ON PRESERVATION OF MODULI OF CONTINUITY BY PARABOLIC EVOLUTION

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2021

Major Subject: Mathematics

ABSTRACT

In this work, we study how Lipschitz continuity propagates by a certain class of nonlinear, nonlocal parabolic equations. This work draws inspiration from ideas developed in recent years by Kiselev, Nazarov, Volberg and Shterenberg to address issues relating to the regularity of solutions of critical active scalar equations such as the the surface quasi-geostrophic equation and Burgers model. Namely, we will extend and improve on such techniques in order for them to be applicable to combustion models as well as other fluid equations such as the incompressible Navier-Stokes system and Burgers-Hilbert flow.

The main problem we address here is proving a global regularity result relating to a slight modification of the so called Michelson-Sivashinsky equation. We also give outlines of how can one use similar ideas to obtain various new regularity and partial regularity criteria for the incompressible Navier-Stokes system, as well as provide a different proof to a known criterion in terms of critical Hölder-type norms. We also outline how to extend the technique to a viscous, multi-dimensional Burgers-Hilbert problem in order to prove global regularity for this model.

DEDICATION

To my mom, for all the sacrifices.

ACKNOWLEDGMENTS

I would like to start by thanking my PhD adviser, Professor Titi, for his guidance and feedback. His passion and enthusiasm for mathematics is contagious, and was a driving force for me during the early days of my research. I thank him for introducing me to the beautiful world of PDE analysis, and for presenting me with an interesting problem to work on. I also would like to thank him for supporting my travels to conferences and workshops around the world while a student here, as well as providing me with research funds during the summer of 2018 and 2019. I extend my gratitude to the department of mathematics at Texas A&M for supporting my research and my education. I also would like to thank the research group of Professor Rupert Klein at the Freie Univerität Berlin (where part of this work was completed) for their kind hospitality.

Next, I would like to thank members of my PhD committee (Professors Guermond, Kuchment and Panetta) for taking the time to review my work. A special thanks is in order to Professor Kuchment for his support at the personal level, I would not have made it this far without him wisely and calmly resolving various issues I ran into. To every story there is an unsung hero, and in this case, that person is Professor Howard: this dissertation would not have seen the light of day without his unconditional, selfless support.

I consider myself lucky to have had the chance to sit down in the several classes and learn from the brilliant faculty members in the mathematics department at Texas A&M. In particular, I thoroughly enjoyed the classes I took under Professors Guermond, Johnson, Kuchment and Narcowich. I thank them for bringing out the analyst hidden within me. I also am glad I got the chance to meet Dr. Lührmann during my final year here and attend his special topics course.

I thank David Manuel, Angela Allen and Jenn Whitfield for what they do. Thanks to Monique Stewart, Sherry Floyd and the rest of the administrative staff for making our lives that much easier. Graduate students would be absolutely lost without Monique, she is a superwoman!

From outside the math department at Texas A&M, my gratitude extends to Professors Anna Mazzucato (Penn State) and Luis Silvestre (UChicago) for their support and for several interesting discussions I've had with them. I also would like to thank Professors Peter Constantin (Princeton), Theodore Drivas (Stony Brook) and Tarek Elgindi (Duke) for their useful feedback on my work. I extend my gratitude to my undergraduate professors: David Radnell, Ayman Badawi, Mahmoud Anabtawi, and Hana Suleiman for their support.

Of course graduate school would have been unbearable without the company of many of my classmates and friends. I thank Dustin, Kash, Konrad, Ola, Nida, John, Justin, Kari, Juihua and Sheagan for the good times and the possibly uncountable number of laughs we had together. Thanks to Quyuan Lin for several discussions, and a special thanks is in order to my office mate Jintao Deng for putting up with my nonsense. From back home, I thank my dear friends Hamza, Amir, Mohammed, Rabie, Adnan, Waleed, Wahib and Younan for their support and words of encouragement. I am grateful to have had Hassan Armali as a friend and physics teacher in high-school, I thank him for his encouragement.

Though words are never enough, I want to thank my mom for everything she's done. I have never, nor will I ever, meet anyone with a kinder heart and gentler soul. I also want to thank my brother, Abdulrahman, for his support and encouragement throughout the years. I consider myself lucky to have the light-hearted cousins I do. Their support means a lot and I cannot wait to be reunited with them and their satirical comments.

Finally, to the person dearest to my heart, Daoud Araki. You were the first person to teach me mathematics, and you always believed in me no matter what. I have never met anyone who provided me with the unconditional love and support that you did. You were a father when I needed one, a friend when I needed one, and most importantly, you were, and always will be, a teacher that I am lucky to have. You truly are an inspiration, and I can only aspire to become half the person that you are.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Professors Edriss Titi [advisor], Jean-Luc Guermond of the Department of Mathematics, Peter Kuchment of the Department of Mathematics, and Professor R. Lee Panetta of the Department of Atmospheric Sciences.

The global regularity problem related to the Michelson-Sivashinsky equation was suggested to the student by his adviser, Professor Edriss Titi. Professor Edriss Titi also suggested the argument presented in Chapter 5. All other work conducted for the dissertation was completed by the student independently.

Funding Sources

Graduate study was supported by a teaching assistantship from the department of mathematics at Texas A&M University, as well as in part by a research assistanship via the student's adviser (Edriss Titi) research funds. The student's adviser also provided funding for attending conferences, workshops and a research visit abroad.

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1. INTRODUCTION

A recurring theme in the analysis of time-dependent partial differential equations (PDEs) is the idea of looking for *a-priori* estimates. Those are norms that are more or less under control (and in many cases are preserved) by the underlying evolution equation. One then would hope to "bootstrap" such control (specially in the case of parabolic, or more generally, "dissipative" equations) to show that stronger norms are also under control, and hence obtain a "regularity result". This is a very useful and powerful approach (especially when studying nonlinear equations), since such a strategy tells us that in order to control all derivatives of the solution, we would only need to control certain ones. What a-priori estimate we can obtain (or need to go to higher regularity) depends purely on the equation being studied.

This is best demonstrated via the viscous Burgers equation, where given a smooth enough vector field $u_0 : \mathbb{R}^d \to \mathbb{R}^d$, one looks for a vector field solution to

$$\partial_t u(t,x) - \Delta u(t,x) = (u \cdot \nabla) u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

that satisfies $u(0,x) = u_0(x)$ (we assume that the boundary conditions are either the periodic ones, or sufficient decay at spatial infinity). If we set $v(t,x) := |u(t,x)|^2$, then direct calculations tell us that

$$\partial_t v(t,x) - \Delta v(t,x) - u \cdot \nabla v(t,x) \leq 0,$$

from which we get $\|v(t,\cdot)\|_{L^\infty} = \|u(t,\cdot)\|_{L^\infty}^2 \leq \|v(0,\cdot)\|_{L^\infty} = \|u_0\|_{L^\infty}^2$, with $\|\cdot\|_{L^\infty}$ is the standard supremum norm (see [16] for instance). Such a bound allows us to control all higher order derivatives as well [36]. The main idea is that controlling the supremum norm allows us to treat equation (1.1) as a perturbation of the heat equation. Difficulties arise when the only a-priori estimates available do not guarantee regularity, or worse, when they are not available. Perhaps the most famous example of a system where the available a-priori bounds are not sufficient to deduce regularity is the three dimensional incompressible Navier-Stokes system,

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = (u\cdot\nabla)u(t,x) + \nabla p(t,x), & \forall (t,x)\in(0,\infty)\times\mathbb{R}^3, \\ \nabla\cdot u(t,x) = 0, & \forall (t,x)\in[0,\infty)\times\mathbb{R}^3, \\ u(0,x) = u_0(x), & u_0\in C^\infty(\mathbb{R}^3). \end{cases} \tag{1.2}$$

We again complement (1.2) with either a periodic boundary condition or require the solution (and initial data) to decay sufficiently rapidly at spatial infinity. From the incompressibility (divergence free) condition $\nabla\cdot u(t,x) = 0$, one can integrate by parts in space (assuming we have sufficient regularity) to get the identity

$$\int (u\cdot\nabla)u(t,x)\cdot u(t,x)\,dx = 0.$$

Thus, if we multiply the PDE in (1.2) by u, and integrate over the spatial domain by parts we get

$$\frac{1}{2}\frac{d}{dt}(\|u(t,\cdot)\|_{L^2}^2) + \|\nabla u(t,\cdot)\|_{L^2}^2 = 0,$$

from which

$$\|u(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla u(s,\cdot)\|_{L^2}^2\,ds = \|u_0\|_{L^2}^2.$$

In particular, $\|u(t,\cdot)\|_{L^2} \le \|u_0\|_{L^2}$ for every $t\ge 0$. Such a control can be bootstrapped to control higher order norms when the spatial dimension is two [12,37], but whether the same is true in three or higher dimensions is still an open problem.

An example of a PDE where one has no available a-priori bounds is the Kuramoto-Sivashinsky equation. This equation reads as

$$\partial_t u + (-\Delta)^2 u = (u\cdot\nabla)u - \Delta u. \tag{1.3}$$

Due to the lack of incompressibility constraints, we cannot apply the previous argument to control the L^2 norm of the solution. Furthermore, we do not have a maximum principle as in the case of Burgers equation, since the bi-Laplacian $(-\Delta)^2$ is not necessarily non-negative at a point of maximum. At the time of writing this dissertation, to our knowledge the global regularity problem associated with the Kuramoto-Sivashinsky equation is still open. Partial results are available in special cases, for instance when the initial data is symmetric in an annular region with homogenous Neumann boundary conditions [4], or in thin periodic domains [1, 6, 44, 51].

So how can we study whether regularity propagates by the corresponding evolution equation from smooth initial data or not in such scenarios? There is no definite answer to that question; as Klainerman discusses in his brilliant review [30], each PDE is a world on its own, and it is practically hopeless to try and come up with an analysis tool that would work for a wide class of evolution equations. Thus, in general one would need to exploit the fine structure of the equation being analyzed to understand issues such as global regularity.

That being said, this dissertation draws inspiration from the work of Kiselev, Nazarov, Volberg and Shterenberg, where they were studying the propagation of Lipschitz moduli of continuity by the critically dissipative surface quasi-geostrophic equation [29] and Burgers equation [28]. The main result of this dissertation is to extend this technique and address the propagation of regularity under the following equation

$$\begin{cases} \partial_t \theta - \nu \Delta \theta = \lambda |\nabla \theta|^p + \mu (-\Delta)^\alpha \theta, \\ \theta(0, x) = \theta_0(x), \end{cases} \tag{1.4}$$

where $\nu > 0$, $\alpha \in (0, 1/2)$, $p \in [1, \infty)$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $\theta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is a scalar. Here, the operator $(-\Delta)^\alpha$ is called the fractional Laplacian, a nonlocal operator whose Fourier symbol is given by $|k|^{2\alpha}$ and has the representation

$$(-\Delta)^\alpha \theta(x) = C_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{\theta(x) - \theta(x - z)}{|z|^{d+2\alpha}} \, dz,$$

for some normalizing constant $C_{d,\alpha}$. The above integral is understood in the principal value sense.

Equation (1.4) is a modification to the Michelson-Sivashinsky equation, where the latter corresponds to the case when $p = 2$ and $\alpha = 1/2$. The Michelson-Sivashinsky equation is a combustion model that is related to the Kuramoto-Sivashinsky equation (1.3), we give more extensive background information about this model in Chapter 2.

Let us now highlight the challenges involved when analyzing (1.4) and how they relate to the incompressible NSE (1.2). If we drop the nonlocal term (i.e. choose $\mu = 0$), then the resulting equation does have a maximum principle, see for instance [36], which can be bootstrapped to obtain higher regularity. One the other hand, if we ignore the nonlinearity (choose $\lambda = 0$), one can do energy estimates and to show that higher order norms grow at most exponentially in time. The presence of a nonlinear and nonlocal term in the equation at the same time complicates matters. A similar situation is encountered when analyzing the incompressible Navier-Stokes system (1.2). Namely, if we drop the pressure term and incompressibility constraints, one ends up with the viscous Burgers equation, which has a maximum principle that can be bootstrapped. Adding a nonlocal feedback term (the pressure) seems to dramatically change the equation and the result is a problem that is arguably labelled as one of the most challenging mathematical problems one could encounter.

To explicitly demonstrate the nonlocal structure of the incompressible NSE, let us recall that by taking the divergence of the PDE in (1.2) and using the divergence free condition, one recovers the pressure from the velocity-field u via solving the elliptic problem

$$-\Delta p = \text{div}\left[(u \cdot \nabla)u\right].$$

Using the divergence-free condition one more time, one realizes that in terms of Fourier symbols, the pressure could be defined as

$$p := \sum_{m,n=1}^{d} \partial_m \partial_n (-\Delta)^{-1}(u_n u_m) = \sum_{m,n=1}^{d} R_m R_n (u_m u_n),$$

where $\{R_n\}_{n=1}^d$ are the Riesz transforms on $\mathbb{R}^d$, the nonlocal operators with symbol $-ik_n/|k|$, with

$i := \sqrt{-1}$. In particular, the nonlocal term, $\nabla R_m R_n$, that is present in (1.2) is of order one. That is, the PDE reduces to

$$\partial_t u(t,x) - \Delta u(t,x) = (u \cdot \nabla) u(t,x) + \sum_{m,n=1}^{d} \nabla R_m R_n (u_m u_n).$$

Recalling that the square root of the Laplacian has the representation

$$(-\Delta)^{1/2} \theta = \sum_{m=1}^{d} R_m \partial_m \theta,$$

one sees how that original Michelson-Sivashinsky (equation (2.7) with $\alpha = 1/2$ and $p = 2$) could potentially serve as a toy model for the incompressible NSE: both involve nonlocal terms of order one coupled with a quadratic nonlinear term involving the gradient. In fact, when $p = 2$ the $v := \nabla \theta$ satisfies a viscous Burgers equation, perturbed by $(-\Delta)^{\alpha} v$

The general idea in studying the evolution of moduli of continuity is as follows: given some evolution equation, can one construct a smooth enough non-negative, non-decreasing function Ω such that the solution to the evolution equation being analyzed obeys Ω in the sense that

$$|\theta(t,x) - \theta(t,y)| \leq \Omega(t, |x-y|),$$

whenever $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$? If this is true, then one gets a bound on the gradient of the form $\|\nabla \theta(t,\cdot)\|_\infty \leq \partial_\xi \Omega(t,0)$, and so if $\partial_\xi \Omega(t,0)$ doesn't blowup, then neither will the solution. In order to derive conditions that Ω has to satisfy, one should be able to obtain continuity estimates on all terms appearing in the equation in terms of Ω. Thus, in order to at least initiate the study of propagating moduli of continuity by the NSE or the Michelson-Sivashinsky equation, one needs to obtain continuity estimates on an operator of order one. In general, one should not expect to obtain a continuity estimate for $\mathcal{N}\theta$ from one known for θ when $\mathcal{N}$ is an order one operator.

That last remark is the main reason why our proof strategy for (1.4) fails when $\alpha = 1/2$. So we cannot even initiate the study of propagation of moduli of continuity for the original Michelson-

Sivashinsky equation, and we shouldn't expect to be able to do so for the NSE as well. However, rather remarkably, one *can* obtain continuity estimates on ∇p from those known on the velocity field u, without any a-priori knowledge regarding the continuity of ∇u. The reason why this is possible for the NSE is because of incompressibility: in an unpublished work, Silvestre [53] was able to show that $C_x^{0,\beta}$ incompressible velocity fields have $C_x^{0,2\beta-1}$ pressure terms. When $\beta \in (1/2, 1)$, this translates to a Hölder condition on ∇p. Such an observation allows us to at least derive conditions that Ω must satisfy in order to guarantee the preservation of the modulus of continuity by the evolution equation in (1.2). However, the resulting inequality that Ω needs to satisfy is quite complicated, and it is unclear at this point whether one can prove that this inequality has a solution. The complications arise in two forms: the inequality is highly nonlinear and involves an integral with a high degree of singularity near the origin.

That being said, we consider various simplifications to the NSE in the paper [22]. For instance, we replace incompressibility and the pressure term with a simple nonlocal term of order zero (a Riesz transform). Such a model is called the Burgers-Hilbert equation, and was introduced by Marsden and Weinstein [39] as an approximate model for the dynamics of free boundary, two dimensional vortex patches. Biello and Hunter [7] also proposed the same equation as a surface wave model. We show that regularity does persist in this model, regardless of the spatial dimension we are working in. Another simplification that we analyzed in the same paper [22] is linearizing the NSE. Namely, we replace the term $(u \cdot \nabla)u$ with $(b \cdot \nabla)u$, where b is a given divergence-free vector field. Roughly speaking, the assumptions that we make on the drift-velocity b is $b \in L_t^p C_x^{0,\beta}$, and we show that for various values $p > 0$ and any $\beta \in (0, 1)$, one gets either regularity or partial regularity. We refer the reader to Chapter 6 for more precise statements and outlines of the proofs. The details of the results described in this paragraph are omitted from this dissertation for the sake of brevity, but we refer the reader to the paper [22] for rigorous justifications.

This dissertation is organized as follows. In Chapter 2 we provide background information regarding the combustion model (1.4) in §2.1, while we precisely formulate the main results in §2.2. Chapter 3 concerns itself with some preliminary results that will be needed. Namely, §3.1 contains various results about moduli of continuity, while in §3.2 we derive some pointwise estimates on the fractional Laplacian. In Chapter 4 we prove the global regularity result for (1.4), while in Chapter 5 we mainly outline a strategy suggested by my adviser Professor Titi that could potentially help address the regularity problem to (1.4) with $\alpha = 1/2$, as well as other models including the NSE. Chapters 2 through 5 mainly consists of results that appeared in [23]. We move on to briefly describe the results obtained in the paper [22] in Chapter 6, while in Chapter 7 we discuss future directions in this research program. This will be followed by a list of references and the appendix, where we include miscellaneous estimates and identities that are used throughout this work.

2. COMBUSTION MODEL*

2.1 Background

Let us recall the equation (1.4)

$$\begin{aligned} &\partial_t\theta - \nu\Delta\theta = \lambda|\nabla\theta|^p + \mu(-\Delta)^\alpha\theta, \\ &\theta(0,x) = \theta_0(x), \end{aligned} \tag{2.1}$$

where $\nu > 0$, $\alpha \in (0, 1/2)$, $p \in [1, \infty)$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $\theta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is a scalar. We will prove that if $\theta_0 \in W^{1,\infty}(\mathbb{R}^d)$ and is periodic (with arbitrary period $L > 0$ in every direction) or vanishes at infinity, then there is a unique globally regular solution to (2.1) satisfying the bound

$$\|\nabla\theta(t,\cdot)\|_{L^\infty} \le Be^{C_0 t},$$

where B depends only on $\|\theta_0\|_{W^{1,\infty}}$ and C_0 depends on B, ν, α, d, μ (see Theorems 2.1 and 2.2, below).

Let us start by discussing the motivation behind this work and provide some background information. One of the outstanding questions in the analysis of partial differential equations is whether the Kuramoto-Sivashinsky (KS) equation develops a singularity in finite time or whether solutions arising from smooth enough initial data remain smooth for all time (in spatial dimension $d \ge 2$). In its scalar form, this equation reads

$$\partial_t\theta(t,x) + \Delta^2\theta(t,x) + \frac{1}{2}|\nabla\theta(t,x)|^2 + \Delta\theta(t,x) = 0. \tag{2.2}$$

In spatial dimension $d = 1$, the solution to the initial value problem associated with (2.2) (in the periodic or whole space setting) does not develop any singularities in finite time starting from

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smooth enough initial data θ_0, see for instance [46, 60]. In dimensions $d = 2, 3$, and under the assumption of radially symmetric initial data in an annular region with homogenous Neumann boundary conditions, global regularity was proven in [4]. Nevertheless, the question of global well-posedness of the IVP associated with (2.2) remains open, in the large, for arbitrary smooth enough initial data when the spatial dimension d is larger than one.

The KS equation was derived independently by Sivashinsky [56] as a model for flame propagation (see also [43]), and by Kuramoto [33] in the context of a diffusion-induced chaos in a chemical reaction system (see also [34, 35]). The original model derived by Sivashinsky in [56] and discussed in [43] reads

$$\partial_t\theta + 4(1+\epsilon)^2\Delta^2\theta + \epsilon\Delta\theta + \frac{1}{2}|\nabla\theta|^2 = (1-\sigma)(-\Delta)^{1/2}\theta, \tag{2.3}$$

where $\sigma \in (0, 1)$ is the coefficient of thermal expansion of a gas, $\epsilon = (L_0 - L)/(1 - L_0)$, with L being the Lewis number (the ratio of thermal and molecular/mass diffusivities) of the component of the combustible mixture limiting the reaction, and $L_0 < 1$ being the critical Lewis number for a flame instability that depends on various physical properties of the mixture. Here, θ models the perturbation in position of the flame front location, and $(-\Delta)^\alpha$, $\alpha \in (0, 1)$, is the nonlocal operator whose Fourier symbol is given by $|k|^{2\alpha}$. Equivalently, it can be represented in terms of the singular integral

$$(-\Delta)^\alpha\theta(x) = C_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{\theta(x) - \theta(x-z)}{|z|^{d+2\alpha}}\, dz, \tag{2.4}$$

for $\alpha \in (0, 1)$, sufficiently regular θ, and $C_{d,\alpha} > 0$ being a normalizing constant, degenerating (goes to zero) as $\alpha \to 0^+$ or 1^-. When $\epsilon > 0$, upon rescaling, one formally recovers equation (2.2) from (2.3) by setting $\sigma = 1$. Much of the analysis done in the literature has been carried out for the case when $\epsilon > 0$ and $\sigma = 1$. To the best of our knowledge, no rigorous mathematical treatment for the case $\sigma \neq 1$ has been done. Furthermore, when $L > L_0$ ($\epsilon < 0$), asymptotic analysis leads to dropping out the hyperviscous term Δ^2 in (2.3), and the instabilities in the flame front in this case arise as a consequence of thermal expansion on its own [43, 56], and one gets (upon rescaling)

$$\partial_t \theta - \Delta\theta - (-\Delta)^{1/2}\theta + \frac{1}{2}|\nabla\theta|^2 = 0. \tag{2.5}$$

In other words, it is physically possible to have $\epsilon < 0$; we refer the reader to the survey articles [40, 57] for further insight regarding the physical role of the parameters in (2.3) in the theory of combustion.

Equation (2.5) is called the Michelson-Sivashinsky (MS) equation. It is a refined combustion model based on the Darrieus–Landau flame stability analysis, and was also recently derived in [45, 63]. Several computational studies were performed on the periodic one-dimensional version of (2.5), see for instance [19, 42, 43, 48], where typical turbulence induced chaotic behavior was noted (see the previous references for details). Numerical observations have led several authors to consider special solutions of (2.5) in the one-dimensional case (see, for instance, [32,47,50,61] and the references therein). However, the global regularity of the one-dimensional version of (2.5) does not present any mathematical challenges. Indeed, one has a-priori control over the H^1 norm of the solution, which can be bootstrapped to control higher order Sobolev norms, with the nonlocal part causing at most growth in time but not blow up.

In dimensions higher than one, one runs into the same technical difficulties as in the KS equation. Namely, no a-priori bound, not even in L^2, can be obtained, due to the nonlinear term. Thus one can only prove short-time existence, uniqueness, and regularity via standard arguments for smooth enough initial data. On the other hand, because the dissipative operator in the KS, Δ^2, is replaced by the standard Laplacian, $(-\Delta)$, in the MS equation (2.5), there might be hope to control the Lipschitz constant of the solution to (2.5) (i.e., prove a "maximum principle" for the gradient of the solution to (2.5)), which can then be bootstrapped to control higher order derivatives, as in the case of the viscous Burgers equation. This was also the basis of the recent work [38], where the authors propose a modification of the KSE in its vectorial form. Namely, by replacing hyperviscosity with standard Laplacian in one component, they were able to bootstrap the resulting maximum principle and show that smoothness persists under evolution.

A rather ingenious method developed fairly recently by Kiselev, Nazarov and Volberg [29] (see also [28]) was used to obtain a maximum principle for the critically dissipative surface quasi-geostrophic (SQG) equation (and the fractal Burgers equation). Evolution under the critically dissipative SQG equation (when $d = 2$) is described by

$$\begin{cases} \partial_t\theta + (-\Delta)^{1/2}\theta + (u\cdot\nabla)\theta = 0, \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta), \end{cases} \tag{2.6}$$

where R_1, R_2 are the usual Riesz transforms in $\mathbb{R}^2$. Even though (2.6) has a maximum principle of the form $\|\theta(t,\cdot)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$, this control, although useful, does not necessarily prevent blowup in general; one would require control of a stronger norm in order to address the global existence of smooth solutions in the positive direction. The elegant work in [28, 29] introduced techniques that allow one to compare dissipation, (gradient) nonlinearity and nonlocality in the local (pointwise) setting, without any a-priori assumptions other than short time existence and regularity. The main idea is to show that if the initial data has a certain modulus of continuity (see Definition 3.1, below), and if the solution is guaranteed to be smooth for short time, then preservation of the modulus of continuity on some non-degenerate time interval $[0, T]$ implies control of the Lipschitz constant of the solution on that interval, which in many cases is sufficient to prevent blowup of higher order norms. The difficulty lies in constructing a modulus of continuity that is able to (locally) balance dissipation with the instabilities that may arise from nonlinearity and nonlocality for all time. In many cases this is not a trivial task, see for instance [14,26,27,41] and the references therein where this program was expanded and built upon in several other scenarios.

Such techniques rely upon pointwise estimates, and so it is crucial to be able to

(1) make sense of the PDE in the classical way,

(2) make sure the solution enjoys parabolic regularity $C^1_t C^2_x$,

(3) obtain pointwise estimates of all terms in the PDE, preferably via quantifying continuity of

such terms in terms of Hölder estimates, or the modulus of continuity itself,

(4) have a regularity criterion in terms of the Lipschitz constant of the solution.

That being said, in order to study the evolution of moduli of continuity under (2.5) (or even formally obtain a maximum principle), a pointwise upper bound for the nonlocal part must be obtained, ideally in terms of the modulus of continuity being studied. In fact, as will be demonstrated later on, what one really needs is a bound that does not exceed a constant multiple of $\|\nabla\theta\|_{L^\infty}$. However, this does not seem to be possible: the square root of the Laplacian has the representation

$$(-\Delta)^{1/2}\theta = \sum_{i=1}^{d} R_i \partial_i \theta,$$

with $\{R_i\}_{i=1}^d$ being the standard Riesz transforms, and it is well known that L^∞ is a bad space for those operators, see for instance [59]. That is, even when θ has a modulus of continuity and its Lipschitz constant is under control, no information can be obtained about $(-\Delta)^{1/2}\theta$ in terms of the controlled quantities; we refer the reader to [58, 64] for a classical characterization of the singular integral (2.4), and [10] for a more recent one. Nevertheless, see Chapter 5 for further remarks about a possible remedy to this situation. This has led us to consider a slightly weaker model than (2.5), namely equation (2.7), below.

2.2 Main Results.

With the previous remarks in mind, replacing the nonlocal part of equation (2.5) with $(-\Delta)^\alpha$, where $\alpha \in (0, 1/2)$, allows one not only to locally bound the corresponding nonlocality (Lemma 3.4, below), but also to obtain a continuity estimate. Indeed, if $\theta \in C^{0,\beta}$ with $0 < 2\alpha < \beta \leq 1$, then $(-\Delta)^\alpha\theta \in C^{0,\beta-2\alpha}$ [52]. Similarly we show in Lemma 3.6 below, that while the operator $(-\Delta)^\alpha$ doesn't quite preserve an abstract modulus of continuity, it doesn't distort it too much either. This allows us to control the nonlocality and prove that dissipation will prevail, thereby proving that strong (and hence classical) solutions exist and are unique for all time. The nonlinearity does not seem to introduce any extra complications in the proof in the absence of physical boundaries: see for instance [49] for various blowup results for viscous Hamilton-Jacobi equations in the presence of boundaries. Thus, in this work we study the initial value problem associated with

$$\partial_t\theta(t,x) - \nu\Delta\theta(t,x) = \lambda\,|\nabla\theta(t,x)|^p + \mu(-\Delta)^\alpha\theta(t,x), \tag{2.7}$$

where $\nu > 0$, $\alpha \in (0,1/2)$, $p \in [1,\infty)$, $\mu > 0$ and $\lambda \in \mathbb{R}$, with no further restrictions on these parameters. We study evolution under equation (2.7) starting from a $\theta_0 \in W^{1,\infty}(\mathbb{R}^d)$ and we look for strong solutions on an interval of time $[T_1,T_2]$. By a strong solution, we mean

Definition 2.1. *Let $T_2 > T_1$, and suppose $\theta_0 \in W^{1,\infty}\left(\mathbb{R}^d\right)$. We say θ is a strong solution to* (2.7) *on $[T_1,T_2]$ corresponding to θ_0 if $\theta \in C\left([T_1,T_2]; W^{1,\infty}\left(\mathbb{R}^d\right)\right)$ and*

$$\begin{aligned}\theta(t,x) = &\int_{\mathbb{R}^d}\Psi(t-T_1,x-y)\theta_0(y)\,dy + \lambda\int_{T_1}^t\int_{\mathbb{R}^d}\Psi(t-s,x-y)\,|\nabla\theta(s,y)|^p\,dyds\\ &+\mu\int_{T_1}^t\int_{\mathbb{R}^d}\Psi(t-s,x-y)\,(-\Delta)^\alpha\,\theta(s,y)\,dy\,ds, \quad (t,x)\in[T_1,T_2]\times\mathbb{R}^d,\end{aligned}$$

where Ψ is the d-dimensional heat kernel,

$$\Psi(s,y) := (4\pi\nu s)^{-d/2}\exp\left(\frac{-|y|^2}{4\nu s}\right), \quad (s,y)\in\mathbb{R}^+\times\mathbb{R}^d.$$

Remark 2.1. Lemma 3.4, below, allows us to make sense of $(-\Delta)^\alpha\theta$ as an L^∞ function.

Using standard (classical) properties of the heat kernel, one can show that strong solutions satisfy the initial condition in the sense

$$\lim_{t\to T_1^+}\|\theta(t,\cdot)-\theta_0\|_{W^{1,\infty}} = 0, \tag{2.8}$$

and are classical solutions to the PDE (2.7). By classical, we mean that they are once continuously differentiable in time and twice in space on the set $(T_1,T_2]\times\mathbb{R}^d$ and satisfy (2.7) in the pointwise sense. In addition, their time derivatives have the regularity $\partial_t\theta \in L^1([T_1,T_2]; L^\infty(\mathbb{R}^d))$, and a regularity criterion in terms of the Lipschitz constant of the solution should not be surprising. That is, we first establish the following local well-posedness result.

Theorem 2.1. *Let $d \in \mathbb{N}$ be a positive integer, $\nu > 0$, $\alpha \in (0, 1/2)$, $\lambda \in \mathbb{R}$, $\mu > 0$, $p \in [1, \infty)$ and $\theta_0 \in W^{1,\infty}\left(\mathbb{R}^d\right)$ with no further restrictions. Then there is a $T_0 = T_0(\theta_0, d, \alpha, p, \nu, \mu, \lambda) > 0$ and a unique, strong solution θ to* (2.7) *on $[0, T_0]$ corresponding to θ_0 and depending continuously on the initial data in the $W^{1,\infty}\left(\mathbb{R}^d\right)$ norm. Furthermore, if θ is the strong solution corresponding to θ_0 on an arbitrary interval of time $[0, T]$, then $\partial_t \theta \in L^1\left([0, T]; L^\infty\left(\mathbb{R}^d\right)\right)$, $\theta \in C_t^1 C_x^2\left((0, T] \times \mathbb{R}^d\right)$,*

$$\lim_{t \to 0^+} \|\theta(t, \cdot) - \theta_0\|_{W^{1,\infty}} = 0,$$

and

$$\partial_t \theta(t, x) - \nu \Delta \theta(t, x) = \lambda \left|\nabla \theta(t, x)\right|^p + \mu \left(-\Delta\right)^\alpha \theta(t, x),$$

holds true in the classical (pointwise) sense for every $(t, x) \in (0, T] \times \mathbb{R}^d$. If $[0, T_*)$ is the maximal interval of existence of the strong solution, then we must have

$$T_* = \sup\left\{T > 0 : \|\nabla \theta(t, \cdot)\|_{L^\infty} < \infty \; \forall t \in [0, T]\right\}.$$

Remark 2.2. One can obtain a result analogous to Theorem 2.1 for any $\alpha \in (0, 1)$. We restrict ourselves to the case $\alpha \in (0, 1/2)$ for the sake of simplicity. See discussion at the end of §4.1 for more details.

Thus, in order to go from local to global well-posedness, it is sufficient to prevent a gradient blowup scenario (in the L^∞ norm) in finite time. This will be guaranteed if we impose either a periodicity hypothesis on the initial data or require it to vanish at at infinity, i.e., we further assume that either

$$\theta_0(x + L e_j) = \theta_0(x), \quad \forall j \in \{1, 2, \cdots, d\}, \quad x \in \mathbb{R}^d,$$

where $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$ and $L > 0$, or

$$\lim_{|x| \to \infty} |\theta_0(x)| = 0.$$

In this case, we show that the (unique) strong solution arising from such initial data (as defined in Definition 2.1) automatically inherits those properties. Moreover, we are able to control its Lipschitz constant for all time by constructing a strong modulus of continuity (Definition 3.1, below) that must be obeyed by the solution. That is, we establish

Theorem 2.2. *Assume the hypotheses of Theorem 2.1 and suppose further that θ_0 is either periodic with period $L > 0$ in every spatial direction or vanishes at infinity. Then there exists a strong solution θ to* (2.7) *on $[0, \infty)$ corresponding to θ_0, which is periodic if θ_0 is (with the same period) or vanishes at infinity if θ_0 does. Furthermore, θ is unique in the class of strong solutions and we have the following estimate valid for every $t \geq 0$,*

$$\|\nabla\theta(t, \cdot)\|_{L^\infty} \leq Be^{C_0 t}, \tag{2.9}$$

where B depends only on $\|\theta_0\|_{W^{1,\infty}}$ and C_0 depends on B, ν, α, d, μ, with C_0 blowing up as $\alpha \to 1/2$ or $\nu \to 0$. In particular, B and C_0 do not depend on the period L if θ_0 is periodic, nor on p or λ.

Remark 2.3. One can certainly allow for more singular initial data by considering the periodic and whole space scenario separately, and modifying the definition of a strong solution accordingly; see discussion at the end of §4.1 for more details. Essentially, one only needs to guarantee that the solution immediately experiences parabolic regularity (satisfy the PDE (2.7) in the pointwise sense on $(0, T] \times \mathbb{R}^d$). We chose the space $W^{1,\infty}(\mathbb{R}^d)$ and define strong solutions as in Definition 2.1 in order to handle both scenarios in a simple, unified fashion. That is to say, a direct corollary is that we establish the global well-posedness of regular solutions to the initial value problem associated with (2.7) when posed with "periodic boundary conditions".

It is unclear at this stage whether the growth in time observed in (2.9) is sharp or is simply a technical difficulty arising from the proof. Equation (2.7) does not have any scale invariance, and so our modulus of continuity will be customized for each initial data, complicating the construction. Furthermore, in order to "absorb" the instabilities arising from the nonlocality without allowing time dependence, the second derivative of the modulus should be bounded from above by a negative constant, a scenario that might lead the modulus of continuity to be negative. This will be made clear at the technical level in §4.2, and touched upon in Chapter 5. Moreover, such growth in time is also expected for the linear equation, that is equation (2.7) with $\lambda = 0$.

3. PRELIMINARIES*

In this Chapter, we list some preliminary results and estimates that will be used throughout this work. We summarize the main ingredients introduced in [28, 29] when studying the evolution of moduli of continuity in §3.1 . In §3.2, we obtain some (elementary) pointwise upper bounds for the nonlocal operator $(-\Delta)^\alpha$ that we will need in the analysis to follow. In particular, Lemma 3.6 (a generalization of [52, Proposition 2.5]) is the crucial estimate that will be used to prove the long-time existence of strong solutions, and is the key ingredient that fails when trying to obtain similar results for $\alpha \geq 1/2$.

3.1 Moduli of Continuity.

Definition 3.1. *We say a function $\omega : [0,\infty) \to [0,\infty)$ is a modulus of continuity if $\omega \in C([0,\infty)) \cap C^2(0,\infty)$, is nondecreasing and concave, and $\omega(0) = 0$. A modulus of continuity ω is said to be strong if in addition $0 < \omega'(0) < \infty$ and $\lim_{\xi \to 0^+} \omega''(\xi) = -\infty$.*

Definition 3.2. *Let ω be a modulus of continuity. We say a scalar function $\theta \in C(\mathbb{R}^d)$ has modulus of continuity ω if $|\theta(x) - \theta(y)| \leq \omega(|x-y|)$. We say θ has strict modulus of continuity ω if $|\theta(x) - \theta(y)| < \omega(|x-y|)$ whenever $x \neq y$.*

To avoid cumbersome notation, in the proof of the following two Lemmas, we drop the subscript L^∞ from $\|\cdot\|_{L^\infty}$. Even though they are discussed in [28,29], we prove them again here for the sake of completeness and convenience. Moreover, we find it necessary to rigorously prove Lemma 3.1 in order to verify that the control on the Lipschitz constant of the solution is independent of the period length $L > 0$ when θ_0 is chosen to be periodic.

Lemma 3.1. *Let $\theta \in W^{1,\infty}\left(\mathbb{R}^d\right)$ be bounded and Lipschitz scalar, and suppose ω is an unbounded modulus of continuity. Then there exists $B_\theta \geq 1$ depending only on $\|\theta\|_{L^\infty}$ and $\|\nabla\theta\|_{L^\infty}$ such that θ has strict modulus of continuity $\omega(B|x-y|)$ whenever $B \geq B_\theta$.*

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Proof. Chose $B_\theta > 0$ such that $\omega(B_\theta) > \max\{2\|\theta\| + 1, \|\nabla\theta\| + 1\}$, which is possible as ω is unbounded. As ω is nondecreasing, we must have $\omega(B) \geq \omega(B_\theta)$ for any $B \geq B_\theta$. Let $\xi := |x - y| > 0$. For $\xi \geq 1$, we write:

$$\omega(B\xi) = \omega(B) + \int_B^{B\xi} \omega'(\eta)\, d\eta \geq \omega(B) > 2\|\theta\| \geq |\theta(x) - \theta(y)|,$$

meaning $|\theta(x) - \theta(y)| < \omega(B|x - y|)$ whenever $|x - y| \geq 1$. When $\xi \in (0, 1)$, we first write

$$|\theta(x) - \theta(y)| \leq \|\nabla\theta\||x - y|,$$

and note that due to the concavity of ω, the function

$$h(\xi) := \|\nabla\theta\| - \frac{\omega(B\xi)}{\xi}$$

is increasing and so must be negative on $(0, 1)$, as $h(1) < 0$ by choice of B. $\square$

Lemma 3.2. *Suppose $\theta \in C^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ and has a strong modulus of continuity ω. Then θ is Lipschitz and $\|\nabla\theta\|_{L^\infty} < \omega'(0)$.*

Remark 3.1. That θ is Lipschitz and $\|\nabla\theta\|_{L^\infty} \leq d^{1/2}\omega'(0)$ follows from Definition 3.2 and the limit definition of a derivative. The important part is the strict inequality, for which we need $\omega''(0) = -\infty$, and $\theta \in C^2$.

Proof. Let $x \in \mathbb{R}^d$ be arbitrary, let $\xi \in (0, 1]$ and let $y = x + \xi e$, where e is any unit vector. From the first order Taylor expansion of θ about x we see that

$$|\theta(y) - \theta(x)| \geq |\nabla\theta(x)|\xi - \frac{C\xi^2}{2}\|\nabla^2\theta\|,$$

here $\|\nabla^2\theta\|$ is just the maximum of all second order derivatives, and C is a combinatorial constant.

The left hand side is at most $\omega(\xi)$, and so after rearranging we get for any $\xi \in (0,1]$,

$$|\nabla\theta(x)| \leq \frac{\omega(\xi)}{\xi} + \frac{C\xi}{2}\|\nabla^2\theta\|. \tag{3.1}$$

Since ω is C^2 on $(0,\infty)$, and $\lim_{\xi\to 0^+} \omega''(\xi) = -\infty$, it follows from the Taylor expansion of ω around $\xi/2$ that

$$\omega(\xi) = \omega(\xi/2) + \frac{\omega'(\xi/2)}{2}\xi - \rho(\xi)\xi^2,$$

where $\lim_{\xi\to 0^+} \rho(\xi) = \infty$. Plugging this into (3.1) and taking the supremum over all x we get

$$\|\nabla\theta\| \leq \frac{\omega(\xi/2)}{\xi} + \frac{\omega'(\xi/2)}{2} + \xi\left(C\|\nabla^2\theta\| - \rho(\xi)\right).$$

The result now follows by choosing $\xi \in (0,1]$ small enough that $C\|\nabla^2\theta\| - \rho(\xi) < 0$ and noting that

$$\frac{\omega(\xi/2)}{\xi} + \frac{\omega'(\xi/2)}{2} < \frac{\omega'(0)}{2} + \frac{\omega'(0)}{2} = \omega'(0),$$

where we again used the concavity of ω. □

The following Lemma is crucial in handling the nonlinear part of the equation, as well as extracting local dissipation from the Laplacian. See [26, Proposition 2.4] for further insight, and a slightly different proof. We relax the assumptions on the modulus of continuity and only assume it is continuous on $[0,\infty)$, and piecewise C^2 on $(0,\infty)$, with finite one sided derivatives, except for the condition $\omega''(0) = -\infty$.

Lemma 3.3. *Suppose θ is $C^2(\mathbb{R}^d)$ and has modulus of continuity ω. If $\theta(x) - \theta(y) = \omega(|x-y|)$ for some $x \neq y$, with $x - y = (\xi, 0, \cdots, 0)$, $\xi > 0$, then*

$$\begin{cases} \omega'(\xi^-) \leq \partial_1\theta(x) = \partial_1\theta(y) \leq \omega'(\xi^+), \\ \partial_j\theta(x) = \partial_j\theta(y) = 0, \quad j > 1 \end{cases} \tag{3.2}$$

and

$$\Delta\theta(x) - \Delta\theta(y) \le 4\omega''(\xi^-). \tag{3.3}$$

Proof. We start by showing $\partial_j\theta(x) = \partial_j\theta(y)$ and $\partial_j^2\theta(x) - \partial_j^2\theta(y) \le 0$. Let $\epsilon > 0$ and define:

$$\begin{aligned}
d_\epsilon^+ &:= \theta(x+\epsilon e_j) - \theta(y+\epsilon e_j) - [\theta(x) - \theta(y)],\\
d_\epsilon^- &:= \theta(x) - \theta(y) + [\theta(y-\epsilon e_j) - \theta(x-\epsilon e_j)],\\
d_\epsilon &:= [\theta(x+\epsilon e_j) - 2\theta(x) + \theta(x-\epsilon e_j)] - [\theta(y+\epsilon e_j) - 2\theta(y) + \theta(y-\epsilon e_j)],
\end{aligned}$$

where $\{e_j\}_{j=1}^d$ is the standard unit basis of $\mathbb{R}^d$. It is sufficient to show $d_\epsilon^+ \le 0$, $d_\epsilon^- \ge 0$ and $d_\epsilon \le 0$. But this follows immediately from the fact that $\theta(x) - \theta(y) = \omega(\xi)$ and $|\theta(z_0) - \theta(z_1)| \le \omega(|z_0 - z_1|)$ for any z_0, z_1. Next, we define

$$\begin{aligned}
d_{\epsilon,j}^+ &:= \theta(x+\epsilon e_j) - \theta(x) = \theta(x+\epsilon e_j) - \theta(y) - \omega(\xi),\\
d_{\epsilon,j}^- &:= \theta(x) - \theta(x-\epsilon e_j) = \omega(\xi) + \theta(y) - \theta(x-\epsilon e_j).
\end{aligned}$$

Notice that for $j = 1$, we have $|x^0 + \epsilon e_1 - y| = \xi + \epsilon$, and $|y - x + \epsilon e_1| = \xi - \epsilon$ whenever $\epsilon \in (0, \xi/2)$, while for $j > 1$, $|x + \epsilon e_j - y| = |y - x + \epsilon e_j| = \sqrt{\xi^2 + \epsilon^2}$. Hence,

$$d_{\epsilon,j}^+ \le \begin{cases} \omega(\xi+\epsilon) - \omega(\xi), & j = 1,\\ \omega(\sqrt{\xi^2+\epsilon^2}) - \omega(\xi), & j > 1 \end{cases}, \tag{3.4}$$

$$d_{\epsilon,j}^- \ge \begin{cases} \omega(\xi) - \omega(\xi-\epsilon), & j = 1,\\ \omega(\xi) - \omega(\sqrt{\xi^2+\epsilon^2}), & j > 1 \end{cases}, \tag{3.5}$$

from which (3.2) follows immediately upon dividing (3.4) and (3.5) by $\epsilon > 0$ and letting $\epsilon \to 0^+$, since ω is continuous and have one-sided derivatives. Finally, let $x' := (x_2, \cdots, x_d) \in \mathbb{R}^{d-1}$ be the other coordinates, and define

$$h(s) := \theta(s, x') - \theta(x_1 + y_1 - s, x') - \omega(2s - x_1 - y_1),\ s > \frac{x_1 + y_1}{2}.$$

Suppose by way of contradiction that $\partial_1^2\theta(x) - \partial_1^2\theta(y) > 4\omega''(\xi^-)$. As ω is piecewise C^2, there exists some small enough $\epsilon > 0$ such that $h(s)$ is C^2 on $[x_1 - \epsilon, x_1]$ and $-h''(s) < 0$ on that interval. On the one hand, a Lemma of Hopf (or simple calculus) tells us that we must have $h'(x_1^-) > 0$. On the other hand, owing to (3.2), we must have

$$h'(x_1^-) = 2\left(\partial_1\theta(x) - \omega'(\xi^-)\right) \leq 2\left(\omega'(\xi^+) - \omega'(\xi^-)\right),$$

which leads to a contradiction under the concavity assumption of ω. □

Remark 3.2. Under the concavity assumption of ω, from (3.2) we see that the modulus of continuity cannot be violated at a point where ω' has a jump discontinuity.

3.2 Pointwise Estimates for $(-\Delta)^\alpha$.

This section is devoted to deriving some simple pointwise upper bounds for the fractional Laplacian. Lemma 3.4 is used in proving local well-posedness in a simple manner, regardless of whether we are in the periodic or whole space setting, while Lemma 3.5 is required when handling the whole space setting. We remark that one can do without Lemma 3.4 by specializing to the periodic or whole space scenario, where short time existence and regularity can be proven by standard energy techniques and, in the periodic case, Galerkin approximations. Lemma 3.4 simply allows us to prove local-well-posedness and regularity for either scenario, and arbitrary dimension d in a simple, unified fashion. On the other hand, we emphasize again, that Lemma 3.6 is the key ingredient that allows one to control the nonlocal (destabilizing) part by the local diffusive term, and is the key estimate that is missing when trying to prove similar results for $\alpha \geq 1/2$. We remark that some version of Lemma 3.6 was proven very recently in [41] for a special class of Fourier multipliers of order strictly less than one. However, the class of operators considered does not include the fractional Laplacian, since the argument in [41] requirs the kernel to have a zero average, a property that is not satisfied by the operator $(-\Delta)^\alpha$.

Recall the singular integral definition of $(-\Delta)^\alpha$

$$(-\Delta)^\alpha\theta(x) = C_{d,\alpha}P.V.\int_{\mathbb{R}^d}\frac{\theta(x)-\theta(x-z)}{|z|^{d+2\alpha}}\,dz, \tag{3.6}$$

which is known to be equivalent to the Fourier multiplier definition (in the whole space)

$$\widehat{(-\Delta)^\alpha\theta}(\zeta) := |\zeta|^{2\alpha}\hat{\theta}(\zeta).$$

We remark that for periodic functions (assuming the period is 2π for simplicity), it is common to instead use the following pointwise formula

$$(-\Delta)^\alpha\theta(x) = C_{d,\alpha}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^d}\frac{\theta(x)-\theta(x-z)}{|z+k|^{d+2\alpha}}\,dz, \tag{3.7}$$

with (3.7) known to be equivalent to the (periodic) Fourier multiplier definition

$$(-\Delta)^\alpha\theta(x) = \sum_{k\in\mathbb{Z}^d}|k|^{2\alpha}\hat{\theta}(k)e^{ik\cdot x}, \tag{3.8}$$

see for instance [13]. We show in Appendix A the equivalency of (3.6) and (3.8) when θ is smooth and periodic. We prefer to work with the representation (3.6), as it allows us to easily obtain the required bounds and continuity estimates, regardless of whether the function is periodic or not.

Lemma 3.4. *Let $\alpha\in(0,1/2)$, $\gamma\in(2\alpha,1]$, $\theta\in L^\infty(\mathbb{R}^d)\cap C^{0,\gamma}\left(\mathbb{R}^d\right)$ and define*

$$[\theta]_{C^{0,\gamma}} := \sup_{x\neq y}\frac{|\theta(x)-\theta(y)|}{|x-y|^\gamma}.$$

Then $(-\Delta)^\alpha\theta \in L^\infty(\mathbb{R}^d)$ *and*

$$\|(-\Delta)^\alpha\theta\|_{L^\infty} \leq \frac{\gamma C_{d,\alpha}|\mathbb{S}^{d-1}|}{\alpha(\gamma-2\alpha)}\, \|\theta\|_{L^\infty}^{1-\frac{2\alpha}{\gamma}}\, [\theta]_{C^{0,\gamma}}^{\frac{2\alpha}{\gamma}}. \tag{3.9}$$

Similarly, if $\alpha \in [1/2,1)$, $\gamma \in (2\alpha-1,1]$, *and* $\theta \in L^\infty(\mathbb{R}^d)\cap C^{1,\gamma}\left(\mathbb{R}^d\right)$, *we must have* $(-\Delta)^\alpha\theta \in L^\infty(\mathbb{R}^d)$ *and*

$$\|(-\Delta)^\alpha\theta\|_{L^\infty} \leq \frac{(1+\gamma)C_{d,\alpha}|\mathbb{S}^{d-1}|}{\alpha(1+\gamma-2\alpha)}\, \|\theta\|_{L^\infty}^{1-\frac{2\alpha}{1+\gamma}}\, [\nabla\theta]_{C^{0,\gamma}}^{\frac{2\alpha}{1+\gamma}}. \tag{3.10}$$

Proof. For $\alpha \in (0,1/2)$, the singular integral (3.6) is absolutely convergent when $\theta \in L^\infty(\mathbb{R}^d)\cap C^{0,\gamma}\left(\mathbb{R}^d\right)$, $\beta \in (2\alpha,1]$. Moreover, if θ is constant, the result is trivial, so we assume otherwise. For fixed $R>0$, we have

$$\begin{aligned}
|(-\Delta)^\alpha\theta(x)| &\leq C_{d,\alpha}\int_{|z|\leq R}\frac{|\theta(x)-\theta(x-z)|}{|z|^{d+2\alpha}}dz + C_{d,\alpha}\int_{|z|>R}\frac{|\theta(x)-\theta(x-z)|}{|z|^{d+2\alpha}}dz \\
&\leq 2C_{d,\alpha}|\mathbb{S}^{d-1}|\left([\theta]_{C^{0,\gamma}}\int_0^R \rho^{\gamma-2\alpha-1}\,d\rho + \|\theta\|_{L^\infty}\int_R^\infty \rho^{-2\alpha-1}\,d\rho\right) \\
&\leq 2C_{d,\alpha}|\mathbb{S}^{d-1}|\left(\frac{R^{\gamma-2\alpha}}{\gamma-2\alpha}[\theta]_{C^{0,\gamma}} + \frac{R^{-2\alpha}}{2\alpha}\,\|\theta\|_{L^\infty}\right).
\end{aligned}$$

Bound (3.9) now follows by choosing $R := \left(\|\theta\|_{L^\infty}[\theta]_{C^{0,\gamma}}^{-1}\right)^{1/\gamma}$. When $\alpha \in [1/2,1)$ we use the mean value theorem to get

$$|2\theta(x)-\theta(x-z)-\theta(x+z)| \leq C_d[\nabla\theta]_{C^{0,\gamma}}|z|^{1+\gamma},$$

and so if $\theta \in L^\infty(\mathbb{R}^d)\cap C^{1,\gamma}\left(\mathbb{R}^d\right)$ with $\gamma \in (2\alpha-1,1]$, we can use the regularization (see Appendix A for details)

$$(-\Delta)^\alpha\theta(x) = C_{d,\alpha}\int_{\mathbb{R}^d}\frac{2\theta(x)-\theta(x-z)-\theta(x+z)}{|z|^{d+2\alpha}}\,dz,$$

to obtain

$$
\begin{aligned}
|(-\Delta)^\alpha \theta(x)| &\le C_{d,\alpha}|\mathbb{S}^{d-1}| \left([\nabla\theta]_{C^{0,\gamma}} \int_0^R \rho^{\gamma-2\alpha}\, d\rho + \|\theta\|_{L^\infty} \int_R^\infty \rho^{-2\alpha-1}\, d\rho \right) \\
&\le C_{d,\alpha}|\mathbb{S}^{d-1}| \left(\frac{R^{\gamma+1-2\alpha}}{\gamma+1-2\alpha} [\nabla\theta]_{C^{0,\gamma}} + \frac{R^{-2\alpha}}{2\alpha} \|\theta\|_{L^\infty} \right).
\end{aligned}
$$

We conclude by choosing $R := \left(\|\theta\|_{L^\infty} [\nabla\theta]^{-1}_{C^{0,\gamma}} \right)^{1/(1+\gamma)}$. ☐

Lemma 3.5. *For integer $k \ge 0$, denote by $C_0^k(\mathbb{R}^d) \subset W^{k,\infty}(\mathbb{R}^d)$ the space of all $C^k(\mathbb{R}^d)$ functions such that all derivatives up to order k are bounded and vanish at infinity, i.e.,*

$$
\lim_{|x|\to\infty} |D^\beta \theta(x)| = 0, \quad \forall |\beta| \le k.
$$

If $\alpha \in (0, 1/2)$, then $(-\Delta)^\alpha \theta \in C_0^{k-1}(\mathbb{R}^d)$ when $k \ge 1$. If $\alpha \in [1/2, 1)$, then $(-\Delta)^\alpha \theta \in C_0^{k-2}(\mathbb{R}^d)$ when $k \ge 2$.

Proof. It suffices to prove the results for $k = 1, 2$, when $\alpha \in (0, 1/2)$ and $[1/2, 1)$, respectively. For $\alpha \in (0, 1)$, we regularize the singular integral (3.6) by

$$
(-\Delta)^\alpha \theta(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{\theta(x) - \theta(x-y) - y \cdot \nabla\theta(x) \chi_{|y|\le 1}(y)}{|y|^{d+2\alpha}}\, dy,
$$

making the above integral absolutely convergent for $\theta \in C^1$ if $\alpha \in (0, 1/2)$, and for $\theta \in C^2$ if $\alpha \in [1/2, 1)$. We start by splitting the integral into a singular part, intermediate part, and decaying part as follows

$$
\begin{aligned}
I_S &:= \int_{|y|\le 1} \frac{\theta(x) - \theta(x-y) - y \cdot \nabla\theta(x)}{|y|^{d+2\alpha}}\, dy, \\
I_M &:= \int_{1\le |y| \le R} \frac{\theta(x) - \theta(x-y)}{|y|^{d+2\alpha}}\, dy, \\
I_R &:= \int_{|y|\ge R} \frac{\theta(x) - \theta(x-y)}{|y|^{d+2\alpha}}\, dy.
\end{aligned}
$$

In what follows, $C_{d,\alpha}$ always denotes a positive constant depending on d, α, degenerating (vanishing) as $\alpha \to 0^+$ or 1^-, and whose value may change from line to line. For any given $\epsilon > 0$, we start by choosing a large enough $R > 1$ such that

$$\int_{|y|\geq R} |y|^{-d-2\alpha}\, dz < \frac{\epsilon}{6C_{d,\alpha}\|\theta\|_{L^\infty}},$$

making $I_R < \epsilon/3$. Next, we chose a large enough $N_0 > R$ such that

$$|\theta(z)| \leq \frac{\epsilon}{6C_{d,\alpha}},$$

whenever $|z| \geq N_0 - R$, rendering $I_M < \epsilon/3$ provided $|x| > N_0$. To handle I_S, notice that given $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, by the mean value theorem, we can find some $\lambda = \lambda(x, y) \in (0, 1)$ such that

$$\theta(x) - \theta(x - y) = y \cdot \nabla\theta\left(x + (\lambda - 1)y\right),$$

implying the singular integrand of I_S is bounded from above by

$$\frac{|\theta(x) - \theta(x - y) - y \cdot \nabla\theta(x)|}{|y|^{d+2\alpha}} \leq \frac{|\nabla\theta\left(x + (\lambda - 1)y\right) - \nabla\theta(x)|}{|y|^{d+2\alpha-1}}. \tag{3.11}$$

For $\alpha \in (0, 1/2)$, we can chose a large enough $N_1 > 1$ such that, whenever $|z| \geq N_1 - 1$,

$$|\nabla\theta(z)| < \frac{\epsilon}{6C_{d,\alpha}},$$

making $I_S < \epsilon/3$ when $|x| \geq N_1$. This concludes the case when $\alpha \in (0, 1/2)$. For $\alpha \in [1/2, 1)$, we apply the mean value theorem once again to (3.11) to get a $\sigma \in (0, 1)$ and conclude that the singular integrand is now dominated by

$$|y|^{2-2\alpha-d}\left|\nabla^2\theta(x + \sigma(\lambda - 1)y)\right|,$$

allowing us to conclude by choosing a large enough N_1 such that

$$\left|\nabla^2\theta(z)\right| < \frac{\epsilon}{3C_{d,\alpha}},$$

whenever $|z| \geq N_1 - 1$. Hence $I_S < \epsilon/3$ when $|x| \geq N_1$. □

Lemma 3.6. *Suppose $\theta \in C(\mathbb{R}^d)$ has a strong modulus of continuity ω, and let $\alpha \in (0, 1/2)$. Then $(-\Delta)^\alpha\theta$ has modulus of continuity*

$$\widetilde{\omega}(\xi) := C_{d,\alpha}|\mathbb{S}^{d-1}|\alpha^{-1}\int_0^\xi \frac{\omega'(\eta)}{\eta^{2\alpha}}\,d\eta. \tag{3.12}$$

Remark 3.3. The modulus of continuity ω need not be strong. All that is required is for the integral (3.12) to be convergent, that is we require $\omega(\xi) = O(\xi^\beta)$ some $\beta \in (2\alpha, 1]$ when ξ is small.

Proof. Following Remark 3.1, we must have $\theta \in W^{1,\infty}(\mathbb{R}^d)$, and so for $\alpha \in (0, 1/2)$, the singular integral (3.6) is absolutely convergent. Therefore, for arbitrary $\rho > 0$, $(x, z) \in \mathbb{R}^d \times \mathbb{R}^d$, we must have

$$|(-\Delta)^\alpha\theta(x) - (-\Delta)^\alpha\theta(z)| \leq C_{d,\alpha}(I_1 + I_2),$$

where

$$\begin{aligned} I_1 &:= \left|\int_{|y|\leq\rho} \frac{\theta(x) - \theta(x-y) - (\theta(z) - \theta(z-y))}{|y|^{d+2\alpha}}\,dy\right|, \\ I_2 &:= \left|\int_{|y|>\rho} \frac{\theta(x) - \theta(z) - (\theta(x-y) - \theta(z-y))}{|y|^{d+2\alpha}}\,dy\right|. \end{aligned}$$

For I_1, we estimate from above by

$$I_1 \leq 2\int_{|y|\leq\rho} \frac{\omega(|y|)}{|y|^{d+2\alpha}}\,dy = 2|\mathbb{S}^{d-1}|\int_0^\rho \frac{\omega(\eta)}{\eta^{2\alpha+1}}\,d\eta = |\mathbb{S}^{d-1}|\alpha^{-1}\int_0^\rho \frac{\omega'(\eta)}{\eta^{2\alpha}}\,d\eta - |\mathbb{S}^{d-1}|\frac{\omega(\rho)}{\alpha\rho^{2\alpha}},$$

where we integrated by parts in the last step. For I_2, we have

$$I_2 \leq 2\omega(|x-z|) \int_{|y|\geq\rho} |y|^{-d-2\alpha}\, dy = |\mathbb{S}^{d-1}|\alpha^{-1} \frac{\omega(|x-z|)}{\rho^{2\alpha}},$$

from which we conclude by choosing $\rho = |x-z|$. □

4. PROOF OF MAIN RESULTS *

4.1 Proof of Theorem 2.1

In this section, ∇ always denotes the gradient vector acting on spatial coordinates, while $C_{d,\alpha} \geq 1$ always denotes an absolute constant depending on the dimension d and α. The constant $C_{d,\alpha}$ may blow up as $\alpha \to 1/2$, and its value may change from line to line. Let us start by recalling some properties of the heat kernel $\Psi(s,y) := (4\pi\nu s)^{-d/2} \exp\left(\frac{-|y|^2}{4\nu s}\right)$ (defined for $(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$):

$$\int_{\mathbb{R}^d} \Psi(s,y)\, dy = 1, \tag{4.1}$$

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y)|\ dy = \frac{C_d}{\sqrt{\nu s}}, \tag{4.2}$$

$$\int_{\mathbb{R}^d} |x-y|^\gamma |\partial_s \Psi(s,x-y)|\, dy \leq C_d \nu^{\gamma/2} s^{\gamma/2-1}, \tag{4.3}$$

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\ dy \leq \frac{C_d}{\nu s}|x-z|, \tag{4.4}$$

where $s, \gamma > 0$, and $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d$ are arbitrary. From (4.2) and (4.4) we get

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\ dy \leq \frac{C_{d,\beta}|x-z|^\beta}{(\nu s)^{\frac{1}{2}(1+\beta)}}, \tag{4.5}$$

where $\beta \in (0,1)$ is arbitrary. We refer the reader to Appendix B where (4.1)-(4.5) are proven. We will also make use of the following Gronwall-type inequality, which we prove in Appendix C.

Lemma 4.1. *Let $q \in [1,\infty)$, $1/q + 1/r = 1$, $T_2 \geq T_1$, $C_0 \geq 0$ and assume that $g \in L^q(T_1,T_2)$, $f \in L^r(0,T_2-T_1)$ are both non-negative. If*

$$g(t) \leq \int_{T_1}^t f(t-s)g(s)\, ds + C_0, \quad a.e.\ t \in [T_1,T_2],$$

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then

$$g(t) \le C_0 \left[2 \left(\int_0^{t-T_1} |f(s)|^r ds \right)^{1/r} \left(\int_{T_1}^t e^{h(t)-h(s)} ds \right)^{1/q} + 1 \right], \quad a.e.\ t \in [T_1, T_2],$$

where

$$h(t) := 2^q \int_{T_1}^t \left(\int_0^{s-T_1} |f(\sigma)|^r d\sigma \right)^{q/r} ds.$$

The proof of Theorem 2.1 closely follows the ideas presented in [2,5], and will be broken down into several propositions. We begin by constructing in Proposition 4.1 strong solutions that exist at least for a short time and which inherit periodicity and decay properties from the initial data. This is followed by proving that strong solutions depend continuously on initial data and hence are unique in their own class (Proposition 4.2). Those two propositions give us a local well-posedness result in the space $W^{1,\infty}(\mathbb{R}^d) \cap X$, where X is either the space of continuous periodic functions defined on $\mathbb{R}^d$ or the space of functions that vanish at infinity. We conclude by showing that those solutions have parabolic regularity (that is, they are classical) and derive a regularity criterion in Propositions 4.3 and 4.4, respectively.

Proposition 4.1. *Let $d \in \mathbb{N}$, $\nu > 0$, $\alpha \in (0, 1/2)$, $\lambda \in \mathbb{R}$, $\mu > 0$, $p \in [1, \infty)$ and $\theta_0 \in W^{1,\infty}\left(\mathbb{R}^d\right)$ with no further restrictions. Then there is a $T_0 = T_0(\theta_0, d, \alpha, p, \nu, \mu, \lambda) > 0$ and a strong solution θ to (2.7) on $[0, T_0]$ corresponding to θ_0. Furthermore, if θ_0 is periodic with period $L > 0$, then so is $\theta(t, \cdot)$, and if $\theta_0 \in C_0(\mathbb{R}^d)$, then so is $\theta(t, \cdot)$ for $t \in [0, T_0]$.*

Proof. For $T > 0$, let X_T be the Banach space $X_T := C\left([0, T]; W^{1,\infty}(\mathbb{R}^d)\right)$ with the norm

$$\|f\|_{X_T} := \max_{t \in [0,T]} \|f(t)\|_{W^{1,\infty}}.$$

We will construct a strong solution by choosing a small enough $T_0 > 0$ that the inductively defined sequence of functions $\{\theta_k\}_{k=1}^{\infty}$

$$
\begin{aligned}
\theta_1(t,x) &:= \int_{\mathbb{R}^d} \Psi(t,y)\theta_0(x-y)\, dy, \\
\theta_k(t,x) &:= \theta_1(t,x) + \lambda \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-y)\, |\nabla \theta_{k-1}(s,y)|^p \; dy\, ds \\
&\qquad + \mu \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-y)\, (-\Delta)^{\alpha}\, \theta_{k-1}(s,y)\, dy\, ds, \quad k \geq 2
\end{aligned}
$$

is Cauchy in X_{T_0}. We start by obtaining some uniform bounds. Let

$$
M_0 := 1 + \|\theta_0\|_{L^\infty}, \quad M_1 := 1 + \|\nabla \theta_0\|_{L^\infty},
$$

$$
\kappa_0 := C_{d,\alpha} \left(2p|\lambda| M_1^p + \mu M_0^{1-2\alpha} M_1^{2\alpha} \right),
$$

and set

$$
T_0 := \frac{1}{16} \min \left\{ \nu \kappa_0^{-2}, \kappa_0^{-1} \right\} > 0.
$$

We obtain the following bounds, uniform in $k \in \mathbb{N}$, $t \in [0, T_0]$,

$$
\|\theta_k(t,\cdot)\|_{L^\infty} \leq M_0, \quad \|\nabla \theta_k(t,\cdot)\|_{L^\infty} \leq M_1, \tag{4.6}
$$

via an inductive argument: they hold trivially for θ_1, and assuming they are true for θ_{k-1}, we get, by using (4.1) along with bound (3.9) with $\gamma = 1$ from Lemma 3.4, that

$$
\begin{aligned}
|\theta_k(t,x)| \leq& \|\theta_0\|_{L^\infty} + |\lambda| \int_0^t \|\nabla \theta_{k-1}(s,\cdot)\|_{L^\infty}^p \, ds \\
&+ C_{d,\alpha}\mu \int_0^t \|\theta_{k-1}(s,\cdot)\|_{L^\infty}^{1-2\alpha} \|\nabla \theta_{k-1}(s,\cdot)\|_{L^\infty}^{2\alpha} \, ds \\
&\leq \|\theta_0\|_{L^\infty} + \left(|\lambda| M_1^p + \mu C_{d,\alpha} M_0^{1-2\alpha} M_1^{2\alpha} \right) T_0 \leq \|\theta_0\|_{L^\infty} + \kappa_0 T_0,
\end{aligned}
$$

By choice of T_0, the right hand side is bounded from above by M_0. Similarly, except now using

(4.2), we get that

$$|\nabla\theta_k(t,x)| \le \|\nabla\theta_0\|_{L^\infty} + C_{d,\alpha}\nu^{-1/2}\left(|\lambda|M_1^p + \mu M_0^{1-2\alpha}M_1^{2\alpha}\right)\int_0^t s^{-1/2}ds$$
$$\le \|\nabla\theta_0\|_{L^\infty} + \sqrt{\frac{\kappa_0^2 T_0}{\nu}},$$

and the right-hand side is bounded by M_1 by choice of T_0, closing the inductive argument. To show that the sequence is Cauchy in X_{T_0}, it is sufficient to show that

$$\|\theta_k - \theta_{k-1}\|_{X_{T_0}} \le \frac{1}{2^{k-1}}, \quad k \ge 2. \tag{4.7}$$

To begin, notice that by choice of T_0 and bound (3.9) with $\gamma = 1$ from Lemma 3.4, we have for $(t,x) \in [0,T_0]\times\mathbb{R}^d$,

$$|\theta_2(t,x) - \theta_1(t,x)| \le |\lambda|M_1^p T_0 + \mu C_{d,\alpha}M_0^{1-2\alpha}M_1^{2\alpha}T_0 \le \kappa_0 T_0 \le \frac{1}{4},$$
$$|\nabla\theta_2(t,x) - \nabla\theta_1(t,x)| \le C_d|\lambda|M_1^p\sqrt{\frac{T_0}{\nu}} + \mu C_{d,\alpha}M_0^{1-2\alpha}M_1^{2\alpha}\sqrt{\frac{T_0}{\nu}} \le \sqrt{\frac{\kappa_0^2 T_0}{\nu}} \le \frac{1}{4},$$

implying $\|\theta_2 - \theta_1\|_{X_{T_0}} \le 1/2$. Since $|a^p - b^p| \le p|a-b|(a^{p-1}+b^{p-1})$, for $p \ge 1$ and $a,b \ge 0$, similar calculations yield, for $k \ge 3$ and $(t,x)\in[0,T_0]\times\mathbb{R}^d$,

$$|\theta_k(t,x) - \theta_{k-1}(t,x)| \le \left(2p|\lambda|M_1^{p-1} + \mu C_{d,\alpha}\right)T_0\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}}$$
$$\le \kappa_0 T_0\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}} \le \frac{1}{4}\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}},$$
$$|\nabla\theta_k(t,x) - \nabla\theta_{k-1}(t,x)| \le \left(2C_d p|\lambda|M_1^{p-1} + \mu C_{d,\alpha}\right)\sqrt{\frac{T_0}{\nu}}\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}},$$
$$\le \sqrt{\frac{\kappa_0^2 T_0}{\nu}}\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}} \le \frac{1}{4}\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}}.$$

Thus,

$$\|\theta_k - \theta_{k-1}\|_{X_{T_0}} \le \frac{1}{2}\|\theta_{k-1}-\theta_{k-2}\|_{X_{T_0}}, \quad k \ge 3,$$

making (4.7) true. It follows that $\{\theta_k\}_{k=1}^{\infty}$ converges to some θ in the norm topology of X_{T_0} and so, by utilizing Lemma 3.4 one more time,

$$\theta(t,x) = \int_{\mathbb{R}^d} \Psi(t,y)\theta_0(x-y)\,dy + \lambda \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-y)\,|\nabla\theta(s,y)|^p\,dy\,ds$$
$$+ \mu \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-y)\,(-\Delta)^\alpha\,\theta(s,y)\,dy\,ds, \quad (t,x) \in [0,T_0] \times \mathbb{R}^d.$$

Hence θ is a strong solution on $[0,T_0]$ corresponding to θ_0, with the extra regularity $C_t C_x^1\left((0,T_0] \times \mathbb{R}^d\right)$.

It is clear that if θ_0 is periodic with period $L > 0$, then so is each $\theta_k(t,\cdot)$, and so the same can be said of the limiting function. We now argue that if $\theta_0 \in C_0(\mathbb{R}^d)$, then $\theta_k(t,\cdot) \in C_0^1(\mathbb{R}^d)$ for each fixed $t > 0$. Since the estimate

$$|\Psi(t,y)\theta_0(x-y)| + |\nabla\Psi(t,y)\theta_0(x-y)| \le \|\theta_0\|_{L^\infty}\left[\Psi(t,y) + |\nabla\Psi(t,y)|\right],$$

holds uniformly in $x \in \mathbb{R}^d$ and since $\Psi(t,\cdot)$ and $|\nabla\Psi(t,\cdot)|$ are both integrable for $t > 0$, we conclude that $\theta_1(t,\cdot) \in C_0^1(\mathbb{R}^d)$ for $t > 0$. Assuming $\theta_{k-1}(t,\cdot) \in C_0^1(\mathbb{R}^d)$, by virtue of the following bound holding uniformly in $x \in \mathbb{R}^d$, $s \in [0,t]$,

$$|\Psi(s,y)||\nabla\theta_{k-1}(t-s,x-y)|^p \le M_1^p\left[\Psi(t,y) + |\nabla\Psi(t,y)|\right],$$

coupled with the observation that Ψ and $|\nabla\Psi|$ are both integrable on $(0,t] \times \mathbb{R}^d$, we get

$$\lim_{|x|\to\infty} \left| \int_0^t \int_{\mathbb{R}^d} \Psi(s,y)|\nabla\theta_{k-1}(t-s,x-y)|^p\,dy\,ds \right|$$
$$= \lim_{|x|\to\infty} \left| \int_0^t \int_{\mathbb{R}^d} \nabla\Psi(s,y)|\nabla\theta_{k-1}(t-s,x-y)|^p\,dy\,ds \right| = 0.$$

Similarly, utilizing Lemmas 3.4 and 3.5, we conclude that

$$\lim_{|x|\to\infty}\left|\int_0^t\int_{\mathbb{R}^d}\Psi(s,y)(-\Delta)^\alpha\theta_{k-1}(t-s,x-y)\,dy\,ds\right|$$
$$=\lim_{|x|\to\infty}\left|\int_0^t\int_{\mathbb{R}^d}\nabla\Psi(s,y)(-\Delta)^\alpha\theta_{k-1}(t-s,x-y)\,dy\,ds\right|=0,$$

meaning $\theta_k(t,\cdot)\in C_0^1(\mathbb{R}^d)$ for every $t>0$. By virtue of the convergence in the norm topology of X_{T_0}, we must have $\theta(t,\cdot)\in C_0^1\left(\mathbb{R}^d\right)$ for $t\in(0,T_0]$. □

Proposition 4.2. *Let $T_2\geq T_1$, $\theta_0\in W^{1,\infty}(\mathbb{R}^d)$, and suppose θ is a strong solution corresponding to θ_0 on $[T_1,T_2]$. It follows that*

$$\lim_{t\to T_1^+}\|\theta(t,\cdot)-\theta_0\|_{W^{1,\infty}}=0.$$

Furthermore, if $\theta_1\in W^{1,\infty}\left(\mathbb{R}^d\right)$ and φ is a strong solution corresponding to θ_1 on $[T_1,T_2]$, then

$$\|\theta(t,\cdot)-\varphi(t,\cdot)\|_{W^{1,\infty}}\leq\|\theta_0-\theta_1\|_{W^{1,\infty}}\,\gamma(t),\quad t\in[T_1,T_2],$$

where $\gamma\in C[T_1,T_2]$ is a positive, increasing function depending on α,p,ν,λ,μ, as well as the $L^\infty\left([T_1,T_2];W^{1,\infty}\left(\mathbb{R}^d\right)\right)$ norms of θ and φ.

Proof. From the uniform continuity of θ_0, it is clear that

$$\lim_{t\to T_1^+}\|\theta(t,\cdot)-\theta_0\|_{L^\infty}=0, \tag{4.8}$$

and so it remains to show that

$$\lim_{t\to T_1^+}\|\nabla\theta(t,\cdot)-\nabla\theta_0\|_{L^\infty}=0.$$

To do so, first of all notice that because $\theta\in C([T_1,T_2];W^{1,\infty}(\mathbb{R}^d))$, $\nabla\theta(t,x)$ converges to some vector $g(x)$ as $t\to T_1^+$ in the norm topology of $L^\infty(\mathbb{R}^d)$, so what must be shown is that $g(x)=$

$\nabla\theta_0(x)$ for almost every $x \in \mathbb{R}^d$. This can be done as follows: let $x_0 \in \mathbb{R}^d$ and $R > 0$ be arbitrary, and let χ_R be a smooth function compactly supported in a ball of radius R centered at x_0. Then we must have

$$\begin{aligned}
&\left| \int_{|x-x_0|\le R} (g(x) - \nabla\theta_0(x))\, \chi_R(x) dx \right| = \lim_{t\to T_1^+} \left| \int_{|x-x_0|\le R} (\nabla\theta(t,x) - \nabla\theta_0(x))\, \chi_R(x) dx \right| \\
&= \lim_{t\to T_1^+} \left| \int_{|x-x_0|\le R} (\theta(t,x) - \theta_0(x))\, \nabla\chi_R(x) dx \right| = 0,
\end{aligned}$$

owing to (4.8) and the fact that χ_R is compactly supported.

Now, let $w(t,x) := \theta(t,x) - \varphi(t,x)$, and notice that

$$\begin{aligned}
w(t,x) = &\int_{\mathbb{R}^d} \Psi(t-T_1,y) w_0(x-y) dy + \mu \int_{T_1}^t \int_{\mathbb{R}^d} \Psi(t-s,x-y) \left(-\Delta\right)^\alpha w(s,y) dy ds \\
&+ \lambda \int_{T_1}^t \int_{\mathbb{R}^d} \Psi(t-s,x-y) \left(|\nabla\theta(s,y)|^p - |\nabla\varphi(s,y)|^p\right) dy ds. \qquad (4.9)
\end{aligned}$$

Since $|a^p - b^p| \le p|a-b|(a^{p-1} + b^{p-1})$ when $p \in [1,\infty)$, we see that

$$\begin{aligned}
|w(t,x)| \le &\|w_0\|_{L^\infty} + \mu C_{d,\alpha} \int_{T_1}^t \|w(s,\cdot)\|_{L^\infty}^{1-2\alpha} \|\nabla w(s,\cdot)\|_{L^\infty}^{2\alpha}\, ds \\
&+ p|\lambda| \max_{s\in[T_1,T_2]} \left[\|\nabla\theta(s,\cdot)\|_{L^\infty}^{p-1} + \|\nabla\varphi(s,\cdot)\|_{L^\infty}^{p-1} \right] \int_{T_1}^t \|\nabla w(s,\cdot)\|_{L^\infty} ds \\
\le &\|w_0\|_{L^\infty} + A \int_{T_1}^t \|w(s,\cdot)\|_{W^{1,\infty}} ds, \qquad (4.10)
\end{aligned}$$

where

$$A := 2\left(\mu C_{d,\alpha} + p|\lambda| \max_{s\in[T_1,T_2]} \left(\|\nabla\theta(s,\cdot)\|_{L^\infty}^{p-1} + \|\nabla\varphi(s,\cdot)\|_{L^\infty}^{p-1} \right) \right).$$

Similarly, applying ∇ to (4.9) and using (4.2), we obtain

$$\begin{aligned}
|\nabla w(t,x)| \leq & \|\nabla w_0\|_{L^\infty} + \frac{A}{2\sqrt{\nu}} \int_{T_1}^t (t-s)^{-1/2} \|w(s,\cdot)\|_{L^\infty}^{1-2\alpha} \|\nabla w(s,\cdot)\|_{L^\infty}^{2\alpha} \, ds \\
& + \frac{A}{2\sqrt{\nu}} \int_{T_1}^t (t-s)^{-1/2} \|\nabla w(s,\cdot)\|_{L^\infty} \, ds \\
& \leq \|\nabla w_0\|_{L^\infty} + A\nu^{-1/2} \int_{T_1}^t (t-s)^{-1/2} \|w(s,\cdot)\|_{W^{1,\infty}} \, ds. \qquad (4.11)
\end{aligned}$$

Adding inequalities (4.10) and (4.11), while setting $g(t) := \|w(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^d)}$ and $f(\sigma) := A\left((\nu\sigma)^{-1/2} + 1\right)$ we obtain, for every $t \in [T_1, T_2]$,

$$g(t) \leq \int_{T_1}^t f(t-s) g(s) \, ds + \|w_0\|_{W^{1,\infty}} .$$

The result now follows from Lemma 4.1, by choosing, for instance, $q = 3, r = 3/2$. $\square$

Remark 4.1. A direct consequence of Propositions 4.1 and 4.2 is that if θ_0 is periodic or vanishes at infinity, and if θ is the strong solution corresponding to θ_0 on $[T_1, T_2]$, then $\theta(t,\cdot)$ is periodic or vanishes at infinity if θ_0 has one of these properties.

To avoid cumbersome notation, in Proposition 4.3 we work with strong solutions posed on $[0,T]$, without any loss in generality. Further, we write $a \lesssim b$ whenever there exists a constant $C > 0$, depending (possible nonlinearly) on $d, \alpha, p, \beta \in (0,1), \nu, \mu, \lambda$, and the $L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^d))$ norm of θ such that $a \leq Cb$ uniformly in $t \in [0,T]$. This notation is only used in the proof of Proposition 4.3.

Proposition 4.3. *Let $T > 0$ and suppose θ is the strong solution on $[0,T]$ corresponding to some $\theta_0 \in W^{1,\infty}$. Then*

$$|\nabla\theta(t,x) - \nabla\theta(t,z)| \lesssim \left(t^{-1/2(1+\beta)} + t^{1/2(1-\beta)}\right) |x-z|^\beta, \qquad (4.12)$$

where $\beta \in (0,1)$, $t \in (0,T]$ are arbitrary. Consequently, we get that $\partial_t\theta \in L^1\left([0,T]; L^\infty(\mathbb{R}^d)\right)$, $\theta \in C_t^1 C_x^2\left((0,T] \times \mathbb{R}^d\right)$, and

$$\partial_t\theta(t,x) - \nu\Delta\theta(t,x) - \lambda|\nabla\theta(t,x)|^p - \mu(-\Delta)^\alpha\theta(t,x) = 0,$$

holds true in the classical (pointwise) sense $\forall(t,x)\in(0,T]\times\mathbb{R}^d$.

Proof. We have

$$\nabla\theta(t,x) = \int_{\mathbb{R}^d}\nabla\Psi(t,x-y)\theta_0(y)dy + \lambda\int_0^t\int_{\mathbb{R}^d}\nabla\Psi(t-s,x-y)\,|\nabla\theta(s,y)|^p\,dyds$$
$$+\,\mu\int_0^t\int_{\mathbb{R}^d}\nabla\Psi(t-s,x-y)\,(-\Delta)^\alpha\,\theta(s,y)\,dy\,ds,$$

and so (4.12) follows from estimate (4.5), bound (3.9) with $\gamma=1$ from Lemma 3.4, and straightforward bounds.

To prove that $\partial_t\theta\in L^1\left([0,T];L^\infty(\mathbb{R}^d)\right)$, it is sufficient to show, for any $t_0\in[\epsilon,T/2]$,

$$\|\partial_t\theta(2t_0,\cdot)\|_{L^\infty}\lesssim t_0^{-1/2}+1, \tag{4.13}$$

with $\epsilon>0$ being arbitrary small. First of all, notice that by virtue of Proposition 4.2, we must have, for any $t\in[t_0,T]$,

$$\theta(t,x)=\varphi_0(t,x)+\varphi_1(t,x),$$

where

$$\varphi_0(t,x) := \int_{\mathbb{R}^d}\Psi(t-t_0,x-y)\theta(t_0,y)\,dy, \tag{4.14}$$
$$\varphi_1(t,x) := \mu\int_{t_0}^t\int_{\mathbb{R}^d}\Psi(t-s,x-y)\,(-\Delta)^\alpha\,\theta(s,y)\,dy\,ds$$
$$+\,\lambda\int_{t_0}^t\int_{\mathbb{R}^d}\Psi(t-s,x-y)\,|\nabla\theta(s,y)|^p\,dy\,ds. \tag{4.15}$$

Differentiating φ_0 once in time, integrating by parts, bounding $|\nabla\theta(t_0,y)|\leq\|\nabla\theta(t_0,\cdot)\|_{L^\infty}$ and using (4.2) we see that, for $t \in (t_0, T]$,

$$\|\partial_t \varphi_0(t,\cdot)\|_{L^\infty} \lesssim (t-t_0)^{-1/2}. \tag{4.16}$$

Since $\theta(t,\cdot)$ is Lipschitz, owing to Lemma 3.6, we must have that

$$|(-\Delta)^\alpha \theta(s,x) - (-\Delta)^\alpha \theta(s,y)| \lesssim |x-y|^{1-2\alpha}, \tag{4.17}$$

for $s \in [0,T]$ and $(x,y) \in \mathbb{R}^d$. Furthermore, estimate (4.12), along with $|a^p - b^p| \le p(a^{p-1} + b^{p-1})|a-b|$, imply that

$$||\nabla\theta(s,y)|^p - |\nabla\theta(s,x)|^p| \lesssim |x-y|^\beta \left(s^{-\frac{1}{2}(1+\beta)} + s^{1/2(1-\beta)}\right) \lesssim |x-y|^\beta \left(t_0^{-\frac{1}{2}(1+\beta)} + 1\right), \tag{4.18}$$

for $s \in [t_0, T]$, $(x,y) \in \mathbb{R}^d$, and $\beta \in (0,1)$. Hölder estimates (4.17)-(4.18) allow us to differentiate the volume potentials (4.15) once in time (see for instance [18]) to get

$$\begin{aligned}
\partial_t \varphi_1(t,x) =& \mu\,(-\Delta)^\alpha \theta(t,x) + \lambda\,|\nabla\theta(t,x)|^p \\
&+\mu \int_{t_0}^t \int_{\mathbb{R}^d} \partial_t \Psi(t-s,x-y)\left((-\Delta)^\alpha \theta(s,y) - (-\Delta)^\alpha \theta(s,x)\right) dy\, ds \\
&+ \lambda \int_{t_0}^t \int_{\mathbb{R}^d} \partial_t \Psi(t-s,x-y)\left(|\nabla\theta(s,y)|^p - |\nabla\theta(s,x)|^p\right) dy\, ds.
\end{aligned}$$

Choosing $\beta = 1-2\alpha$, bounding from above, and utilizing (4.3) we get

$$\begin{aligned}
|\partial_t \varphi_1(t,x)| &\lesssim \left(t_0^{\alpha-1} + 1\right) \int_{t_0}^t \int_{\mathbb{R}^d} |x-y|^{1-2\alpha} |\partial_t \Psi(t-s,x-y)| dy\, ds + 1 \\
&\lesssim \left(t_0^{\alpha-1} + 1\right)(t-t_0)^{\frac{1}{2}(1-2\alpha)} + 1.
\end{aligned} \tag{4.19}$$

From (4.16) and (4.19), we obtain (4.13). It is clear that φ_0 is smooth and solves the homogenous heat equation on $[t_0, T] \times \mathbb{R}^d$, while the Hölder estimates (4.17)-(4.18) allow us to differentiate the volume potentials (4.15) twice in space to conclude that $\varphi_1 \in C_t^1 C_x^2 \left([t_0, T] \times \mathbb{R}^d\right)$ and that

$$\partial_t \varphi_1(t,x) - \nu \Delta \varphi_1(t,x) = \mu \left(-\Delta\right)^{\alpha} \theta(t,x) + \lambda |\nabla \theta(t,x)|^p,$$

holds in the pointwise sense on $[t_0, T] \times \mathbb{R}^d$, with $t_0 \geq \epsilon > 0$ being arbitrary small. □

Proposition 4.4. *Suppose θ is the strong solution on $[T_1, T_2]$ corresponding to some $\theta_0 \in W^{1,\infty}$. If*

$$T_* := \sup \left\{ T \geq T_1 : \|\theta(t,\cdot)\|_{W^{1,\infty}\left(\mathbb{R}^d\right)} < \infty,\ \forall t \in [T_1, T] \right\},$$

is the maximal interval of existence of the strong solution, then

$$T_* = \sup \left\{ T \geq T_1 : \|\nabla \theta(t,\cdot)\|_{L^\infty} < \infty,\ \forall t \in [T_1, T] \right\}.$$

Proof. Set

$$T_0 := \sup \left\{ T \geq T_1 : \|\theta(t,\cdot)\|_{L^\infty} < \infty,\ \forall t \in [T_1, T] \right\},$$
$$\widetilde{T}_0 := \sup \left\{ T \geq T_1 : \|\nabla \theta(t,\cdot)\|_{L^\infty} < \infty,\ \forall t \in [T_1, T] \right\},$$

and suppose by way of contradiction that $T_0 < \widetilde{T}_0$. We must have

$$\limsup_{t \to T_0^-} \|\theta(t,\cdot)\|_{L^\infty} = \infty,$$

while for $t \in [T_1, T_0)$,

$$\theta(t,x) = \int_{\mathbb{R}^d} \Psi(t - T_1, y) \theta_0(x-y)\, dy + \lambda \int_{T_1}^t \int_{\mathbb{R}^d} \Psi(t-s, x-y) \left|\nabla \theta(s,y)\right|^p dyds$$
$$+ \mu \int_{T_1}^t \int_{\mathbb{R}^d} \Psi(t-s, x-y) \left(-\Delta\right)^{\alpha} \theta(s,y)\, dy\, ds.$$

Setting

$$A := \max_{t\in[T_1,T_0]} \|\nabla\theta(t,\cdot)\|_{L^\infty} < \infty$$

and bounding from above, while utilizing Lemma 3.4, we get, whenever $t \in [T_1, T_0)$,

$$\|\theta(t,\cdot)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + |\lambda| A^p (t - T_1) + \mu C_{d,\alpha} A^{2\alpha} \int_{T_1}^{t} \|\theta(s,\cdot)\|_{L^\infty}^{1-2\alpha}\, ds.$$

Since $1 - 2\alpha \in (0,1)$, we must have

$$\|\theta(s,\cdot)\|_{L^\infty}^{1-2\alpha} \leq (1-2\alpha)\, \|\theta(s,\cdot)\|_{L^\infty} + 2\alpha,$$

allowing us to conclude the proof by Gronwall's inequality. □

We conclude this section with a few comments. First, to prove an analogous result when $\alpha \in [1/2, 1)$, we can use the same technique. One way of proceeding is to require the initial data to be in $W^{2,\infty}(\mathbb{R}^d)$ and find a fixed point in the space $X_T := C\left([0,T]; W^{2,\infty}(\mathbb{R}^d)\right)$ while utilizing bound (3.10) instead of (3.9) in Lemma 3.4. The only part of the proof that has to be significantly changed is the regularity criterion (Proposition 4.4), and we could instead specialize to the periodic or whole space setting separately and work with energy estimates rather than pointwise.

The requirement that $\theta_0 \in W^{1,\infty}(\mathbb{R}^d)$ is not optimal: from Lemma 3.4 and the above proof, it shouldn't be too hard to work with $\theta_0 \in L^\infty \cap C^{0,\gamma}$ with $\gamma \in (2\alpha, 1]$. Of course we have to appropriately modify the definition of "strong solutions" along with the proof of Proposition 4.1 in order to make sense of the nonlinearity. In fact, using heat kernel properties, one can show that $\|\nabla\theta(t,\cdot)\|_{L^\infty} \lesssim t^{(\gamma-1)/2}[\theta_0]_{C^{0,\gamma}}$ for t close to 0. We only need to make sure that the initial data has sufficiently high regularity to treat the nonlinear equation as a perturbation of the heat equation. If a nonlinear evolution equation is invariant under some scaling, a general rule of thumb is that one should expect a good local well-posedness theory in spaces with norms that are invariant (critical) or are monotone (subcritical) with respect to the scaling. Since equation (2.7) with $\mu = 0$ does have a scale invariance, it is natural to expect a good local well-posedness theory in spaces that respect such invariance. Indeed, the nonlocal term is linear and of order less than that of dissipation, so it is not expected to dramatically change the local theory. We do not pursue that direction here.

4.2 Proof of Theorem 2.2

4.2.1 Strategy of the Proof.

By Theorem 2.1 and Proposition 4.4, we only need to show that $\|\nabla\theta(t,\cdot)\|_{L^\infty} < \infty$ for any $t \geq 0$. This will be achieved by constructing a time dependent strong modulus of continuity (an $\Omega(t,\xi)$ such that $\Omega(t,\cdot)$ is a strong modulus of continuity for any $t \geq 0$ according to Definition 3.1) such that $\theta(t,\cdot)$ has $\Omega(t,\cdot)$ as a strict modulus of continuity for all $t \geq 0$, thereby ruling out gradient blowup. As a byproduct, we are able to obtain an explicit bound on the gradient in terms of $\partial_\xi\Omega(t,0)$.

Time dependent moduli of continuity have been studied before in [26], mainly in the context of eventual regularization of active scalars. Hölder time dependent moduli of continuity were also considered in [55], where a drift-diffusion equation with a pressure term was considered, and the solution was shown to remain Hölder continuous as long as the drift velocity is under control. Following (and slightly generalizing) the work in [26], a time dependent modulus $\Omega(t,\xi)$ will be constructed such that the initial data has strict modulus of continuity $\Omega(0,\cdot)$ and

$$\partial_t\Omega(t,\xi) - 4\nu\partial_\xi^2\Omega(t,\xi) > h(t,\xi), \quad (t,\xi) \in (0,\infty)\times\mathbb{R}^+, \tag{4.20}$$

where h will represent any "local" instabilities that may arise from the nonlocal and nonlinear part of the equation. Here, h may depend linearly, nonlinearly and/or nonlocally on Ω.

As will be shown below, the nonlinear term will not be of any concern and will in fact vanish; we only need to treat the nonlocal term. As is easily observed for the linear equation (that is, equation (2.7) with $\lambda = 0$), this will cause at most exponential growth in time, but not blowup. As opposed to [26], where $h(t,\xi)$ is a "nonlocal" Burgers type nonlinear term, in our case, since the nonlinearity will vanish, $h(t,\xi)$ is a linear term in Ω, allowing us to solve the "heat inequality"

(4.20) by a simple separation of variables. That is, we seek a modulus of continuity of the form

$$\Omega(t,\xi) = f(t)\omega(\xi).$$

In our case,

$$h(t,\xi) = C_{d,\alpha} \int_0^{\xi} \frac{\partial_\eta \Omega(t,\eta)}{\eta^{2\alpha}}\, d\eta,$$

and owing to the fact that

$$\lim_{\xi \to 0^+} \partial_\xi^2 \Omega(t,\xi) = -\infty,$$

we see that the local dissipation from the Laplacian (the term $-4\nu\partial_\xi^2\Omega$) will balance h when ξ is small. Time dependence on the other hand is necessary, since the above integral cannot be made to vanish as $\xi \to \infty$, while local dissipation from the Laplacian must go to 0 as $\xi \to \infty$; otherwise the modulus of continuity will become negative for large ξ. Therefore, we need to rely on the time derivative term $\partial_t\Omega$ to neutralize those instabilities when ξ is large. This will be clear in §4.2.2 below.

Before constructing the modulus of continuity, let us recall the main ideas introduced in [28,29] and slightly modify them for problem (2.7). Let us suppose that $\Omega(t,\cdot)$ is an unbounded strong modulus of continuity for each $t \geq 0$, and assume that θ_0 has $\Omega(0,\xi)$ as a strict modulus of continuity. Furthermore, suppose that $\Omega \in C\left([0,\infty) \times [0,\infty)\right)$, and that $\Omega(\cdot,\xi)$ is non-decreasing as a function of time for each $\xi \in [0,\infty)$. Let us now define

$$T_* := \sup\left\{T \geq 0 : \|\theta(t,\cdot)\|_{W^{1,\infty}} < \infty,\ \forall t \in [0,T]\right\}, \tag{4.21}$$

$$\tau := \sup\left\{T \geq 0 : |\theta(t,x) - \theta(t,y)| < \Omega\left(t,|x-y|\right),\ \forall t \in [0,T],\ x \neq y\right\}, \tag{4.22}$$

where $x,y \in \mathbb{R}^d \times \mathbb{R}^d$ are arbitrary, and assume for the moment that $\tau > 0$. It is clear that if $T_* < \infty$, then $\tau < T_*$: by virtue of Proposition 4.4, θ must exhibit gradient blowup at T_*, while the fact that $\partial_\xi\Omega(t,0) < \infty$ would lead to a uniform bound for the gradient on the interval $[0,T_*]$. It follows that by continuity, $\theta(\tau,\cdot)$ has $\Omega(\tau,\cdot)$ as a modulus of continuity, albeit not necessarily strict. The idea is then to construct Ω such that if $\tau > 0$, then $\tau = \infty$, contradicting the fact that $T_* < \infty$ and providing the explicit bound

$$\|\nabla\theta(t,\cdot)\|_{L^\infty} < \partial_\xi\Omega(t,0),\ \forall t \geq 0.$$

To show that $\tau = \infty$, it will be sufficient to rule out the "breakthrough" scenario

$$|\theta(\tau,x) - \theta(\tau,y)| = \Omega(\tau,|x-y|), \quad \text{any } x \neq y. \tag{4.23}$$

Indeed, suppose scenario (4.23) is not possible. As $\tau < T_*$, the solution is still C^2 in space at time τ, and for a short time beyond that. It follows that Lemma 3.2 is applicable, and so $\|\nabla\theta(\tau,\cdot)\|_{L^\infty} < \partial_\xi\Omega(\tau,0)$. This guarantees that the strict modulus of continuity can never be violated in a neighborhood of the diagonal $x = y$, and so the same must be true for a short time beyond time τ and small $|x-y|$, say $|x-y| \in (0,\delta)$ some $\delta > 0$. Since the solution is bounded in space in a neighborhood of time τ, while Ω is unbounded in space and nondecreasing in time, we know that the strict modulus of continuity is not violated for a short time beyond τ and large $|x-y|$, say $|x-y| \geq K$ for some $K > \delta$. The only concern then is extending the time τ when $|x-y| \in [\delta, K]$ without assuming any bound on x or y. This can be done under the assumption that the solution is either periodic or vanishes at spatial infinity (both properties which are inherited from the initial data, Remark 4.1). Let us now make this rigorous. No C^2 or concavity assumptions on $\Omega(t,\cdot)$ are necessary for the proof of the next Proposition (see also [26, Lemma 2.3]).

Proposition 4.5. *Suppose $\Omega \in C([0,\infty) \times [0,\infty))$ is such that $\Omega(t,\cdot)$ is an unbounded strong modulus of continuity for each $t \geq 0$, and that $\Omega(\cdot,\xi)$ is nondecreasing as a function of time for each $\xi \geq 0$. Suppose θ_0 has $\Omega(0,\cdot)$ as a strict modulus of continuity, and let θ be the (short-time) strong solution to* (2.7) *corresponding to θ_0. Assume further that either θ_0 is periodic with period $L > 0$ or vanishes at infinity, and let T_* and τ be as defined in* (4.21) *and* (4.22)*, respectively. Then $\tau > 0$ and if $\tau < \infty$, we must have $|\theta(\tau,x) - \theta(\tau,y)| = \Omega(\tau,|x-y|)$ for some $x \neq y$.*

Proof. By virtue of Theorem 2.1 we may, without any loss in generality, assume $\theta_0 \in C^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. That the second derivative is bounded follows from $\partial_t \theta(t,\cdot) \in L^\infty(\mathbb{R}^d)$ and the fact that we have a classical solution. In this case, Lemma 3.2 tells us that $\|\nabla\theta_0\|_{L^\infty} < \partial_\xi \Omega(0,0)$, and by continuity of the function $\|\nabla\theta(t,\cdot)\|_{L^\infty}$ this remains true for $t \in [0,\epsilon_0]$, some $\epsilon_0 > 0$. Set

$$M_0 := \max_{t\in[0,\epsilon_0]} \|\nabla\theta(t,\cdot)\|_{L^\infty} < \partial_\xi \Omega(0,0),$$
$$M_1 := \max_{t\in[0,\epsilon_0]} \|\theta(t,\cdot)\|_{L^\infty},$$

and for $\xi > 0$, consider the function

$$h(\xi) := M_0 - \frac{\Omega(0,\xi)}{\xi}.$$

Clearly, $h(\xi) < 0$ for $\xi \in (0,\delta)$, some $\delta > 0$. It follows that for $t \in [0,\epsilon_0]$ and $|x-y| \in (0,\delta)$, since $\Omega(\cdot,\xi)$ is nondecreasing as a function of time for each fixed $\xi \geq 0$, we must have

$$|\theta(t,x) - \theta(t,y)| \leq M_0|x-y| < \Omega(t,|x-y|).$$

As $\Omega(0,\cdot)$ is unbounded and nondecreasing, there exists some $K >> \delta$ such that $\Omega(0,\xi) \geq 3M_1$ when $\xi \geq K$. It follows that for $t \in [0,\epsilon_0]$ and $|x-y| \in [K,\infty)$,

$$|\theta(t,x) - \theta(t,y)| \leq 2M_1 < 3M_1 \leq \Omega(0,|x-y|) \leq \Omega(t,|x-y|).$$

It remains to handle the case $|x-y| \in [\delta, K]$. If θ_0 is $L-$ periodic, then so is $\theta(t,\cdot)$ (Remark 4.1) and in this case, we first define

$$\mathcal{A} := \left\{(x,y) \in [0,L]^d \times \mathbb{R}^d : |x-y| \in [\delta,K]\right\},$$

and note that since the set $[0, \epsilon_0] \times \mathcal{A}$ is compact, the function

$$R(t, x, y) := |\theta(t, x) - \theta(t, y)| - \Omega(t, |x - y|),$$

is uniformly continuous on it. Since $R(0, x, y) < 0$, the same must be true on $[0, \epsilon] \times \mathcal{A}$ for some $\epsilon \in (0, \epsilon_0]$. As $\theta(t, \cdot)$ is $L-$ periodic, this proves that $\tau \geq \epsilon > 0$.

On the other hand, if θ_0 vanishes at spatial infinity, one can chose a large enough K_0 and a small enough $\epsilon_1 \in (0, \epsilon_0]$ such that $|\theta(t, z)| \leq \Omega(0, \delta)/4$ when $|z| \geq K_0$, $t \in [0, \epsilon_1]$, owing to the fact that $\partial_t \theta \in L^1([0, T_*]; L^\infty(\mathbb{R}^d))$. We now decompose the set

$$\mathcal{B} := \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \in [\delta, K]\right\}$$

into $\mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_1 := \{(x, y) \in \mathcal{B} : \min\{|x|, |y|\} > K_0\},$$

and $\mathcal{B}_2 := \mathcal{B} \backslash \mathcal{B}_1$ is the complement of the set $\mathcal{B}_1$. By choice of K_0 and ϵ_1, we have

$$|\theta(t, x) - \theta(t, y)| < \Omega(0, \delta) \leq \Omega(t, |x - y|), \ (t, x, y) \in [0, \epsilon_1] \times \mathcal{B}_1,$$

since Ω is nondecreasing in both variables. It is fairly straightforward to verify that $\mathcal{B}_2$ is compact, and so as in the periodic case one can find a small enough $\epsilon \in (0, \epsilon_1]$ such that Ω is not violated on $[0, \epsilon]$.

The second part of the proposition follows by similar arguments. Because the solution has not exhibited any blowup on $[0, \tau]$, $\theta(\tau, \cdot)$ has modulus of continuity $\Omega(\tau, \cdot)$, albeit not necessarily strict. Therefore, Lemma 3.2 can still be applied and we have the strict bound $\|\nabla \theta(\tau, \cdot)\|_{L^\infty} < \partial_\xi \Omega(\tau, 0)$. Since $\theta(\tau, \cdot) \in W^{1,\infty}\left(\mathbb{R}^d\right)$, by Theorem 2.1 and Remark 4.1, the solution is smooth for a short time beyond τ, and is periodic or vanishes at infinity if θ_0 is. Therefore, assuming that

$$|\theta(\tau, x) - \theta(\tau, y)| < \Omega(\tau, |x - y|), \ \forall x \neq y,$$

we may repeat the above argument and prolong the time τ, by some $\epsilon > 0$, contradicting the definition of τ. Hence, if $\tau < \infty$, we must have

$$|\theta(\tau, x) - \theta(\tau, y)| = \Omega(\tau, |x - y|),$$

for some $x \neq y$. □

4.2.2 Constructing the Modulus of Continuity.

We start by analyzing the breakthrough scenario described in Proposition 4.5, i.e. we assume τ as defined in (4.22) is positive and finite, so that

$$|\theta(\tau, x^0) - \theta(\tau, y^0)| = \Omega(\tau, |x^0 - y^0|),$$

for some $x^0 \neq y^0$. Because of rotation and translation invariance, we may assume that the strict modulus of continuity is violated at some $x^0 \neq y^0$, with $x^0 - y^0 = (\xi, 0, \cdots, 0)$ for some $\xi > 0$. Further, it is sufficient to assume

$$\theta(\tau, x^0) - \theta(\tau, y^0) = \Omega(\tau, \xi);$$

the case when $\theta(\tau, x^0) < \theta(\tau, y^0)$ is handled similarly. To rule out this scenario, we consider the function

$$g(t) := \theta(t, x^0) - \theta(t, y^0) - \Omega(t, \xi),$$

on the interval $[0, \tau + \epsilon]$, for some small enough ϵ, and we construct Ω such that $g'(\tau) < 0$. Since we do not know what the value of $\xi > 0$ is, we would need to ensure that $g'(\tau) < 0$ for any $\xi > 0$ and any $\tau > 0$. Furthermore, we also need to guarantee that θ_0 strictly obeys $\Omega(0, \cdot)$. To do so, we start by obtaining a bound on $g'(\tau)$ by using the fact that the PDE holds pointwise to get that

$$\begin{aligned} g'(\tau) =& \nu\left(\Delta\theta(\tau,x^0)-\Delta\theta(\tau,y^0)\right)-\partial_t\Omega(\tau,\xi) \\ &+\lambda\left|\nabla\theta(\tau,x^0)\right|^p-\lambda\left|\nabla\theta(\tau,y^0)\right|^p+\mu\left(-\Delta\right)^\alpha\theta(\tau,x^0)-\mu\left(-\Delta\right)^\alpha\theta(\tau,y^0). \end{aligned} \tag{4.24}$$

Terms on the first line of equation (4.24) are stabilizing, while those on the second line may cause instabilities. From (3.2), we see that

$$\left|\nabla\theta(\tau,x^0)\right|^p-\left|\nabla\theta(\tau,y^0)\right|^p=0,$$

while (3.3) and (3.12) give us

$$\begin{aligned} &\nu\left(\Delta\theta(\tau,x^0)-(\Delta\theta(\tau,y^0)\right)+\mu(-\Delta)^\alpha\theta(\tau,x^0)-\mu(-\Delta)^\alpha\theta(\tau,y^0) \\ &\leq 4\nu\partial_\xi^2\Omega(\tau,\xi)+\mu C_{d,\alpha}\int_0^\xi\frac{\partial_\eta\Omega(\tau,\eta)}{\eta^{2\alpha}}\,d\eta. \end{aligned}$$

Therefore we obtain

$$g'(\tau)\leq 4\nu\partial_\xi^2\Omega(\tau,\xi)-\partial_t\Omega(\tau,\xi)+\mu C_{d,\alpha}\int_0^\xi\frac{\partial_\eta\Omega(\tau,\eta)}{\eta^{2\alpha}}\,d\eta.$$

Our aim now is to construct an Ω that satisfies the hypothesis of Proposition 4.5 and for which

$$\partial_t\Omega(t,\xi)-4\nu\partial_\xi^2\Omega(t,\xi)-\mu C_{d,\alpha}\int_0^\xi\frac{\partial_\eta\Omega(t,\eta)}{\eta^{2\alpha}}\,d\eta>0, \tag{4.25}$$

for every $(t,\xi)\in(0,\infty)\times(0,\infty)$. To do that, we start by defining

$$\omega(\xi):=\frac{\xi}{1+\xi^{1-\alpha}},\quad \xi\geq 0.$$

Clearly, ω is an unbounded strong modulus of continuity: it is concave, grows like ξ^α, $\omega'(0)=1$ and $\omega''(\xi)=-O(\xi^{-\alpha})$ near $\xi=0$. Next, we choose a sufficiently large $B=B\left(\|\theta_0\|_{W^{1,\infty}}\right)$

such that θ_0 has $\omega_B(\xi) := \omega(B\xi)$ as a strict modulus of continuity (owing to Lemma 3.1), and let $\delta_0 = \delta_0(B, \nu, \alpha, \mu, d) > 0$ be a small number to be determined later. Set

$$C_0 := \frac{\mu C_{d,\alpha}}{\omega_B(\delta_0)} \left(\frac{B^{2\alpha-1}}{\delta_0} + \frac{B^\alpha}{\delta_0^\alpha} + \frac{B\delta_0^{1-2\alpha}}{(1-2\alpha)} \right),$$

and let $f(t) = \exp(C_0 t)$, which solves

$$f'(t) - C_0 f(t) = 0, \quad f(0) = 1. \tag{4.26}$$

Finally, define

$$\Omega(t,\xi) := f(t)\omega_B(\xi), \quad (t,\xi) \in [0,\infty) \times [0,\infty),$$

and note that Ω satisfies the hypothesis of Proposition 4.5. Now, δ_0 will be chosen small enough that the dissipative term alone will balance the local instabilities arising from the nonlocal part for $\xi \in (0, \delta_0]$, while the time dependent part of Ω will balance those instabilities away from δ_0. To see this, as ω is concave, we have $\omega_B'(\xi) \leq B$, and so

$$\int_0^\xi \frac{\partial_\eta \Omega(t,\eta)}{\eta^{2\alpha}} \, d\eta \leq f(t)(1-2\alpha)^{-1} B \xi^{1-2\alpha},$$

and as $\lim\limits_{\xi \to 0^+} \omega_B''(\xi) = -\infty$, one can choose a $\delta_0 = \delta_0(B, \nu, \alpha, \mu, d) > 0$ such that

$$4\nu\omega_B''(\xi) + \mu C_{d,\alpha}(1-2\alpha)^{-1} B \xi^{1-2\alpha} < 0$$

when $\xi \in (0, \delta_0]$. A straightforward calculation yields that

$$\delta_0 \leq \min \left\{ B^{-1}, \left(\frac{\nu(1-2\alpha)}{8\mu C_{d,\alpha}} \right)^{\frac{1}{(1-\alpha)}} \right\}.$$

This immediately implies that (4.25) is true for any $(t, \xi) \in [0, \infty) \times (0, \delta_0]$, since f is positive and nondecreasing. When $\xi \geq \delta_0$ our aim is to bound (3.12) uniformly in ξ, and so we use the bound

$$\omega_B'(\eta) \leq B^{2\alpha-1}\eta^{2\alpha-2} + \alpha B^\alpha \eta^{\alpha-1},$$

to get

$$\int_{\delta_0}^{\xi} \frac{\partial_\eta \Omega(t,\eta)}{\eta^{2\alpha}}\, d\eta \leq f(t)\left(\frac{B^{2\alpha-1}}{\delta_0} + \frac{B^\alpha}{\delta_0^\alpha}\right).$$

Therefore, for $(t, \xi) \in [0, \infty) \times (\delta_0, \infty)$, because ω_B is concave, we can bound the left-hand side of (4.25) from below by

$$f'(t)\omega_B(\delta_0) - f(t)\mu C_{d,\alpha}\left(\frac{B^{2\alpha-1}}{\delta_0} + \frac{B^\alpha}{\delta_0^\alpha} + \frac{B\delta_0^{1-2\alpha}}{(1-2\alpha)}\right),$$

making (4.25) true for $(t, \xi) \in [0, \infty) \times (\delta_0, \infty)$ by choice of f (4.26). This concludes the proof of Theorem 2.2.

5. FURTHER REMARKS ON THE COMBUSTION MODEL*

Let us start by commenting on the exponential growth observed in bound (2.9). As opposed to the scenario in the SQG and critical Burgers equation analyzed in [28, 29], the instabilities in our case manifest themselves in estimate (3.12), which cannot be made to decay in ξ. This is the main technical difficulty that forces us to allow the modulus to depend on time, as the best we could do is construct a modulus such that (3.12) is bounded. Therefore, if we do not "absorb" that term by a function of time, one possibility would be to require the concavity of the modulus to be bounded from above by some fixed negative constant, since this will be the only positive quantity in inequality (4.25). But this immediately would imply that at some large enough ξ, the modulus would become decreasing, and in fact negative at even larger ξ. We might be able to overcome this in the periodic setting by constructing a more sophisticated modulus, since we only need to rule out the "breakthrough" scenario for ξ in some compact set.

A possible approach to prove that regularity persists under evolution when $\alpha = 1/2$, and to eliminate time dependence in (2.9), is the following. Recall that the main difficulty in studying evolution of moduli of continuity under the original MS model (2.5) is the lack of pointwise control of $(-\Delta)^{1/2}\theta$. However, one can bootstrap control of the Lipschitz constant and obtain, via energy techniques, a bound on a high enough Sobolev norm. Owing to the Sobolev embedding theorem, we obtain a pointwise bound or even a continuity (Hölder) estimate for the term $(-\Delta)^{1/2}\theta$ in (4.24). Nevertheless, the time dependent part of the modulus will now have to satisfy a first order ODE whose solution blows up in finite time, rendering the separation of variables approach useless.

On the other hand, Lemma 3.2 is still valid for moduli of continuity of the form

$$\Omega(t,\xi) := \begin{cases} \omega_{\mathcal{L}}(\xi), & \xi \in [0,\delta], \\ \omega_{\mathcal{R}}(t,\xi), & \xi \in (\delta,\infty), \end{cases}$$

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allowing for uniform in time control over the Lipschitz constant. However, solving the heat equation by a simple separation of variables for the right part of the modulus, $\omega_{\mathcal{R}}$, now results in a jump discontinuity at δ, introducing various technical difficulties in the proof. In a discussion of the above remarks and results with my doctoral adviser Edriss Titi, the latter suggested that in order to adapt the approach for the MS model (2.5), we should instead try to solve the heat equation implicitly to patch the break in the modulus at δ [62]. That is, we should solve a boundary value problem for $\omega_{\mathcal{R}}$. Ideally, we would want the modulus to be at least C^1 in space, and so this amounts to prescribing Cauchy data to the forced heat equation at $\xi = \delta$, resulting in an overdetermined problem; the so called "lateral Cauchy problem". That is, we seek to find a solution to

$$\begin{cases} \partial_t \omega_{\mathcal{R}}(t,\xi) - 4\partial_\xi^2 \omega_{\mathcal{R}}(t,\xi) \geq C_{T,\omega'_{\mathcal{L}}(0)}\xi^\beta, & \forall (t,\xi) \in [0,T]\times(\delta,\infty), \\ \omega_{\mathcal{R}}(t,\delta) = \omega_{\mathcal{L}}(\delta), & \forall t \in [0,T], \\ \partial_\xi \omega_{\mathcal{R}}(t,\delta) = \omega'_{\mathcal{L}}(\delta), & \forall t \in [0,T], \\ \omega_{\mathcal{R}}(0,\xi) = \omega_0(\xi), & \forall \xi \in [\delta,\infty). \end{cases}$$

Such equation has a solution of the form

$$\omega_{\mathcal{R}}(t,\xi) = \omega_{\mathcal{T}}(t,\xi) + \omega_{\mathcal{H}}(t,\xi),$$

where

$$\begin{cases} \partial_t \omega_{\mathcal{T}}(t,\xi) - 4\partial_\xi^2 \omega_{\mathcal{T}}(t,\xi) \geq C_{T,\omega'_{\mathcal{L}}(0)}\xi^\beta, & \forall (t,\xi) \in [0,T]\times(\delta,\infty), \\ \omega_{\mathcal{T}}(t,\delta) = \omega_{\mathcal{L}}(\delta), & \forall t \in [0,T], \\ \omega_{\mathcal{T}}(0,\xi) = \omega_0(\xi), & \forall \xi \in [\delta,\infty), \end{cases}$$

and

$$\begin{cases} \partial_t \omega_{\mathcal{H}}(t,\xi) - 4\partial_\xi^2 \omega_{\mathcal{H}}(t,\xi) = 0, & \forall (t,\xi) \in [0,T]\times(\delta,\infty), \\ \omega_{\mathcal{H}}(t,\delta) = 0, & \forall t \in [0,T], \\ \partial_\xi \omega_{\mathcal{H}}(t,\delta) = \omega'_{\mathcal{L}}(\delta) - \partial_\xi \omega_{\mathcal{T}}(t,\delta), & \forall t \in [0,T]. \end{cases} \tag{5.1}$$

Notice that problem (5.1) has no initial condition: the Cauchy data is prescribed on the line $\xi = \delta$ rather than $t = 0$. A solution to the (5.1) could be written down in the form

$$\omega_{\mathcal{H}}(t, \xi) = \sum_{n=0}^{\infty} f_n(t)(\xi - \delta)^{2n+1},$$

where the functions f_n are defined recursively by plugging this into the equation and taking care of the boundary conditions. Such analytic solutions were considered by Tychonov to prove non-uniqueness of the Cauchy problem related to heat propagation in the whole space without restricting the growth at spatial infinity [25].

The main technical difficulty in solving this so called lateral Cauchy problem is the lack of a minimum principle, in particular one can no longer guarantee positivity of the modulus of continuity. To solve this, it was also recommended by Titi to relax one of the boundary conditions instead of trying to solve the overdetermined problem. That being said, it is natural to relax the Neumann condition in order to guarantee concavity of the modulus of continuity, at least near 0. This is remaining faithful to the spirit of the ideas presented in [26–29], mainly in order to be able to deal with the nonlinearity and extract dissipation at δ (apply Lemma 3.3). In simple words, this translates to showing that the Dirichlet to Neumann map is not increasing in time, at least on an arbitrary interval of time $[0, T]$. Such a technique would also work for other equations that include the incompressible NSE. However, we were not able to successfully execute this strategy.

In light of trying to address the case when $\alpha = 1/2$, we briefly discuss the recent work of Miao and Xue [41] (building upon [26,27]). In [41], the authors analyzed the following one-dimensional dissipative-dispersive perturbation of Burgers equation

$$\partial_t u + (-\Delta)^{\gamma/2} u = u \partial_x u + L_\beta u, \quad \gamma \in [\beta, 2], \tag{5.2}$$

with L_β being an operator of order at most β with an odd kernel. For the main results of their work, they assumed $\beta \in (0, 1)$. However a key estimate that was derived (and utilized for the case when $\gamma = 1$) in [41] is the following bound: for any $\gamma \in [\beta, 2)$ (including $\beta = 1$), if $\xi := |x - y|$ and

$u(t,x) - u(t,y) = \Omega(t,\xi)$, then for some positive $C > 0$ depending on γ and β,

$$L_\beta u(t,x) - L_\beta u(t,y) \le -C\xi^{\gamma-\beta} D_\gamma + C\xi \int_\xi^\infty \frac{\Omega(t,\eta)}{\eta^{2+\beta}} d\eta + C\frac{\Omega(t,\xi)}{\xi^\beta}, \tag{5.3}$$

where $D_\gamma = (-\Delta)^{\gamma/2}u(t,y) - (-\Delta)^{\gamma/2}u(t,x)$. Notice that, as in Lemma 3.3 in this paper, D_γ is strictly negative. The estimate (5.3) is indeed surprising when $\beta = 1$: one should not expect a continuity estimate on Lu from u when L is an operator of order 1 or higher. Estimate (5.3) does not violate this general rule, since it is valid only when we are at the breakthrough scenario depicted by Proposition 4.5, and not for any pair (x, y). Since we only care about the breakthrough scenario in our analysis, it is natural to ask if we can make use of such an estimate by replacing classical dissipation with fractional and then adopting the approach of [41].

Unfortunately, we do not believe this can be done, regardless of whether dissipation is fractional or classical. Indeed, the strategy of [41] is to upgrade the regularity in steps: from $L_t^\infty L_x^2$ to $L_t^\infty L_x^\infty$ to $L_t^\infty C_x^{0,\delta}$, for some $\delta \in (0,1]$, with the propagation of moduli of continuity (and utilizing bound (5.3)) being applied in going from $L_t^\infty L_x^\infty$ to $L_t^\infty C_x^{0,\delta}$. If we ignore the term L_β, we end up with the critically dissipative fractional Burgers equation when $\gamma = 1$, and so the bound $L_t^\infty L_x^\infty$ is not sufficient to deduce regularity, which is why the authors of [41] needed to go from $L_t^\infty L_x^\infty$ to $L_t^\infty C_x^{0,\delta}$. As a consequence of the $L_t^\infty L_x^\infty$ bound, it is only necessary to rule out the equality $u(t,x) - u(t,y) = \Omega(t,\xi)$ for ξ in some bounded set, and by carefully constructing a stationary (independent of time) Ω that depends on the $L_t^\infty L_x^\infty$ bound, the authors were able to absorb the term $\xi^{\gamma-\beta} D_\gamma$ in the viscous one.

We do not have such luxury here, in particular, there is no a-priori $L_t^\infty L_x^2$ bound to bootstrap to $L_t^\infty L_x^\infty$ as in [41, Lemma 2.6] (even if we consider the evolution of $u = \nabla\theta$ and $p = 2$ to end up with the transport version of Burgers equation in multi-dimensions). In fact, one of the reasons that we were interested in the MS model is the lack of any obvious a-priori bounds. Unless the approach of [41] can be modified to bypass the $L_t^\infty L_x^2$ estimate, it does not seem to be applicable here. Actually, if the proof of [41, Lemma 2.6] can be modified to bypass the energy estimate, we can probably apply it directly to the evolution of $u = \nabla\theta$ to obtain the required estimate in our scenario without having to propagate moduli of continuity. In fact, using similar arguments utilized in deducing regularity to the subcritical Burgers and SQG equation from the $L_t^\infty L_x^\infty$ bound, it will also be possible to deduce global regularity for the model

$$\partial_t u + (-\Delta u)^{\gamma/2} = (u \cdot \nabla)u + (-\Delta)^{1/2}u, \quad \gamma \in (1, 2),$$

even when u is not conservative ($u \neq \nabla\theta$), if one can improve on [41, Lemma 2.6] as described above. This is because the moment we get an L^∞ bound, we will be able to close the H^s energy estimates (provided $\gamma > 1$) by standard product and interpolation inequalities, and the linear, nonlocal one would not introduce any noteworthy difficulties (again, as long as $\gamma > 1$).

So the next question is whether bound (5.3) can be utilized directly in studying the evolution of Ω, keeping in mind that we need to rule out the breakthrough scenario for $\xi \in (0, \infty)$ and not just in some bounded set, due to the lack of an $L_t^\infty L_x^\infty$ bound. The viscous term (fractional or classical) is very powerful over small distances, so the difficulty is when $\xi \in (1, \infty)$. The term $-C\xi^{\gamma-\beta} D_\gamma$ cannot be absorbed by the viscous one for ξ in this region (even with fractional dissipation, D_γ). It also cannot be absorbed by $\partial_t\Omega$, since we need an *upper bound* on $-D_\gamma$ in terms of Ω with $\gamma > 1$, which is not available (what we have is a lower bound on $-D_\gamma$, see [26]). That is to say, we are back to square one: lack of a continuity estimate on an operator of order one or higher. That being said, a possible scenario where one can use bound (5.3) to address the case $\alpha = 1/2$ in our work is if we replace standard dissipation with fractional, restrict ourselves to the periodic setting, and impose a smallness condition on the period. In this scenario, although we still do not have an $L_t^\infty L_x^\infty$ bound, periodicity implies that we only need to worry about $\xi \in (0, \kappa]$, where κ would depend on the period. Then we can use bound (5.3) with $\beta = 1$ and $\gamma \in (1, 2)$ to show that if the period is small enough, we can absorb the term $-C\xi^{\gamma-\beta} D_\gamma$ in the dissipative one, provided the latter comes from $(-\Delta)^{\gamma/2}$. The remaining terms can be handled by the time derivative if necessary. We also want to point out that the fractional Laplacian does not have an odd kernel, an assumption that was made on the operator L_β in [41]. But this probably is not the main issue at hand: one only has to verify that all the key estimates do not rely on this cancellation property (which they probably do not). The main issue is the lack of energy bounds.

While we are on the topic of fractional dissipation, our final remark is that one would expect results analogous to those obtained in this work to hold even if the dissipative operator, $(-\Delta)$ is replaced by a fractional one $(-\Delta)^{\gamma/2}$, where $\gamma \in (1,2]$. Indeed, it was shown in [26], that the local dissipative power of $(-\Delta)^{\gamma/2}$ for small ξ is, roughly speaking (note that in [26], the power $\gamma/2$ is replaced by γ),

$$\xi^{2-\gamma}\omega''(\xi),$$

and as long as $\gamma \in (1,2]$, one can still construct a modulus of continuity, according to Definition 3.1, such that

$$\lim_{\xi \to 0^+} \xi^{1-\gamma+2\alpha}\omega''(\xi) = -\infty.$$

To obtain a local well-posedness result and regularity criteria in terms of $\|\nabla\theta\|_{L^\infty}$ in this case, one should be able to adapt the ideas in [15, 24, 54] and the references therein.

6. DRIFT-DIFFUSION SYSTEMS

6.1 Incompressible NSE

As mentioned in Chapter 1, we only describe the main results obtained in [22] and outline the key ideas. We turn our attention to drift-diffusion systems, and the incompressible Navier-Stokes system in particular:

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = (u\cdot\nabla)u(t,x) + \nabla p(t,x), & \forall (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ \nabla\cdot u(t,x) = 0, & \forall (t,x)\in[0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x), & u_0\in C^\infty(\mathbb{R}^d). \end{cases} \tag{6.1}$$

One can develop vector-field analogue of the previous analysis and track the evolution of moduli of continuity by vector-fields. That is, we assume that the solution strictly obeys Ω on some time interval $[0,\tau)$ and that at time τ

$$|u(\tau,x_0) - u(\tau,y_0)| = \Omega(\tau, |x_0-y_0|), \quad x_0\neq y_0.$$

Using the rotation invariance of (6.1), we may without any loss in generality assume that the first component of the vector u breaks the strict inequality

$$|u_1(\tau,x) - u_1(\tau,y)| < \Omega(\tau,|x-y|),$$

at some $x_0 - y_0 = \xi e_1$. Thus, we study

$$\gamma(t) := u_1(t,x_0) - u_1(t,y_0) - \Omega(t,|x_0-y_0|), \tag{6.2}$$

on the interval $[0, \tau]$, with our task being reduced to deriving conditions on Ω in order to guarantee that $\gamma'(\tau) \leq 0$, which would lead to a contradiction. To do so, we use the PDE in (6.1):

$$\begin{aligned}\gamma'(\tau) =& \Delta u_1(\tau, x^0) - \Delta u_1(\tau, y^0) - \partial_t \Omega(\tau, \xi) \\ &+ (u \cdot \nabla)\, u_1(\tau, x^0) - (u \cdot \nabla)\, u_1(\tau, y^0) + \partial_1 p(\tau, x^0) - \partial_1 p(\tau, y^0).\end{aligned} \tag{6.3}$$

Using Lemma 3.3, we see that

$$\begin{aligned}\Delta u_1(\tau, x^0) - \Delta u_1(\tau, y^0) + (u \cdot \nabla)\, u_1(\tau, x^0) - (u \cdot \nabla)\, u_1(\tau, y^0) \\ \leq 4\partial_\xi^2 \Omega(\tau, \xi) + \Omega(\tau, \xi)\partial_\xi \Omega(\tau, \xi),\end{aligned}$$

which leads us to

$$\gamma'(\tau) \leq 4\partial_\xi^2 \Omega(\tau, \xi) + \Omega(\tau, \xi)\partial_\xi \Omega(\tau, \xi) - \partial_t \Omega(\tau, \xi) + |\nabla p(\tau, x_0) - \nabla p(\tau, y_0)|.$$

As was mentioned previously, since ∇p is recovered from u via an order one operator, in general we should not expect to obtain an estimate on $|\nabla p(\tau, x_0) - \nabla p(\tau, y_0)|$ in terms of Ω. However, let us recall that the pressure is a solution to the elliptic equation

$$-\Delta p = \sum_{i,j} \partial_i \partial_j (u_i u_j).$$

The observation that Silvestre [53] made was following. Due to the incompressibility constraint, we have the identity

$$\sum_{i,j=1}^{d} \partial_{y_i} \partial_{y_j} \left[u_i(x-y) u_j(x-y)\right] = \sum_{i,j=1}^{d} \partial_{y_i} \partial_{y_j} \left[\left(u_i(x-y) - u_i(x)\right)\left(u_j(x-y) - u_j(x)\right)\right]. \tag{6.4}$$

Thus if $\Phi(y) := C_d|y|^{2-d}$ is the fundamental solution to Laplace equation in $d \geq 3$, we get the following representation for ∇p (after integrating by parts):

$$\nabla p(x) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \nabla \partial_i \partial_j \Phi(y) \varphi_{i,j}(x,y)\, dy, \quad \forall k \in \{1, \cdots, d\}, \tag{6.5}$$

where

$$\varphi_{i,j}(x,y) := \left(u_i(x-y) - u_i(x)\right)\left(u_j(x-y) - u_j(x)\right), \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{6.6}$$

That is, we gain an extra cancellation near the origin that will help us absorb the singularity coming from $\nabla \partial_i \partial_j \Phi(y)$, which is of order $|y|^{-d-1}$. This means that if u has sufficient Hölder regularity, in particular if $u(t,\cdot) \in C_x^{0,\beta}$ with $\beta \in (1/2, 1)$, then the singularity can be absorbed and we get a continuity estimate on ∇p in terms of continuity estimates known only on u. We can generalize this to an abstract modulus of continuity and get the estimate

$$\frac{1}{C_d}|\nabla p(t,x) - \nabla p(t,y)| \leq \int_0^{\xi} \frac{\Omega^2(t,\eta)}{\eta^2} d\eta + \Omega(t,\xi) \int_{\xi}^{\infty} \frac{\Omega(t,\eta)}{\eta^2} d\eta$$

for some constant positive C_d depending only on the dimension d. Thus, in order to guarantee the preservation of Ω by a solution to the NSE we must guarantee that

$$\partial_t \Omega(t,\xi) - 4\partial_\xi^2 \Omega(t,\xi) \geq \Omega \partial_\xi \Omega(t,\xi) + C_d \int_0^{\xi} \frac{\Omega^2(t,\eta)}{\eta^2} d\eta + C_d \Omega(t,\xi) \int_{\xi}^{\infty} \frac{\Omega(t,\eta)}{\eta^2} d\eta. \tag{6.7}$$

It is unclear at this stage whether we can construct a modulus of continuity that satisfies the above inequality on $[0,\infty) \times (0,\infty)$. The difficulty stems from the quadratic terms, in addition to the singularity that is present in the integrands. A natural question we may ask is whether we need Ω to depend on time or not. To answer that, let us consider the stationary viscous Burgers equation:

$$\omega''(\xi) + \omega\omega'(\xi) = 0. \tag{6.8}$$

This has a solution that (roughly speaking) behaves like the hyperbolic tangent function. If we perturb it and look instead for solutions to

$$\omega''(\xi) + \omega\omega'(\xi) + \epsilon = 0, \quad \epsilon > 0,$$

then upon employing the Cole-Hopf transformation, we end up solving an Airy type equation. This is bad news, since such solutions exhibit oscillatory behavior, a property that is quite undesirable for moduli of continuity: moduli of continuity need to be non-decreasing. Thus, if there is any hope in solving inequality (6.7) we need to depend on the entire parabolic operator $\partial_t - 4\partial_\xi^2$, not just the elliptic one.

A natural way to introduce a time variable is to rescale a stationary solution to (6.8) in a matter that will not disturb the balance between the dissipative and nonlinear advective terms. Specifically, consider $\Omega(t,\xi) := \lambda(t)\omega(\lambda(t)\xi)$, where for instance

$$\omega(\sigma) := \begin{cases} 2\sigma - \sigma^{3/2}, & \sigma \in [0, \delta_0], \\ \tanh\left((\sigma - \delta_0) + \mu_0\right), & \sigma > \delta_0, \end{cases}$$

for some $\delta_0, \mu_0 \in (0,1)$. This gives some power to the time derivative without disturbing the balance between dissipation and advection. However, due to the presence of quadratic term in (6.7), as well as a singularity that corresponds to an order one operator, when ξ is away from zero we would need $\lambda' \gtrsim \lambda^3$, which blows up in finite time. To expand on this last remark, notice that due to the condition $\partial_\xi^2\Omega(t, 0^+) = -\infty$, viscosity will always be the dominant term in (6.7) whenever $\xi\lambda \lesssim 1$. The issues arise when trying to control the integrals over "intermediate" distances, i.e. when $\xi \in [\lambda^{-1}, 1]$. One can show that in this region, since we can no longer rely on viscosity to absorb the integrals, we would need to make use of the time derivative. The second integral, for instance, would be of the order $\lambda^3\omega(\lambda\xi)$ (since ω is the hyperbolic tangent, which is bounded), while $\partial_t\Omega(t,\xi)$ is roughly $\lambda'\omega(\lambda\xi)$, and so we would need $\lambda' \gtrsim \lambda^3$.

That being said, we could ask about the preservation of such moduli of continuity in a simpler, linear model. Namely,

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = (b\cdot\nabla)u(t,x) + \nabla p(t,x), & \forall (t,x) \in (0,T]\times\mathbb{R}^d, \\ \nabla\cdot u(t,x) = 0, & \forall (t,x)\in[0,T]\times\mathbb{R}^d, \\ u(0,x) = u_0(x), & u_0 \in C^\infty(\mathbb{R}^d), \end{cases} \tag{6.9}$$

where in this case $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is a given divergence-free vector field. Since we have to deal with Riesz transforms, we need to assume a Hölder condition (in the spatial variable) on the drift-velocity, that is we assume the existence of some function $g : [0,T] \to [1,\infty)$ such that

$$[b(t,\cdot)]_{C^{0,\beta}} := \sup_{x\neq y} \frac{|b(t,x)-b(t,y)|}{|x-y|^\beta} \leq g(t), \quad a.e.\ t\in[0,T].$$

What we were able to show is that if we assume g is non-decreasing, then we have the bound

$$\|\nabla u(t,\cdot)\|_{L^\infty} \leq B g^\gamma(t)\exp\left(C_{d,\beta}B^{1-\beta}\int_0^t g^{2\gamma}(s)ds\right), \quad a.e.\ t\in[0,T],$$

where

$$\gamma := \frac{1}{1+\beta},$$

$B \geq 1$ is a constant depending on the initial data, and $C_{d,\beta} > 0$ is a constant depending only on the dimension d and $\beta \in (0,1)$. In particular, we see that if $g \in L^{2\gamma}([0,T])$, then

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^\infty} dt < \infty,$$

which owing to a weaker version of the Beale-Kato-Majda criterion [3], means that solutions to the nonlinear system (6.1) are not going to develop singularities on $[0,T]\times\mathbb{R}^d$. We remark that such a regularity criterion is slightly weaker than simply assuming $u \in L^{2\gamma}_t C^{0,\beta}_x$, since we assumed that the bounding function g is nondecreasing. Nevertheless, instead of working with the standard

$L^p_t C^{0,\beta}_x$ semi-norm, we could work with the quantity

$$\|b\| := \left(\int_0^T \sup_{s\in[0,t]} [b(s,\cdot)]^p_{C^{0,\beta}_x} dt \right)^{1/p},$$

which would make the non-decreasing assumption on g quite natural. Not much is lost when working with the above quantity rather than the standard (weaker) semi-norm on $L^p_t C^{0,\beta}_x$, since both of them scale in exactly the same way. That is to say, both are critical with respect to the natural scale invariance of (6.1) and (6.9), when $p = 2\gamma$. Such a result compliments the previous work of Silvestre and Vicol [55], where they show that whenever $b \in L^{2\gamma}_t C^{0,\beta}_x$ (without a nondecreasing assumption on the bounding function g), solutions to (6.9) lie in the space $L^\infty_t C^{0,\alpha}_x([0,T] \times \mathbb{R}^d)$, provided the initial data lies in $C^{0,\alpha}_x$, any $\alpha \in (0,1)$. Their precise result is actually stronger than that: they allow for the drift velocity b to lie in certain Morrey spaces that include the Hölder ones, but still at the critical scaling level. They do explicitly mention that their technique would not work when $\alpha = 1$, and so our results fill in this gap. Furthermore, we also prove in [22] that under the supercritical assumption with $p = 1/(1+\beta)$, we obtain a partial regularity result in the sense that

$$\int_0^T \log\left(\|\nabla u(t,\cdot)\|_{L^\infty}\right) dt < \infty.$$

Details can be found in [22], where such arguments are also extended to the case when classical dissipation is replaced by fractional.

6.2 Burgers-Hilbert Equation

Another simplification to the NSE one could look into is the following. Before trying to analyze a nonlinear, nonlocal order one term, let us try to address a simpler problem. Namely,

$$\partial_t u - \Delta u = (u \cdot \nabla) u + \mathcal{N}[u], \tag{6.10}$$

where $\mathcal{N}$ is a linear order zero operator, for instance a linear combination of Riesz transforms. This is a generalization of the so called Burgers-Hilbert model, where the latter is the one dimensional, inviscid version of (6.10), with $\mathcal{N}$ being the one dimensional Hilbert transform. As was mentioned in Chapter 1, this model was introduced by Marsden and Weinstein [39] as an approximate model for the dynamics of free boundary, two dimensional vortex patches. Biello and Hunter [7] also proposed the same equation as a surface wave model. Since then, it has attracted much attention in the literature, see [8, 9, 11, 20, 21, 31]. All those papers deal with the one-dimensional inviscid version of (6.10), which was shown to exhibit singularity formation in finite time in [11]. Of course, in dimensions higher than one, equation (6.10) has no a-priori bounds, and so it is unclear what happens. We outline below a strategy to show global regularity for this model if we add viscosity, regardless of the spatial dimension.

As was done for NSE, we track the evolution of moduli of continuity by (6.10) and end up with

$$\gamma'(\tau) \leq 4\partial_\xi^2\Omega(\tau,\xi) + \Omega(\tau,\xi)\partial_\xi\Omega(\tau,\xi) - \partial_t\Omega(\tau,\xi) + |\mathcal{N}[u](\tau,x_0) - \mathcal{N}[u](\tau,y_0)|,$$

where γ is as defined in (6.2). We then need to show that the right-hand side of the above inequality is non-positive. To estimate the non-local part, we show that whenever $\mathcal{N}$ is a non-local operator given by

$$\mathcal{N}\theta(x) := P.V. \int_{\mathbb{R}^d} K(x-z)\theta(z)\, dz,$$

where K is a kernel of the form

$$K(z) := \begin{cases} \dfrac{\Phi\left(z/|z|\right)}{|z|^d}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

and $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ is a Hölder continuous function that satisfies the following zero average condition

$$\int_{\mathbb{S}^{d-1}} \Phi(y)\, dy = 0,$$

we have

$$|\mathcal{N}[u](t,x) - \mathcal{N}[u](t,y)| \leq C_{\mathcal{N},d} \left[\int_0^{3\xi} \frac{\Omega(t,\eta)}{\eta} \, d\eta + \xi^\alpha \int_{3\xi}^\infty \frac{\Omega(t,\eta)}{\eta^{1+\alpha}} \, d\eta \right].$$

Thus, our task reduces to showing that

$$\partial_t \Omega(t,\xi) - 4\partial_\xi^2 \Omega(t,\xi) \geq \Omega(t,\xi)\partial_\xi \Omega(t,\xi) + C_{\mathcal{N},d} \left[\int_0^{3\xi} \frac{\Omega(t,\eta)}{\eta} \, d\eta + \xi^\alpha \int_{3\xi}^\infty \frac{\Omega(t,\eta)}{\eta^{1+\alpha}} \, d\eta \right].$$

Comparing with (6.7), the nonlocal part is much simpler: it is linear, and the singularity in the integrand is milder. Thus, if we do the same "dynamic" rescaling trick: $\Omega(t,\xi) := \lambda(t)\omega(\lambda(t)\xi)$, where ω is a solution to

$$\omega''(\sigma) + \omega'\omega(\sigma) \leq 0,$$

we see that λ needs to satisfy $\lambda' \gtrsim \lambda \log \lambda$. We do want to point out that this would only work in the periodic setting, i.e. when $\xi \in (0, L]$, where L is some multiple of the period. This is due to the fact that the hyperbolic tangent function is bounded, while the first integral at best grows logarithmically in ξ. That is, we can only balance the first integral with $\partial_t \Omega(t,\xi)$ when $\xi \in (0, L]$, if we are to use the "dynamic rescaling" trick.

7. CONCLUSION AND CURRENT/FUTURE WORK

In our work, we analyzed the global regularity problem related to the following system of equations:

$$\partial_t u - \Delta u = (u \cdot \nabla)u + \mathcal{N}[u],$$

where $\mathcal{N}$ is a non-local operator. We showed that when the vector-field u is conservative (that is, $u = \nabla\theta$, for some scalar θ) and if $\mathcal{N} := (-\Delta)^\alpha$, with $\alpha \in (0, 1/2)$, then the solution will not develop a singularity in finite time provided the initial data is smooth enough. On the other hand, if u is not conservative, we can show that whenever $\mathcal{N}$ is a well-defined zero order operator, then regularity does persist. The third result we were able to obtain is conditional regularity for the incompressible Navier-Stokes system, which corresponds to the case when the initial data is divergence free and

$$\mathcal{N}[u] = \sum_{i,j=1}^{d} \nabla R_i R_j u_i u_j.$$

This leads to the following questions, some of which were partially addressed in [22]:

(i) **Supercritical Hölder assumptions**: Model (6.9) is linear, and the assumption that $b(t,\cdot) \in C_x^{0,\beta}$ (along with incompressibility) makes the term ∇p mimic the behavior of an operator of order $1-\beta$ at the continuity level (see [22] for more precise details). Now, due to the natural scale invariance of the equation (6.9), the $L_t^p C_x^\beta$ semi-norm of the drift velocity b becomes supercritical if $p < 2/(1+\beta)$. What we have shown in [22] that if $p = 1$, then the Lipschitz constant of the solution is logarithmically integrable in time. Of course, the same is true for any $p \in (1, 2/(1+\beta))$ in the supercritical range. It will be interesting to see if one can improve on this when $p \in (1, 2/(1+\beta))$, that is, by assuming more regularity on b while remaining in the supercritical regime. In fact, some progress have already been made in this direction, see [22].

(ii) **Fractional Dissipation**: It is known that the Burgers equation with fractional dissipation

$(-\Delta)^{\alpha}$ does not develop singularities when $\alpha \in [1/2, 1]$, but does blowup in finite time if $\alpha \in [0, 1/2)$, see for instance [28]. Blowup has been proven for the one-dimensional inviscid Burgers-Hilbert problem [11]. Thus, it is natural to expect solutions to the multi-dimensional Burgers-Hilbert problem with fractional dissipation to be regular when $\alpha \in [1/2, 1)$, and to blowup when $\alpha \in (0, 1/2)$. This seems to be an interesting problem to consider. It will also be interesting to consider model (6.9) with fractional dissipation.

(iii) **Gradient form of Michelson-Sivashinsky**: The viscous Burgers-Hilbert model is an equation where part of the full nonlinear structure of the NSE is preserved; we assumed that the nonlocal operator acts linearly on the solution, while retaining $(u \cdot \nabla)u$. Moreover, we assumed that the nonlocal operator is of order zero. Thus, it is natural to see what happens when one increases the order to $\gamma \in (0, 1)$ in the presence of an advective nonlinearity and study the regularity of solutions to

$$\partial_t u - \Delta u = (u \cdot \nabla u) + (-\Delta)^{\alpha} u, \quad \alpha \in (0, 1/2). \tag{7.1}$$

If $u(t, \cdot)$ happens to be a conservative vector field, then Theorem 2.2 tells us that u remains smooth for all time. On the other hand, the proof of outlined in §6.10 cannot be directly applied to equation (7.1). Loosely speaking this happens because in general, when trying to obtain continuity estimates for an operator of order $\beta \in [0, 1)$, one loses some regularity; for instance $(-\Delta)^{\alpha} : C^{0,\gamma} \to C^{0,\gamma-2\alpha}$ when $\alpha \in (0, 1/2)$. It is worthwhile noting that this is not a problem in the scalar equation (2.7) with a purely gradient nonlinearity. It seems that coupling advective nonlinearities to such operators is troublesome.

REFERENCES

[1] D. M. Ambrose and A. L. Mazzucato. Global existence and analyticity for the 2D Kuramoto-Sivashinsky equation. *J. Dynam. Differential Equations*, 31(3):1525–1547, 2019.

[2] L. Amour and M. Ben-Artzi. Global existence and decay for viscous Hamilton-Jacobi equations. *Nonlinear Anal.*, 31(5-6):621–628, 1998.

[3] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[4] H. Bellout, S. Benachour, and E. S. Titi. Finite-time singularity versus global regularity for hyper-viscous Hamilton-Jacobi-like equations. *Nonlinearity*, 16(6):1967–1989, 2003.

[5] M. Ben-Artzi, J. Goodman, and A. Levy. Remarks on a nonlinear parabolic equation. *Trans. Amer. Math. Soc.*, 352(2):731–751, 2000.

[6] S. Benachour, I. Kukavica, W. Rusin, and M. Ziane. Anisotropic estimates for the two-dimensional Kuramoto-Sivashinsky equation. *J. Dynam. Differential Equations*, 26(3):461–476, 2014.

[7] J. Biello and J. K. Hunter. Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities. *Comm. Pure Appl. Math.*, 63(3):303–336, 2010.

[8] A. Bressan and K. T. Nguyen. Global existence of weak solutions for the Burgers-Hilbert equation. *SIAM J. Math. Anal.*, 46(4):2884–2904, 2014.

[9] A. Bressan and T. Zhang. Piecewise smooth solutions to the Burgers-Hilbert equation. *Commun. Math. Sci.*, 15(1):165–184, 2017.

[10] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[11] A. Castro, D. Córdoba, and F. Gancedo. Singularity formations for a surface wave model. *Nonlinearity*, 23(11):2835–2847, 2010.

[12] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[13] A. Córdoba and D. Córdoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249(3):511–528, 2004.

[14] M. Dabkowski, A. Kiselev, L. Silvestre, and V. Vicol. Global well-posedness of slightly supercritical active scalar equations. *Anal. PDE*, 7(1):43–72, 2014.

[15] J. Droniou and C. Imbert. Fractal first-order partial differential equations. *Arch. Ration. Mech. Anal.*, 182(2):299–331, 2006.

[16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[17] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[18] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[19] S. Gutman and G. Sivashinsky. The cellular nature of hydrodynamic flame instability. *Phys. D*, 43(1):129 – 139, 1990.

[20] J. K. Hunter and M. Ifrim. Enhanced life span of smooth solutions of a Burgers-Hilbert equation. *SIAM J. Math. Anal.*, 44(3):2039–2052, 2012.

[21] J. K. Hunter, M. Ifrim, D. Tataru, and T. K. Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proc. Amer. Math. Soc.*, 143(8):3407–3412, 2015.

[22] H. Ibdah. Lipschitz continuity of solutions to drift-diffusion equations in the presence of nonlocal terms. Preprint (submitted), arXiv:2006.01859 [math.AP], 2021.

[23] H. Ibdah. Strong solutions to a modified Michelson-Sivashinsky equation. *Commun. Math. Sci.*, 19(4):1071–1100, 2021.

[24] C. Imbert. A non-local regularization of first order Hamilton-Jacobi equations. *J. Differential Equations*, 211(1):218–246, 2005.

[25] F. John. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition, 1991.

[26] A. Kiselev. Nonlocal maximum principles for active scalars. *Adv. Math.*, 227(5):1806–1826, 2011.

[27] A. Kiselev and F. Nazarov. Global regularity for the critical dispersive dissipative surface quasi-geostrophic equation. *Nonlinearity*, 23(3):549–554, 2010.

[28] A. Kiselev, F. Nazarov, and R. Shterenberg. Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.*, 5(3):211–240, 2008.

[29] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.

[30] S. Klainerman. PDE as a unified subject. Number Special Volume, Part I, pages 279–315. 2000. GAFA 2000 (Tel Aviv, 1999).

[31] S. G. Krupa and A. F. Vasseur. Stability and uniqueness for piecewise smooth solutions to a nonlocal scalar conservation law with applications to Burgers-Hilbert equation. *SIAM J. Math. Anal.*, 52(3):2491–2530, 2020.

[32] O. Kupervasser and Z. Olami. Random noise and pole-dynamics in unstable front propagation. *Combustion, Explosion, and Shock Waves*, 49(2):141–152, Mar 2013.

[33] Y. Kuramoto. Diffusion-induced chaos in reaction systems. *Progress of Theoretical Physics Supplement*, 64:346–367, 02 1978.

[34] Y. Kuramoto and T. Tsuzuki. On the formation of dissipative structures in reaction-diffusion systems: Reductive perturbation approach. *Progress of Theoretical Physics*, 54(3):687–699, 09 1975.

[35] Y. Kuramoto and T. Tsuzuki. Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. *Progress of Theoretical Physics*, 55(2):356–369, 02 1976.

[36] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[37] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[38] A. Larios and K. Yamazaki. On the well-posedness of an anisotropically-reduced two-dimensional Kuramoto-Sivashinsky equation. *Phys. D*, 411:132560, 14, 2020.

[39] J. Marsden and A. Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. volume 7, pages 305–323. 1983. Order in chaos (Los Alamos, N.M., 1982).

[40] M. Matalon. Intrinsic flame instabilities in premixed and nonpremixed combustion. *Annual Review of Fluid Mechanics*, 39(1):163–191, 2007.

[41] Q. Miao and L. Xue. Regularity and singularity results for the dissipative Whitham equation and related surface wave equations. *Commun. Math. Sci.*, 17(8):2141–2190, 2019.

[42] D. Michelson and G. Sivashinsky. Thermal-expansion induced cellular flames. *Combustion and Flame*, 48:211 – 217, 1982.

[43] D. M. Michelson and G. I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames. II. Numerical experiments. *Acta Astronaut.*, 4(11-12):1207–1221, 1977.

[44] L. Molinet. Local dissipativity in L^2 for the Kuramoto-Sivashinsky equation in spatial dimension 2. *J. Dynam. Differential Equations*, 12(3):533–556, 2000.

[45] F. Navin. Interplay between background turbulence and darrieus-landau instability in premixed flames via a model equation. Master's thesis, University of Illinois at Urbana-Champaign, May 2010.

[46] B. Nicolaenko and B. Scheurer. Remarks on the Kuramoto-Sivashinsky equation. *Phys. D*, 12(1-3):391–395, 1984.

[47] Z. Olami, B. Galanti, O. Kupervasser, and I. Procaccia. Random noise and pole dynamics in unstable front propagation. *Phys. Rev. E*, 55:2649–2663, Mar 1997.

[48] A. Pumir. Equation describing wrinkled flame fronts. *Phys. Rev. A*, 31:543–546, Jan 1985.

[49] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, 2019. Blow-up, global existence and steady states, Second edition of [MR2346798].

[50] M. Renardy. A model equation in combustion theory exhibiting an infinite number of secondary bifurcations. *Phys. D*, 28(1-2):155–167, 1987.

[51] G. R. Sell and M. Taboada. Local dissipativity and attractors for the Kuramoto-Sivashinsky equation in thin $2\mathrm{D}$ domains. *Nonlinear Anal.*, 18(7):671–687, 1992.

[52] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[53] L. Silvestre. A non obvious estimate for the pressure. Unpublished material. Retrieved from http://www.math.uchicago.edu/ luis/preprints/pressureestimate.pdf, 2010.

[54] L. Silvestre. On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion. *Adv. Math.*, 226(2):2020–2039, 2011.

[55] L. Silvestre and V. Vicol. Hölder continuity for a drift-diffusion equation with pressure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(4):637–652, 2012.

[56] G. I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations. *Acta Astronaut.*, 4(11-12):1177–1206, 1977.

[57] G. I. Sivashinsky. Instabilities, pattern formation, and turbulence in flames. *Annual Review of Fluid Mechanics*, 15(1):179–199, 1983.

[58] E. M. Stein. The characterization of functions arising as potentials. *Bull. Amer. Math. Soc.*, 67:102–104, 1961.

[59] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[60] E. Tadmor. The well-posedness of the Kuramoto-Sivashinsky equation. *SIAM J. Math. Anal.*, 17(4):884–893, 1986.

[61] O. Thual, U. Frisch, and M. Hénon. Application of pole decomposition to an equation governing the dynamics of wrinkled flame fronts. In P. Pelcé, editor, *Dynamics of Curved Fronts*, pages 489 – 498. Academic Press, San Diego, 1988.

[62] E. S. Titi. Private Communication, 2019.

[63] A. Trucchia and G. Pagnini. Restoring property of the Michelson–Sivashinsky equation. *Combustion Science and Technology*, 191(9):1734–1741, 2019.

[64] R. L. Wheeden. On hypersingular integrals and Lebesgue spaces of differentiable functions. *Trans. Amer. Math. Soc.*, 134:421–435, 1968.

APPENDIX A

PERIODIC FRACTIONAL LAPLACIAN

Proposition A.1. *Let $\theta \in C^2(\mathbb{R}^d)$ and suppose that $\theta(x + 2\pi e_j) = \theta(x)$ for every $x \in \mathbb{R}^d$. Denote the k^{th} Fourier mode of θ by $\widehat{\theta}_k$ so that its Fourier series*

$$\theta(x) = \sum_{k\in\mathbb{Z}^d} \widehat{\theta}_k e^{ik\cdot x},$$

converges absolutely and uniformly on any compact set. Then given any $\alpha \in (0, 1)$, there exists a constant $C_{d,\alpha}$ depending only on the dimension d and α such that if we define

$$(-\Delta)^\alpha \theta(x) := \sum_{k\in\mathbb{Z}^d} |k|^{2\alpha} \widehat{\theta}_k e^{ik\cdot x},$$

then

$$(-\Delta)^\alpha \theta(x) = \lim_{\epsilon\to 0^+} C_{d,\alpha} \int_{|z|\geq\epsilon} \frac{\theta(x) - \theta(x - z)}{|z|^{d+2\alpha}}\, dz.$$

Proof. For $\epsilon \in (0, 1)$, we let

$$I_\epsilon(x) := \int_{|z|\geq\epsilon} \frac{\theta(x) - \theta(x - z)}{|z|^{d+2\alpha}}\, dz,$$

and we note that by making a change of variable $y = -z$ we get

$$I_\epsilon(x) = \int_{|y|\geq\epsilon} \frac{\theta(x) - \theta(x + y)}{|y|^{d+2\alpha}}\, dy = \int_{|z|\geq\epsilon} \frac{\theta(x) - \theta(x + z)}{|z|^{d+2\alpha}}\, dz.$$

Thus, we get that

$$I_\epsilon(x) = \frac{1}{2} \int_{|z|\geq\epsilon} \frac{2\theta(x) - \theta(x - z) - \theta(x + z)}{|z|^{d+2\alpha}}\, dz.$$

From there, we note that the tail end of the integral is controlled by

$$\left| \int_{|z| \geq \epsilon^{-1}} \frac{2\theta(x) - \theta(x-z) - \theta(x+z)}{|z|^{d+2\alpha}} \, dz \right| \leq \frac{3\epsilon}{2\alpha} |\mathbb{S}^{d-1}| \|\theta\|_{L^\infty}. \tag{A.1}$$

On the other hand, using the fact that the Fourier series of θ converges absolutely and uniformly on compact sets, we can write for fixed $x \in \mathbb{R}^d$ and $|z| \leq \epsilon^{-1}$

$$2\theta(x) - \theta(x-z) - \theta(x+z) = 2 \sum_{k \in \mathbb{Z}^d} \widehat{\theta}_k e^{ik \cdot x} \left(1 - \cos\left(k \cdot z\right)\right),$$

and so

$$\frac{1}{2} \int_{\epsilon \leq |z| \leq \epsilon^{-1}} \frac{2\theta(x) - \theta(x-z) - \theta(x+z)}{|z|^{d+2\alpha}} \, dz = \sum_{k \in \mathbb{Z}^d} \widehat{\theta}_k e^{ik \cdot x} \int_{\epsilon \leq |z| \leq \epsilon^{-1}} \frac{1 - \cos\left(k \cdot z\right)}{|z|^{d+2\alpha}} \, dz.$$

Let us now invoke a change of variable $z = y/|k|$ for $k \neq 0$ to get

$$\int_{\epsilon \leq |z| \leq \epsilon^{-1}} \frac{1 - \cos\left(k \cdot z\right)}{|z|^{d+2\alpha}} \, dz = |k|^{2\alpha} \int_{|k|\epsilon \leq |y| \leq |k|\epsilon^{-1}} \frac{1 - \cos\left(\frac{k}{|k|} \cdot y\right)}{|y|^{d+2\alpha}} \, dy.$$

From the Taylor expansion of $\cos(\cdot)$ near 0 and boundedness at infinity, we clearly see that the limit

$$\lim_{\epsilon \to 0^+} \int_{|k|\epsilon \leq |y| \leq |k|\epsilon^{-1}} \frac{1 - \cos\left(\frac{k}{|k|} \cdot y\right)}{|y|^{d+2\alpha}} \, dy < \infty,$$

and so we have

$$\lim_{\epsilon \to 0^+} \int_{|z| \geq \epsilon} \frac{\theta(x) - \theta(x-z)}{|z|^{d+2\alpha}} \, dz = \sum_{k \in \mathbb{Z}^d} |k|^{2\alpha} \widehat{\theta}_k e^{ik \cdot x} \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{k}{|k|} \cdot y\right)}{|y|^{d+2\alpha}} \, dy.$$

Finally, we note that the function

$$G(k) := \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{k}{|k|} \cdot y\right)}{|y|^{d+2\alpha}} \, dy, \quad k \in \mathbb{R} \backslash \{0\}$$

is rotation invariant. Indeed, let M be an orthogonal matrix, and notice that since $|Mk| = |k| = |M^T k|$ and $|det(M)| = 1$, we get

$$G(Mk) = \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{Mk}{|k|} \cdot y\right)}{|y|^{d+2\alpha}}\, dy = \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{k}{|k|} \cdot M^T y\right)}{|M^T y|^{d+2\alpha}}\, dy = \int_{\mathbb{R}^d} \frac{1 - \cos\left(\frac{k}{|k|} \cdot z\right)}{|z|^{d+2\alpha}}\, dz,$$

and so $G(Mk) = G(k)$. Hence, if we let $k_0 := (1, 0, \cdots, 0)$ and define $C_{d,\alpha} := (G(k_0))^{-1}$, we get $G(k) = G(k_0)$ for any $k \in \mathbb{R}^d \backslash \{0\}$, meaning that

$$\lim_{\epsilon \to 0^+} C_{d,\alpha} \int_{|z| \geq \epsilon} \frac{\theta(x) - \theta(x - z)}{|z|^{d+2\alpha}}\, dz = \sum_{k \in \mathbb{Z}^d} |k|^{2\alpha} \widehat{\theta}_k e^{ik \cdot x} = (-\Delta)^\alpha \theta(x).$$

□

APPENDIX B

HEAT KERNEL PROPERTIES

We will rigoursly justify the various properties of the heat kernel used previously. We start by obtaining estimates (4.2)-(4.5). Let $d \in \mathbb{N}$ be a positive integer, and let $\nu > 0$. Consider the heat kernel given by

$$\Psi(s,\lambda) := (4\pi\nu s)^{-d/2} \exp\left(\frac{-|\lambda|^2}{4\nu s}\right), \quad (s,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

which satisfies the homogenous heat equation

$$\partial_s \Psi(s,\lambda) - \nu\Delta\Psi(s,\lambda) = 0, \quad \forall (s,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{B.1}$$

The estimates (4.2)-(4.5) are:

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y)| \, dy = \frac{C_d}{\sqrt{\nu s}}, \tag{B.2}$$

$$\int_{\mathbb{R}^d} |x-y|^\gamma |\partial_s \Psi(s,x-y)| \, dy \le C_d \nu^{\gamma/2} s^{\gamma/2-1}, \tag{B.3}$$

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)| \, dy \le \frac{C_d}{\nu s}|x-z|, \tag{B.4}$$

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)| \, dy \le \frac{C_{d,\beta}|x-z|^\beta}{(\nu s)^{\frac{1}{2}(1+\beta)}}, \tag{B.5}$$

where $s, \gamma > 0$, and $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d$ are arbitrary. We start by noting that for any $\beta \ge 0$, if we invoke the change of variable $\lambda := z(4\nu s)^{-1/2}$ we get

$$\int_{\mathbb{R}^d} |z|^\beta \Psi(s,z) \, dy = C_{d,\beta}(\nu s)^{\beta/2}, \quad C_{d,\beta} := \int_{\mathbb{R}^d} 2\pi^{-d/2} |\lambda|^\beta e^{-|\lambda|^2} \, d\lambda. \tag{B.6}$$

From this, (B.2) follows immediately upon realizing that

$$\nabla\Psi(s,\lambda) = \frac{-\lambda}{2\nu s}\Psi(s,\lambda). \tag{B.7}$$

To prove estimate (B.3), from (B.1) and (B.7) we get

$$\partial_s\Psi(s,\lambda) = \frac{\nu|\lambda|^2}{(2\nu s)^2}\Psi(s,\lambda) - \frac{d}{2s}\Psi(s,\lambda),$$

and so for $\gamma > 0$, we take absolute values and integrate to get

$$\begin{aligned}\int_{\mathbb{R}^d} |x-y|^\gamma |\partial_s\Psi(s,x-y)|dy \leq & \frac{1}{4\nu s^2}\int_{\mathbb{R}^d} |x-y|^{2+\gamma}\Psi(s,x-y)\, dy \\ & + \frac{d}{2s}\int_{\mathbb{R}^d} |x-y|^\gamma \Psi(s,x-y)\, dy,\end{aligned}$$

from which we get (B.3) from using (B.6). To obtain estimate (B.4), we start by noting

$$|\partial_i\Psi(s,\lambda) - \partial_i\Psi(s,\sigma)| \leq |\lambda-\sigma|\int_0^1 |\partial_i\nabla\Psi(s,\eta\lambda+(1-\eta)\sigma)|\, d\eta, \quad \forall(\lambda,\sigma)\in\mathbb{R}^d\times\mathbb{R}^d,$$

and so

$$|\nabla\Psi(s,\lambda) - \nabla\Psi(s,\sigma)| \leq C_d|\lambda-\sigma|\sum_{i=1}^d\int_0^1 |\partial_i\nabla\Psi(s,\eta\lambda+(1-\eta)\sigma)|\, d\eta.$$

An application of Fubini-Tonelli thus tells us (dropping the sum for convenience) along with the observation that $\eta(x-y)+(1-\eta)(z-y) = \eta(x-z)+z-y$,

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\, dy \leq C_d|x-z|\int_0^1\int_{\mathbb{R}^d} |\partial_i\nabla\Psi(s,\eta(x-z)+z-y)|dyd\eta.$$

Invoking a change of variable $\lambda := \eta(x-z)+(z-y)$ in the inner integral we get

$$\sum_{i=1}^d\int_0^1\int_{\mathbb{R}^d} |\partial_i\nabla\Psi(s,\eta(x-z)+z-y)|dyd\eta = \sum_{i=1}^d\int_{\mathbb{R}^d} |\partial_i\nabla\Psi(s,\lambda)|\, d\lambda.$$

From (B.7),

$$\sum_{i=1}^{d} |\partial_i \nabla \Psi(s,\lambda)| \leq \frac{C_d}{\nu s}\left(\psi(s,\lambda) + |\lambda||\nabla\Psi(s,\lambda)|\right) = \frac{C_d \Psi(s,\lambda)}{\nu s}\left(1 + \frac{|\lambda|^2}{\nu s}\right),$$

and so by using (B.6) (with $x-y$ being replaced by λ) we obtain

$$\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\,dy \leq \frac{C_d|x-z|}{\nu s}\int_{\mathbb{R}^d}\left(1+|\lambda|^2(\nu s)^{-1}\right)\Psi(s,\lambda)\,d\lambda = \frac{C_d}{\nu s}|x-z|,$$

giving us precisely (B.4). Finally to prove (B.5) we let $\beta \in (0,1)$ and combine estimates (B.2) and (B.4) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\,dy \\
&= \left(\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\,dy\right)^{\beta}\left(\int_{\mathbb{R}^d} |\nabla\Psi(s,x-y) - \nabla\Psi(s,z-y)|\,dy\right)^{1-\beta} \\
&\leq \left(\frac{C_d}{\nu s}|x-z|\right)^{\beta}\left(\frac{2C_d}{\sqrt{\nu s}}\right)^{1-\beta} = \frac{C_{d,\beta}|x-z|^{\beta}}{(\nu s)^{\frac{1}{2}(1+\beta)}}.
\end{aligned}$$

APPENDIX C

GRONWALL INEQUALITY

We provide a proof of Lemma 4.1, which we restate here for convenience.

Lemma C.1. *Let* $q \in [1, \infty)$, $1/q + 1/r = 1$, $T_2 \geq T_1$, $C_0 \geq 0$ *and assume that* $g \in L^q(T_1, T_2)$, $f \in L^r(0, T_2 - T_1)$ *are both non-negative. If*

$$g(t) \leq \int_{T_1}^{t} f(t-s)g(s)\, ds + C_0, \quad a.e.\ t \in [T_1, T_2], \tag{C.1}$$

then

$$g(t) \leq C_0 \left[2 \left(\int_0^{t-T_1} |f(s)|^r ds \right)^{1/r} \left(\int_{T_1}^{t} e^{h(t)-h(s)} ds \right)^{1/q} + 1 \right], \quad a.e.\ t \in [T_1, T_2], \tag{C.2}$$

where

$$h(t) := 2^q \int_{T_1}^{t} \left(\int_0^{s-T_1} |f(\sigma)|^r d\sigma \right)^{q/r} ds. \tag{C.3}$$

Proof. Assume $q \in (1, \infty)$. An application of Hölder's inequality to (C.1) gives us

$$g(t) \leq \left(\int_{T_1}^{t} g^q(s) ds \right)^{1/q} \left(\int_0^{t-T_1} f^r(s) ds \right)^{1/r} + C_0. \tag{C.4}$$

Now setting

$$\eta(t) := \int_{T_1}^{t} g^q(s) ds,$$

we get from inequality (C.4), and by using $(a+b)^q \leq 2^q(a^q + b^q)$ for positive a, b, the relation

$$\eta'(t) = g^q(t) \leq 2^q \eta(t) \left(\int_0^{t-T_1} f^r(s) ds \right)^{q/r} + 2^q C_0.$$

From this, if we set $h(t)$ as defined in (C.3) we see that

$$\frac{d}{ds}\left[e^{-h(s)}\eta(s)\right] \leq 2^q C_0 e^{-h(s)},$$

and so by integrating from $s = 0$ to $s = t$ while using $\eta(0) = 0$ we arrive at

$$\eta(t) \leq 2^q C_0 \int_0^t e^{h(t)-h(s)} ds.$$

Plugging this into (C.4) gives us precisely (C.2) when $q \in (1, \infty)$. If $q = 1$, then by our hypothesis we must have $f \in L^\infty[0, T_2 - T_1] \subset L^r[0, T_2 - T_1]$ for every $r \geq 1$ and so from

$$\lim_{r\to\infty} \|f\|_{L^r} = \|f\|_{L^\infty},$$

(see for instance [17]) we get the desired estimate. □