# GERSTENHABER BRACKET ON HOPF ALGEBRA COHOMOLOGY 

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#### Abstract

M. A. Farinati, A. Solotar, and R. Taillefer showed that the Hopf algebra cohomology of a quasi-triangular Hopf algebra, as a graded Lie algebra under the Gerstenhaber bracket, is abelian. Motivated by the question of whether this holds for nonquasi-triangular Hopf algebras, we calculate the Gerstenhaber bracket on Hopf algebra and Hochschild cohomologies of the Taft algebra $T_{p}$ for any integer $p>2$ which is a nonquasi-triangular Hopf algebra. We show that the bracket is indeed zero on Hopf algebra cohomology of $T_{p}$, as in all known quasi-triangular Hopf algebras. This example is the first known bracket computation for a nonquasi-triangular algebra.

We also show that Gerstenhaber brackets on Hopf algebra cohomology can be expressed via an arbitrary projective resolution using Volkov's homotopy liftings as generalized to some exact monoidal categories. This is a special case of our more general result that a bracket operation on cohomology is preserved under exact monoidal functors-one such functor is an embedding of Hopf algebra cohomology into Hochschild cohomology. As a consequence, we show that this Lie structure on Hopf algebra cohomology is abelian in positive degrees for all quantum elementary abelian groups $\left(T_{p}\right)$, most of which are nonquasi-triangular.

Also, we find a general formula for the bracket on Hopf algebra cohomology of any Hopf algebra with bijective antipode on the bar resolution that is reminiscent of Gerstenhaber's original formula for Hochschild cohomology.


## DEDICATION

To my grandmother

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## TABLE OF CONTENTS

## Page

ABSTRACT ..... ii
DEDICATION ..... iii
ACKNOWLEDGMENTS ..... iv
CONTRIBUTORS AND FUNDING SOURCES ..... v
TABLE OF CONTENTS ..... vi

1. INTRODUCTION ..... 1
2. PRELIMINARIES ..... 4
2.1 Hochschild cohomology ..... 4
2.2 Hopf algebra and Hopf algebra cohomology ..... 6
2.3 G-algebra Structure ..... 9
3. BRACKET ON THE HOCHSCHILD COHOMOLOGY OF A TRUNCATED POLY- NOMIAL RING ..... 12
3.1 A technique to calculate the bracket ..... 12
3.2 Graded Lie structure for the truncated polynomial ring ..... 13
4. BRACKET ON THE HOPF ALGEBRA COHOMOLOGY OF A TAFT ALGEBRA ..... 18
4.1 The bracket on Hochschild cohomology of a Taft algebra ..... 18
4.2 The bracket on Hopf algebra cohomology of a Taft algebra ..... 28
5. THE LIE STRUCTURE OF THE HOPF ALGEBRA COHOMOLOGY OF A TAFT ALGEBRA BY HOMOTOPY LIFTING ..... 30
5.1 Homotopy liftings and exact monoidal categories ..... 30
5.2 Change of exact monoidal categories ..... 33
5.3 Embedding from Hopf algebra cohomology into Hochschild cohomology ..... 37
5.4 A new technique for the bracket on Hopf algebra cohomology ..... 39
5.5 Taft algebras and quantum elementary abelian groups ..... 42
6. G-ALGEBRA STRUCTURE ON HOPF ALGEBRA COHOMOLOGIES ..... 46
7. SUMMARY ..... 52
REFERENCES .............................................................................................................. 53

## 1. INTRODUCTION

Homological algebra was one of the main interest areas in topology from the 1800s to the 1940s. After the 1940s, it became an independent subject which has many applications in differential geometry, algebraic topology, algebraic geometry, and commutative algebra. Hochschild is the mathematician who introduced homology and cohomology of algebras. Almost two decades later, Gerstenhaber saw a Hochschild cohomology ring as a Gerstenhaber algebra that is an algebra with an associative product (cup product) and nonassociative Lie bracket (Gerstenhaber bracket). Although the cup product is defined and can be calculated in several ways, the Gerstenhaber bracket was originally defined on the bar complex which makes the bracket impossible to calculate by the definition.

A Hopf algebra is an algebra that additionally has a coalgebra structure. Hopf algebras were first defined in algebraic topology by Hopf in 1941. Group algebras, tensor algebras, affine group schemes, universal enveloping algebras of Lie algebras, and quantum groups are just a few important examples of Hopf algebras. Hence, Hopf algebras can be seen in different fields of mathematics such as algebraic geometry, representation theory, Lie theory, quantum mechanics, graded ring theory, and combinatorics. A Hopf algebra cohomology ring is also defined in a similar way as Hochschild cohomology is defined and has become an important actor in homological algebra. One of the examples that shows us how crucial the role that Hopf algebra cohomology plays in mathematics is varieties. They are a very useful tool for understanding modules. For an algebra $A$, the support variety of an $A$-module is first defined as the maximal ideal spectrum of a specific quotient of the Hopf algebra cohomology of $A$. Then the definition is extended to Hochschild cohomology of $A$. Moreover, Hopf algebra cohomologies provide tools for constructing spectral sequences, for example a spectral sequence relating Hochschild cohomology of a smash product to cohomology of its components [25, Chapter 9].

The Gerstenhaber bracket was originally defined on Hochschild cohomology by M. Gerstenhaber himself [6, Section 1.1] which makes Hochschild cohomology a G-algebra (graded Lie al-
gebra) together with the cup product. In 1992, M. Gerstenhaber and S. D. Schack conjectured that Hopf algebra cohomology has a G-structure as well as Hochschild cohomology. In 2002, A. Farinati and A. Solotar showed that for any Hopf algebra $A$, Hopf algebra cohomology is a Gerstenhaber algebra [5]. In the same year, R. Taillefer used a different approach and found a bracket on Hopf algebra cohomology [20]. More specifically, Taillefer [20] constructed a Lie bracket on cohomology arising from a category of Hopf bimodule extensions of a Hopf algebra and showed that brackets are always zero. In the finite dimensional case, this corresponds to Hopf algebra cohomology of the opposite of the Drinfeld double, whose Lie structure was also investigated by Farinati and Solotar [5] using different techniques. Farinati and Solotar showed this is indeed the bracket arising from an embedding into Hochschild cohomology. Hermann [9] looked at a more general monoidal category setting and showed that the Lie structure is trivial in case the category is braided [9, Theorem 5.2.7]. However, the bracket structure in the general setting is still unknown.

This dissertation consists of a combination of the articles [10] and [11]. Specifically, we find the bracket structure on Hopf algebra cohomology of a Taft algebra with two different techniques. We also introduce a method that shows how to find a bracket on Hopf algebra cohomology without working on Hochschild cohomology. Lastly, we derive a formula for the Gerstenhaber bracket on Hopf algebra cohomology of any Hopf algebra with bijective antipode that is coming from its definition on Hochschild cohomology.

In Chapter 2, we start by giving some basic definitions and examples in homological algebra. Then, we define Hochschild cohomology of an algebra and state some tools to calculate the bracket on it. At the end of the chapter, we define a Hopf algebra $A$ over a field $k$ and Hopf algebra cohomology, and provide some examples.

In Chapter 3, we compute the Gerstenhaber bracket on Hochschild cohomology of the truncated polynomial ring $k[x] /\left(x^{p}\right)$ where the field $k$ has characteristic 0 and the integer $p>2$. We use the technique introduced by C. Negron and S. Witherspoon [14] who computed the bracket on Hochschild cohomology of $A$ for the case that $k$ has positive characteristic $p$ [14, Section 5].

In Chapter 4, we compute the Gerstenhaber bracket for the Taft algebra $T_{p}$ which is nonquasi-
triangular when $p>2$ [8, Proposition 2.1]. First, we use a similar technique as in [14] to calculate the bracket on Hochschild cohomology of $T_{p}$. It is known that the Hopf algebra cohomology of any Hopf algebra with a bijective antipode can be embedded in Hochschild cohomology of the algebra [25, Theorem 9.4.5 and Corollary 9.4.7]. Since all finite dimensional Hopf algebras (also most of the known infinite dimensional Hopf algebras) have bijective antipode, we can embed the Hopf algebra cohomology of $T_{p}$ into the Hochschild cohomology of $T_{p}$. Then, we use the explicit embedding and find the bracket on the Hopf algebra cohomology of $T_{p}$. As a result of our calculation, we find that the bracket on Hopf algebra cohomology of $T_{p}$ is also trivial. This is the first known example of the Lie structure on Hopf algebra cohomology of a nonquasi-triangular Hopf algebra.

In Chapter 5, we apply a different technique, the homotopy lifting method of Y. Volkov [21], to derive a bracket structure in a more general setting. Homotopy liftings were defined for some exact monoidal categories in [22], and we use them to prove that brackets are preserved under exact monoidal functors. As a consequence, we show that the Lie bracket on Hopf algebra cohomology defined by homotopy liftings agrees with that induced by its embedding into Hochschild cohomology, in case the antipode is bijective. By the homotopy lifting method, we are able to handle the Lie structure independently of choice of projective resolution. A good choice of resolution can facilitate understanding of the Lie structure. We consider the quantum elementary abelian groups (the Taft algebras $T_{p}$ ) as an example and find the bracket on their Hopf algebra cohomology that agrees with the calculation in Chapter 4.

In Chapter 6, we find a general expression of graded Lie bracket on Hopf algebra cohomology of any Hopf algebra with bijective antipode by using an explicit embedding from Hopf algebra cohomology into Hochschild cohomology. However, we should note that the hypothesis that the antipode is bijective is not very restrictive as all finite dimensional Hopf algebras and most of infinite dimensional Hopf algebras have bijective antipodes.

## 2. PRELIMINARIES

### 2.1 Hochschild cohomology

In this section, we give some background of homological algebra and Hochschild cohomology.
Definition 2.1.1. Let $R$ be a ring and $A$ and $B$ be two $R$-modules. A projective resolution of $A$ is an exact sequence of projective $R$-modules $P_{i}$ :

$$
P_{\bullet}: \cdots \xrightarrow{d_{3}} P_{2} \xrightarrow{d_{2}} P_{1} \xrightarrow{d_{1}} P_{0} \xrightarrow{\epsilon} A \longrightarrow 0
$$

and a chain complex of $R$-modules is a sequence of $R$-modules:

$$
C_{\bullet}: \cdots \xrightarrow{d_{3}} C_{2} \xrightarrow{d_{2}} C_{1} \xrightarrow{d_{1}} C_{0} \xrightarrow{d_{0}} C_{-1} \xrightarrow{d_{-1}} C_{-2} \xrightarrow{d_{-2}} \cdots
$$

such that $d_{n} d_{n+1}=0 . \operatorname{Ker}\left(d_{n}\right)$ is called the set of $n$-cycles, $\operatorname{Im}\left(d_{n+1}\right)$ is called the set of $n$ boundaries and $\mathrm{H}_{n}\left(C_{\bullet}\right)=\operatorname{Ker}\left(d_{n}\right) / \operatorname{Im}\left(d_{n+1}\right)$ is called the $n^{\text {th }}$ homology.

Note that the condition $d_{n} d_{n+1}=0$ means that $\operatorname{Im}\left(d_{n+1}\right) \subset \operatorname{Ker}\left(d_{n}\right)$ so that any projective resolution is a chain complex by definition.

Definition 2.1.2. By applying $\operatorname{Hom}_{R}(-, B)$ to the projective resolution $P_{\bullet}$ and dropping the term $\operatorname{Hom}_{R}(A, B)$, we obtain a cochain complex:

$$
\operatorname{Hom}_{R}\left(P_{\bullet}, B\right): 0 \longrightarrow \operatorname{Hom}_{R}\left(P_{0}, B\right) \xrightarrow{d_{1}^{*}} \operatorname{Hom}_{R}\left(P_{1}, B\right) \xrightarrow{d_{2}^{*}} \operatorname{Hom}_{R}\left(P_{2}, B\right) \xrightarrow{d_{3}^{*}} \cdots
$$

where $d_{n}^{*}(f)=f d_{n} . \operatorname{Ker}\left(d_{n+1}^{*}\right)$ is called the set of $n$-cocycles, $\operatorname{Im}\left(d_{n}^{*}\right)$ is called the set of $n$ coboundaries and $\mathrm{H}^{n}\left(\operatorname{Hom}_{R}\left(P_{\bullet}, B\right)\right)=\operatorname{Ker}\left(d_{n+1}^{*}\right) / \operatorname{Im}\left(d_{n}^{*}\right)$ is called the $n^{\text {th }}$ cohomology.

The Ext functor is defined as the cohomology of the cochain complex:

$$
\operatorname{Ext}_{R}^{n}(A, B):=\mathrm{H}^{n}\left(\operatorname{Hom}_{R}\left(P_{\bullet}, B\right)\right)
$$

Definition 2.1.3. Let $k$ be a field and $A$ be a $k$-algebra. The unit map $k \longrightarrow A$ is given by $c \longmapsto c \cdot 1_{A}$. The opposite algebra $A^{o p}$ is given by the multiplication $a \cdot_{o p} b=b a$ and the enveloping algebra $A^{e}=A \otimes_{k} A^{o p}$ is given by the multiplication

$$
\left(a_{1} \otimes_{k} b_{1}\right)\left(a_{2} \otimes_{k} b_{2}\right)=a_{1} a_{2} \otimes_{k} b_{1} \cdot o p b_{2}
$$

Note that we use $\otimes$ instead of $\otimes_{k}$ through Chapters $2,3,4$, and 6.
We can construct the chain complex (also exact)

$$
\cdots \xrightarrow{d_{3}} A^{\otimes 4} \xrightarrow{d_{2}} A^{\otimes 3} \xrightarrow{d_{1}} A^{\otimes 2} \xrightarrow{m} A \longrightarrow 0,
$$

where $m$ is multiplication,

$$
d_{n}\left(a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n+1}\right)=\sum_{i=0}^{n}(-1)^{i} a_{0} \otimes a_{1} \otimes \cdots \otimes a_{i} a_{i+1} \otimes \cdots \otimes a_{n+1},
$$

and the bar complex by

$$
B(A): \cdots \xrightarrow{d_{3}} A^{\otimes 4} \xrightarrow{d_{2}} A^{\otimes 3} \xrightarrow{d_{1}} A^{\otimes 2} \longrightarrow 0 .
$$

Since $A$ is free as a $k$-algebra, $B(A)$ is a free resolution of the $A^{e}$-module $A$, called the bar resolution.

Let $M$ be an $A$-bimodule. If we apply $\operatorname{Hom}_{A^{e}}(-, M)$ to the bar complex $B(A)$, we obtain the complex

$$
\begin{equation*}
0 \longrightarrow \operatorname{Hom}_{A^{e}}\left(A^{\otimes 2}, M\right) \xrightarrow{d_{1}^{*}} \operatorname{Hom}_{A^{e}}\left(A^{\otimes 3}, M\right) \xrightarrow{d_{2}^{*}} \operatorname{Hom}_{A^{e}}\left(A^{\otimes 4}, M\right) \xrightarrow{d_{3}^{*}} \cdots . \tag{2.1.4}
\end{equation*}
$$

Definition 2.1.5. The Hochshild cohomology of $A$ is the cohomology of the complex (2.1.4), i.e.

$$
\mathrm{HH}^{*}(A, M)=\bigoplus_{n \geq 0} \operatorname{Ker}\left(d_{n+1}^{*}\right) / \operatorname{Im}\left(d_{n}^{*}\right) .
$$

Note that we just focus on the case $M=A$ and we use $\operatorname{HH}^{*}(A)$ instead of $\operatorname{HH}^{*}(A, A)$.
Here is a nice example of Hochschild cohomology:

Example 2.1.6. Let $k$ be a field with characteristic 0 , and $A=k[x] /\left(x^{p}\right)$. Then, we have an exact sequence

$$
\begin{equation*}
\mathbb{A}: \cdots \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{m} A \longrightarrow 0, \tag{2.1.7}
\end{equation*}
$$

where $u=x \otimes 1-1 \otimes x$ and $v=x^{p-1} \otimes 1+x^{p-2} \otimes x+\cdots+1 \otimes x^{p-1}$ and it is a free resolution of the $A^{e}$-module $A$.

Once we apply $\operatorname{Hom}_{A^{e}}(-, A)$ to $\mathbb{A}$, by using the isomorphism $\operatorname{Hom}_{A^{e}}\left(A^{e}, A\right) \cong A$, we have the sequence

$$
\operatorname{Hom}_{A^{e}}(\mathbb{A}, A): 0 \longrightarrow A \xrightarrow{0} A \xrightarrow{p x^{p-1}} A \xrightarrow{0} A \xrightarrow{p x^{p-1}} A \longrightarrow \cdots
$$

Since $p$ is not divisible by the characteristic of $k$, $\mathrm{HH}^{0}(A)=A, \mathrm{HH}^{2 i+1}(A) \cong(x)$, and $\mathrm{HH}^{2 i}(A) \cong A /\left(x^{p-1}\right)$.

Hochschild cohomology gives important information about the algebra $A$ in low degrees. For instance, Hochschild cohomology gives the center of the algebra in degree 0 , the derivations in degree 1 , and the infinitesimal deformations in degree 2 .

### 2.2 Hopf algebra and Hopf algebra cohomology

We start this section with the definition of an algebra.

Definition 2.2.1. $A$ is an algebra over the field $k$ if $A$ has two $k$-linear maps:

- $m: A \otimes A \rightarrow A$ (multiplication map)
- $\eta: k \rightarrow A$ (unit map)
which satisfy

1. $m\left(m \otimes i d_{A}\right)=m\left(i d_{A} \otimes m\right)$ (associativity)
2. $m\left(\eta \otimes i d_{A}\right)=i d_{A}$
3. $m\left(i d_{A} \otimes \eta\right)=i d_{A}$
where $i d_{A}$ denotes the identity map on $A$. Moreover, $(A, m, \eta)$ is commutative if $a b=b a$ for all $a, b \in A$.

Definition 2.2.2. An algebra $A$ over the field $k$ is a Hopf algebra with algebra homomorphisms $\Delta: A \rightarrow A \otimes A$ (comultiplication), $\varepsilon: A \rightarrow k$ (counit), and an algebra anti-homomorphism $S: A \rightarrow A$ (antipode) that satisfy:

1. $\left(\Delta \otimes i d_{A}\right) \Delta=\left(i d_{A} \otimes \Delta\right) \Delta$ (coassociativity),
2. $\left(i d_{A} \otimes \varepsilon\right) \Delta=i d_{A}=\left(\varepsilon \otimes i d_{A}\right) \Delta$,
3. $m\left(S \otimes i d_{A}\right) \Delta=\eta \varepsilon=m\left(i d_{A} \otimes S\right) \Delta$

Moreover $(A, \Delta, \varepsilon)$ is cocommutative if $\tau \Delta=\Delta$ where $\tau: A \otimes A \longrightarrow A \otimes A$ is the $k$-linear map such that $\tau(a \otimes b)=b \otimes a$.

We first give a simple example of a Hopf algebra:

Example 2.2.3. Let $G$ be a group. Then $k G=\left\{\sum_{g \in G} a_{g} g: a_{g} \in k\right\}$ is an algebra, called a group algebra with the multiplication

$$
\left(\sum_{g \in G} a_{g} g\right)\left(\sum_{h \in G} b_{h} h\right)=\sum_{g, h \in G}\left(a_{g} b_{h}\right) g h .
$$

Then $k G$ is a Hopf algebra with the structure:

1. $\Delta(g)=g \otimes g$
2. $\varepsilon(g)=1_{k}$
3. $S(g)=g^{-1}$
for all $g, h \in G$. Although a group algebra $k G$ is cocommutative, it is commutative if and only if $G$ is abelian.

A more complicated example of Hopf algebras is

Example 2.2.4. Let $p \geq 2$ and let $T_{p}$ be the $k$-algebra generated by $g$ and $x$ satisfying the relations:

$$
g^{p}=1, x^{p}=0, \text { and } x g=\omega g x
$$

where $\omega$ is a primitive $p^{t h}$ root of unity. Then $T_{p}$, called a Taft algebra, is a Hopf algebra with the structure:

- $\Delta(g)=g \otimes g, \Delta(x)=1 \otimes x+x \otimes g$
- $\varepsilon(g)=1, \varepsilon(x)=0$
- $S(g)=g^{-1}, S(x)=-x g^{-1}$.

It is easy to see that a Taft algebra is a noncocommutative and noncommutative Hopf algebra.

We give the Sweedler notation to define quasi-triangular Hopf algebras:

Definition 2.2.5. Let $A$ be a Hopf algebra over the field $k$. We use Sweedler notation for the coproduct in an Hopf algebra $A$, which is:

$$
\Delta(a)=\sum a_{1} \otimes a_{2}
$$

for $a \in A$ where $a_{1}, a_{2}$ for tensor factors is symbolic.

We recall the Drinfeld's notion of quasi-triangular Hopf algebras [3, Equation (21)]:

Definition 2.2.6. Let $A$ be a Hopf algebra and let $R=\sum a_{1} \otimes a_{2}$ be an invertible element in $A \otimes A$. Define $R_{12}, R_{13}$, and $R_{23}$ as $R_{12}=\sum a_{1} \otimes a_{2} \otimes 1, R_{13}=\sum a_{1} \otimes 1 \otimes a_{2}, R_{23}=\sum 1 \otimes a_{1} \otimes a_{2}$. Then, $A$ is quasi-triangular if the following equations hold:

$$
\begin{align*}
& \tau \Delta(a)=R \Delta(a) R^{-1} \text { for all } a \in A  \tag{2.2.7}\\
& \left(\Delta \otimes i d_{A}\right) R=R_{13} R_{23},\left(i d_{A} \otimes \Delta\right) R=R_{13} R_{12} \tag{2.2.8}
\end{align*}
$$

where $\tau$ is the twisting map on the Definition (2.2.2).

It is known that $T_{p}$ is nonquasi-triangular when $p>2$ [8, Proposition 2.1].

Definition 2.2.9. Let $G$ be a finite group acting by automorphisms on an algebra $A$ and let ${ }^{g} a$ be the result of applying $g \in G$ on $a \in A$. The skew group algebra $A \rtimes G$ is $A \otimes k G$ as a vector space, with the multiplication

$$
\left(a_{1} \otimes g_{1}\right)\left(a_{2} \otimes g_{2}\right)=a_{1}\left({ }^{g_{1}} a_{2}\right) \otimes g_{1} g_{2}
$$

for all $a_{1}, a_{2} \in A$ and $g_{1}, g_{2} \in G$.

Note that $T_{p}$ can be seen as a skew group algebra $A \rtimes G$ for $A=k[x] /\left(x^{p}\right)$ and $G=\mathbb{Z} / p \mathbb{Z}$.

Definition 2.2.10. Let $A$ be a Hopf algebra over a field $k$. Then, the Hopf algebra cohomology ring is

$$
\mathrm{H}^{*}(A, k)=\operatorname{Ext}_{A}^{*}(k, k) .
$$

### 2.3 G-algebra Structure

By its definition, Hochschild cohomology of $A$ is a graded $k$-module. We now give two structures that make Hochschild cohomology a G-algebra.

Let $f \in \operatorname{Hom}_{k}\left(A^{\otimes m}, A\right)$ and $g \in \operatorname{Hom}_{k}\left(A^{\otimes n}, A\right)$. Hochschild cohomology of $A$ is an algebra with the following cup product and the Gerstenhaber bracket structures. The cup product
$f \smile g \in \operatorname{Hom}_{k}\left(A^{\otimes(m+n)}, A\right)$ is defined by

$$
(f \smile g)\left(a_{1} \otimes \cdots \otimes a_{m+n}\right):=(-1)^{m n} f\left(a_{1} \otimes \cdots \otimes a_{m}\right) g\left(a_{m+1} \otimes \cdots a_{m+n}\right)
$$

for all $a_{1}, \cdots, a_{m+n} \in A$. When $m=0$, the equality is

$$
(f \smile g)\left(a_{1} \otimes \cdots \otimes a_{n}\right):=f(1) g\left(a_{1} \otimes \cdots a_{n}\right)
$$

One can see that the cup product is defined at cochain levels. However, it induces a well-defined operation on $\mathrm{HH}^{*}(A)$. The cup product is associative; so, Hochschild cohomology is an associative algebra together with the cup product. Moreover, the cup product is also graded commutative on $\mathrm{HH}^{*}(A)$, i.e. the homogeneous elements commute up to a sign determined by homological degrees [6].

The Gerstenhaber bracket $[f, g]$ is defined as an element of $\operatorname{Hom}_{k}\left(A^{\otimes(m+n-1)}, A\right)$ given by

$$
[f, g]:=f \circ g-(-1)^{(m-1)(n-1)} g \circ f
$$

where the circle product $f \circ g$ is

$$
\begin{aligned}
& (f \circ g)\left(a_{1} \otimes \cdots \otimes a_{m+n-1}\right):= \\
& \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} f\left(a_{1} \otimes \cdots a_{i-1} \otimes g\left(a_{i} \otimes \cdots a_{i+n-1}\right) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}\right)
\end{aligned}
$$

for all $a_{1}, \cdots, a_{m+n-1} \in A$. We note that these definitions directly come from the bar resolution. Moreover, the bracket is also defined at cochain level and it induces a well-defined operation on $\mathrm{HH}^{*}(A)$.

Hochschild cohomology forms a graded Lie algebra with the bracket. Specifically, $\mathrm{HH}^{1}(A)$ is a Lie algebra over the module $\mathrm{HH}^{*}(A)$.

There is an identity between cup product and bracket [6, Section 1]:

$$
\begin{equation*}
\left[f^{*} \smile g^{*}, h^{*}\right]=\left[f^{*}, h^{*}\right] \smile g^{*}+(-1)^{\left|f^{*}\right|\left(\left|h^{*}\right|-1\right)} f^{*} \smile\left[g^{*}, h^{*}\right], \tag{2.3.1}
\end{equation*}
$$

where $f^{*}, g^{*}$, and $h^{*}$ are the images (in Hochschild cohomology) of the cocyles $f, g$, and $h$, respectively.

A $G$-algebra $(A, \smile,[]$,$) is a free graded k$-module where $(A, \smile)$ is a graded commutative associative algebra, $(A,[]$,$) is a graded Lie algebra, and the equation (2.3.1) holds. It is known that$ Hochschild cohomology is a G-algebra together with the cup product and the bracket [25, Theorem 1.4.9]. It is also known that a graded Lie structure can be constructed on Hopf algebra cohomology [5, 20] which makes Hopf algebra cohomology a G-algebra. Moreover, the Lie bracket on Hopf algebra cohomology is trivial if the Hopf algebra is quasi-triangular [5, 9, 20]. Our main purpose on this thesis is introducing some techniques that help us to find the bracket structure for nonquasi-triangular Hopf algebras and illustrating these techniques on Hopf algebra cohomologies of nonquasi-triangular Taft algebras.

# 3. BRACKET ON THE HOCHSCHILD COHOMOLOGY OF A TRUNCATED POLYNOMIAL RING 

### 3.1 A technique to calculate the bracket

Computing the bracket on the bar resolution is not an ideal method. Instead, we can use another resolution, $\mathbb{A} \xrightarrow{\mu} A$, satisfying the following hypotheses [14, (3.1) and Lemma 3.4.1]:
(a) $\mathbb{A}$ admits an embedding $\iota: \mathbb{A} \rightarrow B(A)$ of complexes of $A$-bimodules for which the following diagram commutes

(b) The embedding $\iota$ admits a section $\pi: B \rightarrow \mathbb{A}$, i.e. an $A^{e}$-chain map $\pi$ with $\pi \iota=i d_{\mathbb{A}}$.
(c) There is a diagonal map $\Delta_{\mathbb{A}}: \mathbb{A} \rightarrow \mathbb{A} \otimes_{A} \mathbb{A}$, that is a chain map lifting the canonical isomorphism $A \stackrel{\sim}{\rightarrow} A \otimes_{A} A$, that satisfies $\Delta_{\mathbb{A}}^{(2)}=\left(\pi \otimes_{A} \pi \otimes_{A} \pi\right) \Delta_{B(A)}^{(2)} \iota$ where $\Delta^{(2)}=(i d \otimes \Delta) \Delta$.

We give the following theorem which is a combination of [14, Theorem 3.2.5] and [14, Lemma 3.4.1] that allows us to use a different resolution for the bracket calculation.

Theorem 3.1.1. Suppose $\mathbb{A} \xrightarrow{\mu} A$ is a projective $A$-bimodule resolution of $A$ that satisfies the hypotheses (a)-(c). Let $\phi: \mathbb{A} \otimes_{A} \mathbb{A} \rightarrow \mathbb{A}$ be any contracting homotopy for the chain map $F_{\mathbb{A}}$ : $\mathbb{A} \otimes_{A} \mathbb{A} \rightarrow \mathbb{A}$ defined by $F_{\mathbb{A}}:=\left(\mu \otimes_{A} i d_{\mathbb{A}}-i d_{\mathbb{A}} \otimes_{A} \mu\right)$, i.e.

$$
\begin{equation*}
d(\phi):=d_{\mathbb{A}} \phi+\phi d_{\mathbb{A} \otimes_{A} \mathbb{A}}=F_{\mathbb{A}} . \tag{3.1.2}
\end{equation*}
$$

Then for homogeneous cocycles $f$ and $g$ in $\operatorname{Hom}_{A^{e}}(\mathbb{A}, A)$, the bracket given by

$$
\begin{equation*}
[f, g]_{\phi}=f \circ_{\phi} g-(-1)^{(|f|-1)(|g|-1)} g \circ_{\phi} f \tag{3.1.3}
\end{equation*}
$$

where the circle product is

$$
\begin{equation*}
f \circ_{\phi} g=f \phi\left(i d_{\mathbb{A}} \otimes_{A} g \otimes_{A} i d_{\mathbb{A}}\right) \Delta^{(2)} \tag{3.1.4}
\end{equation*}
$$

agrees with the Gerstenhaber bracket on cohomology.

In general, it is not easy to find a map $\phi$ by the formula (3.1.2). We use an alternative way to find $\phi$.

Let $h$ be any $k$-linear contracting homotopy for the identity map on the extended complex $\mathbb{A} \rightarrow A \rightarrow 0$ where $\mathbb{A}$ is free. A contracting homotopy $\phi_{i}:\left(\mathbb{A} \otimes_{A} \mathbb{A}\right)_{i} \longrightarrow \mathbb{A}_{i+1}$ in Theorem 3.1.1 is constructed by the following formula [14, Lemma 3.3.1]:

$$
\begin{equation*}
\phi_{i}=h_{i}\left(\left(F_{\mathbb{A}}\right)_{i}-\phi_{i-1} d_{\left(\mathbb{A} \otimes_{\mathcal{A}} \mathbb{A}\right)_{i}}\right) . \tag{3.1.5}
\end{equation*}
$$

### 3.2 Graded Lie structure for the truncated polynomial ring

Let $A=k[x] /\left(x^{p}\right)$ where $k$ is a field of characteristic 0 and $p>2$ is an integer. We compute the Lie bracket on Hochschild cohomology of $A$ by Theorem 3.1.1. We work on a smaller resolution of $A$ than the bar resolution of $A$. Consider the following $A^{e}$-module resolution of $A$ :

$$
\begin{equation*}
\mathbb{A}: \cdots \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{\pi} A \longrightarrow 0, \tag{3.2.1}
\end{equation*}
$$

where $u=x \otimes 1-1 \otimes x, v=x^{p-1} \otimes 1+x^{p-2} \otimes x+\cdots+x \otimes x^{p-2}+1 \otimes x^{p-1}$, and $\pi$ is the multiplication.

The bracket on $A$ where $k$ is a field with characteristic $p$, is calculated before [14, Section 5]. We adapt the contracting homotopy $h$ for the identity map from that calculation and obtain a new map $h$ for our setup. Let $\xi_{i}$ be the element $1 \otimes 1$ of $\mathbb{A}_{i}$. The following maps $h_{n}: \mathbb{A}_{n} \longrightarrow \mathbb{A}_{n+1}$ form a contracting homotopy for the identity map, as we can see by direct calculation:

$$
\begin{align*}
h_{-1}\left(x^{i}\right) & =\xi_{0} x^{i}, \\
h_{0}\left(x^{i} \xi_{0} x^{j}\right) & =\sum_{l=0}^{i-1} x^{l} \xi_{1} x^{i+j-1-l}, \\
h_{1}\left(x^{i} \xi_{1} x^{j}\right) & =\delta_{i, p-1} x^{j} \xi_{2},  \tag{3.2.2}\\
h_{2 n}\left(x^{i} \xi_{2 n} x^{j}\right) & =-\sum_{l=0}^{j-1} x^{i+j-1-l} \xi_{2 n+1} x^{l}(n \geq 2), \\
h_{2 n+1}\left(x^{i} \xi_{2 n+1} x^{j}\right) & =\delta_{j, p-1} x^{i} \xi_{2 n+2}(n \geq 2) .
\end{align*}
$$

Then, we take $\phi_{-1}=0$ and find $\phi_{0}$ and $\phi_{1}$ by the formula (3.1.5) as follows:

$$
\begin{align*}
& \phi_{0}\left(\xi_{0} \otimes_{A} x^{i} \xi_{0}\right)=\sum_{l=0}^{i-1} x^{l} \xi_{1} x^{i-1-l} \\
& \phi_{1}\left(\xi_{1} \otimes_{A} x^{i} \xi_{0}\right)=-\delta_{i, p-1} \xi_{2},  \tag{3.2.3}\\
& \phi_{1}\left(\xi_{0} \otimes_{A} x^{i} \xi_{1}\right)=\delta_{i, p-1} \xi_{2} .
\end{align*}
$$

Lastly, we form the following diagonal map $\Delta: \mathbb{A} \longrightarrow \mathbb{A} \otimes_{A} \mathbb{A}$ :

$$
\begin{align*}
& \Delta_{0}\left(\xi_{0}\right)=\xi_{0} \otimes_{A} \xi_{0}, \\
& \Delta_{1}\left(\xi_{1}\right)=\xi_{1} \otimes_{A} \xi_{0}+\xi_{0} \otimes_{A} \xi_{1}, \\
& \Delta_{2 n}\left(\xi_{2 n}\right)=\sum_{i=0}^{n} \xi_{2 i} \otimes_{A} \xi_{2 n-2 i}+\sum_{i=0}^{n-1} \sum_{a+b+c=p-2} x^{a} \xi_{2 i+1} \otimes_{A} x^{b} \xi_{2 n-2 i-1} x^{c}, \text { for } n \geq 1  \tag{3.2.4}\\
& \Delta_{2 n+1}\left(\xi_{2 n+1}\right)=\sum_{i=0}^{2 n+1} \xi_{i} \otimes_{A} \xi_{2 n+1-i}, \text { for } n \geq 1 .
\end{align*}
$$

It can be seen that the map $\Delta$ is a chain map lifting the canonical isomorphism $A \xrightarrow{\sim} A \otimes_{A} A$ by direct calculation.

Now, we are ready to calculate the brackets on cohomology in low degrees. By applying
$\operatorname{Hom}_{A^{e}}(-, A)$ to $\mathbb{A}$, we see that the differentials are all 0 in odd degrees and $\left(p x^{p-1}\right)$. in even degrees. In each degree, the term in the Hom complex is the free $A$-module $\operatorname{Hom}_{A^{e}}\left(A^{e}, A\right) \cong A$. Moreover, since $p$ is not divisible by the characteristic of $k$, we deduce $\operatorname{HH}^{0}(A) \cong A, \operatorname{HH}^{2 i+1}(A) \cong$ $(x)$, and $\operatorname{HH}^{2 i}(A) \cong A /\left(x^{p-1}\right)[25$, Section 1.1].

Let $x^{j} \xi_{i}^{*} \in \operatorname{Hom}_{A^{e}}\left(A^{e}, A\right)$ denote the function that takes $\xi_{i}$ to $x^{j}$. Since the characteristic of $k$ does not divide $p$, the Hochschild cohomology as an $A$-algebra is generated by $\xi_{1}^{*}$ and $\xi_{2}^{*}$ [25, Example 2.2.2]. We only calculate the brackets of the elements of degrees 1 and 2 which can be extended to higher degrees by the formula (2.3.1). Hence, we have the following calculations:

The bracket of the elements of degrees 1 and 1 :

$$
\begin{aligned}
& \left(x^{i} \xi_{1}^{*} \circ_{\phi} x^{j} \xi_{1}^{*}\right)\left(\xi_{1}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(1 \otimes_{A} x^{j} \xi_{1}^{*} \otimes_{A} 1\right) \Delta^{(2)}\left(\xi_{1}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(1 \otimes_{A} x^{j} \xi_{1}^{*} \otimes_{A} 1\right)\left(\xi_{1} \otimes_{A} \xi_{0} \otimes_{A} \xi_{0}+\xi_{0} \otimes_{A} \xi_{1} \otimes_{A} \xi_{0}+\xi_{0} \otimes_{A} \xi_{0} \otimes \xi_{1}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(\xi_{0} \otimes_{A} x^{j} \xi_{0}\right) \\
& =x^{i} \xi_{1}^{*}\left(\xi_{1} x^{j-1}+x \xi_{1} x^{j-2}+\cdots+x^{j-1} \xi_{1}\right) \\
& =j x^{i+j-1}
\end{aligned}
$$

and by symmetry $\left(x^{j} \xi_{1}^{*} \circ_{\phi} x^{i} \xi_{1}^{*}\right)\left(\xi_{1}\right)=i x^{i+j-1}$. Therefore, we have

$$
\left[x^{i} \xi_{1}^{*}, x^{j} \xi_{1}^{*}\right]=(j-i) x^{i+j-1} \xi_{1}^{*} .
$$

The bracket of the elements of degrees 1 and 2 :

$$
\begin{aligned}
& \left(x^{i} \xi_{1}^{*} \circ_{\phi} x^{j} \xi_{2}^{*}\right)\left(\xi_{2}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(1 \otimes_{A} x^{j} \xi_{2}^{*} \otimes_{A} 1\right) \Delta^{(2)}\left(\xi_{2}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(1 \otimes_{A} x^{j} \xi_{2}^{*} \otimes_{A} 1\right)\left(\xi_{0} \otimes_{A} \xi_{0} \otimes_{A} \xi_{2}+\xi_{0} \otimes_{A} \xi_{2} \otimes_{A} \xi_{0}+\xi_{2} \otimes_{A} \xi_{0} \otimes_{A} \xi_{0}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+\xi_{0} \otimes_{A} \sum_{\substack{a+b+c \\
=p-2}}\left(x^{a} \xi_{1} \otimes_{A} x^{b} \xi_{1} x^{c}\right)+\sum_{\substack{a+b+c \\
=p-2}} x^{a} \xi_{1} \otimes_{A} x^{b}\left(\xi_{0} \otimes_{A} \xi_{1}+\xi_{1} \otimes_{A} x_{0}\right) x^{c}\right) \\
& =x^{i} \xi_{1}^{*} \phi\left(\xi_{0} \otimes_{A} x^{j} \xi_{0}\right)=x^{i} \xi_{1}^{*}\left(\xi_{1} x^{j-1}+x \xi_{1} x^{j-2}+\cdots+x^{j-1} \xi_{1}\right)=j x^{i+j-1} .
\end{aligned}
$$

The circle product in the reverse order is

$$
\begin{aligned}
& \left(x^{j} \xi_{2}^{*} \circ_{\phi} x^{p-1} \xi_{1}^{*}\right)\left(\xi_{2}\right) \\
& =x^{j} \xi_{2}^{*} \phi\left(1 \otimes_{A} x^{p-1} \xi_{1}^{*} \otimes_{A} 1\right) \Delta^{(2)}\left(\xi_{2}\right) \\
& =x^{j} \xi_{2}^{*} \phi\left(1 \otimes_{A} x^{p-1} \xi_{1}^{*} \otimes_{A} 1\right)\left(\xi_{0} \otimes_{A} \xi_{0} \otimes_{A} \xi_{2}+\xi_{0} \otimes_{A} \xi_{2} \otimes_{A} \xi_{0}+\xi_{2} \otimes_{A} \xi_{0} \otimes_{A} \xi_{0}\right. \\
& \left.+\xi_{0} \otimes_{A} \sum_{\substack{a+b+c}}\left(x^{a} \xi_{1} \otimes_{A} x^{b} \xi_{1} x^{c}\right)+\sum_{\substack{a+b+c \\
=p-2}} x^{a} \xi_{1} \otimes_{A} x^{b}\left(\xi_{0} \otimes_{A} \xi_{1}+\xi_{1} \otimes_{A} x_{0}\right) x^{c}\right) \\
& =x^{j} \xi_{2}^{*} \phi\left(\sum_{\substack{a+b+c}}^{=p-2}\left(\xi_{0} \otimes_{A} x^{a+b+i} \xi_{1} x^{c}-x^{a} \xi_{1} \otimes_{A} x^{b+i} \xi_{0} x^{c}\right)\right) \\
& =x^{j} \xi_{2}^{*}\left(\sum_{\substack{a+b+c}}^{=p-2}\left(\delta_{a+b+i, p-1} \xi_{2} x^{c}+x^{a} \delta_{b+i, p-1} \xi_{2} x^{c}\right)\right) \\
& =x^{j} \xi_{2}^{*}\left((p-i) \xi_{2} x^{i-1}+\sum_{\substack{a+c \\
=i-1}} x^{a} \xi_{2} x^{c}\right) \\
& =(p-i) x^{i+j-1}+\sum_{\substack{a+c \\
=i-1}} x^{a+c+j}=(p-i) x^{i+j-1}+i x^{i+j-1}=p x^{i+j-1}
\end{aligned}
$$

Therefore, we obtain

$$
\left[x^{i} \xi_{1}^{*}, x^{j} \xi_{2}^{*}\right]=(j-p) x^{i+j-1} \xi_{2}^{*} .
$$

Lastly, the bracket of the elements of degrees 2 and 2:

$$
\begin{aligned}
\left(x^{i} \xi_{2}^{*} \circ_{\phi} x^{j} \xi_{2}^{*}\right)\left(\xi_{3}\right) & =x^{i} \xi_{2}^{*} \phi\left(1 \otimes_{A} x^{j} \xi_{2}^{*} \otimes_{A} 1\right) \Delta^{(2)}\left(\xi_{3}\right) \\
& =x^{i} \xi_{2}^{*} \phi\left(\xi_{1} \otimes_{A} x^{j} \xi_{0}+\xi_{0} \otimes_{A} x^{j} \xi_{1}\right)=x^{i} \xi_{2}^{*}(0)=0
\end{aligned}
$$

and by symmetry $\left(x^{j} \xi_{2}^{*} \circ_{\phi} x^{i} \xi_{2}^{*}\right)\left(\xi_{3}\right)=0$. Therefore, we have

$$
\left[\left(x^{i} \xi_{2}^{*}, x^{j} \xi_{2}^{*}\right)\right]=0
$$

As a consequence, the brackets for the elements of degrees 1 and 2 are

$$
\begin{aligned}
& {\left[\left(x^{i} \xi_{1}^{*}, x^{j} \xi_{1}^{*}\right)\right]=(j-i) x^{i+j-1} \xi_{1}^{*},} \\
& {\left[\left(x^{i} \xi_{1}^{*}, x^{j} \xi_{2}^{*}\right)\right]=(j-p) x^{i+j-1} \xi_{2}^{*},} \\
& {\left[\left(x^{i} \xi_{2}^{*}, x^{j} \xi_{2}^{*}\right)\right]=0 .}
\end{aligned}
$$

Brackets in higher degrees can be determined from these and the identity (2.3.1) since the Hochschild cohomology is generated in degrees 1 and 2 as an $A$-algebra under the cup product [25, Example 2.2.2].
L. Grimley, V. C. Nguyen, and S. Witherspoon [7] calculated Gerstenhaber brackets on Hochschild cohomology of a twisted tensor product of algebras. S. Sanchez-Flores [18] also calculated the bracket on group algebras of a cyclic group over a field of positive characteristic which is isomorphic to $A=k[x] /\left(x^{p}\right)$. C. Negron and S. Witherspoon [14] calculated the bracket on group algebras of a cyclic group over a field of positive characteristic as well with analogous $h, \phi$, and $\Delta$ maps. Our calculation agrees with those except slightly different $\left[\left(x^{i} \xi_{1}^{*}, x^{j} \xi_{2}^{*}\right)\right]$.

## 4. BRACKET ON THE HOPF ALGEBRA COHOMOLOGY OF A TAFT ALGEBRA

We consider a Taft algebra $T_{p}$ with $p>2$ in this chapter. Recall that a Taft algebra $T_{p}$ is a nonquasi-triangular Hopf algebra and it can be seen as the skew group algebra $A \rtimes G$ for $A=$ $k[x] /\left(x^{p}\right)$ and $G=\mathbb{Z} / p \mathbb{Z}$.

The Lie structure on Hopf algebra cohomology is known to be abelian when the Hopf algebra is quasi-triangular [5, 20]. In this chapter, our main goal is to calculate the bracket on Hochschild cohomology of $T_{p}$ with the same technique in Chapter 3 and find the corresponding bracket on Hopf algebra cohomology of $T_{p}$ by using an embedding of $\mathrm{H}^{*}\left(T_{p}, k\right)$ into $\mathrm{HH}^{*}\left(T_{p}, T_{p}\right)$.

### 4.1 The bracket on Hochschild cohomology of a Taft algebra

Let $\mathcal{D}$ be the skew group algebra $A^{e} \rtimes G$ for $A=k[x] /\left(x^{p}\right)$ and $G=\mathbb{Z} / p \mathbb{Z}$ where the action of $G$ on $A^{e}$ is diagonal, i.e. ${ }^{g}(a \otimes b)=\left({ }^{g} a\right) \otimes\left({ }^{g} b\right)$. Then, there is the following algebra isomorphism [1, Section 2]:

$$
\mathcal{D}=A^{e} \rtimes G \xrightarrow{\gamma} \bigoplus_{g \in G} A g \otimes A g^{-1} \subset(A \rtimes G)^{e}
$$

given by $\gamma\left(\left(a_{1} \otimes a_{2}\right) g\right)=a_{1} g \otimes\left(\left(g^{-1} a_{2}\right) g^{-1}\right)$. We just show that $\gamma$ is compatible with multiplication which is not obvious:

$$
\begin{aligned}
\gamma\left(\left(a_{1} \otimes a_{2}\right) g_{1} \cdot\left(a_{3} \otimes a_{4}\right) g_{2}\right) & =\gamma\left(\left(a_{1} \otimes a_{2}\right) \cdot\left({ }^{g_{1}}\left(a_{3} \otimes a_{4}\right)\right) g_{1} g_{2}\right) \\
& \left.\left.=\gamma\left(\left(a_{1}\left({ }^{g_{1}} a_{3}\right) \otimes{ }^{\left(g_{1}\right.} a_{4}\right) a_{2}\right)\right) g_{1} g_{2}\right) \\
& =a_{1}\left({ }^{g_{1}} a_{3}\right) g_{1} g_{2} \otimes\left({ }^{\left(g_{1} g_{2}\right)^{-1}}\left(\left({ }^{g_{1}} a_{4}\right) a_{2}\right)\right)\left(g_{1} g_{2}\right)^{-1} \\
& =a_{1}\left({ }^{g_{1}} a_{3}\right) g_{1} g_{2} \otimes\left({ }^{g_{2}^{-1}} a_{4}\right)\left({ }^{\left(g_{1} g_{2}\right)^{-1}} a_{2}\right)\left(g_{1} g_{2}\right)^{-1}
\end{aligned}
$$

and

$$
\gamma\left(\left(a_{1} \otimes a_{2}\right) g_{1}\right) \cdot \gamma\left(\left(a_{3} \otimes a_{4}\right) g_{2}\right)=\left(a_{1} g_{1} \otimes\left(^{g_{1}^{-1}} a_{2}\right) g_{1}^{-1}\right) \cdot\left(a_{3} g_{2} \otimes\left(^{g_{2}^{-1}} a_{4}\right) g_{2}^{-1}\right)
$$

$$
\begin{aligned}
= & a_{1}\left({ }^{g_{1}} a_{3}\right) g_{1} g_{2} \otimes\left({ }^{g_{2}^{-1}} a_{4}\right)\left({ }^{g_{2}^{-1}}\left(g_{1}^{-1} a_{2}\right)\right) g_{2}^{-1} g_{1}^{-1} \\
& =a_{1}\left({ }^{g_{1}} a_{3}\right) g_{1} g_{2} \otimes\left({ }^{g_{2}^{-1}} a_{4}\right)\left({ }^{\left(g_{1} g_{2}\right)^{-1}} a_{2}\right)\left(g_{1} g_{2}\right)^{-1}
\end{aligned}
$$

Thus $\mathcal{D}$ is isomorphic to a subalgebra of $(A \rtimes G)^{e}$. Note that $A$ is a $\mathcal{D}$-module under the left and right action [1, Section 4]:

$$
\begin{gathered}
\left(a_{1} g \otimes a_{2} g^{-1}\right) a_{3}=a_{1}\left({ }^{g}\left(a_{3} a_{2}\right)\right) \\
a_{3}\left(a_{1} g \otimes a_{2} g^{-1}\right)=a_{2}\left({ }^{g^{-1}}\left(a_{3} a_{1}\right)\right) .
\end{gathered}
$$

Remember the resolution (3.2.1)

$$
\mathbb{A}: \cdots \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{v .} A^{e} \xrightarrow{u .} A^{e} \xrightarrow{\pi} A \longrightarrow 0 .
$$

This is also a $\mathcal{D}$-projective resolution of $A$ where the action of $G$ on $A^{e}$ is given by

- $g \cdot\left(a_{1} \otimes a_{2}\right)=\left({ }^{g} a_{1}\right) \otimes\left({ }^{g} a_{2}\right)$ in even degrees,
- $g \cdot\left(a_{1} \otimes a_{2}\right)=\omega\left({ }^{g} a_{1}\right) \otimes\left({ }^{g} a_{2}\right)$ in odd degrees.

From the resolution $\mathbb{A}$, we construct the following $T_{p}^{e}$-resolution of $T_{p}$ :

$$
\begin{equation*}
T_{p}^{e} \otimes_{\mathcal{D}} \mathbb{A}: \cdots \longrightarrow T_{p}^{e} \otimes_{\mathcal{D}} A^{e} \longrightarrow T_{p}^{e} \otimes_{\mathcal{D}} A^{e} \longrightarrow T_{p}^{e} \otimes_{\mathcal{D}} A^{e} \longrightarrow T_{p}^{e} \otimes_{\mathcal{D}} A \longrightarrow 0 \tag{4.1.1}
\end{equation*}
$$

It is known that, $T_{p} \cong T_{p}^{e} \otimes_{\mathcal{D}} A$ as $T_{p}$-bimodules via the map sending $x^{i} g^{k}$ to $\left(1 \otimes g^{k}\right) \otimes_{\mathcal{D}} x^{i}$ [25, (3.5.4)]. Then we have $A \otimes T_{p} \cong T_{p}^{e} \otimes_{\mathcal{D}} A^{e}$ with the $T_{p}$-bimodule isomorphism given by

$$
\begin{equation*}
\kappa\left(x^{i} \otimes\left(x^{j} g^{k}\right)\right)=\left(1 \otimes g^{k}\right) \otimes_{\mathcal{D}}\left(x^{i} \otimes x^{j}\right) \tag{4.1.2}
\end{equation*}
$$

Then, we obtain the following resolution $\tilde{\mathbb{A}}$ which is isomorphic to the resolution (4.1.1), i.e.

$$
\begin{equation*}
\tilde{\mathbb{A}}: \cdots \xrightarrow{\tilde{u} .} A \otimes T_{p} \xrightarrow{\tilde{v} .} A \otimes T_{p} \xrightarrow{\tilde{u} .} A \otimes T_{p} \xrightarrow{\tilde{\pi} .} T_{p} \longrightarrow 0 \tag{4.1.3}
\end{equation*}
$$

where $\tilde{v}=v \otimes i d_{k G}, \tilde{u}=u \otimes i d_{k G}$, and $\tilde{\pi}=\pi \otimes i d_{k G}$.
The following lemma gives us a contracting homotopy for the identity map on the resolution ヘ̃.

Lemma 4.1.4. Let $h_{n}$ be the contracting homotopy in (3.2.2). Then $\tilde{h}_{n}=h_{n} \otimes 1_{k G}$ forms a contracting homotopy for the identity map on $\tilde{\mathbb{A}}$.

Proof. For $n \geq 0$, the domain of $h_{n} \otimes 1_{k G}$ is $A \otimes A \otimes k G$ which is $A \otimes T_{p}$ as a vector space. Moreover, by definition of contracting homotopy, the maps $h_{n}$ satisfy

$$
h_{i-1} d_{i}+d_{i+1} h_{i}=i d_{\mathbb{A}_{i}} .
$$

Then,

$$
\begin{aligned}
\tilde{h}_{i-1} \tilde{d}_{i}+\tilde{d}_{i+1} \tilde{h}_{i} & =\left(h_{i-1} \otimes i d_{k G}\right)\left(d_{i} \otimes i d_{k G}\right)+\left(d_{i+1} \otimes i d_{k G}\right)\left(h_{i} \otimes i d_{k G}\right) \\
& =\left(h_{i-1} d_{i} \otimes i d_{k G}\right)+\left(d_{i+1} h_{i} \otimes i d_{k G}\right)=\left(h_{i-1} d_{i}+d_{i+1} h_{i}\right) \otimes i d_{k G} \\
& =i d_{\mathbb{A}_{i}} \otimes i d_{k G}=i d_{\tilde{\mathbb{A}}_{i}}
\end{aligned}
$$

and that implies $\tilde{h}_{n}$ is a contracting homotopy for $\tilde{\mathbb{A}}$. The proof is similar for $n=-1$.

By Lemma 4.1.4, we obtain

$$
\begin{aligned}
\tilde{h}_{-1}\left(x^{i} g\right) & =\xi_{0} x^{i} g, \\
\tilde{h}_{0}\left(x^{i} \xi_{0} x^{j} g\right) & =\sum_{l=0}^{i-1} x^{l} \xi_{1} x^{i+j-1-l} g, \\
\tilde{h}_{1}\left(x^{i} \xi_{1} x^{j} g\right) & =\delta_{i, p-1} x^{j} \xi_{2} g, \\
\tilde{h}_{2 n}\left(x^{i} \xi_{2 n} x^{j} g\right) & =-\sum_{l=0}^{j-1} x^{i+j-1-l} \xi_{2 n+1} x^{l} g, \\
\tilde{h}_{2 n+1}\left(x^{i} \xi_{2 n+1} x^{j} g\right) & =\delta_{j, p-1} x^{i} \xi_{2 n+2} g .
\end{aligned}
$$

We need a lemma to have the linear maps $\tilde{\phi}_{i}:\left(\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}\right)_{i} \longrightarrow \tilde{\mathbb{A}}_{i+1}$. However, we first mention that there is an isomorphism

$$
\psi:\left(A \otimes T_{p}\right) \otimes_{T_{p}}\left(A \otimes T_{p}\right) \rightarrow(A \otimes A) \otimes_{A}(A \otimes A) \otimes k G
$$

as $T_{p}^{e}$-modules given by

$$
\begin{equation*}
\psi\left(\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right) \otimes_{T_{p}}\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right)\right)=\omega^{k_{1}\left(i_{2}+j_{2}\right)}\left(x^{i_{1}} \otimes x^{j_{1}}\right) \otimes_{A}\left(x^{i_{2}} \otimes x^{j_{2}}\right) g^{\left(k_{1}+k_{2}\right)} . \tag{4.1.5}
\end{equation*}
$$

Lemma 4.1.6. Let $F_{\mathbb{A}}=\left(\pi \otimes_{A} i d_{\mathbb{A}}-i d_{\mathbb{A}} \otimes_{A} \pi\right)$ be the chain map for the resolution $\mathbb{A}$ in (3.2.1) which is used for calculation of $\phi$ in (3.2.3). Then $F_{\tilde{\mathbb{A}}}: \tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ defined by $\left(\tilde{\pi} \otimes_{T_{p}} i d_{\tilde{\mathbb{A}}}-\right.$ $\left.i d_{\tilde{\mathbb{A}}} \otimes_{T_{p}} \tilde{\pi}\right)$ is exactly $\left(F_{\mathbb{A}} \otimes i d_{k G}\right) \psi$. Moreover $\tilde{\phi}:=\left(\phi \otimes i d_{k G}\right) \psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$.

Proof. Let $\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right) \otimes_{T_{p}}\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right) \in\left(A \otimes T_{p}\right) \otimes_{T_{p}}\left(A \otimes T_{p}\right)$. Note that $F_{\tilde{\mathbb{A}}}$ is zero if the degrees of $\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right)$ and $\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right)$ are both nonzero since $\tilde{\pi}$ is only nonzero on degree zero. Also remember that $\tilde{\pi}=\pi \otimes i d_{k G}$ for the resolution $\tilde{\mathbb{A}}$.

We check the case that the degree of $\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right)$ is zero and the degree of $\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right)$ is nonzero. By using the definition of $F_{\tilde{\mathbb{A}}}$, we obtain

$$
\begin{aligned}
F_{\tilde{\mathbb{A}}}\left(\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right) \otimes_{T_{p}}\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right)\right) & =\left(x^{i_{1}+j_{1}} g^{k_{1}}\right) \otimes_{T_{p}}\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right) \\
& =\omega^{k_{1}\left(i_{2}+j_{2}\right)} x^{i_{1}+i_{2}+j_{1}} \otimes x^{i_{2}} g^{k_{1}+k_{2}}
\end{aligned}
$$

On the other hand, we also have

$$
\begin{aligned}
& \left(F_{\mathbb{A}} \otimes i d_{k G}\right) \psi\left(\left(x^{i_{1}} \otimes x^{j_{1}} g^{k_{1}}\right) \otimes_{T_{p}}\left(x^{i_{2}} \otimes x^{j_{2}} g^{k_{2}}\right)\right) \\
& =\left(F_{\mathbb{A}} \otimes i d_{k G}\right)\left(\omega^{k_{1}\left(i_{2}+j_{2}\right)}\left(x^{i_{1}} \otimes x^{j_{1}}\right) \otimes_{A}\left(x^{i_{2}} \otimes x^{j_{2}}\right) g^{k_{1}+k_{2}}\right) \\
& =\omega^{k_{1}\left(i_{2}+j_{2}\right)} x^{i_{1}+i_{2}+j_{1}} \otimes x^{i_{2}} g^{k_{1}+k_{2}} .
\end{aligned}
$$

The proofs for other cases are similar. Hence $F_{\tilde{\mathbb{A}}}$ and $\left(F_{\mathbb{A}} \otimes i d_{k G}\right) \psi$ are identical.
In order to prove $\tilde{\phi}:=\left(\phi \otimes i d_{k G}\right) \psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$, we need to show that

$$
\tilde{d}_{\tilde{\mathbb{A}}} \tilde{\phi}+\tilde{\phi} \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=F_{\tilde{\mathbb{A}}} .
$$

It is clear that

$$
\begin{equation*}
\tilde{d}_{\mathbb{A}} \tilde{\phi}=\left(d_{\mathbb{A}} \otimes i d_{k G}\right)\left(\phi \otimes i d_{k G}\right) \psi=\left(d_{\mathbb{A}} \phi \otimes i d_{k G}\right) \psi \tag{4.1.7}
\end{equation*}
$$

We now claim that

$$
\begin{equation*}
\psi \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=\left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right) \psi \tag{4.1.8}
\end{equation*}
$$

By definition

$$
\tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=\tilde{d}_{\tilde{\mathbb{A}}} \otimes_{T_{p}} i d_{T_{p}}+(-1)^{*} i d_{T_{p}} \otimes_{T_{p}} \tilde{d}_{\tilde{\mathbb{A}}}
$$

where $*$ is the degree of the element in the left factor $A \otimes T_{p}$. Moreover, $\left(A \otimes T_{p}\right) \otimes_{T_{p}}\left(A \otimes T_{p}\right)$ is generated by $\xi_{m} 1_{G} \otimes_{T_{p}} x^{i} \xi_{n} 1_{G}$ as $T_{p}$-bimodule. First, assume that $m$ and $n$ are odd. Then we have the following calculation:

$$
\begin{aligned}
& \psi \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}\left(\xi_{m} 1_{G} \otimes_{T_{p}} x^{i} \xi_{n} 1_{G}\right) \\
& =\psi\left(\left(x \xi_{m} 1_{G}-\xi_{m} x 1_{G}\right) \otimes_{T_{p}} x^{i} \xi_{n} 1_{G}-\xi_{m} 1_{G} \otimes_{T_{p}}\left(x^{i+1} \xi_{n} 1_{G}-x^{i} \xi_{n} x 1_{G}\right)\right) \\
& =\left(x \xi_{m}-\xi_{m} x\right) \otimes_{A} x^{i} \xi_{n} 1_{G}-\xi_{m} \otimes_{A}\left(x^{i+1} \xi_{n}-x^{i} \xi_{n} x\right) 1_{G}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right) \psi\left(\xi_{m} 1_{G} \otimes_{T_{p}} x^{i} \xi_{n} 1_{G}\right) \\
& =\left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right)\left(\xi_{m} \otimes_{A} x^{i} \xi_{n} 1_{G}\right) \\
& =\left(x \xi_{m}-\xi_{m} x\right) \otimes_{A} x^{i} \xi_{n} 1_{G}-\xi_{m} \otimes_{A}\left(x^{i+1} \xi_{n}-x^{i} \xi_{n} x\right) 1_{G}
\end{aligned}
$$

The calculation is similar for the other cases of $m$ and $n$. Therefore,

$$
\begin{equation*}
\tilde{\phi} \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=\left(\phi \otimes i d_{k G}\right) \psi \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=\left(\phi \otimes i d_{k G}\right)\left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right) \psi=\left(\phi d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right) \psi . \tag{4.1.9}
\end{equation*}
$$

By combining (4.1.7) and (4.1.9), we obtain

$$
\tilde{d}_{\tilde{\mathbb{A}}} \tilde{\phi}+\tilde{\phi} \tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}}=\left(\left(d_{\mathbb{A}} \phi+\phi d_{\mathbb{A} \otimes \otimes_{A} \mathbb{A}}\right) \otimes i d_{k G}\right) \psi=\left(F_{\mathbb{A}} \otimes i d_{k G}\right) \psi=F_{\tilde{\mathbb{A}}}
$$

whence $\tilde{\phi}=\left(\phi \otimes i d_{k G}\right) \psi$ is a contracting homotopy for $F_{\tilde{\mathbb{A}}}$.
We use Lemma 4.1.6 and find the following $T_{p}^{e}$-linear maps $\tilde{\phi}_{i}:\left(\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}\right)_{i} \longrightarrow \tilde{\mathbb{A}}_{i+1}$ :

$$
\begin{aligned}
& \tilde{\phi}_{0}\left(\xi_{0} 1_{G} \otimes_{T_{p}} x^{i} \xi_{0} 1_{G}\right)=\sum_{l=0}^{i-1} x^{l} \xi_{1} x^{i-1-l} 1_{G} \\
& \tilde{\phi}_{1}\left(\xi_{1} 1_{G} \otimes_{T_{p}} x^{i} \xi_{0} 1_{G}\right)=-\delta_{i, p-1} \xi_{2} 1_{G} \\
& \tilde{\phi}_{1}\left(\xi_{0} 1_{G} \otimes_{T_{p}} x^{i} \xi_{1} 1_{G}\right)=\delta_{i, p-1} \xi_{2} 1_{G} .
\end{aligned}
$$

Next, we give a lemma to find the diagonal map.

Lemma 4.1.10. The map $\tilde{\Delta}:=\psi^{-1}\left(\Delta \otimes i d_{k G}\right)$ is a diagonal map on $\tilde{\mathbb{A}}$ where $\Delta$ is in (3.2.4).

Proof. We need to check that $\tilde{\Delta}$ is a chain map. The following equations are straightforward by considering the fact that $\Delta$ is a chain map and (4.1.8):

$$
\begin{aligned}
\tilde{d}_{\tilde{\mathbb{A}} \otimes_{p}} \tilde{\mathbb{A}} & \tilde{\Delta}
\end{aligned}=\tilde{d}_{\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}} \psi^{-1}\left(\Delta \otimes i d_{k G}\right)=\psi^{-1}\left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \otimes i d_{k G}\right)\left(\Delta \otimes i d_{k G}\right) ~\left(\psi^{-1}\left(d_{\mathbb{A} \otimes_{A} \mathbb{A}} \Delta \otimes i d_{k G}\right)=\psi^{-1}\left(\Delta d_{\mathbb{A}} \otimes i d_{k G}\right)=\psi^{-1}\left(\Delta \otimes i d_{k G}\right)\left(d_{\mathbb{A}} \otimes i d_{k G}\right)\right.
$$

Lemma 4.1.10 allows us to compute the $T_{p}$-linear map $\tilde{\Delta}: \tilde{\mathbb{A}}_{i+1} \longrightarrow\left(\tilde{\mathbb{A}} \otimes_{T_{p}} \tilde{\mathbb{A}}\right)_{i}$ as follows:

$$
\begin{aligned}
\tilde{\Delta}_{0}\left(\xi_{0} 1_{G}\right) & =\xi_{0} 1_{G} \otimes_{T_{p}} \xi_{0} 1_{G} \\
\tilde{\Delta}_{1}\left(\xi_{1} 1_{G}\right) & =\xi_{1} 1_{G} \otimes_{T_{p}} \xi_{0} 1_{G}+\xi_{0} 1_{G} \otimes_{T_{p}} \xi_{1} 1_{G}, \\
\tilde{\Delta}_{2 n}\left(\xi_{2 n} 1_{G}\right) & =\sum_{i=0}^{n} \xi_{2 i} 1_{G} \otimes_{T_{p}} \xi_{2 n-2 i} 1_{G} \\
& +\sum_{i=0}^{n-1} \sum_{\substack{a+b+c \\
=p-2}} x^{a} \xi_{2 i+1} 1_{G} \otimes_{T_{p}} x^{b} \xi_{2 n-2 i-1} x^{c} 1_{G}, \text { for } n \geq 1 \\
\tilde{\Delta}_{2 n+1}\left(\xi_{2 n+1} 1_{G}\right) & =\sum_{i=0}^{2 n+1} \xi_{i} 1_{G} \otimes_{T_{p}} \xi_{2 n+1-i} 1_{G}, \text { for } n \geq 1
\end{aligned}
$$

Before computing the bracket on Hochschild cohomology of $T_{p}$, we need to find a basis of $\operatorname{Hom}_{T_{p}^{e}}\left(\tilde{\mathbb{A}}, T_{p}\right)$. In particular, we must find a basis of $\operatorname{Hom}_{T_{p}^{e}}\left(A \otimes T_{p}, T_{p}\right)$ for each degree.

It is known that

$$
\operatorname{HH}^{*}\left(T_{p}\right):=\operatorname{Ext}_{T_{p}^{e}}^{*}\left(T_{p}, T_{p}\right) \cong \operatorname{Ext}_{\mathcal{D}}^{*}\left(A, T_{p}\right) \cong \operatorname{Ext}_{A^{e}}^{*}\left(A, T_{p}\right)^{G}
$$

The Eckmann-Shapiro Lemma (Lemma 5.3.3) and (4.1.2) imply the first isomorphism and see [25, Theorem 3.6.2] for the second isomorphism.

Consider the following resolution

$$
\begin{equation*}
\operatorname{Hom}_{A^{e}}\left(\mathbb{A}, T_{p}\right)^{G}: 0 \longrightarrow \operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)^{G} \longrightarrow \operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)^{G} \longrightarrow \cdots \tag{4.1.11}
\end{equation*}
$$

where the action of $G$ on $\operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)$ is defined by

$$
\begin{equation*}
g \cdot f\left(a_{1} \otimes a_{2}\right)={ }^{g} f\left(g^{-1}\left(a_{1} \otimes a_{2}\right)\right) . \tag{4.1.12}
\end{equation*}
$$

This resolution is clearly isomorphic to

$$
\begin{equation*}
0 \longrightarrow T_{p}^{G} \longrightarrow T_{p}^{G} \longrightarrow T_{p}^{G} \longrightarrow \cdots \tag{4.1.13}
\end{equation*}
$$

with the correspondence

$$
\begin{equation*}
f_{t} \mapsto t \text { where } f_{t}\left(\xi_{*}\right)=t \text { for all } t \in T_{p} \tag{4.1.14}
\end{equation*}
$$

where $\xi_{*}=1 \otimes 1 \in A^{e}$ in degree $*$. The action of $G$ on $T_{p}$ given by (4.1.12) and (4.1.14) depends on degree.
$T_{p}^{G}$ is spanned by $\left\{1, g, \cdots, g^{p-1}\right\}$ in even degrees and $\left\{x, x g, \cdots, x g^{p-1}\right\}$ in odd degrees [15, Section 8.2]. We claim that $\operatorname{Hom}_{T_{p}^{e}}\left(A \otimes T_{p}, T_{p}\right) \cong T_{p}^{G}$. Suppose $x^{i} g^{j} \in T_{p}^{G}$. Then, we have $f_{x^{i} g^{j}} \in \operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)^{G}$ defined by $f_{x^{i} g^{j}}\left(x^{k} \otimes x^{l}\right):=x^{k+l+i} g^{j}$ where $x^{*} \in A$. Now observe that, $f_{x^{i} g^{j}} \in \operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)^{G}$ is a $\mathcal{D}$-module homomorphism since

$$
\begin{aligned}
f_{x^{i} g^{j}}\left(\left(x^{k} \xi_{*} x^{l} g\right)\left(a_{1} \otimes a_{2}\right)\right) & =f_{x^{i} g^{j}}\left(\left(x^{k} \xi_{*} x^{l} 1_{G}\right) g\left(a_{1} \otimes a_{2}\right)\right)=\left(x^{k} \xi_{*} x^{l} 1_{G}\right) f_{x^{i} g^{j}}\left(g\left(a_{1} \otimes a_{2}\right)\right) \\
& =\left(x^{k} \xi_{*} x^{l} 1_{G}\right) g f_{x^{i} g^{j}}\left(a_{1} \otimes a_{2}\right)=\left(x^{k} \xi_{*} x^{l} g\right) f_{x^{i} g^{j}}\left(a_{1} \otimes a_{2}\right)
\end{aligned}
$$

where $x^{k} \xi_{*} x^{l} g \in \mathcal{D}, a_{1} \otimes a_{2} \in A^{e}$. Moreover, if $f \in \operatorname{Hom}_{\mathcal{D}}\left(A^{e}, T_{p}\right)$, then $f$ is $G$-invariant as

$$
g \cdot f\left(a_{1} \otimes a_{2}\right)={ }^{g} f\left({ }^{g^{-1}}\left(a_{1} \otimes a_{2}\right)\right)={ }^{\left(g g^{-1}\right)} f\left(a_{1} \otimes a_{2}\right)=f\left(a_{1} \otimes a_{2}\right)
$$

where $g \in G, a_{1} \otimes a_{2} \in A^{e}$. Hence, the isomorphism from $\operatorname{Hom}_{A^{e}}\left(A^{e}, T_{p}\right)^{G}$ to $\operatorname{Hom}_{\mathcal{D}}\left(A^{e}, T_{p}\right)$ is the identity, so that $f_{x^{i} g^{j}}$ is also in $\operatorname{Hom}_{\mathcal{D}}\left(A^{e}, T_{p}\right)$. We next use the Eckmann-Shapiro Lemma (Lemma 5.3.3) which implies that $\operatorname{Ext}_{\mathcal{D}}^{*}\left(A, T_{p}\right) \cong \operatorname{Ext}_{T_{p}^{e}}^{*}\left(T_{p}^{e} \otimes_{\mathcal{D}} A, T_{p}\right)$ and the isomorphism is given by

$$
\begin{aligned}
\sigma\left(f_{x^{i} g^{j}}\right)\left(x^{m} g^{s} \otimes x^{n} g^{r} \otimes_{\mathcal{D}} x^{k} \otimes x^{l}\right) & =x^{m} g^{s} \otimes x^{n} g^{r} f_{x^{i} g^{j}}\left(x^{k} \otimes x^{l}\right)=x^{m} g^{s} \otimes x^{n} g^{r}\left(x^{k+l+i} g^{j}\right) \\
& =\left(x^{m} g^{s}\right)\left(x^{k+l+i} g^{j}\right)\left(x^{n} g^{r}\right) \\
& =\left(\left(x^{m}\left(g^{s} x^{k+l+i}\right)\right) g^{s+j}\right)\left(x^{n} g^{r}\right) \\
& =\omega^{s(k+l+i)}\left(x^{m+k+l+i} g^{s+j}\right)\left(x^{n} g^{r}\right) \\
& =\omega^{s(k+l+i)}\left(x^{m+k+l+i}\left(g^{s+j} x^{n}\right)\right) g^{j+s+r} \\
& =\omega^{s(k+l+i+n)+j n} x^{i+k+l+m+n} g^{j+s+r}
\end{aligned}
$$

Hence, $\sigma\left(f_{x^{i} g^{j}}\right)$ is in $\operatorname{Hom}_{T_{p}^{e}}\left(T_{p}^{e} \otimes_{\mathcal{D}} A^{e}, T_{p}\right)$. Lastly, recall that $T_{p}^{e} \otimes_{\mathcal{D}} A^{e} \cong A \otimes T_{p}$ via $\kappa$ (4.1.2); so that,

$$
\kappa^{*}\left(\sigma\left(f_{x^{i} g^{j}}\right)\right)\left(x^{k} \otimes x^{l} g^{r}\right)=\sigma\left(f_{x^{i} g^{j}}\right)\left(\left(1_{T_{p}} \otimes \xi_{*} g^{r}\right) \otimes_{\mathcal{D}} x^{k} \otimes x^{l}\right)=x^{i+k+l} g^{j+r}
$$

which implies $\kappa^{*}\left(\sigma\left(f_{x^{i} g^{j}}\right)\right) \in \operatorname{Hom}_{T_{p}^{e}}\left(A \otimes T_{p}, T_{p}\right)$. For simplicity, we define $\tilde{f}_{x^{i} g^{j}}:=\kappa^{*}\left(\sigma\left(f_{x^{i} g^{j}}\right)\right)$.
Recall that $T_{p}^{G}$ is spanned by $\left\{1, g, \cdots, g^{p-1}\right\}$ in even degrees and $\left\{x, x g, \cdots, x g^{p-1}\right\}$ in odd degrees. Hence we have $\left\{\tilde{f}_{1}, \tilde{f}_{g}, \cdots, \tilde{f}_{g^{p-1}}\right\}$ in even degrees and $\left\{\tilde{f}_{x}, \tilde{f}_{x g}, \cdots, \tilde{f}_{x g^{p-1}}\right\}$ in odd degrees as a basis of $\operatorname{Hom}_{T_{p}^{e}}\left(A \otimes T_{p}, T_{p}\right)$.

We only calculate the bracket in degree 1 and 2 as before so we can extend it to higher degrees by the relation between cup product and the bracket. Since $A \otimes T_{p} \cong A^{e} \otimes k G$ as vector spaces, $\xi_{i} 1_{G}$ generates $A \otimes T_{p}$ as a $T_{p}$-bimodule. Through the calculation, $i d$ represents $i d_{A \otimes T_{p}}$ and $\otimes$ represents $\otimes_{T_{p}}$.

The circle product of two elements in degree one is

$$
\begin{aligned}
\left(\tilde{f}_{x g^{i}} \circ_{\tilde{\phi}} \tilde{f}_{x g^{j}}\right)\left(\xi_{1} 1_{G}\right) & =\tilde{f}_{x g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{x g^{j}} \otimes i d\right) \tilde{\Delta}^{(2)}\left(\xi_{1} 1_{G}\right) \\
& =\tilde{f}_{x g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{x g^{j}} \otimes i d\right)\left(\xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{1} 1_{G}+\xi_{0} 1_{G} \otimes \xi_{1} 1_{G} \otimes \xi_{0} 1_{G}\right. \\
& \left.+\xi_{1} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{0} 1_{G}\right) \\
& =\tilde{f}_{x g^{i}} \tilde{\phi}\left(\xi_{0} 1_{G} \otimes x \xi_{0} g^{j}\right)=\tilde{f}_{x g^{i}}\left(\xi_{1} g^{j}\right)=x g^{i+j}
\end{aligned}
$$

Because of the symmetry, $\left(\tilde{f}_{x g^{j}} \circ_{\tilde{\phi}} \tilde{f}_{x g^{i}}\right)\left(\xi_{1} 1_{G}\right)=x g^{i+j}$. Therefore

$$
\left[\tilde{f}_{x g^{i}}, \tilde{f}_{x g^{j}}\right]\left(\xi_{1} 1_{G}\right)=x g^{i+j}-(-1)^{0} x g^{i+j}=0 .
$$

The circle product of the elements of degrees 1 and 2 :

$$
\begin{aligned}
\left(\tilde{f}_{x g^{i}} \circ_{\tilde{\phi}} \tilde{f}_{g^{j}}\right)\left(\xi_{2} 1_{G}\right) & =\tilde{f}_{x g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{g^{j}} \otimes i d\right) \tilde{\Delta}^{(2)}\left(\xi_{2} 1_{G}\right)=\tilde{f}_{x g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{g^{j}} \otimes i d\right) \\
& \left(\xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{2} 1_{G}+\xi_{0} 1_{G} \otimes \xi_{2} 1_{G} \otimes \xi_{0} 1_{G}\right. \\
& +\xi_{0} 1_{G} \otimes \sum_{\substack{a+b+c \\
=p-2}}\left(x^{a} \xi_{1} 1_{G} \otimes x^{b} \xi_{1} x^{c} 1_{G}\right)+\xi_{2} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \\
& \left.+\sum_{\substack{a+b+c \\
=p-2}}\left(x^{a} \xi_{1} 1_{G} \otimes\left(x^{b} \xi_{0} 1_{G} \otimes \xi_{1} x^{c} 1_{G}+x^{b} \xi_{1} 1_{G} \otimes \xi_{0} x^{c} 1_{G}\right)\right)\right) \\
& =\tilde{f}_{x g^{i}} \tilde{\phi}\left(\xi_{0} 1_{G} \otimes \xi_{0} g^{j}\right)=0 .
\end{aligned}
$$

And the circle product in the reverse order:

$$
\begin{aligned}
\left(\tilde{f}_{g^{j}} \circ_{\tilde{\phi}} \tilde{f}_{x g^{i}}\right)\left(\xi_{2} 1_{G}\right) & =\tilde{f}_{g^{j}} \tilde{\phi}\left(i d \otimes \tilde{f}_{x g^{i}} \otimes i d\right) \tilde{\Delta}^{(2)}\left(\xi_{2} 1_{G}\right)=\tilde{f}_{g^{j}} \tilde{\phi}\left(i d \otimes \tilde{f}_{x g^{i}} \otimes i d\right) \\
& \left(\xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{2} 1_{G}+\xi_{0} 1_{G} \otimes \xi_{2} 1_{G} \otimes \xi_{0} 1_{G}\right. \\
& +\xi_{0} 1_{G} \otimes \sum_{\substack{a+b+c \\
=p-2}}\left(x^{a} \xi_{1} 1_{G} \otimes x^{b} \xi_{1} x^{c} 1_{G}\right)+\xi_{2} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \\
& \left.+\sum_{\substack{a+b+c \\
=p-2}}\left(x^{a} \xi_{1} 1_{G} \otimes\left(x^{b} \xi_{0} 1_{G} \otimes \xi_{1} x^{c} 1_{G}+x^{b} \xi_{1} 1_{G} \otimes \xi_{0} x^{c} 1_{G}\right)\right)\right) \\
& =\tilde{f}_{g^{j}} \tilde{\phi}\left(\sum_{\substack{a+b+c}} \omega^{i(b+c)} \xi_{0} 1_{G} \otimes x^{a+b+1} \xi_{1} x^{c} g^{i}+\omega^{i c} x^{a} \xi_{1} 1_{G} \otimes x^{b+1} \xi_{0} x^{c} g^{i}\right)
\end{aligned} \quad \begin{aligned}
& =\tilde{f}_{g^{j}}\left(\sum_{\substack{a+b+c}}^{=p-2} \omega^{i(b+c)} \delta_{a+b+1, p-1} x^{c} \xi_{2} g^{i}-\omega^{i c} \delta_{b+1, p-1} x^{a+c} \xi_{2} g^{i}\right) \\
& =\tilde{f}_{g^{j}}\left(\sum_{\substack{p-2}} \omega^{i b} \xi_{2} g^{i}\right)-\tilde{f}_{g^{j}}\left(\xi_{2} g^{i}\right)
\end{aligned} \quad \begin{array}{ll}
(p-2) g^{j} & \text { if } i=0 \\
& = \begin{cases}-\left(\omega^{-i}+1\right) g^{i+j} & \text { if } i \neq 0\end{cases}
\end{array}
$$

Therefore, we obtain

$$
\left[\tilde{f}_{x g^{i}}, \tilde{f}_{g^{j}}\right]=\left\{\begin{array}{ll}
-(p-2) g^{j} & \text { if } i=0 \\
\left(\omega^{-i}+1\right) g^{i+j} & \text { if } i \neq 0
\end{array} .\right.
$$

Lastly, the bracket of the elements of degrees 2 and 2:

$$
\begin{aligned}
\left(\tilde{f}_{g^{i}} \circ_{\tilde{\phi}} \tilde{f}_{g^{j}}\right)\left(\xi_{3} 1_{G}\right) & =\tilde{f}_{g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{g^{j}} \otimes i d\right) \tilde{\Delta}^{(2)}\left(\xi_{3} 1_{G}\right)=\tilde{f}_{g^{i}} \tilde{\phi}\left(i d \otimes \tilde{f}_{g^{j}} \otimes i d\right) \\
& \left(\xi_{0} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{3} 1_{G}+\xi_{0} 1_{G} \otimes \xi_{1} 1_{G} \otimes \xi_{2} 1_{G}+\xi_{0} 1_{G} \otimes \xi_{2} 1_{G} \otimes \xi_{1} 1_{G}\right. \\
& +\xi_{0} 1_{G} \otimes \xi_{3} 1_{G} \otimes \xi_{0} 1_{G}+\xi_{1} 1_{G} \otimes \xi_{2} 1_{G} \otimes \xi_{0} 1_{G}+\xi_{1} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{2} 1_{G} \\
& \left.+\xi_{2} 1_{G} \otimes \xi_{1} 1_{G} \otimes \xi_{0} 1_{G}+\xi_{2} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{1} 1_{G}+\xi_{3} 1_{G} \otimes \xi_{0} 1_{G} \otimes \xi_{0} 1_{G}\right) \\
& =\tilde{f}_{g^{i}} \tilde{\phi}\left(\xi_{0} 1_{G} \otimes \xi_{1} g^{j}+\xi_{1} 1_{G} \otimes \xi_{0} g^{j}\right)=0
\end{aligned}
$$

and by symmetry $\left(\tilde{f}_{g^{j}} \circ_{\tilde{\phi}} \tilde{f}_{g^{i}}\right)\left(\xi_{3} 1_{G}\right)=0$. Therefore, we have $\left[\tilde{f}_{g^{i}}, \tilde{f}_{g^{j}}\right]=0$. As a consequence, the bracket for the elements of degree 1 and 2 are

$$
\begin{aligned}
{\left[\tilde{f}_{x g^{i}}, \tilde{f}_{x g^{j}}\right] } & =0 \\
{\left[\tilde{f}_{x g^{i}}, \tilde{f}_{g^{j}}\right] } & =\left\{\begin{array}{ll}
-(p-2) g^{j} & \text { if } i=0 \\
\left(\omega^{-i}+1\right) g^{i+j} & \text { if } i \neq 0
\end{array},\right. \\
{\left[\tilde{f}_{g^{i}}, \tilde{f}_{g^{j}}\right] } & =0
\end{aligned}
$$

By the identity (2.3.1), brackets in higher degrees can be determined, since the Hochschild cohomology is generated as an algebra under cup product in degrees 1 and 2.

### 4.2 The bracket on Hopf algebra cohomology of a Taft algebra

The Hopf algebra cohomology of $T_{p}$ is calculated in Section 5.5 and Hochschild cohomology of $T_{p}$ were calculated before by V. C. Nguyen [15, Section 8], i.e. the Hopf algebra cohomology

$$
\mathrm{H}^{n}\left(T_{p}, k\right)= \begin{cases}k & \text { if } n \text { is even } \\ 0 & \text { if } n \text { is odd }\end{cases}
$$

and the Hochschild cohomology

$$
\mathrm{HH}^{n}\left(T_{p}, T_{p}\right)= \begin{cases}k & \text { if } n \text { is even } \\ \operatorname{Span}_{k}\{x\} & \text { if } n \text { is odd }\end{cases}
$$

It is known that for any Hopf algebra with bijective antipode, the Hopf algebra cohomology can be embedded into the Hochschild cohomology. We give a detailed proof in Section 5.3. Since any finite dimensional Hopf algebra has a bijective antipode, the Taft algebra $T_{p}$ is also a Hopf algebra with a bijective antipode. The embedding of $\mathrm{H}^{n}\left(T_{p}, k\right)$ into $\mathrm{HH}^{n}\left(T_{p}, T_{p}\right)$ turns out to be the map that is identity in even degrees and zero on odd degrees. Then, the corresponding bracket in Hopf algebra cohomology is

$$
\left[\tilde{f}_{g^{i}}, \tilde{f}_{g^{j}}\right]=0
$$

so that, the bracket on Hopf algebra cohomology for the elements of all degrees is 0 by the identity (2.3.1). Therefore, we give the following theorem as a summary of this chapter:

Theorem 4.2.1. The Gerstenhaber bracket on the Hopf algebra cohomology of a Taft algebra $T_{p}$ with $p>2$ is trivial.

This is the first example of the Gerstenhaber bracket on the Hopf algebra cohomology of a nonquasi-triangular Hopf algebra and our calculation shows that the bracket on Hopf algebra cohomology of a Taft algebra is zero as it is on the Hopf algebra cohomology of any quasi-triangular algebra. A natural question that arises is whether the bracket structure on the Hopf algebra cohomology is always trivial.

## 5. THE LIE STRUCTURE OF THE HOPF ALGEBRA COHOMOLOGY OF A TAFT ALGEBRA BY HOMOTOPY LIFTING

In this chapter, we give an alternative method, homotopy lifting, to find the bracket structure on Hopf algebra cohomology. Homotopy liftings are defined and constructed for some exact monoidal categories in [22, Section 4] as follows. This turns out to be equivalent to the Gerstenhaber bracket when the category is that of $A$-bimodules.

### 5.1 Homotopy liftings and exact monoidal categories

We refer to [4, Chapter 2] and [12, Appendix A] for definitions, examples, and properties in this section. We start with definitions of an additive category and an exact category.

Definition 5.1.1. An additive category is a category $\mathcal{C}$ satisfying the following axioms:

- Every set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a structure of an abelian group (written additively) such that composition of morphisms is biadditive with respect to this structure.
- There exists a zero object $0 \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(0,0)=0$.
- (Existence of direct sums.) For any objects $X_{1}, X_{2} \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ and morphisms $p_{1}: Y \rightarrow X_{1}, p_{2}: Y \rightarrow X_{2}, i_{1}: X_{1} \rightarrow Y, i_{2}: X_{2} \rightarrow Y$ such that $p_{1} i_{1}=i d_{X_{1}}, p_{2} i_{2}=i d_{X_{2}}$, and $i_{1} p_{1}+i_{2} p_{2}=i d_{Y}$.

Definition 5.1.2. Let $\mathcal{C}$ be an additive category and $\mathcal{E}$ a class of distinguished sequences $X \rightarrow Y \rightarrow Z$ of $\mathcal{C}$. We call $\mathcal{E}$ a class of conflations if for every sequence $X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z$ in $\mathcal{E}$, the morphism $\beta$ is a kernel of $\gamma$ and the morphism $\gamma$ is a cokernel of $\beta$. A morphism $\beta: X \rightarrow Y$ in $\mathcal{E}$ in $\mathcal{C}$ is an inflation if there exists a conflation of the form $X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z$ in $\mathcal{E}$. A morphism $\gamma: Y \rightarrow Z$ in $\mathcal{E}$ in $\mathcal{C}$ is a deflation if there exists a conflation of the form $X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z$ in $\mathcal{E}$. The pair $(\mathcal{C}, \mathcal{E})$ is called an exact category if the following axioms hold:

- $0 \rightarrow 0 \rightarrow 0$ is a conflation;
- the composition of any two deflations is also a deflation;
- if $\gamma: Y \rightarrow Z$ is a deflation and $f: Y^{\prime} \rightarrow Z$ is any morphism, then there exists a pullback

with deflation $\gamma^{\prime}$;
- if $\beta: X \rightarrow Y$ is an inflation and $g: X \rightarrow Y^{\prime}$ is any morphism, then there exists a pushout

with inflation $\beta^{\prime}$.

Definition 5.1.3. A monoidal category $\mathcal{C}$ is a category equipped with

- a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the tensor product,
- a natural isomorphism $\alpha:(-\otimes-) \otimes-\widetilde{\rightarrow}-\otimes(-\otimes-)$, i.e. $\alpha_{X, Y, Z}:(X \otimes Y) \otimes Z \underset{\rightarrow}{\rightarrow} X \otimes(Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$,
- a unit object $\mathbf{1}$ with an isomorphism $\iota: \mathbf{1} \otimes \mathbf{1} \underset{\sim}{\sim} \mathbf{1}$
subject to the following two axioms:

1. The pentagon axiom: The following diagram is commutative for all $W, X, Y, Z \in \mathcal{C}$ :

2. The unit axiom: The functors $L_{1}$ and $R_{1}$ of left and right multiplication by 1 are equivalences $\mathcal{C} \rightarrow \mathcal{C}$.

Example 5.1.4. The category $k$-Vec of all $k$-vector spaces is a monoidal category, where the tensor product is $\otimes_{k}, \mathbf{1}=k$, and the morphisms $\alpha, \iota$ are the obvious ones. The same is true about the category of finite dimensional vector spaces over $k$.

Example 5.1.5. The category of $A$-bimodules (equivalently left $A^{e}$-modules) for an associative algebra $A$ over $k$ is an exact monoidal category, with the tensor product $\otimes_{A}$ and $1=A$. The morphisms $\alpha, \iota$ are the obvious ones.

Example 5.1.6. The category of left modules for a Hopf algebra $A$ over $k$ is an exact monoidal category, with tensor product $\otimes_{k}$ of modules, and $1=k$. Note that for $A$-modules $M, N$, the action of $A$ over $M \otimes N$ is given by

$$
a v=\sum a_{1} v_{1} \otimes_{k} a_{2} v_{2} \text { for all } a \in A, v \in M \otimes N
$$

The morphisms $\alpha, \iota$ are the obvious ones.

Let $\mathcal{C}$ be an exact monoidal category and let 1 be its unit object. As is customary, we will identify $1 \otimes X$ and $X \otimes 1$ with $X$ for all objects $X$ in $\mathcal{C}$, under assumed fixed isomorphisms (for which we will not need notation).

We continue with the definition of power flat resolution [22, Definition 4.3].

Definition 5.1.7. Let $P \rightarrow \mathbf{1}$ be a projective resolution of $\mathbf{1}$ with differential $d$ and let $\mu_{P}: P_{0} \rightarrow \mathbf{1}$ be the corresponding augmentation map. Then, the resolution $\left(P, d, \mu_{P}\right)$ of $\mathbf{1}$ is called $n$-power flat if $\left(P^{\otimes r}, d^{\otimes r}, \mu_{P}^{\otimes r}\right)$ is a projective resolution of 1 for each $r(1 \leq r \leq n)$. If $P$ is $n$-power flat for each $n \geq 2$, then we say that $P$ is power flat.

For the two categories in Example 5.1.5 and Example 5.1.6, projective resolutions of 1 are generally power flat.

Assume 1 has a projective power flat resolution $P$. For a degree $l$ morphism $\psi: P_{i} \rightarrow P_{i-l}$ for all $i$, its differential in the $\operatorname{Hom}$ complex $\operatorname{Hom}_{\mathcal{C}}(P, P)$ is defined to be

$$
\partial(\psi)=d \psi-(-1)^{l} \psi d
$$

Definition 5.1.8. Let $f: P \rightarrow \mathbf{1}$ be an $m$-cocycle. Let $\Delta_{P}: P \rightarrow P \otimes P$ be a diagonal map, i.e. a chain map lifting the isomorphism $1 \xrightarrow{\sim} \mathbf{1} \otimes 1$. A degree $(m-1)$ morphism $\psi_{f}: P \rightarrow P$ is a homotopy lifting of $\left(f, \Delta_{P}\right)$ if

$$
\begin{equation*}
\partial\left(\psi_{f}\right)=\left(f \otimes 1_{P}-1_{P} \otimes f\right) \Delta_{P} \tag{5.1.9}
\end{equation*}
$$

and $\mu_{P} \psi_{f} \sim(-1)^{m+1} f \psi$ for some degree -1 map $\psi: P \rightarrow P$ such that

$$
\begin{equation*}
\partial(\psi)=\left(\mu_{P} \otimes 1_{P}-1_{P} \otimes \mu_{P}\right) \Delta_{P} \tag{5.1.10}
\end{equation*}
$$

The cohomology of the monoidal category $\mathcal{C}$ is $\mathrm{H}^{*}(\mathcal{C})=\mathrm{H}^{*}(\mathcal{C}, \mathbf{1}):=\operatorname{Ext}_{\mathcal{C}}^{*}(\mathbf{1}, \mathbf{1})$. In here, $\operatorname{Ext}_{\mathcal{C}}^{*}(\mathbf{1}, \mathbf{1})$ is indeed the cohomology of the chain complex $\operatorname{Hom}_{\mathcal{C}}(P, \mathbf{1})$ and it has a Lie bracket defined as follows [22, Section 4]:

For an $m$-cocycle $f: P_{m} \rightarrow \mathbf{1}$ and an $n$-cocycle $g: P_{n} \rightarrow \mathbf{1}$, let $\psi_{f}$ and $\psi_{g}$ be homotopy liftings of $\left(f, \Delta_{P}\right)$ and $\left(g, \Delta_{P}\right)$ respectively. Then the cochain $[f, g]$ defined as

$$
\begin{equation*}
[f, g]=f \psi_{g}-(-1)^{(m-1)(n-1)} g \psi_{f} \tag{5.1.11}
\end{equation*}
$$

induces a graded Lie bracket on $\mathrm{H}^{*}(\mathcal{C})$. That is, it induces a well-defined operation on cohomology that is graded alternating and satisfies a graded Jacobi identity (cf. [25, Lemma 1.4.3]).

### 5.2 Change of exact monoidal categories

Let $\mathcal{C}, \mathcal{C}^{\prime}$ be exact monoidal categories for which there exist power flat resolutions of their unit objects $\mathbf{1}, \mathbf{1}^{\prime}$. Let $F: \mathcal{C} \rightarrow \mathcal{C}^{\prime}$ be an exact monoidal functor [4, Definition 2.4.1], that is, $F$ is exact
and there is a natural isomorphism $\eta$ of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}^{\prime}$ given by

$$
\eta_{X, Y}: F(X) \otimes F(Y) \rightarrow F(X \otimes Y)
$$

for all $X, Y$ in $\mathcal{C}, F(\mathbf{1}) \cong \mathbf{1}^{\prime}$ and $(F, \eta)$ satisfies the monoidal structure axiom of [4, Definition 2.4.1], that is, the following diagram commutes for all objects $X, Y, Z$ in $\mathcal{C}$ :

(The horizontal isomorphisms are given by the associativity constraint for $\mathcal{C}^{\prime}$ and the image of the associativity constraint for $\mathcal{C}$ under $F$, respectively. We will not need notation for these isomorphisms.)

Denote the isomorphism from $F(\mathbf{1})$ to $\mathbf{1}^{\prime}$ by $\phi$. Then the following diagrams commute for all objects $X$ by [4, Proposition 2.4.3]; we have chosen to show diagrams involving the inverse maps $\eta_{1, X}^{-1}$ and $\eta_{X, 1}^{-1}$ since we will need these later. The unlabeled isomorphisms in the diagrams are those canonically determined by the fixed isomorphisms given by tensoring with unit objects and the fixed isomorphism $F(\mathbf{1}) \cong \mathbf{1}^{\prime}$.


Example 5.2.2. Let $A$ be a Hopf algebra and $B$ a Hopf subalgebra of $A$. Let $\mathcal{C}$ be the category of left $A$-modules and $\mathcal{C}^{\prime}$ the category of left $B$-modules. Let $F: \mathcal{C} \rightarrow \mathcal{C}^{\prime}$ be the restriction functor, that is on each $A$-module $X$, the action is restricted to $B$ via the inclusion map $B \hookrightarrow A$.

The restriction of a tensor product of modules to $B$ is isomorphic to the tensor product of their restrictions to $B$, and thus there is a natural transformation $\eta$ as required.

In the next section we will apply the following theorem to shed light on the connection between the Lie structures on Hopf algebra cohomology and on Hochschild cohomology.

Let $P$ be a projective resolution of 1 in $\mathcal{C}$ and write $P^{\prime}=F(P)$, which is a projective resolution of $F(\mathbf{1}) \cong \mathbf{1}^{\prime}$ in $\mathcal{C}^{\prime}$ under our assumptions. Let $d$ denote the differential and $\mu_{P}: P \rightarrow \mathbf{1}$ denote the augmentation map of $P$. Write $d^{\prime}=F(d)$ and $\mu_{P^{\prime}}=F\left(\mu_{P}\right)$. Note that $P \otimes P$ is also a projective resolution of 1 in $\mathcal{C}$ with augmentation map $\mu_{P} \otimes \mu_{P}$ followed by the canonical isomorphism $1 \otimes 1 \xrightarrow{\sim} 1$. Let

$$
\Delta_{P^{\prime}}=\eta_{P, P}^{-1} F\left(\Delta_{P}\right)
$$

which is a diagonal map on $P^{\prime}$ under our assumptions.

Theorem 5.2.3. Let $\mathcal{C}, \mathcal{C}^{\prime}$ be exact monoidal categories and let $F: \mathcal{C} \rightarrow \mathcal{C}^{\prime}$ be an exact monoidal functor. Assume there exists a power flat resolution $P$ of $\mathbf{1}$ in $\mathcal{C}$. Let $f \in \operatorname{Hom}_{\mathcal{C}}\left(P_{m}, \mathbf{1}\right)$, an $m$ cocycle. Let $\psi_{f}$ be a homotopy lifting of $f$ with respect to $\Delta_{P}$. Then $F\left(\psi_{f}\right)$ is a homotopy lifting of $F(f)$ with respect to $\Delta_{P^{\prime}}$.

Proof. Since $\psi_{f}$ is a homotopy lifting of $f$,

$$
d \psi_{f}-(-1)^{m-1} \psi_{f} d=(f \otimes 1-1 \otimes f) \Delta_{P}
$$

Set $f^{\prime}=F(f), \psi_{f^{\prime}}=F\left(\psi_{f}\right)$, and apply $F$ to each side of this equation to obtain

$$
\begin{equation*}
d^{\prime} \psi_{f^{\prime}}-(-1)^{m-1} \psi_{f^{\prime}} d^{\prime}=F(f \otimes 1-1 \otimes f) F\left(\Delta_{P}\right) \tag{5.2.4}
\end{equation*}
$$

Since $\eta$ is a natural transformation, under our assumptions (see the above commuting diagrams), the following diagram commutes:


Therefore $F(f \otimes 1) F\left(\Delta_{P}\right)$ can be identified with $(F(f) \otimes 1) \eta_{P, P}^{-1} F\left(\Delta_{P}\right)$, and similarly $F(1 \otimes$ f) $F\left(\Delta_{P}\right)$ with $(1 \otimes F(f)) \eta_{P, P}^{-1} F\left(\Delta_{P}\right)$. So the right side of expression (5.2.4) is equal to

$$
\left(f^{\prime} \otimes 1-1 \otimes f^{\prime}\right) \eta_{P, P}^{-1} F\left(\Delta_{P}\right)
$$

which is in turn equal to $\left(f^{\prime} \otimes 1-1 \otimes f^{\prime}\right) \Delta_{P^{\prime}}$, as desired.
Since $\psi_{f}$ is a homotopy lifting, $\mu_{P} \psi_{f} \sim(-1)^{m+1} f \psi$ for some degree -1 map $\psi: P \rightarrow P$ such that

$$
\partial(\psi)=d \psi+\psi d=\left(\mu_{P} \otimes 1_{P}-1_{P} \otimes \mu_{P}\right) \Delta_{P}
$$

By applying $F$ to both sides of the above equation, we obtain

$$
F(d \psi+\psi d)=F\left(\mu_{P} \otimes 1_{P}-1_{P} \otimes \mu_{P}\right) F\left(\Delta_{P}\right)
$$

and under our identifications, letting $\psi^{\prime}=F(\psi)$, this is

$$
d^{\prime} \psi^{\prime}+\psi^{\prime} d^{\prime}=F\left(\mu_{P} \otimes 1\right) F\left(\Delta_{P}\right)-F\left(1 \otimes \mu_{P}\right) F\left(\Delta_{P}\right) .
$$

Via a commutative diagram such as that above, we see this is equal to

$$
\left(\mu_{P^{\prime}} \otimes 1\right) \Delta_{P^{\prime}}-\left(1 \otimes \mu_{P^{\prime}}\right) \Delta_{P^{\prime}}
$$

Therefore, $\psi^{\prime}: P^{\prime} \rightarrow P^{\prime}$ is a degree -1 map such that

$$
\partial\left(\psi^{\prime}\right)=d^{\prime} \psi^{\prime}+\psi^{\prime} d^{\prime}=\left(\mu_{P^{\prime}} \otimes 1\right) \Delta_{P^{\prime}}-\left(1 \otimes \mu_{P^{\prime}}\right) \Delta_{P^{\prime}}
$$

and $\mu_{P^{\prime}} F\left(\psi_{f}\right) \sim(-1)^{m+1} F(f) \psi^{\prime}$, that is, $\mu_{P^{\prime}} \psi_{f^{\prime}} \sim(-1)^{m+1} f^{\prime} \psi^{\prime}$, as desired.
We have shown that $F\left(\psi_{f}\right)$ is a homotopy lifting of $F(f)$ with respect to $\Delta_{P^{\prime}}$.

Corollary 5.2.5. The functor $F$ induces a graded Lie algebra homomorphism from $\mathrm{H}^{*}(\mathcal{C})$ to $\mathrm{H}^{*}\left(\mathcal{C}^{\prime}\right)$, in positive degrees.

Proof. As a consequence of the theorem and formula (5.1.11), the functor $F$ takes the Lie bracket of two elements of positive degree in the cohomology $\mathrm{H}^{*}(\mathcal{C})$ to the Lie bracket of their images in $\mathrm{H}^{*}\left(\mathcal{C}^{\prime}\right)$ under $F$.

### 5.3 Embedding from Hopf algebra cohomology into Hochschild cohomology

Recall that Hopf algebra cohomology can be embedded into Hochschild cohomology. We give some lemmas which helps us to construct this explicit embedding. For proofs, see [25, Section 9.4].

Lemma 5.3.1. Let $A$ be a Hopf algebra. There is an isomorphism of $A^{e}$-modules,

$$
A \cong A^{e} \otimes_{A} k,
$$

where $A^{e} \otimes_{A} k$ is the tensor induced $A^{e}$-module under the identification of $A$ with the subalgebra of $A^{e}$ that is the image of the embedding $\delta: A \rightarrow A^{e}$ defined for all $a \in A$ by

$$
\delta(a)=\sum a_{1} \otimes_{k} S\left(a_{2}\right)
$$

Lemma 5.3.2. The right $A$-module $A^{e}$, where $A$ acts by right multiplication under its identification with $\delta(A)$, is projective.

Lemma 5.3.3 (Eckmann-Shapiro). Let $A$ be a ring and let $B$ be a subring of $A$ such that $A$ is projective as a right $B$-module. Let $M$ be an $A$-module and $N$ be a $B$-module. Then

$$
\operatorname{Ext}_{B}^{n}(N, M) \cong \operatorname{Ext}_{A}^{n}\left(A \otimes_{B} N, M\right)
$$

Although the proof of Eckmann-Shapiro lemma is not necessary in this chapter, we provide it since it is used in the next chapter.

Proof. Let $P_{\bullet} \rightarrow N$ be a $B$ projective resolution of $N$. Then $A \otimes_{B} P_{n}$ is projective as A-module so that $A \otimes_{B} P_{\bullet} \rightarrow A \otimes_{B} N$ is a projective resolution of $A \otimes_{B} N$ as an $A$-module. Let

$$
\begin{gathered}
\sigma: \operatorname{Hom}_{B}\left(P_{n}, M\right) \rightarrow \operatorname{Hom}_{A}\left(A \otimes_{B} P_{n}, M\right) \text { defined by } \sigma(f)\left(a \otimes_{B} p\right)=a f(p), \\
\tau: \operatorname{Hom}_{A}\left(A \otimes_{B} P_{n}, M\right) \rightarrow \operatorname{Hom}_{B}\left(P_{n}, M\right) \text { defined by } \tau(g)(p)=g\left(1 \otimes_{B} p\right)
\end{gathered}
$$

where $a \in A, p \in P_{n}, f \in \operatorname{Hom}_{B}\left(P_{n}, M\right), g \in \operatorname{Hom}_{A}\left(A \otimes_{B} P_{n}, M\right)$. Since $\sigma$ and $\tau$ are inverse of each other and they are homomorphisms, $\operatorname{Hom}_{A}\left(A \otimes_{B} P_{n}, M\right) \cong \operatorname{Hom}_{B}\left(P_{n}, M\right)$.

We will consider $A$ to be a left $A$-module by the left adjoint action, which is for $a, b \in A$,

$$
a \cdot b=\sum a_{1} b S\left(a_{2}\right)
$$

Denote this $A$-module by $A^{\text {ad }}$.
For any left $A$-module $M$, let $\mathrm{H}^{*}(A, M):=\operatorname{Ext}_{A}^{*}(k, M)$. The following theorem is well known; see, e.g. [25, Theorem 9.4.5]. We sketch a proof since we will need some of the details later.

Theorem 5.3.4. There is an isomorphism of graded $k$-vector spaces

$$
\mathrm{HH}^{*}(A) \cong \mathrm{H}^{*}\left(A, A^{a d}\right)
$$

Proof. By Lemma 5.3.2, $A^{e}$ is projective as a right $A$-module, so we can apply the EckmannShapiro Lemma. We replace $A$ with $A^{e}, B$ with $A$ and take $M=A, N=k$ in the EckmannShapiro Lemma and obtain the isomorphism $\operatorname{Ext}_{A^{e}}^{n}\left(A^{e} \otimes_{A} k, A\right) \cong \operatorname{Ext}_{A}^{n}\left(k, A^{a d}\right)$ as $k$-vector spaces. Lastly, we apply Lemma 5.3.1 and obtain $\operatorname{Ext}_{A^{e}}^{n}(A, A) \cong \operatorname{Ext}_{A}^{n}\left(k, A^{a d}\right)$.

A consequence of the theorem is an embedding of Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ into Hochschild cohomology $\mathrm{HH}^{*}(A)$ : Let $P \rightarrow k$ be a projective resolution of the $A$-module $k$. We can embed $\mathrm{H}^{*}(A, k)$ into $\mathrm{H}^{*}\left(A, A^{a d}\right) \cong \operatorname{HH}^{*}(A)$ via the map $\eta_{*}: \operatorname{Hom}_{A}(P, k) \rightarrow \operatorname{Hom}_{A}\left(P, A^{a d}\right)$ induced by the unit map $\eta: k \rightarrow A$ (see [25, Corollary 9.4.7]). Equivalently, the functor $A^{e} \otimes_{A}-$ induces an embedding of $\mathrm{H}^{*}(A, k)$ into $\mathrm{HH}^{*}(A)$.

### 5.4 A new technique for the bracket on Hopf algebra cohomology

Let $A$ be a Hopf algebra with bijective antipode. Let $\mathcal{C}$ be the category of (left) $A$-modules, and let $\mathcal{C}^{\prime}$ be the category of (left) $A^{e}$-modules. For each $A$-module $U$, let

$$
F(U)=A^{e} \otimes_{A} U
$$

the tensor induced module, where we identify $A$ with the subalgebra $\delta(A)$ of $A^{e}$ as in Lemma 5.3.1. Also by Lemma 5.3.1, $F$ takes the unit object $k$ of $\mathcal{C}$ to an isomorphic copy of the unit object $A$ of $\mathcal{C}^{\prime}$. It takes projective $A$-modules to projective $A^{e}$-modules since $A^{e}$ is projective as an $A$-module by Lemma 5.3.2. For each $A$-module homomorphism $f: U \rightarrow V$, define $F(f)$ by

$$
F(f)\left((1 \otimes 1) \otimes_{A} u\right)=(1 \otimes 1) \otimes_{A} f(u)
$$

for all $u \in U$. Then $F$ may be viewed as the functor providing the embedding of Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ into Hochschild cohomology $\mathrm{HH}^{*}(A)$; see the proof of Theorem 5.3.4 and the subsequent paragraph.

For each pair of $A$-modules $U, V$, we wish to define an $A^{e}$-module homomorphism

$$
\eta_{U, V}: F(U) \otimes_{A} F(V) \rightarrow F(U \otimes V)
$$

that is,

$$
\eta_{U, V}:\left(A^{e} \otimes_{A} U\right) \otimes_{A}\left(A^{e} \otimes_{A} V\right) \rightarrow A^{e} \otimes_{A}(U \otimes V)
$$

For all $a, b \in A, u \in U$, and $v \in V$, set

$$
\eta_{U, V}\left((a \otimes 1) \otimes_{A} u\right) \otimes_{A}\left((1 \otimes b) \otimes_{A} v\right)=(a \otimes b) \otimes_{A}(u \otimes v) .
$$

Note that all elements of $\left(A^{e} \otimes_{A} U\right) \otimes_{A}\left(A^{e} \otimes_{A} V\right)$ can indeed be written as linear combinations of elements of the indicated forms and that the map is well-defined. For example, for all $a, b \in A$ and $u \in U$, letting $b=S\left(b^{\prime}\right)$,

$$
\begin{aligned}
(a \otimes b) \otimes_{A} u & =\sum\left(a S\left(b_{1}^{\prime}\right) b_{2}^{\prime} \otimes S\left(b_{3}^{\prime}\right)\right) \otimes_{A} u \\
& =\sum\left(a S\left(b_{1}^{\prime}\right) \otimes 1\right) \otimes_{A}\left(\left(b_{2}^{\prime} \otimes S\left(b_{3}^{\prime}\right)\right) \cdot u\right) .
\end{aligned}
$$

By its definition, $\eta_{U, V}$ is an $A^{e}$-module homomorphism.
We check that $\eta$ is a natural transformation. That is, the following diagram commutes for all objects $U, V, U^{\prime}, V^{\prime}$ and morphisms $f: U \rightarrow U^{\prime}, g: V \rightarrow V^{\prime}:$


Commutativity follows from the definitions of $\eta_{U, V}, \eta_{U^{\prime}, V^{\prime}}$. To see that $\eta$ is monoidal, that is diagram (5.2.1) commutes, it is easier to check the corresponding diagram associated to $\eta^{-1}$, a straightforward calculation.

For the following theorem, we define the Gerstenhaber bracket of two elements in Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ via the embedding into Hochschild cohomology followed by the Gerstenhaber bracket on Hochschild cohomology. The theorem states that this is the same as their bracket defined by (5.1.11) on Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ via homotopy liftings. Thus the theorem allows us to bypass the need to work with Hochschild cohomology at all, for questions purely about Hopf algebra cohomology.

Theorem 5.4.1. Let A be a Hopf algebra with bijective antipode. Let $P$ be a projective resolution of $k$ and let $f, g$ be cocycles in $\operatorname{Hom}_{A}\left(P_{m}, k\right), \operatorname{Hom}_{A}\left(P_{n}, k\right)$, respectively, representing elements of Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$. Let $\Delta_{P}$ be a diagonal map, and let $\psi_{f}, \psi_{g}$ be homotopy liftings of $f, g$ with respect to $\Delta_{P}$. The Gerstenhaber bracket of the corresponding elements in Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ is represented by

$$
[f, g]=f \psi_{g}-(-1)^{(m-1)(n-1)} g \psi_{f} .
$$

Proof. This is an immediate consequence of Theorem 5.2.3 and expression (5.1.11), since we showed above that the induction functor $F$ is an exact monoidal functor.

One consequence Theorem 5.4.1 is a quick new proof that for a cocommutative Hopf algebra in characteristic not 2, Gerstenhaber brackets on Hopf algebra cohomology in positive degree are always 0 . We state this as Corollary 5.4.2 next. Since cocommutative Hopf algebras are quasitriangular, this is a small special case of the well known results of Farinati, Solotar, Taillefer, and Hermann, but it highlights our completely different approach.

Corollary 5.4.2. Let $k$ be a field of characteristic not 2 , and let $A$ be a cocommutative Hopf algebra. The Lie structure on Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$, given by the Gerstenhaber bracket, is abelian in positive degrees.

Proof. Let $P$ be a projective resolution of $k$ as an $A$-module. Let $\Delta^{\prime}: P \rightarrow P \otimes P$ be a diagonal map. Let $\sigma: P \otimes P \rightarrow P \otimes P$ be the signed transposition map, i.e. $\sigma(x \otimes y)=(-1)^{|x| y \mid} y \otimes x$ for all homogeneous $x, y \in P$. Since $A$ is cocommutative, $\sigma \Delta^{\prime}$ is also an $A$-module homomorphism, and therefore a diagonal map. Let

$$
\Delta=\frac{1}{2}\left(\Delta^{\prime}+\sigma \Delta^{\prime}\right)
$$

a diagonal map as well. Note that $\Delta$ is symmetric in the sense that $\sigma \Delta=\Delta$.
Now, by symmetry, $\left(\mu_{P} \otimes 1_{P}-1_{P} \otimes \mu_{P}\right) \Delta \equiv 0$, and so in (5.1.10), we can take $\psi \equiv 0$. Similarly, in (5.1.9), we can take $\psi_{f} \equiv 0$ for any cocycle $f$. Thus by Theorem 5.4.1, Gerstenhaber
brackets on the Hopf algebra cohomology $\mathrm{H}^{*}(A, k)$ are always 0 in positive degrees.

### 5.5 Taft algebras and quantum elementary abelian groups

Recall that we showed that the graded Lie structure on the Hopf algebra cohomology of a Taft algebra is abelian for positive and zero degrees in Chapter 4. In this section, we illustrate our results in Section 5.4 by showing that the Lie structure on the Hopf algebra cohomology of a Taft algebra, and more generally of a quantum elementary abelian group, is abelian in positive degrees.

Let $k$ be a field of characteristic 0 containing a primitive $p$ th root $\omega$ of 1 . We use $\otimes$ for $\otimes_{k}$ through this section. Recall that the Taft algebra $T_{p}$ is generated by $x$ and $g$ with relations

$$
g x=\omega x g, \quad x^{p}=0, \quad g^{p}=1 .
$$

and the Hopf structure is given by

$$
\begin{array}{rlrl}
\Delta(x) & =x \otimes 1+g \otimes x & \Delta(g) & =g \otimes g \\
\varepsilon(x) & =0 & \varepsilon(g) & =1 \\
S(x) & =-g^{-1} x & S(g) & =g^{-1} .
\end{array}
$$

Let $A=k[x] /\left(x^{p}\right)$, a subalgebra of $T_{p}$, and let $P$ be the following projective resolution of the trivial $A$-module $k$ :

$$
P: \ldots \xrightarrow{x^{p-1} .} A \xrightarrow{x .} A \xrightarrow{x^{p-1}} A \xrightarrow{x} A \xrightarrow{\varepsilon} k \longrightarrow 0
$$

This resolution is pretty close to the resolution (2.1.7); but, recall that the resolution (2.1.7) is $A^{e}$-module resolution.

The action of $A$ on each term of $P$ is by multiplication. Give each component $A$ in the resolution the structure of a $T_{p}$-module by letting $g \cdot x^{i}=\omega^{i} x^{i}$ in even degrees and $g \cdot x^{i}=\omega^{i+1} x^{i}$ in odd degrees. For clarity of notation, in each degree $l$, denote the element $1_{A}$ of $P_{l}=A$ by $\epsilon_{l}$. Note
that $P_{l}$ is projective as a $T_{p}$-module since the characteristic of $k$ is not divisible by $p$ : Specifically, in even degrees, there are $A$-module homomorphisms $P_{l} \rightarrow A\left(x^{i} \mapsto \frac{1}{p} \sum_{j=0}^{p-1} x^{i} g^{j}\right)$ and $A \rightarrow P_{l}$ $\left(x^{i} g^{j} \mapsto x^{i}\right)$ whose composition is the identity map. In odd degrees, a similar statement is true of the maps $P_{l} \rightarrow A$ and $A \rightarrow P_{l}$ given respectively by

$$
x^{i} \mapsto\left\{\begin{array} { l l } 
{ \frac { 1 } { n } \sum _ { j = 0 } ^ { p - 1 } x ^ { i + 1 } g ^ { j } , } & { \text { if } i < p - 1 , } \\
{ \frac { 1 } { p } \sum _ { j = 0 } ^ { p - 1 } g ^ { j } , } & { \text { if } i = p - 1 }
\end{array} \quad \text { and } \quad x ^ { i } g ^ { j } \mapsto \left\{\begin{array}{ll}
x^{i-1}, & \text { if } i \neq 0 \\
x^{p-1}, & \text { if } i=0
\end{array}\right.\right.
$$

Calculations show that the following formulas yield a diagonal map $\Delta: P \rightarrow P \otimes P$, that is for each $l, \Delta_{l}$ is a $T_{p}$-module homomorphism, and $\Delta$ is a chain map lifting the canonical isomorphism $k \xrightarrow{\sim} k \otimes k:$

$$
\begin{aligned}
\Delta_{2 j+1}\left(\epsilon_{2 j+1}\right) & =\sum_{i=0}^{2 j+1} \epsilon_{i} \otimes \epsilon_{2 j+1-i}, \\
\Delta_{2 j}\left(\epsilon_{2 j}\right) & =\sum_{i=0}^{j} \epsilon_{2 i} \otimes \epsilon_{2 j-2 i}+\sum_{i=0}^{j-1} \sum_{a=0}^{p-2}\binom{p-1}{a+1}_{\omega} x^{a} \epsilon_{2 i+1} \otimes x^{p-2-a} \epsilon_{2 j-2 i-1},
\end{aligned}
$$

where $\binom{p-1}{a+1}_{\omega}$ is the $\omega$-binomial coefficient defined for all nonnegative integers $a, b, c$ by

$$
\binom{b}{c}_{\omega}=\frac{(b)_{\omega}(b-1)_{\omega} \cdots(b-c+1)_{\omega}}{(c)_{\omega}(c-1)_{\omega} \cdots(1)_{\omega}} \quad \text { where } \quad(a)_{\omega}=1+\omega+\omega^{2}+\cdots+\omega^{a-1} .
$$

Note that $\Delta \neq \sigma \Delta$ since $\binom{n-1}{a+1}_{\omega} \neq\binom{ n-1}{a}_{\omega}$ in general. However, symmetry does hold after projection onto even degrees, a key property for the proof of the theorem below since the cohomology is concentrated in even degrees as we see next.

The cohomology of $T_{p}$ can be computed directly from the resolution $P$ above, and is

$$
\mathrm{H}^{*}\left(T_{p}, k\right) \cong \mathrm{H}^{*}(A, k)^{G} \cong k[z],
$$

where $\operatorname{deg}(z)=2$. Alternatively, see [19, Corollary 3.4] for the relevant general theory for skew group algebras.

The following theorem was proven in Chapter 4 by different techniques, and more generally there, the elements of degree 0 were included. The homotopy lifting method that we use here was designed for positive degree cohomology.

Theorem 5.5.1. The Lie structure given by the Gerstenhaber bracket on the cohomology $\mathrm{H}^{*}\left(T_{p}, k\right)$ of a Taft algebra $T_{p}$ is abelian in positive degrees.

Proof. Let $P$ be the resolution of $k$ given above. Let $f \in \operatorname{Hom}_{T_{p}}\left(P_{2}, k\right)$ denote the cocycle with $f\left(\epsilon_{2}\right)=1$, a representative of the generator $z$ of the cohomology ring $\mathrm{H}^{*}\left(T_{p}, k\right)$, described above. By [25, Lemma 1.4.7], it will suffice to show that the bracket of $f$ with itself is 0 since $f$ represents an algebra generator of cohomology.

We wish to find a homotopy lifting of $f$. First note that in (5.1.10), we can take $\psi \equiv 0$, the zero map, by symmetry of the image of the diagonal map under the projection onto $\left(P_{0} \otimes P_{i}\right) \oplus\left(P_{i} \otimes P_{0}\right)$ for each $i$. Similarly, by symmetry of the image of the diagonal map under the projection onto $\left(P_{\text {even }} \otimes P\right) \oplus\left(P \otimes P_{\text {even }}\right)$, since $f$ has even degree, in (5.1.9), we can take $\psi_{f} \equiv 0$ and indeed we can take the homotopy lifting of any representative element of cohomology in positive degree to be 0 . Specifically, the map $\psi_{f}$ must satisfy

$$
d \psi_{f}+\psi_{f} d=(f \otimes 1-1 \otimes f) \Delta_{P}
$$

The right side of this equation, evaluated on $\epsilon_{l}$, is

$$
(f \otimes 1-1 \otimes f)\left(\Delta_{P}\left(\epsilon_{l}\right)\right),
$$

and comparing to the formulas for $\Delta_{P}\left(\epsilon_{2 j}\right)$ and $\Delta_{P}\left(\epsilon_{2 j+1}\right)$ above, the only terms that will be nonzero after applying $f \otimes 1-1 \otimes f$ are those having $\epsilon_{2}$ as one of the tensor factors. By symmetry, the resulting terms after applying $f \otimes 1$ and $1 \otimes f$ cancel due to their opposite signs. So we may take $\psi_{f} \equiv 0$ as claimed. Now, by Theorem 5.4.1, $[f, f]=2 f \psi_{f}=0$.

The following theorem is a consequence since the Lie structure of a tensor product of algebras
reduces to that on each factor [13]. Quantum elementary abelian groups are defined to be iterated tensor products of Taft algebras [17].

Theorem 5.5.2. Let $Q$ be a quantum elementary abelian group. The Lie structure of the Hopf algebra cohomology $\mathrm{H}^{*}(Q, k)$, given by Gerstenhaber bracket, is abelian in positive degrees.

## 6. G-ALGEBRA STRUCTURE ON HOPF ALGEBRA COHOMOLOGIES

In this chapter, we explore a general expression for the bracket on the Hopf algebra cohomology that may help us to find the graded Lie structure on Hopf algebra cohomology with a more theoretical perspective in future research. At the end of the chapter, we reach an expression for Gerstenhaber bracket on a Hopf algebra $A$ with a bijective antipode $S$. Note that the hypothesis that the antipode is bijective is not very restrictive as all finite dimensional Hopf algebras and most infinite dimensional Hopf algebras have bijective antipodes.

We give the following lemma which helps us to define the Gerstenhaber bracket on an equivalent resolution to the bar resolution of $A$ as an $A$-bimodule. Once again, in this chapter we use $\otimes$ for $\otimes_{k}$.

Lemma 6.0.1. Let $A$ be a Hopf algebra with bijective antipode. Let $P_{\bullet}$. be the bar resolution of $k$ as a left A-module:

$$
P_{\bullet}: \cdots \xrightarrow{d_{3}} A^{\otimes 3} \xrightarrow{d_{2}} A^{\otimes 2} \xrightarrow{d_{1}} A \xrightarrow{\varepsilon} k \longrightarrow 0,
$$

with differentials
$d_{n}\left(a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n}\right)=\sum_{i=0}^{n-1}(-1)^{i} a_{0} \otimes a_{1} \otimes \cdots \otimes a_{i} a_{i+1} \otimes \cdots \otimes a_{n}+(-1)^{n} \varepsilon\left(a_{n}\right) a_{0} \otimes \cdots \otimes a_{n-1}$

Then $X_{\bullet}=A^{e} \otimes_{A} P_{\bullet}$ is equivalent to the bar resolution of $A$ as an $A$-bimodule.
Proof. Since $S$ is bijective, $A^{e}$ is projective as a right $A$-module [25, Lemma 9.2.9]. Also there is an $A^{e}$-module isomorphism $\rho: A \rightarrow A^{e} \otimes_{A} k$ defined by $\rho(a)=a \otimes 1 \otimes 1$ for all $a \in A$ [25, Lemma 9.4.2].

For each $n$, define $\theta_{n}: X_{n} \rightarrow A^{\otimes(n+2)}$ by

$$
\theta_{n}\left((a \otimes b) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n}\right)\right)=\sum a \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{n}\right) b
$$

for all $a, b, c^{1}, \cdots c^{n} \in A$.

Now, we show that $\theta$ is a chain map:

$$
\begin{aligned}
& \theta_{n-1} d_{n}\left((a \otimes b) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n}\right)\right) \\
= & \theta_{n-1}\left((a \otimes b) \otimes_{A}\left(c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n}\right)\right. \\
& +\sum_{i=1}^{n-1}(-1)^{i}(a \otimes b) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{i} c^{i+1} \otimes \cdots \otimes c^{n}\right) \\
& \left.+(-1)^{n}(a \otimes b) \otimes_{A}\left(\varepsilon\left(c^{n}\right) \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n-1}\right)\right) \\
= & \theta_{n-1}\left(\sum^{n}\left(a c_{1}^{1} \otimes S\left(c_{2}^{1}\right) b\right) \otimes_{A}\left(1 \otimes c^{2} \otimes \cdots \otimes c^{n}\right)\right. \\
& +\sum_{i=1}^{n-1}(-1)^{i}(a \otimes b) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{i} c^{i+1} \otimes \cdots \otimes c^{n}\right) \\
& \left.+(-1)^{n}\left(\varepsilon\left(c^{n}\right) a \otimes b\right) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n-1}\right)\right) \\
= & \sum^{a c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{2} \cdots c_{2}^{n}\right) S\left(c_{2}^{1}\right) b} \\
& +\sum_{i=1}^{n-1}(-1)^{i} \sum a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{i} c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{1} \cdots c_{2}^{n}\right) b \\
& +\sum(-1)^{n} a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{n-1} \otimes \varepsilon\left(c^{n}\right) S\left(c_{2}^{1} \cdots c_{2}^{n-1}\right) b
\end{aligned}
$$

and

$$
\begin{aligned}
& d_{n} \theta_{n}\left((a \otimes b) \otimes_{A}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{n}\right)\right) \\
& =d_{n}\left(\sum a \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{n}\right) b\right) \\
& =\sum a c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{n}\right) b \\
& \quad+\sum \sum_{i=1}^{n-1}(-1)^{i} a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{i} c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{n} \otimes S\left(c_{2}^{1} \cdots c_{2}^{n}\right) b \\
& \quad+\sum(-1)^{n} a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{n-1} \otimes c_{1}^{n} S\left(c_{2}^{1} \cdots c_{2}^{n}\right) b .
\end{aligned}
$$

Since $S$ is an algebra anti-homomorphism that satisfies the third condition in Definition 2.2.2,

$$
\sum c_{1}^{n} S\left(c_{2}^{1} \cdots c_{2}^{n}\right)=\sum c_{1}^{n} S\left(c_{2}^{n}\right) S\left(c_{2}^{n-1}\right) \cdots S\left(c_{2}^{1}\right)=\sum \varepsilon\left(c^{n}\right) S\left(c_{2}^{1} \cdots c_{2}^{n-1}\right)
$$

and

$$
S\left(c_{2}^{2} \cdots c_{2}^{n}\right) S\left(c_{2}^{1}\right)=S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{n}\right)
$$

so that the two expressions are equal from which it follows that $\theta$ is a chain map.
Lastly, one can see that the $A^{e}$-module homomorphism

$$
\psi_{n}\left(a \otimes c^{1} \otimes c^{2} \otimes \cdots c^{n} \otimes b\right)=\sum\left(a \otimes c_{2}^{1} c_{2}^{2} \cdots c_{2}^{n} b\right) \otimes_{A}\left(1 \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{n}\right)
$$

is the inverse of $\theta_{n}$ by using the property that $S$ is an algebra anti-homomorphism that satisfies the third condition in Definition 2.2.2.

Let $f_{x}$ be in $\operatorname{Hom}_{A^{e}}\left(X_{m}, A\right)$ and $g_{x}$ be in $\operatorname{Hom}_{A^{e}}\left(X_{n}, A\right)$. We define the $X$-bracket $\left[f_{x}, g_{x}\right]_{X}$ in $\operatorname{Hom}_{A^{e}}\left(X_{m+n-1}, A\right)$ to be a composition $X \xrightarrow{\theta} B(A) \xrightarrow{\left[\psi^{*} f_{x} \cdot \psi^{*} g_{x}\right]} A$. Then, we have

$$
\left[f_{x}, g_{x}\right]_{X}=\left[\psi^{*} f_{x}, \psi^{*} g_{x}\right] \theta=\left(\psi^{*} f_{x} \circ \psi^{*} g_{x}\right) \theta-(-1)^{(m-1)(n-1)}\left(\psi^{*} g_{x} \circ \psi^{*} f_{x}\right) \theta
$$

We compute one of the circle products:

$$
\begin{aligned}
& \left(\psi^{*} f_{x} \circ \psi^{*} g_{x}\right) \theta_{m+n-1}\left((a \otimes b) \otimes{ }_{A} 1 \otimes c^{1} \otimes \cdots \otimes c^{m+n-1}\right) \\
& =\left(\psi^{*} f_{x} \circ \psi^{*} g_{x}\right)\left(\sum a \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{m+n-1} \otimes S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{m+n-1}\right) b\right) \\
& =\sum \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} f_{x} \psi_{m}\left(a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{i-1} \otimes g_{x} \psi_{n}\left(1 \otimes c_{1}^{i} \otimes \cdots \otimes c_{1}^{i+n-1} \otimes 1\right)\right. \\
& \left.\quad \otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1} \otimes S\left(c_{2}^{1} c_{2}^{2} \cdots c_{2}^{m+n-1}\right) b\right) \\
& =\sum \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} f_{x} \psi_{m}\left(a \otimes c_{1}^{1} \otimes \cdots \otimes c_{1}^{i-1}\right. \\
& \otimes \sum_{x}\left(1 \otimes c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1} \otimes_{A} 1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right) \\
& \left.\otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1} \otimes S\left(c_{2}^{1} \cdots c_{2}^{i-1} c_{3}^{i} \cdots c_{3}^{i+n-1} c_{2}^{i+n} \cdots c_{2}^{m+n-1}\right) b\right) \\
& =\sum_{i=1}^{m}(-1)^{(n-1)(i-1)} f_{x}\left(a \otimes c_{2}^{1} c_{2}^{2} \cdots c_{2}^{i-1} c_{2}^{*} c_{2}^{i+n} \cdots c_{2}^{m+n-1} S\left(c_{3}^{1} c_{3}^{2} \cdots c_{3}^{m+n-1}\right) b\right. \\
& \left.\quad \otimes_{A} 1 \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{i-1} \otimes c_{1}^{*} \otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(c) & =\sum c_{1} \otimes c_{2}, \Delta^{(2)}(c)=\sum c_{1} \otimes c_{2} \otimes c_{3}, \Delta\left(c^{*}\right)=\sum c_{1}^{*} \otimes c_{2}^{*} \text { and } \\
c^{*} & =\sum g_{x}\left(1 \otimes c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1} \otimes_{A} 1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right)
\end{aligned}
$$

Recall that if $A$ is a Hopf algebra over $k$ with bijective antipode, then

$$
\mathrm{HH}^{*}(A) \cong \mathrm{H}^{*}\left(A, A^{a d}\right)
$$

by Theorem 5.3.4.
We already have the Gerstenhaber bracket $[,]_{X}$ on $\operatorname{Ext}_{A^{e}}^{n}\left(A^{e} \otimes_{A} k, A\right)$. Hence, we can use the isomorphisms $\sigma$ and $\tau$ in the proof of Eckmann-Shapiro Lemma combined with Theorem 5.3.4 to find the bracket expression on $\mathrm{H}^{*}\left(A, A^{\text {ad }}\right)$. Now let $\tilde{f} \in \operatorname{Hom}_{A}\left(P_{m}, A^{\text {ad }}\right)$ and $\tilde{g} \in \operatorname{Hom}_{A}\left(P_{n}, A^{\text {ad }}\right)$. Then $[\tilde{f}, \tilde{g}]_{P} \in \operatorname{Hom}_{A}\left(P_{m+n-1}, A^{a d}\right)$ and we have

$$
\begin{aligned}
{[\tilde{f}, \tilde{g}]_{P} } & =\tau[\sigma(\tilde{f}), \sigma(\tilde{g})]_{X} \\
& =\tau\left(\left(\psi^{*}(\sigma(\tilde{f})) \circ \psi^{*}(\sigma(\tilde{g}))\right) \theta\right)-(-1)^{(m-1)(n-1)} \tau\left(\left(\psi^{*}(\sigma(\tilde{g})) \circ \psi^{*}(\sigma(\tilde{f}))\right) \theta\right)
\end{aligned}
$$

For simplification we define

$$
\tilde{f} \circ_{P} \tilde{g}:=\tau\left(\left(\psi^{*}(\sigma(\tilde{f})) \circ \psi^{*}(\sigma(\tilde{g}))\right) \theta\right)
$$

Then, by using the previous circle product formula we obtain:

$$
\begin{aligned}
& \tilde{f} \circ_{P} \tilde{g}\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{m+n-1}\right) \\
& =\tau\left(\left(\psi^{*}(\sigma(\tilde{f})) \circ \psi^{*}(\sigma(\tilde{g}))\right) \theta\right)\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{m+n-1}\right) \\
& =\left(\psi^{*}(\sigma(\tilde{f})) \circ \psi^{*}(\sigma(\tilde{g}))\right) \theta\left((1 \otimes 1) \otimes_{A} 1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{m+n-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} \sigma(\tilde{f})\left(1 \otimes c_{2}^{1} c_{2}^{2} \cdots c_{2}^{i-1} c_{2}^{*} c_{2}^{i+n} \cdots c_{2}^{m+n-1} S\left(c_{3}^{1} c_{3}^{2} \cdots c_{3}^{m+n-1}\right)\right. \\
& \left.\otimes_{A} 1 \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{i-1} \otimes c_{1}^{*} \otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1}\right) \\
= & \sum \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} \tilde{f}\left(1 \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{i-1} \otimes c_{1}^{*} \otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1}\right) \\
& \left.c_{2}^{1} c_{2}^{2} \cdots c_{2}^{i-1} c_{2}^{*} c_{2}^{i+n} \cdots c_{2}^{m+n-1} S\left(c_{3}^{1} c_{3}^{2} \cdots c_{3}^{m+n-1}\right)\right)
\end{aligned}
$$

with $\Delta\left(c^{*}\right)=\sum c_{1}^{*} \otimes c_{2}^{*}$ and

$$
\begin{aligned}
c^{*} & =\sum \sigma(\tilde{g})\left(1 \otimes c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1} \otimes_{A} 1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right) \\
& =\sum\left(1 \otimes c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1}\right) \tilde{g}\left(1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right) \\
& =\sum \tilde{g}\left(1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right) c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1}
\end{aligned}
$$

We now have the Lie bracket $[,]_{P}$ on $\mathrm{H}^{*}\left(A, A^{a d}\right)$. Next, we embed $\mathrm{H}^{*}(A, k)$ into $\mathrm{H}^{*}\left(A, A^{a d}\right)$ [25, Corollary 9.4.7] via the unit map

$$
\eta_{*}: \operatorname{Hom}_{A}\left(P_{\bullet}, k\right) \rightarrow \operatorname{Hom}_{A}\left(P_{\bullet}, A^{a d}\right) .
$$

Let $f \in \operatorname{Hom}_{A}\left(P_{m}, k\right)$ and $g \in \operatorname{Hom}_{A}\left(P_{n}, k\right)$. Then by using the counit map

$$
\varepsilon_{*}: \operatorname{Hom}_{A}\left(P_{\bullet}, A\right) \rightarrow \operatorname{Hom}_{A}\left(P_{\bullet}, k\right),
$$

$\eta_{*}$ and the bracket on $\mathrm{H}^{*}\left(A, A^{a d}\right)$, we derive the formula for $[f, g] \in \operatorname{Hom}_{A}\left(P_{m+n-1}, k\right)$ :

$$
[f, g]=\varepsilon_{*}\left[\eta_{*}(f), \eta_{*}(g)\right]_{P}=\varepsilon_{*}\left(\eta_{*}(f) \circ_{P} \eta_{*}(g)\right)-(-1)^{(m-1)(n-1)} \varepsilon_{*}\left(\eta_{*}(g) \circ_{P} \eta_{*}(f)\right)
$$

where

$$
\begin{aligned}
& \varepsilon_{*}\left(\left(\eta_{*}(f) \circ_{P} \eta_{*}(g)\right)\left(1 \otimes c^{1} \otimes c^{2} \otimes \cdots \otimes c^{m+n-1}\right)\right) \\
& =\varepsilon\left(\sum \sum_{i=1}^{m}(-1)^{(n-1)(i-1)} \eta\left(f\left(1 \otimes c_{1}^{1} \otimes c_{1}^{2} \otimes \cdots \otimes c_{1}^{i-1} \otimes c_{1}^{*} \otimes c_{1}^{i+n} \otimes \cdots \otimes c_{1}^{m+n-1}\right)\right)\right. \\
& \left.c_{2}^{1} c_{2}^{2} \cdots c_{2}^{i-1} c_{2}^{*} c_{2}^{i+n} \cdots c_{2}^{m+n-1} S\left(c_{3}^{1} c_{3}^{2} \cdots c_{3}^{m+n-1}\right)\right)
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta\left(c^{*}\right) & =\sum c_{1}^{*} \otimes c_{2}^{*} \text { and } \\
c^{*} & =\sum \eta\left(g\left(1 \otimes c_{1}^{i} \otimes c_{1}^{i+1} \otimes \cdots \otimes c_{1}^{i+n-1}\right)\right) c_{2}^{i} c_{2}^{i+1} \cdots c_{2}^{i+n-1} .
\end{aligned}
$$

Therefore, the last formula is a general expression of the Gerstenhaber bracket on Hopf algebra cohomology which is indeed inherited from the formula of the bracket on Hochschild cohomology.

## 7. SUMMARY

Understanding the Lie bracket structure of an algebra is one of the main targets of Lie theory and Hochschild cohomology is a graded Lie algebra with Gerstenhaber bracket. However, it is hard to come up with an explicit example of the bracket structure on Hochschild cohomology of even the simplest algebras since the definition of the bracket is on the bar resolution which makes the calculation impossible. Fortunately, some new techniques have been developed that allow us to find the bracket on much simpler resolutions with alternative formulas. One of the results of this work is a new example of the bracket structures on the Hochschild cohomologies of a truncated polynomial ring defined on a field with characteristic 0 and a Taft algebra.

On the other hand, we have known that Hopf algebra cohomology is also a graded Lie algebra and the Lie structure is abelian for quasi-triangular Hopf algebras for almost two decades. Nevertheless, it is even harder to compute the bracket for Hopf algebras as there is not a concrete formula of the bracket on Hopf algebra cohomology. As another result of this thesis, the first example of the bracket on Hopf algebra cohomology of a nonquasi-triangular algebra, a Taft algebra, is found by using the bracket on Hochschild cohomology of the Taft algebra. Furthermore, a new method is developed in order to find the bracket structure on Hopf algebra cohomology without computing the bracket on Hochschild cohomology.

Lastly, starting from the original definition of the Gerstenhaber bracket on Hochschild cohomology, an explicit bracket formula on the Hopf algebra cohomology is explored for a Hopf algebra with a bijective antipode.

Although the question "what is the bracket structure on Hopf algebra cohomology in general? " is still open, we are one more step closer to the answer. We are extremely hopeful that the question will not be open for a long time since there are not many parts stayed in the darkness.

## REFERENCES

[1] S. M. Burciu and S. Witherspoon, Hochschild cohomology of smash products and rank one Hopf algebras, Biblioteca de la Revista Matematica Iberoamericana Actas del "XVI Coloquio Latinoamericano de Algebra" (2005), 153-170.
[2] A. Čap, H. Schichl and J. Vanžura, On twisted tensor products of algebras, Comm. Algebra 23 (1995), no. 12, 4701-4735.
[3] V. Drinfeld, Quantum groups, Proceedings of the International Congress of Mathematics, Berkeley (1987), 798-820.
[4] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, Tensor Categories, Mathematical Surveys and Monographs, Vol. 205, American Mathematical Society, 2015.
[5] M.A. Farinati, A. Solotar, G-structure on the cohomology of Hopf algebras, Proceedings of the American Mathematical Society 132 (2004), no. 10, 2859-2865.
[6] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2), 78:267288, 1963.
[7] L. Grimley, V. C. Nguyen and S. Witherspoon, Gerstenhaber brackets on Hochschild cohomology of twisted tensor products, J. Noncommutative Geometry 11 (2017), no. 4, 13511379.
[8] E. Gunnlaugsdóttir, Monoidal structure of the category of $u_{q}^{+}$-modules, Special issue on linear algebra methods in representation theory, Linear Algebra Appl. 365 (2003), 183-199.
[9] R. Hermann, Monoidal Categories and the Gerstenhaber Bracket in Hochschild Cohomology, American Mathematical Society, 2016.
[10] T. Karadağ, Gerstenhaber bracket on Hopf algebra and Hochschild cohomologies, arXiv:2010.07505.
[11] T. Karadağ and S. Witherspoon, Lie brackets on Hopf algebra cohomology, arXiv:2101.09805.
[12] B. Keller, Chain complexes and stable categories, Manuscripta Math., 67 (1990), 379-417.
[13] J. Le and G. Zhou, On the Hochschild cohomology ring of tensor products of algebras, J. Pure Appl. Algebra 218 (2014), 1463-1477.
[14] C. Negron and S. Witherspoon, An alternate approach to the Lie bracket on Hochschild cohomology, Homology, Homotopy and Applications 18 (2016), no.1, 265-285.
[15] V. C. Nguyen, Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras, J. Pure Appl. Algebra, 217 (2013), 1967-1979.
[16] V. C. Nguyen, X. Wang and S. Witherspoon, Finite generation of some cohomology rings via twisted tensor product and Anick resolutions, to appear in J. Pure Appl. Algebra.
[17] J. Pevtsova and S. Witherspoon, Varieties for modules of quantum elementary abelian groups, Algebras and Rep. Th. 12 (2009), no. 6, 567-595.
[18] S. Sanchez-Flores, The Lie structure on the Hochschild cohomology of a modular group algebra, J. Pure Appl. Algebra 216(3) (2012), 718-733.
[19] D. Ştefan, Hochschild cohomology on Hopf Galois extensions, J. Pure and Applied Algebra 103 (1995), 221-233.
[20] R. Taillefer, Injective Hopf bimodules, Cohomologies of Infinite Dimensional Hopf Algebras and Graded-commutativity of the Yoneda Product, Journal of Algebra. 276 (2004), no.1, 259279.
[21] Y. Volkov, Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution, Proc. Edinburgh Math. Soc. (2) 62 (3) (2019), 817-836.
[22] Y. Volkov, S. Witherspoon, Graded Lie structure on cohomology of some exact monoidal categories, arXiv:2004.06225.
[23] C. Walton and S. Witherspoon, Poincaré-Birkhoff-Witt deformations of smash product algebras from Hopf actions on Koszul algebras, Algebra and Number Theory 8 (2014), no. 7, 1701-1731.
[24] C. A. Weibel, An Introduction to Homological Algebra, volume 38. Cambridge University Press, 1995.
[25] S. Witherspoon, Hochschild Cohomology for Algebras, volume 204. American Mathematical Society, 2019.

