THE QUANTUM UNIQUE ERGODICITY AND THE L^4-NORM PROBLEMS FOR NEWFORM EISENSTEIN SERIES

A Thesis

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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December 2020

Major Subject: Mathematics

ABSTRACT

This thesis deals with two closely related problems about Eisenstein series on varying levels, both of which stem from the Random Wave Conjecture.

The first problem is quantum unique ergodicity for Eisenstein series in the level aspect. With a fixed nice test function, we see equidistribution as the level grows. A new feature for the level aspect is a term of the logarithmic derivative of the Dirichlet L-function, which connects quantum unique ergodicity and Siegel zeroes. Going one step further, we let the test function change with the growth of level in the manner analogous to the recently known results on quantum unique ergodicity on shrinking sets, and surprisingly, we observe some distorting behavior.

The second problem is bounding the regularized L^4-norm for newform Eisenstein series. We manage to express the fourth moment as an average of L-functions.

ACKNOWLEDGMENTS

The author admits, acknowledges, and appreciates the academical, financial, and mental support from Prof. Dr. Matthew Patrick Young. He is also grateful to his course instructors, as well as the administrative staff of the department. In addition, he wishes to express his indebtedness to everyone in the world who ever helped him explore the wonderfulness of number theory.

Moreover, the author insists his years of Americanization is no less important than his education in Texas A&M. There are some rights so valuable that once enjoyed, nobody wants to resell, and he feels lucky for the five year trial on part of them. He therefore sincerely thanks the people of the United States, for granting him this opportunity. He will never forget it.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Professors Young, Jung and Masri of the Department of Mathematics and Professor Zhou of the Department of Statistics.

All work conducted for the dissertation was completed by the student, under the advisement of Matthew Young of the Department of Mathematics.

Funding Sources

Graduate study was supported by a scholarship from Texas A&M University.

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1. INTRODUCTION

1.1 The Random Wave Conjecture

1.1.1 Automorphic forms

Let $N \in \mathbb{Z}_{>0}$ and χ be a Dirichlet character of modulus N. Let $k \in \mathbb{Z}_{\geq 0}$, and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \ (\bmod N) \right\}$$

denote the Hecke congruence subgroup of $SL_2(\mathbb{Z})$ of level N.

Let f be a real analytic functions from $\mathbb{H} := \{(x, y)|x, y \in \mathbb{R}, y > 0\}$ to $\mathbb{C}$. We call f an automorphic form of level N, weight k and central character χ, if for any $\gamma \in \Gamma_0(N)$, there is $f|_\gamma(z) := f(\gamma z) = \chi(\gamma) j^k(\gamma) f(z)$ for all $z \in \mathbb{H}$, where

$$\begin{aligned} (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) z &= \frac{az + b}{cz + d} \\ \chi((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) &:= \chi(d) \\ j((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) &:= \frac{cz + d}{|cz + d|}. \end{aligned}$$

It is well-known that:

- The above linear fractional transformation makes a group action on $\mathbb{H}$, the group action being defined by $f|_{\gamma_1 \gamma_2} = (f|_{\gamma_1})|_{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \Gamma_0(N)$;
- There is $\chi(-1) = (-1)^k$; and
- The action of χ is a group homomorphism from $\Gamma_0(N)$ to $\Big(\mathbb{Z}/N\mathbb{Z}\Big)^\times$.

The group action can be extended to $\mathbb{P}^1(\mathbb{Q})$, the set of *cusps*. We often employ the letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$,..., to denote cusps. We say two cusps $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent on level N and write $\mathfrak{a} \overset{N}{=} \mathfrak{b}$, if there exists $\gamma \in \Gamma_0(N)$ such that $\mathfrak{a} = \gamma\mathfrak{b}$. That is to say, equivalence classes of cusps on level N are the $\Gamma_0(N)$-orbits in $\mathbb{P}^1(\mathbb{Q})$.

By [34, Proposition 2.6], a full set of inequivalent cusps on level N can be written as

$$\begin{aligned} &\mathcal{C}(N) := \{\mathfrak{a} |\ \mathfrak{a} = \tfrac{u}{f}, f \mid N, u = \min \mathcal{R}(v), v \in (\mathbb{Z}/N\mathbb{Z})^{\times}\}, \text{ with} \\ &\mathcal{R}(v) := \{u \equiv v \bmod (f, N/f), u \geq 1\}. \end{aligned} \tag{1.1}$$

We write $L^2(\Gamma_0(N), k, \chi)$ for the L^2-completion of all L^2-integrable automorphic forms of level N, weight k and central character χ, with respect to the inner product

$$\langle f, g \rangle_{N} := \int_{\Gamma_0(N)\backslash\mathbb{H}} y^k f(z)\overline{g}(z) d\mu, \tag{1.2}$$

where f, g are automorphic forms of the same level and central character, and $d\mu = d\mu(z) := y^{-2}dxdy$ is the Poincaré measure.

A Jordan measurable subset $\mathcal{F}_{N}$ of $\mathbb{H}$ is called a fundamental domain of $\Gamma_0(N)\backslash\mathbb{H}$ if $\cup_\gamma \gamma\mathcal{F}_{N} = \mathbb{H}$ and $\Gamma_0(N)z_1 \cap \Gamma_0(N)z_2 = \emptyset$ for any distinct $z_1, z_2 \in \mathcal{F}_{N}$. Specifically, when $N = 1$, we have the standard fundamental domain:

$$\mathcal{D} = \mathcal{F}_{1} := \Big\{(x, y) \in \mathbb{H} \Big| -\frac{1}{2} < x \leq \frac{1}{2}, x^2 + y^2 > 1\Big\} \bigcup \Big\{(\cos\theta, \sin\theta) \Big| \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\Big\}.$$

For convenience, we always assume, unless otherwise specified, that $\mathcal{F}_{N}$ is a tessellation of $\mathcal{D}$, i.e.,

$$\mathcal{F}_{N} = \bigcup_{\gamma} \gamma\mathcal{D},$$

where there are only $\nu(N) = [SL_2(\mathbb{Z}) : \Gamma_0(N)]$ different γ in the union.

Now consider the Beltrami-Laplace operator

$$\Delta_k = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - iky\frac{\partial}{\partial x}.$$

Besides the constant functions, the operator has two types of eigenfunctions in $L^2(\Gamma_0(N), k, \chi)$.

- Hecke-Maass forms u_j of discrete eigenvalues $\lambda_j^2 + \frac{1}{4}$ for $0 \le \lambda_1 \le \lambda_2 \le \ldots$;
- Eisenstein series $E_{\mathfrak{a}}(z, \frac{1}{2} + iT, \chi)$ attached to cusps of eigenvalues $\frac{1}{4} + T^2$ for $T \in \mathbb{R}$.

For general manifolds, we can similarly define the Laplacian. The discrete spectra, namely the family of eigenfunctions associated to isolated eigenvalues of increasing absolute values, often exist and are studied for their limiting behavior. The Random Wave Conjecture is one of them.

1.1.2 The Random Wave Conjecture for automorphic forms

Conceived by M. Berry [9], the Random Wave Conjecture (**RWC** for short) brings light to the randomness of high energy eigenstates on tori. Speaking in the language of automorphic forms, it is a heuristic that a Maass newform of large eigenvalue should behave like a real random wave, i.e., a random linear combination of some other Maass newforms of eigenvalues in a short window around the aforementioned large eigenvalue, where all Maass forms are L^2-renormalized.[1] For automorphic forms defined on arithmetic modular surfaces, if L^2-normalized, the Maass forms should also behave like random waves in the eigenvalue limit, as is evinced by the experimental observations of D. Hejhal and B. Recknar [27] on $SL_2(\mathbb{Z})\backslash\mathbb{H}$.[2]

The number theoretical version of this physical conjecture is still missing, and we walk around by studying its indications. Two of the manifestations of RWCs are discussed in this thesis, namely the Quantum Unique Ergodicity Conjecture and the Gaussian Moments Conjecture.

1.1.3 Benjamini-Schramm convergence

As is mentioned above, RWC is studied by number theorists without a formal definition of it. Nevertheless, it can be formulated via Benjamini-Schramm (**BS** for short) convergence by M. Abért, N. Bergeron and E. Le Masson in a recent preprint [2]. We give a brief introduction about the language, and how it is relevant to the topics in this thesis.

[1] Holomorphic cusp forms should behave like a complex random wave in the weight aspect. See [12] for the comparison.

[2] For general discrete groups Γ, however, Milićević [56] showed that on CM-points random wave functions and the large eigenvalue Hecke-Maass forms behaves quite differently in terms of ∞-norms.

With Petersson inner product defined by (1.2), $L^2(\mathcal{F}_{_N})$ makes a Hilbert space. Given a sequence $\{g_j\}$ in this space, we say the sequence converges weakly to, or has weak* limit, $g \in L^2(\mathcal{F}_{_N})$, if

$$\lim_{j\to\infty} \langle g_j, f\rangle_{_N} = \langle g, f\rangle_{_N},$$

for all $f \in L^2(\mathcal{F}_{_N})$. Note N remains fixed in this process.

For varying N, the weak* convergence can be generalized to the BS convergence, which is inspired by a program in graph theory that was made popular by I. Benjamini and O. Schramm.[3]

Consider $\mathbb{M}$, the space of pointed, connected, complete Riemannian manifolds of dimension 2, up to pointed isometries. For convenience of discussion in this thesis, let us further assume $\mathbb{M}$ only contains manifolds modeled on the symmetric space $SL_2(\mathbb{R})/SO_2(1)$. On $\mathbb{M}$, M. Abért and I. Biringer defined the smooth topology, where two pointed manifolds $(\mathcal{M}, p)$ and $(\mathcal{N}, q)$ are smoothly close if there are compact subsets of $\mathcal{M}$ and $\mathcal{N}$ containing large radius neighborhoods of the base points that are diffeomorphic via a map that is C^∞-close to an isometry.[4] Together with the Chabauty topology on closed subgroups of $SL_2(\mathbb{R})$, the smooth topology induces the product topology in the space of the decorated closed subgroup

$$\widehat{D} := \Big\{(H, g)\Big| H \text{ closed subgroup of } SL_2(\mathbb{R}), g \text{ smooth and } H-\text{invariant}\Big\}.$$

For each $\mathcal{M} = \Gamma\backslash\mathbb{H} \in \mathbb{M}$ and Γ-invariant function $g : \mathcal{M} \to \mathbb{R}$, consider the map

$$\begin{aligned} SL_2(\mathbb{R})/\Gamma &\longrightarrow \widehat{D} \\ \gamma\Gamma &\longmapsto (\gamma\Gamma\gamma^{-1}, g|_{\gamma^{-1}}), \end{aligned}$$

where $g|_{\gamma^{-1}}(z) = g(\gamma^{-1}z)$. Denote by $\mu_{\Gamma,g}$ the push forward of the invariant probability measure on $\mathbb{H}$. Now let $g_n : \mathbb{H} \to \mathbb{R}$ be $\Gamma_0(n)$-invariant for $n \geq 1$, and call the sequence $(\mathcal{F}_{_n}, g_n)$ convergent

[3] See [7] for a reference.
[4] See Sec. A1 of [3].

in the sense of Benjamini-Schramm, if the sequence $\mu_{\Gamma_0(n),g_n}$ converges in the weak* topology of the set of probability measures on $\widehat{D}$.

An equivalent description of BS convergence is as follows due to M. Abért, et al. in Corollary 3.8 of [1]. Recall $d(z,w)$ is the hyperbolic distance between z and w.

Theorem 1. *The sequence $(\mathcal{F}_n, g_n)$ is Benjamini-Schramm convergent if and only if for any $R > 0$,*

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\{z \in \mathcal{F}_n : \mathrm{InjRad}_{\mathcal{F}_n}(z) < R\})}{\mathrm{Vol}(\mathcal{F}_n)} = 0,$$

where

$$\mathrm{InjRad}_{\mathcal{F}_n}(z) := \min_{\gamma\in\Gamma_0(n)-\{\pm Id\}} d(z, \gamma z)$$

is the injectivity radius of $\mathcal{F}_n$ at z.

In this language, M. Frączyk and J. Raimbault [24] showed $\mathcal{F}_N$ converges as $N \to \infty$ in the Benjamini-Schramm sense. On the other hand, BS convergence also contains the scenario of weak convergence for sequences of automorphic functions defined on the same manifold. Thus BS convergence provides a unified language for the level and eigenvalue aspects of automorphic forms.

We write the (real) isotropic monochromatic Gaussian random wave with parameter λ on $SL_2(\mathbb{R})/SO_2(1)$ to be the Gaussian random field F_λ from $SL_2(\mathbb{R})/SO_2(1)$ to $\mathbb{R}$ if its co-variance kernel $\mathbb{E}[F_\lambda(x), F_\lambda(y)]$ equals $\varphi_\lambda(x^{-1}y)$ for any $x, y \in SL_2(\mathbb{R})/SO_2(1)$, where $\varphi_\lambda(\cdot)$ is the only spherical function of eigenvalue λ such that $\varphi_\lambda(\pm\mathrm{Id}) = 1$ given by Harish-Chandra in [26].

Now we can formulate RWC in terms of BS convergence.

Conjecture 1 (Conj. 1, [2]). *Let $\mathcal{M}$ be a compact, negatively curved manifold, and $\mathcal{M}_r$ be the rescaling of $\mathcal{M}$ by $r > 0$, where the only change is the metric in $\mathcal{M}$ gets multiplied by r in $\mathcal{M}_r$. Let $\{u_j\}_j$ be an orthonormal basis of $L^2(\mathcal{M})$ that consists of eigenvectors for the Laplacian, with increasing eigenvalues t_j^2. Then $(\mathcal{M}_{t_j}, u_j)$ converges to the isotropic monochromatic Gaussian* *random Euclidean wave with eigenvalue* 1*, in the Benjamini-Schramm sense, as* $j \to \infty$*.*

1.2 Equidistribution of Automorphic Forms in $\mathbb{H}^2$

1.2.1 The Quantum Ergodicity Theorem

For each of the sequence $\{u_j\}$ of L^2-normalized Maass forms of level 1 and weight 0, we call

$$\mu_j := |u_j(z)|^2 d\mu$$

the mass of u_j. It is obvious that μ_j is a probability measure. For the sequence $\{\mu_j\}_j$ of probability measures, there are two fundamental questions to answer.

Question 1. *What are the weak-* limits of the subsequences of* $\{\mu_j\}_j$*, and for each subsequential limit, what is the density of the respective maximal subsequence?*

Given a topological space X, we call a probability distribution ν on X scar strongly at a point $s \in X$ such that the restricted distribution of any neighborhood of s is not the zero-distribution. By definition, ν scars strongly at the points of discontinuity of its cumulative probability function, if any. On both quantum cat maps [23] and star graphs [8], some subsequences of the eigenfunctions of Laplacian are proved to scar strongly.

Question 2. *Does any of the subsequential weak-* limit of the aforementioned sequence scar strongly?*

These two questions are "almost done" in the sense of the following conclusion. Note that a measure cannot both equidistribute and scar strongly.

Theorem 2 (The Quantum Ergodicity Theorem, [65][17][75][76])**.** *Assume* $X = \mathcal{M}$ *is a manifold. If the geodesic flow on* $S^*(\mathcal{M})$*, the cotangent bundle of* $\mathcal{M}$*, is ergodic, then there exist density-one subsequences of* $\{\mu_j\}_j$ *having the uniform distribution as its weak* limit.*

Note that one sufficient condition for $\mathcal{M}$ to have ergodic geodesic flow is that $\mathcal{M}$ has negative Gaussian curvature. Furthermore, since the hyperbolic plane $\mathbb{H}$ has curvature -1, one can expect to go further with the arithmetic properties of automorphic forms of discrete groups like $SL_2(\mathbb{Z})$.

1.2.2 The Quantum Unique Ergodicity Conjecture

Completely answering Questions 1 and 2, E. Lindenstrauss and K. Soundararajan proved the following theorem.

Theorem 3 (Quantum Unique Ergodicity Theorem on $SL_2(\mathbb{Z})$, [47][67]). *Let $\{\mu_j\}_{j\geq 1}$ be the full sequence of L^2-normalized Maass forms on $\mathcal{D}$. Then $\{\mu_j\}_{j\geq 1}$ has weak-* limit $\frac{3}{\pi}d\mu$.*

On the other hand, however, quantifying the rate of distribution is still yet to be done. In general, the period integral formula of T. Watson[5] converts the triple product integrals to special values of automorphic L-functions, for which any non-trivial (called the subconvexity) bound implies a bound for the error term. However, the subconvexity bounds leading to quantum unique ergodicity are only established for few cases, including holomorphic CM forms by P. Sarnak [64] and primitive dihedral cusp forms, due to V. Blomer [10] and J. Liu and Y. Ye [49]. [6] The main reason for the lack of an unconditional quantum unique ergodicity for arbitrary Maass forms is that we do not have factorization for the central value of the Rankin-Selberg L-function $L(\frac{1}{2}, f \otimes \mathrm{Ad}\, u)$, where u is a cusp form, and f is a Maass form or Eisenstein series.

In general, Z. Rudnick and P. Sarnak have the Quantum Unique Ergodicity Conjecture which still remains open. Note that Maass forms can be defined on general manifolds.

Conjecture 2 (The Quantum Unique Ergodicity Conjecture, [63][64]). *For any manifold $\mathcal{M}$ of negative Gaussian curvature, assume $\{u\}$ be the sequence of L^2-normalized*

- *Maass forms of increasing eigenvalues on general $\mathcal{M}$; or*
- *holomorphic modular forms of increasing weights on $\mathcal{F}_N$.*

Then $\{|u|^2 d\mu\}$ converges to the uniform distribution weakly.

Note that in some respects, the two kinds of cusp forms behave similarly, and hence the proof for one sometimes could be modified into the other without too much work. One similarity is

[5] The formula was later generalized by A. Ichino [33], known as the Ichino-Watson Formula.

[6] See [50][46] for the fixation of a gap of it. P. Humphries and R. Khan [31] bounded the quantum variance.

as follows: if u is a holomorphic form of weight k, then $y^{k/2}u$ is a Laplacian eigenfunction of eigenvalue $i\frac{k-1}{2}$ and it satisfies $\Gamma_0(N)$-automorphy. When $\mathcal{M} = \mathcal{D}$, the weight aspect of the conjecture is proven by R. Holowinsky and K. Soundararajan [28].

It is also known in [2, Theorem 4] that Conjecture 1 implies the Quantum Unique Ergodicity Conjecture in the eigenvalue aspect. On the other hand, we know that Conjecture 1 is stated in a unified language for both the eigenvalue and the level aspects. Thus the Quantum Unique Ergodicity conjecture in the level aspect is also interesting. The level aspect analogue of Conjecture 2 was raised by E. Kowalski, Ph. Michel and J. VanderKam.

Conjecture 3 (The level aspect Quantum Unique Ergodicity for holomorphic forms, [43])**.** *Let $k \geq 2$ be even, and $\{f^{(N)}\}_N$ be the sequence of L^2-normalized holomorphic modular newforms of weight k and level N. As $N \to \infty$, we have*

$$|f^{(N)}|^2 d\mu \sim \nu(N) d\mu.$$

Here $\nu(N) = [SL_2(\mathbb{Z}) : \Gamma_0(N)] = N\prod_{p|N}(1-1/p) = N^{1+o(1)}$ is the group index. The conjecture is proved by P. Nelson, A. Pitale and A. Saha [58][60], who indeed proved a stronger form by obtaining the asymptotic formula hybrid in k and N. Furthermore, they also gave an estimation for the error terms.

1.2.3 Equidistribution of Eisenstein Series

Almost immediately following the announcement of Conjecture 2, the family of classical Eisenstein series was proved to have the mass equidistribution property.

Theorem 4 (Quantum Unique Ergodicity for classical Eisenstein series, [52])**.** *Fix compact set B of $\mathcal{D}$. As $t \to \infty$, we have*

$$\int_B |E(z, \frac{1}{2} + it)|^2 d\mu \sim \frac{\mathrm{Vol}B}{\langle 1, 1 \rangle_1} \log\left(\frac{1}{4} + t^2\right) d\mu.$$

The error term in this approximation was later bounded by M. Young [72], where the corresponding Rankin-Selberg L-function factorizes as

$$L(\frac{1}{2}+it,f)L(\frac{1}{2}-it,f),$$

where f traversing the orthonormal basis of $L^2(\mathcal{D})$.

In the phase space, D. Jacobson [38] proved a similar result for the $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$-Eisenstein series. Moreover, considering an analogue of Conjecture 3, S. Koyama proved the oldform Eisenstein series on growing levels.

Theorem 5 **([45][39]).** *Let N traverse all prime numbers, $\{E_\infty^{(N)}(z,\frac{1}{2}+iT)\}_N$ be the sequence of Eisenstein series of weight 0, level N and trivial central character, and ϕ_0 be a smooth, compactly supported and $SL_2(\mathbb{Z})$-invariant test function. Then as $N\to\infty$, we have*

$$\int_{\mathcal{F}_N}|E_\infty^{(N)}(z,\frac{1}{2}+iT)|^2\phi(z)d\mu\sim 2\frac{\langle 1,\phi_0\rangle_1}{\langle 1,1\rangle_1}\log N.$$

A generalization to Theorem 5 makes the first result of this thesis. Throughout this thesis we let θ be so that the p-th Hecke eigenvalues of Maass newforms are uniformly bounded by $p^\theta+p^{-\theta}$. The value $\theta=7/64$ is allowable by [40].

Theorem 6. *Let $N>1$, $q\mid N$, χ be a Dirichlet character of modulus q, and $E=E_{\mathfrak{a}}^{(N)}(z,\frac{1}{2}+iT,\chi)$ be the Eisenstein series of weight 0, level N and central character χ. Keep all of the rest notations the same as in Theorem 5. We have*

$$\begin{aligned}\langle|E|^2,\phi_0\rangle_N=\frac{\langle 1,\phi_0\rangle_1}{\langle 1,1\rangle_1}\Big(2\log N+4\Re\frac{L'}{L}(1+2iT,\overline{\psi})\Big)&\\+O_{T,\phi_0}((\log\log N)^5)+O_T(N^{-\frac{1}{8}+\varepsilon}(\tfrac{N}{q})^{-\frac{1}{8}+\theta}\|\phi_0\|_2),&\quad(1.3)\end{aligned}$$

where ψ is a primitive Dirichlet character mod q totally determined by χ and $\mathfrak{a}$.

The same is true if the Eisenstein series is of weight 1, as we later prove in Section 3.7.2.

According to the weight shifting operators introduced in [18, Sec 4], our theorem can be extended to the cases of arbitrary positive integer weight. For the similarity of the arguments of the two cases, we write out the proof for weight 0 and transplant the proof to the case of weight 1 by showing how the difference affects.

1.2.4 Quantum Unique Ergodicity on Shrinking Sets

Note that most of the results in quantum unique ergodicity do not contain error term estimation. Applying the regularization method invented by D. Zagier [74] and developed by Ph. Michel and A. Vankatesh [53], M. Young improved Theorem 4 with a power-saving bound for the error. This enabled him to consider a stronger version of convergence, namely quantum unique ergodicity on shrinking sets, by allowing the test function to change.

Theorem 7 ([72]; improved in [30])**.** *For any $z_0 \in \mathbb{H}$, let $\phi = \phi_t$ for $t > 1$ be a family of test functions that are smooth, $SL_2(\mathbb{Z})$-invariant, non-negatively valued, and supported within the ball centered at z_0 of radius $r_t = t^{-\frac{1}{6}+\delta}$ for any $\delta > 0$. Then as $t \to \infty$, we have*

$$\int |E(z, \frac{1}{2} + it)|^2 \phi d\mu \sim \frac{3}{\pi} \log\left(\frac{1}{4} + t^2\right) \int \phi d\mu,$$

the convergence being independent of z_0.

Looking for the quantum unique ergodicity threshold of the rate of shrinkage, P. Humphries find the Planck scale $r_t \asymp t^{-1}$ should be the barrier, via arguments of the probability variance.

1.2.5 The Level Aspect Analogue of Shrinking Sets and its Limitations

For Theorem 6, we let the test function ϕ_0 to be of level 1, while N grows. A mild generalization of (1.3) is to fix a positive integer M and a test function $\phi = \phi^{(M)}$ on $Y_0(M)$, and to confine $N \equiv 0 \pmod{M}$. In analogy to Theorems in Section 1.2.4, where $\phi = \phi_t$ is allowed to change with the spectral parameter t, we are led to consider the much more difficult generalization of letting ϕ depend on N. A natural way to do this is to let M grow with N, constrained by $M|N$, and to choose $\phi = \phi^{(M)}$ on $Y_0(M)$ depending on M. To maintain uniform analytic properties of the test functions $\phi^{(M)}$ of varying levels, we often make the following system of choices.

Convention 1. *Once and for all fix an $SL_2(\mathbb{Z})$-invariant smooth function $\phi_0 = \phi^{(1)}$ with compact and connected support. For simplicity, suppose that the support of ϕ_0, when restricted to the standard fundamental domain $\mathcal{D}$ of $SL_2(\mathbb{Z})$, is contained in its interior. Suppose that $\Gamma_0(1) = \cup_{j=1}^{\nu(M)} \gamma_j \Gamma_0(M)$ as a disjoint coset decomposition. For each positive integer M, choose $\phi^{(M)} = \phi_j^{(M)}$ to be one of the following $\nu(M)$ functions. Set $\phi_j^{(M)}(\gamma_k \Gamma_0(M) z)$ equal to $\phi_0(z)$ if $j = k$, and zero if $j \neq k$, where $k \in \{1, 2, ..., \nu(M)\}$. One can interpret this definition intuitively by noting that $\cup_{j=1}^{\nu(M)} \gamma_j \mathcal{D}$ is a fundamental domain for $Y_0(M)$, and so $\phi_j^{(M)}$ agrees with ϕ_0 on one translate of $\mathcal{D}$ and vanishes at all others.*

The system of test functions satisfying the convention has the following pleasant properties. We have $\phi_0 = \sum_{j=1}^{\nu(M)} \phi_j^{(M)}$, where the supports of these $\phi_j^{(M)}$ are pairwise disjoint. Moreover, we have $\int_{Y_0(M)} \phi_j^{(M)} d\mu = \int_{Y_0(1)} \phi_0 d\mu$, for each j. Since

$$\frac{\mathrm{Vol}(\mathrm{Supp}(\phi^{(M)}))}{\langle 1, 1 \rangle_M} = M^{-1+o(1)},$$

we intuitively see that $\mathrm{Supp}\phi^{(M)}$ "shrinks", if $M \to \infty$ as $N \to \infty$.

Theorem 8. *Let E be as in Theorem 6. Choose a system of test functions according to Convention 1. Then there exists $\mathcal{E} \in \mathcal{A}(Y_0(N))$, such that $|E|^2 - \mathcal{E} \in L^2(Y_0(N))$, and*

$$\langle |E|^2 - \mathcal{E}, \phi \rangle_N \ll_{\varepsilon, T, \phi_0} N^{-\frac{1}{2}+\varepsilon} (\tfrac{N}{q})^{\theta} Q(M, q) \|\phi_0\|_2, \tag{1.4}$$

with

$$Q(M, q) = M^{\frac{1}{4}} q^{\frac{3}{8}} + M^{\frac{1}{2}} (M, q)^{\frac{1}{4}} q^{\frac{1}{4}}.$$

Under the generalized Lindelöf hypothesis, (1.4) *holds with $Q(M, q) = \sqrt{M}$. Finally, we have*

$$\langle \mathcal{E}, \phi \rangle_N = \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_M} \Big(\log \frac{N^2}{M(M, N/q)} + 4\Re \frac{L'}{L}(1 + 2iT, \overline{\psi}) \Big) + O_{T, \phi_0} \Big(\frac{(\log \log N)^5}{\langle 1, 1 \rangle_M} \Big) + \alpha_\phi, \tag{1.5}$$

where α_ϕ is a quantity (see (3.15) for an expression) satisfying

$$|\alpha_\phi| \ll_{\phi_0, T} (\log\log M)^3. \tag{1.6}$$

Note that if $M \ll N^{\frac{1}{10}-\delta}$, then the bound in (1.4) is better than the first displayed main term in (1.5) of size $\approx M^{-1+o(1)}\log N$. This is analogous to the power-saving error term in the QUE problem for Eisenstein series of level 1 in the spectral aspect, as in [72].

In the level aspect, our discussion shows that QUE does not hold for all systems of test functions constructed according to Convention 1. This is in contradiction to the claimed result of Koyama [44, Theorem 1.3], which in our notation would correspond to $N = M$ prime and $q = 1$. Recently, Kaneko and Koyama recast [44] in [39].

1.2.6 Main term discussion

To our surprise, if we construct the system of test functions according to Convention 1, then QUE turns out not to hold for all test functions $\phi = \phi_j^{(M)}$, at least, if $M \gg N^\delta$ for some $\delta > 0$. The problem is that for some choices of ϕ, the contribution of α_ϕ to the main term is dominant and large enough to show that QUE does not hold. In retrospect, one might expect problematic behavior for test functions with support escaping too quickly into a cusp. This is clear in the level 1 case (in the spectral aspect), since very high in the cusp the Eisenstein series is well-approximated by its constant term. In the level aspect, it is a bit tricky to say what it means for a test function to have support escaping into a cusp, not least because the cusp can be changing with the level.

Since α_ϕ is complicated, we will now discuss it in further details in a special case that simplifies the discussion. For more details, see Section 3.5.3. Let $G(z)$ denote the constant term in the Laurent expansion of $E(z,s)$ around $s = 1$ (see [36, (22.69)] for an expression), which is $SL_2(\mathbb{Z})$-invariant, and which satisfies $G(x+iy) \sim y$ for $y \to \infty$. Let $M \mid N$ be prime with $M \gg (\log N)^{1+\delta}$ and $\chi \pmod{N}$ be primitive. Then

$$\langle \mathcal{E}, \phi \rangle_N = c_0 \langle 1, \phi \rangle_M + c_1 \langle G, \phi \rangle_M + c_M \langle G|_M, \phi \rangle_M, \tag{1.7}$$

where

$$c_0 = \frac{1}{\langle 1,1\rangle_M}\Big(\log\frac{N^2}{M} + 4\Re\frac{L'}{L}(1+2iT,\overline{\chi}) + O_{T,\phi_0}(1)\Big), \tag{1.8}$$

$c_1, c_M = M^{-1} + O(M^{-2})$. The term $c_0\langle 1,\phi\rangle_M$ is the naively-expected main term. If $\phi = \phi_j^{(M)}$ is chosen according to Convention 1, then note $\langle G,\phi\rangle_M = \langle G,\phi_0\rangle_1$, which is independent of j and M, so the term $c_1\langle G,\phi\rangle_M$ is bounded acceptably. However, the term $c_M\langle G|_M,\phi\rangle_M$ may be much larger than the expected main term, as we now explain. Suppose that the restriction of ϕ_0 to the standard fundamental domain $\mathcal{D}$ for $Y_0(1)$ has support with $2\le y\le 3$ and that ϕ_0 is non-negative. There exists a fundamental domain $\mathcal{F}_M$ for $Y_0(M)$ so that $\mathcal{D}\subset\mathcal{F}_M$, and there exists a value of j so that $\phi_j^{(M)}(z) = \phi_0(z)$ for $z\in\mathcal{D}$, and $\phi_j^{(M)}(z) = 0$ for $z\in\mathcal{F}_M$, $z\notin\mathcal{D}$. For this value of j, we have

$$c_M\langle G|_M,\phi\rangle_M \approx M^{-1}\int_2^3\int_0^1 G(Mz)\phi(z)\frac{dxdy}{y^2},$$

which can be $\asymp 1$, since $G(Mz)\sim My$ uniformly on the region of integration (see Proposition 6). Note that in this situation, $c_M\langle G|_M,\phi\rangle_M$ is much larger than $c_0\langle 1,\phi\rangle_M \lessapprox M^{-1}\log N$. This choice of $\phi = \phi_j^{(M)}$ should be interpreted as having support high in the cusp ∞.

1.3 The L^4-Norm Problem

1.3.1 Background

One implication of RWC is the Gaussian Moments Conjecture. For any compact Jordan-measurable subset K of $\mathcal{D}$, the conjecture says[7]

$$\lim_{T\to\infty}(\mathrm{Vol}K)^{-1}\int_K\Big|\frac{E(z,\frac{1}{2}+iT)}{(2\log T)^{1/2}}\Big|^p d\mu = \frac{c_p}{(\mathrm{Vol}\mathcal{D})^{p/2}},$$

where c_p is the p-th moment of the random variable of the Gaussian distribution of mean 0 and variance 1, for all even $p\ge 2$. The case $p=2$ has been checked, as we can let ϕ be the indicator

[7] See Sec. 7.3 of [27].

function of K in Theorem 4.[8] For the next case $p = 4$, an upper bound $O(T^\varepsilon)$ was obtained due to F. Spinu [68] and P. Humphries [30]. Conditionally on the Generalized Lindelöf Hypothesis, J. Buttcane and R. Khan [16] proved an asymptotic formula[9] for the fourth moment of dihedral Maass forms with a power-saving bound for the error term. On the level aspect, J. Buttcane and R. Khan [15] obtained an upper bound for holomorphic Hecke newforms.

The reason we confine the integral to a compact set is that Eisenstein series are not in $L^2(\mathcal{D})$, for which we may study the truncated Eisenstein series instead. On the other hand, D. Zagier found powerful tools in computing the regularized integrals, which seems to be a more natural way to define moments of functions of moderate growth. To this end, G. Djanković and R. Khan formulated Random Wave Conjecture for the regularized fourth moment of Eisenstein series, in consistency with two special cases of the old conjecture with $p = 4$.[10]

Theorem 9. *[19][20] Let $E_T = E(z, \frac{1}{2} + iT)$ be the classical Eisenstein series. As $T \to \infty$, we have*

$$\langle |E|^2, |E|^2 \rangle_{\text{reg}} = \frac{72}{\pi} \log^2 T + o(\log^{5/3+\varepsilon}). \tag{1.9}$$

The Gaussian Moments Conjecture is also related to the Quantum Unique Ergodicity Conjecture via the following observation, of which the L^4-norm problems is a special case.

Conjecture 4. *[37] Fix a geodesic ball B of $\mathcal{D}$ and $2 < p \leq \infty$. Assume $\{u\}$ is the same as in Conjecture 2. Then when $\lambda_u \to \infty$, there is*

$$\Big(\int_B |u(z)|^p d\mu \Big)^{1/p} \ll_{p,B,\varepsilon} \lambda_u^\varepsilon \Big(\int_B |u(z)|^2 d\mu \Big)^{1/2}.$$

[8] Although ϕ is assumed to be smooth, we can let it be arbitrarily close to the indicator function.
[9] Also see [51] for an upper bound of it.
[10] See Conj 1.2 of [19]. The conjecture is later proved in [20].

1.3.2 The regularized L^4-norm of newform Eisenstein series

The second main result of this thesis is a level aspect analogue of the computation for the regularized integral in [19]. Recall that $\nu(N) = [SL_2(\mathbb{Z}) : \Gamma_0(N)] = N^{1+o(1)}$.

Theorem 10. *Let $N > 1$, and $E = E_{\mathfrak{a}}(z, s, \chi)$ be a $\Gamma_0(N)$-Eisenstein series attached to an Atkin-Lehner cusp $\mathfrak{a}$, and of nebentypus χ primitive mod N. As $N \to \infty$, we have*

$$\langle |E_{\mathfrak{a}}(\cdot, \frac{1}{2} + iT, \chi)|^2, |E_{\mathfrak{a}}(\cdot, \frac{1}{2} + iT, \chi)|^2 \rangle_{\mathrm{reg}} = I_1 + I_2,$$

where (writing $\mathcal{O}(N)$ for an orthonormal basis of the space of level N Maass forms)

$$\nu(N) I_1 = \sum_{u \in \mathcal{O}(N)} \frac{\cosh(\pi t_u)}{2} \frac{\Lambda^2(\frac{1}{2}, u) |\Lambda(\frac{1}{2} + 2iT, u \otimes \psi)|^2}{|\Lambda(1 + 2iT, \psi)|^2} + \text{continuous spectrum},$$

for some primitive $\psi = \psi(\chi, \mathfrak{a})$ mod N to be defined in the next section, and

$$\nu(N) I_2 = \frac{24}{\pi} \log^2 N + O(\frac{L''}{L}(1 + 2iT, \psi)) + O(\log N \log\log N \frac{L'}{L}(1 + 2iT, \psi)).$$

The multiplication by $\nu(N)$ to I_1 and I_2 is under consideration of L^4-renormalization. That is to say, if we regard $E_{\mathfrak{a}}$ to be "L^2-normalized" (they do have comparable behaviors with the classical Eisenstein series $E(z, \frac{1}{2} + it)$ in the t-aspect, see [52] for a QUE comparison), then we should expect $\int |E_{\mathfrak{a}}|^4$ to have size $\asymp \nu(N)^{-1}$. Assuming GRH, we can see $\nu(N) I_2 \sim \frac{24}{\pi} \log^2 N$, which is in agreement with [19] in the spectral aspect.

2. EISENSTEIN SERIES

2.1 Cusps and their widths

It is well-known that $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) | \, c \equiv 0 \pmod{N}\}$ acts on $\mathbb{H}$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z \mapsto \frac{az+b}{cz+d}$. In this section we introduce some background knowledge of cusps on $\Gamma_0(N)$. We counsel experienced readers to skip this section except for Section 2.1.3 on relative width, and refer other readers to [60, Section 3.4] and [34, Sections 2.1–2.4] for more details.

2.1.1 Cusps

The group action can be extended to $\mathbb{P}^1(\mathbb{Q})$, the set of *cusps*. We often employ the letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$,..., to denote cusps. We say two cusps $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent on level N and write $\mathfrak{a} \overset{N}{=} \mathfrak{b}$, if there exists $\gamma \in \Gamma_0(N)$ such that $\mathfrak{a} = \gamma \mathfrak{b}$. That is to say, equivalence classes of cusps on level N are the $\Gamma_0(N)$-orbits in $\mathbb{P}^1(\mathbb{Q})$.

Recall (1.1) that a full set of inequivalent cusps on level N can be written as

$$\begin{aligned} \mathcal{C}(N) &:= \{\mathfrak{a} | \, \mathfrak{a} = \tfrac{u}{f}, f \mid N, u = \min \mathcal{R}(N, f, v), v \in (\mathbb{Z}/N\mathbb{Z})^\times\}, \text{ with} \\ \mathcal{R}(N, f, v) &:= \{u \equiv v \mathrm{Mod}(f, N/f), u \geq 1\}. \end{aligned} \tag{2.1}$$

Throughout this paper we write $u_\mathfrak{a}$ and $f_\mathfrak{a}$ such that $\mathfrak{a} \overset{N}{=} \frac{u_\mathfrak{a}}{f_\mathfrak{a}} \in \mathcal{C}(N)$, if necessary. Also, if we write $\frac{u}{f} \in \mathcal{C}(N)$, then we always assume that the fraction is in the lowest terms.

Let $\Gamma_\mathfrak{a}^N$ be the stabilizer of $\mathfrak{a}$ in $\Gamma_0(N)$. It is clear that for all N, $\Gamma_\infty^N = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | \, n \in \mathbb{Z}\}$, so we may write Γ_∞ as well. In addition, there are *scaling matrices* $\sigma_{\mathfrak{a},N} \in SL_2(\mathbb{R})$ such that $\sigma_{\mathfrak{a},N}\infty = \mathfrak{a}$, and $\sigma_{\mathfrak{a},N}^{-1} \Gamma_\mathfrak{a}^N \sigma_{\mathfrak{a},N} = \Gamma_\infty$. If the level is clear, we may suppress N in these symbols.

2.1.2 (Absolute) width

If $\tau \in \Gamma = SL_2(\mathbb{Z})$ and $\tau\infty = \mathfrak{a}$, then $\tau^{-1}\Gamma_\mathfrak{a}^N \tau$ is a subgroup of Γ_∞. Since $\tau \Gamma_\infty \tau^{-1} = \Gamma_\mathfrak{a}^1$, we have $[\Gamma_\infty : \tau^{-1}\Gamma_\mathfrak{a}^N \tau] = [\Gamma_\mathfrak{a}^1 : \Gamma_\mathfrak{a}^N]$, which does not depend on the choice of τ. Define this index as the (*absolute*) *width* of $\mathfrak{a}$ on level N and write it $W_N^1(\mathfrak{a})$.

Convention 1. *When there is no ambiguity on levels, we may write the (absolute) width of* $\mathfrak{a}$ *by* $W_{\mathfrak{a}}$ *as well. Width of a cusp is a common terminology, so we add "absolute" only if it is necessary to distinguish it from relative width introduced in the following subsection.*

Remark 1. *For future usage we cite [34, (2.31)] to note that for fixed* $\gamma_{\mathfrak{a}} \in SL_2(\mathbb{Z})$ *sending* ∞ *to* $\mathfrak{a}$, $\gamma_{\mathfrak{a}}\left(\begin{smallmatrix} W_{\mathfrak{a}}^{1/2} & 0 \\ 0 & W_{\mathfrak{a}}^{-1/2} \end{smallmatrix}\right)$ *serves as a scaling matrix* $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{a},N}$.

Lemma 1. *[34, (2.29)] For each* $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}(N)$ *in (1.1), we have*

$$W_{\mathfrak{a}} = \frac{N}{(N, f^2)}.$$

Let $M \mid N$, and $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}(N)$. Then by [41, Proposition 3.1], for all $M \mid N$, $\mathfrak{a}$ is equivalent to a cusp of the form $\frac{u'}{(M,f)} \in \mathcal{C}(M)$, with width

$$W_M^1(\mathfrak{a}) = \frac{M}{(M, (M, f)^2)}.$$

2.1.3 Relative width

Now we fix $\Gamma_0(N)$ but let $\Gamma = \Gamma_0(M)$ for any $M \mid N$ instead. We define the index $[\Gamma_{\mathfrak{a}}^M : \Gamma_{\mathfrak{a}}^N]$ as the *relative width* of $\mathfrak{a} \in \mathcal{C}(N)$ *from level* M, and denote it by $W_N^M(\mathfrak{a})$. Note that the absolute width is a special case of the relative width when $M = 1$.

Remark 2. *From the definition we can also see if* $\mathfrak{a} \overset{N}{=} \mathfrak{b}$, *then* $W_N^M(\mathfrak{a}) = W_N^M(\mathfrak{b})$. *This results from the fact* $\Gamma_{\mathfrak{b}}^* = \tau\Gamma_{\mathfrak{a}}^*\tau^{-1}$, *for any* $\tau \in \Gamma_0(N)$ *with* $\tau\mathfrak{a} = \mathfrak{b}$ *and* $* = M, N$.

The following lemma follows directly from the definition.

Lemma 2. *For each cusp* $\mathfrak{a}$ *on* $Y_0(N)$, *we have*

$$W_N^M(\mathfrak{a}) = \frac{W_N^1(\mathfrak{a})}{W_M^1(\mathfrak{a})}.$$

Lemma 3. *For each cusps $\mathfrak{a}$ and $\mathfrak{b}$ on $Y_0(N)$, we have*

$$\#\{\gamma \in \Gamma_0(N)\backslash\Gamma |\, \gamma\mathfrak{b} \stackrel{N}{=} \mathfrak{a}\} = \begin{cases} W_N^M(\mathfrak{a}) & \text{if } \mathfrak{a} \stackrel{M}{=} \mathfrak{b}; \\ 0 & \text{otherwise}. \end{cases}$$

Proof. If $\mathfrak{a}$ is not Γ-equivalent with $\mathfrak{b}$, then the set is empty. Now assume $\mathfrak{a} \stackrel{M}{=} \mathfrak{b}$ with $\tau\mathfrak{a} = \mathfrak{b}$ for some $\tau \in \Gamma$. We have the following bijective map

$$\{\gamma \in \Gamma_0(N)\backslash\Gamma |\, \gamma\mathfrak{b} \stackrel{N}{=} \mathfrak{a}\} \to \{\gamma \in \Gamma_0(N)\backslash\Gamma |\, \gamma\mathfrak{a} \stackrel{N}{=} \mathfrak{a}\}$$

$$\gamma \mapsto \gamma\tau$$

so it suffices to compute $\#S_\mathfrak{a}$, where $S_\mathfrak{a} = \{\gamma \in \Gamma_0(N)\backslash\Gamma |\, \gamma\mathfrak{a} \stackrel{N}{=} \mathfrak{a}\}$. Note that $\Gamma_\mathfrak{a}^M$ acts transitively on $S_\mathfrak{a}$ (on the right) with stabilizer $\Gamma_\mathfrak{a}^N$. Hence, by the Orbit-Stabilizer Theorem (see e.g., [4, Chapter 5, Proposition (7.2)]), we have $\#S_\mathfrak{a} = [\Gamma_\mathfrak{a}^M : \Gamma_\mathfrak{a}^N] = W_N^M(\mathfrak{a})$. $\square$

2.1.4 Singularity

Given an even Dirichlet character $\chi \pmod N$, i.e., $\chi(-1) = 1$, we define

$$\chi : \Gamma_0(N) \to \mathbb{C}^*$$

by $\chi(\gamma) = \chi(d_\gamma)$, where d_γ stands for the lower-right entry of γ. It is easy to see that χ preserves multiplication of the two sides, and hence it is a group homomorphism.

Convention 2. *We write $\chi_1 \simeq \chi_2$ if they are induced by the same primitive character.*

We say $\mathfrak{a}$ is *singular* for χ, if the kernel of χ contains $\Gamma_\mathfrak{a}^N$. If $\chi_1 \simeq \chi_2$, then the singularity of $\mathfrak{a}$ for χ_1 is equivalent to that for χ_2. For fixed $\chi \pmod N$, singularity and non-singularity of a cusp extends to its $\Gamma_0(N)$-equivalence class, for the same reason as for Remark 2.

Convention 3. *For $\chi \pmod N$, we write the subset of singular cusps for χ by $\mathcal{C}_\chi(N)$. Note $\mathcal{C}_\chi(N) = \mathcal{C}(N)$ if χ is trivial.*

We have a criterion for singularity from [73, Lemma 5.4]. Recall that q is the conductor of χ.

Proposition 1. *The cusp* $\frac{u}{f} \in \mathcal{C}(N)$ *is singular for* χ *if and only if* $q \mid [f, \frac{N}{f}]$.

One interesting case is when χ is primitive $\pmod N$. By Proposition 1, only cusps $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}(N)$ with $(f, N/f) = 1$ are singular for χ. Moreover, from (1.1) we can see $u = 1$. These cusps are known as the *Atkin-Lehner cusps*.

2.2 Eisenstein series of weight zero

This section deals with knowledge about Eisenstein series of weight zero. We suggest advanced readers skip this section with a glance on Propositions 4 and 5 on descriptions of their cuspidal behaviors. Good references include [DS, Chapter 4] and [34].

2.2.1 Two kinds of Eisenstein series

On level N, there are Eisenstein series *attached to cusps* and Eisenstein series *attached to characters*.

The Eisenstein series of central character $\chi \pmod N$ attached to the cusp $\mathfrak{a}$ is

$$E_{\mathfrak{a}}(z, s, \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \overline{\chi}(\gamma)(\operatorname{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z)^s.$$

To make this well-defined, we require χ to be even, and $\mathfrak{a}$ to be singular for χ. The definition does not depend on the choice of $\sigma_{\mathfrak{a}}$. Since $E_{\gamma\mathfrak{a}} = \overline{\chi}(\gamma) E_{\mathfrak{a}}$ for $\gamma \in \Gamma_0(N)$, we can always represent $E_{\mathfrak{a}}$ in terms of $E_{\mathfrak{a}'}$ with $\mathfrak{a}' \in \mathcal{C}_\chi(N)$ (see Convention 3 for definition and Remark 3 for practice).

For Dirichlet characters $\chi_i \pmod{q_i}$ with $i = 1, 2$, having the same parity, the Eisenstein series attached to χ_1, χ_2 is

$$E_{\chi_1,\chi_2}(z, s) = \frac{1}{2} \sum_{(c,d)=1} \frac{(q_2 y)^s \chi_1(c) \chi_2(d)}{|c q_2 z + d|^{2s}}.$$

If both χ_1 and χ_2 are primitive, E_{χ_1,χ_2} is a *newform* Eisenstein series of level $q_1 q_2$.

Both types of Eisenstein series converge absolutely for $\Re s > 1$, with meromorphic continuations to $\mathbb{C}$.

Convention 4. *When $\chi = \chi_{0,N}$, we write $E_{\mathfrak{a}}(z,s)$ in short of $E_{\mathfrak{a}}(z,s,\chi)$. If $N = 1$, then the classical Eisenstein series E is the only one in both types, so we write it in place of $E_{1,1}$. If we want to emphasize $E_{\mathfrak{a}}$ is an Eisenstein series of level N, then we may write $E_{\mathfrak{a}}^{(N)}$ instead.*

These two kinds of Eisenstein series are closely connected. Recently, the second author [73] found the change-of-basis formulas between them, which is also done by Booker, Lee, and Strömbergsson [14].

Theorem 11. *[73, Theorem 6.1] Keeping notations as above and 3, and denoting the Euler totient function by φ, we have for $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}_\chi(N)$*

$$E_{\mathfrak{a}}(z,s,\chi) = \frac{W_{\mathfrak{a}}^{-s} f^{-s}}{\varphi((f,\frac{N}{f}))} \sum_{q_1 | \frac{N}{f}} \sum_{q_2 | f} \sideset{}{^*}\sum_{\chi_1,\chi_2} \overline{\chi_2}(-u) \frac{L(2s,\chi_1\chi_2)}{L(2s,\chi_1\chi_2\chi_{0,N})} \sum_{a|f} \sum_{b|\frac{N}{f}} \frac{\mu(a)\mu(b)\chi_1(b)\chi_2(a)}{(ab)^s} E_{\chi_1,\chi_2}\Big(\frac{bf}{aq_2}z, s\Big),$$

where the asterisked sum is over all primitive $\chi_i \pmod{q_i}, i = 1,2$, satisfying $\chi_1\overline{\chi_2} \simeq \chi$.

Remark 3. *In [73], the cusp choice $\mathfrak{a} = \frac{1}{uf}$ was made, and we transfer it for convenience. It is remarked in [73, Section 5.2], that for all $\frac{u}{f} \in \mathcal{C}(N)$, there is $\gamma \in \Gamma_0(N)$ such that $\gamma\frac{u}{f} = \frac{1}{uf}$, and has lower-right entry equal to $u \pmod N$. Then we have*

$$E_{\frac{u}{f}} = \chi(u) E_{\frac{1}{uf}}.$$

We are interested in two special cases: when $f = N$, and when $q = N$.

Since $\infty \overset{N}{=} \frac{1}{N}$ via $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$, we have $E_\infty = E_{\frac{1}{N}}$. By Theorem 11, we have

$$E_\infty(z,s,\chi) = N^{-s} \frac{L(2s,\overline{\psi})}{L(2s,\overline{\chi})} \sum_{a|N} \frac{\mu(a)\overline{\psi}(a)}{a^s} E_{1,\overline{\psi}}\Big(\frac{N}{aq}z, s\Big). \tag{2.2}$$

If χ is primitive mod N, then only Atkin-Lehner cusps are singular for it, as is discussed in Section 2.1.4. Assuming $\mathfrak{a} = \frac{1}{f} \in \mathcal{C}_\chi(N)$, we have

$$E_{\mathfrak{a}}(z, s, \chi) = N^{-s} E_{\chi_1,\chi_2}(z, s), \tag{2.3}$$

where χ_1 is primitive mod $\frac{N}{f}$ and χ_2 is primitive mod f, with $\chi = \chi_1\overline{\chi_2}$.

Now we see why Theorem 6 implies QUE for all newform Eisenstein series of squarefree levels. If N is squarefree, then by definition, a newform Eisenstein series of level N is $E_{\chi_1,\chi_2}(z, s)$ for some primitive χ_i mod q_i, $i = 1, 2$, with $q_1 q_2 = N$ and $(q_1, q_2) = 1$. Then (2.3) says $E = N^s E_{\frac{1}{q_2}}(z, s, \chi_1\overline{\chi_2})$, to which Theorem 6 applies.

In addition, if we relax the squarefree assumption on N and instead assume $E = E_{\chi_1,\chi_2}$ is a newform Eisenstein series of level N and primitive central character $\chi \simeq \chi_1\overline{\chi_2} \pmod N$, for χ_i mod q_i, $i = 1, 2$, then since $q_1 q_2 = N$, we must have $(q_1, q_2) = 1$. The above argument again shows QUE for $E = N^s E_{\frac{1}{q_2}}(z, s, \chi_1\overline{\chi_2})$.

2.2.2 Fourier expansions

One merit of Eisenstein series attached to primitive characters is their explicit Fourier expansions with multiplicative Fourier coefficients. Define the *completed* Eisenstein series by

$$E^*_{\chi_1,\chi_2}(z, s) := \theta_{\chi_1,\chi_2}(s) E_{\chi_1,\chi_2}(z, s),$$

with χ_i primitive mod q_i, $i = 1, 2$, and

$$\theta_{\chi_1,\chi_2}(s) = \frac{q_2^s \pi^{-s}}{\tau(\chi_2)} \Gamma(s) L(2s, \chi_1\chi_2). \tag{2.4}$$

Then we have

$$E^*_{\chi_1,\chi_2}(z, s) = e^*_{\chi_1,\chi_2}(y, s) + 2\sqrt{y} \sum_{n \neq 0} \lambda_{\chi_1,\chi_2}(n, s) e(nx) K_{s-\frac{1}{2}}(2\pi|n|y), \tag{2.5}$$

where

$$e^*_{\chi_1,\chi_2}(y,s) = \delta_{q_1=1}\theta_{1,\chi_2}(s)(q_2 y)^s + \delta_{q_2=1}\theta_{1,\overline{\chi_1}}(1-s)(q_1 y)^{1-s},$$

$\lambda_{\chi_1,\chi_2}(n,s) = \chi_2(\frac{n}{|n|})\sum_{ab=|n|}\chi_1(a)\overline{\chi_2}(b)(\frac{b}{a})^{s-\frac{1}{2}}$, $\tau(\chi)$ is the Gauss sum of χ, and K_α is the K-Bessel function of order $\alpha \in \mathbb{C}$, so that the series in (2.5) decays exponentially, as $y \to \infty$. See Huxley [32], and Knightly and Li [42, Section 5.6] for more details.

Remark 4. *From the definition we see that when $s = \frac{1}{2} + iT$, $|\lambda_{\chi_1,\chi_2}(n,s)| \leq d(n) \ll n^\varepsilon$.*

Remark 5. *If χ is primitive mod q for $q > 1$, then $E_{\chi,\chi}(z,s)$ is regular at $s = 1$.*

Remark 6. *The newform Eisenstein series are eigenfunctions of all the Hecke operators T_n, and indeed $T_n E_{\chi_1,\chi_2}(z,s) = \lambda_{\chi_1,\chi_2}(n,s)E_{\chi_1,\chi_2}(z,s)$.*

For future application, we write out two special cases. When $\chi_1 = 1$, and $\chi_2 = \overline{\psi}$ primitive mod q, we have

$$E_{1,\overline{\psi}}(z, \tfrac{1}{2}+iT) = e_{1,\overline{\psi}}(y, \tfrac{1}{2}+iT) + 2\rho_{1,\overline{\psi}}(\tfrac{1}{2}+iT)\sqrt{y}\sum_{n\neq 0}\lambda_{1,\overline{\psi}}(n)e(nx)K_{iT}(2\pi|n|y), \tag{2.6}$$

where $e_{\chi_1,\chi_2}(s) = \rho_{\chi_1,\chi_2}(s)e^*_{\chi_1,\chi_2}(y,s)$, $\rho_{\chi_1,\chi_2}(s) = \frac{1}{\theta_{\chi_1,\chi_2}(s)}$, $\lambda_{\chi_1,\chi_2}(n) = \lambda_{\chi_1,\chi_2}(n, \frac{1}{2}+iT)$, and

$$\rho_{1,\overline{\psi}}(\tfrac{1}{2}+iT) = O(q^\varepsilon(1+|T|)^\varepsilon e^{\frac{\pi|T|}{2}}) \tag{2.7}$$

by Stirling's formula, see e.g. [36, (5.73)] and [54, (11.18)]. Another case is when $q_1 q_2 = N$ with $(q_1, q_2) = 1$, and χ_i is primitive mod q_i for $i = 1, 2$. We then have

$$E_{\chi_1,\chi_2}(z, \tfrac{1}{2}+iT) = \rho_{\chi_1,\chi_2}(\tfrac{1}{2}+iT)\sqrt{y}\sum_{n\neq 0}\lambda_{\chi_1,\chi_2}(n)e(nx)K_{iT}(2\pi|n|y), \tag{2.8}$$

and similarly,

$$\rho_{\chi_1,\chi_2}(\tfrac{1}{2}+iT) = O(N^{\varepsilon}(1+|T|)^{\varepsilon}e^{\frac{\pi|T|}{2}}). \tag{2.9}$$

Next we discuss some aspects of the Fourier expansion of $E_{\mathfrak{a}}(z,s,\chi)$. For the following discussion, assume $\mathfrak{a}$, $\mathfrak{b}$ are cusps singular for χ. When $y \to \infty$ (see e.g., [34, (13.15)])

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s,\chi) = \delta_{\mathfrak{ab}}y^s + \varphi_{\mathfrak{ab}}(s,\chi)y^{1-s} + O(y^{-P}), \tag{2.10}$$

for all $P \in \mathbb{N}$, where $\delta_{\mathfrak{ab}} = 1$ if $\mathfrak{a} \overset{N}{=} \mathfrak{b}$, and vanishes otherwise, and $\varphi_{\mathfrak{ab}}$ is meromorphic in $s \in \mathbb{C}$. Iwaniec writes $\varphi_{\mathfrak{ab}}$ as an infinite sum, see [34, (13.16)–(13.18)], and we have an alternative finite expression in Proposition 4 below.

Convention 5. *Analogously to Convention 4, if $\chi = \chi_{0,N}$, then we suppress it from $\varphi_{\mathfrak{ab}}(s,\chi)$; if necessary, we write $\varphi_{\mathfrak{ab}}^{(N)}$ to emphasize it comes from $E_{\mathfrak{a}}^{(N)}$.*

Proposition 2 (Selberg [34] (13.30))**.** *For $\Re s = \frac{1}{2}$, the matrix $\Phi(s,\chi) = \Big(\varphi_{\mathfrak{ab}}(s,\chi)\Big)_{\mathfrak{a},\mathfrak{b}}$ is unitary. In particular, we have $\sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)}|\varphi_{\infty\mathfrak{a}}(s,\chi)|^2 = 1$ for $s = \frac{1}{2}+iT$.*

2.2.3 Functional equations

Eisenstein series attached to Dirichlet characters satisfy the following simple functional equation. Recall $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{a},N}$ is a scaling matrix as in Remark 1.

Proposition 3 (Huxley [32])**.** *For primitive χ_1 and χ_2, we have*

$$E^*_{\chi_1,\chi_2}(z,s) = E^*_{\overline{\chi_2},\overline{\chi_1}}(z,1-s).$$

When $(q_1,q_2)=1$ and $\mathfrak{a} = \frac{1}{q_2}$, Weisinger [70] essentially showed (see also [73, (9.1)])

$$E_{\chi_1,\chi_2}|_{\sigma_{\mathfrak{a}}} = \epsilon_{\chi_1,\chi_2}E_{1,\chi_1\chi_2}, \qquad \text{where} \qquad |\epsilon_{\chi_1,\chi_2}| = 1. \tag{2.11}$$

In addition, we also have in (9.1) of [73] that

$$E^*_{\chi_1,\chi_2}(\sigma_{1/q_2}z,s)=\frac{\tau(\chi_1\chi_2)}{\tau(\chi_2)}(\frac{q_2}{N})^sE^*_{1,\chi_1\chi_2}(z,s);\tag{2.12}$$

$$E^*_{\overline{\chi_2},\overline{\chi_1}}(\sigma_{1/q_1}z,s)=\frac{\tau(\overline{\chi_1\chi_2})}{\tau(\overline{\chi_1})}(\frac{q_1}{N})^sE^*_{1,\overline{\chi_1\chi_2}}(z,s).\tag{2.13}$$

If χ is primitive mod N, then $(q_1,q_2)=1$, and hence χ can uniquely be decomposed as $\chi_1\overline{\chi_2}$ for χ_j primitive mod q_j, $j=1,2$. By [73, (6.2)], we have

$$E_{\frac{1}{q_2}}(z,s,\chi)=N^{-s}\frac{\chi_1(-1)\tau(\chi_2)}{\Lambda(2s,\chi_1\chi_2)}E^*_{\chi_1,\chi_2}(z,s).\tag{2.14}$$

When χ is primitive mod N, $\mathfrak{a}=\frac{1}{q_2}\in\mathcal{C}_\chi$, we denote $\frac{1}{q_1}$ by $\mathfrak{a}^*$. With discussions in Sections 2.2.5, we have the following cuspidal behavior formulas for $\mathfrak{b}\in\mathcal{C}$ and $y\to\infty$:

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s,\chi)=\begin{cases}y^s+O(p^{-\mathbb{N}}) & \text{if } \mathfrak{b}=\mathfrak{a}\\ \overline{\tau}(\overline{\chi_1})\tau(\chi_2)N^{-s}\frac{\Lambda(2-2s,\overline{\chi_1\chi_2})}{\Lambda(2s,\chi_1\chi_2)}y^{1-s}+O(p^{-\mathbb{N}}) & \text{if } \mathfrak{b}=\mathfrak{a}^*\\ O(p^{-\mathbb{N}}) & \text{otherwise}\end{cases}.\tag{2.15}$$

2.2.4 Identifying traced Eisenstein series

Define the *trace operator* $\mathrm{Tr}^N_M:\mathcal{A}(Y_0(N))\to\mathcal{A}(Y_0(M))$ via

$$f\mapsto\sum_{\gamma\in\Gamma_0(N)\backslash\Gamma_0(M)}f|_\gamma.\tag{2.16}$$

Now we can determine the exact shape of $\mathrm{Tr}^N_M E^{(N)}_{\mathfrak{a}}(z,s)$ by (2.10).

Lemma 4. *We have the following equality of meromorphic functions:*

$$\mathrm{Tr}^N_M E^{(N)}_{\mathfrak{a}}(z,s)=(W^M_N(\mathfrak{a}))^{1-s}E^{(M)}_{\mathfrak{a}}(z,s).$$

Remark 7. *We have to point out that when* $\mathfrak{a}$ *is a cusp for* $Y_0(N)$*, there might be ambiguities for the symbol of* $E_{\mathfrak{a}}^{(M)}$*. However, since the central character is trivial, the choice of representative for* $\mathfrak{a}$ *in* $Y_0(M)$ *does not affect the resulted function, as mentioned in Section 2.2.1.*

Proof. Let $\Re s > 1$. By [35, Lemma 6.4], $\operatorname{Tr}_M^N E_{\mathfrak{a}}^{(N)}(z,s)$ is a linear combination of $E_{\mathfrak{b}}^{(M)}(z,s)$ for cusps $\mathfrak{b}$ of level M. Now we compare the y^s-terms to determine this linear combination.

For each $\mathfrak{b}$ pick $\sigma_{\mathfrak{b},M} = \gamma_{\mathfrak{b}}\left(\begin{smallmatrix} W^{1/2} & 0 \\ 0 & W^{-1/2} \end{smallmatrix}\right)$ as in Remark 1, where $\gamma_{\mathfrak{b}} \in SL_2(\mathbb{Z})$, $\gamma_{\mathfrak{b}}\infty = \mathfrak{b}$, and $W = W_M^1(\mathfrak{b})$. As $y \to \infty$, we have by (2.10), Lemmas 3, 2 and Remark 2,

$$\begin{aligned}
\operatorname{Tr}_M^N E_{\mathfrak{a}}^{(N)}(\sigma_{\mathfrak{b},M} z, s) &= \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(M)} E_{\mathfrak{a}}^{(N)}(\gamma\gamma_{\mathfrak{b}} W z, s) = \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(M)} E_{\mathfrak{a}}^{(N)}\Big(\sigma_{\gamma\mathfrak{b},N} \tfrac{W}{W_N^1(\gamma\mathfrak{b})} z, s\Big) \\
&= \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(M)} \delta_{\gamma\mathfrak{b} \overset{N}{=} \mathfrak{a}} \Big(\tfrac{W}{W_N^1(\gamma\mathfrak{b})} y\Big)^s + O(1) \\
&= \delta_{\mathfrak{b} \overset{M}{=} \mathfrak{a}} W_N^M(\mathfrak{a}) \Big(\tfrac{W_M^1(\mathfrak{a})}{W_N^1(\mathfrak{a})} y\Big)^s + O(1) = \delta_{\mathfrak{b} \overset{M}{=} \mathfrak{a}} W_N^M(\mathfrak{a})^{1-s} y^s + O(1).
\end{aligned}$$

On the other hand, $(W_N^M(\mathfrak{a}))^{1-s} E_{\mathfrak{a}}^{(M)}|_{\sigma_{\mathfrak{b}}}$ has exactly the same formula as above by (2.10), which finishes the proof. $\square$

2.2.5 Explicit calculations with scattering matrices and related quantities

As is mentioned previously, we need to study the behavior of $|E_\infty(z,s,\chi)|^2$ at each cusp in $\mathcal{C}(N)$, not just these in $\mathcal{C}_\chi(N)$. The change-of-basis formula, Theorem 11, now helps.

2.2.5.1 Preparation

We begin with proving a lemma.

Lemma 5. *Let* $K \geq 1$*, and* $\gamma = \left(\begin{smallmatrix} u & v \\ f & w \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *with* $f \mid N$*. Then there exist meromorphic* $C_{\chi_1,\chi_2}(s)$ *and* $D_{\chi_1,\chi_2}(s)$ *(depending on* K *and* γ*) such that*

$$E_{\chi_1,\chi_2}(K\gamma z, s) = C_{\chi_1,\chi_2}(s) y^s + D_{\chi_1,\chi_2}(s) y^{1-s} + o(1), \tag{2.17}$$

as $y \to \infty$. Precisely,

$$\begin{aligned} C_{\chi_1,\chi_2}(s) &= \delta_{q_2|f} \frac{(q_2K,f)^{2s}}{q_2^s K^s} \chi_1\Big(\frac{-f}{(q_2K,f)}\Big)\chi_2\Big(\frac{q_2Ku}{(q_2K,f)}\Big), \\ D_{\chi_1,\chi_2}(s) &= \delta_{q_1|f} \frac{\theta_{\overline{\chi_2},\overline{\chi_1}}(1-s)}{\theta_{\chi_1,\chi_2}(s)} \frac{(q_1K,f)^{2-2s}}{q_1^{1-s}K^{1-s}} \overline{\chi_1}\Big(\frac{q_1Ku}{(q_1K,f)}\Big)\overline{\chi_2}\Big(\frac{-f}{(q_1K,f)}\Big). \end{aligned} \tag{2.18}$$

Proof. Observe $E_{\chi_1,\chi_2}(K\gamma z, s)$ is periodic with some integer period. By [35, Proposition 1.5], (2.17) holds. To obtain (2.18), we proceed directly. By definition, we have

$$\begin{aligned} E_{\chi_1,\chi_2}(K\gamma z, s) &= \frac{1}{2}\sum_{(c,d)=1} \frac{(q_2\Im(K\gamma z))^s \chi_1(c)\chi_2(d)}{|cq_2K\gamma z + d|^{2s}} \\ &= \frac{1}{2}\sum_{(c,d)=1} \frac{(q_2Ky)^s \chi_1(c)\chi_2(d)}{|(cq_2Ku+df)z + (cq_2Kv+dw)|^{2s}} = \frac{1}{2}\sum_{\ell\in\mathbb{Z}} \sum_{\substack{(c,d)=1\\ cq_2Ku+df=\ell}} \frac{(q_2Ky)^s \chi_1(c)\chi_2(d)}{|\ell z + (cq_2Kv+dw)|^{2s}}. \end{aligned}$$

For any $\Re s > 1$, we see that as $y \to \infty$, uniform convergence allows us to interchange the limit and the sums, yielding

$$E_{\chi_1,\chi_2}(K\gamma z, s) = C(s)y^s + o(1), \quad \text{for} \quad C(s) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ cq_2Ku+df=0}} \frac{(q_2K)^s \chi_1(c)\chi_2(d)}{|cq_2Kv+dw|^{2s}}.$$

Then (2.17) implies that $C(s) = C_{\chi_1,\chi_2}(s)$, and we can calculate $C_{\chi_1,\chi_2}(s)$ by simplifying the above expression. Solving $cq_2Ku + df = 0$ for $(c,d) = 1$ and $\chi_1(c)\chi_2(d) \neq 0$, we can easily see the solutions exist only if $q_2 \mid f$, and they are

$$\begin{cases} c = \pm\frac{f}{(q_2K,f)} \\ d = \mp\frac{q_2Ku}{(q_2K,f)}. \end{cases}$$

Since $uw - vf = 1$ and $\chi_1\chi_2(-1) = 1$, we arrive at the desired expression for $C_{\chi_1,\chi_2}(s)$. By

Proposition 3, we have

$$D_{\chi_1,\chi_2}(s) = \tfrac{\theta_{\overline{\chi_2},\overline{\chi_1}}(1-s)}{\theta_{\chi_1,\chi_2}(s)} C_{\overline{\chi_2},\overline{\chi_1}}(1-s).$$

Inserting the formula of $C_{\overline{\chi_2},\overline{\chi_1}}$, we complete the proof. $\square$

2.2.5.2 Entries of scattering matrices

Proposition 4. *If* $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_\chi(N)$, *then*

$$\varphi_{\mathfrak{ab}}(s,\chi) = \frac{f_{\mathfrak{a}}^{-1} W_{\mathfrak{a}}^{-s} W_{\mathfrak{b}}^{1-s}}{\varphi((f_{\mathfrak{a}}, \frac{N}{f_{\mathfrak{a}}}))} \sum_{q_1 | (\frac{N}{f_{\mathfrak{a}}}, f_{\mathfrak{b}})} \sum_{q_2 | f_{\mathfrak{a}}} \sideset{}{^*}\sum_{\chi_1,\chi_2} \overline{\chi_1}(u_{\mathfrak{b}}) \overline{\chi_2}(u_{\mathfrak{a}}) \frac{L(2s, \chi_1\chi_2)}{L(2s, \chi_1\chi_2\chi_{0,N})} \frac{\theta_{\overline{\chi_2},\overline{\chi_1}}(1-s)}{\theta_{\chi_1,\chi_2}(s)}$$
$$(\tfrac{q_2}{q_1})^{1-s} \sum_{a|f_{\mathfrak{a}}} \sum_{b|\frac{N}{f_{\mathfrak{a}}}} \frac{\mu(a)\mu(b)\chi_1(b)\chi_2(a)}{a^{2s-1}b} (q_1 \tfrac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})^{2-2s} \overline{\chi_1}\Big(\frac{q_1 \frac{bf_{\mathfrak{a}}}{aq_2}}{(q_1 \frac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})}\Big) \overline{\chi_2}\Big(\frac{f_{\mathfrak{b}}}{(q_1 \frac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})}\Big),$$

where the asterisked sum is over all primitive $\chi_i \pmod{q_i}$ *for* $i = 1, 2$ *with* $\chi_1\overline{\chi_2} \simeq \chi$ *(see Convention 2 for definition).*

Proof. For $\mathfrak{b} = \frac{u_{\mathfrak{b}}}{f_{\mathfrak{b}}}$ as is in (1.1), we have by Theorem 11

$$\varphi_{\mathfrak{ab}}(s,\chi) = \frac{f_{\mathfrak{a}}^{-s} W_{\mathfrak{a}}^{-s}}{\varphi((f_{\mathfrak{a}}, \frac{N}{f_{\mathfrak{a}}}))} \sum_{q_1 | \frac{N}{f_{\mathfrak{a}}}} \sum_{q_2 | f_{\mathfrak{a}}} \sideset{}{^*}\sum_{\chi_1,\chi_2} \overline{\chi_2}(-u_{\mathfrak{a}}) \frac{L(2s, \chi_1\chi_2)}{L(2s, \chi_1\chi_2\chi_{0,N})}$$
$$\sum_{a|f_{\mathfrak{a}}} \sum_{b|\frac{N}{f_{\mathfrak{a}}}} \frac{\mu(a)\mu(b)\chi_1(b)\chi_2(a)}{(ab)^s} \Psi\Big(E_{\chi_1,\chi_2}\Big(\frac{bf_{\mathfrak{a}}}{aq_2}\sigma_{\mathfrak{b}} z, s\Big)\Big),$$

where $\Psi(E_{\chi_1,\chi_2})$ stands for the coefficient of the y^{1-s}-term of E_{χ_1,χ_2}. Since the choice of $\sigma_{\mathfrak{b}}$ does not affect the constant term in the Fourier expansion, we can take

$$\sigma_{\mathfrak{b}} = \gamma_{\mathfrak{b}} \left(\begin{smallmatrix} W_{\mathfrak{b}}^{1/2} & 0 \\ 0 & W_{\mathfrak{b}}^{-1/2} \end{smallmatrix}\right)$$

by Remark 1, where $\gamma_{\mathfrak{b}} = \left(\begin{smallmatrix} u_{\mathfrak{b}} & v \\ f_{\mathfrak{b}} & w \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then for $K = \frac{bf_{\mathfrak{a}}}{aq_2}$, and $\gamma = \gamma_{\mathfrak{b}}$, (2.18) gives

$$\Psi\Big(E_{\chi_1,\chi_2}\Big(\frac{bf_{\mathfrak{a}}}{aq_2}\sigma_{\mathfrak{b}}z, s\Big)\Big) = \delta_{q_1|f_{\mathfrak{b}}}\frac{\theta_{\overline{\chi_2},\overline{\chi_1}}(1-s)}{\theta_{\chi_1,\chi_2}(s)}\frac{(q_1\frac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})^{2-2s}}{q_1^{1-s}(\frac{bf_{\mathfrak{a}}}{aq_2})^{1-s}}$$
$$\overline{\chi_1}\Big(\frac{u_{\mathfrak{b}}q_1\frac{bf_{\mathfrak{a}}}{aq_2}}{(q_1\frac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})}\Big)\overline{\chi_2}\Big(\frac{-f_{\mathfrak{b}}}{(q_1\frac{bf_{\mathfrak{a}}}{aq_2}, f_{\mathfrak{b}})}\Big)W_{\mathfrak{b}}^{1-s}.$$

Then we complete the proof after substitution. □

There are two special cases of Proposition 4 of special interest in this paper.

Firstly, we consider the case $\mathfrak{a} = \infty$. Notice that $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)\mathfrak{a} = \mathfrak{a}' = \frac{1}{N}$, by Remark 3, so we have $\varphi_{\mathfrak{a}\mathfrak{b}} = \chi(1)\varphi_{\mathfrak{a}'\mathfrak{b}} = \varphi_{\mathfrak{a}'\mathfrak{b}}$. In addition, we have the following closed-form formula:

Corollary 1. *For $\mathfrak{b} = \frac{u}{f} \in \mathcal{C}_\chi(N)$ in (1.1), we have*

$$\varphi_{\infty\mathfrak{b}}(s,\chi) = \delta_{f|\frac{N}{q}}\tau(\overline{\psi})\frac{W_{\mathfrak{b}}^{-s}f^{1-2s}}{\varphi((f,\frac{N}{f}))}\frac{\Lambda(2-2s,\psi)}{\Lambda(2s,\overline{\psi})}\prod_{p|N}\Big(1-\frac{\overline{\psi}(p)}{p^{2s}}\Big)^{-1}\prod_{p|f}(1-\frac{1}{p})\prod_{p|\frac{N}{f}}\Big(1-\frac{\overline{\psi}(p)}{p^{2s-1}}\Big),$$

where Λ is the completed Dirichlet L-function. In particular, $\varphi_{\infty\infty}(s,\chi) = 0$ unless $\chi = \chi_{0,N}$.

Sketch of proof. We need to substitute $f_{\mathfrak{a}} = N$, $f_{\mathfrak{b}} = f$ into Proposition 4. Briefly, after some local analysis over different types of prime numbers, we have

$$\sum_{a|N}\frac{\mu(a)\overline{\psi}(a)}{a^{2s-1}}\Big(\frac{N}{aq}, f\Big)^{2-2s}\psi\Big(\frac{f}{(\frac{N}{aq}, f)}\Big) = \delta_{f|\frac{N}{q}}f^{2-2s}\prod_{p|\frac{N}{f}}\Big(1-\frac{\overline{\psi}(p)}{p^{2s-1}}\Big)\prod_{p\nmid\frac{N}{f}}\Big(1-\frac{1}{p}\Big).$$

One can verify the rest easily and complete the proof. □

Secondly, we assume χ is primitive $(\mathrm{mod}\ N)$, where only Atkin-Lehner cusps are singular for χ. Given an Atkin-Lehner cusp $\mathfrak{a} = \frac{1}{f} \in \mathcal{C}(N)$, we call $\mathfrak{a}^* := \frac{1}{N/f} \in \mathcal{C}(N)$ the *Atkin-Lehner complement of* $\mathfrak{a}$ (*on level* N). The following calculation by N. Pitt depicts a special property of Atkin-Lehner complement. Humphries (via personal communication) computed it independently, in full details, and for general weights.

Corollary 2. *[34, Proposition 13.7] If $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(N)$ are Atkin-Lehner, and $\chi = \chi_1\overline{\chi_2}$ with χ_1 primitive mod $\frac{N}{f_\mathfrak{a}}$ and χ_2 primitive mod $f_\mathfrak{a}$, then we have*

$$\varphi_{\mathfrak{a}\mathfrak{b}}(s,\chi) = \begin{cases} \chi_1(-1)\tau(\chi_1)\tau(\chi_2)N^{-s}\frac{\Lambda(2-2s,\overline{\chi_1\chi_2})}{\Lambda(2s,\chi_1\chi_2)} & \textit{if } \mathfrak{b} = \mathfrak{a}^*; \\ 0 & \textit{otherwise}. \end{cases}$$

2.2.5.3 The behavior of Eisenstein series at cusps that are not singular

As we have mentioned in the introduction, the cuspidal behavior of Eisenstein series at cusps not singular for the central character affects the precise description of $\mathcal{E}$.

Proposition 5. *If $\mathfrak{a} \in \mathcal{C}_\chi(N)$, and $\mathfrak{b} \in \mathcal{C}(N)\backslash\mathcal{C}_\chi(N)$, then as $y \to \infty$, we have*

$$E_\mathfrak{a}(\sigma_\mathfrak{b} z, s, \chi) = o_s(1).$$

Selberg proved (yet not published) the proposition for primitive χ; see [66, Thm. 7.1, p.641]. Here we give an alternative proof, for which we need some preparation.

Convention 6. *We denote the p-adic order function by $\nu_p(\cdot)$.*

Lemma 6. *Let χ_i be primitive mod q_i for $i = 1, 2$, and $\chi = \chi_1\overline{\chi_2}$ be induced by primitive ψmod q. Assume there is $f \mid N$ such that $q_1 \mid \frac{N}{f}$ and $q_2 \mid f$, and $K \mid N$ satisfying:*

$$\nu_p(K) \le \begin{cases} \nu_p(N) - \nu_p(q_2) & \textit{if } p \nmid q_1, p \mid q_2; \\ \nu_p(f) - \nu_p(q_2) & \textit{if } p \mid q_1. \end{cases} \tag{2.19}$$

If $E_{\chi_1,\chi_2}(K\sigma_\mathfrak{b} z, s, \chi)$ is unbounded as $y \to \infty$ for some $\mathfrak{b} \in \mathcal{C}(N)$, then $\mathfrak{b} \in \mathcal{C}_\chi(N)$.

Proof. If $E_{\chi_1,\chi_2}|_{K\sigma_\mathfrak{b}}$ is unbounded, then by Lemma 5, either $C_{\chi_1,\chi_2}(s) \ne 0$ or $D_{\chi_1,\chi_2}(s) \ne 0$.

In the **former case**, we have $q_2 \mid f_\mathfrak{b}$, and for all prime numbers $p \mid q_1$,

$$\nu_p(K) \ge \nu_p(f_\mathfrak{b}) - \nu_p(q_2).$$

From (2.19), we know $\nu_p(K) \le \nu_p(f) - \nu_p(q_2)$, which gives $\nu_p(f) \ge \nu_p(f_\mathfrak{b})$. Then by assumption on f, we have

$$\nu_p(q_1) \le \nu_p(N/f) \le \nu_p(N/f_\mathfrak{b}),$$

indicating $q_1 \mid \frac{N}{f_\mathfrak{b}}$. Together with $q_2 \mid f_\mathfrak{b}$, we find $q = [q_1, q_2] \mid [f_\mathfrak{b}, \frac{N}{f_\mathfrak{b}}]$, which means $\mathfrak{b}$ is singular for χ by Proposition 1.

In the **latter case**, we have $q_1 \mid f_\mathfrak{b}$, and for all prime numbers $p \mid q_2$,

$$\nu_p(f_\mathfrak{b}) \le \nu_p(q_1) + \nu_p(K).$$

We want to show

$$\nu_p(q_2) \le \nu_p(N) - \nu_p(f_\mathfrak{b}) \tag{2.20}$$

for all $p \mid q_2$, since this implies $q_2 \mid \frac{N}{f_\mathfrak{b}}$, and hence that $\mathfrak{b}$ is singular for $\chi_1\overline{\chi_2}$ for the same reason in the previous case. We further bifurcate the discussion. Say p also divides q_1. Then

$$\nu_p(f_\mathfrak{b}) \le \nu_p(q_1) + \nu_p(K) \le \nu_p(q_1) + \nu_p(f) - \nu_p(q_2) \le \nu_p(N) - \nu_p(q_2),$$

Thus (2.20) holds. On the contrary, if $p \nmid q_1$, then $\nu_p(f_\mathfrak{b}) \le \nu_p(K) \le \nu_p(N) - \nu_p(q_2)$, giving (2.20) again. $\square$

Proof of Proposition 5. By Theorem 11, $E_\mathfrak{a}(\sigma_\mathfrak{b} z, s, \chi)$ can be written as a linear combination of $E_{\chi_1,\chi_2}(K\sigma_\mathfrak{b} z, s, \chi)$, where χ_i is primitive mod q_i for $i = 1, 2$, $\chi_1\overline{\chi_2} \simeq \chi$, and $K \mid N$ satisfies (2.19). By Lemma 6, none of these $E_{\chi_1,\chi_2}(K\sigma_\mathfrak{b} z, s, \chi)$ contributes any y^s or y^{1-s}-terms, so we have done. $\square$

2.2.6 The formal inner product of Eisenstein series

It is well-known that Eisenstein series are not in L^2. It is nevertheless useful to consider the *formal inner product* of two Eisenstein series. Concretely, if $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(N)$, then the formal inner product of $E_\mathfrak{a}$ and $E_\mathfrak{b}$ is defined by

$$\langle E_\mathfrak{a}(\cdot, s), E_\mathfrak{b}(\cdot, s)\rangle_N^{\mathrm{Eis}} := 4\pi\delta_{\mathfrak{a}\mathfrak{b}},$$

when $s = \frac{1}{2} + iT$. For more details, see Section 3.3, where we adopt newform Eisenstein series to build an alternative orthonormal basis. To accomplish this, we have the following lemma as a special case of [73, Lemma 8.3].

Lemma 7. *For primitive $\psi \pmod q$ with $q^2 \mid N$, we have*

$$\langle E_{\psi,\psi}, E_{\psi,\psi}\rangle_N^{\mathrm{Eis}} = 4\pi N \prod_{p|q}(1 - p^{-1}) \prod_{p|N}(1 + \chi_{0,N}(p)p^{-1}).$$

2.2.7 Laurent expansions of Eisenstein series

Proposition 6. *We have the Laurent expansion*

$$E(z, s) = \frac{3/\pi}{s-1} + G(z) + O(|s-1|),$$

where as $y \to \infty$,

$$G(z) = y + O(\log y). \tag{2.21}$$

Proposition 6 follows directly from [36, (22.66)–(22.69)], so we omit the proof. These formulas also show that $G(z) \in \mathcal{A}(Y_0(1))$ can be expressed in terms of the logarithm of the Dedekind eta function, but all we need for our later purposes is (2.21).

It is also important to explicitly evaluate the Laurent expansion of $E_\mathfrak{a}(z, s)$ around $s = 1$ in terms of the newform Eisenstein series.

Proposition 7. *For $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}(N)$, we have*

$$E_{\mathfrak{a}}(z,s) = \frac{\mathrm{Vol}(Y_0(N))^{-1}}{s-1} + c_{\mathfrak{a},0} + \sum_{g|N} c_{\mathfrak{a},g} G|_g + \sum_{1<r|(f,N/f)} \sideset{}{^*}\sum_{\eta(r)} \overline{\eta}(u) \sum_{g|Nr^{-2}} c_{\mathfrak{a},\eta,g} E_{\eta,\eta}(gz,1) + O(|s-1|),$$

where $c_{\mathfrak{a},\eta,g}$ are independent of u,

$$c_{\mathfrak{a},0} = \frac{1}{\mathrm{Vol}(Y_0(N))}\Big(\log\frac{(f,\frac{N}{f})}{N} + \sum_{p|N}\frac{\log p}{p+1} - \sum_{p|(f,N/f)}\frac{\log p}{p-1}\Big), \tag{2.22}$$

and

$$c_{\mathfrak{a},g} = \frac{(f,N/f)}{N\varphi((f,N/f))}\frac{\zeta(2)}{L(2,\chi_{0,N})}\sum_{a|f}\sum_{b|\frac{N}{f}}\delta_{bf/a=g}\frac{\mu(a)\mu(b)}{ab}. \tag{2.23}$$

Proof. By Theorem 11, $E_{\mathfrak{a}}(z,s)$ can be expressed as a linear combination of $E_{\eta,\eta}|_g$ for primitive η (mod r) with $r \mid (f,N/f)$, and suitable $g|N$. The contribution from $r>1$ is

$$\frac{W_{\mathfrak{a}}^{-s}f^{-s}}{\varphi((f,N/f))}\sum_{1<r|(f,N/f)}\sideset{}{^*}\sum_{\eta}\overline{\eta}(-u)\frac{L(2s,\eta^2)}{L(2s,\eta^2\chi_{0,N})}\sum_{a|f}\sum_{b|\frac{N}{f}}\frac{\mu(a)\mu(b)\eta(ab)}{a^s b^s}E_{\eta,\eta}\Big(\frac{bf}{ar}z,s\Big),$$

which can be expressed as $\sum_{1<r|(f,N/f)}\sum^*_{\eta(r)}\overline{\eta}(-u)\sum_{g|Nr^{-2}}c_{\mathfrak{a},\eta,g}E_{\eta,\eta}(gz,1)$ with $c_{\mathfrak{a},\eta,g}$ independent of u. By Proposition 6, the contribution from $r=1$ equals

$$\frac{W_{\mathfrak{a}}^{-s}f^{-s}}{\varphi((f,N/f))}\frac{\zeta(2s)}{L(2s,\chi_{0,N})}\sum_{a|f}\sum_{b|\frac{N}{f}}\frac{\mu(a)\mu(b)}{a^s b^s}\Big(\frac{3/\pi}{s-1} + G\Big(\frac{bf}{a}z\Big) + O(s-1)\Big).$$

Let

$$F_{\mathfrak{a}}(s) = \frac{W_{\mathfrak{a}}^{-s}f^{-s}}{\varphi((f,N/f))}\frac{\zeta(2s)}{L(2s,\chi_{0,N})}\sum_{a|f}\sum_{b|\frac{N}{f}}\frac{\mu(a)\mu(b)}{a^s b^s}. \tag{2.24}$$

It is well-known that $\mathrm{Res}_{s=1}E_{\mathfrak{a}}(z,s) = (\mathrm{Vol}(Y_0(N)))^{-1}$, so $\frac{3}{\pi}F_{\mathfrak{a}}(1) = \mathrm{Vol}(Y_0(N))^{-1}$; of course, for consistency this can be checked directly from (2.24). Hence the contribution of $r = 1$ to the Laurent expansion of $E_{\mathfrak{a}}(z, s)$ is of the form

$$\frac{\mathrm{Vol}(Y_0(N))^{-1}}{s-1} + \tfrac{3}{\pi}F'_{\mathfrak{a}}(1) + \sum_{g|N} c_{\mathfrak{a},g} G|_g + O(s-1),$$

for $c_{\mathfrak{a},g}$ given by (2.23). The term $F'_{\mathfrak{a}}(1)$ gives rise to $c_{\mathfrak{a},0}$, which is computed by

$$\frac{F'_{\mathfrak{a}}}{F_{\mathfrak{a}}}(1) = -\log N + \log\Big(f, \tfrac{N}{f}\Big) + \sum_{p|N} \frac{\log p}{p+1} - \sum_{p|(f,N/f)} \frac{\log p}{p-1}. \qquad \square$$

Although the level 1 Eisenstein series is an eigenfunction of the Hecke operators, the same is not quite true for the function G.

Lemma 8. *For $n \geq 1$, we have*

$$T_n(G) = \lambda(n)G + \frac{3}{\pi}\sqrt{n}\sum_{a|n} a^{-1}\log\frac{n}{a^2},$$

where T_n is the n-th Hecke operator, and $\lambda(n) = \lambda_{1,1}(n, 1) = n^{1/2}\sum_{b|n} b^{-1}$ as is in (2.6).

Remark 8. *Our normalization of the Hecke operator T_n is so that $T_n u_j = \lambda_j(n)u_j$.*

Proof. Recall that $G(z) = \mathrm{Res}_{s=1}(s-1)^{-1}E(z, s)$, so by Remark 6 we have

$$T_n(G) = \operatorname*{Res}_{s=1}\Big((s-1)^{-1}\lambda(n, s)E(z, s)\Big).$$

By Proposition 6 and since $\lambda(n, s) = \sum_{ab=n}(\frac{b}{a})^{s-1/2}$, we finish the proof. $\square$

2.2.8 Some inequalities

Here we perform some elementary calculations related to $\varphi_{\infty\mathfrak{a}}$, which is critical for future arguments. To begin, we have the following standard lemma.

Lemma 9. *There exists an absolute constant C so that*

$$\sum_{p|N}\frac{1}{p}\leq \log\log\log(N+15)+C, \quad \textit{and} \quad \sum_{p|N}\frac{\log p}{p}\leq \log\log(N+2)+C.$$

Convention 7. *For integers A and B, we denote the greatest divisor of A that divides (is coprime to, respectively) B by A_B ($A_B^{\perp}$, respectively). Notice that $A=A_BA_B^{\perp}$.*

From the fact $N_q^{\perp}\mid\frac{N}{q}$, we have the following corollary.

Corollary 3. *If $s=\frac{1}{2}+iT$ and ψ is primitive $\pmod q$ for $q\mid N$, then*

$$\sum_{p|N}\frac{\psi(p)\log p}{\psi(p)p^{2s}-1}\ll \log\log\Big(\tfrac{N}{q}+2\Big).$$

Then we can bound the coefficients in Proposition 7.

Corollary 4. *For $\mathfrak{a}=\frac{u}{f}\in\mathcal{C}(N)$, we have*

$$c_{\mathfrak{a},0}=\frac{1}{\mathrm{Vol}(Y_0(N))}\Big(\log\frac{(f,N/f)}{N}+O(\log\log N)\Big), \tag{2.25}$$

and

$$\sum_{g|N}|c_{\mathfrak{a},g}|\ll N^{-1}(\log\log N)^3. \tag{2.26}$$

Proof. The equation (2.25) follows from Lemma 9. By (2.23), we have

$$\begin{aligned}\sum_{g|N}|c_{\mathfrak{a},g}| &\leq \frac{(f,N/f)}{N\varphi((f,N/f))}\frac{\zeta(2)}{L(2,\chi_{0,N})}\sum_{a|f}\sum_{b|\frac{N}{f}}\frac{|\mu(a)\mu(b)|}{ab}\\ &= N^{-1}\prod_{p|(f,N/f)}(1-p^{-1})^{-1}\prod_{p|N}(1-p^{-2})^{-1}\prod_{p|f}(1+p^{-1})\prod_{p|\frac{N}{f}}(1+p^{-1}).\end{aligned}$$

Then Lemma 9 completes the proof of (2.26). □

Convention 8. *Given $n\geq 1$, we denote the number of prime divisors of n by $\omega(n)$.*

Proposition 8. *For any positive integers k and L,*

$$\sum_{g|L}\frac{\log g}{g}k^{\omega(g)} \ll_k (\log\log(L+2))^{k+1}.$$

Proof. Decomposing $\log g$ into $\sum_{p|g}\nu_p(g)\log p$, we have

$$\begin{aligned}\sum_{g|L}\frac{\log g}{g}k^{\omega(g)} &= \sum_{p|L}\log p\sum_{\substack{g|L\\ g\equiv 0(p)}}\frac{\nu_p(g)}{g}k^{\omega(g)} = \sum_{p|L}\log p\sum_{i=1}^{\nu_p(L)} i\sum_{\substack{g|L\\ \nu_p(g)=i}}\frac{k^{\omega(g)}}{g}\\ &= \sum_{p|L}\log p\sum_{i=1}^{\nu_p(L)}\frac{ik}{p^i}\sum_{\substack{g|L\\ g\not\equiv 0(p)}}\frac{k^{\omega(g)}}{g} = \underbrace{k\sum_{p|L}\log p\sum_{i=1}^{\nu_p(L)}\frac{i}{p^i}}_{A}\overbrace{\prod_{\substack{p'|L\\ p'\neq p}}(1+k\sum_{j=1}^{\nu_{p'}(L)}\frac{1}{(p')^j})}^{B(p)}.\end{aligned}$$

It is not hard to find that $0<A\ll\sum_{p|L}\frac{\log p}{p}\ll\log\log(L+2)$ by Lemma 9. Since $1\le B(p)\le\prod_{p'|L}(1+k\sum_{j=1}^{\infty}\frac{1}{(p')^j})=:B$, we have again by Lemma 9

$$\log B=\sum_{p|L}\log(1+k\sum_{j=1}^{\infty}\frac{1}{p^j})=k\sum_{p|L}\frac{1}{p}+O_k(1)\le k\log\log\log(L+2)+O_k(1).$$

Then $B\ll_k(\log\log(L+2))^k$ implies $\sum_{g|L}\frac{\log g}{g}k^{\omega(g)}\le AB\ll_k(\log\log(L+2))^{k+1}$. $\square$

Corollary 5. *For $\mathfrak{a}=\frac{u_{\mathfrak{a}}}{f_{\mathfrak{a}}}\in\mathcal{C}(N)$ as in (1.1), and $s=\frac{1}{2}+iT$, we have*

$$\sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(s,\chi)|^2\log\frac{N}{qf_{\mathfrak{a}}}\ll\Big(\log\log\Big(\frac{N}{q}+2\Big)\Big)^5; \tag{2.27}$$

$$\sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(s,\chi)|^2\sum_{p|\frac{N}{f_{\mathfrak{a}}}}\frac{\psi(p)\log p}{\psi(p)p^{2s-1}-1}\ll\Big(\log\log(\frac{N}{q}+2)\Big)^5; \tag{2.28}$$

$$\sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(s,\chi)|^2\log f_{\mathfrak{a}}=\log\frac{N}{q}+O\Big(\Big(\log\log(\frac{N}{q}+2)\Big)^5\Big). \tag{2.29}$$

Proof. Define $S_f(s,\chi) := \sum_{\mathfrak{a}: f_\mathfrak{a}=f} |\varphi_{\infty\mathfrak{a}}(s,\chi)|^2$ for $f \mid \frac{N}{q}$. By Lemma 1, we have

$$S_f(s,\chi) = C_f(s,\chi) \prod_{p \mid \frac{N}{f}} S_f^p(s,\chi),$$

where

$$C_f(s,\chi) = \frac{qf}{N} \prod_{p \mid (f, \frac{N}{f})} (1-p^{-1}) \prod_{p \mid N^{\perp}_{(N/f)}} |1-\overline{\psi}(p)p^{-2s}|^{-2}(1-p^{-1})^2 \leq \frac{qf}{N},$$

and

$$S_f^p(s,\chi) = \Big| \frac{1-\overline{\psi}(p)p^{1-2s}}{1-\overline{\psi}(p)p^{-2s}} \Big|^2 \leq \begin{cases} 4 & \text{if } p \nmid q, \\ 1 & \text{if } p \mid q. \end{cases}$$

There being at most $\omega((\frac{N}{f})^{\perp}_q) \leq \omega(\frac{N}{qf})$ such p that $S_f^p(s,\chi) > 1$ in the last product, we have

$$S_f(s,\chi) \leq \frac{qf}{N} 4^{\omega(\frac{N}{qf})} =: S_f(\chi). \tag{2.30}$$

Then (2.27) follows from Proposition 8 and the fact

$$\Big| \sum_{\mathfrak{a}} |\varphi_{\infty\mathfrak{a}}(s,\chi)|^2 \log \frac{N}{qf_\mathfrak{a}} \Big| \leq \sum_{f \mid \frac{N}{q}} S_f(\chi) \log \frac{N}{qf}.$$

We similarly have

$$\sum_{\mathfrak{a}} |\varphi_{\infty\mathfrak{a}}(s,\chi)|^2 \sum_{p \mid \frac{N}{f_\mathfrak{a}}} \Big| \frac{\psi(p)\log p}{\psi(p)p^{2s-1}-1} \Big| \leq \sum_{f \mid \frac{N}{q}} \sum_{p \mid \frac{N}{qf}} \Big| S_f(s,\chi) \frac{\psi(p)\log p}{\psi(p)p^{2s-1}-1} \Big|.$$

Noticing that $|S_f^p(s,\chi)\frac{1}{\psi(p)p^{2s-1}-1}| = \frac{|1-\psi(p)p^{1-2s}|}{|(1-\psi(p)p^{-2s})|^2} \leq \frac{2}{(1-p^{-1})^2} \leq 8$, we have

$$\Big| \frac{S_f(s,\chi)}{\psi(p)p^{2s-1}-1} \Big| \ll S_f(\chi).$$

Consequently,

$$\sum_{f|\frac{N}{q}}\sum_{p|\frac{N}{qf}}\left|S_f(s,\chi)\frac{\psi(p)\log p}{\psi(p)p^{2s-1}-1}\right| \ll \sum_{f|\frac{N}{q}}S_f(\chi)\sum_{p|\frac{N}{qf}}\log p \leq \sum_{f|\frac{N}{q}}S_f(\chi)\log\frac{N}{qf},$$

and (2.28) follows from Proposition 8. Equation (2.29) results from (2.27) and that $\sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(s)|^2(\log f_{\mathfrak{a}}+\log\frac{N}{qf_{\mathfrak{a}}}) = \log\frac{N}{q}\sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(s)|^2 = \log\frac{N}{q}$ by Proposition 2. $\square$

3. PROOF OF THEOREMS 6 AND 8

3.1 Overall strategy

To expose everything as clearly as possible, we initially prove Theorem 8, which contains Theorem 6, in the case of weight zero. The main body of the proof lies in Sections 3.3–3.5, for which we sketch the argument for (1.4) later in this subsection; the supportive part consists of prerequisites about cusps in Section 2.1, Eisenstein series featured by a comprehensive description of their cuspidal behaviors in Section 2.2, and regularized integrals in Section 3.2. After that, we prove Theorem 6 for $k = 1$ in Section 3.6. Finally, we sketch the proof of the case of weight one in Section 3.7.

The spectral decomposition to $\langle |E|^2 - \mathcal{E}, \phi \rangle_{_N}$ gives

$$\langle |E|^2 - \mathcal{E}, \phi \rangle_{_N} \approx \sum_{t_j \ll T} \sum_{u_j}^{*} \langle |E|^2, u_j \rangle_{_N} \langle u_j, \phi \rangle_{_M} + \text{continuous spectrum},$$

where the inner sum is over all $L^2(Y_0(M))$-normalized Hecke-Maass newforms of level M with spectral parameter t_j, and recall that $E = E_\infty(z, \frac{1}{2} + iT, \chi)$. This regularized spectral decomposition is the topic of Section 3.3, and Section 3.4 mainly focuses on the following estimation.

Proposition 9. *With the above notations, we have*

$$\langle |E|^2, u_j \rangle_{_N} \ll_{T, t_j} N^{-\frac{1}{2}+\varepsilon} M^{-\frac{1}{2}} (\tfrac{N}{q})^{\theta} |L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2} + 2iT, u_j \otimes \overline{\psi})|.$$

The following crucial subconvexity bound for twisted L-functions then finishes the job.

Theorem 12 (Blomer, Harcos [11])**.** *If ψ is primitive $(mod\ q)$ and u_j is a newform of level M, then*

$$L(\tfrac{1}{2} + 2iT, u_j \otimes \psi) \ll (|T|+1)^{\frac{1}{2}} (M^{\frac{1}{4}} q^{\frac{3}{8}} + M^{\frac{1}{2}} (M, q)^{\frac{1}{4}} q^{\frac{1}{4}}).$$

The contribution of the continuous spectrum to $\langle |E|^2 - \mathcal{E}, \phi \rangle_{_N}$ is similar. Section 3.5 addresses the main terms, about which we have briefly discussed earlier in this section.

3.2 Integral renormalization

3.2.1 Equivalent definitions of integral regularizations

We start by recalling Zagier's definition of integral regularizations on $Y_0(1)$. Assume $F(z)$ is $SL_2(\mathbb{Z})$-invariant and satisfies

$$F(z) = \psi_{_F}(y) + O(y^{-P}) \tag{3.1}$$

as $y \to \infty$ for all integers P, where $\psi_{_F} = \sum_{i=1}^{m} c_i y^{\alpha_i}$, with $c_i \in \mathbb{C}^*$, distinct $\alpha_i \in \mathbb{C}\backslash\{1\}$, $i = 1, 2, ..., m$, and $m = m(F) \geq 1$. When $m \neq 0$ and $\Re\alpha_i \geq 1$ for some i, F is not integrable in the usual sense. Nevertheless, F is "renormalizable" (in Zagier's terminology). Write $R.N.(\int Fd\mu)$, the *renormalization* of $\int Fd\mu$, defined by

- $\int_{y<R} Fd\mu + \int_{y\geq R}(F - \psi_{_F})d\mu + \int^R y^{-2}\psi_{_F}(y)dy.$

Here the first two integrals are performed over the standard fundamental domain $\mathcal{F}$ for $SL_2(\mathbb{Z})$, with their displayed additional restrictions, and the third is the "*anti-derivative*" with respect to R, i.e., a linear combination of R-powers without a nonzero constant term. Zagier's definition is independent of R, as we verify in the following subsection. Moreover, as we let $R \to \infty$, the second term tends to zero, giving an alternative definition:

- $\lim\limits_{R\to\infty} (\int_{y<R} Fd\mu - \int^R y^{-2}\psi_{_F}(y)dy).$

The third description is also called the *regularization* of the integral $\int Fd\mu$ by Michel and Venkatesh [53]:

- $\int (F - \sum_{\substack{1\leq i\leq m \\ \Re\alpha_i \geq 1/2}} c_i E(z, \alpha_i))d\mu,$

which is based on $R.N.(\int E(z, s)d\mu) = 0$, a direct result of the following theorem.

Theorem 13 (Zagier [74]). *Assume F is continuous, has Fourier expansion $\sum a_n(y)e(nx)$ and satisfies all above assumptions. Then $E(z, s)F(z)$ is also renormalizable for $\Re s$ large, and for any*

$R > 1$ *the following function*

$$\int_0^R a_0(y)y^{s-2}dy + \int_R^\infty (a_0(y) - \psi_F(y))y^{s-2}dy - \int^R \psi_F(y)y^{s-2}dy \tag{3.2}$$

has meromorphic continuation and equals $R.N.(\int E(z,s)F(z)d\mu)$.

3.2.2 Generalization of Zagier's result to arbitrary level

By [35, Proposition 2.4], there exists a fundamental domain for $Y_0(N)$, whose vertices are $\Gamma_0(N)$-inequivalent cusps. Let $\mathcal{F}$ be such a fundamental domain. For $R > 1$, if we write $\mathcal{F}_\mathfrak{a}(R)$ to be the image of the truncated strip $0 < x < 1, y > R$ under $\sigma_\mathfrak{a}$, and $\mathcal{F}(R) = \mathcal{F} \backslash (\sqcup_\mathfrak{a} \mathcal{F}_\mathfrak{a}(R))$, then we define *the truncated Eisenstein series* by

$$E_\mathfrak{a}^R = \begin{cases} E_\mathfrak{a} & \text{if } z \in \Gamma_0(N)(\mathcal{F}(R)); \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

It is obvious that truncated Eisenstein series are in $\mathcal{L}^2$. Assume $F(z) \in \mathcal{A}(Y_0(N))$ has Fourier expansion $\sum a_n(y)e(nx)$, and at each cusp $\mathfrak{a}$, there is $\psi_\mathfrak{a} = \sum_i c_{\mathfrak{a},i} y^{\alpha_{\mathfrak{a},i}}$, such that $i = 1, 2, ..., m_\mathfrak{a}$ for some $m_\mathfrak{a} \geq 1$, and

$$F(\sigma_\mathfrak{a} z) = \psi_\mathfrak{a}(y) + O(y^{-P}), \tag{3.4}$$

for all integers P as $y \to \infty$, where $c_{\mathfrak{a},i} \in \mathbb{C} \backslash \{0\}$ and $\alpha_{\mathfrak{a},i} \in \mathbb{C} \backslash \{1\}$. Then we call F *renormalizable*, because $\int F d\mu$ can be *renormalized* as follows for all $R > 1$:

$$R.N.\Big(\int_\mathcal{F} F(z)d\mu\Big) := \int_{\mathcal{F}(R)} F d\mu + \sum_\mathfrak{a} \Big(\int_{\mathcal{F}_\mathfrak{a}(R)} (F(z) - \psi_\mathfrak{a}(\text{Im}\,(\sigma_\mathfrak{a}^{-1} z)))d\mu - \int^R \psi_\mathfrak{a} y^{-2} dy \Big).$$

Again, the expression of the renormalized integral is independent of R: pick $1 < R_1 < R_2$, then the difference between the right hand sides of the equation evaluated at R_2 and R_1 is

$$\int_{(\mathcal{F}(R_2)-\mathcal{F}(R_1))} F d\mu - \sum_{\mathfrak{a}} \Big(\int_{\sigma_{\mathfrak{a}}(\mathcal{F}_\infty(R_1)-\mathcal{F}_\infty(R_2))} (F(z) - \psi_{\mathfrak{a}}(\mathrm{Im}\,(\sigma_{\mathfrak{a}}^{-1}z))) d\mu - \int_{R_2}^{R_1} \psi_{\mathfrak{a}}(y) y^{-2} dy \Big)$$
$$= \int_{\mathcal{F}_\infty(R_1)-\mathcal{F}_\infty(R_2)} \sum_{\mathfrak{a}} \psi_{\mathfrak{a}}(y) d\mu - \sum_{\mathfrak{a}} \int_{R_2}^{R_1} \psi_{\mathfrak{a}}(y) y^{-2} dy = 0.$$

Remark 9. *Just as in Zagier's level* 1 *case, if the integrand is integrable already, the renormalized integral agrees with the usual integral.*

Now suppose $F \in \mathcal{A}(Y_0(N), \overline{\chi})$ satisfies (3.4) and has Fourier expansion $\sum a_n^{\mathfrak{a}}(y) e(nx)$ at each $\mathfrak{a}$, with $\sum_{n\neq 0} |a_n^{\mathfrak{a}}(y)| = O(y^{-P})$ as $y \to \infty$ for all $P \geq 1$. Define $R_{\mathfrak{a}}(F; s) := \int_0^\infty (a_0^{\mathfrak{a}}(y) - \psi_{\mathfrak{a}}(y)) y^{s-2} dy$, which converges for $\Re s$ large by work of Dutta-Gupta [22].

Hulse, Kuan, Lowry-Duda and Walker essentially generalized Zagier's theory to higher levels. Their original claim only concerns case χ being trivial, but it takes no extra efforts to see that the same argument works for general central characters.

Theorem 14. *[29, Proposition A3] If* $\Re s$ *sufficiently large, and* $\mathfrak{a} \in \mathcal{C}_\chi(N)$*, then*

$$R.N.(\langle E_{\mathfrak{a}}(\cdot, s, \chi), \overline{F}(\cdot)\rangle_N) = R_{\mathfrak{a}}(F; s).$$

Consequently, the renormalized integral of a single Eisenstein series, attached to any cusp, vanishes, which justifies the third definition in Zagier's work, as well as our generalization:

$$R.N.\Big(\int F d\mu\Big) = \int \Big(F - \sum_{\mathfrak{a}} \sum_{\mathrm{Re}\,\alpha_{\mathfrak{a},i} \geq 1/2} E_{\mathfrak{a}}(z, \alpha_{\mathfrak{a},i})\Big).$$

We also call this the *regularization* of $\langle F, 1\rangle_N$ and write it $\langle F, 1\rangle_N^{\mathrm{reg}}$.

Corollary 6. *For any* $\mathfrak{a}$ *and* $\mathfrak{b}$ *singular for* χ *and* $s_1, s_2 \in \mathbb{C}\backslash\{0, 1\}$*, we have*

$$\langle E_{\mathfrak{a}}(\cdot, s_1, \chi), E_{\mathfrak{b}}(\cdot, s_2, \chi)\rangle_N^{\mathrm{reg}} = 0.$$

Remark 10. *Note the difference between* $\langle\cdot,\cdot\rangle_N^{\mathrm{reg}}$ *above and* $\langle\cdot,\cdot\rangle_N^{Eis}$.

3.3 Spectral decomposition

Here we take the notation in [35] of $\mathcal{B}_\delta(Y_0(N))$ with $\delta \geq 0$, which stands for the space of smooth automorphic functions f on $Y_0(N)$, satisfying

$$f(\sigma_{\mathfrak{a}} z) \ll y^\delta \quad \text{as} \quad y \to \infty,$$

for all $\mathfrak{a} \in \mathcal{C}(N)$. We note that for $\delta < \frac{1}{2}$, $\mathcal{B}_\delta(Y_0(N)) \subset L^2(Y_0(N))$.

3.3.1 Classical theory

For $F \in \mathcal{B}_\delta(Y_0(N))$ with $\delta < 1/2$, we have spectral decomposition:

$$F(z) = \frac{\langle F, 1\rangle_N}{\langle 1, 1\rangle_N} + \sum_{u \in \mathcal{O}(N)} \langle F, u\rangle_N u(z) + \frac{1}{4\pi} \sum_{\mathfrak{a} \in \mathcal{C}(N)} \int_{-\infty}^{\infty} \langle F, E_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + it)\rangle_N E_{\mathfrak{a}}(z, \tfrac{1}{2} + it) dt.$$

Remark 11. *In our work, the choice of* $E_{\mathfrak{a}}$ *as an orthogonal basis in the spectral decomposition is convenient for computations with the main terms, but not for the error terms.*

3.3.2 Regularization for spectral decomposition

To apply the spectral decomposition, we need to regularize $|E|^2$. See [53, Sections 4.3–4.4] for more about the general theory.

Proposition 10. *For* $E = E_\infty(z, \frac{1}{2} + iT, \chi)$ *as in Theorem 8, we have* $|E|^2 - \mathcal{E} \in \mathcal{B}_\varepsilon(Y_0(N))$ *for arbitrarily small* $\varepsilon > 0$ *with*

$$\begin{aligned}\mathcal{E} := 2\Re\Big(&\varphi_{\infty\infty}(\tfrac{1}{2} + iT, \chi) E_\infty(z, 1 - 2iT)\Big) \\ &+ \lim_{\beta \to 0^+} \Big(E_\infty(z, 1 + \beta) + \sum_{\mathfrak{a} \in \mathcal{C}_\chi(N)} \varphi_{\infty\mathfrak{a}}(\tfrac{1}{2} + iT, \chi) \varphi_{\infty\mathfrak{a}}(\tfrac{1}{2} + \beta - iT, \overline{\chi}) E_{\mathfrak{a}}(z, 1 - \beta)\Big).\end{aligned}$$

Remark 12. *We note that as long as* $T \neq 0$, $\mathcal{E}$ *is well-defined as an element in* $\mathcal{B}_\varepsilon(Y_0(N))$.

Proof. This is done by comparing ψ_{F_β} (see (3.1) for definition) with $\psi_{\mathcal{E}_\beta}$ for

$$F_\beta(z,T) = E_\infty(z,\tfrac{1}{2}+iT,\chi)E_\infty(z,\tfrac{1}{2}+\beta-iT,\overline{\chi}) \text{ and}$$

$$\begin{aligned}\mathcal{E}_\beta(z,T) = \varphi_{\infty\infty}(\tfrac{1}{2}+iT,\chi)E_\infty(z,1+\beta-2iT) + \varphi_{\infty\infty}(\tfrac{1}{2}+\beta-iT,\overline{\chi})E_\infty(z,1-\beta+2iT)\\ + E_\infty(z,1+\beta) + \sum_{\mathfrak{a}} \varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}+iT,\chi)\varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}+\beta-iT,\overline{\chi})E_{\mathfrak{a}}(z,1-\beta).\end{aligned}$$

The constant terms in the Fourier expansion of E_∞ can be calculated via (2.2) and (2.6), and that of $E|_{\sigma_{\mathfrak{a}}}$ is computable with Proposition 4. Now that ψ_{F_β} and $\psi_{\mathcal{E}_\beta}$ agree for all sufficiently small $\beta > 0$, their difference lies in $\mathcal{B}_\varepsilon(Y_0(N))$, for all $\varepsilon > \beta$. □

3.3.3 Regularized spectral decomposition in a new choice of orthonormal basis

Define

$$\mathcal{O}_j(M) := \left\{ u_j^{<\ell>}(z) = \sum_{d|\ell} \xi_\ell(d) u_j|_d \quad \middle| \quad u_j \in \mathcal{H}_{it_j}(M_1), \ell \mid M_2, M = M_1M_2 \right\}, \tag{3.5}$$

where $\mathcal{H}_{it_j}(M_1)$ stands for the set of $L^2(Y_0(M))$-normalized Hecke-Maass newforms of level M_1 and spectral parameter t_j, and $\xi_\ell(d)$ are certain coefficients satisfying the bound

$$\xi_\ell(d) \ll \ell^\varepsilon (\ell/d)^{\theta-\frac{1}{2}}, \tag{3.6}$$

as is described in [13, (5.6)]. Here each u_j can be written as $\rho_j u_j^*$, where

$$u_j^*(z) = \sqrt{y}\sum_{n\neq 0} \lambda_j(n) K_{it_j}(2\pi|n|y)e(nx), \tag{3.7}$$

stands for the Hecke-normalized cusp form, and

$$\rho_j = \left\| u_j^* \right\|_2^{-1} = O(M^{-\frac{1}{2}+\varepsilon} e^{\frac{\pi|t_j|}{2}}). \tag{3.8}$$

Blomer and Milićević [1] showed that $\mathcal{O}_j(M)$ is an orthonormal basis of the space of cusp forms of spectral parameter t_j. Consequently, $\mathcal{O}(M) := \sqcup_{j=1}^{\infty}\mathcal{O}_j(M)$ makes an orthonormal basis of Maass cusp forms of level M.

Parallelly, as explained in [73, Section 8.3],

$$\mathcal{O}_t^{\mathrm{Eis}}(M) := \left\{ E_{\eta,\eta}^{<\ell>}(z, \tfrac{1}{2}+it) = \frac{\sum_{d|\ell}\xi_\ell(d)E_{\eta,\eta}(dz,\frac{1}{2}+it)}{\left\|E_{\eta,\eta}^{(M)}\right\|_2^{\mathrm{Eis}}} \;\middle|\; \eta \bmod r, r^2\ell \mid M \right\} \tag{3.9}$$

forms a formal orthonormal basis, with exactly the same $\xi_\ell(d)$. Since

$$\left\|E_{\eta,\eta}^{(M)}\right\|_2^{\mathrm{Eis}} := \sqrt{\langle E_{\eta,\eta}^{(M)}, E_{\eta,\eta}^{(M)}\rangle_M^{\mathrm{Eis}}} = \sqrt{4\pi M}\prod_{p|r}(1-p^{-1})^{\frac{1}{2}}\prod_{p|M_r^{\perp}}(1+p^{-1})^{\frac{1}{2}} = M^{\frac{1}{2}+o(1)}.$$

From the definition of renormalized integral and Corollary 6, we have $\langle |E|^2-\mathcal{E}, 1\rangle_N = 0$. Since $\langle \mathcal{E}, u\rangle_N = 0$, applying the Plancherel formula to $\langle |E|^2-\mathcal{E}, \phi\rangle_N$ yields

$$\langle |E|^2-\mathcal{E}, \phi\rangle_N = \sum_{u\in\mathcal{O}(M)}\langle |E|^2, u\rangle_N\langle u,\phi\rangle_M + \int_{-\infty}^{\infty}\sum_{E_t\in\mathcal{O}_t^{\mathrm{Eis}}(M)}\langle |E|^2, E_t\rangle_N^{\mathrm{reg}}\langle E_t,\phi\rangle_M dt. \tag{3.10}$$

Consequently we can take (3.5) and (3.9) back to (3.10), and obtain

$$\begin{aligned}\langle |E|^2-\mathcal{E}, \phi\rangle_N = &\sum_{j\geq 1}\sum_{M_1M_2=M}\sum_{u_j\in\mathcal{H}_{it_j}(M_1)}\sum_{\ell|M_2}\langle |E|^2, u_j^{<\ell>}\rangle_N\langle u_j^{<\ell>},\phi\rangle_M \\ &+\int_{-\infty}^{\infty}\sum_{r^2L=M}\sideset{}{^*}\sum_{\eta \bmod r}\sum_{\ell|L}\langle |E|^2, E_{\eta,\eta}^{<\ell>}(\cdot,\tfrac{1}{2}+it)\rangle_N^{\mathrm{reg}}\langle E_{\eta,\eta}^{<\ell>}(\cdot,\tfrac{1}{2}+it),\phi\rangle_M dt, \end{aligned} \tag{3.11}$$

where the asterisked sum is over all primitive Dirichlet characters mod r. We estimate the terms in (3.10), or equivalently (3.11), and $\langle \mathcal{E},\phi\rangle_N$ in the following sections.

[1] See https://www.uni-math.gwdg.de/blomer/corrections.pdf for corrections of [13].

3.4 Error term estimation

3.4.1 Calculation with Fourier coefficients

Lemma 10. *Suppose $f \in \mathcal{A}(Y_0(N), \chi)$, $g \in \mathcal{A}(Y_0(N))$ with Fourier expansions*

$$f(z) = a_0(y) + \sqrt{y} \sum_{n \neq 0} \lambda_f(n) a(ny) e(nx)$$
$$g(z) = \sqrt{y} \sum_{n \neq 0} \lambda_g(n) b(ny) e(nx),$$

where λ_f and λ_g are multiplicative and $\lambda_*(-n) = \lambda_*(-1)\lambda_*(n)$ for $* = f$ or g. Then we have

$$\langle E_\infty^{(N)}(\cdot, s, \chi), f \cdot g \rangle_N = (\overline{\lambda_f}(-1) + \overline{\lambda_g}(-1)) h(s) \sum_{n \geq 1} n^{-s} \overline{\lambda_f}(n) \overline{\lambda_g}(n),$$

where $h(s) = \int_0^\infty y^{s-1} \overline{a(y) b(y)} dy$.

Proof. This is easy by unfolding and integration on x. $\square$

Corollary 7. *With the same assumptions as Lemma 10, if we further have $f|_A \in \mathcal{A}(Y_0(N), \chi)$ and $g|_B \in \mathcal{A}(Y_0(N))$ for some $A, B \mid N$, then*

$$\langle E_\infty^{(N)}(\cdot, s, \chi), f|_A \cdot g|_B \rangle_N = (\overline{\lambda_f}(-1) + \overline{\lambda_g}(-1)) h(s) Z_{A,B}(s),$$

with

$$Z_{A,B}(s) = \frac{\sqrt{AB}}{[A,B]^s} \sum_{n \geq 1} n^{-s} \overline{\lambda_f}\Big(\frac{[A,B]}{A} n\Big) \overline{\lambda_g}\Big(\frac{[A,B]}{B} n\Big).$$

3.4.2 Cuspidal contribution

The following corollary is a special case of Corollary 7 with (3.7) and (2.6).

Corollary 8. *For all $A \mid \frac{N}{q}$ and $B \mid N$, we have*

$$\langle E_{\infty}^{(N)}(\cdot, \tfrac{1}{2}+iT, \chi), E_{1,\overline{\psi}}|_A \cdot u_j|_B \rangle_N = F_T(t_j) Z_{A,B}(\tfrac{1}{2}+iT, \psi, u_j),$$

where

$$Z_{A,B}(\tfrac{1}{2}+iT, \psi, u_j) = \frac{\sqrt{AB}}{[A,B]^{\frac{1}{2}+iT}} \sum_{n\geq 1} \frac{\overline{\lambda_{1,\overline{\psi}}}(\frac{[A,B]}{A}n)\lambda_j(\frac{[A,B]}{B}n)}{n^{\frac{1}{2}+iT}}, \text{ and}$$

$$F_T(t_j) = \overline{\rho_{1,\overline{\psi}}(\tfrac{1}{2}+iT)\rho_j}(\overline{\lambda_{1,\overline{\psi}}}(-1)+\lambda_j(-1)) \int_0^{\infty} y^{-\frac{1}{2}+iT} K_{iT}(2\pi y) K_{it_j}(2\pi y) dy.$$

From (2.7), (3.8), and [25, (6.576.4)], we see $F_T(t_j) \ll N^{\varepsilon} M^{-\frac{1}{2}} e^{H_T(t_j)} P(t_j, T)$ for some polynomial $P(x,y)$, where

$$H_T(t_j) = \begin{cases} 0 & \text{if } |t_j| \leq 2|T|, \\ \frac{\pi}{2}(2|T| - |t_j|) & \text{if } |t_j| > 2|T|. \end{cases} \tag{3.12}$$

As for $Z_{A,B}(\frac{1}{2}+iT, \psi, u_j)$, we can rewrite the Dirichlet series as an Euler product

$$\frac{\sqrt{AB}}{[A,B]^{\frac{1}{2}+iT}} \prod_p \Big(\sum_{n\geq 0} \frac{\overline{\lambda_{1,\overline{\psi}}}(p^{n+\nu_p(\frac{[A,B]}{A})})\overline{\lambda_j}(p^{n+\nu_p(\frac{[A,B]}{B})})}{p^{n(\frac{1}{2}+iT)}} \Big) = F_j(A,B) \sum_{n\geq 1} \frac{\overline{\lambda_{1,\overline{\psi}}}(n)\lambda_j(n)}{n^{\frac{1}{2}+iT}},$$

where $F_j(A,B)$ is a finite Euler product over prime divisors of $[A,B]$. Inserting the bounds from Remark 4, we have $F_j(A,B) = O(N^{\varepsilon}(A,B)^{\frac{1}{2}}(A_M^{\perp})^{\theta})$. Applying the Rankin-Selberg method (see e.g. [34, (13.1)]), we have

$$\sum_{n\geq 1} \frac{\overline{\lambda_{1,\overline{\psi}}}(n)\lambda_j(n)}{n^{\frac{1}{2}+iT}} = \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}+2iT, u_j \otimes \overline{\psi})}{L(1+2iT, \overline{\psi}\cdot \chi_{0,M})}.$$

Recalling equation (2.2) and the fact $|L(1+2iT, \overline{\psi})| \gg_T q^{-\varepsilon}$, we have the following lemma.

Lemma 11. *Keeping above notations and $s = \frac{1}{2} + iT$, we have for all $d \mid M$*

$$\langle |E_\infty(\cdot, s, \chi)|^2, u_j|_d \rangle_N \ll_T e^{H_T(t_j)} N^{-\frac{1}{2}+\varepsilon} M^{-\frac{1}{2}} (\tfrac{N}{q}, d)^{\frac{1}{2}} (\tfrac{N}{q})^{\theta} |L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2} + 2iT, u_j \otimes \overline{\psi})|.$$

Notice Lemma 11 implies Proposition 9. Now we can estimate the first part of (3.10).

Proposition 11. *Keeping all notations in Theorems 6 and 8, we have*

$$\sum_{u \in \mathcal{O}(M)} \langle |E_\infty(\cdot, \tfrac{1}{2} + iT, \chi)|^2, u \rangle_N \langle u, \phi \rangle_M \ll_T N^{-\frac{1}{2}+\varepsilon} (\tfrac{N}{q})^{\theta} M^{\frac{1}{2}} q^{\frac{3}{8}} \|\phi\|_2.$$

Before proving Proposition 11, we claim a lemma.

Lemma 12. *We have*

$$\sum_{t_j \le 2|T| + 2\log N} \sum_{u_j \in \mathcal{H}_{it_j}(M_1)} |L(\tfrac{1}{2}, u_j)|^2 \ll_{T,\varepsilon} N^{\varepsilon} M_1.$$

The proof follows from the spectral large sieve inequality, so we omit it. See Motohashi [57, (3.4.4)] for an example on the case $M = 1$.

Remark 13. *A bound of the same quality actually holds for the fourth moment of central values of these L-functions, which follows from the spectral large sieve for $\Gamma_0(M)$ developed by Deshouillers and Iwaniec [DI]. Motohashi [57, Theorem 3.4] shows this for the case $M = 1$.*

Proof of Proposition 11. By (3.5), (3.11) and Cauchy-Schwarz, we have

$$\begin{aligned}\sum_{u \in \mathcal{O}(M)} |\langle |E_\infty|^2, u \rangle_N \langle u, \phi \rangle_M| &= \sum_{j \ge 1} \sum_{u_j \in \mathcal{O}_j(M)} |\langle |E_\infty|^2, u_j \rangle_N \langle u_j, \phi \rangle_M| \\ &\le \Big(\sum_{j \ge 1} \sum_{u_j \in \mathcal{O}_j(M)} |\langle |E_\infty|^2, u_j \rangle_N|^2 \Big)^{\frac{1}{2}} \Big(\sum_{j \ge 1} \sum_{u_j \in \mathcal{O}_j(M)} |\langle u_j, \phi \rangle_M|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

Observe that by Bessel's inequality,

$$\sum_{j\geq 1}\sum_{u_j\in\mathcal{O}_j(M)}|\langle u_j,\phi\rangle_M|^2\leq \|\phi\|_2^2.$$

As for the other factor, we recall (3.5) and (3.6), and apply Cauchy-Schwarz again to see

$$|\langle |E_\infty|^2, u_j^{<\ell>}\rangle_N|\leq \Big(\sum_{d|\ell}|\xi_d^{<\ell>}|^2\Big)^{\frac{1}{2}}\Big(\sum_{d|\ell}|\langle |E_\infty|^2, u_j|_d\rangle_N|^2\Big)^{\frac{1}{2}}\ll \ell^\varepsilon \max_{d|\ell}|\langle |E_\infty|^2, u_j|_d\rangle_N|$$
$$\ll_\varepsilon N^{-\frac{1}{2}+\varepsilon}M^{-\frac{1}{2}}e^{H_T(t_j)}(\tfrac{N}{q},\ell)^{\frac{1}{2}}(\tfrac{N}{q})^\theta\Big|L(\tfrac{1}{2},u_j)L(\tfrac{1}{2}+2iT,u_j\otimes\overline{\psi})\Big|,$$

where $\xi_d^{<\ell>}$ is defined in (3.5).Because of the factor $e^{H_T(t_j)}$ (see (3.12) for its magnitude), we may truncate the sum at $|t_j|\leq 2|T|+2\log N$, with a very small error term.

Furthermore, for all $|t_j|\leq 2|T|+2\log N$, we have

$$\sum_{l|M_2}|\langle |E_\infty|^2, u_j^{<\ell>}\rangle_N|^2\ll_\varepsilon N^{-1+\varepsilon}\frac{\sum_{l|M_2}(\frac{N}{q},\ell)}{M}(\tfrac{N}{q})^{2\theta}|L(\tfrac{1}{2},u_j)L(\tfrac{1}{2}+2iT,u_j\otimes\overline{\psi})|^2$$
$$=N^{-1+\varepsilon}M^{-1}(\tfrac{N}{q})^{2\theta}(\tfrac{N}{q},M_2)|L(\tfrac{1}{2},u_j)L(\tfrac{1}{2}+2iT,u_j\otimes\overline{\psi})|^2,$$

and by Theorem 12 and Lemma 12, we have

$$\sum_{|t_j|\leq 2|T|+2\log N}\ \sum_{u_j\in\mathcal{H}_{it_j}(M_1)}\ \sum_{\ell|M_2}|\langle |E_\infty|^2, u_j^{<\ell>}\rangle_N|^2$$
$$\ll_T N^{-1+\varepsilon}M^{-1}(\tfrac{N}{q})^{2\theta}(\tfrac{N}{q},M_2)M_1\max\{M_1^{\frac{1}{2}}q^{\frac{3}{4}},M_1(M_1,q)^{\frac{1}{2}}q^{\frac{1}{2}}\}.$$

In the summation over $M_1M_2=M$, the term with $M=M_1$ and $M_2=1$ dominates, so

$$\Big(\sum_{|t_j|\leq 2|T|+2\log N}\ \sum_{u_j\in\mathcal{O}_j(M)}|\langle |E_\infty|^2, u_j\rangle_N|^2\Big)^{\frac{1}{2}}\ll_T N^{-\frac{1}{2}+\varepsilon}(\tfrac{N}{q})^\theta\max\{M^{\frac{1}{4}}q^{\frac{3}{8}},M^{\frac{1}{2}}(M,q)^{\frac{1}{4}}q^{\frac{1}{4}}\}.\ \square$$

Remark 14. *Following the same line as Lemma 12 we can similarly have*

$$\sum_{\eta \bmod r}^{*} \int_{-2|T|-2\log N}^{2|T|+2\log N} |L(\tfrac{1}{2}, E_{\eta,\eta}(\cdot, \tfrac{1}{2}+it))|^2 dt \ll_T N^{\varepsilon} r.$$

3.4.3 Eisenstein contribution

Now we estimate the second part in (3.10). It is not hard to see we have made every piece correspond well with that of the first part, in the rewritten formula (3.11), and that is why we choose $\mathcal{O}_t^{\mathrm{Eis}}(M)$ to be the orthonormal basis.

Lemma 13. *Keeping all notations as in (3.11), we have*

$$\langle |E_\infty(\cdot, s, \chi)|^2, E_{\eta,\eta}(d\cdot, \tfrac{1}{2}+it)\rangle_N^{\mathrm{reg}} \ll_T e^{H_T(t)} N^{-\frac{1}{2}+\varepsilon} (\tfrac{N}{q}, d)^{\frac{1}{2}} |L(\tfrac{1}{2}, E_{\eta,\eta}) L(\tfrac{1}{2}+2iT, E_{\eta,\eta} \otimes \overline{\psi})|,$$

where $H_T(t)$ agrees with $H_T(t_j)$ in (3.12).

The proof is almost the same as that of Lemma 11, so we omit the details.

Proposition 12. *Keeping all notations from Theorems 6 and 8, we have*

$$\int_{-\infty}^{\infty} \sum_{E_t \in \mathcal{O}_t^{Eis}(M)} \langle |E|^2, E_t \rangle_N^{reg} \langle E_t, \phi \rangle_M dt \ll_T N^{-\frac{1}{2}+\varepsilon} q^{\frac{3}{8}} M^{\frac{1}{2}} \|\phi\|_2.$$

Sketch of proof. After Lemma 13, the calculation can be reduced to some multiple of

$$\sum_{\eta \bmod r}^{*} \int_{-2|T|-2\log N}^{2|T|+2\log N} |L(\tfrac{1}{2}, E_{\eta,\eta}) L(\tfrac{1}{2}+2iT, E_{\eta,\eta} \otimes \overline{\psi})|^2 dt,$$

with similarly negligible tail. Then we can just perform the same procedure of proving Proposition 11, except for taking the Burgess bound for $|L(\frac{1}{2}, E_{\eta,\eta} \otimes \overline{\psi})|$ instead of that of [13], and putting the equation in Remark 14 in place of Lemma 12. □

3.5 Main term estimation

The main goal of this section is to prove (1.5) and (1.6), which are the main term aspects of Theorem 8. Throughout this section we adopt all notations in previous sections.

3.5.1 Preparation

Recall $W_N^1(\mathfrak{a})$ is the width of $\mathfrak{a}$ (see Section 2.1.2 for definition).

3.5.1.1 Weighted average

Lemma 14. *For $s = \frac{1}{2} + iT$, we have*

$$-\sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}(s,\chi)|^2\Big(\frac{\varphi'_{\infty\mathfrak{a}}(\overline{s},\overline{\chi})}{\varphi_{\infty\mathfrak{a}}(\overline{s},\overline{\chi})} + \log W_N^1(\mathfrak{a})\Big) = 2\log N + 4\Re\frac{L'(1+2iT,\overline{\psi})}{L(1+2iT,\overline{\psi})}$$
$$+ O_T(1) + O((\log\log(\tfrac{N}{q}+2))^5).$$

Proof. According to Lemma 1, for $\mathfrak{a} = \frac{u}{f} \in \mathcal{C}_\chi(N)$ with $f \mid \frac{N}{q}$, we have

$$-\frac{\varphi'_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})}{\varphi_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})} = -(\log\varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}-iT,\overline{\chi}))' = \log\frac{fN}{(f,\frac{N}{f})}$$
$$+ 4\Re\frac{\Lambda'(1+2iT,\overline{\psi})}{\Lambda(1+2iT,\overline{\psi})} + 2\sum_{p\mid N}\frac{\psi(p)p^{-1+2iT}\log p}{1-\psi(p)p^{-1+2iT}} - 2\sum_{p\mid\frac{N}{f}}\frac{\psi(p)p^{2iT}\log p}{1-\psi(p)p^{2iT}},$$

where Λ is the completed L-function. Moreover, by Lemma 1 and Proposition 2, we have

$$\sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} -|\varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}+iT,\chi)|^2\Big(\frac{\varphi'_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})}{\varphi_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})} + \log W_N^1(\mathfrak{a})\Big) = \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}+iT,\chi)|^2$$
$$\cdot\Big(2\log f + 4\Re\frac{\Lambda'(1+2iT,\overline{\psi})}{\Lambda(1+2iT,\overline{\psi})} + 2\sum_{p\mid N}\frac{\psi(p)p^{-1+2iT}\log p}{1-\psi(p)p^{-1+2iT}} - 2\sum_{p\mid\frac{N}{f}}\frac{\psi(p)p^{2iT}\log p}{1-\psi(p)p^{2iT}}\Big).$$

Recalling Corollaries 3 and 5, we arrive at the lemma. $\square$

3.5.1.2 Traced Eisenstein series

Applying the trace operator Tr_M^N (see the definition in (2.16)) to $\mathcal{E}$, we have (see [6, Lemma 12])

$$\langle \mathcal{E}, \phi \rangle_{_N} = \langle \operatorname{Tr}_M^N \mathcal{E}, \phi \rangle_{_M}.$$

To calculate further with this, we need to identify $\operatorname{Tr}_M^N \mathcal{E}$. By Lemma 4 and Proposition 10, we have for all $T \neq 0$

$$\operatorname{Tr}_M^N \mathcal{E} = 2\Re\Big(\varphi_{\infty\infty}^{(N)}(\tfrac{1}{2}+iT,\chi)E_\infty^{(M)}(z,1-2iT)\Big) + \lim_{\beta\to 0^+}\Big(E_\infty^{(M)}(z,1+\beta) + \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} \varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+\beta-iT,\overline{\chi})(W_N^M(\mathfrak{a}))^{\beta}E_{\mathfrak{a}}^{(M)}(z,1-\beta)\Big). \quad (3.13)$$

It is still necessary to simplify (3.13) further.

Proposition 13. *When $T \neq 0$, we have*

$$\operatorname{Tr}_M^N \mathcal{E} = c_0 + \sum_{g\mid M} c_g G|_g + \sum_{g\mid M} c_g' E(\cdot, 1+2iT)|_g,$$

where

$$c_0 = \frac{1}{\langle 1,1\rangle_{_M}}\Big(\log\frac{N^2}{M(M,N/q)} + 4\Re\frac{L'(1+2iT,\overline{\psi})}{L(1+2iT,\overline{\psi})} + O_{_T}((\log\log\Big(\tfrac{N}{q}+2\Big))^5)\Big), \qquad (3.14)$$

and the coefficients c_g, c_g' satisfy $\sum_{g\mid M}|c_g|+|c_g'| \ll M^{-1}(\log\log M)^3$.

Remark 15. *One of the pleasant features in Proposition 13 is that there is no contribution from the newform Eisenstein series with $r > 1$. In addition, by taking $M = N$, Proposition 13 gives an alternative expression for $\mathcal{E}$ itself. Finally, we note from Lemma 1 that $\varphi_{\infty\infty}^{(N)}(s,\chi)$ vanishes unless χ is trivial, which means $c_g' = 0$ for all $g \mid M$ whenever χ is nontrivial.*

Proof. By (3.13), Proposition 2 and Corollary 7, we have

$$\operatorname{Tr}_M^N \mathcal{E} = c_0 + \sum_{g|M} c_g G|_g + \sum_{1<r^2|N} \sideset{}{^*}\sum_{\eta(r)} \sum_{g|Nr^{-2}} c_{\eta,g} E_{\eta,\eta}(gz,1) \\ + 2\Re\Big(\varphi_{\infty\infty}^{(N)}(\tfrac{1}{2}+iT,\chi) E_\infty^{(M)}(z,1-2iT)\Big),$$

where

$$c_0 = c_{\infty,0} + \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)|^2 c_{\mathfrak{a},0} \\ - \frac{1}{\langle 1,1\rangle_M} \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)|^2 \Big(\frac{\varphi'_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})}{\varphi_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})} + \log W_N^M(\mathfrak{a})\Big),$$

$$c_g = c_{\infty,g} + \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)|^2 c_{\mathfrak{a},g},$$

and

$$c_{\eta,g} = \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)|^2 \overline{\eta}(u_\mathfrak{a}) c_{\mathfrak{a},\eta,g}.$$

For clarity, we remark that the coefficients $c_{\mathfrak{a},0}$ and $c_{\mathfrak{a},g}$ correspond to the notation from Proposition 7, but on level M. To simplify, first observe that when $\eta \pmod r$ is primitive with $r > 1$, then $c_{\eta,g} = 0$ for all $g \mid M$. This holds because for each fixed $f \mid N$, $\mathcal{C}_\chi(N)$ contains all cusps $\frac{u}{f}$ with $u \in (\mathbb{Z}/(f,N/f)\mathbb{Z})^\times$. Then, since $|\varphi_{\infty\mathfrak{a}}^{(N)}(\frac{1}{2}+iT,\chi)|^2$ and $c_{\mathfrak{a},\eta,g}$ are independent of $u_\mathfrak{a}$, the sum over $u_\mathfrak{a}$ vanishes.

Next we simplify c_0. Using Lemmas 2 and 1, and Corollary 4, we have $\log W_N^M(\mathfrak{a}) = \log W_N^1(\mathfrak{a}) - \log\Big(\frac{M}{(M,(M,f)^2)}\Big)$, and so

$$\operatorname{Vol}(Y_0(M)) c_0 = -\log M + \sum_{\mathfrak{a}\in\mathcal{C}_\chi(N)} |\varphi_{\infty\mathfrak{a}}^{(N)}(\tfrac{1}{2}+iT,\chi)|^2 \Big(-\log(f,M) \\ - \frac{\varphi'_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})}{\varphi_{\infty\mathfrak{a}}(\frac{1}{2}-iT,\overline{\chi})} - \log W_N^1(\mathfrak{a})\Big) + O(\log\log M).$$

Next we apply some approximations to simplify this further. From Corollary 1, we see that $\varphi_{\infty\mathfrak{a}}(s,\chi) = 0$ unless $f|\frac{N}{q}$, and hence only terms with $(M,f) \mid (M,N/q)$ are in the sum. Moreover, we have $\frac{(M,N/q)}{(M,f)} \mid \frac{N/q}{f}$. By (2.27), we can replace $\log(f,M)$ by $\log(M,N/q)$ with an acceptable error term, which gives the claimed estimation (3.14) for c_0.

The estimation of $\sum_{g|M}|c_g|$ comes from Corollary 4 and the fact that

$$\sum_{g|M}|c_g| \leq \sum_{g|M}|c_{\infty,g}| + \sum_{\mathfrak{a}}|\varphi_{\infty\mathfrak{a}}(\tfrac{1}{2}+iT,\chi)|^2 \sum_{g|M}|c_{\mathfrak{a},g}|.$$

For fixed $T \neq 0$, we have

$$\begin{aligned} E_\infty^{(M)}(z, 1+2iT) &= M^{-1-2iT}\frac{\zeta(2+4iT)}{L(2+4iT,\chi_{0,M})}\sum_{g|M}\frac{\mu(M/g)}{(M/g)^{1+2iT}}E(gz,1+2iT) \\ &= \sum_{g|M} c'_g E(gz, 1+2iT), \end{aligned}$$

with $c'_g = \mu(M/g)M^{-2-2iT}g^{1+iT}\frac{\zeta(2+4iT)}{L(2+4iT,\chi_{0,M})}$. It is obvious that $|c'_g| \leq |c_g|$, so the bound of $\sum_g |c_g|$ applies to $\sum_g |c'_g|$. $\square$

3.5.2 Proof of (1.5) and (1.6)

Recalling Proposition 13, we have

$$\langle \mathcal{E}, \phi\rangle_N = \langle c_0, \phi\rangle_M + \sum_{g|M} c_g \langle G|_g, \phi\rangle_M + \sum_{g|M} c'_g \langle E(g\cdot, 1+2iT), \phi\rangle_M,$$

where c_g and c'_g are the constants from Proposition 13. Define

$$\alpha_\phi = \sum_{g|M} c_g \langle G|_g, \phi\rangle_M + \sum_{g|M} c'_g \langle E(g\cdot, 1+2iT), \phi\rangle_M. \tag{3.15}$$

By Lemma 14 and Remark 2, we have

$$\langle \mathrm{Tr}_M^N \mathcal{E}, \phi\rangle_M = c_0\langle 1, \phi\rangle_M + \alpha_\phi.$$

Then (3.14) gives (1.5), and (1.6) follows from Proposition 13 and (3.15).

3.5.3 Limitations to QUE (continued)

Here we provide the additional details of the example discussed previously. Recall in the example that χ is primitive mod N and M is a prime divisor of N. Then by Proposition 10, Remark 2, Lemma 2 and Corollary 2, we have

$$\operatorname{Tr}_M^N \mathcal{E} = \lim_{\beta \to 0^+} \Big(E_\infty^{(M)}(z, 1+\beta) + (\tfrac{N}{M})^\beta \varphi_{\infty 0}^{(N)}(\tfrac{1}{2} + iT, \chi) \varphi_{\infty 0}^{(N)}(\tfrac{1}{2} + \beta - iT, \overline{\chi}) E_0^{(M)}(z, 1-\beta) \Big).$$

Next, Theorem 11 says

$$E_\infty^{(M)}(z, 1+\beta) = M^{-1-\beta} \frac{\zeta(2+2\beta)}{L(2+2\beta, \chi_{0,M})} \Big(E(Mz, 1+\beta) - M^{-1-\beta} E(z, 1+\beta) \Big),$$

and

$$E_0^{(M)}(z, 1-\beta) = M^{-1+\beta} \frac{\zeta(2-2\beta)}{L(2-2\beta, \chi_{0,M})} \Big(E(z, 1-\beta) - M^{-1+\beta} E(Mz, 1-\beta) \Big).$$

Then since $\langle \mathcal{E}, \phi \rangle_N = \langle \operatorname{Tr}_M^N \mathcal{E}, \phi \rangle_M$, by Proposition 6 we obtain (1.7) with

$$c_1 = c_M = \frac{\zeta(2)}{L(2, \chi_{0,M})} \Big(M^{-1} - M^{-2} \Big) = M^{-1} + O(M^{-2}). \tag{3.16}$$

The estimation (1.8) of c_0 is contained in (3.14).

3.5.4 Comparison of main terms

An astute reader may notice an apparent inconsistency between the main terms displayed in Theorems 6 and 8, and we devote this section to compare these main terms and resolve this paradox. Recall that Theorem 8 estimates $\langle |E|^2, \phi \rangle_N$, where $\phi = \phi_j^{(M)}$ is chosen from the system described in Convention 1. One can recover Theorem 6 in two different ways from Theorem 8; the first way is to simply take $M = 1$ in Theorem 8, which visibly reduces to Theorem 6, and the second is to form ϕ_0 as the sum of $\phi_j^{(M)}$. That is, summing over $\phi = \phi_j^{(M)}$ for $j = 1, 2, ..., \nu(M)$, we have

$$\sum_{\phi} \langle |E|^2, \phi \rangle_{N} = \langle |E|^2, \phi_0 \rangle_{N} \sim \sum_{\phi} \frac{\langle 1, \phi \rangle_{M}}{\langle 1, 1 \rangle_{M}} \Big(\log \frac{N^2}{M(M, N/q)} + 4\Re \frac{L'}{L}(1 + 2iT, \overline{\psi}) \Big) + \sum_{\phi} \alpha_\phi$$
$$= \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_1} \Big(\log \frac{N^2}{M(M, N/q)} + 4\Re \frac{L'}{L}(1 + 2iT, \overline{\psi}) \Big) + \sum_{\phi} \alpha_\phi.$$

This expression has a different shape than that from Theorem 6, which says

$$\langle |E|^2, \phi_0 \rangle_N \sim \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_1} \Big(\log N^2 + 4\Re \frac{L'}{L}(1 + 2iT, \overline{\psi}) \Big).$$

For consistency, we must have

$$\sum_{\phi} \alpha_\phi \sim \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_1} \log(M(M, N/q)). \tag{3.17}$$

We wish to check this directly, at least in some special cases. For simplicity of exposition, we take $q = N$ (i.e., χ is primitive), and M prime.

In (3.15), we have $c'_g = 0$ since $q \neq 1$, whence

$$\sum_{\phi} \alpha_\phi = \sum_{\phi} \sum_{g|M} c_g \langle G|_g, \phi \rangle_M = \sum_{g|M} c_g \langle G|_g, \phi_0 \rangle_M.$$

Since ϕ_0 is $SL_2(\mathbb{Z})$-invariant, we have

$$\sum_{g|M} c_g \langle G|_g, \phi_0 \rangle_M = \sum_{g|M} c_g \frac{\nu(M)}{\nu(g)} \langle \mathrm{Tr}_1^g(G|_g), \phi_0 \rangle_1.$$

On the other hand, one can check directly (see [DS, Sections 5.1–5.4]) that

$$\mathrm{Tr}_1^g(f|_g) = \sqrt{g} T_g(f),$$

for any automorphic function f of level 1. Hence by Lemma 8 and (3.16),

$$\begin{aligned}\sum_{\phi}\alpha_\phi &= \sum_{g|M} c_g \frac{\nu(M)}{\nu(g)}\sqrt{g}\langle T_g(G),\phi_0\rangle_1\\ &= \sum_{g|M} c_g \frac{\nu(M)}{\nu(g)}\sqrt{g}\Big(\lambda(g)\langle G,\phi_0\rangle_1 + \frac{3}{\pi}\sqrt{g}\Big(\sum_{a|g} a^{-1}\log\frac{g}{a^2}\Big)\langle 1,\phi_0\rangle_1\Big)\\ &= \frac{\langle 1,\phi_0\rangle_1}{\langle 1,1\rangle_1}(\log M)(1+O(M^{-1})) + \langle G,\phi_0\rangle_1(2+O(M^{-1})),\end{aligned}$$

which indeed agrees with (3.17).

3.6 QUE for Eisenstein series attached to other cusps

This section concentrates on proving Theorem 6 for $k=1$. Assume χ is primitive modulo N throughout this section. By Proposition 1, $\mathcal{C}_\chi(N)$ consists of Atkin-Lehner cusps. Recall for a cusp $\mathfrak{a}=\frac{1}{f}\in\mathcal{C}_\chi(N)$, we denote the cusp $\frac{1}{N/f}\in\mathcal{C}_\chi(N)$ by $\mathfrak{a}^*$ and call it the Atkin-Lehner conjugate of $\mathfrak{a}$. It is easy to see by Lemma 1 that $W_{\mathfrak{a}}=N/f$, and $W_{\mathfrak{a}^*}=f$.

3.6.1 Identification of $\mathcal{E}$

Corollary 2 and Proposition 5 give the cuspidal behavior of $|E_{\mathfrak{a}}|^2$ at any $\mathfrak{b}\in\mathcal{C}(N)$. The following proposition can be proved similarly as Proposition 10.

Proposition 14. *For $E=E_{\mathfrak{a}}(z,\frac12+iT,\chi)$ as in Theorem 6 for $k=1$, we have $|E|^2-\mathcal{E}\in\mathcal{B}_\varepsilon(Y_0(N))$ for arbitrarily small $\varepsilon>0$ with*

$$\mathcal{E}=\lim_{\beta\to 0^+}\Big(E_{\mathfrak{a}}(z,1+\beta)+\varphi_{\mathfrak{a}\mathfrak{a}^*}(\tfrac12+iT,\chi)\varphi_{\mathfrak{a}\mathfrak{a}^*}(\tfrac12+\beta-iT,\overline{\chi})E_{\mathfrak{a}^*}(z,1-\beta)\Big).$$

The following subsections deal with $\langle|E|^2-\mathcal{E},\phi_0\rangle_N$ and $\langle\mathcal{E},\phi_0\rangle_N$ separately.

3.6.2 Error term

Since $|E|^2-\mathcal{E}\in\mathcal{B}_\varepsilon(Y_0(N))$ and $M=1$, the analog of (3.11) is

$$\langle|E|^2-\mathcal{E},\phi_0\rangle_N=\sum_{j\ge1}\langle|E|^2,u_j\rangle_N\langle u_j,\phi_0\rangle_1+\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle|E|^2,E(\cdot,\tfrac12+it)\rangle_N^{\mathrm{reg}}\langle E(\cdot,\tfrac12+it),\phi_0\rangle_1\,dt.$$

Recall from (2.3) that $E_{\mathfrak{a}}(z,s,\chi) = N^{-s}E_{\chi_1,\chi_2}(z,s)$, where $\chi = \chi_1\overline{\chi_2}$ with χ_1 modulo N/f and χ_2 modulo f. As a result, with (3.7), (2.11) and (2.8) we have for some ϵ with $|\epsilon|= 1$

$$\begin{aligned}\langle |E_{\mathfrak{a}}(\cdot,s,\chi)|^2, u_j\rangle_N &= \chi_1(-1)N^{-\overline{s}}\int_0^1\int_0^\infty y^{s-2}\overline{E_{\chi_1,\chi_2}|_{\sigma_{\mathfrak{a}}}}\overline{u_j}(\sigma_{\mathfrak{a}}z)dxdy\\ &= \epsilon N^{-\overline{s}}\int_0^1\int_0^\infty y^{s-2}\overline{E_{1,\chi_1\chi_2}}\overline{u_j}(\tfrac{N}{f}z)dxdy\\ &= \frac{2\epsilon F_T(t_j)}{N^{\overline{s}}(2\pi)^s\theta_{1,\chi_1\chi_2}}(\overline{\lambda_{1,\chi_1\chi_2}}(-1)+\overline{\lambda_j}(-1))\sum_{n\geq 1}\frac{\overline{\lambda_{1,\chi_1\chi_2}}(\frac{N}{f}n,s)\lambda_j(n)}{n^s}.\end{aligned}$$

Then we can meromorphically continue the above equation to the whole complex plane, and take $s = \frac{1}{2}+iT$, where the Dirichlet series equals a finite Euler product of size $O(N^\varepsilon)$ times

$$\frac{L(\frac{1}{2},u_j)L(\frac{1}{2}+2iT,u_j\otimes\chi_1\chi_2)}{L(1+2iT,\chi_1\chi_2)},$$

which has Burgess bound $N^{\frac{3}{8}+\varepsilon}$. Hence, in total we have

$$\langle |E_{\mathfrak{a}}(\cdot,s,\chi)|^2, u_j\rangle_N \ll_T e^{\frac{\pi}{2}H_T(t_j)}N^{-\frac{1}{8}+\varepsilon},$$

for the same $H_T(t_j)$ as in (3.12). Mimicking the proof of Proposition 11, we have

$$\sum_{u\in\mathcal{O}(1)}\langle |E_{\mathfrak{a}}|^2,u\rangle_N\langle u,\phi_0\rangle_1 = \sum_{j\geq 1}\langle |E_{\mathfrak{a}}|^2,u_j\rangle_N\langle u_j,\phi_0\rangle_1 \ll_T N^{-\frac{1}{8}+\varepsilon}\|\phi_0\|_2,$$

and likewise,

$$\frac{1}{4\pi}\int_{-\infty}^\infty\langle |E_{\mathfrak{a}}|^2, E(\cdot,\tfrac{1}{2}+it)\rangle_N^{\mathrm{reg}}\langle E(\cdot,\tfrac{1}{2}+it),\phi_0\rangle_1 dt \ll_T N^{-\frac{1}{8}+\varepsilon}\|\phi_0\|_2.$$

3.6.3 Main term

Since $W_{\mathfrak{a}^*} = f$ by Lemma 1, we can derive from Lemma 4 and Proposition 14 that

$$\langle \mathcal{E}, \phi_0 \rangle_N = \langle \mathrm{Tr}_1^N \mathcal{E}, \phi_0 \rangle_1 = \lim_{\beta \to 0^+} \Big((\tfrac{N}{f})^{-\beta} \langle E(\cdot, 1+\beta), \phi_0 \rangle_1 \\ + \varphi_{\mathfrak{a}\mathfrak{a}^*}(\tfrac{1}{2} + iT, \chi) \varphi_{\mathfrak{a}\mathfrak{a}^*}(\tfrac{1}{2} + \beta - iT, \overline{\chi}) f^{\beta} \langle E(\cdot, 1-\beta), \phi_0 \rangle_1 \Big).$$

Substituting the Laurent expansion by Proposition 6, we have

$$\langle \mathcal{E}, \phi_0 \rangle_N = \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_1} \Big(-\log \tfrac{N}{f} - \log f - \frac{\varphi'_{\mathfrak{a}\mathfrak{a}^*}}{\varphi_{\mathfrak{a}\mathfrak{a}^*}}(\tfrac{1}{2} - iT, \overline{\chi}) \Big) + 2\langle G, \phi_0 \rangle_1,$$

while from Corollary 2 we see that

$$\frac{\varphi'_{\mathfrak{a}\mathfrak{a}^*}}{\varphi_{\mathfrak{a}\mathfrak{a}^*}}(\tfrac{1}{2} - iT, \overline{\chi}) = -3\log N - 4\Re \frac{L'}{L}(1 + 2iT, \overline{\chi_1 \chi_2}) + O_T(1).$$

After subtraction we arrive at

$$\langle \mathcal{E}, \phi_0 \rangle_N = \frac{\langle 1, \phi_0 \rangle_1}{\langle 1, 1 \rangle_1} \Big(2\log N + 4\Re \frac{L'}{L}(1 + 2iT, \overline{\chi_1 \chi_2}) + O_T(1) \Big) + 2\langle G, \phi_0 \rangle_1.$$

3.7 QUE for Eisenstein Series of Weight One

3.7.1 All differences

From now on we write E by

$$E_{\mathfrak{a}}^{(N)}(z, s, \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \overline{\chi}(\gamma) \frac{|c_\gamma \sigma_{\mathfrak{a}} z + d_\gamma|}{c_\gamma \sigma_{\mathfrak{a}} z + d_\gamma} (\mathrm{Im}\, \gamma \sigma_{\mathfrak{a}}^{-1} z)^s$$

be Eisenstein series of weight one, while all other parameters remains the same as the weight zero case. Then there are three differences in computation:

- the central character of an Eisenstein series is odd instead of even, i.e., $\chi(-1) = -1$;

- the new scattering matrix $(\varphi_{\mathfrak{ab}}(s,\chi))_{\mathfrak{a},\mathfrak{b}}$ has different entries, each $\varphi_{\mathfrak{ab}}(s,\chi)$ being $-\frac{\Gamma(s)\Gamma(\frac{3}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}$ times $\varphi_{\mathfrak{ab}}$ in the weight zero case; and

- for primitive χ_i mod q_i, $i=1,2$, E_{χ_1,χ_2} has a different Fourier expansion:

$$y^s+\varphi_{\chi_1,\chi_2}(s)y^{1-s}+\rho_{\chi_1,\chi_2}(s)\sum_{n\neq 0}|n|^{-\frac{1}{2}}\lambda_{\chi_1,\chi_2}(n,s)e(nx)\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{\mathrm{sgn}(n)}{2})}W_{\frac{|n|}{2n},s-\frac{1}{2}}(4\pi|n|y),$$

  where

  - $\lambda_{\chi_1,\chi_2}(n,s)$ is exactly the same as in the weight zero case;
  - $\varphi_{\chi_1,\chi_2}(s)$ is $-\frac{\Gamma(s)\Gamma(\frac{3}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}$ times the previous $\varphi_{\chi_1,\chi_2}(s)$ in the weight zero case;
  - $\rho_{\chi_1,\chi_2}(s)$ equals $\frac{i\Gamma(s)}{\Gamma(s+\frac{1}{2})}$ multiplies its counterpart in the weight zero case; and
  - $W_{\mu,\nu}(4\pi|n|y)$ is the W-Whittaker function. When $\mu=0$, $|n|^{-\frac{1}{2}}W_{\mu,\nu}(4\pi|n|y)$ together simplifies into $\sqrt{y}K_\nu(2\pi|n|y)$, which is in the weight zero case. When $\mu=\pm\frac{1}{2}$, we have the following lemma.

Lemma 15. *For $y>0$, we have*

$$W_{\frac{1}{2},iT}(2y)=\frac{y}{\sqrt{\pi}}(K_{-\frac{1}{2}+iT}(y)+K_{\frac{1}{2}+iT}(y));$$
$$W_{-\frac{1}{2},iT}(2y)=\frac{y}{\sqrt{\pi}iT}(-K_{-\frac{1}{2}+iT}(y)+K_{\frac{1}{2}+iT}(y)).$$

Lemma 15 is directly obtained from the recurrence relations of the Whittaker function (see e.g. [21, (13,15.10) and (13.15.12)]), and we omit the proof.

3.7.2 Proof of Theorem 6 in the weight one case

To simplify the exposition of showing Theorems 6 and 8 in the new environment, we set χ to be primitive mod N to prove the theorem as an example of such duplication.

3.7.2.1 Identification of $\mathcal{E}$

One quick observation is, $|E|^2$ is still of weight zero, and Proposition 10 functions the same way. Specifically speaking, we have

$$\langle |E|^2, \phi\rangle_N = \langle |E|^2 - \mathcal{E}, \phi\rangle_N + \langle \mathcal{E}, \phi\rangle_N,$$

for some $\mathcal{E}$ of weight 0. Studying cuspidal behaviors, we have $E_\infty(z,s) \sim y^s$, and

$$E_\infty|_{\sigma_0} \sim \varphi_{\infty 0}(s,\chi) y^{1-s},$$

where E_∞ and E_0 are of weight zero and trivial central character, while

$$\varphi_{\infty 0}(s,\chi) = \frac{N^{1-s}}{\tau(\chi)} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(\frac{1}{2}+s)} \frac{\Gamma(s)}{\Gamma(1-s)} \frac{\Lambda(2-2s,\chi)}{\Lambda(2s,\overline{\chi})}.$$

So similarly to the previous case of weight zero, Proposition 10 holds with

$$\mathcal{E} = \lim_{\beta\to 0^+} \Big(E_\infty(z, 1+\beta) + \varphi_{\infty 0}(s,\chi)\varphi_{\infty 0}(\overline{s}+\beta, \overline{\chi}) E_0(z, 1-\beta) \Big).$$

3.7.2.2 Error term

Then we can see Corollary 7 still holds, where $F_T(t_j)$ has a slightly different formula:

$$\overline{\rho_{1,\overline{\chi}}\rho_j} \Big((\overline{\lambda_j}(-1) - \overline{\lambda_{1,\overline{\chi}}}(-1)) \int_0^\infty y^{s-\frac{1}{2}} K_{-\frac{1}{2}+iT}(2\pi y) K_{it_j}(2\pi y) dy + \\ (\overline{\lambda_j}(-1) + \overline{\lambda_{1,\overline{\chi}}}(-1)) \int_0^\infty y^{s-\frac{1}{2}} K_{\frac{1}{2}+iT}(2\pi y) K_{it_j}(2\pi y) dy \Big).$$

From [25, (6.576.,4)], we see the old bound $F_T(t_j) \ll N^\varepsilon e^{H_T(t_j)} P(t_j, T)$ still hold for the new $F_T(t_j)$ with the same $H_T(t_j)$ in (3.12) and perhaps some different polynomial $P(t_j, T)$. Since the Dirichlet series remains the same as in the weight zero case, we arrive at

Proposition 15. *With above notations, we have*

$$\sum_{u\in\mathcal{O}(M)} \langle |E|^2, u\rangle_1 \langle u, \phi_0\rangle_1 = O_T(N^{-\frac{1}{8}+\varepsilon}\|\phi_0\|_1).$$

The Eisenstein contribution to the error terms can be done similarly, so we omit the proof.

3.7.2.3 Main term

Now we complete the proof of Theorem 6 for $k = 1$ by proving the following proposition.

Proposition 16. *With above notations, we have*

$$\langle \mathcal{E}, \phi\rangle_N = \frac{\langle 1, \phi_0\rangle_1}{\langle 1, 1\rangle_1}\Big(2\log N + 4\Re\frac{L'}{L}(1+2iT,\chi) + O_T(1)\Big) + O_\phi(1).$$

Proof. Applying the trace operator, we have

$$\begin{aligned}\langle \mathcal{E}, \phi\rangle_N = \langle \operatorname{Tr}_M^N \mathcal{E}, \phi\rangle_M = \lim_{\beta\to 0^+}\Big(&\langle \operatorname{Tr}_M^N E_\infty(\cdot, 1+\beta), \phi\rangle_M \\ &+ \varphi_{\infty 0}(s,\chi)\varphi_{\infty 0}(\overline{s}+\beta,\overline{\chi})\langle \operatorname{Tr}_M^N E_0(\cdot, 1-\beta), \phi\rangle_M\Big).\end{aligned}$$

Since these Eisenstein series above are of weight zero, everything goes in the same way, which means $\operatorname{Tr}_M^N E_\infty(z, 1+\beta)$ and $\operatorname{Tr}_M^N E_0(z, 1-\beta)$ are same as before, and so are their Laurent expansions. Then

$$\langle \mathcal{E}, \phi\rangle_N = 2\langle G, \phi\rangle_1 - \frac{\langle 1, \phi\rangle_1}{\operatorname{Vol}(\mathcal{F}_1)}\Big(\frac{\varphi'_{\infty 0}(\overline{s},\overline{\chi})}{\varphi_{\infty 0}(\overline{s},\overline{\chi})} + \log W_N^1(0)\Big).$$

Noticing that $\varphi_{\infty 0}(\overline{s},\overline{\chi})$ in the weight one case differs from that in the weight zero case by a factor $-\frac{\Gamma(s)\Gamma(\frac{3}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}$ that is level independent, we know that the new logarithmic derivative will be the same, up to the term of the logarithmic derivative of the additional factor. All differences being absorbed into the $O_T(1)$ error term of the main term asymptotic formula for the weight zero case with $M = 1$ and $q = N$, we arrive at the same conclusion. $\square$

4. PROOF OF THEOREM 10

4.1 General strategy

Just like what happens in [19], [72] and Chapter 3, $|E_{\mathfrak{a}}(z, \frac{1}{2}+iT, \chi)|^2$ is not directly regularizable, because we cannot subtract it by $E_{\mathfrak{a}}(z,1)$, which is not defined. Instead, we need to consider rewriting the L^4-norm as $\langle E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}), \overline{E_{\mathfrak{a}}(\cdot, s_3, \chi)E_{\mathfrak{a}}(\cdot, s_4, \overline{\chi})}\rangle_{\text{reg}}$, and find a path for $(s_1, s_2, s_3, s_4) \in \mathbb{C}^4$, such that it arrives at $(\frac{1}{2}+iT, \frac{1}{2}+iT, \frac{1}{2}+iT, \frac{1}{2}+iT)$ without touching any point of singularity.

As is discussed in Chapter 3, if we further assume $w_1+w_2, w_3+w_4 \neq 1$, then we can regularize $E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi})$ and $E_{\mathfrak{a}}(\cdot, \overline{s_3}, \overline{\chi})E_{\mathfrak{a}}(\cdot, \overline{s_4}, \chi)$ respectively. That is, there exists $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}) - \mathcal{E}_1$ and $E_{\mathfrak{a}}(\cdot, \overline{s_3}, \overline{\chi})E_{\mathfrak{a}}(\cdot, \overline{s_4}, \chi) - \mathcal{E}_2 \in L^2$. Since their product is in L^1, we have

$$\langle E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}), E_{\mathfrak{a}}(\cdot, \overline{s_3}, \overline{\chi})E_{\mathfrak{a}}(\cdot, \overline{s_4}, \chi)\rangle_{\text{reg}} = I_1 + I_2,$$

where

$$I_1 = I_1(s_1, s_2, s_3, s_4) = \langle E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}) - \mathcal{E}_1, E_{\mathfrak{a}}(\cdot, \overline{s_3}, \overline{\chi})E_{\mathfrak{a}}(\cdot, \overline{s_4}, \chi) - \mathcal{E}_2\rangle,$$

and

$$I_2 = I_2(s_1, s_2, s_3, s_4) = \langle E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}), \mathcal{E}_2\rangle_{\text{reg}} + \langle \mathcal{E}_1, E_{\mathfrak{a}}(\cdot, \overline{s_3}, \overline{\chi})E_{\mathfrak{a}}(\cdot, \overline{s_4}, \chi)\rangle_{\text{reg}} + \langle \mathcal{E}_1, \mathcal{E}_2\rangle_{\text{reg}}.$$

In order to decide $\mathcal{E}_1$ and $\mathcal{E}_2$, one needs to study carefully the behavior of $E_{\mathfrak{a}}|_{\sigma_{\mathfrak{b}}}$ for every $\mathfrak{b}$, no matter singular or not for χ. According to Proposition 10, we have

$$\mathcal{E}_1 = \mathcal{E}_1(s_1, s_2) = E_{\mathfrak{a}}(z, s_1+s_2) + \varphi_{\mathfrak{a}\mathfrak{a}^*}(s_1, \chi)\varphi_{\mathfrak{a}\mathfrak{a}^*}(s_2, \overline{\chi})E_{\mathfrak{a}^*}(z, 2-s_1-s_2),$$

and

$$\mathcal{E}_2 = \mathcal{E}_2(s_3, s_4) = E_{\mathfrak{a}}(z, \overline{s_3} + \overline{s_4}) + \varphi_{\mathfrak{a}\mathfrak{a}^*}(\overline{s_3}, \overline{\chi})\varphi_{\mathfrak{a}\mathfrak{a}^*}(\overline{s_4}, \chi)E_{\mathfrak{a}^*}(z, 2 - \overline{s_3} - \overline{s_4}).$$

It is known that $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle_{\text{reg}} = 0$, so it suffices to compute the first two terms of I_2.

One nice feature of the regularized integral is it is easily computable when an Eisenstein series attached to a cusp is a factor of the integrand.

Now we need to introduce the regularized integrals. Roughly speaking, if an $SL_2(\mathbb{Z})$-automorphic function F is of moderate growth, then there always exists $\mathcal{E}$, a linear combination of Eisenstein series, such that $F - \mathcal{E} = O(\sqrt{y})$ as $y \to \infty$, and still maintains automorphy. Then the convergent integral $\int (F - \mathcal{E})$ is defined to be the renormalized integral of $\int F$.

Theorem 15. *For automorphic function F of moderate growth and $w \neq 0, 1$, we have*

$$\langle F, E_{\mathfrak{a}}(\cdot, w) \rangle_{\text{reg}} = \int_0^\infty y^{w-2}(F(\sigma_{\mathfrak{a}} z) - \psi_{\mathfrak{a}}(y))dy,$$

where $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ is any scaling matrix, and $\psi_{\mathfrak{a}}$ is the moderate growth part of $F(\sigma_{\mathfrak{a}} z)$.

Remark 16. *All meromorphic functions in this paper is continuable. So, throughout, we directly assume a function $f(s)$ is defined on $\mathbb{C}$ from the beginning, as long as it is meromorphic on some half plane $\Re s > C$.*

Remark 17. *All implied constants are assumed to be related with ε and T if not specified otherwise.*

4.2 Two consequences of GRH

Lemma 16. *Assume GRH is true, $\Re s = 1$ and χ is primitive mod N, then we have*

$$\frac{L'}{L}(s, \chi) \ll \log\log N; \tag{4.1}$$
$$\frac{L''}{L}(s, \chi) \ll (\log\log N)^{3+\varepsilon}. \tag{4.2}$$

Proof. Inequality (4.1) is in Theorem 5.17 of [36]. As for (4.2), we firstly recall that

$$(\frac{L'}{L})'(s,\chi) = \frac{L''}{L}(s,\chi) - (\frac{L'}{L})^2(s,\chi).$$

By (4.1), we know $(\frac{L'}{L})^2(s,\chi) \ll (\log\log N)^2$; Proposition 5.16 of [36] says that

$$-\frac{L'}{L}(s,\chi) = \sum_p \frac{\chi(p)\log p}{p^s}\phi(\frac{p}{X}) + \sum_\rho \hat{\phi}(\rho - s)X^{\rho - s} + O(1),$$

for $X = \log^{2+\varepsilon} N(|s|+1)$, $\phi(y) = \max\{1-y, 0\}$, and $\hat{\phi}(w) = w^{-1}(w+1)^{-1}$. Taking derivative on both sides, we see

$$\begin{aligned}\Big|(\frac{L'}{L})'(s,\chi)\Big| &\leq \sum_{p\leq X} \frac{\log^2 p}{p} + \log X \sum_\rho \frac{1}{|\rho - s|\cdot|\rho+1-s|} + O(1)\\ &\ll \int_2^X \frac{\log^2 t}{t}dt + \log X \cdot O(\sum_\rho |\rho|^{-2}).\end{aligned}$$

Since $\sum_\rho |\rho|^{-2}$ converges, above can further be bounded by $(\log\log N)^{3+\varepsilon}$. So is $\frac{L''}{L}(s,\chi)$. □

4.3 Proof of Theorem 10

4.3.1 Preparation

Lemma 17. *Assume* $w_1 \neq w_2, 1-w_2$*, and* $w_3 \neq 0, 1$*. Then*

$$\langle E^*_{\chi_1,\chi_2}(\cdot, w_1)E^*_{\overline{\chi_1},\overline{\chi_2}}(\cdot, w_2), E_{\mathfrak{a}}(\cdot, \overline{w_3})\rangle_{\text{reg}} = N^{-w_3}q_1(\frac{q_2}{N})^{w_1+w_2}\frac{1}{\xi(2w_3)}\prod_{p|N}\frac{1-p^{w_1+w_2-w_3-1}}{1-p^{-2w_3}}$$

$$\xi(w_1+w_2+w_3-1)\Lambda(w_1-w_2+w_3, \chi_1\chi_2)\Lambda(-w_1+w_2+w_3, \overline{\chi_1\chi_2})\xi(-w_1-w_2+w_3+1).$$

Proof. Theorem 15 says that the left hand side equals

$$\int_0^\infty y^{w_3-2}(\int_0^1 E^*_{\chi_1,\chi_2}(\sigma_{\mathfrak{a}}z, w_1)E^*_{\chi_1,\chi_2}(\sigma_{\mathfrak{a}}z, w_2))dx - \Psi)dy,$$

where Ψ is the moderate growth part of $F = E^*_{\chi_1,\chi_2}(z, w_1)E^*_{\chi_1,\chi_2}(\cdot, w_2)$. Applying (2.12), we can rewrite it with

$$q_1(\frac{q_2}{N})^{w_1+w_2}\int_0^\infty y^{w_3-2}(\int_0^1 E^*_{1,\chi_1\chi_2}(z, w_1)E^*_{\chi_1,\chi_2}(\cdot, w_2)dx - \Psi)dy.$$

Since the moderate growth part of F is exactly the product of the moderate growth parts of $E^*_{1,\chi_1\chi_2}(z, w_1)$ and $E^*_{1,\overline{\chi_1\chi_2}}(z, w_2)$, we further write it as (note $\lambda_{1,\chi_1\chi_2}(-1) + \lambda_{1,\overline{\chi_1\chi_2}}(-1) = 2$ by evenness of $\chi_1\chi_2$)

$$\begin{aligned} &4q_1(\frac{q_2}{N})^{w_1+w_2}\int_0^\infty y^{w_3-1}\sum_{n\neq 0}\lambda_{1,\chi_1\chi_2}(n, w_1)\lambda_{1,\overline{\chi_1\chi_2}}(-n, w_2)K_{w_1-\frac{1}{2}}(2\pi|n|y)K_{w_2-\frac{1}{2}}(2\pi|n|y)dy \\ &= 8q_1(\frac{q_2}{N})^{w_1+w_2}\sum_{n\geq 1}\frac{\lambda_{1,\chi_1\chi_2}(n, w_1)\lambda_{1,\overline{\chi_1\chi_2}}(n, w_2)}{n^{w_3}}\int_0^\infty y^{w_3-1}K_{w_1-\frac{1}{2}}(2\pi y)K_{w_2-\frac{1}{2}}(2\pi y)dy. \end{aligned}$$

By (6.576.4) of [GR], we know the integral equals

$$\frac{\pi^{-w_3}}{8\Gamma(w_3)}\prod_{\epsilon_1=\pm 1}\prod_{\epsilon_2=\pm 1}\Gamma(\frac{w_3+\epsilon_1(w_1-\frac{1}{2})+\epsilon_2(w_2-\frac{1}{2})}{2}).$$

while the Dirichlet factors into (see (13.1) of [I1])

$$\frac{\zeta(w_1+w_2+w_3-1)L(w_1-w_2+w_3, \chi_1\chi_2)L(-w_1+w_2+w_3, \overline{\chi_1\chi_2})L(-w_1-w_2+w_3+1, \chi_{0,N})}{L(2w_3, \chi_{0,N})}.$$

Completing these L-functions with proper factors, we obtain the right hand side. □

Proposition 17. *We have*

$$\begin{aligned} I_2 =& H_1\xi(s_1+s_2+s_3+s_4-1)\xi(-s_1-s_2+s_3+s_4+1) \\ &+H_2\xi(-s_1-s_2-s_3-s_4+3)\xi(s_1+s_2-s_3-s_4+1) \\ &+H_3\xi(s_1+s_2+s_3+s_4-1)\xi(s_1+s_2-s_3-s_4+1) \\ &+H_4\xi(-s_1-s_2-s_3-s_4+3)\xi(-s_1-s_2+s_3+s_4+1), \end{aligned}$$

where $H_j = H_j(s_1, s_2, s_3, s_4)$ *for* $j = 1, 2, 3, 4$ *with*

$$H_1 = N^{-s_1-s_2-s_3-s_4+1}\frac{\Lambda(s_1 - s_2 + s_3 + s_4, \chi_1\chi_2)\Lambda(-s_1 + s_2 + s_3 + s_4, \overline{\chi_1\chi_2})}{\xi(2s_3 + 2s_4)\Lambda(2s_1, \chi_1\chi_2)\Lambda(2s_2, \overline{\chi_1\chi_2})} \prod_{p|N} \frac{1 - p^{s_1+s_2-s_3-s_4-1}}{1 - p^{-2s_3-2s_4}},$$

$$H_2 = N^{-1}\frac{\Lambda(s_1 - s_2 - s_3 - s_4 + 2, \chi_1\chi_2)\Lambda(-s_1 + s_2 - s_3 - s_4 + 2, \overline{\chi_1\chi_2})}{\xi(-2s_3 - 2s_4 + 4)\Lambda(2s_1, \chi_1\chi_2)\Lambda(2s_2, \overline{\chi_1\chi_2})} \frac{\Lambda(2 - 2s_3, \overline{\chi_1\chi_2})\Lambda(2 - 2s_4, \chi_1\chi_2)}{\Lambda(2s_3, \chi_1\chi_2)\Lambda(2s_4, \overline{\chi_1\chi_2})} \prod_{p|N} \frac{1 - p^{-s_1-s_2+s_3+s_4-1}}{1 - p^{2s_3+2s_4-4}},$$

$$H_3 = N^{-s_1-s_2-s_3-s_4+1}\frac{\Lambda(s_1 + s_2 + s_3 - s_4, \chi_1\chi_2)\Lambda(s_1 + s_2 - s_3 + s_4, \overline{\chi_1\chi_2})}{\xi(2s_1 + 2s_2)\Lambda(2s_3, \chi_1\chi_2)\Lambda(2s_4, \overline{\chi_1\chi_2})} \prod_{p|N} \frac{1 - p^{-s_1-s_2+s_3+s_4-1}}{1 - p^{-2s_1-2s_2}},$$

$$H_4 = N^{-1}\frac{\Lambda(-s_1 - s_2 + s_3 - s_4 + 2, \chi_1\chi_2)\Lambda(-s_1 - s_2 - s_3 + s_4 + 2, \overline{\chi_1\chi_2})}{\xi(-2s_1 - 2s_2 + 4)\Lambda(2s_3, \chi_1\chi_2)\Lambda(2s_4, \overline{\chi_1\chi_2})} \frac{\Lambda(2 - 2s_1, \overline{\chi_1\chi_2})\Lambda(2 - 2s_2, \chi_1\chi_2)}{\Lambda(2s_1, \chi_1\chi_2)\Lambda(2s_2, \overline{\chi_1\chi_2})} \prod_{p|N} \frac{1 - p^{s_1+s_2-s_3-s_4-1}}{1 - p^{2s_1+2s_2-4}}.$$

Proof. Since $E_{\mathfrak{a}}(z, s, \chi) = N^{-s}\chi_1(-1)\rho_{\chi_1,\chi_2}(s)E^*_{\chi_1,\chi_2}(z, s)$ with

$$\rho_{\chi_1,\chi_2}(s) := \frac{q_1^s\tau(\chi_2)}{\Lambda(2s, \chi_1\chi_2)},$$

we see that $\langle E_{\mathfrak{a}}(\cdot, s_1, \chi)E_{\mathfrak{a}}(\cdot, s_2, \overline{\chi}), E_{\mathfrak{a}}(\cdot, s_3 + s_4)\rangle_{\mathrm{reg}}$ equals

$$N^{-s_1-s_2}\rho_{\chi_1,\chi_2}(s_1)\rho_{\overline{\chi_1},\overline{\chi_2}}(s_2)\langle E^*_{\chi_1,\chi_2}(\cdot, s_1)\overline{E^*_{\chi_1,\chi_2}(\cdot, s_2)}, E_{\mathfrak{a}}(\cdot, s_3 + s_4)\rangle_{\mathrm{reg}}.$$

Applying Lemma 17 with $w_1 = s_1, w_2 = s_2$ and $w_3 = s_3 + s_4$, we see above further equals

$$\frac{N^{-s_1-s_2-s_3-s_4}q_1^{s_1+s_2-s_3-s_4+1}\tau(\chi_2)\overline{\tau(\chi_2)}}{\xi(2s_3+2s_4)\Lambda(2s_1,\chi_1\chi_2)\Lambda(2s_2,\overline{\chi_1\chi_2})}\Lambda(s_1-s_2+s_3+s_4,\chi_1\chi_2)\Lambda(-s_1+s_2+s_3+s_4,\overline{\chi_1\chi_2})$$
$$\xi(s_1+s_2+s_3+s_4-1)\xi(-s_1-s_2+s_3+s_4+1)\prod_{p|N}\frac{1-p^{s_1+s_2-s_3-s_4-1}}{1-p^{-2s_3-2s_4}}.$$

This accounts for the first term $H_1\xi(s_1+s_2+s_3+s_4-1)\xi(-s_1-s_2+s_3+s_4+1)$. The other three terms can be obtained in the same way, except that we adopt (2.13) in place of (2.12) for the second and fourth terms. □

4.3.2 Estimation of I_2

Now set $s_1 = s_3 = \frac{1}{2}+iT$, $s_2 = \frac{1}{2}+\eta'-iT$ and $s_4 = \frac{1}{2}+\eta-iT$ with $0<\eta'<\eta<\frac{1}{4}$. Under limit $\eta' \to 0$, I_2 tends to

$$\underbrace{F_1(\eta)\xi^2(1+\eta)}_{\Xi_1}+\underbrace{F_2(\eta)\xi^2(1-\eta)}_{\Xi_2}+\underbrace{F_3(\eta)\xi(1+\eta)\xi(1-\eta)}_{\Xi_3}+\underbrace{F_4(\eta)\xi(1-\eta)\xi(1+\eta)}_{\Xi_4},$$

with $F_j(\eta) = \lim_{\eta'\to 0^+} H_j(\frac{1}{2}+iT,\frac{1}{2}+\eta'-iT,\frac{1}{2}+iT,\frac{1}{2}+\eta-iT)$. The explicit forms are

$$F_1 = N^{-1-\eta}\frac{|\Lambda(1+\eta+2iT,\chi_1\chi_2)|^2}{\xi(2+2\eta)|\Lambda(1+2iT,\chi_1\chi_2)|^2}\prod_{p|N}(1+\frac{1}{p^{1+\eta}})^{-1};$$
$$F_2 = N^{-1}\frac{|\Lambda(1-\eta+2iT,\chi_1\chi_2)|^2\Lambda(1-2\eta+2iT,\chi_1\chi_2)}{\xi(2-2\eta)\Lambda^2(1+2iT,\overline{\chi_1\chi_2})\Lambda(1+2\eta-2iT,\overline{\chi_1\chi_2})}\prod_{p|N}(1+\frac{1}{p^{1-\eta}})^{-1};$$
$$F_3 = N^{-1-\eta}\frac{\Lambda(1-\eta+2iT,\chi_1\chi_2)\Lambda(1+\eta-2iT,\overline{\chi_1\chi_2})}{\xi(2)\Lambda(1+2iT,\chi_1\chi_2)\Lambda(1+2\eta-2iT,\overline{\chi_1\chi_2})}\prod_{p|N}\frac{1-p^{-1+\eta}}{1-p^{-2}};$$
$$F_4 = N^{-1}\frac{\Lambda(1-\eta+2iT,\chi_1\chi_2)\Lambda(1+\eta-2iT,\overline{\chi_1\chi_2})}{\xi(2)\Lambda(1+2iT,\chi_1\chi_2)\Lambda(1+2\eta-2iT,\overline{\chi_1\chi_2})}\prod_{p|N}\frac{1-p^{-1-\eta}}{1-p^{-2}}.$$

Further calculation shows $F_1(0) = F_2(0) = F_3(0) = F_4(0) = (\xi(2)\nu(N))^{-1}$, and

$$\begin{aligned}
F_1'(0) &= F_1(0) \cdot \Big(-\log N + 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) - 2\frac{\xi'}{\xi}(2) + \sum_{p|N}\frac{\log p}{p+1}\Big);\\
F_2'(0) &= F_2(0) \cdot \Big(- 6\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) + 2\frac{\xi'}{\xi}(2) - \sum_{p|N}\frac{\log p}{p+1}\Big);\\
F_3'(0) &= F_3(0) \cdot \Big(-\log N - 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) - \sum_{p|N}\frac{\log p}{p+1}\Big);\\
F_4'(0) &= F_4(0) \cdot \Big(- 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) + \sum_{p|N}\frac{\log p}{p+1}\Big).
\end{aligned}$$

Moreover, by $F_j'' = (F_j \cdot \frac{F_j'}{F_j})' = F_j \cdot ((\frac{F_j'}{F_j})^2 + (\frac{F_j'}{F_j})')$, we see that

$$\begin{aligned}
F_1''(0) &= F_1(0) \cdot \Big(-\log N + 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) - 2\frac{\xi'}{\xi}(2) + \sum_{p|N}\frac{\log p}{p+1}\Big)^2\\
&\quad + F_1(0) \cdot \Big(\log^2 N + 2\Re(\frac{\Lambda'}{\Lambda})'(1+2iT,\chi_1\chi_2) - 4(\frac{\xi'}{\xi})'(2) - \sum_{p|N}\frac{p\log^2 p}{(p+1)^2}\Big);\\
F_2''(0) &= F_2(0) \cdot \Big(- 6\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) + 2\frac{\xi'}{\xi}(2) - \sum_{p|N}\frac{\log p}{p+1}\Big)^2\\
&\quad + F_2(0) \cdot \Big(2(\frac{\Lambda'}{\Lambda})'(1+2iT,\chi_1\chi_2) + 6\Im(\frac{\Lambda'}{\Lambda})'(1+2iT,\chi_1\chi_2) - 4(\frac{\xi'}{\xi})'(2) - \sum_{p|N}\frac{p\log^2 p}{(p+1)^2}\Big);\\
F_3''(0) &= F_3(0) \cdot \Big(-\log N - 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) - \sum_{p|N}\frac{\log p}{p-1}\Big)^2\\
&\quad + F_3(0) \cdot \Big(\log^2 N + 2\Re(\frac{\Lambda'}{\Lambda})'(1+2iT,\chi_1\chi_2) - 4(\frac{\Lambda'}{\Lambda})'(1-2iT,\overline{\chi_1\chi_2}) + \sum_{p|N}\frac{p\log^2 p}{(p-1)^2}\Big);\\
F_4''(0) &= F_4(0) \cdot \Big(- 2\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2) + \sum_{p|N}\frac{\log p}{p-1}\Big)^2\\
&\quad + F_4(0) \cdot \Big(2\Re(\frac{\Lambda'}{\Lambda})'(1+2iT,\chi_1\chi_2) - 4(\frac{\Lambda'}{\Lambda})'(1-2iT,\overline{\chi_1\chi_2}) + \sum_{p|N}\frac{p\log^2 p}{(p-1)^2}\Big).
\end{aligned}$$

Let $\xi(s) = (s-1)^{-1} + a + b(s-1) + O((s-1)^2)$ for some a, b, then for η around 0, we have

$$\begin{aligned}
\Xi_1 &= +\frac{F_1(0)}{\eta^2} + \frac{F_1'(0) + 2aF_1(0)}{\eta} + \frac{F_1''(0)}{2} + 2aF_1'(0) + (a^2+2b)F_1(0) + O(\eta);\\
\Xi_2 &= +\frac{F_2(0)}{\eta^2} + \frac{F_2'(0) - 2aF_2(0)}{\eta} + \frac{F_2''(0)}{2} - 2aF_2'(0) + (a^2+2b)F_2(0) + O(\eta);\\
\Xi_3 &= -\frac{F_3(0)}{\eta^2} - \frac{F_3'(0)}{\eta} - \frac{F_3''(0)}{2} + (a^2-2b)F_3(0) + O(\eta);\\
\Xi_4 &= -\frac{F_4(0)}{\eta^2} - \frac{F_4'(0)}{\eta} - \frac{F_4''(0)}{2} + (a^2-2b)F_4(0) + O(\eta).
\end{aligned}$$

Thus, the coefficients of the $\frac{1}{\eta^2}$ and $\frac{1}{\eta}$ of I_2 vanish by cancellation, and its constant term equals

$$\begin{aligned}
&\frac{1}{2}(F_1''(0) + F_2''(0) - F_3''(0) - F_4''(0)) + 2a(F_1'(0) - F_2'(0)) + 4a^2F_1(0)\\
&\quad = \frac{1}{2}(F_1''(0) + F_2''(0) - F_3''(0) - F_4''(0)) + \frac{4a^2 - 2a\log N + 16a\Re\frac{\Lambda'}{\Lambda}(1+2iT, \chi_1\chi_2)}{\xi(2)\nu(N)}.
\end{aligned}$$

A well-known fact being $\sum_{p|N} \frac{\log p}{p} = O(\log\log N)$, we have

$$\begin{aligned}
\xi(2)\nu(N)I_2 &= 4\Re(\frac{\Lambda'}{\Lambda})'(1+2iT, \chi_1\chi_2) + 16\Re(\frac{\Lambda'}{\Lambda})^2(1+2iT, \chi_1\chi_2) + 16\Big|\frac{\Lambda'}{\Lambda}\Big|^2(1+2iT, \chi_1\chi_2)\\
&\quad - 4\log N\Re\frac{\Lambda'}{\Lambda}(1+2iT, \chi_1\chi_2) + O\Big(\log N + \log\log N\Big|\Re\frac{\Lambda'}{\Lambda}(1+2iT, \chi_1\chi_2)\Big|\Big).
\end{aligned}$$

Since $(\frac{\Lambda'}{\Lambda})' = \frac{\Lambda''}{\Lambda} - (\frac{\Lambda'}{\Lambda})^2$, and

$$\begin{aligned}
PY\frac{\Lambda''}{\Lambda}(s, \chi_1\chi_2) &= \frac{1}{4}\log^2\frac{N}{\pi} + \frac{1}{4}\frac{\Gamma''}{\Gamma}(\frac{s}{2}) + \frac{L''}{L}(s, \chi_1\chi_2)\\
&\quad + \frac{1}{2}\log\frac{N}{\pi}\frac{\Gamma'}{\Gamma}(\frac{s}{2}) + \log\frac{N}{\pi}\frac{L'}{L}(s, \chi_1\chi_2) + \frac{\Gamma'}{\Gamma}(\frac{s}{2})\frac{L'}{L}(s, \chi_1\chi_2),
\end{aligned}$$

we can see that

$$\begin{aligned}
\xi(2)\nu(N)I_2 &= \log^2 N + 4\Re\frac{L''}{L}(1+2iT, \chi_1\chi_2) + 12\Re(\frac{\Lambda'}{\Lambda})^2(1+2iT, \chi_1\chi_2) + 16\Big|\frac{\Lambda'}{\Lambda}\Big|^2(1+2iT, \chi_1\chi_2)\\
&- 4\log N\Re\frac{\Lambda'}{\Lambda}(1+2iT, \chi_1\chi_2) + O\Big(\log N\Big|\frac{L'}{L}(s, \chi_1\chi_2)\Big| + \log\log N\Big|\Re\frac{\Lambda'}{\Lambda}(1+2iT, \chi_1\chi_2)\Big|\Big),
\end{aligned}$$

and after substituting $\frac{\Lambda'}{\Lambda}(s,\chi_1\chi_2) = \frac{1}{2}\log N + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}) + \frac{L'}{L}(s,\chi_1\chi_2)$, we arrive at

$$\begin{aligned}\xi(2)\nu(N)I_2 &= 4\log^2 N + 4\Re\frac{L''}{L}(1+2iT,\chi_1\chi_2)\\ &\quad + O\Big(\log N\Big|\frac{L'}{L}(s,\chi_1\chi_2)\Big| + \Big|\frac{L'}{L}(s,\chi_1\chi_2)\Big|^2 + \log\log N\Big|\Re\frac{\Lambda'}{\Lambda}(1+2iT,\chi_1\chi_2)\Big|\Big).\end{aligned}$$

Now we assume GRH. With Lemma 16, we have (note $\xi(2) = \frac{\pi}{6}$)

$$\nu(N)I_2 = \frac{24}{\pi}\log^2 N + O(\log N\log\log N).$$

5. SUMMARY

As we can see from the previous chapters, we have proved QUE and completed the reduction to L-functions for all newform Eisenstein series of primitive central characters. In the process, the simple structure of such Eisenstein series are heavily relied on. To generalize our results to arbitrary newform Eisenstein series, we will need to study the adelic language of the $GL(2)$-automorphic forms. We may do this as time permits in the near future.

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