BERGMAN KERNELS, HARTOGS DOMAINS, AND THE WIEGERINCK PROBLEM

A Dissertation

by

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ABSTRACT

Let $L^2_h(\Omega,\mu)$ denote the space of holomorphic functions on Ω which are square-integrable with respect to the weight $\mu : \Omega \to [0,\infty)$, where Ω is a domain in $\mathbb{C}^n$. When μ is sufficiently well-behaved, the space $L^2_h(\Omega,\mu)$ possesses a unique sesqui-holomorphic function $K_{\Omega,\mu} : \Omega \times \Omega \to \mathbb{C}$ such that

$$f(z) = \int_\Omega f(\zeta) K_{\Omega,\mu}(z,\zeta)\mu(\zeta)\text{dVolume}(\zeta)$$

known as the Bergman kernel.

This dissertation contains a variety of results concerning Bergman spaces. The Bergman kernel and Wiegerinck problem (whether a nontrivial Bergman space must have infinite dimension) have particular focus.

In Chapter 2, we show that, by changing the weight, one may create zeroes in the Bergman kernel without changing the associated space of holomorphic functions. We also provide a construction of a weight on $\mathbb{C}$ whose Bergman kernel has an infinite number of zeroes.

In Chapter 3, we expand some of the results of Jucha [23] to show that a complete N-circled Hartogs domain has infinite-dimensional Bergman space whenever its associated plurisubharmonic function has a neighborhood on which it is strictly plurisubharmonic. This agrees with a work of Gallagher et al. [16]. We follow this up with some sufficient conditions for the infinite-dimensionality of a complete N-circled Hartogs domain based on various forms of the Ohsawa-Takegoshi extension theorem. We also address a question of Pflug and Zwonek [35].

In Chapter 4, we directly compute a coefficient which relates the L^2-norm of a holomorphic function on a complete N-circled Hartogs domain to the weighted L^2-norm of an associated function over the base domain. We then use this relationship to compute explicit formulae of the Bergman kernel for generalized Hartogs triangles with rational index, in an alternative manner to Edholm and McNeal [13], [14]. We also provide an alternative proof of the well-known "inflation" identity of Boas, Fu, and Straube [6]. Other relationships of this type are presented.

DEDICATION

To my wife and family

Kristen, Francis, Louise, Christopher, and Erin

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First, I would like to thank my beautiful wife for her constant love and support. She has made many sacrifices so that I can be where I am today, and I am eternally grateful. I would also like to thank my family, who, although they cannot understand why a person would choose to stay in school for such a long time, have nonetheless supported my decisions. I would especially like to thank my parents for their support, as well as for the occasional financial assistance. I hope I have made you proud.

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The idea behind the proof of the main theorem in Chapter 2 was provided by Professor Harold Boas.

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1. INTRODUCTION AND LITERATURE REVIEW

1.1 Preliminaries

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, let

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \bar{w}_j$$

denote the standard Hermitian inner product on $\mathbb{C}^n$, with norm given by $\|z\| := \langle z, z\rangle^{1/2}$. If $a \in \mathbb{C}^n$ and $r > 0$, let $B(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ denote the ball of radius r. To avoid confusion we will often write $|z| := (z\bar{z})^{1/2}$ whenever $z \in \mathbb{C}$ as well as $D(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ whenever $a \in \mathbb{C}$ to denote the disk of radius $r > 0$ in the complex plane. We will also use the convention that $B(r) := B(0, r)$ and $\mathbb{B} := B(0, 1)$; likewise we set $D(r) := D(0, r)$ and $\mathbb{D} := D(0, 1)$ whenever $n = 1$.

A *polydisk* is a product of disks in $\mathbb{C}^n$, i.e. a domain of the form $\prod_{k=1}^{n} D(a_k, r_k)$, where $a_k \in \mathbb{C}$ and $r_k > 0$, $k = 1, \ldots, n$. Sometimes we may (carelessly) refer to *the unit polydisk*; this is simply the n-fold product the unit disk in $\mathbb{C}^n$.

Here are some other conventional notations which we make use of.

- Let $\mathrm{d}V_n$ denote Lebesgue measure on $\mathbb{C}^n$; when there is no chance of confusion we will omit the subscript.

- Let $\mathbb{Z}_+^n$ denote the set of nonnegative multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_k is a nonnegative integer for each $1 \leq k \leq n$. We will use the notation that $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ for a multi-index α.

- Let $\mathrm{USC}(\Omega)$ denote the set of upper semi-continuous functions on a set $\Omega \subseteq \mathbb{C}^M$; that is, functions $f : \Omega \to [-\infty, \infty)$ such that

$$\limsup_{z \to z_0} f(z) \leq f(z_0)$$

for each $z_0 \in \Omega$.

We will also make use of the so-called *Wirtinger derivatives*. If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and $z_k = x_k + iy_k$ for $x_k, y_k \in \mathbb{R}$, $1 \leq k \leq n$, we define the operators

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right).$$

These operators are very convenient in complex analysis. Here are a few examples of their utility.

- The Cauchy-Riemann equations of a complex-valued function $f(z) = u(z) + iv(z)$ in $\mathbb{C}^n$ can be written simply as
$$\frac{\partial f}{\partial \bar{z}_k}(z) = 0$$
for each $k = 1, \ldots, n$.

- If f is a holomorphic function on $\mathbb{C}^n$, then
$$\frac{\partial f}{\partial z_k}(z)$$
is the holomorphic derivative of f with respect to the variable z_k. In the classical theory of one complex variable this is typically written as $f'(z)$.

- If $u(z)$ is a function defined on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, then
$$4\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k}(z) = \left(\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(z) + \frac{\partial^2 u}{\partial y_k^2}(z)\right) = \Delta u(z),$$
where Δ denotes the classical Laplacian in $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

1.1.1 Domains in $\mathbb{C}^n$

Let $\Omega \subseteq \mathbb{C}^n$ be a *domain*, that is, Ω is a connected open subset of $\mathbb{C}^n$. Let $\mathcal{O}(\Omega)$ denote the set of holomorphic functions on Ω. We use the standard convention in complex analysis that $\mathrm{b}\Omega$

denotes the topological boundary of Ω (as the arguably more traditional symbol "∂" is reserved for the valuable $\overline{\partial}$ operator).

Occasionally, we will require the boundary of Ω to have some regularity. In order to make this precise, we introduce the notion of a defining function.

Definition 1. *Let $k \in \mathbb{N} \cup \{\infty\}$. A bounded set Ω is said to have C^k boundary if there exists a real-valued C^k function ρ, defined in a neighborhood U of $\overline{\Omega}$, such that*

(1) $\Omega = \{z \in U \,:\, \rho(z) < 0\}$*;*

(2) $b\Omega = \{z \in U \,:\, \rho(z) = 0\}$*;*

(3) $d\rho(z) \neq 0$ *for all* $z \in b\Omega$*.*

The function ρ is called a defining function for Ω.

Sometimes ρ is casually referred to as *the* defining function for Ω. This is somewhat justified as it can be shown that any other defining function is locally of the form $h\rho$ on bΩ for some positive C^{k-1} function h. Consequently many properties of ρ do not depend on the particular defining function chosen.

1.1.2 Plurisubharmonic Functions and Pseudoconvexity

Definition 2. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^1 boundary. The complex tangent space to $b\Omega$ at $p \in b\Omega$ is given by*

$$T_p^{1,0}(b\Omega) := \left\{ \sum_{k=1}^{n} v_k \frac{\partial}{\partial z_k}\bigg|_p \,:\, v_k \in \mathbb{C} \text{ and } \sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(p) v_k = 0 \right\},$$

where ρ is a defining function for Ω. Often the space $T_p^{1,0}(b\Omega)$ is referred to as the space of $(1,0)$-tangent vectors to $b\Omega$ at p.

This definition should be compared to the traditional definition of a *real* tangent space.

Likewise, we may define the space of $(0,1)$-vectors to bΩ at p by

$$T_p^{0,1}(\mathrm{b}\Omega) := \left\{ \sum_{k=1}^n v_k \frac{\partial}{\partial \bar{z}_k}\bigg|_p \ : \ \sum_{k=1}^n \frac{\partial \rho}{\partial \bar{z}_k}(p) v_k = 0 \right\}.$$

It should be noted that the direct sum $T_p^{1,0}(\mathrm{b}\Omega) \oplus T_p^{0,1}(\mathrm{b}\Omega)$ is the complexification of the real tangent space to $b\Omega$ at p.

Define the spaces $\Lambda^{1,0}(T_p^* \mathrm{b}\Omega)$ and $\Lambda^{0,1}(T_p^* \mathrm{b}\Omega)$ to be the corresponding dual spaces to $T_p^{1,0}(\mathrm{b}\Omega)$ and $T_p^{0,1}(\mathrm{b}\Omega)$, respectively. Denote by $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ and $\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n$ the corresponding bases of $\Lambda^{1,0}T_p^*(\mathrm{b}\Omega)$ and $\Lambda^{0,1}T_p^*(\mathrm{b}\Omega)$, respectively.

We are now in a position to define Levi pseudoconvexity.

Definition 3. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^2 boundary and defining function ρ. Ω is Levi pseudoconvex at a point $p \in b\Omega$ if*

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k \geq 0$$

for all $v \in T_p^{1,0}(b\Omega) \setminus \{0\}$. Likewise, Ω is called strictly Levi pseudoconvex at a point $p \in b\Omega$ if

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k > 0$$

for all $v \in T_p^{1,0}(b\Omega) \setminus \{0\}$.

In other words, Ω is called Levi pseudoconvex (resp. strictly Levi pseudoconvex) if the Hermitian form relating to the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)_{j,k=1}^n$ is positive semi-definite (resp. positive definite) on the complex tangent space of bΩ at the point p. For emphasis we may sometimes refer to the former condition in Definition 3 as *weak pseudoconvexity*.

A similar notion is that of a plurisubharmonic function, but first we must define subharmonic functions.

Definition 4. *An upper semi-continuous function $u : \Omega \to [-\infty, \infty)$, not identically equal to*

$-\infty$, on a domain $\Omega \subseteq \mathbb{C}$ is subharmonic if for every $p \in \Omega$ there exists an $r_0 > 0$ such that $B(p, r_0) \subseteq \Omega$, and

$$u(p) \leq \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{i\theta}) d\theta \qquad \textit{for all } r \in (0, r_0).$$

This is known as the submean value property.

We denote the class of subharmonic functions on a domain $\Omega \subseteq \mathbb{C}$ by $\mathrm{SH}(\Omega)$.

Subharmonicity can be characterized in terms of the Laplacian in the case that u is a twice-differentiable function [37, Proposition II.4.8].

Proposition 1. *A twice-differentiable function u on a domain $\Omega \subseteq \mathbb{C}$ is subharmonic if and only if*

$$4\frac{\partial^2 u}{\partial z \partial \bar{z}} = \Delta u(z) \geq 0.$$

If u is not twice-differentiable, then subharmonicity may still be characterized in terms of the Laplacian, interpreted as a Riesz measure [38, Section 3.7].

With subharmonic functions in hand we are in a position to define plurisubharmonic functions.

Definition 5. *An upper semi-continuous function $u : \Omega \to [-\infty, \infty)$, not identically equal to $-\infty$, is called plurisubharmonic on a domain $\Omega \subseteq \mathbb{C}^n$ if its restriction to any complex line is either subharmonic on equal to $-\infty$; that is, $\lambda \mapsto u(p + \lambda v)$ is either a subharmonic function or equal to the constant function $-\infty$ on $\{\lambda \in \mathbb{C} \,:\, p + \lambda v \in \Omega\}$ for all $p \in \Omega$ and $v \in \mathbb{C}^n$. u is called strictly plurisubharmonic at a point $p \in \Omega$ if there exists a $c > 0$ such that $u - c\|\cdot\|^2$ is plurisubharmonic in a neighborhood of p.*

Similarly to Levi pseudoconvexity, a plurisubharmonic function that is *not* strictly plurisubharmonic a point p is sometimes said to be *weakly plurisubharmonic* at p.

We denote the class of plurisubharmonic functions on a domain $\Omega \subseteq \mathbb{C}^n$ by $\mathrm{PSH}(\Omega)$. Note that $\mathrm{PSH}(\Omega) = \mathrm{SH}(\Omega)$ whenever $n = 1$.

In the case that u is twice-differentiable, plurisubharmonicity can be similarly characterized in terms of the complex Hessian [37, Proposition II.4.9].

Proposition 2. *A twice-differentiable function u on a domain $\Omega \subseteq \mathbb{C}^n$ is plurisubharmonic if and only if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k \geq 0$$

for all $v \in \mathbb{C}^n \setminus \{0\}$ and $p \in \Omega$. Further, u is strictly plurisubharmonic at $p \in \Omega$ if the inequality is strict. (That is, the complex Hessian is positive-definite).

Note the difference between Levi pseudoconvexity and plurisubharmonicity: u being plurisubharmonic requires the complex Hessian of u to be positive semi-definite at points of its domain, while Levi pseudoconvexity of Ω requires the complex Hessian of the defining function of Ω to be positive semi-definite on the complex tangent space at each point of bΩ.

Because plurisubharmonic functions are allowed to attain the value "$-\infty$" at points of their domain, it is often useful to understand the strength of the singularity at such points. For this we define the *Lelong number* of a plurisubharmonic function at a point.

Definition 6. *The Lelong number of $\varphi \in PSH(\Omega)$ at a point $a \in \Omega$ is given by*

$$\nu(\varphi, a) = \lim_{r \to 0^+} \frac{(2\pi)^{-1} \Delta\varphi(B(a,r))}{dV_{n-1}\big(B(a,r) \cap \mathbb{C}^{n-1}\big)}.$$

In other words, the Lelong number of φ at $z = a$ is the $(n-1)$-dimensional density of the Riesz measure $(2\pi)^{-1}\Delta\varphi$ (interpreted in the sense of distributions) at that point. In the particular case that $n = 1$, we use the convention that $\mathbb{C}^0 = \{a\}$ and $\mathrm{d}V_0 = \delta_a$, the unit delta mass, so that

$$\mathrm{d}V_{n-1}\big(B(a,r) \cap \mathbb{C}^{n-1}\big) = \mathrm{d}V_0(\{a\}) = 1.$$

Therefore, when $n = 1$ and $f \in \mathcal{O}(G)$, $(2\pi)^{-1}\Delta(\log|f|)$ is a just a sum of point masses, one at each zero of f whose weight is the multiplicity of the zero at that point; e.g.

$$\frac{1}{2\pi}\Delta(\log|z\cdot(z-a)^2|) = 2\delta_a + \delta_0,$$

in which case it follows that the Lelong numbers of the plurisubharmonic function $\log|z\cdot(z-a)^2|$ at the points $z = a$ and $z = 0$ are 2 and 1, respectively.

We plurisubharmonic functions in hand, we may now briefly turn our attention back to pseudoconvexity.

Weak pseudoconvexity may still be defined in a meaningful way when the condition that $\mathrm{b}\Omega$ has C^2 boundary is dispensed with, as follows.

Definition 7. *Let Ω be a domain in $\mathbb{C}^n$. Ω is pseudoconvex if there exists a smooth plurisubharmonic exhaustion function φ for Ω; that is, there exists an infinitely-differentiable plurisubharmonic function $\varphi : \Omega \to \mathbb{R}$ with the property that $\{z \in \Omega : \varphi(z) \leq c\}$ is compact in Ω for every $c \in \mathbb{R}$.*

It should come as no surprise that the two notions are equivalent when Ω is bounded with C^2 boundary (see Krantz [25, Theorem 3.3.5] or Range [37, Theorem II.5.8] for a long list of equivalent conditions for pseudoconvexity):

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. Ω is pseudoconvex if and only if it is Levi pseudoconvex.*

1.2 Weighted and Unweighted Bergman Spaces

A nonnegative measurable function μ on Ω is called a *weight*. To each of these weights μ there corresponds a Hilbert space $L^2(\Omega,\mu)$ of functions that are square-integrable with respect to μ, that is,

$$L^2(\Omega,\mu) = \left\{ f : \Omega \to \mathbb{C} \text{ is measurable } : \|f\|^2_{\Omega,\mu} := \int_\Omega |f(\zeta)|^2 \mu(\zeta)\mathrm{d}V < \infty \right\}.$$

L^2-spaces, as these are called, are extremely useful in complex analysis, as well as function theory as a whole, because of their Hilbert space structure.

We denote by $L^2_h(\Omega,\mu)$ the subspace of $L^2(\Omega,\mu)$ consisting of those functions that are also holomorphic on Ω; in other words, $L^2_h(\Omega,\mu) = L^2(\Omega,\mu)\cap\mathcal{O}(\Omega)$. For simplicity, we will use the convenient shorthands

- $L^2(\Omega) := L^2(\Omega, 1)$,
- $L^2_h(\Omega) := L^2_h(\Omega, 1)$,
- and $\|f\|_\Omega := \|f\|_{\Omega,1}$.

The space $L^2_h(\Omega)$ is known as the *Bergman space* of Ω. Likewise, the space $L^2_h(\Omega,\mu)$ is known as the *weighted Bergman space* of Ω with respect to the weight μ. When emphasis is necessary to distinguish it from its weighted counterparts, $L^2_h(\Omega)$ is sometimes called the *unweighted* Bergman space of Ω. Bergman spaces (weighted and unweighted) will be, broadly speaking, the underlying focus of this dissertation.

We will also briefly make use of L^2-spaces of forms. We define $L^2_{(0,1)}(\Omega,\mu)$ to be those (0,1)-forms (defined above) whose coefficients, written in the standard basis $\mathrm{d}\bar{z}_1,\ldots,\mathrm{d}\bar{z}_n$, are square-integrable with respect to the weight μ. More precisely,

$$L^2_{(0,1)}(\Omega,\mu) = \left\{ u = \sum_{k=1}^{n} u_k \mathrm{d}\bar{z}_k \in \Lambda^{0,1}(T^*\Omega) \ : \ \|u\|^2_{\Omega,\mu} := \sum_{k=1}^{n} \|u_k\|^2_{\Omega,\mu} < \infty \right\}.$$

It should be noted that the space $L^2_{(0,1)}(\Omega,\mu)$ will not change if the forms are represented in a different frame, as any two norms on the finite-dimensional vector space $\Lambda^{0,1}T^*_p(\Omega)$ are equivalent [15, §5.1 Exercise 6].

The $\bar{\partial}$ *operator* is one of the most powerful tools in complex analysis, which we denote by "$\bar{\partial}$". Though the $\bar{\partial}$ operator may be defined on (p,q)-forms, for our purposes it suffices to view it as the densely-defined operator from $L^2(\Omega)$ to $L^2_{(0,1)}(\Omega)$ given by

$$\bar{\partial}u = \sum_{k=1}^{n} \frac{\partial u}{\partial \bar{z}_k}\mathrm{d}\bar{z}_k.$$

Note that the kernel of this operator is precisely those functions which are holomorphic on Ω (though this does not remain true when the $\bar{\partial}$ operator is defined as an operator on forms).

As far as Bergman spaces are concerned, we are particularly interested in weights that do not alter the space of functions induced.

Definition 8. *We say that two weights μ_1 and μ_2 on Ω are equivalent if $L^2_h(\Omega, \mu_1) = L^2_h(\Omega, \mu_2)$ as sets.*

In fact, Definition 8 implies a much stronger relationship between Bergman spaces of equivalent weights (Proposition 8 below).

Biholomorphic mappings, i.e., surjective holomorphic mappings with holomorphic inverse, induce isometric isomorphisms between unweighted Bergman spaces.

Proposition 3. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$, and let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a biholomorphic mapping from Ω_1 onto Ω_2. Then $L^2_h(\Omega_1)$ and $L^2_h(\Omega_2)$ are isometrically isomorphic as Hilbert spaces.*

Proof. Define the mapping $T : L^2_h(\Omega_2) \to L^2_h(\Omega_1)$ by

$$f(z) \mapsto f\big(\Phi(z)\big) \cdot \det\left(\frac{\partial \Phi_j}{\partial z_k}(z)\right)_{j,k=1}^{n}.$$

It is straightforward to see that this map is well-defined. Indeed, the $\frac{\partial \Phi_j}{\partial z_k}(z)$, $1 \le j, k \le n$, are holomorphic, and the determinant function is simply a polynomial of the matrix entries, so $T(f)$ is holomorphic for any $f \in L^2_h(\Omega_2)$. Further, because $\left|\det\left(\frac{\partial \Phi_j}{\partial z_k}(z)\right)_{j,k=1}^{n}\right|^2$ is equivalent to the Jacobian determinant of the differential of Φ in the underlying $2n$ real variables [37, Lemma I.2.1], we see that

$$\|T(f)\|^2_{\Omega_1} = \int_{\Omega_1} |f\big(\Phi(z)\big)|^2 \cdot \left|\det\left(\frac{\partial \Phi_j}{\partial z_k}(z)\right)_{j,k=1}^{n}\right|^2 \mathrm{d}V = \int_{\Omega_1} |f(w)|^2 dV = \|f\|^2_{\Omega_1}.$$

T_Φ is bijective with $(T_\Phi)^{-1} = T_{\Phi^{-1}}$. $\square$

1.2.1 Admissible Weights and Bergman Kernels

We are interested in those weighted Bergman spaces on which a unique reproducing kernel can be defined. More precisely, we are interested in conditions on the weight $\mu : \Omega \to [0, \infty)$ which give rise to a function $K_{\Omega,\mu} : \Omega \times \Omega \to \mathbb{C}$ with the following properties:

(1) for every $\zeta \in \Omega$, the function $z \mapsto K_{\Omega,\mu}(z, \zeta)$ belongs to $L^2_h(\Omega, \mu)$, and

(2) $K_{\Omega,\mu}$ has the *reproducing property*, that is

$$f(z) = \int_\Omega f(\zeta) K_{\Omega,\mu}(z, \zeta)\mu(\zeta)dV$$

for all $f \in L^2_h(\Omega, \mu)$.

If it exists, $K_{\Omega,\mu}$ is known as a *reproducing kernel* for Ω with respect to the weight μ. If $\mu \equiv 1$, then $K_\Omega := K_{\Omega,1}$ is simply known as the *Bergman kernel* of Ω.

Aronszajn [1, Part I, Section 2] provides us with necessary and sufficient conditions for the existence of a reproducing kernel.

Proposition 4. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\mu : \Omega \to [0, \infty)$ be a weight. Further suppose that $L^2_h(\Omega, \mu)$ is a Hilbert space. Then the following hold:*

(1) If a reproducing kernel exists for $L^2_h(\Omega, \mu)$ then it is unique.

(2) For the existence of a reproducing kernel $K_{\Omega,\mu}$ it is a necessary and sufficient condition that for every $z \in \Omega$, the evaluation functional $E_z : L^2_h(\Omega, \mu) \to \mathbb{C}$ given by $E_z(f) \to f(z)$ is continuous.

This motivates the following definition.

Definition 9. *Let $\Omega \subseteq \mathbb{C}$ be a domain. A weight $\mu : \Omega \to [0, \infty)$ is admissible if the evaluation functionals $E_z : L^2_h(\Omega, \mu) \to \mathbb{C}$ given by $E_z(f) = f(z)$ on $L^2_h(\Omega, \mu)$ are continuous for every $z \in \Omega$, and $L^2_h(\Omega, \mu)$ is a closed subspace of $L^2(\Omega, \mu)$.*

The requirement that $L_h^2(\Omega, \mu)$ be a closed subspace of $L^2(\Omega, \mu)$ is simply to guarantee that $L_h^2(\Omega, \mu)$ is a Hilbert space.

So, in order to have a Bergman kernel function, a restriction must be imposed on the associated weight. Fortunately, this is not much of a restriction. For instance [31, Theorem 3.1],

Proposition 5. *If μ is a weight on Ω and the function $1/\mu$ is locally integrable on Ω, then μ is an admissible weight.*

Or more generally [32, Corollary 3.1],

Proposition 6. *Let μ be a weight on Ω. Assume that for each $z \in \Omega$ there exists a compact set $Y \subset \Omega$ which contains z and has the following property: for any $p \in bY$ there exists a neighborhood U_p of p in Ω and a number $a_p > 0$ such that the function μ^{-a_p} is integrable on U_p with respect to the Lebesgue measure. Then μ is admissible. If in particular, the function μ^{-a} is locally integrable on Ω for some $a > 0$ then μ is admissible.*

Bergman kernels possess many useful properties for the analysis of weighted and unweighted Bergman spaces [31, Theorem 2.1].

Theorem 2. *Let μ be an admissible weight on Ω and let $K_{\Omega,\mu}$ be the associated Bergman kernel. Then*

(1) for any complete orthonormal system $\{\varphi_k\}_{k=1}^{\infty}$ in $L_h^2(\Omega, \mu)$

$$K_{\Omega,\mu}(z, \zeta) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(\zeta)}, \qquad z, w \in \Omega,$$

where the convergence is uniform on compact subsets of $\Omega \times \Omega$;

(2) for any $z, w \in \Omega$

$$K_{\Omega,\mu}(z, \zeta) = \overline{K_{\Omega,\mu}(\zeta, z)};$$

(3) the orthogonal projection $P_{\Omega,\mu}$ from $L^2(\Omega,\mu)$ onto the closed subspace $L^2_h(\Omega,\mu)$ is given by

$$P_{\Omega,\mu}(f) = \int_\Omega K_{\Omega,\mu}(z,\zeta) f(\zeta)\mu(\zeta)dV.$$

The unweighted Bergman kernel also induces a biholomorphically-invariant Hermitian metric.

Definition 10. *Let Ω be a domain in $\mathbb{C}^n$ whose associated (unweighted) Bergman space is non-trivial. Then*

$$\sum_{j,k=1}^{n} g_{jk}(z) dz_j \otimes d\bar{z}_k,$$

where

$$g_{jk}(z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_\Omega(z,z), \tag{1.1}$$

is a biholomorphically Hermitian metric known as the Bergman metric.

1.3 Hartogs Domains, Balanced Domains, and More

1.3.1 Hartogs Domains with Complete N-circled Fibers

Let $G \subseteq \mathbb{C}^M$ be a domain. Consider a domain of the form

$$D = D_\varphi(G) = D(G) = \left\{(z,w) \in G \times \mathbb{C}^N \,:\, \|w\| < e^{-\varphi(z)}\right\} \subseteq \mathbb{C}^M \times \mathbb{C}^N, \tag{1.2}$$

where $\varphi : G \to [-\infty,\infty)$ is an upper semi-continuous function. D is known as a *Hartogs domain over G with complete N-circled fibers*.

It can be shown that $D_\varphi(G)$ is pseudoconvex if and only if G is pseudoconvex and φ is plurisubharmonic [22, Proposition 2.2.22].

Because of the rotational symmetry in the second variable, many useful examples and counterexamples in several complex variables can be found within this class of domains. There is a close relationship between the Bergman space of $D_\varphi(G)$ and weighted Bergman spaces over the base domain G. We start with a lemma [23, Lemma 3.1].

Lemma 1. *Let $D = D_\varphi(G) \subseteq G \times \mathbb{C}^N$ be a Hartogs domain over $G \subseteq \mathbb{C}^N$ with complete N-circled fibers.*

(1) If $f \in \mathcal{O}(D)$, then there exist $f_n \in \mathcal{O}(D)$, $n \in \mathbb{Z}_+^N$, such that

$$f(z,w) = \sum_{n\in\mathbb{Z}_+^N} f_n(z)w^n, \qquad (z,w) \in D, \tag{1.3}$$

and the series is uniformly convergent on compact subsets of D.

(2) If $f \in L_h^2(D)$, then $f_n(z)w^n \in L_h^2(D)$ for every $n \in \mathbb{Z}_+^N$ and the series (1.3) is convergent in $L_h^2(D)$.

In fact, more can be said. Let $f(z,w) = \sum_{n\in\mathbb{Z}_+^N} f_n(z)w^n \in L_h^2(D)$, where D is given by (1.2). Then

$$\begin{aligned}
\|f\|_D^2 &= \int_D |f(z,w)|^2 \mathrm{d}V_{M+N}(z,w) \\
&= \sum_{n\in\mathbb{Z}_+^N} \int_D |f_n(z)|^2 |w^n|^2 \mathrm{d}V_{M+N}(z,w) \\
&= \sum_{n\in\mathbb{Z}_+^N} \left(\int_{\mathbb{S}^N} \prod_{k=1}^N \left(\frac{|w_k|}{\|w\|} \right)^{n_k} \mathrm{d}\sigma(w) \right) \int_G |f_n(z)|^2 \left(\int_0^{e^{-\varphi(z)}} r^{2N+2|n|-1} \mathrm{d}r \right) \mathrm{d}V_M(z) \\
&= \sum_{n\in\mathbb{Z}_+^N} C(n) \int_G |f_n(z)|^2 e^{-2(N+|n|)\varphi(z)} \mathrm{d}V_M(z),
\end{aligned} \tag{1.4}$$

where

$$C(n) := \frac{1}{2N+2|n|} \int_{\mathbb{S}^N} \prod_{k=1}^N \left(\frac{|w_k|}{\|w\|} \right)^{n_k} \mathrm{d}\sigma(w).$$

We will compute (a more general version of) $C(n)$ explicitly in Chapter 4.

Put simply: the (unweighted) L^2-norm of a function $f_n(z)w^n \in L_h^2(D)$ is equivalent to the L^2-norm of f_n over G with weight $e^{-2(N+|n|)\varphi}$, where the constant depends on N and $n \in \mathbb{Z}_+^N$. This has useful implications for the Bergman kernel, as we will see.

1.3.2 Balanced Domains

Let $h : \mathbb{C}^n \to [0, \infty)$ be an upper semi-continuous function. Further, let h be homogeneous; that is, let h satisfy $h(\lambda z) = |\lambda| h(z)$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Consider a domain of the form

$$D = D_h = \{z \in \mathbb{C}^n \, : \, h(z) < 1\}.$$

D is known as a *balanced domain*. Note that $D = \mathbb{C}^n$ is equivalent to $h \equiv 0$. It can be shown that D_h is pseudoconvex if and only if $\log h$ is plurisubharmonic [22, Proposition 2.2.22].

Equivalently, a domain D is balanced if $\lambda z \in D$ whenever $\lambda \in \mathbb{D}$ and $z \in D$; the unique upper semi-continuous function h associated to D can be defined by

$$h(z) := \inf \{r \in (0, \infty) \, : \, z \in rD\}. \tag{1.5}$$

This function h is known as the *Minkowski function* of D.

The Bergman space of a balanced domain is related to the Bergman space of a Hartogs domain with complete 1-circled fibers in the following way [35]. Consider the mapping Φ on $D_h \setminus \{z \in \mathbb{C}^n \, : \, z_n = 0\}$ given by

$$(z_1, \ldots, z_{n-1}, z_n) \mapsto \left(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}, z_n \right). \tag{1.6}$$

Observe that Φ is a biholomorphic mapping of $D_h \setminus \{z \in \mathbb{C}^n \, : \, z_n = 0\}$ onto the complete 1-circled Hartogs domain given by

$$D_{\log h(\cdot,1)}(\mathbb{C}^{n-1}) = \left\{(\zeta, \eta) \in \mathbb{C}^{n-1} \times \mathbb{C} \, : \, |\eta| < e^{-\log h(\zeta,1)}\right\}.$$

By Proposition 3, the Bergman space of $D_{\log h(\cdot,1)}(\mathbb{C}^{n-1})$ has the same dimension as the Bergman space over $D_h \setminus \{z \in \mathbb{C}^n \, : \, z_n = 0\}$.

Actually, the Bergman space of $D_h \setminus \{z_n = 0\}$ is isometrically isomorphic to the Bergman space of D_h [2, p. 687]. We conclude that the Bergman space of D_h has the same dimension as the Bergman space of $D_{\log h(\cdot,1)}(\mathbb{C}^{n-1})$, a Hartogs domain with complete 1-circled fibers with $\mathbb{C}^{n-1}$ as its base domain.

A particular class of balanced domains we are interested in is the class of *elementary balanced domains*. These are balanced domains $D_h = \{z \in \mathbb{C}^n \,:\, h(z) < 1\}$ where h is of the form

$$h(z) = \prod_{k=1}^{n} |A_k z|^{t_k}.$$

Here $A_1, \ldots, A_n : \mathbb{C}^n \to \mathbb{C}$ are nontrivial linear mappings and the $t_1, \ldots, t_n > 0$ are such that $t_1 + \ldots + t_n = 1$.

1.3.3 Hartogs Domains Over G with k-Dimensional Balanced Fibers

We say that D is a *Hartogs domain over G with k-dimensional balanced fibers* if for any $z \in G$ the fiber

$$D_z := \{w \in \mathbb{C}^k \,:\, (z, w) \in D\}$$

is a balanced domain in $\mathbb{C}^k$.

We may define a function $H : G \times \mathbb{C}^k \to [0, \infty)$ by

$$H(z, w) := h_{D_z}(w),$$

where $h_{D_z}(w)$ is the Minkowski function (1.5) on the balanced domain D_z. Note [21, Remark 1.6.4] that

$$D = \{(z, w) \in G \times \mathbb{C}^k \,:\, H(z, w) < 1\},$$

$H(z, \lambda w) = |\lambda| H(z, w)$ for $\lambda \in \mathbb{C}$, and H is upper semi-continuous on $\Omega \times \mathbb{C}^k$. Similarly to balanced domains, the pseudoconvexity of D is equivalent to the plurisubharmonicity of $\log H$ in addition to the pseudoconvexity of G [21, Proposition 4.1.14].

1.4 The Lu Qi-Keng Conjecture

The Riemann mapping theorem states that any simply connected domain properly contained in the plane is biholomorphically equivalent to the unit disk [10], [18], [46]. This fundamental result and its generalizations [40] show that domains in the plane are reasonably understood up to biholomorphism.

Poincaré [36] showed that the unit ball and the unit bidisk in $\mathbb{C}^2$ are biholomorphically inequivalent. This indicates that the situation in $\mathbb{C}^n$, $n \geq 2$, is much more delicate.

In an attempt to classify domains biholomorphic to the ball in higher dimension, Lu Qi-Keng showed [27] that any bounded domain whose associated Bergman metric has constant holomorphic sectional curvature is biholomorphically equivalent to the ball. To do this, Lu Qi-Keng took advantage of the so-called *Bergman representative coordinates*

$$\sum_{k=1}^{n} g_{kj}^{-1}(p) \frac{\partial}{\partial \bar{\zeta}_k} \log \frac{K_\Omega(z,\zeta)}{K_\Omega(\zeta,\zeta)}\bigg|_{\zeta=p}, \qquad j = 1, \ldots, n,$$

where the g_{kj}^{-1} are the entries of the inverse matrix to $(g_{j,k})_{j,k=1}^{n}$ defined above (1.1). (To see that this matrix is indeed invertible, refer to Range [37, Exercise IV.4.1(ii)].) The Bergman representative coordinates are well-defined locally near $p \in \Omega$, as the function $z \mapsto K_\Omega(z,z)$ is real and positive for any bounded Ω [25, Proposition 1.4.13]. Consequently, by continuity one may find a product neighborhood U of (p,p) so that $K_\Omega(z,\zeta) \neq 0$ on U.

These coordinates are especially useful because biholomorphic mappings become complex-linear when written in them.

A clear obstruction to the *global* definition of these coordinates is if K_Ω has a zero on $\Omega \times \Omega$. At the time of Lu Qi-Keng's paper, the only domains for which the Bergman kernel had explicit formulae were the ball and the polydisk (which have no zeroes), so Lu Qi-Keng naturally wondered whether this was true for every domain. In other words: *Does there exist a domain whose Bergman kernel has a zero?* This became known as the Lu Qi-Keng conjecture.

Swarzyński [42], Rosenthal [39], as well Suita and Yamada [45] found examples of domains whose Bergman kernel function has zeroes, however these examples all had nonzero genus. This led some to believe that zeroes of the Bergman kernel depended on the topological structure of the domain in some way. Twenty years after Lu Qi-Keng's original paper, Boas [4] found an example of a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^2$, diffeomorphic to the ball, whose Bergman kernel has zeroes. Furthermore, Boas [5] showed that domains which fail the Lu Qi-Keng conjecture were the norm rather than the exception: two reasonable topologies in which such domains are dense were exhibited.

Regarding the class of weighted Bergman spaces of $\mathbb{D}$, Perälä [34] presented a method for generating Bergman kernels with arbitrary, but finitely many zeroes. In particular, it was shown that an integrable weight which induces a Bergman kernel having zeroes may be, in some instances, replaced with an equivalent weight (Definition 8), whose induced Bergman kernel is zero-free. It was posed whether an integrable weight μ on a domain Ω whose induced Bergman kernel has a zero at a point $(p, q) \in \Omega \times \Omega$ could always be replaced by an equivalent weight $\tilde{\mu}$ whose Bergman kernel does not have a zero at (p, q). An alternative question was also posed: *Given a Bergman kernel $K_{\Omega,\mu}$ that is nonzero at a point $(p, q) \in \Omega \times \Omega$, is it possible to replace μ with an equivalent weight $\tilde{\mu}$ so that $K_{\Omega}(p, q) = 0$?*

Additionally, Perälä showed that a Bergman kernel induced by a radial weight on $\mathbb{D}$ cannot have infinitely many zeroes; it was also posed whether this persisted for radial weights on the entire plane.

1.5 The Wiegerinck Problem

The Bergman space of a domain $\Omega \subseteq \mathbb{C}^n$ with weight μ is in particular a vector space, so it is natural to ask related questions about it. For instance, is the vector space dimension of a nontrivial Bergman space always infinite?

Wiegerinck addressed this question in 1984 [47]. First, Wiegerinck showed that the unweighted Bergman space for a domain in the complex plane cannot have finite dimension unless it is trivial, answering the question affirmatively in the one-dimensional case. Second it was shown that for each natural number k there exists a domain $\Omega_k \subset \mathbb{C}^2$ whose unweighted Bergman space has dimension k. However, these latter domains Ω_k are not pseudoconvex. Taking into consideration that every domain in the complex plane is vacuously pseudoconvex (for instance, the complex tangent space of a point on the boundary of a domain in the plane is trivial), the conjecture was adjusted accordingly: *Does there exist a pseudoconvex domain with finite but nontrivial (unweighted) Bergman space?*

The calculation (1.4) above indicates that a natural next step in solving the Wiegerinck problem may be to solve the Wiegerinck problem on complete pseudoconvex N-circled Hartogs domains with base in the complex plane. This case was largely solved by Jucha [23], with the exception of Hartogs domains whose one-dimensional base has polar complement and whose associated plurisubharmonic function is harmonic.

The main result of Jucha [23, Theorem 4.1] is a necessary and sufficient condition for the nontriviality or infinite-dimensionality of domains of the form

$$\left\{(z,w) \in \mathbb{C} \times \mathbb{C}^N \,:\, \|w\| < e^{-\varphi(z)} \text{ and } \varphi \in \mathrm{SH}(\mathbb{C})\right\} \subseteq \mathbb{C} \times \mathbb{C}^N$$

in terms of the Riesz measure $\Delta\varphi$. A critical step of the associated proof is the following.

Theorem 3. *Let a domain $G \subseteq \mathbb{C}$ and a function $\varphi \in SH(G)$ be such that there exists a $D(z_0,\delta)$, relatively compact in G, with the following property:*

$$\begin{aligned} \Delta\varphi &\not\equiv 0 \qquad \text{on} \qquad D(z_0,\delta), \\ \nu(\varphi,z) &\equiv 0 \qquad \text{on} \qquad D(z_0,\delta). \end{aligned}$$

Then the dimension of $L^2_h(D_\varphi(G))$ is infinite.

Pflug and Zwonek [35, Theorem 7] used the biholomorphism (1.6) to solve the Wiegerinck problem in the class of balanced domains in $\mathbb{C}^2$. In particular they showed that a pseudoconvex balanced domain $D_h \subseteq \mathbb{C}^2$ has trivial Bergman space if and only if it is an elementary balanced domain or all of $\mathbb{C}^2$. They then posed the question of whether this dichotomy persists in higher dimensions. That is, "*Do there exist pseudoconvex balanced domains in* $\mathbb{C}^n$, $n > 2$, *with trivial Bergman space that are not elementary balanced domains?*

2. EQUIVALENT BERGMAN SPACES WITH INEQUIVALENT WEIGHTS*

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and μ be a weight on Ω. We now address the following questions of Perälä [34].

(1) Is it possible to replace μ with an equivalent weight $\tilde{\mu}$ so that $K_{\Omega,\tilde{\mu}}$ has zeroes?

(2) Does there exist an integrable, radial weight on $\mathbb{C}$ whose induced Bergman kernel function has infinitely many zeroes?

Both questions will be answered in the affirmative. The results of this chapter have been published in the *Journal of Geometric Analysis* [8].

Before proceeding, we should verify that our definition of an admissible weight coincides with another standard definition [31].

Proposition 7. *Let μ be a weight on a domain $\Omega \subseteq \mathbb{C}^n$. Then μ is admissible if and only if the norm of the point evaluation functional $E_z : f \to f(z)$ is locally bounded (if the operator norm of E_z is thought of as a real-valued function on Ω).*

Proof. Suppose that μ is admissible and let V_z be an open set containing z with $\overline{V_z} \subseteq D$. Since

$$\sup_{w \in V_z} |E_w(f)| = \sup_{w \in V_z} |f(w)| < \infty$$

for every $f \in L_h^2(D, \mu)$, an application of the uniform boundedness principle [15, Proposition 5.13] to the family $\{E_w : w \in V_z\}$ of continuous linear functionals shows that

$$\sup_{w \in V_z} \|E_w\| < \infty.$$

The converse, including the condition that $L_h^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, follows from known results [31, Proposition 2.1]. $\square$

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2.1 Creating Zeroes in the Bergman Kernel

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and μ be an admissible weight on Ω. In this section we will present a procedure generating an admissible weight $\tilde{\mu}$, equivalent to μ, with the property that $K_{\Omega,\tilde{\mu}}$ has a zero. In fact, we can specify the location of the zero on $\Omega \times \Omega$ up to some arbitrarily small error.

For the proof, we require a generalization of Ramadanov's Theorem [33].

Theorem 4 (Weighted generalization of Ramadanov's theorem). *Let $\{D_k\}_{k=1}^{\infty}$ be a sequence of domains in $\mathbb{C}^n$ and set $D := \bigcup_{k=1}^{\infty} D_k$. Let μ and $\mu_1, \mu_2, \ldots$ be admissible weights on D and $D_1, D_2, \ldots$, respectively (extend μ_k by μ on D). Assume moreover that*

(1) for any $m \in \mathbb{N}$ there is an $M = M(m)$ such that $D_m \subseteq D_\ell$ and $\mu_m(z) \leq \mu_\ell(z) \leq \mu(z)$ for $\ell \geq M(m)$, $z \in D_m$;

(2) $\mu_m \xrightarrow[m\to\infty]{} \mu$ pointwise almost everywhere on D.

Then

$$\lim_{m\to\infty} K_{D_m,\mu_m} = K_{D,\mu}$$

locally uniformly on $D \times D$.

The procedure for creating zeroes in the Bergman kernel is most easily seen in the special case that Ω is bounded and μ is continuous: Specify a point $p \in \Omega$ with $\mu(p) \neq 0$; such a point must exist, for otherwise μ is identically zero and the admissibility of μ is violated. By continuity of μ, there is a neighborhood of p where μ is nonzero. Then the weight $\mu(z) \cdot \|z-p\|^{-2n}$ is not integrable in this neighborhood. Therefore every f in the Bergman space with weight $\mu(z) \cdot \|z-p\|^{-2n}$ must be have a zero at p; in particular, the Bergman kernel function induced by this weight must be zero whenever the point p is in either input. Now the weights $\min\left(k, \|z-p\|^{-2n}\right) \cdot \mu(z)$ are each equivalent to μ for each k. Indeed, they are the product of μ and a function that is bounded above and away from zero (recall the assumption that Ω is bounded). Thus Theorem 4 and Hurwitz's Theorem show that for any fixed $\zeta \in \Omega$, the Bergman kernel will have a zero at $(z, \zeta) \in \Omega \times \Omega$ for some z near p whenever k is sufficiently large.

The general case requires dealing with some technicalities which are handled by the following lemma.

Lemma 2. *Let μ_1 be an admissible weight on Ω, and suppose that μ_1 is integrable on a bounded open neighborhood U of some point $z_0 \in \Omega$ with $\overline{U} \subseteq \Omega$. Let μ_2 be a weight on U. Then the weight $\tilde{\mu}_1$ defined by*

$$\tilde{\mu}_1(z) := \begin{cases} \max\big(\mu_1(z), \mu_2(z)\big), & \text{if } z \in U \\ \mu_1(z), & \text{if } z \in \Omega \setminus U \end{cases}$$

is an admissible weight with $L^2_h(\Omega, \tilde{\mu}_1) \subseteq L^2_h(\Omega, \mu_1)$ and continuous inclusion, such that $\mu_2 \leq \tilde{\mu}_1$ on U. Furthermore, if μ_2 is integrable over U as well, then $L^2_h(\Omega, \mu_1)$ and $L^2_h(\Omega, \tilde{\mu}_1)$ are isomorphic as Hilbert spaces.

Proof. Observe that $\|f\|_{\Omega,\mu_1} \leq \|f\|_{\Omega,\tilde{\mu}_1}$ for every measurable function f on Ω. Therefore $L^2(\Omega, \tilde{\mu}_1)$ is a subset of $L^2(\Omega, \mu_1)$ with continuous inclusion. It follows that, for each $z \in \Omega$, the operator norm of E_z, viewed as a functional on $L^2_h(\Omega, \tilde{\mu}_1)$, is at most the operator norm of E_z, viewed as a functional on $L^2_h(\Omega, \mu_1)$. Therefore $\tilde{\mu}_1$ is an admissible weight.

Now suppose that μ_2 is integrable on U as well. By applying the uniform boundedness principle [15, Proposition 5.13] to the family of continuous functionals given by evaluation at each point of U, we may find a $C > 0$ such that

$$\begin{aligned} \|f\|^2_{\Omega,\tilde{\mu}_1} &= \int_U |f(\zeta)|^2 \max\big(\mu_1(\zeta), \mu_2(\zeta)\big) \mathrm{d}V(\zeta) + \int_{\Omega\setminus U} |f(\zeta)|^2 \mu_1(\zeta) \mathrm{d}V(\zeta) \\ &\leq \sup_{z\in U} |f(z)|^2 \cdot \int_U \max\big(\mu_1(\zeta), \mu_2(\zeta)\big) \mathrm{d}V(\zeta) + \|f\|^2_{\Omega,\mu_1} \\ &\leq C \cdot \left(\int_U \max\big(\mu_1(\zeta), \mu_2(\zeta)\big) \mathrm{d}V(\zeta) \right) \cdot \|f\|^2_{\Omega,\mu_1} + \|f\|^2_{\Omega,\mu_1} \\ &\leq \left[C \cdot \left(\int_U \max\big(\mu_1(\zeta), \mu_2(\zeta)\big) \mathrm{d}V(\zeta) \right) + 1 \right] \|f\|^2_{\Omega,\mu_1} \end{aligned}$$

holds for each $f \in L^2_h(\Omega, \mu_1)$. Observe that $\max(\mu_1, \mu_2)$ is integrable over U since both μ_1 and μ_2 are. $\square$

The fact that the equivalent weights induce isomorphic Hilbert spaces is no coincidence. This fact was presented to the author by Straube [43].

Proposition 8. *Let μ_1 and μ_2 be equivalent admissible weights on Ω. Then $L^2_h(\Omega, \mu_1)$ and $L^2_h(\Omega, \mu_2)$ are isomorphic as Hilbert spaces.*

Proof. Let $\iota : L^2_h(\Omega, \mu_1) \to L^2_h(\Omega, \mu_2)$ denote the inclusion map. Suppose that

$$\big\{\big(f_k, \iota(f_k)\big)\big\}_{k=1}^{\infty} \subseteq L^2_h(\Omega, \mu_1) \times L^2_h(\Omega, \mu_2)$$

is a sequence that converges to $(f, g) \in L^2_h(\Omega, \mu_1) \times L^2_h(\Omega, \mu_2)$ in graph norm. By the closed graph theorem [15, Theorem 5.12], it suffices to show that $\iota(f) = g$, or $f = g$. Fix $z_0 \in \Omega$. Because both weights are admissible, and hence point-evaluation functionals are continuous,

$$\lim_{k\to\infty} f_k(z_0) = f(z_0) \qquad \text{and} \qquad \lim_{k\to\infty} f_k(z_0) = \lim_{k\to\infty} \iota\big(f(z_0)\big) = g(z_0).$$

Therefore $f(z_0) = g(z_0)$. Since z_0 was chosen arbitrarily, we see that $f = g$. Thus the inclusion map is continuous as a map from $L^2_h(\Omega, \mu_1)$ to $L^2_h(\Omega, \mu_2)$. The same argument shows the inclusion map in the reverse direction is also continuous. This completes the proof. □

We now have all the necessary tools to prove the main result of this section.

Theorem 5. *Let μ be an admissible weight on a domain $\Omega \subseteq \mathbb{C}^n$. Then there exists an admissible weight μ^*, equivalent to μ, so that K_{Ω,μ^*} has zeroes on $\Omega \times \Omega$.*

Proof. We assume that $L^2_h(\Omega, \mu) \neq \{0\}$; otherwise $K_{\Omega,\mu}$ is identically zero. By translating if necessary, we may assume that $0 \in \Omega$.

If μ is not integrable in any neighborhood of the origin, then $f(0) = 0$ for every $f \in L^2_h(\Omega, \mu)$; in particular, $K_{\Omega,\mu}(0, \zeta) = 0$ for every $\zeta \in \Omega$, and we are done.

Now suppose that μ is integrable in a neighborhood of the origin U. By shrinking U if necessary, we may assume that $\overline{U} \subseteq \Omega$. By replacing μ with the equivalent weight given to us by

Lemma 2 above with μ playing the role of μ_1 and the constant function 1 playing the role of μ_2, we may assume that $\mu \geq 1$ on U.

Define $g(z) = \max(1, \|z\|^{-2n})$ and set

$$\nu(z) := g(z)\mu(z).$$

Note that since $g \geq 1$, we have $\|f\|_{\Omega,\mu} \leq \|f\|_{\Omega,\nu}$, so the inclusion of $L^2_h(\Omega, \nu)$ into $L^2_h(\Omega, \mu)$ is continuous. By composition, then, the evaluation functionals are continuous on $L^2_h(\Omega, \nu)$. We conclude that ν is an admissible weight on Ω, and hence induces a Bergman kernel. By the same argument as in the first paragraph above (recall that we are assuming $\mu \geq 1$ on U and hence ν is not integrable on U), $K_{\Omega,\nu}(0, \zeta) = 0$ for every $\zeta \in \Omega$.

For each natural number k, the function $z \mapsto \min\big(k, g(z)\big)$ is uniformly bounded above and bounded away from zero on Ω, so

$$\mu_k(z) := \min\big(k, g(z)\big)\mu(z)$$

is an equivalent weight to μ for each k. μ_k increases pointwise to ν as $k \to \infty$.

Next we apply Theorem 4. Since $L^2_h(\Omega, \nu)$ is nontrivial (e.g. $z^\alpha f \in L^2_h(\Omega, \nu)$ whenever $f \in L^2_h(\Omega, \mu)$ and α is a multiindex with $|\alpha| = N$), we may find a $\zeta \in \Omega$ so that $z \mapsto K_{\Omega,\nu}(z, \zeta)$ is nontrivial. We claim that $K_{\Omega,\mu_k}(\cdot, \zeta)$ has zeroes for large k. Seeking a contradiction, suppose that $K_{\mu_k}(\cdot, \zeta)$ has no zeroes for every natural number k. We have chosen ζ so that $K_{\Omega,\nu}(\cdot, \zeta)$ is not identically zero, so fix $z_0 \in \Omega$ with $K_{\Omega,\nu}(z_0, \zeta) \neq 0$ close enough to the origin so that z_0 belongs to the connected component of $\{\lambda z_0 \in \Omega : \lambda \in \mathbb{C}\}$ which contains the origin—such a z_0 must exist, for otherwise $K_{\Omega,\nu}(\,\cdot\,, \zeta)$ would be zero in a neighborhood of the origin, and hence identically zero as a function on Ω. Applying Hurwitz's theorem from the classical theory of one complex variable [10], [18], [46] to the connected component of $\{\lambda z_0 \in \Omega : \lambda \in \mathbb{C}\}$ containing the origin shows that $K_{\Omega,\nu}(\lambda z_0, \zeta)$ has no zeroes, contradicting that $K_{\Omega,\nu}(0, \zeta) = 0$. We conclude that $K_{\Omega,\mu_k}(z, \zeta)$ has zeroes for k sufficiently large. Setting μ^* to be μ_k for large k completes the proof. $\square$

Remark. Using the methods of the proof above, something stronger may be shown: that we may—up to any positive error—prescribe the point at which the zero occurs. Furthermore, by carrying out this construction at finitely many points simultaneously, one can show that an equivalent weight exists whose Bergman kernel has zeroes at finitely many predetermined points up to any positive error.

2.2 Radial Weights with Kernel Having Infinitely Many Zeroes in the Plane

It is known [34, Theorem 3.5] that a weighted Bergman kernel function on the unit disk $K_{\mathbb{D},\mu}(z,\zeta)$ cannot have infinitely many zeroes for a fixed $\zeta \in \mathbb{D}$ when μ is radial and integrable.

In this section we show that the analogue in the complex plane fails. In fact, we exhibit an infinite family $\mathcal{W}$ of radial, integrable weights W each inducing a Bergman kernel function $K_{\mathbb{C},W}(z,\zeta)$ having infinitely many zeroes on $\mathbb{C}$ for each fixed nonzero ζ on the plane. This is achieved following a similar construction to Bommier-Hato et al. [7].

Given positive parameters β and γ we may define a holomorphic function $E_{\beta,\gamma}(z)$ by the power series

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}. \tag{2.1}$$

Here, Γ is the classical gamma function represented by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \mathrm{d}t$$

on the right half-plane. The function $E_{\beta,\gamma}$ is known as the Mittag-Leffler function associated to β and γ. It is an entire function of order $1/\beta$ and of type 1 [17].

Theorem 6. *Let $\mathcal{W}$ be the family of integrable weights of the form*

$$W(z) = \frac{1}{2\pi}|z|^n \exp\big(-\alpha|z|^{2m}\big),$$

where $n \in (-2,\infty)$, $\alpha, m \in (0,\infty)$, and $m \notin \mathbb{Z}$. Then every member W of $\mathcal{W}$ is admissible and induces a kernel $K_{\mathbb{C},W}(\cdot,\zeta)$ having infinitely many zeroes for each non-zero ζ in the plane.

Proof. Fix $W \in \mathcal{W}$. We first show that W is admissible. By Proposition 6, it suffices to show that there exists a $c > 0$ such that W^{-c} is locally integrable; setting $c = 1/n$ if $n > 0$ and $c = 1$ otherwise does the trick.

Since W is radial, the monomials are orthogonal in $L^2_h(\mathbb{C}, W)$. Furthermore,

$$\begin{aligned}\|z^k\|^2_{\mathbb{C},W} &= \frac{1}{2\pi}\int_{\mathbb{C}} |z|^{2k}|z|^n e^{-\alpha|z|^{2m}} \mathrm{d}V \\ &= \int_0^\infty r^{2k+n+1}e^{-\alpha r^{2m}}\mathrm{d}r \\ &= \frac{1}{2m}\alpha^{-\frac{2k+n+2}{2m}} \cdot \Gamma\left(\frac{2k+2+n}{2m}\right).\end{aligned}$$

Therefore

$$\left\{ \frac{z^k}{\sqrt{\frac{1}{2m}\alpha^{-\frac{2k+n+2}{2m}} \cdot \Gamma\left(\frac{2k+2+n}{2m}\right)}} \right\}_{k=0}^\infty$$

denotes an orthonormal basis for $L^2_h(\mathbb{C}, W)$, and hence we may apply Theorem 2 above to see that

$$\begin{aligned}K_{\mathbb{C},W}(z,\zeta) &= 2m\alpha^{\frac{n+2}{2m}} \sum_{k=0}^\infty \alpha^{k/m}\frac{(z\bar{\zeta})^k}{\Gamma(\frac{2k+2+n}{2m})} \\ &= 2m\alpha^{\frac{n+2}{2m}} \sum_{k=0}^\infty \frac{\left(\alpha^{1/m}(z\bar{\zeta})\right)^k}{\Gamma\left(\frac{k}{m}+\frac{2+n}{2m}\right)}.\end{aligned}$$

We may write this in terms of a Mittag-Leffler function (2.1) as

$$K_{\mathbb{C},W}(z,\zeta) = 2m\alpha^{\frac{n+2}{2m}} E_{\frac{1}{m},\frac{2+n}{2m}}\left(\alpha^{1/m}(z\bar{\zeta})\right).$$

Fix a nonzero ζ in the plane. It follows from the representation above (by properties of Mittag-Leffler functions) that $K_{\mathbb{C},W}$ is an entire function of order m. Since $m \notin \mathbb{Z}$ by assumption, it is a consequence of the Hadamard factorization theorem [10, Theorem XI.3.7] that the function $K_{\mathbb{C},W}(\cdot,\zeta)$ has infinitely many zeroes. $\square$

3. ON THE DIMENSION OF THE BERGMAN SPACE OF SOME HARTOGS DOMAINS WITH HIGHER DIMENSIONAL BASES*

Let $G \subseteq \mathbb{C}^M$ be a domain. In this section, we expand on the work of Jucha [23] and present sufficient conditions for $L_h^2(D_\varphi(G))$ to have infinite dimension, where we recall that

$$D_\varphi(G) = \left\{(z, w) \in G \times \mathbb{C}^N \,:\, \|w\| < e^{-\varphi(z)}\right\} \subseteq \mathbb{C}^M \times \mathbb{C}^N.$$

We also analyze the Wiegerinck problem on balanced domains and Hartogs domains with k-dimensional fibers. In addition we address a question of Pflug and Zwonek [35].

The results of this section have been published in the *Journal of Geometric Analysis* [9].

3.1 Sufficient Conditions for Infinite-Dimensionality of Some Hartogs Domains

Let $D_\varphi(G)$ be defined as above. Our first result is a generalization of Theorem 3 [23, Proposition 4.3]. It shows in particular that if φ has a point of strong plurisubharmonicity, then the Bergman space of $D_\varphi(G)$ is infinite-dimensional. This is interesting as it is not clear that a local condition should be sufficient for such a global property. A much more general version of this result comes from the work of Gallagher et al. [16], however a particularly simple and direct proof results in the context of complete N-circled Hartogs domains.

Theorem 7. *Let $G \subset \mathbb{C}^M$ be a pseudoconvex domain and $\varphi : G \to [-\infty, \infty)$ be a plurisubharmonic function. Assume that φ is strictly plurisubharmonic on some open set $U \subset G$, and $\nu(\varphi, \cdot) = 0$ on U. Then the Bergman space of $D_\varphi(G)$ has infinite dimension.*

Before proceeding, we require a result implicit in Hörmander [20, Theorem 2.2.1′] which was made explicit by Gallagher et al. [16, Theorem 5]. It is stated as follows.

Theorem 8. *Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain and let $\Phi : \Omega \to [-\infty, \infty)$ be plurisubharmonic. Assume that*

(1) $U \subset \Omega$ is open such that $\Phi - c\| \cdot \|^2$ is plurisubharmonic on U for some constant $c > 0$, and

(2) $v \in L^2_{(0,1)}\left(\Omega, e^{-\Phi}\right)$ is a smooth form such that $\overline{\partial} v = 0$ and supp $v \subset U$.

Then there exists a smooth function $u : \Omega \to \mathbb{R}$ such that $\overline{\partial} u = v$ and

$$\int_\Omega |u|^2 e^{-\Phi} dV \leq \frac{1}{c} \int_\Omega |v|^2 e^{-\Phi} dV.$$

Note that condition (1) means that Φ is strictly plurisubharmonic on U by Definition 5 above.

Proof of Theorem 7. We follow closely a lemma of Gallagher et al. [16, Lemma 6].

By Lemma 1 and the computation (1.4) that follows, it suffices to find for infinitely many $n \in \mathbb{Z}^N_+$ a nontrivial holomorphic function f_n on G such that

$$\int_G |f_n(z)|^2 e^{-2(N+|n|)\varphi(z)} \mathrm{d}V_M < \infty.$$

Indeed, if such $f_n \in \mathcal{O}(G) \setminus \{0\}$ are found, then the set of functions consisting of $f_n w^n \in L^2_h\big(D_\varphi(G)\big)$ for each such n constitutes an infinite set of mutually orthogonal (and hence linearly independent) members of $L^2_h\big(D_\varphi(G)\big)$.

Accordingly, we fix a $n \in \mathbb{Z}^N_+$. Fix $p \in U$. By shrinking U if necessary, we may assume that U is relatively compact in G. Further, since $\nu(\varphi, \cdot) = 0$ on U we may apply a result of Kiselman [24, Theorem 3.4] to see that $\exp(-(N + |n|)\varphi)$ is locally integrable on U. Consequently, we may choose $\varepsilon > 0$ small enough so that $B(p, \varepsilon)$ is relatively compact in U and $\exp(-(N + |n|)\varphi)$ is integrable on $B(p, \varepsilon)$. Also, choose a smooth function $\chi : \mathbb{C}^M \to [0, 1]$ such that $\chi(z) = 1$ when $\|z\| \leq \varepsilon/3$ and $\chi(z) = 0$ when $\|z\| \geq 2\varepsilon/3$.

Set

$$v(z) := \bar{\partial}\chi(z - p)$$

and

$$\Phi_n(z) := 2(N+|n|)\varphi + 2M \cdot \chi(z-p)\log\|z-p\|.$$

Observe that for $n \in \mathbb{Z}_+^N$ with large $|n|$, Φ_n is plurisubharmonic on G and $\Phi_n(z) - \|z\|^2$ is plurisubharmonic on $B(p,\varepsilon)$. Now, $v \in L^2_{(0,1)}\big(G, \exp(-\Phi_n)\big)$, and so applying Theorem 8 yields a smooth function u_n on G such that $\bar\partial u_n = v$ and

$$\int_G |u_n|^2 e^{-\Phi_n} \mathrm{d}V_M \le \int_G |v|^2 e^{-\Phi_n} \mathrm{d}V_M.$$

Furthermore,

$$\begin{aligned}\int_G |u_n(z)|^2 e^{-2(N+|n|)\varphi(z)} \mathrm{d}V &= \int_G |u_n(z)|^2 e^{-\Phi_n(z)+2M\cdot\chi(z-p)\log|z-p|} \mathrm{d}V \\ &= \int_G |u_n(z)|^2 e^{-\Phi_n(z)} e^{2M\cdot\chi(z-p)\log|z-p|} \mathrm{d}V \\ &\lesssim \int_G |u_n(z)|^2 e^{-\Phi_n(z)} \mathrm{d}V < \infty\end{aligned}$$

since the function $\chi(z-p)\log\|z-p\|$ is bounded from above (recall that $\chi(z)$ has compact support). Thus $u_n \in L^2\left(G, \exp\left(2(N+|n|)\varphi\right)\right)$ as well.

Now, $\exp(-\Phi_n)$ is not integrable near p, so u_n must have a zero at p. Define

$$f_n(z) = \chi(z-p) - u_n(z).$$

Since $\bar\partial f_n = \bar\partial\chi(z-p) - \bar\partial u_n(z) = 0$, f_n is holomorphic on G; moreover $f_n(p) = 1$, so f_n is nontrivial.

The argument above works for all n with $|n|$ sufficiently large, so we conclude that there are infinitely many multiindices n possessing a nontrivial $f_n \in L^2_h\left(G, \exp\left(-2(N+|n|)\varphi\right)\right)$, thus completing the proof. □

Remark. Theorem 7 implies in particular that if φ is such that the Monge-Ampère operator of φ, $(dd^c)^M\varphi$, exists (e.g. if φ is locally bounded on G), and $\dim L^2_h(D_\varphi(G)) < \infty$, then $(dd^c)^M\varphi$

is a sum of point masses. Viewing Theorem 7 in this manner yields a more direct analogue to Proposition 4.3 of Jucha [23].

The next result leverages a generalization of the Ohsawa-Takegoshi extension theorem (see [30] for the original work, and see [44] for a statement in the setting of bounded domains in $\mathbb{C}^n$). The generalization is due to Dinew [11] and is stated as follows.

Theorem 9. *Suppose $D \subset \Omega \times \mathbb{C}^{M-1}$ is a pseudoconvex domain, where $\Omega \subset \mathbb{C}$ is a planar domain with nonpolar complement. Let $\varphi \in PSH(D)$. Then, there exists a constant $C > 0$, depending only on Ω, such that for any $z_0 \in \Omega$ and $f \in L^2_h\big(D \cap (\{z_0\} \times \mathbb{C}^{M-1})\big)$ there exists $\tilde{f} \in L^2_h(D)$ with $\tilde{f}(z_0, \cdot) = f$ such that*

$$\int_D |\tilde{f}|^2 e^{-\varphi} dV_M \leq C \int_{D \cap (\{z_0\} \times \mathbb{C}^{M-1})} |f|^2 e^{-\varphi} dV_{M-1}.$$

Theorem 10. *Let $G \subset \mathbb{C}^M$ be a pseudoconvex domain containing the origin, and suppose that $A : \mathbb{C}^M \to \mathbb{C}$ is a nonzero linear mapping whose image $A(G)$ has nonpolar complement in $\mathbb{C}$. Then for any $\varphi \in PSH(G)$, $\dim L^2_h(D_\varphi(G)) = \infty$ whenever $L^2_h\big(D_\varphi(\ker(A) \cap G)\big)$ is infinite-dimensional.*

Proof. After making a complex-linear change of coordinates, we may assume that A is the projection $z \mapsto z_1$ so that G satisfies the conditions of Theorem 9.

Suppose $L^2_h\big(D_\varphi(\ker(A) \cap G)\big)$ has infinite dimension. Then we may assume that there is an infinite set of mutually linearly independent $f_n w^{b(n)} \in L^2_h\big(D_\varphi(\ker(A) \cap G)\big) \setminus \{0\}$, where $b : \mathbb{N} \to \mathbb{Z}^N_+$. (We introduce the function $b(n)$ as it may happen that there are only finitely many powers of w; e.g. we are given $f_1 w^2$, $f_2 w^2$, $f_3 w^2$, and so on.) By the computation (1.4) above,

$$\int_{D_\varphi(G \cap \ker(A))} |f_n(z) w^{b(n)}|^2 \mathrm{d}V_{M-1+N} \cong \int_{G \cap \ker(A)} |f_n(z)|^2 e^{-2(N+|b(n)|)\varphi(z)} \mathrm{d}V_{M-1} < \infty.$$

Applying Theorem 9 above we may holomorphically extend each f_n to a function $\tilde{f}_n$, defined on all of G, with the property that

$$\int_G |\tilde{f}_n|^2 e^{-2(N+|b(n)|)\varphi} \mathrm{d}V_M \lesssim \int_{G\cap\ker(A)} |f_n|^2 e^{-2(N+|b(n)|)\varphi} \mathrm{d}V_{M-1} < \infty.$$

Note that the extensions $\tilde{f}_n(z)w^{b(n)}$ remain linearly independent and nontrivial. Thus $L^2_h(D_\varphi(G))$ has infinite dimension as desired. $\square$

An example of the utility of Theorem 10 is the following.

Corollary 1. *Suppose $G \subset \mathbb{C}^M$ is a pseudoconvex domain and $\varphi \in PSH(G)$. Suppose further that there exists a nonconstant affine linear map $B : \mathbb{C}^M \to \mathbb{C}$ whose restriction to G is bounded. Then $L^2_h(D_\varphi(G))$ has infinite dimension whenever $L^2_h(D_\varphi(G \cap \ker B))$ does.*

Remark. By applying the results of Jucha [23] or Theorem 7, Theorem 10 may be used in the following situations to show that $L^2_h(D_\varphi(G))$ has infinite dimension (below we consider $G\cap\ker(A)$ a subset of $\mathbb{C}^{M-1}$).

- when $\varphi|_{G\cap\ker(A)}$ has a point of strong plurisubharmonicity;
- when $G \cap \ker(A)$ is bounded;
- When $M = 2$ and $G \cap \ker(A)$ has nonpolar complement.

After reading Corollary 1, one might wonder if it is possible to replace the nonzero mapping $B : \mathbb{C}^M \to \mathbb{C}$ with a nontrivial bounded holomorphic function, and $\ker B$ replaced by the zero set of f. This turns out to be, in essence, true. However we must first make the appropriate definitions.

Whenever Y is the zero set of some nontrivial holomorphic function g on a domain $G \subseteq \mathbb{C}^M$, let $Y_0 \subseteq Y$ be the set of regular points of g, that is, points p with $\partial g/\partial z_k(p) \neq 0$ for some $k = 1, \ldots, M$. Note that Y_0 is a complex hypersurface in G. Now let $\psi \in$ PSH(G) and $\iota : Y \hookrightarrow G$ denote the inclusion map. We analogously define $L^2_h(Y_0, \exp(-\psi))$ to be the space of square-integrable holomorphic function with respect to the $(M-1)$-form $\exp(-\psi)\iota^*\mathrm{d}V_M$, where $\iota^*\mathrm{d}V_M$

denotes the pullback of the volume form $\mathrm{d}V_M$ through ι; i.e.,

$$L_h^2(Y_0, e^{-\psi}) = \left\{ f \in \mathcal{O}(Y_0) \;:\; \int_{Y_0} |f|^2 e^{-\psi} \iota^* \mathrm{d}V_M < \infty \right\}.$$

Theorem 11. *Let $G \subseteq \mathbb{C}^M$ be a pseudoconvex domain with a holomorphic function $g \in \mathcal{O}(G)$ whose image Im$(g) \subseteq \mathbb{C}$ has nonpolar complement, and let φ be a plurisubharmonic function on G. Furthermore, suppose there is no irreducible component of $g^{-1}(\{0\}) = Y$ on which $\partial g/\partial z_k$ is identically zero for every $k = 1, \ldots, M$. Then $\dim L_h^2(D_\varphi(G)) = \infty$ whenever there are infinitely many $n \in \mathbb{Z}_+^N$ such that the weighted space $L_h^2\big(Y_0, \exp(-2(N + |n|))\varphi\big)$ is nontrivial.*

The proof of Theorem 11 is an application of an extension result of Ohsawa [28, Theorem 1.1], stated as follows.

Theorem 12. *Let $G \subset \mathbb{C}^M$ be a pseudoconvex domain, and let φ and ψ be plurisubharmonic functions on G, and let g be a holomorphic function on G such that $\sup(\psi + 2\log|g|) \leq 0$ and $\partial g/\partial z_k$ is not identically zero for $k = 1, \ldots, M$ on every irreducible component of $g^{-1}(\{0\}) = Y$. Then, for any holomorphic $(M-1)$-form F on Y_0 satisfying*

$$\left| \int_{Y_0} e^{-\varphi} F \wedge \bar{F} \right| < \infty,$$

there exists a holomorphic M-form $\tilde{F}$ on G such that $\tilde{F} = dg \wedge F$ at any point of Y_0 and

$$\left| \int_D e^{-\varphi+\psi} \tilde{F} \wedge \overline{\tilde{F}} \right| \leq 2\pi \left| \int_{Y_0} e^{-\varphi} F \wedge \bar{F} \right|.$$

Let $G_\Omega(z, \cdot)$ be the negative Green function of a domain $\Omega \subseteq \mathbb{C}$ with pole at z, and let $\Omega' = g(G)$, the image of G under g (where g is as above). As in the discussion before Theorem 3 in Dinew [11], we may substitute

$$2(G_{\Omega'}(0, g(z)) - \log|g(z)|)$$

for ψ in Theorem 12 to get the following corollary.

Corollary 2. *Let $G \subseteq \mathbb{C}^M$ be a pseudoconvex domain and φ be plurisubharmonic. Suppose that the image of $g \in \mathcal{O}(G)$ has nonpolar complement in $\mathbb{C}$, and that $\partial g/\partial z_k$, $k = 1, \ldots, M$ is not identically zero on every irreducible component of $g^{-1}(\{0\}) = Y$. Then, for any holomorphic $(M-1)$-form F on Y_0 satisfying*

$$\left| \int_{Y_0} e^{-\varphi} F \wedge \bar{F} \right| < \infty,$$

there exists a holomorphic M-form $\tilde{F}$ on G such that $\tilde{F} = dg \wedge F$ at any point of Y_0 and

$$\left| \int_{D} e^{-\varphi} \tilde{F} \wedge \overline{\tilde{F}} \right| \leq 2\pi \left| \int_{Y_0} e^{-\varphi} F \wedge \bar{F} \right|.$$

Proof of Theorem 11. By assumption, there are infinitely many $f_n \in \mathcal{O}(Y_0) \setminus \{0\}$, indexed by $n \in \mathbb{Z}_+^N$, such that

$$\int_{Y_0} |f_n|^2 e^{-2(N+|n|)\varphi} \iota^* \mathrm{d}V_M < \infty.$$

Fix such a f_n and let $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_M$. Then $f_n \iota^* \mathrm{d}z$ is a holomorphic $(M-1)$-form on Y_0 with

$$\left| \int_{Y_0} e^{-2(N+|n|)\varphi} (f_n \iota^* \mathrm{d}z) \wedge \overline{f_n \iota^* \mathrm{d}z} \right| \leq \frac{1}{2^M} \int_{Y_0} |f_n|^2 e^{-2(N+|n|)\varphi} \iota^* \mathrm{d}V_M < \infty.$$

We have used that $\mathrm{d}V_M = \frac{(-1)^{M(M-1)/2}}{(2i)^M} \mathrm{d}\bar{z} \wedge \mathrm{d}z$ [37, §III.3.3]. By Corollary 2, there exists a holomorphic function $F_n \in \mathcal{O}(D)$ such that $F_n \mathrm{d}z = (f_n \iota^* \mathrm{d}z) \wedge \mathrm{d}g$ and

$$\begin{aligned}
2^M \int_D |F_n|^2 e^{-2(N+|n|)\varphi} \mathrm{d}V_M &\cong \left| \int_D e^{-2(N+|n|)\varphi} (F_n \mathrm{d}z) \wedge \overline{(F_n \mathrm{d}z)} \right| \\
&\leq 2\pi \left| \int_{Y_0} e^{-2(N+|n|)\varphi} (f_n \iota^* \mathrm{d}z) \wedge \overline{(f_n \iota^* \mathrm{d}z)} \right| \\
&\leq \frac{\pi}{2^{M-1}} \int_{Y_0} |f_n|^2 e^{-2(N+|n|)\varphi} \iota^* \mathrm{d}V_M < \infty.
\end{aligned}$$

It follows that the set $\{F_n w^n\}$ is an infinite family of pairwise orthogonal members of $L_h^2(D_\varphi(G))$

as desired. $\square$

So far, most conditions for the infinite-dimensionality of a Hartogs domain over a hyperplane to propagate to the full Hartogs domain (Corollary 1 is the simplest example of this phenomenon) were conditions on the base domain G. We will now introduce a condition that is a restriction on the plurisubharmonic function φ. We first require another extension theorem of Ohsawa [29, Theorem 4.1].

Theorem 13. *For any pseudoconvex domain G in $\mathbb{C}^M$, for any plurisubharmonic function φ on G, for any $\alpha > 0$, and for any holomorphic function f on $G' = G \cap \{z : z_M = 0\}$, there exists a holomorphic function $\tilde{f}$ on G that extends f with*

$$\int_G e^{-\alpha|z_M|^2-\varphi(z)}|\tilde{f}(z)|^2 dV_M \leq \frac{\pi}{\alpha}\int_{G'} e^{-\varphi(z)}|f(z)|^2 dV_{M-1}.$$

For the following, let $H_p(\psi, v)$ denote the complex Hessian of ψ evaluated at the point p applied to the vector $v \in \mathbb{C}^M$.

Proposition 9. *Suppose that $G \subset \mathbb{C}^M$ is pseudoconvex and $\varphi \in PSH(G) \cap C^2(G)$. Further suppose that there exists a complex hyperplane $A \subset \mathbb{C}^M$ with the property that*

$$\inf_{p\in A} H_p(\varphi, N_p) > 0, \tag{3.1}$$

where N_p is the unit complex normal vector to A at $p \in A$. Then $L^2_h(D_\varphi(G))$ is infinite-dimensional whenever $L^2_h(D_{\varphi|_{A\cap G}}(A \cap G))$ is.

Proof. After applying a translation and unitary transformation if necessary, we may assume that $A = \{z \in \mathbb{C}^M : z_M = 0\}$. By (3.1), there exists a $c > 0$ such that $\varphi(z) - c\|z_M\|^2 \in \mathrm{PSH}(G)$. If we suppose further that $\dim L^2_h(D_{\varphi|_{G\cap A}}(G \cap A)) = \infty$, then by Lemma 1 we may assume there are infinitely many linearly independent $f_n w^{b(n)} \in L^2_h(D_\varphi(G \cap A))$, where $b : \mathbb{N} \to \mathbb{Z}^N_+$. Hence

$$\int_{D_\varphi(G\cap A)} |f_n(z)w^{b(n)}|^2 \mathrm{d}V_{M-1+N} \cong \int_{G\cap A} |f_n(z)|^2 e^{-2(N+|b(n)|)\varphi(z)} \mathrm{d}V_{M-1} < \infty.$$

Then we may apply Theorem 13 above with $\alpha = 2(N + |b(n)|)c$ to find $\tilde{f}_n \in \mathcal{O}(G)$ that extends f_n with

$$\begin{aligned}\int_G |\tilde{f}(z)|^2 e^{-2(N+|b(n)|)\varphi(z)} \mathrm{d}V_M &\leq \frac{\pi}{2(N+|b(n)|)c} \int_{G\cap A} |f_n(z)|^2 e^{-2(N+|b(n)|)\varphi(z)+2c(N+|b(n)|)|z_M|^2} \mathrm{d}V_{M-1} \\ &= \frac{\pi}{2(N+|b(n)|)c} \int_{G\cap A} |f_n(z)|^2 e^{-2(N+|b(n)|)\varphi(z)} \mathrm{d}V_{M-1} < \infty.\end{aligned}$$

By the same reasoning as before, this completes the proof. $\square$

3.2 Balanced Domains and Hartogs Domains with k-Dimensional Balanced Fibers

As we have observed in §1.3.2, there is an isomorphism between the Bergman space of balanced domains in $\mathbb{C}^n$ and the Bergman space of certain Hartogs domains with complete 1-circled fibers having $\mathbb{C}^{n-1}$ as base. Therefore, to solve the Wiegerinck problem on balanced domains in $\mathbb{C}^n$, it suffices to understand the problem on complete 1-circled Hartogs domains with base $\mathbb{C}^{n-1}$.

In this sense, Hartogs domains with k-dimensional balanced fibers (§1.3.3) are similar to balanced domains: There is a biholomorphism from $D \setminus \{(z,w) \in G \times \mathbb{C}^k \, : \, w_k = 0\}$ onto the complete 1-circled Hartogs domain with base G given by

$$\left\{(\zeta,\eta) \in G \times \mathbb{C}^k \, : \, |\eta_k| < e^{-\log H(\zeta,\eta_1,\ldots,\eta_{k-1},1)}\right\} \tag{3.2}$$

via the mapping

$$(z, w_1, \ldots, w_{k-1}, w_k) \mapsto \left(z, \frac{w_1}{w_k}, \ldots, \frac{w_{k-1}}{w_k}, w_k\right),$$

and there is a natural isometry [2, p. 687]

$$L^2_h\left(D \setminus \{(z,w) \in G \times \mathbb{C}^k \, : \, w_k = 0\}\right) \hookrightarrow L^2_h(D).$$

So to solve the Wiegerinck problem on Hartogs domains with k-dimensional balanced fibers, it suffices to solve the problem on complete 1-circled Hartogs domains of the form (3.2).

The above discussion may be summarized by the following:

Theorem 14. *Let $D = \{z \in \mathbb{C}^M \,:\, H(z) < 1\}$ be a pseudoconvex balanced domain in $\mathbb{C}^M$ and let $\tilde{D} = \{(z, w) \in G \times \mathbb{C}^k \,:\, \tilde{H}(z, w) < 1\}$ be a pseudoconvex Hartogs domain with k-dimensional balanced fibers. Then the dimension of $L^2_h(D)$ is equal to the dimension of*

$$L^2_h\left(\left\{w \in \mathbb{C}^M \,:\, |w_M| < e^{-\log H(w_1,\dots,w_{M-1},1)}\right\}\right)$$

and the dimension of $L^2_h(\tilde{D})$ is equal to the dimension of

$$L^2_h\left(\left\{(\zeta, \eta) \in G \times \mathbb{C}^k \,:\, |\eta_k| < e^{-\log \tilde{H}(\zeta,\eta_1,\dots,\eta_{k-1},1)}\right\}\right).$$

Remark. By leveraging the methods of the previous subsections, we have a few immediate consequences of Theorem 14.

(1) Suppose $D = \{(z, w) \in G \times \mathbb{C} \,:\, \tilde{H}(z, w) < 1\}$ is a pseudoconvex Hartogs domain with 1-dimensional balanced fibers. If G is in the plane and has nonpolar complement [23, Corollary 3.6], or $\log \tilde{H}(z, 1)$ has a point of strong plurisubharmonicity (Theorem 7 above), then $L^2_h(D)$ has infinite dimension.

(2) If $D = \{(z, w) \in G \times \mathbb{C}^2 \,:\, \tilde{H}(z, w) < 1\} \subset \mathbb{C}^3$ is a Hartogs domain with 2-dimensional balanced fibers and G has nonpolar complement, then $L^2_h(D)$ is either trivial or infinite-dimensional. Indeed, by Theorem 14 it suffices to consider the domain

$$\tilde{D} = \left\{(z, w) \in G \times \mathbb{C}^2 \,:\, |w_2| < e^{-\log \tilde{H}(z,w_1,1)}\right\}.$$

By the work of Jucha [23, Theorem 4.1], for each $z \in G$ the Bergman space of the Hartogs domain

$$\tilde{D}_z = \left\{w \in \mathbb{C}^2 \,:\, |w_2| < e^{-\log \tilde{H}(z,w_1,1)}\right\}$$

is either trivial or has infinite dimension. If the dimension is infinite for some $z \in G$, then Theorem 10 shows that $L^2_h(D)$ has infinite dimension. On the other hand, if the Bergman space of $\tilde{D}_z$ is trivial for all $z \in G$, then Fubini's theorem asserts that any $f(z, w_1)w_2^n \in L_h^2(\tilde{D})$ satisfies

$$\int_G \int_{\mathbb{C}} |f(z, w_1)|^2 e^{-2(1+n) \log \tilde{H}(z,w_1,1)} \mathrm{d}V(z, w_1) < \infty,$$

so $f(z, w_1)w_2^n \in L_h^2(D_z)$ for almost every z, and hence $f(z, \cdot)$ is trivial for almost every, and thus every, z belonging to G.

Let us now turn our attention to pseudoconvex balanced domains $D = D_h$ in $\mathbb{C}^M$ with trivial Bergman space. If $M = 1$, the only balanced domains are disks centered at the origin, along with $\mathbb{C}$ itself. Therefore the Bergman space of a balanced domain $D \subseteq \mathbb{C}$ is trivial if and only if $D = \mathbb{C}$, otherwise it has infinite dimension.

Pflug and Zwonek [35] showed that, if $D \subseteq \mathbb{C}^2$, then $L^2(D)$ is trivial if and only if D is an elementary balanced domain (as defined in §1.3.2).

We next introduce a family of pseudoconvex balanced domains in $\mathbb{C}^M$, $M > 2$, each with trivial Bergman space.

Theorem 15. *Let $D = D_H = \{z \in \mathbb{C}^M : H(z) < 1\}$ be a pseudoconvex balanced domain in $\mathbb{C}^M$, $M > 2$, where H is of the form $H(z) = |z_1|^t \big(u(z_2, \ldots, z_M)\big)^{1-t}$, where $u : \mathbb{C}^{M-1} \to [0, \infty)$ is upper semi-continuous and homogeneous, and $t \in (0, 1)$. Then $L_h^2(D)$ is trivial.*

Proof. By Theorem 14 it suffices to consider the domain

$$G_\varphi := \left\{(z, w) \in \mathbb{C}^{M-1} \times \mathbb{C} : |w| < e^{-\varphi(z)}\right\},$$

where $\varphi(z) := \log H(z, 1)$. Accordingly, suppose $f \in L_h^2(G_\varphi)$. Then we may assume f is of the form $f(z, w) = g(z)w^n$. Here $z \in \mathbb{C}^{M-1}$, w is in the plane, and n is some nonnegative integer. Since $f \in L^2_h(G_\varphi)$, the computation (1.4) in §1.3.1 shows that

$$\int_{\mathbb{C}^{M-1}} |g(z)|^2 e^{-2(1+n)\varphi(z)} \mathrm{d}V_{M-1}(z) < \infty.$$

By Fubini's theorem,

$$\int_{\mathbb{C}^{M-2}} \int_{\mathbb{C}} |g(z_1, \eta)|^2 e^{-2(1+n)\varphi(z_1,\eta)} \mathrm{d}V_1(z_1) \mathrm{d}V_{M-2}(\eta) < \infty.$$

Therefore, for almost every fixed $\eta \in \mathbb{C}^{M-2}$, we have

$$\int_{\mathbb{C}} |g(\zeta, \eta)|^2 e^{-2(1+n)\varphi(\zeta,\eta)} \mathrm{d}V_1(\zeta) < \infty.$$

But since

$$\varphi(\zeta, \eta) = t \log |\zeta| + (1 - t) \log u(\eta, 1),$$

for almost every fixed $\eta \in \mathbb{C}^{M-2}$ we see that

$$\int_{\mathbb{C}} |g(\zeta, \eta)|^2 e^{-2(1+n)t \log |\zeta|} \mathrm{d}V_1(\zeta) < \infty.$$

This implies that $g(\cdot, \eta) w^n \in L^2_h\big(D_{t \log|\cdot|}(\mathbb{C})\big)$ for almost every fixed η, where recall that

$$D_{t \log |\cdot|}(\mathbb{C}) = \left\{(z, w) \in \mathbb{C}^2 \,:\, |w| < e^{-t \log |z|}\right\}.$$

Now, the work of Jucha [23, Theorem 4.1] shows that $g(\cdot, \eta) \equiv 0$ for almost every fixed η, and hence g is identically zero (now thought of again as a function of both parameters). We conclude that $L^2_h(D)$ is trivial. $\square$

Remark. Pflug and Zwonek [35] raised the question: *Do there exist pseudoconvex, non-elementary, balanced domains in $\mathbb{C}^M$, $M > 2$, with trivial Bergman space?* Special cases of Theorem 15 answer this question in the affirmative. For instance, one may set $M = 3$ and

$u(z_2, z_3) = \max(|z_2|, |z_3|)$.

Alternatively, this example could be confirmed by applying the work of Zwonek [48, Lemma 2.2.1], which contains a geometric characterization of which monomials are square integrable in a pseudoconvex Reinhardt domain.

4.1 Hartogs Domains and Weighted Bergman Spaces

From the expression (1.4) in §1.3.3, one can see there is a direct relationship between complete N-circled Hartogs domains G and weighted Bergman spaces over G. This relationship may be generalized, but we must first broaden our definition of a Hartogs domain.

Definition 11. *Let $G \subseteq \mathbb{C}^M$ be a domain, $\varphi \in USC(G)$, and $p \in [1,\infty]$. Define*

$$D^p_\varphi(G) = D^p_\varphi = D^p = \{(z,w) \in G \times \mathbb{C}^N \subseteq \mathbb{C}^{M+N} \,:\, \|w\|_p < e^{-\varphi(z)}\}.$$

Here $\|\cdot\|_p$ denotes the ℓ^p norm, given by $\|w\|_\infty = \max_{1\le k\le N} |w_k|$ whenever $p=\infty$ and $\|w\|_p^p = |w_1|^p + \cdots + |w_N|^p$ otherwise. We call $D^p_\varphi(G)$ a complete N-circled p-Hartogs domain with base G.

Lemma 3. *Let $D^p_\varphi = D^p \subseteq G \times \mathbb{C}^N$ be a p-Hartogs domain over $G \subseteq \mathbb{C}^M$ with complete N-circled fibers. If $f(z,w) = \sum_{n\in\mathbb{Z}^N_+} f_n(z)w^n \in L^2_h(D^p_\varphi)$, then there exists a $C = C(p;n)$ such that $\|f_n w^n\|_{D^p_\varphi} = C\|f_n\|_{G,\exp(-2(|n|+N)\varphi)}$. Furthermore,*

$$C(p;n) = \left(\frac{2}{p}\right)^{N-1} \frac{\pi^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)}{(N+|n|)\cdot\Gamma\left(\frac{2N+2|n|}{p}\right)}$$

when $p \in [1,\infty)$, and

$$C(\infty;n) = \frac{\pi^N}{\prod_{k=1}^N (n_k+1)}.$$

Before beginning the proof of Lemma 3, it is useful to define generalized polar coordinates on $\mathbb{C}^N$. We follow a similar construction to [15, §2.7]. Denote the unit ℓ^p-sphere $\{w \in \mathbb{C}^N \,:\, \|w\|_p = 1\}$ by $\mathbb{S}^N_p$. If $w \neq 0$, the *p-polar coordinates* of w are

$$r_p = \|w\|_p \in (0,\infty) \qquad \text{and} \qquad w'_p = \frac{w}{\|w\|_p} \in \mathbb{S}^N_p.$$

The subscript p will often be omitted if there is no risk of confusion. The map $\Phi(w) = (r_p, w'_p)$ is a homeomorphism from $\mathbb{C}^N \setminus \{0\}$ to $(0,\infty) \times \mathbb{S}_p^N$. If m is Lebesgue measure on $\mathbb{C}^N$, we denote by m_* the Borel measure on $(0,\infty) \times \mathbb{S}_p^N$ induced by Φ from Lebesgue measure on $\mathbb{C}^N$; in other words, we set $m_*(E) = m(\Phi^{-1}(E))$. Also, we define the measure $\rho = \rho_N$ on $(0,\infty)$ by $\rho(E) = \int_E r_p^{2N-1}\mathrm{d}r_p$.

It follows similarly to [15, Theorem 2.49] that there is a unique Borel measure $\sigma_p = \sigma_{p,N}$ on $\mathbb{S}_p^N$ such that $m_* = \rho \times \sigma_p$, and if f is Borel measurable on $\mathbb{C}^N$ and $f \geq 0$ or $f \in L^1(\mathbb{C}^N)$, then

$$\int_{\mathbb{C}^N} f(w)\mathrm{d}V = \int_0^\infty \int_{\mathbb{S}_p^N} f(r_p w') r_p^{2N-1}\mathrm{d}\sigma_p(w')\mathrm{d}r_p. \tag{4.1}$$

Proof of Lemma 3. By (4.1),

$$\begin{aligned}
&\|f_n(z)w^n\|^2_{D^p_\varphi} \\
&\quad= \int_D |f_n(z)|^2|w^n|^2\mathrm{d}V_{M+N}(z,w) \\
&\quad= \int_G |f_n(z)|^2 \int_{\|w\|_p<e^{-\varphi(z)}} \left(|w_1|^{n_1}\cdot|w_2|^{n_2}\cdots|w_N|^{n_N}\right)^2 \mathrm{d}V_N(w)\mathrm{d}V_M(z) \\
&\quad= \int_G |f_n(z)|^2 \left(\int_0^{e^{-\varphi(z)}} r_p^{2N+2|n|-1}\mathrm{d}r_p\right)\left(\int_{\mathbb{S}_p^N}\prod_{k=1}^N\left(\frac{|w_k|}{\|w\|_p}\right)^{2n_k}\mathrm{d}\sigma_p\right)\mathrm{d}V_M(z) \\
&\quad= C(p;n)\int_G |f_n(z)|^2 e^{-2(N+|n|)\varphi}\mathrm{d}V_M(z),
\end{aligned}$$

where

$$C(p;n) = \frac{1}{2N+2|n|}\int_{\mathbb{S}_p^N}\left(\frac{|w_1|}{\|w\|_p}\right)^{2n_1}\cdots\left(\frac{|w_N|}{\|w\|_p}\right)^{2n_N}\mathrm{d}\sigma_p.$$

Furthermore, since

$$\begin{aligned}
\int_{\|w\|_\infty < e^{-\varphi(z)}} |w_1|^{2n_1} \cdots |w_N|^{2n_N} \mathrm{d}V(w) &= \prod_{k=1}^{N} \int_{|w_k| < e^{-\varphi(z)}} |w_k|^{2n_k} \mathrm{d}V_1(w_k) \\
&= (2\pi)^N \prod_{k=1}^{N} \int_0^{e^{-\varphi(z)}} r^{2n_k+1} \mathrm{d}r \\
&= \pi^N \prod_{k=1}^{N} \frac{1}{n_k+1} e^{-2(n_k+1)\varphi(z)} \\
&= \pi^N e^{-2\varphi(z) \sum_{k=1}^{N}(n_k+1)} \prod_{k=1}^{N} \frac{1}{n_k+1} \\
&= \pi^N e^{-2(|n|+N)\varphi(z)} \prod_{k=1}^{N} \frac{1}{n_k+1},
\end{aligned}$$

we see that

$$C(\infty; n) = \frac{\pi^N}{\prod_{k=1}^{N}(n_k+1)}.$$

To find $C(p;n)$ when $p \in [1, \infty)$, first observe that

$$\int_{\mathbb{C}^N} e^{-\|w\|_p^p} \prod_{k=1}^{N} |w_k|^{2n_k} \mathrm{d}V(w) = \int_{\mathbb{S}_p^N} \prod_{k=1}^{N} \left(\frac{|w_k|}{\|w\|_p} \right)^{2n_k} \int_0^\infty e^{-r^p} r^{2N-1+2\sum_{k=1}^{N} n_k} \mathrm{d}r \mathrm{d}\sigma. \tag{4.2}$$

By a change of variables $s = r^p$, the inner integral on the right side of (4.2) becomes

$$\int_0^\infty e^{-r^p} r^{2N+2|n|-1} dr = \frac{1}{p} \int_0^\infty e^{-s} s^{\frac{2N}{p} + \frac{2|n|}{p} - 1} ds = \frac{1}{p} \Gamma\left(\frac{2N+2|n|}{p} \right).$$

Similarly, the left side of (4.2) becomes

$$\begin{aligned}\int_{\mathbb{C}^N} e^{-\|w\|_p^p} \prod_{k=1}^N |w_k|^{2n_k} dV(w) &= \prod_{k=1}^N \int_{\mathbb{C}} e^{-|w_k|^p} |w_k|^{2n_k} \mathrm{d}V_1(w_k) \\ &= (2\pi)^N \prod_{k=1}^N \int_0^\infty e^{-r^p} r^{2n_k+1} \mathrm{d}r \\ &= \left(\frac{2\pi}{p}\right)^N \prod_{k=1}^N \int_0^\infty e^{-s} s^{\frac{2n_k}{p}+\frac{2}{p}-1} \mathrm{d}s \\ &= \left(\frac{2\pi}{p}\right)^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right).\end{aligned}$$

Putting this together reveals that

$$\int_{\mathbb{S}_p^N} \left(\frac{|w_1|}{\|w\|_p}\right)^{2n_1} \cdots \left(\frac{|w_N|}{\|w\|_p}\right)^{2n_N} d\sigma_p = \frac{(2\pi)^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)}{p^{N-1}\Gamma\left(\frac{2N+2|n|}{p}\right)}$$

and hence

$$C(p;n) = \left(\frac{2}{p}\right)^{N-1} \frac{\pi^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)}{(N+|n|)\cdot\Gamma\left(\frac{2N+2|n|}{p}\right)}. \qquad \square$$

The first theorem of this section is fundamental to the rest of the chapter; it relates the Bergman kernel of D to weighted Bergman kernels on G. This was first shown in [26, Prop. 0], however only the case $p = 2$ was discussed, and $C(2;n)$ was not computed explicitly.

Theorem 16. *If $G \subseteq \mathbb{C}^M$ is a domain and $\varphi \in USC(G)$, then the Bergman kernel $K_D(z,\zeta,w,\eta)$ of $D = D_\varphi^p(G)$ can be written as*

$$\begin{cases} \left(\frac{p}{2}\right)^{N-1} \sum_{n\in\mathbb{Z}_+^N} \frac{(N+|n|)\cdot\Gamma\left(\frac{2N+2|n|}{p}\right)}{\pi^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)} K_{G,\exp(-2(N+|n|)\varphi)}(z,\zeta)(w\bar{\eta})^n, & \textit{when } p \in [1,\infty) \\ \frac{1}{\pi^N} \sum_{n\in\mathbb{Z}_+^N} \prod_{k=1}^N (n_k+1) K_{G,\exp(-2(N+|n|)\varphi)}(z,\zeta)(w\bar{\eta})^n, & \textit{when } p = \infty \end{cases}$$

with uniform convergence on compact subsets of $D \times D$.

The particular case of $N = 1$ and $p = 2$ will be used often, so it will be stated separately.

Corollary 3. *Let G and φ be defined as above. Let $N = 1$ and $p = 2$. Then*

$$K_{D^2_\varphi}(z, \zeta, w, \eta) = \frac{1}{\pi} \sum_{n=0}^{\infty} (1+n) K_{G,\exp(-2(1+n)\varphi)}(z, \zeta)(w\bar{\eta})^n.$$

Proof of Theorem 16. Suppose that $\{\chi_{n,j}\}_{j=0}^{\infty}$ is a orthonormal basis for $L^2_h(G, \exp(-2(N+|n|)\varphi))$ for each multiindex n. Then each $g_n \in L^2_h(G, \exp(-2(N + |n|)\varphi))$ has a decomposition $g_n = \sum_{j=0}^{\infty} c_{n,j}\chi_{n,j}$ and hence each $f \in L^2_h(D)$ has a decomposition

$$f(z, w) = \sum_{n \in \mathbb{Z}^N_+} f_n(z) w^n = \sum_{j=0}^{\infty} \sum_{n \in \mathbb{Z}^N_+} c_{n,j}\chi_{n,j}(z) w^n.$$

Thus $\{\chi_{n,j}(z)w^n\}_{(j,n)\in\mathbb{N}\times\mathbb{Z}^N_+}$ is an orthogonal basis for D. By Lemma 3,

$$\|\chi_{n,j}(z)w^n\|^2_D = \begin{cases} \left(\frac{2}{p}\right)^{N-1} \frac{\pi^N \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)}{(N+|n|)\cdot\Gamma\left(\frac{2N+2|n|}{p}\right)}, & \text{when } p \in [1, \infty) \\ \frac{\pi^N}{\prod_{k=1}^N (n_k+1)}, & \text{when } p = \infty, \end{cases}$$

so dividing out this quantity and taking square roots yields an orthonormal basis for $L^2_h(D)$.

Therefore, by the representation of the Bergman kernel by an orthonormal basis [37, Section 4.2],

$$\begin{aligned} K_{D^p_\varphi}(z, \zeta, w, \eta) &= \frac{1}{2^{N-1}\pi^N} \sum_{j=0}^{\infty} \sum_{n \in \mathbb{Z}^N_+} \frac{p^{N-1}(N + |n|) \cdot \Gamma\left(\frac{2N+2|n|}{p}\right)}{\prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)} \chi_{n,j}(z)\overline{\chi_{n,j}(\zeta)}(w\bar{\eta})^n \\ &= \frac{1}{2^{N-1}\pi^N} \sum_{n \in \mathbb{Z}^N_+} \frac{p^{N-1}(N + |n|) \cdot \Gamma\left(\frac{2N+2|n|}{p}\right)}{\prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)} K_{G,\exp(-2(1+n)\varphi)}(w\bar{\eta})^n, \end{aligned}$$

when $p \in [1, \infty)$, and

$$K_{D_\varphi^\infty}(z, \zeta, w, \eta) = \frac{1}{\pi^N} \sum_{j=0}^{\infty} \sum_{n \in \mathbb{Z}_+^N} \prod_{k=1}^{N} (n_k + 1) \chi_{n,j}(z) \overline{\chi_{n,j}(\zeta)} (w\bar{\eta})^n$$

$$= \frac{1}{\pi^N} \sum_{n \in \mathbb{Z}_+^N} \prod_{k=1}^{N} (n_k + 1) K_{G, \exp(-2(n+|n|)\varphi)}(z, \zeta)(w\bar{\eta})^n$$

with uniform convergence on compact subsets of $D \times D$. □

This result gives an alternative proof of the well-known “inflation” identity presented in [6, Section 2.2].

Corollary 4. *Let $G \subseteq \mathbb{C}^M$ be a domain. Consider two Hartogs domains:*

$$D = D_\varphi = \{(z, w) \in G \times \mathbb{C} \,:\, |w| < \exp(-\varphi(z))\}$$

and

$$\tilde{D} = \tilde{D}_\varphi = \{(z, W) \in G \times \mathbb{C}^N \,:\, \|W\|_2 < \exp(-\varphi(z))\}.$$

Let $K_D(z, \zeta, w, \eta)$ and $\tilde{K}_D(z, Z, w, W)$ be the Bergman kernels for D and $\tilde{D}$, respectively. Because of the circular symmetry in the one-dimensional variable, $K(z, \zeta, w, \eta)$ can be written as $L(z, \zeta, w\bar{\eta})$. Then

$$\tilde{K}_D(z, Z, w, W) = \frac{1}{\pi^{N-1}} \frac{\partial^{N-1}}{\partial t^{N-1}} L(z, w, t) \Big|_{t = \langle Z, W \rangle}.$$

Proof. Using Corollary 3, we see that

$$L(z, \zeta, t) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1) K_{G, \exp(-2(1+n)\varphi)}(z, \zeta) t^n.$$

By taking derivatives within the power series, we get

$$\frac{\partial^{N-1}}{\partial t^{N-1}} L(z, \zeta, t) = \frac{1}{\pi} \sum_{n=N-1}^{\infty} \frac{(n+1)!}{(n-N+1)!} K_{G, \exp(-2(1+n)\varphi)}(z, \zeta) t^{n-N+1},$$

and by the change of variables $k = n - N + 1$, the right side of the above is equivalent to

$$\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(k+N)!}{k!}K_{G,\exp(-2(N+k)\varphi)}(z,\zeta)t^k.$$

Therefore by the multinomial theorem and changing into multiindex notation,

$$\begin{aligned}\left.\frac{\partial^{N-1}}{\partial t^{N-1}}L(z,\zeta,t)\right|_{t=\langle Z,W\rangle} &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(k+N)!}{k!}K_{G,\exp(-2(N+k)\varphi)}(z,\zeta)\left(\sum_{j=1}^{N}Z_j\bar{W}_j\right)^k \\ &= \frac{1}{\pi}\sum_{\alpha\in\mathbb{Z}_+^N}\frac{(|\alpha|+N)!}{\alpha!}K_{G,\exp(-2(N+|\alpha|)\varphi)}(z,\zeta)(Z\overline{W})^\alpha \\ &= \frac{1}{\pi}\sum_{\alpha\in\mathbb{Z}_+^N}\frac{(|\alpha|+N)\Gamma(|\alpha|+N)}{\prod_{k=1}^{N}\Gamma(\alpha_k+1)}K_{G,\exp(-2(N+|\alpha|)\varphi)}(z,\zeta)(Z\overline{W})^\alpha \\ &= \pi^{N-1}K_{\tilde{D}}(z,\zeta,Z,W).\end{aligned}$$

The last equality comes from a final use of Theorem 16. □

Remark. It would be interesting to see if one could develop a generalized version of the inflation identity for $p \in [1,\infty)$, instead of just when $p = 2$.

4.2 Explicit Formulae for the Bergman Kernels of Certain Hartogs Domains

For this section, we will only consider Hartogs domains of the form

$$D_\varphi^2(G) = D = \{(z,w) \in G \times \mathbb{C}^N \subset \mathbb{C}^{M+N} \,:\, \|w\|_2 < \exp(-\varphi(z))\},$$

i.e. only those whose symmetry is in the ℓ^2-norm. Because of this we can write $\|\cdot\|_2 = \|\cdot\|$ without any confusion.

A second consequence of Theorem 16 is an alternative derivation of the Bergman kernel for a certain type of "egg-shaped" domain:

$$\{(z,w) \in \mathbb{C}^2 \,:\, |z|^2 + |w|^{2q} < 1\},$$

where $q \in (0, \infty)$. This was first computed by Bergman in 1936 [3, Formula (2, 3)].

Corollary 5. *The Bergman kernel of the domain* $D_q = \{(z, w) \in \mathbb{C}^2 \,:\, |z|^2 + |w|^{2q} < 1\}$, $q > 0$, *is given by*

$$K_{D_q}(z, \zeta, w, \eta) = \frac{(q-1)w\bar{\eta} - (q+1)(1 - z\bar{\zeta})^{1/q}}{q\pi^2(1 - z\bar{\zeta})^{(2q-1)/q}(w\bar{\eta} - (1 - z\bar{\zeta})^{1/q})^3}.$$

Proof. Observe that we may rewrite D_q as

$$\left\{(z, w) \in \mathbb{D} \times \mathbb{C} \,:\, |w| < e^{\frac{1}{2q}\log(1-|z|^2)}\right\},$$

so by Corollary 3,

$$K_{D_q}(z, \zeta, w, \eta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)K_{\mathbb{D},\exp(q^{-1}(1+n)\log(1-|z|^2))}(z, \zeta)(w\bar{\eta})^n.$$

From [19, p. 6], we know that

$$K_{\mathbb{D},\exp(q^{-1}(1+n)\log(1-|z|^2))}(z, \zeta) = \frac{n+q+1}{q\pi}\frac{1}{(1 - z\bar{\zeta})^{\frac{n+2q+1}{q}}},$$

so it suffices to get a closed-form expression for

$$\sum_{n=0}^{\infty}(n+1)\frac{n+q+1}{q\pi}\frac{1}{(1 - z\bar{\zeta})^{\frac{n+2q+1}{q}}}(w\bar{\eta})^n.$$

The power series

$$\sum_{n=0}^{\infty}(n+q+1)(n+1)x^n = \frac{qx - q - x - 1}{(x-1)^3}$$

converges for $|x| < 1$, so

$$\begin{aligned} K_{D_q}(z,\zeta,w,\eta) &= \frac{1}{q\pi^2(1-z\bar{\zeta})^{2+q^{-1}}}\sum_{n=0}^{\infty}(n+q+1)(n+1)\left(\frac{w\bar{\eta}}{(1-z\bar{\zeta})^{1/q}}\right)^n \\ &= \frac{q\cdot\frac{w\bar{\eta}}{(1-z\bar{\zeta})^{1/q}} - q - \frac{w\bar{\eta}}{(1-z\bar{\zeta})^{1/q}} - 1}{q\pi^2(1-z\bar{\zeta})^{2+q^{-1}}\left(\frac{w\bar{\eta}}{(1-z\bar{\zeta})^{1/q}}-1\right)^3} \\ &= \frac{(q-1)w\bar{\eta}-(q+1)(1-z\bar{\zeta})^{1/q}}{q\pi^2(1-z\bar{\zeta})^{(2q-1)/q}(w\bar{\eta}-(1-z\bar{\zeta})^{1/q})^3}, \end{aligned}$$

where $(1-z\bar{\zeta})^{1/q}$ in the expression above is defined via the principal branch of the logarithm. $\square$

For another application, we consider the generalized Hartogs triangle

$$\mathbb{H}_q = \left\{(z,w)\in\mathbb{C}^2 \,:\, |w|^q < |z| < 1\right\},$$

where $q \in \mathbb{Q}_+$. Explicit formulae for the Bergman kernel of $\mathbb{H}_k$ and $\mathbb{H}_{1/k}$, $k \in \mathbb{N}$, were exhibited by Edholm [13]. More generally, we present explicit formulae for $\mathbb{H}_q$ when q is a positive rational number. This has been done previously by Edholm and McNeal [14] through a different method.

Theorem 17. *The Bergman kernel of the domain $\mathbb{H}_q$, $q \in \mathbb{Q}_+$, is given by*

$$\begin{aligned} &\sum_{r=0}^{a-2}\left\{\left(\frac{\big((a-b(r+1))z\bar{\zeta}+b(r+1)\big)(w\bar{\eta})^r}{a\pi^2 z\bar{\zeta}(z\bar{\zeta}-1)^2}\right)\right. \\ &\qquad\qquad \left.\cdot\left(\frac{a(w\bar{\eta})^a(z\bar{\zeta})^b+(1+r)(z\bar{\zeta})^b\big((z\bar{\zeta})^b-(w\bar{\eta})^a\big)}{\big((z\bar{\zeta})^b-(w\bar{\eta})^a\big)^2}\right)\right\} \\ &\qquad\qquad\qquad\qquad + \frac{a(w\bar{\eta})^{a-1}(z\bar{\zeta})^b}{\pi^2(z\bar{\zeta}-1)^2((w\bar{\eta})^a-(z\bar{\zeta})^b)^2}, \end{aligned}$$

where $q = a/b$ is written in lowest terms. If $a = 1$, then the sum on the left is taken to be identically zero.

It can be shown that this representation is the equivalent to the one presented by Edholm and McNeal [14]. In the particular case of $q = 1/k$, then $a = 1$, the sum on the left vanishes, and one is left with the formula in Theorem 1.4 of Edholm [13].

$\mathbb{H}_q$ can be rewritten as

$$\left\{(z,w)\in\mathbb{D}^*\times\mathbb{C}\,:\,|w|<\exp(\tfrac{1}{q}\log|z|)\right\},$$

so one must first investigate the Bergman kernel of the weighted space $L^2_h(\mathbb{D}^*,\exp(2\alpha\log|z|))$, $\alpha>0$.

Lemma 4. *The Bergman kernel of the weighted space $L^2_h(\mathbb{D}^*,\exp(2\alpha\log|z|))$, $\alpha>0$, is given by*

$$K_{\mathbb{D}^*,\exp(2\alpha\log|\cdot|)}(z,\zeta)=\frac{(z\bar{\zeta})^{-\lfloor\alpha\rfloor-1}\big((1-\text{frac}(\alpha))z\bar{\zeta}+\text{frac}(\alpha)\big)}{\pi(z\bar{\zeta}-1)^2}.$$

In particular,

$$K_{\mathbb{D}^*,\exp(2\alpha\log|\cdot|)}(z,\zeta)=\frac{1}{\pi(z\bar{\zeta})^{\alpha}(z\bar{\zeta}-1)^2}$$

whenever $\alpha\in\mathbb{N}$. Here $\lfloor x\rfloor$ represents the greatest integer less than or equal to x and frac$(x)=x-\lfloor x\rfloor$ is the fractional part of x.

Proof. Since the weight is radial, the monomials are orthogonal in $L^2_h(\mathbb{D}^*,\exp(2\alpha\log|z|))$. Furthermore, for integers n such that $n>-\alpha-1$ we have

$$\int_{\mathbb{D}^*}|z|^{2n}|z|^{2\alpha}dV=2\pi\int_0^1 r^{2n+2\alpha+1}dr=\frac{\pi}{\alpha+n+1},$$

and so

$$K_{\mathbb{D}^*,\exp(2\alpha\log|\cdot|)}(z,\zeta)=\frac{1}{\pi}\sum_{n>-\alpha-1}(\alpha+n+1)(z\bar{\zeta})^n.$$

We now now separate into two cases. If α is an integer, then

$$K_{\mathbb{D}^*,\exp(2\alpha\log|\cdot|)}(z,\zeta)=\frac{1}{\pi(z\bar{\zeta})^{\alpha}(z\bar{\zeta}-1)^2}.$$

If α is not an integer, then

$$\begin{aligned}
K_{\mathbb{D}^*,\exp(2\alpha\log|\cdot|)}(z,\zeta) &= \frac{1}{\pi}\sum_{n=-\lfloor\alpha+1\rfloor}^{\infty}(\alpha+n+1)(z\bar{\zeta})^n \\
&= \frac{(z\bar{\zeta})^{-\lfloor\alpha\rfloor-1}\left(z\bar{\zeta}\lfloor\alpha\rfloor-\lfloor\alpha\rfloor-\alpha z\bar{\zeta}+\alpha+z\bar{\zeta}\right)}{\pi(z\bar{\zeta}-1)^2} \\
&= \frac{(z\bar{\zeta})^{-\lfloor\alpha\rfloor-1}\big((1-\mathrm{frac}(\alpha))z\bar{\zeta}+\mathrm{frac}(\alpha)\big)}{\pi(z\bar{\zeta}-1)^2}. \qquad \square
\end{aligned}$$

Proof of Theorem 17. Suppose that $q\in\mathbb{Q}^+$. Write q in lowest terms as a/b, and notice that $q^{-1}(n+1)\in\mathbb{N}$ if and only if a divides $(n+1)$. Set $\varphi_{n,q}=2q^{-1}(n+1)\log|\cdot|$. Using Corollary 3 and the division algorithm, we see that

$$\begin{aligned}
K_{\mathbb{H}_q}(z,\zeta,w,\eta) &= \frac{1}{\pi}\sum_{n=0}^{\infty}(1+n)K_{\mathbb{D}^*,\exp(\varphi_{n,q})}(z,\zeta)(w\bar{\eta})^n \\
&= \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{r=0}^{a-1}(1+ka+r)K_{\mathbb{D}^*,\exp(\varphi_{ka+r,q})}(z,\zeta)(w\bar{\eta})^{ka+r} \\
&= \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{r=0}^{a-2}(1+ka+r)K_{\mathbb{D}^*,\exp(\varphi_{ka+r,q})}(z,\zeta)(w\bar{\eta})^{ka+r} \\
&\qquad + \frac{1}{\pi}\sum_{k=0}^{\infty}(ka+a)K_{\mathbb{D}^*,\exp(\varphi_{ka+a-1,q})}(z,\zeta)(w\bar{\eta})^{ka+a-1}. \qquad (4.3)
\end{aligned}$$

The second sum on the right represents precisely the case when $q^{-1}(n+1)\in\mathbb{N}$, so we may use Lemma 4 to write it explicitly as

$$\begin{aligned}
\frac{1}{\pi^2}\sum_{k=0}^{\infty}\frac{(ka+a)}{(z\bar{\zeta})^{b(k+1)}(z\bar{\zeta}-1)^2}(w\bar{\eta})^{ka+a-1} &= \frac{a(w\bar{\eta})^{a-1}}{\pi^2(z\bar{\zeta})^b(z\bar{\zeta}-1)^2}\sum_{k=0}^{\infty}(k+1)\left(\frac{(w\bar{\eta})^a}{(z\bar{\zeta})^b}\right)^k \\
&= \frac{a(w\bar{\eta})^{a-1}(z\bar{\zeta})^b}{\pi^2(z\bar{\zeta}-1)^2((w\bar{\eta})^a-(z\bar{\zeta})^b)^2}. \qquad (4.4)
\end{aligned}$$

Now we turn our attention to the first sum on the right. Fix $0\le r\le a-2$. By Lemma 4 again,

$$K_{\mathbb{D}^*,\exp(\varphi_{ka+r,q})}(z,\zeta) = \frac{(z\bar{\zeta})^{-kb-1}\big((1-\frac{b}{a}(r+1))z\bar{\zeta}+\frac{b}{a}(r+1)\big)}{\pi(z\bar{\zeta}-1)^2},$$

so that

$$\frac{1}{\pi}\sum_{k=0}^{\infty}(1+ka+r)K_{\mathbb{D}^*,\exp(\varphi_{ka+r,q})}(z,\zeta)(w\bar{\eta})^{ka+r}$$
$$=\sum_{k=0}^{\infty}(1+ka+r)\frac{(z\bar{\zeta})^{-kb-1}\big((a-b(r+1))z\bar{\zeta}+b(r+1)\big)}{a\pi^2(z\bar{\zeta}-1)^2}(w\bar{\eta})^{ka+r},$$

since $\lfloor q^{-1}(ka+r+1)\rfloor = kb$ and $\operatorname{frac}(q^{-1}(ka+r+1)) = \frac{b}{a}(r+1)$. The right side of the above is equivalent to

$$\frac{\big((a-b(r+1))z\bar{\zeta}+b(r+1)\big)(w\bar{\eta})^r}{a\pi^2 z\bar{\zeta}(z\bar{\zeta}-1)^2}\sum_{k=0}^{\infty}(1+ka+r)\left(\frac{(w\bar{\eta})^a}{(z\bar{\zeta})^b}\right)^k$$
$$=\left(\frac{\big((a-b(r+1))z\bar{\zeta}+b(r+1)\big)(w\bar{\eta})^r}{a\pi^2 z\bar{\zeta}(z\bar{\zeta}-1)^2}\right)$$
$$\cdot\left(\frac{a(w\bar{\eta})^a(z\bar{\zeta})^b+(1+r)(z\bar{\zeta})^b\big((z\bar{\zeta})^b-(w\bar{\eta})^a\big)}{\big((z\bar{\zeta})^b-(w\bar{\eta})^a\big)^2}\right). \quad (4.5)$$

Lastly, combining (4.3), (4.4), and (4.5) yields the formula. □

4.3 Other Relationships Between Weighted Bergman Spaces and Bergman Spaces of Some Hartogs Domains

There is a well-known relationship between the (unweighted) Bergman kernels of two domains which are biholomorphic [37, Section IV.4.6]. This motivates the following question: *what if the weight changes but the domain remains the same?* As it turns out, there is an analogous formula in the case that the quotient of the two weights is the modulus-squared of a meromorphic function.

Proposition 10. *Let $D \subseteq \mathbb{C}^M$ be a domain and $\varphi_1,\varphi_2 \in USC(D)$. Suppose that $\varphi_1-\varphi_2 = 2\log|f|-2\log|g|$ for some nontrivial holomorphic functions f and g on D. Then*

$$g(z)K_{D,\exp(-\varphi_1)}(z,\zeta)\overline{g(\zeta)} = f(z)K_{D,\exp(-\varphi_2)}(z,\zeta)\overline{f(\zeta)}$$

Proof. We claim that the map T from $L^2_h(D,\exp(-\varphi_2))$ to $L^2_h(D,\exp(-\varphi_1))$ given by $\lambda \mapsto \frac{f}{g}\lambda$ is an isometric isomorphism. First,

$$\int_D |T(\lambda)|e^{-\varphi_1}dV = \int_D \left|\frac{f}{g}\lambda\right|^2 e^{-\varphi_1}dV = \int_D \left|\frac{f}{g}\lambda\right|^2 e^{\varphi_2-\varphi_1}e^{-\varphi_2}dV = \int_D |\lambda|^2 e^{-\varphi_2}dV,$$

so $T(\lambda)$ is square-integrable with respect to the weight $\exp(-\varphi_1)$. Since φ_1 is upper-semicontinuous, it is bounded from above on compact sets, and hence $T(\lambda)$ is locally square-integrable with respect to no weight. Now a theorem of Bell [2] shows that $T(\lambda) \in L^2_h(D, \exp(-\varphi_1))$ and hence T is well-defined.

Clearly T is injective with inverse $k \mapsto \frac{g}{f}k$, and from the computation above one sees that T is an isometry.

It follows that $\{T(\chi_j)\}_{j=0}^{\infty}$ is an orthonormal basis of $L^2_h(D, \exp(-\varphi_1))$ whenever $\{\chi_j\}_{j=0}^{\infty}$ is an orthonormal basis of $L^2_h(D, \exp(-\varphi_2))$. Thus

$$K_{D,\exp(-\varphi_1)}(z,\zeta) = \sum_{j=0}^{\infty} \frac{f(z)}{g(z)}\chi_j(z)\overline{\chi_j(\zeta)}\overline{\frac{f(\zeta)}{g(\zeta)}} = \frac{f(z)}{g(z)}K_{D,\exp(-\varphi_2)}(z,\zeta)\overline{\frac{f(z)}{g(z)}}. \qquad \square$$

This result, in combination with Theorem 16, yields a formula for certain special types of Hartogs domains.

Theorem 18. *Let f and g be nontrivial holomorphic functions on a domain $G \subseteq \mathbb{C}^M$, and let D be the Hartogs domain*

$$D = \left\{(z,w) \in G \times \mathbb{C}^N \,:\, \|wf(z)\| < |g(z)|\right\}.$$

Then

$$K_D(z,\zeta,w,\eta) = \frac{N!f(z)^{N-1}g(z)^2 K_{G,|g/f|^2}(z,\zeta)\overline{f(\zeta)^{N-1}g(\zeta)^2}}{\pi^N\left(g(z)\overline{g(\zeta)} - \langle w,\eta\rangle f(z)\overline{f(\zeta)}\right)^{N+1}} \tag{4.6}$$

and

$$K_D(z,\zeta,w,\eta) = \frac{N!f(z)^N g(z)K_G(z,\zeta)\overline{f(\zeta)^N g(\zeta)}}{\pi^N\left(g(z)\overline{g(\zeta)} - \langle w,\eta\rangle\, f(z)\overline{f(\zeta)}\right)^{N+1}}. \tag{4.7}$$

Proof. Let us first suppose that $N = 1$. By Theorem 16, then, we know that

$$K_D(z,\zeta,w,\eta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(1+n)K_{G,\exp(-2(1+n)\log|f/g|)}(z,\zeta)(w\bar{\eta})^n.$$

However, $2(1+n)\log|f/g| - 2\log|f/g| = 2n\log|f| - 2n\log|g|$ for each n, so Proposition 10 yields

$$g(z)^n K_{G,\exp(-2(1+n)\log|f/g|)}\overline{g(\zeta)}^n = f(z)^n K_{G,\exp(-2\log|f/g|)}(z,\zeta)\overline{f(\zeta)}^n.$$

Therefore

$$\begin{aligned} K_D(z,\zeta,w,\eta) &= \frac{1}{\pi}K_{G,\exp(-2\log|f/g|)}(z,\zeta)\sum_{n=0}^{\infty}(n+1)\left(w\bar{\eta}\frac{f(z)\overline{f(\zeta)}}{g(z)\overline{g(\zeta)}}\right)^n \\ &= \frac{g(z)^2 K_{G,\exp(-2\log|f/g|)}(z,\zeta)\overline{g(\zeta)}^2}{\pi\left(g(z)\overline{g(\zeta)} - w\bar{\eta}f(z)\overline{f(\zeta)}\right)^2}. \end{aligned}$$

Now suppose that $N \geq 1$. By the inflation identity (Corollary 4), we see that

$$\begin{aligned} K_D(z,\zeta,w,\eta) &= \frac{1}{\pi^N}\frac{\partial^{N-1}}{\partial t^{N-1}}\frac{g(z)^2 K_{G,\exp(-2\log|f/g|)}(z,\zeta)\overline{g(\zeta)}^2}{\left(g(z)\overline{g(\zeta)} - tf(z)\overline{f(\zeta)}\right)^2}\Bigg|_{t=\langle w,\eta\rangle} \\ &= \frac{N!f(z)^{N-1}g(z)^2 K_{G,\exp(-2\log|f/g|)}(z,\zeta)\overline{f(\zeta)^{N-1}g(\zeta)^2}}{\pi^N\left(g(z)\overline{g(\zeta)} - \langle w,\eta\rangle f(z)\overline{f(\zeta)}\right)^{N+1}}. \end{aligned}$$

This shows (4.6).

(4.7) can be shown by applying Proposition 10 with $\varphi_1 = 2\log|f/g|$ and $\varphi_2 \equiv 0$ to (4.6). $\square$

Corollary 6. *Let f, g, G, and D be as above. Then $K_D(z,\zeta,w,\eta)$ has a zero whenever z or ζ is a zero of f.*

Setting $g(z) = z^k$ for some $k \in \mathbb{N}$ yields an analogous relationship for a “twisted” generalized Hartogs triangle with inflated second coordinate:

Corollary 7. *Let f be a holomorphic function on the punctured unit disc $\mathbb{D}^* := \mathbb{D}\setminus\{0\}$ and let k*

be a natural number. Set

$$D = \left\{(z,w) \in \mathbb{D}^* \times \mathbb{C}^N \, : \, \|wf(z)\|^{1/k} < |z| < 1\right\}.$$

Then

$$K_D(z,\zeta,w,\eta) = \frac{N! f(z)^{N-1} z^{2k} K_{\mathbb{D}^*,|\cdot|^k/|f|^2}(z,\zeta)\overline{f(\zeta)}^{N-1}\bar{\zeta}^{2k}}{\pi^N \left(z^k\bar{\zeta}^k - \langle w,\eta\rangle f(z)\overline{f(\zeta)}\right)^{N+1}}$$

and

$$\begin{aligned} K_D(z,\zeta,w,\eta) &= \frac{N! f(z)^N z^k K_{\mathbb{D}^*}(z,\zeta)\overline{f(\zeta)}^N \bar{\zeta}^k}{\pi^N \left(z^k\bar{\zeta}^k - \langle w,\eta\rangle\, f(z)\overline{f(\zeta)}\right)^{N+1}} \\ &= \frac{N! f(z)^N z^k \overline{f(\zeta)}^N \bar{\zeta}^k}{\pi^{N+1} \left(z^k\bar{\zeta}^k - \langle w,\eta\rangle\, f(z)\overline{f(\zeta)}\right)^{N+1} \left(1 - z\bar{\zeta}\right)^2}. \end{aligned}$$

In the last equality we have used that $K_{\mathbb{D}^*} = K_{\mathbb{D}}$ on $\mathbb{D}^* \times \mathbb{D}^*$, which is a consequence of $L_h^2(\mathbb{D}^*) = L_h^2(\mathbb{D})$ [2, p. 687], in addition to the well-known explicit formula for the Bergman kernel of the unit disk [25], [37].

4.4 A Reproducing Kernel for Hartogs Domains

It is well-known that the Bergman kernel of a product domain is given by the product of the Bergman kernels of the domains [37]. Since Hartogs domains are similar to product domains in the sense that their fibers are disks over some base domain, one might expect something similar for Hartogs domains. What one gets is a reproducing kernel, however it is not conjugate-holomorphic in one of the variables.

Theorem 19. *Let $G \subset \mathbb{C}^M$, $\varphi \in USC(G)$, and*

$$D = \{(z,w) \in G \times \mathbb{C}^N \, : \, \|w\| < e^{-\varphi(z)}\}.$$

Suppose that φ is bounded above on G. Set

$$\tilde{K}(z,\zeta,w,\eta) = \frac{N!}{\pi^N} K_G(z,\zeta) \frac{e^{-2\varphi(\zeta)}}{(e^{-2\varphi(\zeta)} - \langle w,\eta\rangle)^{N+1}},$$

where K_G is the Bergman kernel for G. Then $\tilde{K}$ is a reproducing kernel for D; that is,

$$f(z,w) = \int_D f(\zeta,\eta)\tilde{K}(z,\zeta,w,\eta)dV(\zeta,\eta). \qquad \textit{for all} \qquad f \in L^2_h(D) \tag{4.8}$$

Proof. Because of Lemma 1, it suffices to check (4.8) on elements of the form $f(z)w^n$, where $n \in \mathbb{Z}^N_+$ and $f \in L^2_h(G, \exp(-2(N+|n|)\varphi))$. Accordingly, we compute

$$\begin{aligned}
&\int_D f(\zeta)\eta^n \tilde{K}(z,\zeta,w,\eta)\mathrm{d}V(\zeta,\eta) \\
&\qquad = \frac{N!}{\pi^N}\int_D f(\zeta)\eta^n K_G(z,\zeta)\frac{e^{-2\varphi(\zeta)}}{\left(e^{-2\varphi(\zeta)} - \langle w,\eta\rangle\right)^{N+1}}\mathrm{d}V(\zeta,\eta) \\
&\qquad = \frac{N!}{\pi^N}\int_G f(\zeta)K_G(z,\zeta)\int_{\|\eta\|<e^{-\varphi(\zeta)}} \frac{\eta^n e^{-2\varphi(\zeta)}}{\left(e^{-2\varphi(\zeta)} - \langle w,\eta\rangle\right)^{N+1}}\mathrm{d}V(\eta)\mathrm{d}V(\zeta) \\
&\qquad = \int_G f(\zeta)K_G(z,\zeta)w^n \mathrm{d}V(\zeta) = f(z)w^n.
\end{aligned}$$

We have used that $\frac{N!e^{-2\varphi(\zeta)}}{\pi^N(e^{-2\varphi(\zeta)}-\langle w,\eta\rangle)^{N+1}}$ is the Bergman kernel for the ball in $\mathbb{C}^N$ with radius $e^{-\varphi(\zeta)}$, and $f \in L^2_h(G)$ because φ is uniformly bounded above on G. $\square$

5. SUMMARY AND REMAINING QUESTIONS

In this chapter, we will spend some time exploring questions that remain. This is not intended to be an exhaustive list of open questions relating to the content of the previous three chapters; rather, we bring attention to open questions which are directly related to problems addressed in this dissertation. Most often, these questions have been directly investigated by the author to no avail.

5.1 Questions Pertaining to Chapter 2

In Theorem 5, we show that any admissible weight may be exchanged for an equivalent, admissible weight whose Bergman kernel has a zero near a specified point. Another question posed by Perälä [34] is the following.

Question 1. *Consider a Bergman kernel $K_{\Omega,\mu}(z,w)$ on a domain $\Omega \in \mathbb{C}^n$ with admissible weight μ. Further suppose $K_{\Omega,\mu}$ has a zero at a point $(z_0,w_0) \in \Omega \times \Omega$. Is it possible to replace μ with an equivalent, admissible weight $\hat{\mu}$ so that $K_{\Omega,\hat{\mu}}(z_0,w_0)$ is nonzero?*

If μ is not integrable in any neighborhood of some point $a \in \Omega$, then $f(a) = 0$ for every $f \in L^2_h(D,\mu)$. Consequently, it is clear that any admissible weight equivalent to μ induces a Bergman kernel that is zero whenever a is in either input. Therefore Question 1 should be modified so that only locally integrable weights are considered. On the other hand, Question 1 was answered affirmatively in some cases [34, Remark 3.2].

The construction of the family of weights in Theorem 6 requires solving a Stieltjes moment problem whose solution is absolutely continuous which respect to Lebesgue measure. There is much known about solving the Stieltjes moment problem [12], [41], so it would be interesting to see if one could characterize the entire functions f for which this holds. More precisely,

Question 2. *Under what conditions of an entire function f does there correspond an admissible radial weight μ_f on the plane such that $K_{\mathbb{C},\mu_f}(z,\zeta) = f(z\bar{\zeta})$?*

For instance, it is clear that a necessary condition on such a function f is that its Maclaurin series coefficients be all real and positive.

5.2 Questions Pertaining to Chapter 3

In §1.5, we discuss a work of Jucha in relation to the Wiegerinck problem [23]. The one case with base in the plane which is not concluded from this work of Jucha is the case where the base has polar complement in $\mathbb{C}$ and the corresponding subharmonic function is in fact harmonic.

Question 3. *Suppose $G \subseteq \mathbb{C}$ has polar complement and φ is harmonic on G. Does $L^2_h\big(D_\varphi(G)\big)$ have infinite dimension whenever it is nontrivial?*

It would be interesting to know if Theorem 13 could be generalized to the zero set of a holomorphic function.

Question 4. *For any $\alpha > 0$ and holomorphic function f on $G' = \{z \in G : g(z) = 0\}$ with $g \in \mathcal{O}(G)$, does there exist a holomorphic function $\tilde{f}$ on G that extends f with*

$$\int_G e^{-\alpha|g|^2-\varphi}|\tilde{f}|dV_M \leq \frac{\pi}{\alpha}\int_{G'} e^{-\varphi}|f|^2 dV_{M-1}?$$

If so, Proposition 9 could be appropriately generalized as well.

5.3 Questions Pertaining to Chapter 4

In §4.2, we used Theorem 16 to derive explicit representations of the Bergman kernel for some domains. However all representations presented in this dissertation are previously known.

Question 5. *Can the methods of Chapter 4 be used to develop a previously unknown, explicit representation of the Bergman kernel for some domain Ω?*

If so, there are many questions which can be potentially answered regarding the holomorphic function theory on Ω; e.g. whether the Bergman kernel has zeroes, or whether the associated Bergman projection is continuous as a mapping from $L^p(\Omega)$ to $L^q(\Omega)$ for some $p, q \in [1, \infty]$.

In §4.4, a reproducing kernel for Hartogs domains is introduced. The reproducing kernel is similar to the Bergman kernel, with the exception that it is, in general, not conjugate-holomorphic in one of the variables.

Question 6. *Can the reproducing kernel $\tilde{K}$ of Theorem 19 be modified so that it is conjugate-holomorphic in the ζ-variable?*

If so, then $\tilde{K}$ is in fact identical to the Bergman kernel by uniqueness. It may be possible to approach Question 6 by $\bar{\partial}$-methods, as was done in Theorem 7.

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