"] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] & M \arrow[r] \arrow[d, "f"] \arrow[dl, "r_{-1}"] & 0 \arrow[dl, "r_{-2}"] \\
\cdots \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & N \arrow[r] & 0
\end{tikzcd}
$$

yields $g_{-1} - f_{-1} = f - f = 0 = q_0 r_{-1} + r_{-2} p_{-1}$. Suppose now that we have the following commutative diagram, as before exactness of the bottom sequence gives $\operatorname{im}(q_{i+2}) = \ker(q_{i+1})$.

$$
\begin{tikzcd}
 & P_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, shift right, "g_{i+1}"'] \arrow[d, shift left, "f_{i+1}"] & P_i \arrow[r, "p_i"] \arrow[d, shift right, "g_i"'] \arrow[d, shift left, "f_i"] \arrow[dl, "r_i"'] & P_{i-1} \arrow[d, shift right, "g_{i-1}"'] \arrow[d, shift left, "f_{i-1}"] \arrow[dl, "r_{i-1}"'] \\
Q_{i+2} \arrow[r, "q_{i+2}"] & Q_{i+1} \arrow[r, "q_{i+1}"] & Q_i \arrow[r, "q_i"] & Q_{i-1}
\end{tikzcd}
$$

The induction hypothesis $g_i - f_i = q_{i+1} r_i + r_{i-1} p_i$ gives $\operatorname{im}(g_{i+1} - f_{i+1} - r_i p_{i+1}) \subseteq \operatorname{im}(q_{i+2})$ since

$$
\begin{aligned}
q_{i+1}(g_{i+1} - f_{i+1} - r_i p_{i+1}) &= q_{i+1}(g_{i+1} - f_{i+1}) - q_{i+1} r_i p_{i+1} \\
&= (g_i - f_i) p_{i+1} - (g_i - f_i - r_{i-1} p_i) p_{i+1} = 0.
\end{aligned}
$$

Consider the following diagram.

$$
\begin{tikzcd}
 & P_{i+1} \arrow[d, "g_{i+1} - f_{i+1} - r_i p_{i+1}"] \arrow[dl, dashed, "r_{i+1}"'] & \\
Q_{i+2} \arrow[r, "q_{i+2}"] & \operatorname{im}(q_{i+2}) \arrow[r] & 0
\end{tikzcd}
$$

Indeed, since P_{i+1} is projective we obtain the existence of an A-morphism $r_{i+1} : P_{i+1} \to Q_{i+2}$ with $q_{i+1} r_{i+1} = g_{i+1} - f_{i+1} - r_i p_{i+1}$. We then have $g_i - f_i = q_{i+1} r_i + r_{i-1} p_i$ for all $i \in \mathbb{N} \cup \{-1\}$. Thus $g_\bullet$ is homotopic to $f_\bullet$. $\square$

As a corollary, we obtain that Ext and Tor are well defined.

Proposition 2.37. *Let M, N be (left or right if necessary) A-modules, let $(\boldsymbol{P}_\bullet, \boldsymbol{p}_\bullet)$ and $(\boldsymbol{Q}_\bullet, \boldsymbol{q}_\bullet)$ be two projective resolutions of M. Then for all $i \in \mathbb{N}$*

$$H^i(\mathrm{Hom}_A(\boldsymbol{P}_\bullet, N)) \cong H^i(\mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N)),$$
$$H_i(\boldsymbol{P}_\bullet \otimes_A N) \cong H_i(\boldsymbol{Q}_\bullet \otimes_A N).$$

Proof. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
\cdots \arrow[r, "q_3"] & Q_2 \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & M \arrow[r] & 0,
\end{tikzcd}$$

by the Comparison Theorem 2.36 there is a chain map $f_\bullet : \boldsymbol{P}_\bullet \to \boldsymbol{Q}_\bullet$ making the completed diagram commute. Applying the functors $\mathrm{Hom}_A(?, N)$ and $? \otimes_A N$ we obtain chain maps

$$\mathrm{Hom}_A(f_\bullet, N) : \mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N) \to \mathrm{Hom}_A(\boldsymbol{P}_\bullet, N),$$
$$f_\bullet \otimes_A N : \boldsymbol{P}_\bullet \otimes_A N \to \boldsymbol{Q}_\bullet \otimes_A N,$$

where $\mathrm{Hom}_A(f_{-1}, N) = 1_M^* = 1_{\mathrm{Hom}_A(M,N)}$ and $f_{-1} \otimes_A N = 1_M \otimes 1_N = 1_{M \otimes_A N}$. These chain maps induce A-morphisms in homology for every $i \in \mathbb{N}$

$$\mathrm{Hom}_A(f_\bullet, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N)) \to H^i(\mathrm{Hom}_A(\boldsymbol{P}_\bullet, N)),$$
$$(f_\bullet \otimes_A N)_{*_i} : H_i(\boldsymbol{P}_\bullet \otimes_A N) \to H_i(\boldsymbol{Q}_\bullet \otimes_A N).$$

The same procedure permuting the roles of $(\boldsymbol{P}_\bullet, \boldsymbol{p}_\bullet)$ and $(\boldsymbol{Q}_\bullet, \boldsymbol{q}_\bullet)$ gives chain maps

$$g_\bullet : \boldsymbol{Q}_\bullet \to \boldsymbol{P}_\bullet,$$
$$\mathrm{Hom}_A(g_\bullet, N) : \mathrm{Hom}_A(\boldsymbol{P}_\bullet, N) \to \mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N),$$
$$g_\bullet \otimes_A N : \boldsymbol{Q}_\bullet \otimes_A N \to \boldsymbol{P}_\bullet \otimes_A N,$$

and A-morphisms in homology for every $i \in \mathbb{N}$

$$\mathrm{Hom}_A(g_\bullet, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)) \to H^i(\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N)),$$

$$(g_\bullet \otimes_A N)_{*_i} : H_i(\boldsymbol{Q_\bullet} \otimes_A N) \to H_i(\boldsymbol{P_\bullet} \otimes_A N).$$

The uniqueness statement of the Comparison Theorem gives $g_\bullet f_\bullet \simeq 1_{\boldsymbol{P_\bullet}}$ and $f_\bullet g_\bullet \simeq 1_{\boldsymbol{Q_\bullet}}$, so

$$\mathrm{Hom}_A(g_\bullet f_\bullet, N) \simeq \mathrm{Hom}_A(1_{\boldsymbol{P_\bullet}}, N) = 1_{\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)},$$

$$\mathrm{Hom}_A(f_\bullet g_\bullet, N) \simeq \mathrm{Hom}_A(1_{\boldsymbol{Q_\bullet}}, N) = 1_{\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N)},$$

$$g_\bullet f_\bullet \otimes_A N \simeq 1_{\boldsymbol{P_\bullet}} \otimes_A N = 1_{\boldsymbol{P_\bullet} \otimes_A N},$$

$$f_\bullet g_\bullet \otimes_A N \simeq 1_{\boldsymbol{Q_\bullet}} \otimes_A N = 1_{\boldsymbol{Q_\bullet} \otimes_A N}.$$

Thus taking homology and using Proposition 2.10 we have for every $i \in \mathbb{N}$

$$1_{H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N))} = \mathrm{Hom}_A(g_\bullet f_\bullet, N)_{*_i} = \mathrm{Hom}_A(f_\bullet, N)_{*_i} \mathrm{Hom}_A(g_\bullet, N)_{*_i},$$

$$1_{H^i(\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N))} = \mathrm{Hom}_A(f_\bullet g_\bullet, N)_{*_i} = \mathrm{Hom}_A(g_\bullet, N)_{*_i} \mathrm{Hom}_A(g f_\bullet, N)_{*_i},$$

$$1_{H_i(\boldsymbol{P_\bullet} \otimes_A N)} = (g_\bullet f_\bullet \otimes_A N)_{*_i} = (g_\bullet \otimes_A N)_{*_i} (f_\bullet \otimes_A N)_{*_i},$$

$$1_{H_i(\boldsymbol{Q_\bullet} \otimes_A N)} = (f_\bullet g_\bullet \otimes_A N)_{*_i} = (f_\bullet \otimes_A N)_{*_i} (g_\bullet \otimes_A N)_{*_i}.$$

Hence $\mathrm{Hom}_A(f_\bullet, N)_{*_i}$ and $(f_\bullet \otimes_A N)_{*_i}$ are A-isomorphisms for all $i \in \mathbb{N}$. $\square$

Given L a (left or right if necessary) A-module, $(\boldsymbol{R_\bullet}, \boldsymbol{r_\bullet})$ a projective resolution of L, and $f : M \to L$ an A-morphism, applying the Comparison Theorem 2.36 to the diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{p_3} & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow & 0 \\
 & & & & & & & & \big\downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow{r_3} & R_2 & \xrightarrow{r_2} & R_1 & \xrightarrow{r_1} & R_0 & \xrightarrow{r_0} & L & \longrightarrow & 0
\end{array}$$

yields a chain map $f_{\boldsymbol{P_\bullet}}^{\boldsymbol{R_\bullet}} : \boldsymbol{P_\bullet} \to \boldsymbol{R_\bullet}$. Proceeding as in the proof of Proposition 2.37 we obtain

A-morphisms in homology for every $i \in \mathbb{N}$.

$$\mathrm{Hom}_A(f_{\boldsymbol{P_\bullet}}^{\boldsymbol{R_\bullet}}, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{R_\bullet}, N)) \to H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)),$$
$$(f_{\boldsymbol{P_\bullet}}^{\boldsymbol{R_\bullet}} \otimes_A N)_{*_i} : H_i(\boldsymbol{P_\bullet} \otimes_A N) \to H_i(\boldsymbol{R_\bullet} \otimes_A N).$$

Proposition 2.38. *Let L, M, N be (left or right if necessary) A-modules, let $f : M \to L$ be an A-morphism, let $(\boldsymbol{R_\bullet}, \boldsymbol{r_\bullet})$ and $(\boldsymbol{S_\bullet}, \boldsymbol{s_\bullet})$ be projective resolutions of L, let $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ and $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ be projective resolutions of M, for all $i \in \mathbb{N}$ let*

$$\mathrm{Hom}_A(1_M{}_{\boldsymbol{Q_\bullet}}^{\boldsymbol{P_\bullet}}, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)) \cong H^i(\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N)),$$
$$(1_M{}_{\boldsymbol{P_\bullet}}^{\boldsymbol{Q_\bullet}} \otimes_A N)_{*_i} : H_i(\boldsymbol{P_\bullet} \otimes_A N) \cong H_i(\boldsymbol{Q_\bullet} \otimes_A N).$$

be the isomorphisms of Proposition 2.37. Then for all $i \in \mathbb{N}$ the following diagrams commute.

$$\begin{tikzcd}
H^i(\mathrm{Hom}_A(\boldsymbol{R_\bullet}, N)) \arrow[r, "\mathrm{Hom}_A(1_L{}_{\boldsymbol{S_\bullet}}^{\boldsymbol{R_\bullet}}, N)_{*_i}"] \arrow[d, "\mathrm{Hom}_A(f_{\boldsymbol{P_\bullet}}^{\boldsymbol{R_\bullet}}, N)_{*_i}"'] & H^i(\mathrm{Hom}_A(\boldsymbol{S_\bullet}, N)) \arrow[d, "\mathrm{Hom}_A(f_{\boldsymbol{Q_\bullet}}^{\boldsymbol{S_\bullet}}, N)_{*_i}"] \\
H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)) \arrow[r, "\mathrm{Hom}_A(1_M{}_{\boldsymbol{Q_\bullet}}^{\boldsymbol{P_\bullet}}, N)_{*_i}"] & H^i(\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N))
\end{tikzcd}$$

$$\begin{tikzcd}
H_i(\boldsymbol{Q_\bullet} \otimes_A N) \arrow[r, "(1_M{}_{\boldsymbol{Q_\bullet}}^{\boldsymbol{P_\bullet}} \otimes_A N)_{*_i}"] \arrow[d, "(f_{\boldsymbol{Q_\bullet}}^{\boldsymbol{S_\bullet}} \otimes_A N)_{*_i}"'] & H_i(\boldsymbol{P_\bullet} \otimes_A N) \arrow[d, "(f_{\boldsymbol{P_\bullet}}^{\boldsymbol{S_\bullet}} \otimes_A N)_{*_i}"] \\
H_i(\boldsymbol{S_\bullet} \otimes_A N) \arrow[r, "(1_L{}_{\boldsymbol{S_\bullet}}^{\boldsymbol{R_\bullet}} \otimes_A N)_{*_i}"] & H_i(\boldsymbol{R_\bullet} \otimes_A N)
\end{tikzcd}$$

Proof. Applying the Comparison Theorem 2.36 to the diagrams

$$\begin{tikzcd}
\cdots \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "s_1"] & S_0 \arrow[r, "s_0"] & L \arrow[r] \arrow[d, "1_L"] & 0 \\
\cdots \arrow[r, "r_1"] & R_0 \arrow[r, "r_0"] & L \arrow[r] & 0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdots \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
\cdots \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "r_1"] & R_0 \arrow[r, "r_0"] & L \arrow[r] & 0
\end{tikzcd}$$

yields homotopic chain maps $g_\bullet, h_\bullet : \boldsymbol{Q_\bullet} \to \boldsymbol{R_\bullet}$. Using again the uniqueness statement of the

Comparison Theorem 2.36, we have $g_\bullet \simeq 1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}} f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}}$ and $h_\bullet \simeq f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} 1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}}$. Hence

$$\begin{aligned}
\operatorname{Hom}_A(1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}} f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}}, N) \simeq \operatorname{Hom}_A(g_\bullet, N) \simeq \operatorname{Hom}_A(h_\bullet, N) \simeq \operatorname{Hom}_A(f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} 1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}}, N),\\
1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}} f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N \simeq g_\bullet \otimes_A N \simeq g_\bullet \otimes_A N \simeq f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} 1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N
\end{aligned}$$

so taking homology and using Proposition 2.10 we have for every $i \in \mathbb{N}$

$$\begin{aligned}
\operatorname{Hom}_A(1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}}, N)_{*_i} \operatorname{Hom}_A(f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}}, N)_{*_i} &= (\operatorname{Hom}_A(f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}} 1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}}, N))_{*_i} = \operatorname{Hom}_A(g_\bullet, N)_{*_i}\\
&= \operatorname{Hom}_A(h_\bullet, N)_{*_i} = \operatorname{Hom}_A(f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} 1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}}, N)_{*_i}\\
&= \operatorname{Hom}_A(1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}}, N)_{*_i} \operatorname{Hom}_A(f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}}, N)_{*_i},\\
(1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}} \otimes_A N)_{*_i} (f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N)_{*_i} &= (1_{L}{}^{\boldsymbol{R_\bullet}}_{\boldsymbol{S_\bullet}} f^{\boldsymbol{S_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N)_{*_i} = (g_\bullet \otimes_A N)_{*_i} = (h_\bullet \otimes_A N)_{*_i}\\
&= (f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} 1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N)_{*_i} = (f^{\boldsymbol{R_\bullet}}_{\boldsymbol{P_\bullet}} \otimes_A N)_{*_i} (1_{M}{}^{\boldsymbol{P_\bullet}}_{\boldsymbol{Q_\bullet}} \otimes_A N)_{*_i}.
\end{aligned}$$

□

Although we explicitly used Ext and Tor, our reasoning in Proposition 2.37 and Proposition 2.38 in fact holds for derived functors between abelian categories. The definitions of Ext and Tor seem to be asymmetric with respect to their components. There are however alternative ways of computing them that clarify this perceived asymmetry. Moreover, an important internal characterization of Ext is in terms of exact sequences of finite length. It turns out that for each $i \in \mathbb{N}$ and pair of A-modules M, N, we can see $\operatorname{Ext}^i_A(M, N)$ as the equivalence classes of i-extensions of M by N.

It is natural to ask what happens when Ext and Tor are applied to short exact sequences. The answer is the following four long exact sequences.

Theorem 2.39 (First long exact sequence for Ext)**.** *Let K, L, M, N be A-modules, and $0 \to K \to L \to M \to 0$ be an exact sequence. Then there is a long exact sequence of $\mathbb{Z}$-modules*

$$\begin{aligned}
0 \to \operatorname{Hom}_A(N, K) \longrightarrow \operatorname{Hom}_A(N, L) \longrightarrow \operatorname{Hom}_A(N, M) \longrightarrow \operatorname{Ext}^1_A(N, K) \to \cdots\\
\to \operatorname{Ext}^i_A(N, K) \longrightarrow \operatorname{Ext}^i_A(N, L) \longrightarrow \operatorname{Ext}^i_A(N, M) \longrightarrow \operatorname{Ext}^{i+1}_A(N, K) \to \cdots
\end{aligned}$$

Theorem 2.40 (Second long exact sequence for Ext)**.** *Let K, L, M, N be A-modules, and $0 \to K \to L \to M \to 0$ be an exact sequence. Then there is a long exact sequence of $\mathbb{Z}$-modules*

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M,N) &\longrightarrow \mathrm{Hom}_A(L,N) \longrightarrow \mathrm{Hom}_A(K,N) \longrightarrow \mathrm{Ext}^1_A(M,N) \to \cdots \\ \to \mathrm{Ext}^i_A(M,N) &\longrightarrow \mathrm{Ext}^i_A(L,N) \longrightarrow \mathrm{Ext}^i_A(K,N) \longrightarrow \mathrm{Ext}^{i+1}_A(M,N) \to \cdots \end{aligned}$$

Theorem 2.41 (First long exact sequence for Tor)**.** *Let K, L, M be right A-modules, N be a left A-module, and $0 \to K \to L \to M \to 0$ be an exact sequence. Then there is a long exact sequence of $\mathbb{Z}$-modules*

$$\begin{aligned} \cdots \to \mathrm{Tor}^A_{i+1}(M,N) &\longrightarrow \mathrm{Tor}^A_i(K,N) \longrightarrow \mathrm{Tor}^A_i(L,N) \longrightarrow \mathrm{Tor}^A_i(M,N) \to \\ \cdots \to \mathrm{Tor}^A_1(M,N) &\longrightarrow K \otimes_A N \longrightarrow L \otimes_A N \longrightarrow M \otimes_A N \to 0. \end{aligned}$$

Theorem 2.42 (Second long exact sequence for Tor)**.** *Let K, L, M be left A-modules, N be a right A-module, and $0 \to K \to L \to M \to 0$ be an exact sequence. Then there is a long exact sequence of $\mathbb{Z}$-modules*

$$\begin{aligned} \cdots \to \mathrm{Tor}^A_{i+1}(N,M) &\longrightarrow \mathrm{Tor}^A_i(N,K) \longrightarrow \mathrm{Tor}^A_i(N,L) \longrightarrow \mathrm{Tor}^A_i(N,M) \to \\ \cdots \to \mathrm{Tor}^A_1(N,M) &\longrightarrow N \otimes_A K \longrightarrow N \otimes_A L \longrightarrow N \otimes_A M \to 0. \end{aligned}$$

As an application of these long exact sequences, we have a similar result to Proposition 2.32, where we now require flatness of a different pair of modules.

Proposition 2.43. *Let $f : M \to N$ and $g : P \to Q$ be left and right A-morphisms respectively such that $\mathrm{im}(f)$ and $\mathrm{im}(g)$ are flat. Then $\ker(f \otimes g) = (\ker(f) \otimes_A P) \oplus (M \otimes_A \ker(g))$ as $\mathbb{Z}$-modules.*

Proof. Proceeding as in the proof of Proposition 2.32, we have the exact sequences

$$\begin{aligned} 0 \to \ker(f) &\xhookrightarrow{\quad\iota\quad} M \xrightarrow{\ f\ } \mathrm{im}(f) \longrightarrow 0, \\ 0 \to \ker(g) &\xhookrightarrow{\quad\kappa\quad} P \xrightarrow{\ g\ } \mathrm{im}(g) \longrightarrow 0, \\ (\ker(f) \otimes_A P) \oplus (M \otimes_A \ker(g)) &\xrightarrow{\iota \otimes 1_P + 1_M \otimes \kappa} M \otimes_A P \xrightarrow{f \otimes g} \mathrm{im}(f) \otimes \mathrm{im}(g) \longrightarrow 0, \end{aligned}$$

with $\ker(f \otimes g) = \mathrm{im}(\iota \otimes 1_P) \oplus \mathrm{im}(1_M \otimes \kappa) = \mathrm{im}(\iota \otimes 1_P + 1_M \otimes \kappa)$. Moreover by Theorem 2.41 and Theorem 2.42 we have the exact sequences

$$\operatorname{Tor}_1^A(\operatorname{im}(f), P) \longrightarrow \ker(f) \otimes_A P \xrightarrow{\iota\otimes 1_P} M \otimes_A P \xrightarrow{f\otimes 1_P} \operatorname{im}(f) \otimes_A P \longrightarrow 0,$$
$$\operatorname{Tor}_1^A(M, \operatorname{im}(g)) \longrightarrow M \otimes_A \ker(g) \xrightarrow{1_M\otimes\kappa} M \otimes_A P \xrightarrow{1_M\otimes g} M \otimes_A \operatorname{im}(g) \longrightarrow 0.$$

Since $\operatorname{im}(f)$ and $\operatorname{im}(g)$ are flat, then $\operatorname{Tor}_1^A(\operatorname{im}(f), P) = 0$ and $\operatorname{Tor}_1^A(M, \operatorname{im}(g)) = 0$ by Theorem 2.35, so $\iota \otimes 1_P : \ker(f) \otimes_A P \to M \otimes_A P$ and $1_M \otimes \kappa : M \otimes_A \ker(g) \to M \otimes_A P$ are injective, respectively. Whence $\operatorname{im}(\iota \otimes 1_P) \cong \ker(f) \otimes_A P$ and $\operatorname{im}(1_M \otimes \kappa) \cong M \otimes_A \ker(g)$ so $\ker(f \otimes g) \cong (\ker(f) \otimes_A P) \oplus (M \otimes_A \ker(g))$ as $\mathbb{Z}$-modules. □

Note that letting any choice of $\operatorname{im}(f)$ or M together with $\operatorname{im}(g)$ or P to be flat will yield, through a combination of the reasoning in Proposition 2.32 and Proposition 2.43, that $\ker(f \otimes g) = (\ker(f) \otimes_A P) \oplus (M \otimes_A \ker(g))$ as $\mathbb{Z}$-modules.

Another natural question to ask is whether the homology of the tensor product of complexes, as described in Example 2.21, can be understood in terms of the homologies of the respective complexes. This is known as the Künneth Theorem 2.44, and is a more elaborate application of Theorem 2.41.

Theorem 2.44 (Künneth Theorem)**.** *Let $(\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$, $(\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet)$ be complexes of left and right A-modules respectively such that M_i and $m_j(M_j)$ are flat for all $j \in \mathbb{Z}$. Then for each $i \in \mathbb{Z}$ there is an exact sequence of $\mathbb{Z}$-modules*

$$0 \to \bigoplus_{r+s=i} (H_r(\boldsymbol{M}_\bullet) \otimes_A H_s(\boldsymbol{N}_\bullet)) \to H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \to \bigoplus_{r+s=i-1} \operatorname{Tor}_1^A(H_r(\boldsymbol{M}_\bullet), H_s(\boldsymbol{N}_\bullet)) \to 0.$$

An important consequence arises when concentrating one of the complexes in degree zero.

Theorem 2.45 (Universal Coefficient Theorem)**.** *Let $(\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ be a complex of A-modules such that M_i and $m_j(M_j)$ are flat for all $j \in \mathbb{Z}$, and N be a left A-module. Then for each $i \in \mathbb{Z}$ there is an exact sequence of $\mathbb{Z}$-modules*

$$0 \to H_i(\boldsymbol{M}_\bullet) \otimes_A N \to H_i(\boldsymbol{M}_\bullet \otimes_A N) \to \operatorname{Tor}_1^A(H_{i-1}(\boldsymbol{M}_\bullet), N) \to 0.$$

This allows to reduce computations of $H_i(\boldsymbol{M}_\bullet \otimes_A N)$, lying in homological degree i, to computations of $\mathrm{Tor}_1^A(H_{i-1}(\boldsymbol{M}_\bullet), N)$, in homological degree 1. It is especially useful when working over fields or semisimple rings, where the above sequence splits and Tor can be readily found.

3. RELATIVE HOMOLOGICAL ALGEBRA

In this chapter we will focus on relative homological algebra over categories of modules of rings. A general setup (that may limit some of the results that can be obtained) can be found in [10, 11]. For the readers familiar with [27], most of our setup will turn out to be the relative abelian category $A \otimes_B ? : \mathcal{B} \to \mathcal{A}$ where $\mathcal{A}$, $\mathcal{B}$ are the categories of left A, B modules respectively. Remaining in the realm of relative homological algebra, we explicitly develop all the necessary tools. Moreover, we will exploit some explicit constructions that this setting allows and that are not within reach of other theories.

Throughout the chapter we let A be an associative unitary ring and B a subring of A with $1_A \in B$. The modules over A and B will be unitary, namely 1_A will act as the identity element. Moreover, sometimes we will see an A-module as a B-module by restriction.

3.1 Definitions and basic properties

In Chapter 2 we outlined some properties of A-modules. We are now interested in the same modules when we forget some of their structure over A, and instead we know some structure over B.

Definition 3.1. Let

$$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

be a sequence of A-modules. It is called (A, B)-*exact* whenever it is exact as A-modules and for all $i \in \mathbb{Z}$ we have $\ker(d_i)$ is a direct summand of M_i as a B-module.

While this is the original definition given by Hochschild in [24], it is not the most useful when trying to prove results. For this, it is convenient to see (A, B)-exactness as two complementary conditions: Exactness when seen as A-modules and splitting when seen as B-modules.

Proposition 3.2. *For* $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ *a sequence of* A*-modules, the following are equivalent:*

1. $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ *is* (A, B)*-exact.*

2. $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ *as a sequence of* B*-modules satisfies:*

   (a) $d_i d_{i+1} = 0$ *for all* $i \in \mathbb{Z}$*, and*

   (b) *the identity* $1_{\boldsymbol{M_\bullet}}$ *is null homotopic. That is, there exists a sequence*

$$\cdots \xleftarrow[s_{i+1}]{} M_{i+1} \xleftarrow[s_i]{} M_i \xleftarrow[s_{i-1}]{} \cdots$$

   of B*-morphisms such that* $d_{i+1}s_i + s_{i-1}d_i = 1_{M_i}$ *for all* $i \in \mathbb{Z}$*.*

3. $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ *is split exact as a sequence of* B*-modules.*

Proof. (1.⇒2.) Since the sequence is (A, B)-exact, in particular it is exact as a sequence of A-modules. This implies that it is exact as a sequence of B-modules: The condition $\mathrm{im}(d_i) = \ker(d_{i-1})$ is independent of the module structure. This immediately yields that $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. By (A, B)-exactness we also have $\ker(d_i)$ is a direct summand of M_i as a B-module, namely $M_i \cong Q_i \oplus \ker(d_i)$ for Q_i some B-module. Without loss of generality, we may take $M_i = Q_i \oplus \ker(d_i)$. We then have the short exact sequence

$$0 \longrightarrow \ker(d_i) \hookrightarrow Q_i \oplus \ker(d_i) \xrightarrow{d_i} d_i(Q_i) \longrightarrow 0$$

and the First Isomorphism Theorem implies that $Q_i \cong d_i(Q_i) \subseteq M_{i-1}$ as B-modules, and in particular we have

$$M_i \cong d_i(Q_i) \oplus \ker(d_i) = \mathrm{im}(d_i) \oplus \ker(d_i) = \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}).$$

Notice that this isomorphism is $d_i \oplus 1_{\mathrm{im}(d_{i+1})}$, simply applying d_i on the first component of M_i. Consider now the clearly exact sequence of B-modules

$$\cdots \xrightarrow{d_{i+1}} \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) \xrightarrow{d_i} \mathrm{im}(d_{i-1}) \oplus \mathrm{im}(d_i) \xrightarrow{d_{i-1}} \cdots$$

where $d_i : \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) \to \mathrm{im}(d_{i-1}) \oplus \mathrm{im}(d_i)$ is the B-morphism that sends $\mathrm{im}(d_i)$ to itself and $\mathrm{im}(d_{i+1})$ to zero. We have a chain isomorphism of complexes

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{d_{i+1}} & M_i & \xrightarrow{d_i} & M_{i-1} & \xrightarrow{d_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle d_i \oplus 1_{\mathrm{im}(d_{i+1})}} & & \downarrow{\scriptstyle d_{i-1} \oplus 1_{\mathrm{im}(d_i)}} & & \\
\cdots & \xrightarrow{d_{i+1}} & \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) & \xrightarrow{d_i} & \mathrm{im}(d_{i-1}) \oplus \mathrm{im}(d_i) & \xrightarrow{d_{i-1}} & \cdots
\end{array}
$$

so proving the existence of a B-homotopy for the second complex is enough to finish the implication. Set

$$h_i : \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) \to \mathrm{im}(d_{i+1}) \oplus \mathrm{im}(d_{i+2})$$

the B-morphism that sends $\mathrm{im}(d_{i+1})$ to itself and $\mathrm{im}(d_i)$ to zero. Now $d_{i+1}h_i(\mathrm{im}(d_i)\oplus\mathrm{im}(d_{i+1})) = 0 \oplus \mathrm{im}(d_{i+1})$ and $h_{i-1}d_i(\mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1})) = \mathrm{im}(d_i) \oplus 0$, so $d_{i+1}h_i + h_{i-1}d_i = 1_{\mathrm{im}(d_i)\oplus\mathrm{im}(d_{i+1})}$ for all $i \in \mathbb{Z}$ as desired.

(2.⇒3.) Since $d_i d_{i+1} = 0$ makes $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ a complex, this follows from Proposition 2.16.

(3.⇒1.) Again using that the condition $\mathrm{im}(d_i) = \ker(d_{i-1})$ is independent of the module structure, having exactness as B-modules yields exactness as A-modules. Following the proof of Proposition 2.16, whenever $(\boldsymbol{M_\bullet}, \boldsymbol{d_\bullet})$ is split exact as B-modules we have

$$M_i \cong \mathrm{im}(d_i) \oplus \mathrm{im}(d_{i+1}) = \mathrm{im}(d_i) \oplus \ker(d_i)$$

as B-modules, making $\ker(d_i)$ a direct summand of M_i as B-modules, as desired.

$\square$

Note how the construction used for (1.⇒2.) closely resembles the proof of Proposition 2.16. We provide the complete proof for clarity, but in fact that implication can be seen as specific setup of the aforementioned result. In fact, using the forward direction of that result it is also possible to prove the above as (1.⇒3.⇒2.⇒1.), where analogously (1.⇒2.) can be seen as specific setup of Proposition 2.16.

The following useful result states that we can extract shorter (A, B)-exact sequences from longer (A, B)-exact sequences.

Lemma 3.3. *Let L, M, N, P, Q be A-modules such that*

$$\begin{tikzcd} L \arrow[r, "f"] & M \arrow[l, bend left, "r"] \arrow[r, "g"] & N \arrow[l, bend left, "s"] \arrow[r, "h"] & P \arrow[l, bend left, "t"] \arrow[r, "v"] & Q \arrow[l, bend left, "u"] \end{tikzcd}$$

is an (A, B)-exact sequence. Then there is a short (A, B)-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(f) \arrow[r, "\alpha"] & N \arrow[l, bend left, "\gamma"] \arrow[r, "\beta"] & \ker(v) \arrow[l, bend left, "\delta"] \arrow[r] & 0. \end{tikzcd}$$

Proof. The exactness of the sequence is well known, as follows. Define $\alpha : \operatorname{coker}(f) \to N$ as $\alpha(\overline{m}) = g(m)$ for all $m \in M$. To see that it is well defined, let $m_1, m_2 \in M$ with $\overline{m_1} = \overline{m_2} \in \operatorname{coker}(f)$. This means that there is $m_3 \in \operatorname{im}(f)$ with $m_1 = m_2 + m_3$, and moreover there is $l \in L$ with $f(l) = m_3$. Hence $g(m_1) = g(m_2 + f(l)) = g(m_2) + gf(l) = g(m_2)$ because $gf = 0$ by exactness. Define $\beta : N \to \ker(v)$ as $\beta(n) = h(n)$ for all $n \in N$. This is well defined since $\operatorname{im}(h) = \ker(v)$ by exactness.

($\ker(\alpha) = 0$) Let $\overline{m} \in \ker(\alpha)$ so that $g(m) = \alpha(\overline{m}) = 0$ in N. Hence $m \in \ker(g) = im(f)$ by exactness, so $\overline{m} = \overline{0}$ in $\operatorname{coker}(f)$.

($\ker(\beta) \subseteq \operatorname{im}(\alpha)$) Let $n \in \ker(\beta)$ so that $h(n) = \beta(n) = 0$ in $\ker(v) \subseteq N$. Hence $n \in \ker(h) = \operatorname{im}(g)$ by exactness, so there is $m \in M$ with $g(m) = n$. Now $\alpha(\overline{m}) = g(m) = n$ so $n \in \operatorname{im}(\alpha)$.

($\operatorname{im}(\alpha) \subseteq \ker(\beta)$) Let $n \in \operatorname{im}(\alpha)$, then there is $m \in M$ with $g(m) = \alpha(\overline{m}) = n$. Hence $\beta(n) = h(n) = hg(m) = 0$ because $hg = 0$ by exactness, so $n \in \ker(\beta)$.

($\ker(v) \subseteq \operatorname{im}(\beta)$) Let $p \in \ker(v) = \operatorname{im}(h)$ by exactness, then there is $n \in N$ with $h(n) = p$. Hence $\beta(n) = h(n) = p$ so $p \in \operatorname{im}(\beta)$.

($\operatorname{im}(\beta) \subseteq \ker(v)$) Let $p \in \operatorname{im}(\beta)$, then there is $n \in N$ with $h(n) = \beta(n) = p$. Hence $v(p) = vh(n) = 0$ because $vh = 0$ by exactness, so $p \in \ker(v)$.

The splitting as a sequence of B-modules can now be checked as follows. Define $\gamma : N \to \operatorname{coker}(f)$ as $\gamma(n) = \overline{s(n)}$ for all $n \in N$ and $\delta : \ker(v) \to N$ as $\delta(p) = r(p)$ for all $p \in \ker(v)$.

($\alpha\gamma\alpha = \alpha$) Let $\overline{m} \in \operatorname{coker}(f)$, then $\alpha\gamma\alpha(\overline{m}) = \alpha\gamma(g(m)) = \alpha(\overline{sg(m)}) = gsg(m) = g(m) = \alpha(\overline{m})$.

$(\beta\delta\beta = \beta)$ Let $n \in N$, then $\beta\delta\beta(n) = \beta\delta(h(n)) = \beta(rh(n)) = hrh(n) = h(n) = \beta(n)$.

$\square$

Definition 3.4. An A-module U is said to be (A, B)*-free* if there is X a B-submodule of U such that for every A-module M and every B-morphism $g : X \to M$ there is a unique A-morphism $h : U \to M$ with $h|_X = g$ as B-morphisms. We say that X is a *basis* of U.

$$\begin{tikzcd} U \arrow[dr, dashed, "h_A"] & \\ X \arrow[u, hook] \arrow[r, "g_B"] & M \end{tikzcd}$$

A sequence of A-modules $(\boldsymbol{U_\bullet}, \boldsymbol{d_\bullet})$ is said to be an (A, B)*-free resolution* of an A-module M when it is an (A, B)-exact sequence bounded on the right by M and U_i is (A, B)-free for all $i \in \mathbb{N}$

$$\begin{tikzcd} \cdots \arrow[r, "d_{i+1}"] & U_i \arrow[l, bend left, "s_i"] \arrow[r, "d_i"] & U_{i-1} \arrow[l, bend left, "s_{i-1}"] \arrow[r, "d_{i-1}"] & \cdots \arrow[l, bend left, "s_{i-2}"] \arrow[r, "d_1"] & U_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] & 0. \end{tikzcd}$$

The above definition is motivated by Proposition 2.23. This aligns with the treatment Alperin gave in [1], and we believe that translating the usual concept of a free module to the relative case provides valuable insight. However, this is quite unique to our approach, since other authors in for example [7, 8] prefer to use *extended A-module*, a nomenclature native of representation theory. Moreover, Definition 3.4 will recover what is known as the *induced representation*, see for example [3, 4]. For this, let G be a group having H as a subgroup, set $A = k[G]$ and $B = k[H]$, then the (A, B)-free module with basis the B-module X is precisely $k[G] \otimes_{k[H]} X$ by Proposition 3.7.

Lemma 3.5. *Let X be a B-module. Then $A \otimes_B X$ is an (A, B)-free module with basis X.*

Proof. Let M be an A-module and $g : X \to M$ a B-morphism. Define $h' : A \times X \to M$ as $h'(a, x) = ag(x)$, which is B-balanced since for all $a_1, a_2, a \in A$, $b \in B$, $x_1, x_2, x \in X$ we have

$$h'(a, x_1 + x_2) = ag(x_1 + x_2) = a(g(x_1) + g(x_2)) = ag(x_1) + ag(x_2) = h'(a, x_1) + h'(a, x_2),$$

$$h'(a_1 + a_2, x) = (a_1 + a_2)g(x) = a_1g(x) + a_2g(x) = h'(a_1, x) + h'(a_2, x),$$

$$h'(ab, x) = (ab)g(x) = a(bg(x)) = ag(bx) = h'(a, bx).$$

Hence there is a unique A-morphism $h : A \otimes_B X \to M$ satisfying $h(a \otimes x) = h'(a, x) = ag(x)$. Define a B-morphism $\iota : X \to A \otimes_B X$ as $\iota(x) = 1 \otimes x$, we can now identify $X \cong \mathrm{im}(\iota) \subseteq A \otimes_B X$. Moreover, we have $h(1 \otimes x) = h'(1, x) = g(x)$ and thus using the identification above we indeed have $h|_X = h|_{\mathrm{im}(\iota)} = g$ as B-morphisms. Suppose further that there is $f : A \otimes_B X \to M$ another A-morphism satisfying $f|_{\mathrm{im}(\iota)} = f|_X = g$ as B-morphisms, we then have $f = h$ since

$$f(a \otimes x) = f(a(1 \otimes x)) = af(1 \otimes x) = ag(x) = h(a \otimes x).$$

□

Notice how the isomorphism $X \cong \mathrm{im}(\iota) \subseteq A \otimes_B X$ above is in fact representing the canonical isomorphism of B-modules $X \cong B \otimes_B X$.

Lemma 3.6. *Let U, V be (A, B)-free modules with basis X, Y, respectively. If X is isomorphic to Y as B-modules, then U is isomorphic to V as A-modules.*

Proof. Let $g_{XY} : X \to Y$ be a B-isomorphism with inverse $g_{YX} : Y \to X$. Since U, V are (A, B)-free there are h_{UV}, h_{VU} two A-morphisms making the following diagrams commute

$$\begin{tikzcd} U \arrow[rrd, dashed, "h_{UV}"] & & \\ X \arrow[u, hook] \arrow[r, "g_{XY}"] & Y \arrow[r, hook] & V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[rrd, dashed, "h_{VU}"] & & \\ Y \arrow[u, hook] \arrow[r, "g_{YX}"] & X \arrow[r, hook] & U \end{tikzcd}$$

so by the following commutative diagrams we have $(h_{VU}h_{UV})|_X = 1_X$ and $(h_{VU}h_{UV})|_Y = 1_Y$.

$$\begin{tikzcd} U \arrow[r, "h_{UV}"] & V \arrow[r, "h_{VU}"] & U \\ X \arrow[u, hook] \arrow[r, "g_{XY}"] & Y \arrow[u, hook] \arrow[r, "g_{YX}"] & X \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "h_{VU}"] & U \arrow[r, "h_{UV}"] & V \\ Y \arrow[u, hook] \arrow[r, "g_{YX}"] & X \arrow[u, hook] \arrow[r, "g_{XY}"] & Y \arrow[u, hook] \end{tikzcd}$$

Since $1_U|_X = 1_X$ and $1_V|_Y = 1_Y$, by the uniqueness of Definition 3.4 we have $(h_{VU}h_{UV}) = 1_U$ and $(h_{VU}h_{UV}) = 1_V$. Hence h_{UV} has inverse h_{VU} and $U \cong V$ as A-modules. □

The following characterization of (A, B)-free modules states that our definition is suitable, since it now aligns with [27, Chapter IX, Section 6].

Proposition 3.7. *An A-module U is (A, B)-free with basis X if and only if U is isomorphic to $A \otimes_B X$ as A-modules.*

Proof. ($\Rightarrow$) If U is (A, B)-free with respect to X then by Lemma 3.5 we have both U and $A \otimes_B X$ are (A, B)-free with respect to X. Then by Lemma 3.6 we have U and $A \otimes_B X$ are isomorphic as A-modules.

($\Leftarrow$) If U is isomorphic with $A \otimes_B X$ as A-modules, then by Lemma 3.5 we have U is (A, B)-free with respect to X.

□

Because of the result above, as in the case of free modules over a ring, there is no need for specific examples of relative free modules: All (A, B)-free modules are isomorphic to $A \otimes_B X$ for X some B-module.

Remark 3.8. A free A-module U is (A, B)-free for every subring B of A. Namely if $U \cong \bigoplus_{x \in X} A_x$ for some set X, then $A \otimes_B (\bigoplus_{x \in X} B_x) \cong \bigoplus_{x \in X} (A \otimes_B B_x) \cong \bigoplus_{x \in X} A_x \cong U$ whence U has basis the free B-module $\bigoplus_{x \in X} B_x$.

Example 3.9. Consider the $\mathbb{Z}[x]$-module $(\mathbb{Z}/(i))[x]$ for a fixed $i \in \mathbb{N}$. We claim that it is $(\mathbb{Z}[x], \mathbb{Z})$-free with basis the $\mathbb{Z}$-module $\mathbb{Z}/(i)$, namely $\mathbb{Z}[x] \otimes_{\mathbb{Z}} (\mathbb{Z}/(i)) \cong (\mathbb{Z}/(i))[x]$. To see this, consider the $\mathbb{Z}[x]$-morphism $f : \mathbb{Z}[x] \otimes_{\mathbb{Z}} (\mathbb{Z}/(i)) \to (\mathbb{Z}/(i))[x]$ defined via $f(p(x) \otimes \overline{j}) = \overline{p(x)j}$ for all $p(x) \in \mathbb{Z}[x]$ and $\overline{j} \in \mathbb{Z}/(i)$, and the $\mathbb{Z}[x]$-morphism $g : (\mathbb{Z}/(i))[x] \to \mathbb{Z}[x] \otimes_{\mathbb{Z}} (\mathbb{Z}/(i))$ defined via $g(\overline{p(x)}) = p(x) \otimes \overline{1}$ for all $\overline{p(x)} \in (\mathbb{Z}/(i))[x]$. Now f is well defined since whenever $\overline{j} = \overline{t}$ in $\mathbb{Z}/(i)$ then $f(p(x) \otimes \overline{j}) = \overline{p(x)j} = \overline{p(x)}\overline{j} = \overline{p(x)}\overline{t} = \overline{p(x)t} = f(p(x)\overline{t})$ for all $p(x) \in \mathbb{Z}[x]$. To see that g is well defined, pick $p(x), q(x) \in \mathbb{Z}[x]$ with $\overline{p(x)} = \overline{q(x)}$ in $(\mathbb{Z}/(i))[x]$. Then $\overline{p(x)} = \overline{p_0} + \cdots + \overline{p_s}x^s$ and $\overline{q(x)} = \overline{q_0} + \cdots + \overline{q_s}x^s$ for certain $s \in \mathbb{Z}$, and $\overline{p_r} = \overline{q_r}$ for all $r = 0, \ldots, s$. Whence

$$\begin{aligned} g(\overline{p(x)}) &= p(x)\overline{1} = \sum_{r=0}^{s} p_r x^r \otimes \overline{1} = \sum_{r=0}^{s} x^r \otimes \overline{p_r} \\ &= \sum_{r=0}^{s} x^r \otimes \overline{q_r} = \sum_{r=0}^{s} q_r x^r \otimes \overline{1} = q(x) \otimes \overline{1} = g(\overline{q(x)}) \end{aligned}$$

so g is well defined. Moreover, $fg(\overline{p(x)}) = f(p(x) \otimes \overline{1}) = \overline{p(x)} = 1_{(\mathbb{Z}/(i))[x]}$ and $gf(p(x) \otimes \overline{j}) = g(\overline{p(x)j}) = p(x)j \otimes \overline{1} = p(x) \otimes \overline{j} = 1_{\mathbb{Z}[x] \otimes_{\mathbb{Z}} (\mathbb{Z}/(i))}$, so they are isomorphisms, so $(\mathbb{Z}/(i))[x]$ is $(\mathbb{Z}[x], \mathbb{Z})$-free. However, $(\mathbb{Z}/(i))[x]$ is not $\mathbb{Z}[x]$-free, since if it were then $(\mathbb{Z}/(i))[x] \cong \bigoplus_{y \in Y} \mathbb{Z}[x]$ for some set Y. Now $i \in \mathbb{Z}[x]$ acts on $(\mathbb{Z}/(i))[x]$ as zero, but no element in $\mathbb{Z}[x]$ acts on $\bigoplus_{y \in Y} \mathbb{Z}[x]$ as zero, which is a contradiction. Similarly, $(\mathbb{Z}/(i))[x]$ is not $\mathbb{Z}$-free since $i \in \mathbb{Z}$ acts on it as zero, but $i \in \mathbb{Z}$ does not act as zero on $\bigoplus_{y \in Y} \mathbb{Z}$.

Definition 3.10. An A-module P is said to be (A, B)-*projective* if for every (A, B)-exact sequence $M \xrightarrow{g} N \to 0$ and every A-morphism $h : P \to N$ there is an A-morphism $h' : P \to M$ with $gh' = h$.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h'_A"'] \arrow[d, "h_A"] & \\
M \arrow[r, "g_A"] & N \arrow[l, bend left, "s_B"] \arrow[r] & 0
\end{tikzcd}$$

A sequence of A-modules $(\boldsymbol{P_\bullet}, \boldsymbol{d_\bullet})$ is said to be an (A, B)-*projective resolution* of an A-module M when it is an (A, B)-exact sequence bounded on the right by M and P_i is (A, B)-projective for all $i \in \mathbb{N}$

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+1}"] & P_i \arrow[l, bend left, "s_i"] \arrow[r, "d_i"] & P_{i-1} \arrow[l, bend left, "s_{i-1}"] \arrow[r, "d_{i-1}"] & \cdots \arrow[l, bend left, "s_{i-2}"] \arrow[r, "d_1"] & P_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] & 0.
\end{tikzcd}$$

Remark 3.11. A projective A-module P is (A, B)-projective for every subalgebra B of A. However, we now illustrate how a projective resolution of an A-module M need not be (A, B)-projective.

Let $p \in \mathbb{Z}$ be a prime, $1 \leq i < j \in \mathbb{Z}$, and consider $A = B = \mathbb{Z}/(p^j)$ with $M = \mathbb{Z}/(p^i)$ as an $\mathbb{Z}/(p^j)$-module. We have a short exact sequence $0 \to \mathbb{Z}/(p^i) \xrightarrow{p^{j-i}\cdot} \mathbb{Z}/(p^j) \xrightarrow{\pi} \mathbb{Z}/(p^i) \to 0$ where $\pi : \mathbb{Z}/(p^j) \xrightarrow{\pi} \mathbb{Z}/(p^i)$ is the canonical projection. Notice that $\mathbb{Z}/(p^i) \oplus \mathbb{Z}/(p^i) \ncong \mathbb{Z}/(p^j)$ and thus by Remark 2.17 the short exact sequence does not split. The following diagram with diagonals this short exact sequence gives a free (hence projective) resolution of $\mathbb{Z}/(p^i)$ as a $\mathbb{Z}/(p^j)$-module.

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathbb{Z}/(p^i) \arrow[ur] \arrow[dr, hook, "p^{j-i}\cdot"] & & & \\
\cdots \arrow[rr, "p^{j-i}\cdot"] \arrow[dr, two heads, "\pi"'] & & \mathbb{Z}/(p^j) \arrow[ur, two heads, "\pi"] \arrow[rr, "p^{j-i}\cdot"] & & \mathbb{Z}/(p^j) \arrow[r, two heads, "\pi"] \arrow[dr, two heads, "\pi"] & \mathbb{Z}/(p^i) \arrow[r] \arrow[d, equal] & 0 \arrow[dd, equal] \\
 & \mathbb{Z}/(p^i) \arrow[ur, hook, "p^{j-i}\cdot"] \arrow[dr] & & & & \mathbb{Z}/(p^i) \arrow[dr] & \\
0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}
$$

The middle row of this diagram is exact by construction. Suppose that this middle row does split, then by the proof of Proposition 2.16 the diagonals should be split and satisfy that $\mathbb{Z}/(p^i) \oplus \mathbb{Z}/(p^i) \cong \mathbb{Z}/(p^j)$, which is a contradiction. Hence this middle row does not split as $\mathbb{Z}/(p^j)$-modules. The middle row is then indeed a free (hence projective) resolution of $\mathbb{Z}/(p^i)$ as a $\mathbb{Z}/(p^j)$-module, but not splitting over $\mathbb{Z}/(p^j)$ means that it is not a $(\mathbb{Z}/(p^j), \mathbb{Z}/(p^j))$-projective resolution.

Proposition 3.12. *Let U be an (A, B)-free module. Then U is an (A, B)-projective module.*

Proof. Given U an (A, B)-free module, by Proposition 3.7 there is a B-module $X \subseteq U$ such that $U \cong A \otimes_B X$ as A-modules. Let $M \xrightarrow{g} N \to 0$ be an (A, B)-exact sequence with $s : N \to M$ its B-splitting, $h : A \otimes_B X \to N$ an A-morphism. We define $h'' : A \times X \to M$ as $h''(a, x) = a(sh(1 \otimes x))$, which is B-balanced since for all $a_1, a_2, a \in A$, $b \in B$, $x_1, x_2, x \in X$ we have

$$
\begin{aligned}
h''(a, x_1 + x_2) &= a(sh(x_1 + x_2)) = a(sh(x_1) + sh(x_2)) \\
&= a(sh(x_1)) + a(sh(x_2)) = h''(a, x_1) + h''(a, x_2) \\
h''(a_1 + a_2, x) &= (a_1 + a_2)(sh(x)) = a_1(sh(x)) + a_2(sh(x)) = h''(a_1, x) + h''(a_2, x) \\
h''(ab, x) &= (ab)(sh(x)) = a(b(sh(x))) = a(s(bh(x))) = a(sh(bx)) = h''(a, bx).
\end{aligned}
$$

Hence there is a unique A-morphism $h' : A \otimes_B X \to M$ with $h'(a \otimes x) = h''(a, x) = a(sh(1 \otimes x))$. Now $gh'(a \otimes x) = gh''(a, x) = g(a(sh(1 \otimes x))) = a(gsh(1 \otimes x)) = ah(1 \otimes x) = h(a \otimes x)$ so

$$\begin{tikzcd}
 & A \otimes_B X \arrow[dl, "h'"'] \arrow[d, "h"] & \\
M \arrow[r, "g"] & N \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

is a commutative diagram. $\square$

Note how, in contrast with Remark 2.25, we do not require the axiom of choice for (A, B)-free modules to be (A, B)-projective.

Remark 3.13. A more straightforward proof attempt is to note that the diagram

$$\begin{tikzcd}
 & U \arrow[d, "h_A"] & \\
M \arrow[r, "g_A"] & N \arrow[r] \arrow[l, bend left, "s_B"] & 0
\end{tikzcd} \quad \text{induces the commutative diagram} \quad \begin{tikzcd}
U \arrow[dr, dashed, "h'_A"] & \\
X \arrow[u, hook] \arrow[r, "s_B h_A"] & M.
\end{tikzcd}$$

Unfortunately it cannot be guaranteed that $h' : U \to M$ satisfies $gh' = h$, only that $gh'|_X = h|_X$.

Corollary 3.14. *Let M be an A-module. Then there is U an (A, B)-free module and an (A, B)-exact sequence $U \xrightarrow{g} M \to 0$.*

Proof. Given M an A-module, we can see it as a B-module and consider the (A, B)-free module $A \otimes_B M$. We now define $g' : A \times M \to M$ as $g'(a, m) = am$ which is B-balanced since for all $a_1, a_2, a \in A$, $b \in B$, $m_1, m_2, m \in M$ we have

$$g'(a, m_1 + m_2) = a(m_1 + m_2) = am_1 + am_2 = g'(a, m_1) + g'(a, m_2)$$
$$g'(a_1 + a_2, m) = (a_1 + a_2)m = a_1 m + a_2 m = g'(a_1, m) + g'(a_2, m)$$
$$g'(ab, m) = (ab)m = a(bm) = g'(a, bm).$$

Hence there is a unique A-morphism $g : A \otimes_B M \to M$ with $g(a \otimes m) = g'(a, m) = am$, which is surjective. Define a B-morphism $s : M \to A \otimes_B M$ as $s(m) = 1 \otimes m$ which is a B-splitting since we have $gs(m) = g(1 \otimes m) = m = 1_M(m)$. Thus $A \otimes_B M \xrightarrow{g} M \to 0$ is (A, B)-exact. $\square$

An alternative way of stating the above corollary is that we have enough (A, B)-free modules, in particular we have enough (A, B)-projective modules. This means that we can construct (A, B)-projective resolutions of any A-module.

In fact, (A, B)-free modules have further properties. In Theorem 3.16 we will see that the following one turns out to be equivalent to being (A, B)-projective.

Lemma 3.15. *Let U be an (A, B)-free module. For every exact sequence $M \xrightarrow{f} N \to 0$, every A-morphism $g : U \to N$, and every B-morphism $h : U \to M$ with $fh = g$ there is an A-morphism $h' : U \to M$ with $fh' = g$.*

$$\begin{tikzcd} & U \arrow[dl, dashed, bend right, "h'_A"'] \arrow[dl, "h_B"] \arrow[d, "g_A"] & \\ M \arrow[r, "f_A"'] & N \arrow[r] & 0 \end{tikzcd}$$

Proof. Given U an (A, B)-free module, by Proposition 3.7 there is a B-module $X \subseteq U$ such that $U \cong A \otimes_B X$ as A-modules. We define $h'' : A \times X \to M$ as $h''(a, x) = ah(1 \otimes x)$, which is B-balanced since for all $a_1, a_2, a \in A$, $b \in B$, $x_1, x_2, x \in X$ we have

$$h''(a, x_1 + x_2) = a(h(x_1 + x_2)) = ah(x_1) + ah(x_2) = h''(a, x_1) + h''(a, x_2)$$

$$h''(a_1 + a_2, x) = (a_1 + a_2)h(x) = a_1h(x) + a_2h(x) = h''(a_1, x) + h''(a_2, x)$$

$$h''(ab, x) = (ab)h(x) = a(bh(x)) = h''(a, bx).$$

Hence there is a unique A-morphism $h' : A \otimes_B X \to M$ with $h'(a \otimes x) = h''(a, x) = ah(1 \otimes x)$. Now $fh'(a \otimes x) = fh''(a, x) = f(ah(1 \otimes x)) = af(h(1 \otimes x)) = ag(1 \otimes x) = g(a \otimes x)$. □

Note that the uniqueness of the constructed A-morphism h' above does not guarantee the uniqueness of a morphism making the completed diagram commute. In particular, our construction relied on the B-morphism h and the fact that $fh = g$, but an A-morphism $\tilde{h} : U \to M$ may satisfy $f\tilde{h} = g$ for different reasons that do not involve h. We then may have multiple different $\tilde{h} : U \to M$ making the completed diagram commute.

Theorem 3.16. *For P an A-module, the following are equivalent:*

1. P is (A,B)-projective.

2. Every (A,B)-exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} P \to 0$ splits as a sequence of A-modules.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "g_A"] & N \arrow[r, shift left, "f_A"] & P \arrow[l, shift left, "h_B"] \arrow[l, dashed, bend left=60, "h'_A"] \arrow[r] & 0
\end{tikzcd}$$

3. P is a direct summand of an (A,B)-free module.

4. For every exact sequence $M \xrightarrow{f} N \to 0$, every A-morphism $g : P \to N$, and every B-morphism $h : P \to M$ with $fh = g$ there is an A-morphism $h' : P \to M$ with $fh' = g$.

$$\begin{tikzcd}
 & P \arrow[d, "g_A"] \arrow[dl, "h_B"] \arrow[dl, dashed, bend right, "h'_A"'] & \\
M \arrow[r, "f_A"'] & N \arrow[r] & 0
\end{tikzcd}$$

Proof. (1.$\Rightarrow$2.) We can fit the (A,B)-exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} P \to 0$ in the diagram

$$\begin{tikzcd}
 & & & P \arrow[d, "1_P"] \arrow[dl, dashed, "h'_A"'] & \\
0 \arrow[r] & M \arrow[r, "g_A"] & N \arrow[r, shift left, "f_A"] & P \arrow[l, shift left, "h_B"] \arrow[r] & 0
\end{tikzcd}$$

where h is a B-morphism. Since P is (A,B)-projective there is an A-morphism $h' : P \to N$ with $fh' = 1_P$, and thus by Remark 2.15 the short exact sequence splits as A-modules.

(2.$\Rightarrow$3.) We define $f' : A \times P \to P$ as $f'(a,p) = ap$, which is B-balanced since for all $a_1, a_2, a \in A$, $b \in B$, $p_1, p_2, p \in P$ we have

$$\begin{aligned}
f'(a, p_1 + p_2) &= a(p_1 + p_2) = ap_1 + ap_2 = f'(a,p_1) + f'(a,p_2) \\
f'(a_1 + a_2, x) &= (a_1 + a_2)p = a_1 p + a_2 p = f'(a_1,p) + h''(a_2,p) \\
f'(ab, x) &= (ab)p = a(bp) = f'(a, bx).
\end{aligned}$$

Hence there is a unique A-morphism $f : A \otimes_B P \to P$ with $f(a \otimes p) = f'(a,p) = ap$. We define $h : P \to A \otimes_B P$ as $h(p) = 1 \otimes p$, which is a B-morphism. Consider the exact sequence of A-modules $0 \to \ker(f) \hookrightarrow A \otimes_B P \xrightarrow{f} P \to 0$, since $fh = 1_P$ by Remark 2.15 it splits as B-modules, and thus it is an (A, B)-exact sequence. Hence by hypothesis it splits as A-modules, inducing an A-morphism $h' : P \to A \otimes_B P$ with $fh' = 1_P$. Then Remark 2.17 yields that P is a direct summand of the (A, B)-free module $A \otimes_B P$.

(3.⇒4.) Let P be a direct summand of an (A, B)-free module U, let $\pi : U \to P$, $\iota : P \to U$ be the canonical projection and inclusion respectively. Given an exact sequence $M \xrightarrow{f} N \to 0$, an A-morphism $g : P \to N$, and a B-morphism $h : P \to M$ with $fh = g$, we can fit them in the following diagram.

$$\begin{tikzcd}
 & U \arrow[dl, dashed, bend right, "h''_A"'] \arrow[dl, "h_B\pi"] \arrow[d, shift left, "\pi"] & \\
 & P \arrow[u, shift left, "\iota"] \arrow[dl, "h_B"] \arrow[d, "g_A"] & \\
M \arrow[r, "f_A"'] & N \arrow[r] & 0
\end{tikzcd}$$

By Lemma 3.15 applied to the exact sequence $M \xrightarrow{f} N \to 0$, the A-morphism $g\pi : U \to N$, and the B-morphism $h\pi : U \to M$, there exists an A-morphism $h'' : U \to M$ with $fh'' = g\pi$. Now $h' = h''\iota$ is an A-morphism $h' : P \to M$ with $fh' = fh''\iota = g\pi\iota = g1_P = g$.

(4.⇒2.) We can fit the (A, B)-exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} P \to 0$ in the diagram

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed, bend right, "h'_A"'] \arrow[dl, "h_B"] \arrow[d, "1_P"] & \\
0 \arrow[r] & M \arrow[r, "g_A"] & N \arrow[r, "f_A"] & P \arrow[r] & 0
\end{tikzcd}$$

where h is a B-morphism and $fh = 1_P$. By hypothesis, there is an A-morphism $h' : P \to N$ with $fh' = 1_P$, and thus by Remark 2.15 the short exact sequence splits as A-modules.

(3.⇒1.) Let P be a direct summand of an (A, B)-free module U, let $\pi : A \otimes_B X \to P$, $\iota : P \to A \otimes_B X$ be the canonical projection and inclusion respectively. Given an (A, B)-exact sequence $M \xrightarrow{f} N \to 0$ and an A-morphism $g : P \to N$, we can fit them in the following diagram.

$$\begin{tikzcd}
& U \arrow[d, shift left, "\pi"] \arrow[ddl, dashed, "h''_A"'] & \\
& P \arrow[u, shift left, "\iota"] \arrow[d, "g_A"] & \\
M \arrow[r, shift left, "f_A"] & N \arrow[l, shift left, "h_B"] \arrow[r] & 0
\end{tikzcd}$$

Since (A, B)-free modules are (A, B)-projective, the (A, B)-exact sequence $M \xrightarrow{f} N \to 0$ and the A-morphism $g\pi : U \to N$ imply the existence of an A-morphism $h'' : U \to M$ with $fh'' = g\pi$. Now $h' = h''\iota$ is an A-morphism $h' : P \to M$ with $fh' = fh''\iota = g\pi\iota = g1_P = g$, hence P is an (A, B)-projective module.

$\square$

Example 3.17. We now present several behaviors of (A, B)-projective modules.

1. Consider the $\mathbb{Z}[i]$-module $\mathbb{Q}[i]$. Let $0 \to M \xrightarrow{g} N \xrightarrow{f} \mathbb{Q}[i] \to 0$ be a $(\mathbb{Z}[i], \mathbb{Z})$-exact sequence. In particular, there is a $\mathbb{Z}$-morphism $h : \mathbb{Q}[i] \to N$ such that $fh = 1_{\mathbb{Q}[i]}$. Define

$$\begin{aligned}
\mathbb{Q}[i] &\xrightarrow{\quad h' \quad} N \\
\tfrac{p}{q} + \tfrac{r}{s}i &\longmapsto h\left(\tfrac{p}{q}\right) + ih\left(\tfrac{r}{s}\right)
\end{aligned}$$

which is a $\mathbb{Z}[i]$-morphism since for all $a, b, p, q, r, s, p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 \in \mathbb{Z}$ we have

$$\begin{aligned}
h'\left(\frac{p_1}{q_1} + \frac{r_1}{s_1}i + \frac{p_2}{q_2} + \frac{r_2}{s_2}i\right) &= h'\left(\frac{p_1q_2 + p_2q_1}{q_1q_2} + \frac{r_1s_2 + r_2s_1}{s_1s_2}i\right) = h\left(\frac{p_1q_2 + p_2q_1}{q_1q_2}\right) \\
&+ ih\left(\frac{r_1s_2 + r_2s_1}{s_1s_2}\right) = h\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) + ih\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right) = h\left(\frac{p_1}{q_1}\right) + h\left(\frac{p_2}{q_2}\right) \\
&+ ih\left(\frac{r_1}{s_1}\right) + ih\left(\frac{r_2}{s_2}\right) = h'\left(\frac{p_1}{q_1} + \frac{r_1}{s_1}i\right) + h'\left(\frac{p_2}{q_2} + \frac{r_2}{s_2}i\right) \\
h'\left((a + bi)\left(\frac{p}{q} + \frac{r}{s}i\right)\right) &= h'\left(\frac{ap}{q} + \frac{ar}{s}i + \frac{bp}{q}i - \frac{br}{s}\right) = h\left(\frac{ap}{q}\right) + ih\left(\frac{ar}{s}\right) \\
&+ ih\left(\frac{bp}{q}\right) - h\left(\frac{br}{s}\right) = ah\left(\frac{p}{q}\right) + aih\left(\frac{r}{s}\right) + bih\left(\frac{p}{q}\right) - bh\left(\frac{r}{s}\right) \\
&= (a + bi)\left(h\left(\frac{p}{q}\right) + ih\left(\frac{r}{s}\right)\right) = (a + bi)h'\left(\frac{p}{q} + \frac{r}{s}i\right).
\end{aligned}$$

Moreover for all $p, q, r, s \in \mathbb{Z}$ we have the equality of $\mathbb{Z}[i]$-morphisms

$$fh'\left(\frac{p}{q}+\frac{r}{s}i\right) = f\left(h\left(\frac{p}{q}\right)+ih\left(\frac{r}{s}\right)\right) = fh\left(\frac{p}{q}\right)+ifh\left(\frac{r}{s}\right) = \frac{p}{q}+\frac{r}{s}i = 1_{\mathbb{Q}[i]}\left(\frac{p}{q}+\frac{r}{s}i\right)$$

and thus $\mathbb{Q}[i]$ is $(\mathbb{Z}[i], \mathbb{Z})$-projective by Theorem 3.16. This is a particular case of Proposition 3.12 since $\mathbb{Q}[i] \cong \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Z}[i]$-modules, and thus $\mathbb{Q}[i]$ is $(\mathbb{Z}[i], \mathbb{Z})$-free. To see this, consider the $\mathbb{Z}[i]$-morphism $m : \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[i]$ given by $m((a+bi) \otimes (p/q)) = p(a+bi)/q$ for all $a, b, p, q \in \mathbb{Z}$. A generic element of $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by $\sum_{r=1}^{s} (a_r + b_r i) \otimes (p_r/q_r)$ with $a_r, b_r, p_r, q_r \in \mathbb{Z}$ for all $r = 1, \ldots, s$. Now

$$\begin{aligned}
\sum_{r=1}^{s} (a_r + b_r i) \otimes \frac{p_r}{q_r} &= \sum_{r=1}^{s} (a_r + b_r i) \otimes \frac{q_1 \cdots q_{r-1} p_r q_{r+1} \cdots q_s}{q_1 \cdots q_s} \\
&= \sum_{r=1}^{s} q_1 \cdots q_{r-1} p_r q_{r+1} \cdots q_s (a_r + b_r i) \otimes \frac{1}{q_1 \cdots q_s} \\
&= \left(\sum_{r=1}^{s} q_1 \cdots q_{r-1} p_r q_{r+1} \cdots q_s (a_r + b_r i)\right) \otimes \frac{1}{q_1 \cdots q_s} = (a+bi) \otimes \frac{1}{q}
\end{aligned}$$

is a pure tensor since $a = \sum_{r=1}^{s} q_1 \cdots q_{r-1} p_r q_{r+1} \cdots q_s a_r$, $b = \sum_{r=1}^{s} q_1 \cdots q_{r-1} p_r q_{r+1} \cdots q_s b_r$, and $q = q_1 \cdots q_s$ are in $\mathbb{Z}$. Then m is injective since $0 = m((a+bi) \otimes (1/q)) = (a+bi)/q$ if and only if $a = 0$ and $b = 0$, whence $(a+bi) \otimes (1/q) = 0$ in $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q}$. Also m is surjective since $m((ps + rqi) \otimes (1/qs)) = (p/q) + (r/s)i$, so m is indeed an isomorphism.

2. Let $i, j \in \mathbb{N}$ be coprime. Consider $A = \mathbb{Z}/(ij)$, fix $B \subseteq A$ a subring, and the polynomial ring $(\mathbb{Z}/(i))[x]$. Note that $(\mathbb{Z}/(ij))[x]$ is an (A, B)-free module with basis $B[x]$ since $(\mathbb{Z}/(ij))[x] \cong \bigoplus_{r \in \mathbb{N}} \mathbb{Z}/(ij) \cong \bigoplus_{r \in \mathbb{N}} \mathbb{Z}/(ij) \otimes_B B \cong \mathbb{Z}/(ij) \otimes_B (\bigoplus_{r \in \mathbb{N}} B) \cong \mathbb{Z}/(ij) \otimes_B B[x]$. Moreover, note that $(\mathbb{Z}/(ij))[x] \cong (\mathbb{Z}/(i))[x] \oplus (\mathbb{Z}/(j))[x]$ as $\mathbb{Z}/(ij)$-modules, whence $(\mathbb{Z}/(i))[x]$ is (A, B)-projective by Theorem 3.16. In fact, the above reasoning shows that $(\mathbb{Z}/(ij))[x]$ is free as a $\mathbb{Z}/(ij)$-module, and thus $(\mathbb{Z}/(i))[x]$ is projective as a $\mathbb{Z}/(ij)$-module by Proposition 2.27, whence it must be (A, B)-projective by Remark 3.11. However, $(\mathbb{Z}/(i))[x]$ is not (A, B)-free, since if it were then as $\mathbb{Z}/(ij)$-modules we would

have $(\mathbb{Z}/(i))[x] \cong \mathbb{Z}/(ij) \otimes_B X$ for some B-module X. Now $i \in \mathbb{Z}/(ij)$ acts on $(\mathbb{Z}/(i))[x]$ as zero, but $i \in \mathbb{Z}/(ij)$ does not act as zero on $\mathbb{Z}/(ij)$ and thus does not act as zero on $\mathbb{Z}/(ij) \otimes_B X$, which is a contradiction. As in Example 3.9, $(\mathbb{Z}/(i))[x]$ is not $\mathbb{Z}/(ij)$-free since $i \in \mathbb{Z}/(ij)$ acts on it as zero, but $i \in \mathbb{Z}/(ij)$ does not act as zero on $\bigoplus_{y\in Y} \mathbb{Z}/(ij)$ for all non-empty sets Y.

3. Consider $B = k$ and the polynomial ring $A = k[x_1, \ldots, x_i]$ for some $i \in \mathbb{N}$. Let J be a non-trivial ideal of A, namely $J \neq \{0\}$ and $1 \notin J$. Notice that there is no A-module M such that $J \oplus M \cong R$ as A-modules, because otherwise there would be non-zero $j \in I$ and $m \in M$ such that $1 = j + m$, but then $j = j^2 + jm = j^2$ since $jm \in I \cap M = \{0\}$, whence $(j-1)j = 0$ contradicting that A is an integral domain. This means that the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \longrightarrow 0$$

does not split as A-modules, since if it were split we would have $A \cong (A/J) \oplus J$ by Remark 2.17 and we have seen that this cannot happen. However, the short exact sequence does split as k-modules because k is a field, and thus it is $(k[x_1, \ldots, x_i], k)$-exact. Hence A/J is not (A, B)-projective by Theorem 3.16.

This reasoning applies more generally when A is a k-algebra as well as an integral domain.

4. Let D be an integral domain that is not a field and Q its field of fractions. Every (D, D)-exact sequence $0 \to M \xrightarrow{g} N \xrightarrow{f} Q \to 0$ splits as a sequence of D-modules by hypothesis, and thus Q is (D, D)-projective by Theorem 3.16. However, Q is not D-projective. Suppose otherwise, then by Proposition 2.27 there is P a D-module with $P \oplus Q \cong \bigoplus_{x\in X} D_x$ for a set X. In particular, there is a D-morphism $h' : Q \to \bigoplus_{x\in X} D_x$ which induces a D-morphism $h : Q \to D$. Pick a non-invertible $d \in D$, then

$$dh(1)h\left(\frac{1}{dh(1)}\right) = h\left(\frac{dh(1)}{dh(1)}\right) = h(1) \quad \text{so} \quad dh\left(\frac{1}{dh(1)}\right) = 1$$

since D is an integral domain. Whence d is invertible, a contradiction.

5. Consider the $\mathbb{Z}[x]$-module $\mathbb{Q}$. Note that the $\mathbb{Z}[x]$-morphism $m : \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[x]$ given by $m(p(x) \otimes (a/b)) = p(x)a/q$ for all $p(x) \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$ is an isomorphism. A generic element of $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by $\sum_{r=1}^{s} p_r(x) \otimes (a_r/b_r)$ with $p_r(x) \in \mathbb{Z}[x]$ and $a_r, b_r \in \mathbb{Z}$ for all $r = 1, \ldots, s$. Now

$$\begin{aligned}\sum_{r=1}^{s} p_r(x) \otimes \frac{a_r}{b_r} &= \sum_{r=1}^{s} p_r(x) \otimes \frac{b_1 \cdots b_{r-1} a_r b_{r+1} \cdots b_s}{b_1 \cdots b_s} \\ &= \sum_{r=1}^{s} b_1 \cdots b_{r-1} a_r b_{r+1} \cdots b_s p_r(x) \otimes \frac{1}{b_1 \cdots b_s} \\ &= \left(\sum_{r=1}^{s} b_1 \cdots b_{r-1} a_r b_{r+1} \cdots b_s p_r(x) \right) \otimes \frac{1}{b_1 \cdots b_s} = p(x) \otimes \frac{1}{b}\end{aligned}$$

is a pure tensor since $p(x) = \sum_{r=1}^{s} b_1 \cdots b_{r-1} a_r b_{r+1} \cdots b_s p_r(x) \in \mathbb{Z}[x]$ and $b = b_1 \cdots b_s \in \mathbb{Z}$. Then m is injective since $0 = m(p(x) \otimes (1/b)) = p(x)/b$ if and only if $p(x) = 0$, whence $p(x) \otimes (1/b) = 0$ in $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q}$. Also m is surjective since given a generic

$$\sum_{r=1}^{s} \frac{a_r}{b_r} x^r \in \mathbb{Q}[x] \quad \text{then} \quad m\left(\left(\sum_{r=1}^{s} b_0 \cdots b_{r-1} a_r b_{r+1} \cdots b_s x^r \right) \otimes \frac{1}{b_0 \cdots b_s} \right) = \sum_{r=1}^{s} \frac{a_r}{b_r} x^r$$

so m is indeed an isomorphism. Moreover, $\mathbb{Q}[x] \cong \mathbb{Q} \oplus x\mathbb{Q}[x]$ as $\mathbb{Z}[x]$-modules, and thus $\mathbb{Q}$ is $(\mathbb{Z}[x], \mathbb{Z})$-projective by Theorem 3.16. However, $\mathbb{Q}$ is not $\mathbb{Z}$-projective as we saw above. Also, $\mathbb{Q}$ is not $\mathbb{Z}[x]$-projective, since otherwise by Proposition 2.27 there would be P a $\mathbb{Z}[x]$-module with $P \oplus Q \cong \bigoplus_{x \in X} D_x$ for a set X. As above, this induces a $\mathbb{Z}[x]$-morphism $h : Q \to \mathbb{Z}[x]$ which implies that every $i \in \mathbb{Z}$ is invertible with inverse $h(1/(ih(1)))$, a contradiction. Furthermore, $\mathbb{Q}$ is not $(\mathbb{Z}[x], \mathbb{Z})$-free since $x \in \mathbb{Z}[x]$ acts on $\mathbb{Q}$ as zero, but $x \in \mathbb{Z}[x]$ does not act as zero on $\mathbb{Z}[x]$ and thus does not act as zero on $\mathbb{Z}[x] \otimes_B \mathbb{Q}$.

As before, for completeness we include the situation arising when we reverse all the arrows.

Definition 3.18. An A-module I is said to be (A, B)-*injective* if for every (A, B)-exact sequence $0 \to M \xrightarrow{g} N$ and every A-morphism $h : M \to I$ there is an A-morphism $h' : N \to I$ with

$h'g = h$.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & M \arrow[u, "h_A"] \arrow[r, "g_A"] & N \arrow[ul, dashed, "h'_A"'] \arrow[l, bend left, "s_B"] \end{tikzcd}$$

A sequence of A-modules $(\boldsymbol{I_\bullet}, \boldsymbol{d_\bullet})$ is said to be an (A, B)-*injective resolution* of an A-module M when it is an (A, B)-exact sequence bounded on the left by M and I_i is (A, B)-injective for all $i \in \mathbb{N}$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "d_{-1}"] & I_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & \cdots \arrow[l, bend left, "s_1"] \arrow[r, "d_{i-2}"] & I_{i-1} \arrow[l, bend left, "s_{i-1}"] \arrow[r, "d_{i-1}"] & I_i \arrow[l, bend left, "s_i"] \arrow[r, "d_i"] & \cdots . \arrow[l, bend left, "s_{i+1}"] \end{tikzcd}$$

We now prove that we have enough (A, B)-injective modules, and thus we can construct (A, B)-injective resolutions of any A-module.

Proposition 3.19. *Let L be a B-module. Then $\mathrm{Hom}_B(A, L)$ is an (A, B)-injective module.*

Proof. Given L a B-module, recall that $\mathrm{Hom}_B(A, L)$ is an A-module via $(a'f)(a) = f(aa')$ for all $f \in \mathrm{Hom}_B(A, N)$ and $a, a' \in A$. Let $0 \to M \xrightarrow{g} N$ be an (A, B)-exact sequence with $s : N \to M$ its B-splitting, $h : M \to \mathrm{Hom}_B(A, L)$ an A-morphism. We define

$$\begin{array}{rcl} N & \xrightarrow{h'} & \mathrm{Hom}_B(A, L) \\ n & \longmapsto & h'(n) \end{array} \quad \text{via} \quad \begin{array}{rcl} A & \xrightarrow{h'(n)} & L \\ a & \longmapsto & h(s(an))(1) \end{array}$$

where $s(an) \in M$ gives the B-morphism $h(s(an)) : A \to L$ so $h(s(an))(1) \in L$. Now $h'(n) \in \mathrm{Hom}_B(A, L)$ for all $n \in N$ since for all $a, a_1, a_2 \in A$, $b \in B$ we have

$$\begin{aligned} h'(n)(a_1 + a_2) &= h(s((a_1 + a_2)n))(1) = h(s(a_1 n + a_2 n)(1) = h(s(a_1 n) + s(a_2 n))(1) \\ &= h(s(a_1 n))(1) + h(s(a_2 n))(1) = h'(n)(a_1) + h'(n)(a_2) \\ h'(n)(ba) &= h(s(ban))(1) = h(bs(an))(1) = ((bh)(s(an)))(1) = h(s(an))(b) \\ &= b(h(s(an))(1)) = bh'(n)(a). \end{aligned}$$

Also $h' : N \to \mathrm{Hom}_B(A, L)$ is an A-morphism since for all $n, n_1, n_2 \in N$, $a, a' \in A$ we have

$$\begin{aligned} h'(n_1 + n_2)(a) &= h(s(a(n_1 + n_2)))(1) = h(s(an_1 + an_2))(1) = h(s(an_1) + s(an_2))(1) \\ &= h(s(an_1))(1) + h(s(an_2))(1) = h'(n_1)(a) + h'(n_2)(a) = (h'(n_1) + h'(n_2))(a) \\ h'(a'n)(a) &= h(s(aa'n))(1) = h'(n)(aa') = (a'(h'(n))(a). \end{aligned}$$

Moreover $h'g = h$ since for all $m \in M$, $a \in A$ we have

$$h'(g(m))(a) = h(s(ag(m)))(1) = h(s(g(am)))(1) = h(am)(1) = (ah(m))(1) = h(m)(a)$$

so

$$\begin{tikzcd} & \mathrm{Hom}_B(A, L) & \\ 0 \arrow[r] & M \arrow[u, "h"] \arrow[r, "g"] & N \arrow[ul, "h'"'] \arrow[l, bend left, "s"] \end{tikzcd}$$

is a commutative diagram. $\square$

Corollary 3.20. *Let M be an A-module. Then there is I an (A, B)-injective module and an (A, B)-exact sequence $0 \to M \xrightarrow{g} I$.*

Proof. Given M an A-module, we can see it as a B-module and consider the (A, B)-injective module $\mathrm{Hom}_B(A, M)$. We now define

$$\begin{array}{rcl} M & \xrightarrow{g} & \mathrm{Hom}_B(A, M) \\ m & \longmapsto & g(m) \end{array} \quad \text{via} \quad \begin{array}{rcl} A & \xrightarrow{g(m)} & M \\ a & \longmapsto & am \end{array}$$

where $g(m) : A \to M$ is a B-morphism for all $m \in M$ since for all $a, a_1, a_2 \in A$, $b \in B$ we have

$$\begin{aligned} g(m)(a_1 + a_2) &= (a_1 + a_2)m = a_1 m + a_2 m = g(m)(a_1) + g(m)(a_2) \\ g(m)(ba) &= bam = bg(m)(a), \end{aligned}$$

and $g : M \to \operatorname{Hom}_B(A, M)$ is an A-morphism since for all $m, m_1, m_2 \in M$, $a, a' \in A$ then

$$g(m_1 + m_2)(a) = a(m_1 + m_2) = am_1 + am_2 = g(m_1)(a) + g(m_2)(a) = (g(m_1) + g(m_2))(a)$$
$$g(m)(a'a) = a'am = a'g(m)(a).$$

Define $s : \operatorname{Hom}_B(A, M) \to M$ as $s(f) = f(1)$ for all $f \in \operatorname{Hom}_B(A, M)$, which is a B-morphism since for all $f, f_1, f_2 \in \operatorname{Hom}_B(A, L)$ and $b \in B$ we have

$$s(f_1 + f_2) = (f_1 + f_2)(1) = f_1(1) + f_2(1) = s(f_1) + s(f_2)$$
$$s(bf) = (bf)(1) = f(b) = bf(1) = bs(f),$$

and it is a B-splitting since we have $(sg)(m) = s(g(m)) = g(m)(1) = m = 1_M(m)$. Thus $0 \to M \xrightarrow{g} \operatorname{Hom}_B(A, L)$ is (A, B)-exact. $\square$

The above corollary states that we have enough (A, B)-injective modules, meaning that we can construct (A, B)-injective resolutions of any A-module.

Definition 3.21. An A-module F is said to be (A, B)-*flat* if for every (A, B)-exact sequence of right A-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ then

$$0 \longrightarrow L \otimes_A F \xrightarrow{f \otimes 1_F} M \otimes_A F \xrightarrow{g \otimes 1_F} N \otimes_A F \longrightarrow 0$$

is a splitting short exact sequence of $\mathbb{Z}$-modules, that is, a $(\mathbb{Z}, \mathbb{Z})$-exact sequence.

A sequence of A-modules $(\boldsymbol{F_\bullet}, \boldsymbol{d_\bullet})$ is said to be an (A, B)-*flat resolution* of an A-module M when it is an (A, B)-exact sequence bounded on the right by M and F_i is (A, B)-flat for all $i \in \mathbb{N}$

$$\begin{tikzcd} \cdots \arrow[r, "d_{i+1}"] & F_i \arrow[l, bend left, "s_i"] \arrow[r, "d_i"] & F_{i-1} \arrow[l, bend left, "s_{i-1}"] \arrow[r, "d_{i-1}"] & \cdots \arrow[l, bend left, "s_{i-2}"] \arrow[r, "d_1"] & F_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] & 0. \end{tikzcd}$$

A similar definition with a right A-module F yields right (A, B)-flat modules.

Remark 3.22. Note that if

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{L_\bullet} \arrow[r, "f_\bullet"] & \boldsymbol{M_\bullet} \arrow[l, bend left, "r_\bullet"] \arrow[r, "g_\bullet"] & \boldsymbol{N_\bullet} \arrow[l, bend left, "s_\bullet"] \arrow[r] & 0 \end{tikzcd}$$

is an (A, A)-exact sequence of complexes of right A-modules then for every A-module F

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{L}_\bullet \otimes_A F \arrow[r, "f_\bullet \otimes 1_F"] & \boldsymbol{M}_\bullet \otimes_A F \arrow[l, bend left, "r_\bullet \otimes 1_F"] \arrow[r, "g_\bullet \otimes 1_F"] & \boldsymbol{N}_\bullet \otimes_A F \arrow[l, bend left, "s_\bullet \otimes 1_F"] \arrow[r] & 0 \end{tikzcd}$$

is a splitting short exact sequence of complexes of $\mathbb{Z}$-modules. Namely, since for all $i \in \mathbb{N}$ we have that $0 \to L_i \stackrel{f_i}{\to} M_i \stackrel{g_i}{\to} N_i \to 0$ splits as a sequence of A-modules, this is equivalent by Proposition 2.16 to the identity map being null homotopic, hence $r_i f_i = 1_{L_i}$, $f_i r_i + s_i g_i = 1_{M_i}$, and $g_i s_i = 1_{N_i}$. Now for all $i \in \mathbb{Z}$

$$\begin{aligned}
&(r_i \otimes 1_F)(f_i \otimes 1_F) = (r_i f_i) \otimes 1_{N_i} = 1_{L_i} \otimes 1_F \\
&(f_i \otimes 1_F)(r_i \otimes 1_F) + (s_i \otimes 1_F)(g_i \otimes 1_F) = (f_i r_i) \otimes 1_F + (s_i g_i) \otimes 1_F \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = (f_i r_i + s_i g_i) \otimes 1_F = 1_{M_i} \otimes 1_F \\
&(g_i \otimes 1_F)(s_i \otimes 1_F) = (g_i s_i) \otimes 1_F = 1_{N_i} \otimes 1_F
\end{aligned}$$

and thus the identity map in $0 \to L_i \otimes_A F \stackrel{f_i \otimes 1_F}{\longrightarrow} M_i \otimes_A F \stackrel{g_i \otimes 1_F}{\longrightarrow} N_i \otimes_A F \to 0$ is null homotopic, which by by Proposition 2.16 is equivalent to the sequence being split exact as $\mathbb{Z}$-modules.

Moreover, relative flat modules preserve relative exact sequences.

Remark 3.23. Given $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ a right or left (A, B)-exact sequence, and F a right or left (A, B)-flat module, then $(\boldsymbol{M}_\bullet \otimes_A F, \boldsymbol{d}_\bullet \otimes 1_F)$ or $(F \otimes_A \boldsymbol{M}_\bullet, 1_F \otimes \boldsymbol{d}_\bullet)$ is a $(\mathbb{Z}, \mathbb{Z})$-exact sequence, respectively. We will now prove this for $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ a sequence of right A-modules and F a left A-module, the other statement follows analogously. Recall that by the proof of Proposition 2.16 we have $M_i \cong \operatorname{im}(d_i) \oplus \ker(d_i)$ as B-modules for each $i \in \mathbb{Z}$, and thus $0 \to \ker(d_i) \stackrel{\iota_i}{\to} M_i \stackrel{\pi_i}{\to} \operatorname{im}(d_i) \to 0$ is (A, B)-exact. Since F is (A, B)-flat, for each $i \in \mathbb{Z}$ we have the $(\mathbb{Z}, \mathbb{Z})$-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(d_i) \otimes_A F \arrow[r, hook, "\iota_i \otimes 1_F"] & M_i \otimes_A F \arrow[l, bend left, "r_i"] \arrow[r, two heads, "\pi_i \otimes 1_F"] & \operatorname{im}(d_i) \otimes_A F \arrow[l, bend left, "t_i"] \arrow[r] & 0 \end{tikzcd}$$

As we have seen before, we can construct the sequence $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ from these short exact sequences.

$$
\begin{tikzcd}
 & 0 & 0 \arrow[d] & \\
 & \mathrm{im}(d_i) \otimes_A F \arrow[u] \arrow[r, equal] \arrow[d, bend left, "t_i"] & \ker(d_{i-1}) \otimes_A F \arrow[d, "\iota_{i-1}\otimes 1_F"] & \\
\cdots \arrow[r, "d_{i+1}\otimes 1_F"] & M_i \arrow[u, two heads, "d_i\otimes 1_F"] \arrow[r, "\pi_i\otimes 1_F"] \arrow[d, bend left, "r_i"] & M_{i-1} \arrow[u, bend left, "r_{i-1}"] \arrow[r, "d_{i-1}\otimes 1_F"] \arrow[d, two heads, "\pi_{i-1}\otimes 1_F"] & \cdots \\
\cdots \arrow[r, equal] & \ker(d_i) \otimes_A F \arrow[u, hook, "\iota\otimes 1_F"] & \mathrm{im}(d_{i-1}) \otimes_A F \arrow[u, bend left, "t_{i-1}"] \arrow[r, equal] \arrow[d] & \cdots \\
 & 0 \arrow[u] & 0 &
\end{tikzcd}
$$

Here we have used $\ker(d_{i-1}) = \mathrm{im}(d_i)$ by exactness of $\boldsymbol{M_\bullet}$, and $d_i \otimes 1_F = (\iota_{i-1} \otimes 1_F)(\pi_i \otimes 1_F)$. Consider $s_{i-1} : M_{i-1} \otimes_A F \to M_i \otimes_A F$ the B-morphism given via $s_i = t_i r_{i-1}$. Now

$$
\begin{aligned}
(d_i \otimes 1_F)s_{i-1}(d_i \otimes 1_F) &= (\iota_{i-1} \otimes 1_F)(\pi_i \otimes 1_F)(t_i r_{i-1})(\iota_{i-1} \otimes 1_F)(\pi_i \otimes 1_F) \\
&= (\iota_{i-1} \otimes 1_F)(1_{\mathrm{im}(d_i)\otimes_A F})(1_{\ker(d_{i-1})\otimes_A F})(\pi_i \otimes 1_F) = d_i \otimes 1_F
\end{aligned}
$$

because $(\pi_i \otimes 1_F)t_i = 1_{\ker(d_{i-1})\otimes_A F}$ and $r_{i-1}(\iota_{i-1} \otimes 1_F) = 1_{\ker(d_{i-1})\otimes_A F}$ by Remark 2.15, whence the sequence $(\boldsymbol{M_\bullet} \otimes_A F, \boldsymbol{d_\bullet} \otimes 1_F)$ is $(\mathbb{Z}, \mathbb{Z})$-exact.

An important observation is that the splitting sequence $0 \to \ker(d_i) \otimes_A F \overset{\iota_i \otimes 1_F}{\to} M_i \otimes_A F \overset{\pi_i \otimes 1_F}{\to} \mathrm{im}(d_i) \otimes_A F \to 0$ is short. If this were to be longer than three terms, we would not be able to guarantee that the splitting morphisms $t_i : \mathrm{im}(d_i) \otimes_A F \to M_i \otimes_A F$ and $s_i : M_i \otimes_A F \to \ker(d_i) \otimes_A F$ compose to the respective identities, only that $(\pi_i \otimes 1_F)t_i(\pi_i \otimes 1_F) = (\pi_i \otimes 1_F)$ and $(\iota_i \otimes 1_F)r_i(\iota_i \otimes 1_F) = (\iota_i \otimes 1_F)$. In that case, the sequence we constructed may not split.

Proposition 3.24. *Let U be an (A, B)-free module. Then U is (A, B)-flat.*

Proof. Given U an (A, B)-free module, by Proposition 3.7 there is a B-module $Y \subseteq U$ such that $U \cong A \otimes_B Y$ as A-modules. Given now an (A, B)-exact sequence of right A-modules

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] \arrow[l, bend left, "r"] & N \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}
$$

then tensoring on the right with $A \otimes_B Y$ we obtain the complex

$$0 \longrightarrow L \otimes_A A \otimes_B Y \xrightarrow{f \otimes 1_A \otimes 1_Y} M \otimes_A A \otimes_B Y \xrightarrow{g \otimes 1_A \otimes 1_Y} N \otimes_A A \otimes_B Y \longrightarrow 0$$

which via the canonical isomorphism $A \otimes_A A \cong A$ becomes

$$0 \longrightarrow L \otimes_B Y \xrightarrow{f \otimes 1_Y} M \otimes_B Y \xrightarrow{g \otimes 1_Y} N \otimes_B Y \longrightarrow 0.$$

We now have the well defined $\mathbb{Z}$-morphisms $s \otimes 1_Y : M \otimes_B Y \to L \otimes_B Y$ and $r \otimes 1_Y : N \otimes_B Y \to M \otimes_B Y$ given respectively by $(s \otimes 1_Y)(m \otimes y) = s(m) \otimes y$ and $(r \otimes 1_Y)(n \otimes y) = r(n) \otimes y$ for all $m \in M$, $n \in N$, and $y \in Y$. Now as in Remark 2.17 we have $rf = 1_L$, $fr + sg = 1_M$, and $gs = 1_N$, yielding $(r \otimes 1_Y)(f \otimes 1_Y) = 1_{L \otimes_B Y}$, $(f \otimes 1_Y)(r \otimes 1_Y) + (s \otimes 1_Y)(g \otimes 1_Y) = 1_{M \otimes_B Y}$, and $(g \otimes 1_Y)(s \otimes 1_Y) = 1_{N \otimes_B Y}$.

$$\begin{tikzcd} 0 \arrow[r] & L \otimes_B Y \arrow[r, "f \otimes 1_Y"] & M \otimes_B Y \arrow[r, "g \otimes 1_Y"] \arrow[l, bend left, "r \otimes 1_Y"] & N \otimes_B Y \arrow[r] \arrow[l, bend left, "s \otimes 1_Y"] & 0 \end{tikzcd}$$

Hence the above complex of $\mathbb{Z}$-modules splits as $\mathbb{Z}$-modules, giving a $(\mathbb{Z}, \mathbb{Z})$-exact sequence. $\square$

Remark 3.25. Understanding the splitting of $0 \to L \otimes_A F \xrightarrow{f \otimes 1_F} M \otimes_A F \xrightarrow{g \otimes 1_F} N \otimes_A F \to 0$ in terms of the splitting of $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ yields, in general, limited results. Suppose

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] \arrow[l, bend left, "\tilde{r}"] & N \arrow[r] \arrow[l, bend left, "\tilde{s}"] & 0 \end{tikzcd} \quad \text{and}$$

$$\begin{tikzcd} 0 \arrow[r] & L \otimes_A F \arrow[r, "f \otimes 1_F"] & M \otimes_A F \arrow[r, "g \otimes 1_F"] \arrow[l, bend left, "u"] & N \otimes_A 1_F \arrow[r] \arrow[l, bend left, "v"] & 0 \end{tikzcd}$$

split as B-modules and $\mathbb{Z}$-modules respectively. Focusing first on $u : M \otimes_A F \to L \otimes_A F$, we have $u(f \otimes 1_F) = 1_{L \otimes_A F}$. This means that for all $x \in L$ and $y \in F$ then $x \otimes y = 1_{L \otimes_A F}(x \otimes y) = u(f \otimes 1_F)(x \otimes y) = u(f(x) \otimes y)$. Hence we can factor $u = r \otimes 1_F$ for some $\mathbb{Z}$-morphism $r : M \to L$ satisfying $rf = 1_L$ as Z-morphisms. Moreover, since $r \otimes 1_F : M \otimes_A F \to L \otimes_A F$ is well defined, for all $a \in A$ we have $r(xa) \otimes y = (r \otimes 1_F)(xa \otimes y) = (r \otimes 1_F)(x \otimes ay) = r(x) \otimes ay = r(x)a \otimes y$. However, as illustrated in Proposition 3.24 and shown in Example 3.28, this is not enough for $r : M \to L$ to be an A-morphism in general.

A similar reasoning follows for $v : N \otimes_A F \to M \otimes_A F$, it factors as $v = s \otimes 1_F$ for some $\mathbb{Z}$-morphism $s : N \to M$, which is not an A-morphism in general, satisfying $gs = 1_N$

as Z-morphisms. In addition, we have $1_{M\otimes_A F} = (f\otimes 1_F)(r\otimes 1_F) + (s\otimes 1_F)(g\otimes 1_F) = (fr\otimes 1_F)+(sg\otimes 1_F) = (fr+sg)\otimes 1_F$ as $\mathbb{Z}$-morphisms, and thus $fr+sg = 1_M$ as $\mathbb{Z}$-morphisms. Again, this is not enough for an equality of A-morphisms.

We have then recovered the $(A,\mathbb{Z})$-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[l, bend left, "r"] \arrow[r, "g"] & N \arrow[l, bend left, "s"] \arrow[r] & 0. \end{tikzcd}$$

Proposition 3.26. *Let X be a (not necessarily finite) set and $\{F_x\}_{x\in X}$ be a family of A-modules. Then $\bigoplus_{x\in X} F_x$ is (A,B)-flat if and only if F_x is (A,B)-flat for all $x\in X$.*

Proof. Given an (A,B)-exact sequence of right A-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ then tensoring on the right with $\bigoplus_{x\in X} F_x$ yields the canonically isomorphic complexes

$$\begin{tikzcd} 0 \to L\otimes_A \left(\bigoplus_{x\in X} F_x\right) \arrow[r, "f\otimes(\oplus_{x\in X}1_{F_x})"] \arrow[d, "\cong"] & M\otimes_A \left(\bigoplus_{x\in X} F_x\right) \arrow[r, "g\otimes(\oplus_{x\in X}1_{F_x})"] \arrow[d, "\cong"] & N\otimes_A \left(\bigoplus_{x\in X} F_x\right) \to 0 \arrow[d, "\cong"] \\ 0 \to \bigoplus_{x\in X}(L\otimes_A F_x) \arrow[r, "\oplus_{x\in X}(f\otimes 1_{F_x})"] & \bigoplus_{x\in X}(M\otimes_A F_x) \arrow[r, "\oplus_{x\in X}(g\otimes 1_{F_x})"] & \bigoplus_{x\in X}(N\otimes_A F_x) \to 0. \end{tikzcd}$$

Moreover, if $g : M \to N$ is surjective then $g\otimes 1_{F_x} : M\otimes_A F_x \to N\otimes_A F_x$ is surjective for all $x\in X$, whence both $g\otimes(\oplus_{x\in X}1_{F_x}) : M\otimes_A \left(\bigoplus_{x\in X} F_x\right) \to N\otimes_A \left(\bigoplus_{x\in X} F_x\right)$ and $\oplus_{x\in X}(g\otimes 1_{F_x}) : \bigoplus_{x\in X}(M\otimes_A F_x) \to \bigoplus_{x\in X}(N\otimes_A F_x)$ are surjective. Also note that, by construction, $\oplus_{x\in X}(f\otimes 1_{F_x}) : \bigoplus_{x\in X}(L\otimes_A F_x) \to \bigoplus_{x\in X}(M\otimes_A F_x)$ is injective if and only if $f\otimes 1_{F_x} : L\otimes_A F_x \to M\otimes_A F_x$ is injective for all $x\in X$.

($\Rightarrow$) If $\bigoplus_{x\in X} F_x$ is (A,B)-flat then the top row in the above diagram is $(\mathbb{Z},\mathbb{Z})$-exact, which means that the bottom row is also $(\mathbb{Z},\mathbb{Z})$-exact. Then by Remark 3.25 we can write

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{x\in X}(L\otimes_A F_x) \arrow[r, "\oplus_{x\in X}(f\otimes 1_{F_x})"] & \bigoplus_{x\in X}(M\otimes_A F_x) \arrow[l, bend left, "\oplus_{x\in X}(r\otimes 1_{F_x})"] \arrow[r, "\oplus_{x\in X}(g\otimes 1_{F_x})"] & \bigoplus_{x\in X}(N\otimes_A F_x) \arrow[l, bend left, "\oplus_{x\in X}(s\otimes 1_{F_x})"] \arrow[r] & 0. \end{tikzcd}$$

Fix $x\in X$, then $f\otimes 1_{F_x} : L\otimes_A F_x \to M\otimes_A F_x$ is injective by the reasoning above and thus

$$\begin{tikzcd} 0 \arrow[r] & L\otimes_A F_x \arrow[r, "f\otimes 1_{F_x}"] & M\otimes_A F_x \arrow[l, bend left, "r\otimes 1_{F_x}"] \arrow[r, "g\otimes 1_{F_x}"] & N\otimes_A F_x \arrow[l, bend left, "s\otimes 1_{F_x}"] \arrow[r] & 0 \end{tikzcd}$$

is a $(\mathbb{Z}, \mathbb{Z})$-exact sequence. Hence F_x is (A, B)-flat for all $x \in X$.

$(\Leftarrow)$ If F_x is (A, B)-flat for all $x \in X$, then $0 \to L \otimes_A F_x \xrightarrow{f \otimes 1_{F_x}} M \otimes_A F_x \xrightarrow{g \otimes 1_{F_x}} N \otimes_A F_x \to 0$ is $(\mathbb{Z}, \mathbb{Z})$-exact for all $x \in X$. Taking the direct sum of these sequences we obtain that the bottom row in the above diagram is $(\mathbb{Z}, \mathbb{Z})$-exact, which means that the top row is also $(\mathbb{Z}, \mathbb{Z})$-exact. Hence $\bigoplus_{x \in X} F_x$ is (A, B)-flat.

□

Theorem 3.27. *Let P be an (A, B)-projective module. Then P is (A, B)-flat.*

Proof. Given P an (A, B)-projective module, then P is a direct summand of U an (A, B)-free module by Theorem 3.16. Now U is (A, B)-flat by Proposition 3.24, so its direct summands are (A, B)-flat by Proposition 3.26. Hence P is an (A, B)-flat module. □

Example 3.28. We now present several behaviors of (A, B)-flat modules.

1. Consider the A-module $U = A \otimes_B B$. Given an (A, B)-exact sequence of right A-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, we have the canonically isomorphic complexes of $\mathbb{Z}$-modules

$$\begin{array}{ccccc}
0 \to L \otimes_A U & \xrightarrow{f \otimes 1_U} & M \otimes_A U & \xrightarrow{g \otimes 1_U} & N \otimes_A U \to 0 \\
\| & & \| & & \| \\
0 \to L \otimes_A (A \otimes_B B) & \xrightarrow{f \otimes (1_A \otimes 1_B)} & M \otimes_A (A \otimes_B B) & \xrightarrow{g \otimes (1_A \otimes 1_B)} & N \otimes_A (A \otimes_B B) \to 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 \to L & \xrightarrow{f} & M & \xrightarrow{g} & N \to 0.
\end{array}$$

Since the bottom row is split exact as B-modules, it is also split exact as $\mathbb{Z}$-modules. Hence $U \cong A$ is (A, B)-flat. This is a particular case of Proposition 3.24.

2. Consider $B = k$, $A = k[x_1, \ldots, x_i]$ for some $i \in \mathbb{N}$, and J a non-trivial ideal of A as in Example 3.17. We saw that A/J is not $(k[x_1, \ldots, x_i], k)$-projective because the canonical sequence $0 \to J \to A \to A/J \to 0$ is $(k[x_1, \ldots, x_i], k)$-exact but it does not split as $k[x_1, \ldots, x_i]$-modules. We claim that A/J is not (A, B)-flat, and we will see this by contradiction. If A/J was $(k[x_1, \ldots, x_i], k)$-flat, we would have the $(\mathbb{Z}, \mathbb{Z})$-exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J \otimes_A A/J & \xrightarrow{\iota\otimes 1_{A/J}} & A \otimes_A A/J & \xrightarrow{\pi\otimes 1_{A/J}} & A/J \otimes_A A/J & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & J/J^2 & \xrightarrow{\overline{\iota}} & A/J & \xrightarrow{\overline{\pi}} & A/J & \longrightarrow & 0
\end{array}$$

where $\overline{\iota} : J/J^2 \to A/J$ and $\overline{\pi} : A/J \to A/J$ are the $\mathbb{Z}$-morphisms obtained from $\iota : J \to A/J$ and $\pi : A/J \to J$, respectively, at the quotient. In particular $\overline{\iota} : J/J^2 \to A/J$ would be injective, whereas $\overline{\iota}(\overline{j}) = \overline{j} = \overline{0}$ for all $j \in J$, meaning that $J/J^2 = 0$ so $J = J^2$. Since $k[x_1, \dots, x_i]$ is Noetherian then J is finitely generated, so by Nakayama's Lemma there exists $a \in A$ and $j \in J$ such that $as = 0$ for all $s \in J$ and $r + j = 1$. Then for all $s \in J$ we have $s = (a+j)s = as + js = js$, whence $J = (j)$ is a principal ideal and $j = j^2$. Since $k[x_1, \dots, x_i]$ is an integral domain, $j(j-1) = 0$ implies $j = 0$, contradicting that $J \neq \{0\}$.

This reasoning applies more generally when A is a k-algebra as well as a Noetherian domain.

3. Consider the $\mathbb{Z}[x]$-module $\mathbb{Z}[x]/(x^i)$ for some fixed $2 \leq i \in \mathbb{N}$, which is not $\mathbb{Z}[x]$-flat and is not $\mathbb{Z}$-flat. The canonical sequence $0 \to (x) \xrightarrow{\iota} \mathbb{Z}[x] \to \mathbb{Z}[x]/(x) \to 0$ is $(\mathbb{Z}[x], \mathbb{Z})$-exact because it is exact as $\mathbb{Z}[x]$-modules, $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$ as $\mathbb{Z}$-modules, and the isomorphism of $\mathbb{Z}$-modules $\mathbb{Z}[x] \cong \mathbb{Z} \oplus (x)$ respects the canonical inclusion and projection. Now $\iota \otimes 1_{\mathbb{Z}[x]/(x^i)} : (x) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^i) \to \mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^i) \cong \mathbb{Z}[x]/(x^i)$ is given by $(\iota \otimes 1_{\mathbb{Z}[x]/(x^i)})(p \otimes q) = \overline{pq}$ for all $p \in (x)$ and $\overline{q} \in \mathbb{Z}[x]/(x^i)$. Since $x \otimes \overline{x^{i-1}} \neq 0$ in $(x) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^i)$ but $(\iota \otimes 1_{\mathbb{Z}[x]/(x^i)})(x \otimes \overline{x^{i-1}}) = \overline{x^i} = \overline{0}$, then $\iota \otimes 1_{\mathbb{Z}[x]/(x^i)}$ is not injective, hence $\mathbb{Z}[x]/(x^i)$ is not $(\mathbb{Z}[x], \mathbb{Z})$-flat.

4. Consider the $\mathbb{Z}$-module $\mathbb{Z}/(i)$ for a fixed $i \in \mathbb{N}$. Recall that since $\mathbb{Z}$ is commutative and Noetherian, finitely generated flat modules are projective. In particular, by Proposition 2.27 they are a direct summand of a free $\mathbb{Z}$-module $\mathbb{Z}^j$ for some $j \in \mathbb{N}$, and thus finitely generated flat modules do not have zero divisors. Since $\mathbb{Z}/(i)$ is finitely generated and has zero divisors, it is not a $\mathbb{Z}$-flat module.

   Given an $(\mathbb{Z}, \mathbb{Z})$-exact sequence of right $\mathbb{Z}$-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, we have the canonically isomorphic complexes of $\mathbb{Z}$-modules

$$\begin{tikzcd}
0 \arrow[r] & L \otimes_{\mathbb{Z}} \mathbb{Z}/(i) \arrow[r, "f\otimes 1_{\mathbb{Z}/(i)}"] \arrow[d, "\cong"] & M \otimes_{\mathbb{Z}} \mathbb{Z}/(i) \arrow[r, "g\otimes 1_{\mathbb{Z}/(i)}"] \arrow[d, "\cong"] & N \otimes_{\mathbb{Z}} \mathbb{Z}/(i) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L/iL \arrow[r, "\overline{f}"] \arrow[d, "\cong"] & M/iM \arrow[r, "\overline{g}"] \arrow[d, "\cong"] & N/iN \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L/iL \arrow[r, "\iota_1"] & L/iL \oplus N/iN \arrow[r, "\pi_2"] & N/iN \arrow[r] & 0.
\end{tikzcd}$$

Here $\overline{f} : L/iL \to M/iM$ and $\overline{g} : M/iM \to N/iN$ are the $\mathbb{Z}$-morphisms obtained from $f : L \to M$ and $g : M \to N$, respectively, at the quotient. Moreover we have used that $M \cong N \oplus L$ by Remark 2.17, as well as the notation $\iota_1 : L/iL \to L/iL \oplus N/iN$ and $\pi_2 : L/iL \oplus N/iN \to N/iN$ for the inclusion on the first component and the projection on the second component, respectively. Since the bottom row in the above diagram is split exact as $\mathbb{Z}$-modules, then $\mathbb{Z}/(i)$ is $(\mathbb{Z}, \mathbb{Z})$-flat. This is a particular case of Remark 3.22.

5. Consider $B = k$ and the polynomial ring $A = k[x_1, \ldots, x_j]$ for some $j \in \mathbb{N}$. Let $Q = k(x_1, \ldots, x_j)$ be its field of fractions, we claim that Q is (A, B)-flat but not (A, B)-projective.

   First, note that Q is flat as an A-module. To see this, take I an ideal of A via the inclusion $\iota : I \to A$, and consider the $\mathbb{Z}$-morphism $\iota \otimes 1_Q : I \otimes_A Q \to A \otimes_A Q \cong Q$ given by $(\iota \otimes 1_Q)(i \otimes q) = iq$ for all $i \in I$ and $q \in Q$. A generic element of $I \otimes_A Q$ is given by $\sum_{r=1}^{s} i_r \otimes q_r$ for $i_1, \ldots, i_s \in I$ and $q_1, \ldots, q_s \in Q$. Now

$$\begin{aligned}
\sum_{r=1}^{s} i_r \otimes q_r &= \sum_{r=1}^{s} i_r \otimes q_r \frac{i_1 \cdots i_{r-1} i_{r+1} \cdots i_s}{i_1 \cdots i_{r-1} i_{r+1} \cdots i_s} = \sum_{r=1}^{s} i_1 \cdots i_s \otimes \frac{q_r}{i_1 \cdots i_{r-1} i_{r+1} \cdots i_s} \\
&= i_1 \cdots i_s \otimes \sum_{r=1}^{s} \frac{q_r}{i_1 \cdots i_{r-1} i_{r+1} \cdots i_s} = i \otimes q
\end{aligned}$$

   is a pure tensor because $i = i_1 \cdots i_s \in I$ and $q = \sum_{r=1}^{s} \frac{q_r}{i_1 \cdots i_{r-1} i_{r+1} \cdots i_s} \in Q$. Moreover $0 = (\iota \otimes 1_Q)(i \otimes q) = iq$ if and only if $i = 0$ or $q = 0$ in Q, whence $i \otimes q = 0$ in $I \otimes_A Q$. This means that $\iota \otimes 1_Q$ is injective, so by Remark 2.30 we have that Q is A-flat.

   The above now implies that Q is $(k[x_1, \ldots, x_j], k)$-flat, since given an $(k[x_1, \ldots, x_j], k)$-exact short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ it is exact as a sequence of $k[x_1, \ldots, x_j]$-

modules. Then Q flat means that

$$0 \longrightarrow L \otimes_A F \xrightarrow{f \otimes 1_F} M \otimes_A F \xrightarrow{g \otimes 1_F} N \otimes_A F \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules. Note that since A is commutative, all the modules above are left A-modules and all the morphisms are A-morphisms. In particular, this is an exact sequence of k-modules, and thus it splits because k is a field. We then obtained a $(\mathbb{Z}, \mathbb{Z})$-exact sequence, whence Q is (A, B)-flat.

Second, note that Q is a divisible module, namely for every $q \in Q$ and every non-zero divisor $a \in A$ then $p/a \in Q$ and $a(p/a) = p$. However, projective modules over A are not divisible. To see this, take I a non-zero ideal of A and note that since A is not a field there is a non-zero and non-invertible $a \in I$. Here a is not divisible by a^2, because if there is a non-zero $d \in A$ with $a = a^2 d$ then $1 = ad$ since A is an integral domain, contradicting that a is not invertible. Thus I is not a divisible A-module since there is a non-zero element $a^2 \in A$ that does not divide $a \in I$. Suppose that P is a projective A-module, then by Proposition 2.27 it is a direct summand of a free A-module, say $U = \bigoplus_{y \in Y} A_y$ with basis Y. Pick a non-zero $u \in P \subseteq U$, then $u = u_{y_1} + \cdots + u_{y_r}$ for some $r \in \mathbb{N}$, $u_{y_1}, \ldots, u_{y_r} \in A$, and distinct $y_1, \ldots, y_r \in Y$. Then $I = (u_{y_1}, \ldots, u_{y_r})$ is a non-zero ideal of A, whence we have seen that there is a non-zero $a \in I$ that is not divisible by $a^2 \in A$. If there is a non-zero $v \in P$ with $a^2 v = x$, we must have $v = v_{y_1} + \cdots + v_{y_r}$ for some $v_{y_1}, \ldots, v_{y_r} \in A$ because A is an integral domain. Then $a^2 v_{y_s} = u_{y_s}$ for all $1 \leq s \leq r$, so every element of I is divisible by a^2, a contradiction with $a \in I$. This means that there is $u \in P$ that cannot be divided by $a^2 \in A$, whence P is not divisible. This means that Q is not a projective A-module.

We saw an alternative proof of Q not being a projective A-module in Example 3.17.

The above now implies that Q is not $(k[x_1, \ldots, x_j], k)$-projective. If it was, then by Theorem 3.16 we would have that it is a direct summand of an $(k[x_1, \ldots, x_j], k)$-free module, namely there is P an A-module and T a k-module such that $P \oplus Q \cong A \otimes_k T$ as A-modules. Since k is a field then $T = \bigoplus_{y \in Y} k_y$ is a free k-module with basis Y, so

$P \oplus Q \cong A \otimes_k T \cong A \otimes_k \left(\bigoplus_{y\in Y} k_y\right) \cong \bigoplus_{y \in Y} (A \otimes_k k_y) \cong \bigoplus_{y\in Y} (A_y)$. Then Q would be projective because it would be a direct summand of an A-free module, a contradiction. Whence Q is not $(k[x_1, \dots, x_j], k)$-projective.

This reasoning applies more generally when A is a k-algebra, an integral domain, and not a field. In particular, for the ring of formal power series $A = k[[x_1, \dots, x_j]]$ for some $j \in \mathbb{N}$.

We have then seen that (A, B)-flat modules need not be A-flat, B-flat, nor (A, B)-projective.

3.2 Relative homology, relative Ext, and relative Tor

In Section 2.3 we recalled the working definitions of Ext and Tor. We now translate them to our current setup and see some of the analogous results that can be obtained.

Definition 3.29. Let M, N be A-modules, $(\boldsymbol{P_\bullet}, \boldsymbol{d_\bullet})$ an (A, B)-projective resolution of M

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] \arrow[l, bend left, "s_i"] & P_{i-1} \arrow[r, "d_{i-1}"] \arrow[l, bend left, "s_{i-1}"] & \cdots \arrow[r, "d_1"] \arrow[l, bend left, "s_{i-1}"] & P_0 \arrow[r, "d_0"] \arrow[l, bend left, "s_0"] & M \arrow[r] \arrow[l, bend left, "s_{-1}"] & 0
\end{tikzcd}$$

truncating at M and applying $\operatorname{Hom}_A(?, N)$ we obtain

$$\begin{tikzcd}
\cdots \arrow[r, bend right, "s_i^*"'] & \operatorname{Hom}_A(P_i, N) \arrow[l, "d_{i+1}^*"'] \arrow[r, bend right, "s_{i-1}^*"'] & \operatorname{Hom}_A(P_{i-1}, N) \arrow[l, "d_i"'] \arrow[r, bend right, "s_{i-2}^*"'] & \cdots \arrow[l, "d_{i-1}^*"'] \arrow[r, bend right, "s_0^*"'] & \operatorname{Hom}_A(P_0, N) \arrow[l, "d_1^*"'] & 0 \arrow[l]
\end{tikzcd}$$

where we define $\operatorname{Ext}^0_{(A,B)}(M, N) = \ker(d_1^*)$, $\operatorname{Ext}^i_{(A,B)}(M, N) = \ker(d_{i+1}^*)/\operatorname{im}(d_i^*)$ for all $i \in \mathbb{N}$, the *relative* Ext *groups*. We denote $\operatorname{Ext}^\bullet_{(A,B)}(M, N) = \bigoplus_{i\in\mathbb{N}} \operatorname{Ext}^i_{(A,B)}(M, N)$.

Definition 3.30. Let M be a right A-module, N a left A-module, $(\boldsymbol{P_\bullet}, \boldsymbol{d_\bullet})$ an (A, B)-projective resolution of M

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] \arrow[l, bend left, "s_i"] & P_{i-1} \arrow[r, "d_{i-1}"] \arrow[l, bend left, "s_{i-1}"] & \cdots \arrow[r, "d_1"] \arrow[l, bend left, "s_{i-1}"] & P_0 \arrow[r, "d_0"] \arrow[l, bend left, "s_0"] & M \arrow[r] \arrow[l, bend left, "s_{-1}"] & 0
\end{tikzcd}$$

truncating at M and applying $? \otimes_A N$ we obtain

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+1}\otimes 1_N"] & P_i \otimes_A N \arrow[r, "d_i \otimes 1_N"] \arrow[l, bend left, "s_i \otimes 1_N"] & P_{i-1} \otimes_A N \arrow[r, "d_{i-1}\otimes 1_N"] \arrow[l, bend left, "s_{i-1}\otimes 1_N"] & \cdots \arrow[r, "d_1 \otimes 1_N"] \arrow[l, bend left, "s_{i-2}\otimes 1_N"] & P_0 \otimes_A N \arrow[r] \arrow[l, bend left, "s_0 \otimes 1_N"] & 0.
\end{tikzcd}$$

We define $\mathrm{Tor}_0^{(A,B)}(M,N) = P_0 \otimes_A N / \mathrm{im}(d_1 \otimes 1_N)$, $\mathrm{Tor}_i^{(A,B)}(M,N) = \ker(d_i \otimes 1_N)/\mathrm{im}(d_{i+1} \otimes 1_N)$ for all $i \in \mathbb{N}$, the *relative* Tor *groups*. We denote $\mathrm{Tor}_\bullet^{(A,B)}(M,N) = \bigoplus_{i\in\mathbb{N}} \mathrm{Tor}_i^{(A,B)}(M,N)$.

We have a result completely analogous to Theorem 2.36.

Theorem 3.31 (Relative Comparison Theorem)**.** *Let M, N be A-modules, $f : M \to N$ an A-morphism, $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ a sequence of (A,B)-projective modules bounded on the right by M, $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ an (A,B)-exact sequence of modules bounded on the right by N. Then there exists a chain map $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q_\bullet}$ making the completed diagram commute. This chain map is unique up to homotopy.*

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[l, bend left, "s_2"] \arrow[r, "p_2"] \arrow[d, dashed, "f_2"] & P_1 \arrow[l, bend left, "s_1"] \arrow[r, "p_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[l, bend left, "s_0"] \arrow[r, "p_0"] \arrow[d, dashed, "f_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "q_3"] & Q_2 \arrow[l, bend left, "t_2"] \arrow[r, "q_2"] & Q_1 \arrow[l, bend left, "t_1"] \arrow[r, "q_1"] & Q_0 \arrow[l, bend left, "t_0"] \arrow[r, "q_0"] & N \arrow[l, bend left, "t_{-1}"] \arrow[r] & 0
\end{tikzcd}$$

Proof. The existence of $f_i : P_i \to Q_i$ for all $i \in \mathbb{N}$ follows by induction. For $i = 0$ consider

$$\begin{tikzcd}
P_0 \arrow[r, "p_0"] \arrow[d, dashed, "f_0"'] & M \arrow[d, "f"] \\
Q_0 \arrow[r, "q_0"] & N \arrow[l, bend left, "t_{-1}"] \arrow[r] & 0,
\end{tikzcd}$$

since P_0 is (A,B)-projective the diagram guarantees the existence of an A-morphism $f_0 : P_0 \to Q_0$ with $q_0 f_0 = f p_0$. Suppose now for induction that we have the following commutative square.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "p_{i+1}"] & P_i \arrow[r, "p_i"] \arrow[d, "f_i"] & P_{i-1} \arrow[d, "f_{i-1}"] \\
Q_{i+1} \arrow[r, "q_{i+1}"] & Q_i \arrow[r, "q_i"] & Q_{i-1}
\end{tikzcd}$$

Exactness of the bottom sequence gives $\mathrm{im}(q_{i+1}) = \ker(q_i)$ and hence $q_i f_i p_{i+1} = f_{i-1} p_i p_{i+1} = 0$ yields $\mathrm{im}(f_i p_{i+1}) \subseteq \mathrm{im}(q_{i+1})$. Consider the following diagram.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, dashed, "f_{i+1}"'] & P_i \arrow[d, "f_i"] \\
Q_{i+1} \arrow[r, "q_{i+1}"] & \mathrm{im}(q_{i+1}) \arrow[l, bend left, "t_i|_{\mathrm{im}(q_{i+1})}"] \arrow[r] & 0
\end{tikzcd}$$

Indeed, since P_{i+1} is (A, B)-projective we obtain the existence of an A-morphism $f_{i+1} : P_{i+1} \to Q_{i+1}$ with $q_{i+1}f_{i+1} = f_i p_i$.

Suppose that $g_\bullet : \boldsymbol{P}_\bullet \to \boldsymbol{Q}_\bullet$ is another chain map making the completed diagram commute, the uniqueness up to a homotopy $r_i : P_i \to Q_{i+1}$ follows from an explicit construction of the homotopy by induction on $i \in \mathbb{N} \cup \{-1\}$. For $i = -1$, set $P_{-1} = M$, $P_{-2} = 0$, $Q_{-1} = N$, $r_{-1} : M \to Q_0$, $f_{-1} = g_{-1} = f$, and $r_{-1} = r_{-2} = 0$. The commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "f_1"] & P_0 \arrow[r, "p_0"] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] & M \arrow[r] \arrow[dl, "s_{-1}"] \arrow[d, "f"] & 0 \arrow[dl, "s_{-2}"] \\
\cdots \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & N \arrow[r] & 0
\end{tikzcd}$$

yields $g_{-1} - f_{-1} = f - f = 0 = q_0 s_{-1} + s_{-2} p_{-1}$. Suppose now that we have the following commutative diagram, as before exactness of the bottom sequence gives $\operatorname{im}(q_{i+2}) = \ker(q_{i+1})$.

$$\begin{tikzcd}
 & P_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, shift right, "g_{i+1}"'] \arrow[d, shift left, "f_{i+1}"] & P_i \arrow[r, "p_i"] \arrow[dl, "r_i"'] \arrow[d, shift right, "g_i"'] \arrow[d, shift left, "f_i"] & P_{i-1} \arrow[dl, "r_{i-1}"'] \arrow[d, shift right, "g_{i-1}"'] \arrow[d, shift left, "f_{i-1}"] \\
Q_{i+2} \arrow[r, "q_{i+2}"] & Q_{i+1} \arrow[r, "q_{i+1}"] & Q_i \arrow[r, "q_i"] & Q_{i-1}
\end{tikzcd}$$

The induction hypothesis $g_i - f_i = q_{i+1}s_i + s_{i-1}p_i$ gives $\operatorname{im}(g_{i+1} - f_{i+1} - r_i p_{i+1}) \subseteq \operatorname{im}(q_{i+2})$ since

$$\begin{aligned}
q_{i+1}(g_{i+1} - f_{i+1} - r_i p_{i+1}) &= q_{i+1}(g_{i+1} - f_{i+1}) - q_{i+1} r_i p_{i+1} \\
&= (g_i - f_i)p_{i+1} - (g_i - f_i - s_{i-1}p_i)p_{i+1} = 0.
\end{aligned}$$

Consider the following diagram.

$$\begin{tikzcd}
 & P_{i+1} \arrow[dl, dashed, "r_{i+1}"'] \arrow[d, "g_{i+1} - f_{i+1} - r_i p_{i+1}"] & \\
Q_{i+2} \arrow[r, "q_{i+2}"] & \operatorname{im}(q_{i+2}) \arrow[r] \arrow[l, bend left, "t_{i+1}|_{\operatorname{im}(q_{i+2})}"] & 0
\end{tikzcd}$$

Indeed, since P_{i+1} is (A, B)-projective we obtain the existence of an A-morphism $r_{i+1} : P_{i+1} \to Q_{i+2}$ with $q_{i+1} r_{i+1} = g_{i+1} - f_{i+1} - r_i p_{i+1}$. We then have $g_i - f_i = q_{i+1} r_i + r_{i-1} p_i$ for all $i \in \mathbb{N} \cup \{-1\}$. Thus $g_\bullet$ is homotopic to $f_\bullet$. $\square$

As in Section 2.3, this implies that relative Ext and relative Tor are independent of the resolution.

Proposition 3.32. *Let M, N be (left or right if necessary) A-modules, let $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ and $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ be two (A, B)-projective resolutions of M. Then for all $i \in \mathbb{N}$*

$$H^i(\mathrm{Hom}_A(\boldsymbol{P_\bullet}, N)) \cong H^i(\mathrm{Hom}_A(\boldsymbol{Q_\bullet}, N)),$$

$$H_i(\boldsymbol{P_\bullet} \otimes_A N) \cong H_i(\boldsymbol{Q_\bullet} \otimes_A N).$$

Proof. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[l, bend left, "s_2"] \arrow[r, "p_2"] & P_1 \arrow[l, bend left, "s_1"] \arrow[r, "p_1"] & P_0 \arrow[l, bend left, "s_0"] \arrow[r, "p_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] \arrow[d, "1_M"] & 0 \\
\cdots \arrow[r, "q_3"] & Q_2 \arrow[l, bend left, "t_2"] \arrow[r, "q_2"] & Q_1 \arrow[l, bend left, "t_1"] \arrow[r, "q_1"] & Q_0 \arrow[l, bend left, "t_0"] \arrow[r, "q_0"] & M \arrow[l, bend left, "t_{-1}"] \arrow[r] & 0,
\end{tikzcd}$$

by the Relative Comparison Theorem 3.31 there is a chain map $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{Q_\bullet}$ making the completed diagram commute. Permuting the roles of $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ and $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ gives a chain map $g_\bullet : \boldsymbol{Q_\bullet} \to \boldsymbol{P_\bullet}$. Moreover, by the uniqueness statement of the Relative Comparison Theorem 3.31 we have $g_\bullet f_\bullet \simeq 1_{\boldsymbol{P_\bullet}}$ and $f_\bullet g_\bullet \simeq 1_{\boldsymbol{Q_\bullet}}$. We are now in the exact same situation as in the proof of Proposition 2.37, and we can proceed in the exact same way to obtain the desired isomorphisms. $\square$

Although it will not concern us here, the above result can be refined to relative flat resolutions.

Remark 3.33. A more straightforward proof attempt is to try using Proposition 2.37 directly. However, as a consequence of Example 3.17 an (A, B)-projective resolution need not be a projective resolution of A-modules. In particular, such a reduction to the absolute case can not be achieved.

Note that given L a (left or right if necessary) A-module, $(\boldsymbol{R_\bullet}, \boldsymbol{r_\bullet})$ an (A, B)-projective resolution of L, and $f : M \to L$ an A-morphism, we can proceed as in the absolute case and apply the Relative Comparison Theorem 3.31 to the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p_3"] & P_2 \arrow[r, "p_2"] \arrow[l, bend left, "s_2"] & P_1 \arrow[r, "p_1"] \arrow[l, bend left, "s_1"] & P_0 \arrow[r, "p_0"] \arrow[l, bend left, "s_0"] & M \arrow[r] \arrow[l, bend left, "s_{-1}"] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "r_3"] & R_2 \arrow[r, "r_2"] \arrow[l, bend left, "u_2"] & R_1 \arrow[r, "r_1"] \arrow[l, bend left, "u_1"] & R_0 \arrow[r, "r_0"] \arrow[l, bend left, "u_0"] & L \arrow[r] \arrow[l, bend left, "u_{-1}"] & 0
\end{tikzcd}$$

yielding a chain map $f_{\boldsymbol{P}_\bullet}^{\boldsymbol{R}_\bullet} : \boldsymbol{P}_\bullet \to \boldsymbol{R}_\bullet$. Proceeding as in the proof of Proposition 2.37 or Proposition 3.32 we obtain A-morphisms in homology for every $i \in \mathbb{N}$.

$$\mathrm{Hom}_A(f_{\boldsymbol{P}_\bullet}^{\boldsymbol{R}_\bullet}, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{R}_\bullet, N)) \to H^i(\mathrm{Hom}_A(\boldsymbol{P}_\bullet, N)),$$
$$(f_{\boldsymbol{P}_\bullet}^{\boldsymbol{R}_\bullet} \otimes_A N)_{*_i} : H_i(\boldsymbol{P}_\bullet \otimes_A N) \to H_i(\boldsymbol{R}_\bullet \otimes_A N).$$

Proposition 3.34. *Let L, M, N be (left or right if necessary) A-modules, let $f : M \to L$ be an A-morphism, let $(\boldsymbol{R}_\bullet, \boldsymbol{r}_\bullet)$ and $(\boldsymbol{S}_\bullet, \boldsymbol{s}_\bullet)$ be (A, B)-projective resolutions of L, let $(\boldsymbol{P}_\bullet, \boldsymbol{p}_\bullet)$ and $(\boldsymbol{Q}_\bullet, \boldsymbol{q}_\bullet)$ be (A, B)-projective resolutions of M, for all $i \in \mathbb{N}$ let*

$$\mathrm{Hom}_A({1_M}_{\boldsymbol{Q}_\bullet}^{\boldsymbol{P}_\bullet}, N)_{*_i} : H^i(\mathrm{Hom}_A(\boldsymbol{P}_\bullet, N)) \cong H^i(\mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N)),$$
$$({1_M}_{\boldsymbol{P}_\bullet}^{\boldsymbol{Q}_\bullet} \otimes_A N)_{*_i} : H_i(\boldsymbol{P}_\bullet \otimes_A N) \cong H_i(\boldsymbol{Q}_\bullet \otimes_A N).$$

be the isomorphisms of Proposition 3.32. Then for all $i \in \mathbb{N}$ the following diagrams commute.

$$\begin{tikzcd}
H^i(\mathrm{Hom}_A(\boldsymbol{R}_\bullet, N)) \arrow[r, "\mathrm{Hom}_A({1_L}_{\boldsymbol{S}_\bullet}^{\boldsymbol{R}_\bullet}, N)_{*_i}"] \arrow[d, "\mathrm{Hom}_A(f_{\boldsymbol{P}_\bullet}^{\boldsymbol{R}_\bullet}, N)_{*_i}"'] & H^i(\mathrm{Hom}_A(\boldsymbol{S}_\bullet, N)) \arrow[d, "\mathrm{Hom}_A(f_{\boldsymbol{Q}_\bullet}^{\boldsymbol{S}_\bullet}, N)_{*_i}"] \\
H^i(\mathrm{Hom}_A(\boldsymbol{P}_\bullet, N)) \arrow[r, "\mathrm{Hom}_A({1_M}_{\boldsymbol{Q}_\bullet}^{\boldsymbol{P}_\bullet}, N)_{*_i}"] & H^i(\mathrm{Hom}_A(\boldsymbol{Q}_\bullet, N))
\end{tikzcd}$$

$$\begin{tikzcd}
H_i(\boldsymbol{Q}_\bullet \otimes_A N) \arrow[r, "({1_M}_{\boldsymbol{Q}_\bullet}^{\boldsymbol{P}_\bullet} \otimes_A N)_{*_i}"] \arrow[d, "(f_{\boldsymbol{Q}_\bullet}^{\boldsymbol{S}_\bullet} \otimes_A N)_{*_i}"'] & H_i(\boldsymbol{P}_\bullet \otimes_A N) \arrow[d, "(f_{\boldsymbol{P}_\bullet}^{\boldsymbol{S}_\bullet} \otimes_A N)_{*_i}"] \\
H_i(\boldsymbol{S}_\bullet \otimes_A N) \arrow[r, "({1_L}_{\boldsymbol{S}_\bullet}^{\boldsymbol{R}_\bullet} \otimes_A N)_{*_i}"] & H_i(\boldsymbol{R}_\bullet \otimes_A N)
\end{tikzcd}$$

Proof. Applying the Relative Comparison Theorem 3.31 to the diagrams

$$\begin{tikzcd}
\cdots \arrow[r, "q_1"] & Q_0 \arrow[l, bend left, "t_0"] \arrow[r, "q_0"] & M \arrow[l, bend left, "t_{-1}"] \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "s_1"] & S_0 \arrow[l, bend left, "v_0"] \arrow[r, "s_0"] & L \arrow[l, bend left, "v_{-1}"] \arrow[r] \arrow[d, "1_L"] & 0 \\
\cdots \arrow[r, "r_1"] & R_0 \arrow[l, bend left, "u_0"] \arrow[r, "r_0"] & L \arrow[l, bend left, "v_{-1}"] \arrow[r] & 0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdots \arrow[r, "q_1"] & Q_0 \arrow[l, bend left, "t_0"] \arrow[r, "q_0"] & M \arrow[l, bend left, "t_{-1}"] \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "p_1"] & P_0 \arrow[l, bend left, "w_0"] \arrow[r, "p_0"] & M \arrow[l, bend left, "w_{-1}"] \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "r_1"] & R_0 \arrow[l, bend left, "u_0"] \arrow[r, "r_0"] & L \arrow[l, bend left, "v_{-1}"] \arrow[r] & 0
\end{tikzcd}$$

yields homotopic chain maps $g_\bullet, h_\bullet : \boldsymbol{Q_\bullet} \to \boldsymbol{R_\bullet}$. Using again the uniqueness statement of the Relative Comparison Theorem 3.31, we have $g_\bullet \simeq 1_{L\,\boldsymbol{S_\bullet}}^{\,\boldsymbol{R_\bullet}} f_{\boldsymbol{Q_\bullet}}^{\boldsymbol{S_\bullet}}$ and $h_\bullet \simeq f_{\boldsymbol{P_\bullet}}^{\boldsymbol{R_\bullet}} 1_{M\,\boldsymbol{Q_\bullet}}^{\,\boldsymbol{P_\bullet}}$. We are now in the exact same situation as in the proof of Proposition 2.38, and we can proceed in the exact same way to obtain the desired isomorphisms. $\square$

Note that Proposition 3.34 proves the functoriality of relative Ext and relative Tor. Moreover, similarly as in Remark 3.33, we are unable to prove this by reduction to the absolute case because (A, B)-projective resolutions need not be A-projective. We now embark on the quest for a long exact sequence for relative Tor, which will take us what remains of this section. There are also long exact sequences for relative Ext, but we will not be concerned by that here. We begin with the following remark.

Remark 3.35. Let $f_\bullet : (\boldsymbol{L_\bullet}, \boldsymbol{l_\bullet}) \to (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet})$ and $g_\bullet : (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet}) \to (\boldsymbol{N_\bullet}, \boldsymbol{n_\bullet})$ be chain maps such that for all $i \in \mathbb{Z}$ the following is an (A, B)-exact sequence

$$\begin{tikzcd}
0 \arrow[r] & L_i \arrow[r, "f_i"] & M_i \arrow[l, bend left, "r_i"] \arrow[r, "g_i"] & N_i \arrow[l, bend left, "s_i"] \arrow[r] & 0,
\end{tikzcd}$$

then the B-splittings are also chain maps. To begin with, for each $i \in \mathbb{Z}$ there are B-morphisms $r_i : M_i \to L_i$ and $s_i : N_i \to M_i$ such that $f_i r_i f_i = f_i$ and $g_i s_i g_i = g_i$. By the proof of Proposition 2.16 we have $M_i \cong N_i \oplus L_i$ as B-modules. Moreover $f_i : L_i \to M_i$ is the inclusion of B-modules in the second component, $g_i : M_i \to N_i$ is the projection of B-modules in the first component, $r_i : M_i \to L_i$ is the projection of B-modules in the second component, and $s_i : N_i \to M_i$ is the inclusion of B-modules in the first component. In particular we have $r_i f_i = 1_{L_i}$, $f_i r_i|_{L_i} = 1_{L_i}$, $g_i s_i = 1_{N_i}$, $s_i g_i|_{N_i} = 1_{N_i}$, and $1_{M_i} = s_i g_i + f_i r_i$. These imply $r_i s_i = 0$ and $\ker(r_i) = \mathrm{im}(s_i)$, whence $0 \leftarrow L_i \overset{r_i}{\leftarrow} M_i \overset{s_i}{\leftarrow} N_i \leftarrow 0$ is an exact sequence of B-modules.

Since $f_\bullet$ and $g_\bullet$ are chain maps, we have the commutative diagram of A-morphisms

$$\begin{tikzcd}
 & \vdots \arrow[d, "l_{i+1}"] & \vdots \arrow[d, "m_{i+1}"] & \vdots \arrow[d, "n_{i+1}"] & \\
0 \arrow[r] & L_i \arrow[r, "f_i"] \arrow[d, "l_i"] & M_i \arrow[r, "g_i"] \arrow[d, "m_i"] & N_i \arrow[r] \arrow[d, "n_i"] & 0 \\
0 \arrow[r] & L_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "l_{i-1}"] & M_{i-1} \arrow[r, "g_{i-1}"] \arrow[d, "m_{i-1}"] & N_{i-1} \arrow[r] \arrow[d, "n_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where $m_i f_i = f_{i-1} l_i$ and $n_i g_i = g_{i-1} m_i$ for all $i \in \mathbb{Z}$. Using the above paragraph and seeing everything as B-morphisms, the isomorphism $M_i \cong N_i \oplus L_i$ allows us to decompose $m_i = n_i + l_i$ because $\operatorname{im}(m_i) = \operatorname{im}(n_i) \oplus \operatorname{im}(l_i)$ and $\ker(m_i) = \ker(n_i) \oplus \ker(l_i)$. Under this isomorphism, the chain map conditions become then $m_i f_i = f_{i-1} l_i$ and $n_i g_i = g_{i-1} m_i$ since the horizontal B-morphisms are just projections or injections. We can rewrite the commutative diagram as

$$\begin{tikzcd}
L_i \arrow[r, "f_i"] \arrow[d, "l_i"] & L_i \oplus N_i \arrow[r, "g_i"] \arrow[d, shift right=3, "l_i"'] \arrow[d, shift left=3, "n_i"] & N_i \arrow[d, "n_i"] \\
L_{i-1} \arrow[r, "f_{i-1}"] & L_{i-1} \oplus N_{i-1} \arrow[r, "g_{i-1}"] & N_{i-1}
\end{tikzcd}$$

Hence as B-morphisms we have $r_{i-1} m_i = r_{i-1}(n_i + l_i) = r_{i-1} n_i + r_{i-1} l_i = r_{i-1} l_i = l_i r_i$ and $m_i s_i = (n_i + l_i) s_i = n_i s_i + l_i s_i = n_i s_i = s_{i-1} n_i$. In particular, we can complete the commutative diagram with the B-splittings

$$\begin{tikzcd}
 & \vdots \arrow[d, "l_{i+1}"] & \vdots \arrow[d, "m_{i+1}"] & \vdots \arrow[d, "n_{i+1}"] & \\
0 \arrow[r] & L_i \arrow[r, "f_i"] \arrow[d, "l_i"] & M_i \arrow[r, "g_i"] \arrow[d, "m_i"] \arrow[l, bend left, "r_i"] & N_i \arrow[r] \arrow[d, "n_i"] \arrow[l, bend left, "s_i"] & 0 \\
0 \arrow[r] & L_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "l_{i-1}"] & M_{i-1} \arrow[r, "g_{i-1}"] \arrow[d, "m_{i-1}"] \arrow[l, bend left, "r_{i-1}"] & N_{i-1} \arrow[r] \arrow[d, "n_{i-1}"] \arrow[l, bend left, "s_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This means that seeing $f_\bullet : \boldsymbol{L_\bullet} \to \boldsymbol{M_\bullet}$ and $g_\bullet : \boldsymbol{M_\bullet} \to \boldsymbol{N_\bullet}$ as chain maps between complexes of B-modules, the families of B-morphisms $\{r_i : M_i \to L_i\}_{i\in\mathbb{Z}}$ and $\{s_i : N_i \to M_i\}_{i\in\mathbb{Z}}$ are chain maps. They then deserve to be denoted by $r_\bullet : \boldsymbol{M_\bullet} \to \boldsymbol{L_\bullet}$ and $s_\bullet : \boldsymbol{N_\bullet} \to \boldsymbol{M_\bullet}$.

Recall that from a short exact sequence of complexes we can produce a long exact sequence from their homology modules. The following results are an analogous construction, where from a short (A, B)-exact sequence of complexes we build a long (A, B)-exact sequence.

Proposition 3.36. *Let $f_\bullet : (\boldsymbol{L_\bullet}, \boldsymbol{l_\bullet}) \to (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet})$ and $g_\bullet : (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet}) \to (\boldsymbol{N_\bullet}, \boldsymbol{n_\bullet})$ be chain maps such that for all $i \in \mathbb{Z}$ the following is an (A, B)-exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & L_i \arrow[r, "f_i"] & M_i \arrow[l, bend left, "r_i"] \arrow[r, "g_i"] & N_i \arrow[l, bend left, "s_i"] \arrow[r] & 0. \end{tikzcd}$$

Then for each $i \in \mathbb{Z}$ there is an A-morphism $\partial_i : H_i(\boldsymbol{N_\bullet}) \to H_i(\boldsymbol{L_\bullet})$ given by

$$\begin{aligned} \partial_i : H_i(\boldsymbol{N_\bullet}) &\longrightarrow H_{i-1}(\boldsymbol{L_\bullet}) \\ \overline{z} &\longmapsto \overline{f_{i-1}^{-1} m_i g_i^{-1}(z)} \end{aligned}$$

and an A-morphism $\int_{i-1} : H_{i-1}(\boldsymbol{L_\bullet}) \to H_i(\boldsymbol{N_\bullet})$ given by

$$\begin{aligned} \textstyle\int_i : H_{i-1}(\boldsymbol{L_\bullet}) &\longrightarrow H_i(\boldsymbol{N_\bullet}) \\ \overline{z} &\longmapsto \overline{g_i m_i^{-1} f_{i-1}(z)} \end{aligned}$$

Proof. The diagram chase for both ∂_i and $\int_{i-1}$ yields the desired result. Consider first ∂_i. To see that it is well defined, let $z \in \ker(n_i) \subseteq N_i$. Since g_i is surjective, there is $x \in M_i$ with $g_i(x) = z$. Since $g_{i-1} m_i(x) = n_i g_i(x) = n_i(z) = 0$ we have $m_i(x) \in \ker(g_{i-1}) = \operatorname{im}(f_{i-1})$. Hence, since f_{i-1} is injective, there is a unique $y \in L_{i-1}$ with $f_{i-1}(y) = m_i(x)$. In fact $f_{i-2} l_{i-1}(y) = m_{i-1} f_{i-1}(y) = m_{i-1}(m_i(x)) = 0$, so by injectivity of f_{i-2} we have $l_{i-1}(y) = 0$ and $y \in \ker(l_{i-1})$. Hence $f_i^{-1} m_i g_i^{-1}(z) \in \ker(l_{i-1})$ so $\partial_i(\overline{z}) = \overline{f_i^{-1} m_i g_i^{-1}(z)} \in H_{i-1}(\boldsymbol{L_\bullet})$. To see that this is independent of the choice $x = g_i^{-1}(z)$, suppose that in the above reasoning there is another $x' \in M_i$ with $g_i(x') = z$, to which there is an unique $y' \in L_{i-1}$ with $f_{i-1}(y') = m_i(x')$. Then $g_i(x - x') = g_i(x) - g_i(x') = 0$ so $x - x' \in \ker(g_i) = \operatorname{im}(f_i)$, so there is $\tilde{y} \in L_i$ with $f_i(\tilde{y}) = x - x'$. Hence $f_{i-1} l_i(\tilde{y}) = m_i f_i(\tilde{y}) = m_i(x - x') = m_i(x) - m_i(x') = f_{i-1}(y) - f_{i-1}(y') = f_{i-1}(y - y')$, so by injectivity of f_{i-1} we have $l_i(\tilde{y}) = y - y'$ and thus $\overline{y} = \overline{y'}$ in $H_{i-1}(\boldsymbol{L_\bullet})$. To see that this factors through homology, suppose in the above reasoning that $z \in \operatorname{im}(n_{i+1})$, so there is $\tilde{z} \in N_{i+1}$ with $n_{i+1}(\tilde{z}) = z$. Since g_{i+1} is surjective, there is $\tilde{x} \in M_{i+1}$ with $g_{i+1}(\tilde{x}) = \tilde{z}$. Now $g_i m_{i+1}(\tilde{x}) = n_{i+1} g_{i+1}(\tilde{x}) = n_{i+1}(\tilde{z}) = z$, so we can choose $x = m_{i+1}(\tilde{x})$. Then $y = 0 \in L_{i-1}$ satisfies $f_{i-1}(y) = 0 = m_i m_{i+1}(\tilde{x}) = m_i(x)$, whence $\partial_i(\overline{z}) = \overline{0}$ in $H_{i-1}(\boldsymbol{L_\bullet})$. Finally, to see that it is an A-morphism, whenever we have $a \in A$ we can choose $g_i^{-1}(az) = a g_i^{-1}(z)$, and whenever we have $z' \in \ker(n_i)$ we can choose $g_i^{-1}(z + z') = g_i^{-1}(z) + g_i^{-1}(z')$. Then $\partial_i(\overline{az}) = \overline{f_{i-1}^{-1} m_i g_i^{-1}(az)} = \overline{f_{i-1}^{-1} m_i(a g_i^{-1}(z))} = \overline{a f_{i-1}^{-1} m_i g_i^{-1}(z)} = a\partial_i(\overline{z})$ and $\partial_i(\overline{z + z'}) = \overline{f_{i-1}^{-1} m_i g_i^{-1}(z + z')} = \overline{f_{i-1}^{-1} m_i(g_i^{-1}(z) + g_i^{-1}(z))} = \overline{f_{i-1}^{-1} m_i g_i^{-1}(z) + f_{i-1}^{-1} m_i g_i^{-1}(z')} = \partial_i(\overline{z}) + \partial_i(\overline{z'})$.

Consider then $\int_{i-1}$. To see that it is well defined, let $z \in \ker(l_{i-1}) \subseteq L_{i-1}$. Since $m_{i-1} f_{i-1}(z) = f_{i-2} l_{i-1}(z) = 0$ we have $f_{i-1}(z) \in \ker(m_{i-1}) = \operatorname{im}(m_i)$. Hence there is $x \in M_i$ with $m_i(x) = f_{i-1}(z)$. Since $n_i g_i(x) = g_{i-1} m_i(x) = g_{i-1} f_{i-1}(x) = 0$ we have $g_i(x) \in \ker(n_i)$. Hence $g_i m_i^{-1} f_{i-1}(z) \in \ker(n_i)$ so $\int_{i-1}(\overline{z}) = \overline{g_i m_i^{-1} f_{i-1}(z)} \in H_i(\boldsymbol{N_\bullet})$. To see that this is independent of the choice $x = m_i^{-1}(z)$, suppose that in the above reasoning there is another $x' \in M_i$ with $m_i(x') = f_{i-1}(z)$. Then $m_i(x - x') = m_i(x) - m_i(x') = 0$ so $x - x' \in \ker(m_i) = \operatorname{im}(m_{i+1})$, so there is $y \in M_{i+1}$ with $m_{i+1}(y) = x - x'$. Hence $g_i(x) - g_i(x') = g_i(x - x') = g_i m_{i+1}(y) = n_i g_{i+1}(y) \in \operatorname{im}(n_i)$, so $\overline{g_i(x)} = \overline{g_i(x')}$ in $H_i(\boldsymbol{N_\bullet})$. To see that this factors through homology, suppose in the above reasoning that $z \in \operatorname{im}(l_i)$, so there is $\tilde{z} \in L_i$ with $l_i(\tilde{z}) = z$. Now $m_i f_i(\tilde{z}) = f_{i-1} l_i(\tilde{z}) = f_{i-1}(z)$, so we can choose $x = f_i(\tilde{z})$. Now $g_i(x) = g_i f_i(\tilde{z}) = 0 \in \operatorname{im}(n_{i+1})$ hence $\int_{i-1}(\overline{z}) = \overline{0}$ in $H_i(\boldsymbol{N_\bullet})$. Finally, to see that it is an A-morphism, whenever we have $a \in A$ we can choose $m_i^{-1}(f_{i-1}(az)) = a m_i^{-1}(f_{i-1}(z))$, and whenever we have $z' \in \ker(l_{i-1})$ we can choose $m_i^{-1}(f_{i-1}(z + z')) = m_i^{-1}(f_{i-1}(z)) + m_i^{-1}(f_{i-1}(z'))$. Then $\int_{i-1}(\overline{az}) = \overline{g_i m_i^{-1} f_{i-1}(az)} = \overline{g_i(a m_i^{-1} f_{i-1}(z))} = \overline{a g_i m_i^{-1} f_{i-1}(z)} = a \int_{i-1}(\overline{z})$ and $\int_{i-1}(\overline{z + z'}) = \overline{g_i m_i^{-1} f_{i-1}(z + z')} = \overline{g_i(m_i^{-1} f_{i-1}(z) + m_i^{-1} f_{i-1}(z'))} = \overline{g_i m_i^{-1} f_{i-1}(z) + g_i m_i^{-1} f_{i-1}(z')} = \int_{i-1}(\overline{z}) + \int_{i-1}(\overline{z'})$. $\square$

Notice that we only used the exactness of the short exact sequences $0 \to L_i \xrightarrow{f_i} M_i \xrightarrow{g_i} N_i \to 0$, the B-splittings were not involved. The above indeed holds in the absolute case, and in fact can be used to prove a relative version of the Snake Lemma.

Theorem 3.37. *Let $f_\bullet : (\boldsymbol{L_\bullet}, \boldsymbol{l_\bullet}) \to (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet})$ and $g_\bullet : (\boldsymbol{M_\bullet}, \boldsymbol{m_\bullet}) \to (\boldsymbol{N_\bullet}, \boldsymbol{n_\bullet})$ be chain maps such that for all $i \in \mathbb{Z}$ the following is an (A, B)-exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & L_i \arrow[r, "f_i"] & M_i \arrow[l, bend left, "r_i"] \arrow[r, "g_i"] & N_i \arrow[l, bend left, "s_i"] \arrow[r] & 0. \end{tikzcd}$$

Then there is a long (A, B)-exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "g_{*i+1}"] & H_{i+1}(\boldsymbol{N_\bullet}) \arrow[l, bend left, "s_{*i+1}"] \arrow[r, "\partial_{i+1}"] & H_i(\boldsymbol{L_\bullet}) \arrow[l, bend left, "\int_{i+1}"] \arrow[r, "f_{*i}"] & H_i(\boldsymbol{M_\bullet}) \arrow[l, bend left, "r_{*i}"] \arrow[r, "g_{*i}"] & H_i(\boldsymbol{N_\bullet}) \arrow[l, bend left, "s_{*i}"] \arrow[r, "\partial_i"] & H_{i-1}(\boldsymbol{L_\bullet}) \arrow[l, bend left, "\int_i"] \arrow[r, "f_{*i-1}"] & \cdots \arrow[l, bend left, "r_{*i-1}"] \end{tikzcd}$$

Proof. Let $i \in \mathbb{Z}$ throughout. By Proposition 3.2, it is enough to check that the sequence is split exact as a sequence of B-modules. The exactness of the sequence is well known, as follows.

($\ker(f_{*_i}) \subseteq \operatorname{im}(\partial_{i+1})$) Let $\overline{z} \in \ker(f_{*_i})$ so that $\overline{f_i(z)} = f_{*_i}(\overline{z}) = \overline{0}$ in $H_i(\boldsymbol{M_\bullet})$. Hence $f_i(z) \in \operatorname{im}(m_{i+1})$ so there is $x \in M_{i+1}$ with $m_{i+1}(x) = f_i(z)$. Then $n_i g_{i+1}(x) = g_i m_i(x) = g_i f_i(z) = 0$ so $g_{i+1}(x) \in \ker(n_i)$. Now $\partial_{i+1}(\overline{g_{i+1}(x)}) = \overline{f_i^{-1} m_{i+1} g_{i+1}^{-1} g_{i+1}(x)} = \overline{f_i^{-1} m_{i+1}(x)} = \overline{f_i^{-1} f_i(z)} = \overline{z}$ so $\overline{z} \in \operatorname{im}(\partial_{i+1})$.

($\operatorname{im}(\partial_{i+1}) \subseteq \ker(f_{*_i})$) Let $\overline{z} \in H_{i+1}(\boldsymbol{N_\bullet})$, then $f_{*_i}\partial_{i+1}(\overline{z}) = \overline{f_i f_i^{-1} m_{i+1} g_{i+1}^{-1}(z)} = \overline{m_{i+1} g_{i+1}^{-1}(z)} = \overline{0}$ in $H_i(\boldsymbol{M_\bullet})$.

($\ker(g_{*_i}) \subseteq \operatorname{im}(f_{*_i})$) Let $\overline{z} \in \ker(g_{*_i})$, in particular $z \in \ker(m_i)$, so that $\overline{g_i(z)} = g_{*_i}(\overline{z}) = \overline{0}$ in $H_i(\boldsymbol{N_\bullet})$. Hence $g_i(z) \in \operatorname{im}(n_{i+1})$ so there is $y \in N_{i+1}$ with $n_{i+1}(x) = g_i(z)$. Since g_i is surjective, there is $x \in M_{i+1}$ with $g_{i+1}(x) = y$. Now $g_i(z - m_{i+1}(x)) = g_i(z) - g_i m_{i+1}(x) = 0$ since $g_i m_{i+1}(x) = n_{i+1} g_{i+1}(x) = n_{i+1}(y) = g_i(z)$, so $z - m_{i+1}(x) \in \ker(g_i) = \operatorname{im}(f_i)$ Hence there is $\tilde{z} \in L_i$ with $f_i(\tilde{z}) = z - m_{i+1}(x)$. Since $f_{i-1} l_i(\tilde{z}) = m_i f_i(\tilde{z}) = m_i(z) - m_i m_{i+1}(x) = 0$ and f_{i-1} is injective, then $l_i(\tilde{z}) = 0$ so $\tilde{z} \in \ker(l_i)$ and $\overline{\tilde{z}} \in H_i(\boldsymbol{L_\bullet})$. Now $f_{*_i}(\overline{\tilde{z}}) = \overline{f_i(\tilde{z})} = \overline{z - m_{i+1}(x)} = \overline{z}$ so $\overline{z} \in \operatorname{im}(f_{*_i})$.

$(\mathrm{im}(f_{*_i}) \subseteq \ker(g_{*_i}))$ Let $\overline{z} \in H_i(\boldsymbol{L_\bullet})$, then $g_{*_i} f_{*_i}(\overline{z}) = (gf)_{*_i}(\overline{z}) = (0)_{*_i}(\overline{z}) = \overline{0}$ in $H_i(\boldsymbol{N_\bullet})$.

$(\ker(\partial_i) \subseteq \mathrm{im}(g_{*_i}))$ Let $\overline{z} \in \ker(\partial_i)$ and set $y = f_{i-1}^{-1} m_i g_i^{-1}(z)$ so that $\overline{y} = \partial_i(\overline{z}) = \overline{0}$ in $H_{i-1}(\boldsymbol{L_\bullet})$. Hence $y \in \mathrm{im}(l_i)$ so there is $x \in L_i$ with $l_i(x) = y$. Now $m_i(g_i^{-1}(z) - f_i(x)) = m_i g_i^{-1}(z) - m_i f_i(x) = 0$ since $m_i g_i^{-1}(z) = f_{i-1}(y) = f_{i-1} l_i(x) = m_i f_i(x)$, so $g_i^{-1}(z) - f_i(x) \in \ker(m_i)$ and $\overline{g_i^{-1}(z) - f_i(x)} \in H_i(\boldsymbol{M_\bullet})$. Now $g_{*_i}(\overline{g_i^{-1}(z) - f_i(x)}) = \overline{g_i(g_i^{-1}(z) - f_i(x))} = \overline{z - g_i f_i(x)} = \overline{z}$ so $\overline{z} \in \mathrm{im}(g_{*_i})$.

$(\mathrm{im}(g_{*_i}) \subseteq \ker(\partial_i))$ Let $\overline{z} \in H_i(\boldsymbol{M_\bullet})$, in particular $z \in \ker(m_i)$, then $\partial_i g_{*_i}(\overline{z}) = \overline{f_{i-1}^{-1} m_i g_i^{-1} g_i(z)} = \overline{f_{i-1}^{-1} m_i(z)} = \overline{0}$ in $H_i(\boldsymbol{M_\bullet})$.

The splitting as a sequence of B-modules can now be checked as follows.

$(f_{*_i} r_{*_i} f_{*_i} = f_{*_i})$ Let $\overline{z} \in H_i(\boldsymbol{L_\bullet})$, then $f_{*_i} r_{*_i} f_{*_i}(\overline{z}) = f_{*_i}(rf)_{*_i}(\overline{z}) = f_{*_i}(1_{\boldsymbol{L_\bullet}})_{*_i}(\overline{z}) = f_{*_i}(\overline{z})$.

$(g_{*_i} s_{*_i} g_{*_i} = g_{*_i})$ Let $\overline{z} \in H_i(\boldsymbol{M_\bullet})$, then $g_{*_i} s_{*_i} g_{*_i}(\overline{z}) = (gs)_{*_i} g_{*_i}(\overline{z}) = (1_{\boldsymbol{N_\bullet}})_{*_i} g_{*_i}(\overline{z}) = g_{*_i}(\overline{z})$.

$(\partial_i \int_i \partial_i = \partial_i)$ Let $\overline{z} \in H_i(\boldsymbol{N_\bullet})$, then $\int_i \partial_i(\overline{z}) = \overline{g_i m_i^{-1} f_{i-1} f_{i-1}^{-1} m_i g_i^{-1}(z)} = \overline{z}$ so $\int_i \partial_i = 1_{H_i(\boldsymbol{N_\bullet})}$. Let $\overline{z'} \in H_{i-1}(\boldsymbol{L_\bullet})$, then $\partial_i \int_i(\overline{z'}) = \overline{f_{i-1}^{-1} m_i g_i^{-1} g_i m_i^{-1} f_{i-1}(z')} = \overline{z'}$ so $\partial_i \int_i = 1_{H_{i-1}(\boldsymbol{L_\bullet})}$. Hence $\partial_i \int_i \partial_i = \partial_i$ because we are composing ∂_i with the identity, either on the left or the right.

□

There is also a relative version of the Horseshoe Lemma, which can be used to construct relative resolutions.

Lemma 3.38 (Relative Horseshoe Lemma)**.** *Let L, M, N be A-modules, $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$ and $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$ be (A, B)-projective resolutions of L and N respectively, $f : L \to M$, $g : M \to N$ be A-morphisms such that the following is an (A, B)-exact sequence.*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[l, bend left, "r"] \arrow[r, "g"] & N \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

Then:

1. *There exists $(\boldsymbol{T_\bullet}, \boldsymbol{t_\bullet})$ an (A, B)-projective resolution of M.*

2. *There exist chain maps of A-modules $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{T_\bullet}$ and $g_\bullet : \boldsymbol{T_\bullet} \to \boldsymbol{Q_\bullet}$.*

3. *There exist chain maps of B-modules $r_\bullet : \boldsymbol{T_\bullet} \to \boldsymbol{P_\bullet}$ and $s_\bullet : \boldsymbol{Q_\bullet} \to \boldsymbol{T_\bullet}$ such that*

$$\begin{tikzcd}
0 \arrow[r] & P_i \arrow[r, "f_i"] & T_i \arrow[l, bend left, "r_i"] \arrow[r, "g_i"] & Q_i \arrow[l, bend left, "s_i"] \arrow[r] & 0
\end{tikzcd}$$

   is an (A, B)-exact sequence for all $i \in \mathbb{N}$.

4. *Denote by $\{p'_i : P_i \to P_{i+1}\}_{i\in\mathbb{Z}}$, $\{q'_i : Q_i \to Q_{i+1}\}_{i\in\mathbb{Z}}$, $\{t'_i : T_i \to T_{i+1}\}_{i\in\mathbb{Z}}$ the B-splittings of $(\boldsymbol{P_\bullet}, \boldsymbol{p_\bullet})$, $(\boldsymbol{Q_\bullet}, \boldsymbol{q_\bullet})$, $(\boldsymbol{T_\bullet}, \boldsymbol{t_\bullet})$ respectively. Then as B-morphisms $f_i p'_{i-1} = t'_{i-1} f_{i-1}$ and $g_i t'_{i-1} = q'_{i-1} g_{i-1}$ for all $i \in \mathbb{N}$.*

In particular, we can complete the following commutative diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d, "p_2"] & \vdots \arrow[d, dashed, "t_2"] & \vdots \arrow[d, "q_2"] & \\
0 \arrow[r, dashed] & P_1 \arrow[r, dashed, "f_1"] \arrow[d, "p_1"] & T_1 \arrow[l, dashed, bend left, "r_1"] \arrow[r, dashed, "g_1"] \arrow[d, dashed, "t_1"] & Q_1 \arrow[l, dashed, bend left, "s_1"] \arrow[r, dashed] \arrow[d, "q_1"] & 0 \\
0 \arrow[r, dashed] & P_0 \arrow[r, dashed, "f_0"] \arrow[d, "p_0"] & T_0 \arrow[l, dashed, bend left, "r_0"] \arrow[r, dashed, "g_0"] \arrow[d, dashed, "t_0"] & Q_0 \arrow[l, dashed, bend left, "s_0"] \arrow[r, dashed] \arrow[d, "q_0"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[l, bend left, "r"] \arrow[r, "g"] \arrow[d, dashed] & N \arrow[l, bend left, "s"] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. We begin by constructing $(\boldsymbol{T_\bullet}, \boldsymbol{t_\bullet})$. Set $T_{-1} = M$, for all $i \in \mathbb{N}$ define $T_i = P_i \oplus Q_i$, the A-morphisms $t_i : T_i \to T_{i-1}$ via $t_i(m + n) = p_i(l) + q_i(n)$ for all $m \in P_i$ and $n \in Q_i$, and the the B-morphisms $t'_{i-1} : T_{i-1} \to T_i$ defined via $t'_{i-1}(m + n) = p'_{i-1}(l) + q'_{i-1}(n)$ for all $m \in P_{i-1}$ and $n \in Q_{i-1}$. Since for all $i \in \mathbb{N}$ both P_i and Q_i are (A, B)-projective, T_i is also (A, B)-projective by Theorem 3.16. Moreover

$$\ker(t_i) = \ker(p_i) \oplus \ker(q_i) = \operatorname{im}(p_{i+1}) \oplus \operatorname{im}(q_{i+1}) = \operatorname{im}(t_{i+1})$$

for all $i \in \mathbb{N}$ by exactness of $\boldsymbol{P_\bullet}$ and $\boldsymbol{Q_\bullet}$, so $\boldsymbol{T_\bullet}$ is also exact. Since

$$t_i t'_{i-1} t_i = (p_i + q_i)(p'_{i-1} + q'_{i-1})(p_i + q_i) = p_i p'_{i-1} p_i + q_i q'_{i-1} q_i = p_i + q_i = t_i$$

because $p_i p'_{i-1} p_i$ and $q_i q'_{i-1} q_i = q_i$ by the B-splitting of $\boldsymbol{P_\bullet}$ and $\boldsymbol{Q_\bullet}$, whence $\boldsymbol{T_\bullet}$ is also B-split. Hence $(\boldsymbol{T_\bullet}, \boldsymbol{t_\bullet})$ is indeed an (A, B)-projective resolution of M.

We now construct the chain maps between $\boldsymbol{P_\bullet}$, $\boldsymbol{T_\bullet}$, and $\boldsymbol{Q_\bullet}$. For all $i \in \mathbb{N}$ define $f_i : P_i \to T_i$ as the inclusion of A-modules in the first component and $g_i : T_i \to Q_i$ be the projection of A-modules in the second component. Then $f_i t_i = f_i(p_i + q_i) = f_i p_i = p_i f_{i-1}$ and $g_{i-1} t_i = g_{i-1}(p_i + q_i) = g_{i-1} q_i = q_i g_i$ and thus $f_\bullet : \boldsymbol{P_\bullet} \to \boldsymbol{T_\bullet}$ and $g_\bullet : \boldsymbol{T_\bullet} \to \boldsymbol{Q_\bullet}$ are chain maps of A-modules.

We now construct the chain maps between $\boldsymbol{Q_\bullet}$, $\boldsymbol{T_\bullet}$, and $\boldsymbol{P_\bullet}$. For all $i \in \mathbb{N}$, by Remark 2.17,

$$0 \longrightarrow P_i \xrightarrow{f_i} T_i \xrightarrow{g_i} Q_i \longrightarrow 0$$

is an exact sequence that splits as B-modules via $r_i : T_i \to P_i$ the projection of B-modules in the first component and $s_i : Q_i \to T_i$ the inclusion of B-modules in the second component. Moreover, we have that $r_\bullet : \boldsymbol{T_\bullet} \to \boldsymbol{P_\bullet}$ and $s_\bullet : \boldsymbol{Q_\bullet} \to \boldsymbol{T_\bullet}$ are chain maps of B-modules by Remark 3.35.

Finally, for all $i \in \mathbb{N}$ we have $t'_{i-1} f_{i-1} = (p'_{i-1} + q'_{i-1}) f_{i-1} = p'_{i-1} f_{i-1} = f_i p'_{i-1}$ and $g_i t'_{i-1} = g_i(p'_{i-1} + q'_{i-1}) = g_i q'_{i-1} = q'_{i-1} g_{i-1}$. □

In particular, in the following squares any two paths with different source and target coincide.

$$
\begin{tikzcd}
 & \vdots \arrow[d, "p_2"] & & \vdots \arrow[d, "t_2"] & & \vdots \arrow[d, "q_2"] & \\
0 \arrow[r] & P_1 \arrow[u, bend left, "p'_1"] \arrow[r, "f_1"] \arrow[d, "p_1"] & P_1 \oplus Q_1 \arrow[u, bend left, "t'_1"] \arrow[l, bend left, "r_1"] \arrow[r, "g_1"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "q_1"] & Q_1 \arrow[u, bend left, "q'_1"] \arrow[l, bend left, "s_1"] \arrow[r] \arrow[d, "q_1"] & 0 \\
0 \arrow[r] & P_0 \arrow[u, bend left, "p'_0"] \arrow[r, "f_0"] \arrow[d, "p_0"] & P_0 \oplus Q_0 \arrow[u, bend left, "p'_0"] \arrow[u, bend right, "q'_0"'] \arrow[l, bend left, "r_0"] \arrow[r, "g_0"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "q_0"] & Q_0 \arrow[u, bend left, "q'_1"] \arrow[l, bend left, "s_0"] \arrow[r] \arrow[d, "q_0"] & 0 \\
0 \arrow[r] & L \arrow[u, bend left, "p'_{-1}"] \arrow[r, "f"] \arrow[d] & L \oplus N \arrow[u, bend left, "p'_{-1}"] \arrow[u, bend right, "q'_{-1}"'] \arrow[l, bend left, "r"] \arrow[r, "g"] \arrow[d] & N \arrow[u, bend left, "q'_{-1}"] \arrow[l, bend left, "s"] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Putting together the last few pages, we achieve the claimed constructions.

Theorem 3.39 (Relative first long exact sequence for Tor)**.** *Let K, L, M be right A-modules, N be a left A-module, and $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be an (A,B)-exact sequence. Then there is a splitting long exact sequence of $\mathbb{Z}$-modules*

$$
\begin{gathered}
\cdots \rightleftarrows \operatorname{Tor}_{i+1}^{(A,B)}(M,N) \rightleftarrows \operatorname{Tor}_{i}^{(A,B)}(K,N) \rightleftarrows \operatorname{Tor}_{i}^{(A,B)}(L,N) \rightleftarrows \operatorname{Tor}_{i}^{(A,B)}(M,N) \rightleftarrows \\
\cdots \rightleftarrows \operatorname{Tor}_{1}^{(A,B)}(M,N) \rightleftarrows K \otimes_A N \rightleftarrows L \otimes_A N \rightleftarrows M \otimes_A N \to 0.
\end{gathered}
$$

Proof. Let $(\boldsymbol{P}_\bullet, \boldsymbol{p}_\bullet)$ and $(\boldsymbol{Q}_\bullet, \boldsymbol{q}_\bullet)$ be (A,B)-projective resolutions via right A-modules of K and M respectively. Then by the Relative Horseshoe Lemma 3.38 we have $(\boldsymbol{T}_\bullet, \boldsymbol{t}_\bullet)$ an (A,B)-projective resolution of L, chain maps of A-modules $f_\bullet : \boldsymbol{P}_\bullet \to \boldsymbol{T}_\bullet$ and $g_\bullet : \boldsymbol{T}_\bullet \to \boldsymbol{Q}_\bullet$, and chain maps of B-modules $\tilde{r}_\bullet : \boldsymbol{T}_\bullet \to \boldsymbol{P}_\bullet$ and $\tilde{s}_\bullet : \boldsymbol{Q}_\bullet \to \boldsymbol{T}_\bullet$ with $\tilde{r}_\bullet f_\bullet = 1_{\boldsymbol{P}_\bullet}$ and $g_\bullet \tilde{s}_\bullet = 1_{\boldsymbol{Q}_\bullet}$ such that

$$
\begin{tikzcd}
0 \arrow[r] & \boldsymbol{P}_\bullet \arrow[r, "f_\bullet"] & \boldsymbol{T}_\bullet \arrow[l, bend left, "\tilde{r}_\bullet"] \arrow[r, "g_\bullet"] & \boldsymbol{Q}_\bullet \arrow[l, bend left, "\tilde{s}_\bullet"] \arrow[r] & 0
\end{tikzcd}
$$

is an (A,B)-exact sequence of complexes of right A-modules. Since Q_i is (A,B)-projective for all $i \in \mathbb{N}$, by Theorem 3.16 we have that the (A,B)-exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & P_i \arrow[r, "f_i"] & T_i \arrow[l, bend left, "\tilde{r}_i"] \arrow[r, "g_i"] & Q_i \arrow[l, bend left, "\tilde{s}_i"] \arrow[r] & 0
\end{tikzcd}
$$

splits as a sequence of A-modules. Hence by Remark 2.15 we have that there are A-morphisms $s_i : Q_i \to T_i$ and $r_i : T_i \to P_i$ such that the following sequence is split exact as A-modules.

$$\begin{tikzcd} 0 \arrow[r] & P_i \arrow[r, "f_i"] & T_i \arrow[l, bend left, "r_i"] \arrow[r, "g_i"] & Q_i \arrow[l, bend left, "s_i"] \arrow[r] & 0 \end{tikzcd}$$

We have that $\tilde{s}_\bullet : Q_i \to T_i$ and $\tilde{r}_\bullet : T_i \to P_i$ are chain maps by Remark 3.35. Hence

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{P}_\bullet \arrow[r, "f_\bullet"] & \boldsymbol{T}_\bullet \arrow[l, bend left, "r_\bullet"] \arrow[r, "g_\bullet"] & \boldsymbol{Q}_\bullet \arrow[l, bend left, "s_\bullet"] \arrow[r] & 0 \end{tikzcd}$$

is an (A, A)-exact sequence of complexes. Applying $? \otimes_A N$ we obtain

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{P}_\bullet \otimes_A N \arrow[r, "f_\bullet \otimes 1_N"] & \boldsymbol{T}_\bullet \otimes_A N \arrow[l, bend left, "r_\bullet \otimes 1_N"] \arrow[r, "g_\bullet \otimes 1_N"] & \boldsymbol{Q}_\bullet \otimes_A N \arrow[l, bend left, "s_\bullet \otimes 1_N"] \arrow[r] & 0 \end{tikzcd}$$

which is a splitting short exact sequence of complexes of $\mathbb{Z}$-modules by Remark 3.22. Then by Theorem 3.37 we have the splitting long exact sequence of $\mathbb{Z}$-modules

$$\begin{tikzcd}
\cdots \arrow[r, "(g_\bullet \otimes 1_N)_{*_{i+1}}"] & H_{i+1}(\boldsymbol{Q}_\bullet \otimes_A N) \arrow[l, bend left, "(s_\bullet \otimes 1_N)_{*_{i+1}}"] \arrow[r, "\partial_{i+1}"] & H_i(\boldsymbol{P}_\bullet \otimes_A N) \arrow[l, bend left, "\int_{i+1}"] \arrow[r, "(f_\bullet \otimes 1_N)_{*_i}"] & H_i(\boldsymbol{T}_\bullet \otimes_A N) \arrow[l, bend left, "(r_\bullet \otimes 1_N)_{*_i}"] \arrow[d, shift right, "(g_\bullet \otimes 1_N)_{*_i}"'] & \\
H_1(\boldsymbol{Q}_\bullet \otimes_A N) \arrow[r, bend right, "(s_\bullet \otimes 1_N)_{*_1}"'] \arrow[d, shift right, "\partial_1"'] & \cdots \arrow[l, "(g_\bullet \otimes 1_N)_{*_1}"'] \arrow[r, bend right, "(r_\bullet \otimes 1_N)_{*_{i-1}}"'] & H_{i-1}(\boldsymbol{P}_\bullet \otimes_A N) \arrow[l, "(f_\bullet \otimes 1_N)_{*_{i-1}}"'] \arrow[r, bend right, "\int_i"'] & H_i(\boldsymbol{Q}_\bullet \otimes_A N) \arrow[l, "\partial_i"'] \arrow[u, shift right, "(s_\bullet \otimes 1_N)_{*_i}"'] & \\
H_0(\boldsymbol{P}_\bullet \otimes_A N) \arrow[u, shift right, "\int_1"'] \arrow[r, "(f_\bullet \otimes 1_N)_{*_0}"] & H_0(\boldsymbol{T}_\bullet \otimes_A N) \arrow[l, bend left, "(r_\bullet \otimes 1_N)_{*_0}"] \arrow[r, "(g_\bullet \otimes 1_N)_{*_0}"] & H_0(\boldsymbol{Q}_\bullet \otimes_A N) \arrow[l, bend left, "(s_\bullet \otimes 1_N)_{*_0}"] \arrow[rr] & & 0.
\end{tikzcd}$$

By construction, this is the desired splitting long exact sequence of $\mathbb{Z}$-modules

$$\cdots \rightleftarrows \operatorname{Tor}_{i+1}^{(A,B)}(M, N) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(K, N) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(L, N) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(M, N) \rightleftarrows$$
$$\cdots \rightleftarrows \operatorname{Tor}_1^{(A,B)}(M, N) \rightleftarrows K \otimes_A N \rightleftarrows L \otimes_A N \rightleftarrows M \otimes_A N \to 0.$$

□

In fact, we can compute Tor using a resolution of the module in the first component, so reasoning similarly as above yields the following.

Theorem 3.40 (Relative second long exact sequence for Tor)**.** *Let K, L, M be left A-modules, N be a right A-module, and $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be an (A, B)-exact sequence. Then there is a splitting long exact sequence of $\mathbb{Z}$-modules*

$$\cdots \rightleftarrows \operatorname{Tor}_{i+1}^{(A,B)}(N,M) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(N,K) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(N,L) \rightleftarrows \operatorname{Tor}_i^{(A,B)}(N,M) \rightleftarrows$$

$$\cdots \rightleftarrows \operatorname{Tor}_1^{(A,B)}(N,M) \rightleftarrows N \otimes_A K \rightleftarrows N \otimes_A L \rightleftarrows N \otimes_A M \to 0.$$

We will now immediately harvest a bounty of results. An immediate consequence of these long exact sequences is a characterization of flat modules that can be used to give a relationship between flatness and short exact sequences.

Theorem 3.41. *For F a right A-module, the following are equivalent:*

1. *F is right (A,B)-flat.*

2. *$\operatorname{Tor}_i^{(A,B)}(F,M) = 0$ for all A-modules M and for all positive $i \in \mathbb{N}$.*

3. *$\operatorname{Tor}_1^{(A,B)}(F,M) = 0$ for all A-modules M.*

Proof. (1.⇒2.) Let $(\boldsymbol{P_\bullet}, \boldsymbol{d_\bullet})$ be an (A,B)-projective resolution of M a given A-module.

$$\begin{tikzcd} \cdots \arrow[r, "d_{i+1}"] & P_i \arrow[l, bend left, "s_i"] \arrow[r, "d_i"] & P_{i-1} \arrow[l, bend left, "s_{i-1}"] \arrow[r, "d_{i-1}"] & \cdots \arrow[l, bend left, "s_{i-2}"] \arrow[r, "d_1"] & P_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] & 0 \end{tikzcd}$$

Since F is (A,B)-flat, by Remark 3.23 the following sequence is $(\mathbb{Z},\mathbb{Z})$-exact.

$$\cdots \xrightarrow{1_F \otimes d_{i+1}} F \otimes_A P_i \xrightarrow{1_F \otimes d_i} F \otimes_A P_{i-1} \xrightarrow{1_F \otimes d_{i-1}} \cdots \xrightarrow{1_F \otimes d_1} F \otimes_A P_0 \xrightarrow{1_F \otimes d_0} F \otimes_A M \longrightarrow 0$$

Hence, by definition, $\operatorname{Tor}_i^{(A,B)}(F,M) = 0$ for all positive $i \in \mathbb{N}$.

(2.⇒3.) We already have that $\operatorname{Tor}_1^{(A,B)}(F,M) = 0$ for all A-modules M by hypothesis.

(3.⇒1.) Given and (A,B)-exact sequence of A-modules $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$, by Theorem 3.40 we have the long $(\mathbb{Z},\mathbb{Z})$-exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_1^{(A,B)}(F,N) \longrightarrow F \otimes_A L \xrightarrow{1_F \otimes f} F \otimes_A M \xrightarrow{1_F \otimes g} F \otimes_A N \longrightarrow 0.$$

Since $\operatorname{Tor}_1^{(A,B)}(F,N) = 0$ by hypothesis, we obtain the $(\mathbb{Z},\mathbb{Z})$-exact sequence $0 \to F \otimes_A L \xrightarrow{1_F \otimes f} F \otimes_A M \xrightarrow{1_F \otimes g} F \otimes_A N \to 0$ and thus F is (A,B)-flat.

$\square$

When F is a left (A, B)-flat module, an analogous result holds: F is (A, B)-flat if and only if $\operatorname{Tor}_1^{(A,B)}(M, F) = 0$ for all A-modules M, or equivalently $\operatorname{Tor}_i^{(A,B)}(M, F) = 0$ for all A-modules M and for all positive $i \in \mathbb{N}$.

Corollary 3.42. *Let K, L, M be right A-modules such that M is (A, B)-flat and $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ is an (A, B)-exact sequence. Then K is (A, B)-flat if and only if L is (A, B)-flat.*

Proof. Theorem 3.39 applied to $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ and to any left A-module N gives the splitting long exact sequence of $\mathbb{Z}$-modules

$$\cdots \to \operatorname{Tor}_{i+1}^{(A,B)}(M, N) \to \operatorname{Tor}_i^{(A,B)}(K, N) \to \operatorname{Tor}_i^{(A,B)}(L, N) \to \operatorname{Tor}_i^{(A,B)}(M, N) \to \cdots$$

where M is (A, B)-flat, so $\operatorname{Tor}_i^{(A,B)}(M, N) = 0$ for all positive $i \in \mathbb{N}$ by Theorem 3.41. Hence

$$0 \to \operatorname{Tor}_i^{(A,B)}(K, N) \to \operatorname{Tor}_i^{(A,B)}(L, N) \to 0$$

is an exact sequence for any positive $i \in \mathbb{N}$. Then $\operatorname{Tor}_i^{(A,B)}(K, N) = 0$ if and only if $\operatorname{Tor}_i^{(A,B)}(L, N)$, so by Theorem 3.41 we have K is (A, B)-flat if and only if L is (A, B)-flat. $\square$

The relative long exact sequences of Theorem 3.39 and Theorem 3.40 can be generalized to total complexes. Namely given a left or right (A, B)-exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{K}_\bullet \arrow[r, "f_\bullet"] & \boldsymbol{L}_\bullet \arrow[r, "g_\bullet"] \arrow[l, bend left, "r_\bullet"] & \boldsymbol{M}_\bullet \arrow[r] \arrow[l, bend left, "s_\bullet"] & 0, \end{tikzcd}$$

a complex of left or right A-modules $\boldsymbol{N}_\bullet$, then

$$\cdots \rightleftarrows \operatorname{Tor}_1^{(A,B)}(\boldsymbol{M}_\bullet, \boldsymbol{N}_\bullet) \rightleftarrows \boldsymbol{K}_\bullet \otimes_A \boldsymbol{N}_\bullet \rightleftarrows \boldsymbol{L}_\bullet \otimes_A \boldsymbol{N}_\bullet \rightleftarrows \boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet \to 0,$$
$$\cdots \rightleftarrows \operatorname{Tor}_1^{(A,B)}(\boldsymbol{N}_\bullet, \boldsymbol{M}_\bullet) \rightleftarrows \boldsymbol{N}_\bullet \otimes_A \boldsymbol{K}_\bullet \rightleftarrows \boldsymbol{N}_\bullet \otimes_A \boldsymbol{L}_\bullet \rightleftarrows \boldsymbol{N}_\bullet \otimes_A \boldsymbol{M}_\bullet \to 0,$$

are splitting long exact sequences of $\mathbb{Z}$-modules (for a suitable interpretation of $\operatorname{Tor}_i^{(A,B)}(?, ?)$). However, since we do not require the full strength of this result, we will only consider the following particular case.

Corollary 3.43. *Let $(\boldsymbol{L}_\bullet, \boldsymbol{l}_\bullet)$, $(\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$, $(\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet)$ be complexes of right A-modules, $(\boldsymbol{P}_\bullet, \boldsymbol{p}_\bullet)$ be a complex of left A-modules, $f_\bullet : (\boldsymbol{L}_\bullet, \boldsymbol{l}_\bullet) \to (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$, $g_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet) \to (\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet)$ be chain maps of A-morphisms, and $\tilde{r}_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet) \to (\boldsymbol{L}_\bullet, \boldsymbol{l}_\bullet)$, $\tilde{s}_\bullet : (\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet) \to (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ be chain maps of B-morphisms such that the following is an (A, B)-exact sequence of complexes.*

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{L}_\bullet \arrow[r, "f_\bullet"] & \boldsymbol{M}_\bullet \arrow[l, bend left, "\tilde{r}_\bullet"] \arrow[r, "g_\bullet"] & \boldsymbol{N}_\bullet \arrow[l, bend left, "\tilde{s}_\bullet"] \arrow[r] & 0 \end{tikzcd}$$

Suppose that N_i is (A, B)-flat for all $i \in \mathbb{Z}$. Then the following is a split exact sequence of complexes of $\mathbb{Z}$-modules.

$$0 \to \operatorname{Tot}_\bullet(\boldsymbol{L}_\bullet \otimes_A \boldsymbol{P}_\bullet) \xrightarrow{\operatorname{Tot}_\bullet(f_\bullet \otimes 1_{\boldsymbol{P}_\bullet})} \operatorname{Tot}_\bullet(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{P}_\bullet) \xrightarrow{\operatorname{Tot}_\bullet(g_\bullet \otimes 1_{\boldsymbol{P}_\bullet})} \operatorname{Tot}_\bullet(\boldsymbol{N}_\bullet \otimes_A \boldsymbol{P}_\bullet) \to 0$$

Proof. Fix $i, j \in \mathbb{Z}$. We have the (A, B)-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L_j \arrow[r, "f_j"] & M_j \arrow[l, bend left, "\tilde{r}_j"] \arrow[r, "g_j"] & N_j \arrow[l, bend left, "\tilde{s}_j"] \arrow[r] & 0. \end{tikzcd}$$

Applying Theorem 3.39 with P_{i-j}, the relative first long exact sequence for Tor is

$$\cdots \rightleftarrows \operatorname{Tor}_1^{(A,B)}(N_j, P_{i-j}) \rightleftarrows L_j \otimes_A P_{i-j} \rightleftarrows M_j \otimes_A P_{i-j} \rightleftarrows N_j \otimes_A P_{i-j} \to 0.$$

Since N_i is (A, B)-flat, by Theorem 3.41 we have $\operatorname{Tor}_1^{(A,B)}(N_j, P_{i-j}) = 0$, and thus using the notation of the proof of Theorem 3.39 we obtain the splitting exact sequence of $\mathbb{Z}$-modules

$$\begin{tikzcd} 0 \arrow[r] & L_j \otimes_A P_{i-j} \arrow[r, "f_j \otimes 1_{P_{i-j}}"] & M_j \otimes_A P_{i-j} \arrow[l, bend left, "r_j \otimes 1_{P_{i-j}}"] \arrow[r, "g_j \otimes 1_{P_{i-j}}"] & N_j \otimes_A P_{i-j} \arrow[l, bend left, "s_j \otimes 1_{P_{i-j}}"] \arrow[r] & 0 \end{tikzcd}$$

where $r_j : M_j \to L_j$ and $s_j : N_j \to M_j$ are A-morphisms. In fact $r_\bullet : M_\bullet \to L_\bullet$ and $s_\bullet : N_\bullet \to M_\bullet$ are chain maps by Remark 3.35. We immediately obtain that

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{j\in\mathbb{Z}}(L_j \otimes_A P_{i-j}) \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(f_j \otimes 1_{P_{i-j}})"] & \bigoplus\limits_{j\in\mathbb{Z}}(M_j \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(r_j \otimes 1_{P_{i-j}})"] \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(g_j \otimes 1_{P_{i-j}})"] & \bigoplus\limits_{j\in\mathbb{Z}}(N_j \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(s_j \otimes 1_{P_{i-j}})"] \arrow[r] & 0 \end{tikzcd}$$

is a split exact sequence of $\mathbb{Z}$-modules. We now combine all possible $i, j \in \mathbb{Z}$ into two commutative squares. Unraveling the definitions of $\operatorname{Tot}_\bullet(\boldsymbol{L}_\bullet \otimes_A \boldsymbol{P}_\bullet)$, $\operatorname{Tot}_\bullet(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{P}_\bullet)$, $\operatorname{Tot}_\bullet(\boldsymbol{N}_\bullet \otimes_A \boldsymbol{P}_\bullet)$, $\operatorname{Tot}_\bullet(f_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, $\operatorname{Tot}_\bullet(g_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, $\operatorname{Tot}_\bullet(r_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, and $\operatorname{Tot}_\bullet(s_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, give the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bigoplus_{j\in\mathbb{Z}}(L_j \otimes_A P_{i-j}) \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(f_j\otimes 1_{P_{i-j}})"] \arrow[d, "\bigoplus_{j\in\mathbb{Z}}\substack{(l_j\otimes 1_{P_{i-j}}+\\ 1_{L_j}\otimes p_{i-j})}"] & \bigoplus_{j\in\mathbb{Z}}(M_j \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(r_j\otimes 1_{P_{i-j}})"] \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(g_j\otimes 1_{P_{i-j}})"] \arrow[d, "\bigoplus_{j\in\mathbb{Z}}\substack{(m_j\otimes 1_{P_{i-j}}+\\ 1_{M_j}\otimes p_{i-j})}"] & \bigoplus_{j\in\mathbb{Z}}(N_j \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(s_j\otimes 1_{P_{i-j}})"] \arrow[d, "\bigoplus_{j\in\mathbb{Z}}\substack{(n_j\otimes 1_{P_{i-j}}+\\ 1_{N_j}\otimes p_{i-j})}"] \\
\bigoplus_{j\in\mathbb{Z}}(L_{j-1} \otimes_A P_{i-j}) \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(f_{j-1}\otimes 1_{P_{i-j}})"] & \bigoplus_{j\in\mathbb{Z}}(M_{j-1} \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(r_{j-1}\otimes 1_{P_{i-j}})"] \arrow[r, "\bigoplus_{j\in\mathbb{Z}}(g_{j-1}\otimes 1_{P_{i-j}})"] & \bigoplus_{j\in\mathbb{Z}}(N_{j-1} \otimes_A P_{i-j}) \arrow[l, bend left, "\bigoplus_{j\in\mathbb{Z}}(s_{j-1}\otimes 1_{P_{i-j}})"]
\end{tikzcd}$$

Since $f_\bullet : \boldsymbol{L}_\bullet \to \boldsymbol{M}_\bullet$ is a chain map, we have

$$\begin{aligned}
(f_{j-1} \otimes 1_{P_{i-j}})(l_j \otimes 1_{P_{i-j}}) &= (f_{j-1}l_j) \otimes 1_{P_{i-j}} = (m_j f_j) \otimes 1_{P_{i-j}} = (m_{j-1} \otimes 1_{P_{i-j}})(f_j \otimes 1_{P_{i-j}}) \\
(f_j \otimes 1_{P_{i-j}})(1_{L_j} \otimes p_{i-j}) &= f_j \otimes p_{i-j} = (1_{M_j} \otimes p_{i-j})(f_j \otimes 1_{P_{i-j}}).
\end{aligned}$$

Since $r_\bullet : \boldsymbol{M}_\bullet \to \boldsymbol{L}_\bullet$ is a chain map, we have

$$\begin{aligned}
(l_j \otimes 1_{P_{i-j}})(r_j \otimes 1_{P_{i-j}}) &= (l_j r_j) \otimes 1_{P_{i-j}} = (r_{j-1} m_j) \otimes 1_{P_{i-j}} = (r_{j-1} \otimes 1_{P_{i-j}})(m_j \otimes 1_{P_{i-j}}) \\
(1_{L_j} \otimes p_{i-j})(r_j \otimes 1_{P_{i-j}}) &= r_j \otimes p_{i-j} = (r_j \otimes 1_{P_{i-j}})(1_{M_j} \otimes p_{i-j}).
\end{aligned}$$

The left square is thus commutative. Similarly, since $g_\bullet : \boldsymbol{M}_\bullet \to \boldsymbol{N}_\bullet$ and $s_\bullet : \boldsymbol{N}_\bullet \to \boldsymbol{M}_\bullet$ are chain maps, the right square is commutative. Hence $\operatorname{Tot}_\bullet(f_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, $\operatorname{Tot}_\bullet(g_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, $\operatorname{Tot}_\bullet(r_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$, and $\operatorname{Tot}_\bullet(s_\bullet \otimes 1_{\boldsymbol{P}_\bullet})$ are chain maps of complexes of $\mathbb{Z}$-modules. This can be rewritten as

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tot}_\bullet(\boldsymbol{L}_\bullet \otimes_A \boldsymbol{P}_\bullet) \arrow[r, "\operatorname{Tot}_\bullet(f_\bullet\otimes 1_{\boldsymbol{P}_\bullet})"] & \operatorname{Tot}_\bullet(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{P}_\bullet) \arrow[l, bend left, "\operatorname{Tot}_\bullet(r_\bullet\otimes 1_{\boldsymbol{P}_\bullet})"] \arrow[r, "\operatorname{Tot}_\bullet(g_\bullet\otimes 1_{\boldsymbol{P}_\bullet})"] & \operatorname{Tot}_\bullet(\boldsymbol{N}_\bullet \otimes_A \boldsymbol{P}_\bullet) \arrow[l, bend left, "\operatorname{Tot}_\bullet(s_\bullet\otimes 1_{\boldsymbol{P}_\bullet})"] \arrow[r] & 0
\end{tikzcd}$$

being a split exact sequence of complexes of $\mathbb{Z}$-modules, proving the desired result. □

In fact, we do not require that $r_\bullet : (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet) \to (\boldsymbol{L}_\bullet, \boldsymbol{l}_\bullet)$ and $s_\bullet : (\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet) \to (\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ are chain maps of B-morphisms, by Remark 3.35 it suffices that the induced short exact sequences are B-split in each degree.

3.3 The relative Künneth theorem

We now have all the necessary tools to prove a relative version of the Künneth Theorem 2.44.

However, before stating the result, we need to be aware of the compatibility requirements of $(\boldsymbol{M}_\bullet, \boldsymbol{d}_\bullet)$ an exact sequence of A-modules with respect to the relative structure induced by A and B. Namely, if we require that $\boldsymbol{M}_\bullet$ is (A, B)-exact, we will not have interesting homology since the sequence $\boldsymbol{M}_\bullet$ will be forced to be exact by Proposition 3.2. Because of this, we will instead require that the two exact sequences induced by the canonical surjections $N_i \to \operatorname{im}(d_i)$ and inclusions $\operatorname{im}(d_{i+1}) \to \ker(d_i)$ implied by the equalities $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$ are (A, B)-exact.

Theorem 3.44 (Relative Künneth Theorem)**.** *Let* $(\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ *be a complex of right* A*-modules such that* M_j *and* $\operatorname{im}(m_j)$ *are* (A, B)*-flat and* $0 \to \ker(m_j) \xrightarrow{\iota_j} M_j \xrightarrow{m_j} \operatorname{im}(m_j) \to 0$ *are* (A, B)*-exact sequences for all* $j \in \mathbb{Z}$*. Let* $(\boldsymbol{N}_\bullet, \boldsymbol{n}_\bullet)$ *be a complex of left* A*-modules such that* $0 \to \ker(n_j) \xrightarrow{\kappa_j} N_j \xrightarrow{n_j} \operatorname{im}(n_j) \to 0$ *and* $0 \to \operatorname{im}(n_{j+1}) \to \ker(n_j) \to \ker(n_j)/\operatorname{im}(n_{j+1}) \to 0$ *are* (A, B)*-exact sequences for all* $j \in \mathbb{Z}$*. Then for each* $i \in \mathbb{Z}$ *there is a* $(\mathbb{Z}, \mathbb{Z})$*-exact sequence*

$$\begin{tikzcd}
\bigoplus\limits_{r+s=i} (H_r(\boldsymbol{M}_\bullet) \otimes_A H_s(\boldsymbol{N}_\bullet)) \arrow[r, hook] & H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, two heads] \arrow[l, bend left] & \bigoplus\limits_{r+s=i-1} \operatorname{Tor}_1^{(A,B)}(H_r(\boldsymbol{M}_\bullet), H_s(\boldsymbol{N}_\bullet)) \arrow[l, bend left]
\end{tikzcd}$$

Proof. Since for each $i \in \mathbb{Z}$

$$0 \longrightarrow \ker(m_i) \xhookrightarrow{\iota_i} M_i \xrightarrow{m_i} \operatorname{im}(m_i) \longrightarrow 0$$

is an (A, B)-exact sequence, we can arrange them in a short (A, B)-exact sequence of A-modules

$$\begin{tikzcd}
 & \vdots \arrow[d, "m_{i+1}"] & \vdots \arrow[d, "m_{i+1}"] & \vdots \arrow[d, "m_{i+1}"] & \\
0 \arrow[r] & \ker(m_i) \arrow[r, hook, "\iota_i"] \arrow[d, "m_i|_{\ker(m_i)}"] & M_i \arrow[r, two heads, "m_i"] \arrow[d, "m_i"] & \operatorname{im}(m_i) \arrow[r] \arrow[d, "m_{i-1}|_{\operatorname{im}(m_i)}"] & 0 \\
0 \arrow[r] & \ker(m_{i-1}) \arrow[r, hook, "\iota_{i+1}"] \arrow[d, "m_{i-1}"] & M_{i-1} \arrow[r, two heads, "m_{i-1}"] \arrow[d, "m_{i-1}"] & \operatorname{im}(m_{i-1}) \arrow[r] \arrow[d, "m_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

because the outermost arrows are zero, and thus all the inner squares commute. For simplicity, we can rewrite this using Definition 2.4 as the (A,B)-exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{Z}_\bullet \arrow[r, hook, "\iota_\bullet"] & \boldsymbol{M}_\bullet \arrow[r, two heads, "m_\bullet"] \arrow[l, bend left] & \boldsymbol{B}[-\mathbf{1}]_\bullet \arrow[r] \arrow[l, bend left] & 0, \end{tikzcd}$$

where the shift appears because $\operatorname{im}(m_i) \subseteq M_{i-1}$ for all $i \in \mathbb{Z}$, changing the homological degree. We can apply Corollary 3.43 because $B[-1]_{j+1} = \operatorname{im}(m_j)$ is (A,B)-flat for all $j \in \mathbb{Z}$ by hypothesis. Hence taking the tensor product of the above with $\boldsymbol{N}_\bullet$ yields the $(\mathbb{Z},\mathbb{Z})$-exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tot}_\bullet(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "\operatorname{Tot}_\bullet(\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})"] & \operatorname{Tot}_\bullet(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "\operatorname{Tot}_\bullet(m_\bullet \otimes 1_{\boldsymbol{N}_\bullet})"] \arrow[l, bend left] & \operatorname{Tot}_\bullet(\boldsymbol{B}[-\mathbf{1}]_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

where we will simplify notation as in Example 2.20. Applying Theorem 3.37 to this gives the long $(\mathbb{Z},\mathbb{Z})$-exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_{i+1}(\boldsymbol{B}[-\mathbf{1}]_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "\partial_{i+1}"] \arrow[l, bend left] & H_i(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "(\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}"] \arrow[l, bend left, "\int_{i+1}"] & H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[d, "(m_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}"] \arrow[l, bend left] \\ & \cdots & H_{i-1}(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[l] \arrow[l, bend right, leftarrow] & H_i(\boldsymbol{B}[-\mathbf{1}]_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[l, "\partial_i"] \arrow[u, bend left] \arrow[l, bend right, leftarrow, "\int_i"] \end{tikzcd}$$

where we observe that $m_i|_{\ker(m_i)} = 0 = m_{i-1}|_{\operatorname{im}(m_i)}$ for all $i \in \mathbb{Z}$. This means that $\boldsymbol{Z}_\bullet$ and $\boldsymbol{B}[-\mathbf{1}]_\bullet$ have zero differentials and are in fact $(\boldsymbol{Z}_\bullet, 0)$ and $(\boldsymbol{B}[-\mathbf{1}]_\bullet, 0)$ respectively. This allows us to reinterpret $H_{i+1}(\boldsymbol{B}[-\mathbf{1}]_\bullet \otimes_A \boldsymbol{N}_\bullet)$ and $H_i(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)$ in terms of the differentials of $\boldsymbol{N}_\bullet$, since the differentials of the total complexes $\operatorname{Tot}_\bullet(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)$ and $\operatorname{Tot}_\bullet(\boldsymbol{B}[-\mathbf{1}]_\bullet \otimes_A \boldsymbol{N}_\bullet)$ are then direct sums of $1_{Z_\bullet} \otimes_A n_\bullet$ and $1_{B[-1]_\bullet} \otimes_A n_\bullet$ respectively.

For this reinterpretation, note first that since $0 \to \ker(m_j) \overset{\iota_j}{\to} M_j \overset{m_j}{\to} \operatorname{im}(m_j) \to 0$ are (A,B)-exact sequences for all $j \in \mathbb{Z}$, and both M_j and $\operatorname{im}(m_j)$ are (A,B)-flat, then $\ker(m_j)$ is also (A,B)-flat by Corollary 3.42. Since $\operatorname{im}(m_{j+1})$ and $\ker(m_j)$ are (A,B)-flat and $0 \to \operatorname{im}(n_{i+1}) \to \ker(n_i) \to \ker(n_i)/\operatorname{im}(n_{i+1}) \to 0$ are (A,B)-exact sequences for all $i \in \mathbb{Z}$, then $0 \to \operatorname{im}(m_{j+1}) \otimes_A \operatorname{im}(n_{i+1}) \to \operatorname{im}(m_{j+1}) \otimes_A \ker(n_i) \to \operatorname{im}(m_{j+1}) \otimes_A (\ker(n_i)/\operatorname{im}(n_{i+1})) \to 0$ and $0 \to \ker(m_j) \otimes_A \operatorname{im}(n_{i+1}) \to \ker(m_j) \otimes_A \ker(n_i) \to \ker(m_j) \otimes_A (\ker(n_i)/\operatorname{im}(n_{i+1})) \to 0$

are $(\mathbb{Z},\mathbb{Z})$-exact. In particular as $\mathbb{Z}$-modules we have

$$(\operatorname{im}(m_{j+1}) \otimes_A \ker(n_i))/(\operatorname{im}(m_{j+1}) \otimes_A \operatorname{im}(n_{i+1})) \cong \operatorname{im}(m_{j+1}) \otimes_A (\ker(n_i)/\operatorname{im}(n_{i+1}))$$
$$(\ker(m_j) \otimes_A \ker(n_i))/(\ker(m_j) \otimes_A \operatorname{im}(n_{i+1})) \cong \ker(m_j) \otimes_A (\ker(n_i)/\operatorname{im}(n_{i+1})).$$

Second, for each $i \in \mathbb{Z}$ take

$$0 \longrightarrow \ker(n_{i-j}) \xhookrightarrow{\kappa_{i-j}} N_{i-j} \xrightarrow{n_{i-j}} \operatorname{im}(n_{i-j}) \longrightarrow 0$$

which is an (A,B)-exact sequence by hypothesis. Third, since $\operatorname{im}(m_{j+1})$ and $\ker(m_j)$ are (A,B)-flat, for all $i \in \mathbb{Z}$ we have the are $(\mathbb{Z},\mathbb{Z})$-exact sequences

$$\operatorname{im}(m_{j+1}) \otimes_A \ker(n_{i-j}) \xhookrightarrow{1_{\operatorname{im}(m_{j+1})}\otimes\kappa_{i-j}} \operatorname{im}(m_{j+1}) \otimes_A N_{i-j} \xrightarrow{1_{\operatorname{im}(m_{j+1})}\otimes n_{i-j}} \operatorname{im}(m_{j+1}) \otimes_A \operatorname{im}(n_{i-j})$$

$$\ker(m_j) \otimes_A \ker(n_{i-j}) \xhookrightarrow{1_{\ker(m_j)}\otimes\kappa_{i-j}} \ker(m_j) \otimes_A N_{i-j} \xrightarrow{1_{\ker(m_j)}\otimes n_{i-j}} \ker(m_j) \otimes_A \operatorname{im}(n_{i-j}).$$

We then obtain the $(\mathbb{Z},\mathbb{Z})$-exact sequences

$$\bigoplus_{j\in\mathbb{Z}}(B[-1]_{j+1} \otimes_A \ker(n_{i-j})) \xhookrightarrow{\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}}\otimes\kappa_{i-j})} \bigoplus_{j\in\mathbb{Z}}(B[-1]_{j+1} \otimes_A N_{i-j}) \xrightarrow{\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}}\otimes n_{i-j})} \bigoplus_{j\in\mathbb{Z}}(B[-1]_{j+1} \otimes_A \operatorname{im}(n_{i-j}))$$

$$\bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A \ker(n_{i-j})) \xrightarrow{\bigoplus_{j\in\mathbb{Z}}(1_{Z_j}\otimes\kappa_{i-j})} \bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A N_{i-j}) \xrightarrow{\bigoplus_{j\in\mathbb{Z}}(1_{Z_j}\otimes n_{i-j})} \bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A \operatorname{im}(n_{i-j}))$$

where $\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}}\otimes n_{i-j}) : \operatorname{Tot}_{i+1}(\boldsymbol{B[-1]_\bullet} \otimes_A \boldsymbol{N_\bullet}) \to \operatorname{Tot}_i(\boldsymbol{B[-1]_\bullet} \otimes_A \boldsymbol{N_\bullet})$ and $\bigoplus_{j\in\mathbb{Z}}(1_{Z_j}\otimes n_{i-j}) : \operatorname{Tot}_i(\boldsymbol{Z_\bullet} \otimes_A \boldsymbol{N_\bullet}) \to \operatorname{Tot}_{i-1}(\boldsymbol{Z_\bullet} \otimes_A \boldsymbol{N_\bullet})$ are the differentials of the total complexes

$\mathrm{Tot}_\bullet(\boldsymbol{B[-1]}_\bullet \otimes_A \boldsymbol{N}_\bullet)$ and $\mathrm{Tot}_\bullet(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)$ respectively. They thus satisfy

$$\begin{aligned}
\ker\left(\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}} \otimes n_{i-j})\right) &= \bigoplus_{j\in\mathbb{Z}}(B[-1]_{j+1} \otimes_A \ker(n_{i-j})),\\
\mathrm{im}\left(\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}} \otimes n_{i-j})\right) &= \bigoplus_{j\in\mathbb{Z}}(B[-1]_{j+1} \otimes_A \mathrm{im}(n_{i-j})),\\
\ker\left(\bigoplus_{j\in\mathbb{Z}}(1_{Z_j} \otimes n_{i-j})\right) &= \bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A \ker(n_{i-j})),\\
\mathrm{im}\left(\bigoplus_{j\in\mathbb{Z}}(1_{Z_j} \otimes n_{i-j})\right) &= \bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A \mathrm{im}(n_{i-j})).
\end{aligned}$$

Hence for all $i \in \mathbb{Z}$

$$\begin{aligned}
H_{i+1}(\boldsymbol{B[-1]}_\bullet \otimes_A \boldsymbol{N}_\bullet) &\cong \frac{\ker\left(\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}} \otimes n_{i-j})\right)}{\mathrm{im}\left(\bigoplus_{j\in\mathbb{Z}}(1_{B[-1]_{j+1}} \otimes n_{i-j+1})\right)} \cong \bigoplus_{j\in\mathbb{Z}}\left(\frac{B_j \otimes_A \ker(n_{i-j})}{B_j \otimes_A \mathrm{im}(n_{i-j+1})}\right)\\
&\cong \bigoplus_{j\in\mathbb{Z}}\left(B_j \otimes_A \frac{\ker(n_{i-j})}{\mathrm{im}(n_{i-j+1})}\right) \cong \bigoplus_{j\in\mathbb{Z}}(B_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet)) \cong (\boldsymbol{B}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_i
\end{aligned}$$

and

$$\begin{aligned}
H_i(\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet) &\cong \frac{\ker\left(\bigoplus_{j\in\mathbb{Z}}(1_{Z_j} \otimes n_{i-j})\right)}{\mathrm{im}\left(\bigoplus_{j\in\mathbb{Z}}(1_{Z_j} \otimes n_{i-j+1})\right)} \cong \bigoplus_{j\in\mathbb{Z}}\left(\frac{Z_j \otimes_A \ker(n_{i-j})}{Z_j \otimes_A \mathrm{im}(n_{i-j+1})}\right)\\
&\cong \bigoplus_{j\in\mathbb{Z}}\left(Z_j \otimes_A \frac{\ker(n_{i-j})}{\mathrm{im}(n_{i-j+1})}\right) \cong \bigoplus_{j\in\mathbb{Z}}(Z_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet)) \cong (\boldsymbol{Z}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_i
\end{aligned}$$

We can then rewrite the previous long $(\mathbb{Z}, \mathbb{Z})$-exact sequence as

$$
\begin{tikzcd}
\cdots \arrow[r] & (\boldsymbol{B}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_i \arrow[r, "\partial_{i+1}"] \arrow[l, bend left] & (\boldsymbol{Z}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_i \arrow[l, bend left, "\int_{i+1}"] \arrow[dr, "(\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}"] & \\
 & & & H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[ul, bend left] \arrow[dl, "(m_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}"'] \\
\cdots \arrow[r, bend right] & (\boldsymbol{Z}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_{i-1} \arrow[l] \arrow[r, bend right, "\int_i"'] & (\boldsymbol{B}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_{i-1} \arrow[l, "\partial_i"'] \arrow[ur, bend right] &
\end{tikzcd}
$$

so by Lemma 3.3 we have the short $(\mathbb{Z}, \mathbb{Z})$-exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}(\partial_{i+1}) \arrow[r, "\alpha_i"] & H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "\beta_i"] \arrow[l, bend left, "\gamma_i"] & \ker(\partial_i) \arrow[r] \arrow[l, bend left, "\delta_i"] & 0
\end{tikzcd}
$$

where $\alpha_i : \operatorname{coker}(\partial_{i+1}) \to H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet)$ is given by $\alpha_i(\sum_{j \in \mathbb{Z}} z_j \otimes \overline{y_{i-j}}) = \sum_{j \in \mathbb{Z}} \overline{\iota_j(z_j) \otimes y_{i-j}}$ and $\beta_i : H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \to \ker(\partial_i)$ is given by $\beta_i(\sum_{j \in \mathbb{Z}} \overline{x_j} \otimes \overline{y_{i-j}}) = \sum_{j \in \mathbb{Z}} m_j(x_j) \otimes \overline{y_{i-j}}$ for all $z_j \in Z_j$, $x_j \in M_j$, $y_{i-j} \in \ker(n_{i-j})$, and $i, j \in \mathbb{Z}$.

Consider now the canonical inclusion $\theta_i : B_i = \operatorname{im}(m_{i+1}) \to \ker(m_i) = Z_i$, we clearly have that $(\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_\bullet : \boldsymbol{B}_\bullet \otimes_A \boldsymbol{N}_\bullet \to \boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet$ is a chain map, and we claim that $\partial_i = (\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_{i-1}}$. Pick a generic element $\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \in (\boldsymbol{B}_\bullet \otimes_A \boldsymbol{N}_\bullet)_{i-1}$ with $z_{j-1} \in \operatorname{im}(m_j)$, $y_{i-j} \in N_{i-j}$ for all $i, j \in \mathbb{Z}$. In particular, for each $j \in \mathbb{Z}$ there are $x_j \in M_j$ with $m_j(x_j) = z_{j-1}$, and $z_{j-1} \in \operatorname{im}(m_j) \subseteq \ker(m_{j-1})$ implies $\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \in (\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)_{i-1}$, so by Proposition 3.36 we have

$$
\begin{aligned}
\partial_i \left(\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \right) &= (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} (m_\bullet \otimes n_\bullet)(m_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} \left(\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \right) \\
&= (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} (m_\bullet \otimes n_\bullet) \left(\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \right) \\
&= (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} \left(\sum_{j \in \mathbb{Z}} m_j(x_j) \otimes y_{i-j} + (-1)^j x_j \otimes n_{i-j}(y_{i-j}) \right) \\
&= (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} \left(\sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} \right) + (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} \left(\sum_{j \in \mathbb{Z}} (-1)^j x_j \otimes n_{i-j}(y_{i-j}) \right) \\
&= \sum_{j \in \mathbb{Z}} z_{j-1} \otimes y_{i-j} + (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1} \left(\sum_{j \in \mathbb{Z}} (-1)^j x_j \otimes n_{i-j}(y_{i-j}) \right).
\end{aligned}
$$

Note that the preimage of $\sum_{j\in\mathbb{Z}}(-1)^j x_j \otimes n_{i-j}(y_{i-j})$ under $(\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})$ will be of the form

$$\sum_{j\in\mathbb{Z}}(-1)^j x'_j \otimes n_{i-j}(y_{i-j})$$

for some $x'_j \in \ker(m_j)$. The differentials are $\bigoplus_{j\in\mathbb{Z}}(1_{Z_j}\otimes n_{i-j}) : (\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)_i \to (\boldsymbol{Z}_\bullet \otimes_A \boldsymbol{N}_\bullet)_{i-1}$ as remarked above, and thus

$$\begin{aligned}\bigoplus_{j\in\mathbb{Z}}(1_{Z_j}\otimes n_{i-j})\left(\sum_{j\in\mathbb{Z}}(-1)^j x'_j \otimes y_{i-j}\right) &= \sum_{j\in\mathbb{Z}}(-1)^j x'_j \otimes n_{i-j}(y_{i-j}) \\ &= (\iota_\bullet \otimes 1_{\boldsymbol{N}_\bullet})^{-1}\left(\sum_{j\in\mathbb{Z}}(-1)^j x_j \otimes n_{i-j}(y_{i-j})\right)\end{aligned}$$

so this preimage is a boundary, making it zero in homology. Hence if our generic element is a cycle in $(\boldsymbol{B}_\bullet \otimes_A \boldsymbol{N}_\bullet)_{i-1}$, so it represents an element of $(\boldsymbol{B}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_{i-1}$, in $(\boldsymbol{Z}_\bullet \otimes_A H_\bullet(\boldsymbol{N}_\bullet))_{i-1}$ we obtain the equality

$$\partial_i\left(\overline{\sum_{j\in\mathbb{Z}} z_{j-1}\otimes y_{i-j}}\right) = \overline{\sum_{j\in\mathbb{Z}} z_{j-1}\otimes y_{i-j}} = (\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_{i-1}}\left(\overline{\sum_{j\in\mathbb{Z}} z_{j-1}\otimes y_{i-j}}\right)$$

so $\partial_i = (\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_{i-1}}$ for all $i \in \mathbb{Z}$. We can then rewrite the short $(\mathbb{Z},\mathbb{Z})$-exact sequences as

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}((\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}) \arrow[r, "\alpha_i"] & H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \arrow[r, "\beta_i"] \arrow[l, bend left, "\gamma_i"] & \ker((\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_{i-1}}) \arrow[r] \arrow[l, bend left, "\delta_i"] & 0. \end{tikzcd}$$

Consider for each $i, j \in \mathbb{Z}$ the short (A,B)-exact sequence

$$0 \longrightarrow \operatorname{im}(m_{j+1}) \xhookrightarrow{\theta_j} \ker(m_j) \longrightarrow\!\!\!\!\rightarrow \ker(m_j)/\operatorname{im}(m_{j+1}) \longrightarrow 0,$$

and the A-module $H_{i-j}(\boldsymbol{N}_\bullet)$. Since $\ker(m_j)$ is (A,B)-flat then $\operatorname{Tor}_1^{(A,B)}(\ker(m_j), H_{i-j}(\boldsymbol{N}_\bullet)) = 0$, so by Theorem 3.39 we have the $(\mathbb{Z},\mathbb{Z})$-exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tor}_1^{(A,B)}(H_j(\boldsymbol{M}_\bullet), H_{i-j}(\boldsymbol{N}_\bullet)) \arrow[r, hook] & B_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \arrow[d, "\theta_j\otimes 1_{H_{i-j}(\boldsymbol{N}_\bullet)}"] \\ 0 & H_j(\boldsymbol{M}_\bullet)\otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \arrow[l] & Z_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \arrow[l, two heads] \end{tikzcd}$$

which we can combine into the $(\mathbb{Z},\mathbb{Z})$-exact sequence

$$\begin{array}{ccc} 0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} \operatorname{Tor}_1^{(A,B)}(H_j(\boldsymbol{M}_\bullet), H_{i-j}(\boldsymbol{N}_\bullet)) & \hookrightarrow & \bigoplus_{j\in\mathbb{Z}} B_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \\ & & \downarrow{\scriptstyle \bigoplus_{j\in\mathbb{Z}}(\theta_j\otimes 1_{H_{i-j}(\boldsymbol{N}_\bullet)})} \\ 0 \longleftarrow \bigoplus_{j\in\mathbb{Z}} H_j(\boldsymbol{M}_\bullet)\otimes_A H_{i-j}(\boldsymbol{N}_\bullet) & \twoheadleftarrow & \bigoplus_{j\in\mathbb{Z}} Z_j \otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \end{array}$$

which using $(\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i} = \bigoplus_{j\in\mathbb{Z}}(\theta_j \otimes 1_{H_{i-j}(\boldsymbol{N}_\bullet)})$ implies

$$\operatorname{coker}((\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}) = \bigoplus_{j\in\mathbb{Z}} H_j(\boldsymbol{M}_\bullet)\otimes_A H_{i-j}(\boldsymbol{N}_\bullet),$$
$$\ker((\theta_\bullet \otimes 1_{\boldsymbol{N}_\bullet})_{*_i}) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Tor}_1^{(A,B)}(H_j(\boldsymbol{M}_\bullet), H_{i-j}(\boldsymbol{N}_\bullet)).$$

Finally, we can rewrite the above $(\mathbb{Z},\mathbb{Z})$-exact sequences as

$$0 \to \bigoplus_{j\in\mathbb{Z}} H_j(\boldsymbol{M}_\bullet)\otimes_A H_{i-j}(\boldsymbol{N}_\bullet) \underset{\gamma_i}{\overset{\alpha_i}{\rightleftarrows}} H_i(\boldsymbol{M}_\bullet \otimes_A \boldsymbol{N}_\bullet) \underset{\delta_i}{\overset{\beta_i}{\rightleftarrows}} \bigoplus_{j\in\mathbb{Z}} \operatorname{Tor}_1^{(A,B)}(H_j(\boldsymbol{M}_\bullet), H_{i-j}(\boldsymbol{N}_\bullet)) \to 0$$

obtaining the desired result. □

An important consequence arises when concentrating one of the complexes in degree zero.

Theorem 3.45 (Relative Universal Coefficient Theorem)**.** *Let $(\boldsymbol{M}_\bullet, \boldsymbol{m}_\bullet)$ be a complex of right A-modules such that M_j and $\operatorname{im}(m_j)$ are (A,B)-flat and $0 \to \ker(m_j) \xrightarrow{\iota_j} M_j \xrightarrow{m_j} \operatorname{im}(m_j) \to 0$ are (A,B)-exact sequences for all $j \in \mathbb{Z}$. Let N be a left A-module. Then for each $i \in \mathbb{Z}$ there is a $(\mathbb{Z},\mathbb{Z})$-exact sequence*

$$0 \to H_i(\boldsymbol{M}_\bullet)\otimes_A N \rightleftarrows H_i(\boldsymbol{M}_\bullet \otimes_A N) \rightleftarrows \operatorname{Tor}_1^{(A,B)}(H_{i-1}(\boldsymbol{M}_\bullet), N) \to 0.$$

As in the absolute case, this enables us to reduce computations of arbitrary homological degree to computations of homological degree 1.

4. SUMMARY

In this dissertation we have extended relative homological algebra in the sense of Hochschild [24]. Our main insight lies in the realization that relative flat modules preserve relative exact sequences, and our treatment of the subject showcases how this an extension can also be carried out to the viewpoint presented by Buchsbaum [6] and concisely exposed in Mac Lane [27]. We now summarize our improvements over the state of the art.

The definitions of (A, B)-projective and (A, B)-injective were well known, but a useful characterization was not. We showcased a direct analogy from the absolute case to the relative setup, in which (A, B)-free modules played a key role and could be formulated in terms of a universal property. Although some of the implications in the characterization of (A, B)-projective modules had appeared in the literature, we are the first to provide the complete picture. This systematic treatment also extends to (A, B)-flat modules, whose definition we rephrase. We also provide a wealth of examples dealing with the behavior of relative free, relative projective, and relative flat modules, including several infinite families of objects exhibiting behaviors native to the relative setup. We must point out that the established definition of (A, B)-flat modules does not include splitting conditions on the resulting sequence after tensoring, which severely hinders the strength of the results that can be obtained. Namely, it was known that relative Tor formed a long exact sequence, but it was not known that this sequence always splits. Our reformulation enables us to prove this splitting, and the price we have to pay is the observation that the connecting homomorphism has a section. This section had, to the best knowledge of the author, not been noticed before. The culmination of the dissertation is the Relative Künneth Theorem. While this is a well known result in the absolute setup [12] and has been generalized to multiple other settings, we emphasize that the short exact sequences obtained through our techniques are split, a fact that was not previously known. This is an unprecedented result in the relative setup.

The splitting obtained in the Relative Künneth Theorem was avidly sought by the author for its potential in the cohomology theory of associative algebras. More specifically, in upcoming work [31] the author applies it to establish the equivalence of several ways of computing the cup product in relative Hochschild cohomology. This establishes an analogous ring structure as the one existing in the absolute Hochschild cohomology, and enables the study of the finite generation of the relative cohomology. It is clear that for very specific cases of pairs of algebras (A, B) the relative setup reduces to the absolute one, but a complete characterizations of these pairs remain unknown, and would be of great interest. Further future directions include the study of a support theory arising from relative Hochschild cohomology, which is expected to yield connections with the established theory of support varieties for Hochschild, Hopf, and group cohomologies.

As a final remark, the author would like to comment on the fact that in the current standard references for relative homological algebra in the sense treated in this dissertations, proofs are seldom presented. This is attributed to the fact that the results presented in the literature do not substantially improve on the absolute results. That is, while the hypothesis of the statements are adapted to the relative setup, the conclusions are the same as in the absolute one. In this context, proofs are indeed somewhat redundant since they follow the known theory. However, our stronger conclusions require adapted techniques, and we can no longer claim that proofs follow by the exact same reasoning. In this dissertation we presented at least three different possibilities on how to proceed to prove the desired results. Sometimes, as in the case of the functoriality of relative Ext and Tor, proofs are indeed identical to the absolute case. Often, as in the case of the Relative Comparison Theorem, proofs follow a similar reasoning but they do not immediately reduce to the absolute case. In other occasions, as in the case of the Relative Horseshoe Lemma, the proof can be simplified using tools native to the relative setup. We hope that this vindicates the use of relative tools and emphasizes their utility.

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