ASYMPTOTICS FOR TRACES OF WEAK MAASS FORMS AND APPLICATIONS

A Dissertation

by

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ABSTRACT

In this dissertation, we give two main results. First, we use the Bruinier-Ono formula to give an asymptotic formula for the partition function $p(n)$ with an effective bound on the error term. Second, we give an asymptotic formula with a power saving error term for traces of a generic class of weight 0 weak Maass forms. Those forms are images of weight $-2k$ weakly holomorphic modular forms of squarefree level N under the differential operator $\mathcal{D}^k$. We apply this result to study the asymptotic distribution of several arithmetic functions, including the partition function $p(n)$, the Andrews' smallest parts function spt(n), and the coefficients $\alpha(n)$ of Ramanujan's $f(q)$ mock theta function.

DEDICATION

This dissertation is dedicated to my grandmother, Taengthai Jaidee.

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NOMENCLATURE

$\mathbb{H}$	The complex upper half plane
q	$e(z) := e^{2\pi iz}$
$GL_2(\mathbb{R})$	The general linear group of 2-by-2 matrices with real entries and non-zero determinant
$GL_2^+(\mathbb{R})$	The general linear group of 2-by-2 matrices with real entries and positive determinant
$SL_2(\mathbb{R})$	The special linear group of 2-by-2 matrices with real entries and determinant 1
$SL_2(\mathbb{Z})$	The special linear group of 2-by-2 matrices with integer entries and determinant 1
Δ_k	Weight k hyperbolic Laplacian
R_k	Maass weight raising operator on non-holomorphic modular forms of weight k
L_k	Maass weight lowering operator on non-holomorphic modular forms of weight k
$H_k(N)$	The space of harmonic weak Maass forms of weight k on $\Gamma_0(N)$
$M_k^!(N)$	The space of weakly holomorphic modular forms of weight k on $\Gamma_0(N)$
C^∞	The set of all infinitely differentiable functions
$L^1(S)$	The space of all measurable functions $f : S \to \mathbb{C}$ such that $\int_S \|f\| d\mu < \infty$

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1. INTRODUCTION

1.1 Overview

1.1.1 Motivation

The traces of weak Maass forms have been studied extensively for the last 20 years. Much of this work was inspired by Zagier's seminal paper [28] which realized traces of singular moduli as Fourier coefficients of half-integral weight weakly holomorphic modular forms. There have been many additional works which realize traces of weak Maass forms as coefficients of q-series and modular forms. Among other things, such formulas provide a way to study the asymptotic distribution of these arithmetically rich coefficients.

In order to provide some examples, we recall some important arithmetic functions. A *partition* λ of a positive integer n is a non-increasing sequence $\lambda_1 \geq \cdots \geq \lambda_k$ of positive integers such that $n = \lambda_1 + \cdots + \lambda_k$. Each λ_i is called a *part* of λ. The classical Hardy-Ramanujan *partition function* $p(n)$ counts the number of partitions of n. The Andrews *smallest parts function* $\mathrm{spt}(n)$ counts the number of smallest parts appearing among the partitions of n. Finally, the *rank* of a partition λ is defined by $\mathrm{rank}(\lambda) = \lambda_1 - k$, the largest part minus the total number of parts.

Now, let $\mathcal{Q}_{-24n+1,6,1}$ be the set of positive definite, integral binary quadratic forms $Q = [a_Q, b_Q, c_Q]$ of discriminant $-24n+1$ with $a_Q \equiv 0 \pmod{6}$ and $b_Q \equiv 1 \pmod{12}$. Also, let τ_Q denote the Heegner point associated to the form $Q \in \mathcal{Q}_{-24n+1,6,1}$ (see Section 1.4 for details).

In 2013, Bruinier and Ono [5] gave a remarkable formula for $p(n)$. They expressed $p(n)$ as the trace of CM values of a weight 0 weak Maass form. They proved the following formula. For all $n \geq 1$,

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_{-24n+1,6,1}/\Gamma_0(6)} P(\tau_Q),$$

where P is a weak Maass form defined by (2.2) and the sum is over the $\Gamma_0(6)$ equivalence classes of $\mathcal{Q}_{-24n+1,6,1}$.

Ahlgren and Andersen [1] obtained an algebraic formula for $\mathrm{spt}(n)$ in term of the trace of $f-P$, when f is a weakly holomorphic modular form defined by (3.52) and P is a weak Maass form defined by (2.2). They proved the following formula. For all $n \geq 1$,

$$\mathrm{spt}(n) = \frac{1}{12} \sum_{Q \in \mathcal{Q}_{-24n+1,6,1}/\Gamma_0(6)} (f(\tau_Q) - P(\tau_Q)).$$

They used this formula to prove the exact formula for $\mathrm{spt}(n)$.

In a recent paper, Bruinier and Schwagenscheidt [8] gave a formula for the coefficients $\alpha(n)$ of Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = 1 + \sum_{n=1}^{\infty} \alpha(n) q^n, \quad q := e(z) = e^{2\pi i z}$$

in terms of the traces of CM values of a weight 0 weakly holomorphic modular form on $\Gamma_0(6)$. They proved that for $n \geq 1$,

$$\alpha(n) = \frac{-1}{\sqrt{24n-1}} \mathrm{Im} \left(\sum_{Q \in \mathcal{Q}_{-24n+1,6,1}/\Gamma_0(6)} \frac{F(\tau_Q)}{\omega_Q} \right),$$

where F is a weakly holomorphic modular form on $\Gamma_0(6)$ defined by (3.54) and ω_Q is half the order of the stabilizer of Q in $\Gamma_0(6)$.

1.1.2 Goals and Outline of this dissertation

In this dissertation, we investigate the asymptotic distribution of traces of weight 0 weak Maass forms. We focuses on two main results.

The first main result, which will be given with details in Section 2, is an asymptotic formula for $p(n)$ with an effective bound on the error terms.

The second main result, which will be given with details in Section 3, is an asymptotic formula with a power saving error term for traces of a generic class of weak Maass forms. More precisely, let $k \geq 0$ be an integer and $M^!_{-2k}(N)$ be the space of weakly holomorphic modular forms of weight $-2k$ and level N. We define the differential operator $\mathcal{D}^k$ by

$$\mathcal{D}^k f := \frac{1}{(4\pi)^k} R_{-2} R_{-4} \cdots R_{-2k} f, \quad k \geq 1,$$

and $\mathcal{D}^0 f = f$, where R_t is the Maass weight raising operator

$$R_t f := 2i \frac{\partial}{\partial z} + \frac{t}{y}, \quad t \in \mathbb{Z}.$$

The operator $\mathcal{D}^k$ maps $M^!_{-2k}(N)$ to the space of weight 0 weak Maass forms of level N. We study asymptotic distribution of the trace of $\mathcal{D}^k f$ when $f \in M^!_{-2k}(N)$ is an eigenfunction for the Atkin-Lehner operators of squarefree level N (see Theorem 3.2). We apply this result to give asymptotic formulas with power saving error terms for $p(n)$, spt(n), and $\alpha(n)$, improving significantly upon some previous results.

The remaining part of this section is devoted to background and notation, which will be used in the subsequent sections. Sections 2 and 3 are constituted of two papers [16, 18], which are joint works of Riad Masri, Wei-Lun Tsai and the author.

1.2 Quotients of the upper half plane

The group $SL_2(\mathbb{R})$ acts on the complex upper half plane $\mathbb{H}$ by *linear fractional transformations*

$$\gamma z := \frac{az+b}{cz+d}, \quad \text{for } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{1.1}$$

Define the action of $SL_2(\mathbb{R})$ on extended upper half plane $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ by the same formula (1.1) in addition to the rules $\gamma\infty = a/c$ and $\gamma(-d/c) = \infty$ if $c \neq 0$, and $\gamma\infty = \infty$ if $c = 0$.

A *fundamental domain* for the action of a discrete subgroup Γ of $SL_2(\mathbb{R})$ is a subset $F \subset \mathbb{H}$ satisfying the following conditions:

- F is an open subset in $\mathbb{H}$.

- The closure $\bar{F}$ of F in $\mathbb{C}$ intersects every orbit of Γ, i.e. for all $z \in \mathbb{H}$ there is $\gamma \in \Gamma$ with $\gamma z \in \bar{F}$.

- No two points in F are Γ-equivalent.

One example of a discrete subgroup of $SL_2(\mathbb{R})$ is $SL_2(\mathbb{Z})$. We let $\mathcal{F}$ denote the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ defined by

$$\mathcal{F} := \{z \in \mathbb{H} : |\mathrm{Re}(z)| < 1/2, |z| > 1\}.$$

If Γ is a subgroup of $SL_2(\mathbb{Z})$, then for each $z \in \mathbb{H}$ we let Γ_z denote the *stabilizer subgroup* of z given by

$$\Gamma_z := \{\gamma \in \Gamma : \gamma z = z\}.$$

Now, let N be a positive integer. The *Hecke congruence subgroup* of level N is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

The subgroup $\Gamma_0(N)$ have finite index in $SL_2(\mathbb{Z})$, in particular,

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + 1/p)$$

where the product is taken over all prime numbers dividing N. The action of $\Gamma_0(N)$ on $\mathbb{H}$ by linear fractional transformations induces an equivalence relation; two element $z_1, z_2 \in \mathbb{H}$ are equivalent if and only if their orbits are the same. The $\Gamma_0(N)$-equivalence classes of $\mathbb{Q} \cup \{\infty\}$ is called the *cusps* of $\Gamma_0(N)$. Define the *modular curve* $Y_0(N)$ as the quotient space of orbits under $\Gamma_0(N)$,

$$Y_0(N) := \Gamma_0(N) \backslash \mathbb{H} = \{\Gamma_0(N)z : z \in \mathbb{H}\}.$$

The modular curve $Y_0(N)$ can be compactified. Let $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. We define the *modular curve* $X_0(N)$ corresponding to $\Gamma_0(N)$ by the quotient

$$X_0(N) := \Gamma_0(N) \backslash \mathbb{H}^*.$$

The modular curve $X_0(N)$ is compact.

1.3 Weak Maass forms

Let $k \in \mathbb{Z}$. For $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, we define the *slash operator* $|_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f|_k\gamma(z) = \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z)$$

for the multiplier system $j(\gamma, z) := cz + d$. The *weight k hyperbolic Laplacian* is defined by

$$\Delta_k := -y^2 \left(\partial_x^2 + \partial_y^2\right) + iky(\partial_x + i\partial_y).$$

A *weak Maass form of weight k on* $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) $f|_k M = f$ for all $M \in \Gamma_0(N)$.

(2) There is a complex number λ such that $\Delta_k f = \lambda f$.

(3) There is a constant $C > 0$ such that $f(z) = O(e^{Cy})$ as $y \to \infty$. An analogous condition is required at all cusps.

A weak Maass form f is *harmonic* if $\Delta_k f = 0$. Every harmonic Maass form has a Fourier expansion

$$f(z) = f^-(z) + f^+(z)$$

with non-holomorphic part

$$f^{-}(z) := \sum_{n<0} c_f^{-}(n)\Gamma(1-k, 4\pi|n|y)q^n$$

and holomorphic part

$$f^{+}(z) := \sum_{n=1}^{N_\infty} c_f^{+}(-n)q^{-n} + \sum_{n=0}^{\infty} c_f^{+}(n)q^n,$$

where $N_\infty \geq 1$ is an integer and $\Gamma(a,t)$ is the incomplete Gamma function given by

$$\Gamma(a,t) := \int_t^\infty x^{a-1}e^{-x}dx.$$

A harmonic Maass form with trivial non-holomorphic part is called a *weakly holomorphic modular form*, in which case

$$f(z) = \sum_{n=1}^{N_\infty} a(-n)q^{-n} + \sum_{n=0}^{\infty} a(n)q^n. \tag{1.2}$$

Let

$$M_k^!(N) \subset H_k(N)$$

denote the spaces of weakly holomorphic modular forms and harmonic Maass forms of weight k on $\Gamma_0(N)$, respectively.

1.4 Binary quadratic forms, Heegner points and Traces

Given a binary quadratic form

$$Q(X,Y) = [a_Q, b_Q, c_Q](X,Y) = a_Q X^2 + b_Q XY + c_Q Y^2,$$

the *discriminant* of Q is defined by $\operatorname{disc}(Q) := b_Q^2 - 4a_Q c_Q$.

Let N be a positive integer and $D < 0$ be a negative discriminant. Let $\mathcal{Q}_{D,N}$ be the set of positive definite, integral binary quadratic forms $Q = [a_Q, b_Q, c_Q]$ of discriminant D with $a_Q \equiv 0 \pmod{N}$. There is a right action of $\Gamma_0(N)$ on $\mathcal{Q}_{D,N}$ defined by

$$Q = [a_Q, b_Q, c_Q] \longmapsto Q \circ \sigma = [a_Q^\sigma, b_Q^\sigma, c_Q^\sigma] \quad \text{for} \quad \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N),$$

where

$$\begin{aligned} a_Q^\sigma &:= a_Q\alpha^2 + b_Q\alpha\gamma + c_Q\gamma^2, \\ b_Q^\sigma &:= 2a_Q\alpha\beta + b_Q(\alpha\delta + \beta\gamma) + 2c_Q\gamma\delta, \\ c_Q^\sigma &:= a_Q\beta^2 + b_Q\beta\delta + c_Q\delta^2. \end{aligned}$$

Given a solution $\rho \pmod{2N}$ of $\rho^2 \equiv D \pmod{4N}$, we define the subset of forms

$$\mathcal{Q}_{D,N,\rho} := \{Q = [a_Q, b_Q, c_Q] \in \mathcal{Q}_{D,N} : b_Q \equiv \rho \pmod{2N}\}.$$

Then the group $\Gamma_0(N)$ also acts on $\mathcal{Q}_{D,N,\rho}$. The number of $\Gamma_0(N)$ equivalence classes in $\mathcal{Q}_{D,N,\rho}$ is given by the Hurwitz-Kronecker class number $H(D)$.

A binary quadratic form $Q = [a_Q, b_Q, c_Q]$ is called *primitive* if $(a_Q, b_Q, c_Q) = 1$. The preceding facts remain true if we restrict to the subset

$$\mathcal{Q}_{D,N}^{\text{prim}} := \{Q \in \mathcal{Q}_{D,N} : Q \text{ is primitive}\}$$

In this case, the number of $\Gamma_0(N)$ equivalence classes in $\mathcal{Q}_{D,N,\rho}^{\text{prim}}$ is given by the class number $h(D)$. Recall that a form $Q \in \mathcal{Q}_{D,1}$ is *reduced* if

$$|b_Q| \leq a_Q \leq c_Q,$$

and if, in addition, $|b_Q| = a_Q$ or $a_Q = c_Q$, then $b_Q \geq 0$. Let $\mathcal{Q}_D^{\text{red}}$ denote a set of primitive, reduced forms representing the equivalence classes in $\mathcal{Q}_{D,1}^{\text{prim}}/SL_2(\mathbb{Z})$. To each form $Q \in \mathcal{Q}_{D,N}$, we associate a Heegner point τ_Q which is the root of $Q(X, 1)$ given by

$$\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.$$

We write $x_Q := \text{Re}(\tau_Q)$ and $y_Q := \text{Im}(\tau_Q)$.

The Heegner points τ_Q are compatible with the action of $\Gamma_0(N)$ in the sense that if $\sigma \in \Gamma_0(N)$, then

$$\sigma(\tau_Q) = \tau_{Q\circ\sigma^{-1}}. \tag{1.3}$$

We now give definition of traces. Given a $\Gamma_0(N)$-invariant function $f : \mathbb{H} \to \mathbb{C}$, we define the trace

$$\text{Tr}_D(f) := \sum_{Q \in \mathcal{Q}_{D,N,\rho}/\Gamma_0(N)} f(\tau_Q).$$

1.5 The Differential operator

The Maass weight raising operator R_k and the Maass weight lowering operator L_k on non-holomorphic modular forms of weight k are defined by

$$R_k f := 2i\frac{\partial}{\partial z} + \frac{k}{y},$$
$$L_k f := -2iy^2\frac{\partial}{\partial \bar{z}}.$$

The raising operator R_k raise the weight of an automorphic form by 2, while the lowering operator L_k lowers it by 2. With respect to the slash operator, these operators R_k and L_k satisfy the intertwining properties

$$R_k(f|_k\gamma) = (R_k f)|_{k+2}\gamma,$$
$$L_k(f|_k\gamma) = (L_k f)|_{k-2}\gamma,$$

for any $\gamma \in SL_2(\mathbb{R})$. The hyperbolic Laplacian Δ_k can be expressed in terms of R_k and L_k as follows

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k.$$

Definition 1.1. *Let $k \geq 0$ be an interger. We define the differential operator $\mathcal{D}^k$ on $f \in M^!_{-2k}(N)$ by*

$$\mathcal{D}^k f := \frac{1}{(4\pi)^k} R_{-2}R_{-4}\cdots R_{-2k}f, \ \ k \geq 1,$$

and $\mathcal{D}^0 f = f$. We let $\mathcal{D}$ denote the differential operator $\mathcal{D}^1$.

Since the Maass weight raising operators raise the weight of an automorphic form by 2, the differential operator $\mathcal{D}^k$ raises the weight of an automorphic form by $2k$.

1.6 Notation

Let m be a positive integer and $k \in \mathbb{Z}$. Define the Poincaré series (see [5])

$$F_m(z, s, k) := \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \left[\mathcal{M}_{s,k}(4\pi m y)e(-mx)\right]|_k\gamma$$

where $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Here

$$\mathcal{M}_{s,k}(y) := y^{-k/2}M_{-\frac{k}{2}, s-\frac{1}{2}}(y),$$

where $M_{\kappa,\mu}$ is the usual Whittaker function. This Poincaré series converges for Re$(s) > 1$, and is an eigenfunction for Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$. Its specialization at $s_0 = 1 - k/2$ is a harmonic Maass form.

The *Kloosterman sum of modulus* c is defined by

$$S(a, b; c) := \sum_{\substack{d \pmod{c} \\ (c,d)=1}} e\left(\frac{a\bar{d} + bd}{c}\right) \tag{1.4}$$

where $\bar{d}$ is the multiplicative inverse of $d \pmod{c}$.

Throughout this dissertation, we let I_ν, J_ν, K_ν denote the I, J, K-Bessel functions of order ν, respectively.

2. AN EFFECTIVE BOUND FOR THE PARTITION FUNCTION*

2.1 The partition function

In 1937, Rademacher [26] showed that $p(n)$ can be expressed as an absolutely convergent infinite series

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)}{\sqrt{n-1/24}} \right),$$

where

$$A_k(n) := \sum_{\substack{h \pmod{k} \\ (h,k)=1}} \omega_{h,k}\, e^{-2\pi i h n/k},$$

and

$$\omega_{h,k} := \exp\left(\pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right) \right).$$

By truncating Rademacher's series and bounding the remainder termwise, Lehmer [20, 21] obtained an asymptotic formula for $p(n)$ with an effective bound on the error term.

Bruinier and Ono [5] gave a remarkable formula which expresses $p(n)$ as a finite sum of algebraic numbers. These numbers are singular moduli for a weak Maass form. Let E_2 be the weight 2 quasimodular Eisenstein series given by

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} d q^n$$

and $\eta(z)$ be the Dedekind's eta function given by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Then define the weight -2 weakly holomorphic modular form for $\Gamma_0(6)$ by

$$F_p(z) := \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} - 10 - 29q - \cdots . \tag{2.1}$$

By applying the differential operator $\mathcal{D}$ defined by Definition 1.1 to F_p, Bruinier and Ono obtained the following weight 0 weak Maass form for $\Gamma_0(6)$,

$$P(z) := \mathcal{D}F_p(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right) F_p(z). \tag{2.2}$$

Then, using the theory of harmonic weak Maass forms and theta lifts, they expressed $p(n)$ as the trace of $P(z)$ over a Galois orbit of Heegner points of discriminant $-24n + 1$ and level 6.

Theorem 2.1 (Bruinier-Ono)**.** *Let* $D_n := -24n + 1$*. Then for all* $n \geq 1$

$$p(n) = \frac{1}{24n - 1} \sum_{Q \in \mathcal{Q}_{D_n,6,1}/\Gamma_0(6)} P(\tau_Q).$$

2.2 Asymptotic formula for $p(n)$ with an effective bound on the error term

We now give our first main result, which is an asymptotic formula for $p(n)$ with an effective bound on the error term. To state our main results, we require the following definitions and notation.

The group $\Gamma_0(6)$ has index 12 in $SL_2(\mathbb{Z})$. Following [10], we choose the following right coset representatives for $\Gamma_0(6)\backslash SL_2(\mathbb{Z})$.

Definition 2.2. *Let* $\mathbf{C}_6$ *be the set of right coset representatives of* $\Gamma_0(6)\backslash SL_2(\mathbb{Z})$ *given by*

$$\begin{aligned}
\gamma_\infty &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\gamma_{1/3,u} &:= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \text{ for } u = 0, 1 \\
\gamma_{1/2,v} &:= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \text{ for } v = 0, 1, 2 \\
\gamma_{0,t} &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \text{ for } t = 0, 1, 2, 3, 4, 5.
\end{aligned}$$

Given a form $Q \in \mathcal{Q}_D^{\text{red}}$, there is a unique coset representative $\gamma_Q \in \mathbf{C_6}$ such that

$$[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}_{D,6,1}^{\text{prim}}/\Gamma_0(6).$$

An explicit list of the matrices $\gamma_Q \in \mathbf{C_6}$ for each $Q \in \mathcal{Q}_D^{\text{red}}$ can be found in [10, Lemma 3]. Let h_Q denote the width of the cusp $\gamma_Q(\infty)$ of $\Gamma_0(6)$ (see Table 2.3). Also, we assign to each γ_Q a certain sixth root of unity ζ_Q (see Table 2.5).

We prove the following asymptotic formula for $p(n)$.

Theorem 2.3. *Let* $D_n := -24n + 1$*. Then for all* $n \geq 1$ *we have*

$$p(n) = M(n) + E(n),$$

where

$$M(n) := \frac{1}{24n-1} \sum_{\substack{d>0 \\ d^2|D_n}} \epsilon(d) \sum_{Q \in \mathcal{Q}_{D_n/d^2}^{\text{red}}} \zeta_Q \left(1 - \frac{h_Q}{2\pi \text{Im}(\tau_Q)}\right) e\left(\frac{-\tau_Q}{h_Q}\right),$$

with

$$\epsilon(d) := \begin{cases} 1, & \textit{if } d \equiv \pm 1 \pmod{12} \\ -1, & \textit{otherwise}, \end{cases}$$

and

$$|E(n)| \leq \left(5.6044 \times 10^{23}\right) \frac{H(D_n)}{24n-1}.$$

In Table 2.1, we display a list of values of $M(n)$ and

$$|E(n)| = |p(n) - M(n)|$$

which we computed using `SageMath` [27]. Our code is available on GitHub [17]. As the reader can observe, the main term $M(n)$ approximates $p(n)$ very closely.

n	$M(n)$	$p(n)$	$\|E(n)\|$
1	$0.5282 - 0.3143i$	1	0.5669
10	$41.8519 + 0.0449i$	42	0.1548
50	$204226.0156 + 0.0481i$	204226	0.0506
100	$190569291.9356 + 0.0176i$	190569292	0.0663
500	230016503257432399 5026.9770 $-0.0092i$	2300165032574323995027	0.0248
1000	24061467864032622473692149727990.9880 $+ 0.0078i$	2406146786403262247369214972799 1	0.0143
5000	169820168825442121851975101689306431361757683049829233322203824652329144348.9971 $- 0.0037i$	16982016882544212185197510168930643136175768304982923332220382465232914434 9	0.0047
10000	36167251325636293988820471890953695495016030339315650422081868605887952568754066420592310556052906916435144.0024 $+ 0.0014i$	36167251325636293988820471890953695495016030339315650422081868605887952568754066420592310556052906916435144	0.0028
50000	3626186097141667844592140891595633728165383082527785049015872755414109904256712082718122747316610565824630881772910217544261659239432670671532413858378256188987333877121891586607957389750538447474712592979263719012461858719791627302489739548262.9989 $+ 0.0002i$	3626186097141667844592140891595633728165383082527785049015872755414109904256712082718122747316610565824630881772910217544261659239432670671532413858378256188987333877121891586607957389750538447474712592979263719012461858719791627302489739548263	0.0011

Table 2.1: $M(n)$ and $|E(n)|$

2.3 The Fourier expansion of $P(z)$

In this subsection, we compute the Fourier expansion of $P(z)$ defined by (2.2).

Proposition 2.4. *We have*

$$P(z) = \left(1 - \frac{1}{2\pi y}\right) e(-z) + \frac{5}{\pi y} + \sum_{n=1}^{\infty} \frac{b(n)}{\sqrt{n}} \left(1 + \frac{1}{2\pi y n}\right) e(nz)$$

where

$$b(n) := 2\pi \sum_{l|6} \mu(l) \sum_{\substack{0<c\equiv 0 \pmod{6/l} \\ (c,s)=1}} \frac{S(-\bar{l}, n; c)}{c\sqrt{l}} I_3\left(\frac{4\pi\sqrt{n}}{c\sqrt{l}}\right).$$

The remaining part of this subsection is devoted to the proof of Proposition 2.4. Let $\{W_l : l|6\}$ be the group of Atkin-Lehner involutions for $\Gamma_0(6)$ defined by

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}, \quad W_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}. \tag{2.3}$$

By [5, p. 213], we can write $F_p(z)$ as a linear combination of Poincaré series[1]

$$F_p(z) = 2 \sum_{l|6} \mu(l) F_1(z, 2, -2)|_{-2} W_l, \tag{2.4}$$

where μ is the Möbius function.

Lemma 2.5. *The weak Maass form $P(z)$ can be expressed as*

$$P(z) = 2 \sum_{l|6} \mu(l) F_1(z, 2, 0)|_0 W_l. \tag{2.5}$$

[1]The factor 2 is missing on the right hand side of the identity for $F_p(z)$ in [5, p. 213].

Proof. By (2.4) and the fact that the raising operator R_{-2} and the slash operator commutes, we have

$$P(z) = \frac{1}{4\pi} R_{-2}(F_p(z)) = 2\sum_{l|6} \mu(l) \frac{1}{4\pi} \left(R_{-2}F_1(z, 2, -2)\right)|_0 W_l.$$

Also, by [5, Proposition 2.2], we have

$$\frac{1}{4\pi} R_{-2}F_1(z, 2, -2) = F_1(z, 2, 0).$$

Hence, we complete the proof. □

Lemma 2.6. *For $z = x + iy \in \mathbb{H}$, define*

$$\phi_t(y) := 2\pi\sqrt{y} I_{t-\frac{1}{2}}(2\pi y)$$

and

$$g(z) := \phi_2(y) \exp(-2\pi y) e(-z).$$

Then

$$P(z) = \sum_{l|6} \mu(l) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} g(\gamma W_l z).$$

Proof. Let $\Gamma(\nu)$ denote the Gamma function. Using the following relation between the I-Bessel function and the Whittaker function $M_{0,\nu}$ (see e.g. [25, Eq. 10.39.7]),

$$I_\nu(z) = \frac{(2z)^{-1/2} M_{0,\nu}(2z)}{2^{2\nu}\Gamma(\nu + 1)} \qquad \text{whenever} \quad 2\nu \neq -1, -2, -3, ...,$$

we have

$$\mathcal{M}_{t,0}(4\pi y) = M_{0,t-\frac{1}{2}}(4\pi y) = 2^{2\left(t-\frac{1}{2}\right)}\Gamma\left(t+\frac{1}{2}\right) I_{t-\frac{1}{2}}(2\pi y)(4\pi y)^{1/2}.$$

So, by the definition of $F_1(z,t,0)$, we have

$$\begin{aligned} F_1(z,t,0) &= \frac{1}{2\Gamma(2t)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\mathcal{M}_{t,0}(4\pi y)e(-x)\right]|_0\gamma \\ &= \frac{2^{2t}\Gamma\left(t+\frac{1}{2}\right)(4\pi)^{1/2}}{4\Gamma(2t)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\sqrt{y}I_{t-\frac{1}{2}}(2\pi y)e(-x)\right]|_0\gamma. \end{aligned}$$

Since

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we have

$$\frac{\Gamma\left(t+\frac{1}{2}\right)}{\Gamma(2t)} = \frac{(2\pi)^{1/2}2^{-(2t-1/2)}}{\Gamma(t)} = \frac{2^{-2t+1}\sqrt{\pi}}{\Gamma(t)}.$$

Hence

$$\begin{aligned} F_1(z,t,0) &= \frac{\pi}{\Gamma(t)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\sqrt{y}I_{t-\frac{1}{2}}(2\pi y)e(-x)\right]|_0\gamma \\ &= \frac{1}{2\Gamma(t)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\phi_t(y)e(-x)\right]|_0\gamma \\ &= \frac{1}{2\Gamma(t)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\phi_t(\mathrm{Im}\gamma z)e(-\mathrm{Re}\gamma z)\right]. \end{aligned}$$

Thus when $t=2$,

$$F_1(z,2,0) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} \left[\phi_2(\mathrm{Im}\gamma z)e(-\mathrm{Re}\gamma z)\right].$$

Therefore by Lemma 2.5,

$$\begin{aligned}P(z) &= 2\sum_{l|6}\mu(l)F_1(z,2,0)|_0 W_l \\ &= 2\sum_{l|6}\mu(l)F_1\left(W_l z,2,0\right) \\ &= \sum_{l|6}\mu(l)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)}[\phi_2(\text{Im}\gamma W_l z)e(-\text{Re}\gamma W_l z)].\end{aligned}$$

By the definition of $g(z)$, we have

$$g(\gamma W_l z) = \phi_2(\text{Im}\gamma W_l z)e(-\text{Re}\gamma W_l z).$$

Hence

$$P(z) = \sum_{l|6}\mu(l)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma W_l z). \tag{2.6}$$

□

Now, we need to switch our choice of Atkin-Lehner operators $\{W_l : l|6\}$ to another choice of Atkin-Lehner operators defined in [19, Eq. (2.9)] which are scaling matrices for the Atkin-Lehner cusps. This will enable us to use the set of allowed moduli in [19, Proposition 2.6].

For each $l|6$, let $r := 6/l$. For such r, define the matrix $\sigma_{1/r}$ as in [19, Eq. (2.9)],

$$\sigma_{1/r} = \begin{pmatrix} 1 & (\bar{l}l-1)/r \\ r & \bar{l}l \end{pmatrix}\begin{pmatrix}\sqrt{l} & 0 \\ 0 & 1/\sqrt{l}\end{pmatrix} = \frac{1}{\sqrt{l}}\begin{pmatrix} l & (\bar{l}l-1)/r \\ rl & \bar{l}l\end{pmatrix},$$

where $\bar{l}$ is the multiplicative inverse of $l \pmod r$ given by

r	1	2	3	6
l	6	3	2	1
$\bar{l}$	1	1	2	1

Table 2.2: Multiplicative inverse of $l \pmod{r}$

Then we obtain the following choice for the matrices $\sigma_{1/r}$:

$$\sigma_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 6 & 5 \\ 6 & 6 \end{pmatrix}, \quad \sigma_{1/2} = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad \sigma_{1/3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}, \quad \sigma_{1/6} = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}.$$

Observe that for each $l|6$ there is a matrix $M_l \in \Gamma_0(6)$ such that $\sigma_{1/r} = M_l W_l$. More concretely, we have

$$M_1 = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -5 & 2 \\ -18 & 7 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} -5 & 1 \\ -6 & 1 \end{pmatrix}.$$

So, by substituting $W_l = M_l^{-1}\sigma_{1/r}$ and changing the index of the sum, we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma W_l z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma M_l^{-1}\sigma_{1/r} z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma\sigma_{1/r} z). \tag{2.7}$$

Thus (2.6) becomes

$$P(z) = \sum_{l|6} \mu(l) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma\sigma_{1/r} z). \tag{2.8}$$

Let

$$\psi(y) := \phi_2(y)\exp(-2\pi y).$$

Then

$$g(z) = \psi(y)e(-z).$$

Consider the Poincaré series

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} g(\gamma \sigma_{1/r} z).$$

Since the matrix $\sigma_{1/r}$ satisfies the conditions $\sigma_{1/r} \in SL_2(\mathbb{R}),\ \sigma_{1/r}(\infty) = 1/r$, and

$$\sigma_{1/r}^{-1} \Gamma_0(6)_{1/r} \sigma_{1/r} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

it is a scaling matrix for the Atkin-Lehner cusp $1/r$ in the sense of [15]. So, we can use the same argument as in [15, pages 59-60] to compute the Fourier expansion

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} g(\gamma \sigma_{1/r} z) = \delta_{\infty, \frac{1}{r}} e(-z)\psi(y) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c > 0} S_{\infty, 1/r}(-1, n; c) \alpha(y, c, n), \tag{2.9}$$

where $\delta_{\infty, \frac{1}{r}} = 1$ if the cusps ∞ and $1/r$ are equivalent and $\delta_{\infty, \frac{1}{r}} = 0$ otherwise, the general Kloosterman sum to modulus c is defined by

$$S_{\infty, 1/r}(m_1, m_2; c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(6) \sigma_{1/r} / \Gamma_\infty} e\left(\frac{am_1 + dm_2}{c}\right),$$

and

$$\alpha(y, c, n) := \int_{-\infty}^{\infty} \psi\left(\frac{yc^{-2}}{t^2 + y^2}\right) e\left(\frac{c^{-2}}{t + iy} - nt\right) dt.$$

By [19, Definition 2.3], the set of allowed moduli for the cusps ∞ and $1/r$ is

$$\mathbf{C}_{\infty,1/r} := \left\{ c' > 0 : \begin{pmatrix} * & * \\ c' & * \end{pmatrix} \in \Gamma_0(6)\sigma_{1/r} \right\}.$$

Using [19, Proposition 2.6], we obtain that

$$\mathbf{C}_{\infty,1/r} = \{c' = c\sqrt{l} > 0 : c \equiv 0 \pmod{r}, (c, l) = 1\} \tag{2.10}$$

and for such $c' = c\sqrt{l} \in \mathbf{C}_{\infty,1/r}$ the Kloosterman sum to modulus c' is given by

$$S_{\infty,1/r}(m_1, m_2; c\sqrt{l}) = S(\bar{l}m_1, m_2; c) \tag{2.11}$$

where $S(\bar{l}m_1, m_2; c)$ is the ordinary Kloosterman sum defined by (1.4).

Using (2.10) and changing c to $c\sqrt{l}$ in (2.9), we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma\sigma_{1/r}z) = \delta_{\infty,\frac{1}{r}}e(-z)\psi(y) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \alpha(y, c\sqrt{l}, n) S_{\infty,1/r}(-1, n; c\sqrt{l}).$$

Then by (2.11), we obtain

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(6)} g(\gamma\sigma_{1/r}z) = \delta_{\infty,\frac{1}{r}}e(-z)\psi(y) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \alpha(y, c\sqrt{l}, n) S(-\bar{l}, n; c). \tag{2.12}$$

We now compute $\alpha(y, c\sqrt{l}, n)$. By short calculation, we have

$$\alpha(y, c\sqrt{l}, n) = \frac{2\pi\sqrt{y}}{c\sqrt{l}} L, \tag{2.13}$$

where

$$L := \int_{-\infty}^{\infty} I_{3/2}\left(\frac{2\pi y c^{-2} l^{-1}}{t^2+y^2}\right)\frac{1}{\sqrt{t^2+y^2}} e\left(\frac{t}{c^2 l(t^2+y^2)} - nt\right) dt.$$

Making the changes of variables $t = yu$, $A = \frac{1}{lc^2 y}$, and $B = -ny$, we get

$$L = \int_{-\infty}^{\infty} \left(u^2+1\right)^{-1/2} I_{3/2}\left(\frac{2\pi A}{u^2+1}\right) e\left(\frac{Au}{u^2+1} + Bu\right) du.$$

Then the same argument as in [1, pages 426-427] yields

$$L = \begin{cases} 2K_{3/2}(2\pi B) J_3\left(4\pi\sqrt{AB}\right) & \text{if } B > 0 \\ \dfrac{2\pi^2}{3} A^{3/2} & \text{if } B = 0 \\ 2K_{3/2}\left(2\pi|B|\right) I_3\left(4\pi\sqrt{A|B|}\right) & \text{if } B < 0 \end{cases}$$

After substituting A and B, we get

$$L = \begin{cases} 2K_{3/2}\left(-2\pi n y\right) J_3\left(\dfrac{4\pi\sqrt{-n}}{c\sqrt{l}}\right) & \text{if } n < 0 \\ \dfrac{2\pi^2}{3} l^{-3/2} c^{-3} y^{-3/2} & \text{if } n = 0 \\ 2K_{3/2}\left(2\pi n y\right) I_3\left(\dfrac{4\pi\sqrt{n}}{c\sqrt{l}}\right) & \text{if } n > 0. \end{cases}$$

Now, substituting (2.13) in (2.12) yields

$$\begin{aligned} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} g(\gamma \sigma_{1/r} z) &= \delta_{\infty, \frac{1}{r}} e(-z)\psi(y) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} S(-\bar{l}, n; c) \frac{2\pi\sqrt{y}}{c\sqrt{l}} L \\ &= \delta_{\infty, \frac{1}{r}} e(-z)\psi(y) + A_0 + A_1 + A_2, \end{aligned} \tag{2.14}$$

where

$$A_0 := \frac{4\pi^3}{3} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l},0;c)}{c^4 l^2 y},$$

$$A_1 := \sum_{n<0} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{4\pi\sqrt{y} S(-\bar{l},n;c)}{c\sqrt{l}} K_{3/2}(-2\pi n y) J_3\left(\frac{4\pi\sqrt{|n|}}{c\sqrt{l}}\right),$$

$$A_2 := \sum_{n>0} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{4\pi\sqrt{y} S(-\bar{l},n;c)}{c\sqrt{l}} K_{3/2}(2\pi n y) I_3\left(\frac{4\pi\sqrt{n}}{c\sqrt{l}}\right).$$

Then by (2.8) and (2.14), we have

$$\begin{aligned}
P(z) =& \sum_{l|6} \mu(l) \left[\delta_{\infty,\frac{1}{r}} e(-z)\psi(y) + A_0 + A_1 + A_2\right] \\
=& e(-z)\psi(y) + \sum_{l|6} \mu(l) A_0 + \sum_{l|6} \mu(l) A_1 + \sum_{l|6} \mu(l) A_2 \\
=& 2\pi\sqrt{y} e(-x) I_{3/2}(2\pi y) + \frac{4\pi^3}{3} \sum_{l|6} \mu(l) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l},0;c)}{c^4 l^2 y} \\
&+ \sum_{l|6} \mu(l) \sum_{n\neq 0} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{4\pi\sqrt{y} S(-\bar{l},n;c)}{c\sqrt{l}} K_{3/2}(2\pi|n|y) B(n,c,l),
\end{aligned} \tag{2.15}$$

where

$$B(n,c,l) := \begin{cases} J_3\left(\frac{4\pi\sqrt{|n|}}{c\sqrt{l}}\right) & \text{if } n<0 \\ I_3\left(\frac{4\pi\sqrt{n}}{c\sqrt{l}}\right) & \text{if } n>0. \end{cases}$$

Now, using the identity(see e.g. [25, Eq. 10.47.7 and 10.49.9])

$$I_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\cosh(z) - \frac{\sinh(z)}{z}\right) \quad \text{for } z \in \mathbb{C} - (-\infty, 0],$$

we have

$$2\pi\sqrt{y}e(-x)I_{3/2}(2\pi y) = 2\pi\sqrt{y}e(-x)\sqrt{\frac{1}{\pi^2 y}}\left(\frac{e^{2\pi y}+e^{-2\pi y}}{2} - \frac{e^{2\pi y}-e^{-2\pi y}}{4\pi y}\right)$$

Then a short calculation yields

$$2\pi\sqrt{y}e(-x)I_{3/2}(2\pi y) = e(-z) + e(-\bar{z}) - \left(\frac{e(-z)-e(-\bar{z})}{2\pi y}\right). \tag{2.16}$$

Similarly, using the identities (see e.g. [25, Eq. 10.27.4])

$$K_{3/2}(z) = \frac{\pi}{2}\left(I_{3/2}(z) - I_{-3/2}(z)\right) \quad \text{for } z \in \mathbb{C} - (-\infty, 0]$$

and (see e.g. [25, Eqs. 10.47.8 and 10.49.11])

$$I_{-3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\sinh(z) - \frac{\cosh(z)}{z}\right) \quad \text{for } z \in \mathbb{C} - (-\infty, 0],$$

we obtain

$$\begin{aligned} K_{3/2}(2\pi yn) &= \frac{\pi}{2}\left[\frac{1}{\pi\sqrt{yn}}\left(\cosh(2\pi yn) - \frac{\sinh(2\pi yn)}{2\pi yn}\right) - \frac{1}{\pi\sqrt{yn}}\left(\sinh(2\pi yn) - \frac{\cosh(2\pi yn)}{2\pi yn}\right)\right] \\ &= \frac{1}{2\sqrt{yn}}\left(e^{-2\pi yn} + \frac{e^{-2\pi yn}}{2\pi yn}\right). \end{aligned}$$

Hence

$$4\pi\sqrt{y}K_{3/2}(2\pi yn)e(nx) = \frac{2\pi}{\sqrt{n}}\left(e(nz) + \frac{e(nz)}{2\pi yn}\right)$$

and

$$4\pi\sqrt{y}K_{3/2}(2\pi yn)e(-nx) = \frac{2\pi}{\sqrt{n}}\left(e(-n\bar{z}) + \frac{e(-n\bar{z})}{2\pi yn}\right).$$

By combining the preceding calculations, we can write (2.15) as

$$P(z) = P_1(z) + P_2(z), \tag{2.17}$$

where

$$\begin{aligned}
P_1(z) &:= \left(1 - \frac{1}{2\pi y}\right) e(-z) + \frac{b(0)}{y} + \sum_{n=1}^{\infty} \frac{b(n)}{\sqrt{n}} \left(1 + \frac{1}{2\pi yn}\right) e(nz), \\
P_2(z) &:= \left(1 + \frac{1}{2\pi y}\right) e(-\bar{z}) + \sum_{n=1}^{\infty} \frac{b(-n)}{\sqrt{n}} \left(1 + \frac{1}{2\pi yn}\right) e(-n\bar{z}), \\
b(0) &:= \frac{4\pi^3}{3} \sum_{l|6} \mu(l) \sum_{\substack{0<c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l}, 0; c)}{c^4 l^2}, \\
b(n) &:= 2\pi \sum_{l|6} \mu(l) \sum_{\substack{0<c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l}, n; c)}{c\sqrt{l}} J_3\left(\frac{4\pi\sqrt{|n|}}{c\sqrt{l}}\right) \quad \text{if } n < 0, \\
b(n) &:= 2\pi \sum_{l|6} \mu(l) \sum_{\substack{0<c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l}, n; c)}{c\sqrt{l}} I_3\left(\frac{4\pi\sqrt{n}}{c\sqrt{l}}\right) \quad \text{if } n > 0.
\end{aligned}$$

Proposition 2.7. *Let $a(n) \in \mathbb{Z}$ be the Fourier coefficients of the weakly holomorphic modular form $F_p(z)$ defined by (2.1). Then we have $b(n) = -n^{3/2}a(n)$ for $n \geq 1$, $b(0) = 5/\pi, b(-1) = -1$, and $b(-n) = 0$ for $n \geq 2$. In particular, $P_2(z) = 0$.*

Proof. The weakly holomorphic modular form $F_p(z)$ has the Fourier expansion

$$F_p(z) = \sum_{n=-1}^{\infty} a(n)e(nz)$$

for some integers $a(n)$. Then, recalling that the weak Maass form $P(z)$ is defined by

$$P(z) := -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right) F_p(z),$$

we get the Fourier expansion

$$P(z) = \sum_{n=-1}^{\infty} -\left(n + \frac{1}{2\pi y}\right) a(n)e(nz). \tag{2.18}$$

Since the Fourier expansion of $P(z)$ is unique, by comparing the Fourier expansion of $P(z)$ in (2.18) and (2.17), we get $b(-1) = -1$, $b(-n) = 0$ for $n \geq 2$, $b(0) = -a(0)/2\pi = 5/\pi$, and

$$\frac{b(n)}{\sqrt{n}}\left(1 + \frac{1}{2\pi y n}\right) = -\left(n + \frac{1}{2\pi y}\right) a(n)$$

for $n \geq 1$. Take the limit of both sides as $y \to \infty$ to get $b(n) = -n^{3/2}a(n)$ for $n \geq 1$. $\square$

Proposition 2.4 now follows from (2.17) and Proposition 2.7.

2.4 The Fourier expansion of $P(z)$ in the cusps

We now compute the Fourier expansion of $P(z)$ in the cusps of $\Gamma_0(6)$. Recall that $\mathbf{C_6}$ is the set of right coset representatives of $\Gamma_0(6)\backslash SL_2(\mathbb{Z})$ defined by Definition 2.2. First, note that each matrix $\gamma \in \mathbf{C_6}$ maps the cusp ∞ to one of the cusps $\{\infty, 1/3, 1/2, 0\}$ of the modular curve $X_0(6)$ as follows:

matrix $\gamma \in \mathbf{C_6}$	γ_∞	$\gamma_{1/3,u}$	$\gamma_{1/2,v}$	$\gamma_{0,t}$
cusp $\gamma(\infty)$	$\infty \simeq 1/6$	$1/3$	$1/2$	$0 \simeq 1$
width of the cusp $\gamma(\infty)$	1	2	3	6

Table 2.3: The width of the cusps of $\Gamma_0(6)$

Recall that $\mathcal{Q}_D^{\text{red}}$ denote a set of primitive, reduced forms representing the equivalence classes in $\mathcal{Q}_{D,1}^{\text{prim}}/SL_2(\mathbb{Z})$. For each $Q \in \mathcal{Q}_D^{\text{red}}$, there is a unique choice of coset representative $\gamma_Q \in \mathbf{C_6}$ such that

$$[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}_{D,6,1}^{\text{prim}}/\Gamma_0(6).$$

This induces a bijection

$$\mathcal{Q}_D^{\text{red}} \to \mathcal{Q}_{D,6,1}^{\text{prim}}/\Gamma_0(6) \tag{2.19}$$

$$Q \mapsto [Q \circ \gamma_Q^{-1}];$$

see the proposition in [13, p. 505], or more concretely [10, Lemma 3], where an explicit list of the matrices $\gamma_Q \in \mathbf{C_6}$ is given.

We will compute the Fourier expansion of $P(z)$ with respect to the matrices in $\mathbf{C_6}$. Recall that $\{W_l : l|6\}$ is the group of Atkin-Lehner operators defined by (2.3). For each $l|6$ and $r = 6/l$, let $V_l := \sqrt{l}W_l$ and

$$A_l := \begin{pmatrix} \frac{1}{\text{width of the cusp } 1/r} & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

cusp $1/r$	$\infty \simeq 1/6$	$1/3$	$1/2$	$0 \simeq 1$
l	1	2	3	6
V_l	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$
A_l	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/6 & 0 \\ 0 & 1 \end{pmatrix}$
$V_l A_l$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Table 2.4: The matrices V_l and A_l

Let $\gamma \in SL_2(\mathbb{Z})$ be any matrix such that $\gamma(\infty) = 1/r$. Observe from Table 2.4 that $V_l A_l \in SL_2(\mathbb{Z})$ and $V_l A_l$ maps ∞ to the cusp $1/r$. Hence

$$(V_l A_l)^{-1} \gamma(\infty) = \infty,$$

so that $(V_l A_l)^{-1} \gamma \in SL_2(\mathbb{Z})_\infty$ stabilizes the cusp ∞. In particular, there is an integer $n \in \mathbb{Z}$ such that

$$\gamma = V_l A_l \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \tag{2.20}$$

Since $\gamma_\infty, \gamma_{1/3,u}, \gamma_{1/2,v}$, and $\gamma_{0,t}$ map the cusp ∞ to the cusps $\infty, 1/3, 1/2$, and 0, respectively, it follows that these matrices satisfy (2.20). By solving for n in each case, we get

$$\begin{aligned}
\gamma_\infty &= V_1 A_1, \\
\gamma_{1/3,u} &= V_2 A_2 \begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix}, \\
\gamma_{1/2,v} &= V_3 A_3 \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \\
\gamma_{0,t} &= V_6 A_6 \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

A calculation using (2.5) and the group law on $\{W_l : l|6\}$ shows that

$$P(W_l z) = \lambda_l P(z), \tag{2.21}$$

where the eigenvalues $\lambda_l = 1$ for $l = 1, 6$ and $\lambda_l = -1$ for $l = 2, 3$. Thus we obtain

$$\begin{aligned}
P|_0\gamma_\infty(z) &= P(z), \\
P|_0\gamma_{1/3,u}(z) &= P(\gamma_{1/3,u}z) = P\left(V_2 A_2 \begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix} z \right) = -P\left(\frac{z+u+1}{2}\right), \\
P|_0\gamma_{1/2,v}(z) &= P(\gamma_{1/2,v}z) = P\left(V_3 A_3 \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} z \right) = -P\left(\frac{z+v}{3}\right), \\
P|_0\gamma_{0,t}(z) &= P(\gamma_{0,t}z) = P\left(V_6 A_6 \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} z \right) = P\left(\frac{z+t}{6}\right).
\end{aligned}$$

Let $\zeta_6 = e(1/6)$ be a primitive sixth root of unity. Then by Proposition 2.4 the Fourier expansion of P with respect to the matrices $\gamma_{1/3,u}, \gamma_{1/2,v}, \gamma_{0,t}$ is

$$P|_0\gamma_{1/3,u}(z) = \zeta_6^{3u}\left(1 - \frac{2}{2\pi y}\right) e\left(\frac{-z}{2}\right) - \frac{2\times 5}{\pi y} + \sum_{n=1}^{\infty} \frac{\zeta_6^{3+3n(u+1)} b(n)}{\sqrt{n}} \left(1 + \frac{2}{2\pi y n}\right) e\left(\frac{nz}{2}\right)$$
$$P|_0\gamma_{1/2,v}(z) = \zeta_6^{3-2v}\left(1 - \frac{3}{2\pi y}\right) e\left(\frac{-z}{3}\right) - \frac{3\times 5}{\pi y} + \sum_{n=1}^{\infty} \frac{\zeta_6^{3+2nv} b(n)}{\sqrt{n}} \left(1 + \frac{3}{2\pi y n}\right) e\left(\frac{nz}{3}\right)$$
$$P|_0\gamma_{0,t}(z) = \zeta_6^{-t}\left(1 - \frac{6}{2\pi y}\right) e\left(\frac{-z}{6}\right) + \frac{6\times 5}{\pi y} + \sum_{n=1}^{\infty} \frac{\zeta_6^{nt} b(n)}{\sqrt{n}} \left(1 + \frac{6}{2\pi y n}\right) e\left(\frac{nz}{6}\right).$$

More generally, given a form $Q \in \mathcal{Q}_D^{\text{red}}$ and corresponding coset representative $\gamma_Q \in \mathbf{C}_6$, let h_Q be the width of the cusp $\gamma_Q(\infty)$ and let $\zeta_Q, \phi_{n,Q}$ be sixth roots of unity defined as follows:

coset representative γ_Q	γ_∞	$\gamma_{\frac{1}{3},u}$	$\gamma_{\frac{1}{2},v}$	$\gamma_{0,t}$
cusp $\gamma_Q(\infty)$	$\infty \simeq 1/6$	$1/3$	$1/2$	$0 \simeq 1$
ζ_Q	1	${\zeta_6}^{3u}$	${\zeta_6}^{3-2v}$	${\zeta_6}^{-t}$
$\phi_{n,Q}$	1	${\zeta_6}^{3+3n(u+1)}$	${\zeta_6}^{3+2nv}$	${\zeta_6}^{nt}$

Table 2.5: Sixth roots of unity assigned to γ_Q

Then

$$P(\gamma_Q z) = \zeta_Q\left(1 - \frac{h_Q}{2\pi y}\right) e\left(\frac{-z}{h_Q}\right) + \frac{5\mu(h_Q)h_Q}{\pi y} + \sum_{n=1}^{\infty} \frac{b(n)\phi_{n,Q}}{\sqrt{n}} \left(1 + \frac{h_Q}{2\pi y n}\right) e\left(\frac{nz}{h_Q}\right). \tag{2.22}$$

2.5 Effective bounds for the number $b(n)$

In this subsection, we give effective bounds for the number $b(n)$ defined in Proposition 2.4.

Lemma 2.8. *For $n \geq 1$, let $b(n)$ be defined as in Proposition 2.4. Then*

$$|b(n)| \leq n^{1/2}\left(c_1 e^{4\pi\sqrt{n}} + \frac{c_1}{e} + c_2\right),$$

where

$$c_1 := 192\sqrt{6}\pi^{3/2} \quad \textit{and} \quad c_2 := \frac{4\sqrt{3}}{9}\pi^3.$$

Proof. We first establish some bounds we will need for $I_3(x)$.

If $0 < x < 1$, we use the series expansion (see [25, Eq. 10.25.2])

$$I_3(x) = \left(\frac{x}{2}\right)^3 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}k!(k+3)!}$$

to get

$$|I_3(x)| \leq \frac{x^3}{36}. \tag{2.23}$$

Next, assume that $x \geq 1$. Using the identity [25, Eq. 10.34.5] with $m = 1$, we have

$$I_3(x) \;=\; \frac{i}{\pi}K_3(-x) + \frac{i}{\pi}K_3(x),$$

which yields

$$|I_3(x)| \;\leq\; \frac{1}{\pi}\left(|K_3(-x)| + |K_3(x)|\right). \tag{2.24}$$

Now, by [24, Eq. (14)], we have the identities

$$K_3(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(\sum_{j=0}^{2} \frac{a_j(3)}{x^j} + R_3^{(K)}(x,3) \right),$$
$$K_3(-x) = -i\sqrt{\frac{\pi}{2x}} e^{x} \left(\sum_{j=0}^{2} \frac{(-1)^j a_j(3)}{x^j} + R_3^{(K)}(-x,3) \right)$$

where

$$a_j(3) := \frac{(-1)^{j+1}\Gamma\left(j+\frac{7}{2}\right)\Gamma\left(j-\frac{5}{2}\right)}{\pi 2^j \Gamma(j+1)}, \quad j \geq 0$$

and $R_3^{(K)}(\pm x, 3)$ is a remainder function which we estimates below.

First, we compute

$$a_0(3) = 1, \quad a_1(3) = \frac{35}{8}, \quad a_2(3) = \frac{945}{128}, \quad a_3(3) = \frac{3465}{1024}.$$

Also, by [24, Theorem 1.5, Eq. (26)] and [24, Proposition B.1, Eq. (96)], we have

$$\begin{aligned} \left| R_3^{(K)}(\pm x, 3) \right| &\leq |a_3(3)| \sup_{r \geq 1} |\Lambda_3(\pm 2xr)| \\ &\leq \frac{3465}{1024}\sqrt{\frac{7e}{2}} \\ &\leq 11. \end{aligned}$$

Then

$$\begin{aligned} |K_3(\pm x)| &\leq \sqrt{\frac{\pi}{2x}} e^{\mp x} \left(a_0(3) + \frac{a_1(3)}{x} + \frac{a_2(3)}{x^2} + \left| R_3^{(K)}(\pm x, 3) \right| \right) \\ &\leq \sqrt{\frac{\pi}{2x}} e^{\mp x} \left(1 + \frac{35}{8} + \frac{945}{128} + 11 \right) \\ &\leq 24\sqrt{\frac{\pi}{2x}} e^{\mp x}. \end{aligned}$$

After applying the preceding bound in (2.24), we get

$$|I_3(x)| \leq \frac{24}{\sqrt{2\pi x}} \left(e^x + e^{-x}\right). \tag{2.25}$$

We now estimate $|b(n)|$ for $n \geq 1$. Recall that

$$b(n) = 2\pi \sum_{l|6} \mu(l) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{S(-\bar{l}, n; c)}{c\sqrt{l}} I_3\left(\frac{4\pi\sqrt{n}}{c\sqrt{l}}\right).$$

Let $M := \frac{4\pi\sqrt{n}}{\sqrt{l}}$. Then we have

$$|b(n)| \leq 2\pi \sum_{l|6} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{6/l} \\ (c,l)=1}} \frac{\left|S(-\bar{l}, n; c)\right|}{c\sqrt{l}} \left|I_3\left(\frac{M}{c}\right)\right|.$$

Using the Weil bound :

$$|S(a, b; c)| \leq \tau(c)(a, b, c)^{1/2} c^{1/2},$$

where $\tau(c)$ is the *divisor function*, which counts the number of positive divisors of c, and the fact that $(\bar{l}, n, c) = 1$ for $(c, l) = 1$ (see Table 2.2), we get

$$|S(-\bar{l}, n; c)| \leq \tau(c) c^{1/2} \qquad \text{for } (c, l) = 1. \tag{2.26}$$

Then applying the following effective bound for the divisor function (see [9, p. 12])

$$\tau(c) \leq \sqrt{3} c^{1/2}$$

yields

$$\frac{|S(-\bar{l}, n, ; c)|}{c\sqrt{l}} \leq \frac{\sqrt{3}}{\sqrt{l}} \qquad \text{for } (c, l) = 1. \tag{2.27}$$

Using the bound (2.27), we get

$$|b(n)| \leq 2\pi \sum_{l|6} \sum_{0<c\leq M} \frac{\sqrt{3}}{\sqrt{l}} \left| I_3\left(\frac{M}{c}\right)\right| + 2\pi \sum_{l|6} \sum_{c>M} \frac{\sqrt{3}}{\sqrt{l}} \left| I_3\left(\frac{M}{c}\right)\right|.$$

If $0 < c \leq M$, we use (2.25) to obtain

$$\begin{aligned}\left| I_3\left(\frac{M}{c}\right)\right| &\leq \frac{24}{\sqrt{2\pi M/c}} \left(e^{M/c} + e^{-M/c}\right) \\ &\leq \frac{12c^{1/2}l^{1/4}}{\sqrt{2}\pi n^{1/4}} \left(e^{\frac{4\pi\sqrt{n}}{c\sqrt{l}}} + e^{-1}\right) \\ &\leq \frac{12c^{1/2}l^{1/4}}{\sqrt{2}\pi n^{1/4}} \left(e^{4\pi\sqrt{n}} + e^{-1}\right).\end{aligned}$$

On the other hand, if $c > M$, we use (2.23) to obtain

$$\left| I_3\left(\frac{M}{c}\right)\right| \leq \frac{1}{36}\left(\frac{M}{c}\right)^3 \leq \frac{16\pi^3 n^{3/2}}{9c^3 l^{3/2}}.$$

Hence

$$|b(n)| \leq 12\sqrt{6} n^{-1/4} \sum_{l|6} l^{-1/4} \sum_{0<c\leq M} c^{1/2} \left(e^{4\pi\sqrt{n}} \; + \; \frac{1}{e}\right) + \frac{32\sqrt{3}}{9}\pi^4 n^{3/2} \sum_{l|6} l^{-2} \sum_{c>M} c^{-3}.$$

Now, observe that

$$\begin{aligned}\sum_{0<c\leq M} c^{1/2}\left(e^{4\pi\sqrt{n}} + \frac{1}{e}\right) &\leq M^{3/2}\left(e^{4\pi\sqrt{n}} + \frac{1}{e}\right) \\ &\leq \frac{8\pi^{3/2} n^{3/4}}{l^{3/4}} \left(e^{4\pi\sqrt{n}} + \frac{1}{e}\right).\end{aligned}$$

Also, we have

$$\sum_{c>M} c^{-3} \leq \int_{M-1}^{\infty} c^{-3} dc = \frac{1}{2(M-1)^2} = \frac{1}{2}\left(\frac{l}{16\pi^2 n - 8\pi\sqrt{nl} + l}\right).$$

Since $0 < l \leq 6$ and $\sqrt{n} \geq 1 \geq \frac{\sqrt{6}}{2\pi - 1}$, it follows that

$$\begin{aligned} 16\pi^2 n - 8\pi\sqrt{nl} + l &\geq 16\pi^2 n - 8\pi\sqrt{6n} \\ &= 8\pi\sqrt{n}(2\pi\sqrt{n} - \sqrt{6}) \\ &\geq 8\pi n. \end{aligned}$$

Hence

$$\sum_{c>M} c^{-3} \leq \frac{l}{16\pi n}. \tag{2.28}$$

Finally, applying these estimate yields

$$\begin{aligned} |b(n)| &\leq 96\sqrt{6}\pi^{3/2} n^{1/2}\left(e^{4\pi\sqrt{n}} + \frac{1}{e}\right)\sum_{l|6} l^{-1} + \frac{2\sqrt{3}}{9}\pi^3 n^{1/2}\sum_{l|6} l^{-1} \\ &= 192\sqrt{6}\pi^{3/2} n^{1/2}\left(e^{4\pi\sqrt{n}} + \frac{1}{e}\right) + \frac{4\sqrt{3}}{9}\pi^3 n^{1/2} \\ &= n^{1/2}\left(c_1 e^{4\pi\sqrt{n}} + \frac{c_1}{e} + c_2\right). \end{aligned}$$

□

2.6 Proof of Theorem 2.3

In this subsection, we give an asymptotic formula for the trace of the function $P(z)$ with an effective bound on the error term. Then, we combine it with Theorem 2.1 to prove Theorem 2.3.

For $n \geq 1$, Let $D_n := -24n + 1$ and define the trace of $P(z)$ by

$$S(n) := \sum_{[Q] \in \mathcal{Q}_{D_n,6,1}/\Gamma_0(6)} P(\tau_Q).$$

First, we decompose $S(n)$ as a linear combination of traces involving primitive forms. Let $D < 0$ be any discriminant with $D \equiv 1 \pmod{24}$ and define the class polynomials

$$H_n(X) := \prod_{[Q] \in \mathcal{Q}_{D_n,6,1}/\Gamma_0(6)} (X - P(\tau_Q))$$

and

$$\hat{H}_D(X) := \prod_{[Q] \in \mathcal{Q}^{\text{prim}}_{D,6,1}/\Gamma_0(6)} (X - P(\tau_Q))\,. \tag{2.29}$$

By (2.21), we have $P|_0 W_l(z) = P(z)$ for $l = 1, 6$ and $P|_0 W_l(z) = -P(z)$ for $l = 2, 3$. Then, using the same argument as the proof in [7, Lemma 3.7], we can express $H_n(X)$ as a product of class polynomials in (2.29) as follows:

$$H_n(X) = \prod_{\substack{u>0 \\ d^2 | D_n}} \epsilon(d)^{h\left(D_n/d^2\right)} \hat{H}_{D_n/d^2}\left(\epsilon(d) X\right), \tag{2.30}$$

where $\epsilon(d) = 1$ if $d = \pm 1 \pmod{12}$ and $\epsilon(d) = -1$ otherwise. Comparing terms on both sides of (2.30) yields the class number relation

$$H(D_n) = \sum_{\substack{d>0 \\ d^2 | D_n}} h\left(D_n/d^2\right) \tag{2.31}$$

and the decomposition

$$S(n) = \sum_{\substack{d>0 \\ d^2 | D_n}} \epsilon(d) \sum_{[Q] \in \mathcal{Q}^{\text{prim}}_{D_n/d^2,6,1}/\Gamma_0(6)} P(\tau_Q).$$

Now, using the bijection (2.19) and the compatibility relation (1.3) we have

$$\sum_{[Q]\in\mathcal{Q}^{\text{prim}}_{D_n/d^2,6,1}/\Gamma_0(6)} P(\tau_Q) = \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} P(\gamma_Q(\tau_Q)).$$

It follows that

$$S(n) = \sum_{\substack{d>0\\ d^2|D_n}} \epsilon(d) \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} P(\gamma_Q(\tau_Q)).$$

Then using (2.22), we get

$$S(n) = \mathrm{MT(n)} + \mathrm{ET(n)},$$

where the main term is given by

$$\mathrm{MT}(n) := \sum_{\substack{d>0\\ d^2|D_n}} \epsilon(d) \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} \zeta_Q \left(1 - \frac{h_Q}{2\pi y_Q}\right) e\left(\frac{-\tau_Q}{h_Q}\right)$$

with $y_Q := \mathrm{Im}(\tau_Q)$, and the error term is given by

$$\mathrm{ET}(n) := \beta_1 + \beta_2,$$

where

$$\beta_1 := \frac{5}{\pi} \sum_{\substack{d>0\\ d^2|D_n}} \epsilon(d) \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} \mu(h_Q)\frac{h_Q}{y_Q},$$

$$\beta_2 := \sum_{m=1}^{\infty} \sum_{\substack{d>0\\ d^2|D_n}} \epsilon(d) \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} C_{m,Q}$$

with

$$C_{m,Q} := \phi_{m,Q} \frac{b(m)}{\sqrt{m}} \left(1 + \frac{h_Q}{2\pi m y_Q}\right) e\left(\frac{m\tau_Q}{h_Q}\right).$$

First, we estimate $|\beta_2|$. Since $Q \in \mathcal{Q}^{\text{red}}_{D_n/d^2}$ is reduced, the corresponding Heegner point

$$\tau_Q = \frac{-b_Q + \sqrt{D_n/d^2}}{2a_Q}$$

lies in the standard fundamental domain for $SL_2(\mathbb{Z})$. Hence

$$\frac{\sqrt{|D_n|/d^2}}{2a_Q} = \text{Im}(\tau_Q) \geq \frac{\sqrt{3}}{2},$$

which implies that

$$\frac{m\pi\sqrt{|D_n|/d^2}}{h_Q a_Q} \geq \frac{m\pi\sqrt{3}}{h_Q}.$$

Then, since $h_Q \leq 6$, we have

$$\frac{-m\pi\sqrt{|D_n|/d^2}}{h_Q a_Q} \leq \frac{-m\pi\sqrt{3}}{h_Q} \leq \frac{-m\pi}{2\sqrt{3}},$$

so that

$$\left|e\left(\frac{m\tau_Q}{h_Q}\right)\right| = \exp\left(\frac{-m\pi\sqrt{|D_n|/d^2}}{h_Q a_Q}\right) \leq \exp\left(\frac{-m\pi}{2\sqrt{3}}\right).$$

Using this estimate, we get

$$|\beta_2| \leq \sum_{m=1}^{\infty} |b(m)| \exp\left(\frac{-m\pi}{2\sqrt{3}}\right) \sum_{\substack{d>0 \\ d^2|D_n}} \sum_{Q\in\mathcal{Q}^{\text{red}}_{D_n/d^2}} \left(\frac{1}{\sqrt{m}} + \frac{h_Q}{2\pi m^{3/2} y_Q}\right).$$

Now, since $h_Q \leq 6$ and $y_Q \geq \sqrt{3}/2$, we have

$$\frac{1}{\sqrt{m}} + \frac{h_Q}{2\pi m^{3/2} y_Q} \leq \frac{1}{\sqrt{m}} + \frac{2\sqrt{3}}{\pi m^{3/2}}.$$

Hence using (2.31), we get

$$|\beta_2| \leq H(D_n) \sum_{m=1}^{\infty} |b(m)| \exp\left(\frac{-m\pi}{2\sqrt{3}}\right)\left(\frac{1}{\sqrt{m}} + \frac{2\sqrt{3}}{\pi m^{3/2}}\right).$$

Finally, inserting the bound for $b(m)$ from Lemma 2.8 yields

$$|\beta_2| \leq H(D_n)\left[c_1 S_1 + \left(\frac{c_1}{e} + c_2\right) S_2\right],$$

where

$$S_1 := \sum_{m=1}^{\infty} \left(1 + \frac{2\sqrt{3}}{\pi m}\right) \exp\left(4\pi\sqrt{m} - \frac{m\pi}{2\sqrt{3}}\right) \leq 2.14 \times 10^{20}$$
$$S_2 := \sum_{m=1}^{\infty} \left(1 + \frac{2\sqrt{3}}{\pi m}\right) \exp\left(\frac{-m\pi}{2\sqrt{3}}\right) \leq 1.5.$$

Thus

$$\begin{aligned}|\beta_2| &\leq H(D_n)\left(\left(2.14 \times 10^{20}\right) 192\sqrt{6}\pi^{3/2} + (1.5)\left(\frac{192\sqrt{6}\pi^{3/2}}{e} + \frac{4\sqrt{3}}{9}\pi^3\right)\right) \\ &\leq \left(5.6043 \times 10^{23}\right) H(D_n).\end{aligned}$$

Next, we estimate $|\beta_1|$. We have

$$|\beta_1| \leq \frac{5}{\pi} \sum_{\substack{d>0 \\ d^2 | D_n}} \sum_{Q \in \mathcal{Q}^{\text{red}}_{D_n/d^2}} \frac{h_Q}{|y_Q|}.$$

Again, using $h_Q \le 6$ and $y_Q \ge \sqrt{3}/2$, we get

$$\begin{aligned}|\beta_1| &\le \frac{5}{\pi} \sum_{\substack{d>0 \\ d^2 | D_n}} \sum_{Q \in \mathcal{Q}^{\mathrm{red}}_{D_n/d^2}} \frac{12}{\sqrt{3}} \\ &\le H(D_n) \frac{20\sqrt{3}}{\pi}.\end{aligned}$$

We have now shown that

$$\begin{aligned}\mathrm{ET}(n) &\le \left(5.6043 \times 10^{23} + \frac{20\sqrt{3}}{\pi}\right) H(D_n) \\ &\le \left(5.6044 \times 10^{23}\right) H(D_n),\end{aligned}$$

To complete the proof, we combine these results with Theorem 2.1 to get

$$p(n) = M(n) + E(n),$$

where

$$M(n) := \frac{\mathrm{MT}(n)}{24n-1}$$

and

$$E(n) := \frac{\mathrm{ET}(n)}{24n-1} \le \left(5.6044 \times 10^{23}\right) \frac{H(D_n)}{24n-1}.$$

3. ASYMPTOTIC DISTRIBUTION WITH POWER SAVING ERROR TERMS

Throughout this section, we assume that N is a squarefree positive integer and the discriminant D satisfies the following Heegner hypothesis:

- $D < -4$ is an odd fundamental discriminant coprime to N such that every prime divisor of N splits in $\mathbb{Q}(\sqrt{D})$.

The aim of this section is to give the asymptotic formula with a power-saving error term for the trace of $\mathcal{D}^k f$ when $f \in M^!_{-2k}(N)$ is an eigenfunction for the Atkin-Lehner operators of level N. We then apply the result to give asymptotic formulas with power-saving error terms for the partition function $p(n)$, Andrews' smallest parts function spt(n), and the coefficients $\alpha(n)$ of Ramanujan's $f(q)$ mock theta function.

3.1 Asymptotic formula with a power saving error terms for the trace

Recall that the trace of a $\Gamma_0(N)$-invariant function $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$\mathrm{Tr}_D(f) := \sum_{Q \in \mathcal{Q}_{D,N,\rho}/\Gamma_0(N)} f(\tau_Q).$$

Since D is an odd fundamental discriminant, D is squarefree. Because any quadratic form of squarefree discriminant is primitive, we have

$$\mathcal{Q}_{D,N,\rho} = \mathcal{Q}^{\text{prim}}_{D,N,\rho}$$

and hence

$$\mathrm{Tr}_D(f) = \sum_{Q \in \mathcal{Q}^{\text{prim}}_{D,N,\rho}/\Gamma_0(N)} f(\tau_Q).$$

Next, we fix a distinguished set of right coset representatives of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$.

Definition 3.1. *Let $l|N$. For each $t \in \{0, 1, \dots, l-1\}$, we define*

$$\gamma_{\frac{l}{N},t} := \begin{pmatrix} 1 & 0 \\ \frac{N}{l} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ \frac{N}{l} & \frac{N}{l}t+1 \end{pmatrix}. \tag{3.1}$$

Let

$$\mathcal{C}_N := \left\{ \gamma_{\frac{l}{N},t} : l|N,\ t = 0, 1, \dots, l-1 \right\}$$

denote the set of all matrices defined by (3.1).

Then $\mathcal{C}_N$ is a set of right coset representatives of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ (see Proposition 3.8).

Given $Q \in \mathcal{Q}_D^{\text{red}}$, there is a unique coset representative $\gamma_Q \in \mathcal{C}_N$ such that $[Q \circ \gamma_Q^{-1}]_N \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N)$. This gives a bijection

$$\begin{aligned} \mathcal{Q}_D^{\text{red}} &\longrightarrow \mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N) \\ Q &\longmapsto [Q \circ \gamma_Q^{-1}]_N \end{aligned} \tag{3.2}$$

(see Section 3.4). It follows from (3.2) and (1.3) that

$$\text{Tr}_D(f) = \sum_{Q \in \mathcal{Q}_D^{\text{red}}} f(\tau_{Q \circ \gamma_Q^{-1}}) = \sum_{Q \in \mathcal{Q}_D^{\text{red}}} f(\gamma_Q \tau_Q).$$

Finally, define

$$S_{l,D,N} := \bigsqcup_{t \in \{0,1,\dots,l-1\}} S_{l,t,D,N}$$

where

$$S_{l,t,D,N} := \{Q \in \mathcal{Q}_D^{\text{red}} : \gamma_Q = \gamma_{\frac{l}{N},t}\}.$$

Then

$$\mathcal{Q}_D^{\mathrm{red}} = \bigsqcup_{l|N} S_{l,D,N}$$

so that

$$\mathrm{Tr}_D(f) = \sum_{Q\in\mathcal{Q}_D^{\mathrm{red}}} f(\gamma_Q\tau_Q) = \sum_{l|N}\sum_{t=0}^{l-1}\sum_{Q\in S_{l,tD,N}} f(\gamma_Q\tau_Q).$$

We are now in a position to state our main result, which is the following asymptotic formula with a power saving error term.

Theorem 3.2. *Let $k \geq 0$ be an integer. Suppose that $f \in M^!_{-2k}(N)$ has the Fourier expansion*

$$f(z) = \sum_{m=0}^{N_\infty} a(-m)q^{-m} + \sum_{m=1}^{\infty} a(m)q^m, \quad q := e(z) = e^{2\pi iz}$$

and that f is an eigenfunction for the Atkin-Lehner operators W_l defined in (3.6)*, i.e.*

$$f|_{-2k}W_l(z) = \lambda(l)f(z), \tag{3.3}$$

where the eigenvalues $\lambda(l)$ satisfy the following properties:

(P.1) $\lambda(l) \in \{1,-1\}$ *for any integer $l|N$,*

(P.2) $\lambda\left(\frac{l_1l_2}{(l_1,l_2)^2}\right) = \lambda(l_1)\lambda(l_2)$ *for $l_1|N, l_2|N$.*

Then

$$\mathrm{Tr}_D(\mathcal{D}^k f) = \sum_{l|N}\lambda(l)\sum_{t=0}^{l-1}\sum_{\substack{Q\in S_{l,t,D,N}\\ y_Q>l\left(\frac{2N}{\sqrt{3}}+|D|^{-\frac{1}{80}}\right)}}\sum_{m=0}^{N_\infty} a(-m)c_k\left(m,\frac{y_Q}{l}\right)e\left(\frac{-m\tau_Q}{l}\right)\zeta_N^{-m\left(\frac{N}{l}t+1-l\tilde{l}\right)}$$
$$+ h(D)\beta_k(f) + O_{\epsilon,N}(|D|^{\frac{1}{2}-\frac{1}{80}+\epsilon})$$

as $|D| \to \infty$*. Here*

$$c_k(m,y) := \sum_{j=0}^{k} \frac{(-1)^j (k+j)! m^{k-j}}{(4\pi y)^j j!(k-j)!},$$

$\zeta_N := e(1/N)$ *is a primitive* N*-th root of unity,* $\tilde{l}$ *is any integer that satisfies* $\tilde{l}l \equiv 1 \pmod{N/l}$*, and*

$$\beta_k(f) := \int_{\text{reg}} \mathcal{D}^k f(z) d\mu$$

is the regularized integral defined in Section 3.9*.*

Remark 3.3. Note that if a multiplicative function $\lambda : \mathbb{N} \to \mathbb{R}$ satisfies (P.1), then it automatically satisfies (P.2).

3.2 Applications and Discussion

3.2.1 Applications of Theorem 3.2

We illustrate the utility of Theorem 3.2 by applying it to give asymptotic formulas with power saving error terms for some important arithmetic functions.

We will establish the following asymptotic formula for $p(n)$.

Theorem 3.4. *As* $n \to \infty$ *through a sequence of integers such that* $D_n := -24n + 1$ *is squarefree, we have*

$$p(n) = \frac{1}{24n-1} \sum_{l|6} \mu(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \left[\frac{5l}{\pi y_Q} + \left(1 - \frac{l}{2\pi y_Q}\right) e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-(\frac{6}{l}t+1-l\tilde{l})}\right] + O_\epsilon(n^{-\frac{41}{80}+\epsilon})$$

where μ *is the Möbius function.*

Similarly, we will establish the following asymptotic formula for $\text{spt}(n)$.

Theorem 3.5. *As $n \to \infty$ through a sequence of integers such that $D_n := -24n+1$ is squarefree, we have*

$$\operatorname{spt}(n) = \sum_{l|6} \mu(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \left[1 - \frac{5l}{12\pi y_Q} + \frac{l}{24\pi y_Q} \zeta_6^{-(\frac{6}{l}t+1-l\tilde{l})} e\left(\frac{-\tau_Q}{l}\right)\right] + O_\epsilon(n^{\frac{1}{2}-\frac{1}{80}+\epsilon}).$$

Next, recall that Ramanujan's $f(q)$ mock theta function is defined by the q-series

$$f(q) := 1 + \sum_{n=1}^{\infty} \alpha(n) q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

We will establish the following asymptotic formula for the coefficients $\alpha(n)$.

Theorem 3.6. *As $n \to \infty$ through a sequence of integers such that $D_n := -24n+1$ is squarefree, we have*

$$\alpha(n) = \frac{-1}{\sqrt{24n-1}} \sum_{l|6} \beta(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} e^{2\pi y_Q/l} \sin\left(-2\pi\left(\frac{x_Q}{l} + \frac{\frac{6}{l}t+1-l\tilde{l}}{6}\right)\right) + O_\epsilon(n^{-\frac{1}{80}+\epsilon})$$

where $\beta(l) = 1$ if $l = 1, 2$ and $\beta(l) = -1$ if $l = 3, 6$.

Recall that the rank of a partition is the largest part minus the total number of parts. The coefficients $\alpha(n)$ encode important information about the distribution of partition ranks. Namely, let $N_e(n)$ and $N_o(n)$ denote the number of partitions of n with even and odd rank, respectively. It is well known that

$$N_e(n) = \frac{p(n)+\alpha(n)}{2} \quad \text{and} \quad N_o(n) = \frac{p(n)-\alpha(n)}{2}.$$

By combining Theorem 3.4 and 3.6, we immediately obtain the following asymptotic formulas for $N_e(n)$ and $N_o(n)$.

Corollary 3.7. *As $n \to \infty$ through a sequence of integers such that $D_n := -24n+1$ is squarefree,* *we have*

$$N_e(n) = M^{+}_{p,\alpha}(n) + O_\epsilon(n^{-\frac{1}{80}+\epsilon}),$$
$$N_o(n) = M^{-}_{p,\alpha}(n) + O_\epsilon(n^{-\frac{1}{80}+\epsilon})$$

where

$$M^{+}_{p,\alpha}(n) := \sum_{l|6} \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \frac{A_p(l,t,\tau_Q,n) + A_\alpha(l,t,\tau_Q,n)}{2},$$
$$M^{-}_{p,\alpha}(n) := \sum_{l|6} \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \frac{A_p(l,t,\tau_Q,n) - A_\alpha(l,t,\tau_Q,n)}{2},$$
$$A_p(l,t,\tau_Q,n) := \frac{\mu(l)}{24n-1}\left[\frac{5l}{\pi y_Q} + \left(1 - \frac{l}{2\pi y_Q}\right) e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-(\frac{6}{l}t+1-l\tilde{l})}\right],$$
$$A_\alpha(l,t,\tau_Q,n) := \frac{-\beta(l)}{\sqrt{24n-1}}\left[e^{2\pi y_Q/l} \sin\left(-2\pi\left(\frac{x_Q}{l} + \frac{\frac{6}{l}t+1-l\tilde{l}}{6}\right)\right)\right].$$

3.2.2 Discussion

Here we compare our results to some previous work.

1. Building on work of Rademacher [26], Lehmer [20] gave an asymptotic formula for $p(n)$ with an error term which is $\ll n^{-1/2+\varepsilon}$. Folsom and Masri [11] gave an asymptotic formula for $p(n)$ with an error term which is $\ll n^{-(1/2+\delta)}$ for some $\delta > 0$. By employing the Bruinier/Ono [5] formula for $p(n)$, this result was refined by Masri in [22, Theorem 1.6]. Theorem 3.4 refines the main term and sharpens the exponent in the error term of [22, Theorem 1.6].

2. In [4], Bringmann gave an asymptotic formula for spt(n) with an error term which is $\ll n^{1+\varepsilon}$. Using the Ahlgren/Andersen [1] formula for spt(n), Banks, Barquero-Sanchez, Sheng and Masri [3, Theorem 1.3] gave an asymptotic formula for spt(n) with an error term which is $\ll n^{1/2-\delta}$ for some $\delta > 0$. Theorem 3.5 refines the main term and sharpens the exponent in the error term of [3, Theorem 1.3].

3. In [23, Theorem 1.3], Masri gave an asymptotic formula for $\alpha(n)$ with a power saving error term. An asymptotic formula with a power saving error term was given by Ahlgren and Dunn [2] using different methods. Theorem 3.6 refines the main term and sharpens the exponent in the error term of [23, Theorem 1.3]. An important new input here is the Bruinier/Schwagenscheidt [8] formula for $\alpha(n)$.

3.3 Atkin-Lehner operators for $\Gamma_0(N)$

In this subsection we make a choice of Atkin-Lehner operators for $\Gamma_0(N)$ and find a relation between the coset representatives in $\mathcal{C}_N$ and our choice of Atkin-Lehner operators.

First, we verify the following.

Proposition 3.8. *$\mathcal{C}_N$ is a set of right coset representatives of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$.*

Proof. Recall that the group $\Gamma_0(N)$ has index

$$[SL_2(\mathbb{Z}):\Gamma_0(N)] = N\prod_{p|N}(1+1/p).$$

Since N is squarefree,

$$N\prod_{p|N}(1+1/p) = \sum_{d|N} d = |\mathcal{C}_N|.$$

So it suffices to show that two different matrices in $\mathcal{C}_N$ are not in the same coset. For convenience, for each $l|N$, let $r := N/l$. Then each matrix $\gamma_{\frac{l}{N},t} \in \mathcal{C}_N$ can be written as

$$\gamma_{\frac{l}{N},t} = \gamma_{\frac{1}{r},t} = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ r & rt+1 \end{pmatrix}, \quad t = 0, 1, \cdots, N/r-1. \tag{3.4}$$

Observe that for $r_1|N$ and $r_2|N$, the matrices

$$\gamma_{\frac{1}{r_1},m_1} := \begin{pmatrix} 1 & m_1 \\ r_1 & r_1m_1+1 \end{pmatrix}, \quad 0 \le m_1 \le N/r_1 - 1$$

and

$$\gamma_{\frac{1}{r_2},m_2} := \begin{pmatrix} 1 & m_2 \\ r_2 & r_2m_2+1 \end{pmatrix}, \quad 0 \le m_2 \le N/r_2 - 1$$

are in the same coset if there is $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ such that

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 & m_1 \\ r_1 & r_1m_1+1 \end{pmatrix} = \begin{pmatrix} 1 & m_2 \\ r_2 & r_2m_2+1 \end{pmatrix}.$$

Using this identity and the fact that $ad - bcN = 1$, we can solve for a, b, c, d. We have

$$\begin{aligned} a &= 1 - r_1(m_2 - m_1), \\ b &= m_2 - m_1, \\ d &= r_2(m_2 - m_1) + 1, \\ cN &= r_2 - r_1 - r_1r_2(m_2 - m_1). \end{aligned} \tag{3.5}$$

Since $r_1|N$ and $r_2|N$, it follows by (3.5) that $r_1 = r_2$. So $c\frac{N}{r_1} = -r_1(m_2 - m_1)$. Since N is squarefree, r_1 and N/r_1 are coprime, hence $m_1 \equiv m_2 \pmod{N/r_1}$. Since $0 \le m_1, m_2 \le N/r_1 - 1$, it follows that $m_1 = m_2$. □

Now, we make a choice of Atkin-Lehner operators. Define the matrices

$$\tau_r := \begin{pmatrix} 1 & (\tilde{l}l - 1)/r \\ r & \tilde{l}l \end{pmatrix}, \quad \nu_l := \begin{pmatrix} \sqrt{l} & 0 \\ 0 & 1/\sqrt{l} \end{pmatrix},$$

where $\tilde{l}$ is any integer that satisfies $\tilde{l}l \equiv 1 \pmod{r}$. By [19], $\tau_r \nu_l$ is an acceptable choice for an Atkin-Lehner operator. Let $\{W_l : l|N\}$ be the group of Atkin-Lehner operators for $\Gamma_0(N)$ defined by

$$W_l := \tau_r \nu_l = \frac{1}{\sqrt{l}} \begin{pmatrix} l & (\tilde{l}l-1)/r \\ rl & \tilde{l}l \end{pmatrix}. \tag{3.6}$$

Further, let $V_l := \sqrt{l} W_l$ and

$$A_l := \begin{pmatrix} \frac{1}{\text{width of the cusp } 1/r} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/l & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$V_l A_l = \begin{pmatrix} 1 & (\tilde{l}l-1)/r \\ r & \tilde{l}l \end{pmatrix} \in SL_2(\mathbb{Z})$$

and this matrix maps ∞ to the cusp $1/r$. Since the matrix $\gamma_{\frac{1}{r},t} \in \mathcal{C}_N$ maps the cusp ∞ to the cusp $1/r$, it follows that

$$(V_l A_l)^{-1} \gamma_{\frac{1}{r},t}(\infty) = \infty,$$

hence $(V_l A_l)^{-1} \gamma_{\frac{1}{r},t} \in SL_2(\mathbb{Z})_\infty$ stabilizes the cusp ∞. In particular, there is an integer $n \in \mathbb{Z}$ such that

$$\gamma_{\frac{1}{r},t} = V_l A_l \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

By solving for n for each matrix $\gamma_{\frac{1}{r},t}$, we obtain

$$\gamma_{\frac{1}{r},t} = V_l A_l T_{l,t} = \sqrt{l} W_l A_l T_{l,t}, \tag{3.7}$$

where

$$T_{l,t} := \begin{pmatrix} 1 & t + \frac{1-\tilde{l}l}{r} \\ 0 & 1 \end{pmatrix}.$$

3.4 The bijection from $\mathcal{Q}_D^{\text{red}}$ to $\mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N)$

In this subsection we show that the map defined by (3.2) is a bijection. First, observe that by reduction theory, given $Q \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}$ there is a matrix $m_Q \in SL_2(\mathbb{Z})$ such that $Q \circ m_Q \in \mathcal{Q}_D^{\text{red}}$. In particular, $m_Q = A_Q\gamma_Q$ for some $A_Q \in \Gamma_0(N)$ and $\gamma_Q \in \mathcal{C}_N$, so that $Q \circ A_Q\gamma_Q \in \mathcal{Q}_D^{\text{red}}$.

Lemma 3.9. *If* $Q_1, Q_2 \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}$ *and* $Q_1 \equiv Q_2 \pmod{\Gamma_0(N)}$*, then* $\gamma_{Q_1} = \gamma_{Q_2}$.

Proof. Since D and N are coprime, we have $\left(N, \rho, \frac{\rho^2-D}{4N}\right) = 1$. Hence by the Proposition on page 505 of [13], there is a bijection

$$\mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N) \longrightarrow \mathcal{Q}_{D,1}^{\text{prim}}/SL_2(\mathbb{Z}) \longrightarrow \mathcal{Q}_D^{\text{red}} \tag{3.8}$$

given by $[Q]_N \longmapsto [Q]_1 \longmapsto \widetilde{Q}$, where the first arrow is the natural map and $\widetilde{Q}$ is the unique reduced form in the class $[Q]_1$.

Now, since $[Q_1]_N = [Q_2]_N$ there is a matrix $B \in \Gamma_0(N)$ such that $Q_2 = Q_1 \circ B$. Also, by (3.8) there is a unique reduced form $\widetilde{Q} \in \mathcal{Q}_D^{\text{red}}$ such that $\widetilde{Q} = Q_1 \circ A_{Q_1}\gamma_{Q_1}$ and $\widetilde{Q} = Q_2 \circ A_{Q_2}\gamma_{Q_2}$. Hence

$$Q_1 \circ A_{Q_1}\gamma_{Q_1} = Q_2 \circ A_{Q_2}\gamma_{Q_2} = (Q_1 \circ B) \circ A_{Q_2}\gamma_{Q_2},$$

which yields

$$Q_1 = Q_1 \circ BA_{Q_2}\gamma_{Q_2}\gamma_{Q_1}^{-1}A_{Q_1}^{-1}.$$

Since $D < -4$, the stabilizer subgroup of $Q \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}$ is $SL_2(\mathbb{Z})_Q = \{\pm I\}$ where I is the identity matrix. Hence

$$BA_{Q_2}\gamma_{Q_2}\gamma_{Q_1}^{-1}A_{Q_1}^{-1} = \pm I$$

which yields $\gamma_{Q_1} = \pm A_{Q_1}^{-1}BA_{Q_2}\gamma_{Q_2}$. Therefore $\gamma_{Q_1} \equiv \gamma_{Q_2} \pmod{\Gamma_0(N)}$ which implies $\gamma_{Q_1} = \gamma_{Q_2}$. □

By Lemma 3.9 and (3.8), given $\widetilde{Q} \in \mathcal{Q}_D^{\text{red}}$ there is a unique class $[Q]_N \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N)$ and a unique coset representative $\gamma_{\widetilde{Q}} := \gamma_{[Q]_N} = \gamma_Q \in \mathcal{C}_N$ such that

$$\widetilde{Q} \circ \gamma_{\widetilde{Q}}^{-1} = (Q \circ A_Q\gamma_{\widetilde{Q}}) \circ \gamma_{\widetilde{Q}}^{-1} = Q \circ A_Q \in [Q]_N \in \mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N).$$

This gives a map

$$\begin{aligned}\alpha : \mathcal{Q}_D^{\text{red}} &\longrightarrow \mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N)\\ \widetilde{Q} &\longmapsto [\widetilde{Q} \circ \gamma_{\widetilde{Q}}^{-1}]_N.\end{aligned}$$

Proposition 3.10. *The map α is a bijection.*

Proof. By (3.8) we have $|\mathcal{Q}_D^{\text{red}}| = |\mathcal{Q}_{D,N,\rho}^{\text{prim}}/\Gamma_0(N)|$. Hence it suffices to prove that α is injective. Let $\widetilde{Q}_1, \widetilde{Q}_2 \in \mathcal{Q}_D^{\text{red}}$ and suppose that $\alpha(\widetilde{Q}_1) = \alpha(\widetilde{Q}_2)$. Then $[\widetilde{Q}_1 \circ \gamma_{\widetilde{Q}_1}^{-1}]_N = [\widetilde{Q}_2 \circ \gamma_{\widetilde{Q}_2}^{-1}]_N$ so there is $B \in \Gamma_0(N)$ such that $\widetilde{Q}_1 \circ \gamma_{\widetilde{Q}_1}^{-1} = (\widetilde{Q}_2 \circ \gamma_{\widetilde{Q}_2}^{-1}) \circ B$. Thus $\widetilde{Q}_1 = \widetilde{Q}_2 \circ \gamma_{\widetilde{Q}_2}^{-1}B\gamma_{\widetilde{Q}_1}$ which implies $\widetilde{Q}_1 \equiv \widetilde{Q}_2 \pmod{SL_2(\mathbb{Z})}$. Hence $\widetilde{Q}_1 = \widetilde{Q}_2$, so α is injective. □

3.5 The Fourier expansion of $f \in M_{-2k}^!(N)$

In this subsection, we compute the Fourier expansion of a weakly holomorphic modular form f of weight $-2k$ which is an eigenfunction for the Atkin-Lehner operators. We first prove the following lemma, which shows that if we know the Fourier expansion at the cusp ∞ of f , then we can obtain the Fourier expansion of f with respect to the matrices $\gamma_{\frac{1}{r},t} \in \mathcal{C}_N$.

Lemma 3.11. *Let $k \geq 0$ be an integer and suppose that $f : \mathbb{H} \to \mathbb{C}$ is an eigenfunction for the Atkin-Lehner operators W_l defined in* (3.6)*, i.e.*

$$f|_{-2k} W_l(z) = \lambda(l) f(z) \tag{3.9}$$

for some $\lambda(l) \in \mathbb{C}$. Then for $\gamma_{\frac{1}{r},t} \in \mathcal{C}_N$,

$$f|_{-2k}\gamma_{\frac{1}{r},t}(z) = l^k \lambda(l) f\left(\frac{z}{l} + \frac{t}{l} + \frac{1 - l\tilde{l}}{N}\right)$$

and if, in addition, $f \in M^!_{-2k}(N)$, then

$$f|_{-2k}\gamma_{\frac{1}{r},t}(z) = l^k \lambda(l) \sum_{n=-N_\infty}^{\infty} a(n) e\left(\frac{nz}{l}\right) \zeta_N^{n(rt+1-l\tilde{l})}. \tag{3.10}$$

Proof. Using the relation (3.7) and the property

$$j(AB, z) = j(A, Bz) j(B, z) \tag{3.11}$$

for $A, B \in GL_2(\mathbb{R})$ we have

$$\begin{aligned} f|_{-2k}\gamma_{\frac{1}{r},t}(z) &= j(\gamma_{\frac{1}{r},t}, z)^{2k} f(\gamma_{\frac{1}{r},t} z) = j(\sqrt{l} W_l A_l T_{l,t}, z)^{2k} f(\sqrt{l} W_l A_l T_{l,t} z) \\ &= j(\sqrt{l} W_l, A_l T_{l,t} z)^{2k} j(A_l T_{l,t}, z)^{2k} f(\sqrt{l} W_l A_l T_{l,t} z). \end{aligned}$$

Note that $j(A_l T_{l,t}, z) = 1$ and $j(\sqrt{l} W_l, A_l T_{l,t} z) = \sqrt{l} j(W_l, A_l T_{l,t} z)$. Then by (3.9),

$$f|_{-2k}\gamma_{\frac{1}{r},t}(z) = l^k f|_{-2k} W_l(A_l T_{l,t} z) = l^k \lambda(l) f(A_l T_{l,t} z) = l^k \lambda(l) f\left(\frac{z}{l} + \frac{t}{l} + \frac{1 - l\tilde{l}}{N}\right).$$

Finally, if $f \in M^!_{-2k}(N)$, we apply the Fourier expansion (1.2) to $f\left(\frac{z}{l} + \frac{t}{l} + \frac{1-l\tilde{l}}{N}\right)$. □

The following lemma shows how to construct linear combinations of the Poincaré series F_m

which are automorphic for $\Gamma_0(N)$ and eigenfunctions for the Atkin-Lehner operators.

Lemma 3.12. *Let $\lambda(l) \in \mathbb{C}$ for each $l|N$ and let $\lambda := (\lambda(l))_{l|N}$. Let m be a positive integer and $k \in \mathbb{Z}$. Define*

$$P_{m,\lambda}(z,s,k) := \sum_{l|N} \lambda(l) F_m(z,s,k)|_k W_l, \quad \mathrm{Re}(s) > 1.$$

Then for $M \in \Gamma_0(N)$,

$$P_{m,\lambda}(z,s,k)|_k M = P_{m,\lambda}(z,s,k).$$

Moreover, if the eigenvalues $\lambda(l)$ satisfy properties (P.1) *and* (P.2)*, then*

$$P_{m,\lambda}(z,s,k)|_k W_l = \lambda(l) P_{m,\lambda}(z,s,k).$$

Proof. Let $M \in \Gamma_0(N)$. Since $W_l M W_l^{-1} \in \Gamma_0(N)$, there is a matrix $M' \in \Gamma_0(N)$ such that $W_l M = M' W_l$. Hence

$$\begin{aligned} (F_m(z,s,k)|_k W_l)\,|_k M &= F_m(z,s,k)|_k (W_l M) \\ &= F_m(z,s,k)|_k (M' W_l) \\ &= (F_m(z,s,k)|_k M')\,|_k W_l. \end{aligned} \tag{3.12}$$

By definition of F_m, we have

$$F_m(z,s,k)|_k M' = F_m(z,s,k).$$

So (3.12) becomes

$$(F_m(z,s,k)|_k W_l)\,|_k M = F_m(z,s,k)|_k W_l,$$

and we get

$$\begin{aligned} P_{m,\lambda}(z,s,k)|_k M &= \sum_{l|N} \lambda(l) \left(F_m(z,s,k)|_k W_l\right)|_k M \\ &= \sum_{l|N} \lambda(l) F_m(z,s,k)|_k W_l \\ &= P_{m,\lambda}(z,s,k). \end{aligned}$$

Next, suppose that $\lambda(l)$ satisfy properties (P.1) and (P.2). We have

$$P_{m,\lambda}(z,s,k)|_k W_l = \left(\sum_{l_0|N} \lambda(l_0) F_m(z,s,k)|_k W_{l_0}\right)|_k W_l = \sum_{l_0|N} \lambda(l_0) F_m(z,s,k)|_k (W_{l_0} W_l).$$

For each $l_0|N$, let $d := \frac{ll_0}{(l,l_0)^2}$. Using properties (P.1) and (P.2), we obtain $\lambda(d) = \lambda(l)\lambda(l_0)$ and

$$\lambda(d)\lambda(l) = \lambda(l)^2\lambda(l_0) = \lambda(l_0).$$

So by the group law on $\{W_l : l|N\}$, i.e. $W_{l_1}W_{l_2} \equiv W_{\frac{l_1 l_2}{(l_1,l_2)^2}} \pmod{\Gamma_0(N)}$, we obtain

$$P_{m,\lambda}(z,s,k)|_k W_l = \lambda(l) \sum_{d|N} \lambda(d) F_m(z,s,k)|_k W_d = \lambda(l) P_{m,\lambda}(z,s,k).$$

This completes the proof. □

Recall that $S(a,b;c)$ is the Kloosterman sum of modulus c is defined by (1.4) and I_ν, J_ν, K_ν denote the I, J, K-Bessel functions of order ν, respectively.

Proposition 3.13. *Let k be a positive integer and $f \in M^!_{-2k}(N)$ be an eigenfunction for the Atkin-Lehner operators W_l where the eigenvalues $\lambda(l)$ satisfy properties* (P.1) *and* (P.2)*. Then*

$$f(z) = \mathcal{P}_f(z)$$

where

$$\mathcal{P}_f(z) := 2\sum_{m=1}^{N_\infty} a(-m)\sum_{l|N}\lambda(l)F_m(z,k+1,-2k)|_{-2k}W_l. \tag{3.13}$$

Moreover, the Fourier coefficients $a(n)$ *for* $n\geq 0$ *in* (1.2) *are given by*

$$a(0) = \frac{(-1)^k 2^{2k+1}\pi^{2k+2}}{(k+1/2)\Gamma(2k+1)}\sum_{l|N}\frac{\lambda(l)}{l^{k+1}}\sum_{\substack{c>0\\ c\equiv 0 \pmod{N/l}\\ (c,l)=1}} c^{-2k-2}\sum_{m=1}^{N_\infty} a(-m)S(-m\tilde{l},0;c)m^{2k+1},$$

and

$$a(n) = \frac{2\pi(-1)^k}{n^{k+1/2}}\sum_{l|N}\frac{\lambda(l)}{\sqrt{l}}\sum_{\substack{c>0\\ c\equiv 0 \pmod{N/l}\\ (c,l)=1}} c^{-1}\sum_{m=1}^{N_\infty} a(-m)S(-m\tilde{l},n;c)m^{k+1/2}I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right), \quad n\geq 1.$$

Proof. We first compute the Fourier expansion of $\mathcal{P}_f$. Write

$$F_m(z,k+1,-2k) = \frac{1}{2\Gamma(2k+2)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\left[\psi_{m,k}(y)e(-mz)\right]|_{-2k}\gamma$$

where

$$\psi_{m,k}(y) := (4\pi my)^k M_{k,k+\frac{1}{2}}(4\pi my)e^{-2\pi my}.$$

Then by (3.11) we have

$$F_m(z,k+1,-2k)|_{-2k}W_l = \frac{1}{2\Gamma(2k+2)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\psi_{m,k}(\mathrm{Im}(\gamma W_l z))e(-m(\gamma W_l z))j(\gamma W_l,z)^{2k}.$$

Hence

$$\mathcal{P}_f(z) = \frac{1}{\Gamma(2k+2)}\sum_{l|N}\lambda(l)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g(\gamma W_l z)j(\gamma W_l,z)^{2k} \tag{3.14}$$

where

$$g(z) := \sum_{m=1}^{N_\infty} a(-m)\psi_{m,k}(\mathrm{Im}(z))e(-mz).$$

Now, let B denote the group of integral translations

$$B := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}$$

and let

$$w_{d/c} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)W_l.$$

Since the matrix W_l satisfies the conditions $W_l \in SL_2(\mathbb{R})$, $W_l(\infty) = 1/r$, and

$$W_l^{-1}\Gamma_0(N)_{1/r}W_l = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

then W_l is a scaling matrix for the Atkin-Lehner cusp $1/r$ in the sense of [15]. So we can use the double coset decomposition in [15, (2.21)] to obtain

$$\begin{aligned}
\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g(\gamma W_l z) j(\gamma W_l, z)^{2k} &= \sum_{\tau \in B \backslash \Gamma_0(N) W_l} g(\tau z) j(\tau, z)^{2k} \\
&= \delta_{\infty, \frac{1}{r}} g(z) + \sum_{c>0} \sum_{d \pmod c} \sum_{n \in \mathbb{Z}} g(w_{d/c}(z+n)) j\left(w_{d/c} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, z \right)^{2k} \\
&= \delta_{\infty, \frac{1}{r}} g(z) + \sum_{c>0} \sum_{d \pmod c} \sum_{n \in \mathbb{Z}} g(w_{d/c}(z+n))(cz + cn + d)^{2k},
\end{aligned}$$

where $\delta_{\infty,1/r} = 1$ if the cusps ∞ and $1/r$ are equivalent and 0 otherwise. Applying the Poisson summation formula to the innermost sum, we get

$$\sum_{n\in\mathbb{Z}} g(w_{d/c}(z+n))(cz+cn+d)^{2k} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} g(w_{d/c}(z+t))(cz+ct+d)^{2k} e(-nt)dt.$$

Note that

$$w_{d/c}(z+t) = \frac{a}{c} - \frac{1}{c^2(t+x+d/c+iy)}.$$

By the change of variables $t \to t - x - d/c$, the integral

$$\int_{-\infty}^{\infty} g(w_{d/c}(z+t))(cz+ct+d)^{2k} e(-nt)dt$$

becomes

$$c^{2k} e(nx+nd/c) \int_{-\infty}^{\infty} g\left(\frac{a}{c} - \frac{1}{c^2(t+iy)}\right)(t+iy)^{2k} e(-nt)dt.$$

So

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g(\gamma W_l z) j(\gamma W_l, z)^{2k} = \delta_{\infty,\frac{1}{r}} g(z) \\ + \sum_{n\in\mathbb{Z}} e(nx) \sum_{c>0} c^{2k} \sum_{m=1}^{N_\infty} a(-m) S_{\infty,\frac{1}{r}}(-m,n;c)\alpha_k(m,c,y,n),$$

where

$$\alpha_k(m,c,y,n) := \int_{-\infty}^{\infty} \psi_{m,k}\left(\frac{y}{c^2(t^2+y^2)}\right) e\left(\frac{m}{c^2(t+iy)} - nt\right)(t+iy)^{2k} dt$$

and $S_{\infty,1/r}$ is the general Kloosterman sum defined by

$$S_{\infty,\frac{1}{r}}(m_1,m_2;c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)W_l/\Gamma_\infty} e\left(\frac{am_1+dm_2}{c}\right).$$

Thus equation (3.14) becomes

$$\mathcal{P}_f(z) = \sum_{l|N} \lambda(l)\delta_{\infty,\frac{1}{r}} \frac{g(z)}{\Gamma(2k+2)} + \sum_{l|N} \lambda(l) \sum_{n\in\mathbb{Z}} e(nx) \sum_{c>0} c^{2k} \sum_{m=1}^{N_\infty} a(-m) S_{\infty,\frac{1}{r}}(-m,n;c) \frac{\alpha_k(m,c,y,n)}{\Gamma(2k+2)}.$$

Observe that $\delta_{\infty,1/r} = 1$ only when $r = N$ (so $l = 1$). Also, properties (P.1) and (P.2) of $\lambda(l)$ imply that $\lambda(1) = 1$. So

$$\mathcal{P}_f(z) = \frac{g(z)}{\Gamma(2k+2)} + \sum_{n\in\mathbb{Z}} e(nx) \sum_{l|N} \lambda(l) \sum_{c>0} c^{2k} \sum_{m=1}^{N_\infty} a(-m) S_{\infty,\frac{1}{r}}(-m,n;c) \frac{\alpha_k(m,c,y,n)}{\Gamma(2k+2)}. \tag{3.15}$$

We compute $g(z)/\Gamma(2k+2)$. Using equation 13.14.2 in [25], we obtain

$$M_{k,k+1/2}(4\pi my) = e^{-2\pi my}(4\pi my)^{k+1} M(1, 2k+2, 4\pi my)$$

where $M(a, b, z)$ is the Kummer confluent hypergeometric function. So

$$\frac{\psi_{m,k}(y)}{\Gamma(2k+2)} = \frac{e^{-4\pi my}(4\pi my)^{2k+1}}{\Gamma(2k+2)} M(1, 2k+2, 4\pi my).$$

Using equation 13.2.2 in [25], we obtain

$$M(1, 2k+2, 4\pi my) = 1 + \frac{1}{2k+2} 4\pi my + \frac{1\cdot 2}{(2k+2)(2k+3)} \frac{(4\pi my)^2}{2!} + \cdots .$$

So

$$\begin{aligned}\frac{\psi_{m,k}(y)}{\Gamma(2k+2)} &= \frac{e^{-4\pi my}(4\pi my)^{2k+1}}{(2k+1)!}\left[1+\frac{1}{2k+2}4\pi my+\frac{1\cdot 2}{(2k+2)(2k+3)}\frac{(4\pi my)^2}{2!}+\cdots\right] \\ &= e^{-4\pi my}\left[\frac{(4\pi my)^{2k+1}}{(2k+1)!}+\frac{(4\pi my)^{2k+2}}{(2k+2)!}+\cdots\right] \\ &= e^{-4\pi my}\left[e^{4\pi my}-\left(1+\frac{4\pi my}{1!}+\frac{(4\pi my)^2}{2!}+\cdots+\frac{(4\pi my)^{2k}}{(2k)!}\right)\right] \\ &= 1-e^{-4\pi my}\sum_{j=0}^{2k}\frac{(4\pi my)^j}{j!}.\end{aligned}$$

Therefore by definition of g we obtain

$$\frac{g(z)}{\Gamma(2k+2)} = \sum_{m=1}^{N_\infty} a(-m)e(-mz) - \sum_{m=1}^{N_\infty} a(-m)e(-m\bar{z})\sum_{j=0}^{2k}\frac{(4\pi my)^j}{j!}. \tag{3.16}$$

Now, by [19, Definition 2.3], the set of allowed moduli for the cusps ∞ and $1/r$ is given by

$$\mathbf{C}_{\infty,\frac{1}{r}} = \left\{\gamma > 0 : \begin{pmatrix} * & * \\ \gamma & * \end{pmatrix} \in \Gamma_0(N)W_l\right\}.$$

Using [19, Proposition 2.6], we obtain that

$$\mathbf{C}_{\infty,\frac{1}{r}} = \{\gamma = c\sqrt{l} > 0 : c \equiv 0 \ (\mathrm{mod}\ r), (c,l) = 1)\},$$

and for such $\gamma = c\sqrt{l} \in \mathbf{C}_{\infty,1/r}$, the Kloosterman sum of modulus γ is given by

$$S_{\infty,\frac{1}{r}}(m_1, m_2; c\sqrt{l}) = S(\tilde{l}m_1, m_2; c).$$

By the above argument and (3.16), equation (3.15) becomes

$$\mathcal{P}_f(z) = \sum_{m=1}^{N_\infty} a(-m)e(-mz) - \sum_{m=1}^{N_\infty} a(-m)e(-m\bar{z}) \sum_{j=0}^{2k} \frac{(4\pi m y)^j}{j!}$$
$$+ \sum_{n\in\mathbb{Z}} e(nx) \sum_{l|N} \lambda(l) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{2k} l^k \sum_{m=1}^{N_\infty} a(-m) S(-m\tilde{l}, n; c) \frac{\alpha_k(m, c\sqrt{l}, y, n)}{\Gamma(2k+2)}. \qquad (3.17)$$

By straightforward calculation, we have

$$\alpha_k(m, c\sqrt{l}, y, n) = \frac{(4\pi m y)^k}{c^{2k} l^k} \int_{-\infty}^{\infty} \frac{M_{k,k+1/2}\left(\frac{4\pi m y}{c^2 l (t^2+y^2)}\right)}{(t^2+y^2)^k} e\left(\frac{mt}{c^2 l (t^2+y^2)} - nt\right)(t+iy)^{2k} dt.$$

Using equation 13.14.2 in [25], we obtain

$$M_{k,k+1/2}\left(\frac{4\pi m y}{c^2 l(t^2+y^2)}\right) = \exp\left(\frac{-2\pi m y}{c^2 l(t^2+y^2)}\right)\left(\frac{4\pi m y}{c^2 l(t^2+y^2)}\right)^{k+1} M\left(1, 2k+2, \frac{4\pi m y}{c^2 l(t^2+y^2)}\right).$$

Using the change of variables $t = yu$ and setting $A = \frac{1}{c^2 l y}, B = -ny$, we get

$$\frac{\alpha_k(m, c\sqrt{l}, y, n)}{\Gamma(2k+2)} = \frac{(-4\pi m)^k y^{k+1}}{c^{2k} l^k} L_n \qquad (3.18)$$

where

$$L_n := \int_{-\infty}^{\infty} G\left(\frac{4\pi m A}{1+u^2}\right) e^{2\pi i\left(\frac{muA}{1+u^2}+Bu\right)} \left(\frac{1-iu}{1+iu}\right)^k du$$

and

$$G(z) := \frac{z^{k+1} e^{-z/2} M(1, 2k+2, z)}{\Gamma(2k+2)}.$$

Using Lemma 5.5 on page 357 of [14], we have

$$L_n = \begin{cases} \dfrac{2\pi(-4\pi ny)^{k+1}}{\Gamma(2k+1)} e^{2\pi ny} U(1, 2k+2, -4\pi ny) \sqrt{\dfrac{m}{-nlc^2y^2}} J_{2k+1}\left(\dfrac{4\pi\sqrt{-mn}}{c\sqrt{l}}\right) & \text{if } n < 0 \\ \dfrac{2\pi^{k+2}}{(k+1/2)\Gamma(2k+1)} \left(\dfrac{m}{c^2ly}\right)^{k+1} & \text{if } n = 0 \\ 2\pi(4\pi n|y|)^{k+1} e^{-2\pi n|y|} U(2k+1, 2k+2, 4\pi n|y|) \sqrt{\dfrac{m}{nlc^2y^2}} I_{2k+1}\left(\dfrac{4\pi\sqrt{mn}}{c\sqrt{l}}\right) & \text{if } n > 0, \end{cases}$$

where $U(a, b, z)$ is the Tricomi confluent hypergeometric function. Therefore by (3.17) and (3.18), the Fourier expansion of $\mathcal{P}_f$ is

$$\begin{aligned} \mathcal{P}_f(z) = &\sum_{m=1}^{N_\infty} a(-m)e(-mz) + C_k(0) - \sum_{m=1}^{N_\infty} a(-m)e(-m\bar{z}) \sum_{j=0}^{2k} \frac{(4\pi my)^j}{j!} \\ &+ \sum_{n>0} C_k(-n, y)e(-n\bar{z}) + \sum_{n>0} C_k(n, y)e(nz), \end{aligned}$$

where

$$C_k(0) := \frac{(-1)^k 2^{2k+1} \pi^{2k+2}}{(k+1/2)\Gamma(2k+1)} \sum_{l|N} \frac{\lambda(l)}{l^{k+1}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-2k-2} \sum_{m=1}^{N_\infty} a(-m) S(-m\tilde{l}, 0; c) m^{2k+1},$$

and for $n \geq 1$,

$$C_k(-n, y) := \frac{2\pi(4\pi y)^{2k+1}(-1)^k n^{k+\frac{1}{2}}}{\Gamma(2k+1)} \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) m^{k+\frac{1}{2}} \mathrm{SUJ}(-n, l, c, m, y),$$

$$C_k(n, y) := 2\pi(4\pi y)^{2k+1}(-1)^k n^{k+\frac{1}{2}} \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) m^{k+\frac{1}{2}} \mathrm{SUI}(n, l, c, m, y),$$

$$\mathrm{SUJ}(-n, l, c, m, y) := S(-m\tilde{l}, -n; c) U(1, 2k+2, 4\pi ny) J_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right),$$

$$\mathrm{SUI}(n, l, c, m, y) := S(-m\tilde{l}, n; c) U(2k+1, 2k+2, 4\pi ny) I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right).$$

By equation 13.6.4 in [25], we have

$$U(2k+1, 2k+2, 4\pi ny) = (4\pi ny)^{-2k-1}.$$

It follows that for $n \geq 1$,

$$C_k(n,y) = \frac{2\pi(-1)^k}{n^{k+1/2}} \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) S(-m\tilde{l}, n; c) m^{k+1/2} I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right) =: C_k(n),$$

which does not depend on y. Thus

$$\begin{aligned} \mathcal{P}_f(z) &= \sum_{m=1}^{N_\infty} a(-m)e(-mz) + C_k(0) + \sum_{n>0} C_k(n)e(nz) \\ &\quad + \sum_{n=1}^{N_\infty} e(-n\bar{z}) \left(C_k(-n,y) - a(-n) \sum_{j=0}^{2k} \frac{(4\pi ny)^j}{j!} \right) + \sum_{n>N_\infty} C_k(-n,y)e(-n\bar{z}). \end{aligned} \quad (3.19)$$

Now, observe that

$$\mathcal{P}_f(z) = 2\sum_{m=1}^{N_\infty} a(-m) P_{m,\lambda}(z, k+1, -2k).$$

Thus by Lemma 3.12 we have

$$\mathcal{P}_f(z)|_{-2k} M = \mathcal{P}_f(z) \quad \text{for all} \quad M \in \Gamma_0(N) \quad (3.20)$$

and

$$\mathcal{P}_f(z)|_{-2k} W_l = \lambda(l)\mathcal{P}_f(z).$$

Hence by Lemma 3.11 and (3.19) we obtain

$$
\begin{aligned}
\mathcal{P}_f|_{-2k}\gamma_{\frac{1}{r},t}(z) &= l^k\lambda(l)\mathcal{P}_f\left(\frac{z}{l}+\frac{t}{l}+\frac{1-l\tilde{l}}{N}\right) \\
&= l^k\lambda(l)\sum_{m=1}^{N_\infty} a(-m)e\left(\frac{-mz}{l}\right)\zeta_N^{-m(rt+1-l\tilde{l})} + l^k\lambda(l)C_k(0) \\
&\quad + l^k\lambda(l)\sum_{n>0} C_k(n)e\left(\frac{nz}{l}\right)\zeta_N^{n(rt+1-l\tilde{l})} \\
&\quad + l^k\lambda(l)\sum_{n=1}^{N_\infty} e\left(\frac{-n\bar{z}}{l}\right)\zeta_N^{-n(rt+1-l\tilde{l})}\left(C_k\left(-n,\frac{y}{l}\right) - a(-n)\sum_{j=0}^{2k}\frac{(4\pi ny)^j}{l^j j!}\right) \\
&\quad + l^k\lambda(l)\sum_{n>N_\infty} C_k\left(-n,\frac{y}{l}\right)e\left(\frac{-n\bar{z}}{l}\right)\zeta_N^{-n(rt+1-l\tilde{l})}. \qquad (3.21)
\end{aligned}
$$

From Lemma 3.11 and (3.21), we find that the Fourier expansions of f and $\mathcal{P}_f$ with respect to the matrices $\gamma_{\frac{1}{r},t}$ have the same principal parts. Furthermore, since

$$\Delta_{-2k}(F_m|_{-2k}\gamma) = (\Delta_{-2k}F_m)|_{-2k}\gamma$$

for all $\gamma \in GL_2^+(\mathbb{R})$ (see [12]) and $\Delta_{-2k}F_m(z, k+1, -2k) = 0$, it follows that $\mathcal{P}_f$ is harmonic. Hence $f - \mathcal{P}_f$ is a bounded harmonic function on the compact Riemann surface $X_0(N)$ and is thus constant. Write $f = \mathcal{P}_f + C_f$ for some constant C_f. We will show that $C_f = 0$.

Let $M := \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma_0(N)$. Then by (3.20) we have

$$f(z) = f(z)|_{-2k}M = \mathcal{P}_f(z)|_{-2k}M + C_f|_{-2k}M = \mathcal{P}_f(z) + (Nz+1)^{2k}C_f.$$

It follows that $C_f = (Nz+1)^{2k}C_f$ for all $z \in \mathbb{H}$. Since k is non-zero, $C_f = 0$, and hence $f = \mathcal{P}_f$.

Finally, by comparing the Fourier expansion of f and $\mathcal{P}_f$, we get $a(0) = C_k(0)$, $a(n) = C_k(n)$

for $n \geq 1$,

$$C_k(-n,y) - a(-n)\sum_{j=0}^{2k}\frac{(4\pi ny)^j}{j!} = 0$$

for $1 \leq n \leq N_\infty$, and $C_k(-n,y) = 0$ for $n > N_\infty$. This completes the proof. $\square$

Proposition 3.14. *Let $f \in M_0^!(N)$ be an eigenfunction for the Atkin-Lehner operators W_l where the eigenvalues $\lambda(l)$ satisfy properties* (P.1) *and* (P.2). *Then*

$$f(z) = \mathcal{P}_f(z,1) + C_f$$

where $\mathcal{P}_f(z,1)$ is the value at $s=1$ of the analytic continuation of the Poincaré series

$$\mathcal{P}_f(z,s) := 2\sum_{m=1}^{N_\infty} a(-m)\sum_{l|N}\lambda(l)F_m(z,s,0)|_0 W_l$$

to $\mathrm{Re}(s) > 3/4$ *and*

$$C_f = a(0) - 4\pi^2\sum_{l|N}\frac{\lambda(l)}{l}\sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-2}\sum_{m=1}^{N_\infty} a(-m)mS(-m\tilde{l},0;c).$$

Moreover, the Fourier coefficients $a(n)$ for $n \geq 1$ in (1.2) *are given by*

$$a(n) = \frac{2\pi}{\sqrt{n}}\sum_{l|N}\frac{\lambda(l)}{\sqrt{l}}\sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1}\sum_{m=1}^{N_\infty}\sqrt{m}a(-m)S(-m\tilde{l},n;c)I_1\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right).$$

Proof. We will give only the idea of the proof here. The details of the proof can be found in [18]. We first compute the Fourier expansion of $\mathcal{P}_f(z,s)$ using an argument similar to Proposition 3.13, and use it to obtain the analytic continuation of $\mathcal{P}_f(z,s)$ to $\mathrm{Re}(s) > 3/4$. Then we compute the Fourier expansion of $\mathcal{P}_f(z,1)$ by letting $s = 1$ in the the Fourier expansion of $\mathcal{P}_f(z,s)$. Finally, we use the argument similar to Proposition 3.13 to complete the proof. $\square$

3.6 The Fourier expansion of $\mathcal{D}^k f$

In this subsection, we compute the Fourier expansion of $\mathcal{D}^k f$ for those $f \in M^!_{-2k}(N)$ which satisfy the conditions in Proposition 3.13. We first prove the following lemma.

Lemma 3.15. *Let $k \geq 0$ be an integer. Then for any $c \in \mathbb{C}$ we have*

$$\mathcal{D}^k c = \frac{(-1)^k (2k)! c}{(4\pi)^k k! y^k},$$

and for $n \in \mathbb{Z}$

$$\mathcal{D}^k e(nz) = g_n(y) e(nz)$$

for some rational function $g_n(y)$.

Proof. It is obvious that the statement is true for $k = 0$. Let $k \geq 1$. We first prove the second statement. We claim that for $0 \leq t \leq k-1$,

$$R_{-2k+2t} R_{-2k+2(t-1)} \cdots R_{-2k} e(nz) = g_n(y) e(nz)$$

for some rational function $g_n(y)$. We will prove the claim by induction. Note that for $t = 0$, we have

$$R_{-2k} e(nz) = 2i \frac{\partial e(nz)}{\partial z} + \frac{(-2k)}{y} e(nz) = \left(-4\pi n - \frac{2k}{y} \right) e(nz).$$

Suppose that the claim is true for $t - 1$, i.e. there is a rational function $g_n(y)$ such that

$$R_{-2k+2(t-1)} \cdots R_{-2k} e(nz) = g_n(y) e(nz).$$

Then a straightforward calculation yields that

$$R_{-2k+2t}R_{-2k+2(t-1)}\cdots R_{-2k}e(nz) = 2i\frac{\partial g_n(y)e(nz)}{\partial z} + \frac{(-2k+2t)}{y}g_n(y)e(nz)$$
$$= e(nz)\left(g_n'(y) - 4\pi n g_n(y) + \frac{(-2k+2t)}{y}g_n(y)\right).$$

This proves the claim. Thus by letting $t = k-1$ in the claim, we complete the proof of the second statement.

Similarly, the first statement can be proved using the following claim: for $0 \le t \le k-1$,

$$R_{-2k+2t}R_{-2k+2(t-1)}\cdots R_{-2k}c = \frac{c(-1)^{t+1}(2k)(2k-1)\cdots(2k-t)}{y^{t+1}},$$

which can be proved by induction. Then we let $t = k-1$ to complete the proof. □

Proposition 3.16. *Let k be a positive integer and $f \in M^!_{-2k}(N)$ be an eigenfunction for the Atkin-Lehner operators W_l where the eigenvalues $\lambda(l)$ satisfy properties* (P.1) *and* (P.2). *Then*

$$\mathcal{D}^k f(z) = \sum_{n=0}^{N_\infty} a(-n)e(-nz)c_k(n,y) + \sum_{n=1}^{\infty} B_k(n,y)e(nz),$$

where for $0 \le n \le N_\infty$

$$c_k(n,y) := \sum_{j=0}^{k} \frac{(-1)^j(k+j)!n^{k-j}}{(4\pi y)^j j!(k-j)!},$$

and for $n \geq 1$

$$\begin{aligned}
B_k(n,y) &:= \frac{2\pi}{\sqrt{n}} d_k^*(n,y) S_k(n), \\
d_k^*(n,y) &:= \sum_{j=0}^{k} \frac{(k+j)!}{(4\pi n y)^j j!(k-j)!}, \\
S_k(n) &:= \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) m^{k+1/2} S(-m\tilde{l}, n; c) I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right).
\end{aligned}$$

Proof. By Proposition 3.13, $f = \mathcal{P}_f$ where $\mathcal{P}_f$ is defined by (3.13). Hence

$$\mathcal{D}^k f(z) = 2\sum_{m=1}^{N_\infty} a(-m) \sum_{l|N} \lambda(l) \mathcal{D}^k \left(F_m(z, k+1, -2k)|_{-2k} W_l\right).$$

Now, by Proposition 2.2 in [5] we have that for $j \in \mathbb{Z}$,

$$\frac{1}{4\pi} R_{-2j} F_m(z, k+1, -2j) = m(k+1-j) F_m(z, k+1, -2j+2). \tag{3.22}$$

Since the raising operator R_{-2j} and the slash operator commute, (3.22) implies that for $W \in SL_2(\mathbb{Z})$ and $t = 0, 1, \ldots, k-1$,

$$\begin{aligned}
&\frac{1}{4\pi} R_{-2(k-t)} \left(F_m(z, k+1, -2(k-t))|_{-2(k-t)} W\right) \\
&\quad = \left(\frac{1}{4\pi} R_{-2(k-t)} F_m(z, k+1, -2(k-t))\right) |_{-2(k-t)+2} W \\
&\quad = m(t+1) F_m(z, k+1, -2(k-t-1))|_{-2(k-t-1)} W.
\end{aligned}$$

Hence

$$\mathcal{D}^k f(z) = 2\sum_{m=1}^{N_\infty} a(-m) k! m^k \sum_{l|N} \lambda(l) F_m(z, k+1, 0)|_0 W_l. \tag{3.23}$$

We have

$$F_m(z,k+1,0) = \frac{1}{2\Gamma(2k+2)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \left[M_{0,k+1/2}(4\pi my)e(-mx)\right]|_0\gamma$$

where $M_{0,k+1/2}$ is the usual Whittaker function. Then using the equation 10.39.7 in [25] we get

$$F_m(z,k+1,0) = \frac{2^{2k}\Gamma(k+3/2)}{\Gamma(2k+2)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} I_{k+1/2}(2\pi my)\sqrt{4\pi my}e(-mx)|_0\gamma.$$

Further, by equation 5.5.5 in [25] we have

$$\frac{2^{2k}\Gamma(k+3/2)}{\Gamma(2k+2)} = \frac{\sqrt{\pi}}{2\Gamma(k+1)}. \tag{3.24}$$

Hence

$$\mathcal{D}^k f(z) = 2\sum_{m=1}^{N_\infty} a(-m)m^{k+1/2}\sum_{l|N}\lambda(l) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} I_{k+1/2}(2\pi my)\pi\sqrt{y}e(-mx)|_0\gamma W_l.$$

Let

$$g_{\mathcal{D}}(z) := \sum_{m=1}^{N_\infty} a(-m)\psi_{m,\mathcal{D}}(y)e(-mz)$$

where

$$\psi_{m,\mathcal{D}}(y) := m^{k+1/2}\pi\sqrt{y}I_{k+1/2}(2\pi my)e^{-2\pi my}.$$

Then

$$\mathcal{D}^k f(z) = 2\sum_{l|N}\lambda(l) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g_{\mathcal{D}}(\gamma W_l z).$$

For $l|N$, let $r := N/l$. Then arguing as in the proof of Proposition 3.13, we obtain

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g_{\mathcal{D}}(\gamma W_l z) = \delta_{\infty,\frac{1}{r}} g_{\mathcal{D}}(z) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} \sum_{m=1}^{N_\infty} a(-m) S(-\tilde{l}m, n; c) \alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n)$$

where

$$\alpha_{\mathcal{D}}(m,c,y,n) := \int_{-\infty}^{\infty} \psi_{m,\mathcal{D}}\left(\frac{y}{c^2(t^2+y^2)}\right) e\left(\frac{m}{c^2(t+iy)} - nt\right) dt.$$

Hence

$$\begin{aligned}\mathcal{D}^k f(z) = {} & 2\sum_{l|N} \lambda(l) \delta_{\infty,\frac{1}{r}} g_{\mathcal{D}}(z) \\ & + 2\sum_{l|N} \lambda(l) \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} \sum_{m=1}^{N_\infty} a(-m) S(-\tilde{l}m, n; c) \alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n).\end{aligned}$$

Since $\delta_{\infty,1/r} = 1$ only when $r = N$ (so $l = 1$), it follows that

$$\mathcal{D}^k f(z) = 2g_{\mathcal{D}}(z) + 2\sum_{l|N} \lambda(l) \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} \sum_{m=1}^{N_\infty} a(-m) S(-\tilde{l}m, n; c) \alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n). \tag{3.25}$$

We compute $2g_{\mathcal{D}}$. By equations 10.47.7, 10.49.8 and 10.49.1 in [25], we have

$$I_{k+1/2}(2\pi ym) = \frac{1}{2\pi\sqrt{ym}} \left(e^{2\pi ym} d_k(m,y) + (-1)^{k+1} e^{-2\pi ym} d_k^*(m,y)\right),$$

where

$$d_k(m,y) := \sum_{j=0}^{k} \frac{(-1)^j (k+j)!}{(4\pi ym)^j j!(k-j)!} \quad \text{and} \quad d_k^*(m,y) := \sum_{j=0}^{k} \frac{(k+j)!}{(4\pi ym)^j j!(k-j)!}.$$

Then by definition of $\psi_{m,\mathcal{D}}$ we obtain

$$\psi_{m,\mathcal{D}}(y) = \frac{m^k}{2}\left(d_k(m,y) + (-1)^{k+1}e^{-4\pi y m}d_k^*(m,y)\right).$$

Hence

$$2g_{\mathcal{D}}(z) = \sum_{m=1}^{N_\infty} a(-m)e(-mz)c_k(m,y) + \sum_{m=1}^{N_\infty} a(-m)e(-m\bar{z})(-1)^{k+1}c_k^*(m,y), \tag{3.26}$$

where

$$c_k(m,y) := \sum_{j=0}^{k}\frac{(-1)^j(k+j)!m^{k-j}}{(4\pi y)^j j!(k-j)!} \quad \text{and} \quad c_k^*(m,y) := \sum_{j=0}^{k}\frac{(k+j)!m^{k-j}}{(4\pi y)^j j!(k-j)!}.$$

Now, by definition of $\alpha_{\mathcal{D}}$ and straightforward calculation, we get

$$\alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n) := \int_{-\infty}^{\infty} m^{k+1/2}\pi\sqrt{\frac{y}{c^2l(t^2+y^2)}}I_{k+1/2}\left(\frac{2\pi y m}{c^2l(t^2+y^2)}\right)e\left(\frac{mt}{c^2l(t^2+y^2)} - nt\right)dt.$$

Then using the change of variables $t = yu$ and setting $A = \frac{1}{c^2ly}$, $B = -ny$, we obtain

$$2\alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n) := 2\pi m^{k+1/2}y\int_{-\infty}^{\infty}\sqrt{\frac{A}{1+u^2}}I_{k+1/2}\left(\frac{2\pi m A}{1+u^2}\right)e\left(\frac{muA}{1+u^2} + Bu\right)du.$$

Let

$$G(z) := \frac{z^{k+1}e^{-z/2}M(k+1, 2k+2, z)}{\Gamma(2k+2)}$$

and

$$L_{\mathcal{D},n} := \int_{-\infty}^{\infty} G\left(\frac{4\pi m A}{1+u^2}\right)e\left(\frac{mAu}{1+u^2} + Bu\right)du.$$

Using (3.24) and the following property obtained from (xiv) in [14],

$$M\left(k+1, 2k+2, \frac{4\pi mA}{1+u^2}\right) = \Gamma\left(k+\frac{3}{2}\right)\left(\frac{\pi mA}{1+u^2}\right)^{-k-1/2} \exp\left(\frac{2\pi mA}{1+u^2}\right) I_{k+1/2}\left(\frac{2\pi mA}{1+u^2}\right),$$

it follows that

$$2\alpha_{\mathcal{D}}(m, c\sqrt{l}, y, n) = y\Gamma(k+1)m^k L_{\mathcal{D},n}. \tag{3.27}$$

We compute $L_{\mathcal{D},n}$ using Lemma 5.5 in [14] to get

$$L_{\mathcal{D},n} = \begin{cases} \frac{2\sqrt{\pi}(-4\pi ny)^{1/2}}{\Gamma(k+1)} K_{k+1/2}(-2\pi ny)\sqrt{\frac{-m}{n}}\frac{1}{yc\sqrt{l}} J_{2k+1}\left(\frac{4\pi\sqrt{-mn}}{c\sqrt{l}}\right) & \text{if } n<0 \\ \frac{1}{y^{k+1}}\frac{2\pi^{k+2}m^{k+1}}{(k+1/2)(\Gamma(k+1))^2 c^{2k+2} l^{k+1}} & \text{if } n=0 \\ \frac{2\sqrt{\pi}(4\pi ny)^{1/2}}{\Gamma(k+1)} K_{k+1/2}(2\pi ny)\sqrt{\frac{m}{n}}\frac{1}{yc\sqrt{l}} I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right) & \text{if } n>0. \end{cases}$$

By equations 10.27.4, 10.47.7, 10.47.8, 10.49.8, and 10.49.10 in [25], we have for $n>0$

$$K_{k+1/2}(2\pi ny) = \pi\sqrt{ny}e^{-2\pi ny}\sum_{j=0}^{k}\frac{(k+j)!}{(2\pi ny)^{j+1}2^j j!(k-j)!}.$$

Therefore by inserting (3.27) and (3.26) into (3.25), we obtain the Fourier expansion

$$\begin{aligned} \mathcal{D}^k f(z) = &\sum_{n=1}^{N_\infty} a(-n)e(-nz)c_k(n,y) + \frac{B_k(0)}{y^k} + \sum_{n=1}^{\infty} B_k(n,y)e(nz) \\ &+ \sum_{n=1}^{N_\infty}\left(B_k(-n,y) + a(-n)(-1)^{k+1}c_k^*(n,y)\right)e(-n\bar{z}) + \sum_{n>N_\infty} B_k(-n,y)e(-n\bar{z}), \end{aligned} \tag{3.28}$$

where

$$B_k(0) := \frac{2\pi^{k+2}}{(k+1/2)k!} \sum_{l|N} \frac{\lambda(l)}{l^{k+1}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-2k-2} \sum_{m=1}^{N_\infty} a(-m) m^{2k+1} S(-m\tilde{l}, 0; c),$$

and for $n \geq 1$

$$B_k(-n, y) := \frac{2\pi}{\sqrt{n}} d_k^*(n, y) S_k(-n) \quad \text{and} \quad B_k(n, y) := \frac{2\pi}{\sqrt{n}} d_k^*(n, y) S_k(n),$$

where

$$S_k(-n) := \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) m^{k+1/2} S(-m\tilde{l}, -n; c) J_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right),$$

$$S_k(n) := \sum_{l|N} \frac{\lambda(l)}{\sqrt{l}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} c^{-1} \sum_{m=1}^{N_\infty} a(-m) m^{k+1/2} S(-m\tilde{l}, n; c) I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{l}}\right).$$

We now simplify the Fourier expansion. By applying the differential operator $\mathcal{D}^k$ to f directly and using Lemma 3.15, we obtain

$$\mathcal{D}^k f(z) = \sum_{n=1}^{N_\infty} a(-n) g_{-n}(y) e(-nz) + \frac{(-1)^k (2k)! a(0)}{(4\pi)^k k! y^k} + \sum_{n=1}^{\infty} a(n) g_n(y) e(nz) \tag{3.29}$$

for some rational functions $g_{-n}(y)$ and $g_n(y)$. By comparing the Fourier expansion of $\mathcal{D}^k f$ in (3.28) and (3.29), we get

$$B_k(0) = \frac{(-1)^k (2k)! a(0)}{(4\pi)^k k!}.$$

In particular, by inserting this identity in (3.28) and observing that

$$c_k(0,y) = \frac{(-1)^k (2k)!}{(4\pi y)^k k!},$$

we complete the proof. □

By (3.23), $\mathcal{D}^k f$ is a linear combination of the $P_{m,\lambda}(z, k+1, 0)$. So by Lemma 3.12,

$$(\mathcal{D}^k f(z))|_0 W_l = \lambda(l) \mathcal{D}^k f(z).$$

Hence by Lemma 3.11 and Proposition 3.16, we get

$$\begin{aligned}
&\mathcal{D}^k f(\gamma_{\frac{1}{r},t} z) = \lambda(l) \mathcal{D}^k f\left(\frac{z}{l} + \frac{t}{l} + \frac{1 - l\tilde{l}}{N}\right) \\
&= \lambda(l) \sum_{n=0}^{N_\infty} a(-n) e\left(\frac{-nz}{l}\right) \zeta_N^{-n(rt+1-l\tilde{l})} c_k\left(n, \frac{y}{l}\right) + \lambda(l) \sum_{n=1}^{\infty} B_k\left(n, \frac{y}{l}\right) e\left(\frac{nz}{l}\right) \zeta_N^{n(rt+1-l\tilde{l})}.
\end{aligned} \tag{3.30}$$

3.7 Regularization of $\mathcal{D}^k f$

In this subsection, we construct a function which regularizes the function $\mathcal{D}^k f$ in the cusps of $\Gamma_0(N)$ where $k \geq 0$ and $f \in M^!_{-2k}(N)$ satisfies the conditions in Theorem 3.2.

Let $\lambda_0 : \mathbb{R} \to [0, 1]$ be a C^∞ function such that

$$\lambda_0(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1. \end{cases}$$

Let $k \geq 0$. Let $C_N := \frac{2N}{\sqrt{3}}$ and $\eta > 0$. Define

$$f_{k,\eta}(z) := \sum_{l|N} \lambda(l) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l z)$$

where

$$g_{k,\eta}(z) := \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}(\mathrm{Im}(z))e(-mz),$$
$$\psi_{m,k,\eta}(y) := \lambda_0\left(\frac{y-C_N}{\eta}\right)c_k(m,y)$$

and $c_k(m,y)$ is defined in Theorem 3.2. Then define the regularized function

$$f_{k,\eta}^{\mathrm{reg}}(z) := \mathcal{D}^k f(z) - f_{k,\eta}(z).$$

Proposition 3.17. *Let $\gamma_{\frac{1}{r},t} \in \mathcal{C}_N$. Then for $y \geq N(C_N+\eta)$, we have*

$$f_{k,\eta}^{\mathrm{reg}}(\gamma_{\frac{1}{r},t}z) = \lambda(l)\sum_{n=1}^{\infty} b_k\left(n,\frac{y}{l}\right)e\left(\frac{nz}{l}\right)\zeta_N^{n(rt+1-l\tilde{l})}$$

where $b_k(n,y) := B_k(n,y)$ if $k \geq 1$ and $b_k(n,y) := a(n)$ if $k = 0$.

Proof. Arguing as in the proof of Proposition 3.13, we obtain

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g_{k,\eta}(\gamma W_l z) = \delta_{\infty,\frac{1}{r}}g_{k,\eta}(z) + \sum_{n\in\mathbb{Z}} e(nx)\sum_{\substack{c>0\\ c\equiv 0 \pmod{N/l}\\ (c,l)=1}}\sum_{m=0}^{N_\infty} a(-m)S(-\tilde{l}m,n;c)\alpha_{k,\eta}(m,c\sqrt{l},y,n), \tag{3.31}$$

where

$$\alpha_{k,\eta}(m,c,y,n) := \int_{-\infty}^{\infty}\psi_{m,k,\eta}\left(\frac{y}{c^2(t^2+y^2)}\right)e\left(\frac{m}{c^2(t+iy)}-nt\right)dt.$$

Hence

$$f_{k,\eta}(z) = \sum_{l|N}\lambda(l)\delta_{\infty,\frac{1}{r}}g_{k,\eta}(z) + \sum_{l|N}\lambda(l)\sum_{n\in\mathbb{Z}} e(nx)\sum_{\substack{c>0\\ c\equiv 0 \pmod{N/l}\\ (c,l)=1}}\sum_{m=0}^{N_\infty} a(-m)S(-\tilde{l}m,n;c)\alpha_{k,\eta}(m,c\sqrt{l},y,n).$$

Since $\delta_{\infty,1/r} = 1$ only when $r = N$ (so $l = 1$), it follows from the definition of $g_{k,\eta}$ that

$$f_{k,\eta}(z) = \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}(y)e(-mz) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{l|N} \lambda(l) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l} \\ (c,l)=1}} \sum_{m=0}^{N_\infty} a(-m)S(-\tilde{l}m, n; c)\alpha_{k,\eta}(m, c\sqrt{l}, y, n). \tag{3.32}$$

Using the definition of $f_{k,\eta}$, properties (P.1) and (P.2) of $\lambda(l)$, and the group law on $\{W_l : l|N\}$, we have

$$f_{k,\eta}(W_l z) = \lambda(l) f_{k,\eta}(z).$$

Thus by Lemma 3.11 and (3.32), we obtain

$$\begin{aligned} f_{k,\eta}(\gamma_{\frac{1}{r},t} z) &= \lambda(l) f_{k,\eta}\left(\frac{z}{l} + \frac{t}{l} + \frac{1-l\tilde{l}}{N}\right) \\ &= \lambda(l) \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}\left(\frac{y}{l}\right) e\left(\frac{-mz}{l}\right) \zeta_N^{-m(rt+1-l\tilde{l})} \\ &\quad + \lambda(l) \sum_{n\in\mathbb{Z}} e\left(\frac{nx}{l}\right) \zeta_N^{n(rt+1-l\tilde{l})} \sum_{l_0|N} \lambda(l_0) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l_0} \\ (c,l_0)=1}} \sum_{m=0}^{N_\infty} a(-m)S(-\tilde{l}_0 m, n; c)\alpha_{k,\eta}\left(m, c\sqrt{l_0}, \frac{y}{l}, n\right). \end{aligned} \tag{3.33}$$

Now, by definition of $\psi_{m,k,\eta}$ we have

$$\psi_{m,k,\eta}(y) = \begin{cases} 0 & \text{if } y \leq C_N \\ c_k(m,y) & \text{if } y \geq C_N + \eta. \end{cases} \tag{3.34}$$

Moreover, by definition of $\alpha_{k,\eta}$ we have

$$\alpha_{k,\eta}\left(m, c\sqrt{l_0}, \frac{y}{l}, n\right) = \int_{-\infty}^{\infty} \psi_{m,k,\eta}\left(\frac{y/l}{c^2 l_0(t^2+(y/l)^2)}\right) e\left(\frac{m}{c^2 l_0(t+iy/l)} - nt\right) dt.$$

Observe that for $y \geq \frac{\sqrt{3}}{2}$,

$$\frac{y/l}{c^2 l_0(t^2+(y/l)^2)} \leq \frac{y/l}{t^2+(y/l)^2} \leq \frac{y/l}{(y/l)^2} = \frac{l}{y} \leq \frac{2l}{\sqrt{3}} \leq \frac{2N}{\sqrt{3}} = C_N,$$

which yields that

$$\psi_{m,k,\eta}\left(\frac{y/l}{c^2 l_0(t^2+(y/l)^2)}\right) = 0$$

and hence

$$\alpha_{k,\eta}\left(m, c\sqrt{l_0}, \frac{y}{l}, n\right) = 0.$$

So by (3.33), for $y \geq \frac{\sqrt{3}}{2}$,

$$f_{k,\eta}(\gamma_{\frac{1}{r},t} z) = \lambda(l) \sum_{m=0}^{N_\infty} a(-m) \psi_{m,k,\eta}\left(\frac{y}{l}\right) e\left(\frac{-mz}{l}\right) \zeta_N^{-m(rt+1-l\tilde{l})}. \tag{3.35}$$

Next, let $y \geq N(C_N+\eta)$. Since $\frac{y}{l} \geq \frac{y}{N} \geq C_N + \eta$ for every $l|N$,

$$\psi_{m,k,\eta}\left(\frac{y}{l}\right) = c_k\left(m, \frac{y}{l}\right).$$

Thus for $y \geq N(C_N+\eta)$,

$$f_{k,\eta}(\gamma_{\frac{1}{r},t} z) = \lambda(l) \sum_{m=0}^{N_\infty} a(-m) e\left(\frac{-mz}{l}\right) \zeta_N^{-m(rt+1-l\tilde{l})} c_k\left(m, \frac{y}{l}\right). \tag{3.36}$$

Since

$$f^{\text{reg}}_{k,\eta}(\gamma_{\frac{1}{r},t}z) = \mathcal{D}^k f(\gamma_{\frac{1}{r},t}z) - f_{k,\eta}(\gamma_{\frac{1}{r},t}z),$$

the proposition follows from (3.30) and (3.36) in the case $k \geq 1$, and it follows from (3.10) and (3.36) in the case $k = 0$. □

For future reference, we will require the following inequalities.

Lemma 3.18. *Let $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$, $\{W_l : l|N\}$ be the group of Atkin-Lehner operators defined in* (3.6)*, $\gamma_{\frac{1}{R},t} \in \mathcal{C}_N$, and $z \in \mathbb{H}$. Then*

$$\operatorname{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z) \leq \max\left\{\frac{\operatorname{Im}(z)}{l}, \frac{1}{l\operatorname{Im}(z)}\right\}.$$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \gamma\sqrt{l}W_l\gamma_{\frac{1}{R},t}$. Then $a, b, c, d \in \mathbb{Z}$ and

$$\frac{1}{\sqrt{l}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma W_l \gamma_{\frac{1}{R},t} \in SL_2(\mathbb{R}).$$

So

$$\operatorname{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z) = \operatorname{Im}\left(\frac{1}{\sqrt{l}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \frac{\operatorname{Im}(z)}{\left|\frac{c}{\sqrt{l}}z + \frac{d}{\sqrt{l}}\right|^2} = \frac{\operatorname{Im}(z)l}{|cz+d|^2}. \tag{3.37}$$

Let $\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 N & \gamma_4 \end{pmatrix} := \gamma$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 N & \gamma_4 \end{pmatrix}\begin{pmatrix} l & (\tilde{l}l-1)/r \\ rl & \tilde{l}l \end{pmatrix}\begin{pmatrix} 1 & t \\ R & Rt+1 \end{pmatrix}.$$

A straightforward calculation yields

$$\begin{aligned} c &= N\gamma_3 \left[l + R\left(\frac{\tilde{l}l - 1}{r}\right)\right] + \gamma_4(rl + R\tilde{l}l), \\ d &= ct + N\gamma_3\left(\frac{\tilde{l}l-1}{r}\right) + \gamma_4\tilde{l}l. \end{aligned} \tag{3.38}$$

First, we consider the case $c = 0$. Then

$$d = N\gamma_3\left(\frac{\tilde{l}l-1}{r}\right) + \gamma_4\tilde{l}l = l\left[r\gamma_3\left(\frac{\tilde{l}l-1}{r}\right) + \gamma_4\tilde{l}\right],$$

which implies that $l|d$. Since

$$l = \det(\gamma\sqrt{l}W_l\gamma_{\frac{1}{R},t}) = \det\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) = ad,$$

we have $d|l$. Thus $|d| = l$. Therefore (3.37) becomes

$$\text{Im}(\gamma W_l\gamma_{\frac{1}{R},t}z) = \frac{\text{Im}(z)}{l}.$$

Now, suppose that $c \neq 0$. By (3.38) we have $l|c$ and hence $|c| \geq l$. So

$$|cz+d|^2 = (cx+d)^2 + (cy)^2 \geq c^2y^2 \geq l^2y^2.$$

Thus by (3.37),

$$\text{Im}(\gamma W_l\gamma_{\frac{1}{R},t}z) \leq \frac{\text{Im}(z)l}{l^2(\text{Im}(z))^2} = \frac{1}{l\text{Im}(z)}.$$

□

The next lemma follows directly from Lemma 3.18 and the fact that $\text{Im}(z) \geq \frac{\sqrt{3}}{2}$ when $z \in \mathcal{F}$.

Lemma 3.19. *Let $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$, $\{W_l : l|N\}$ be the group of Atkin-Lehner operators defined in (3.6), $\gamma_{\frac{1}{R},t} \in \mathcal{C}_N$, and $z \in \mathcal{F}$. Then*

$$\mathrm{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z) \leq \max\left\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\right\} \leq \max\left\{\mathrm{Im}(z), \frac{2}{\sqrt{3}}\right\}.$$

3.8 Proof of Theorem 3.2

We now give the proof of Theorem 3.2.

Let

$$\Delta := \Delta_0 = -y^2(\partial_x^2 + \partial_y^2)$$

be the weight 0 hyperbolic Laplacian. A fundamental domain for $\Gamma_0(N)$ is given by

$$\mathcal{F}(N) := \bigcup_{\sigma \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} \sigma \mathcal{F}. \tag{3.39}$$

Assume that $0 < \eta < 1$. By Proposition 3.17, we may apply [22, Proposition 6.1] to obtain

$$\mathrm{Tr}_D(f_{k,\eta}^{\mathrm{reg}}) = h(D) \int_{\mathcal{F}(N)} f_{k,\eta}^{\mathrm{reg}}(z) d\mu + O(\|\Delta^2 f_{k,\eta}^{\mathrm{reg}}\| |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}) \tag{3.40}$$

where $d\mu$ is the normalized hyperbolic measure on $\mathcal{F}(N)$ given by

$$d\mu := \frac{1}{\mathrm{vol}(\mathcal{F}(N))} \frac{dxdy}{y^2}.$$

Since $\mathcal{D}^k f = f_{k,\eta}^{\mathrm{reg}} + f_{k,\eta}$, we have

$$\mathrm{Tr}_D(\mathcal{D}^k f) = \mathrm{Tr}_D(f_{k,\eta}^{\mathrm{reg}}) + \mathrm{Tr}_D(f_{k,\eta}).$$

Let

$$\text{Trace1} := \sum_{l|N} \lambda(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D,N} \\ \frac{2N}{\sqrt{3}} l < y_Q \leq l\left(\frac{2N}{\sqrt{3}} + \eta\right)}} \sum_{m=0}^{N_\infty} a(-m) \psi_{m,k,\eta}\left(\frac{y_Q}{l}\right) e\left(\frac{-m\tau_Q}{l}\right) \zeta_N^{-m(\frac{N}{l}t+1-l\tilde{l})}$$

and

$$\text{Trace2} := \sum_{l|N} \lambda(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D,N} \\ y_Q > l\left(\frac{2N}{\sqrt{3}} + \eta\right)}} \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q/l) e\left(\frac{-m\tau_Q}{l}\right) \zeta_N^{-m(\frac{N}{l}t+1-l\tilde{l})}.$$

Then by Lemma 3.20, $\text{Tr}_D(f_{k,\eta}) = \text{Trace1} + \text{Trace2}$. So (3.40) yields

$$\text{Tr}_D(\mathcal{D}^k f) = h(D) \int_{\mathcal{F}(N)} f_{k,\eta}^{\text{reg}}(z) d\mu + O(\|\Delta^2 f_{k,\eta}^{\text{reg}}\| |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + \text{Trace1} + \text{Trace2}. \tag{3.41}$$

Define

$$R_{N,k,\eta}(D) := \text{Tr}_D(\mathcal{D}^k f) - \text{Trace2}.$$

Then (3.41) can be written as

$$R_{N,k,\eta}(D) = h(D) \int_{\mathcal{F}(N)} f_{k,\eta}^{\text{reg}}(z) d\mu + O(\|\Delta^2 f_{k,\eta}^{\text{reg}}\| |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + \text{Trace1}.$$

In Lemma 3.25, we will show that

$$\int_{\mathcal{F}(N)} f_{k,\eta}^{\text{reg}}(z) d\mu = \int_{\text{reg}} \mathcal{D}^k f(z) d\mu =: \beta_k(f).$$

Note that the integral on the right hand side is independent of η. We now give a bound for Trace1.

A straightforward estimate yields

$$|\text{Trace1}| \le \sum_{l|N} \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D,N} \\ \frac{2N}{\sqrt{3}} l < y_Q \le l\left(\frac{2N}{\sqrt{3}}+\eta\right)}} \sum_{m=0}^{N_\infty} |a(-m)||c_k(m, y_Q/l)| \left| e\left(\frac{-m\tau_Q}{l}\right)\right|$$

$$\le \sum_{l|N} c_l \# \Lambda_{l,\eta}(D),$$

where

$$c_l := \sum_{m=0}^{N_\infty} |a(-m)| \sum_{j=0}^{k} \frac{(k+j)! l^j m^{k-j}}{(4\pi)^j j!(k-j)!} e^{2\pi m\left(\frac{2N}{\sqrt{3}}+1\right)}$$

and

$$\Lambda_{l,\eta}(D) := \left\{ Q \in \mathcal{Q}_D^{\text{red}} : \frac{2N}{\sqrt{3}} l < y_Q \le l\left(\frac{2N}{\sqrt{3}} + \eta\right)\right\}.$$

By Lemma 3.21, we have

$$\# \Lambda_{l,\eta}(D) = O(\eta h(D)) + O(\eta^{-1} |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}).$$

So

$$|\text{Trace1}| = O(\eta h(D)) + O(\eta^{-1} |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}).$$

By Lemma 3.22, we have the estimate

$$\|\Delta^2 f_{k,\eta}^{\text{reg}}\| = O(\eta^{-4}).$$

Combining the preceding estimates yields

$$R_{N,k,\eta}(D) = h(D)\beta_k(f) + O(\eta^{-4} |D|^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O(\eta h(D)).$$

Let $\eta = |D|^{-b}$ for some $b > 0$. Then we use

$$h(D) \ll_\epsilon |D|^{1/2+\epsilon}$$

to obtain

$$R_{N,k,\eta}(D) = h(D)\beta_k(f) + O(|D|^{\frac{1}{2}-(\frac{1}{16}-4b)+\epsilon}) + O(|D|^{\frac{1}{2}-b+\epsilon}).$$

The exponent is optimized when $\frac{1}{16} - 4b = b$, or $b = \frac{1}{80}$. Thus

$$R_{N,k,\eta}(D) = h(D)\beta_k(f) + O(|D|^{\frac{1}{2}-\frac{1}{80}+\epsilon}).$$

This completes the proof of Theorem 3.2. □

Lemma 3.20. *We have*

$$\mathrm{Tr}_D(f_{k,\eta}) = \mathrm{Trace1} + \mathrm{Trace2}.$$

Proof. By definition of $f_{k,\eta}$, we have

$$\begin{aligned}
\mathrm{Tr}_D(f_{k,\eta}) &= \sum_{Q \in \mathcal{Q}_D^{\mathrm{red}}} f_{k,\eta}(\gamma_Q \tau_Q) \\
&= \sum_{Q \in \mathcal{Q}_D^{\mathrm{red}}} \sum_{l|N} \lambda(l) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_Q \tau_Q) \\
&= \sum_{l|N} \lambda(l) \sum_{Q \in \mathcal{Q}_D^{\mathrm{red}}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_Q \tau_Q).
\end{aligned}$$

Using Lemma 3.19, we know that if $\mathrm{Im}(\tau_Q) \le \frac{2N}{\sqrt{3}}$, then

$$\mathrm{Im}(\gamma W_l \gamma_Q \tau_Q) \le \frac{2N}{\sqrt{3}} = C_N,$$

which yields that $\psi_{m,k,\eta}(\mathrm{Im}(\gamma W_l \gamma_Q \tau_Q)) = 0$ and hence $g_{k,\eta}(\gamma W_l \gamma_Q \tau_Q) = 0$. Thus

$$\mathrm{Tr}_D(f_{k,\eta}) = \sum_{l|N} \lambda(l) \sum_{\substack{Q \in \mathcal{Q}_D^{\mathrm{red}} \\ y_Q \geq \frac{2N}{\sqrt{3}}}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_Q \tau_Q). \tag{3.42}$$

Recall that for each $Q \in \mathcal{Q}_D^{\mathrm{red}}$, we have $Q \in S_{L,t,D,N}$ for some $L|N$ and $t \in \{0, 1, \ldots, L-1\}$ where $\gamma_Q = \gamma_{\frac{L}{N},t} = \begin{pmatrix} 1 & t \\ \frac{N}{L} & \frac{N}{L}t + 1 \end{pmatrix}$. For each $L|N$, let $R := N/L$. So (3.42) can be written as

$$\mathrm{Tr}_D(f_{k,\eta}) = \sum_{l|N} \lambda(l) \sum_{L|N} \sum_{t=0}^{L-1} \sum_{\substack{Q \in S_{L,t,D,N} \\ y_Q \geq \frac{2N}{\sqrt{3}}}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_{\frac{1}{R},t} \tau_Q).$$

By (3.7), we know

$$\gamma_{\frac{1}{R},t} = \sqrt{L} W_L A_L \begin{pmatrix} 1 & n_{R,t} \\ 0 & 1 \end{pmatrix},$$

where $n_{R,t} := t + \frac{1 - L\tilde{L}}{R}$. So for each $l|N$ and $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$, we have

$$\gamma W_l \gamma_{\frac{1}{R},t} z = \sqrt{L} \gamma W_l W_L A_L \begin{pmatrix} 1 & n_{R,t} \\ 0 & 1 \end{pmatrix} z = \sqrt{L} \gamma B W_{l_0} \left(\frac{z + n_{R,t}}{L} \right)$$

for some $B \in \Gamma_0(N)$ where $l_0 := \frac{lL}{(l,L)^2}$. So

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_{\frac{1}{R},t} z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta} \left(\gamma W_{l_0} \left(\frac{z + n_{R,t}}{L} \right) \right). \tag{3.43}$$

Let $r_0 := N/l_0$. Using the Fourier expansion in (3.31), we obtain

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g_{k,\eta}(\gamma W_{l_0} z) =$$
$$\delta_{\infty,\frac{1}{r_0}} g_{k,\eta}(z) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l_0} \\ (c,l_0)=1}} \sum_{m=0}^{N_\infty} a(-m) S(-\tilde{l_0}m, n; c)\alpha_{k,\eta}(m, c\sqrt{l_0}, y, n). \tag{3.44}$$

So by (3.43) and (3.44),

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} g_{k,\eta}(\gamma W_l \gamma_{\frac{1}{R},t} z) = \delta_{\infty,\frac{1}{r_0}} g_{k,\eta}\left(\frac{z+n_{R,t}}{L}\right)$$
$$+ \sum_{n\in\mathbb{Z}} e\left(\frac{n(x+n_{R,t})}{L}\right) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l_0} \\ (c,l_0)=1}} \sum_{m=0}^{N_\infty} a(-m) S(-\tilde{l_0}m, n; c)\alpha_{k,\eta}(m, c\sqrt{l_0}, \frac{y}{L}, n).$$

Thus

$$\mathrm{Tr}_D(f_{k,\eta}) = \sum_{l|N} \lambda(l) \sum_{L|N} \sum_{t=0}^{L-1} \sum_{\substack{Q\in S_{L,t,D,N} \\ y_Q \geq \frac{2N}{\sqrt{3}}}} \delta_{\infty,\frac{1}{r_0}} g_{k,\eta}\left(\frac{\tau_Q+n_{R,t}}{L}\right)$$
$$+ \sum_{l|N} \lambda(l) \sum_{L|N} \sum_{t=0}^{L-1} \sum_{\substack{Q\in S_{L,t,D,N} \\ y_Q \geq \frac{2N}{\sqrt{3}}}} \sum_{n\in\mathbb{Z}} e\left(\frac{n(x_Q+n_{R,t})}{L}\right)$$
$$\times \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N/l_0} \\ (c,l_0)=1}} \sum_{m=0}^{N_\infty} a(-m) S(-\tilde{l_0}m, n; c)\alpha_{k,\eta}(m, c\sqrt{l_0}, \frac{y_Q}{L}, n). \tag{3.45}$$

Observe that for all $c \geq 1$ and $t \in \mathbb{R}$,

$$\frac{y_Q/L}{c^2 l_0(t^2 + (y_Q/L)^2)} \leq \frac{y_Q/L}{l_0((y_Q/L)^2)} = \frac{L}{l_0 y_Q}.$$

If $y_Q \geq \frac{2N}{\sqrt{3}}$, then

$$\frac{L}{l_0 y_Q} = \frac{L}{y_Q}\frac{(l,L)^2}{lL} = \frac{(l,L)}{l}\frac{(l,L)}{y_Q} \leq \frac{N}{y_Q} \leq \frac{\sqrt{3}}{2} \leq C_N,$$

which yields

$$\psi_{m,k,\eta}\left(\frac{y_Q/L}{c^2 l_0(t^2+(y_Q/L)^2)}\right) = 0,$$

and hence

$$\alpha_{k,\eta}\left(m, c\sqrt{l_0}, \frac{y_Q}{L}, n\right) = 0.$$

Thus (3.45) becomes

$$\mathrm{Tr}_D(f_{k,\eta}) = \sum_{l|N}\lambda(l)\sum_{L|N}\sum_{t=0}^{L-1}\sum_{\substack{Q\in S_{L,t,D,N}\\ y_Q\geq\frac{2N}{\sqrt{3}}}}\delta_{\infty,\frac{1}{r_0}} g_{k,\eta}\left(\frac{\tau_Q+n_{R,t}}{L}\right).$$

Since

$$\delta_{\infty,\frac{1}{r_0}} = 1 \iff \text{the cusp } 1/r_0 \text{ is equivalent to the cusp } \infty$$
$$\iff r_0 = N \iff l_0 = 1 \iff lL = (l,L)^2 \iff l = L$$

and $\delta_{\infty,1/r_0} = 0$ otherwise, it follows that

$$\begin{aligned}\mathrm{Tr}_D(f_{k,\eta}) &= \sum_{l|N}\lambda(l)\sum_{t=0}^{l-1}\sum_{\substack{Q\in S_{l,t,D,N}\\ y_Q\geq\frac{2N}{\sqrt{3}}}} g_{k,\eta}\left(\frac{\tau_Q+n_{r,t}}{l}\right)\\ &= \sum_{l|N}\lambda(l)\sum_{t=0}^{l-1}\sum_{\substack{Q\in S_{l,t,D,N}\\ y_Q\geq\frac{2N}{\sqrt{3}}}}\sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}\left(\frac{y_Q}{l}\right)e\left(\frac{-m\tau_Q}{l}\right)\zeta_N^{-m(rt+1-l\tilde{l})}.\end{aligned}$$

By (3.34), we have

$$\psi_{m,k,\eta}(y_Q/l) = 0 \text{ if } y_Q \leq lC_N,$$

and

$$\psi_{m,k,\eta}(y_Q/l) = c_k(m, y_Q/l) \text{ if } y_Q \geq l(C_N + \eta).$$

Therefore the lemma follows. □

Lemma 3.21. *Fix* $l|N$. *For each number* $0 < \eta < 1$ *we have*

$$\#\Lambda_{l,\eta}(D) = O(\eta h(D)) + O(\eta^{-1}|D|^{\frac{1}{2}-\frac{1}{16}+\epsilon}).$$

Proof. Let $0 < \eta < 1$ and $\phi_{l,\eta} : \mathbb{R} \to [0,1]$ be a C^∞ function such that

(1) $\phi_{l,\eta}$ is supported on $\left(\frac{2N}{\sqrt{3}}l - \eta, l(\frac{2N}{\sqrt{3}} + \eta) + \eta\right)$

(2) $\phi_{l,\eta} = 1$ on $\left[\frac{2N}{\sqrt{3}}l, l(\frac{2N}{\sqrt{3}} + \eta)\right]$

(3) $\phi_{l,\eta}$ satisfies the bound

$$\phi_{l,\eta}^{(A)} \ll \eta^{-A}, \quad A = 0, 1, 2. \tag{3.46}$$

Define

$$G_{l,\eta}(z) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \phi_{l,\eta}(\text{Im}(\gamma z)).$$

Then

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}_D^{\text{red}}} G_{l,\eta}(\tau_Q) &= \sum_{Q\in\mathcal{Q}_D^{\text{red}}} \sum_{\substack{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})\\ \gamma\neq \bar{I}}} \phi_{l,\eta}(\text{Im}(\gamma\tau_Q)) + \sum_{Q\in\mathcal{Q}_D^{\text{red}}} \phi_{l,\eta}(\text{Im}(\tau_Q))\\
&\geq \sum_{Q\in\mathcal{Q}_D^{\text{red}}} \phi_{l,\eta}(\text{Im}(\tau_Q))\\
&= \sum_{\substack{Q\in\mathcal{Q}_D^{\text{red}}\\ Q\in\Lambda_{l,\eta}(D)}} 1 \;+ \sum_{\substack{Q\in\mathcal{Q}_D^{\text{red}}\\ Q\notin\Lambda_{l,\eta}(D)}} \phi_{l,\eta}(\text{Im}(\tau_Q))\\
&\geq \#\Lambda_{l,\eta}(D).
\end{aligned}$$

We have

$$G_{l,\eta}(z) = \frac{1}{\text{vol}(\mathcal{F})}\hat{\phi}_{l,\eta}(1) + \frac{1}{2\pi i}\int_{\mathbb{R}} \hat{\phi}_{l,\eta}\left(1/2+it\right) E_\infty\left(z, 1/2+it\right) dt,$$

where $E_\infty(z,s)$ is the real-analytic Eisenstein series associated to the cusp ∞ and

$$\hat{\phi}_{l,\eta}(s) := \int_0^\infty \phi_{l,\eta}(u)u^{-s-1}du.$$

Thus

$$\sum_{Q\in\mathcal{Q}_D^{\text{red}}} G_{l,\eta}(\tau_Q) = \frac{1}{\text{vol}(\mathcal{F})}\hat{\phi}_{l,\eta}(1)h(D) + \frac{1}{2\pi i}\int_{\mathbb{R}} \hat{\phi}_{l,\eta}\left(1/2+it\right) W_\infty(D,t)dt,$$

where

$$W_\infty(D,t) := \sum_{Q\in\mathcal{Q}_D^{\text{red}}} E_\infty\left(\tau_Q, 1/2+it\right).$$

Now, a straightforward estimate yields

$$\hat{\phi}_{l,\eta}(1) = \int_{\frac{2N}{\sqrt{3}}l-\eta}^{\frac{2N}{\sqrt{3}}l+l\eta+\eta} \phi_{l,\eta}(u)u^{-2}du = O(\eta).$$

By (6.5) of [22], we have

$$W_\infty(D,t) \ll \left(1/4+t^2\right)^{\frac{17}{96}+\epsilon} |D|^{\frac{1}{2}-\frac{1}{16}+\epsilon},$$

hence we obtain

$$\sum_{Q\in\mathcal{Q}_D^{\mathrm{red}}} G_{l,\eta}(\tau_Q) = O(\eta h(D)) + O(c_\eta |D|^{\frac{1}{2}-\frac{1}{16}+\epsilon}),$$

where

$$c_\eta := \int_{\mathbb{R}} \left|\hat{\phi}_{l,\eta}\left(1/2+it\right)\right| \left(1/4+t^2\right)^{\frac{17}{96}+\epsilon} dt.$$

We integrate by parts 2 times and use the bound (3.46) to obtain

$$\hat{\phi}_{l,\eta}\left(1/2+it\right) \ll \frac{\eta^{-1}}{\frac{1}{4}+t^2},$$

which yields that $c_\eta \ll \eta^{-1}$. Thus the lemma follows. □

Lemma 3.22. *For each $0<\eta<1$, we have*

$$\|\Delta^2 f_{k,\eta}^{\mathrm{reg}}\| = O(\eta^{-4}).$$

Proof. We choose $\mathcal{C}_N$ as the right coset representatives of $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$. Then using the $SL_2(\mathbb{R})$-

invariance of the normalized hyperbolic measure $d\mu$ on $\mathcal{F}(N)$, we find that

$$\begin{aligned}\|\Delta^2 f_{k,\eta}^{\text{reg}}\|^2 &= \int_{\mathcal{F}(N)} |\Delta^2 f_{k,\eta}^{\text{reg}}(z)|^2 d\mu \\ &= \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int_{\gamma_{\frac{1}{r},t}\mathcal{F}} |\Delta^2 f_{k,\eta}^{\text{reg}}(z)|^2 d\mu \\ &= \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int_{\mathcal{F}} |\Delta^2 f_{k,\eta}^{\text{reg}}(\gamma_{\frac{1}{r},t}z)|^2 d\mu.\end{aligned}$$

Let

$$P\left(\frac{2N}{\sqrt{3}}\right) := \{z = x+iy \in \mathcal{F} : y \geq \frac{2N}{\sqrt{3}}\}.$$

Then

$$\|\Delta^2 f_{k,\eta}^{\text{reg}}\|^2 \leq \text{I} + \text{II},$$

where

$$\text{I} := \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int_{\mathcal{F}-P\left(\frac{2N}{\sqrt{3}}\right)} |\Delta^2 f_{k,\eta}^{\text{reg}}(\gamma_{\frac{1}{r},t}z)|^2 d\mu$$

and

$$\text{II} := \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int_{P\left(\frac{2N}{\sqrt{3}}\right)} |\Delta^2 f_{k,\eta}^{\text{reg}}(\gamma_{\frac{1}{r},t}z)|^2 d\mu.$$

First, we estimate II. Let $z \in \mathcal{F}$. Then $y = \text{Im}(z) \geq \frac{\sqrt{3}}{2}$. If $y \leq \frac{2N}{\sqrt{3}}$, then $\frac{y}{l} \leq \frac{2N}{\sqrt{3}}$ which yields that $\psi_{m,k,\eta}\left(\frac{y}{l}\right) = 0$, hence by (3.35) we have

$$f_{k,\eta}(\gamma_{\frac{1}{r},t}z) = 0. \tag{3.47}$$

By (3.35), (3.47) and Proposition 3.17, we obtain

$$f_{k,\eta}^{\mathrm{reg}}(\gamma_{\frac{1}{r},t}z) =$$
$$\begin{cases} \mathcal{D}^k f(\gamma_{\frac{1}{r},t}z) & \text{if } \frac{\sqrt{3}}{2} \leq y \leq \frac{2N}{\sqrt{3}} \\ \mathcal{D}^k f(\gamma_{\frac{1}{r},t}z) - \lambda(l) \sum\limits_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}\left(\frac{y}{l}\right) e\left(\frac{-mz}{l}\right) \zeta_N^{-m(rt+1-l\tilde{l})} & \text{if } \frac{2N}{\sqrt{3}} < y \leq N(\frac{2N}{\sqrt{3}} + \eta) \\ \lambda(l) \sum\limits_{n=1}^{N_\infty} b_k\left(n, \frac{y}{l}\right) e\left(\frac{nz}{l}\right) \zeta_N^{n(rt+1-l\tilde{l})} & \text{if } y > N(\frac{2N}{\sqrt{3}} + \eta). \end{cases} \tag{3.48}$$

By splitting the y-integral in II into the different ranges considered in (3.48), we obtain

$$\mathrm{II} = \sum_{\gamma_{\frac{1}{r},t} \in \mathcal{C}_N} \int_{\frac{2N}{\sqrt{3}}}^{\infty} \int_{-1/2}^{1/2} |\Delta^2 f_{k,\eta}^{\mathrm{reg}}(\gamma_{\frac{1}{r},t}z)|^2 d\mu = \mathrm{III} + O(1)$$

where

$$\mathrm{III} := \sum_{\gamma_{\frac{1}{r},t} \in \mathcal{C}_N} \int_{\frac{2N}{\sqrt{3}}}^{N(\frac{2N}{\sqrt{3}}+\eta)} \int_{-1/2}^{1/2} |\Delta^2 f_{k,\eta}^{\mathrm{reg}}(\gamma_{\frac{1}{r},t}z)|^2 d\mu.$$

By linearity of Δ and the triangle inequality, we obtain

$$\mathrm{III} \leq \mathrm{IV} + \mathrm{V} + O(1)$$

where

$$\mathrm{IV} :=$$
$$2 \sum_{\gamma_{\frac{1}{r},t} \in \mathcal{C}_N} \int_{\frac{2N}{\sqrt{3}}}^{N(\frac{2N}{\sqrt{3}}+1)} \int_{-1/2}^{1/2} \left|\Delta^2 \mathcal{D}^k f(\gamma_{\frac{1}{r},t}z)\right| \left|\sum_{m=0}^{N_\infty} a(-m)\zeta_N^{-m(rt+1-l\tilde{l})} \Delta^2 \left(\psi_{m,k,\eta}\left(\frac{y}{l}\right) e\left(\frac{-mz}{l}\right)\right)\right| d\mu$$

and

$$\mathrm{V} := \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int\limits_{\frac{2N}{\sqrt{3}}}^{N(\frac{2N}{\sqrt{3}}+1)} \int\limits_{-1/2}^{1/2} \left|\sum_{m=0}^{N_\infty} a(-m)\zeta_N^{-m(rt+1-l\tilde{l})}\Delta^2\left(\psi_{m,k,\eta}\left(\frac{y}{l}\right)e\left(\frac{-mz}{l}\right)\right)\right|^2 d\mu.$$

Using the estimate

$$\max_{(x,y)\in[-1/2,1/2]\times[\frac{2N}{\sqrt{3}},N(\frac{2N}{\sqrt{3}}+1)]} \left|\Delta^2\left(\psi_{m,k,\eta}\left(\frac{y}{l}\right)e\left(\frac{-mz}{l}\right)\right)\right| \ll \eta^{-4},$$

we obtain

$$\mathrm{IV} \ll \eta^{-4} \quad \text{and} \quad \mathrm{V} \ll \eta^{-8}.$$

Thus

$$\mathrm{II} \ll \eta^{-8}.$$

Next, we estimate I. Observe that $\mathcal{F} - P\left(\frac{2N}{\sqrt{3}}\right)$ can be contained in a rectangle

$$R_N := [-1/2, 1/2] \times [\sqrt{3}/2, 2N/\sqrt{3}].$$

Then

$$\mathrm{I} \leq \sum_{\gamma_{\frac{1}{r},t}\in\mathcal{C}_N} \int_{R_N} |\Delta^2\mathcal{D}^k f(\gamma_{\frac{1}{r},t}z)|^2 d\mu = O(1).$$

Therefore by combining the estimates for I and II we obtain

$$\|\Delta^2 f_{k,\eta}^{\mathrm{reg}}\|^2 = O(\eta^{-8})$$

hence

$$\|\Delta^2 f_{k,\eta}^{\text{reg}}\| = O(\eta^{-4}).$$

□

3.9 Regularized integrals

For a fixed $Y > \frac{2}{\sqrt{3}}$, define the truncated domain

$$\mathcal{F}_Y := \{z \in \mathcal{F} : \text{Im}(z) \leq Y\}$$

and

$$\mathcal{F}_Y(N) := \bigcup_{\sigma \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} \sigma \mathcal{F}_Y.$$

Let $f \in M^!_{-2k}(N)$ and suppose that f satisfies the conditions in Theorem 3.2. We define the regularized integral

$$\beta_k(f) := \int_{\text{reg}} \mathcal{D}^k f(z) d\mu := \lim_{Y \to \infty} \int_{\mathcal{F}_Y(N)} \mathcal{D}^k f(z) d\mu.$$

For each $l|N$ and $\eta > 0$, let

$$g_{k,\eta,Y/l}(z) := \sum_{m=0}^{N_\infty} a(-m) \psi_{m,k,\eta,Y/l}(\text{Im}(z)) e(-mz)$$

where

$$\psi_{m,k,\eta,Y/l}(t) := \begin{cases} \psi_{m,k,\eta}(t) & \text{if } t \leq \frac{Y}{l} \\ 0 & \text{if } t > \frac{Y}{l}. \end{cases}$$

Define the Poincaré series

$$f_{k,\eta,Y}(z) := \sum_{l|N} \lambda(l) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g_{k,\eta,Y/l}(\gamma W_l z).$$

So

$$f_{k,\eta,Y}(z) = \sum_{l|N} \lambda(l) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta,Y/l}(\text{Im}(\gamma W_l z))e(-m\gamma W_l z). \tag{3.49}$$

Lemma 3.23. *For $z \in \mathcal{F}_Y(N)$, we have $f_{k,\eta,Y}(z) = f_{k,\eta}(z)$.*

Proof. By (3.49) and the definition of $\psi_{m,k,\eta,Y/l}$, we obtain

$$f_{k,\eta}(z) = f_{k,\eta,Y}(z) + \sum_{l|N} \lambda(l) \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma_0(N) \\ \text{Im}(\gamma W_l z) > \frac{Y}{l}}} \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}(\text{Im}(\gamma W_l z))e(-m\gamma W_l z). \tag{3.50}$$

Let $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$ and $z \in \mathcal{F}_Y(N)$. Then $\gamma W_l z = \gamma W_l \gamma_{\frac{1}{R},t} z'$ for some $\gamma_{\frac{1}{R},t} \in \mathcal{C}_N$ and $z' \in \mathcal{F}_Y$. By Lemma 3.18, we have

$$\text{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z') \leq \max\left\{\frac{\text{Im}(z')}{l}, \frac{1}{\text{Im}(z')l}\right\}.$$

Since $z' \in \mathcal{F}_Y$, $\frac{\sqrt{3}}{2} \leq \text{Im}(z') \leq Y$. So $\frac{\text{Im}(z')}{l} \leq \frac{Y}{l}$ and $\frac{1}{\text{Im}(z')l} \leq \frac{2}{l\sqrt{3}}$. Thus

$$\text{Im}(\gamma W_l z) = \text{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z') \leq \max\left\{\frac{Y}{l}, \frac{2}{l\sqrt{3}}\right\} = \frac{Y}{l}.$$

Hence

$$\#\left\{\gamma \in \Gamma_\infty \backslash \Gamma_0(N) : \text{Im}(\gamma W_l z) > \frac{Y}{l}\right\} = 0.$$

Therefore the lemma follows from (3.50). $\square$

Lemma 3.24. *For $z \in \mathcal{F}(N) \backslash \mathcal{F}_Y(N)$, we have $f_{k,\eta,Y}(z) = 0$.*

Proof. Let $z \in \mathcal{F}(N)\backslash\mathcal{F}_Y(N)$. Then $z = \gamma_{\frac{1}{R},t}z'$ for some $\gamma_{\frac{1}{R},t} \in \mathcal{C}_N$ and $z' \in \mathcal{F}\backslash\mathcal{F}_Y$, i.e., $z' \in \mathcal{F}$ with $\mathrm{Im}(z') > Y$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \gamma\sqrt{l}W_l\gamma_{\frac{1}{R},t}$. Then using the same argument as in the proof of Lemma 3.18, we obtain that if $c = 0$, then

$$\mathrm{Im}(\gamma W_l z) = \mathrm{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z') = \frac{\mathrm{Im}(z')}{l} > \frac{Y}{l},$$

which yields that $\psi_{m,k,\eta,Y/l}(\mathrm{Im}(\gamma W_l z)) = 0$, and if $c \neq 0$, then

$$\mathrm{Im}(\gamma W_l z) = \mathrm{Im}(\gamma W_l \gamma_{\frac{1}{R},t} z') \leq \frac{1}{\mathrm{Im}(z')l} \leq \frac{1}{lY} < \min\left\{\frac{Y}{l}, 1\right\},$$

which yields that $\psi_{m,k,\eta,Y/l}(\mathrm{Im}(\gamma W_l z)) = \psi_{m,k,\eta}(\mathrm{Im}(\gamma W_l z))$. Since $\psi_{m,k,\eta}(t) = 0$ for $t < 1$, $\psi_{m,k,\eta,Y/l}(\mathrm{Im}(\gamma W_l z)) = 0$. Therefore $f_{k,\eta,Y}(z) = 0$. □

Lemma 3.25. *We have*

$$\int_{\mathcal{F}(N)} f^{\mathrm{reg}}_{k,\eta}(z)d\mu = \lim_{Y\to\infty} \int_{\mathcal{F}_Y(N)} \mathcal{D}^k f(z)d\mu.$$

Proof. Since $f^{\mathrm{reg}}_{k,\eta} := \mathcal{D}^k f - f_{k,\eta} \in L^1(\mathcal{F}(N))$, we have

$$\int_{\mathcal{F}(N)} f^{\mathrm{reg}}_{k,\eta}(z)d\mu = \lim_{Y\to\infty} \int_{\mathcal{F}_Y(N)} (\mathcal{D}^k f(z) - f_{k,\eta}(z))d\mu.$$

By Lemma 3.23,

$$\int_{\mathcal{F}_Y(N)} f_{k,\eta}(z)d\mu = \int_{\mathcal{F}_Y(N)} f_{k,\eta,Y}(z)d\mu.$$

Then using Lemma 3.24, we get

$$\int_{\mathcal{F}_Y(N)} f_{k,\eta,Y}(z)d\mu = \int_{\mathcal{F}(N)} f_{k,\eta,Y}(z)d\mu.$$

Hence by (3.49),

$$\int_{\mathcal{F}(N)} f_{k,\eta,Y}(z)d\mu = \sum_{l|N}\lambda(l) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \sum_{m=0}^{N_\infty} a(-m)\int_{\mathcal{F}(N)} \psi_{m,k,\eta,Y/l}(\text{Im}(\gamma W_l z))e(-m\gamma W_l z)d\mu$$
$$= \sum_{l|N}\lambda(l) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \sum_{m=0}^{N_\infty} a(-m)\int_{W_l\mathcal{F}(N)} \psi_{m,k,\eta,Y/l}(\text{Im}(\gamma z))e(-m\gamma z)d\mu.$$

Using the fact that $W_l\mathcal{F}(N)$ is a fundamental domain for $\Gamma_0(N)$, we obtain

$$\int_{\mathcal{F}(N)} f_{k,\eta,Y}(z)d\mu = \sum_{l|N}\lambda(l) \sum_{m=0}^{N_\infty} a(-m) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \int_{\Gamma_0(N)\backslash\mathbb{H}} \psi_{m,k,\eta,Y/l}(\text{Im}(\gamma z))e(-m\gamma z)d\mu$$
$$= \sum_{l|N}\lambda(l) \sum_{m=0}^{N_\infty} a(-m) \int_{\Gamma_\infty\backslash\mathbb{H}} \psi_{m,k,\eta,Y/l}(\text{Im}(z))e(-mz)d\mu$$
$$= \frac{1}{\text{vol}(\mathcal{F}(N))}\sum_{l|N}\lambda(l) \sum_{m=0}^{N_\infty} a(-m) \int_0^{Y/l}\int_0^1 \psi_{m,k,\eta,Y/l}(y)e^{-2\pi i m(x+iy)}\frac{dxdy}{y^2}$$
$$= 0.$$

Thus we conclude that

$$\int_{\mathcal{F}(N)} f_{k,\eta}^{\text{reg}}(z)d\mu = \lim_{Y\to\infty}\int_{\mathcal{F}_Y(N)} \mathcal{D}^k f(z)d\mu.$$

□

We use an argument similar to [6, Theorem 4.1] to deduce the following two propositions. We omit the proofs, which can be found in [18].

Proposition 3.26. *Suppose that $f \in M_0^!(N)$ is an eigenfunction for the Atkin-Lehner operators W_l with eigenvalues $\lambda(l)$. Then*

$$\beta_0(f) = \frac{1}{[SL_2(\mathbb{Z}):\Gamma_0(N)]}\left(a(0)\sum_{l|N} l\lambda(l) - 24\sum_{n=1}^{N_\infty} a(-n)\sum_{\substack{l|N\\ l|n}} \sigma_1\left(\frac{n}{l}\right) l\lambda(l)\zeta_N^{-n(1-l\tilde{l})}\right)$$

where $\sigma_1(n)$ is the sum of all positive divisors of n.

Proposition 3.27. *Suppose that $f \in M^!_{-2}(N)$ is an eigenfunction for the Atkin-Lehner operators W_l with eigenvalues $\lambda(l)$. Then*

$$\begin{aligned}\beta_1(f) = &-\frac{24}{[SL_2(\mathbb{Z}):\Gamma_0(N)]}\sum_{l|N} l\lambda(l)\sum_{\substack{n=1\\ l|n}}^{N_\infty} a(-n)\zeta_N^{-n(1-l\tilde{l})} n\sigma_1\left(\frac{n}{l}\right)\\ &-\frac{a(0)}{12[SL_2(\mathbb{Z}):\Gamma_0(N)]}\sum_{l|N} l^2\lambda(l)\\ &+\frac{4}{[SL_2(\mathbb{Z}):\Gamma_0(N)]}\sum_{l|N} l^2\lambda(l)\sum_{\substack{n=1\\ l|n}}^{N_\infty} a(-n)\zeta_N^{-n(1-l\tilde{l})}\sigma_1\left(\frac{n}{l}\right)\\ &-\frac{48}{[SL_2(\mathbb{Z}):\Gamma_0(N)]}\sum_{l|N} l^2\lambda(l)\sum_{\substack{n=1\\ l|n}}^{N_\infty} a(-n)\zeta_N^{-n(1-l\tilde{l})}\sum_{k=1}^{n/l-1}\sigma_1(k)\sigma_1\left(\frac{n}{l}-k\right)\end{aligned}$$

where

$$\sum_{k=1}^{n/l-1}\sigma_1(k)\sigma_1\left(\frac{n}{l}-k\right) = 0 \quad \text{if} \quad n/l-1 \le 0.$$

3.10 Proof of Theorems 3.4, 3.5 and 3.6

In this subsection, we give proofs of Theorem 3.4, 3.5 and 3.6.

Proof of Theorem 3.4. Recall that

$$F_p(z) := \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} - 10 + \cdots$$

is a weight -2 weakly holomorphic modular form for $\Gamma_0(6)$ which satisfies $F_p|_{-2}W_l = \mu(l)F_p$ for all $l|6$. Then applying Theorem 3.2 with $N = 6, k = 1, N_\infty = 1, a(0) = -10$, $a(-1) = 1$ and

$\lambda(l) = \mu(l)$ for all $l|6$, we obtain

$$\mathrm{Tr}_{D_n}(\mathcal{D}F_p) = \sum_{l|6} \mu(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \left[\frac{5l}{\pi y_Q} + \left(1 - \frac{l}{2\pi y_Q}\right) e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-\left(\frac{6}{l}t+1-l\tilde{l}\right)}\right]$$
$$+ h(D_n)\beta_1(F_p) + O_\epsilon(n^{\frac{1}{2}-\frac{1}{80}+\epsilon}) \tag{3.51}$$

as $n \to \infty$ through a sequence of integers such that $D_n := -24n + 1$ is squarefree. We compute the regularized integral $\beta_1(F_p)$ by applying Proposition 3.27 with $[SL_2(\mathbb{Z}) : \Gamma_0(6)] = 12$ to get $\beta_1(F_p) = 0$. To complete the proof, we combine (3.51) with Theorem 2.1. □

Proof of Theorem 3.5. Let E_4 denote the normalized Eisenstein series of weight 4 for $SL_2(\mathbb{Z})$ given by

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n.$$

Define

$$f(z) := \frac{1}{24} \frac{E_4(z) - 4E_4(2z) - 9E_4(3z) + 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} + 12 + \cdots . \tag{3.52}$$

The function f is a weight 0 weakly holomorphic modular form for $\Gamma_0(6)$ which satisfies $f|_0 W_l = \mu(l) f$ for all $l|6$. Then applying Theorem 3.2 with $N = 6, k = 0, N_\infty = 1, a(0) = 12, a(-1) = 1$ and $\lambda(l) = \mu(l)$ for all $l|6$, we obtain

$$\mathrm{Tr}_{D_n}(f) = \sum_{l|6} \mu(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3}+|D_n|^{-\frac{1}{80}}\right)}} \left[12 + e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-\left(\frac{6}{l}t+1-l\tilde{l}\right)}\right] + h(D_n)\beta_0(f) + O_\epsilon(n^{\frac{1}{2}-\frac{1}{80}+\epsilon}) \tag{3.53}$$

as $n \to \infty$ through a sequence of integers such that $D_n := -24n + 1$ is squarefree. We compute the regularized integral $\beta_0(f)$ by applying Proposition 3.26 with $[SL_2(\mathbb{Z}) : \Gamma_0(6)] = 12$ to get

$\beta_0(f) = 0$. To complete the proof, we combine (3.53) and (3.51) with the following formula for spt(n) due to Ahlgren and Andersen [1],

$$\text{spt}(n) = \frac{1}{12} \sum_{Q \in \mathcal{Q}_{D_n,6,1}/\Gamma_0(6)} (f(\tau_Q) - \mathcal{D}F_p(\tau_Q)).$$

□

Proof of Theorem 3.6. Define

$$F(z) := -\frac{1}{40} \frac{E_4(z) + 4E_4(2z) - 9E_4(3z) - 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} = q^{-1} - 4 + \cdots . \tag{3.54}$$

By page 47 of [8], F is a weight 0 weakly holomorphic modular form for $\Gamma_0(6)$ which satisfies $F|_0 W_l = \beta(l)F$ for all $l|6$, where $\beta(l) := 1$ if $l = 1, 2$ and $\beta(l) := -1$ if $l = 3, 6$. By a straightforward calculation, one can show that $\beta(l)$ satisfies properties (P.1) and (P.2) in Theorem 3.2. So, by applying Theorem 3.2 with $N = 6, k = 0, N_\infty = 1, a(0) = -4, a(-1) = 1$ and $\lambda(l) = \beta(l)$ for all $l|6$, we obtain

$$\text{Tr}_{D_n}(F) = \sum_{l|6} \beta(l) \sum_{t=0}^{l-1} \sum_{\substack{Q \in S_{l,t,D_n,6} \\ y_Q > l\left(4\sqrt{3} + |D_n|^{-\frac{1}{80}}\right)}} \left[-4 + e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-(\frac{6}{l}t + 1 - l\tilde{l})}\right] + h(D_n)\beta_0(F) + O_\epsilon(n^{\frac{1}{2} - \frac{1}{80} + \epsilon}) \tag{3.55}$$

as $n \to \infty$ through a sequence of integers such that $D_n := -24n + 1$ is squarefree. We compute the regularized integral $\beta_0(F)$ by applying Proposition 3.26 with $[SL_2(\mathbb{Z}) : \Gamma_0(6)] = 12$ to get $\beta_0(F) = 0$. Finally, observe that

$$e\left(\frac{-\tau_Q}{l}\right) \zeta_6^{-(\frac{6}{l}t + 1 - l\tilde{l})} = e^{2\pi y_Q / l} e^{-2\pi i \left(\frac{x_Q}{l} + \frac{\frac{6}{l}t + 1 - l\tilde{l}}{6}\right)},$$

and hence

$$\operatorname{Im}\left(e\left(\frac{-\tau_Q}{l}\right)\zeta_6^{-\left(\frac{6}{l}t+1-l\tilde{l}\right)}\right) = e^{2\pi y_Q/l}\sin\left(-2\pi\left(\frac{x_Q}{l}+\frac{\frac{6}{l}t+1-l\tilde{l}}{6}\right)\right).$$

Then to complete the proof, we combine (3.55) with the following formula for $\alpha(n)$ due to Bruinier and Schwagenscheidt [8],

$$\alpha(n) = \frac{-1}{\sqrt{24n-1}}\operatorname{Im}\left(\sum_{Q\in\mathcal{Q}_{-24n+1,6,1}/\Gamma_0(6)} F(\tau_Q)\right).$$

□

4. SUMMARY AND FUTURE WORKS

In this dissertation, we give two main results. Our first main result is an asymptotic formula for the partition function $p(n)$ with an effective bound on the error term. The second one is an asymptotic formula with a power saving error term for traces of a generic class of weak Maass forms. We apply the second result to obtain the asymptotic distribution of $p(n)$, spt(n), $\alpha(n)$, $N_o(n)$, and $N_e(n)$.

In the near future, we would like to extend our study in Section 2 to a more general setting. Namely, we will study an asymptotic distribution with an effective bound on the error term for the trace of a generic class of weak Maass forms. More precisely, let $k \geq 0$ be an integer and N be a squarefree positive integer. We would like to give an asymptotic formula with an effective bound on the error term for the trace of $\mathcal{D}^k f$ when $f \in M^!_{-2k}(N)$ is an eigenfunction for the Atkin-Lehner operators of level N. We would like to obtain the formula

$$\mathrm{Tr}_D(\mathcal{D}^k f) = M_f(D) + E_f(D),$$

where the error term $E_f(D)$ is bounded by

$$|E_f(D)| \leq C_f H(D),$$

for some effective constant C_f that does not depend on D. We will apply the formula to establish asymptotic formulas for several arithmetic functions.

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