RIGID ANALYTIC TRIVIALIZATIONS AND PERIODS OF DRINFELD MODULES AND THEIR TENSOR PRODUCTS

A Dissertation

by

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ABSTRACT

The purpose of this research is to study Drinfeld modules, tensor product of Drinfeld modules, their rigid analytic trivializations, and their periods. A formula for rigid analytic trivializations for Drinfeld modules was originally given by Pellarin. In this research, we provide a new method to construct a rigid analytic trivialization for Drinfeld modules. Unlike Pellarin's formula, our method does not require periods of Drinfeld modules. Given a rank r Drinfeld module, we provide a recursive process that produce a convergent t-division sequence. Consequently we use the t-division sequence to construct a sequence of matrices $(\Upsilon_n)_{n\geq 1}$ and by computing the limit of $(\Upsilon_n)_{n\geq 1}$, we obtain our rigid analytic trivialization for a Drinfeld module. Using the function $\mathcal{L}_\phi(\xi;t)$ introduced by El-Guindy and Papanikolas, we are able to find an explicit formula for our rigid analytic trivialization. Furthermore, in the second part of our research, we investigate tensor products of two Drinfeld modules ϕ_1 and ϕ_2. Using the theory of t-motives, we define a t-action for $\phi_1 \otimes \phi_2$. Inspired by a formula for periods of the tensor product of Carlitz module by Anderson and Thakur, we discover a formula for periods of the tensor product $\phi_1 \otimes \phi_2$. Moreover, we provide a formula for Anderson generating functions for the tensor product $\phi_1 \otimes \phi_2$.

DEDICATION

To my family.

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NOMENCLATURE

$\mathbb{F}_q$	finite field with $q = p^n$ elements.
A	$\mathbb{F}_q[\theta]$, the polynomial ring in θ over $\mathbb{F}_q$.
k	$\mathbb{F}_q(\theta)$, the fraction field of A.
k_∞	$\mathbb{F}_q((1/\theta))$, the completion of k with respect to $\lvert \cdot \rvert$.
$\mathbb{K}$	the completion of an algebraic closure of k_∞.
$\overline{k}$	the algebraic closure of k inside $\mathbb{K}$.
$\mathbf{A}$	$\mathbb{F}_q[t]$, the polynomial ring in t over $\mathbb{F}_q$, t independent from θ.
$\mathrm{Mat}_{m\times n}(R)$	for a ring R, the left R-module of $m \times n$ matrices.
$\mathrm{Mat}_d(R)$	$\mathrm{Mat}_{d\times d}(R)$.
R^d	$\mathrm{Mat}_{d\times 1}(R)$.
M^{T}	the transpose of a matrix M.

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1. INTRODUCTION

1.1 Background and motivation

The theory of Drinfeld modules was established by V. G. Drinfeld [9] in 1974. A higher dimensional version of Drinfeld modules, called t-*modules*, was introduced in 1986 by Anderson [1]. In particular, a Drinfeld module is a 1-dimensional t-module. Anderson defined a new object called t-*motives*, whose category is anti-equivalent to the category of t-modules. Anderson also gave a notion of the rigid analytic triviality of t-motives which play an important role in studying the uniformization of t-modules. He proved that a t-module associated to an abelian t-motive M is uniformizable if and only if M is rigid analytically trivial.

The rigid analytic trivialization is also useful for finding periods and quasi-periods of t-modules. Periods arise as the kernel of the exponential function associated to the t-module, and its quasi-periods arise as values of quasi-periodic functions coming from biderivations associated to the t-motive of the t-module. The exponential function was developed by Anderson [1] (see also [15], [31]). The theory of the de Rham module and quasi-periodic functions for Drinfeld modules was developed by Anderson, Deligne, Gekeler, and Yu [12], [14], [32], and this was extended by Brownawell and Papanikolas to higher dimensional t-modules [5] (see also [18, §2.5]). Anderson first observed that quasi-periods of Drinfeld modules could be obtained by specializations and residues of what are now called Anderson generating functions (see [14]), and it was observed by Pellarin that Anderson generating functions were crucial ingredients to constructing the rigid analytic trivialization of a Drinfeld module [26] (see also [7], [8]). Anderson generating functions have subsequently arisen in many other contexts for Drinfeld modules and general t-modules (e.g., see [11], [16], [17], [21], [22], [27], [28], [29], [30]). In the present dissertation we explore these connections in depth for the tensor product of two Drinfeld modules (see Theorem 4.26 and Theorem 4.32).

The method to construct a rigid analytic trivialization for a Drinfeld module was originally given by Pellarin [26]. In his method, he fixed a basis $\pi_1, \ldots, \pi_r$ of the period lattice and used the Anderson generating function associated to each π_i to define a rigid analytic trivialization for a Drinfeld module. See also [7], [8] for further developments in these directions.

In this dissertation, we provide a new method to construct a rigid analytic trivialization for a Drinfeld module (see Theorem 3.15). Given a rank r Drinfeld module, we provide a recursive process that produces a convergent t-division sequence. Then we use the t-division sequence to construct a sequence of matrices $(\Upsilon_n)_{n\geq 1}$ and by computing the limit of $(\Upsilon_n)_{n\geq 1}$, we obtain our rigid analytic trivialization for a Drinfeld module. Adopting the function $\mathcal{L}_\phi(\xi; t)$ introduced by El-Guindy and Papanikolas [11], we are able to find an explicit formula for our rigid analytic trivialization. Furthermore we show that the rigid analytic trivialization derived from our approach coincides with the one obtained by using Pellarin's method. The benefit of our construction is that it is effective in the sense that our construction requires only a finite amount of initial computation.

Moreover, we investigate the tensor products of two Drinfeld modules. The tensor powers of Carlitz modules, which are Drinfeld modules of rank 1, are well studied by Anderson and Thakur [3]. They showed that a generator of the period lattice of the tensor power $C^{\otimes n}$ has a final coordinate equal to the n-th power of the Carlitz period. For more details about tensor power $C^{\otimes n}$, the reader is directed to Goss [15, §5] and Thakur [31, §7]. In the second part of our research, our goal is to expand the results by Anderson and Thakur by studying the tensor product of two Drinfeld modules with arbitrary rank. Using the theory of t-motives, we express the t-action of the tensor products of two Drinfeld modules. As a consequence, we provide a formula for periods of the tensor product of two Drinfeld modules with arbitrary rank. Moreover, using our formula for the periods, we obtain a formula for Anderson generating functions for the tensor product of two Drinfeld modules.

1.2 An outline of this dissertation

In §2, we will give preliminary definitions and results on Drinfeld modules, t-modules and t-motives, which will be used to state and prove our results in §3 and §4.

We then give some details about rigid analytic trivializations for Drinfeld modules in §3. First, we recall Pellarin's method to construct a rigid analytic trivialization in §3.1. Then we provide our method to construct a rigid analytic trivialization in §3.2. We finish the section by providing an application of our method and an example on a specific rank 2 Drinfeld module in §3.3.

In §4, we investigate the tensor product of two Drinfeld modules of arbitrary rank. In §4.1, we state some results from Anderson and Thakur. Then we give a definition of a tensor product of two Drinfeld modules $\phi_1 \otimes \phi_2$ in §4.2. In §4.3, we state the main result in Theorem 4.26, which provide a formula for the periods of $\phi_1 \otimes \phi_2$. Moreover, we provide a formula for the Anderson generating functions for the tensor product in §4.4. Finally, we give an example in §4.5, where we consider a tensor product of Drinfeld modules of rank 2.

2. PRELIMINARIES

2.1 Drinfeld modules

Let $\mathbb{F}_q$ denote the field with q elements and let $k = \mathbb{F}_q(\theta)$ be the rational function field in the variable θ over $\mathbb{F}_q$. Let $k_\infty = \mathbb{F}_q((1/\theta))$ be the completion of k at ∞, with absolute value $|\cdot|$ chosen so that $|\theta| = q$. Let v_∞ be the valuation at ∞ with $v_\infty(\theta) = -1$, and let $\deg := -v_\infty$. Let $\mathbb{K}$ denote the completion of an algebraic closure of k_∞.

Consider the q-th power Frobenius map $\tau : \mathbb{K} \to \mathbb{K}$ defined by $z \mapsto z^q$. Let $\mathbb{K}[\tau]$ be the ring of twisted polynomials in τ subject to the relation

$$\tau a = a^q \tau, \quad a \in \mathbb{K}.$$

Let $\mathbf{A} = \mathbb{F}_q[t]$ be the polynomial ring in a variable t independent from θ. A *Drinfeld module of rank* r over $\mathbb{K}$ is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : \mathbf{A} \to \mathbb{K}[\tau]$$

such that

$$\phi_t = \theta + A_1\tau + \cdots + A_r\tau^r, \quad A_r \neq 0. \tag{2.1}$$

We obtain an $\mathbf{A}$-module structure on $\mathbb{K}$ induced by ϕ by the action

$$a \cdot x = \phi_a(x), \quad a \in \mathbf{A}, \quad x \in \mathbb{K}.$$

For any $a \in \mathbf{A}$, the *a-torsion* of ϕ is the $\mathbf{A}$-submodule $\phi[a] = \{x \in \mathbb{K} : \phi_a(x) = 0\}$. The *exponential* of ϕ is the $\mathbb{F}_q$-linear power series in z,

$$\exp_\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^{q^n}, \quad \alpha_0 = 1, \quad \alpha_n \in \mathbb{K},$$

satisfying $\exp_\phi(a(\theta)z) = \phi_a(\exp_\phi(z))$ for any $a \in \mathbf{A}$. This power series defines an entire function $\exp_\phi : \mathbb{K} \to \mathbb{K}$. The *logarithm* of ϕ is the formal inverse of $\exp_\phi(z)$, which can be written as

$$\log_\phi(z) = \sum_{n=0}^{\infty} \beta_n z^{q^n}, \quad \beta_0 = 1, \quad \beta_n \in \mathbb{K},$$

and has a finite radius of convergence, denoted by R_ϕ, see [15, Prop. 4.14.2].

Let Λ_ϕ be the kernel of $\exp_\phi(z)$. We call Λ_ϕ the *period lattice* of ϕ and call any element of Λ_ϕ a *period* of ϕ. Then $\Lambda_\phi \subset \mathbb{K}$ is a free A-module of rank r.

2.2 Anderson generating functions

Define the Tate algebra

$$\mathbb{T} = \left\{ \sum_{i=0}^{\infty} c_i t^i \in \mathbb{K}[[t]] : |c_i| \to 0 \right\}.$$

We use the Gauss norm $\|\cdot\|$ on $\mathbb{T}$ defined by $\|\sum c_i t^i\| = \sup_i |c_i| = \max_i |c_i|$. For any $f = \sum_{i=0}^{\infty} c_i t^i \in \mathbb{T}$ and any $n \in \mathbb{Z}$, let

$$f^{(n)} = \sum_{i=0}^{\infty} c_i^{q^n} t^i \in \mathbb{T}.$$

For any matrix $M = (f_{ij}) \in \mathrm{Mat}_{r\times s}(\mathbb{T})$ and any $n \in \mathbb{Z}$, we define the matrix $M^{(n)} = (f_{ij}^{(n)}) \in \mathrm{Mat}_{r\times s}(\mathbb{T})$ and we set $\|M\| = \max_{i,j} \|f_{ij}\|$. Assume that we have a Drinfeld module ϕ of rank r given as in equation (2.1). For $u \in \mathbb{K}$, the *Anderson generating function* associated to u is defined by

$$f_\phi(u;t) = \sum_{m=0}^{\infty} \exp_\phi\left(\frac{u}{\theta^{m+1}}\right) t^m.$$

Pellarin [26, §4.2] gave a formula for Anderson generating functions in the following proposition.

Proposition 2.1 (Pellarin)**.** *For $u \in \mathbb{K}$,*

$$f_\phi(u;t) = \sum_{n=0}^{\infty} \frac{\alpha_n u^{q^n}}{\theta^{q^n} - t} \in \mathbb{T},$$

where α_n are the coefficients of $\exp_\phi$. Furthermore, $f_\phi(u;t)$ extends to a meromorphic function on $\mathbb{K}$ with simple poles at $t = \theta^{q^n}$, $n = 0, 1, \ldots$, and with residues

$$\mathrm{Res}_{t=\theta^{q^n}} f_\phi(u;t) = -\alpha_n u^{q^n}.$$

As an example, we consider the Carlitz module C. We know that Λ_C is an $\mathbf{A}$-module of rank 1. Fix a nonzero element $\widetilde{\pi} \in \Lambda_C$. The Anderson generating function associated to $\widetilde{\pi}$ is

$$f_C(\widetilde{\pi};t) = \sum_{m=0}^{\infty} \exp_C\left(\frac{\widetilde{\pi}}{\theta^{m+1}}\right)t^m.$$

Using Proposition 2.1 of Pellarin, we know that $\mathrm{Res}_{t=\theta} f_C(\widetilde{\pi};t) = -\widetilde{\pi}$. Let

$$\omega_C(t) = (-\theta)^{1/(q-1)} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1},$$

which is defined by Anderson and Thakur [3, §2] and is nowadays called the *Anderson-Thakur function*. It is known from [3] that $f_C(\widetilde{\pi};t) = \omega_C(t)$. In other words, $\omega_C(t)$ is a formula for the Anderson generating function that does not require $\widetilde{\pi}$ to define it. This gives us the benefit of finding a formula for $\widetilde{\pi}$ by comparing the residues of both functions, which gives

$$\widetilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

We call $\widetilde{\pi}$ the Carlitz period.

Anderson generating functions are useful tools for finding periods of a Drinfeld module. In the recent work of El-Guindy and Papanikolas [11], they expressed Anderson generating functions in terms of the defining polynomial of the Drinfeld module. They defined a series $\mathcal{L}_\phi(\xi;t)$ by using shadowed partitions as follows. For $n, r \in \mathbb{N}$, $P_r(n)$ is the set of r-tuples $(S_1, S_2, \ldots, S_r)$ such that for each i, $S_i \subseteq \{0, 1, \ldots, n-1\}$ and the set $\{S_i + j : 1 \le i \le r, 0 \le j \le i-1\}$ forms a partition of $\{0, 1, \ldots, n-1\}$. They defined the series

$$\mathcal{L}_\phi(\xi; t) = \sum_{n=0}^{\infty} \mathcal{B}_n(t) \xi^{q^n} \in \mathbb{T}, \quad |\xi| < R_\phi,$$

where R_ϕ is the radius of convergence of $\log_\phi(z)$ and

$$\mathcal{B}_n(t) = \sum_{\mathbf{S} \in P_r(n)} \prod_{i=1}^{r} \prod_{j \in S_i} \frac{A_i^{q^j}}{t - \theta^{q^{i+j}}}$$

The series $\mathcal{L}_\phi(\xi; t)$ is related to the Anderson generating function as one can see in the following theorem proved by El-Guindy and Papanikolas [11, Thm. 6.13]. Moreover, it also appears in our formula for rigid analytic trivialization in section 3.2.

Theorem 2.2 (El-Guindy and Papanikolas)**.** *Let $u \in \mathbb{K}$ with $|u| < R_\phi$ and $\xi = \exp_\phi(u)$. Suppose that $|\xi| < R_\phi$. Then $\mathcal{L}_\phi(\xi; \theta) = \log_\phi(\xi) = u$ and $\mathcal{L}_\phi(\xi; t) = -(t - \theta) f_\phi(u; t)$.*

2.3 t-motives for Drinfeld modules

The ring $\mathbb{K}[t, \tau]$ is the polynomial ring in t and τ with coefficients in $\mathbb{K}$ subject to the following relations,

$$tc = ct, \;\; t\tau = \tau t, \;\; \tau c = c^q \tau, \;\; c \in \mathbb{K}.$$

A *t-motive* M is a left $\mathbb{K}[t, \tau]$-module that is free and finitely generated over $\mathbb{K}[\tau]$ and for which there is $\ell \in \mathbb{N}$ with

$$(t - \theta)^\ell (M/\tau M) = \{0\}.$$

Suppose we have a Drinfeld module $\phi : \mathbb{F}_q[t] \to \mathbb{K}[\tau]$, given as in equation (2.1). The *t-motive associated to* ϕ, denoted $M(\phi)$, is defined as follows: let $M(\phi) = \mathbb{K}[\tau]$ and make $M(\phi)$ into a left $\mathbb{K}[t, \tau]$-module by setting

$$ct^i \cdot m := cm\phi_{t^i}, \quad m \in M(\phi), \quad c \in \mathbb{K}[\tau].$$

2.4 t-modules

Developing on the theory of Drinfeld modules, Anderson introduced t-modules which are a higher dimensional version of Drinfeld modules. Most concepts in the theory of Drinfeld modules still appear in the theory of t-modules. A *t-module* over $\mathbb{K}$ is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$$

such that ϕ_t is given by

$$\phi_t = B_0 + B_1\tau + \cdots + B_\ell \tau^\ell, \tag{2.2}$$

where $B_0, \ldots, B_\ell \in \mathrm{Mat}_d(\mathbb{K})$ and $B_0 = \theta I_d + N$ for some nilpotent matrix N. We denote $\mathrm{d}\phi_t = B_0$ and we say that ϕ has *dimension* d. Every t-module ϕ induces an $\mathbf{A}$-module structure on $\mathbb{K}^d$ by setting

$$a \cdot x = \phi_a(x), \quad a \in \mathbf{A}, x \in \mathbb{K}^d.$$

Given two t-modules ϕ and ψ with dimensions d and e, respectively, a *morphism* $\gamma : \phi \to \psi$ is a matrix $\gamma \in \mathrm{Mat}_{e\times d}(\mathbb{K}[\tau])$ such that

$$\gamma\phi_a = \psi_a\gamma,$$

for any $a \in \mathbf{A}$. If $d = e$ and $\gamma \in \mathrm{GL}_d(\mathbb{K}[\tau])$, then we call γ an *isomorphism* of t-modules. An *exponential function* $\mathrm{Exp}_\phi : \mathbb{K}^d \to \mathbb{K}^d$ is defined via a power series

$$\mathrm{Exp}_\phi(z) = z + \sum_{i\geq 1} C_i z^{(i)}, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}, \quad C_i \in \mathrm{Mat}_d(\mathbb{K}),$$

such that, for all $a \in \mathbf{A}$,

$$\mathrm{Exp}_\phi(\mathrm{d}\phi_a z) = \phi_a(\mathrm{Exp}_\phi(z)).$$

This functional equation uniquely determines the coefficients C_i. The exponential function is an entire function. We say that ϕ is *uniformizable* if Exp_ϕ is surjective. The kernel of the exponential function is denoted by $\Lambda_\phi = \ker(\mathrm{Exp}_\phi) \subset \mathbb{K}^d$, and we call it a *period lattice* of ϕ. It is well-known that Λ_ϕ is a free, finitely generated discrete $\mathbf{A}$-submodule.

2.5 t-motives and dual t-motives for t-modules

Recall the definition of t-motives given in §2.3. For a t-motive M, we call $\mathrm{rank}_{\mathbb{K}[\tau]} M$ the *dimension* of M, which will be denoted by $d(M)$. If M is free and finitely generated as a $\mathbb{K}[t]$-module, we say that M is *abelian* and we call $\mathrm{rank}_{\mathbb{K}[t]} M$ the *rank* of M, which will be denoted by $r(M)$. Setting

$$M((1/t)) := M \otimes_{\mathbb{K}[t]} \mathbb{K}((1/t)),$$

we say that an abelian t-motive M is *pure* if there is a finitely generated $\mathbb{K}[[1/t]]$-submodule H in $M((1/t))$ such that $t^u H = \tau^v H$ for some positive integers u, v. In this case, we define the *weight* of M to be

$$w(M) := u/v.$$

It is evident that $w(M) = d(M)/r(M)$. Moreover, for every Drinfeld module ϕ of rank r, its t-motive is pure of dimension 1, rank r, weight $1/r$ (see [31, §7] or [6, §4] for more details).

For any t-module $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$, the *$t$-motive associated to* ϕ, denoted $M(\phi)$, is defined as follows: let $M(\phi) = \mathrm{Mat}_{1\times d}(\mathbb{K}[\tau])$ and make $M(\phi)$ into a left $\mathbb{K}[t,\tau]$-module by setting

$$ct^i \cdot m := cm\phi_{t^i}, \quad m \in M(\phi), \quad c \in \mathbb{K}[\tau].$$

The ring $\mathbb{K}[t,\sigma]$ is the polynomial ring in t and σ with coefficients in $\mathbb{K}$ subject to the following relations,

$$tc = ct, \quad t\sigma = \sigma t, \quad \sigma c = c^{1/q}\sigma, \quad c \in \mathbb{K}.$$

In this way for any $f \in \mathbb{K}[t]$,

$$\sigma f = f^{(-1)}\sigma.$$

A *dual t-motive* N is a left $\mathbb{K}[t,\sigma]$-module that is free and finitely generated over $\mathbb{K}[\sigma]$ and for which there is $\ell \in \mathbb{N}$ with

$$(t-\theta)^{\ell}(N/\sigma N) = \{0\}.$$

We call $\operatorname{rank}_{\mathbb{K}[\sigma]} N$ the *dimension* of N. If N is free and finitely generated as a $\mathbb{K}[t]$-module, we say that N is $\mathbf{A}$-*finite* and we call $\operatorname{rank}_{\mathbb{K}[t]} N$ the *rank* of N. The map $* : \mathbb{K}[\tau] \to \mathbb{K}[\sigma]$ is the anti-isomorphism given by

$$(\sum a_i \tau^i)^* = \sum a_i^{(-i)} \sigma^i.$$

This map induces a map $* : \operatorname{Mat}_{\ell \times k}(\mathbb{K}[\tau]) \to \operatorname{Mat}_{k \times \ell}(\mathbb{K}[\sigma]), (b_{ij}) \mapsto (b_{ij}^*)^{\mathsf{T}}$. For a t-module ϕ as before, the *dual t-motive associated to* ϕ, denoted $N(\phi)$, is defined as follows: let $N(\phi) = \operatorname{Mat}_{1\times d}(\mathbb{K}[\sigma])$ and make $N(\phi)$ into a left $\mathbb{K}[t,\sigma]$-module by setting

$$ct^i \cdot n := cn(\phi_{t^i})^*, \quad n \in N(\phi), \quad c \in \mathbb{K}[\sigma].$$

If a t-module ϕ is uniformizable, and $M(\phi)$ is abelian and $N(\phi)$ is $\mathbf{A}$-finite, then

$$\operatorname{rank}_{\mathbf{A}} \Lambda_\phi = \operatorname{rank}_{\mathbb{K}[t]} M(\phi) = \operatorname{rank}_{\mathbb{K}[t]} N(\phi).$$

Every morphism of t-modules $\gamma : \phi \to \psi$ induces a morphism of t-motives

$$\gamma^+ : M(\psi) \to M(\phi), \quad m \mapsto m\gamma,$$

and a morphism of dual t-motives

$$\gamma_+ : N(\phi) \to N(\psi), \quad n \mapsto n\gamma^*.$$

2.6 Anderson generating function for a t-module

Suppose $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$ is a t-module and y is an element in $\mathbb{K}^d$. Then the *Anderson generating function* for ϕ with respect to y is the column vector of power series,

$$\mathcal{G}_y(t) = \sum_{n=0}^{\infty} \mathrm{Exp}_\phi((\mathrm{d}\phi_t)^{-n-1} \cdot y)t^n \in \mathbb{T}^d.$$

The properties of the Anderson generating functions for a t-module are investigated by Anderson and Thakur [3, §2], Green [16, §6], Green and Papanikolas [17, §5], and Namoijam and Papanikolas [22, §4]. Anderson and Thakur defined the Anderson generating function for a t-module $C^{\otimes n}$ and provided a residue formula in this case. For a vector of meromorphic function $h = (h_1, \ldots, h_d)^{\mathsf{T}} \in \mathbb{T}^d$ and $x \in \mathbb{K}$, its residue is defined by

$$\mathrm{Res}_{t=x}\, h = (\mathrm{Res}_{t=x}\, h_1, \ldots, \mathrm{Res}_{t=x}\, h_d)^{\mathsf{T}}.$$

Proposition 2.3 (Namoijam and Papanikolas; see [22, §4.2])**.** *Let $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$ be a t-module and let $y \in \mathbb{K}^d$. Then*

$$\mathrm{Res}_{t=\theta}(\mathcal{G}_y(t)) = -y.$$

Furthermore, if $\lambda \in \Lambda_\phi$ and $a \in \mathbf{A}$, then

$$\phi_{a(t)}(\mathcal{G}_\lambda(t)) = a(t)\mathcal{G}_\lambda(t).$$

3. RIGID ANALYTIC TRIVIALIZATIONS FOR DRINFELD MODULES

3.1 Pellarin's method

We begin this section by providing a definition of rigid analytic trivialization. First of all, we set M to be an abelian t-motive. Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(\mathbb{K}[\tau])$ comprise a basis for M as a $\mathbb{K}[t]$-module, and let $\Theta \in \mathrm{Mat}_r(\mathbb{K}[t])$ represent multiplication by τ on M with respect to $\mathbf{m}$ i.e.,

$$\tau\mathbf{m} = \Theta\mathbf{m}.$$

We say that M is *rigid analytically trivial* if there exists $\Upsilon \in \mathrm{GL}_r(\mathbb{T})$ that satisfies

$$\Upsilon^{(1)} = \Theta\Upsilon. \tag{3.1}$$

Anderson [1] called such a matrix Υ a *rigid analytic trivialization* for M.

There is also a notion of rigid analytic trivialization for a dual t-motive, which is defined using the similar concept (see [2, §4.4]). Suppose that H is an $\mathbf{A}$-finite dual t-motive and the σ-action on H is represented by the matrix $\Phi \in \mathrm{Mat}_d(\mathbb{K}[t])$. We say that H is *rigid analytically trivial* if there exists $\Psi \in \mathrm{GL}_d(\mathbb{T})$ that satisfies

$$\Psi^{(-1)} = \Phi\Psi. \tag{3.2}$$

We call Ψ a *rigid analytic trivialization* for H. The reader is directed to [6, §4] for more details about rigid analytic trivialization.

In this section, we mainly focus on a rigid analytic trivialization for a t-motive associated to a Drinfeld module. In other words, a rigid analytic trivialization in this section refers to a matrix Υ satisfying equation (3.1). First, we consider a Drinfeld module ϕ of rank r defined by $\phi_t = \theta + A_1\tau + \cdots + A_r\tau^r$, $A_r \neq 0$, and its associated t-motive $M(\phi) = \mathbb{K}[\tau]$ as in the previous section. Recall that $t \cdot 1 = 1 \cdot \phi_t = \theta + A_1\tau + \cdots + A_{r-1}\tau^{r-1} + A_r\tau^r$, so

$$\tau^r \cdot 1 = \left(\frac{t-\theta}{A_r} - \frac{A_1}{A_r}\tau - \cdots - \frac{A_{r-1}}{A_r}\tau^{r-1} \right) \cdot 1.$$

By using the right division algorithm on $\mathbb{K}[\tau]$, we see that $1, \tau, \ldots, \tau^{r-1}$ form a $\mathbb{K}[t]$-basis for $M(\phi)$, and note that

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{t-\theta}{A_r} & \frac{-A_1}{A_r} & \frac{-A_2}{A_r} & \cdots & \frac{-A_{r-1}}{A_r} \end{pmatrix} \begin{pmatrix} 1 \\ \tau \\ \vdots \\ \tau^{r-1} \end{pmatrix} = \begin{pmatrix} \tau \\ \tau^2 \\ \vdots \\ \tau^r \end{pmatrix} = \tau \begin{pmatrix} 1 \\ \tau \\ \vdots \\ \tau^{r-1} \end{pmatrix}.$$

Therefore multiplication by τ on $M(\phi)$ is represented by the matrix

$$\Theta := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{t-\theta}{A_r} & \frac{-A_1}{A_r} & \frac{-A_2}{A_r} & \cdots & \frac{-A_{r-1}}{A_r} \end{pmatrix}. \tag{3.3}$$

The method to construct a rigid analytic trivialization for a Drinfeld module was given by Pellarin [26, §4.2] using Anderson generating functions as follows (see also [14, §2]). Let $\pi_1, \ldots, \pi_r$ be an A-basis of the period lattice $\Lambda := \Lambda_\phi$ and for $i = 1, \ldots, r$ let $f_i(t) = f_\phi(\pi_i; t)$. He defined the matrix

$$\Upsilon = \begin{pmatrix} f_1 & f_2 & \cdots & f_r \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_r^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(r-1)} & f_2^{(r-1)} & \cdots & f_r^{(r-1)} \end{pmatrix},$$

and then he proved that $\det(\Upsilon) \in \mathbb{T}^{\times}$ and $\Upsilon^{(1)} = \Theta\Upsilon$, i.e., Υ is a rigid analytic trivialization for $M(\phi)$. However, in order to construct Υ this way, we need to know an A-basis of the period lattice, which inherently are transcendental quantities. Therefore, our motivation is to introduce a new method to find a rigid analytic trivialization that does not require the periods. In our research, we provide a procedure to construct a rigid analytic trivialization Υ by simply finding roots of finitely many polynomials.

3.2 Our method

In this section, we will state our results which are from our first paper [20]. However, only some of the proofs will be included in this dissertation. Throughout this section, we consider a Drinfeld module ϕ of rank r defined by $\phi_t = \theta + A_1\tau + \cdots + A_r\tau^r$ and its associated t-motive $M(\phi) = \mathbb{K}[\tau]$. By using the right division algorithm on $\mathbb{K}[\tau]$, one can see that $1, \tau, \ldots, \tau^{r-1}$ form a $\mathbb{K}[t]$-basis for $M(\phi)$, and the matrix representing multiplication by τ on $M(\phi)$ with respect to this basis is the matrix Θ given in (3.3), and so we want to find a matrix Υ such that $\Upsilon^{(1)} = \Theta\Upsilon$. Furthermore, our main goal is to construct Υ directly from ϕ in an effective manner, i.e., by requiring only a finite amount of initial computation. Recall from §2.2 about the Gauss norm $\|\cdot\|$ on $\mathbb{T}$. Our idea is to build a matrix $M \in \mathrm{Mat}_r(\mathbb{T})$ satisfying

$$\|M^{-1}\Theta^{-1}M^{(1)} - I\| < 1, \tag{3.4}$$

where I is the $r \times r$ identity matrix. Then by letting $F = M^{-1}\Theta^{-1}M^{(1)}$, we obtain that $F^{(n)} \to I$ as $n \to \infty$, with respect to the Gauss norm. Thus the infinite product $\prod_{n=1}^{\infty} F^{(n)}$ converges with respect to the Gauss norm. Then by defining $\Upsilon = MFF^{(1)}F^{(2)}\cdots$, we will observe that

$$\Upsilon^{(1)} = M^{(1)}F^{(1)}F^{(2)}\cdots = \Theta MFF^{(1)}F^{(2)}\cdots = \Theta\Upsilon,$$

which means Υ is the rigid analytic trivialization for $M(\phi)$. Our construction of a rigid analytic trivialization consists of 3 major steps as follows.

Step 1: Starting with a finite amount of data, we give a procedure to find a convergent t-division sequence $y_1, y_2, y_3, \ldots$, and by using this procedure, we obtain a positive integer N and torsion elements $\xi_1, \ldots, \xi_r \in \phi[t^N]$. In this step, one important tool is the Newton polygon of the polynomial $\phi_t(x) = \theta x + A_1 x^q + \cdots + A_r x^{q^r}$.

Step 2: We use N and $\xi_1, \ldots, \xi_r$ from step 1 to construct polynomials $h_1(t), \ldots, h_r(t) \in \mathbb{K}[t]$ and then we use them to create a matrix $M \in \mathrm{Mat}_r(\mathbb{K}[t])$ which satisfies $\|M^{-1}\Theta^{-1}M^{(1)} - I\| < 1$.

Step 3: We use the matrix M from step 2 to define the sequence of matrices $(\Upsilon_n)_{n\geq 1}$. Then we let

$$\Upsilon = \lim_{n\to\infty} \Upsilon_n.$$

Our main result is Theorem 3.15, in which we show that Υ is a rigid analytic trivialization and we also give an explicit formula for Υ.

3.2.1 Step 1

First of all, we recall the theory of the Newton polygon (see [15, §2]). The Newton polygon of a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is defined to be the lower convex hull in $\mathbb{R}^2$ of the set of points

$$(i, v_\infty(a_i)), \ \ i = 0, 1, \ldots, n.$$

Let $\lambda_1, \ldots, \lambda_s$ be the slopes of the lower edges of the Newton polygon of $f(x)$ arranged in increasing order, and let $\ell_1, \ldots, \ell_s$ be the corresponding horizontal lengths of these line segments projected onto the x-axis. Then for each integer $1 \leq n \leq s$, $f(x)$ has exactly ℓ_n roots with valuation $-\lambda_n$.

Now consider the Newton polygon of the polynomial

$$\phi_t(x) = \theta x + A_1 x^q + \cdots + A_{r-1} x^{q^{r-1}} + A_r x^{q^r}.$$

Denote the vertices of the lower convex hull by $(q^{d_j}, -\deg(A_{d_j})), j = 1, 2, \ldots, s$ for some $s \in \mathbb{N}$. Note that $0 = d_0 < d_1 < \cdots < d_s = r$. For $n, m \in \{0, 1, \ldots, r\}$ with $n < m$, define $L_{n,m}$ to be the line segments connecting points $(q^n, -\deg(A_n))$ and $(q^m, -\deg(A_m))$ and let $w_{n,m}$ be its slope. For $j = 1, 2, \ldots, s$, we set $\lambda_j = w_{d_{j-1},d_j}$. It is not hard to see that $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ and the line segments $L_{d_{j-1},d_j}, j = 1, 2, \ldots, s$ form the Newton polygon of $\phi_t(x)$ as shown in Figure 3.1.

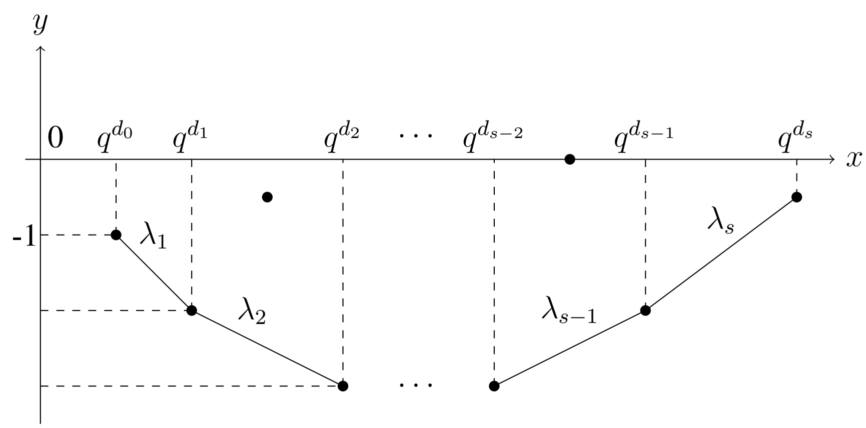

Figure 3.1: Newton polygon of $\phi_t(x)$

Suppose the Newton polygon consists of s edges. Let $\lambda_1, \ldots, \lambda_s$ be the slopes of the edges of the Newton polygon of $\phi_t(x)$. Let $(q^{d_0}, *), \ldots, (q^{d_s}, *)$ be the vertices of the Newton polygon of $\phi_t(x)$.

Let $N(\phi) := \{1 \leq i \leq r : A_i \neq 0\}$. For each $n \in N(\phi)$, we set

$$\mu_n = \frac{\deg(A_n) - q^n}{q^n - 1}. \tag{3.5}$$

According to the result by El-Guindy and Papanikolas [11, Prop. 6.10], we know that

$$R_\phi = |\theta|^{-\mu_m}, \tag{3.6}$$

where R_ϕ is the radius of convergence of logarithm $\log_\phi$ and m is the smallest index in $N(\phi)$ such that $\mu_m \geq \mu_i$ for every $i \in N(\phi)$. In the following lemma, we investigate useful properties on the

Newton polygon of $\phi_t(x)$, which contains information about the radius of convergence R_ϕ.

Lemma 3.1 (Khaochim and Papanikolas [20]). *For $j = 1, 2, \dots, s$, let a_j be the y-intercept of the line containing L_{d_{j-1},d_j}. The following hold.*

1. $a_1 = \mu_m$, *where μ_m is from equation* (3.6).
2. $a_1 > a_2 > \cdots > a_s$.
3. $-a_j > -\frac{\deg(A_{d_j})}{q^{d_j}-1}$ *for every $j = 1, 2, \dots, s$.*

Given a nonzero element in $\phi[t]$, by using properties of the Newton polygon, we provide a recursive procedure to find a convergent t-division sequence in the following proposition.

Proposition 3.2 (Khaochim and Papanikolas [20]). *Let $y_1 \in \phi[t]$ be nonzero. Then there exist a sequence $y_1, y_2, \dots$ and a positive integer N such that*

1. $\phi_t(y_k) = y_{k-1}$ *for all $k = 2, 3, \dots$*
2. $\deg(y_1) > \deg(y_2) > \deg(y_3) > \cdots$
3. $|y_N| < R_\phi$
4. $\lim_{k\to\infty} \deg(y_k) = -\infty$.

Proof. First, for $k \in N(\phi)$, we define a function

$$u_k(z) = \frac{z - \deg(A_k)}{q^k}.$$

Since y_1 is a root of $\phi_t(x)$, we see that $\deg(y_1) \leq \lambda_s$. For $k \geq 1$, we perform the following recursive process. Suppose $\deg(y_k) \leq \lambda_s$ and set $y = y_k$. Consider the Newton polygon of $\phi_t(x) - y$ which is obtained from the Newton polygon of $\phi_t(x)$ by adding one more point $(0, -\deg(y))$. We observe that $-\deg(y)$ must belong to one of the following intervals:

$$I_1 := (a_1, \infty), I_2 := (a_2, a_1], \dots, I_s := (a_s, a_{s-1}],$$

where $a_1, a_2 \ldots, a_s$ is defined in Lemma 3.1. To see why $-\deg(y) > a_s$, one can use the technique in the proof of Lemma 3.1 and see that $a_s < -\lambda_s$. So $-\deg(y) \geq -\lambda_s > a_s$.

- If $-\deg(y) \in (a_1, \infty)$, then the Newton polygon of $\phi_t(x) - y$ is obtained from the Newton polygon of $\phi_t(x)$ by adding the line segment from $(0, -\deg(y))$ to $(1, -1)$. This new line has slope $\deg(y) - 1 = u_0(\deg(y))$, so there is one root of $\phi_t(x) - y$ with degree equal to $u_0(\deg(y))$.

- If $-\deg(y) \in (a_{j+1}, a_j]$ for some $1 \leq j \leq s-1$, then the Newton polygon of $\phi_t(x)-y$ is obtained from the Newton polygon of $\phi_t(x)$ by replacing line segments $L_{d_0,d_1}, L_{d_1,d_2}, \ldots L_{d_{j-1},d_j}$ by line segment from $(0, -\deg(y))$ to $(q^{d_j}, -\deg(A_{d_j}))$. This new line has slope $\frac{\deg(y)-\deg(A_{d_j})}{q^{d_j}} = u_{d_j}(\deg(y))$, so there are q^{d_j} roots of $\phi_t(x) - y$ with degree equal to $u_{d_j}(\deg(y))$.

Choose y_{k+1} to be a root of $\phi_t(x) - y$ with

$$\deg(y_{k+1}) = \begin{cases} u_0(\deg(y)) & \text{if } -\deg(y) \in (a_1, \infty) \\ u_{d_j}(\deg(y)) & \text{if } -\deg(y) \in (a_{j+1}, a_j] \end{cases}$$

We claim that $\deg(y_{k+1}) < \deg(y)$. To prove this, we observe by the definition above that $\deg(y_{k+1})$ is either $u_0(\deg(y))$ or $u_{d_j}(\deg(y))$ for some j. It is clear that $u_0(\deg(y)) = \deg(y) - 1 < \deg(y)$. Note that $\deg(y_{k+1}) = u_{d_j}(\deg(y))$ only if $-\deg(y) \in (a_{j+1}, a_j]$. In this case, we compute using Lemma 3.1 that

$$\deg(y) \geq -a_j > -\frac{\deg(A_{d_j})}{q^{d_j} - 1}.$$

This implies that

$$\frac{\deg(y) - \deg(A_{d_j})}{q^{d_j}} < \deg(y).$$

That is, $\deg(y_{k+1}) = u_{d_j}(\deg(y)) < \deg(y)$. In conclusion, we obtain y_{k+1} to be a root of $\phi_t(x)-y$, which satisfies $\deg(y_{k+1}) < \deg(y) \leq \lambda_s$.

This recursion provides us the sequence $y_1, y_2, \ldots$ satisfying (1) and (2). Our next step is to prove (3). We claim that there exist a positive integer N such that $\deg(y_N) < -a_1$. To prove this, first we compute a limit of $u_k^{\circ n}(z)$ when $n \to \infty$, where we use the notation $u_k^{\circ n}$ for the n composition of the function u_k. We observe that

$$u_k^{\circ n}(z) = \frac{z}{q^{nk}} - \left(\frac{q^{nk}-1}{q^k-1}\right)\frac{\deg(A_k)}{q^{nk}}$$

for any $z \in \mathbb{R}$ and $k \in N(\phi)$, which means we have the pointwise limit

$$\lim_{n\to\infty} u_k^{\circ n}(z) = -\frac{\deg(A_k)}{q^k-1}.$$

If $-\deg(y_1) \in I_1 = (a_1, \infty)$, then we already have $\deg(y_1) < -a_1$ and we can choose $N = 1$. Now assume that $-\deg(y_1) \in I_{j+1}$ for some $1 \leq j \leq s-1$. By Lemma 3.1,

$$\lim_{n\to\infty} u_{d_j}^{\circ n}(\deg(y_1)) = -\frac{\deg(A_{d_j})}{q^{d_j}-1} < -a_j,$$

so there must be a positive integer n such that $u_{d_j}^{\circ n}(\deg(y_1)) < -a_j$. We choose k_1 to be the smallest such integer. Then k_1 is the smallest integer such that after applying u_{d_j} to $\deg(y_1)$ for k_1 times, we have $u_{d_j}^{\circ k_1}(\deg(y_1)) < -a_j$. Therefore $-\deg(y_{k_1+1}) = -u_{d_j}^{\circ k_1}(\deg(y_1)) \in I_1 \cup \cdots \cup I_j$. Repeating the same argument, we can choose the smallest integer $k_2 \geq 0$ that makes $u_{d_{j-1}}^{\circ k_2}(\deg(y_{k_1+1})) < -a_{j-1}$, and thus

$$-\deg(y_{k_2+k_1+1}) = -u_{d_{j-1}}^{\circ k_2}(\deg(y_{k_1+1})) \in I_1 \cup \cdots \cup I_{j-1}.$$

Continuing this, we finally get $k_j \geq 0$ that makes $u_{d_1}^{\circ k_j}(\deg(y_{k_{j-1}+\cdots+k_1+1})) < -a_1$, i.e.,

$$-\deg(y_{k_j+\cdots+k_1+1}) = -u_{d_j}^{\circ k_j}(\deg(y_{k_{j-1}+\cdots+k_1+1})) \in I_1.$$

Letting $N = 1 + k_1 + \cdots + k_j$, we obtain that $-\deg(y_N) \in I_1$, i.e., $\deg(y_N) < -a_1$. This proves the claim and (3) follows from the definition of R_ϕ. Furthermore, since $\deg(y_N) < -a_1$, we see that

$$\deg(y_{N+k}) = u_0^{\circ k}(\deg(y_N)) = \deg(y_N) - k$$

for every $k \geq 1$. It follows that $\lim_{k\to\infty} \deg(y_k) = -\infty$. □

Remark 3.3. The integer N in the above proposition can be expressed as follows. Let j be such that $-\deg(y_1) \in I_{j+1}$. Define $k_0 = 1$ and for $1 \leq i \leq j$, define k_i to be the smallest integer such that $u_{d_{j-i+1}}^{\circ k_i}(\deg(y_{k_{i-1}})) < -a_{j-i+1}$. Then

$$N = k_0 + k_1 + \cdots + k_j.$$

Moreover, we observe that

$$\deg(y_N) = u_{d_1}^{\circ k_j} \circ \cdots \circ u_{d_{j-1}}^{\circ k_2} \circ u_{d_j}^{\circ k_1}(\deg(y_1)).$$

Fix an $\mathbb{F}_q$-basis $x_1, \ldots, x_r$ of $\phi[t]$. By applying the algorithm in Proposition 3.2 to each x_i, we obtain a positive integer N and torsion elements $\xi_1, \ldots, \xi_r \in \phi[t^N]$ with applicable properties as follows.

Proposition 3.4 (Khaochim and Papanikolas [20]). *Let $x_1, \ldots, x_r$ be a basis of $\phi[t]$. Then there exist $N \geq 1$ and $\xi_1, \ldots, \xi_r \in \phi[t^N]$ such that for each i,*

1. $|\xi_i| < R_\phi$
2. $\phi_{t^{N-1}}(\xi_i) = x_i$
3. $\deg(\phi_{t^{N-1}}(\xi_i)) > \ldots > \deg(\phi_t(\xi_i)) > \deg(\xi_i)$.

Proof. Fix a basis $x_1, \ldots, x_r$ of $\phi[t]$. Using Proposition 3.2 for each x_i, we obtain a sequence $x_{i,1}, x_{i,2}, x_{i,3}, \ldots$ such that

- $x_{i,1} = x_i$ and $\phi_t(x_{i,k}) = x_{i,k-1}$ for all $k = 2, 3, \ldots$

- $x_{i,n} \in \phi[t^n]$ for every n
- $\deg(x_{i,1}) > \deg(x_{i,2}) > \deg(x_{i,3}) > \ldots$
- there exist a positive integer $N(i)$ such that $|x_{i,N(i)}| < R_\phi$.

Let $N = \max\{N(1), N(2), \ldots, N(r)\}$ and let $\xi_i = x_{i,N}$. Then, for every $i = 1, 2, \ldots, r$, we obtain that

- $\xi_i \in \phi[t^N]$
- $|\xi_i| < R_\phi$
- $\deg(\phi_{t^{N-1}}(\xi_i)) > \ldots > \deg(\phi_t(\xi_i)) > \deg(\xi_i)$.

Moreover,

$$\phi_{t^{N-1}}(\xi_i) = \phi_{t^{N-1}}(x_{i,N}) = \phi_{t^{N-2}}(\phi_t(x_{i,N})) = \phi_{t^{N-2}}(x_{i,N-1}) = \ldots = \phi_t(x_{i,2}) = x_{i,1} = x_i.$$

□

Recall that $\lambda_1, \ldots, \lambda_s$ represent the slopes in the Newton polygon of $\phi_t(x)$ as we defined in the beginning of this section. Now we define a strict basis for $\phi[t]$ as follows. Also we prove in the next lemma that, for any Drinfeld module, a strict basis always exists.

Definition 3.5. An $\mathbb{F}_q$-basis $x_1, \ldots, x_r$ of $\phi[t]$ is *strict* if for $1 \leq n \leq r$, we have $\deg(x_n) = \lambda_j$, where j is the positive integer such that $d_{j-1} + 1 \leq n \leq d_j$.

Lemma 3.6 (Khaochim and Papanikolas [20])**.** *There exists a strict basis $x_1, \ldots, x_r$ of $\phi[t]$.*

Proof. For $1 \leq j \leq s$, define

$$Q_j = \{x \in \phi[t] : \deg(x) \leq \lambda_j\}$$

and

$$R_j = \{x \in \phi[t] : \deg(x) = \lambda_j\}.$$

We observe that Q_j is an $\mathbb{F}_q$-subspace of $\phi[t]$ for $1 \leq j \leq s$. Since $Q_1 = R_1 \sqcup \{0\}$ and the set Q_1 has q^{d_1} elements, there exists $x_1, \ldots, x_{d_1} \in R_1$ such that

$$Q_1 = \mathbb{F}_q x_1 \oplus \cdots \oplus \mathbb{F}_q x_{d_1}.$$

We note that $Q_1 \subset Q_2$ and the set Q_2 has q^{d_2} elements, so we can pick $x_{d_1+1}, \ldots, x_{d_2}$ such that

$$Q_2 = \mathbb{F}_q x_1 \oplus \cdots \oplus \mathbb{F}_q x_{d_1} \oplus \mathbb{F}_q x_{d_1+1} \oplus \cdots \oplus \mathbb{F}_q x_{d_2}.$$

We claim that for every $d_1 + 1 \leq n \leq d_2$, $\deg(x_n) = \lambda_2$. To prove this, fix $d_1 + 1 \leq n \leq d_2$ and suppose that $\deg(x_n) \leq \lambda_1$. Thus $x_n \in Q_1$ which implies that $x_n = c_1 x_1 + \cdots + c_{d_1} x_{d_1}$ for some $c_1, \ldots, c_{d_1} \in \mathbb{F}_q$. This cannot happen because $x_1, \ldots, x_{d_2}$ are linearly independent over $\mathbb{F}_q$. Therefore $\deg(x_n) = \lambda_2$. We note that $Q_2 \subset Q_3$ and the set Q_3 has q^{d_3} elements, so we can pick $x_{d_2+1}, \ldots, x_{d_3}$ such that

$$Q_3 = \mathbb{F}_q x_1 \oplus \cdots \oplus \mathbb{F}_q x_{d_2} \oplus \mathbb{F}_q x_{d_2+1} \oplus \cdots \oplus \mathbb{F}_q x_{d_3}.$$

By the same reason as above, we obtain that for every $d_2 + 1 \leq n \leq d_3$, $\deg(x_n) = \lambda_3$. We can continue this process until we get $x_{d_{s-1}+1}, \ldots, x_{d_s}$ such that

$$Q_s = \mathbb{F}_q x_1 \oplus \cdots \oplus \mathbb{F}_q x_{d_{s-1}} \oplus \mathbb{F}_q x_{d_{s-1}+1} \oplus \cdots \oplus \mathbb{F}_q x_{d_s}$$

and for every $d_{s-1} + 1 \leq n \leq d_s$, we obtain that $\deg(x_n) = \lambda_s$. Therefore

$$\phi[t] = \mathbb{F}_q x_1 \oplus \cdots \oplus \mathbb{F}_q x_r$$

and for every $1 \leq j \leq s$ and $d_{j-1} + 1 \leq n \leq d_j$, we obtain that $\deg(x_n) = \lambda_j$. Thus we obtain a strict basis of $\phi[t]$. $\square$

3.2.2 Step 2

In this step, we will use the t-division sequence from Proposition 3.4 to create a matrix M in Theorem 3.12. Suppose we have a strict basis $x_1, \ldots, x_r$ of $\phi[t]$ as in Lemma 3.6. We choose $N \geq 1$ and $\xi_1, \ldots, \xi_r \in \phi[t^N]$ as in Proposition 3.4. For $1 \leq i \leq r$, we define

$$h_i = \phi_{t^{N-1}}(\xi_i) + \phi_{t^{N-2}}(\xi_i)t + \cdots + \xi_i t^{N-1}.$$

Observe that

$$\|h_i\| = \max_{0 \leq m \leq N-1} \{|\phi_{t^{N-1-m}}(\xi_i)|\} = |x_i|.$$

Then we let $M = (h_j^{(i-1)}) \in \mathrm{Mat}_r(\mathbb{K}[t])$. We dedicate the rest of this section to prove that the matrix M defined this way satisfies (3.4). First, we need more information about $\det(M)$.

Proposition 3.7 (Khaochim and Papanikolas [20])**.** *Suppose that for $1 \leq i \leq r$, $h_i = x_i + y_i t$ for some $x_i \in \mathbb{K}$ and $y_i \in \mathbb{K}[t]$. Let $M = (h_j^{(i-1)})$ and let $X = (x_j^{q^{i-1}})$. Then*

$$\det(M) = \det(X) + yt$$

for some $y \in \mathbb{K}[t]$.

The above proposition allows us to express the determinant of M in terms of the determinant of another matrix $X \in \mathrm{Mat}_r(\mathbb{K})$ plus the term with t. Thus we want to gather information about the determinant of X. Using the following lemma and properties of Moore determinant (see [15, §1.3]), we get the formula for the degree of $\det(X)$ in Proposition 3.9.

Lemma 3.8 (Khaochim and Papanikolas [20])**.** *Let $x_1, \ldots, x_r$ be a strict basis of $\phi[t]$. If $1 \leq j \leq s$ and $d_{j-1} + 1 \leq n \leq d_j$, then*

$$\deg(c_1 x_1 + \cdots + c_{n-1} x_{n-1} + x_n) = \lambda_j$$

for every $c_1, \ldots, c_{n-1} \in \mathbb{F}_q$.

Proposition 3.9 (Khaochim and Papanikolas [20]). *Let $x_1, \dots, x_r$ be a strict basis for $\phi[t]$ and let $X = (x_j^{q^{i-1}})$. Then*

$$\deg(\det(X)) = \sum_{n=1}^{r} q^{n-1} \deg(x_n).$$

In the next lemma, we provide a bound for a determinant of a matrix in terms of its entries (see also [4, §2-3] and [25, §2] for similar types of formulas). This formula will serve as a tool to prove our main result.

Lemma 3.10 (Khaochim and Papanikolas [20]). *Let $n \geq 1$. For $i = 1, 2, \dots, n$, let $b_i \in \mathbb{T}$. Suppose that $\|b_1\| \leq \|b_2\| \leq \cdots \leq \|b_n\|$ and let $N = (b_j^{(i-1)})$. Then*

$$\|\det(N)\| \leq \|b_1\| \|b_2\|^q \cdots \|b_n\|^{q^{n-1}}.$$

Our main goal in this section is to prove Theorem 3.12, which means we need to prove that $\|M^{-1}\Theta^{-1}M^{(1)} - I\| < 1$. In the following proposition, we give a formula for $M^{-1}\Theta^{-1}M^{(1)} - I$ which is more practical when we compute its norm. Then we finish this section by stating and proving Theorem 3.12.

Proposition 3.11 (Khaochim and Papanikolas [20]). *Let $\ell \in \mathbb{N}$ and for $i = 1, 2, \dots, r$, let $h_i = \sum_{m=0}^{\ell-1} \phi_{t^{\ell-1-m}}(\xi_i) t^m$ where $\xi_i \in \phi[t^\ell]$. Let*

$$M = \begin{pmatrix} h_1 & h_2 & \dots & h_r \\ h_1^{(1)} & h_2^{(1)} & \dots & h_r^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(r-1)} & h_2^{(r-1)} & \dots & h_r^{(r-1)} \end{pmatrix}.$$

Then

$$M^{-1}\Theta^{-1}M^{(1)} - I = \frac{-t^\ell}{t-\theta} M^{-1} W,$$

where

$$W = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.7}$$

Theorem 3.12 (Khaochim and Papanikolas [20])**.** *Let $x_1, \ldots, x_r$ be a strict basis of $\phi[t]$. Choose $N \geq 1$ and $\xi_1, \ldots, \xi_r \in \phi[t^N]$ as in Proposition 3.4 and for $1 \leq i \leq r$, define*

$$h_i = \phi_{t^{N-1}}(\xi_i) + \phi_{t^{N-2}}(\xi_i)t + \cdots + \xi_i t^{N-1} \in \mathbb{K}[t].$$

Let $M = (h_j^{(i-1)}) \in \mathrm{Mat}_r(\mathbb{K}[t])$. Then

$$\|M^{-1}\Theta^{-1}M^{(1)} - I\| < 1.$$

Proof. By Proposition 3.11, we have $\|M^{-1}\Theta^{-1}M^{(1)} - I\| = \|\frac{-t^\ell}{t-\theta}M^{-1}W\|$. We further observe that $\|\frac{-t^\ell}{t-\theta}\| = 1/q$, so proving $\|M^{-1}\Theta^{-1}M^{(1)} - I\| < 1$ is equivalent to showing

$$\|M^{-1}W\| < q, \tag{3.8}$$

where W is defined in (3.7). We denote the (i,j)-entry of M^{-1} by m_{ij} and observe that

$$M^{-1}W = \begin{pmatrix} m_{11}\xi_1 & m_{11}\xi_2 & \cdots & m_{11}\xi_r \\ m_{21}\xi_1 & m_{21}\xi_2 & \cdots & m_{21}\xi_r \\ \vdots & \vdots & \ddots & \vdots \\ m_{r1}\xi_1 & m_{r1}\xi_2 & \cdots & m_{r1}\xi_r \end{pmatrix}.$$

That is, $M^{-1}W = (m_{i1}\xi_j)$. It follows that $\|M^{-1}W\| = \max\{\|m_{i1}\xi_j\| : 1 \leq i,j \leq r\}$. To show that $\|M^{-1}W\| < q$, it suffices to show that $\|m_{i1}\xi_j\| < q$ for all $1 \leq i,j \leq r$. Now fix i and j.

Using a formula for the inverse of a matrix,

$$M^{-1} = \frac{1}{\det(M)} \operatorname{Adj}(M),$$

we see that

$$m_{i1} = \frac{(-1)^{i+1}}{\det(M)} \det(M^*),$$

where

$$M^* = \begin{pmatrix} h_1^{(1)} & \cdots & h_{i-1}^{(1)} & h_{i+1}^{(1)} & \cdots & h_r^{(1)} \\ h_1^{(2)} & \cdots & h_{i-1}^{(2)} & h_{i+1}^{(2)} & \cdots & h_r^{(2)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h_1^{(r-1)} & \cdots & h_{i-1}^{(r-1)} & h_{i+1}^{(r-1)} & \cdots & h_r^{(r-1)} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{r-1} \\ b_1^{(1)} & b_2^{(1)} & \cdots & b_{r-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{(r-2)} & b_2^{(r-2)} & \cdots & b_{r-1}^{(r-2)} \end{pmatrix}$$

and $b_1 = h_1^{(1)}, \ldots, b_{i-1} = h_{i-1}^{(1)}$ and $b_i = h_{i+1}^{(1)}, \ldots, b_{r-1} = h_r^{(1)}$. By the definition of a strict basis, we know that $|x_1| \leq |x_2| \leq \cdots \leq |x_r|$. Moreover, by Proposition 3.4, we know that

$$|\xi_i| < |\phi_t(\xi_i)| < \cdots < |\phi_{t^{N-1}}(\xi_i)| = |x_i|,$$

which implies that $\|h_i\| = |x_i|$ for all $1 \leq i \leq r$. Thus, we have $\|h_1\| \leq \|h_2\| \leq \cdots \leq \|h_r\|$. Using Lemma 3.10, we obtain that

$$\begin{aligned} \|\det(M^*)\| &\leq \|b_1\| \|b_2\|^q \cdots \|b_{r-1}\|^{q^{r-2}} \\ &= \|h_1\|^q \cdots \|h_{i-1}\|^{q^{i-1}} \|h_{i+1}\|^{q^i} \cdots \|h_r\|^{q^{r-1}} \\ &\leq \|h_2\|^q \cdots \|h_i\|^{q^{i-1}} \|h_{i+1}\|^{q^i} \cdots \|h_r\|^{q^{r-1}} \\ &= \prod_{k=2}^{r} \|h_k\|^{q^{k-1}} \\ &= \prod_{k=2}^{r} |x_k|^{q^{k-1}}. \end{aligned}$$

It follows that

$$\begin{aligned}\log_q \|m_{i1}\xi_j\| &= \log_q \left(\frac{\|\det(M^*)\||\xi_i|}{\|\det(M)\|}\right)\\ &= \log_q \|\det(M^*)\| + \log_q |\xi_i| - \log_q \|\det(M)\|\\ &\leq \sum_{k=2}^{r} q^{k-1}\deg(x_k) + \log_q |\xi_i| - \log_q \|\det(M)\|.\end{aligned}$$

Since $\xi_1, \ldots, \xi_r$ are chosen so that $|\xi_i| < R_\phi = q^{-\mu_m}$, we obtain that

$$\log_q |\xi_i| < -\mu_m.$$

It follows from Lemma 3.1 that $\mu_m = a_1 = -1 - \lambda_1$, which implies that

$$\log_q |\xi_i| < 1 + \lambda_1 = 1 + \deg(x_1).$$

Therefore

$$\begin{aligned}\log_q \|m_{i1}\xi_j\| &< \sum_{k=2}^{r} q^{k-1}\deg(x_i) + \deg(x_1) + 1 - \log_q \|\det(M)\|\\ &= 1 + \sum_{k=1}^{r} q^{k-1}\deg(x_k) - \log_q \|\det(M)\|.\end{aligned}$$

Combining Proposition 3.7 and Proposition 3.9, we obtain that

$$\log_q \|\det(M)\| \geq \deg(\det(X)) = \sum_{k=1}^{r} q^{k-1}\deg(x_k),$$

where $X = (x_j^{q^{i-1}})$. It follows that $\log_q \|m_{i1}\xi_j\| < 1$. Therefore $\|M^{-1}W\| < q$. □

3.2.3 Step 3

In this section, we will use the matrix M from step 2 to construct a rigid analytic trivialization Υ. First of all, we set $F = M^{-1}\Theta^{-1}M^{(1)}$. It follows directly from Theorem 3.12 that $F^{(n)} \to I$

as $n \to \infty$ with respect to the Gauss norm. Recall that the space of matrices with the Gauss norm $\|\cdot\|$ is a complete normed space, so $\prod_{n=0}^{\infty} F^{(n)}$ converges with respect to $\|\cdot\|$. We define

$$\Upsilon_n = MFF^{(1)}F^{(2)}\cdots F^{(n)}$$

and

$$\Upsilon = \lim_{n\to\infty} \Upsilon_n = MFF^{(1)}F^{(2)}\cdots .$$

Since $M^{(1)} = \Theta MF$, we obtain that

$$\Upsilon^{(1)} = M^{(1)}F^{(1)}F^{(2)}\cdots = (\Theta MF)F^{(1)}F^{(2)}\cdots = \Theta\Upsilon.$$

Our next goal is to compute Υ explicitly. We start by computing Υ_n. In the following lemma, we provide a formula for Υ_n as a summation of matrices.

Lemma 3.13 (Khaochim and Papanikolas [20]). *Let W be the matrix defined in* (3.7) *and define*

$$R_m = \frac{\Theta^{-1}(\Theta^{-1})^{(1)}\cdots(\Theta^{-1})^{(m-1)}}{t-\theta^{q^m}}, \quad m \geq 1.$$

Then for $n \geq 1$,

$$\Upsilon_n = M - \frac{t^N}{t-\theta}W - t^N \sum_{m=1}^{n} R_m W^{(m)}.$$

We recall the functions $\mathcal{B}_n(t)$ and $\mathcal{L}_\phi(\xi;t)$ from §2.2. Using the formula in Lemma 3.13, we are able to express each coordinate in the matrix Υ_n as follows.

Proposition 3.14 (Khaochim and Papanikolas [20]). *For $n \geq 1$ and for $1 \leq i, j \leq r$, the element in (i,j)-coordinate of Υ_n is*

$$\left(h_j - \frac{t^N}{t-\theta}\sum_{m=0}^{n-(i-1)} \mathcal{B}_m \xi_j^{q^m}\right)^{(i-1)}.$$

Finally, we use a formula for each entry of Υ_n given in Proposition 3.14 to get the main result which is the following theorem.

Theorem 3.15 (Khaochim and Papanikolas [20])**.** *Let M be the matrix defined in Theorem 3.12 and let $F = M^{-1}\Theta^{-1}M^{(1)}$ and $\Upsilon_n = MFF^{(1)}F^{(2)}\cdots F^{(n)}$. Let*

$$\Upsilon = \lim_{n\to\infty} \Upsilon_n = MFF^{(1)}F^{(2)}\cdots .$$

Then

1. *the element in (i, j)-coordinate of Υ is*

$$\left(h_j - \frac{t^N}{t-\theta}\mathcal{L}_\phi(\xi_j;t)\right)^{(i-1)}$$

2. *Υ is a rigid analytic trivialization for $M(\phi)$.*

We finish this section with the following proposition, in which we observe that the rigid analytic trivialization from our method matches the one from Pellarin's method for a particular choice of basis.

Proposition 3.16 (Khaochim and Papanikolas [20])**.** *Let $x_1, \ldots, x_r$ be a strict basis of $\phi[t]$. Choose $N \geq 1$ and $\xi_1, \ldots, \xi_r \in \phi[t^N]$ as in Proposition 3.4 and for $1 \leq j \leq r$, let*

$$\pi_j = \theta^N \log_\phi(\xi_j). \tag{3.9}$$

Then $\pi_1, \ldots, \pi_r$ is an A-basis of Λ_ϕ. Moreover,

$$h_j - \frac{t^N}{t-\theta}\mathcal{L}_\phi(\xi_j;t) = f_\phi(\pi_j;t).$$

3.3 Application and example

Proposition 3.16 from the previous section can be viewed as a tool to find periods. Suppose we know a formula for $\xi_1, , \ldots, \xi_r$ and N. Then we can apply equation (3.9) and get the periods

$\pi_1, \ldots, \pi_r \in \Lambda_\phi$. Even if we do not know the explicit formula for ξ_i, it is not difficult to compute the degree of ξ_i. In this section, we demonstrate how to compute the positive integer N and the degrees of ξ_1 and ξ_2 defined in Proposition 3.4 for rank 2 Drinfeld modules. Then we compute degree of the periods directly from that information. Consider a Drinfeld module ϕ of rank 2 defined by

$$\phi_t = \theta + A_1\tau + A_2\tau^2, \quad A_2 \neq 0. \tag{3.10}$$

We categorize Drinfeld modules into 2 cases depending on the Newton polygon of $\phi_t(x)$ as follows.

- Case 1 The Newton polygon of $\phi_t(x)$ has one lower edge with slope λ_1.
- Case 2 The Newton polygon of $\phi_t(x)$ has two lower edges with slopes λ_1 and λ_2.

The reader can see the Newton polygons of Drinfeld modules of rank 2 in Figure 3.2.

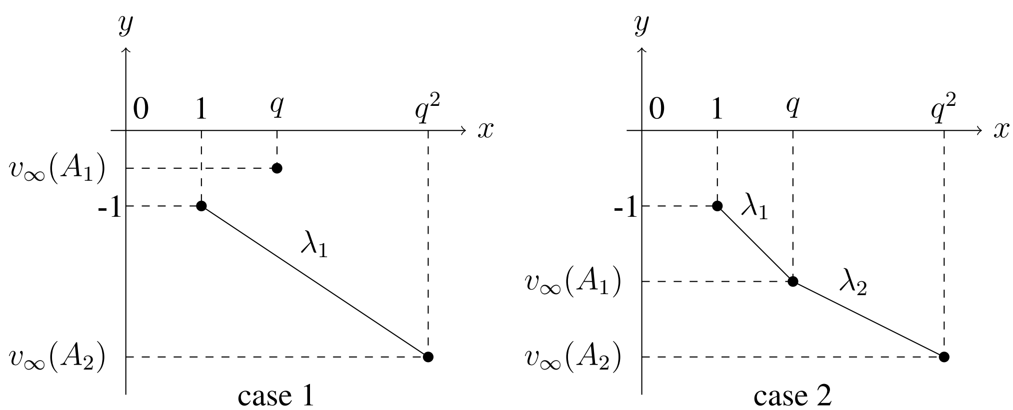

Figure 3.2: Newton polygons of Drinfeld modules of rank 2

Using our observation in Remark 3.3 and some parts of the proof of Proposition 3.4, we obtain the following theorem.

Theorem 3.17. *Let ϕ be a rank 2 Drinfeld module defined as in 3.10. Consider the following cases.*

(1) $\deg(A_1) \leq \frac{q+\deg(A_2)}{q+1}$

(2) $\deg(A_1) > \frac{q+\deg(A_2)}{q+1}$

(2.1) $\deg(A_1) < \frac{q^2+\deg(A_2)}{q+1}$

(2.2) There is a unique integer $\ell \geq 2$ *such that* $\frac{q^\ell+\deg(A_2)}{q+1} \leq \deg(A_1) < \frac{q^{\ell+1}+\deg(A_2)}{q+1}$.

Then the positive integer N *and the degrees of* ξ_1, ξ_2 *in Proposition 3.4 are determined by*

case	N	$\deg(\xi_1)$	$\deg(\xi_2)$
1	1	$\frac{1-\deg(A_2)}{q^2-1}$	$\frac{1-\deg(A_2)}{q^2-1}$
2.1	1	$\frac{1-\deg(A_1)}{q-1}$	$\frac{\deg(A_1)-\deg(A_2)}{q^2-q}$
2.2	ℓ	$\frac{1-\deg(A_1)}{q-1} - (\ell-1)$	$\frac{(-q^\ell+q+1)\deg(A_1)-\deg(A_2)}{q^\ell(q-1)}$

Table 3.1: N and degrees of ξ_i for rank 2 Drinfeld module

Theorem 3.17 allows us to compute degrees of the periods π_1, π_2 in Proposition 3.16 as follows. For $j = 1, 2$, we know from equation (3.9) that $\pi_j = \theta^N \log_\phi(\xi_j)$. We then compute that $\deg(\pi_j) = N + \deg(\log_\phi(\xi_j))$. Using a result from El-Guindy and Papanikolas [10, Cor. 4.3], we know that $\deg(\log_\phi(\xi_j)) = \deg(\xi_j)$. As a consequence, we have

$$\deg(\pi_j) = N + \deg(\xi_j), \quad j = 1, 2.$$

Finally, we apply Theorem 3.17 to a specific case when $q = 3$ and ϕ is a Drinfeld module of rank 2 defined by

$$\phi_t = \theta + y(\theta^3 - \theta)\tau + \tau^2,$$

where $y \in \mathbb{K}$ satisfies $y^2 = \theta^3 - \theta - 1$ (see [10, §7] for comparison). First, we observe that

$$\deg(A_1) = \frac{9}{2}, \quad \deg(A_2) = 0.$$

Thus

$$\frac{q^2 + \deg(A_2)}{q+1} \le \deg(A_1) < \frac{q^3 + \deg(A_2)}{q+1},$$

i.e. ϕ belongs to case 2.2, with $\ell = 2$. Using Theorem 3.17, we obtain that $N = 2$ and

$$\deg(\xi_1) = -\frac{11}{4}, \quad \deg(\xi_2) = -\frac{5}{4}.$$

Then we apply the formula $\deg(\pi_j) = N + \deg(\xi_j)$ and compute that

$$\deg(\pi_1) = 2 + \deg(\xi_1) = -\frac{3}{4}, \quad \deg(\pi_2) = 2 + \deg(\xi_2) = \frac{3}{4}.$$

4. TENSOR PRODUCTS OF DRINFELD MODULES

4.1 Tensor product

Throughout this section, we suppose that ϕ_1 and ϕ_2 are Drinfeld modules of rank r and s with the corresponding t-motives M_1 and M_2, respectively. When M_1 and M_2 are pure t-motives, the tensor product of their t-motives $M_1 \otimes_{\mathbb{K}[t]} M_2$ is also a pure t-motive, on which τ acts diagonally, i.e.

$$\tau(m \otimes m') = \tau(m) \otimes \tau(m'),$$

with weight

$$w(M_1 \otimes_{\mathbb{K}[t]} M_2) = w(M_1) + w(M_2)$$

and rank

$$r(M_1 \otimes M_2) = r(M_1)r(M_2).$$

For more details, the reader is directed to [1, §1] or [31, §7.3]. Recall from §2.5 that every Drinfeld module of rank r is pure of weight $1/r$. In our case, we then have

$$w(M_1 \otimes_{\mathbb{K}[t]} M_2) = w(M_1) + w(M_2) = \frac{1}{r} + \frac{1}{s},$$

and

$$r(M_1 \otimes M_2) = r(M_1)r(M_2) = rs.$$

We will begin our investigation by reviewing the definition and important properties of the tensor powers of the Carlitz module. The theory of the tensor powers of the Carlitz module was introduced by Anderson and Thakur, and has been well-studied (see [3], [6] and [24]). The t-module $C^{\otimes n}$ is given by

$$C_t^{\otimes n} = (\theta I + N) + E\tau,$$

where $I \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ is the identity matrix,

$$N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix}.$$

The t-motive $M(C^{\otimes n})$ of $C^{\otimes n}$ has rank 1, dimension n, and weight n. In fact, it is given by the n-th tensor power of the t-motive of the Carlitz module. Namely,

$$M(C^{\otimes n}) = \mathbb{K}[\tau] \otimes_{\mathbb{K}[t]} \cdots \otimes_{\mathbb{K}[t]} \mathbb{K}[\tau],$$

on which τ acts diagonally. For example, we consider $n = 2$. In this case, the $C^{\otimes 2}$ is a t-module given by

$$C_t^{\otimes 2} = \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tau.$$

In this case, $\mathbf{m} := (1 \otimes 1, \tau \otimes 1)^{\mathsf{T}}$ is a $\mathbb{K}[\tau]$-basis for its t-motive $M(C^{\otimes 2})$, and

$$t \cdot \mathbf{m} = t \cdot \begin{pmatrix} 1 \otimes 1 \\ \tau \otimes 1 \end{pmatrix} = \begin{pmatrix} \theta & 1 \\ \tau & \theta \end{pmatrix} \begin{pmatrix} 1 \otimes 1 \\ \tau \otimes 1 \end{pmatrix} = C_t^{\otimes 2}\mathbf{m}.$$

Anderson and Thakur [3, Cor. 2.5.8] proved that a generator in the period lattice of $C^{\otimes n}$ can be chosen so that its last coordinate is equal to the n-th power of the Carlitz period.

Theorem 4.1 (Anderson and Thakur)**.** *Let Λ_n be the period lattice of $C^{\otimes n}$. Then there is a vector $\Pi_n \in \mathrm{Mat}_{n\times 1}(\mathbb{K})$ so that $\Lambda_n = \{(\mathrm{d}C_a^{\otimes n})\Pi_n : a \in \mathbf{A}\}$. Moreover, Π_n can be chosen to have the* *form*

$$\Pi_n = \begin{pmatrix} * \\ \vdots \\ * \\ \tilde{\pi}^n \end{pmatrix}, \tag{4.1}$$

where $\tilde{\pi}$ is the Carlitz period.

4.2 Tensor product of two Drinfeld modules

In this section, we consider two Drinfeld modules ϕ_1 and ϕ_2 defined by

$$\phi_1(t) = \theta + A_1\tau + \ldots + A_r\tau^r,\ \phi_2(t) = \theta + B_1\tau + \ldots + B_s\tau^s, \tag{4.2}$$

where $r \leq s$, and both A_r and B_s are not zero. To simplify the notation, we denote $T = t - \theta$ and

$$[m, n] = A_m^{(-n)}, \quad (m, n) = B_m^{(-n)}, \quad m, n \in \mathbb{N}.$$

Recall that the associated t-motives of ϕ_1 and ϕ_2 are $M_1 = \mathbb{K}[\tau]$ with the action of $\mathbb{K}[t, \tau]$ given by

$$ct^i \cdot m = c \cdot m \cdot \phi_1(t^i),$$

and $M_2 = \mathbb{K}[\tau]$ with the action of $\mathbb{K}[t, \tau]$ given by

$$ct^i * m = c \cdot m \cdot \phi_2(t^i),$$

respectively. One can see that the basis of M_1 as a $\mathbb{K}[t]$-module is $\{1, \tau, \ldots, \tau^{r-1}\}$ and the basis of M_2 as a $\mathbb{K}[t]$-module is $\{1, \tau, \ldots, \tau^{s-1}\}$. In fact, the reader may go back to section 3.1 for a review of this result. Also, we let

$$\Theta_1 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ T/A_r & -A_1/A_r & \dots & -A_{r-1}/A_r \end{pmatrix},\ \Theta_2 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ T/B_s & -B_1/B_s & \dots & -B_{s-1}/B_s \end{pmatrix}.$$

Then $\tau(1,\dots,\tau^{r-1})^\intercal = \Theta_1(1,\dots,\tau^{r-1})^\intercal$ and $\tau(1,\dots,\tau^{s-1})^\intercal = \Theta_2(1,\dots,\tau^{s-1})^\intercal$. That is, Θ_1 and Θ_2 are the matrices representing τ-action of ϕ_1 and ϕ_2 with respect to the bases $\{1,\dots,\tau^{r-1}\}$ and $\{1,\dots,\tau^{s-1}\}$, respectively. It is not hard to see that the following is a basis of $M_1 \otimes_{\mathbb{K}[t]} M_2$ as a $\mathbb{K}[t]$-module

$$\{\tau^i \otimes \tau^j : 0 \leq i \leq r-1, 0 \leq j \leq s-1\}.$$

It is fundamental to find a matrix representing τ-action of $\phi_1 \otimes \phi_2$ with respect to the basis of $M_1 \otimes_{\mathbb{K}[t]} M_2$ above.

For any matrices $A = (a_{i,j}) \in \mathrm{Mat}_{n\times m}(R)$ and $B = (b_{i,j}) \in \mathrm{Mat}_{\ell\times k}(R)$, where R is a commutative ring, the *Kronecker product* $A \otimes B$ is the $n\ell \times mk$ block matrix defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{pmatrix}.$$

Theorem 4.2. *Let $\Theta = \Theta_1 \otimes \Theta_2$ be the Kronecker product of matrices and let*

$$\mathbf{m} = \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes \tau \\ \vdots \\ 1 \otimes \tau^{s-1} \\ \vdots \\ \tau^{r-1} \otimes 1 \\ \tau^{r-1} \otimes \tau \\ \vdots \\ \tau^{r-1} \otimes \tau^{s-1} \end{pmatrix}.$$

Then Θ is a matrix representing the τ-action with respect to $\mathbf{m}$, i.e. $\tau\mathbf{m} = \Theta\mathbf{m}$.

Now, we consider the dual t-motives of ϕ_1 and ϕ_2. For each $i = 1, 2$, we denote the dual t-motive of ϕ_i by N_i. One can see that the basis of N_1 as a $\mathbb{K}[t]$-module is $\{1, \sigma, \dots, \sigma^{r-1}\}$ and the basis of N_2 as a $\mathbb{K}[t]$-module is $\{1, \sigma, \dots, \sigma^{s-1}\}$. As s result, the following is a basis of $N_1 \otimes_{\mathbb{K}[t]} N_2$ as a $\mathbb{K}[t]$-module

$$\{\sigma^i \otimes \sigma^j : 0 \le i \le r-1, 0 \le j \le s-1\}.$$

Let

$$\Phi_1 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \frac{T}{[r,r]} & -\frac{[1,1]}{[r,r]} & \dots & -\frac{[r-1,r-1]}{[r,r]} \end{pmatrix}, \; \Phi_2 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \frac{T}{(s,s)} & -\frac{(1,1)}{(s,s)} & \dots & -\frac{(s-1,s-1)}{(s,s)} \end{pmatrix}.$$

Then Φ_1 and Φ_2 are the matrices representing σ-action of ϕ_1 and ϕ_2 with respect to the bases $\{1, \sigma, \dots, \sigma^{r-1}\}$ and $\{1, \sigma, \dots, \sigma^{s-1}\}$, respectively.

Theorem 4.3. *Let $\Phi = \Phi_1 \otimes \Phi_2$ the a Kronecker product of matrices and let*

$$\mathbf{n} = \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes \sigma \\ \vdots \\ 1 \otimes \sigma^{s-1} \\ \vdots \\ \sigma^{r-1} \otimes 1 \\ \sigma^{r-1} \otimes \sigma \\ \vdots \\ \sigma^{r-1} \otimes \sigma^{s-1} \end{pmatrix}.$$

Then Φ is a matrix representing the σ-action with respect to $\mathbf{n}$, i.e. $\sigma\mathbf{n} = \Phi\mathbf{n}$.

Next, we want to define the tensor product $\phi_1 \otimes \phi_2$ as a t-module in a similar way as $C^{\otimes n}$ is defined in section 4.1. Hamahata defined the tensor product $\phi_1 \otimes \phi_2$ in Definition 2.1 in [19]. We denote $\rho = \phi_1 \otimes \phi_2$ as a t-module, so $\rho : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$, where d is a dimension. The first main goal is to determine the value of ρ_t. We know that $w(\rho) = w(\phi_1) + w(\phi_2) = (r+s)/rs$ and that $r(\rho) = \mathrm{rank}_{\mathbb{K}[t]}\, \rho = rs$. This implies that $\dim_{\mathbb{K}[\tau]}(\rho) = d(\rho) = r + s$. As a result, we are able to find a $\mathbb{K}[\tau]$-basis for $M_1 \otimes M_2$ as follows.

Lemma 4.4. *For $1 \leq i \leq s$ and $1 \leq j \leq r$, let*

$$x_i = 1 \otimes \tau^{i-1}, \quad y_j = \tau^j \otimes 1.$$

Then $\{x_1, \ldots, x_s, y_1, \ldots, y_r\}$ is a basis of $M_1 \otimes_{\mathbb{K}[t]} M_2$ as a $\mathbb{K}[\tau]$-module.

Proof. We observe that $\mathrm{rank}_{\mathbb{K}[\tau]}(M_1 \otimes_{\mathbb{K}[t]} M_2) = r + s$, so it suffices to show that the set $X := \{x_1, \ldots, x_s, y_1, \ldots, y_r\}$ spans $M_1 \otimes_{\mathbb{K}[t]} M_2$ as a $\mathbb{K}[\tau]$-module. Since $\{\tau^i \otimes \tau^j : 0 \leq i \leq r-1, 0 \leq$

$j \le s-1\}$ is a $\mathbb{K}[t]$-basis for $M_1 \otimes_{\mathbb{K}[t]} M_2$, any element $m \in M_1 \otimes_{\mathbb{K}[t]} M_2$ can be written as

$$m = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} a_{i,j}(\tau^i \otimes \tau^j), \quad a_{i,j} \in \mathbb{K}[t].$$

The idea of the proof is to show that, for every $k \ge 0$, $0 \le i \le r-1$, and $0 \le j \le s-1$,

$$t^k(\tau^i \otimes \tau^j) \in \operatorname{Span}_{\mathbb{K}[\tau]} X. \tag{4.3}$$

This can be proved by using induction on k as follows. For $k=0$, we observe that $\tau^i \otimes \tau^j$ is equal to either $\tau^i(1 \otimes \tau^{j-i})$ or $\tau^j(\tau^{i-j} \otimes 1)$, and both of them belong to $\operatorname{Span}_{\mathbb{K}[\tau]} X$. Now suppose that the statement is true for k. Fix $0 \le i \le r-1$, and $0 \le j \le s-1$. By the hypothesis,

$$t^k(\tau^i \otimes \tau^j) = \sum_{\ell=1}^{s} a_\ell x_\ell + \sum_{\ell=1}^{r} b_\ell y_\ell, \quad a_\ell, b_\ell \in \mathbb{K}[\tau].$$

We compute that, for each ℓ,

$$\begin{aligned} t x_\ell &= (t \cdot 1) \otimes \tau^{\ell-1} = (\theta + \ldots + A_r \tau^r) \otimes \tau^{\ell-1} \in \operatorname{Span}_{\mathbb{K}[\tau]} X, \\ t y_\ell &= \tau^\ell \otimes (t \cdot 1) = \tau^\ell \otimes (\theta + \ldots + B_s \tau^s) \in \operatorname{Span}_{\mathbb{K}[\tau]} X. \end{aligned}$$

Therefore,

$$t^{k+1}(\tau^i \otimes \tau^j) = \sum_{\ell=1}^{s} a_\ell t x_\ell + \sum_{\ell=1}^{r} b_\ell t y_\ell \in \operatorname{Span}_{\mathbb{K}[\tau]} X.$$

□

Now we have a $\mathbb{K}[\tau]$-basis for $M_1 \otimes_{\mathbb{K}[t]} M_2$. We can use it to determine the value of ρ_t. After a calculation, one can see that the t-module ρ can be defined in the following way.

Definition 4.5. We define a t-module $\rho := \phi_1 \otimes \phi_2 : \mathbf{A} \to \operatorname{Mat}_{r+s}(\mathbb{K}[\tau])$ given by

$$\rho_t = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right),$$

where $X_1 \in \mathrm{Mat}_{s\times s}(\mathbb{K}[\tau]), X_2 \in \mathrm{Mat}_{s\times r}(\mathbb{K}[\tau]), X_3 \in \mathrm{Mat}_{r\times s}(\mathbb{K}[\tau]), X_4 \in \mathrm{Mat}_{r\times r}(\mathbb{K}[\tau])$ are defined by

$$X_1 = \begin{pmatrix} \theta & & & & & & \\ A_1\tau & \theta & & & & & \\ \vdots & \ddots & \ddots & & & & \\ A_{r-1}\tau^{r-1} & \dots & A_1\tau & \theta & & & \\ A_r\tau^r & & \dots & A_1\tau & \theta & & \\ & \ddots & & & \ddots & \ddots & \\ & & A_r\tau^r & & \dots & A_1\tau & \theta \end{pmatrix}, X_2 = \begin{pmatrix} A_1 & \dots & A_{r-1} & A_r \\ A_2\tau & \dots & A_r\tau & \\ \vdots & \iddots & & \\ A_r\tau^{r-1} & & & \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} B_1\tau & \dots & \dots & B_{s-1}\tau & B_s\tau \\ B_2\tau^2 & \dots & \dots & B_s\tau^2 & \\ \vdots & & \iddots & & \\ B_r\tau^r & \dots & B_s\tau^r & & \end{pmatrix}, X_4 = \begin{pmatrix} \theta & & & \\ B_1\tau & \theta & & \\ \vdots & \ddots & \ddots & \\ B_{r-1}\tau^{r-1} & \dots & B_1\tau & \theta \end{pmatrix}.$$

Remark 4.6. We observe that ρ_t can be expressed as a polynomial in τ as

$$\rho_t = \theta I_{r+s} + N + F_1\tau + \cdots + F_r\tau^r, \tag{4.4}$$

where $N, F_1, \dots, F_r \in \mathrm{Mat}_{r+s}(\mathbb{K})$ and

$$N = \begin{pmatrix} 0 & \dots & 0 & A_1 & A_2 & \dots & A_{r-1} & A_r \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

As an example, we take the tensor product of two Carlitz modules. From the definition above, we have

$$C_t^{\otimes 2} = \begin{pmatrix} \theta & 1 \\ \tau & \theta \end{pmatrix},$$

which gives the same definition of $C^{\otimes 2}$ as the one Anderson and Thakur defined; see §4.1.

Now we consider another example. Suppose that ϕ_1 and ϕ_2 are Drinfeld modules both with rank 3, and given as in (4.2). Then from Definition 4.5, the tensor product $\rho = \phi_1 \otimes \phi_2$ is a t-module given by

$$\rho_t = \begin{pmatrix} \theta & 0 & 0 & A_1 & A_2 & A_3 \\ A_1\tau & \theta & 0 & A_2\tau & A_3\tau & 0 \\ A_2\tau^2 & A_1\tau & \theta & A_3\tau^2 & 0 & 0 \\ B_1\tau & B_2\tau & B_3\tau & \theta & 0 & 0 \\ B_2\tau^2 & B_3\tau^2 & 0 & B_1\tau & \theta & 0 \\ B_3\tau^3 & 0 & 0 & B_2\tau^2 & B_1\tau & \theta \end{pmatrix}.$$

Recall from §4.1 that in the case of the t-module $C^{\otimes 2}$, we can choose a $\mathbb{K}[\tau]$-basis $\mathbf{m}$ of its t-motive so that $t \cdot \mathbf{m} = C_t^{\otimes 2}\mathbf{m}$. We want our definition of ρ to have the same property. We actually obtain from this property that ρ is the tensor product.

Theorem 4.7. *Let* $\mathbf{m} \in \mathrm{Mat}_{(r+s)\times 1}(M_1 \otimes_{\mathbb{K}[t]} M_2)$ *consist of the* $\mathbb{K}[\tau]$*-basis of* $M_1 \otimes_{\mathbb{K}[t]} M_2$ *from Lemma 4.4 and let* ρ *be the* t*-module defined in Definition 4.5. Then the* t*-action with respect to this basis is represented by the matrix* ρ_t*, which means* $t \cdot \mathbf{m} = \rho_t\mathbf{m}$*.*

Remark 4.8. One can see that

$$\rho_t \mathbf{m} = \begin{pmatrix} \theta x_1 + A_1 y_1 + A_2 y_2 + \ldots + A_r y_r \\ \theta x_2 + A_1 \tau x_1 + A_2 \tau y_1 + \ldots + A_r \tau y_{r-1} \\ \vdots \\ \theta x_r + A_1 \tau x_{r-1} + \ldots A_{r-1} \tau^{r-1} x_1 + A_r \tau^{r-1} y_1 \\ \theta x_{r+1} + A_1 \tau x_r + A_2 \tau^2 x_{r-1} + \ldots + A_r \tau^r x_1 \\ \vdots \\ \theta x_s + A_1 \tau x_{s-1} + A_2 \tau^2 x_{s-2} + \ldots + A_r \tau^r x_{s-r} \\ \theta y_1 + B_1 \tau x_1 + B_2 \tau x_2 + \ldots + B_s \tau x_s \\ \theta y_2 + B_1 \tau y_1 + B_2 \tau^2 x_1 + \ldots + B_s \tau^s x_{s-1} \\ \vdots \\ \theta y_r + B_1 \tau y_{r-1} + \ldots + B_{r-1} \tau^{r-1} y_1 + B_r \tau^r x_1 + B_{r+1} \tau^r x_2 + \ldots + B_s \tau^r x_{s-r+1} \end{pmatrix}.$$

Comparing to the result by Y. Hamahata [19], we see that our definition of $\phi_1 \otimes \phi_2$ is actually equal to $\phi_2 \otimes \phi_1$ in his definition. However, he also proved that $\phi_1 \otimes \phi_2$ is isomorphic to $\phi_2 \otimes \phi_1$, as the tensor product of t-motives is symmetric. Also, the t-motive $M(\phi_1 \otimes \phi_2)$ is isomorphic to $M_1 \otimes_{\mathbb{K}[t]} M_2$.

4.3 Periods for the tensor product of two Drinfeld modules

First of all, we will introduce useful notation and results on rigid analytic trivializations and periods of a t-module. More details about these results can be found in [13], [18] and [22, §3]. Suppose $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$ is an $\mathbf{A}$-finite t-module with its dual t-motive $N_\phi = \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma])$. For $\mathbf{n} = \sum_{i=0}^{\ell} \mathbf{a}_i \sigma^i \in N_\phi$, we set

$$\epsilon_0(\mathbf{n}) = \mathbf{d}\mathbf{n}^\mathsf{T} = \mathbf{a}_0^\mathsf{T},$$

where d denote the projection onto the constant term. Suppose that $\{\mathbf{n}_1, \ldots, \mathbf{n}_r\}$ is a $\mathbb{K}[t]$-basis of N_ϕ and $\Phi \in \mathrm{Mat}_r(\mathbb{K}[t])$ is the unique matrix such that $\sigma \mathbf{n} = \Phi \mathbf{n}$, where $\mathbf{n} := (\mathbf{n}_1, \ldots, \mathbf{n}_r)^\mathsf{T}$. We define a map

$$\iota : \mathrm{Mat}_{1\times r}(\mathbb{K}[t]) \to N_\phi,$$

by setting

$$\iota(\alpha) = \alpha \cdot \mathbf{n},$$

for any $\alpha \in \mathrm{Mat}_{1\times r}(\mathbb{K}[t])$. We call the pair (ι, Φ) a *t-frame* for ϕ.

Lemma 4.9 (Anderson; see [13, Rem. 4.4.4] and [18, Prop. 2.5.8])**.** *There exists a unique bounded $\mathbb{K}$-linear map*

$$\mathcal{E}_0 : (\mathrm{Mat}_{1\times r}(\mathbb{T}_\theta), \|\cdot\|_\theta) \to (\mathbb{K}^d, |\cdot|_\infty)$$

of normed vector spaces such that $\mathcal{E}_0|_{\mathrm{Mat}_{1\times r}(\mathbb{K}[t])} = \epsilon_0 \circ \iota$.

The map $\mathcal{E}_0$ in Lemma 4.9 is a tool for finding periods of a t-module as one can see in the following theorem (see [13, Thm. 4.5.14] or [22, Thm. 3.4.7]).

Theorem 4.10 (Anderson)**.** *Let $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$ be an $\mathbf{A}$-finite t-module . Then*

1. *ϕ is uniformizable if and only if it has rigid analytic trivialization.*

2. *If (ι, Φ, Ψ) is a rigid analytic trivialization of ϕ in the sense of* (3.2)*, then*

$$\Lambda_\phi = \mathcal{E}_0((\mathrm{Mat}_{1\times r}\,\mathbf{A}) \cdot \Psi^{-1}).$$

For $\ell \in \mathbb{N}$, we denote the standard basis vectors of $\mathrm{Mat}_{1\times \ell}(\mathbb{K}[\tau])$ by $s_1, \ldots, s_\ell$, and we denote the standard basis vectors of $\mathrm{Mat}_{1\times \ell}(\mathbb{K}[t])$ by $e_1, \ldots, e_\ell$. In the next proposition, we state a useful result from Namoijam and Papanikolas [22, §3.5] in which they demonstrated how to calculate $\mathcal{E}_0$ when the matrix $\mathrm{d}\phi_t$ is in Jordan normal form.

Proposition 4.11 (Namoijam and Papanikolas)**.** *Let $\phi : \mathbf{A} \to \mathrm{Mat}_d(\mathbb{K}[\tau])$ be a t-module of rank r with t-frame (ι, Φ). Suppose that the following conditions hold.*

(i) There exists $C \in \mathrm{GL}_r(\mathbb{K}[t])$ so that

$$D := C\Phi = \begin{pmatrix} T^{\ell_1} & & \\ & \ddots & \\ & & T^{\ell_r} \end{pmatrix}$$

is a diagonal matrix. Furthermore, for some m with $1 \leq m \leq r$, we have $\ell_1, \ldots, \ell_m > 0$, $\ell_{m+1} = \cdots = \ell_r = 0$, and $\ell_1 + \cdots + \ell_m = d$.

(ii) For $1 \leq i \leq m$ and $1 \leq j \leq \ell_i$,

$$\mathrm{d}\iota((t-\theta)^{j-1} \cdot e_i) = s_{\ell_1+\cdots+\ell_i-j+1}.$$

Then for $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$,

$$\mathcal{E}_0(\alpha) = \begin{pmatrix} \partial_t^{\ell_1-1}(\alpha_1) \\ \vdots \\ \partial_t^1(\alpha_1) \\ \alpha_1 \\ \vdots \\ \partial_t^{\ell_m-1}(\alpha_m) \\ \vdots \\ \partial_t^1(\alpha_m) \\ \alpha_m \end{pmatrix}\Bigg|_{t=\theta}. \tag{4.5}$$

From now on, we let ρ be the tensor product $\phi_1 \otimes \phi_2$, which is defined in Definition 4.5. Our goal in this section is to find periods for ρ using Theorem 4.10. First, we need to find an explicit formula for $\mathcal{E}_0$ using Proposition 4.11. Namoijam and Papanikolas ([22, Rem. 3.5.9]) also explained that conditions (i) and (ii) imply that the matrix $\mathrm{d}\phi_t$ is in the following Jordan normal form:

$$\mathrm{d}\phi_t = \begin{pmatrix} \mathrm{d}_{\theta,\ell_1}[\theta] & & \\ & \ddots & \\ & & \mathrm{d}_{\theta,\ell_m}[\theta] \end{pmatrix} \in \operatorname{Mat}_d \mathbb{K}, \tag{4.6}$$

where, for any $k \in \mathbb{N}$, we define

$$\mathrm{d}_{\theta,k}[\theta] = \begin{pmatrix} \theta & 1 & & \\ & \ddots & \ddots & \\ & & \theta & 1 \\ & & & \theta \end{pmatrix} \in \operatorname{Mat}_k \mathbb{K}.$$

Returning to our formula for ρ_t from equation (4.4), we recall that the matrix $\mathrm{d}\rho_t$ does not have Jordan normal form. Thus we want to pursue our investigation by finding a t-module ρ' such that ρ' is isomorphic to ρ and $\mathrm{d}\rho'_t$ is in Jordan normal form as in equation (4.6).

Theorem 4.12. *Let ρ be a t-module defined as in equation 4.4 and let*

$$\gamma = \begin{pmatrix} & & & & & 1 & 0 \\ & & & & \iddots & & \vdots \\ & & & 1 & & & 0 \\ & & 1 & & & & 0 \\ & \iddots & & & & & \vdots \\ 1 & & & & & & 0 \\ -\frac{A_{r-1}}{A_r} & \dots & -\frac{A_1}{A_r} & 0 & \dots & 0 & \frac{1}{A_r} \end{pmatrix} \in \operatorname{Mat}_{r+s}(\mathbb{K}).$$

Then γ induces an isomorphism of t-modules $\gamma : \rho' \to \rho$, where the t-module ρ' is given by $\rho'_t = \gamma^{-1}\rho_t\gamma$. Moreover,

$$\rho'_t = \left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right) \in \operatorname{Mat}_{r+s} \mathbb{K}[\tau],$$

where $Y_1 \in \mathrm{Mat}_{r-1}(\mathbb{K}[\tau]), Y_2 \in \mathrm{Mat}_{(r-1)\times(s+1)}(\mathbb{K}[\tau]), Y_3 \in \mathrm{Mat}_{(s+1)\times(r-1)}(\mathbb{K}[\tau]), Y_4 \in \mathrm{Mat}_{s+1}(\mathbb{K}[\tau])$ *are defined by*

$$Y_1 = \begin{pmatrix} \theta & B_1\tau & B_2\tau^2 & \dots & \dots & B_{r-2}\tau^{r-2} \\ & \theta & B_1\tau & \dots & \dots & B_{r-3}\tau^{r-3} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \theta & B_1\tau & B_2\tau^2 \\ & & & & \theta & B_1\tau \\ & & & & & \theta \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} & & & B_s\tau^{r-1} & \dots & B_r\tau^{r-1} & B_{r-1}\tau^{r-1} & 0 \\ & & \iddots & & & \vdots & \vdots & \vdots \\ & B_s\tau^2 & \dots & \dots & \dots & B_3\tau^2 & B_2\tau^2 & 0 \\ B_s\tau & B_{s-1}\tau & \dots & \dots & \dots & B_2\tau & B_1\tau & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & & \dots & 0 \\ \vdots & & & \vdots \\ & & & A_r\tau^{r-1} \\ & & \iddots & \vdots \\ & A_r\tau^2 & \dots & A_3\tau^2 \\ A_r\tau & A_{r-1}\tau & \dots & A_2\tau \\ 0 & & \dots & 0 \\ d_1 & d_2 & \dots & d_{r-1} \end{pmatrix},$$

$$
Y_4 = \begin{pmatrix}
\theta & A_1\tau & A_2\tau^2 & \dots & A_r\tau^r & & & 0 \\
 & \ddots & \ddots & & & \ddots & & \vdots \\
 & & \theta & A_1\tau & A_2\tau^2 & \dots & A_r\tau^r & 0 \\
 & & & \theta & A_1\tau & \dots & A_{r-1}\tau^{r-1} & 0 \\
 & & & & \ddots & \ddots & \vdots & \vdots \\
 & & & & & \theta & A_1\tau & 0 \\
 & & & & & & \theta & 1 \\
c_1 & c_2 & \dots & c_r & e_1 & \dots & e_{s-r} & \theta
\end{pmatrix},
$$

and

$$
d_\ell = \sum_{i=1}^{\ell} A_{(r-\ell)+i} B_i \tau^i, \quad 1 \le \ell \le r-1,
$$

$$
c_\ell = \sum_{i=1}^{\ell} A_i B_{(s-\ell)+i} \tau^i, \quad 1 \le \ell \le r,
$$

$$
e_\ell = \sum_{i=1}^{r} A_i B_{(s-r-\ell)+i} \tau^i, \quad 1 \le \ell \le s-r.
$$

Proof. We compute that $\det(\gamma) = 1/A_r$. This implies that γ is an isomorphism of t-modules. The second statement follows from a direct computation; we leave the details to the reader. $\square$

Next, we want to find a $\mathbb{K}[t]$-basis for the dual t-motive of ρ' defined in Theorem 4.12. Recall that the dual t-motive is a left $\mathbb{K}[t,\sigma]$-module $N_{\rho'} = \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma])$ with $t \cdot h = h(\rho'_t)^*$.

Lemma 4.13. *Let ρ' be a t-module defined in Theorem 4.12. Then*

$$
(\rho'_t)^* = \left(\begin{array}{c|c} Y_1^* & Y_3^* \\ \hline Y_2^* & Y_4^* \end{array}\right) \in \mathrm{Mat}_{r+s}(\mathbb{K}[\tau]),
$$

$$
Y_1^* = \begin{pmatrix}
\theta & & & & \\
\sigma B_1 & \theta & & & \\
\sigma^2 B_2 & \sigma B_1 & \theta & & \\
\vdots & & \ddots & \ddots & \\
\sigma^{r-2} B_{r-2} & \dots & \dots & \sigma B_1 & \theta
\end{pmatrix} \mathrm{Mat}_{r-1}(\mathbb{K}[\tau]),
$$

$$
Y_2^* = \begin{pmatrix}
 & & & \sigma B_s \\
 & & \sigma^2 B_s & \sigma B_{s-1} \\
 & \iddots & & \vdots \\
\sigma^{r-1} B_s & \dots & \dots & \sigma B_{s-r+2} \\
\vdots & & & \vdots \\
\sigma^{r-1} B_{r-1} & \dots & \dots & \sigma B_1 \\
0 & \dots & \dots & 0
\end{pmatrix} \in \mathrm{Mat}_{(s+1)\times(r-1))}(\mathbb{K}[\tau]),
$$

$$
Y_3^* = \begin{pmatrix}
0 & \dots & 0 & & & & \sigma A_r & 0 & d_1^* \\
0 & \dots & 0 & & & \sigma^2 A_r & \sigma A_{r-1} & 0 & d_2^* \\
\vdots & & \vdots & & \iddots & & \vdots & \vdots & \vdots \\
0 & \dots & 0 & \sigma^{r-1} A_r & \dots & \dots & \sigma A_2 & 0 & d_{r-1}^*
\end{pmatrix} \in \mathrm{Mat}_{(r-1)\times(s+1))}(\mathbb{K}[\tau]),
$$

and

$$
Y_4^* = \begin{pmatrix}
\theta & & & & & & & c_1^* \\
\sigma A_1 & & & & & & & c_2^* \\
\vdots & \ddots & \theta & & & & & c_3^* \\
\vdots & & \sigma A_1 & & & & & \vdots \\
\sigma^r A_r & & \vdots & \ddots & & & & e_1^* \\
 & \ddots & \vdots & & \ddots & & & \vdots \\
 & & \sigma^r A_r & \dots & & \sigma A_1 & \theta & e_{s-r}^* \\
 & & & & & & 1 & \theta
\end{pmatrix} \in \mathrm{Mat}_{s+1}(\mathbb{K}[\tau]).
$$

Proof. This formula is computed directly from the definition of ρ_t and the notation of $*$ defined in section 2.5 . □

Lemma 4.14. *Let $N_{\rho'}$ be the dual t-motive associated to ρ' from Theorem 4.12. For $1 \leq j \leq r$, let*

$$N_j = \{\sigma^{j-1}s_1, \ldots, \sigma^{j-1}s_{r-j}, \sigma^{j-1}s_r, \ldots, \sigma^{j-1}s_{r+s-(j+1)}, \sigma^{j-1}s_{r+s}\}.$$

Then $N := \cup_{j=1}^{r} N_j$ is a $\mathbb{K}[t]$-basis of $N_{\rho'}$.

Proof. For each $1 \leq j \leq r$, we observe that $\#N_j = r+s-j-(j-1) = r+s+1-2j$, so

$$\#N = \sum_{j=1}^{r}(r+s+1-2j) = r(r+s+1) - 2(r/2)(r+1) = rs.$$

Recall that $\operatorname{rank}_{\mathbb{K}[t]} N_{\rho'} = rs$, so we just need to show the $\operatorname{Span}_{\mathbb{K}[t]} N = N_{\rho'}$. Recall that $N_{\rho'} = \operatorname{Mat}_{1\times(r+s)} \mathbb{K}[\sigma]$, so every element $\alpha \in N_{\rho'}$ is a sum of monomials in the form $a\sigma^{\ell}s_k$ where $a \in \mathbb{K}$ $\ell \geq 0$ and $1 \leq k \leq r+s$. Thus it suffices to show that $\sigma^{\ell}s_k \in \operatorname{Span}_{\mathbb{K}[t]} N$ for all ℓ, k. For each ℓ, let

$$X_\ell = \{\sigma^{\ell-1}s_k : k = 1, \ldots, r+s\},$$
$$P_\ell = \{\sigma^{\ell-1}s_k : k = r-\ell+1, \ldots, r-1\},$$
$$Q_\ell = \{\sigma^{\ell-1}s_k : k = r+s-\ell, \ldots, r+s-1\}.$$

Then $X_\ell = N_\ell \cup P_\ell \cup Q_\ell$ and we want to show that $X_\ell \subset \operatorname{Span}_{\mathbb{K}[t]} N$ for all ℓ. This can be proved using induction on ℓ as follows. For $\ell = 1$, we already have $s_1, \ldots, s_{r+s-2}, s_{r+s} \in \operatorname{Span}_{\mathbb{K}[t]} N$. We also compute directly from the formula for $(\rho'_t)^*$ that

$$t \cdot s_{r+s} = s_{r+s}(\rho'_t)^* = s_{r+s-1} + \theta s_{r+s},$$

so $s_{r+s-1} = (t-\theta)s_{r+s} \in \operatorname{Span}_{\mathbb{K}[t]} N$. Suppose that $X_1 \cup \ldots \cup X_\ell \subset \operatorname{Span}_{\mathbb{K}[t]} N$. Let $x \in X_{\ell+1}$. If $x \in N_{\ell+1}$, then $x \in \operatorname{Span}_{\mathbb{K}[t]} N$ and we are done. Now suppose that $x \in P_{\ell+1}$. Then $x = \sigma^{\ell}s_k$ for some $k \in \{r-\ell, \ldots, r-1\}$. Case 1: Suppose that $x = \sigma^{\ell}s_{r-\ell}$. By the definition of $(\rho'_t)^*$, we observe that

$$B_s^{(-\ell)}\sigma^\ell s_{r-\ell} = (t-\theta)s_{r-1+\ell} + \sum_{k=1}^{3}\sum_{z\in D_{k,\ell}} a(z)z,$$

where $a(z)\in\mathbb{K}$ depends only on z and

$$\begin{aligned}
D_{1,\ell} &= \{\sigma^i s_{r-i} : i = 1,2,\dots,\ell-1\},\\
D_{2,\ell} &= \{\sigma^i s_{r+l-1-i} : i = 1,2,\dots,\ell-1\},\\
D_{3,\ell} &= \{\sigma^i s_{r+s} : i = 1,2,\dots,r-\ell\}.
\end{aligned}$$

It is clear that $D_{1,\ell}\cup D_{2,\ell}\in X_{\ell-1}\subset \operatorname{Span}_{\mathbb{K}[t]} N$ and that $D_{3,\ell}\in N$. So $D_{1,\ell}\cup D_{2,\ell}\cup D_{3,\ell}\in \operatorname{Span}_{\mathbb{K}[t]} N$. This implies that $x\in \operatorname{Span}_{\mathbb{K}[t]} N$. Case 2: Suppose that $x=\sigma^\ell s_i$ for some $r-\ell\le i\le r-1$. Then $x=\sigma^{\ell-(r-i)}(\sigma^{r-i}s_i)=\sigma^{\ell-u}(\sigma^u s_{r-u})$, where $u=r-i$. By the definition of $(\rho_t')^*$, we observe that

$$B_s^{(-u)}\sigma^u s_{r-u} = (t-\theta)s_{r-1+u} + \sum_{k=1}^{3}\sum_{z\in D_{k,u}} a(z)z,$$

so

$$\begin{aligned}
x &= \sigma^{\ell-u}\frac{1}{B_s^{(-u)}}\left((t-\theta)s_{r-1+u} + \sum_{k=1}^{3}\sum_{z\in D_{k,u}} a(z)z\right)\\
&= \frac{1}{B_s^{(-\ell)}}\left((t-\theta^{q^{u-\ell}})\sigma^{\ell-u}s_{r-1+u} + \sum_{k=1}^{3}\sum_{z\in D_{k,u}} a(z)^{q^{u-\ell}}\sigma^{\ell-u}z\right).
\end{aligned}$$

We observe that, since $u\ge 1$, $\sigma^{\ell-u}s_{r-1+u}\in X_1\cup\ldots\cup X_\ell$ and $\sigma^{\ell-u}z\in X_1\cup\ldots\cup X_\ell$ for all $z\in D_{1,u}\cup D_{2,u}$. Also $\sigma^{\ell-u}z\in N$ for all $z\in D_{3,u}$. It follows that $x\in\operatorname{Span}_{\mathbb{K}[t]} N$. When $x\in Q_{\ell+1}$, by using the similar argument as above, we can see that $x\in\operatorname{Span}_{\mathbb{K}[t]} N$. □

Let $\mathbf{n}'\in \operatorname{Mat}_{rs\times 1}(N_{\rho'})$ be a column vector consisting of $\mathbb{K}[t]$-basis for the dual t-motive $N_{\rho'}$

from Lemma 4.14 with the following order:

$$\mathbf{n}' = (s_1, \ldots, s_{r+s-2}, s_{r+s}, \sigma s_1, \ldots, \sigma s_{r-2}, \sigma s_r, \ldots, \sigma s_{r+s-3}, \ldots, \sigma s_{r+s}, \ldots, \sigma^{r-1} s_{r+s})^{\mathsf{T}}.$$

Let (ι', Φ') be the t-frame induced by $\mathbf{n}'$ and let Ψ' be a rigid analytic trivialization corresponding Φ'. This means $\sigma\mathbf{n}' = \Phi'\mathbf{n}'$ and $\Psi'^{(-1)} = \Phi'\Psi'$. Next, we want to prove that the t-frame (ι', Φ') satisfies the condition (i) and (ii) in Proposition 4.11. First, we need to define a matrix B which will serve as a matrix representing the changing of $\mathbb{K}[t]$-basis.

Definition 4.15. 1. For $1 \le j \le r-1$ and $1 \le \ell \le rs$, define $b_{\ell,j}$ as follows:

$$b_{\ell,j} = \begin{cases} \frac{1}{[r,j]} & \text{if } \ell = js+1 \\ -\frac{1}{[r,k]} \sum_{i=j}^{k-1} b_{is+1,j}[r-(k-i), i] & \text{if } \ell = ks+1 \text{ for some k} \ge j+1 \\ 0 & \text{otherwise} \end{cases}$$

2. For $r \le j \le r+s-2$ and $1 \le \ell \le rs$, define $b_{\ell,j}$ as follows:

$$b_{\ell,j} = \begin{cases} \frac{1}{(s,j-r+1)} & \text{if } \ell = j-r+2 \\ -\frac{1}{(s,\ell-1)} \sum_{m=j-r+2}^{\ell-1} b_{m,j}(s-\ell+m, m-1) & \text{if } j-r+3 \le \ell \le s \\ 0 & \text{otherwise} \end{cases}$$

3. For $1 \le \ell \le r^2$, define

$$b_{\ell,r+s-1} = \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 4.16. Define a matrix $B \in \mathrm{Mat}_{rs}(\mathbb{K})$ as follows. For each $0 \le \ell \le r-1$ and $1 \le m \le s$, we define

$$\operatorname{row}_{\ell s+m} B = \sum_{k=r}^{r+m-(\ell+2)} b^{(-\ell)}_{m-\ell,k} s_{k+\ell[r+s-(\ell+1)]}, \quad \text{if } \ell+2 \leq m, \tag{4.7}$$

$$\operatorname{row}_{\ell s+m} B = s_{m(r+s-m)}, \quad \text{if } m = \ell+1, \tag{4.8}$$

$$\operatorname{row}_{\ell s+m} B = \sum_{k=1}^{\ell-(m-1)} b^{(-m+1)}_{[\ell-(m-1)]s+1,k} s_{k+(m-1)[r+s-(m-1)]}, \quad \text{if } m \leq \ell, \tag{4.9}$$

where $b_{i,j}$ is defined in definition 4.15.

In the following lemma, we show that after rearranging columns of B, we obtain the new matrix which is a lower triangular matrix. Furthermore, we show that $B \in \mathrm{GL}_{rs}(\mathbb{K})$.

Lemma 4.17. *Let B be the matrix from Definition 4.16. Then there is a matrix $X \in \mathrm{GL}_{rs}(\mathbb{K})$ such that $B' := BX$ is a lower triangular matrix with nonzero diagonal terms. Moreover $\det(B) \neq 0$.*

Proof. Let $X \in \mathrm{GL}_{rs}(\mathbb{K})$ be the matrix given as follows: for each $1 \leq j \leq rs$, we can write j uniquely as $j = \ell s + m$ for some $0 \leq \ell \leq r-1$ and $1 \leq m \leq s$. We denote a column vector $\delta_k = (0, \ldots, 1, \ldots, 0)^\mathsf{T} \in \mathrm{Mat}_{rs\times 1}(\mathbb{K})$, where 1 is in the k-th coordinate, and let

$$\operatorname{col}_j X = \begin{cases} \delta_{r+m-2+\ell(r+s-\ell-2)} & \text{if } \ell+2 \leq m \\ \delta_{m(r+s-m)} & \text{if } m = \ell+1 \\ \delta_{(\ell-m+1)-(m-1)(r+s-m+1)} & \text{if } m \leq \ell \end{cases} \cdot$$

The matrix X is a permutation matrix defined so that when we multiply B by X, it acts like we are rearranging columns of B. Then $B' := BX$ is a lower triangular matrix with the diagonal entries $d_1, \ldots, d_{rs}$, where

$$d_{\ell s+m} = \begin{cases} b^{(-\ell)}_{m-\ell,r-2+m-\ell} & \text{if } \ell+2 \leq m \\ 1 & \text{if } m = \ell+1 \\ b^{(-m+1)}_{(\ell-m+1)s+1,\ell-m+1} & \text{if } m \leq \ell \end{cases} \cdot$$

for $0 \leq \ell \leq r-1$ and $1 \leq m \leq s$. By the definition of $b_{i,j}$, we see that $d_1, \ldots, d_{rs}$ are all nonzero. Thus, $\det(B) = \det(B') \neq 0$. For example, in case $r = s = 3$, we set $X = (\delta_5, \delta_3, \delta_4, \delta_1, \delta_8, \delta_7, \delta_2, \delta_6, \delta_9) \in \mathrm{Mat}_{9\times 9}(\mathbb{K})$. Then

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{3,3} & b_{3,4} & 0 & 0 & 0 & 0 & 0 \\ b_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{2,3}^{(-1)} & 0 & 0 \\ b_{7,1} & b_{7,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{4,1}^{(-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$B' = BX = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{3,3} & b_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{4,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{2,3}^{(-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{7,1} & 0 & 0 & b_{7,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{4,1}^{(-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Lemma 4.18. *Let $\Phi = \Phi_1 \otimes \Phi_2$. After rearranging rows of Φ, we obtain the new matrix which is an upper triangular matrix. In other words, there is a matrix Y such that $Y\Phi$ is an upper triangular matrix, and all diagonal entries of $Y\Phi$ are nonzero.*

Proof. Let $Y \in \mathrm{Mat}_{rs}(\mathbb{K})$ be given by

$$\begin{aligned}
(\mathrm{row}_1\, Y, \ldots, \mathrm{row}_s\, Y) &= (s_{rs}, s_{rs-(s-1)}, \ldots, s_{rs-1}), \\
(\mathrm{row}_{s+1}\, Y, \ldots, \mathrm{row}_{2s}\, Y) &= (s_s, s_1, \ldots, s_{s-1}), \\
(\mathrm{row}_{2s+1}\, Y, \ldots, \mathrm{row}_{3s}\, Y) &= (s_{2s}, s_{s+1}, \ldots, s_{2s-1}), \\
&\vdots \\
(\mathrm{row}_{(r-1)s+1}\, Y, \ldots, \mathrm{row}_{rs}\, Y) &= (s_{(r-1)s}, s_{(r-2)s+1}, \ldots, s_{(r-2)s+s-1}).
\end{aligned}$$

The matrix Y is a permutation matrix defined so that when we multiply Φ by Y, it acts like we are rearranging rows of Φ, and it follows that $Y\Phi$ is upper triangular. Moreover, we observe that the diagonal entries of $Y\Phi$ are

$$\underbrace{\frac{T^2}{[r,r](s,s)}, \frac{T}{[r,r]}, \ldots, \frac{T}{[r,r]}}_{\text{s terms}}, \underbrace{\frac{T}{(s,s)}, 1, \ldots, 1}_{\text{s terms}}, \ldots, \underbrace{\frac{T}{(s,s)}, 1, \ldots, 1}_{\text{s terms}}$$

For example, when $r = s = 3$,

$$Y\Phi = \begin{pmatrix}
\frac{T^2}{[3,3](3,3)} & * & * & * & * & * & * & * & * \\
 & \frac{T}{[3,3]} & * & * & * & * & * & * & * \\
 & & \frac{T}{[3,3]} & * & * & * & * & * & * \\
 & & & \frac{T}{(3,3)} & * & * & * & * & * \\
 & & & & 1 & * & * & * & * \\
 & & & & & 1 & * & * & * \\
 & & & & & & \frac{T}{(3,3)} & * & * \\
 & & & & & & & 1 & * \\
 & & & & & & & & 1
\end{pmatrix}.$$

$\square$

Lemma 4.19. *Let*

$$D = \begin{pmatrix} T & & & & \\ & \ddots & & & \\ & & T & & \\ & & & T^2 & \\ & & & & I_{rs-(r+s-1)} \end{pmatrix}.$$

Then there exists a matrix P such that

$$P\Phi B = D.$$

Proof. Let B' and X be the matrices from Lemma 4.17. Let $D' = X^{-1}DX$. Using the definition of the permutation matrix X, one can see that the

$$\mathrm{diag}(D') = (\underbrace{T^2, T, \ldots, T}_{\text{s terms}}, \underbrace{T, 1, \ldots, 1}_{\text{s terms}}, \ldots, \underbrace{T, 1, \ldots, 1}_{\text{s terms}})$$

Since B' is lower triangular, we know that $(B')^{-1}$ is lower triangular. Recall that the product of two lower triangular matrices $U = (u_{i,j})$ and $V = (v_{i,j})$ is lower triangular. Moreover, the diagonal entry of UV is equal to $u_{i,i}v_{i,i}$. Using this property, we compute that

$$\mathrm{diag}(D'(B')^{-1}) = (T^2 z_1, Tz_2, \ldots, Tz_{s+1}, \ldots),$$

for some $z_1, \ldots, z_{s+1} \in \mathbb{K}$. Let Y be the matrix from Lemma 4.18. Since the matrix $Y\Phi$ is invertible, there is a matrix $V \in \mathrm{GL}_{rs}(\mathbb{K}[t])$ so that $V(Y\Phi) = D'(B')^{-1}$. Now, we let $P = XVY$. Then

$$VY\Phi B' = D',$$

$$VY\Phi BX = D',$$

$$XVY\Phi B = XD'X^{-1},$$

$$P\Phi B = D.$$

□

Lemma 4.20. *Let D be the matrix given in Lemma 4.19. Then there exists a matrix C such that $C\Phi' = D$.*

Proof. Recall that $\Phi = B^{(-1)}\Phi' B^{-1}$, so $\Phi B = B^{(-1)}\Phi'$. Let $C = PB^{(-1)}$. Then

$$C\Phi' = PB^{(-1)}\Phi' = P\Phi B = D.$$

□

Theorem 4.21. *Let ρ' be the t-module from Theorem 4.12 and let $\mathcal{E}_0$ be the function from Proposition 4.11 associated to ρ'. Then*

$$\mathcal{E}_0(\alpha) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+s-2} \\ \partial_t^1 \alpha_{r+s-1} \\ \alpha_{r+s-1} \end{pmatrix}\Bigg|_{t=\theta}, \tag{4.10}$$

for any $\alpha = (\alpha_1, \ldots, \alpha_{rs}) \in \mathrm{Mat}_{1\times rs}(\mathbb{T}_\theta)$.

Proof. In Theorem 4.12, we create the t-module ρ' isomorphic our original t-module ρ such that $\mathrm{d}\rho'_t$ is in Jordan normal form, in particular,

$$\mathrm{d}\rho'_t = \begin{pmatrix} \theta & & & \\ & \ddots & & \\ & & \theta & 1 \\ & & & \theta \end{pmatrix} = \begin{pmatrix} d_{\theta,1}[\theta] & & & \\ & \ddots & & \\ & & d_{\theta,1}[\theta] & \\ & & & d_{\theta,2}[\theta] \end{pmatrix}.$$

Comparing this matrix to equation (4.6), we know that $m = r+s-1$ and $\ell_1 = \ldots = \ell_{m-1} = 1,\ \ell_m = 2$. By Lemma 4.20, we know that the t-frame (ι', Φ') induced by the basis $\mathbf{n}'$ satisfies condition (i) in Proposition 4.11. We observe that for each $1 \leq i \leq m-1$, we have $\ell_i = 1$ and

$$\mathrm{d}\iota'((t-\theta)^0 e_i) = \mathbf{n}'_i = s_i = s_{\ell_1+\ldots+\ell_i}.$$

Also, we see that $\ell_1 + \ldots + \ell_m = r + s$, and

$$\mathrm{d}\iota'((t-\theta)^0 e_m) = \mathbf{n}'_m = s_{r+s} = s_{\ell_1+\ldots+\ell_m},$$
$$\mathrm{d}\iota'((t-\theta) e_m) = (t-\theta)\mathbf{n}'_m = (t-\theta)s_{r+s} = s_{r+s-1} = s_{\ell_1+\ldots+\ell_m-1}.$$

This shows that $\mathbf{n}'$ satisfies condition (ii) in Proposition 4.11. Therefore, by substituting in equation (4.5), we obtain the formula for $\mathcal{E}_0$. □

Now we set $\Phi = \Phi_1 \otimes \Phi_2$, which is the Kronecker product of matrices. Furthermore, for each $i = 1, 2$, we let Ψ_i be the rigid analytic trivialization corresponding to Φ_i in the sense of equation (3.2), i.e.

$$\Psi_i^{(-1)} = \Phi_i \Psi_i, \quad i = 1, 2.$$

One can show that

$$(\Psi_1 \otimes \Psi_2)^{(-1)} = \Phi(\Psi_1 \otimes \Psi_2).$$

Let Ψ be the Kronecker product of matrices given by

$$\Psi = \Psi_1 \otimes \Psi_2. \tag{4.11}$$

Therefore Ψ is the rigid analytic trivialization corresponding to Φ. Also, there is a well-known formula for Ψ_i, which we will state soon. Thus the formula for Ψ is easier to obtain comparing to Ψ'. Because of this, we will use Ψ to compute Ψ'. In the following proposition, we state a useful result on rigid analytic trivialization. We also refer the reader to [8, §3.4] and [23, §4.2] for more details. Recall the matrix Υ from §3.

Proposition 4.22. *Suppose that ϕ is a Drinfeld module of rank r given in equation* (2.1). *Let*

$\pi_1, \ldots, \pi_r \in \Lambda_\phi$ be an A-basis of the period lattice and let $f_1, \ldots, f_r$ be the corresponding Anderson generating functions for ϕ. Let

$$\Upsilon = \begin{pmatrix} f_1 & \ldots & f_r \\ \vdots & \ddots & \vdots \\ f_1^{(r-1)} & \ldots & f_r^{(r-1)} \end{pmatrix}$$

and

$$V = \begin{pmatrix} A_1 & A_2^{(-1)} & \ldots & A_{r-1}^{(-r+2)} & A_r^{(-r+1)} \\ A_2 & A_3^{(-1)} & \ldots & A_r^{(-r+2)} & \\ \vdots & \vdots & \iddots & & \\ A_{r-1} & A_r^{(-1)} & & & \\ A_r & & & & \end{pmatrix}.$$

Then $\Psi = ((\Upsilon^{(1)})^{\mathsf{T}} V)^{-1}$ is a rigid analytic trivialization for ϕ.

Remark 4.23. Fix an A-basis $\pi_1, \ldots, \pi_r$ of the period lattice Λ_{ϕ_1} and an A-basis $\mu_1, \ldots, \mu_s$ of the period lattice Λ_{ϕ_2}. For each $1 \leq i \leq r$, let f_i be the Anderson generating function for ϕ_1 with respect to π_i. Similarly, for each $1 \leq j \leq s$, let g_j be the Anderson generating function for ϕ_2 with respect to μ_j. According to the result by Pellarin, which is mentioned in Proposition 2.1, we can write

$$f_i(t) = -\frac{\pi_i}{t-\theta} + u_i + \text{higher order terms in } t-\theta,$$

and

$$g_j(t) = -\frac{\mu_j}{t-\theta} + v_j + \text{higher order terms in } t-\theta,$$

for some $u_i, v_j \in \mathbb{K}$. Let

$$\Upsilon_1 = \begin{pmatrix} f_1 & \ldots & f_r \\ \vdots & \ddots & \vdots \\ f_1^{(r-1)} & \ldots & f_r^{(r-1)} \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} g_1 & \ldots & g_s \\ \vdots & \ddots & \vdots \\ g_1^{(s-1)} & \ldots & g_s^{(s-1)} \end{pmatrix},$$

$$V_1 = \begin{pmatrix} A_1 & A_2^{(-1)} & \dots & A_{r-1}^{(-r+2)} & A_r^{(-r+1)} \\ A_2 & A_3^{(-1)} & \dots & A_r^{(-r+2)} & \\ \vdots & \vdots & \iddots & & \\ A_{r-1} & A_r^{(-1)} & & & \\ A_r & & & & \end{pmatrix}, V_2 = \begin{pmatrix} B_1 & B_2^{(-1)} & \dots & B_{s-1}^{(-s+2)} & B_s^{(-s+1)} \\ B_2 & B_3^{(-1)} & \dots & B_s^{(-s+2)} & \\ \vdots & \vdots & \iddots & & \\ B_{s-1} & B_s^{(-1)} & & & \\ B_s & & & & \end{pmatrix}.$$

By Proposition 4.22, we have

$$\Psi_i^{-1} = (\Upsilon_i^{(1)})^{\mathsf{T}} V_i, \quad i = 1, 2. \tag{4.12}$$

As we previously explained, we want to compute Ψ' from Ψ. Therefore, we want to know the relation between them. Let $\mathbf{n}'$ be a column vector consisting of the $\mathbb{K}[t]$-basis for $N_{\rho'}$ from Lemma 4.14 with a t-frame (ι', Φ') and a corresponding rigid analytic trivialization Ψ'. Picking another $\mathbb{K}[t]$-basis $\mathbf{n}$ for $N_{\rho'}$, with a t-frame (ι, Φ) and a corresponding rigid analytic trivialization Ψ is equivalent to picking a matrix $B \in \mathrm{GL}_{rs}(\mathbb{K})$ so that $\mathbf{n} = B\mathbf{n}'$. In the next theorem, we use the matrix B given in Definition 4.16 to pick another $\mathbb{K}[t]$-basis for $N_{\rho'}$.

Theorem 4.24. *Let $\mathbf{n}'$ be a column vector consisting of the $\mathbb{K}[t]$-basis for $N_{\rho'}$ from Lemma 4.14. Let $B = (b_{ij}) \in \mathrm{GL}_{rs}(\mathbb{K})$ be the matrix defined in Definition 4.16. Let $\mathbf{n} = B\mathbf{n}'$. Then $\mathbf{n}$ is a column vector consisting of the $\mathbb{K}[t]$-basis for $N_{\rho'}$ such that $\sigma\mathbf{n} = (\Phi_1 \otimes \Phi_2)\mathbf{n}$.*

Proof. First, we denote

$$\mathbf{n} = (n_1, \dots, n_{rs})^{\mathsf{T}}, \quad \mathbf{n}' = (n_1', \dots, n_{rs}')^{\mathsf{T}}.$$

It is clear that every integer $1 \le j \le rs$ can be written uniquely as $j = \ell s + m$ for some $0 \le \ell \le r-1$ and $1 \le m \le s$. Therefore, in order to prove that $\sigma\mathbf{n} = (\Phi_1 \otimes \Phi_2)\mathbf{n}$, we need to show that, for such ℓ and m,

$$\sigma n_{\ell s+m} = (\mathrm{row}_{\ell s+m}(\Phi_1 \otimes \Phi_2)) \cdot \mathbf{n}. \tag{4.13}$$

Fix $0 \leq \ell \leq r-1$ and $1 \leq m \leq s$. To prove the equation (4.13), we divide it into 3 cases depending on ℓ and m, namely $\ell \leq m-2$, $\ell = m-1$ and $\ell \geq m$. Then we use (4.7), (4.8) and (4.9) to compute $\sigma n_{\ell s+m}$ as follows. **case 1** Suppose $\ell \leq m-2$. Using equation (4.7), we compute that

$$\begin{aligned} n_{\ell s+m} &= (\mathrm{row}_{\ell s+m} B) \cdot \mathbf{n}' \\ &= \left(\sum_{k=r}^{r+m-(\ell+2)} b_{m-\ell,k}^{(-\ell)} s_{k+\ell[r+s-(\ell+1)]} \right) \cdot \mathbf{n}' \\ &= \sum_{k=r}^{r+m-(\ell+2)} b_{m-\ell,k}^{(-\ell)} n'_{k+\ell[r+s-(\ell+1)]} \\ &= \sum_{k=r}^{r+m-(\ell+2)} b_{m-\ell,k}^{(-\ell)} \sigma^{\ell} s_k. \end{aligned}$$

So

$$\sigma n_{\ell s+m} = \sum_{k=r}^{r+m-(\ell+2)} b_{m-\ell,k}^{(-\ell-1)} \sigma^{\ell+1} s_k.$$

case 2 Suppose that $\ell = m-1$. Using equation (4.8), we compute that

$$n_{\ell s+m} = (row_{\ell s+m} B) \cdot \mathbf{n}' = n'_{m(r+s-m)} = \sigma^{m-1} s_{r+s}.$$

So $\sigma n_{\ell s+m} = \sigma^m s_{r+s}$. **case 3** Suppose that $\ell \geq m$. Using equation (4.9), we compute that

$$\begin{aligned} n_{\ell s+m} &= (row_{\ell s+m} B) \cdot \mathbf{n}' \\ &= \sum_{k=1}^{\ell-(m-1)} b_{(\ell-(m-1))s+1,k}^{(-m+1)} n'_{k+(m-1)(r+s-(m-1))} \\ &= \sum_{k=1}^{\ell-(m-1)} b_{(\ell-(m-1))s+1,k}^{(-m+1)} \sigma^{m-1} s_k. \end{aligned}$$

So

$$\sigma n_{\ell s+m} = \sum_{k=1}^{\ell-(m-1)} b^{(-m)}_{(\ell-(m-1))s+1,k}\sigma^m s_k.$$

Then we consider the Kronecker product $\Phi_1 \otimes \Phi_2$ and compute $(\mathrm{row}_{\ell s+m}(\Phi_1 \otimes \Phi_2)) \cdot \mathbf{n}$, which also depends on ℓ and m. To compute $(\mathrm{row}_{\ell s+m}(\Phi_1 \otimes \Phi_2)) \cdot \mathbf{n}$, we divide each case into 4 sub-cases, including

(i) $\ell = r-1, m = s$,

(ii) $\ell = r-1, m \leq s-1$,

(iii) $\ell \leq r-2, m = s$,

(iv) $\ell \leq r-2, m \leq s-1$.

Then in each case, using the definition of the basis $\mathbf{n}'$ and the matrix B, we can show that (4.13) holds. For example, in case 1(iv), we compute that

$$\begin{aligned}
\sigma n_{\ell s+m} &= \sum_{k=r}^{r+m-(\ell+2)} b^{(-\ell-1)}_{m-\ell,k}\sigma^{\ell+1} s_k \\
&= \sum_{k=r}^{r+m-(\ell+2)} b^{(-\ell-1)}_{m-\ell,k} n'_{k+(\ell+1)[r+s-(\ell+2)]} \\
&= \left(\sum_{k=r}^{r+m-(\ell+2)} b^{(-\ell-1)}_{m-\ell,k} s_{k+(\ell+1)[r+s-(\ell+2)]} \right) \cdot \mathbf{n}' \\
&= \big(\mathrm{row}_{(\ell+1)s+(m+1)}\, B\big) \cdot \mathbf{n}' \\
&= n_{(\ell+1)s+(m+1)} \\
&= (\mathrm{row}_{\ell s+m}(\Phi_1 \otimes \Phi_2)) \cdot \mathbf{n}.
\end{aligned}$$

□

At this point, we have two bases for $N_{\rho'}$, namely $\mathbf{n}$ and $\mathbf{n}'$ with the corresponding rigid analytic trivializations (ι, Φ, Ψ) and (ι', Φ', Ψ'), respectively. Since $\mathbf{n} = B\mathbf{n}'$, we know that

$\Phi = B^{(-1)}\Phi' B^{-1}$ and $\Psi = B\Psi'$ (see [22, §3.5] or [23, §3.2] for more details). So the matrix B is a key to convert from Ψ to Ψ'. Moreover, we know that $\Psi = \Psi_1 \otimes \Psi_2$. Using this formula together with the relation between Ψ and Ψ', we derive a formula for Ψ' in the following Theorem.

Theorem 4.25. *Let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be the Anderson generating functions in Remark 4.23. Suppose that Ψ' is the rigid analytic trivialization corresponding to the t-frame (ι', Φ'). Then $(\Psi')^{-1}$ has the following form,*

$$(\Psi')^{-1} = \begin{pmatrix} f_1^{(r-1)}Tg_1 & \ldots & f_1^{(1)}Tg_1 & Tf_1g_1^{(s-1)} & \ldots & Tf_1g_1^{(1)} & T^2f_1g_1 & * & \ldots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_1^{(r-1)}Tg_s & \ldots & f_1^{(1)}Tg_s & Tf_1g_s^{(s-1)} & \ldots & Tf_1g_s^{(1)} & T^2f_1g_s & * & \ldots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_r^{(r-1)}Tg_1 & \ldots & f_r^{(1)}Tg_1 & Tf_rg_1^{(s-1)} & \ldots & Tf_rg_1^{(1)} & T^2f_rg_1 & * & \ldots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_r^{(r-1)}Tg_s & \ldots & f_r^{(1)}Tg_s & Tf_rg_s^{(s-1)} & \ldots & Tf_rg_s^{(1)} & T^2f_rg_s & * & \ldots & * \end{pmatrix}.$$

Proof. Let B be the matrix defined in Definition 4.16. From Theorem 4.24, we have two bases for $N_{\rho'}$, namely $\mathbf{n}$ and $\mathbf{n}'$ with the corresponding rigid analytic trivializations (ι, Φ, Ψ) and (ι', Φ', Ψ'), respectively. Since $\mathbf{n} = B\mathbf{n}'$, it follows that $\Psi = B\Psi'$. Thus

$$(\Psi')^{-1} = \Psi^{-1}B.$$

We will derive the formula for the first row of the matrix $(\Psi')^{-1}$. The other rows can be derived using the same method by changing the subscripts from f_1, g_1 to f_1, g_2, and then f_1, g_3, and so on. We denote $\mathrm{row}_1(\Psi_1^{-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $\mathrm{row}_1(\Psi_2^{-1}) = (\beta_1, \beta_2, \ldots, \beta_s)$. It follows from a straightforward computation that

$$\alpha_1 = Tf_1, \quad \alpha_k = \sum_{i=1}^{r+1-k} [i+k-1, k-1] f_1^{(i)}, \quad 2 \leq k \leq r,$$

$$\beta_1 = Tg_1, \quad \beta_k = \sum_{i=1}^{s+1-k} (i+k-1, k-1) g_1^{(i)}, \quad 2 \le k \le s,$$

where $[m,n] := A_m^{(-n)}$ and $(m,n) := B_m^{(-n)}$ for $m, n \in \mathbb{N}$. Since $\Psi^{-1} = (\Psi_1^{-1}) \otimes (\Psi_2^{-1})$, we have

$$\text{row}_1(\Psi^{-1}) = \text{row}_1(\Psi_1^{-1}) \otimes \text{row}_1(\Psi_2^{-1}) = (\alpha_1\beta_1, \ldots, \alpha_1\beta_s, \ldots, \alpha_r\beta_1, \ldots, \alpha_r\beta_s)$$

Fix $j \in \{1, \ldots, r-1\}$. It follows from Definition 4.15 (1) that

$$\begin{aligned}
(\Psi')^{-1}_{1j} &= \sum_{\ell=1}^{rs} (\Psi^{-1})_{1,\ell} b_{\ell,j} \\
&= \sum_{m=j}^{r-1} (\Psi^{-1})_{1,ms+1} b_{ms+1,j} \\
&= \sum_{m=j}^{r-1} \alpha_{m+1}\beta_1 b_{ms+1,j} \\
&= \sum_{m=j}^{r-1} \left(\sum_{d=1}^{r-m} [d+m, m] f_1^{(d)} \right) \beta_1 b_{ms+1,j} \\
&= \beta_1 \left(\sum_{m=j}^{r-1} \sum_{d=1}^{r-m} b_{ms+1,j} [d+m, m] f_1^{(d)} \right).
\end{aligned}$$

Let

$$\sum_{m=j}^{r-1} \sum_{d=1}^{r-m} b_{ms+1,j} [d+m, m] f_1^{(d)} = c_1 f_1^{(1)} + c_2 f_1^{(2)} + \ldots + c_{r-j} f_1^{(r-j)}.$$

Using the definition of b_{ij}, we compute that $c_1 = \cdots = c_{r-j-1} = 0$ and $c_{r-j} = 1$. Therefore,

$$(\Psi')^{-1}_{1j} = \beta_1 f_1^{(r-j)} = f_1^{(r-j)} Tg_1, \quad 1 \le j \le r-1.$$

Next, we fix $r \leq j \leq r+s-2$. It follows from Definition 4.15 (2) that

$$\begin{aligned}
(\Psi')^{-1}_{1j} &= \sum_{m=1}^{rs} (\Psi^{-1})_{1,m} b_{m,j} \\
&= \sum_{m=j-r+2}^{s} (\Psi^{-1})_{1,m} b_{m,j} \\
&= \sum_{m=j-r+2}^{s} \alpha_1 \beta_m b_{m,j} \\
&= \sum_{m=j-r+2}^{s} \alpha_1 \left(\sum_{d=1}^{s+1-m} (d+m-1, m-1) g_1^{(d)} \right) b_{m,j} \\
&= \alpha_1 \left(\sum_{m=j-r+2}^{s} \sum_{d=1}^{s+1-m} b_{m,j} (d+m-1, m-1) g_1^{(d)} \right).
\end{aligned}$$

Let

$$\sum_{m=j-r+2}^{s} \sum_{d=1}^{s+1-m} b_{m,j} (d+m-1, m-1) g_1^{(d)} = a_1 g_1^{(1)} + a_2 g_1^{(2)} + \ldots + a_{r+s-1-j} g_1^{(r+s-1-j)}.$$

Using the definition of b_{ij}, we compute that $a_1 = \cdots = a_{r+s-j-2} = 0$ and $a_{r+s-j-1} = 1$. So

$$(\Psi')^{-1}_{1j} = \alpha_1 g_1^{(r+s-1-j)} = g_1^{(r+s-1-j)} T f_1, \quad r \leq j \leq r+s-2.$$

From Definition 4.15 (3), we compute that

$$(\Psi')^{-1}_{1,r+s-1} = \sum_{m=1}^{rs} (\Psi^{-1})_{1,m} b_{m,r+s-1} = (\Psi^{-1})_{1,1} b_{1,r+s-1} = \alpha_1 \beta_1 = T^2 f_1 g_1.$$

□

Recall that we have the t-module ρ' and the t-frame (ι', Φ'). In the following theorem, we use the formula for the map $\mathcal{E}_0$ from Theorem 4.21 and the formula for Ψ' from Theorem 4.25, associated to the t-frame (ι', Φ') to find the periods for ρ'.

Theorem 4.26. *Let $\mathcal{E}_0$ be the map given in Theorem 4.21 and let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be* *the Anderson generating functions in Remark 4.23. Let Ψ' be the matrix in Theorem 4.25. For $1 \leq i \leq r$ and $1 \leq j \leq s$, let*

$$\lambda_{i,j} = \mathcal{E}_0(\mathrm{row}_{s(i-1)+j}(\Psi')^{-1}). \tag{4.14}$$

Then $\{\lambda_{i,j} : 1 \leq i \leq r, 1 \leq j \leq s\}$ forms an $\mathbf{A}$-basis for the period lattice $\Lambda_{\rho'}$ and

$$\lambda_{i,j} = -\begin{pmatrix} f_i^{(r-1)}(\theta)\mu_j \\ \vdots \\ f_i^{(1)}(\theta)\mu_j \\ g_j^{(s-1)}(\theta)\pi_i \\ \vdots \\ g_j^{(1)}(\theta)\pi_i \\ \pi_i v_j + \mu_j u_i \\ -\pi_i\mu_j \end{pmatrix}.$$

Proof. By Theorem 4.10, we know that

$$\Lambda_{\rho'} = \mathcal{E}_0((\mathrm{Mat}_{1\times rs}\,\mathbf{A}) \cdot (\Psi')^{-1}).$$

It follows that $\lambda_{i,j} \in \Lambda_{\rho'}$ for every $1 \leq i \leq r, 1 \leq j \leq s$. By the proof of Proposition 4.5.9 (b) in [13], we know that $\mathrm{row}_1(\Psi')^{-1}, \ldots, \mathrm{row}_{rs}(\Psi')^{-1}$ forms a basis for $V := (\mathrm{Mat}_{1\times rs}\,\mathbf{A})(\Psi')^{-1}$. By Theorem 4.5.14 in [13],

$$\mathcal{E}_0|_V : V \to \Lambda_{\rho'}$$

is a bijection. It follows that $\lambda_1, \ldots, \lambda_{rs}$ forms a basis for $\Lambda_{\rho'}$. The formula for $\lambda_{i,j}$ can be derived as follows. Fix i and j. We denote $\mathrm{row}_{(i-1)s+j}(\Psi')^{-1} = (\alpha_1, \ldots, \alpha_{rs})$. Then $\lambda_{i,j} = \mathcal{E}_0(\mathrm{row}_{(i-1)s+j}(\Psi')^{-1}) = \mathcal{E}_0(\alpha_1, \ldots, \alpha_{rs})$. Using the formula for $\mathrm{row}_{(i-1)s+j}(\Psi')^{-1}$ from Theorem

4.25, we get

$$\alpha_\ell = f_i^{(r-\ell)} T g_j, \quad \ell = 1, \dots, r-1,$$
$$\alpha_{r-1+k} = T f_i g_j^{(s-k)}, \quad k = 1, \dots, s-1,$$
$$\alpha_{r+s-1} = T^2 f_i g_j.$$

The last equation implies that

$$\partial_t(\alpha_{r+s-1}) = (T f_i)\partial_t(T g_j) + (T g_j)\partial_t(T f_i).$$

Evaluating at $t = \theta$, we obtain that

$$\alpha_\ell|_{t=\theta} = -f_i^{(r-\ell)}(\theta)\mu_j, \quad \ell = 1, \dots, r-1,$$
$$\alpha_{r-1+k}|_{t=\theta} = -\pi_i g_j^{(s-k)}(\theta), \quad k = 1, \dots, s-1,$$
$$\partial_t(\alpha_{r+s-1})|_{t=\theta} = -\pi_i v_j - \mu_j u_i,$$
$$\alpha_{r+s-1}|_{t=\theta} = \pi_i \mu_j.$$

Substituting in equation (4.10), we obtain the formula for $\lambda_{i,j}$.

□

Remark 4.27. We recall our observation in Remark 4.6, where we investigate $C^{\otimes 2}$. Using Theorem 4.26, we acquire the period of $C^{\otimes 2}$ as follows. First, we take an Anderson generating function $f(t)$ associated to $\tilde{\pi}$, which is equal to the Anderson-Thakur function $\omega_C(t)$ from §2,

$$f(t) = \omega_C(t) = -\frac{\tilde{\pi}}{t-\theta} + u + \text{higher order terms in } t-\theta.$$

Then the period we get from Theorem 4.26 is

$$\lambda = \begin{pmatrix} -2\tilde{\pi} u \\ \tilde{\pi}^2 \end{pmatrix}.$$

This coincides with the result by Anderson and Thakur in Theorem 4.1. Moreover, u can be computed as follows. Recall that $-\tilde{\pi} = \operatorname{Res}_{t=\theta} \omega_C = ((t-\theta)\omega_c)|_{t=\theta} = \omega_c^{(1)}|_{t=\theta}$. So

$$-\partial_\theta^1(\tilde{\pi}) = \partial_\theta^1(\omega_c^{(1)}|_{t=\theta}).$$

By applying the chain rule (see [24, Cor. 2.4.6]), we see that

$$\partial_\theta^1(\omega_c^{(1)}|_{t=\theta}) = \partial_\theta^1(\omega_C^{(1)})|_{t=\theta} + \partial_t^1(\omega_C^{(1)})|_{t=\theta}.$$

We also observe that $\partial_\theta^1(\omega_C^{(1)})|_{t=\theta} = 0$, and $\partial_t^1(\omega_C^{(1)})|_{t=\theta} = \partial_t^1((t-\theta)\omega_C)|_{t=\theta} = u$. Therefore

$$u = -\partial_\theta^1(\tilde{\pi}).$$

4.4 Anderson generating functions for the tensor product of two Drinfeld modules

Recall the definition of Anderson generating functions for a t-module given in §2.6. In this section, we provide a formula for the Anderson generating functions for ρ' with respect to the periods $\lambda_{i,j}$ in Theorem 4.26. For $1 \le \ell \le r$ and $1 \le k \le s$, we let $\lambda_{(\ell-1)s+k} := \lambda_{\ell,k}$, where $\lambda_{\ell,k}$ is defined in Theorem 4.26. For $1 \le j \le rs$, let

$$\mathcal{G}_{\lambda_j}(t) = (h_{1,j}, \ldots, h_{r+s,j})^{\mathsf{T}}$$

be the Anderson generating function for ρ' associated to λ_j. Recall from Proposition 2.3 that

$$\operatorname{Res}_{t=\theta} \mathcal{G}_{\lambda_j}(t) = -\lambda_j.$$

We will use this property along with a formula for λ_j to find a formula for $\mathcal{G}_{\lambda_j}(t)$.

Lemma 4.28. *Let $M_{\rho'}$ be a t-motive associated to ρ' from Theorem 4.12. For $1 \leq j \leq r$, let*

$$M_j = \{\tau^{j-1} s_k | 1 \leq k \leq r+s-1, k \neq 1, \ldots, j-1, r, \ldots, r+j-2\}.$$

Then $M := \cup_{j=1}^r M_j$ is a $\mathbb{K}[t]$-basis of $M_{\rho'}$.

Proof. For each $1 \leq j \leq r$, we observe that $\# M_j = r+s-1-2(j-1) = r+s+1-2j$, so

$$\# M = \sum_{j=1}^{r} (r+s+1-2j) = r(r+s+1) - 2(r/2)(r+1) = rs.$$

Recall that $\operatorname{rank}_{\mathbb{K}[t]} M_{\rho'} = rs$, so we just need to show the $\operatorname{Span}_{\mathbb{K}[t]} M = M_{\rho'}$. We follows the same idea as the proof of Lemma 4.14. We leave the details to the reader. □

Fix a $\mathbb{K}[t]$-basis vector $\mathbf{m}$ of $M_{\rho'}$ as in Lemma 4.28 with

$$\begin{aligned}
(m_1, \ldots, m_s) &= (s_{r+s-1}, \ldots, s_r), \\
(m_{s+1}, \ldots, m_{2s}) &= (s_{r-1}, \tau s_{r+s-1}, \ldots, \tau s_{r+1}), \\
(m_{2s+1}, \ldots, m_{3s}) &= (s_{r-2}, \tau s_{r-1}, \tau^2 s_{r+s-1}, \ldots, \tau^2 s_{r+2}), \\
&\vdots \\
(m_{(r-1)s+1}, \ldots, m_{rs}) &= (s_1, \tau s_2, \tau^2 s_3, \ldots, \tau^{r-1} s_{s+r-1}).
\end{aligned} \tag{4.15}$$

Then we compute that

$$\tau \mathbf{m} = \Theta \mathbf{m},$$

where $\Theta = \Theta_1 \otimes \Theta_2$. For any $\beta = \sum_{k=0}^{\ell} X_k \tau^k \in \operatorname{Mat}_{m \times n}(\mathbb{K}[\tau])$ and $M \in \operatorname{Mat}_{n \times s}(\mathbb{T})$, we define

$$\langle \beta \mid M \rangle = \sum_{k=0}^{\ell} X_k M^{(k)} \in \operatorname{Mat}_{m \times s}(\mathbb{T}).$$

Let $R \in \mathrm{Mat}_{rs}(\mathbb{T})$ be the matrix given by

$$R_{i,j} = \left\langle \tau m_i \mid \mathcal{G}_{\lambda_j} \right\rangle . \tag{4.16}$$

From the result by Namoijam and Papanikolas [22, §4.3], we know that $R = \Theta R^{(-1)}$. Let $\Gamma = R^{(-1)}$. Then $\Gamma^{(1)} = \Theta\Gamma$. Also, recall that $\Upsilon := \Upsilon_1 \otimes \Upsilon_2$ satisfies the same equation as Γ, so $\Gamma = \Upsilon X$ for some $X \in \mathrm{GL}_{rs}(\mathbb{F}_q[t])$. Thus

$$\Upsilon X = \Gamma = R^{(-1)}. \tag{4.17}$$

First, we consider λ_1 and we denote the Anderson generating function for $\phi_1 \otimes \phi_2$ associated to λ_1

$$\mathcal{G}_{\lambda_1}(t) = (h_1, \ldots, h_{r+s})^{\mathsf{T}}.$$

Lemma 4.29. *Let $h_1, \ldots, h_{r+s} \in \mathbb{T}$ be such that $\mathcal{G}_{\lambda_1}(t) = (h_1, \ldots, h_{r+s})^{\mathsf{T}}$. Let $X = (x_{i.j}) \in \mathrm{GL}_{rs}(\mathbb{F}_q[t])$ be the matrix satisfying equation* (4.17). *Then*

$$\begin{aligned}
h_\ell &= \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1} f_i^{(r-\ell)} g_j, \quad 1 \le \ell \le r-1,\\
h_{r+\ell} &= \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1} f_i g_j^{(s-1-\ell)}, \quad 0 \le \ell \le s-1,\\
h_{r+s} &= T h_{r+s-1}.
\end{aligned}$$

Proof. We observe that $\Gamma_{i,j} = R_{i,j}^{(-1)} = \langle m_i \mid \mathcal{G}_{\lambda_j}(t) \rangle$. Using definition of $\mathbf{m}$ in (4.15), we have

$$\begin{aligned} (\Gamma_{1,1}, \dots, \Gamma_{s,1}) &= (h_{r+s-1}, \dots, h_r), \\ (\Gamma_{s+1,1}, \dots, \Gamma_{2s,1}) &= (h_{r-1}, h_{r+s-1}^{(1)}, \dots, h_{r+1}^{(1)}), \\ (\Gamma_{2s+1,1}, \dots, \Gamma_{3s,1}) &= (h_{r-2}, h_{r-1}^{(1)}, h_{r+s-1}^{(2)}, \dots, h_{r+2}^{(2)}), \\ &\vdots \\ (\Gamma_{(r-1)s+1,1}, \dots, \Gamma_{rs,1}) &= (h_1, h_2^{(1)}, h_3^{(2)}, \dots, h_{s+r-1}^{(r-1)}). \end{aligned} \tag{4.18}$$

For $0 \leq \ell \leq s-1$, we then have

$$h_{r+\ell} = \Gamma_{\ell+1,1} = \sum_{k=1}^{rs} \Upsilon_{\ell+1,k} x_{k,1} = \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1} f_i g_j^{(s-1-\ell)}.$$

Similarly, for $1 \leq \ell \leq r-1$,

$$h_\ell = \Gamma_{(r-\ell)s+1,1} = \sum_{k=1}^{rs} \Upsilon_{(r-\ell)s+1,k} x_{k,1} = \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1} f_i^{(r-\ell)} g_j.$$

It follows from Proposition 2.3 that $\rho_t'(\mathcal{G}_{\lambda_1}) = t\mathcal{G}_{\lambda_1}$. Comparing the $(r+s-1)$-th coordinate of both sides, we see that $\theta h_{r+s-1} + h_{r+s} = t h_{r+s-1}$. This proves the last equality. $\square$

Lemma 4.30. *Let $h_1, \dots, h_{r+s} \in \mathbb{T}$ be such that $\mathcal{G}_{\lambda_1}(t) = (h_1, \dots, h_{r+s})^{\mathsf{T}}$. Let $X = (x_{i.j}) \in \mathrm{GL}_{rs}(\mathbb{F}_q[t])$ be the matrix satisfying equation* (4.17). *Then*

$$\begin{aligned} \operatorname{Res}_{t=\theta} h_\ell &= \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1}(\theta) f_i^{(r-\ell)}(\theta) \mu_j, \quad 1 \leq \ell \leq r-1, \\ \operatorname{Res}_{t=\theta} h_{r+\ell} &= \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1}(\theta) \pi_i g_j^{(s-1-\ell)}(\theta), \quad 0 \leq \ell \leq s-2, \\ \operatorname{Res}_{t=\theta} h_{r+s-1} &= \sum_{i=1}^{r} \sum_{j=1}^{s} \left(-x_{(i-1)s+j,1}(\theta)(\pi_i v_j + \mu_j u_i) + x'_{(i-1)s+j,1}(\theta) \pi_i \mu_j \right) \\ \operatorname{Res}_{t=\theta} h_{r+s} &= \sum_{i=1}^{r} \sum_{j=1}^{s} x_{(i-1)s+j,1}(\theta) \pi_i \mu_j. \end{aligned}$$

Proof. This follows from Lemma 4.29. The first two equations is obtained by a direct computation. To compute the residue of h_{r+s-1} and h_{r+s}, we recall that

$$f_i = -\frac{\pi_i}{T} + u_i + \text{higher order terms}, \quad g_j = -\frac{\mu_j}{T} + v_j + \text{higher order terms},$$

which makes

$$f_i g_j = \frac{\pi_i \mu_j}{T^2} - \frac{\pi_i v_j + \mu_j u_i}{T} + \text{higher order terms}.$$

We write

$$x_{(i-1)s+j,1}(t) = a_0 + a_1 T + \cdots + a_m T^m, \quad a_0, \ldots, a_m \in \mathbb{F}_q.$$

Then

$$x_{(i-1)s+j,1}(t) f_i g_j = \frac{a_0(\pi_i \mu_j)}{T^2} + \frac{a_1(\pi_i \mu_j) - a_0(\pi_i v_j + \mu_j u_i)}{T} + \text{higher order terms}.$$

Recall that $a_0 = x_{(i-1)s+j,1}(\theta)$ and $a_1 = x'_{(i-1)s+j,1}(\theta)$. So

$$\begin{aligned}
\mathrm{Res}_{t=\theta}(x_{(i-1)s+j,1}(t) f_i g_j) &= x'_{(i-1)s+j,1}(\theta)(\pi_i \mu_j) - x_{(i-1)s+j,1}(\theta)(\pi_i v_j + \mu_j u_i), \\
\mathrm{Res}_{t=\theta}(T x_{(i-1)s+j,1}(t) f_i g_j) &= x_{(i-1)s+j,1}(\theta)(\pi_i \mu_j),
\end{aligned}$$

and the results follow. □

Lemma 4.31. *Let $X = (x_{i.j}) \in \mathrm{GL}_{rs}(\mathbb{F}_q[t])$ be the matrix satisfying* (4.17). *The following hold.*

1. *For $1 \leq k \leq rs$,*
$$\sum_{j=1}^{rs} x_{j,k}(t)\lambda_j = \sum_{j=1}^{rs} \mathrm{d}\rho'_{x_{j,k}(t)}\lambda_j = -\lambda_k.$$

2. *For $1 \leq j, k \leq rs$, $x_{j,k} = -1$ if $j = k$; and $x_{j,k} = 0$ otherwise.*

3. *$X = -I$.*

Proof. (2) follows from (1) and the fact that $\lambda_1, \ldots, \lambda_{rs}$ are linearly independent over $\mathbb{F}_q[t]$. Moreover, (3) follows directly from (2). We just need to prove (1). We fix $k = 1$. For $k \geq 2$, the proof is the same. First, we observe that

$$\mathrm{d}\rho'_{x(t)} = \begin{pmatrix} x(\theta) & 0 & \dots & 0 & 0 \\ 0 & x(\theta) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x(\theta) & x'(\theta) \\ 0 & 0 & \dots & 0 & x(\theta) \end{pmatrix} \in \mathrm{Mat}_{r+s}(\mathbb{K}).$$

For $1 \leq i \leq r$ and $1 \leq j \leq s$, it follows from the formula in Theorem 4.26 that

$$\mathrm{d}\rho'_{x_{(i-1)s+j,1}(t)}\lambda_{(i-1)s+j} = - \begin{pmatrix} x_{(i-1)s+j,1}(\theta) f_i^{(r-1)}(\theta)\mu_j \\ \vdots \\ x_{(i-1)s+j,1}(\theta) f_i^{(1)}(\theta)\mu_j \\ x_{(i-1)s+j,1}(\theta) g_j^{(s-1)}(\theta)\pi_i \\ \vdots \\ x_{(i-1)s+j,1}(\theta) g_j^{(1)}(\theta)\pi_i \\ x_{(i-1)s+j,1}(\theta)(\pi_i v_j + \mu_j u_i) - x'_{(i-1)s+j,1}(\theta)\pi_i\mu_j \\ -x_{(i-1)s+j,1}(\theta)\pi_i\mu_j \end{pmatrix}.$$

It follows from Lemma 4.30 that

$$\begin{aligned}
\sum_{j=1}^{rs} x_{j,1}(t)\lambda_j &= \sum_{j=1}^{rs} \mathrm{d}\rho'_{x_{j,1}(t)}\lambda_j \\
&= \sum_{i=1}^{r}\sum_{j=1}^{s} \mathrm{d}\rho'_{x_{(i-1)s+j,1}(t)}\lambda_{(i-1)s+j} \\
&= \begin{pmatrix} \mathrm{Res}_{t=\theta}(h_1) \\ \vdots \\ \mathrm{Res}_{t=\theta}(h_{r+s}) \end{pmatrix} \\
&= \mathrm{Res}_{t=\theta}\,\mathcal{G}_{\lambda_1}(t) \\
&= -\lambda_1.
\end{aligned}$$

□

By the above lemma, we obtain that $-\Upsilon = R^{(-1)} = \Gamma$. Recall that $\Upsilon = \Upsilon_1 \otimes \Upsilon_2$, which means

$$\begin{aligned}
(\Upsilon_{1,1}, \ldots, \Upsilon_{s,1}) &= (f_1 g_1, f_1 g_1^{(1)}, \ldots, f_1 g_1^{(s-1)}), \\
(\Upsilon_{s+1,1}, \ldots, \Upsilon_{2s,1}) &= (f_1^{(1)} g_1, f_1^{(1)} g_1^{(1)}, \ldots, f_1^{(1)} g_1^{(s-1)}) \\
&\vdots \\
(\Upsilon_{(r-1)s+1,1}, \ldots, \Upsilon_{rs,1}) &= (f_1^{(r-1)} g_1, f_1^{(r-1)} g_1^{(1)}, \ldots, f_1^{(r-1)} g_1^{(s-1)}).
\end{aligned}$$

Using (4.18) to compare each coordinate of $-\Gamma$ and Υ, we derive

$$\mathcal{G}_{\lambda_1}(t) = \begin{pmatrix} h_1 \\ \vdots \\ h_{r+s} \end{pmatrix} = -\begin{pmatrix} f_1^{(r-1)} g_1 \\ \vdots \\ f_1^{(1)} g_1 \\ f_1 g_1^{(s-1)} \\ \vdots \\ f_1 g_1^{(1)} \\ f_1 g_1 \\ f_1 g_1 T \end{pmatrix}.$$

We compute $\mathcal{G}_{\lambda_2}, \ldots, \mathcal{G}_{\lambda_{rs}}$ using the same technique and then we obtain the following result.

Theorem 4.32. *Let* $\{\lambda_{i,j} : 1 \leq i \leq r, 1 \leq j \leq s\}$ *be a basis in* $\Lambda_{\rho'}$ *from Theorem 4.26. Then*

$$\mathcal{G}_{\lambda_{i,j}}(t) = -\begin{pmatrix} f_i^{(r-1)} g_j \\ \vdots \\ f_i^{(1)} g_j \\ f_i g_j^{(s-1)} \\ \vdots \\ f_i g_j^{(1)} \\ f_i g_j \\ f_i g_j T \end{pmatrix}.$$

Recall the rigid analytic trivialization Ψ' from Theorem 4.25. The following formula shows the relation between Ψ' and the Anderson generating functions $\mathcal{G}_{\lambda_{i,j}}$ from Theorem 4.32.

Proposition 4.33. *Let* R *be the matrix defined in* (4.16) *and let* B *be the matrix from Definition*

4.16. Let $V = V_1 \otimes V_2$, where V_1 and V_2 are the matrices from (4.12). *Then*

$$(\Psi')^{-1} = -R^{\mathsf{T}} V B.$$

Proof. Recall that $\Upsilon X = R^{(-1)}$, where Υ and X are the matrices in (4.17). It follows from Lemma 4.31 that $-\Upsilon = R^{(-1)}$. Therefore $\Psi^{-1} = (\Upsilon^{(1)})^{\mathsf{T}} V = -R^{\mathsf{T}} V$. Then the result follows from the relation $(\Psi')^{-1} = \Psi^{-1} B$. □

4.5 Examples

In this section, we consider a tensor product of two Drinfeld modules ϕ_1 and ϕ_2 defined by

$$(\phi_1)_t = \theta + A_1\tau + A_2\tau^2, \quad (\phi_2)_t = \theta + B_1\tau + B_2\tau^2, \quad A_2 \neq 0, B_2 \neq 0.$$

Recall the t-module $\rho = \phi_1 \otimes \phi_2$ from Definition 4.5 and the t-module ρ' from Theorem 4.12. In this case, the t-modules ρ and ρ' are defined by

$$\rho_t = \begin{pmatrix} \theta & 0 & A_1 & A_2 \\ A_1\tau & \theta & A_2\tau & 0 \\ B_1\tau & B_2\tau & \theta & 0 \\ B_2\tau^2 & 0 & B_1\tau & \theta \end{pmatrix}, \quad \rho'_t = \begin{pmatrix} \theta & B_2\tau & B_1\tau & 0 \\ A_2\tau & \theta & A_1\tau & 0 \\ 0 & 0 & \theta & 1 \\ A_2B_1\tau & A_1B_2\tau & A_1B_1\tau + A_2B_2\tau^2 & \theta \end{pmatrix}.$$

Also, the $\mathbb{K}[t]$-basis $\mathbf{n}'$ in Lemma 4.14 is $(s_1, s_2, s_4, \sigma s_4)^{\mathsf{T}}$ and the matrix B in Definition 4.16 is given by

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{B_2^{(-1)}} & 0 & 0 \\ \frac{1}{A_2^{(-1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using Theorem 4.26, we acquire the periods of ρ' as follows. First, we fix periods $\pi_1, \pi_2 \in \Lambda_{\phi_1}$ and take the Anderson generating functions $f_1(t), f_2(t)$ associated to π_1, π_2. For each i, we expand

$$f_i(t) = -\frac{\pi_i}{t-\theta} + u_i + \text{higher order terms in } t-\theta.$$

Similarly, we fix periods $\mu_1, \mu_2 \in \Lambda_{\phi_2}$ and take the Anderson generating functions $g_1(t), g_2(t)$ associated to μ_1, μ_2. For each i, we expand

$$g_i(t) = -\frac{\mu_i}{t-\theta} + v_i + \text{higher order terms in } t-\theta.$$

Then, using our formula in Theorem 4.26, we obtain the following periods in $\Lambda_{\rho'}$:

$$\lambda_{11} = \begin{pmatrix} -f_1^{(1)}(\theta)\mu_1 \\ -g_1^{(1)}(\theta)\pi_1 \\ -\pi_1 v_1 - \mu_1 u_1 \\ \pi_1\mu_1 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} -f_1^{(1)}(\theta)\mu_2 \\ -g_2^{(1)}(\theta)\pi_1 \\ -\pi_1 v_2 - \mu_2 u_1 \\ \pi_1\mu_2 \end{pmatrix},$$
$$\lambda_{21} = \begin{pmatrix} -f_2^{(1)}(\theta)\mu_1 \\ -g_1^{(1)}(\theta)\pi_2 \\ -\pi_2 v_1 - \mu_1 u_2 \\ \pi_2\mu_1 \end{pmatrix}, \quad \lambda_{22} = \begin{pmatrix} -f_2^{(1)}(\theta)\mu_2 \\ -g_2^{(1)}(\theta)\pi_2 \\ -\pi_2 v_2 - \mu_2 u_2 \\ \pi_2\mu_2 \end{pmatrix}.$$

To find the Anderson generating functions associated to $\lambda_{i,j}$, we apply our formula in Theorem 4.32 and then we get

$$\mathcal{G}_{\lambda_{1,1}}(t) = -\begin{pmatrix} f_1^{(1)} g_1 \\ g_1^{(1)} f_1 \\ f_1 g_1 \\ f_1 g_1 T \end{pmatrix}, \quad \mathcal{G}_{\lambda_{1,2}}(t) = -\begin{pmatrix} f_1^{(1)} g_2 \\ g_2^{(1)} f_1 \\ f_1 g_2 \\ f_1 g_2 T \end{pmatrix},$$

$$\mathcal{G}_{\lambda_{2,1}}(t) = -\begin{pmatrix} f_2^{(1)}g_1 \\ g_1^{(1)}f_2 \\ f_2g_1 \\ f_2g_1T \end{pmatrix}, \quad \mathcal{G}_{\lambda_{2,2}}(t) = -\begin{pmatrix} f_2^{(1)}g_2 \\ g_2^{(1)}f_2 \\ f_2g_2 \\ f_2g_2T \end{pmatrix}.$$

Moreover, we compute the inverse of the rigid analytic trivialization Ψ' using Proposition 4.33 as follows. Let R be the matrix obtaining from the coordinates of $\mathcal{G}_{\lambda_{i,j}}$ given by

$$R = -\begin{pmatrix} f_1^{(1)}g_1^{(1)} & f_1^{(1)}g_2^{(1)} & f_2^{(1)}g_1^{(1)} & f_2^{(1)}g_2^{(1)} \\ f_1^{(1)}g_1^{(2)} & f_1^{(1)}g_2^{(2)} & f_2^{(1)}g_1^{(2)} & f_2^{(1)}g_2^{(2)} \\ f_1^{(2)}g_1^{(1)} & f_1^{(2)}g_2^{(1)} & f_2^{(2)}g_1^{(1)} & f_2^{(2)}g_2^{(1)} \\ f_1^{(2)}g_1^{(2)} & f_1^{(2)}g_2^{(2)} & f_2^{(2)}g_1^{(2)} & f_2^{(2)}g_2^{(2)} \end{pmatrix}.$$

Let

$$V = V_1 \otimes V_2 = \begin{pmatrix} A_1B_1 & A_1B_2^{(-1)} & A_2^{(-1)}B_1 & A_2^{(-1)}B_2^{(-1)} \\ A_1B_2 & 0 & A_2^{(-1)}B_2 & 0 \\ A_2B_1 & A_2B_2^{(-1)} & 0 & 0 \\ A_2B_2 & 0 & 0 & 0 \end{pmatrix}.$$

Then it follows from Proposition 4.33 that

$$(\Psi')^{-1} = -R^{\mathsf{T}}VB = \begin{pmatrix} Tf_1^{(1)}g_1 & Tf_1g_1^{(1)} & T^2f_1g_1 & A_2^{(-1)}B_2^{(-1)}f_1^{(1)}g_1^{(1)} \\ Tf_1^{(1)}g_2 & Tf_1g_2^{(1)} & T^2f_1g_2 & A_2^{(-1)}B_2^{(-1)}f_1^{(1)}g_2^{(1)} \\ Tf_2^{(1)}g_1 & Tf_2g_1^{(1)} & T^2f_2g_1 & A_2^{(-1)}B_2^{(-1)}f_2^{(1)}g_1^{(1)} \\ Tf_2^{(1)}g_2 & Tf_2g_2^{(1)} & T^2f_2g_2 & A_2^{(-1)}B_2^{(-1)}f_2^{(1)}g_2^{(1)} \end{pmatrix}.$$

5. CONCLUSION

In this dissertation, we provided a method to construct a rigid analytic trivialization for a Drinfeld module. Moreover, we study a tensor product of two Drinfeld modules ϕ_1 and ϕ_2, and then we find the periods of $\phi_1 \otimes \phi_2$. Furthermore, we provide a formula for the Anderson generating functions associated to the tensor product $\phi_1 \otimes \phi_2$, which can be expressed via the Anderson generating functions of ϕ_1 and ϕ_2.

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