ON THE LIE ALGEBRA STRUCTURE ON HOCHSCHILD COHOMOLOGY OF KOSZUL QUIVER ALGEBRAS

A Dissertation

by

TOLULOPE NATHANEAL OKE

Submitted to the Graduate and Professional School of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Sarah Witherspoon
Committee Members,	Eric Rowell
	Frank Sottile
	Andreas Klappenecker
Head of Department,	Sarah Witherspoon

August 2021

Major Subject: Mathematics

ABSTRACT

We present the Gerstenhaber algebra structure on Hochschild cohomology of Koszul algebras defined by quivers and relations using the idea of homotopy liftings. There is a canonical way of constructing a minimal (graded) projective resolution $\mathbb{K}$ of a Koszul quiver algebra over its enveloping algebra. This resolution was shown to have a comultiplicative structure. Our presentation involves the use of the resolution $\mathbb{K}$ and the comultiplicative structure on it. We present general forms of homotopy lifting maps for cocycles defined on Hochschild cohomology using $\mathbb{K}$. To demonstrate the theory, we study the Hochschild cohomology ring of a family of quiver algebras and present explicit examples of homotopy lifting maps for cocycles of degrees 1 and 2. As an application to the theory of deformation of algebras, we specify Hochschild 2-cocycles satisfying the Maurer-Cartan equation.

DEDICATION

This work is dedicated to the glory of God - who called me out of darkness, and translated me into the Kingdom of His dear Son.

ACKNOWLEDGMENTS

First and foremost is my deepest gratitude to God who made this a reality and gave me strength to travel and complete the Ph.D. journey.

I would like to offer my sincere appreciation to following families, friends and teachers whose support, encouragement, mentorship and prayers strengthened me during my program at Texas A&M University.

I am most grateful to my wife Oluwafolakemi for her unflinching love and support throughout this journey and to my son ObalOluwa for giving me a rare opportunity to combine fatherhood, teaching and research as a graduate student.

I thank my extended family members: the Oke family - Daddy and Mummy, Victor, Emmanuel and Precious and the Odejinmi family – Daddy and Mummy, Olalekan and Sinmiloluwa.

I am thankful to Pastor Martins Kuyoro and his wife, Pastor Praise Apampa and his wife, the family of Olalekan Faromiki and every past and present member of the Redeemed Christian Church of God (FORA), College Station for the opportunity to serve together in God's vineyard these past years.

I also appreciate Elias and Ohemaa Adanu, Ayomikun and Uche Adeniran, Moffi and Adetayo Ige, Jacqueline Antwi-Danso and every member of the Friday Bible Study group for the opportunity to fellowship and study God's words together. I am especially thankful to Adedayo Adebayo for hosting my family each time we travel to Austin.

I thank every member of DLBC and the DLCF Campus Connect group for providing a community of believers where we shared our burdens and prayed together – the families of Segun & Tolu Ahmed, Emmanuel & Iveth Afolayan, Marvelous & Gbenga Ogunyemi, Segun & Precious Ofe, Wisdom & Ololade, Tamunonye West, Sis Omene Ojeme and the family of Uyiosa Abusomwan.

I am grateful to all my colleagues in the department of Mathematics and Texas A&M University who have contributed to making this journey a fruitful endeavor – Alexander Ruysdeperez, Supun Samarakoon and Kio Adaiyibo.

I sincerely thank Drs Eric Rowell, Andrea Klappenecker and Frank Sottile for serving on my committee. Dr Sottile was instrumental to my coming into graduate school.

Finally, my utmost gratitude goes to my thesis advisor, Dr Sarah Witherspoon for her guidance and support throughout the period of this program. I am especially grateful for her patience with me as I learn to be a professional mathematician and for showing commitment to my professional goals and aspirations.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis committee consisting of Professors Sarah Witherspoon [advisor], Eric Rowell and Frank Sottile of the Department of Mathematics and Professor Andrea Klappenecker of the Department of Computer Science, Texas A&M University.

All other work conducted for the thesis was completed by the student independently.

Funding Sources

The author's graduate study was supported by funding from the Mathematics Department of Texas A&M University and partially from the NSF grants 1665286 and 2001163 during summer 2020, spring and summer 2021 .

TABLE OF CONTENTS

LIST OF TABLES

TABLE | Page

1. INTRODUCTION

While S. Eilenberg and S. Mac Lane introduced homology and cohomology of groups in the 1940s, G. Hochschild introduced homology and cohomology of algebras around the same time. Hochschild cohomology conveys meaningful information about rings and algebras. It has become a very useful tool in the theory of deformation of algebras and in their representations. The zero dimensional Hochschild cohomology of an associative algebra corresponds with the center of the algebra. The one dimensional Hochschild cohomology of an associative algebra is isomorphic to the space of derivations modulo inner derivations on the algebra. A derivation is a function on an algebra which generalizes certain features of the derivative operator. Specifically, given an algebra A over a ring or a field k, a k-derivation is a k-linear map $D : A \to A$ that satisfies the *Leibniz law*. The space of Hochschild 2-cocycles contains information about formal and infinitesimal deformations of the algebra. As an algebraic object, there are two binary operations on Hochschild cohomology: the *cup product* and the *Gerstenhaber bracket*. The bracket plays an important role in the theory of deformation of algebras.

The cup product has several equivalent definitions some of which are presented in Chapter 4. There is a definition involving direct application of module homomorphisms on elements for which they can be applied; the cup product of an m-cocycle and an n-cocycle is an $m + n$-cocycle such that when it is applied to a homogeneous element of degree $m+n$, the degree m-cocycle applies to the first m components while the degree n cocycle applies to the remaining n components. There is another definition involving composition of maps. This technique is known as the diagonal approximation of the cup product and one of the maps used is the diagonal map.

The Gerstenhaber bracket is a type of Lie bracket. It is a bracket of degree -1 i.e. the bracket of an m-cocycle and an n-cocycle is a cocycle of degree $m + n - 1$. It was initially defined by M. Gerstenhaber using the bar resolution. The bracket makes Hochschild cohomology into a Gerstenhaber algebra. The initial definition of the Gerstenhaber bracket is useful theoretically but not easily accessible for computational purposes. For instance to compute the bracket using an arbitrary resolution, appropriate chain maps between the resolution and the bar resolution would have to be constructed. Morphisms defined on the resolution would then have to be carried over to the bar resolution and vice versa. This process of coming up with comparison morphisms is not always easy.

To overcome the challenge of determining Gerstenhaber brackets using comparison morphisms, several works have been carried out on interpreting the initial definition of the bracket given by M. Gerstenhaber. For example, in 2004 [9], B. Keller realized Hochschild cohomology as the Lie algebra of the derived Picard group. In [11], C. Negron and S. Witherspoon introduced the idea of a contracting homotopy which works for resolutions that are differential graded coalgebras. M. Suárez-Álvarez showed that the bracket of a cocycle of degree 1 and any degree n cocycle can be realized from derivation operators associated to the degree 1 cocycle expressed on an arbitrary projective resolution [15]. In [16], Y. Volkov generalized the method introduced in [11] to arbitrary resolutions by defining the bracket of any two cocycles in terms of homotopy lifting maps.

We present the Gerstenhaber algebra structure for Koszul algebras defined by quivers and relations using homotopy liftings. We construct examples of homotopy lifting maps for some cocycles. We also present a general form of these homotopy lifting maps under certain conditions. Our cocycles and homotopy lifting maps are defined using the resolution $\mathbb{K}$ introduced by E.L. Green, G. Hartman, E.N. Marcos, Ø. Solberg in [5]. The resolution $\mathbb{K}$ is a differential graded coalgebra. A homotopy lifting map is a map between two chain complexes satisfying two conditions. These conditions are presented in Equation (2.17). For Koszul algebras, the Koszul complex has certain characteristics for which the second condition of Equation (2.17) is easily satisfied. Moreover, because there is an algorithmic approach for constructing projective resolutions such as $\mathbb{K}$ in the literature [4], perhaps it might be possible to employ our method in finding an algorithmic approach for computing the Gerstenhaber bracket on Hochschild cohomology for Koszul quiver algebras.

In Chapter 2, we give a brief introduction to Hochschild cohomology and the two binary operations on it. We give equivalent definitions of these binary operations in Sections 2.2 and 2.3.

Since our examples will come from a family of quiver algebras, we introduce quiver algebras and Koszul algebras in Chapter 3. We discuss the construction of the projective resolution $\mathbb{K}$ in detail in Section 3.3. We give a generalized cup product formula on Hochschild cohomology of the family of quiver algebras in Section 4.4. With the cup product formula, we determine the set of nilpotent cocycles and hence the structure of Hochschild cohomology modulo nilpotents. In Section 4.3, we present F. Xu's counterexample to the Snashall-Solberg finite generation conjecture using the generalized cup product formula. After deriving the generalized cup product formula, we give an explicit description of a diagonal map which we use in the computations of homotopy lifting maps in Section 5.2. Detailed calculations that these maps are indeed homotopy lifting maps are given in Subsections 5.2.1 and 5.2.2. Our main ideas with respect to the bracket structure are given in Chapter 5. In Section 5.1 we specify a general form of homotopy lifting map demonstrated by the examples of Subsections 5.2.1 and 5.2.2. We use these results to give the Gerstenhaber algebra structure on Hochschild cohomology of Koszul algebras defined by quivers and relations under certain conditions. We discuss an application to specifying solutions to the Maurer-Cartan equation in Section 5.3.

2. HOCHSCHILD COHOMOLOGY OF ASSOCIATIVE ALGEBRAS

In this chapter, we give a brief description of Hochschild cohomology of associative algebras and the two binary operations on it. Throughout, we take k to be a field, Λ to be a unital associative k-algebra and take $\otimes = \otimes_k$ unless otherwise specified. This means that Λ is a k-vector space with a bilinear map $\Lambda \times \Lambda \to \Lambda$ that is associative and has a unit element denoted by 1. Also, denote by Λ^{op} the opposite algebra of Λ with the same elements as Λ and multiplication $a \circ b = ba$. We denote the enveloping algebra of Λ by $\Lambda^e = \Lambda \otimes \Lambda^{op}$ with multiplication given by $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (b \circ b') = aa' \otimes b'b$, for all $a, a', b, b' \in \Lambda$.

A Λ-bimodule M can be viewed as a left Λ^e-module via the map $\Lambda^e \times M \to M$ taking $(a \otimes b) \cdot m \mapsto amb$. The algebra Λ is itself a left Λ^e-module and more generally, the n-fold tensor product $\Lambda^{\otimes n}$ of Λ is a left module over the enveloping algebra via $\Lambda^e \times \Lambda^{\otimes n} \to \Lambda^{\otimes n}$ defined by $(a \otimes b) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n) \mapsto aa_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n b$.

2.1 Introduction to Hochschild cohomology

Hochschild cohomology was originally defined using the bar resolution. The bar resolution consists of Λ^e-modules $\mathbb{B}_n := \Lambda^{\otimes(n+2)}$ which are the $(n+2)$-fold tensor products of the algebra over the field k:

$$\mathbb{B}_\bullet : \qquad \cdots \to \Lambda^{\otimes(n+2)} \xrightarrow{\delta_n} \Lambda^{\otimes(n+1)} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} \Lambda^{\otimes 3} \xrightarrow{\delta_1} \Lambda^{\otimes 2} \; (\xrightarrow{\mu} \Lambda) \tag{2.1}$$

where μ is the multiplication map and the differentials δ_n are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{2.2}$$

for all $a_0, a_1, \ldots, a_{n+1} \in \Lambda$. The map μ is sometimes called the *augmentation map* and we write $\mathbb{B}_\bullet \xrightarrow{\mu} \Lambda$ for short. The n-th homology of $\mathbb{B}_\bullet$ is given by $\mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n+1})$ and is equal to 0 for all n except at $n = 0$ where it is Λ. This is because by direct computation, $\delta_n \delta_{n+1} = 0$, showing that $\operatorname{Ker}(\delta_n) \supseteq \operatorname{Im}(\delta_{n+1})$ and the existence of a contracting homotopy $s_n : \mathbb{B}_n \to \mathbb{B}_{n+1}$ satisfying $s_{n-1}\delta_n + \delta_{n+1}s_n = 1$ implies that $\operatorname{Ker}(\delta_n) \subseteq \operatorname{Im}(\delta_{n+1})$. A contracting homotopy s_n is defined by $(a_0 \otimes \cdots \otimes a_{n+1}) \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$. Since Λ is free as a k-module, each $\mathbb{B}_n$ is a free Λ^e-module and the bar resolution is a free resolution. For a left Λ^e-module M, any module homomorphism $f : \mathbb{B}_n \to M$ gives rise to a module homomorphism $\delta_{n+1}^*(f) = f\delta_{n+1} : \mathbb{B}_{n+1} \to M$. This means that we can apply the Hom functor $\operatorname{Hom}_{\Lambda^e}(-, M)$ to the bar resolution to obtain the following cochain complex:

$$\begin{aligned} \operatorname{Hom}_{\Lambda^e}(\mathbb{B}_\bullet, M): \quad & 0 \to \operatorname{Hom}_{\Lambda^e}(\Lambda^{\otimes 2}, M) \xrightarrow{\delta_1^*} \operatorname{Hom}_{\Lambda^e}(\Lambda^{\otimes 3}, M) \xrightarrow{\delta_2^*} \cdots \\ & \xrightarrow{\delta_{n-1}^*} \operatorname{Hom}_{\Lambda^e}(\Lambda^{\otimes (n+1)}, M) \xrightarrow{\delta_n^*} \operatorname{Hom}_{\Lambda^e}(\Lambda^{\otimes (n+2)}, M) \xrightarrow{\delta_{n+1}^*} \cdots \end{aligned} \tag{2.3}$$

The n-th cohomology of this chain complex, also referred to as the space of Hochschild n-cochains, is given by

$$\operatorname{HH}^n(\Lambda, M) = \mathbf{H}^n(\operatorname{Hom}_{\Lambda^e}(\mathbb{B}_n, M)) = \operatorname{Ker}(\delta_{n+1}^*)/\operatorname{Im}(\delta_n^*).$$

Hochschild cohomology of the algebra Λ with coefficients in M is defined to be

$$\operatorname{HH}^*(\Lambda, M) := \bigoplus_{n \geq 0} \mathbf{H}^n(\operatorname{Hom}_{\Lambda^e}(\mathbb{B}_n, M)).$$

If $M = \Lambda$, we write $\operatorname{HH}^*(\Lambda) := \operatorname{HH}^*(\Lambda, \Lambda)$. We next discuss two binary operations on Hochschild cohomology.

Remark 2.4. *By applying the functor* $\operatorname{Hom}_{\Lambda^e}(-, \Lambda)$ *to any projective bimodule resolution* $(\mathbb{P}_\bullet, \bar{\delta})$ *of* Λ*, we obtain the cochain complex similar to the one given by* (2.3) *with* M *replaced by* Λ *and the* Λ^e*-modules* $\Lambda^{\otimes(n+2)}$ *replaced by the* Λ^e*-modules* $\mathbb{P}_n$. $\operatorname{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$ *is defined as the* n*-th cohomology group of this cochain complex, and it is also given by* $\operatorname{Ker}(\bar{\delta}_{n+1}^*)/\operatorname{Im}(\bar{\delta}_n^*)$. *Since* Ext *is independent of the choice of projective resolution, we obtain an isomorphism* $H^n(\operatorname{Hom}_{\Lambda^e}(\mathbb{B}_n, M)) \cong \operatorname{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$ *of abelian groups. See for example [7, Chapter IV, Section 7],[18, Appendix A.3] or [17, Chapter 3] for details. Hochschild cohomology can therefore be realized from the* Ext *functor,* *that is:*

$$\mathrm{HH}^*(\Lambda) = \bigoplus_{n\geq 0} \boldsymbol{H}^n(\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_n, M)) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda,\Lambda).$$

2.2 The cup product on Hochschild cohomology

The cup product makes Hochschild cohomology into a graded commutative ring. Furthermore, it is by definition associative and has several equivalent definitions provided by researchers over the years. The definitions we present here are not self contained. We refer the reader to [7, 17, 18] for further reading. A Λ^e-module homomorphism $\hat{f} : \mathbb{B}_m \to \Lambda$ can be seen as a k-module homomorphism $f : \Lambda^{\otimes m} \to \Lambda$ by defining $\hat{f}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m \otimes b) = af(a_1 \otimes a_2 \otimes \cdots \otimes a_m)b$. This is in fact an isomorphism of k-modules i.e. $\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_m, M) \cong \mathrm{Hom}_k(\Lambda^{\otimes m}, M)$.

Definition 2.5. *Let $f \in \mathrm{Hom}_k(\Lambda^{\otimes m},\Lambda)$ and $g \in \mathrm{Hom}_k(\Lambda^{\otimes n},\Lambda)$. The cup product $f \smile g$ at the chain level is an element of $\mathrm{Hom}_k(\Lambda^{\otimes(m+n)},\Lambda)$ defined by*

$$f \smile g(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \tag{2.6}$$

for all $a_1, a_2 \ldots, a_{m+n} \in \Lambda$.

We present the following equivalent definition because we will later refer to it in Chapter 4. Let $\mathbb{P}_\bullet$ be a projective resolution of Λ as an Λ^e-module with differential $d^{\mathbb{P}}$. The total complex Tot($\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet$) is also an Λ^e-projective resolution of Λ with its n-th module Tot($\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet)_n$ given as $\sum_{i+j=n} \mathbb{P}_i \otimes_\Lambda \mathbb{P}_j$ and differentials $d^{\mathbb{P}} \otimes 1 + 1 \otimes d^{\mathbb{P}}$ [18, page 33]. By the comparison theorem (see [17, 7]), there is a chain map $\Delta_{\mathbb{P}} : \mathbb{P}_\bullet \to \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet$ lifting the canonical isomorphism from Λ to $\Lambda \otimes_\Lambda \Lambda$. In particular, the diagram

$$\begin{tikzcd}
\mathbb{P}_\bullet \arrow[r] \arrow[d, "\Delta_{\mathbb{P}}"'] & \Lambda \arrow[d, "\cong"] \\
\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \arrow[r] & \Lambda \otimes_\Lambda \Lambda
\end{tikzcd}$$

is commutative. The map $\Delta_{\mathbb{P}}$ is called a *diagonal* map or a *comultiplication* map (whenever we consider $\mathbb{P}_\bullet$ as a graded coalgebra) and it is unique up to chain homotopy. Throughout, we use these terms interchangeably. There is a standard way of defining the tensor product of two maps on the total complex. Suppose that $f \in \mathrm{Hom}_{\Lambda^e}(\mathbb{P}_m, \Lambda)$ and $g \in \mathrm{Hom}_{\Lambda^e}(\mathbb{P}_n, \Lambda)$. The tensor product $f \otimes g$ can be viewed as a map $f \otimes g : \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \to \Lambda \otimes_\Lambda \Lambda \cong \Lambda$ defined as $(f \otimes g)(u \otimes v) = (-1)^{|g||u|} f(u) \otimes g(v)$ provided $u \in \mathbb{P}_m$ and $v \in \mathbb{P}_n$ and 0 otherwise. The symbol $|g|$ is used to denote the degree of g. In this case $|g| = n$ and $|f| = m$.

Definition 2.7. *Let $f \in \mathrm{Hom}_k(\mathbb{P}_m, \Lambda)$ and $g \in \mathrm{Hom}_k(\mathbb{P}_n, \Lambda)$. The cup product of f and g is an element of $\mathrm{Hom}_k(\mathbb{P}_{m+n}, \Lambda)$ defined by*

$$f \smile g = \pi(f \otimes g)\Delta_{\mathbb{P}}, \tag{2.8}$$

where π is the multiplication map and $\Delta_{\mathbb{P}}$ is a diagonal map.

Definition 2.7 is also known as the *diagonal approximation* definition of the cup product. At the chain level, Equation (2.8) depends on the choice of $\Delta_{\mathbb{P}}$ but does not depend on the choice at cohomology level. If $\mathbb{P}_\bullet$ is taken to be $\mathbb{B}_\bullet$, the bar resolution, one definition of a diagonal map $\Delta : \mathbb{B}_\bullet \to \mathbb{B}_\bullet \otimes_\Lambda \mathbb{B}_\bullet$ is the following:

$$\begin{aligned} &\Delta(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = (1 \otimes 1) \otimes_\Lambda (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \\ &+ \Big[\sum_{i=1}^{n} (1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes_\Lambda (1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes 1)\Big] + (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes_\Lambda (1 \otimes 1). \end{aligned} \tag{2.9}$$

Given Equation (2.9), it is easy to verify that Definitions 2.5 and 2.7 of the cup product are equivalent. For example if $f \in \mathrm{Hom}_k(\Lambda^{\otimes 2}, \Lambda)$ and $g \in \mathrm{Hom}_k(\Lambda^{\otimes 1}, \Lambda)$, Definition 2.5 yields

$$f \smile g(a_1 \otimes a_2 \otimes a_3) = (-1)^2 f(a_1 \otimes a_2) g(a_3)$$

while Definition 2.7 yields $\pi(\hat{f}\otimes\hat{g})\Delta(1\otimes a_1\otimes a_2\otimes a_3\otimes 1)$ which is the same as

$$\pi(\hat{f}\otimes\hat{g})\Big((1\otimes 1)\otimes_\Lambda(1\otimes a_1\otimes a_2\otimes a_3\otimes 1)+(1\otimes a_1\otimes 1)\otimes_\Lambda(1\otimes a_2\otimes a_3\otimes 1)$$
$$+(1\otimes a_1\otimes a_2\otimes 1)\otimes_\Lambda(1\otimes a_3\otimes 1)+(1\otimes a_1\otimes a_2\otimes a_3\otimes 1)\otimes_\Lambda(1\otimes 1)\Big)$$
$$=\pi((\hat{f}\otimes\hat{g})(1\otimes a_1\otimes a_2\otimes 1)\otimes_\Lambda(1\otimes a_3\otimes 1))=(-1)^2 f(a_1\otimes a_2)g(a_3).$$

2.3 The Gerstenhaber bracket on Hochschild cohomology

The second binary operation on Hochschild cohomology was introduced by M. Gerstenhaber in 1962 [3]. This binary operation makes Hochschild cohomology into a Gerstenhaber algebra. The usual Lie bracket on graded Lie algebras is of degree zero i.e. the bracket of a degree m and a degree n element is an element of degree $m+n$. The Gerstenhaber bracket can be viewed as a Lie bracket of degree -1, i.e. the bracket of a degree m and a degree n cocycle is a degree $m+n-1$ cocycle. The following definition, originally given by M. Gerstenhaber in [3], is presented as reformulated in [18].

Let $\mathbb{B}_\bullet$ be the bar resolution and use the previously defined isomorphism $\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_m,\Lambda)\cong\mathrm{Hom}_k(\Lambda^{\otimes m},\Lambda)$ of abelian groups.

Definition 2.10. *Let $f\in\mathrm{Hom}_k(\Lambda^{\otimes m},\Lambda)$ and $g\in\mathrm{Hom}_k(\Lambda^{\otimes n},\Lambda)$. The Gerstenhaber bracket of f and g is defined as*

$$[f,g]=f\circ g-(-1)^{(m-1)(n-1)}g\circ f \tag{2.11}$$

where $f\circ g=\sum_{j=1}^{m}(-1)^{(n-1)(j-1)}f\circ_j g$ and

$$(f\circ_j g)(a_1\otimes\cdots\otimes a_{m+n-1})$$
$$=f(a_1\otimes\cdots\otimes a_{j-1}\otimes g(a_j\otimes\cdots\otimes a_{j+n-1})\otimes a_{j+n}\otimes\cdots\otimes a_{m+n-1})$$

and it induces a well defined operation on cohomology.

This definition has been quite difficult to interpret when the resolution is not the bar resolution.

In practice, to compute the Gerstenhaber bracket using resolutions other than the bar resolution, one often uses the comparison morphism technique. That is, one has to define appropriate chain maps between the choice resolution and the bar resolution. Then morphisms defined on the resolution would have to be carried over to the bar resolution and vice versa. Recently, the idea of a contracting homotopy was introduced by C. Negron and S. Witherspoon in [11] for resolutions with certain properties. The idea of contracting homotopy was generalized in [16] by Y. Volkov using homotopy lifting maps for arbitrary projective resolutions. We briefly present these ideas next.

2.3.1 Brackets via contracting homotopy

The idea of a contracting homotopy was introduced to efficiently compute Gerstenhaber brackets for algebras using resolutions that are differential graded coalgebras.

Definition 2.12. *Let $(\mathcal{Q}_\bullet, d^{\mathcal{Q}})$ be a projective Λ^e-module resolution of Λ. Let $\mathcal{Q}_\bullet \xrightarrow{\nu} \Lambda$ be the augmentation map. Then $\mathcal{Q}_\bullet$ is a differential graded coalgebra over Λ if there is a comultiplicative map $\Delta_{\mathcal{Q}} : \mathcal{Q}_\bullet \to \mathcal{Q}_\bullet \otimes_\Lambda \mathcal{Q}_\bullet$ lifting the identity map on $\Lambda \cong \Lambda \otimes_\Lambda \Lambda$, satisfying $(d^{\mathcal{Q}} \otimes 1 + 1 \otimes d^{\mathcal{Q}})\Delta_{\mathcal{Q}} = \Delta_{\mathcal{Q}} d^{\mathcal{Q}}$ and is coassociative i.e. $(\Delta_{\mathcal{Q}} \otimes 1)\Delta_{\mathcal{Q}} = (1 \otimes \Delta_{\mathcal{Q}})\Delta_{\mathcal{Q}}$. Furthermore, the resolution $\mathcal{Q}_\bullet$ also written as a triple $(\mathcal{Q}_\bullet, \Delta_{\mathcal{Q}}, \nu)$, is counital if the augmentation map also satisfies $(\nu \otimes 1)\Delta_{\mathcal{Q}} = (1 \otimes \nu)\Delta_{\mathcal{Q}} = 1$.*

The bar resolution $(\mathbb{B}_\bullet, \Delta, \mu)$ of the algebra Λ is a counital differential graded coalgebra over Λ for which the diagonal map of Equation (2.9) is the comultiplication and the augmentation map is the multiplication map μ of Equation (2.1).

Let $\mathbb{P}_\bullet$ be a projective resolution of Λ and $\mu_{\mathbb{P}}$ the augmentation map. Suppose further that there is a comultiplicative chain map $\Delta_{\mathbb{P}} : \mathbb{P}_\bullet \to \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet$ making $\mathbb{P}_\bullet$ into a counital differential graded coalgebra over Λ. Denote by $\Delta_{\mathbb{P}}^{(2)}$ the composition map $(\Delta_{\mathbb{P}} \otimes 1)\Delta_{\mathbb{P}} = (1 \otimes \Delta_{\mathbb{P}})\Delta_{\mathbb{P}}$. We take $\mathbf{d}$ to be the differential on the Hom complex $\mathrm{Hom}_{\Lambda^e}(\mathbb{P}_\bullet, \mathbb{P}_\bullet)$ defined for all Λ^e maps $\rho : \mathbb{P}_\bullet \to \mathbb{P}_\bullet[-n]$ as

$$\mathbf{d}(\rho) := d^{\mathbb{P}}\rho - (-1)^n \rho d^{\mathbb{P}}$$

where $\mathbb{P}_\bullet[-n]$ is a shift in homological dimension with $\mathbb{P}_m[-n] = \mathbb{P}[-n]_m = \mathbb{P}_{m-n}$. It was observed that the chain map $\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}} : \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \to \mathbb{P}_\bullet$ is a coboundary in the Hom complex $\operatorname{Hom}_{\Lambda^e}(\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet, \mathbb{P}_\bullet)$. This is a justification:

$$\begin{aligned}\mathbf{d}(\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}}) &= d^{\mathbb{P}}(\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}}) - (\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}})(d^{\mathbb{P}} \otimes 1 + 1 \otimes d^{\mathbb{P}}) \\ &= \mu_{\mathbb{P}} \otimes d^{\mathbb{P}} - d^{\mathbb{P}} \otimes \mu_{\mathbb{P}} - \mu_{\mathbb{P}} d^{\mathbb{P}} \otimes 1 - \mu_{\mathbb{P}} \otimes d^{\mathbb{P}} + d^{\mathbb{P}} \otimes \mu_{\mathbb{P}} + 1 \otimes \mu_{\mathbb{P}} d^{\mathbb{P}} \\ &= \mu_{\mathbb{P}} d^{\mathbb{P}} \otimes 1 + 1 \otimes \mu_{\mathbb{P}} d^{\mathbb{P}} = 0,\end{aligned}$$

since $\mu_{\mathbb{P}} d^{\mathbb{P}} = 0$. The quasi-isomorphism $(\mu_{\mathbb{P}})_* : \operatorname{Hom}_{\Lambda^e}(\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet, \mathbb{P}_\bullet) \to \operatorname{Hom}_{\Lambda^e}(\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet, \Lambda)$, which takes cocycles to cocycles and coboundaries to coboundaries, takes $\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}}$ to 0, that is,

$$\mu_{\mathbb{P}}(\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}}) = \mu_{\mathbb{P}} \otimes \mu_{\mathbb{P}} - \mu_{\mathbb{P}} \otimes \mu_{\mathbb{P}} = 0,$$

so the map $(\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}})$ is a cocycle that is a coboundary with respect to **d** in the Hom complex. The consequence of this is that there is a degree 1 map $\phi : \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \to \mathbb{P}_\bullet[1]$ such that

$$\mathbf{d}(\phi) := d^{\mathbb{P}}\phi + \phi(d^{\mathbb{P}} \otimes 1 + 1 \otimes d^{\mathbb{P}}) = \mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}}. \tag{2.13}$$

The map ϕ of Equation (2.13) is called a *contracting homotopy*. The following definition of the Gerstenhaber bracket is equivalent to the one presented in Definition 2.10.

Definition 2.14. *Let $\mathbb{P}_\bullet$ be a projective resolution of Λ that is a differential graded coalgebra. Let $f \in \operatorname{Hom}_k(\mathbb{P}_m, \Lambda)$, and $g \in \operatorname{Hom}_k(\mathbb{P}_n, \Lambda)$ be cocycles. The Gerstenhaber bracket of f and g may be defined at the chain level to be*

$$[f, g] = f \circ_\phi g - (-1)^{(m-1)(n-1)} g \circ_\phi f \tag{2.15}$$

where $f \circ_\phi g = f(\phi(1 \otimes g \otimes 1)\Delta_{\mathbb{P}}^{(2)})$, and ϕ is the contracting homotopy of Equation (2.13)*.*

The map $\mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet \xrightarrow{1\otimes g\otimes 1} \mathbb{P}_\bullet \otimes_\Lambda \Lambda \otimes_\Lambda \mathbb{P}_\bullet \cong \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet$ acts in such a way that $(1 \otimes g \otimes 1)(x \otimes y \otimes z) = 0$ for all homogeneous x, y, z unless $y \in \mathbb{P}_n$. See [18, Theorem 6.4.5] and [11, Definition 2.1.1] for more on contracting homotopy. Furthermore, these ideas were used in [6] and [8] to present the Gerstenhaber algebra structure on the quantum complete intersection and the Jordan plane respectively.

2.3.2 Brackets via homotopy liftings

Homotopy lifting maps are a generalization of contracting homotopy. They were introduced by Y. Volkov [16] for handling the Gerstenhaber algebra structure on Hochschild cohomology using arbitrary projective bimodule resolutions. We give explicit examples of homotopy lifting maps in Subsections 5.2.1 and 5.2.2 .

Let $\mathbb{P}_\bullet, \mu_{\mathbb{P}}$ and $\Delta_{\mathbb{P}}$ be as defined in Subsection 2.3.1, but the resolution is not necessarily a differential graded coalgebra i.e. we only require $\Delta_{\mathbb{P}}$ to be a chain map lifting the isomorphism $\Lambda \cong \Lambda \otimes_\Lambda \Lambda$. The following is a definition of a homotopy lifting map for f.

Definition 2.16. *Let $f \in \mathrm{Hom}_{\Lambda^e}(\mathbb{P}_n, \Lambda)$ be a cocycle. A Λ^e-module homomorphism $\psi_f : \mathbb{P}_\bullet \to \mathbb{P}_\bullet[1-n]$ is called a homotopy lifting map of f with respect to $\Delta_{\mathbb{P}}$ if*

$$\begin{aligned} \boldsymbol{d}(\psi_f) &= (f \otimes 1 - 1 \otimes f)\Delta_{\mathbb{P}} \qquad \textit{and} \\ \mu_{\mathbb{P}}\psi_f &\sim\ (-1)^{n-1} f\psi \end{aligned} \tag{2.17}$$

for some $\psi : \mathbb{P}_\bullet \to \mathbb{P}_\bullet[1]$ for which $\boldsymbol{d}(\psi) = d^{\mathbb{P}}\psi - \psi d^{\mathbb{P}} = (\mu_{\mathbb{P}} \otimes 1 - 1 \otimes \mu_{\mathbb{P}})\Delta_{\mathbb{P}}$.

In the above definition, the notation $\sim$ is used for two cocycles that are cohomologous, that is, they differ by a coboundary. For Koszul algebras, the Koszul resolution $\mathcal{K}_\bullet$ is a differential graded coalgebra. Furthermore, the augmentation map $\bar{\mu} : \mathcal{K}_\bullet \to \Lambda$ on the Koszul resolution is a counit i.e. $(\bar{\mu} \otimes 1)\Delta_{\mathcal{K}} - (1 \otimes \bar{\mu})\Delta_{\mathcal{K}} = 0$. We can therefore take $\psi = 0$, so that we now require $\bar{\mu}\psi_f \sim 0$. By setting $\psi_f(\mathcal{K}_{n-1}) = 0$, the second hypothesis of Equation (2.17) is satisfied. It is therefore sufficient to verify only the first condition of Equation (2.17) when the resolution is Koszul. We use these notions in Subsections 5.2.1 and 5.2.2 when giving examples of homotopy lifting maps.

The following definition of the Gerstenhaber bracket is equivalent to the ones presented in Definitions 2.10 and 2.14. See [18, Section 6.3] for details and proofs.

Definition 2.18. *Let $\mathbb{P}_\bullet$ be any Λ^e-projective resolution of Λ and let $\Delta_\mathbb{P} : \mathbb{P}_\bullet \to \mathbb{P}_\bullet \otimes_\Lambda \mathbb{P}_\bullet$ be a diagonal map. Let $f \in \operatorname{Hom}_{\Lambda^e}(\mathbb{P}_m, \Lambda)$, and $g \in \operatorname{Hom}_{\Lambda^e}(\mathbb{P}_n, \Lambda)$ be cocycles. The Gerstenhaber bracket of f and g represented by $[f, g] \in \operatorname{Hom}_{\Lambda^e}(\mathbb{P}_{n+m-1}, \Lambda)$ may be defined at the chain level by*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f \tag{2.19}$$

where ψ_f, ψ_g are homotopy lifting maps associated to the cocycles f and g respectively.

Example 2.20. *Let $\mathbb{P}_\bullet$ be the bar resolution $\mathbb{B}_\bullet$. Suppose that $g \in \operatorname{Hom}_{\Lambda^e}(\mathbb{B}_n, \Lambda) \cong \operatorname{Hom}_k(\Lambda^{\otimes n}, \Lambda)$. Then one way to define $\psi_g : \mathbb{B}_{m+n-1} \longrightarrow \mathbb{B}_m$ is*

$$\psi_g(1 \otimes a_1 \otimes \cdots \otimes a_{m+n-1} \otimes 1) = \sum_{i=1}^{m} (-1)^{(m-1)(i-1)} 1 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes$$
$$g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1} \otimes 1.$$

Notice that $f\psi_g = f \circ g$, where $f \circ g$ is that which was given in Definition 2.10. Suppose that $g \in \operatorname{Hom}_k(\Lambda^{\otimes 2}, \Lambda)$ and using the differentials on the bar resolution given by Equation (2.2)*, we have*

$$\delta\psi_g(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = \delta(1 \otimes g(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes a_1 \otimes g(a_2 \otimes a_3) \otimes 1)$$
$$= g(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes g(a_1 \otimes a_2)a_3 \otimes 1 + 1 \otimes g(a_1 \otimes a_2) \otimes a_3$$
$$- a_1 \otimes g(a_2 \otimes a_3) \otimes 1 + 1 \otimes a_1 g(a_2 \otimes a_3) \otimes 1 - 1 \otimes a_1 \otimes g(a_2 \otimes a_3) \qquad \textit{and}$$
$$\psi_g\delta(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = \psi_g(a_1 \otimes a_2 \otimes a_3 \otimes 1 - 1 \otimes a_1a_2 \otimes a_3 \otimes 1$$
$$+ 1 \otimes a_1 \otimes a_2a_3 \otimes 1 - 1 \otimes a_1 \otimes a_2 \otimes a_3) = a_1 \otimes g(a_2 \otimes a_3) \otimes 1 - 1 \otimes g(a_1a_2 \otimes a_3) \otimes 1$$
$$+ 1 \otimes g(a_1 \otimes a_2a_3) \otimes 1 - 1 \otimes g(a_1 \otimes a_2) \otimes a_3.$$

Therefore $(\delta\psi_g + \psi_g\delta)(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = g(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes a_1 \otimes g(a_2 \otimes a_3)$. On *the other hand,* $(\hat{g}\otimes 1 - 1\otimes \hat{g})\Delta(1\otimes a_1\otimes a_2\otimes a_3\otimes 1)$ *is equal to*

$$\begin{aligned}
&(\hat{g}\otimes 1 - 1\otimes \hat{g})\Big((1\otimes 1)\otimes_\Lambda (1\otimes a_1\otimes a_2\otimes a_3\otimes 1) + (1\otimes a_1\otimes 1)\otimes_\Lambda (1\otimes a_2\otimes a_3\otimes 1)\\
&+ (1\otimes a_1\otimes a_2\otimes 1)\otimes_\Lambda (1\otimes a_3\otimes 1) + (1\otimes a_1\otimes a_2\otimes a_3\otimes 1)\otimes_\Lambda (1\otimes 1)\Big)\\
&= g(a_1\otimes a_2)\otimes a_3\otimes 1 - 1\otimes a_1\otimes g(a_2\otimes a_3).
\end{aligned}$$

So we see that Equation (2.17) *holds in degree 3.*

2.4 Summary remarks on Hochschild cohomology

This section summarizes the fact that for an associative algebra Λ, the Hochschild cohomology is a Gerstenhaber algebra $(\mathrm{HH}^*(\Lambda), \smile, [\ ,\ ])$ with two binary operations. Let $(\mathbb{B}_\bullet, \delta)$ be the bar resolution given in Equation (2.1) with differential given by Equation (2.2). Let $f \in \mathrm{Hom}_k(\Lambda^{\otimes m}, \Lambda)$, $g \in \mathrm{Hom}_k(\Lambda^{\otimes n}, \Lambda)$, and $h \in \mathrm{Hom}_k(\Lambda^{\otimes u}, \Lambda)$. The following are properties of Hochschild cohomology with respect to the cup product and the Gerstenhaber bracket. Their proofs can be found in the literature.

(I). Cup product on Hochschild cohomology is graded commutative:

M. Gerstenhaber in [3, Theorem 3] showed that

$$f\circ \delta^*(g) - \delta^*(f\circ g) + (-1)^{n-1}\delta^*(f)\circ g = (-1)^{n-1}[g\smile f - (-1)^{mn} f\smile g].$$

If f and g are cocycles, then the cup products $(-1)^{mn} f\smile g$ and $g\smile f$ differ by the coboundary $(-1)^n\delta^*(f\circ g)$ i.e.

$$(g\smile f) \sim (-1)^{mn}(f\smile g).$$

The circle product $f\circ g$ is given in Definition 2.10. We have already noted that the cup product inducing the map $\smile\colon \mathrm{HH}^m(\Lambda)\times \mathrm{HH}^n(\Lambda) \to \mathrm{HH}^{(m+n)}(\Lambda)$ has several equivalent definitions some of which were expressed in Equations (2.6) and (2.8).

(II). Hochschild cohomology is a differential graded algebra:

Let us define $\hat{d}(f) := (-1)^m \delta^*_{m+1}(f)$. Notice then that

$$\delta^*_{m+n+1}(f \smile g)(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n+1}) = (f \smile g)\delta_{m+n+1}(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n+1})$$
$$= (f \smile g)\Big[(a_1 a_2 \otimes \cdots \otimes a_{m+1}) \otimes (a_{m+2} \otimes \cdots \otimes a_{m+n+1}) + \cdots +$$
$$(-1)^{m+1}(a_1 \otimes a_2 \otimes \cdots \otimes a_m a_{m+1}) \otimes (a_{m+2} \otimes \cdots \otimes a_{m+n+1}) +$$
$$(a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} a_{m+2} \otimes \cdots \otimes a_{m+n+1}) + \cdots +$$
$$(-1)^{m+n+1}(a_1 \otimes a_2 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_{m+n} a_{m+n+1})\Big]$$

Applying Definition 2.5 of the cup product yields

$$(-1)^{mn}\Big[f(a_1 a_2 \otimes \cdots \otimes a_{m+1}) + \cdots + (-1)^{m+1} f(a_1 \otimes a_2 \otimes \cdots \otimes a_m a_{m+1})\Big] g(a_{m+2} \otimes \cdots \otimes a_{m+n+1})$$
$$+ (-1)^{mn+m} f(a_1 \otimes \cdots \otimes a_m)\Big[(-1)^m g(a_{m+1} a_{m+2} \otimes \cdots \otimes a_{m+n+1}) + \cdots +$$
$$(-1)^{n+1} g(a_{m+1} \otimes \cdots \otimes a_{m+n} a_{m+n+1})\Big]$$
$$= (-1)^{mn}\Big[f\delta_{m+1}(a_1 a_2 \otimes \cdots \otimes a_{m+1}) g(a_{m+2} \otimes \cdots \otimes a_{m+n+1})\Big] +$$
$$(-1)^{mn+m}\Big[f(a_1 \otimes \cdots \otimes a_m) g\delta_{n+1}(a_{m+1} \otimes \cdots \otimes a_{m+n+1})\Big]$$

$$= (-1)^{mn+m}\Big[(-1)^m \delta^*_{m+1} f(a_1 a_2 \otimes \cdots \otimes a_{m+1}) g(a_{m+2} \otimes \cdots \otimes a_{m+n+1})\Big] +$$
$$(-1)^{mn+m+n}\Big[f(a_1 \otimes \cdots \otimes a_m)(-1)^n \delta^*_{n+1} g(a_{m+1} \otimes \cdots \otimes a_{m+n+1})\Big]$$
$$= (-1)^{(m+1)n+m(n+1)}\Big[(-1)^m \delta^*_{m+1}(f) \smile g\Big](a_1 a_2 \otimes \cdots \otimes a_{m+1} \otimes a_{m+2} \otimes \cdots \otimes a_{m+n+1}) +$$
$$(-1)^{m(n+1)+(m+1)n}\Big[(-1)^m f \smile (-1)^{n+1} \delta^*_{n+1} g\Big](a_1 \otimes \cdots \otimes a_m \otimes a_{m+1} \otimes \cdots \otimes a_{m+n+1}).$$

After multiplying by $(-1)^{m+n}$, the constant term $(-1)^{m(n+1)+(m+1)n}$ becomes 1, so we obtain

$$
\begin{aligned}
&(-1)^{m+n}\delta^*_{m+n+1}(f \smile g)(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n+1}) \\
&= \Big[(-1)^m \delta^*_{m+1}(f) \smile g + (-1)^m f \smile (-1)^{n+1}\delta^*_{n+1} g\Big](a_1 a_2 \otimes \cdots \otimes a_{m+1} \otimes a_{m+2} \otimes \cdots \otimes a_{m+n+1}).
\end{aligned}
$$

The above equation implies that when restricted to cochains, Hochschild cohomology is a differential graded algebra with respect to the cup product, i.e.

$$\hat{d}(f \smile g) = (\hat{d}(f)) \smile g + (-1)^m f \smile (\hat{d}(g)). \tag{2.21}$$

(III). Hochschild cohomology is a differential graded Lie algebra:

The Gerstenhaber bracket $[\cdot\, ,\cdot] : \mathrm{HH}^m(\Lambda) \times \mathrm{HH}^n(\Lambda) \to \mathrm{HH}^{m+n-1}(\Lambda)$, is graded anti-symmetric i.e. $[f, g] = (-1)^{(m-1)(n-1)+1}[g, f]$ and satisfies the Jacobi identity i.e.

$$(-1)^{(m-1)(u-1)}[f, [g, h]] + (-1)^{(n-1)(m-1)}[g, [h, f]] + (-1)^{(u-1)(n-1)}[h, [f, g]] = 0$$

M. Gerstenhaber also showed in [3, Theorem 4] that the bracket makes Hochschild cohomology into a differential graded Lie algebra and that

$$\delta^*[f, g] = (-1)^{n-1}[\delta^*(f), g] + [f, \delta^*(g)]. \tag{2.22}$$

A modification of Equation (2.22) as presented in [18, Lemma 1.4.3] is given by $\bar{d}([f, g]) = [\bar{d}(f), g] + (-1)^{(m-1)}[f, \bar{d}(g)]$, where $\bar{d}(f) = (-1)^{(m-1)} f \delta_{m+1}$.

(IV). Hochschild cohomology is a Gerstenhaber algebra:

Definition 2.23. *A Gerstenhaber algebra is a graded commutative algebra with a Lie bracket of* *degree* -1 *satisfying the Poisson identity of Equation* (2.24) *below.*

We have already noted that $(\mathrm{HH}^*(\Lambda), \smile)$ is a graded commutative algebra and $(\mathrm{HH}^*(\Lambda), [\ ,\ ])$ is a graded Lie algebra with the degree of the bracket equal to -1. We can also see from **(III)** and [18, Lemma 1.4.7], that the bracket is anti-symmetric, satisfies the Jacobi identity and the Poisson identity given below:

$$[h, f \smile g] = [h, f] \smile g + (-1)^{m(u-1)} f \smile [h, g]. \tag{2.24}$$

Therefore Hochschild cohomology is a Gerstenhaber algebra.

3. KOSZUL ALGEBRAS DEFINED BY QUIVERS AND RELATIONS

In this chapter, we present relevant definitions of Koszul algebras and quiver algebras. We discuss the construction of a graded free resolution $\mathbb{K}$ of Koszul quiver algebras first introduced by E.L. Green, G. Hartman, E.N. Marcos and Ø. Solberg in [5]. Subsequent sections of this work depend on the preliminary results that are presented here. We start by defining a quiver algebra.

3.1 Quiver algebras

A *quiver* is a directed graph with the allowance of loops and multiple arrows. A quiver Q is sometimes denoted as a quadruple (Q_0, Q_1, o, t) where Q_0 is the set of vertices in Q, Q_1 is the set of arrows in Q, and $o, t : Q_1 \longrightarrow Q_0$ are maps which assign to each arrow $a \in Q_1$, its origin vertex $o(a)$ and terminal vertex $t(a)$ in Q_0. A path in Q is a sequence of arrows $a = a_1 a_2 \cdots a_{n-1} a_n$ such that the terminal vertex of a_i is the same as the origin vertex of a_{i+1}, using the convention of concatenating paths from left to right. The path algebra kQ is defined as the vector space having all paths in Q as a basis. Vertices are regarded as paths of length 0, an arrow is a path of length 1, and so on. We take multiplication on kQ as concatenation of paths. Two paths a and b satisfy $ab = 0$ if $t(a) \neq o(b)$. This multiplication defines an associative algebra over k. By taking kQ_i to be the k-vector subspace of kQ with paths of length i as basis, $kQ = \bigoplus_{i \geq 0} kQ_i$ can be viewed as an $\mathbb{N}$-graded algebra. A relation on a quiver Q is a linear combination of paths from Q each having the same origin vertex and terminal vertex. A quiver together with a set of relations is called a quiver with relations. Let I be an ideal of the path algebra kQ generated by some relations. We denote by (Q, I) the quiver Q with relations I. The quotient $\Lambda = kQ/I$ is called the *quiver algebra* associated with (Q, I).

Example 3.1.

1. *Let Q be the quiver with one vertex and no arrows. We associate to this vertex a trivial path or the idempotent e_1 of length 0. Then $kQ \cong k$, where $e_1 \mapsto 1$.*

2. *Let Q be the quiver with two vertices and an arrow: $1 \xrightarrow{\alpha} 2$. There are two trivial paths e_1*

and e_2 associated with the vertices $1, 2$. The defining property of the algebra kQ is $e_1\alpha = e_1\alpha e_2 = \alpha e_2$. Now define a map $kQ \to \mathbb{M}_2(k)$ given by $e_1 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, where $\mathbb{M}_2(k)$ is the set of 2×2 matrices with coefficients in k. The map defines an isomorphism of algebras $kQ \cong \{A \in \mathbb{M}_2(k) : A_{12} = 0\}$.

3. *Let $Q :=$*
$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[loop right, "c"] \end{tikzcd}$$
be the quiver with a vertex and 3 paths. The defining relations on kQ are the same as the set of all words on $\{a, b, c\}$, with the empty word being e_1. Multiplication is the same as concatenation of words that is, multiplication in the free monoid over $\{a, b, c\}$. The path algebra kQ is therefore isomorphic to the free associative algebra in three non-commuting indeterminates. That is $kQ \cong k\langle x, y, z\rangle$, where $e_1 \mapsto 1$, $a \mapsto x$, $b \mapsto y$ and $c \mapsto z$.

3.2 Koszul algebras

We now present the notion of graded Koszul algebras and their connections to quiver algebras, as presented in the first chapter of [10].

Definition 3.2. *A k-algebra $\mathcal{A}$ is said to be positively graded if*

- $\mathcal{A} = \oplus_{i \geq 0} \mathcal{A}_i$
- $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ *for all i, j,*
- $\mathcal{A}_0 \cong k \times k \times k \times \cdots \times k$, *and*
- $\mathcal{A}_1$ *is a finite dimensional k-vector space.*

If for each i, $\mathcal{A}_i$ is finite dimensional as a k-vector space, then $\mathcal{A}$ is said to be *locally finite*. Every positively graded algebra can be associated with a quiver. To do this, start with a positively graded k-algebra $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ and let $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents of Λ. Now let Q be a quiver whose vertex set Q_0 is in one to one correspondence with the idempotents e_i, that is, label the elements of Q_0 as $\{v_i\}_i$ for some i with $v_i \leftrightarrow e_i$. Furthermore, define the arrows $v_i \xrightarrow{a} v_j$ of Q to correspond with elements $\{a_{ij}\}$ of some basis of $e_i \Lambda_1 e_j$ for each i, j. There is an induced homomorphism of graded k-algebras,

$$\Phi : kQ \longrightarrow \Lambda$$

defined by $\Phi(v_i) = e_i$ and $\Phi(a) = a_{ij}$. The homomorphism Φ is surjective if and only if $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for all i and j. We call kQ the quiver or path algebra associated with the positively graded algebra Λ. Every path of length $i + j$ in kQ can be expressed as a product of paths of length i and length j.

We denote by J the ideal of kQ generated by all arrows. A homogeneous ideal I of kQ is *admissible* if there is an m such that $J^m \subseteq I \subseteq J^2$. An element $\alpha \in I$ is called a *uniform relation* if $\alpha = \sum_{j=1}^{n} \lambda_j w_j$ where each scalar λ_j is nonzero, and for each j, w_j are all of equal length having the same origin vertex and terminal vertex. Such a uniform relation $\alpha = \sum_{j=1}^{n} \lambda_j w_j$ is *minimal* if there is no proper subset $S \subset \{1, 2, \ldots, n\}$ for which $\alpha = \sum_{j \in S} \lambda'_j w_j$ is a uniform relation of the ideal I. Every admissible ideal can be generated by a set of uniform relations [10, page 2]. We can conclude that a k-algebra Λ is called a graded quiver algebra if and only if there exists a finite quiver Q and a homogeneous admissible ideal I of kQ such that $\Lambda \cong kQ/I$. The algebra Λ is quadratic if all the uniform relations of I are homogeneous of degree two.

For a quiver Q, we denote by Q^{op} the quiver obtained by reversing the arrows of the quiver Q. Let kQ^{op} be the path algebra generated from Q^{op} and kQ_2^{op} the vector subspace of kQ^{op} spanned by all paths of length 2. As a vector space, kQ_2^{op} is the dual of kQ_2 such that for any path $\alpha \in kQ_2$, there is an opposite path (i.e. a path with arrows reversed) $\alpha^o \in kQ_2^{op}$. As a result, there is a k-linear pairing on $kQ \times kQ^{op}$ such that for every path $x \in kQ$ and opposite path $\alpha^o \in kQ^{op}$

$$\langle x, \alpha^o \rangle = \begin{cases} 1 & \text{if } x = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For the ideal I, we define $I^\perp$ as the vector subspace of kQ^{op} generated by the set

$$\{\beta^o \in kQ_2^{op} \mid \langle a, \beta^o \rangle = 0,\ a \in I\}.$$

The *quadratic dual* $\Lambda^!$ of Λ is given by $\Lambda^! := kQ^{op}/I^\perp$.

We now present two definitions of a Koszul algebra. Denote by $E(\Lambda) := \operatorname{Ext}^*_\Lambda(\Lambda_0, \Lambda_0)$, the Yoneda algebra of the k-algebra Λ which is a direct sum of the extension groups that is $\operatorname{Ext}^*_\Lambda(\Lambda_0, \Lambda_0) := \bigoplus_{m\geq 0} \operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$. Note that the extension group $\operatorname{Ext}^m_\Lambda(\Lambda_0, \Lambda_0)$ consists of equivalence classes of m extensions of Λ_0 by Λ_0 such as $0 \to \Lambda_0 \to N_m \to N_{m-1} \to \cdots \to N_2 \to N_1 \to \Lambda_0 \to 0$, a long exact sequence of Λ-modules N_i for each i.

Definition 3.3. *Let Λ be a quadratic graded quiver algebra. That is, $\Lambda = kQ/I$, for some finite quiver Q and an admissible ideal I generated by homogeneous elements of degree 2. Then Λ is Koszul if its Yoneda algebra $E(\Lambda)$ is generated in degrees 0 and 1 and isomorphic to the opposite algebra of Λ. That is $E(\Lambda) \cong \Lambda^!$.*

The following is another equivalent definition of a Koszul algebra.

Definition 3.4. *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a quadratic graded quiver algebra. We say Λ is Koszul if the Λ-module Λ_0 has a linear (minimal) graded free resolution*

$$\mathbb{L}_\bullet : \qquad \cdots \to \mathbb{L}_n \xrightarrow{d_n} \mathbb{L}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{L}_1 \xrightarrow{d_1} \mathbb{L}_0 (\to \Lambda_0).$$

The resolution $\mathbb{L}_\bullet$ is minimal in the sense that the differentials have entries in $\Lambda_+ = \bigoplus_{i>0} \Lambda_i$ or equivalently $\operatorname{Im}(d_n) \subseteq \Lambda_+ \mathbb{L}_{n-1}$. It is linear in the sense that the differentials have entries in Λ_1 or equivalently each Λ-module $\mathbb{L}_n$ is generated in degree n.

Remark 3.5.

1. *For Koszul algebras with $M = \Lambda_0 \cong k$, e.g. $\Lambda = k[x_1, x_2 \dots, x_n]$, there is a standard way of constructing the resolution $\mathbb{L}$. The resolution is referred to as the Koszul complex and we refer the reader to [18, Theorem 3.4.6] and the next section for more.*

2. *For Koszul algebras such as $\Lambda = kQ/I$ with $\Lambda_0 \cong k \times k \times \cdots \times k$ (n-copies, the number of vertices of Q), we present an algorithmic way of constructing a (minimal) graded projective resolution of Λ_0 as a right Λ-module as well as a minimal graded projective resolution $\mathbb{K}$ of Λ as a module over the enveloping algebra Λ^e in the next section.*

3.3 Construction of the resolutions $\mathbb{L}_\bullet$ and $\mathbb{K}_\bullet$

Let $\Lambda = kQ/I$, and assume that Λ is Koszul. Then Λ_0 has a graded (minimal) projective resolution. An algorithmic approach to find such a minimal projective resolution $\mathbb{L}_\bullet \to \Lambda_0$ of right Λ-modules was given in [4] and we briefly describe it.

Letting $R = kQ$, it was shown in [5] that there are integers $\{t_n\}_{n\geq 0}$ and uniform elements $\{f_i^n\}_{i=0}^{t_n}$ such that the minimal right projective resolution $\mathbb{L}_\bullet \to \Lambda_0$ of Λ_0, is obtained from a filtration of R. The filtration is given by the following nested family of right ideals:

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_1} f_i^1 R \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R,$$

where for each n, $\mathbb{L}_n = \bigoplus_{i=0}^{t_n} f_i^n R / \bigoplus_{i=0}^{t_n} f_i^n I$ and the differentials d^L on $\mathbb{L}$ are induced by the inclusions $\bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R$. The existence of these inclusions imply that there are elements $h_{ji}^{n-1,n}$ in R such that

$$f_i^n = \sum_{j=0}^{t_{n-1}} f_j^{n-1} h_{ji}^{n-1,n}$$

for all $i = 0, 1, \dots, t_n$ and all $n \geq 1$. The differentials $d_n^L : \mathbb{L}_n \to \mathbb{L}_{n-1}$ are given by

$$d_n^L(f_i^n) = \begin{pmatrix} h_{0i}^{n-1,n} & h_{1i}^{n-1,n} & \cdots & h_{t_{n-1}i}^{n-1,n} \end{pmatrix}$$

for all $n \geq 1$.

In [5], it was shown that with some choice of scalars, the uniform elements $\{f_i^n\}_{i=0}^{t_n}$ satisfy a comultiplicative relation given in (3.6) and this choice in fact makes $\mathbb{L}_\bullet$ minimal. That is, for each positive integer r and $0 \leq i \leq t_n$, there are scalars $c_{pq}(n,i,r)$ such that

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n,i,r) f_p^r f_q^{n-r}. \tag{3.6}$$

For example, we can take $\{f_i^0\}_{i=0}^{t_0}$ to be the set of vertices, $\{f_i^1\}_{i=0}^{t_1}$ to be the set of arrows, $\{f_i^2\}_{i=0}^{t_2}$ to be the set of uniform relations generating the ideal I, and define $\{f_i^n\}(n \geq 3)$ recursively in terms of f_i^{n-1} and f_j^1. As an example, we give a detailed description in Section 4.2 of this comultiplicative structure for a family of quiver algebras introduced in Chapter 4.

The resolution $\mathbb{L}_\bullet$ and the comultiplicative structure (3.6) were used to construct a minimal projective resolution $\mathbb{K}_\bullet \to \Lambda$ of modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$ on which we now define Hochschild cohomology. This minimal projective resolution $\mathbb{K}_\bullet$ of Λ^e-modules associated to $\Lambda = kQ/I$ was given by the following theorem.

Theorem 3.7. *[5, Theorem 2.1] Let $\Lambda = KQ/I$ and assume that Λ is Koszul. Let $\{f_i^n\}_{i=0}^{t_n}$ define a minimal resolution of Λ_0 as a right Λ-module. A minimal projective resolution $(\mathbb{K}, d)$ of Λ over Λ^e is given by*

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n)\Lambda$$

for $n \geq 0$, where the differential $d_n : \mathbb{K}_n \to \mathbb{K}_{n-1}$ applied to $\varepsilon_i^n = (0,\ldots,0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$, $0 \leq i \leq t_n$ with $o(f_i^n) \otimes_k t(f_i^n)$ in the i-th position, is given by

$$d_n(\varepsilon_i^n) = \sum_{j=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} c_{p,j}(n,i,1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{j,q}(n,i,n-1) \varepsilon_j^{n-1} f_q^1 \Big) \tag{3.8}$$

and $d_0 : K_0 \to \Lambda$ is the multiplication map. In particular, Λ is a linear module over Λ^e.

We note that for each n and i, $\{\varepsilon_i^n\}_{i=0}^{t_n}$ is a free basis of $\mathbb{K}_n$ as a Λ^e-module. The scalars $c_{p,j}(n,i,r)$ are those appearing in Equation (3.6) and f_*^1 is taken to be $\overline{f_*^1}$, the residue class of

f_*^1 in $\bigoplus_{i=0}^{t_1} f_i^1 R / \bigoplus_{i=0}^{t_n} f_i^1 I$. Using the comultiplicative structure of Equation (3.6), it was shown in [2] that the cup product on the Hochschild cohomology ring of a Koszul quiver algebra has the following description.

Theorem (See [2], Theorem 2.3)**.** *Let $\Lambda = kQ/I$ and assume that Λ is Koszul. Suppose that Q is a finite quiver and $I \subseteq J^2$, where J is the ideal generated by all paths in Q. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_i^m) = \lambda_i'$ for $i = 0, 1, \ldots, t_m$. Then $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$ can be expressed as*

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, i, n)\lambda_p \lambda_q',$$

for $j = 0, 1, 2, \ldots, t_{n+m}$.

We give the proof of this theorem in Section 4.2 and present a similar formula for the generalized Gerstenhaber bracket on Hochschild cohomology of Koszul algebras defined by quivers and relations in Section 5.1.

3.3.1 The reduced bar resolution of algebras:

We now recall the definition of the reduced bar resolution of an algebra Λ as presented in [2, Section 1]. Let $\mathfrak{a}$ be an ideal of Λ and set $\Lambda_0 = \Lambda/\mathfrak{a}$. Assume that Λ_0 is isomorphic to a finite number of copies of k and that the natural homomorphism $\Lambda \to \Lambda_0$ is a split k-algebra homomorphism. For example, if $\Lambda = kQ/I$ and $I \subseteq J^2$, with J the ideal generated by all paths and $\mathfrak{a} = J/I$, then Λ satisfies these conditions.

Now assume that Λ_0 is isomorphic to m copies of k. In this case, take $\{e_1, e_2, \ldots, e_m\}$ to be a complete set of primitive orthogonal central idempotents of Λ. If Λ_0 is isomorphic to k, then Λ is an algebra over Λ_0. The reduced bar resolution $(\mathcal{B}_\bullet, \delta)$, where $\mathcal{B}_n := \Lambda^{\otimes_{\Lambda_0}(n+2)}$, the $(n+2)$-fold tensor product of Λ over Λ_0, uses the same differential as the usual bar resolution i.e. the differential presented in Equation (2.2). The resolution $\mathbb{K}_\bullet$ can be embedded naturally into the reduced bar resolution $\mathcal{B}_\bullet$. There is a map

$$\iota : \mathbb{K}_\bullet \to \mathcal{B}_\bullet \quad \text{defined by} \quad \iota(\varepsilon_r^n) = 1 \otimes \widetilde{f_r^n} \otimes 1 \tag{3.9}$$

such that $\delta\iota = \iota d$, where

$$\widetilde{f_j^n} = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 \otimes f_{j_2}^1 \otimes \cdots \otimes f_{j_n}^1 \quad \text{if} \quad f_j^n = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 f_{j_2}^1 \cdots f_{j_n}^1 \tag{3.10}$$

for some scalars $c_{j_1 j_2 \cdots j_n}$. See [2, Proposition 2.1] for a proof that ι is indeed an embedding. By taking $\Delta : \mathcal{B}_\bullet \to \mathcal{B}_\bullet \otimes_\Lambda \mathcal{B}_\bullet$ to be the diagonal map of Equation (2.9) on the bar resolution, it was also shown and given explicitly in [2, Proposition 2.2] that there is a diagonal map $\Delta_{\mathbb{K}} : \mathbb{K}_\bullet \to \mathbb{K}_\bullet \otimes_\Lambda \mathbb{K}_\bullet$ defined by

$$\Delta_{\mathbb{K}}(\varepsilon_r^n) = \sum_{v=0}^{n} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n-v}} c_{p,q}(n,r,v) \varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n-v} \tag{3.11}$$

on the complex $\mathbb{K}_\bullet$ compatible with ι. This means that $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota$ where $(\iota \otimes \iota)(\mathbb{K}_\bullet \otimes_\Lambda \mathbb{K}_\bullet) = \iota(\mathbb{K}_\bullet) \otimes_\Lambda \iota(\mathbb{K}_\bullet) \subseteq \mathcal{B}_\bullet \otimes_\Lambda \mathcal{B}_\bullet$.

4. CUP PRODUCT STRUCTURE

In this chapter we introduce and study a family of quiver algebras and present the cup structure on their Hochschild cohomology. We show that for some members of the family, Hochschild cohomology modulo the set of homogeneous nilpotent elements is not finitely generated as an algebra. We do this by providing a cup product formula defined using Definition 2.7. In the last section of this chapter, we provide a proof of the generalized multiplicative structure or cup product structure on Hochschild cohomology of Koszul quiver algebras.

4.1 Introduction

We begin with the following finite quiver

$$Q := \quad \begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

with two vertices and three arrows a, b, c. We denote by e_1 and e_2 the idempotents associated with vertices 1 and 2. Let kQ be the path algebra associated with Q and take for each $q \in k$, $I_q = \langle a^2, b^2, ab - qba, ac \rangle$ to be ideals of kQ. Let

$$\{\Lambda_q = kQ/I_q\}_{q \in k} \tag{4.1}$$

be the family of quiver algebras corresponding to the quiver Q. The resolution $\mathbb{K}_\bullet \to \Lambda_q$ given in Definition 3.7 has free basis elements $\{\varepsilon_i^n\}_{i=0}^{t_n}$ such that for each n,

$$\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0).$$

To concretely define the free basis elements ε_i^n for each module $\mathbb{K}_n$, we need to define f_i^n for each n and i as defined in [5]. The choice of these notations comes from [5]. We start by letting kQ_0 be the subspace of kQ generated by the vertices of Q with basis $\{e_1, e_2\}$. Let $f_0^0 = e_1$ and $f_1^0 = e_2$. Next, set kQ_1 to be the subspace generated by paths of length 1. A free basis of kQ_1 is $\{a, b, c\}$. So define $f_0^1 = a$, $f_1^1 = b$ and $f_2^1 = c$. We let the set $\{f_j^2\}_{j=0}^3$ be the set of paths of length 2 which generates the ideal I_q, that is, $f_0^2 = a^2$, $f_1^2 = ab - qba$, $f_2^2 = b^2$, $f_3^2 = ac$. We continue in this way and define for each $n > 2$,

$$\begin{cases} f_0^n = a^n, \\ f_s^n = f_{s-1}^{n-1} b + (-q)^s f_s^{n-1} a, \quad (0 < s < n), \\ f_n^n = b^n, \\ f_{n+1}^n = a^{(n-1)} c. \end{cases} \tag{4.2}$$

We recall that each f_i^n is a uniform relation therefore the origin vertex $o(f_i^n)$ and the terminal vertex $t(f_i^n)$ exist. Moreover $o(f_r^n) = e_1 = t(f_r^n)$ for $r = 0, 1, \ldots, n$, $o(f_{n+1}^n) = e_1$ and $t(f_{n+1}^n) = e_2$. Therefore the notation $o(f_i^n) \otimes_k t(f_i^n)$ in the definition of ε_i^n makes sense. The differentials on $\mathbb{K}_n$ are given explicitly for this family by

$$\begin{aligned} d_1(\varepsilon_2^1) &= c\varepsilon_1^0 - \varepsilon_0^0 c \\ d_n(\varepsilon_r^n) &= (1 - \partial_{n,r})[a\varepsilon_r^{n-1} + (-1)^{n-r} q^r \varepsilon_r^{n-1} a] \\ &\quad + (1 - \partial_{r,0})[(-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b], \quad \text{for } \ r \le n \\ d_n(\varepsilon_{n+1}^n) &= a\varepsilon_n^{n-1} + (-1)^n \varepsilon_0^{n-1} c, \quad \text{when } \ n \ge 2, \end{aligned} \tag{4.3}$$

where $\partial_{r,s} = 1$ when $r = s$ and 0 when $r \neq s$. It can be verified that the differentials satisfy $d_{n-1} d_n = 0$. For a general proof that the resolution described above, and its general form presented in Equation (3.7) is minimal and that $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$ for all n, see [5, Theorem 2.1].

Recall from Subsection 3.3.1 that the reduced bar resolution $(\mathcal{B}_\bullet, \delta)$ of Λ_q for each q is given by $\mathcal{B}_n = \Lambda_q \otimes_{(\Lambda_q)_0} \Lambda_q \otimes_{(\Lambda_q)_0} \cdots \otimes_{(\Lambda_q)_0} \Lambda_q$, the $(n+2)$-fold tensor product of Λ_q over $(\Lambda_q)_0$, where $(\Lambda_q)_0 = \Lambda_q/\mathfrak{a}$ and is isomorphic to two copies of k in this case. The ideal $\mathfrak{a}$ is isomorphic to J/I where J is the ideal generated by all paths and $I \subseteq J^2$. The differentials on the reduced bar resolution are the same as that on the usual bar resolution given by Equation (2.2). We later give an explicit description of the diagonal map $\Delta_{\mathbb{K}} : \mathbb{K}_\bullet \to \mathbb{K}_\bullet \otimes_{\Lambda_q} \mathbb{K}_\bullet$ on the resolution $\mathbb{K}_\bullet$ in Remark 4.11. We also recall that the resolution $\mathbb{K}_\bullet$ embeds into $\mathcal{B}_\bullet$ via the map $\iota : \mathbb{K}_n \to \mathcal{B}_n$ defined by Equation (3.9). For Λ_q for example, $\iota(\varepsilon_i^2) = 1 \otimes f_i^2 \otimes 1$ where $\widetilde{f_0^2} = f_0^1 \otimes f_0^1 = a \otimes a$, $\widetilde{f_1^2} = f_0^1 \otimes f_1^1 - q f_1^1 \otimes f_0^1 = a \otimes b - qb \otimes a$, $\widetilde{f_2^2} = f_1^1 \otimes f_1^1 = b \otimes b$ and $\widetilde{f_3^2} = f_0^1 \otimes f_2^1 = a \otimes c$. It is clear from Equation (4.2) that the following holds;

$$\begin{cases} \widetilde{f_0^n} = f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1, & n \text{ times}, \\ \widetilde{f_s^n} = \widetilde{f_{s-1}^{n-1}} \otimes f_1^1 + (-q)^s \widetilde{f_s^{n-1}} \otimes f_0^1, & 0 < s < n, \\ \widetilde{f_n^n} = f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1, & n \text{ times}, \\ \widetilde{f_{n+1}^n} = f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1 \otimes f_2^1, & f_0^1 \text{ appearing } (n-1) \text{ times}. \end{cases} \tag{4.4}$$

In case $0 < s < n$, it was shown in [1] that
$f_s^n = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} f_j^t f_{s-j}^{n-t}$, hence,

$$\widetilde{f_s^n} = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} \widetilde{f_j^t} \otimes \widetilde{f_{s-j}^{n-t}}. \tag{4.5}$$

4.2 Cup product structure on Hochschild cohomology of Λ_q.

In order to give an explicit cup product formula, we start with the following:

Remark 4.6. ***(Notation)*** *Throughout, we will use the following notation which is standard. Since the set* $\{\varepsilon_r^n\}_{r=0}^{n+1}$, *forms a basis for* $\mathbb{K}_n$, *for any module homomorphism* $\theta : \mathbb{K}_n \to \Lambda_q$ *taking* ε_i^n *to* λ_i, $i = 0, \ldots, t_n$, *we use the notation* $\theta = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{t_n} \end{pmatrix}$ *to encode this information. Furthermore, if* θ *takes* ε_i^n *to* λ_p, *and every other basis element to* 0, *we write* $\theta =$

$\begin{pmatrix} 0 & \cdots & 0 & (\lambda_p)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$.

We recall that for any member Λ_q of the family, if $\phi \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_m, \Lambda_q)$, and $\eta \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_n, \Lambda_q)$ are two cocycles, then one equivalent definition of the cup product as presented in Definition 2.7 is the composition of the following maps

$$\phi \smile \eta : \mathbb{K}_\bullet \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K}_\bullet \otimes_{\Lambda_q} \mathbb{K}_\bullet \xrightarrow{\phi \otimes \eta} \Lambda_q \otimes_{\Lambda_q} \Lambda_q \overset{\pi}{\simeq} \Lambda_q$$

where $(\phi \otimes \eta)(\varepsilon_i^m \otimes \varepsilon_j^n) = (-1)^{mn}\phi(\varepsilon_r^m) \otimes \eta(\varepsilon_j^n)$, and the comultiplicative map $\Delta_{\mathbb{K}}$ of Equation (3.11) is such that the diagram

$$\begin{array}{ccc} \mathbb{K}_\bullet & \xrightarrow{\Delta_{\mathbb{K}}} & \mathbb{K}_\bullet \otimes_{\Lambda_q} \mathbb{K}_\bullet \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota\otimes\iota} \\ \mathcal{B}_\bullet & \xrightarrow{\Delta} & \mathcal{B}_\bullet \otimes_{\Lambda_q} \mathcal{B}_\bullet. \end{array}$$

is commutative i.e.

$$(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota, \tag{4.7}$$

where Δ is the diagonal map on the bar resolution given in Equation (2.9).

Let $\phi : \mathbb{K}_m \to \Lambda_q$ and $\eta : \mathbb{K}_n \to \Lambda_q$ be two cocycles of homological degrees m and n respectively. Suppose that ϕ takes ε_i^m to ϕ_i^m, $i = 0, \ldots, m+1$ and η takes ε_i^n to η_j^n, $j = 0, \ldots, n+1$. We have from Remark 4.6 that $\phi = \begin{pmatrix} \phi_0^m & \phi_1^m & \cdots & \phi_m^m & \phi_{m+1}^m \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_0^n & \eta_1^n & \eta_2^n & \cdots & \eta_n^n & \eta_{n+1}^n \end{pmatrix}$. We denote the cup product of ϕ and η by

$$\phi \smile \eta := \begin{pmatrix} (\phi\eta)_0^{m+n} & (\phi\eta)_1^{m+n} & (\phi\eta)_2^{m+n} & \cdots & (\phi\eta)_{m+n}^{m+n} & (\phi\eta)_{m+n+1}^{m+n} \end{pmatrix},$$

where $(\phi \smile \eta)(\varepsilon_i^{m+n}) = (\phi\eta)_i^{m+n}$.

Proposition 4.8. *Let $\phi : \mathbb{K}_m \to \Lambda_q$ and $\eta : \mathbb{K}_n \to \Lambda_q$ be two representatives of some classes in $\mathrm{HH}^m(\Lambda_q)$ and $\mathrm{HH}^n(\Lambda_q)$ respectively. The following gives a formula for the cup product $\phi \smile \eta$:*

$\mathbb{K}_{m+n} \to \Lambda_q$ *of* ϕ *and* η.

$$(\phi \smile \eta)(\varepsilon_i^{m+n}) = (\phi\eta)_i^{m+n} = \begin{cases} (-1)^{mn}\phi_0^m\eta_0^n, & when\ i = 0 \\ (-1)^{mn}T_i^{m+n} & when\ 0 < i < m+n \\ (-1)^{mn}\phi_m^m\eta_n^n, & when\ i = m+n \\ (-1)^{mn}\phi_0^m\eta_{n+1}^n, & when\ i = m+n+1 \end{cases} \tag{4.9}$$

$$where \qquad T_i^{m+n} = \sum_{j=max\{0,i-n\}}^{min\{m,i\}} (-q)^{j(n-i+j)}\phi_j^m\eta_{i-j}^n, \qquad 0 < i < m+n.$$

Before we give a proof, we make the following remark:

Remark 4.10. *The result of Proposition 4.8 is very specific to the family of quiver algebras* $\{\Lambda_q\}_{q\in k}$ *under study. Theorem 4.22, which is presented in Section 4.4, is a generalization of this proposition.*

Proof. (of Proposition 4.8). We will find an explicit description of the diagonal map $\Delta_{\mathbb{K}}$ for which Equation (4.7) holds. We will first find the image of the basis elements $\{\varepsilon_r^{m+n}\}_{r=0}^{m+n+1}$ under the diagonal map. We will then use the formula $(\phi \smile \eta)(\varepsilon_r^{m+n}) = \pi(\phi\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^{m+n})$ as the definition of the cup product.

When $r = 0$,

$$\begin{aligned}
&(\iota\otimes\iota)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \Delta\iota(\varepsilon_0^{m+n}) \\
&= \Delta(1\otimes \widetilde{f_0^{m+n}}\otimes 1) = \Delta(1\otimes \overbrace{f_0^1\otimes f_0^1\otimes\cdots\otimes f_0^1}^{m+n\ times}\otimes 1) \\
&= \sum_{s=0}^{m+n}(1\otimes \widetilde{f_0^s}\otimes 1)\otimes(1\otimes \widetilde{f_0^{m+n-s}}\otimes 1) = (\iota\otimes\iota)(\sum_{s=0}^{m+n}\varepsilon_0^s\otimes\varepsilon_0^{m+n-s}).
\end{aligned}$$

We take $\iota(1\otimes\widetilde{f_0^0}\otimes 1) = 1\otimes 1$, so we have $\boxed{\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = (\sum_{s=0}^{m+n}\varepsilon_0^s\otimes\varepsilon_0^{m+n-s}).}$ Since ϕ is a cocycle of degree m, we can evaluate $\phi(\varepsilon_*^m)$ by specializing to the case where the index $s = m$. In a similar way, we evaluate $\eta(\varepsilon_*^n)$ to obtain

$$(\phi \smile \eta)(\varepsilon_0^{m+n}) = \pi(\phi \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \pi(\phi \otimes \eta)(\varepsilon_0^m \otimes \varepsilon_0^n)$$
$$= \pi(-1)^{mn}\phi(\varepsilon_0^m) \otimes \eta(\varepsilon_0^n) = (-1)^{mn}\phi_0^m\eta_0^n.$$

In case $r = m + n$,

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \Delta\iota(\varepsilon_{m+n}^{m+n}) \\
&= \Delta(1 \otimes \widetilde{f_{m+n}^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1}^{m+n\ times} \otimes 1) \\
&= \sum_{s=0}^{m+n} (1 \otimes \widetilde{f_s^s} \otimes 1) \otimes (1 \otimes \widetilde{f_{m+n-s}^{m+n-s}} \otimes 1) = (\iota \otimes \iota)(\sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s}),
\end{aligned}$$

so $\boxed{\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s},}$ and

$$(\phi \smile \eta)(\varepsilon_{m+n}^{m+n}) = \pi(-1)^{mn}\phi(\varepsilon_m^m) \otimes \eta(\varepsilon_n^n) = (-1)^{mn}\phi_m^m\eta_n^n.$$

A similar result holds with $r = m + n + 1$, i.e.

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_{m+n+1}^{m+n}) = \Delta(1 \otimes \widetilde{f_{m+n+1}^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1}^{m+n-1\ times} \otimes f_2^1 \otimes 1) \\
&= \sum_{s=0}^{m+n-1} (1 \otimes \widetilde{f_0^s} \otimes 1) \otimes (1 \otimes \widetilde{f_{m+n-s+1}^{m+n-s}} \otimes 1) + (1 \otimes f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1 \otimes f_2^1 \otimes 1) \otimes (1 \otimes 1) \\
&= (\iota \otimes \iota)(\sum_{s=0}^{m+n-1} \varepsilon_0^s \otimes \varepsilon_{m+n-s+1}^{m+n-s} + \varepsilon_{m+n+1}^{m+n} \otimes \varepsilon_0^0),
\end{aligned}$$

hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon_{m+n+1}^{m+n}) = (\sum_{s=0}^{m+n-1} \varepsilon_0^s \otimes \varepsilon_{m+n-s+1}^{m+n-s}) + \varepsilon_{m+n+1}^{m+n} \otimes \varepsilon_0^0}$. Therefore, when $s = m + n + 1$, we obtain

$$(\phi \smile \eta)(\varepsilon_{m+n+1}^{m+n}) = \pi(-1)^{mn}\phi(\varepsilon_0^m) \otimes \eta(\varepsilon_{n+1}^n) = (-1)^{mn}\phi_0^m\eta_{n+1}^n.$$

It was shown in [1] that for $r = 1, 2, \cdots, n-1$,

$$f_r^n = \sum_{j=max\{0,r+t-n\}}^{min\{t,r\}} (-q)^{j(n-r+j-t)} f_j^t f_{r-j}^{n-t}.$$

Therefore

$$\iota(\varepsilon_r^{m+n}) = 1 \otimes \Big[\sum_{j=max\{0,r+t-m-n\}}^{min\{t,r\}} (-q)^{j(m+n-r+j-t)} \widetilde{f_j^t} \otimes \widetilde{f_{r-j}^{m+n-t}} \Big] \otimes 1,$$

and by letting $t = m$, the above expression equals
$\sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} 1 \otimes \widetilde{f_j^m} \otimes \widetilde{f_{r-j}^n} \otimes 1.$

Applying the comultiplication Δ to the above expression yields

$$\begin{aligned}(\Delta\iota)(\varepsilon_r^{m+n}) &= \sum_{u=-m}^{n} \sum_{j=max\{0,r-n+u\}}^{min\{m+u,r\}} (-q)^{j(n-u-r+j)} (1 \otimes \widetilde{f_j^{m+u}} \otimes 1) \otimes (1 \otimes \widetilde{f_{r-j}^{n-u}} \otimes 1) \\ &= \sum_{u=-m}^{n} \sum_{j=max\{0,r-n+u\}}^{min\{m+u,r\}} (-q)^{j(n-u-r+j)} (\iota \otimes \iota)(\varepsilon_j^{m+u} \otimes \varepsilon_{r-j}^{n-u}).\end{aligned}$$

Using the relation $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota$ we obtain
$\boxed{\Delta_{\mathbb{K}}(\varepsilon_r^{m+n}) = \sum_{u=-m}^{n} \sum_{j=max\{0,r-n+u\}}^{min\{m+u,r\}} (-q)^{j(n-u-r+j)} (\iota \otimes \iota)(\varepsilon_j^{m+u} \otimes \varepsilon_{r-j}^{n-u})}$. Setting $u = 0$ and applying $\pi(\phi \otimes \eta)$ we obtain

$$\begin{aligned}(\phi \smile \eta)(\varepsilon_r^{m+n}) &= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi(\varepsilon_j^m)\eta(\varepsilon_{r-j}^n) \\ &= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi_j^m \eta_{r-j}^n \\ &= (-1)^{mn} T_r^{m+n},\end{aligned}$$

which is the result. $\square$

Remark 4.11. *We can infer from all the boxed equations in the proof Proposition 4.8, which was also given in [12, Proposition 3.7] that*

$$\Delta_{\mathbb{K}}(\varepsilon_s^n) = \begin{cases} \displaystyle\sum_{r=0}^{n} \varepsilon_0^r \otimes \varepsilon_0^{n-r}, & s = 0 \\ \displaystyle\sum_{w=0}^{n} \sum_{j=max\{0,s+w-n\}}^{min\{w,s\}} (-q)^{j(n-s+j-w)} \varepsilon_j^w \otimes \varepsilon_{s-j}^{n-w}, & 0 < s < n \\ \displaystyle\sum_{t=0}^{n} \varepsilon_t^t \otimes \varepsilon_{n-t}^{n-t}, & s = n \\ \Big[\displaystyle\sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_{n-t+1}^{n-t}\Big] + \varepsilon_{n+1}^n \otimes \varepsilon_0^0, & s = n+1 \end{cases}$$

where in the expansion of $\Delta_{\mathbb{K}}(\varepsilon_s^n), 0 < s < n$, the index w is such that there are no repeated terms.

4.3 Hochschild cohomology modulo nilpotents not finitely generated

The theory of support varieties has been well developed for finite groups using group cohomology. Several efforts were made to develop similar theories for finitely generated modules over finite dimensional algebras using Hochschild cohomology. If the characteristic $char(k) \neq 2$, then each homogeneous element of odd degree is nilpotent. Let $\mathcal{N}$ be the ideal generated by homogeneous nilpotent elements of $\mathrm{HH}^*(\Lambda)$. The Hochschild cohomology ring of Λ modulo nilpotents $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is a commutative k-algebra.

Let M, N be two Λ-modules and $\mathrm{Ext}_\Lambda^*(M, N)$ their extension group. There is an action of Hochschild cohomology on the extension group defined in the following way. Let $\mathbb{P}_\bullet \to \Lambda$ be a projective bimodule resolution of Λ. Let $f \in \mathrm{Hom}_{\Lambda^e}(\mathbb{P}_m, \Lambda)$ be a representative of a class in the cohomology group $\mathrm{HH}^m(\Lambda)$. We can also think of f as a representative of an equivalence class of m-extensions of Λ by Λ in $\mathrm{Ext}_{\Lambda^e}^m(\Lambda, \Lambda)$ because of the isomorphism between Hochschild cohomology and the Ext group. Now define a map $\Phi : \mathrm{Ext}_{\Lambda^e}^m(\Lambda, \Lambda) \to \mathrm{Ext}_\Lambda^m(M, M)$ taking the equivalence class $[f]$ to the equivalence class $[f \otimes_\Lambda 1_M]$ which by abuse of notation is written as $\Phi(f) = f \otimes_\Lambda 1_M$. For any representative $g \in \mathrm{Ext}_\Lambda^n(M, N)$, the Yoneda product of $[f \otimes_\Lambda 1_M]$ and

$[g]$ gives an element representing a class in $\mathrm{Ext}_\Lambda^{m+n}(M,N)$. This induces the left action

$$\mathrm{HH}^*(\Lambda) \times \mathrm{Ext}_\Lambda^*(M,N) \to \mathrm{Ext}_\Lambda^*(M,N)$$

defined by taking any pair (f,g) to the Yoneda product of $\Phi(f)$ and g written as $\Phi(f)\cdot g$.

For some finite dimensional algebras, it is well known that the Hochschild cohomology ring modulo nilpotents is finitely generated as an algebra. Furthermore, when M, N are finite-dimensional modules and H a subalgebra of $\mathrm{HH}^*(\Lambda)$, define

$$I_H(M,N) = \{f \in H \mid \Phi(f)\cdot g = 0, \text{ for all } g \in \mathrm{Ext}_\Lambda^*(M,N)\}$$

to be the annihilator of $\mathrm{Ext}_\Lambda^*(M,N)$ in H. $I_H(M,N)$ is obviously an ideal of H. This theory of support varieties is built on the following definition of a variety.

Definition 4.12. *Let M, N be finite-dimensional Λ-modules. The support variety of the pair M, N is*

$$V_H(M,N) = V_H(I_H(M,N)) = \mathit{Max}(H/I_H(M,N)),$$

the maximal ideal spectrum of the quotient ring $H/I_H(M,N)$. The variety of M is defined as $V_H(M) = V_H(M,M)$.

For this theory to have all the nice properties that one would like, (i) H has to be a finitely generated algebra and (ii) $\mathrm{Ext}_\Lambda^*(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ has to be finitely generated as an H-module. This leads to the conjecture in [14] that for finite dimensional algebras, Hochschild cohomology modulo nilpotents is always finitely generated as an algebra. For instance, we can take $H = \mathrm{HH}^{ev}(\Lambda)$ the subalgebra of $\mathrm{HH}^*(\Lambda)$ generated by homogeneous elements of even degrees.

The first counterexample to this conjecture appeared in [19] where over a field of characteristic 2, F. Xu used certain techniques in category theory to construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated. Furthermore, N. Snashall presented F. Xu's counterexample over any field of characteristic 0 in [14]. The example of N. Snashall corresponds to the case $q = 1$ for the family of quiver algebras under study. Using the generalized cup product formula of Proposition 4.8, we now prove that for $q = \pm 1$, $\mathrm{HH}^*(\Lambda_q)/\mathcal{N}$ is not finitely generated. Starting with 0-cocycles, we find solutions to different sets of equations in order to determine nilpotent and non-nilpotent cocycles.

The 0th Hochschild cohomology ($\mathrm{HH}^0(\Lambda) = \frac{\ker d_1^*}{\mathrm{Im}(0)}$).

Let $\phi \in \ker d_1^* \subseteq \hat{\mathbb{K}}_0 = \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_0, \Lambda)$, such that $\phi = (\lambda_0^0 \ \lambda_1^0)$, for some $\lambda_1^0, \lambda_1^0 \in \Lambda$. We solve for the λ_i^0 $(i = 0, 1)$ for which $d_1^*\phi(\varepsilon_i^1) = 0$ as follows

$$d_1^*\phi(\varepsilon_0^1) = \phi d_1(\varepsilon_0^1) = \phi(a(\varepsilon_0^0) + (-1)^1 q^0(\varepsilon_0^0)a) = a\lambda_0^0 - \lambda_0^0 a = 0$$
$$d_1^*\phi(\varepsilon_1^1) = \phi d_1(\varepsilon_1^1) = \phi((-q)^0 b(\varepsilon_0^0) - (\varepsilon_0^0)b) = b\lambda_0^0 - \lambda_0^0 b = 0$$
$$d_1^*\phi(\varepsilon_2^1) = \phi d_1(\varepsilon_2^1) = \phi(c(\varepsilon_1^0) - (\varepsilon_0^0)c) = c\lambda_1^0 - \lambda_0^0 c = 0$$

By taking $q = 1$, we have the relation $ab - ba = 0$. Whenever we take $\phi = (a \ 0)$, $(ab \ 0)$,$(0 \ a)$, $(0 \ b)$, $(e_1 \ e_2)$ or $(0 \ e_1)$, the above equations hold. Note that ϕ is a Λ^e-module homomorphism defined by $\varepsilon_0^0 \mapsto \lambda_0^0$ and $\varepsilon_1^0 \mapsto \lambda_1^0$. The fact that $o(f_i^0)\varepsilon_i^0 t(f_i^0) = \varepsilon_i^0$, $i = 0, 1$ means that we need to identify each solution $(\lambda_0^0 \ \ \lambda_1^0)$ with $(o(f_0^0)\lambda_0^0 t(f_0^0) \ \ o(f_1^0)\lambda_1^0 t(f_1^0)) = (e_1\lambda_0^0 e_1 \ \ e_2\lambda_1^0 e_2)$. We should have $o(\lambda_0^0) = t(\lambda_0^0) = e_1$ and $o(\lambda_1^0) = t(\lambda_1^0) = e_2$. This leads us to eliminate nonsolutions and we are left with $\phi_1 = (a \ 0), \phi_2 = (ab \ 0)$ and $\phi_3 = (e_1 \ e_2)$.

When $q = -1$, we have the relation $ab + ba = 0$, and we get these solutions: $\phi_1 = (a \ 0)$ and $\phi_3 = (e_1 \ e_2)$.

If $q \neq \pm 1$, then $ab - qba = 0$. We get $\phi_2 = (ab \ 0)$ and $\phi_3 = (e_1 \ e_2)$ as solutions. Therefore, the Λ^e-module homomorphisms ϕ_1, ϕ_2, ϕ_3 form a basis for the kernel of d_1^* as a k-vector space. That is,

$$\ker d_1^* = span_k\{\phi_1, \phi_2, \phi_3\}.$$

In summary we obtain for any $q \in k$ that,

$$\mathrm{HH}^0(\Lambda) = \frac{\ker d_1^*}{\mathrm{Im}(0)} = \begin{cases} span_k\{(a\ 0), (ab\ 0), (e_1\ e_2)\}, & \text{if } q = 1 \\ span_k\{(ab\ 0), (e_1\ e_2)\}, & \text{if } q \neq 1. \end{cases}$$

Remark 4.13. *We note that the Hochschild 0-cocycles $\phi_1 = (a\ 0)$ and $\phi_2 = (ab\ 0)$ correspond to elements a and ab respectively. These elements are in the center of the algebra $\Lambda_q, q = 1$ and are nilpotent. We will later prove that they are nilpotent using the cup product formula of Proposition 4.8. The 0-cocycle $\phi_3 = (e_1\ e_2)$ is not nilpotent, since e_1 and e_2 are idempotent elements. It is obvious that ϕ_3 generates $\mathrm{HH}^0(\Lambda_q)/\mathcal{N}$. Now because $e_1 + e_2 = 1$, we make the following deduction for any $q \in k$:*

$$\mathrm{HH}^0(\Lambda)/\mathcal{N} = \frac{\ker d_1^*}{\mathrm{Im}(0)} = span_k\{(e_1\ e_2)\} \cong k. \tag{4.14}$$

Higher Hochschild cocycles

Let $\phi \in ker\ \ d_{n+1}^*$, with $\phi = \begin{pmatrix} \phi_0^n & \phi_1^n & \cdots & \phi_n^n & \phi_{n+1}^n \end{pmatrix}$. The elements $\phi_i^n = \phi(\varepsilon_i^n), i = 0, \cdots, n+1$ are obtained by solving the following set of equations for all n and q. In the first two equations of (4.15), we begin to consider possible values of ϕ_r^n when $r = 0$ and $n+1$. In the last two equations, we consider possible values of ϕ_r^n when $r = 1, 2, \ldots, n$. Notice that in order to capture all these values, we must solve $d_{n+1}^*\phi(\varepsilon_r^{n+1}) = 0$ and $d_{n+1}^*\phi(\varepsilon_{r+1}^{n+1}) = 0$ simultaneously. We now solve:

$$\begin{aligned}
d_{n+1}^*\phi(\varepsilon_0^{n+1}) &= a\phi(\varepsilon_0^n) + (-1)^{n+1}\phi(\varepsilon_0^n)a = a\phi_0^n \pm \phi_0^n a = 0 \\
d_{n+1}^*\phi(\varepsilon_{n+2}^{n+1}) &= a\phi(\varepsilon_{n+1}^n) + (-1)^{n+1}\phi(\varepsilon_0^n)c = a\phi_{n+1}^n \pm \phi_0^n c = 0 \\
d_{n+1}^*\phi(\varepsilon_r^{n+1}) &= a\phi(\varepsilon_r^n) + (-1)^{n+1-r}q^r\phi(\varepsilon_r^n)a + (-q)^{n+1-r}b\phi(\varepsilon_{r-1}^n) + (-1)^{n+1}\phi(\varepsilon_{r-1}^n)b \\
&= a\phi_r^n + (-1)^{n+1-r}q^r\phi_r^n a + (-q)^{n+1-r}b\phi_{r-1}^n + (-1)^{n+1}\phi_{r-1}^n b = 0, \\
d_{n+1}^*\phi(\varepsilon_{r+1}^{n+1}) &= a\phi_{r+1}^n + (-1)^{n-r}q^{r+1}\phi_{r+1}^n a + (-q)^{n-r}b\phi_r^n + (-1)^{n+1}\phi_r^n b = 0.
\end{aligned} \tag{4.15}$$

Now letting n, r be even, we obtain

$$d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - q^r\phi^n_r a + (-q)b\phi^n_{r-1} - \phi^n_{r-1}b = 0,$$
$$d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = a\phi^n_{r+1} + q^{r+1}\phi^n_{r+1}a + (q)^2 b\phi^n_r - \phi^n_r b = 0,$$

and for the specific cases of $q = \pm 1$, we obtain

$$d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - \phi^n_r a \pm b\phi^n_{r-1} - \phi^n_{r-1}b = 0,$$
$$d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = a\phi^n_{r+1} \pm \phi^n_{r+1}a + b\phi^n_r - \phi^n_r b = 0.$$

If we set $\phi^n_r = e_1$ and $\phi^n_{r-1}, \phi^n_{r+1}$ equal to 0, then $\phi = \begin{pmatrix} 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 \end{pmatrix}$ is a solution of Equations (4.15) under these conditions when r is not 0 or $n+1$. As given in Remark 4.6, we will use the notation $\phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$ to specify the position of e_1 when it is obvious that ϕ is an n-cocycle. Table 4.1 shows all the possible solutions of $\phi(\varepsilon^n_i)$, $i = 0, 1, \ldots, n+1$ for different values of q, n and r realized from Equation (4.15).

Remark 4.16. *If* $\phi = \begin{pmatrix} 0 & \cdots & 0 & \phi^n_r & 0 & \cdots & 0 \end{pmatrix}$ *is any solution of Equations* (4.15) *such that* $\phi^n_r \neq e_1$*, then* ϕ *is nilpotent. This is because from Proposition 4.8*

$$(\phi \smile \phi)(\varepsilon^{m+n}_r) = (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi^m_j \phi^n_{r-j} \tag{4.17}$$

where $\phi^m_j \phi^n_{r-j}$ *is a product of any two elements in the set* $\{a, b, ab, c, bc\}$ *which is equal to 0 in the algebra except in few instances. If it is not a zero, we simply take a triple cup product using the*

$q=1$	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	a, b, ab, e_1	ab	a, ab	b, ab
ϕ_{n+1}^n	c, bc		c, bc	

$q=-1$	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	ab, e_1	b, ab	a, b, ab	a, ab
ϕ_{n+1}^n	c, bc		c, bc	

$q\neq\pm1$	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	ab	ab	ab	ab
ϕ_{n+1}^n	c, bc		c, bc	

Table 4.1: Possible values of $\phi(\varepsilon_r^n)$ for different q, n and r.

following;

$$
\begin{aligned}
&(\phi\smile\phi\smile\phi)(\varepsilon_r^{n+n+n})\\
&=(\mu\smile\phi)(\varepsilon_r^{m+n}) \quad (\textit{take } \mu=\phi\smile\phi, m=n+n)\\
&=(-1)^{mn}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}\mu(\varepsilon_j^m)\phi(\varepsilon_{r-j}^n)\\
&=(-1)^{mn}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}[\phi\smile\phi(\varepsilon_j^{n+n})]\phi(\varepsilon_{r-j}^n)\\
&=(-1)^{mn}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}\Big[(-1)^{n^2}\sum_{i=max\{0,l-n\}}^{min\{n,l\}}(-q)^{i(n-l+i)}\phi(\varepsilon_i^n)\phi(\varepsilon_{l-i}^n)\Big]\phi(\varepsilon_{r-j}^n)\\
&=(-1)^{3n^2}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}\sum_{i=max\{0,l-n\}}^{min\{n,l\}}(-q)^{ij(n-r+j)(n-l+i)}\phi(\varepsilon_i^n)\phi(\varepsilon_{l-i}^n)\phi(\varepsilon_{r-j}^n).
\end{aligned}
$$

The product $\phi(\varepsilon_i^n)\phi(\varepsilon_{l-i}^n)\phi(\varepsilon_{r-j}^n)=\phi_i^n\phi_{l-i}^n\phi_{r-j}^n$ is always 0 in Λ_q by the defining relations in I_q.

Therefore a cocycle $\phi : \mathbb{K}_m \to \Lambda$ is non-nilpotent if and only if $\phi_i^m = \phi_{l-i}^m = \phi_{r-j}^m = e_1$ for some i, j, l, r. Accordingly, this is the case if and only if $q = \pm 1, n$ is even and i is even.

We now present the following corollary.

Corollary 4.18. *Let $\phi : \mathbb{K}_n \to \Lambda_q$ be an n-cocycle. Then ϕ is non-nilpotent if, and only if $q = \pm 1, n$ and r are even, $r \neq 0$ and $\phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$.*

Proof. This follows from Remarks 4.16 and the tables of solutions in Table 4.1. □

Let $Z^n(\Lambda_q, \Lambda_q) := \mathrm{HH}^n(\Lambda_q)/\mathcal{N}$, where $\mathrm{HH}^n(\Lambda_q) = \ker(d_{n+1}^*)/\mathrm{Im}(d_{n-1}^*)$. For each n, representatives of classes in $Z^n(\Lambda_q, \Lambda_q)$ are the distinct non-nilpotent elements given by Corollary 4.18. In order to show that each element given by Corollary 4.18 constitutes its own class with respect to modding out by nilpotent elements, we do the following: For a fixed n, let ϕ, β be two distinct $2n$-cocycles such that $\phi(\varepsilon_r^{2n}) = \phi_r^{2n} = e_1$ and $\beta(\varepsilon_s^{2n}) = \beta_s^{2n} = e_1$ with $r < s$ and both r and s are even. Suppose there is an α such that

$$d^*(\alpha) = \phi - \beta = \begin{pmatrix} 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 \end{pmatrix}$$

where the idempotent e_1 is in the r-th and s-th positions. This α does not exist because by considering for example at the position r,

$$e_1 = (\phi - \beta)(\varepsilon_r^{2n}) = d^*(\alpha)(\varepsilon_r^{2n}) = \alpha(d(\varepsilon_r^{2n})), \quad \text{implies that}$$
$$e_1 = a\alpha(\varepsilon_r^{2n-1}) + (-1)^{2n-r}q^r\alpha(\varepsilon_r^{2n-1})a + (-q)^{2n-r}b\alpha(\varepsilon_{r-1}^{2n-1}) + (-1)^{2n}\alpha(\varepsilon_{r-1}^{2n-1})b.$$

There is no way to define $\alpha(\varepsilon_r^{2n-1})$ and $\alpha(\varepsilon_{r-1}^{2n-1})$ so that equality holds in the above expression. Another way to look at this is that if $d^*(\alpha) = \phi - \beta$ for some α, then α has to be a non-nilpotent element of odd homological degree. However, there are no non-nilpotents of odd degree. Therefore there is no such α. Therefore each distinct non-nilpotent n-cocycle constitutes its own class in $Z^n(\Lambda_q, \Lambda_q)$.

We now define a canonical map from $Z^*(\Lambda_q, \Lambda_q) = \bigoplus_{n>0} Z^n(\Lambda_q, \Lambda_q)$ to the polynomial ring in two indeterminates $k[x, y]$. We can recall from Table 4.1 and Corollary 4.18 that ϕ_0^n and ϕ_{n+1}^n are never equal to e_1. We define this map by

$$\begin{pmatrix} 0 & 0 & (e_1)^2 & 0 & \cdots & 0 \end{pmatrix} \mapsto x^{2(n-1)}y^2,$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & (e_1)^4 & 0 & \cdots & 0 \end{pmatrix} \mapsto x^{2(n-2)}y^4,$$
$$\vdots$$
$$\begin{pmatrix} 0 & \cdots & 0 & (e_1)^r & 0 & \cdots & 0 \end{pmatrix} \mapsto x^{2n-r}y^r,$$
$$\vdots$$
$$\begin{pmatrix} 0 & 0 & \cdots & 0 & (e_1)^{2n} & 0 \end{pmatrix} \mapsto y^{2n}$$

This map is well defined as the kernel contains only the zero map. Under this map, the image of $Z^*(\Lambda_q, \Lambda_q)$ is the subalgebra $k[x^2, y^2]y^2$ of $k[x, y]$. Notice $k[x^2, y^2]y^2$ is generated by the set $\{y^2, x^2y^2, y^4, x^4y^2, x^2y^4, y^6, \ldots\}$, hence not finitely generated. Moreso, for each n, $x^{2(n-1)}y^2$ cannot be generated by lower degree elements. Also note how the cup product corresponds with multiplication in $k[x, y]$, that is, given even positive integers r, s, we have

$$\begin{tikzcd}
(0 \ \cdots \ 0 \ (e_1)^r \ 0 \ \cdots \ 0) \smile (0 \ \cdots \ 0 \ (e_1)^s \ 0 \ \cdots \ 0) \arrow[r] \arrow[d, dashed] & (x^{2n-r}y^r) \cdot (x^{2m-s}y^s) \arrow[d, dashed] \\
(0 \ \cdots \ 0 \ (e_1)^{r+s} \ 0 \ \cdots \ 0) \arrow[r] & x^{2(n+m)-(r+s)}y^{r+s}
\end{tikzcd}$$

At each degree n, the element $\begin{pmatrix} 0 & 0 & e_1 & 0 & \cdots & 0 \end{pmatrix}$ identified with $x^{2(n-1)}y^2$ cannot be generated as a cup product of any two elements of lower homological degrees. Since this map is one-to-one, we conclude that $Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$. The next proposition formalizes this idea whereas the next example is an illustration.

Proposition 4.19. *For $q = \pm 1$, $Z^*(\Lambda_q, \Lambda_q)$ is graded with respect to the cup product and is canon-*

ically isomorphic to the subalgebra $k[x^2, y^2]y^2$ of $k[x, y]$. That is $Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$ where the degree of y^2 is 2 and that of x^2y^2 is 4.

Example 4.20. *To show that,*

$$x^2y^2 \cdot y^2 \cong \begin{pmatrix} 0 & 0 & e_1 & 0 & 0 & 0 \end{pmatrix} \smile \begin{pmatrix} 0 & 0 & e_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_1 & 0 & 0 & 0 \end{pmatrix} \cong x^2 \cdot y^4,$$

take $\phi = x^2y^2 \leftrightarrow \begin{pmatrix} \phi_0^4 & \phi_1^4 & \phi_2^4 & \phi_3^4 & \phi_4^4 & \phi_5^4 \end{pmatrix}$ and $\mu = y^2 \leftrightarrow \begin{pmatrix} \mu_0^2 & \mu_1^2 & \mu_2^2 & \mu_3^2 \end{pmatrix}$. Then

$$\begin{aligned}
(\phi \smile \mu)(\varepsilon_0^6) &= \phi_0^4 \mu_0^2 = 0 \\
(\phi \smile \mu)(\varepsilon_1^6) &= \sum_{j=0}^{1} (-1)^{j(1+j)} \phi_j^4 \mu_{1-j}^2 = \phi_0^4 \mu_1^2 + \phi_1^4 \mu_0^2 = 0 \\
(\phi \smile \mu)(\varepsilon_2^6) &= \sum_{j=0}^{2} (-1)^{j^2} \phi_j^4 \mu_{2-j}^2 = \phi_0^4 \mu_2^2 - \phi_1^4 \mu_1^2 + \phi_2^4 \mu_0^2 = 0 \\
(\phi \smile \mu)(\varepsilon_3^6) &= \sum_{j=1}^{3} (-1)^{j(-1+j)} \phi_j^4 \mu_{3-j}^2 = \phi_1^4 \mu_2^2 + \phi_2^4 \mu_1^2 + \phi_3^4 \mu_0^2 = 0 \\
(\phi \smile \mu)(\varepsilon_4^6) &= \sum_{j=2}^{4} (-1)^{j(-2+j)} \phi_j^4 \mu_{4-j}^2 = \phi_2^4 \mu_2^2 - \phi_3^4 \mu_1^2 + \phi_4^4 \phi_0^2 = e_1 \\
(\phi \smile \mu)(\varepsilon_5^6) &= \sum_{j=3}^{4} (-1)^{j(-3+j)} \phi_j^4 \mu_{5-j}^2 = \phi_3^4 \mu_2^2 + \phi_4^4 \mu_1^2 = 0 \\
(\phi \smile \mu)(\varepsilon_6^6) &= \phi_4^4 \mu_4^2 = 0 \\
(\phi \smile \mu)(\varepsilon_7^6) &= \phi_0^4 \mu_3^2 = 0
\end{aligned}$$

As we have previously mentioned, N. Snashall showed in [14] that when $q = 1$, the Hochschild cohomology ring modulo nilpotents for Λ_q is not finitely generated as an algebra. We present the following theorem which realizes this result by showing that Hochschild cohomology modulo nilpotents is not finitely generated when $q = \pm 1$.

Theorem 4.21. *[12, Theorem 3.13] Let k ($char(k) \neq 2$) be a field, $q \in k$ and consider the quiver* *algebra* Λ_q *of* (4.1)*. Let* $\mathcal{N}$ *be the set of nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$*. Then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong Z^0(\Lambda_q, \Lambda_q) \cong k, & \text{if } q \neq \pm 1 \\ Z^0(\Lambda_q, \Lambda_q) \oplus Z^*(\Lambda_q, \Lambda_q) \cong k \oplus k[x^2, y^2]y^2, & \text{if } q = \pm 1 \end{cases}$$

where the degree of y^2 *is 2, and that of* x^2y^2 *is 4.*

Proof. If $q \neq \pm 1$, and $n > 0$, then all $\phi : \mathbb{K}_n \to \Lambda_q$ are nilpotent elements by Remark 4.16 and Table 4.1. From Remark 4.13, we have that

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong Z^0(\Lambda_q, \Lambda_q) \cong k.$$

If $q = \pm 1$, the non-nilpotent elements are described by Corollary 4.18. From Remarks 4.13 and Proposition 4.19 the Hochschild cohomology ring modulo nilpotent elements of the quiver algebra $\Lambda_q, q = \pm 1$ is spanned by graded copies of non-nilpotent cocycles which are in one to one correspondence with $k[x^2, y^2]y^2$. This means that

$$\begin{aligned} &\mathrm{HH}^*(\Lambda_q)/\mathcal{N} \cong Z^0(\Lambda, \Lambda) \oplus Z^*(\Lambda, \Lambda) \\ &\cong k \oplus \Big(\bigoplus_{n>0} span_k \Big\{\phi : \mathbb{K}_{2n} \to \Lambda_q \mid \phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}, r \text{ is even} \Big\}\Big) \\ &= k \oplus k[x^2, y^2]y^2. \end{aligned}$$

□

4.4 Generalized cup product formula in the literature

We now give a general proof of the cup product structure on the Hochschild cohomology ring of a Koszul quiver algebra which was originally presented in [2, Theorem 2.3]. The proof uses the diagonal map of Equation (3.11) and the cup product definition of (2.7) that was presented in Section 2.2.

Theorem 4.22. *Let $\Lambda = kQ/I$ and assume that Λ is Koszul. Suppose that Q is a finite quiver and $I \subseteq J^2$, where J is the ideal generated by all paths. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$ for each m. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in* $\mathrm{HH}^*(\Lambda)$ *and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_i^m) = \lambda_i'$ for $i = 0, 1, \ldots, t_m$. Then $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$ can be expressed as*

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, j, n)\lambda_p \lambda_q',$$

for $j = 0, 1, 2, \ldots, t_{n+m}$, and the scalars $c_{pq}(n+m, j, n)$ are coming from the comultiplicative structure of Equation (3.6).

Proof. We recall Equation (3.11) that

$$\Delta_{\mathbb{K}}(\varepsilon_r^n) = \sum_{v=0}^{n} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n-v}} c_{p,q}(n, r, v)\varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n-v}.$$

Applying the cup product definition given in 2.7, we realize

$$\begin{aligned}
(\eta \smile \theta)(\varepsilon_j^{n+m}) &= \pi(\eta \otimes \theta)\Delta_{\mathbb{K}}(\varepsilon_j^{n+m}) \\
&= \pi(\eta \otimes \theta)\Big(\sum_{v=0}^{n+m} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n+m-v}} c_{p,q}(n+m, j, v)\varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n+m-v}\Big) \\
&= \pi \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{p,q}(n+m, j, n)\eta(\varepsilon_p^n) \otimes_\Lambda \theta(\varepsilon_q^m) \\
&= \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{p,q}(n+m, j, n)\lambda_p \lambda_q^{'}.
\end{aligned}$$

$\square$

5. GERSTENHABER ALGEBRA STRUCTURE

In this chapter, we present a general Gerstenhaber algebra structure on Hochschild cohomology of Koszul algebras defined by quivers and relations. We use the idea of homotopy liftings that were introduced in Subsection 2.3.2. We also present explicit examples of homotopy lifting maps for degree 1 and degree 2 cocycles coming from a family of quiver algebras that was introduced in Chapter 4. We present an application in specifying solutions to the Maurer-Cartan equation in Section 5.3. Our proof uses the minimal (graded) projective resolution $\mathbb{K}$ that has been discussed extensively in Section 3.3. For a quick review of the resolution $\mathbb{K}$, see Theorem 3.7. We recall that this resolution possesses a comultiplicative structure with which we now present the Gerstenhaber algebra structure.

5.1 Generalized Gerstenhaber bracket structure on Koszul algebras

In this section, we present a general way to define homotopy lifting maps on the free basis elements of the resolution $\mathbb{K}$. We show that defining it in certain ways enables us to obtain new scalars and some equations in the field k. These equations draw relationships between the new scalars we obtain and those scalars coming from the comultiplicative structure $\Delta_{\mathbb{K}}$ on the resolution $\mathbb{K}$. In order to handle the differential on the resolution $\mathbb{K}$, we introduce new maps and emphasize special situations in which these maps coincide with the differentials.

The main results are Theorems 5.3, 5.14 and 5.23 where we specify how to define homotopy lifting maps for cocycles taking free basis elements to an idempotent, a path of length 1 and a path of length 2. Under certain conditions, we present a combinatorial Gerstenhaber algebra structure in Theorem 5.24 and a general Gerstenhaber algebra structure in Theorem 5.25. Let us first recall the Definition of a homotopy lifting map (first given as Definition 2.16) with respect to the resolution $\mathbb{K}$ and the comultiplication $\Delta_{\mathbb{K}}$ given by Equation (3.11). The augmentation map $\mathbb{K} \xrightarrow{\mu} \Lambda$ is the same as the multiplication d_0 given in Theorem 3.7.

Definition 5.1. *Let $\theta \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_n, \Lambda)$ be a cocycle. A Λ^e-module homomorphism $\psi_\theta : \mathbb{K}_\bullet \to$*

$\mathbb{K}_\bullet[1-n]$ *is called a homotopy lifting map of* θ *with respect to* $\Delta_{\mathbb{K}}$ *if*

$$\begin{aligned} \boldsymbol{d}(\psi_\theta) &= d\psi_\theta - (-1)^{n-1}\psi_\theta d = (\theta\otimes 1 - 1\otimes\theta)\Delta_{\mathbb{K}} \qquad \textit{and} \\ \mu\psi_\theta &\sim\ (-1)^{n-1}\theta\psi \end{aligned} \tag{5.2}$$

for some $\psi : \mathbb{K}_\bullet \to \mathbb{K}_\bullet[1]$ *for which* $\boldsymbol{d}(\psi) = d\psi - \psi d = (\mu\otimes 1 - 1\otimes\mu)\Delta_{\mathbb{K}}$.

Notation: We recall the standard notation earlier given in Remark 4.6. Since the set $\{\varepsilon_r^n\}_{r=0}^{t_n}$, forms a basis for $\mathbb{K}_n$, for any module homomorphism $\theta : \mathbb{K}_n \to \Lambda_q$ taking ε_i^n to λ_i, $i = 0, 1, \ldots, t_n$, we use the notation $\theta = \begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{t_n}\end{pmatrix}$ to encode this information. Furthermore, if $\bar{\theta}^j$ takes ε_j^n to λ, and every other basis element to 0, we write $\bar{\theta}^j = \begin{pmatrix}0 & \cdots & 0 & (\lambda)^{(j)} & 0 & \cdots & 0\end{pmatrix}$. Notice that we can write $\theta = \sum_{j=1}^{t_n} \bar{\theta}^j$. It is therefore enough to consider maps such as $\bar{\theta}^j$ where λ is an idempotent, a path of length 1, or a path of length 2. We start by considering the case where λ is an idempotent.

In what follows, we use the hypothesis of Theorem 3.11 to show that maps taking basis elements ε_r^m of $\mathbb{K}_m$ to idempotents e_j associated to a vertex j in the quiver Q have certain properties.

Theorem 5.3. *Let* $\Lambda = kQ/I$ *be a quiver algebra that is Koszul. Suppose that* Q *is a finite quiver and* $I \subseteq J^2$. *Denote by* $\{f_r^m\}_{r=0}^{t_m}$ *elements of* kQ *defining a minimal projective resolution of* Λ_0 *as a right* Λ*-module. Let* $\mathbb{K}$ *be the projective bimodule resolution of* Λ *with free basis consisting of all* $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$, *where* $\varepsilon_r^m = (0,\cdots,0, o(f_r^m)\otimes_k t(f_r^m), 0, \cdots, 0)$. *Suppose further that* $\eta : \mathbb{K}_n \to \Lambda$ *is a map such that for some* i, j, $\eta = \begin{pmatrix}0 & \cdots & 0 & (e_j)^{(i)} & 0 & \cdots & 0\end{pmatrix}$, *the following results hold for all* m *and* r.

(i) *If for all* $0 \le p \le t_{m-n}$, $o(f_p^{m-n}) \neq e_j$ *and* $t(f_p^{m-n}) \neq e_j$, *then* $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^m) = 0$.

(ii) *If for all* $0 \le p' \le t_{m-n}$, $o(f_{p'}^{m-n}) = e_j$ *and* $t(f_{p'}^{m-n}) = e_j$, *then* $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^m) = 0$ *holds provided* $c_{i,p'}(m,r,n) = (-1)^{n(m-n)}c_{p',i}(m,r,m-n)$, *where* $c_{*,*}(m,r,*)$ *are the scalars appearing in the comultiplicative relations on* $\mathbb{K}$.

(iii) If for all $0 \le p'' \le t_{m-n}$*,* $o(f^{m-n}_{p''}) = e_j$ *and* $t(f^{m-n}_{p''}) \neq e_j$*, then* $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon^m_r) \neq 0$*.*

If for all $0 \le p'' \le t_{m-n}$*,* $o(f^{m-n}_{p''}) \neq e_j$ *and* $t(f^{m-n}_{p''}) = e_j$*, then* $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon^m_r) \neq 0$*.*

Proof. The comultiplication on the resolution $\mathbb{K}$ is given by $\Delta_{\mathbb{K}}(\varepsilon^m_r) = \sum_{v=0}^{m}\sum_{p=0}^{t_v}\sum_{q=0}^{t_{m-v}} c_{p,q}(m,r,v)\varepsilon^v_p \otimes_\Lambda \varepsilon^{m-v}_q$. The right hand side of Equation (5.2) which is $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon^m_r)$ therefore becomes

$$
\begin{aligned}
&(\eta\otimes 1 - 1\otimes\eta)\sum_{v=0}^{m}\sum_{p=0}^{t_v}\sum_{q=0}^{t_{m-v}} c_{p,q}(m,r,v)\varepsilon^v_p \otimes_\Lambda \varepsilon^{m-v}_q \\
&= \sum_{v=0}^{m}\sum_{p=0}^{t_v}\sum_{q=0}^{t_{m-v}} c_{p,q}(m,r,v)(\eta\otimes 1)(\varepsilon^v_p \otimes_\Lambda \varepsilon^{m-v}_q) \\
&- \sum_{v=0}^{m}\sum_{p=0}^{t_v}\sum_{q=0}^{t_{m-v}} c_{p,q}(m,r,v)(1\otimes \eta)(\varepsilon^v_p \otimes_\Lambda \varepsilon^{m-v}_q).
\end{aligned}
$$

Whenever $v = n, p = i$ in the first summation and $m - v = n, q = i$ in the second summation, the above expression yields

$$
\begin{aligned}
&\sum_{q=0}^{t_{m-n}} c_{i,q}(m,r,n)(\eta\otimes 1)(\varepsilon^n_i \otimes_\Lambda \varepsilon^{m-n}_q) - \sum_{p=0}^{t_{m-n}} c_{p,i}(m,r,m-n)(1\otimes\eta)(\varepsilon^{m-n}_p \otimes_\Lambda \varepsilon^n_i) \\
&= \sum_{q=0}^{t_{m-n}} c_{i,q}(m,r,n)\eta(\varepsilon^n_i)\varepsilon^{m-n}_q - (-1)^{n(m-n)}\sum_{p=0}^{t_{m-n}} c_{p,i}(m,r,m-n)\varepsilon^{m-n}_p\eta(\varepsilon^n_i) \\
&= \sum_{q=0}^{t_{m-n}} c_{i,q}(m,r,n)e_j\varepsilon^{m-n}_q - (-1)^{n(m-n)}\sum_{p=0}^{t_{m-n}} c_{p,i}(m,r,m-n)\varepsilon^{m-n}_p e_j. \qquad (5.4)
\end{aligned}
$$

If for every p, with $0 \le p \le t_{m-n}$, $o(f^{m-n}_p) \neq e_j$ and $t(f^{m-n}_p) \neq e_j$, then the above expression is equal to 0. Now suppose for every p' $(0 \le p' \le t_{m-n})$ we have $o(f^{m-n}_{p'}) = e_j$ and $t(f^{m-n}_{p'}) = e_j$, then $e_j\varepsilon^{m-n}_{p'}$ is equal to

$$
\begin{aligned}
&e_j(0,\cdots,0,o(f^{m-n}_{p'})\otimes_k t(f^{m-n}_{p'}),0,\cdots,0) = (0,\cdots,0,e_j o(f^{m-n}_{p'})\otimes_k t(f^{m-n}_{p'}),0,\cdots,0) \\
&= (0,\cdots,0,e_j^2\otimes_k t(f^{m-n}_{p'}),0,\cdots,0) = (0,\cdots,0,e_j\otimes_k t(f^{m-n}_{p'}),0,\cdots,0) \\
&= (0,\cdots,0,o(f^{m-n}_{p'})\otimes_k t(f^{m-n}_{p'}),0,\cdots,0) = \varepsilon^{m-n}_{p'}
\end{aligned}
$$

and $\varepsilon_{p'}^{m-n}e_j = \varepsilon_{p'}^{m-n}$. The expression $(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^m)$ which is equal to Equation (5.4) therefore becomes

$$\sum_{p'}[c_{i,p'}(m,r,n) - (-1)^{n(m-n)}c_{p',i}(m,r,m-n)]\varepsilon_{p'}^{m-n}.$$

The above expression is 0 if $c_{i,p'}(m,r,n) = (-1)^{n(m-n)}c_{p',i}(m,r,m-n)$ for all such p'. Now suppose for every p'', $o(f_{p''}^{m-n}) = e_j$ and $t(f_{p''}^{m-n}) \neq e_j$, then by similar argument we would have $\varepsilon_{p''}^{m-n}e_j = 0$ and $e_j\varepsilon_{p''}^{m-n} = \varepsilon_{p''}^{m-n}$. The expression $(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^m)$ which is equal to Equation (5.4) therefore becomes

$$\sum_{p''}^{t_{m-n}} c_{i,p''}(m,r,n)\varepsilon_{p''}^{m-n}.$$

Since the scalars $c_{i,p''}(m,r,n)$ are not all zero, $(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^m) \neq 0$. If $o(f_{p''}^{m-n}) \neq e_j$ and $t(f_{p''}^{m-n}) = e_j$, Equation (5.4) becomes $\displaystyle\sum_{p''}^{t_{m-n}} c_{p''i}(m,r,m-n)\varepsilon_{p''}^{m-n}$, so the expression $(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^m) \neq 0$. □

Remark 5.5. *A special case of Theorem 5.3 occurs when $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle and $m = n+1$. In this case, the associated homotopy lifting map ψ_η satisfies $d\psi_\eta - (-1)^{n-1}\psi_\eta d = 0$ if all paths have origin and terminal vertex as e_j. The map η being a cocycle means that $\eta(\varepsilon_i^n) = e_j$ and $\eta(\varepsilon_r^n) = 0$ for any $r \neq i$ and $0 = d^*\eta(\varepsilon_r^{n+1}) = \eta d(\varepsilon_r^{n+1})$ which is equal to*

$$\begin{aligned}
&\eta\sum_{j=0}^{t_n}\Big(\sum_{p=0}^{t_1} c_{p,j}(n+1,r,1)f_p^1\varepsilon_j^n + (-1)^{n+1}\sum_{q=0}^{t_1} c_{j,q}(n+1,r,n)\varepsilon_j^n f_q^1\Big)\\
&= \sum_{p=0}^{t_1} c_{p,i}(n+1,r,1)f_p^1\eta(\varepsilon_i^n) + (-1)^{n+1}\sum_{q=0}^{t_1} c_{i,q}(n+1,r,n)\eta(\varepsilon_i^n)f_q^1\\
&= \sum_{p=0}^{t_1} c_{p,i}(n+1,r,1)f_p^1 e_j + (-1)^{n+1}\sum_{q=0}^{t_1} c_{i,q}(n+1,r,n)e_j f_q^1.
\end{aligned}$$

Since all paths have origin and terminal vertex as e_j, the above expression becomes $\sum_p[c_{p,i}(n+1,r,1) + (-1)^{n+1}c_{i,p}(n+1,r,n)]f_p^1$ and hence $c_{p,i}(n+1,r,1) = (-1)^n c_{i,p}(n+1,r,n)$ for all

$0 \leq p \leq t_1$.

We recall from Definition 2.17 of Subsection 2.3.2 that for Koszul algebras, we can take the degree 0 part of the homotopy lifting map i.e. $(\psi_\eta)_{n-1} : \mathbb{K}_{n-1} \to \mathbb{K}_0$ to be the zero map. From the result of Theorem 5.3, $d(\psi_\eta)_n = (-1)^{n-1}(\psi_\eta)_{n-1}d = 0$, so we see that we can define all homotopy lifting maps ψ_η to be the zero map for all n.

Corollary 5.6. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Assume that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of all $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that for some i, j,*

$$\eta = \begin{pmatrix} 0 & \cdots & 0 & (e_j)^{(i)} & 0 & \cdots & 0 \end{pmatrix}.$$

Then a homotopy lifting map associated to η can be taken to be the zero map.

Proof. Take $m = n + 1$. If for all $0 \leq p \leq t_1, o(f_p^1) \neq e_j$ and $t(f_p^1) \neq e_j$, we obtain $d\psi_\eta - (-1)^{n-1}\psi_\eta d(\varepsilon_r^{n+1}) = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^{n+1}) = 0$ according to Theorem 5.3. We can then set ψ_η to be 0.

If for all $0 \leq p \leq t_1, o(f_p^1) = e_j$ and $t(f_p^1) = e_j$, according to Remark 5.5, some scalars match up, that is $c_{p,i}(n+1, r, 1) = (-1)^n c_{i,p}(n+1, r, n)$. This case also yields $d\psi_\eta - (-1)^{n-1}\psi_\eta d(\varepsilon_r^{n+1}) = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^{n+1}) = 0$ and we can set ψ_η to be 0.

We are now left with the case where $o(f_p^1) = e_j, t(f_p^1) \neq e_j$ and $o(f_p^1) \neq e_j, t(f_p^1) = e_j$. According to Theorem 5.3, these scenarios yield $(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^{n+1}) = \sum_p c_{*,*}(n+1, r, *)\varepsilon_p^1$. The differentials map basis elements ε_r^{n+1} to a linear combination of $f_j^1 \varepsilon_q^n$ and $\varepsilon_q^n f_j^1$ while the homotopy lifting map $\psi_\eta : \mathbb{K}_n \to \mathbb{K}_1$ maps ε_q^n to a k-linear combination of $\varepsilon_p^1 \in \mathbb{K}_1$. The expression $d\psi_\eta - (-1)^{n-1}\psi_\eta d(\varepsilon_r^{n+1})$ therefore yields a linear combination of $f_j^1 \varepsilon_p^1$ and $\varepsilon_p^1 f_j^1$ for all r, j, q, and p. Equation 5.2 is therefore given by

$$\sum_j \sum_p c_{j,p}(n+1, r, n) f_j^1 \varepsilon_p^1 + \sum_j \sum_p c_{p,j}(n+1, r, n)\varepsilon_p^1 f_j^1 = \sum_p c_{*,*}(n+1, r, *)\varepsilon_p^1$$

which is contradictory because the right hand side contains no paths. It must be that there is no p for which $o(f_p^1) = e_j, t(f_p^1) \neq e_j$ and $o(f_p^1) = e_j, t(f_p^1) = e_j$ holds whenever η is a cocycle. So we have the previous two cases. If this is not the case i.e. there are such p, then we must have that $\sum_p c_{*,*}(m,r,*)\varepsilon_p^1 = 0$ and therefore $d\psi_\eta - (-1)^{n-1}\psi_\eta d = 0$. Then set $\psi_\eta = 0$ and we are done.

$\square$

Moving on to the case where a free basis element is mapped to a path of length 1, we start with the following definition.

Definition 5.7. *For each fixed n, r, let $< \ >_{n,r}: \mathbb{K}_{n-1} \to \mathbb{K}_{n-1}$ be a map defined on ε_j^{n-1} for each j by*

$$< \varepsilon_j^{n-1} >_{n,r} = \sum_{v=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} w_{pv}^{(j)}(n,r,1) f_p^1 \varepsilon_v^{n-1} + \sum_{q=0}^{t_1} w_{vq}^{(j)}(n,r,n-1) \varepsilon_v^{n-1} f_q^1 \Big)$$

for scalars $w_{pv}^{(j)}(n,r,1)$ and $w_{vq}^{(j)}(n,r,n-1)$. Then extend to all of $\mathbb{K}_{n-1}$ by requiring it to be a Λ^e-module homomorphism.

Remark 5.8. *Whenever $w_{pv}^{(j)}(n,r,1) = 0 = w_{vq}^{(j)}(n,r,n-1)$ for all $v \neq j$ and $w_{pj}^{(j)}(n,r,1) = c_{pj}(n,r,1)$, $w_{jq}^{(j)}(n,r,n-1) = (-1)^n c_{jq}(n,r,n-1)$, where $c_{*,*}(n,r,*)$ are the scalars appearing in the comultiplicative relations given by Equation* (3.6)*, we obtain a special case of the module homomorphism which is defined as*

$$< \varepsilon_j^{n-1} >_{n,r} = \sum_{p=0}^{t_1} c_{pj}(n,r,1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{jq}(n,r,n-1) \varepsilon_j^{n-1} f_q^1. \tag{5.9}$$

The differential factors through this map i.e. there is a component of the differential map d_n^j, $j = 0, 1, \ldots, t_{n-1}$ taking every basis element ε_r^n to a free basis element ε_j^{n-1} such that in the special case defined above, the following diagram

$$\begin{tikzcd}
\mathbb{K}_n \arrow[r, "d_n^j"] \arrow[dr, "d_n"'] & \mathbb{K}_{n-1} \arrow[d, "< \ >_{n,r}"] \\
& \mathbb{K}_{n-1}
\end{tikzcd}$$

commutes i.e. $d(\varepsilon_r^n) = \sum_{j=0}^{t_{n-1}} < d_n^j(\varepsilon_r^n) >_{n,r} = \sum_{j=0}^{t_{n-1}} < \varepsilon_j^{n-1} >_{n,r}$. *The subscript* (n,r) *in* $< \ >_{n,r}$ *indicates that the scalars* $c_{pj}(n,r,1)$ *and* $(-1)^n c_{jq}(n,r,n-1)$ *are coming from or associated with the basis element* ε_r^n. *For example, in the expansion of* $< \varepsilon_j^{n-1} >_{n,r}$, $< \varepsilon_{j+1}^{n-1} >_{n,r}$ *and* $< \varepsilon_j^{n-1} >_{n,r+1}$, *the associated scalars to* $< \varepsilon_j^{n-1} >_{n,r}$ *will be* $c_{pj}(n,r,1)$ *and* $(-1)^n c_{jq}(n,r,n-1)$ *and they come from the expansion of* $d_n(\varepsilon_r^n)$, *the associated scalars to* $< \varepsilon_{j+1}^{n-1} >_{n,r}$ *will be* $c_{p,j+1}(n,r,1)$ *and* $(-1)^n c_{j+1,q}(n,r,n-1)$ *and they come from the expansion of* $d_n(\varepsilon_r^n)$, *while the associated scalars to* $< \varepsilon_j^{n-1} >_{n,r+1}$ *will be* $c_{pj}(n,r+1,1)$ *and* $(-1)^n c_{jq}(n,r+1,n-1)$ *and they come from the expansion of* $d_n(\varepsilon_{r+1}^n)$. *We immediately see that under these conditions,*

$$d_n(\varepsilon_r^n) = \sum_{j=0}^{t_{n-1}} < \varepsilon_j^{n-1} >_{n,r} = \sum_{j=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} c_{pj}(n,r,1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{jq}(n,r,n-1) \varepsilon_j^{n-1} f_q^1 \Big).$$

Henceforth, we will make use of the special case module homomorphism $< \ >_{n,r}$ *because of its connection to the differentials* d.

We now give series of results that will be a basis for defining homotopy lifting maps for cocycles taking free basis elements to paths of length 1. Suppose that η is an n-cocycle, our goal is to define ψ_η such that Equation (5.2) holds. Now define for all $m \geq 1$ and for all j, $\psi_\eta : \mathbb{K}_{m-1} \to \mathbb{K}_{m-n}$ by $\psi_\eta(\varepsilon_j^{m-1}) = \sum_{r=0}^{t_{m-n}} b_{m-1,j}(m-n,r)\varepsilon_r^{m-n}$, where $b_{m-1,j}(m-n,r)$ are scalars and extend it to all of $\mathbb{K}_{m-1}$ by requiring it to be a Λ^e-module homomorphism. Now consider the special case where for all m and each j, there exists an integer j' depending on j such that $b_{m-1,j}(m-n,r) = 0$ for all $r \neq j'$ that is

$$\psi_\eta(\varepsilon_j^{m-1}) = b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}. \tag{5.10}$$

The next series of results show that this special case is indeed a homotopy lifting map under certain conditions on the scalars $b_{m,r}(*,*)$, $c_{pj}(m,r,*)$, $c_{jq}(m,r,*)$. Furthermore, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{K}_{m-1} \arrow[r, "< >_{m,r}"] \arrow[d, "\psi_\eta"] & \mathbb{K}_{m-1} \arrow[d, "\psi_\eta"] \\
\mathbb{K}_{m-n} \arrow[r, "< >_{m,r}"] & \mathbb{K}_{m-n}
\end{tikzcd}$$

and the equality $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = b_{m-1,j}(m-n,j')<\varepsilon_{j'}^{m-n}>_{m-n+1,r}$ holds if the scalars $c_{pj}(n,r,1), c_{jq}(n,r,n-1)$ of the comultiplicative structure (3.6) satisfy

Equation 5.11 (which we denote by $j \leftrightarrow j'$) below for all p, q, m and r:

$$c_{pj}(m,r,1) = c_{pj'}(m-n+1,r,1) \quad \text{and}$$
$$(-1)^m c_{jq}(m,r,m-1) = (-1)^{m-n+1} c_{j'q}(m-n+1,r,m-n). \tag{5.11}$$

The following Lemma captures these ideas.

Lemma 5.12. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Suppose that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of all $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$, where*

$$\varepsilon_r^m = (0, \cdots, 0, o(f_r^m) \otimes_k t(f_r^m), 0, \cdots, 0).$$

Suppose $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle defined by $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$. The special Λ^e-module map defined by $\psi_\eta(\varepsilon_j^{m-1}) = b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}$ for all j satisfies $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) =<\psi_\eta(\varepsilon_j^{m-1})>_{m,r}$ for all m, r. Furthermore, the last equation implies that

$$\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = b_{m-1,j}(m-n,j')<\varepsilon_{j'}^{m-n}>_{m,r}= b_{m-1,j}(m-n,j')<\varepsilon_{j'}^{m-n}>_{m-n+1,r}$$

provided Equation (5.11) *holds.*

Proof. We first observe that if $\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1,s)\varepsilon_s^{m-n+1}$, then for $0 \le w \le t_1$, $f_w^1\psi_\eta(\varepsilon_r^m) = f_w^1 b_{m,r}(m-n+1,s)\varepsilon_s^{m-n+1}$. This is the same as $b_{m,r}(m-n+1,s)f_w^1\varepsilon_s^{m-n+1} = \psi_\eta(f_w^1\varepsilon_s^{m-n+1})$ since ψ_η is a Λ^e-module homomorphism. $\psi_\eta(\varepsilon_r^m)f_w^1 = \psi_\eta(\varepsilon_r^m f_w^1)$ holds similarly.

Taking $\psi_\eta(\varepsilon_j^{m-1}) = b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}$, we will have

$$\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = \psi_\eta\Big(\sum_{p=0}^{t_1} c_{pj}(m,r,1)f_p^1\varepsilon_j^{m-1} + (-1)^m\sum_{q=0}^{t_1} c_{jq}(m,r,m-1)\varepsilon_j^{m-1}f_q^1\Big)$$
$$= \sum_{p=0}^{t_1} c_{pj}(m,r,1)f_p^1\psi_\eta(\varepsilon_j^{m-1}) + (-1)^m\sum_{q=0}^{t_1} c_{jq}(m,r,m-1)\psi_\eta(\varepsilon_j^{m-1})f_q^1$$
$$= \sum_{p=0}^{t_1} c_{pj}(m,r,1)b_{m-1,j}(m-n,j')f_p^1\varepsilon_{j'}^{m-n}$$
$$+ (-1)^m\sum_{q=0}^{t_1} c_{jq}(m,r,m-1)b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}f_q^1.$$

On the other hand

$$<\psi_\eta(\varepsilon_j^{m-1})>_{m,r} =< b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}>_{m,r}$$
$$= \sum_{p=0}^{t_1} c_{pj}(m,r,1)b_{m-1,j}(m-n,j')f_p^1\varepsilon_{j'}^{m-n}$$
$$+ (-1)^m\sum_{q=0}^{t_1} c_{jq}(m,r,m-1)b_{m-1,j}(m-n,j')\varepsilon_{j'}^{m-n}f_q^1.$$

It is now established that $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) =< \psi_\eta(\varepsilon_j^{m-1})>_{m,r}$. Also, you can factor out the scalars $b_{m-1,j}(m-n,j')$ from the last expansion so that $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = b_{m-1,j}(m-n,j') <\varepsilon_{j'}^{m-n}>_{m,r}$. From the expansion of $(<\varepsilon_{j'}^{m-n}>_{m-n+1,r})$ the equality $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = b_{m-1,j}(m-n,j') <\varepsilon_{j'}^{m-n}>_{m,r}= b_{m-1,j}(m-n,j') <\varepsilon_{j'}^{m-n}>_{m-n+1,r}$ holds provided that Equation (5.11) is satisfied: that is whenever $c_{pj}(m,r,1) = c_{pj'}(m-n+1,r,1)$ and $(-1)^m c_{jq}(m,r,m-1) = (-1)^{m-n+1}c_{j'q}(m-n+1,r,m-n)$. In addition, since ψ_η maps basis elements $\{\varepsilon_r^{m-1}\}_{r=0}^{t_{m-1}}$ of $\mathbb{K}_{m-1}$ to basis elements $\{\varepsilon_r^{m-n}\}_{r=0}^{t_{m-n}}$ of $\mathbb{K}_{m-n}$, over a sum, we have the following expression

$$\sum_{j=0}^{t_{m-1}}\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) = \sum_{j=0}^{t_{m-1}} <\psi_\eta(\varepsilon_j^{m-1})>_{m,r} = \sum_{j'=0}^{t_{m-n}} b_{m-1,j}(m-n,j') <\varepsilon_{j'}^{m-n}>_{m-n+1,r}\ . \tag{5.13}$$

$\square$

We now give an important result stating conditions on the scalars $b_{m,r}(*,*)$, $c_{pj}(m,r,*)$ and $c_{jq}(m,r,*)$ for which the map of Equation (5.10) is a homotopy lifting map for the cocycle η.

Theorem 5.14. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Suppose that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of all $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ for some $0 \leq w \leq t_1$, and Equation* (5.11) *holds for all j with $j \leftrightarrow j'$. For all m, r and some s depending on r, assume there are scalars $b_{m,r}(m-n+1, s)$ such that*

(i). $B = c_{i,j'}(m,r,n)$ when $p = w$, $B = 0$ when $p \neq w$, and

(ii). $B' = -(-1)^{n(m-n)}c_{j',i}(m,r,m-n)$ when $p = w$, $B' = 0$ when $p \neq w$

where $B = b_{m,r}(m-n+1,s)c_{pj'}(m-n+1,s,1) + (-1)^n b_{m-1,j}(m-n,j')c_{pj'}(m-n+1,r,1)$, and $B' = (-1)^{m+1}[(-1)^n b_{m,r}(m-n+1,s)c_{j'p}(m-n+1,s,m-n)$ $+ b_{m-1,j}(m-n,j')c_{j'p}(m-n+1,r,m-n)]$. Then a homotopy lifting map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ associated to η can be defined by

$$\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1,s)\varepsilon_s^{m-n+1}.$$

Proof. We have to show that under the stated conditions (i) and (ii), the equation

$$(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^m)$$

holds. We have from the left hand side that

$$
\begin{aligned}
&(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = d\psi_\eta(\varepsilon_r^m) - (-1)^{n-1}\psi_\eta d(\varepsilon_r^m) \\
&= d(b_{m,r}(m-n+1,s)\varepsilon_s^{m-n+1}) - (-1)^{n-1}\psi_\eta(\sum_{j=0}^{t_{m-1}} < \varepsilon_j^{m-1} >_{m,r}) \\
&= b_{m,r}(m-n+1,s)d(\varepsilon_s^{m-n+1}) - (-1)^{n-1}\sum_{j=0}^{t_{m-1}} \psi_\eta(< \varepsilon_j^{m-1} >_{m,r}).
\end{aligned}
$$

Using Equation (5.13) of Lemma 5.12, and applying the definition of the differential, we get

$$
b_{m,r}(m-n+1,s)\sum_{\alpha=0}^{t_{m-n}} < \varepsilon_\alpha^{m-n} >_{m-n+1,s} -(-1)^{n-1}\sum_{j'=0}^{t_{m-n}} b_{m-1,j}(m-n,j') < \varepsilon_{j'}^{m-n} >_{m-n+1,r},
$$

then applying the definition of the map $< \cdot >_{*,*}$, the last expression equals

$$
\begin{aligned}
&\sum_{\alpha=0}^{t_{m-n}}\sum_{p=0}^{t_1} b_{m,r}(m-n+1,s)c_{p\alpha}(m-n+1,s,1)f_p^1\varepsilon_\alpha^{m-n} \\
&+ (-1)^{m-n+1}\sum_{\alpha=0}^{t_{m-n}}\sum_{q=0}^{t_1} b_{m,r}(m-n+1,s)c_{\alpha q}(m-n+1,s,m-n)\varepsilon_\alpha^{m-n}f_q^1 \\
&+ (-1)^n\sum_{j'=0}^{t_{m-n}}\sum_{p=0}^{t_1} b_{m-1,j}(m-n,j')c_{pj'}(m-n+1,r,1)f_p^1\varepsilon_{j'}^{m-n} \\
&+ (-1)^{m+1}\sum_{j'=0}^{t_{m-n}}\sum_{q=0}^{t_1} b_{m-1,j}(m-n,j')c_{j'q}(m-n+1,r,m-n)\varepsilon_{j'}^{m-n}f_q^1.
\end{aligned}
$$

Collecting like terms and re-indexing $\alpha = j'$, we get

$$= \sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[b_{m,r}(m-n+1,s) c_{p\alpha}(m-n+1,s,1)$$

$$+ (-1)^n b_{m-1,j}(m-n,\alpha) c_{p\alpha}(m-n+1,r,1) \Big] f_p^1 \varepsilon_\alpha^{m-n}$$

$$+ (-1)^{m+1} \sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[(-1)^n b_{m,r}(m-n+1,s) c_{\alpha p}(m-n+1,s,m-n)$$

$$+ b_{m-1,j}(m-n,\alpha) c_{\alpha p}(m-n+1,r,m-n) \Big] \varepsilon_\alpha^{m-n} f_p^1$$

which is succinctly expressed as

$$= \sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[B \Big] f_p^1 \varepsilon_\alpha^{m-n} + \sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[B' \Big] \varepsilon_\alpha^{m-n} f_p^1.$$

After applying the definitions of B and B' given by (i) and (ii) of the theorem, that is substitute $B = c_{i,\alpha}(m,r,n)$, $B' = -(-1)^{n(m-n)} c_{\alpha,i}(m,r,m-n)$ when $p = w$ and 0 otherwise, we get

$$= \sum_{\alpha=0}^{t_{m-n}} c_{i,\alpha}(m,r,n) f_w^1 \varepsilon_\alpha^{m-n} - (-1)^{n(m-n)} \sum_{\alpha=0}^{t_{m-n}} c_{\alpha,i}(m,r,m-n) \varepsilon_\alpha^{m-n} f_w^1.$$

On the other hand, the comultiplication on the resolution $\mathbb{K}$ is given by $\Delta_{\mathbb{K}}(\varepsilon_r^m) = \sum_{v=0}^{m} \sum_{x=0}^{t_v} \sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v) \varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v}$. Applying $(\eta \otimes 1 - 1 \otimes \eta)$, we obtain

$$(\eta \otimes 1 - 1 \otimes \eta) \Delta_{\mathbb{K}}(\varepsilon_r^m) = (\eta \otimes 1 - 1 \otimes \eta) \sum_{v=0}^{m} \sum_{x=0}^{t_v} \sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v) \varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v}$$

$$= \sum_{v=0}^{m} \sum_{x=0}^{t_v} \sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v) (\eta \otimes 1)(\varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v})$$

$$- \sum_{v=0}^{m} \sum_{x=0}^{t_v} \sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v) (1 \otimes \eta)(\varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v}).$$

Whenever $v = n, x = i$ in the first summation and $m - v = n, y = i$ in the second summation, and changing the indices x, y to α later on, and using Koszul signs convention in the expansion of $(1 \otimes \eta)(\varepsilon_x^{m-n} \otimes \varepsilon_y^n)$ to obtain $(-1)^{|\eta|(m-n)}\varepsilon_x^{m-n} \cdot \eta(\varepsilon_y^n)$, where $|\eta| = n$ is the degree of η, the above expression yields

$$\begin{aligned}
&\sum_{y=0}^{t_{m-n}} c_{i,y}(m,r,n)(\eta \otimes 1)(\varepsilon_i^n \otimes_\Lambda \varepsilon_y^{m-n}) - \sum_{x=0}^{t_{m-n}} c_{x,i}(m,r,m-n)(1 \otimes \eta)(\varepsilon_x^{m-n} \otimes_\Lambda \varepsilon_i^n) \\
&= \sum_{y=0}^{t_{m-n}} c_{i,x}(m,r,n)\eta(\varepsilon_i^n)\varepsilon_y^{m-n} - (-1)^{n(m-n)} \sum_{x=0}^{t_{m-n}} c_{x,i}(m,r,m-n)\varepsilon_x^{m-n}\eta(\varepsilon_i^n) \\
&= \sum_{\alpha=0}^{t_{m-n}} c_{i,\alpha}(m,r,n)f_w^1\varepsilon_\alpha^{m-n} - (-1)^{n(m-n)} \sum_{\alpha=0}^{t_{m-n}} c_{\alpha,i}(m,r,m-n)\varepsilon_\alpha^{m-n}f_w^1 \\
&= (d\psi_\eta - \psi_\eta d)(\varepsilon_r^n).
\end{aligned}$$

□

We will next consider the case where free basis elements of $\mathbb{K}_m$ are mapped to paths of length 2 by an n-cocycle. We start with the following definition.

Definition 5.15. *Let f_w^1, f_{w+1}^1 be paths of length 1 in $\Lambda = kQ/I$. For a fixed n and for all $m \geq 1$, define a map $\psi : \mathbb{K}_{m-1} \to \mathbb{K}_{m-n}$ by*

$$\psi(\varepsilon_j^{m-1}) = \sum_{v=0}^{t_{m-n}} b_{m-1,j}(m-n,v)f_w^1\varepsilon_v^{m-n} + \sum_{v'=0}^{t_{m-n}} b_{m-1,j}(m-n,v')\varepsilon_{v'}^{m-n}f_{w+1}^1,$$

for all j where $b_{m-1,j}(m-n,*)$ are scalars and extend it to all of $\mathbb{K}_{m-1}$ as a Λ^e-module homomorphism.

Remark 5.16. *For a cocycle $\eta : \mathbb{K}_n \to \Lambda$, Y. Volkov showed that there are homotopy lifting maps $\psi_\eta : \mathbb{K} \to \mathbb{K}[1-n]$, as presented in Definition 2.17. Let the cocycle η be defined by $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1 f_{w+1}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$. Now consider a special case of the module homomorphism ψ where for all m and each j, there exist integers j' and j'' depending on j such that $b_{m-1,j}(m-n+1,v) = 0$ for all $v \neq j'$ and $b_{m-1,j}(m-n,v') = 0$ for all $v' \neq j''$. We obtain the* *following special case:*

$$\psi_\eta(\varepsilon_j^{m-1}) = b_{m-1,j}(m-n,j')f_w^1\varepsilon_{j'}^{m-n+1} + b_{m-1,j}(m-n,j'')\varepsilon_{j''}^{m-n}f_{w+1}^1. \tag{5.17}$$

We will show that under certain conditions on the scalars $b_{m-1,j}(m-n,*)$ *and the scalars* $c_{pj}(m,r,*)$*, this special case is a homotopy lifting map for* η*. We do not assume that* $j' = j''$ *all the time. However, in Section 5.2 where we give examples, we note that the indices* j' *and* j'' *are consecutive, i.e.* $j' = j'' + 1$.

Before presenting another major theorem (Theorem 5.23), we present two lemmas. Since we will be expanding $\psi_\eta(<\varepsilon_r^m>_{m,r})$, Lemma 5.18 gives information on this expansion much like Lemma 5.12 and Lemma 5.20 helps to give a succinct way to express the sum $d\psi_\eta(\varepsilon_r^m)+\psi_\eta d(\varepsilon_r^m)$. We say that Equation (5.11) holds with $j \leftrightarrow j''$ by changing all j' to j'' in Equation (5.11).

Lemma 5.18. *Let* $\Lambda = kQ/I$ *be a quiver algebra that is Koszul. Suppose that* Q *is a finite quiver and* $I \subseteq J^2$*. Denote by* $\{f_r^m\}_{r=0}^{t_m}$ *elements of* kQ *defining a minimal projective resolution of* Λ_0 *as a right* Λ*-module. Let* $\mathbb{K}$ *be the projective bimodule resolution of* Λ *with free basis consisting of* $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$*. Suppose* $\eta : \mathbb{K}_n \to \Lambda$ *is a cocycle defined by* $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1f_{w+1}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ *and Equation* (5.11) *holds with* $j \leftrightarrow j'$ *and* $j \leftrightarrow j''$*. The module homomorphism defined by*

$$\psi_\eta(\varepsilon_j^{m-1}) = b_{m,j}(m-n,j')f_w^1\varepsilon_{j'}^{m-n} + b_{m,j}(m-n,j'')\varepsilon_{j''}^{m-n}f_{w+1}^1$$

for all j *satisfies* $\psi_\eta(<\varepsilon_j^{m-1}>_{m,r}) =<\psi_\eta(\varepsilon_j^{m-1})>_{m,r}$ *for all* m *and* r *and this equation implies that*

$$\begin{aligned}<\psi_\eta(\varepsilon_j^{m-1})>_{m,r} &= b_{m-1,j}(m-n,j') <f_w^1\varepsilon_{j'}^{m-n}>_{m-n+1,r} \\ &\quad + b_{m-1,j}(m-n,j'') <\varepsilon_{j''}^{m-n}f_{w+1}^1>_{m-n+1,r}\ .\end{aligned}$$

Proof. We use the fact that ψ_η and $<\ \cdot\ >_{m,r}$ are Λ^e-module homomorphisms. Using $\psi_\eta(\varepsilon_j^{m-1}) =$

$b_{m-1,j}(m-n,j')f_w^1\varepsilon_{j'}^{m-n}+b_{m-1,j}(m-n,j'')\varepsilon_{j''}^{m-n}f_{w+1}^1$, we have

$$\begin{aligned}
&\psi_\eta(<\varepsilon_j^{m-1}>_{m,r})\\
&=\psi_\eta\Big(\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1\varepsilon_j^{m-1}+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)\varepsilon_j^{m-1}f_q^1\Big)\\
&=\sum_{p=0}^{t_1}c_{pj}(m,r,1)\psi_\eta(f_p^1\varepsilon_j^{m-1})+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)\psi_\eta(\varepsilon_j^{m-1}f_q^1)\\
&=\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1\psi_\eta(\varepsilon_j^{m-1})+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)\psi_\eta(\varepsilon_j^{m-1})f_q^1=<\psi_\eta(\varepsilon_j^{m-1})>_{m,r}
\end{aligned}$$

$$\begin{aligned}
&=\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1\Big[b_{m-1,j}(m-n,j')f_w^1\varepsilon_{j'}^{m-n}+b_{m-1,j}(m-n,j'')\varepsilon_{j''}^{m-n}f_{w+1}^1\Big]\\
&+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)\Big[b_{m-1,j}(m-n,j')f_w^1\varepsilon_{j'}^{m-n}+b_{m-1,j}(m-n,j'')\varepsilon_{j''}^{m-n}f_{w+1}^1\Big]f_q^1
\end{aligned}$$

$$\begin{aligned}
&=b_{m-1,j}(m-n,j')\Big[\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1(f_w^1\varepsilon_{j'}^{m-n})+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)(f_w^1\varepsilon_{j'}^{m-n})f_q^1\Big]\\
&+b_{m-1,j}(m-n,j'')\Big[\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1(\varepsilon_{j''}^{m-n}f_{w+1}^1)+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)(\varepsilon_{j''}^{m-n}f_{w+1}^1)f_q^1\Big].
\end{aligned}$$

We now recall that the first equality of Equation (5.11) matching $j\leftrightarrow j''$ implies that

$$\begin{aligned}
&<\varepsilon_{j''}^{m-n}f_{w+1}^1>_{m-n+1,r}\\
&=\sum_{p=0}^{t_1}c_{pj''}(m-n+1,r,1)f_p^1(\varepsilon_{j''}^{m-n}f_{w+1}^1)\\
&+(-1)^{m-n+1}\sum_{q=0}^{t_1}c_{j''q}(m-n+1,r,m-n)(\varepsilon_{j''}^{m-n}f_{w+1}^1)f_q^1\\
&=\sum_{p=0}^{t_1}c_{pj}(m,r,1)f_p^1(\varepsilon_{j''}^{m-n}f_{w+1}^1)+(-1)^m\sum_{q=0}^{t_1}c_{jq}(m,r,m-1)(\varepsilon_{j''}^{m-n}f_{w+1}^1)f_q^1,
\end{aligned}$$

and the second equality of Equation (5.11) matching $j \leftrightarrow j'$ implies that

$$
\begin{aligned}
&< f_w^1 \varepsilon_{j'}^{m-n} >_{m-n+1,r} \\
&= \sum_{p=0}^{t_1} c_{p,j'}(m-n+1,r,1) f_p^1 (f_w^1 \varepsilon_{j'}^{m-n}) \\
&+ (-1)^{m-n+1} \sum_{q=0}^{t_1} c_{j',q}(m-n+1,r,m-n)(f_w^1 \varepsilon_{j'}^{m-n}) f_q^1 \\
&= \sum_{p=0}^{t_1} c_{pj}(m,r,1) f_p^1 (f_w^1 \varepsilon_{j'}^{m-n}) + (-1)^m \sum_{q=0}^{t_1} c_{jq}(m,r,m-1)(f_w^1 \varepsilon_{j'}^{m-n}) f_q^1 .
\end{aligned}
$$

Putting all these together, we get the desired result:

$$
\begin{aligned}
&\psi_\eta(< \varepsilon_j^{m-1} >_{m,r}) =< \psi_\eta(\varepsilon_j^{m-1}) >_{m,r} \\
&=< b_{m-1,j}(m-n,j') f_w^1 \varepsilon_{j'}^{m-n} + b_{m-1,j}(m-n,j'') \varepsilon_{j''}^{m-n} f_{w+1}^1 >_{m-n+1,r} \\
&= b_{m-1,j}(m-n,j') < f_w^1 \varepsilon_{j'}^{m-n} >_{m-n+1,r} + b_{m-1,j}(m-n,j'') < \varepsilon_{j''}^{m-n} f_{w+1}^1 >_{m-n+1,r} .
\end{aligned}
$$

$\square$

The differentials map free basis elements ε_r^m to a linear combination of $f_*^1 \varepsilon_{**}^{m-1}$ and $\varepsilon_{**}^{m-1} f_*^1$ and the map ψ of Definition 5.15 maps free basis elements ε_*^{m-1} to linear combination of $f_{**}^1 \varepsilon_{***}^{m-n}$ and $\varepsilon_{***}^{m-n} f_{**}^1$. Combining these two maps means ψd and $d\psi$ will map free basis elements ε_r^m to a linear combination of $f_*^1 f_{**}^1 \varepsilon_{***}^{m-n}$,$f_*^1 \varepsilon_{***}^{m-n} f_{**}^1$, $f_{**}^1 \varepsilon_{***}^{m-n} f_*^1$ and $\varepsilon_{***}^{m-n} f_{**}^1 f_*^1$. The map $J : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ given in the next definition describes all the possible k-linear combinations there are and the next lemma shows that after suitable substitution of certain scalars, $J = d\psi_\eta - (-1)^{n-1}\psi_\eta d$, where ψ_η is the special case map given by Equation (5.17).

Definition 5.19. *For $0 \le \nu \le t_{m-n}$, define a map $J : \mathbb{K}_m \to \mathbb{K}_{m-n}$ on the basis elements ε_r^m of $\mathbb{K}_m$ by*

$$
J(\varepsilon_r^m) = \sum_{\nu=0}^{t_{m-n}} \sum_{i=0}^{t_1} \sum_{j=0}^{t_1} \left[\sigma_{m,r}(i,j,\nu) f_i^1 f_j^1 \varepsilon_\nu^{m-n} + \sigma_{m,r}(i,\nu,j) f_i^1 \varepsilon_\nu^{m-n} f_j^1 + \sigma_{m,r}(\nu,i,j) \varepsilon_\nu^{m-n} f_i^1 f_j^1 \right]
$$

for some scalars $\sigma_{m,r}(*,*,*)$ and extend to all of $\mathbb{K}_m$ by requiring it to be a Λ^e-module homomorphism.

Lemma 5.20. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Assume that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle defined by $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1 f_{w+1}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ and Equation (5.11) holds with $j \leftrightarrow j'$ and $j \leftrightarrow j''$. Then for all m and r, there are scalars $\sigma_{*,*}(*,*,*)$ such that whenever*

$$\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1,s) f_w^1 \varepsilon_s^{m-n+1} + b_{m,r}(m-n+1,s')\varepsilon_{s'}^{m-n+1} f_{w+1}^1,$$

for some s and s' depending on r,

$$(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = J(\varepsilon_r^m). \tag{5.21}$$

Proof. We begin the proof with direct evaluation of these maps on the free basis elements.

$$\begin{aligned}
&(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = d\psi_\eta(\varepsilon_r^m) - (-1)^{n-1}\psi_\eta d(\varepsilon_r^m) \\
&= d\Big(b_{m,r}(m-n+1,s) f_w^1 \varepsilon_s^{m-n+1} + b_{m,r}(m-n+1,s')\varepsilon_{s'}^{m-n+1} f_{w+1}^1\Big) \\
&- (-1)^{n-1}\psi_\eta\Big(\sum_{j=0}^{t_{m-1}} < \varepsilon_j^{m-1} >_{m,r}\Big) \\
&= b_{m,r}(m-n+1,s) f_w^1 d(\varepsilon_s^{m-n+1}) + b_{m,r}(m-n+1,s') d(\varepsilon_{s'}^{m-n+1}) f_{w+1}^1 \\
&- (-1)^{n-1}\sum_{j=0}^{t_{m-1}} \psi_\eta(< \varepsilon_j^{m-1} >_{m,r}).
\end{aligned}$$

Summing over the indices j' and j'' after applying ψ_η to ε_j^{m-1} and applying the result of Lemma 5.18, we obtain

$$
\begin{aligned}
b_{m,r}(m-n+1,s)f_w^1 \sum_{\alpha=0}^{t_{m-n}} < \varepsilon_\alpha^{m-n} >_{m-n+1,s} \\
+ b_{m,r}(m-n+1,s') \sum_{\beta=0}^{t_{m-n}} < \varepsilon_\beta^{m-n} >_{m-n+1,s'} f_{w+1}^1 \\
- (-1)^{n-1} \sum_{j'=0}^{t_{m-n}} b_{m-1,j}(m-n,j') < f_w^1 \varepsilon_{j'}^{m-n} >_{m-n+1,r} \\
- (-1)^{n-1} \sum_{j''=0}^{t_{m-n}} b_{m-1,j}(m-n,j'') < \varepsilon_{j''}^{m-n} f_{w+1}^1 >_{m-n+1,r} .
\end{aligned}
$$

Applying the definition of $< \varepsilon_*^* >_{*,*}$ at the appropriate places, we obtain

$$
\begin{aligned}
b_{m,r}(m-n+1,s)f_w^1 \sum_{\alpha=0}^{t_{m-n}} \Big[\sum_{p=0}^{t_1} c_{p\alpha}(m-n+1,s,1) f_p^1 \varepsilon_\alpha^{m-n} \\
+ (-1)^{m-n+1} \sum_{q=0}^{t_1} c_{\alpha q}(m-n+1,s,m-n) \varepsilon_\alpha^{m-n} f_q^1 \Big] \\
+ b_{m,r}(m-n+1,s') \sum_{\beta=0}^{t_{m-n}} \Big[\sum_{p=0}^{t_1} c_{p\beta}(m-n+1,s',1) f_p^1 \varepsilon_\beta^{m-n} \\
+ (-1)^{m-n+1} \sum_{q=0}^{t_1} c_{\beta q}(m-n+1,s',m-n) \varepsilon_\beta^{m-n} f_q^1 \Big] f_{w+1}^1
\end{aligned}
$$

$$
\begin{aligned}
- (-1)^{n-1} \sum_{j'=0}^{t_{m-n}} b_{m-1,j}(m-n,j') f_w^1 \Big[\sum_{p=0}^{t_1} c_{p,j'}(m-n+1,r,1) f_p^1 \varepsilon_{j'}^{m-n} \\
+ (-1)^{m-n+1} \sum_{q=0}^{t_1} c_{j',q}(m-n+1,r,m-n) \varepsilon_{j'}^{m-n} f_q^1 \Big] \\
- (-1)^{n-1} \sum_{j''=0}^{t_{m-n}} b_{m-1,j}(m-n,j'') \Big[\sum_{p=0}^{t_1} c_{pj''}(m-n+1,r,1) f_p^1 \varepsilon_{j''}^{m-n} \\
+ (-1)^{m-n+1} \sum_{q=0}^{t_1} c_{j''q}(m-n+1,r,m-n) \varepsilon_{j''}^{m-n} f_q^1 \Big] f_{w+1}^1 .
\end{aligned}
$$

After re-arranging and bringing together similar terms, we get

$$= \sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} b_{m,r}(m-n+1,s) c_{p\alpha}(m-n+1,s,1) f_w^1 f_p^1 \varepsilon_\alpha^{m-n} \tag{a1}$$

$$- (-1)^{n-1} \sum_{j'=0}^{t_{m-n}} \sum_{p=0}^{t_1} b_{m-1,j}(m-n,j') c_{p,j'}(m-n+1,r,1) f_w^1 f_p^1 \varepsilon_{j'}^{m-n} \tag{a2}$$

$$+ (-1)^{m-n+1} \sum_{\alpha=0}^{t_{m-n}} \sum_{q=0}^{t_1} b_{m,r}(m-n+1,s) c_{\alpha q}(m-n+1,s,m-n) f_w^1 \varepsilon_\alpha^{m-n} f_q^1 \tag{b1}$$

$$- (-1)^{m} \sum_{j'=0}^{t_{m-n}} \sum_{q=0}^{t_1} b_{m-1,j}(m-n,j') c_{j',q}(m-n+1,r,m-n) f_w^1 \varepsilon_{j'}^{m-n} f_q^1 \tag{b2}$$

$$+ \sum_{\beta=0}^{t_{m-n}} \sum_{p=0}^{t_1} b_{m,r}(m-n+1,s') c_{p\beta}(m-n+1,s',1) f_p^1 \varepsilon_\beta^{m-n} f_{w+1}^1 \tag{c1}$$

$$- (-1)^{n-1} \sum_{j''=0}^{t_{m-n}} \sum_{p=0}^{t_1} b_{m-1,j}(m-n,j'') c_{pj''}(m-n+1,r,1) f_p^1 \varepsilon_{j''}^{m-n} f_{w+1}^1 \tag{c2}$$

$$+ (-1)^{m-n+1} \sum_{\beta=0}^{t_{m-n}} \sum_{q=0}^{t_1} b_{m,r}(m-n+1,s') c_{\beta q}(m-n+1,s',m-n) \varepsilon_\beta^{m-n} f_q^1 f_{w+1}^1 \tag{d1}$$

$$(-1)^{m} \sum_{j''=0}^{t_{m-n}} \sum_{q=0}^{t_1} b_{m-1,j}(m-n,j'') c_{j''q}(m-n+1,r,m-n) \varepsilon_{j''}^{m-n} f_q^1 f_{w+1}^1. \tag{d2}$$

Next we combine Expressions (a1) and (a2) and re-index by setting $j' = \alpha$, combine Expressions (b1) and (b2) and re-index $\alpha = j'$. In a similar way, we combine Expressions (c1) and (c2) and also combine Expressions (d1) and (d2) and re-index by setting $j'' = \beta$. We obtain

$$\sum_{\alpha=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[A_\alpha\Big] f_w^1 f_p^1 \varepsilon_\alpha^{m-n} + \sum_{\alpha=0}^{t_{m-n}} \sum_{q=0}^{t_1} \Big[B_\alpha\Big] f_w^1 \varepsilon_\alpha^{m-n} f_q^1$$
$$+ \sum_{\beta=0}^{t_{m-n}} \sum_{p=0}^{t_1} \Big[C_\beta\Big] f_p^1 \varepsilon_\beta^{m-n} f_{w+1}^1 + \sum_{\beta=0}^{t_{m-n}} \sum_{q=0}^{t_1} \Big[D_\beta\Big] \varepsilon_\beta^{m-n} f_q^1 f_{w+1}^1 \tag{5.22}$$

where

$$\begin{aligned}
A_\alpha = b_{m,r}(m-n+1,s)c_{p\alpha}(m-n+1,s,1)& \\
+ (-1)^n b_{m-1,j}(m-n,\alpha)c_{p,\alpha}(m-n+1,r,1)& \\
B_\alpha = (-1)^{m-n+1} b_{m,r}(m-n+1,s)c_{\alpha q}(m-n+1,s,m-n)& \\
- (-1)^m b_{m-1,j}(m-n,\alpha)c_{\alpha,q}(m-n+1,r,m-n)& \\
C_\beta = b_{m,r}(m-n+1,s')c_{p\beta}(m-n+1,s',1)& \\
+ (-1)^n b_{m-1,j}(m-n,\beta)c_{p\beta}(m-n+1,r,1)& \\
D_\beta = (-1)^{m-n+1} b_{m,r}(m-n+1,s')c_{\beta q}(m-n+1,s',m-n)& \\
+ (-1)^m b_{m-1,j}(m-n,\beta)c_{\beta,q}(m-n+1,r,m-n).&
\end{aligned}$$

Now re-write Equation (5.22) as follows:

$$\sum_{\alpha=0}^{t_{m-n}} \sum_{i=0}^{t_1} \sum_{p=0}^{t_1} \Big[A_\alpha\Big] f_i^1 f_p^1 \varepsilon_\alpha^{m-n} + \sum_{\alpha=0}^{t_{m-n}} \sum_{i=0}^{t_1} \sum_{q=0}^{t_1} \Big[B_\alpha\Big] f_i^1 \varepsilon_\alpha^{m-n} f_q^1$$
$$+ \sum_{\beta=0}^{t_{m-n}} \sum_{j=0}^{t_1} \sum_{p=0}^{t_1} \Big[C_\beta\Big] f_p^1 \varepsilon_\beta^{m-n} f_j^1 + \sum_{\beta=0}^{t_{m-n}} \sum_{j=0}^{t_1} \sum_{q=0}^{t_1} \Big[D_\beta\Big] \varepsilon_\beta^{m-n} f_q^1 f_j^1$$

such that $A_\alpha = B_\alpha = 0$ for all $i \neq w$ and $C_\beta = D_\beta = 0$ for all $j \neq w+1$. This expression is in the form of $J(\varepsilon_r^m)$ given by Definition 5.19. More specifically, when we substitute

$$\begin{cases} \sigma_{m,r}(i,j,\nu) = A_\alpha & \text{when } i = w \text{ (fixed)}, p = j, \nu = \alpha, \\ \sigma_{m,r}(i,\nu,j) = B_\alpha & \text{when } i = w \text{ (fixed)}, q = j, \nu = \alpha, \\ \sigma_{m,r}(i,\nu,j) = C_\beta & \text{when } \ i = p, j = w+1 \text{ (fixed)}, \nu = \beta, \\ \sigma_{m,r}(\nu,i,j) = D_\beta & \text{when } \ i = q, j = w+1 \text{ (fixed)}, \nu = \beta, \end{cases}$$

and 0 otherwise, we obtain $(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = J(\varepsilon_r^m)$. □

Theorem 5.23. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Assume that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1 f_{w+1}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ for some $0 \leq w \leq t_1$ and Equation (5.11) holds with $j \leftrightarrow j'$ and $j \leftrightarrow j''$. For all r and some integers s, s' depending on r, assume there are scalars $b_{m,r}(m-n+1,s)$, $b_{m,r}(m-n+1,s')$ such that*

(i). $A_{j'} = c_{ij'}(m,r,n)$ *if* $p = w+1, A_{j'} = 0,$ *if* $p \neq w+1,$ *and*

$D_{j''} = (-1)^{n(m-n)+1} c_{j''i}(m,r,m-n),$ *if* $q = w, D_{j''} = 0$ *if* $q \neq w$ *and*

(ii). $B_{j'} = 0, C_{j''} = 0$ *for all* j' *and* j'' *where*

$A_{j'} = b_{m,r}(m-n+1,s)c_{pj'}(m-n+1,s,1) + (-1)^n b_{m-1,j}(m-n,j')c_{pj'}(m-n+1,r,1),$

$B_{j'} = (-1)^{m-n+1} b_{m,r}(m-n+1,s)c_{j'p}(m-n+1,s,m-n)$

$\quad - (-1)^m b_{m-1,j}(m-n,j')c_{j'p}(m-n+1,r,m-n),$

$C_{j''} = b_{m,r}(m-n+1,s')c_{pj''}(m-n+1,s',1) + (-1)^n b_{m-1,j}(m-n,j'')c_{pj'}(m-n+1,r,1)$ *and*

$D_{j''} = (-1)^{m-n+1} b_{m,r}(m-n+1,s')c_{j''p}(m-n+1,s',m-n)$

$\quad + (-1)^m b_{m-1,j}(m-n,j'')c_{j'',p}(m-n+1,r,m-n).$

Then a homotopy lifting map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ associated to η can be defined by

$$\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1,s)f_w^1\varepsilon_s^{m-n+1} + b_{m,r}(m-n+1,s')\varepsilon_{s'}^{m-n+1}f_{w+1}^1.$$

Proof. We first note that all the conditions of Lemma 5.20 are satisfied. We can therefore establish from Lemma 5.20 that whenever $\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1,s)f_w^1\varepsilon_s^{m-n+1} + b_{m,r}(m-n+1,s')\varepsilon_{s'}^{m-n+1}f_{w+1}^1$,

$$(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = J(\varepsilon_r^m)$$

holds. Applying condition (ii) of the theorem, that is substitute $B_\alpha = 0$ and $C_\beta = 0$ into Equation 5.22 of Lemma 5.20, and re-index by substituting q as p we get

$$(d\psi_\eta - (-1)^{n-1}\psi_\eta d)(\varepsilon_r^m) = \sum_{\alpha=0}^{t_{m-n}}\sum_{p=0}^{t_1}\Big[A_\alpha\Big]f_w^1f_p^1\varepsilon_\alpha^{m-n} + \sum_{\beta=0}^{t_{m-n}}\sum_{p=0}^{t_1}\Big[D_\beta\Big]\varepsilon_\beta^{m-n}f_p^1f_{w+1}^1.$$

After applying the definition of A_α and D_β of condition (i) of the theorem into the above expression, we obtain

$$\sum_{\alpha=0}^{t_{m-n}} c_{i,\alpha}(m,r,n)f_w^1f_{w+1}^1\varepsilon_\alpha^{m-n} - (-1)^{n(m-n)}\sum_{\beta=0}^{t_{m-n}} c_{\beta,i}(m,r,m-n)\varepsilon_\beta^{m-n}f_w^1f_{w+1}^1.$$

On the other hand, using the multiplicative structure on $\mathbb{K}$, we get $\Delta_{\mathbb{K}}(\varepsilon_r^m) = \sum_{v=0}^{m}\sum_{x=0}^{t_v}\sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v)\varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v}$. Applying $(\eta\otimes 1 - 1\otimes\eta)$, we obtain

$$\begin{aligned}(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^m) = &\sum_{v=0}^{m}\sum_{x=0}^{t_v}\sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v)(\eta\otimes 1)(\varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v})\\ &- \sum_{v=0}^{m}\sum_{x=0}^{t_v}\sum_{y=0}^{t_{m-v}} c_{x,y}(m,r,v)(1\otimes\eta)(\varepsilon_x^v \otimes_\Lambda \varepsilon_y^{m-v}).\end{aligned}$$

Whenever $v = n, x = i$ in the first summation and $m-n = v, y = i$ in the second summation, the above expression will become

$$= \sum_{y=0}^{t_{m-n}} c_{i,y}(m,r,n)(\eta \otimes 1)(\varepsilon_i^n \otimes_\Lambda \varepsilon_y^{m-n}) - \sum_{x=0}^{t_{m-n}} c_{x,i}(m,r,m-n)(1 \otimes \eta)(\varepsilon_x^{m-n} \otimes_\Lambda \varepsilon_i^n)$$
$$= \sum_{y=0}^{t_{m-n}} c_{i,y}(m,r,n)\eta(\varepsilon_i^n)\varepsilon_y^{m-n} - (-1)^{n(m-n)} \sum_{x=0}^{t_{m-n}} c_{x,i}(m,r,m-n)\varepsilon_x^{m-n}\eta(\varepsilon_i^n)$$

which after applying the definition of η and re-indexing by taking $y = \alpha, x = \beta$, we get

$$= \sum_{\alpha=0}^{t_{m-n}} c_{i,\alpha}(m,r,n) f_w^1 f_{w+1}^1 \varepsilon_\alpha^{m-n} - (-1)^{n(m-n)} \sum_{\beta=0}^{t_{m-n}} c_{\beta,i}(m,r,m-n)\varepsilon_\beta^{m-n} f_w^1 f_{w+1}^1.$$

□

The following theorem gives a combinatorial description of what we obtain when the Gerstenhaber bracket of any two Hochschild cochains is applied to free basis elements.

Theorem 5.24. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Assume that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta = \begin{pmatrix} 0 & \cdots & 0 & (\lambda_i)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ and $\theta = \begin{pmatrix} 0 & \cdots & 0 & (\lambda_j)^{(j)} & 0 & \cdots & 0 \end{pmatrix}$ for all i, j where $0 \leq i \leq t_n$ and $0 \leq j \leq t_m$. Assume also that there are scalars $b_{m-n+1,r}(n,i)$ and $b_{m-n+1,r}(m,j)$ associated with the homotopy lifting maps $\psi_{\theta^{(j)}}$ and $\psi_{\eta^{(i)}}$ respectively that satisfy the conditions stated in Theorems 5.14 and 5.23. Then the bracket $[\eta, \theta] : \mathbb{K}_{n+m-1} \to \Lambda$ has the property that*

$$[\eta,\theta](\varepsilon_r^{m+n-1}) \in \begin{cases} kQ_1 & \text{if } \lambda_i = f_i^1 \text{ and } \lambda_j = f_j^1, \\ kQ_2 & \text{if } \lambda_i = f_i^1 f_{i+1}^1 \text{ and } \lambda_j = f_j^1 \\ kQ_3 & \text{if } \lambda_i = f_i^1 f_{i+1}^1 \text{ and } \lambda_j = f_j^1 f_{j+1}^1. \end{cases}$$

Proof. Using the definition of Gerstenhaber bracket of Definition 2.18, we get

$$\begin{aligned}
[\eta,\theta](\varepsilon_r^{m+n-1}) &= (\eta\psi_\theta - (-1)^{(m-1)(n-1)}\theta\psi_\eta)(\varepsilon_r^{m+n-1}) \\
&= \eta\psi_\theta(\varepsilon_r^{m+n-1}) - (-1)^{(m-1)(n-1)}\theta\psi_\eta(\varepsilon_r^{m+n-1})
\end{aligned}$$

We now apply the definition of a homotopy lifting map given by Theorems 5.14 and 5.23 in the following scenarios.

(1) Suppose that both are paths of length 1, i.e. $\lambda_i = f_w^1, \lambda_j = f_p^1$. We will get $\eta(b_{m+n-1,r}(n,r'')\cdot \varepsilon_{r''}^n) - (-1)^{(m-1)(n-1)}\theta(b_{m+n-1,r}(m,r')\cdot\varepsilon_{r'}^m)$. This expression will give 0 or a non-zero path. We are interested in the non-zero case i.e. when $r''=i$ (or $\eta(\varepsilon_{r''}^n)=\lambda_i$) and $r'=j$ (or $\theta(\varepsilon_{r'}^m)=\lambda_j$). This yields $b_{m+n-1,r}(n,i)\lambda_i - (-1)^{(m-1)(n-1)}b_{m+n-1,r}(m,j)\lambda_j$ which is a k-linear combination of paths of length 1.

(2) Suppose that one of them is a path of length 2, i.e. $\lambda_i = f_w^1 f_{w+1}^1,\ \lambda_j = f_p^1$. The expression will yield $\eta(b_{m+n-1,r}(n,r'')\cdot\varepsilon_{r''}^n) - (-1)^{(m-1)(n-1)}\theta[b_{m+n-1,r}(m,s)f_w^1\cdot\varepsilon_s^m + b_{m+n-1,r}(m,s')\cdot\varepsilon_{s'}^m f_{w+1}^1]$. Now consider only the cases in which we obtain a non-zero i.e. either one or all of $r''=i, s=j, s'=j$ holds. We obtain

$$\begin{aligned}
&b_{m+n-1,r}(n,i)\lambda_i - (-1)^{(m-1)(n-1)}b_{m+n-1,r}(m,j)f_w^1\lambda_j - (-1)^{(m-1)(n-1)}b_{m+n-1,r}(m,j)\lambda_j f_{w+1}^1 \\
&= b_{m+n-1,r}(n,i)f_w^1 f_{w+1}^1 - (-1)^{(m-1)(n-1)}b_{m+n-1,r}(m,j)f_w^1 f_p^1 \\
&- (-1)^{(m-1)(n-1)}b_{m+n-1,r}(m,j)f_p^1 f_{w+1}^1,
\end{aligned}$$

which is a k-linear combination of paths of length 2.

(3) Suppose that both are paths of length 2. Let $\lambda_i = f_w^1 f_{w+1}^1,\ \lambda_j = f_p^1 f_{p+1}^1$, we obtain for the bracket expression

$$\begin{aligned}
&\eta[b_{m+n-1,r}(n,v)f_p^1\cdot\varepsilon_v^n + b_{m+n-1,r}(n,v')\varepsilon_{v'}^n\cdot f_{p+1}^1] \\
&\qquad - (-1)^{(m-1)(n-1)}\theta[b_{m+n-1,r}(m,s)f_w^1\cdot\varepsilon_s^m + b_{m+n-1,r}(m,s')\cdot\varepsilon_{s'}^m f_{w+1}^1].
\end{aligned}$$

We consider only the cases in which we obtain a non-zero i.e. either one or all of $v' = i, v' = i, s = j, s' = j$ holds. We obtain

$$
\begin{aligned}
& b_{m+n-1,r}(n,i) f_p^1 \lambda_i + b_{m+n-1,r}(n,i) \lambda_i f_{p+1}^1 \\
& - (-1)^{(m-1)(n-1)} \Big[b_{m+n-1,r}(m,j) f_w^1 \lambda_j + b_{m+n-1,r}(m,j) \lambda_j f_{w+1}^1 \Big] \\
& = b_{m+n-1,r}(n,i) f_p^1 f_w^1 f_{w+1}^1 + b_{m+n-1,r}(n,i) f_w^1 f_{w+1}^1 f_{p+1}^1 \\
& - (-1)^{(m-1)(n-1)} \Big[b_{m+n-1,r}(m,j) f_w^1 f_p^1 f_{p+1}^1 + b_{m+n-1,r}(m,j) f_p^1 f_{p+1}^1 f_{w+1}^1 \Big]
\end{aligned}
$$

which is a linear combination of paths of length 3. □

Any cocycle η of degree n can be thought of as a sum of maps $\eta = \sum_{i=0}^{t_n} \eta^{(i)}$ where $\eta^{(i)}$ is the map taking the i-th basis element ε_i^n to λ, a non-zero element of the algebra, and all other basis elements ε_j^n to 0, $i \neq j$. Consistent with our notation, this map is written $\eta^{(i)} = \begin{pmatrix} 0 & \cdots & 0 & (\lambda)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, and we use this notation in the following theorem.

Theorem 5.25. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Suppose that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis consisting of $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Assume that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_j^m) = \beta_j$ for $j = 0, 1, \ldots, t_m$, with each λ_i and β_j paths of length of 1. Then the r-component of the bracket $[\eta, \theta] : \mathbb{K}_{n+m-1} \to \Lambda$ denoted by $[\eta, \theta]^{(r)}$ can be expressed on the r-th basis element ε_r^{m+n-1} as*

$$
[\eta, \theta]^{(r)}(\varepsilon_r^{m+n-1}) = \sum_{i=0}^{t_n} \sum_{j=0}^{t_m} b_{m-n+1,r}(n,i)\lambda_i - (-1)^{(m-1)(n-1)}(b_{m-n+1,r}(m,j)\beta_j
$$

where the scalars $b_{m-n+1,r}(n,i)$ and $b_{m-n+1,r}(m,j)$ are coming from homotopy lifting maps $\psi_{\theta^{(j)}}$ and $\psi_{\eta^{(i)}}$ respectively and satisfy the conditions of Theorem 5.14.

Proof. Write $\eta = \sum_{i=0}^{t_n} \eta^{(i)}$ where $\eta^{(i)} = \begin{pmatrix} 0 & \cdots & 0 & (\lambda_i)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ and $\theta = \sum_{j=0}^{t_m} \theta^{(j)}$

where $\theta^{(j)} = \begin{pmatrix} 0 & \cdots & 0 & (\beta_j)^{(j)} & 0 & \cdots & 0 \end{pmatrix}$. Using the definition of Gerstenhaber bracket of 2.18, we get for $0 \leq r \leq t_{m+n-1}$,

$$[\eta, \theta]^{(r)}(\varepsilon_r^{m+n-1}) = [\sum_{i=0}^{t_n} \eta^{(i)}, \sum_{j=0}^{t_m} \theta^{(j)}]^r(\varepsilon_r^{m+n-1}) = \sum_{i=0}^{t_n} \sum_{j=0}^{t_m} [\eta^{(i)}, \theta^{(j)}]^r(\varepsilon_r^{m+n-1})$$
$$= \sum_{i=0}^{t_n} \sum_{j=0}^{t_m} (\eta^{(i)} \psi_{\theta^{(j)}} - (-1)^{(m-1)(n-1)} \theta^{(j)} \psi_{\eta^{(i)}})(\varepsilon_r^{m+n-1})$$

Since $\lambda_i = f^1_{w_i}$ for all i, and $\beta_j = f^1_{p_j}$ for all j, then from Theorem 5.14, the homotopy lifting maps associated to $\psi_{\theta^{(j)}}$ and $\psi_{\eta^{(i)}}$ can be defined as $\psi_{\theta^{(j)}}(\varepsilon_r^{m+n-1}) = b_{m-n+1,r}(n, i)\varepsilon_i^n$ for some i and $\psi_{\eta^{(i)}}(\varepsilon_r^{m+n-1}) = b_{m-n+1,r}(m, j)\varepsilon_j^m$ for some j. Applying this, we get

$$\sum_{i=0}^{t_n} \sum_{j=0}^{t_m} \eta^{(i)}(b_{m-n+1,r}(n, i)\varepsilon_i^n) - (-1)^{(m-1)(n-1)} \theta^{(j)}(b_{m-n+1,r}(m, j)\varepsilon_j^m)$$
$$= \sum_{i=0}^{t_n} \sum_{j=0}^{t_m} b_{m-n+1,r}(n, i)\lambda_i - (-1)^{(m-1)(n-1)}(b_{m-n+1,r}(m, j)\beta_j$$

□

5.2 Working examples

Any non-zero module homomorphism $\mathbb{K}_\bullet \to \Lambda$ maps basis elements to an idempotent, paths or linear combination of paths of length one or paths and linear combination of paths of length two or a mixture of any of these. We now present examples of homotopy lifting maps coming from cocycles on Hochschild cohomology of members of the family of quiver algebras introduced by Equation (4.1). Two examples involve cocycles of degrees 1 and 2 taking basis elements to a path of length 1. The other two examples involve degrees 1 and 2 cocycles mapping ε_r^n to a path of length 2. Any other cocycle will take basis elements to a linear combination of paths of length 1 or 2. The case where cocycles take basis elements to idempotents yields homotopy lifting maps equal to zero as we see in Remark 5.5. Our choice of these four cocycles were arbitrary but it illustrates the general theory presented in the previous section.

While a member of this family was introduced in [13] as a counterexample to the Snashall-Solberg finite generation conjecture, the Hochschild cohomology modulo nilpotent cocycles of this family as a whole was studied in [12]. Since the algebras of this family are of infinite global dimension, the resolution $\mathbb{K}$ does not terminate. We are able to define homotopy lifting maps associated to two degree 1 cocycles on the whole of $\mathbb{K}$. Furthermore, to illustrate the theory presented in the previous section, we are able to find three associated homotopy lifting maps for some degree two cocycles as well. We recall that a map $\psi_f : \mathbb{K}_\bullet \to \mathbb{K}_\bullet[1-n]$ is a homotopy lifting map associated with the degree n cocycle $f : \mathbb{K}_n \to \Lambda$ if it satisfies the conditions of Equation (2.17) and (5.2) given below for easy reference.

$$\begin{aligned} \mathbf{d}(\psi_f) &= (f \otimes 1 - 1 \otimes f)\Delta_{\mathbb{K}} \qquad \text{and} \\ \mu\psi_f &\sim\ (-1)^{n-1} f\psi \end{aligned}$$

for some $\psi : \mathbb{K}_\bullet \to \mathbb{K}_\bullet[1]$ for which $\mathbf{d}(\psi) = d\psi - \psi d = (\mu \otimes 1 - 1 \otimes \mu)\Delta_{\mathbb{K}}$.

5.2.1 Homotopy liftings from degree 2 cocycles

Suppose that the Λ_q^e-module homomorphism $\eta : \mathbb{K}_2 \to \Lambda_q$ defined by $\eta = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}$ is a cocycle, that is $d^*\eta = 0$, with $\lambda_i \in \Lambda_q$ for all i. Since $d^*\eta : \mathbb{K}_3 \to \Lambda_q$, we obtain $d^*\eta(\varepsilon_r^3) = \eta d(\varepsilon_r^3)$ expressed in the following way.

$$\eta d(\varepsilon_i^3) = \eta \begin{cases} a\varepsilon_0^2 - \varepsilon_0^2 a & \text{if } i = 0 \\ a\varepsilon_1^2 + q\varepsilon_1^2 a + q^2 b\varepsilon_0^2 - \varepsilon_0^2 b & \text{if } i = 1 \\ a\varepsilon_2^2 - q^2\varepsilon_2^2 a - qb\varepsilon_1^2 - \varepsilon_1^2 b & \text{if } i = 2 \\ b\varepsilon_2^2 - \varepsilon_2^2 b & \text{if } i = 3 \\ a\varepsilon_3^2 - \varepsilon_0^2 c & \text{if } i = 4 \end{cases}$$

solutions	1	2	3	4	5	6	7	8	9
λ_0	a	ab	0	0	0	0	0	0	0
λ_1	0	0	0	0	0	0	ab	0	0
λ_2	0	0	a	b	ab	e_1	0	0	0
λ_3	0	0	0	0	0	0	0	c	bc

Table 5.1: Table of solutions for $\lambda_i, i = 0, 1, 2, 3$

ηd may then be identified with the 1×5 row matrix

$$\begin{pmatrix} a\lambda_0 - \lambda_0 a & a\lambda_1 + q\lambda_1 a + q^2 b\lambda_0 - \lambda_0 b & a\lambda_2 - q^2\lambda_2 a - qb\lambda_1 - \lambda_1 b & b\lambda_2 - \lambda_2 b & a\lambda_3 - \lambda_0 c \end{pmatrix}$$

which will be equated to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and solved. We solve this system of equations with the following in mind. There is an isomorphism of Λ_q^e-modules $\mathrm{Hom}_{\Lambda^e}(\Lambda o(f_i^n) \otimes_k t(f_i^n)\Lambda, \Lambda) \simeq o(f_i^n)\Lambda\, t(f_i^n)$ ensuring that

$$\begin{aligned} o(f_i^2)\lambda_i t(f_i^2) &= o(f_i^2)\eta(\varepsilon_i^2)t(f_i^2) = o(f_i^2)\eta(o(f_i^2) \otimes_k t(f_i^2))t(f_i^2) \\ &= \phi(o(f_i^2)^2 \otimes_k t(f_i^2)^2) = \phi(o(f_i^2) \otimes_k t(f_i^2)) = \lambda_i. \end{aligned}$$

This means that for $i = 0, 1, 2$ each λ_i should satisfy $e_1\lambda_i e_1 = \lambda_i$ since the origin and terminal vertices of f_0^2, f_1^2, f_2^2 are e_1, and $e_1\lambda_3 e_2 = \lambda_3$ since the origin and terminal vertex of f_3^2 is e_1 and e_2 respectively. We obtain the following 9 solutions (which is a basis for the solution set). We present them in Table 5.1. For the rest of this subsection, we are interested in the first and fifth cocycles, that is

$$\bar{\eta} = \begin{pmatrix} a & 0 & 0 & 0 \end{pmatrix} \text{ and } \bar{\chi} = \begin{pmatrix} 0 & 0 & ab & 0 \end{pmatrix}.$$

We note that homotopy lifting maps corresponding to other cocycles obtained in Table 5.1 can be defined in a similar fashion as generally given by Theorems 5.14 and 5.23.

Homotopy lifting for the first and fifth maps $\bar{\eta}$ and $\bar{\chi}$ will be maps $\psi_{\bar{\eta}}, \psi_{\bar{\chi}} : \mathbb{K} \to \mathbb{K}[-1]$ such that $d\psi_{\bar{\eta}} + \psi_{\bar{\eta}} d = (\bar{\eta} \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes \bar{\eta})\Delta_{\mathbb{K}}$, and $d\psi_{\bar{\chi}} + \psi_{\bar{\chi}} d = (\bar{\chi} \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes \bar{\chi})\Delta_{\mathbb{K}}$. We track the left hand side of the above equations using the following diagram, which is not necessarily commutative.

$$\begin{array}{ccccccccccc}
\mathbb{K} := \cdots & \xrightarrow{d_5} & \mathbb{K}_4 & \xrightarrow{d_4} & \mathbb{K}_3 & \xrightarrow{d_3} & \mathbb{K}_2 & \xrightarrow{d_2} & \mathbb{K}_1 & \xrightarrow{d_1} & \mathbb{K}_0 \\
\downarrow{\scriptstyle \psi_{\bar\eta},\, \psi_{\bar\chi}} & & \downarrow{\scriptstyle \psi_{\bar\eta_4},\, \psi_{\bar\chi_4}} & & \downarrow{\scriptstyle \psi_{\bar\eta_3},\, \psi_{\bar\chi_3}} & & \downarrow{\scriptstyle \psi_{\bar\eta_2},\, \psi_{\bar\chi_2}} & & \downarrow{\scriptstyle \psi_{\bar\eta_1},\, \psi_{\bar\chi_1}} & & \\
\mathbb{K} := \cdots & \xrightarrow{d_4} & \mathbb{K}_3 & \xrightarrow{d_3} & \mathbb{K}_2 & \xrightarrow{d_2} & \mathbb{K}_1 & \xrightarrow{d_1} & \mathbb{K}_0 & &
\end{array}$$

We now define $\psi_{\bar\eta_1}, \psi_{\bar\eta_2}, \psi_{\bar\eta_3}$ and $\psi_{\bar\chi_1}, \psi_{\bar\chi_2}, \psi_{\bar\chi_3}$ only, just to point out that homotopy lifting maps can be defined in certain ways (as is generalized in Theorem (5.14) and (5.14)). Calculations show that

$$\psi_{\bar\eta_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\bar\eta_2}(\varepsilon_i^2) = \begin{cases} \varepsilon_0^1, & \text{if } i = 0 \\ 0, & \text{if } i = 1,2,3 \end{cases},\quad \psi_{\bar\eta_3}(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ \varepsilon_1^2, & \text{if } i = 1 \\ 0, & \text{if } i = 2 \\ 0, & \text{if } i = 3 \\ \varepsilon_3^2, & \text{if } i = 4 \end{cases} \tag{5.26}$$

defines an homotopy lifting associated to the cocycle $\bar\eta$ while

$$\psi_{\bar\chi_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\bar\chi_2}(\varepsilon_i^2) = \begin{cases} 0 & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ a\varepsilon_1^1 + \varepsilon_0^1 b & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases},\quad \psi_{\bar\chi_3}(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ -a\varepsilon_1^2, & \text{if } i = 2 \\ \varepsilon_1^2 b, & \text{if } i = 3 \\ 0, & \text{if } i = 4 \end{cases} \tag{5.27}$$

defines an homotopy lifting map associated to the cocycle $\bar\chi$. According to Theorems 5.14 and 5.23, we see that for $\bar\chi$, $b_{1,i}(0,j) = 0$ for all i, j, $b_{2,i}(1,j) = 1$ when $(i,j) = (2,1), (i,j) = (2,0)$ and $b_{2,i}(1,j) = 0$ otherwise. Also, $b_{3,i}(2,j) = -1$ when $(i,j) = (2,1)$, $b_{3,i}(2,j) = 1$ when $(i, j+1) = (3,1)$ and $b_{3,i}(2,j) = 0$ otherwise. The proof that these maps are homotopy lifting maps is by direct computations. We illustrate with just one of the numerous computations i.e. we will show that for the cocycle $\bar{\eta}$ and $\bar{\chi}$ with $q = 1$,

$$(d_2\psi_{\bar{\eta}_3} + \psi_{\bar{\eta}_2}d_3)(\varepsilon_i^3) = (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\Delta_{\mathbb{K}}(\varepsilon_i^3), \quad i = 0,1,2,3,4, \qquad \text{and}$$
$$(d_2\psi_{\bar{\chi}_3} + \psi_{\bar{\chi}_2}d_3)(\varepsilon_i^3) = (\bar{\chi}\otimes 1 - 1\otimes\bar{\chi})\Delta_{\mathbb{K}}(\varepsilon_i^3), \quad i = 0,1,2,3,4.$$

The case of $\bar{\eta}$:

$$(d_2\psi_{\bar{\eta}_3} + \psi_{\bar{\eta}_2}d_3)(\varepsilon_i^3) = \begin{cases} d_2(0) + \psi_{\bar{\eta}_2}(a\varepsilon_0^2 - \varepsilon_0^2 a) \\ d_2(\varepsilon_1^2) + \psi_{\bar{\eta}_2}(a\varepsilon_1^2 + q\varepsilon_1^2 a + q^2 b\varepsilon_0^2 - \varepsilon_0^2 b) \\ d_2(0) + \psi_{\bar{\eta}_2}(a\varepsilon_2^2 + q^2\varepsilon_2^2 a - qb\varepsilon_1^2 - \varepsilon_1^2 b) \\ d_2(0) + \psi_{\bar{\eta}_2}(b\varepsilon_2^2 - \varepsilon_2^2 b) \\ d_2(\varepsilon_3^2) + \psi_{\bar{\eta}_2}(a\varepsilon_3^2 - \varepsilon_0^2 c) \end{cases}$$
$$= \begin{cases} 0 + a\varepsilon_0^1 - \varepsilon_0^1 a \\ a\varepsilon_1^1 - q\varepsilon_1^1 a - qb\varepsilon_0^1 + \varepsilon_0^1 b + q^2 b\varepsilon_0^1 - \varepsilon_0^1 b \\ 0 + 0 \\ 0 + 0 \\ a\varepsilon_2^1 + \varepsilon_0^1 c - \varepsilon_0^1 c \end{cases} = \begin{cases} a\varepsilon_0^1 - \varepsilon_0^1 a & \text{if } i = 0 \\ a\varepsilon_1^1 - \varepsilon_1^1 a & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 0 & \text{if } i = 3 \\ a\varepsilon_2^1 & \text{if } i = 4 \end{cases} \cdot$$

On the other hand, and using Koszul signs in the expansion of $(1\otimes\bar{\eta})(\varepsilon_r^n\otimes\varepsilon_s^m)$ to obtain $(-1)^{|\bar{\eta}|n}\varepsilon_r^n\bar{\eta}(\varepsilon_s^m)$, we get

$$(\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\Delta_{\mathbb{K}}(\varepsilon_i^3)$$

$$= \begin{cases} (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\big[\varepsilon_0^0\otimes\varepsilon_0^3 + \varepsilon_0^1\otimes\varepsilon_0^2 + \varepsilon_0^2\otimes\varepsilon_0^1 + \varepsilon_0^3\otimes\varepsilon_0^0\big] \\ (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\big[\varepsilon_0^0\otimes\varepsilon_1^3 + \varepsilon_0^1\otimes\varepsilon_1^2 + q^2\varepsilon_1^1\otimes\varepsilon_0^2 + \varepsilon_0^2\otimes\varepsilon_1^1 - q\varepsilon_1^2\otimes\varepsilon_0^1 + \varepsilon_1^3\otimes\varepsilon_0^0\big] \\ (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\big[\varepsilon_0^0\otimes\varepsilon_2^3 + \varepsilon_0^1\otimes\varepsilon_2^2 + q\varepsilon_1^1\otimes\varepsilon_1^2 - \varepsilon_1^2\otimes\varepsilon_1^1 + \varepsilon_2^2\otimes\varepsilon_0^1 + \varepsilon_2^3\otimes\varepsilon_0^0\big] \\ (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\big[\varepsilon_0^0\otimes\varepsilon_3^3 + \varepsilon_1^1\otimes\varepsilon_2^2 + \varepsilon_2^2\otimes\varepsilon_1^1 + \varepsilon_3^3\otimes\varepsilon_0^0\big] \\ (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\big[\varepsilon_0^0\otimes\varepsilon_4^3 + \varepsilon_0^1\otimes\varepsilon_3^2 + \varepsilon_0^2\otimes\varepsilon_2^1 + \varepsilon_4^3\otimes\varepsilon_0^0\big] \end{cases}$$

$$= \begin{cases} a\varepsilon_0^1 - \varepsilon_0^1 a & \text{if } i = 0 \\ a\varepsilon_1^1 - q^2\varepsilon_1^1 a & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 0 & \text{if } i = 3 \\ a\varepsilon_2^1 & \text{if } i = 4. \end{cases}$$

So we see that $(d_2\psi_{\bar{\eta}_3} + \psi_{\bar{\eta}_2}d_3)(\varepsilon_i^3) = (\bar{\eta}\otimes 1 - 1\otimes\bar{\eta})\Delta_{\mathbb{K}}(\varepsilon_i^3), \quad i = 0,1,2,3,4.$

The case of $\bar{\chi}$:

$$(d_2\psi_{\chi_3} + \psi_{\bar{\chi}_2} d_3)(\varepsilon_i^3) = \begin{cases} d_2(0) + \psi_{\bar{\chi}_2}(a\varepsilon_0^2 - \varepsilon_0^2 a) \\ d_2(0) + \psi_{\bar{\chi}_2}(a\varepsilon_1^2 + q\varepsilon_1^2 a + q^2 b\varepsilon_0^2 - \varepsilon_0^2 b) \\ d_2(-a\varepsilon_1^2) + \psi_{\chi_2}(a\varepsilon_2^2 + q^2\varepsilon_2^2 a - qb\varepsilon_1^2 - \varepsilon_1^2 b) \\ d_2(\varepsilon_1^2 b) + \psi_{\bar{\chi}_2}(b\varepsilon_2^2 - \varepsilon_2^2 b) \\ d_2(0) + \psi_{\bar{\chi}_2}(a\varepsilon_3^2 - \varepsilon_0^2 c) \end{cases}$$

$$= \begin{cases} 0 + 0 \\ 0 + 0 \\ qa\varepsilon_1^1 a + qab\varepsilon_0^1 - a\varepsilon_0^1 b + a\varepsilon_0^1 b - q^2 a\varepsilon_1^1 a - q^2\varepsilon_0^1 ba \\ a\varepsilon_1^1 b - q\varepsilon_1^1 ab - qb\varepsilon_0^1 b + ba\varepsilon_1^1 - b\varepsilon_0^1 b - a\varepsilon_1^1 b \\ 0 + 0 \end{cases} = \begin{cases} 0 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ ab\varepsilon_0^1 - \varepsilon_0^1 ab & \text{if } i = 2 \\ ab\varepsilon_1^1 - \varepsilon_1^1 ab & \text{if } i = 3 \\ 0 & \text{if } i = 4. \end{cases}$$

On the other hand, and using Koszul signs convention as done in the previous example, we get

$$(\bar{\chi} \otimes 1 - 1 \otimes \bar{\chi})\Delta_{\mathbb{K}}(\varepsilon_i^3) = (\bar{\chi} \otimes 1 - 1 \otimes \bar{\chi})\Bigg($$

$$\begin{cases} \varepsilon_0^0 \otimes \varepsilon_0^3 + \varepsilon_0^1 \otimes \varepsilon_0^2 + \varepsilon_0^2 \otimes \varepsilon_0^1 + \varepsilon_0^3 \otimes \varepsilon_0^0 \\ \varepsilon_0^0 \otimes \varepsilon_1^3 + \varepsilon_0^1 \otimes \varepsilon_1^2 + q^2\varepsilon_1^1 \otimes \varepsilon_0^2 + \varepsilon_0^2 \otimes \varepsilon_1^1 - q\varepsilon_1^2 \otimes \varepsilon_0^1 + \varepsilon_1^3 \otimes \varepsilon_0^0 \\ \varepsilon_0^0 \otimes \varepsilon_2^3 + \varepsilon_0^1 \otimes \varepsilon_2^2 + q\varepsilon_1^1 \otimes \varepsilon_1^2 - \varepsilon_1^2 \otimes \varepsilon_1^1 + \varepsilon_2^2 \otimes \varepsilon_0^1 + \varepsilon_2^3 \otimes \varepsilon_0^0 \\ \varepsilon_0^0 \otimes \varepsilon_3^3 + \varepsilon_1^1 \otimes \varepsilon_2^2 + \varepsilon_2^2 \otimes \varepsilon_1^1 + \varepsilon_3^3 \otimes \varepsilon_0^0 \\ \varepsilon_0^0 \otimes \varepsilon_4^3 + \varepsilon_0^1 \otimes \varepsilon_3^2 + \varepsilon_0^2 \otimes \varepsilon_2^1 + \varepsilon_4^3 \otimes \varepsilon_0^0 \end{cases} \Bigg) = \begin{cases} 0 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ ab\varepsilon_0^1 - \varepsilon_0^1 ab & \text{if } i = 2 \cdot \\ ab\varepsilon_1^1 - \varepsilon_1^1 ab & \text{if } i = 3 \\ 0 & \text{if } i = 4 \end{cases}$$

So we see again that $(d_2\psi_{\bar{\chi}_3} + \psi_{\bar{\chi}_2} d_3)(\varepsilon_i^3) = (\bar{\chi} \otimes 1 - 1 \otimes \bar{\chi})\Delta_{\mathbb{K}}(\varepsilon_i^3), \quad i = 0, 1, 2, 3, 4.$

solutions	1	2	3	4	5	6
λ_0	a	ab	0	0	0	0
λ_1	0	0	b	ab	0	0
λ_2	0	0	0	0	c	bc

Table 5.2: Table of solutions to $\lambda_i, i = 0, 1, 2$.

5.2.2 Homotopy liftings from degree 1 cocycles

In this section, we will find Hochschild 1 cocycles and their associated homotopy lifting maps. The process of solving equations to obtain cocycles is similar to what was done in the previous Subsection (5.2.1), so we just state the results. We then give explicit maps ψ_{η_n} and ψ_{χ_n} for all n.

Suppose that the Λ_q^e-module homomorphism $\eta : \mathbb{K}_1 \to \Lambda_q$ defined by $\eta = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \end{pmatrix}$ is a cocycle, that is $d^*\eta = 0$, with $\lambda_i \in \Lambda_q$ for all i. Since $d^*\eta : \mathbb{K}_2 \to \Lambda_q$, we solve the equation $d^*\eta(\varepsilon_r^2) = \eta d_2(\varepsilon_r^2) = 0$ and present the solutions in Table 5.2. Let us consider the first and second cocycles, i.e.

$$\eta = \begin{pmatrix} a & 0 & 0 \end{pmatrix} \text{ and } \quad \chi = \begin{pmatrix} ab & 0 & 0 \end{pmatrix}.$$

There are homotopy lifting maps $\psi_\eta, \psi_\chi : \mathbb{K}_\bullet \to \mathbb{K}_\bullet$ associated to η and χ respectively satisfying

$$(d\psi_\eta - \psi_\eta d)(\varepsilon_r^n) = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^n) \tag{5.28}$$

and

$$(d\psi_\chi - \psi_\chi d)(\varepsilon_r^n) = (\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_r^n). \tag{5.29}$$

We will prove that for each n and r,

$$\psi_{\eta_n}(\varepsilon_r^n) = \begin{cases} (n-r)\varepsilon_r^n & \text{when } r = 0, 1, 2, \ldots, n \\ (n+1)\varepsilon_r^n & \text{when } r = n+1, \end{cases} \tag{5.30}$$

is a homotopy lifting map associated to the cocycle η and whenever $q = 1$,

$$\psi_{\chi_n}(\varepsilon_r^n) = \begin{cases} (\frac{1+(-1)^r}{2})(-1)^{n+1}a\varepsilon_{r+1}^n + (n-r)\varepsilon_r^n b & r = 0, 1, 2, \ldots, n-1, \\ 0 & r = n, \\ (n-1)(b\varepsilon_r^n + \varepsilon_1^n c) & r = n+1, \end{cases} \tag{5.31}$$

is a homotopy lifting map associated to the cocycle χ. We present proofs of these claims as follows.

The case for η: For $r = 0, 1, \ldots, n$, we get for the left hand side of Equation (5.28)

$$\begin{aligned}
&(d\psi_{\eta_n} - \psi_{\eta_{n-1}} d)(\varepsilon_r^n) \\
&= d\big\{(n-r)\varepsilon_r^n\big\} - \psi_{\eta_{n-1}}\big\{\bar{\partial}_{n,r}[a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a] + \bar{\partial}_{r,0}[(-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b]\big\} \\
&= (n-r)\big\{\bar{\partial}_{n,r}[a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a] + \bar{\partial}_{r,0}[(-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b]\big\} \\
&- \bar{\partial}_{n,r}[(n-r-1)a\varepsilon_r^{n-1} + (-1)^{n-r}q^r(n-r-1)\varepsilon_r^{n-1}a] \\
&+ \bar{\partial}_{r,0}[(-q)^{n-r}(n-r)b\varepsilon_{r-1}^{n-1} + (-1)^n(n-r)\varepsilon_{r-1}^{n-1}b] \\
&= \bar{\partial}_{n,r}\big[(n-r) - (n-r-1)a\varepsilon_r^{n-1} + (-1)^{n-r}q^r((n-r) - (n-r-1))\varepsilon_r^{n-1}a\big] \\
&+ \bar{\partial}_{r,0}\big[(-q)^{n-r}((n-r) - (n-r))b\varepsilon_{r-1}^{n-1} + (-1)^n((n-r) - (n-r))\varepsilon_{r-1}^{n-1}b\big] \\
&= \bar{\partial}_{n,r}\big[a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a\big] \\
&= a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a, \quad \text{if} \quad r \neq n.
\end{aligned}$$

If $r = n$, the expression $(d\psi_{\eta_n} - \psi_{\eta_{n-1}} d)(\varepsilon_r^n)$ equals 0. For the case $r = n+1$, we obtain

$$\begin{aligned}
&(d\psi_{\eta_n} - \psi_{\eta_{n-1}} d)(\varepsilon_{n+1}^n) = d\big\{(n+1)\varepsilon_{n+1}^n\big\} - \psi_{\eta_{n-1}}\big\{[a\varepsilon_n^{n-1} + (-1)^n\varepsilon_0^{n-1}c]\big\} \\
&= (n+1)[a\varepsilon_n^{n-1} + (-1)^n\varepsilon_0^{n-1}c] + [na\varepsilon_n^{n-1} + (-1)^n(n-1)\varepsilon_0^{n-1}c] \\
&= a\varepsilon_n^{n-1}.
\end{aligned}$$

On the right hand side of Equation (5.28), we obtain after using the Koszul signs convention,

$$(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^n) = (\eta \otimes 1 - 1 \otimes \eta)$$
$$\begin{cases} \displaystyle\sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_0^{n-t}, & \text{if } r = 0 \\ \displaystyle\sum_{w=0}^{n} \sum_{j=max\{0,s+w-n\}}^{min\{w,s\}} (-q)^{j(n-s+j-w)} \varepsilon_j^w \otimes \varepsilon_{s-j}^{n-w}, & \text{if } 0 < r < n \\ \displaystyle\sum_{t=0}^{n} \varepsilon_t^t \otimes \varepsilon_{n-t}^{n-t}, & \text{if } r = n \\ \varepsilon_0^0 \otimes \varepsilon_{n+1}^n + \Big[\displaystyle\sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_{n-t+1}^{n-t}\Big] + \varepsilon_{n+1}^n \otimes \varepsilon_0^0, & \text{if } r = n+1. \end{cases}$$

In case $r = 0$, substitute 1 for the index t when applying $\eta \otimes 1$ and substitute $n-1$ for the index t when applying $1 \otimes \eta$. Similarly for the case $0 < r < n$, substitute $1, 0, r$ respectively for the indices w, j, s when applying $\eta \otimes 1$ and substitute $n-1, r, r$ respectively for the indices w, j, s when applying $1 \otimes \eta$. When $r = n$, everything is zero since $\eta(\varepsilon_i^1) = 0$ if $i \neq 0$ and finally when $r = n+1$ substitute 1 for the index t. What we then have is equal to the following

$$\begin{cases} \eta(\varepsilon_0^1)\varepsilon_0^{n-1} - (-1)^{n-1}\varepsilon_0^{n-1}\eta(\varepsilon_0^1) \\ \eta(\varepsilon_0^1)\varepsilon_r^{n-1} - (-1)^{n-1}(-q)^r\varepsilon_r^{n-1}\eta(\varepsilon_0^n) \\ \eta(\varepsilon_1^1)\varepsilon_0^{n-1} - (-1)^{n-1}\varepsilon_{n-1}^{n-1}\eta(\varepsilon_1^1) \\ \eta(\varepsilon_0^1)\varepsilon_n^{n-1} \end{cases} = \begin{cases} a\varepsilon_0^{n-1} + (-1)^n\varepsilon_0^{n-1}a & \text{if } r = 0 \\ a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a & \text{if } 0 < r < n \\ 0 & \text{if } r = n \\ a\varepsilon_n^{n-1} & \text{if } r = n+1 \end{cases}$$
$$= (d\psi_{\eta_n} - \psi_{\eta_{n-1}}d)(\varepsilon_r^n).$$

Thus we have shown that for $r = 0, 1, \ldots, n+1$, Equation (5.28) holds.

The case for χ: When $r = 0$, the left hand side of Equation (5.29) is

$$
\begin{aligned}
&(d\psi_{\chi_n} - \psi_{\chi_{n-1}} d)(\varepsilon_0^n) \\
&= d((-1)^{n+1} a\varepsilon_1^n + n\varepsilon_0^n b) - \psi_\chi(a\varepsilon_0^{n-1} + (-1)^n \varepsilon_0^{n-1} a) \\
&= (-1)^{n+1} a[a\varepsilon_1^{n-1} + (-1)^{n-1} q\varepsilon_1^{n-1} a + (-q)^{n-1} b\varepsilon_0^{n-1} + (-1)^n \varepsilon_0^{n-1} b] \\
&+ n[a\varepsilon_0^{n-1} + (-1)^n \varepsilon_0^{n-1} a]b - a[(-1)^n a\varepsilon_1^{n-1} + (n-1)\varepsilon_0^{n-1} b] \\
&- (-1)^n[(-1)^n a\varepsilon_1^{n-1} + (n-1)\varepsilon_0^{n-1} b]a.
\end{aligned}
$$

Whenever $q = 1, ab = ba$, so we obtain $ab\varepsilon_0^{n-1} + (-1)^n \varepsilon_0^{n-1} ab$ which is equal to the right hand side of (5.29). Therefore $(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^n)$ becomes

$$
(\chi \otimes 1 - 1 \otimes \chi) \sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_0^{n-t} = \sum_{t=0}^{n} \chi(\varepsilon_0^t)\varepsilon_0^{n-t} - \sum_{t=0}^{n} (-1)^t \varepsilon_0^t \chi(\varepsilon_0^{n-t}).
$$

When $t = 1$ in the first summation and $t = n-1$ in the second summation, the last expression is equal to $\chi(\varepsilon_0^1)\varepsilon_0^{n-1} - (-1)^{n-1}\varepsilon_0^{n-1}\eta(\varepsilon_0^1) = ab\varepsilon_0^{n-1} + (-1)^n \varepsilon_0^{n-1} ab$.

When $0 < r < n$ and r is even, the left hand side of (5.29) that is $(d\psi_{\chi_n} - \psi_{\chi_{n-1}} d)(\varepsilon_r^n)$ is equal to

$$
\begin{aligned}
&d((-1)^{n+1} a\varepsilon_{r+1}^n + (n-r)\varepsilon_r^n b) \\
&- \psi_\chi(a\varepsilon_r^{n-1} + (-1)^{n-r} q^r \varepsilon_r^{n-1} a + (-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b) \\
&= (-1)^{n+1} a[a\varepsilon_{r+1}^{n-1} + (-1)^{n-r-1} q^{r+1} \varepsilon_{r+1}^{n-1} a + (-q)^{n-r-1} b\varepsilon_r^{n-1} + (-1)^n \varepsilon_r^{n-1} b] \\
&+ (n-r)[a\varepsilon_r^{n-r} + (-1)^{n-1} q^r \varepsilon_r^{n-1} a + (-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b]b \\
&- a[(-1)^n a\varepsilon_{r+1}^{n-1} + (n-r-1)\varepsilon_r^{n-1} b] - (-1)^{n-r} q^r[(-1)^n a\varepsilon_{r+1}^{n-1} + (n-r-1)\varepsilon_r^{n-1} b]a \\
&- (-q)^{n-r} b[(n-r)\varepsilon_{r-1}^{n-1} b] - (-1)^n[(n-r)\varepsilon_{r-1}^{n-1} b]b \\
&= [(-1)^{2n-r} q^{r+1} - (-1)^{2n-r} q^r] a\varepsilon_{r+1}^{n-1} a + [(-1)^{n+1}(-q)^{n-r-1}] ab\varepsilon_r^{n-1} \\
&+ [(-1)^{2n+1} + (n-r) - (n-r-1)] a\varepsilon_r^{n-1} b + [(-1)^{n-r} q^r (n-r)]\varepsilon_r^{n-1} ab \\
&+ [-(-1)^{n-r} q^r (n-r-1)]\varepsilon_r^{n-1} ba + [(-q)^{n-r}(n-r) - (-q)^{n-r}(n-r)] b\varepsilon_{r-1}^{n-1} b
\end{aligned}
$$

which with $q = 1$ becomes $ab\varepsilon_r^{n-1} + (-1)^{n-r}\varepsilon_r^{n-1}ab$ and all other terms vanish. When r is odd and $0 < r < n$, we obtain the following for the left hand side of (5.29)

$$\begin{aligned}
&(d\psi_{\chi_n} - \psi_{\chi_{n-1}}d)(\varepsilon_r^n)\\
&= d((n-r)\varepsilon_r^n b) - \psi_\chi(a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a + (-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b)\\
&= (n-r)[a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a + (-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b]b\\
&- a[(n-r-1)\varepsilon_r^{n-1}b] - (-1)^{n-r}q^r[(n-r-1)\varepsilon_r^{n-1}b]a\\
&- (-q)^{n-r}b[(-1)^n a\varepsilon_r^{n-1} + (n-r)\varepsilon_{r-1}^{n-1}b] - (-1)^n[(-1)^n a\varepsilon_r^{n-1} + (n-r)\varepsilon_{r-1}^{n-1}b]b\\
&= [-(-1)^n(-q)^{n-r}]ba\varepsilon_r^{n-1} + [(n-r) - (n-r-1) - (-1)^{2n}]a\varepsilon_r^{n-1}b + [(-1)^{n-r}q^r(n-r)]\varepsilon_r^{n-1}ab\\
&+ [-(-1)^{n-r}q^r(n-r-1)]\varepsilon_r^{n-1}ba + [(-q)^{n-r}(n-r) - (-q)^{n-r}(n-r)]b\varepsilon_{r-1}^{n-1}b
\end{aligned}$$

which with $q = 1$, we get the result obtained previously: $ab\varepsilon_r^{n-1} + (-1)^{n-r}\varepsilon_r^{n-1}ab$. On the other hand, the right hand side of (5.29) when $0 < r < n$ becomes

$$(\chi \otimes 1 - 1 \otimes \chi)\Big[\sum_{w=0}^{n}\sum_{j=max\{0,r+w-n\}}^{min\{w,r\}}(-q)^{j(n-r+j-w)}\varepsilon_j^w \otimes \varepsilon_{r-j}^{n-w}\Big].$$

To get a non-zero term, substitute $w = 1, j = 0$ then apply $\chi \otimes 1$, and substitute $w = n-1, j = r$ and apply $1 \otimes \chi$:

$$(-q)^{0(n-r+1)}\chi(\varepsilon_0^1)\varepsilon_r^{n-1} - (-1)^{n-1}(-q)^{r(n-r+r-n+1)}\varepsilon_r^{n-1}\chi(\varepsilon_0^1) = ab\varepsilon_r^{n-1} + (-1)^{n-r}\varepsilon_r^{n-1}ab.$$

When $r = n$, the left hand side of (5.29) becomes

$$(d\psi_\chi - \psi_\chi d)(\varepsilon_n^n) = d(0) - \psi_\chi(b\varepsilon_{n-1}^{n-1} + (-1)^n\varepsilon_{n-1}^{n-1}b) = 0 - b\cdot 0 + (-1)^{n+1}0\cdot b = 0,$$

while the right hand side $(\chi\otimes 1-1\otimes\chi)\Delta_{\mathbb{K}}(\varepsilon_n^n)$ becomes

$$(\chi\otimes 1-1\otimes\chi)\sum_{t=0}^{n}\varepsilon_1^t\otimes\varepsilon_{n-t}^{n-t}=\chi(\varepsilon_1^1)\varepsilon_1^{n-1}-(-1)^{n-1}\varepsilon_1^{n-1}\chi(\varepsilon_1^1)=0$$

and they are equal. It is also true whenever $r=n+1$:

$$\begin{aligned}
&(d\psi_\chi-\psi_\chi d)(\varepsilon_{n+1}^n)=d((n-1)\varepsilon_1^n c+b\varepsilon_{n+1}^n)-\psi_\chi(a\varepsilon_n^{n-1}+(-1)^n\varepsilon_0^{n-1}c)\\
&=[(n-1)-(n-2)+(-1)^{2n-1}]a\varepsilon_1^{n-1}c+(n-1)[(-q)^{n-1}+(-1)^n]b\varepsilon_0^{n-1}c\\
&+(n-1)[(-1)^n+(-1)^{n-1}]\varepsilon_0^{n-1}bc+(n-1)ba\varepsilon_n^{n-1}-(n-2)ab\varepsilon_n^{n-1}\\
&=(n-1-n+2)ab\varepsilon_n^{n-1}=ab\varepsilon_n^{n-1}
\end{aligned}$$

is equal to

$$\begin{aligned}
&(\chi\otimes 1-1\otimes\chi)\Delta_{\mathbb{K}}(\varepsilon_{n+1}^n)=(\chi\otimes 1-1\otimes\chi)\sum_{t=0}^{n}\varepsilon_0^t\otimes\varepsilon_{n-t+1}^{n-t}+\varepsilon_{n+1}^n\otimes\varepsilon_0^0\\
&=\chi\otimes 1(\varepsilon_0^1\otimes\varepsilon_n^{n-1})-1\otimes\chi((-1)^{n-1}\varepsilon_0^{n-1}\otimes\varepsilon_2^1)+(\chi\otimes 1-1\otimes\chi)(\varepsilon_{n+1}^n\otimes\varepsilon_0^0)\\
&=ab\varepsilon_n^{n-1}.
\end{aligned}$$

We have therefore shown that for $r=0,1,\ldots,n+1$, Equation (2.17) holds.

Remark 5.32. ***Some bracket computations based on these examples****: Now take $\eta=\begin{pmatrix}a & 0 & 0\end{pmatrix}$ and $\chi=\begin{pmatrix}ab & 0 & 0\end{pmatrix}$ to be the degree 1 cocycles with homtopy lifting maps given in* (5.30) *and* (5.31) *respectively. Also take $\overline{\eta}=\begin{pmatrix}a & 0 & 0 & 0\end{pmatrix}$ and $\overline{\chi}=\begin{pmatrix}0 & 0 & ab & 0\end{pmatrix}$ to be the degree 2 cocycles whose homotopy lifting maps were given in* (5.26) *and* (5.27) *respectively. Take $\theta=\begin{pmatrix}ab & 0 & 0 & 0\end{pmatrix}$ to be the degree 2 cocycle appearing in the table of solutions in Subsection 5.2.1. The following bracket structure can be verified by direct computations.*

$[\cdot\,,\cdot]$	η	χ	$\overline{\eta}$	$\overline{\chi}$
η	0	0	$-\overline{\eta}$	$\overline{\chi}$
χ	0	0	$-\theta$	0
$\overline{\eta}$	$\overline{\eta}$	θ	0	0
$\overline{\chi}$	-$\overline{\chi}$	0	0	0

Table 5.3: Some bracket computations

5.3 An application

In this section, we specify solutions to the Maurer-Cartan equation using Theorems 5.14 and 5.23. This is particularly useful in the theory of deformation of algebras. In particular, since the second Hochschild cohomology group of an algebra contains information about infinitesimal deformations of the algebra, the result of this section could be useful in determining infinitesimal deformations of algebras defined by quivers and relations. We have already remarked in Section 2.4 that Hochschild cohomology is a differential graded Lie algebra i.e.

$$\bar{d}([f,g]) = [\bar{d}(f), g] + (-1)^{(m-1)}[f, \bar{d}(g)],$$

where $\bar{d}(f) = (-1)^{(m-1)} f\delta_{m+1}$. Since the resolution $(\mathbb{K}, d)$ embeds into the bar resolution $(\mathbb{B}, \delta)$ via $\mathbb{K} \xrightarrow{\iota} \mathbb{B}$, with $\iota d = \delta\iota$, there are no sign changes, hence we take $\bar{d}(\eta) = (-1)^{(m-1)} d^*_{m+1}\eta = (-1)^{(m-1)}\eta d_{m+1}$. An Hochschild 2-cocycle η is then said to satisfy the Maurer-Cartan equation if

$$\bar{d}(\eta) + \frac{1}{2}[\eta, \eta] = 0. \tag{5.33}$$

Applying the definition of the bracket we obtain the following version of the Maurer-Cartan equation

$$-\, d^*_3(\eta) = -\frac{1}{2}(\eta\psi_\eta + \eta\psi_\eta) = -\eta\psi_\eta \tag{5.34}$$

Theorem 5.35. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Assume that Q is a finite quiver and $I \subseteq J^2$. Denote by $\{f^m_r\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as* *a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis $\{\varepsilon_r^m\}_{r=0}^{t_m} \in \mathbb{K}_m$. Suppose $\eta : \mathbb{K}_2 \to \Lambda$ is a cocycle defined by $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_u^1 f_v^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, for some $0 \leq i \leq t_2$. The cocycle η satisfies the Maurer-Cartan equation if*

$$\sum_{p=0}^{t_1} c_{pi}(3,r,1) f_p^1 f_u^1 f_v^1 - c_{ip}(3,r,2) f_u^1 f_v^1 f_p^1 - b_{3,r}(2,i) f_u^1 f_u^1 f_v^1 = 0,$$

for some scalars $c_{p,i}(3,r,*)$ coming from the comultiplicative structure and some scalars $b_{3,r}(2,*)$ coming from the homotopy lifting map ψ_η and satisfying the conditions of Theorem 5.23.

Proof. The left hand side of Equation (5.34) is given by

$$\begin{aligned} d_3^*(\eta)(\varepsilon_r^3) &= \eta d_3(\varepsilon_r^3) = \eta\Big[\sum_{j=0}^{t_2} < \varepsilon_j^2 >_{3,r} \Big] = \eta \sum_{j=0}^{t_2} \Big[\sum_{p=0}^{t_1} c_{pj}(3,r,1) f_p^1 \varepsilon_j^2 - \sum_{q=0}^{t_1} c_{jq}(3,r,2) \varepsilon_j^2 f_q^1\Big] \\ &= \sum_{p=0}^{t_1} c_{pi}(3,r,1) f_p^1 \eta(\varepsilon_i^2) - \sum_{q=0}^{t_1} c_{iq}(3,r,2) \eta(\varepsilon_i^2) f_q^1 \\ &= \sum_{p=0}^{t_1} c_{pi}(3,r,1) f_p^1 f_u^1 f_v^1 - \sum_{q=0}^{t_1} c_{iq}(3,r,2) f_u^1 f_v^1 f_q^1 \end{aligned}$$

which is in kQ_3. On the other hand we can conclude from the combinatorial description of the Gerstenhaber bracket given in Theorem 5.24 that $[\eta, \eta](\varepsilon_r^3) \in kQ_3$ since λ is a path of length 2. In particular,

$$\begin{aligned} \frac{1}{2}[\eta,\eta](\varepsilon_r^3) &= \eta\psi_\eta(\varepsilon_r^3) = \eta(b_{3,r}(2,i) f_u^1 \varepsilon_i^2 + b_{3,r}(2,j) \varepsilon_j^2 f_v^1) \\ &= b_{3,r}(2,i) f_u^1 \eta(\varepsilon_i^2) = b_{3,r}(2,i) f_u^1 f_u^1 f_v^1. \end{aligned}$$

The Maurer-Cartan Equation of (5.34) becomes

$$\sum_{p=0}^{t_1} c_{pi}(3,r,1) f_p^1 f_u^1 f_v^1 - \sum_{q=0}^{t_1} c_{iq}(3,r,2) f_u^1 f_v^1 f_q^1 = b_{3,r}(2,i) f_u^1 f_u^1 f_v^1.$$

Since both left and right hand side of the above equation are in kQ_3 we get

$$\sum_{p=0}^{t_1} c_{pi}(3,r,1) f_p^1 f_u^1 f_v^1 - c_{ip}(3,r,2) f_u^1 f_v^1 f_p^1 - b_{3,r}(2,i) f_u^1 f_u^1 f_v^1 = 0. \tag{5.36}$$

□

Remark 5.37. *If $c_{ui}(3,r,1) = b_{3,r}(2,i)$ and the ideal I of relations is such that $f_i^1 f_j^1 = \kappa(i,j) f_j^1 f_i^1$, for some scalars $\kappa(i,j)$, Equation* (5.36) *becomes*

$$\sum_{p \neq u} c_{pi}(3,r,1) f_p^1 f_u^1 f_v^1 - c_{ip}(3,r,2) f_u^1 f_v^1 f_p^1 = \sum_{p \neq u} [\kappa(p,u)\kappa(p,v) c_{pi}(3,r,1) - c_{ip}(3,r,2)] f_u^1 f_v^1 f_p^1$$

which is zero whenever $\kappa(p,u)\kappa(p,v)c_{pi}(3,r,1) = c_{ip}(3,r,2)$ for all p, u, v and i. If

$$\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_u^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix},$$

then such a η cannot satisfy the Maurer-Cartan equation. This is because while the left hand side of the Maurer-Cartan Equation 5.34 yields a linear combination of paths of length 2, the right hand side yields a linear combination of paths of length 1, so equality does not hold.

REFERENCES

[1] R.O. BUCHWEITZ, E.L. GREEN, D. MADSEN, Ø. SOLBERG, *Hochschild cohomology without finite global dimension*, Math. Res. Lett., vol. 12, no. 6, pp. 805-816, 2005.

[2] R.O. BUCHWEITZ, E.L. GREEN, N. SNASHALL, Ø. SOLBERG, *Multiplicative structures for Koszul algebras*, The Quarterly Journal of Mathematics, vol. 59, no. 4, pp. 441-454, 2008.

[3] M. GERSTENHABER, *The Cohomology Structure of an Associative Ring*, Annals of Mathematics, Second Series, vol. 78, no. 2, pp. 267-288, 1963.

[4] E.L. GREEN, Ø. SOLBERG, *An Algorithmic approach to resolutions*, J. Symbolic Comput., vol. 42, pp. 1012-1033, 2007.

[5] E. L. GREEN, G. HARTMAN, E. N. MARCOS, Ø. SOLBERG, *Resolutions over Koszul algebras*, Arch. Math. vol. 85, pp. 118-127, 2005.

[6] L. GRIMLEY, V. C. NGUYEN AND S. WITHERSPOON, *Gerstenhaber brackets on Hochschild cohomology of twisted tensor products*, J. Noncommutative Geometry, vol.11, no. 4, pp. 1351-1379, 2017.

[7] P.J. HILTON, U. STAMMBACH, *A Course in Homological Algebra*, Graduate Text in Mathematics 4, Springer 1997.

[8] T. KARADAĞ, D. MCPHATE, P.S. OCAL, T. OKE, S. WITHERSPOON, *Gerstenhaber brackets on Hochschild cohomology of general twisted tensor products*, J. of Pure and Applied Algebra, vol. 225, no. 6, 2021.

[9] B. KELLER, *Hochschild cohomology and derived Picard group*, J. of Pure and Applied Algebra, vol. 190, no. 1–3, pp. 177-196, 2004.

[10] R. MARTÍNEZ-VILLA, *Introduction to Koszul algebras*, Rev. Un. Mat. Argentina, vol. 58, no.2, pp. 67-95, 2008.

[11] C. NEGRON AND S. WITHERSPOON, *An alternate approach to the Lie bracket on Hochschild cohomology*, Homology, Homotopy and Applications, vol. 18, no.1, pp. 265-285, 2016.

[12] T. OKE, *Cup product and Gerstenhaber bracket on Hochschild cohomology of a family of quiver algebras*, ArXiv.2004.00780.

[13] N. SNASHALL, *Support varieties and the Hochschild cohomology ring modulo nilpotence*, Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Tsukuba, pp. 68–82, 2009.

[14] N. SNASHALL, Ø. SOLBERG, *Support varieties and the Hochschild cohomology rings*, Proc. London Maths. Soc., vol. 88, no. 3, pp. 705–732, 2004.

[15] M. SUÁREZ-ÁLVAREZ, *A little bit of Extra functoriality for Ext and the computation of the Gerstenhaber brackets*, J. of Pure and Applied Algebra, vol. 221, no. 8, pp. 1981-1998, 2017.

[16] Y. VOLKOV, *Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution*, Proc. of the Edinburgh Math. Soc., vol. 62, no. 3, pp. 817-836, 2019.

[17] C. WEIBEL, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press 1994.

[18] S. WITHERSPOON, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics 204, American Mathematical Society, 2019.

[19] F. XU, *Hochschild and ordinary cohomology rings of small categories*, Adv Math., vol. 219, pp. 1872-1893, 2008.

[20] Y. XU, C. ZHANG, *More counterexamples to Happel's question and Snashall-Solberg's conjecture*, ArXiv:1109.3956.