HYPERDERIVATIVES OF PERIODS AND LOGARITHMS OF ANDERSON t-MODULES, AND ALGEBRAIC INDEPENDENCE

A Dissertation

by

CHANGNINGPHAABI NAMOIJAM

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Matthew A. Papanikolas
Committee Members,	Riad Masri
	Huiyan Sang
	Matthew P. Young
Head of Department,	Sarah Witherspoon

August 2020

Major Subject: Mathematics

ABSTRACT

In this dissertation, we study algebraic relations among periods, quasi-periods, logarithms and quasi-logarithms of Drinfeld modules. This work is motivated by the Tannakian theory for t-motives especially the function field analogue, proved by Papanikolas, of Grothendieck's conjecture for periods of abelian varieties. Papanikolas' theorem shows that the dimension of the Galois group associated to a t-motive is equal to the transcendence degree of the entries of the period matrix of the t-motive. In recent work, Papanikolas and the author proved that the period matrix of the prolongation t-motives, introduced by Maurischat, of t-motives associated to t-modules entail hyperderivatives of periods and quasi-periods. Computing the Galois group of these prolongations, we prove that the algebraic relations among the hyperderivatives of periods and quasi-periods of a Drinfeld module are the ones induced by the endomorphisms of the Drinfeld module. Furthermore, we construct a new t-motive using these prolongations and compute its Galois group, using which we investigate hyperderivatives of Drinfeld logarithms and quasi-logarithms, and prove transcendence results about them.

DEDICATION

This dissertation is dedicated to my parents and my brother.

ACKNOWLEDGMENTS

I would like to thank my doctoral adviser, Matthew Papanikolas, for his guidance, encouragement and patience throughout my graduate studies. His invaluable advice and reassurances have been influential in the completion of this dissertation.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Professor Matthew A. Papanikolas, Professor Riad Masri and Professor Matthew P. Young of the Department of Mathematics and Professor Huiyan Sang of the Department of Statistics.

All work conducted for the dissertation was completed by the student, under the supervision of Matthew A. Papanikolas of the Department of Mathematics.

Funding Sources

Graduate study was supported by a fellowship from Texas A&M University and partially by NSF Grant DMS-1501362.

NOMENCLATURE

$\mathbb{F}_q$	finite field with $q = p^n$ elements.
A	$\mathbb{F}_q[\theta]$, the polynomial ring in θ over $\mathbb{F}_q$.
k	$\mathbb{F}_q(\theta)$, the fraction field of A.
k_∞	$\mathbb{F}_q(\!(1/\theta)\!)$, the completion of k with respect to $\lvert \cdot \rvert_\infty$.
$\mathbb{K}$	the completion of an algebraic closure of k_∞.
$\overline{k}$	the algebraic closure of k inside $\mathbb{K}$.
$\mathbf{A}$	$\mathbb{F}_q[t]$, the polynomial ring in t over $\mathbb{F}_q$, t independent from θ.
$\mathbf{k}$	$\mathbb{F}_q(t)$, the fraction field of $\mathbf{A}$.
$\mathbb{T}$	$\{\sum_{h=0}^\infty a_h t^h \in \mathbb{K}[\![t]\!] \mid \lim_{h\to\infty} \lvert a_h \rvert_\infty = 0\}$, the Tate algebra of the closed unit disk of $\mathbb{K}$.
$\mathbb{L}$	the fraction field of $\mathbb{T}$.
F^{sep}	a separable algebraic closure of a field F.
GL_r/F	for the field F, the F-group scheme of invertible $r \times r$ matrices.

TABLE OF CONTENTS

1. INTRODUCTION

1.1 Introduction

Drinfeld modules were introduced as “elliptic modules" by Drinfeld, as a function field analogue of elliptic curves [18]. Later, Anderson defined higher dimensional generalizations of Drinfeld modules, called t-modules [1]. Yu proved several transcendence results concerning periods and logarithms of Drinfeld modules [41], [42], and proved the Sub-t-module theorem [44], a remarkable result regarding linear independence among logarithms of Anderson t-modules. Brownawell [4], and Brownawell and Papanikolas [6] proved linear independence results concerning logarithms and quasi-logarithms of certain t-modules. Thiery [39] proved algebraic independence results among periods and quasi-periods of rank 2 Drinfeld modules with complex multiplication. Concerning hyperderivatives, algebraic independence results among hyperderivatives of the period of the Carlitz module (rank 1 Drinfeld module) were proved by Denis [15], [16], [17] and Maurischat [28]. Further work in this direction was also done in unpublished work by Brownawell and van der Poorten. Utilizing Yu’s sub-t-module theorem, Brownawell and Denis proved linear independence of hyperderivatives of logarithms and quasi-logarithms of Drinfeld modules [3], [5]. Shortly after this, Brownawell proved linear independence results among hyperderivatives of several logarithms and quasi-logarithms of Drinfeld modules [4].

In this dissertation, we prove all algebraic independence results among hyperderivatives of periods, quasi-periods, logarithms and quasi-logarithms of Drinfeld modules. To prove our results, we employ Papanikolas’ theorem [34, Thm. 1.1.7] on the transcendence degree of the period matrix of a t-motive and the dimension of its Galois group. Using this result, Chang and Papanikolas have subsequently proved algebraic independence results among periods, quasi-periods, logarithms and quasi-logarithms of Drinfeld modules [11], [12].

1.2 Hyperderivatives of periods and logarithms

Let $\mathbb{F}_q$ be a finite field, where q is a positive power of a prime number p. Letting θ be an indeterminate, we set $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$ and $k_\infty := \mathbb{F}_q((1/\theta))$, the completion of k at its infinite place. We further set $\mathbb{K}$ to be the completion of an algebraic closure of k_∞ and let $\overline{k}$ and k^{sep} be the algebraic closure and the separable closure respectively of k inside $\mathbb{K}$. For a variable t independent from θ, we further define $\mathbf{A} := \mathbb{F}_q[t]$ and $\mathbf{k} := \mathbb{F}_q(t)$.

For $n \in \mathbb{Z}$, we define the n-fold Frobenius twist $\tau^n : \mathbb{K}(\!(t)\!) \to \mathbb{K}(\!(t)\!)$ by setting for $f = \sum_i a_i t^i$,

$$\tau^n(f) := f^{(n)} = \sum_i a_i^{q^n} t^i. \tag{1.1}$$

For a field $K \subseteq \mathbb{K}$, we define the twisted power series ring $K[\![\tau]\!]$ such that $\tau f = f^q \tau$ for all $f \in K$. Then, we define the twisted polynomial ring $K[\tau]$ as the subring of $K[\![\tau]\!]$, where $K[\tau]$ is viewed as a subalgebra of the $\mathbb{F}_q$-linear endomorphisms of the additive group of K.

For a field $k \subseteq K \subseteq \mathbb{K}$, a *Drinfeld* $\mathbf{A}$*-module* of rank r defined over K is an $\mathbb{F}_q$-algebra homomorphism

$$\rho : \mathbf{A} \to K[\tau]$$

uniquely determined by $\rho_t = \theta + \kappa_1\tau + \cdots + \kappa_r\tau^r$ such that $\kappa_r \neq 0$. The exponential function associated to ρ is given by $\mathrm{Exp}_\rho(z) = z + \sum_{h\geq 1} \alpha_h z^{q^h} \in K[\![z]\!]$ and it satisfies the functional equation $\mathrm{Exp}_\rho(\theta z) = \rho_t(\mathrm{Exp}_\rho(z))$. The *period lattice* of ρ is the kernel Λ_ρ of Exp_ρ, which is a free and finitely generated discrete $\mathbf{A}$-submodule of rank r inside $\mathbb{K}$ (see §3.1 for details about general Anderson t-modules).

The de Rham cohomology theory for Drinfeld modules was developed by Anderson, Deligne, Gekeler and Yu [20], [42]. A ρ*-biderivation* is an $\mathbb{F}_q$-linear map $\delta : \mathbf{A} \to \mathbb{K}[\tau]\tau$ satisfying

$$\delta_{ab} = a(\theta)\delta_b + \delta_a\rho_b \quad \forall a, b \in \mathbf{A}.$$

Let $u \in \mathbb{K}[\tau]$. Then, the ρ-biderivation $\delta^{(u)}$ defined by $\delta^{(u)}_a = u\rho_a - a(\theta)u,\ \forall\, a \in \mathbf{A}$ is called an *inner biderivation*. If $u \in \mathbb{K}[\tau]\tau$, then $\delta^{(u)}$ is said to be *strictly inner*. The set of ρ-biderivations $\operatorname{Der}(\rho)$ forms a $\mathbb{K}$-vector space. The set of inner biderivations $\operatorname{Der}_{\mathrm{in}}(\rho)$ and the set of strictly inner biderivations $\operatorname{Der}_{\mathrm{si}}(\rho)$ are also $\mathbb{K}$-vector subspaces. We define the *de Rham module* for ρ to be $\mathrm{H}^1_{\mathrm{DR}}(\rho) := \operatorname{Der}(\rho)/\operatorname{Der}_{\mathrm{si}}(\rho)$.

For each $\delta \in \operatorname{Der}(\rho)$ there is a unique $\mathbb{F}_q$-linear and entire power series $\mathrm{F}_\delta(z) = \sum_{i\geq 1} c_i z^{(i)} \in \mathbb{K}[\![z]\!]$ such that

$$\mathrm{F}_\delta(a(\theta)z) = a(\theta)\mathrm{F}_\delta(z) + \delta_a \operatorname{Exp}_\rho(z), \quad \forall\, a \in \mathbf{A}. \tag{1.2}$$

We call F_δ the *quasi-periodic function* associated to δ. For $\lambda \in \Lambda_\rho$, the value $\mathrm{F}_\delta(\lambda)$ is called a *quasi-period of* ρ. For $u \in \mathbb{K}$ satisfying $\operatorname{Exp}_\rho(u) \in K$ the value $F_\delta(u)$, which is a coordinate of logarithms on quasi-periodic extensions, is called a *quasi-logarithm of* ρ (see [6], [32]).

Since every ρ-biderivation δ is uniquely determined by the image δ_t, a $\mathbb{K}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ is represented by $\{\delta_1, \ldots, \delta_r\}$, where δ_1 is the inner biderivation such that $(\delta_1)_t = \rho_t - \theta$, and $\delta_i(t) = \tau^{i-1}$ for $i = 2, \ldots, r$. We see that $F_{\delta^{(1)}}(z) = \operatorname{Exp}_\phi(z) - z$, and so $F_{\delta^{(1)}}(\lambda) = -\lambda$ for all $\lambda \in \Lambda_\rho$. Thus, if we take $\{\lambda_1, \ldots, \lambda_r\}$ to be an $\mathbf{A}$-basis of Λ_ρ and we set $F_{\tau^{i-1}}(z) := F_{\delta_i}(z)$ for $i = 2, \ldots, r$, then we define the *period matrix of* ρ to be

$$\mathbf{P}_\rho := \begin{pmatrix} \lambda_1 & F_\tau(\lambda_1) & \ldots & F_{\tau^{r-1}}(\lambda_1) \\ \lambda_2 & F_\tau(\lambda_2) & \ldots & F_{\tau^{r-1}}(\lambda_2) \\ \vdots & \vdots & & \vdots \\ \lambda_r & F_\tau(\lambda_r) & \ldots & F_{\tau^{r-1}}(\lambda_r) \end{pmatrix}. \tag{1.3}$$

Using Papanikolas' result [34, Thm. 1.1.7], Chang and Papanikolas proved algebraic independence of periods and quasi-periods of Drinfeld modules. We define $\operatorname{End}(\rho) := \{x \in \mathbb{K} \mid x\Lambda_\rho \subseteq \Lambda_\rho\}$ and let K_ρ be its fraction field.

Theorem 1 (Chang-Papanikolas [12, Thm. 1.2.2])**.** *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over $\overline{k}$. Then*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\,(\mathbf{P}_\rho) = r^2/s,$$

where $s = [K_\rho : k]$.

The first goal of the dissertation is to extend this result to algebraic independence of hyperderivatives of periods and quasi-periods of Drinfeld modules. The hyperderivative $\partial_\theta^j : k \to k$ is defined by $\partial_\theta^j(\theta^m) := \binom{m}{j}\theta^{m-j}$ where $j \geq 1$ and $\binom{m}{j}$ is the binomial coefficient, and it satisfies the product rule $\partial_\theta^j(ab) = \sum_{i_1+i_2=j} \partial_\theta^{i_1}(a)\partial_\theta^{i_2}(b)$ for all $a, b \in k$. Moreover, ∂_θ^j extends uniquely to k^{sep} and k_∞^{sep} (see §2.4 for a detailed review of hyperderivatives).

If the Drinfeld $\mathbf{A}$-module ρ is defined over k^{sep}, then Denis [16, p. 6] showed that the periods and quasi-periods of ρ have coordinates in k_∞^{sep} (see also [32, Lem. 4.1.19]). Therefore, for $n \geq 0$ we can consider $\partial_\theta^n(\mathbf{P}_\rho)$, where we take hyperderivatives entrywise. Our first main result is as follows (restated as Theorem 12):

Theorem 2. *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep} and suppose that K_ρ is separable over k. If $s = [K_\rho : k]$, then for $n \geq 1$, we have*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\left(\mathbf{P}_\rho, \partial_\theta^1(\mathbf{P}_\rho), \ldots, \partial_\theta^n(\mathbf{P}_\rho)\right) = (n+1) \cdot r^2/s.$$

By constructing a suitable t-motive and calculating the dimension of its Galois group, Chang and Papanikolas proved the algebraic independence among logarithms and quasi-logarithms of Drinfeld modules.

Theorem 3 (Chang-Papanikolas [12, Thm. 5.1.5])**.** *Let ρ be a Drinfeld $\mathbf{A}$-module defined over $\overline{k}$. Let $u_1, \ldots, u_w \in \mathbb{K}$ with $\operatorname{Exp}_\rho(u_i) = \alpha_i \in \overline{k}$ for each $i = 1, \ldots, w$ and suppose that $\dim_{K_\rho} \operatorname{Span}_{K_\rho}(\lambda_1, \ldots, \lambda_r, u_1, \ldots, u_w) = r/s + w$. Then,*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\left(\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), u_m, F_{\tau^i}(u_m)\}\right) = r^2/s + rw.$$

Denis [16, p. 6] showed that, for a Drinfeld $\mathbf{A}$-module ρ defined over k^{sep} and ρ-biderivation δ defined over k^{sep}, if $u \in \mathbb{K}$ such that $\operatorname{Exp}_\rho(u) \in k^{\mathrm{sep}}$, then $u \in k_\infty^{\mathrm{sep}}$ and $F_\delta(u) \in k_\infty^{\mathrm{sep}}$. (see also [32, Lem. 4.1.19]). Therefore, for $n \geq 0$ we can consider $\partial_\theta^n(u)$ and $\partial_\theta^n(F_\delta(u))$. Building on Theorem 2 and utilizing Theorem 3, we create suitable t-motives and prove algebraic independence results among hyperderivatives of logarithms and quasi-logarithms of Drinfeld modules. Our second main result is as follows (restated as Theorem 14):

Theorem 4. *Let ρ be a Drinfeld $\mathbf{A}$-module defined over k^{sep}. Suppose that K_ρ is separable over k. Let $u_1, \ldots, u_w \in \mathbb{K}$ with $\operatorname{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $i = 1. \ldots, w$ and suppose that $\dim_{K_\rho} \operatorname{Span}_{K_\rho}(\lambda_1, \ldots, \lambda_r, u_1, \ldots, u_w) = r/s + w$. Then,*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\right) = (n+1)\cdot(r^2/s + rw).$$

Returning to an arbitrary basis $\{\delta_1, \ldots, \delta_r\}$ of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}, we further deduce the following corollary.

Corollary 1. *Let ρ be a Drinfeld $\mathbf{A}$-module defined over k^{sep}. Suppose that K_ρ is separable over k. Let $u_1, \ldots, u_w \in \mathbb{K}$ with $\operatorname{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each $i = 1, \ldots, w$. Let $\delta_1, \ldots, \delta_r$ be a basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep}. If $u_1, \ldots, u_w$ are linearly independent over K_ρ, then for $n \geq 1$ the $(n+1)rw$ quantities*

$$\left\{\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\left(\partial_\theta^s(F_{\delta_j}(u_1)), \partial_\theta^s(F_{\delta_j}(u_2)), \ldots, \partial_\theta^s(F_{\delta_j}(u_w))\right)\right\}$$

are algebraically independent over $\overline{k}$.

1.3 Structure of the dissertation

Since, to prove our main theorems, the key result we use is Papanikolas' theorem [34, Thm. 1.1.7] which shows that the dimension of the Galois group associated to a t-motive is equal to the transcendence degree of the entries of the period matrix of the t-motive, in §2.2 and §2.3 we give necessary background concerning t-motives and t-motivic Galois groups. Next, we give a brief review of hyperderivatives and then present a summary of prolongations of dual t-motives as introduced by Maurischat [28].

It proved to be the case that prolongations provide the necessary framework for the study of hyperderivatives of periods and logarithms of Anderson t-modules. The author and Papanikolas [32] showed that using prolongations of t-motives, one can recover hyperderivatives of entries of the period matrix of the t-motive (see Theorem 7), and so to prove Theorem 2, for $n \geq 1$ we calculate the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$, where $\mathrm{P}_n M_\rho$ is the n-prolongation of the t-motive M_ρ associated to the Drinfeld $\mathbf{A}$-module ρ.

For a Drinfeld $\mathbf{A}$-module ρ defined over K, where $k \subseteq K \subseteq \overline{k}$ with $[K : k] < \infty$, there is a representation $\varphi_t : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_r(\mathbb{F}_q[\![t]\!])$ coming from the Galois action on the t-power torsion points $\rho[t^m] := \{x \in \mathbb{K} \mid \rho_{t^m}(x) = 0\}$. In §4, using Anderson generating functions and φ_t we determine the Galois representation on the t-adic Tate module of the n-th prolongation Anderson t-module $\mathrm{P}_n\rho$ associated to the Drinfeld $\mathbf{A}$-module ρ (Lemma 2), and prove that the image of this Galois representation is naturally contained in the $\mathbf{k}_t$-valued points of $\Gamma_{\mathrm{P}_n M_\rho}$ (Theorem 8).

The main difficulty in proving Theorem 2 is that unlike the Drinfeld module case (see [12, §3]) where Pink's theorem [36, Thm. 0.2] on the openness of the Zariski closure of the image of the Galois representation was used to calculate the Galois group Γ_{M_ρ}, the Zariski closure of the image in our case need not be open. However, by building on information from the $n = 0$ case, employing differential algebraic geometry (Theorem 10) and closely examining the elements of $\Gamma_{\mathrm{P}_n M_\rho}$ (Theorem 9), we are able to compute $\Gamma_{\mathrm{P}_n M_\rho}$ explicitly and prove Theorem 2.

In §5, for $u_1, \ldots, u_w \in \mathbb{K}$ satisfying $\mathrm{Exp}_\rho(u_i) \in k^{\mathrm{sep}}$ we build on results of §4 and utilize the ideas of [12, §4 and §5] to construct new t-motives $Y_{1,n}, \ldots, Y_{w,n}$. These t-motives are constructed using the prolongation t-motive $\mathrm{P}_n M_\rho$ such that the entries of the period matrix of $\oplus_{m=1}^{w} Y_{m,n}$ comprise $\bigcup_{s=0}^{n} \bigcup_{i=1}^{r-1} \bigcup_{m=1}^{w} \{\partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}$. We utilize properties of prolongations and hyperderivatives to prove that there is a surjective map from the Galois group of $Y_{m,n}$ to the Galois group of $Y_{m,\ell}$ for $\ell \leq n$ and $m = 1, \ldots, w$ (Lemma 5). Adapting the ideas of the proof of [12, Thm. 5.1.5] and using Lemma 5, we prove Theorem 4.

2. PRELIMINARIES

2.1 Notation

For $n \in \mathbb{Z}$, recall the n-fold Frobenius twist τ^n from (1.1). In some cases, we will write σ for τ^{-1}. For a matrix $M = (m_{ij})$ with entries in $\mathbb{K}(\!(t)\!)$, we define $M^{(n)}$ by setting $M^{(n)} = (m_{ij}^{(n)})$. Let $\overline{k}(t)[\sigma, \sigma^{-1}]$ be the Laurent polynomial ring over $\overline{k}(t)$ in σ subject to the relation

$$\sigma f = f^{(-1)}\sigma, \quad \forall\, f \in \overline{k}(t).$$

For a field $K \subseteq \mathbb{K}$, recall the twisted power series ring $K[\![\tau]\!]$ and the twisted polynomial ring $K[\tau]$ given by $\tau f = f^{(1)}\tau$ for all $f \in K$. Similarly, we define $K[\![\sigma]\!]$ and $K[\sigma]$ when K is a perfect field. For $b = \sum c_i \tau^i \in \mathbb{K}[\tau]$, we define $b^* := \sum b^{(-i)}\sigma^i \in \mathbb{K}[\sigma]$. If $B = (b_{ij}) \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau]) = \operatorname{Mat}_{e\times d}(\mathbb{K})[\tau]$, then $B^* := (b_{ji}^*)$. Thus, if $B \in \operatorname{Mat}_{e\times d}(\mathbb{K}[\tau])$ and $C \in \operatorname{Mat}_{d\times h}(\mathbb{K}[\tau])$, then $(BC)^* = C^*B^*$. Moreover, if $B = \beta_0 + \beta_1\tau + \cdots + \beta_\ell\tau^\ell$, then we set $\mathrm{d}B := \beta_0$.

2.2 Dual t-motives and t-motives

The reader is directed to [34] for details. A *pre-t-motive* M is a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module that is finite dimensional over $\overline{k}(t)$. We denote by $\mathcal{P}$ the category of pre-t-motives whose morphisms are the left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module homomorphisms. Let $\boldsymbol{m} \in \operatorname{Mat}_{r\times 1}(M)$ be so that its entries consist of a $\overline{k}(t)$-basis of M. Then, there is a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ such that

$$\sigma\boldsymbol{m} = \Phi\boldsymbol{m},$$

where the action of σ on $\boldsymbol{m}$ is entry-wise. We say that M is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ such that

$$\Psi^{(-1)} = \Phi\Psi.$$

The matrix Ψ is called a rigid analytic trivialization for Φ. Set $M^\dagger := \mathbb{L} \otimes_{\overline{k}(t)} M$, where we give $M^\dagger$ a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module by letting σ act diagonally:

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad \forall\, f \in \overline{k}(t),\ m \in M.$$

If we let

$$M^B := (M^\dagger)^\sigma := \{\mu \in M^\dagger : \sigma\mu = \mu\},$$

then M^B is a finite dimensional vector space over $\mathbf{k}$, and $M \mapsto M^B$ is a covariant functor from $\mathcal{P}$ to the category of $\mathbf{k}$-vector spaces. The natural map $\mathbb{L} \otimes_{\overline{k}(t)} M^B \to M^\dagger$ is an isomorphism if and only if M is rigid analytically trivial (see [34, §3.3]). If Ψ is a rigid analytic trivialization of Φ, then the entries of $\Psi^{-1}\boldsymbol{m}$ form a $\mathbf{k}$-basis for M^B (see [34, Thm. 3.3.9(b)]). Then, by [34, Thm. 3.3.15], the category of rigid analytically trivial pre-t-motives $\mathcal{R}$ forms a neutral Tannakian category over $\mathbf{k}$ with fiber functor $M \mapsto M^B$. Its trivial object is denoted by $\mathbf{1}$.

We now consider $\mathbf{A}$-finite dual t-motives, which were first introduced in [2] (see also [24], [32]). A *dual t-motive* $\mathcal{M}$ is a left $\overline{k}[t, \sigma]$-module that is free and finitely generated as a left $\overline{k}[\sigma]$-module and such that $(t-\theta)^s\mathcal{M} \subseteq \sigma\mathcal{M}$ for $s \in \mathbb{N}$ sufficiently large. If, in addition, $\mathcal{M}$ is free and finitely generated as a left $\overline{k}[t]$-module, then $\mathcal{M}$ is said to be $\mathbf{A}$-*finite*. Thus, if $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(\mathcal{M})$ is a $\overline{k}[t]$-basis for $\mathcal{M}$, then there is a matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ such that $\sigma(\boldsymbol{m}) = \Phi\boldsymbol{m}$ with $\det\Phi = c(t-\theta)^s$ for some $c \in \overline{k}^\times, s \geq 0$. We say that $\mathcal{M}$ is *rigid analytically trivial* if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$. In [2], the term "dual t-motives" is used for $\mathbf{A}$-finite dual t-motives. We will consider both dual t-motives and $\mathbf{A}$-finite dual t-motives [24].

Given an $\mathbf{A}$-finite dual t-motive $\mathcal{M}$,

$$M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}$$

is a pre-t-motive where $\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m$. Then, $\mathcal{M} \mapsto M$ is a functor from the category of $\mathbf{A}$-finite dual t-motives to the category of pre-t-motives.

The category $\mathcal{A}^I$ of $\mathbf{A}$-*finite dual t-motives up to isogeny*, is the category whose objects are generated by $\mathbf{A}$-finite dual t-motives and morphisms are defined as follows: for $\mathbf{A}$-finite dual t-motives $\mathcal{M}$ and $\mathcal{N}$, $\operatorname{Hom}_{\mathcal{A}^I}(\mathcal{M},\mathcal{N}) := \operatorname{Hom}_{\overline{k}[t,\sigma]}(\mathcal{M},\mathcal{N}) \otimes_{\mathbf{A}} \mathbf{k}$. We further define the full subcategory $\mathcal{AR}^I$ of rigid analytically trivial $\mathbf{A}$-finite dual t-motives up to isogeny by restriction. Then, the functor $\mathcal{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$ is fully faithful (see [34, Thm. 3.4.9]) and we define $\mathcal{T}$, the category of *t-motives*, to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of this functor.

For a t-motive M, we let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{T}$ generated by M. As $\mathcal{T}_M$ is a neutral Tannakian category over $\mathbf{k}$, there is an affine group scheme Γ_M over $\mathbf{k}$, a subgroup of $\operatorname{GL}_r/\mathbf{k}$, so that $\mathcal{T}_M$ is equivalent to the category of finite dimensional representations of Γ_M over $\mathbf{k}$, i.e., $\mathcal{T}_M \approx \mathbf{Rep}(\Gamma_M, \mathbf{k})$ (see [34, §3.5]).

2.3 The difference Galois group

For this section, the reader is directed to [34] for details. A triple of fields $F \subseteq K \subseteq L$ along with an automorphism $\sigma : L \to L$ is said to be *σ-admissible* if σ restricts to automorphisms of F and K; $F = F^\sigma = K^\sigma = L^\sigma$; and L is a separable extension of K. For a fixed σ-admissible triple of fields (F, K, L), we fix $\Phi \in \operatorname{GL}_r(K)$ and suppose that $\Psi \in \operatorname{GL}_r(L)$ satisfies

$$\sigma(\Psi) = \Phi\Psi.$$

We define a K-algebra homomorphism $\nu : K[X, 1/\det X] \to L$ by setting $\nu(X_{ij}) := \Psi_{ij}$, where $X = (X_{ij})$ is an $r \times r$ matrix of independent variables. We let

$$\mathfrak{p} := \ker \nu, \quad \Sigma := \operatorname{Im} \nu = K[\Psi, 1/\det \Psi] \subseteq L.$$

Setting $Z_\Psi = \operatorname{Spec} \Sigma$, we see that Z_Ψ is the smallest closed subscheme of $\operatorname{GL}_{r/K}$ such that $\Psi \in Z_\Psi(L)$.

Set $\Psi_1, \Psi_2 \in \operatorname{GL}_r(L \otimes_K L)$ to be such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let

$\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$. We define an F-algebra homomorphism $\mu : F[X, 1/\det X] \to L \otimes_K L$ by setting $\mu(X_{ij}) := \widetilde{\Psi}_{ij}$. We let

$$\mathfrak{q} := \ker \mu, \quad \Delta := \mathrm{Im}\, \mu.$$

Setting $\Gamma_\Psi = \mathrm{Spec}\ \Delta$, we see that Γ_Ψ is the smallest closed subscheme of $\mathrm{GL}_{r/K}$ such that $\widetilde{\Psi} \in \Gamma_\Psi(L \otimes_K L)$.

Now suppose that M is a t-motive and let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represent multiplication by σ on M. Let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfy $\Psi^{(-1)} = \Phi\Psi$. Using the σ-admissible triple $(F, K, L) = (\mathbf{k}, \overline{k}(t), \mathbb{L})$, the following properties hold.

Theorem 5 (Papanikolas [34, §4])**.** *Let M be a t-motive, and let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represent multiplication by σ on M. Let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfy $\Psi^{(-1)} = \Phi\Psi$.*

(a) Γ_Ψ *is a closed* $\mathbf{k}$*-subgroup scheme of* $\mathrm{GL}_{r/\mathbf{k}}$.

(b) The closed $\overline{k}(t)$-subscheme Z_Ψ is stable under right-multiplication by $\overline{k}(t) \times_{\mathbf{k}} \Gamma_\Psi$ and is a $\overline{k}(t) \times_{\mathbf{k}} \Gamma_\Psi$-torsor over $\overline{k}(t)$. In particular, $\Gamma_\Psi(\overline{\mathbb{L}}) = \Psi^{-1} Z_\Psi(\overline{\mathbb{L}})$.

(c) Z_Ψ is absolutely irreducible and smooth over $\overline{k(t)}$.

(d) The $\mathbf{k}$-scheme Γ_Ψ is absolutely irreducible and smooth over $\overline{\mathbf{k}}$.

(e) $\dim \Gamma_\Psi = \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi$*, where Λ_Ψ is the fraction field of Σ_Ψ.*

(f) $\Gamma_\Psi \cong \Gamma_M$ *over* $\mathbf{k}$.

Furthermore, the main theorem of [34] is as follows.

Theorem 6 (Papanikolas [34, Thm. 1.1.7])**.** *Let M be a t-motive, and let Γ_M be its Galois group. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that* $\det \Phi = c(t-\theta)^s, c \in \overline{k}^\times$. *Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{T})$. Then,* $\mathrm{tr.\,deg}_{\overline{k}}\, \overline{k}(\Psi|_{t=\theta}) = \dim \Gamma_M$.

2.4 Hyperderivatives and Hyperdifferential operators

In this section, we review results concerning hyperderivatives and hyperdifferential operators. The reader is directed to [3], [26] and [32] for more details and proofs. For $m, j \geq 0$, let $\binom{m}{j} \in \mathbb{N}$ be the binomial coefficient. Then, for F a field where θ is transcendental over F, the F-linear map $\partial_\theta^j : F[\theta] \to F[\theta]$ defined by setting $\partial_\theta^j(\theta^m) = \binom{m}{j}\theta^{m-j}$ is called the *j-th hyperdifferential operator with respect to θ*. For each $f \in F[\theta]$, we call $\partial_\theta^j(f)$ the *j-th hyperderivative* of f. The definition of ∂_θ^j extends naturally to $\partial_\theta^j : F[\![\theta]\!] \to F[\![\theta]\!]$. The hyperdifferential operators satisfy various identities including the product rule $\partial_\theta^j(fg) = \sum_{i=0}^{j} \partial_\theta^i(f)\partial_\theta^{j-i}(g)$ and the composition rule $\partial_\theta^i(\partial_\theta^j(f)) = \binom{i+j}{j}\partial_\theta^{i+j}(f)$.

The product rule extends ∂_θ^j to the Laurent series field $F(\!(\theta)\!)$ where as usual for $m > 0$, we have $\binom{-m}{j} = (-1)^j\binom{m+j-1}{j}$. Similarly, for a place v of $F(\theta)$ there is a unique extension $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$. The formulas that we will use are summarized in the following proposition.

Proposition 1 (see Brownawell [3, §7], Jeong [26, §2], [32, Lem. 2.4.3])**.** *Let F be a field of characteristic $p > 0$, and let v be a place of $F(\theta)$. The hyperdifferential operators with respect to θ, $\partial_\theta^j : F(\theta)_v^{\mathrm{sep}} \to F(\theta)_v^{\mathrm{sep}}$, $j \geqslant 0$, satisfy the following.*

(a) For $f_1, \ldots, f_s \in F(\theta)_v^{\mathrm{sep}}$ and $j \geq 0$,

$$\partial_\theta^j(f_1 \cdots f_s) = \sum_{\substack{k_1,\ldots,k_s \geq 0 \\ k_1+\cdots+k_s=j}} \partial_\theta^{k_1}(f_1)\cdots\partial_\theta^{k_s}(f_s).$$

(b) For $f \in F(\theta)_v^{\mathrm{sep}}$, $n \geqslant 0$, and $j \geq 1$,

$$\partial_\theta^j\left(f^{p^n}\right) = \begin{cases} \left(\partial_\theta^k(f)\right)^{p^n} & \text{if } j = kp^n, \\ 0 & \text{if } p^n \nmid j. \end{cases}$$

(c) For $f \in F(\theta)_v^{\mathrm{sep}}$ and $j \geq 1$,

$$\partial_\theta^j\left(\frac{1}{f}\right) = \sum_{k=1}^{j} \binom{j+1}{k+1} \frac{(-1)^k}{f^{k+1}} \partial_\theta^j(f^k).$$

For $f \in F(\theta)_v^{\mathrm{sep}}$ and $n \geq 1$, we define the *d-matrix with respect to* θ, $d_{\theta,n}[f] \in \mathrm{Mat}_n(F(\theta)_v^{\mathrm{sep}})$ to be the upper-triangular $n \times n$ matrix

$$d_{\theta,n}[f] := \begin{pmatrix} f & \partial_\theta^1(f) & \dots & \dots & \partial_\theta^{n-1}(f) \\ & f & \partial_\theta^1(f) & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \partial_\theta^1(f) \\ & & & & f \end{pmatrix}. \tag{2.1}$$

Using the product rule, it is straightforward to see that $d_{\theta,n}[g] \cdot d_{\theta,n}[f] = d_{\theta,n}[gf]$. For a matrix $B := (b_{ij}) \in \mathrm{Mat}_{e\times d}(F(\theta)_v^{\mathrm{sep}})$, we also define the d-matrix with respect to θ, $d_{\theta,n}[B] \in \mathrm{Mat}_{ne\times nd}(F(\theta)_v^{\mathrm{sep}})$ in (2.1), where for $c \geq 0$ we let $\partial_\theta^c(B) := (\partial_\theta^c(b_{ij})) \in \mathrm{Mat}_{e\times d}(F(\theta)_v^{\mathrm{sep}})$.

We further define *partial hyperderivatives* for n independent variables $\theta_1, \theta_2 \dots, \theta_n$ to be the F-linear maps

$$\partial_{\theta_\ell}^j : F(\theta_1, \theta_2, \dots, \theta_n) \to F(\theta_1, \theta_2, \dots, \theta_n), \quad j \geq 0, \;\; \ell = 1, \dots, n$$

such that for $m \in \mathbb{Z}$ and $u, v = 1, \dots, n, u \neq v$, we have $\partial_{\theta_u}^j(\theta_u^m) = \binom{m}{j}\theta_u^{m-j}$ and $\partial_{\theta_u}^j(\theta_v^m) = 0$. Thus, we have that $\partial_{\theta_u} \circ \partial_{\theta_v} = \partial_{\theta_v} \circ \partial_{\theta_u}$. We can define the d-matrix with respect to each independent variable. In this dissertation, we make use of partial hyperderivatives for two variables. For our convenience, we let the two independent variables be t and θ.

2.5 Prolongation of dual t-motives

We review the construction of new dual t-motives and t-modules from old ones, called prolongations, as introduced by Maurischat [28]. For a left $\overline{k}[t, \sigma]$-module $\mathcal{M}$ and $n \geq 0$, we define the

n-th prolongation of $\mathcal{M}$ to be the left $\overline{k}[t]$-module $\mathrm{P}_n\mathcal{M}$ generated by symbols $D_i m$, for $m \in \mathcal{M}$ and $i = 0, \ldots, n$ subject to the relations

(a) $D_i(m_1 + m_2) = D_i m_1 + D_i m_2$,

(b) $D_i(a \cdot m) = \sum_{i=i_1+i_2} \partial_t^{i_1}(a) \cdot D_{i_2} m$,

where $m, m_1, m_2 \in \mathcal{M}$ and $a \in \overline{k}[t]$. The σ action on $\mathrm{P}_n\mathcal{M}$ given by

$$\sigma(a \cdot D_i m) = a^{(-1)} \cdot D_i(\sigma m)$$

where $a \in \overline{k}[t]$, $m \in \mathcal{M}$, is well defined and thus, the n-th prolongation $\mathrm{P}_n\mathcal{M}$ of $\mathcal{M}$ is also a left $\overline{k}[t, \sigma]$-module.

Via $D_0 m \mapsto m$, we see that $\mathrm{P}_0\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$ and for $0 \leq \ell < n$, the ℓ-th prolongation $\mathrm{P}_\ell\mathcal{M}$ is a $\overline{k}[t, \sigma]$-submodule of $\mathrm{P}_n\mathcal{M}$. Thus, we obtain a short exact sequence of $\overline{k}[t, \sigma]$-modules

$$0 \to \mathrm{P}_\ell\mathcal{M} \to \mathrm{P}_n\mathcal{M} \xrightarrow{\mathbf{pr}} \mathrm{P}_{n-\ell-1}\mathcal{M} \to 0 \tag{2.2}$$

where $\mathbf{pr}(D_i m) := D_{i-\ell-1} m$ for $i > \ell$ and $\mathbf{pr}(D_i m) := 0$ for $i \leq \ell$ and $m \in \mathcal{M}$.

If $\mathcal{M}$ is an $\mathbf{A}$-finite dual t-motive, then $\mathrm{P}_n\mathcal{M}$ is an $\mathbf{A}$-finite dual t-motive (see [28, Thm. 3.4]). Thus, if $\boldsymbol{m} = [m_1, \ldots, m_r]^\mathsf{T}$ is a $\overline{k}[t]$-basis of $\mathcal{M}$, then a $\overline{k}[t]$-basis of $\mathrm{P}_n\mathcal{M}$ is given by

$$\boldsymbol{D}_n \boldsymbol{m} := (D_n \boldsymbol{m}^\mathsf{T}, D_{n-1} \boldsymbol{m}^\mathsf{T}, \ldots, \ldots, D_0 \boldsymbol{m}^\mathsf{T})^\mathsf{T} \in \mathrm{Mat}_{r(n+1)\times 1}(\overline{k}[t]), \tag{2.3}$$

where $D_i \boldsymbol{m} := (D_i m_1, \ldots, D_i m_r)^\mathsf{T} \in \mathrm{Mat}_{r\times 1}(\overline{k}[t])$ for each i (see [28, Prop. 4.2]). Also, if $\Phi \in \mathrm{GL}_r(\overline{k}[t])$ represents multiplication by σ on $\boldsymbol{m}$, then

$$\sigma(\boldsymbol{D}_n \boldsymbol{m}) = d_{t,n+1}[\Phi] \cdot \boldsymbol{D}_n \boldsymbol{m}. \tag{2.4}$$

If $\mathcal{M}$ is rigid analytically trivial with $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$, then since twisting commutes differentiation with respect to t, we have

$$(d_{t,n+1}[\Psi])^{(-1)} = d_{t,n+1}[\Psi^{(-1)}] = d_{t,n+1}[\Phi\Psi] = d_{t,n+1}[\Phi]d_{t,n+1}[\Psi] \tag{2.5}$$

Therefore, $\mathrm{P}_n\mathcal{M}$ is rigid analytically trivial. See [28, §3] for detailed proofs.

3. RIGID ANALYTIC TRIVIALIZATIONS, PROLONGATIONS AND HYPERDERIVATIVES

3.1 Anderson t-modules and associated dual t-motives

For a field $K \subseteq \mathbb{K}$, an *Anderson t-module* defined over K is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : \mathbf{A} \to \mathrm{Mat}_d(K[\tau])$$

such that $\phi_t = B_0 + B_1\tau + \cdots + B_\ell \tau^\ell$, where $B_i \in \mathrm{Mat}_d(K)$ and $\mathrm{d}\phi_t = B_0 = \theta I_d + N$ with I_d, the $d \times d$ identity matrix and N, a nilpotent matrix. Then, ϕ defines an $\mathbf{A}$-module structure on $\mathbb{K}^d$ via

$$a \cdot \boldsymbol{x} = \phi_a(\boldsymbol{x}), \quad \forall\, a \in \mathbf{A},\ \boldsymbol{x} \in \mathbb{K}^d.$$

We say that d is the dimension of ϕ. There exists a unique power series $\mathrm{Exp}_\phi(\boldsymbol{z}) = \sum_{i=0}^\infty C_i \boldsymbol{z}^{q^i} \in \mathbb{K}[\![z_1, \ldots, z_d]\!]^d$, $\boldsymbol{z} = [z_1, \ldots, z_d]^\mathsf{T}$ so that $C_0 = I_d$ and satisfies

$$\mathrm{Exp}(\mathrm{d}\phi_a \boldsymbol{z}) = \phi_a(\mathrm{Exp}(\boldsymbol{z}))$$

for all $a \in \mathbf{A}$. Moreover, $\mathrm{Exp}_\phi(\boldsymbol{z})$ defines an entire function $\mathrm{Exp}_\phi : \mathbb{K}^d \to \mathbb{K}^d$. If Exp_ϕ is surjective, then we say that ϕ is *uniformizable*. The kernel $\Lambda_\phi \subseteq \mathbb{K}^d$ of Exp_ϕ is a free and finitely generated discrete $\mathbf{A}$-submodule and it is called the *period lattice* of ϕ. If ϕ is uniformizable, then we have an isomorphism $\mathbb{K}^d/\Lambda_\phi \cong (\mathbb{K}^d, \phi)$, where $(\mathbb{K}^d, \phi)$ denotes $\mathbb{K}^d$ together with the $\mathbf{A}$-module structure defined as above coming from ϕ. If $\phi_t = B_0 \in \mathrm{Mat}_d(K)$, then ϕ is said to be a *trivial* Anderson t-module.

We define the dual t-motive $\mathcal{M}_\phi$ associated to the t-module ϕ in the following way. We let $\mathcal{M}_\phi := \mathrm{Mat}_{1\times d}(\overline{k}[\sigma])$. To give $\mathcal{M}_\phi$ the $\overline{k}[t, \sigma]$-module structure, we set

$$a \cdot m = m\phi_a^*, \quad m \in \mathcal{M}_\phi, a \in \mathbf{A}, \tag{3.1}$$

where ϕ_a^* is defined as in §2.1. Then, as in [7, §4.4], [24] and [32, §2.3], $\mathcal{M}_\phi$ defines a dual t-motive and (3.1) gives a unique correspondence between a t-module and its associated dual t-motive. The reader is directed to [24], [32] for more information on dual t-motives associated to t-modules.

A non-trivial Anderson t-module of dimension 1 is called a *Drinfeld* $\mathbf{A}$*-module*. We now fix a Drinfeld $\mathbf{A}$-module $\rho : \mathbf{A} \to k^{\mathrm{sep}}[\tau]$ such that

$$\rho_t = \theta + \kappa_1\tau + \cdots + \kappa_r\tau^r,$$

where $\kappa_r \neq 0$. We call r the rank of ρ. Drinfeld modules are uniformizable and the rank of the period lattice Λ_ρ of ρ as an $\mathbf{A}$-module is r. As defined above for t-modules, we define $\mathcal{M}_\rho := \overline{k}[\sigma]$. Then $\{m_1, m_2, \ldots, m_r\} = \{1, \sigma, \ldots, \sigma^{r-1}\}$ forms a $\overline{k}[t]$-basis for $\mathcal{M}_\rho$ (see [12, §3.3], [32, Example 3.5.13]) and with respect to this basis, multiplication by σ on $\mathcal{M}_\rho$ is

$$\Phi_\rho := \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ (t-\theta)/\kappa_r^{(-r)} & -\kappa_1^{(-1)}/\kappa_r^{(-r)} & \ldots & -\kappa_{r-1}^{(-r+1)}/\kappa_r^{(-r)} \end{pmatrix}. \tag{3.2}$$

Recall from §2.2 that if $\mathcal{M}_\rho$ is free and finitely generated as a left $\overline{k}[t]$-module, then $\mathcal{M}_\rho$ is said to be $\mathbf{A}$-finite. Thus, $\mathcal{M}_\rho$ is an $\mathbf{A}$-finite dual t-motive. We let $M_\rho := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}_\rho$ be the pre-t-motive associated to $\mathcal{M}_\rho$.

Since we can associate an $\mathbf{A}$-finite dual t-motive to a Drinfeld $\mathbf{A}$-module, by §2.5 we can define an Anderson t-module $\mathrm{P}_n\rho$ for $n \geq 0$ which has as an associated $\mathbf{A}$-finite dual t-motive the n-th prolongation $\mathrm{P}_n\mathcal{M}_\rho$ of $\mathcal{M}_\rho$ (see for details, [28, §5]). The *n-th prolongation t-module* $\mathrm{P}_n\rho : \mathbf{A} \to \mathrm{Mat}_{n+1}(k^{\mathrm{sep}}[\tau])$ of the Drinfeld $\mathbf{A}$-module ρ is of dimension $n+1$ and is defined by

$$(\mathrm{P}_n\rho)_t = \mathrm{d}(\mathrm{P}_n\rho)_t + \mathrm{diag}(\kappa_1)\tau + \cdots + \mathrm{diag}(\kappa_r)\tau^r$$

where

$$
\mathrm{d}(\mathrm{P}_n\rho)_t = \begin{pmatrix} \theta & & & & \\ -1 & \ddots & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & \theta \end{pmatrix}, \tag{3.3}
$$

and $\mathrm{diag}(\kappa_i)$ is the diagonal matrix with diagonal entries all equal to κ_i for each i. We also let $\mathrm{P}_n M_\rho := \overline{k}(t) \otimes_{\overline{k}[t]} \mathrm{P}_n \mathcal{M}_\rho$ be the pre-t-motive associated to $\mathrm{P}_n\mathcal{M}_\rho$. By [32, §5], we have that for $\boldsymbol{z} = [z_0, \dots, z_n]^\mathsf{T} \in \mathbb{K}^{n+1}$,

$$
\mathrm{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{z}) = [\mathrm{Exp}_\rho(z_0), \dots, \mathrm{Exp}_\rho(z_n)]^\mathsf{T}. \tag{3.4}
$$

Set for $z \in \mathbb{K}$, $(z)_j := [0, \dots, 0, z, 0, \dots, 0]^\mathsf{T} \in \mathbb{K}^{n+1}$, where z is in the j-th entry and all other entries are 0. Thus, if $\{\lambda_1, \dots, \lambda_r\}$ is an $\mathbf{A}$-basis of Λ_ρ, then $\{(\lambda_i)_j \mid i = 1, \dots, r \text{ and } 1 \leq j \leq n+1\}$ forms an $\mathbf{A}$-basis of $\Lambda_{\mathrm{P}_n\rho}$, the period lattice of $\mathrm{P}_n\rho$.

For Drinfeld $\mathbf{A}$-modules ρ and ρ' defined over $K \subseteq \mathbb{K}$, a *morphism* $b : \rho \to \rho'$ is a twisted polynomial $b \in \mathbb{K}[\tau]$ such that

$$
b\rho_a = \rho'_a b \quad \forall\, a \in \mathbf{A}.
$$

We say that b is defined over $L \subseteq \mathbb{K}$ if $b \in L[\tau]$. Let $b : \rho \to \rho'$ be a morphism defined over $L \subseteq \overline{k}$. Then, b induces a morphism $B : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ of $\mathbf{A}$-finite dual t-motives in the following way. If $b = \sum c_i \tau^i \in L[\tau]$, then recall from §2.1 that $b^* = \sum c_i^{(-i)} \sigma^i$. As in [11, Lem. 2.4.2], B is the $\overline{k}[\sigma]$-linear map such that $B(1) = b^*$. The map

$$
\mathrm{End}(\rho) \;\to\; \{c \in \mathbb{K} \mid c\Lambda_\rho \subseteq \Lambda_\rho\} : \sum_i c_i\tau^i \mapsto c_0
$$

is an isomorphism. Throughout this dissertation, we identify $\mathrm{End}(\rho)$ with the image of this map and let K_ρ denote its fraction field. The following result is due to Anderson.

Proposition 2 (See Chang-Papanikolas [12, Prop. 3.3.2, Cor. 3.3.3])**.** *The functor $\rho \to \mathcal{M}_\rho$ from the category of Drinfeld $\mathbf{A}$-modules defined over $K \subseteq \overline{k}$ to the category of $\mathbf{A}$-finite dual t-motives is fully faithful. Moreover, for any Drinfeld $\mathbf{A}$-module ρ defined over $K \subseteq \overline{k}$,*

$$\mathrm{End}(\rho) \cong \mathrm{End}_{\overline{k}[t,\sigma]}(\mathcal{M}_\rho), \quad K_\rho \cong \mathrm{End}_{\mathcal{T}}(M_\rho),$$

and M_ρ is a simple left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module.

3.2 Rigid analytic trivializations of Drinfeld modules, and prolongations

We continue with our choice of Drinfeld $\mathbf{A}$-module ρ of rank r defined over k^{sep}. By constructing the rigid analytic trivialization Ψ_ρ, we show that its associated $\mathbf{A}$-finite dual t-motive $\mathcal{M}_\rho$ is rigid analytically trivial, and then extend to the prolongation t-modules $\mathrm{P}_n\rho$. The details regarding Drinfeld modules can be found in [12, §3.4] and [32, §4.4] (see [32, Prop 4.4.11]). For $u \in \mathbb{K}$, we define the *Anderson generating function*

$$f_u(t) := \sum_{m=0}^{\infty} \mathrm{Exp}_\rho\left(\frac{u}{\theta^{m+1}}\right) t^m = \sum_{i=0}^{\infty} \frac{\alpha_i u^{q^i}}{\theta^{q^i} - t} \in \mathbb{T}, \tag{3.5}$$

where $\mathrm{Exp}_\rho(z) = \sum_{i=0}^{\infty} \alpha_i z^{q^i}$ with $\alpha_0 = 1$. The last equality is due to Pellarin [35, §4]. Since $\rho_t(\mathrm{Exp}_\rho(u/\theta^{m+1})) = \mathrm{Exp}_\rho(u/\theta^m)$, we have

$$\kappa_1 f_u^{(1)}(t) + \cdots + \kappa_{r-1} f_u^{(r-1)}(t) + \kappa_r f_u^{(r)}(t) = (t - \theta) f_u(t) + \mathrm{Exp}_\rho(u), \tag{3.6}$$

and so

$$\kappa_1 f_u^{(1)}(\theta) + \cdots + \kappa_{r-1} f_u^{(r-1)}(\theta) + \kappa_r f_u^{(r)}(\theta) = -u + \mathrm{Exp}_\rho(u). \tag{3.7}$$

For $\{\lambda_1, \dots, \lambda_r\}$ an $\mathbf{A}$-basis of Λ_ρ, we set $f_i(t) := f_{\lambda_i}(t)$ for each i. Define the matrix

$$\Upsilon = \begin{pmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(r-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(r-1)} \\ \vdots & \vdots & & \vdots \\ f_r & f_r^{(1)} & \dots & f_r^{(r-1)} \end{pmatrix}. \tag{3.8}$$

For each $1 \le i \le r$ and $1 \le j \le r-1$, it follows from [20, p.194], [32, §4.4] and [38, §6.4] that the quasi-period $F_{\tau^j}(\lambda_i)$ of λ_i associated to the biderivation $\delta_j : t \mapsto \tau^j$ is

$$F_{\tau^j}(\lambda_i) = \sum_{m=0}^{\infty} \mathrm{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right)^{q^j} \theta^m = f_i^{(j)}(t)\ |_{t=\theta}\ . \tag{3.9}$$

Moreover, by [12, Lem. 3.4.4] (see also [32, Lem. 4.3.9]), it follows that $\det \Upsilon \neq 0$. As in [12, §3.4], if we let

$$V = \begin{pmatrix} \kappa_1 & \kappa_2^{(-1)} & \dots & \kappa_{r-1}^{(-r+2)} & \kappa_r^{(-r+1)} \\ \kappa_2 & \kappa_3^{(-1)} & \dots & \kappa_r^{(-r+2)} & \\ \vdots & \vdots & & & \\ \kappa_{r-1} & \kappa_r^{(-1)} & & & \\ \kappa_r & & & & \end{pmatrix},$$

and set

$$\Psi_\rho := V^{-1}[\Upsilon^{(1)}]^{-1}, \tag{3.10}$$

then $\Psi^{(-1)} = \Phi_\rho \Psi_\rho$. Thus, the pre-$t$-motive $M_\rho = \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}_\rho$ is rigid analytically trivial and is in $\mathcal{T}$, the category of t-motives.

By (2.5), the n-th prolongation t-motive $\mathrm{P}_n M_\rho = \overline{k}(t) \otimes_{\overline{k}[t]} \mathrm{P}_n \mathcal{M}_\rho$ is rigid analytically trivial and $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$. Thus,

$$\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[V]^{-1} d_{t,n+1}[\Upsilon^{(1)}]^{-1}.$$

3.3 Hyperderivatives of periods and logarithms, and prolongations

For this section also, we continue with our choice of Drinfeld $\mathbf{A}$-module ρ of rank r defined over k^{sep}. In [32], Papanikolas and the author studied extensively the periods, quasi-periods, logarithms and quasi-logarithms of prolongations of abelian and $\mathbf{A}$-finite Anderson t-modules. For $\delta \in \mathrm{Der}(\phi)$ such that $\delta_t = \sum_{h=1}^{v} b_h \tau^h \in \mathbb{K}[\tau]\tau$, we set $\delta_t(f_u(t)) = \sum_{h=1}^{v} b_h f_u^{(h)}(t)$. The statement of the result from [32] for Drinfeld $\mathbf{A}$-modules is as follows.

Theorem 7 (Namoijam-Papanikolas [32]). *Fix $n \geq 0$. Let ρ be a Drinfeld $\mathbf{A}$-module defined over k^{sep}. If $\{\delta_1, \ldots, \delta_r\}$ is a basis of $\mathrm{H}^1_{\mathrm{DR}}(\rho)$ defined over k^{sep} and $\{\lambda_1, \ldots, \lambda_r\}$ is an $\mathbf{A}$-basis of the period lattice Λ_ρ, then we have*

$$\mathrm{Span}_{\overline{k}}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r}\bigcup_{j=1}^{r}\{\partial_\theta^s(F_{\delta_i}(\lambda_j))\}\right) = \mathrm{Span}_{\overline{k}}\left(d_{t,n+1}[\Psi_\rho]^{-1}|_{t=\theta}\right).$$

Moreover, if $u \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u) \in k^{\mathrm{sep}}$, then

$$\mathrm{Span}_{\overline{k}}\left(\bigcup_{s=1}^{n}\bigcup_{i=1}^{r}\big\{(\delta_i)_t(f_u(t)), \partial_t^s((\delta_i)_t(f_u(t)))\big\}|_{t=\theta}\right) = \mathrm{Span}_{\overline{k}}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r}\big\{\partial_\theta^s(F_{\delta_i}(u))\big\}\right) \quad (3.11)$$

Thus, by Theorem 6, computing the dimension of the Galois group $\Gamma_{\mathrm{P}_n M_\rho}$ for $n \geq 1$ will enable us to prove algebraic independence results concerning hyperderivatives of periods and quasi-periods of abelian and $\mathbf{A}$-finite Anderson t-modules. Moreover, by (3.11), if we are able to create appropriate t-motives and determine the dimension of its associated Galois group, then we can prove algebraic independence results concerning hyperderivatives of logarithms and quasi-logarithms of Abelian and $\mathbf{A}$-finite Anderson t-modules.

4. HYPERDERIVATIVES OF PERIODS AND QUASI-PERIODS

In this chapter, we prove Theorem 2. To prove this theorem, we first show in Theorem 10 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1) \cdot r^2/s$, and in Theorem 11 that $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1) \cdot r^2/s$. Moreover, in Corollary 3 we explicitly compute $\Gamma_{\mathrm{P}_n M_\rho}$ for all $n \geq 1$.

4.1 The t-adic Tate module, Anderson generation functions and prolongations

For parts of this section, we adapt the methods in [12, §3.4]. Let ϕ be a uniformizable, $\mathbf{A}$-finite Anderson t-module of rank r. Then, for any $a \in \mathbf{A}$, the torsion $\mathbf{A}$-module $\phi[a] := \{\boldsymbol{x} \in \mathbb{K}^d \mid \phi_a(\boldsymbol{x}) = 0\}$ is isomorphic to $\big(\mathbf{A}/(a)\big)^{\oplus r}$. We define the *$t$-adic Tate module*

$$T_t(\phi) := \varprojlim_m \phi[t^m] \cong \mathbf{A}_t^{\oplus r}.$$

Now, we fix a Drinfeld $\mathbf{A}$-module ρ of rank r. If ρ is defined over K such that $k \subseteq K \subseteq \overline{k}$ and $[K : k] < \infty$, then every element of $\rho[t^m]$ is separable over K. Thus, the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the separable closure of K inside $\overline{k}$ acts on $T_t(\rho)$, defining a representation

$$\varphi_t : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_t(\rho)) \cong \mathrm{GL}_r(\mathbf{A}_t).$$

We fix an $\mathbf{A}$-basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ_ρ and define

$$\xi_{i,m} := \mathrm{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right) \in \rho[t^{m+1}],$$

for $1 \leq i \leq r$ and $m \geq 0$. We define $\{x_1, \ldots, x_r\}$ to be the $\mathbf{A}_t$-basis of $T_t(\rho)$, where we set $x_i := (\xi_{i,0}, \xi_{i,1}, \xi_{i,2}, \ldots)$. Then, for $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, there exists $g_\epsilon \in \mathrm{GL}_r(\mathbb{F}_q[\![t]\!])$ such that

$$\varphi_t(\epsilon)\mathbf{x} = g_\epsilon \mathbf{x}, \tag{4.1}$$

where $\mathbf{x} = [x_1, \ldots, x_r]^{\mathsf{T}}$.

For the remainder of this section, we fix $n \geq 0$. Recall from §3.1 that for $z \in \mathbb{K}$, $(z)_j = [0,\dots,0,z,0,\dots,0]^{\mathsf{T}} \in \mathbb{K}^{n+1}$, and $\{(\lambda_i)_j \mid i = 1,\dots,r \text{ and } 1 \leq j \leq n+1\}$ is an $\mathbf{A}$-basis of $\Lambda_{\mathrm{P}_n\rho}$. We define

$$\chi_{i,j,m} := \mathrm{Exp}_{\mathrm{P}_n\rho}\big((\mathrm{d}(\mathrm{P}_n\rho)_t)^{-(m+1)}(\lambda_i)_j\big).$$

Similar to the case of the Drinfeld $\mathbf{A}$-module ρ, we define $\{y_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq n+1\}$ to be the $\mathbf{A}_t$-basis of $T_t(\mathrm{P}_n\rho)$, where for each i,j, we set $y_{i,j} := \big(\chi_{i,j,0}, \chi_{i,j,1}, \chi_{i,j,2}, \dots\big)$.

For each $i = 1,\dots,r$, by (3.5) the Anderson generating function of ρ with respect to λ_i is

$$f_i(t) = \sum_{m=0}^{\infty} \mathrm{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1}}\right)t^m = \sum_{m=0}^{\infty} \xi_{i,m}t^m \in K^{\mathrm{sep}}[\![t]\!].$$

The Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $K^{\mathrm{sep}}[\![t]\!]$ by acting on each coefficient. The following lemma shows that the induced Galois action on $f_i(t)$ and its Frobenius twists as elements of $K^{\mathrm{sep}}[\![t]\!]$ are compatible with its action on them as elements of $T_t(\rho)$. Set $\mathbf{f} = [f_1,\dots,f_r]^{\mathsf{T}}$.

Lemma 1 (Chang, Papanikolas [12, Lem. 3.3.2, Cor. 3.2.4]). *For any $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, we have $\epsilon(\mathbf{f}) = g_\epsilon\mathbf{f}$, where $\epsilon(\mathbf{f}) = [\epsilon(f_1),\dots,\epsilon(f_r)]^{\mathsf{T}}$. Moreover, for $1 \leq j \leq r-1$, we have $\epsilon(\mathbf{f}^{(j)}) = g_\epsilon\mathbf{f}^{(j)}$.*

We want to extend Anderson generating functions to Anderson generating functions of the prolongation t-module $\mathrm{P}_n\rho$. Anderson generating functions of general Anderson t-modules have been extensively studied in [21], [27], [32]. For each $i = 1,\dots,r$ and $j = 1,\dots,n+1$, we define the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $(\lambda_i)_j$ to be

$$\mathcal{G}_{i,j}(t) := \sum_{m=0}^{\infty} \mathrm{Exp}_{\mathrm{P}_n\rho}\big((\mathrm{d}(\mathrm{P}_n\rho)_t)^{-(m+1)}(\lambda_i)_j\big)t^m = \sum_{m=0}^{\infty} \chi_{i,j,m}t^m \in K^{\mathrm{sep}}[\![t]\!]^{n+1}.$$

Observe that in (3.3), the subdiagonal entries of $\mathrm{d}(\mathrm{P}_n\rho)_t$ are $-\partial_\theta^1(\theta)$. Also, $0 = (-1)^c\partial_\theta^c(\theta)$ for

$c \geq 2$ and so by the product rule of hyperderivatives, for $h \in \mathbb{Z}$ we have

$$(\mathrm{d}(\mathrm{P}_n\rho)_t)^h = \begin{pmatrix} \theta^h & & & & & \\ -\partial_\theta^1(\theta^h) & \theta^h & & & & \\ \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ (-1)^n\partial_\theta^n(\theta^h) & \dots & \dots & \partial_\theta^2(\theta^h) & -\partial_\theta^1(\theta^h) & \theta^h \end{pmatrix}.$$

From this it follows that for $m \geq 0$

$$(\mathrm{d}(\mathrm{P}_n\rho)_t)^{-(m+1)}(\lambda_i)_j = \sum_{c=0}^{n+1-j} \left((-1)^c\partial_\theta^c(\theta^{-(m+1)})\lambda_i\right)_{j+c},$$

and since $(-1)^c\partial_\theta^c(\theta^{-(m+1)}) = \binom{m+c}{c}\theta^{-(m+1+c)}$ for $c \geq 1$, using (3.4), we have

$$\chi_{i,j,m} = \sum_{c=0}^{n+1-j} \binom{m+c}{c}\left(\mathrm{Exp}_\rho\left(\frac{\lambda_i}{\theta^{m+1+c}}\right)\right)_{j+c} = \sum_{c=0}^{n+1-j} \binom{m+c}{c}(\xi_{i,m+c})_{j+c}, \tag{4.2}$$

and

$$\mathcal{G}_{i,j}(t) = \sum_{m=0}^{\infty}\sum_{c=0}^{n+1-j} \binom{m+c}{c}(\xi_{i,m+c})_{j+c}t^m = \sum_{c=0}^{n+1-j}\left(\sum_{m=c}^{\infty}\binom{m}{c}\xi_{i,m}t^{m-c}\right)_{j+c}.$$

Thus,

$$\mathcal{G}_{i,j}(t) = \sum_{c=0}^{n+1-j}\left(\partial_t^c(f_i)\right)_{j+c}. \tag{4.3}$$

Similar to the case of Drinfeld modules (see Lemma 1), the following lemma and corollary show that the induced Galois action on $\mathcal{G}_{i,j}(t)$ and its Frobenius twists as elements of $K^{\mathrm{sep}}[\![t]\!]^{n+1}$ are compatible with its action on them as elements of $T_t(\mathrm{P}_n\rho)$. Set

$$\mathbf{y} := [y_{1,1},\dots,y_{r,1},\dots,\dots,y_{1,n+1},\dots,y_{r,n+1}]^{\mathsf{T}} \in \left(T_t(\mathrm{P}_n\rho)\right)^{r(n+1)}.$$

Lemma 2. *For each $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, let $g_\epsilon \in \mathrm{GL}_r(\mathbb{F}_q[\![t]\!])$ be defined as in* (4.1). *Let $\mathcal{G} := [\mathcal{G}_{1,1}, \dots, \mathcal{G}_{r,1}, \dots, \dots, \mathcal{G}_{1,n+1}, \dots, \mathcal{G}_{r,n+1}]^{\mathsf{T}} \in \mathrm{Mat}_{r(n+1)\times 1}\big(K^{\mathrm{sep}}[\![t]\!]^{n+1}\big)$. Then,*

(a)

$$\epsilon(\mathbf{y}) = d_{t,n+1}[g_\epsilon](\mathbf{y}),$$

(b)

$$\epsilon(\mathcal{G}) = d_{t,n+1}[g_\epsilon]\mathcal{G},$$

where $\epsilon(\mathcal{G}) = [\epsilon(\mathcal{G}_{1,1}), \dots, \epsilon(\mathcal{G}_{r,1}), \dots, \dots, \epsilon(\mathcal{G}_{1,n+1}), \dots, \epsilon(\mathcal{G}_{r,n+1})]^{\mathsf{T}}$.

Remark 1. For the case of the prolongations of the Carlitz module, this was proved by Maurischat (see [29]).

Proof of Lemma 2. For parts of the proof, we apply methods similar to the ones used in the proof of [12, Lem. 3.2.2]. For $\mathbf{a} = \sum_{\ell=0}^{\infty} a_\ell t^\ell \in \mathbb{F}_q[\![t]\!]$, it is easy to see that for each i, j, we have $\mathbf{a} \cdot y_{i,j} := \big(\mathbf{a} \cdot \chi_{i,j,0}, \mathbf{a} \cdot \chi_{i,j,1}, \mathbf{a} \cdot \chi_{i,j,2}, \dots\big)$, where for each $m \geq 0$,

$$\mathbf{a} \cdot \chi_{i,j,m} = a_m \chi_{i,j,0} + a_{m-1}\chi_{i,j,1} + \cdots + a_0 \chi_{i,j,m} \in (\mathrm{P}_n\rho)[t^{m+1}], \tag{4.4}$$

and

$$\mathbf{a} \cdot \xi_{i,m} = a_m \xi_{i,0} + a_{m-1}\xi_{i,1} + \cdots + a_0 \xi_{i,m} \in \rho[t^{m+1}]. \tag{4.5}$$

For $1 \leq u \leq r$, let $\boldsymbol{h} := [h_{u,1}, \dots, h_{u,r}] \in \mathrm{Mat}_{1\times r}(\mathbb{F}_q[\![t]\!])$ be the u-th row of g_ϵ and let $h_{u,i} = \sum_{\ell=0}^{\infty} \gamma_{i,\ell} t^\ell$ as an element of $\mathbb{F}_q[\![t]\!]$. Then, for each $c \geq 0$, we have

$$\partial_t^c(h_{u,i}) = \sum_{\ell=0}^{\infty} \binom{\ell}{c} \gamma_{i,\ell} t^{\ell-c}. \tag{4.6}$$

Define $\partial_t^c(\boldsymbol{h}) := [\partial_t^c(h_{u,1}), \dots, \partial_t^c(h_{u,r})]$. By definition, $d_{t,n+1}[g_\epsilon]$ is a block upper triangular $r(n+1) \times r(n+1)$ matrix with $(n+1)^2$ blocks of size $r \times r$. Therefore, for $0 \leq v \leq n$ the $(vr+u)$-th row of $d_{t,n+1}[g_\epsilon]$ is $[\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{h}, \partial_t^1(\boldsymbol{h}), \dots, \partial_t^{n+1-v}(\boldsymbol{h})] \in \mathrm{Mat}_{1\times n(r+1)}(\mathbb{F}_q[\![t]\!])$. Thus, the $(vr+u)$-th entry of $d_{t,n+1}[g_\epsilon](\mathbf{y})$ is $\sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\partial_t^c(h_{u,i})\cdot y_{i,v+c}$ and we see that its $(m+1)$-th entry is the following:

$$\begin{aligned}\sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\partial_t^c(h_{u,i})\cdot\chi_{i,v,m} &= \sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\sum_{w=0}^{m+c}\binom{w}{c}\gamma_{i,w}\chi_{i,v+c,m+c-w}\\ &= \sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\sum_{w=0}^{m+c}\sum_{k=0}^{n-(v+c)}\binom{w}{c}\binom{m+c-w+k}{k}\gamma_{i,w}(\xi_{i,m+c-w+k})_{v+c+k},\end{aligned}$$

where we obtain the first equality by using (4.5) and (4.6), and the second equality by using (4.2). Setting $\alpha := c+k$ and changing the order of summations (we also use the fact that for $j>m$, we have $\binom{m}{j}=0$), we obtain

$$\sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\partial_t^c(h_{u,i})\cdot\chi_{i,v,m} = \sum_{\alpha=0}^{n+1-v}\sum_{i=1}^{r}\sum_{w=0}^{m+\alpha}\sum_{k=0}^{\alpha}\binom{w}{\alpha-k}\binom{m+\alpha-w}{k}\gamma_{i,w}(\xi_{i,m+\alpha-w})_{v+\alpha}. \quad (4.7)$$

We now consider the calculation of $\epsilon(\mathbf{y})$. For $1\le u\le r$ and $0\le v\le n$, the $(vr+u)$-th entry of $\epsilon(\mathbf{y})$ is $\epsilon(y_{u,v})$. By using (4.2), the $(m+1)$-th entry of $\epsilon(y_{u,v})$ for $m\ge 0$ is

$$\epsilon(\chi_{u,v,m}) = \sum_{\alpha=0}^{n+1-v}\binom{m+\alpha}{\alpha}\big(\epsilon(\xi_{u,m+\alpha})\big)_{v+\alpha}.$$

Since $\boldsymbol{h}$ is the u-th row of g_ϵ, we see from (4.1) and (4.5) that

$$\epsilon(\chi_{u,v,m}) = \sum_{\alpha=0}^{n+1-v}\sum_{i=1}^{r}\sum_{w=0}^{m+\alpha}\binom{m+\alpha}{\alpha}(\gamma_{i,w}\xi_{i,m+\alpha-w})_{v+\alpha}. \quad (4.8)$$

Using Chu-Vandermonde summation $\sum_{k=0}^{\alpha}\binom{w}{\alpha-k}\binom{m+\alpha-w}{k}=\binom{m+\alpha}{\alpha}$, it follows that the right hand sides of (4.7) and (4.8) are the same and thus, we have that $\epsilon(y_{u,v})=\sum_{c=0}^{n+1-v}\sum_{i=1}^{r}\partial_t^h(h_{u,i})\cdot y_{i.v}$. This proves part (a).

For part (b), we have from Lemma 1 that

$$\epsilon(f_u(t)) = [h_{u,1},\dots,h_{u,r}]\mathbf{f}.$$

Since ϵ commutes with taking hyperderivatives with respect to t, for $c \leq n$ we have

$$\epsilon(\partial_t^c(f_u)) = \partial_t^c([h_{u,1}, \ldots, h_{u,r}]\mathbf{f}),$$

and so by (4.3) we obtain $\epsilon(\mathcal{G}_{u,v}) = \sum_{c=0}^{n+1-v} \big(\partial_t^c([h_{u,1}, \ldots, h_{u,r}]\mathbf{f})\big)_{v+c}$. Therefore, by part (a), it follows that via multiplication of power series in $K^{\mathrm{sep}}[\![t]\!]^{n+1}$, $\epsilon(\mathcal{G}_{u,v})$ is the same as the $(vr+u)$-th entry of $d_{t,n}[g_\epsilon]\mathcal{G}$. $\square$

Corollary 2. *For* $1 \leq i, j \leq r$, *define* $\Upsilon \in \mathrm{Mat}_r(K^{\mathrm{sep}}[\![t]\!])$ *so that* $\Upsilon_{ij} := f_i^{(j-1)}(t)$ *as in* (3.8). *Then, for any* $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ *and* $g_\epsilon \in \mathrm{GL}_r(\mathbb{F}_q[\![t]\!])$ *as in* (4.1)*, we have*

$$\epsilon\big(d_{t,n+1}[\Upsilon]^{(1)}\big) = d_{t,n+1}[g_\epsilon] d_{t,n+1}[\Upsilon]^{(1)}.$$

Proof. Since the j-th column of $\Upsilon^{(1)}$ is $\mathbf{f}^{(j)}$, by Lemma 1 we have that $\epsilon(\Upsilon^{(1)}) = g_\epsilon \Upsilon^{(1)}$. Since for each i, j and $0 \leq c \leq n$, we have $\epsilon(\partial_t^c(f_i^{(j)}))) = \partial_t^c(\epsilon(f_i)^{(j)})$, it follows from Lemma 2 that

$$\epsilon\big(d_{t,n+1}[\Upsilon^{(1)}]\big) = d_{t,n+1}[g_\epsilon \Upsilon^{(1)}] = d_{t,n+1}[g_\epsilon] d_{t,n+1}[\Upsilon^{(1)}] = d_{t,n+1}[g_\epsilon] d_{t,n+1}[\Upsilon]^{(1)}.$$

The last equality follows from the observation that $\partial_t^c(\cdot)$ commutes with twisting. $\square$

We now consider the t-motivic Galois group $\Gamma_{\Psi_{\mathrm{P}_n\rho}}$ and its principal homogeneous space $Z_{\Psi_{\mathrm{P}_n\rho}}$ as in §2.3.

Theorem 8. *Let* ρ *be a Drinfeld* $\mathbf{A}$*-module defined over* K *such that* $k \subseteq K \subseteq \overline{k}$ *and* $[K : k] < \infty$, *and let* $\mathrm{P}_n\rho$ *be its* n*-th prolongation* t*-module. Suppose that* $\mathrm{End}(\rho) \subseteq K[\tau]$ *and* Z_{Ψ_ρ} *is defined over* $K(t)$. *Then, the assignment* $\epsilon \mapsto d_{t,n+1}[g_\epsilon]$ *induces a group homomorphism*

$$\beta_n : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \Gamma_{\Psi_{\mathrm{P}_n\rho}}(\mathbb{F}_q(\!(t)\!)).$$

Proof. The proof uses the ideas of the proof of [12, Thm. 3.5.1]. Since φ_t is a group homomorphism and $d_{t,n+1}[\cdot]$ respects multiplication, it suffices to show that $d_{t,n+1}[g_\epsilon]$ is in $\Gamma_{\Psi_{\mathrm{P}_n\rho}}(\mathbb{F}_q(\!(t)\!))$.

Let $\epsilon \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, and let $g_\epsilon \in \mathrm{GL}_r(\mathbb{F}_q[\![t]\!])$ be defined as in (4.1). By (2.5) and (3.10), the rigid analytic trivialization of $\Phi_{\mathrm{P}_n\rho}$ is $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$. Since $\mathrm{Gal}(K^{\mathrm{sep}}/K) = \mathrm{Aut}(\overline{K}/K)$ and entries of V are purely inseparable over K, we see that $\epsilon(V) = V$. By Corollary 2,

$$\epsilon\big(d_{t,n+1}[\Psi_\rho]\big) = d_{t,n+1}[\epsilon(V^{-1}(\Upsilon^{(1)})^{-1})] = d_{t,n+1}[V^{-1}(g_\epsilon\Upsilon^{(1)})^{-1}] = d_{t,n+1}[\Psi_\rho]d_{t,n+1}[g_\epsilon^{-1}]. \quad (4.9)$$

Let $S \subseteq K(t)[X, 1/\det X]$ denote a finite set of generators of the defining ideal of $Z_{\Psi_{\mathrm{P}_n\rho}}$. Then, for any $h \in S$, we have $h(d_{t,n+1}[\Psi_\rho]) = 0$. Since ϵ fixes the coefficients of h, we have

$$0 = \epsilon(h(d_{t,n+1}[\Psi_\rho])) = h(d_{t,n+1}[\Psi_\rho]d_{t,n+1}[g_\epsilon^{-1}]).$$

Therefore, $d_{t,n+1}[\Psi_\rho]d_{t,n+1}[g_\epsilon^{-1}] \in Z_{\Psi_{\mathrm{P}_n\rho}}(\mathbb{K}(\!(t)\!))$. By Theorem 5, we see that $d_{t,n+1}[g_\epsilon^{-1}] = d_{t,n+1}[g_\epsilon]^{-1} \in \Gamma_{\mathrm{P}_n\rho}(\mathbb{F}_q(\!(t)\!))$. □

4.2 Elements of $\Gamma_{\mathrm{P}_n M_\rho}$

We continue with a Drinfeld $\mathbf{A}$-module ρ defined over k^{sep} and the t-motive M_ρ associated to ρ (see §3.1). In this section, for $n \geq 1$ we determine what the elements of $\Gamma_{\mathrm{P}_n M_\rho}$, the Galois group associated to the n-th prolongation t-motive $\mathrm{P}_n M_\rho$, look like. We let $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ denote the ring of endomorphisms of $\mathrm{P}_n M_\rho$ and set $\mathbf{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$. Recall from (2.3) that if $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ is a $\overline{k}(t)$-basis of M_ρ, then $\boldsymbol{D}_n\boldsymbol{m}$ is a $\overline{k}(t)$-basis of $\mathrm{P}_n M_\rho$. Given $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$, let $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\overline{k}(t))$ be such that $h(\boldsymbol{D}_n\boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n\boldsymbol{m}$. Since $h\sigma = \sigma h$ and $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, we have

$$d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)}d_{t,n+1}[\Phi_\rho].$$

From this, we see that σ fixes $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho]$, and therefore $d_{t,n+1}[\Psi_\rho]^{-1}\mathrm{H}d_{t,n+1}[\Psi_\rho] \in \mathrm{Mat}_{r(n+1)}(\mathbf{k})$. We have thus defined the following injective map:

$$\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho) \to \mathrm{End}((\mathrm{P}_n M_\rho)^B) = \mathrm{Mat}_{r(n+1)}(\mathbf{k}),$$

$$h \mapsto h^B := d_{t,n+1}[\Psi_\rho]^{-1} \mathrm{H} d_{t,n+1}[\Psi_\rho]. \tag{4.10}$$

Since the tautological representation $\varpi_n : \Gamma_{\mathrm{P}_n M_\rho} \to \mathrm{GL}((\mathrm{P}_n M_\rho)^B)$ is functorial in $(\mathrm{P}_n M_\rho)^B$ (see [34, §3.5.2]), for any $\mathbf{k}$-algebra R and $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$, it follows that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n(\mu)} & \mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B \\ \Big\downarrow{\scriptstyle 1\otimes h^B} & & \Big\downarrow{\scriptstyle 1\otimes h^B} \\ \mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B & \xrightarrow{\varpi_n(\mu)} & \mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B. \end{array} \tag{4.11}$$

Proposition 3. *Given $f \in \mathbf{K}_\rho$, let $\mathrm{F} \in \mathrm{Mat}_r(\overline{k}(t))$ satisfy $f(\boldsymbol{m}) = \mathrm{F}\boldsymbol{m}$. Also, for $n \geq 1$ let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that $h(\boldsymbol{D}_n \boldsymbol{m}) = \mathrm{H}\boldsymbol{D}_n \boldsymbol{m}$, where $\mathrm{H} = (\mathrm{H}_{ij}) \in \mathrm{Mat}_{r(n+1)}(\overline{k}(t))$ and each H_{ij} is an $r \times r$ block for $1 \leq i, j \leq n+1$.*

(a) If K_ρ is separable over k, then for $n \geq 1$ there exists $g \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ such that $g(\boldsymbol{D}_n \boldsymbol{m}) = d_{t,n+1}[\mathrm{F}]\boldsymbol{D}_n \boldsymbol{m}$.

(b) For $0 \leq j \leq n-1$, the matrix $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\overline{k}(t))$, $j+1 \leq u \leq n+1, 1 \leq v \leq j+1$ formed by the lower left $r(j+1) \times r(j+1)$ square of H represents an element of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_j M_\rho)$.

Proof. For part (a), since $\mathbf{K}_\rho$ is separable over $\mathbf{k}$ (by hypothesis and Lemma 2), we can take hyperderivatives of entries of F. Since $f\sigma = \sigma f$, we have $\Phi_\rho \mathrm{F} = \mathrm{F}^{(-1)}\Phi_\rho$. Since multiplication by σ on $\mathrm{P}_n M_\rho$ is represented by $\Phi_{\mathrm{P}_n \rho} = d_{t,n+1}[\Phi_\rho]$, the proof of (a) follows from the observation that

$$d_{t,n+1}[\Phi_\rho] d_{t,n+1}[\mathrm{F}] = d_{t,n+1}[\mathrm{F}]^{(-1)} d_{t,n+1}[\Phi_\rho].$$

For part (b), using $d_{t,n+1}[\Phi_\rho]\mathrm{H} = \mathrm{H}^{(-1)} d_{t,n+1}[\Phi_\rho]$ and the definition of d-matrices, we see that for $0 \leq j \leq n-1$,

$$d_{t,j+1}[\Phi_\rho]\mathrm{H}_j = \mathrm{H}_j^{(-1)} d_{t,j+1}[\Phi_\rho],$$

and the result follows. $\square$

Theorem 9. *For each $n \geq 1$ and any $\mathbf{k}$-algebra R, an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form*

$$\begin{pmatrix} \mathrm{A}_0 & \mathrm{A}_1 & \mathrm{A}_2 & \dots & \mathrm{A}_{n-1} & \mathrm{A}_n \\ & \mathrm{A}_0 & \mathrm{A}_1 & \ddots & \ddots & \mathrm{A}_{n-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \mathrm{A}_2 \\ & & & & \ddots & \mathrm{A}_1 \\ & & & & & \mathrm{A}_0 \end{pmatrix},$$

where for each $0 \leq i \leq n$, A_i is an $r \times r$ block. Furthermore, for $0 \leq j \leq n-1$, the matrix formed by the upper left $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$. In particular, the matrix $(\mathrm{A}_0) \in \Gamma_{M_\rho}(\mathrm{R})$.

Proof. Since the prolongation of an $\mathbf{A}$-finite dual t-motive is also an $\mathbf{A}$-finite dual t-motive, by (2.2), for any $n \geq 1$ and $0 \leq j \leq n-1$ we obtain a short exact sequence of t-motives

$$0 \to \mathrm{P}_j M_\rho \xrightarrow{\iota} \mathrm{P}_n M_\rho \xrightarrow{\mathbf{pr}} \mathrm{P}_{n-j-1} M_\rho \to 0, \tag{4.12}$$

where $\mathbf{pr}(D_i m) := D_{i-j-1} m$ for $i > j$ and $\mathbf{pr}(D_i m) := 0$ for $i \leq j$ and $m \in M_\rho$ and ι is the inclusion map. Note that $\mathrm{P}_0 M_\rho \cong M_\rho$ via $D_0 m \mapsto m$ for all $m \in M_\rho$.

For any $\mathbf{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B$ from [34, §4.5]. Recall that $\Psi_{\mathrm{P}_n \rho} = d_{t,n+1}[\Psi_\rho]$. The entries of $\boldsymbol{u}_n := d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m}$ form a $\mathbf{k}$-basis of $(\mathrm{P}_n M_\rho)^B$ (see [34, Prop. 3.3.9]) and similarly for $0 \leq j \leq n-1$, we have that the entries of $\boldsymbol{u}_j := d_{t,j+1}[\Psi_\rho]^{-1} \boldsymbol{D}_j \boldsymbol{m}$ form a $\mathbf{k}$-basis of $(\mathrm{P}_j M_\rho)^B$. For any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $a_h \in \mathrm{Mat}_{1\times r}(\mathrm{R})$, $0 \leq h \leq n$, the action of μ on $(a_0, \dots, a_n) \cdot \boldsymbol{u}_n \in \mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_n M_\rho)^B$ is

$$(a_0, \dots, a_n) \cdot d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} \mapsto (a_0, \dots, a_n) \cdot \mu^{-1} d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} \tag{4.13}$$

Note that $d_{t,n+1}[\Psi_\rho]^{-1} \boldsymbol{D}_n \boldsymbol{m} = \boldsymbol{D}_n(\Psi_\rho^{-1} \boldsymbol{m})$.

We first restrict the action of μ to $\mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_j M_\rho)^B$ via the map ι in (4.12). So, we take $a_0, \ldots, a_{n-j-1} = 0$ and set $\mu^{-1} := (\mathrm{B}_{iw}), 1 \le i, w \le n+1$ where each B_{iw} is an $r \times r$ block. By ι in (4.12), we see that μ leaves $(\mathrm{P}_j M_\rho)^B$ invariant and thus the blocks

$$\mathrm{B}_{n-j+v,1} = \mathrm{B}_{n-j+v,2} = \cdots = \mathrm{B}_{n-j+v,n-j} = \mathbf{0}, \quad \text{for } v = 1, \ldots, j+1.$$

Moreover, since the non-zero a_h's were chosen arbitrarily, we see that the matrix formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that μ^{-1} is a block upper triangular matrix and that the matrices formed by the lower right $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$ for each $j \in \{0, \ldots, n-1\}$.

We return to arbitrary $a_h \in \mathrm{Mat}_{1\times r}(\mathrm{R})$. We restrict the action of μ to $\mathrm{R} \otimes_{\mathbf{k}} (\mathrm{P}_{n-j-1} M_\rho)^B$ via the map $\mathbf{pr}$ in (4.12). Through $\mathbf{pr}$, we see that μ leaves $(\mathrm{P}_{n-j-1} M_\rho)^B$ invariant and so the matrix formed by the upper left $r(n-j) \times r(n-j)$ square of μ is an element of $\Gamma_{\mathrm{P}_{n-j-1} M_\rho}(\mathrm{R})$. Varying j from 0 to $n-1$, we see that the matrices formed by the upper left $r(j+1) \times r(j+1)$ square of μ is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$ for each $j \in \{0, \ldots, n-1\}$.

Now, we let $h \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ be such that for $\mathrm{H} \in \mathrm{Mat}_{r(n+1)}(\overline{k}(t))$ we have $h(\boldsymbol{D}_n \boldsymbol{m}) = \mathrm{H} \boldsymbol{D}_n \boldsymbol{m}$. Let $\mathrm{H} := (\mathrm{H}_{iw})$, where each (H_{iw}) is an $r \times r$ block. For $0 \le j \le n-1$, let $\mathrm{H}_j := (\mathrm{H}_{uv}) \in \mathrm{Mat}_{r(j+1)}(\overline{k}(t)), j+1 \le u \le n+1, 1 \le v \le j+1$ be the matrix formed by the lower left $r(j+1) \times r(j+1)$ square of H. Using the definition of d-matrices, we see that the matrix formed by the lower left $r(j+1) \times r(j+1)$ square of $d_{t,n+1}[\Psi_\rho]^{-1} \mathrm{H} d_{t,n+1}[\Psi_\rho]$ is $d_{t,j+1}[\Psi_\rho]^{-1} \mathrm{H}_j \, d_{t,j+1}[\Psi_\rho]$. By Proposition 3 (b), we have that $d_{t,j+1}[\Psi_\rho]^{-1} \mathrm{H}_j \, d_{t,j+1}[\Psi_\rho]$ is an element in the image of the natural embedding (4.10) for the j-th prolongation. Thus, by using the commutative diagram (4.11) for the n-th and the $(n-1)$-th prolongations, we see that since μ is upper triangular, the matrices formed by the lower right $rn \times rn$ square and the upper left $rn \times rn$ square of μ are equal. Now, comparing each $r \times r$ block in this equality, we get the required result. $\square$

4.3 Lower bound on Dimension of $\Gamma_{\mathrm{P}_n M_\rho}$

For this section, the reader is directed to Appendix A for details about differential algebra and differential algebraic geometry in characteristic $p > 0$. By the properties of hyperderivatives (see §2.4), for a field F of characteristic $p > 0$ with t transcendental over F, we see that $(F(t), \partial_t)$, $(F(\!(t)\!), \partial_t)$ and $(F(t)^{\mathrm{sep}}, \partial_t)$, where ∂_t represents hyperderivative with respect to t, are ∂_t-fields. By Theorem 9, we make the choice to let the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ be

$$\mathbf{X} := \begin{pmatrix} \mathbf{X}_0 & \mathbf{X}_1 & \dots & \dots & \mathbf{X}_n \\ & \mathbf{X}_0 & \ddots & \ddots & \vdots \\ & & \mathbf{X}_0 & \ddots & \vdots \\ & & & \ddots & \mathbf{X}_1 \\ & & & & \mathbf{X}_0 \end{pmatrix}, \tag{4.14}$$

where $\mathbf{X}_h := ((X_h)_{ij})$, an $r \times r$ matrix for $h = 0, 1, \dots, n$. We set $\partial_t^j(\mathbf{X}_h) := (\partial_t^j((X_h)_{i,j}))$ and

$$\mathbf{vec}(\mathbf{X}_h) := [(X_h)_{1,1}, \dots, (X_h)_{r,1}, (X_h)_{1,2}, \dots, (X_h)_{r,2}, \dots, \dots, (X_h)_{1,r}, \dots, (X_h)_{r,r}]^{\mathsf{T}},$$

which consists of all entries of $\mathbf{X}_h$ lined up in a column vector.

Let $0 \le i \le n$. Define $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, \dots, \mathbf{X}_i\}$ to be the ∂_t-polynomial ring, as in Appendix A.2, over $\mathbb{F}_q(\!(t)\!)$ with entries of each $\mathbf{X}_h$ for $h = 0, \dots, i$ as ∂_t-indeterminates. Also, define the ∂_t-polynomial ring $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, \dots, \mathbf{X}_i, 1/\det \mathbf{X}_0\}$ with ∂_t-indeterminates comprising of $1/\det \mathbf{X}_0$ and entries of each $\mathbf{X}_h$ for $h = 0, \dots, i$. Moreover, define $\mathbb{F}_q(\!(t)\!)[\mathbf{X}_0, \dots, \mathbf{X}_i]$ to be the polynomial ring over $\mathbb{F}_q(\!(t)\!)$ with entries of each $\mathbf{X}_h$ for $h = 0, \dots, i$ as indeterminates, and define $\mathbb{F}_q(\!(t)\!)[\mathbf{X}_0, \dots, \mathbf{X}_i, 1/\det \mathbf{X}_0]$ to be the polynomial ring with $1/\det \mathbf{X}_0$ and entries of each $\mathbf{X}_h$ for $h = 0, \dots, i$ as indeterminates.

Theorem 10. *Fix $n \ge 1$. Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its* *associated n-th prolongation t-module. Suppose that $[K_\rho : k] = s$. Then,*

$$\dim \Gamma_{\mathrm{P}_n M_\rho} \geq (n+1)\frac{r^2}{s}.$$

Proof. From Theorem 8, we see that the Zariski closure $\overline{\operatorname{Im}\beta_n}^Z$ of $\operatorname{Im}\beta_n$ is an algebraic subgroup of $\Gamma_{\mathrm{P}_n M_\rho}/\mathbb{F}_q(\!(t)\!)$. Therefore, our task is to prove that $\overline{\operatorname{Im}\beta_n}^Z$ is defined over $\mathbf{k}$ and that $\dim(\overline{\operatorname{Im}\beta_n}^Z/\mathbf{k}) = (n+1)r^2/s$. For any $\mathbf{k}$-algebra R, we define the algebraic group over $\mathbf{k}$,

$$\operatorname{Cent}_{\mathrm{GL}_r/\mathbf{k}}(\mathbf{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{GL}_r(\mathrm{R}) \mid \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\mathbf{k}} \mathbf{K}_\rho \subseteq \operatorname{Mat}_r(\mathrm{R})\}.$$

By [36, Thm. 0.2] and [12, Thm. 3.5.4], we see that $\Gamma_{M_\rho} = \operatorname{Cent}_{\mathrm{GL}_r/\mathbf{k}}(\mathbf{K}_\rho) = \overline{\operatorname{Im}\beta_0}^Z$ with dimension r^2/s. Since the defining polynomials of $\operatorname{Cent}_{\operatorname{Mat}_r(\mathbf{k})}(\mathbf{K}_\rho) = \operatorname{Lie}\Gamma_{M_\rho}$ are homogeneous degree one polynomials, let its defining equations be as follows:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}(b_u)_{ij}(X_0)_{ij} = 0, \quad (b_u)_{ij} \in \mathbf{k}, u = 1, \dots, r^2 - r^2/s, \tag{4.15}$$

which can be written as

$$\mathbf{B} \cdot \mathbf{vec}(\mathbf{X}_0) = \mathbf{0}, \tag{4.16}$$

where we set $\mathbf{B}$ to be the $(r^2 - r^2/s) \times r^2$ matrix with $(b_u)_{ij}$ as the $u \times ((j-1)r + i)$-th entry. We see that $\operatorname{rank}\mathbf{B} = r^2 - \dim\Gamma_{M_\rho} = r^2 - r^2/s$. Therefore, the defining ideal of Γ_{M_ρ} is the ideal generated by the entries of $\mathbf{B} \cdot \mathbf{vec}(\mathbf{X}_0)$ in $\mathbf{k}[\mathbf{X}_0, 1/\det \mathbf{X}_0]$, the coordinate ring of $\mathrm{GL}_r/\mathbf{k}$.

Let $\overline{\operatorname{Im}\beta_0}^\partial$ be the $\mathbb{F}_q(\!(t)\!)$-∂_t-closure of $\operatorname{Im}\beta_0$ in $\mathrm{GL}_r(\overline{\mathbb{F}_q(\!(t)\!)}^\partial)$, where $\overline{\mathbb{F}_q(\!(t)\!)}^\partial$ is the ∂_t-closure of $\mathbb{F}_q(\!(t)\!)$ inside the algebraic closure $\overline{\mathbb{F}_q(\!(t)\!)}$ (see Appendix A.2). Then

$$\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{I}(\overline{\operatorname{Im}\beta_0}^\partial) = \mathfrak{D}(\mathbf{B} \cdot \mathbf{vec}(\mathbf{X}_0)),$$

where $\mathfrak{I}(\operatorname{Im}\beta_0)$ and $\mathfrak{I}(\overline{\operatorname{Im}\beta_0}^\partial)$ are the defining $\mathbb{F}_q(\!(t)\!)$-∂_t-ideal of $\operatorname{Im}\beta_0$ and $\overline{\operatorname{Im}\beta_0}^\partial$ respectively in $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, 1/\det \mathbf{X}_0\}$, and $\mathfrak{D}(\mathbf{B} \cdot \mathbf{vec}(\mathbf{X}_0))$ is the ∂_t-ideal in $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, 1/\det \mathbf{X}_0\}$ generated by the linear homogeneous polynomials given by the entries of $\mathbf{B}\cdot\mathbf{vec}(\mathbf{X}_0)$.

Let T be the radical ∂_t-ideal inside $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0,\ldots,\mathbf{X}_n,1/\det\mathbf{X}_0\}$ generated the entries of $\mathbf{B}\cdot\mathbf{vec}(\mathbf{X}_0),\mathbf{vec}(\partial_t^1(\mathbf{X}_0)-(\mathbf{X}_1)),\mathbf{vec}(\partial_t^2(\mathbf{X}_0)-(\mathbf{X}_2)),\ldots,\mathbf{vec}(\partial_t^n(\mathbf{X}_0)-(\mathbf{X}_n))$, which are linear homogeneous ∂_t-polynomials, that is,

$$T:=\mathfrak{R}(\mathbf{B}\cdot\mathbf{vec}(\mathbf{X}_0),\mathbf{vec}(\partial_t^1(\mathbf{X}_0)-(\mathbf{X}_1)),\mathbf{vec}(\partial_t^2(\mathbf{X}_0)-(\mathbf{X}_2)),\ldots,\mathbf{vec}(\partial_t^n(\mathbf{X}_0)-(\mathbf{X}_n))).$$

We set $\mathfrak{Z}(T)$ to be zero set of T in $\mathrm{GL}_{r(n+1)}(\overline{\mathbb{F}_q(\!(t)\!)}^{\partial})$ and $\overline{\operatorname{Im}\beta_n}^{\partial}$ to be the $\mathbb{F}_q(\!(t)\!)$-∂_t-closure of $\operatorname{Im}\beta_n$. We see from Theorem 8 and (4.16) that $\overline{\operatorname{Im}\beta_n}^{\partial}\subseteq\mathfrak{Z}(T)$. Moreover, Proposition 11 implies that

$$T=\mathfrak{D}(\mathbf{B}\cdot\mathbf{vec}(\mathbf{X}_0),\mathbf{vec}(\partial_t^1(\mathbf{X}_0)-(\mathbf{X}_1)),\mathbf{vec}(\partial_t^2(\mathbf{X}_0)-(\mathbf{X}_2)),\ldots,\mathbf{vec}(\partial_t^n(\mathbf{X}_0)-(\mathbf{X}_n))),\tag{4.17}$$

the ∂_t-ideal generated by the set of linear homogeneous ∂_t-polynomials given by the entries of $\mathbf{B}\cdot\mathbf{vec}(\mathbf{X}_0),\mathbf{vec}(\partial_t^1(\mathbf{X}_0)-(\mathbf{X}_1)),\mathbf{vec}(\partial_t^2(\mathbf{X}_0)-(\mathbf{X}_2)),\ldots,\mathbf{vec}(\partial_t^n(\mathbf{X}_0)-(\mathbf{X}_n))$.

We claim that $\overline{\operatorname{Im}\beta_n}^{\partial}=\mathfrak{Z}(T)$. It suffices to show that $\mathfrak{I}(\overline{\operatorname{Im}\beta_n}^{\partial})\subseteq T$. To do this, we define a monomial order on $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0,\ldots,\mathbf{X}_n\}$ and use the division algorithm [25, Prop. 1.9]. We denote by $\mathbb{Z}_{\geq 0}^{(\infty)}$ the set of all sequences $(a_1,a_2,a_3,\ldots\ldots)$ of non-negative integers such that $a_i=0$ for all but finitely many $i\geq 1$. Note that any monomial in $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0,\ldots,\mathbf{X}_n\}$ can be described uniquely as $\mathbf{X}^{\mathbf{b}}=\prod\partial_t^{\ell}((X_h)_{i,j})^{(b_{h,\ell})_{i,j}}$ for some $\mathbf{b}=(\mathbf{b}_{0,0},\mathbf{b}_{0,1}\ldots,\mathbf{b}_{1,0},\mathbf{b}_{1,1},\ldots,\ldots,\mathbf{b}_{n,0},\mathbf{b}_{n,1},\ldots)\in\mathbb{Z}_{\geq 0}^{(\infty)}$, where for $h=0,\ldots n$ and $\ell\in\mathbb{Z}_{\geq 0}$, we have each $\mathbf{b}_{h,\ell}=\mathbf{vec}(((b_{h,\ell})_{i,j}))^{\mathsf{T}}$ such that $((b_{h,\ell})_{i,j})$ is an $r\times r$ matrix and $(b_{h,\ell})_{i.j}=0$ for all but a finite number of h,ℓ,i,j. We define a monomial order on $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0,\ldots,\mathbf{X}_n\}$ as in [25, Def. 1.1] in the following way:

- we set $\partial_t^{\ell}((X_h)_{1,1})<\cdots<\partial_t^{\ell}((X_h)_{r,1})<\ldots\cdots<\partial_t^{\ell}((X_h)_{1,r}),\cdots<\partial_t^{\ell}((X_h)_{r,r})$,
- we set $\partial_t^{\ell}((X_h)_{i_1,j_1})<\partial_t^{\ell+1}((X_h)_{i_2,j_2})$,
- we set $\partial_t^{\ell_1}((X_h)_{i_1,j_1})<\partial_t^{\ell_2}((X_{h+1})_{i_2,j_2})$,

- we take the pure lexicographic order defined such that $\mathbf{X}^{\mathbf{b}} < \mathbf{X}^{\mathbf{c}}$ if the left-most nonzero component of $\mathbf{b} - \mathbf{c}$ is negative,

where $\mathbf{b}, \mathbf{c} \in \mathbb{Z}_{\geq 0}^{(\infty)}$, $\ell, \ell_1, \ell_2 \in \mathbb{Z}_{\geq 0}$, $i, j, i_1, i_2, j_1, j_2 \in \{0, \ldots, r\}$ and $h = 0, \ldots, n$.

Now, let $F \in \mathfrak{I}(\overline{\operatorname{Im}\beta_n}^{\partial}) \subseteq \mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, \ldots, \mathbf{X}_n, 1/\det \mathbf{X}_0\}$. Note that for $h = 1, \ldots, n$, we have $\partial_t^{\ell}(\partial_t^{h}((X_0)_{i,j})) < \partial_t^{\ell}((X_h)_{i,j})$ and so the leading monomial of each $\partial_t^{\ell}(\partial_t^{h}((X_0)_{i,j}) - (X_h)_{i,j})$ is $\partial_t^{\ell}((X_h)_{i,j})$. Then, by using the division algorithm [25, Prop. 1.9] we see that

$$F = \sum_{h=1}^{n} \sum_{\ell=0}^{m_{h,i,j}} \sum_{i,j=1}^{r} \partial_t^{\ell} \left(\partial_t^{h}((X_0)_{i,j}) - (X_h)_{i,j}\right) \cdot f_{h,i,j} + H, \tag{4.18}$$

where $m_{h,i,j}$ is the largest number such that $\partial_t^{m_{h,i,j}}((X_h)_{i,j})$ occurs as a variable in F, each $f_{h,i,j} \in \mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, \ldots, \mathbf{X}_n, 1/\det \mathbf{X}_0\}$, and the remainder $H = H(\mathbf{X}_0)$ is an element of the ∂_t-polynomial ring $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, 1/\det \mathbf{X}_0\}$. Note that for g_{ϵ}, $\operatorname{Im}\beta_n$ and $\operatorname{Im}\beta_0$ as in Theorem 8, there is a surjective map

$$\operatorname{Im}\beta_n \twoheadrightarrow \operatorname{Im}\beta_0$$

given by

$$d_{t,n+1}[g_{\epsilon}] \mapsto g_{\epsilon}.$$

Moreover $F(d_{t,n+1}[g_{\epsilon}]) = 0$. Since $\sum_{h=1}^{n} \sum_{\ell=0}^{m_{h,i,j}} \sum_{i,j=1}^{n} \partial_t^{\ell} \left(\partial_t^{h}((X_0)_{i,j}) - (X_h)_{i,j}\right) \cdot f_{h,i,j} \in T$ and $\overline{\operatorname{Im}\beta_n}^{\partial} \subseteq \mathfrak{Z}(T)$, we obtain from (4.18) that $H(g_{\epsilon}) = 0$. Thus, $H(\mathbf{X}_0)$ is an element of $\mathfrak{I}(\operatorname{Im}\beta_0) = \mathfrak{D}(\mathbf{B} \cdot \mathbf{vec}(\mathbf{X}_0))$, the ∂_t-ideal in the ∂_t-polynomial ring $\mathbb{F}_q(\!(t)\!)\{\mathbf{X}_0, 1/\det \mathbf{X}_0\}$. Thus, $F \in T$. This proves our claim. Therefore, $\overline{\operatorname{Im}\beta_n}^{\partial} = \mathfrak{Z}(T)$.

We are now ready to compute $\overline{\operatorname{Im}\beta_n}^{Z}$. Based on Lemma 7, we can find the defining equations of $\overline{\operatorname{Im}\beta_n}^{Z}$ if we find

$$\mathbf{T} := T \cap \mathbb{F}_q(\!(t)\!)[\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n, 1/\det \mathbf{X}_0].$$

By (4.17), an element of T is of the form

$$\mathbf{a} := \sum_{u=1}^{r^2-r^2/s} c_u \cdot \partial_t^{v_u}(B_u \cdot \mathbf{vec}(\mathbf{X}_0)) + \sum_{h=1}^{n} \sum_{\ell=0}^{m_{h,i,j}} \sum_{i,j=1}^{r} (w_{h,\ell})_{i,j} \cdot \partial_t^{\ell} \left(\partial_t^h ((X_0)_{i,j}) - (X_h)_{i,j} \right), \quad (4.19)$$

where B_u is the u-th row of $\mathbf{B}$, $(w_{h,\ell})_{i,j}, c_u \in (\mathbb{F}_q(\!(t)\!), \partial_t)$ and $m_{h,i,j} \in \mathbb{Z}_{\geq 0}$ for $h = 1, \ldots, n$, $i, j = 1, \ldots r$, $u = 1, \ldots, r^2 - r^2/s$. Suppose $\mathbf{a} \in \mathbf{T} \subseteq \mathbb{F}_q(\!(t)\!)[\mathbf{X}_0, \ldots, \mathbf{X}_n, 1/\det \mathbf{X}_0]$. Then, since for any $h = 1, \ldots, n$ and for $1 \leq i, j \leq r$, the coordinate $\partial_t^{\ell}((X_h)_{ij})$ appears only once in $\mathbf{a}$, we see that each $m_{h,i,j} = 0$. Note that by the product rule of hyperderivatives, we have $\partial_t^{v_u}(B_u \cdot \mathbf{vec}(\mathbf{X}_0)) = \sum_{h=1}^{v_u} \partial_t^{v_u-h}(B_u) \cdot \mathbf{vec}(\partial_t^h(\mathbf{X}_0)))$. From (4.15), we see that $B_u = (\mathbf{vec}((b_u)_{i,j}))^{\mathsf{T}}$, where $((b_u)_{i,j})$ is the $r \times r$ matrix with $(b_u)_{i,j}$ in the (i, j)-th entry. Since each $m_{h,i,j} = 0$ we see that for $\mathbf{a}$ to be in $\mathbf{T}$, we need

$$\sum_{u=1}^{r^2-r^2/s} c_u \cdot \partial_t^{v_u-h}((b_u)_{i,j}) \partial_t^h((X_0)_{1,1}) + (w_{h,0})_{i,j} \cdot \partial_t^h((X_0)_{1,1}) = 0,$$

that is, $(w_{h,0})_{i,j} = -\sum_{u=1}^{r^2-r^2/s} c_u \cdot \partial_t^{v_u-h}((b_u)_{i,j})$. Moreover, since each $\partial_t^h((X_0)_{ij})$ appears only once in the second sum of $\mathbf{a}$, we need each $v_u \leq n$. Combining all the results above we obtain $\mathbf{a} = -\sum_{u=1}^{r^2-r^2/s} \sum_{h=1}^{v_u} c_u \cdot \left(\partial_t^{v_u-h}(B_u) \cdot \mathbf{vec}(\mathbf{X}_h) \right)$ such that each $c_u \in \mathbb{F}_q(\!(t)\!)$. Varying u from 1 to $r^2 - r^2/s$, varying each v_u from 0 to n and varying c_u over all elements of $\mathbb{F}_q(\!(t)\!)$, we see that the ideal $\mathbf{T} = T \cap \mathbb{F}_q(\!(t)\!)[\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n, 1/\det \mathbf{X}_0]$ is the same as the ideal generated by

$$\left\{ \sum_{h=0}^{n} \partial_t^{n-h}(B_u) \cdot \mathbf{vec}(\mathbf{X}_h), \quad u = 1, \ldots, r^2 - r^2/s \right\},$$

which can be written as

$$d_{t,n+1}[\mathbf{B}] \cdot \mathbf{vec} \begin{pmatrix} \mathbf{X}_n \\ \mathbf{X}_{n-1} \\ \vdots \\ \mathbf{X}_0 \end{pmatrix},$$

where we define $\mathbf{vec}([\mathbf{X}_n, \ldots, \mathbf{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\, \mathbf{X}_n)^{\mathsf{T}}, \ldots, (\mathbf{vec}\, \mathbf{X}_0)^{\mathsf{T}}]^{\mathsf{T}}$. Since, by its definition,

$d_{t,n+1}[\mathbf{B}]$ is a block upper triangular matrix with all diagonal blocks equal to $\mathbf{B}$, we have that $\operatorname{rank} d_{t,n+1}[\mathbf{B}] \geq (n+1) \cdot \operatorname{rank} \mathbf{B} = (n+1) \cdot (r^2 - r^2/s)$. Also, since $d_{t,n+1}[\mathbf{B}]$ is an $(n+1) \cdot (r^2 - r^2/s) \times (n+1) \cdot r^2$ matrix, we have that $\operatorname{rank} d_{t,n+1}[\mathbf{B}] \leq (n+1) \cdot (r^2 - r^2/s)$ and so $\operatorname{rank} d_{t,n+1}[\mathbf{B}] = (n+1) \cdot (r^2 - r^2/s)$. Since $\operatorname{rank} d_{t,n+1}[\mathbf{B}]$ is full, we see that

$$d_{t,n+1}[\mathbf{B}] \cdot \mathbf{vec} \begin{pmatrix} \mathbf{X}_n \\ \mathbf{X}_{n-1} \\ \vdots \\ \mathbf{X}_0 \end{pmatrix} = \mathbf{0} \tag{4.20}$$

are the defining equations of $\overline{\operatorname{Im} \beta_n}^Z$. Since each $(b_u)_{ij}$ is an element of $\mathbf{k}$, we see that each entry of $d_{t,n+1}[\mathbf{B}]$ is an element of $\mathbf{k}$ and so, $\overline{\operatorname{Im} \beta_n}^Z$ is defined over $\mathbf{k}$. Moreover,

$$\dim \overline{\operatorname{Im} \beta_n}^Z = (n+1) \cdot r^2 - \operatorname{rank} d_{t,n+1}[\mathbf{B}] = (n+1) \cdot r^2 - (n+1) \cdot (r^2 - r^2/s) = (n+1) \cdot r^2/s, \tag{4.21}$$

which gives the desired result. □

4.4 Upper bound on Dimension of $\Gamma_{\mathrm{P}_n M_\rho}$

Recall from Theorem 9 that for any $\mathbf{k}$-algebra R and $n \geq 1$, an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form

$$\mu = \begin{pmatrix} \gamma & \mathrm{A}_1 & \dots & \dots & \mathrm{A}_{n-1} & \mathrm{A}_n \\ & \gamma & \mathrm{A}_1 & \ddots & \ddots & \mathrm{A}_{n-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \mathrm{A}_1 \\ & & & & & \gamma \end{pmatrix}, \tag{4.22}$$

where for each $1 \leq i \leq n$, A_i and γ are $r \times r$ blocks, and for $0 \leq j \leq n-1$ the matrix formed by the upper left $r(j+1) \times r(j+1)$ square is an element of $\Gamma_{\mathrm{P}_j M_\rho}(\mathrm{R})$ and $\gamma \in \Gamma_{M_\rho}(\mathrm{R})$.

Since $\mathrm{P}_{n-1}M_\rho$ is a sub-t-motive of P_nM_ρ, we have a short exact sequence of affine group schemes over $\mathbf{k}$,

$$1 \to Q_n \to \Gamma_{\mathrm{P}_nM_\rho} \xrightarrow{\boldsymbol{\pi}_n} \Gamma_{\mathrm{P}_{n-1}M_\rho} \to 1, \tag{4.23}$$

where $\boldsymbol{\pi}_n^{(\mathrm{R})} : \Gamma_{\mathrm{P}_nM_\rho}(\mathrm{R}) \twoheadrightarrow \Gamma_{\mathrm{P}_{n-1}M_\rho}(\mathrm{R})$ maps μ to the matrix formed by the upper left $rn \times rn$ square. An element of $Q_n(\mathrm{R})$ is of the form

$$\nu = \begin{pmatrix} \mathrm{Id}_r & 0 & \dots & 0 & \mathbf{v} \\ & \mathrm{Id}_r & 0 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix} \in \mathrm{GL}_{(n+1)r}(\mathrm{R}).$$

By using the commutative diagram (4.11) and Proposition 3(a), one checks directly that

$$\mathbf{v} \in \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)(\mathrm{R}), \tag{4.24}$$

where $\mathbf{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$ and we set

$$\mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)(\mathrm{R}) := \{\gamma \in \mathrm{Mat}_r(\mathrm{R}) \mid \gamma g = g\gamma \text{ for all } g \in \mathrm{R} \otimes_{\mathbf{k}} \mathbf{K}_\rho \subseteq \mathrm{Mat}_r(\mathrm{R})\}.$$

Moreover, it can easily be checked that

$$\mu\nu\mu^{-1} = \begin{pmatrix} \mathrm{Id}_r & 0 & \dots & 0 & \gamma\mathbf{v}\gamma^{-1} \\ & \mathrm{Id}_r & 0 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & \mathrm{Id}_r \end{pmatrix}. \tag{4.25}$$

Similar to (4.23), since M_ρ is a sub-t-motive of P_nM_ρ, there is a surjective map of affine group schemes over $\mathbf{k}$,

$$\boldsymbol{\pi} : \Gamma_{\mathrm{P}_n M_\rho} \twoheadrightarrow \Gamma_{M_\rho},$$

where $\boldsymbol{\pi}^{(\mathrm{R})} : \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R}) \to \Gamma_{M_\rho}(\mathrm{R})$ is the map given by $\mu \mapsto \gamma$. Thus, via conjugation there is a left action of Γ_{M_ρ} on Q_n given by (4.25).

Theorem 11. *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep} and for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. If Q_n in* (4.23) *is $\mathbf{k}$-smooth, then $\dim \Gamma_{\mathrm{P}_n M_\rho} \leq (n+1) \cdot r^2/s$.*

Proof. By (4.24), for any $\mathbf{k}$-algebra R we have the following well-defined map:

$$\alpha_n^{(\mathrm{R})} : Q_n(\mathrm{R}) \to \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)(\mathrm{R}). \tag{4.26}$$

Since by hypothesis Q_n is $\mathbf{k}$-smooth, regarding $\mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)$ as an additive group scheme of dimension r^2/s over $\mathbf{k}$, we see that the map $\alpha_n : Q_n \to \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)$ defined above is a morphism of group schemes over $\mathbf{k}$. Moreover, by (4.24), α_n is a monomorphism and so $\dim Q_n \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho)$.

Now, by (4.23) our task is to prove that $\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1} M_\rho} \leq (n+1) \cdot r^2/s$, which we show by induction. For the base case $n = 1$, since $\dim \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho) = r^2/s$ (by [19, Thm. 3.15(3)]) and $\dim \Gamma_{M_\rho} = r^2/s$ (by [12, Thm. 3.5.4]) we see that $\dim Q_1 + \dim \Gamma_{M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho) + \dim \Gamma_{M_\rho} = 2 \cdot r^2/s$. Suppose we have shown that $\dim \Gamma_{\mathrm{P}_{n-1} M_\rho} \leq n \cdot r^2/s$. Similar to the base case, we obtain $\dim Q_n + \dim \Gamma_{\mathrm{P}_{n-1} M_\rho} \leq \dim \mathrm{Cent}_{\mathrm{Mat}_r/\mathbf{k}}(\mathbf{K}_\rho) + \dim \Gamma_{\mathrm{P}_{n-1} M_\rho} = (n+1) \cdot r^2/s$. $\square$

Corollary 3. *Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep} and for $n \geq 1$, let $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Let M_ρ and $\mathrm{P}_n M_\rho$ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Let $\overline{\mathrm{Im}\,\beta_n}^Z$ be the Zariski closure of $\mathrm{Im}\,\beta_n$, where β_n is as in Theorem 8. If Q_n in* (4.23) *is $\mathbf{k}$-smooth, then $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1) \cdot r^2/s$ and*

$$\overline{\mathrm{Im}\,\beta_n}^Z/\mathbf{k} = \Gamma_{\mathrm{P}_n M_\rho}.$$

Proof. We obtain $\dim \Gamma_{\mathrm{P}_n M_\rho} = (n+1)\cdot r^2/s$ by combining Theorem 10 and Theorem 11. By (4.21) we see that $\dim \overline{\operatorname{Im}\beta_n}^Z = \dim \Gamma_{\mathrm{P}_n M_\rho}$. Since the defining polynomials of $\overline{\operatorname{Im}\beta_n}^Z$ are degree one polynomials, it is connected and so $\overline{\operatorname{Im}\beta_n}^Z/\mathbf{k} = \Gamma_{\mathrm{P}_n M_\rho}$. $\square$

Lemma 3. *If K_ρ is separable over k, then for $n \geq 1$, Q_n in* (4.23) *is smooth over* $\mathbf{k}$.

Remark 2. This lemma is the reason for the separability hypothesis in Theorem 2 and one of the reasons for the separability hypothesis in Theorem 4. However, suppose to the contrary that K_ρ is not separable over k but the hyperdifferential operator ∂_θ^j can be extended to K_ρ. In this case, if Q_n is $\mathbf{k}$-smooth, then Theorem 2 holds for the Drinfeld $\mathbf{A}$-module ρ defined over k^{sep}.

Proof of Lemma 3. We adapt the ideas of the proof of [11, Prop. 4.1.2] and the proof of a lemma from a preliminary version of [12] (Lemma 5.1.3: arXiv:1005.5120v1). To prove this, by [37, Cor. 12.1.3] it suffices to show that the induced tangent map $\mathrm{d}\boldsymbol{\pi}_n$ at the identity is surjective onto $\operatorname{Lie}\Gamma_{\mathrm{P}_{n-1}M_\rho}$. Since $\mathbf{K}_\rho$ is separable over $\mathbf{k}$ (by hypothesis and Proposition 2), we see from [12, Cor. 3.5.6] and [40, p.61 Problem 14] that through conjugation by some $\mathrm{A} \in \mathrm{GL}_r(\mathbf{k}^{\mathrm{sep}})$, we have an isomorphism

$$\Gamma_{M_\rho} \times_{\mathbf{k}} \mathbf{K}_\rho \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s}/\mathbf{K}_\rho)_i,$$

where

$$\prod_{i=1}^{s} (\mathrm{GL}_{r/s}/\mathbf{K}_\rho)_i := \left\{ \begin{pmatrix} \mathrm{GL}_{r/s} & & \\ & \ddots & \\ & & \mathrm{GL}_{r/s} \end{pmatrix} \right\},$$

and $(\mathrm{GL}_{r/s}/\mathbf{K}_\rho)_i$ is the canonical embedding of $\mathrm{GL}_{r/s}/\mathbf{K}_\rho$ into the i-th diagonal block matrix of $\mathrm{GL}_r/\mathbf{K}_\rho$. Making a change of basis, we obtain

$$\Gamma_{M_\rho} \times_{\mathbf{k}} \overline{\mathbf{k}} \xrightarrow{\cong} \prod_{i=1}^{s} (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i.$$

For $n \geq 1$, it follows that via conjugation by $d_{t,n+1}[\mathrm{A}] \in \mathrm{GL}_{(n+1)r}(\mathbf{k}^{\mathrm{sep}})$ on $\Gamma_{\mathrm{P}_n M_\rho}$, we obtain

$\overline{\Gamma}_{\mathrm{P}_n M_\rho}$, an algebraic subgroup of $\mathrm{GL}_{(n+1)r}/\overline{\mathbf{k}}$, such that there is an isomorphism

$$a : \Gamma_{\mathrm{P}_n M_\rho} \times_{\mathbf{k}} \overline{\mathbf{k}} \xrightarrow{\cong} \overline{\Gamma}_{\mathrm{P}_n M_\rho}. \tag{4.27}$$

Moreover, the projection $\overline{\boldsymbol{\pi}}_n : \overline{\Gamma}_{\mathrm{P}_n M_\rho} \to \overline{\Gamma}_{\mathrm{P}_{n-1} M_\rho}$ induced by $\boldsymbol{\pi}_n$ is surjective. Thus, we are reduced to proving that the induced tangent map $\mathrm{d}\overline{\boldsymbol{\pi}}_n : \mathrm{Lie}\,\overline{\Gamma}_{\mathrm{P}_n M_\rho} \to \mathrm{Lie}\,\overline{\Gamma}_{\mathrm{P}_{n-1} M_\rho}$ is surjective.

Similar to the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$ in (4.14), we make the choice to let the coordinates of $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$ be

$$\mathbf{Y} := \begin{pmatrix} \mathbf{Y}_0 & \mathbf{Y}_1 & \dots & \dots & \mathbf{Y}_n \\ & \mathbf{Y}_0 & \ddots & \ddots & \vdots \\ & & \mathbf{Y}_0 & \ddots & \vdots \\ & & & \ddots & \mathbf{Y}_1 \\ & & & & \mathbf{Y}_0 \end{pmatrix},$$

where $\mathbf{Y}_h := ((Y_h)_{ij})$, an $r \times r$ matrix for $h = 0, 1, \dots, n$. Recall $\mathbf{X}$, the coordinates of $\Gamma_{\mathrm{P}_n M_\rho}$, from (4.14). Then by construction we have $\mathbf{X} = d_{t,n+1}[\mathrm{A}]\mathbf{Y} d_{t,n+1}[\mathrm{A}]^{-1}$ and so for each $w = 0, \dots, n$, we obtain

$$\mathbf{X}_w = \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \sum_{h=0}^{w_1} \partial_t^{w_1-h}(\mathrm{A}) \cdot \mathbf{Y}_{w_2} \cdot (\partial_t^h(\mathrm{A}))^{-1},$$

where the hyperderivatives are taken entrywise. Then, we have

$$\begin{aligned} \mathbf{vec}(\mathbf{X}_w) &= \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \sum_{h=0}^{w_1} \left([(\partial_t^h(\mathrm{A}))^{-1}]^{\mathsf{T}} \otimes \partial_t^{w_1-h}(\mathrm{A}) \right) \cdot \mathbf{vec}(\mathbf{Y}_{w_2}) \\ &= \sum_{\substack{w_1+w_2=w \\ w_1,w_2 \geq 0}} \partial_t^{w_1} \left((\mathrm{A}^{-1})^{\mathsf{T}} \otimes \mathrm{A} \right) \cdot \mathbf{vec}(\mathbf{Y}_{w_2}), \end{aligned}$$

where we obtain the first equality by using properties of the Kronecker product and the second equality by further applying the product rule for hyperderivatives. This implies

$$\mathbf{vec}\begin{pmatrix}\mathbf{X}_n\\ \mathbf{X}_{n-1}\\ \vdots\\ \mathbf{X}_0\end{pmatrix} = d_{t,n+1}[(\mathrm{A}^{-1})^{\mathsf{T}}\otimes \mathrm{A}]\cdot \mathbf{vec}\begin{pmatrix}\mathbf{Y}_n\\ \mathbf{Y}_{n-1}\\ \vdots\\ \mathbf{Y}_0\end{pmatrix} \tag{4.28}$$

where we set

$$\mathbf{vec}([\mathbf{X}_n,\ldots,\mathbf{X}_0]^{\mathsf{T}}) := [(\mathbf{vec}\,\mathbf{X}_n)^{\mathsf{T}},\ldots,(\mathbf{vec}\,\mathbf{X}_0)^{\mathsf{T}}]^{\mathsf{T}},$$

and we define $\mathbf{vec}([\mathbf{Y}_n,\ldots,\mathbf{Y}_0]^{\mathsf{T}})$ similarly. Recall $\overline{\operatorname{Im}\beta_n}^Z$ from Theorem 3. Note that by Theorem 8, we have $\overline{\operatorname{Im}\beta_n}^Z \subseteq \Gamma_{\mathrm{P}_n M_\rho}$. For $i=0,\ldots,n$, let $\overline{\mathbf{k}}[\mathbf{Y}_0,\ldots,\mathbf{Y}_i,1/\det\mathbf{Y}_0]$ be the polynomial ring over $\overline{\mathbf{k}}$ with $1/\det\mathbf{Y}_0$ and the entries of each $\mathbf{Y}_h$ for $h=0,\ldots,i$ as indeterminates. Then, by (4.20), (4.27) and (4.28), the defining ideal of $a(\overline{\operatorname{Im}\beta_n}^Z \times_{\mathbf{k}} \overline{\mathbf{k}})$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_0,\ldots,\mathbf{Y}_n,1/\det\mathbf{Y}_0]$ generated by the entries of

$$d_{t,n+1}[\mathbf{B}\cdot\big((\mathrm{A}^{-1})^{\mathsf{T}}\otimes \mathrm{A}\big)]\cdot\mathbf{vec}\begin{pmatrix}\mathbf{Y}_n\\ \mathbf{Y}_{n-1}\\ \vdots\\ \mathbf{Y}_0\end{pmatrix}. \tag{4.29}$$

First suppose $n=1$. We consider the following short exact sequence of linear algebraic groups

$$1\to \overline{Q}_1 \to \overline{\Gamma}_{\mathrm{P}_1 M_\rho} \xrightarrow{\overline{\boldsymbol{\pi}}_1} \prod_{i=1}^{s}(\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i \to 1.$$

Our task is to prove that the induced tangent map at the identity $\mathrm{d}\overline{\boldsymbol{\pi}}_1$ is surjective. It is clear by observing $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ that for $\mathbf{Y}_0 := ((Y_0)_{i,j})$, the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_0,1/\det\mathbf{Y}_0]$ generated by

$$\{(Y_0)_{i,j} \mid (i,j)\neq(ur/s+v_1, ur/s+v_2), u=0,\ldots,s-1 \text{ and } v_1,v_2=1,\ldots,r/s\}. \tag{4.30}$$

Moreover, by (4.16) and (4.28), the defining ideal of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ is also generated by the entries of

$$(\mathbf{B}\cdot((\mathrm{A}^{-1})^{\mathsf{T}}\otimes \mathrm{A}))\cdot \mathbf{vec}(\mathbf{Y}_0). \tag{4.31}$$

By (4.30) and (4.31), we see that the entries of $\mathbf{B}\cdot((A^{-1})^{\mathsf{T}}\otimes A)$ that give relations among

$$\{(Y_0)_{i,j} \mid (i,j) = (ur/s+v_1, ur/s+v_2), u = 0,\dots,s-1 \text{ and } v_1, v_2 = 1,\dots,r/s\}$$

are all zero. Therefore, the hyperderivatives of these entries are also all zero. Using this and using (4.30), for $\gamma_0 \in \prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ we see that

$$\partial_t^1\left(\mathbf{B}\cdot((\mathrm{A}^{-1})^{\mathsf{T}}\otimes \mathrm{A})\right)\cdot\gamma_0 = \mathbf{0}. \tag{4.32}$$

By using (4.31) and (4.32), we have

$$d_{t,2}[\mathbf{B}\cdot((\mathrm{A}^{-1})^{\mathsf{T}}\otimes \mathrm{A})]\cdot\mathbf{vec}\begin{pmatrix}\mathbf{0}\\ \gamma_0\end{pmatrix} = \mathbf{0}$$

and so by (4.29) for $n=1$, we see that

$$\overline{\gamma}_0 = \left(\begin{array}{c|c}\gamma_0 & \mathbf{0}\\ \hline \mathbf{0} & \gamma_0\end{array}\right) \in \overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}})$$

is a pre-image of γ_0 under the map $\overline{\boldsymbol{\pi}}_1$. Let $\overline{Q}_{1,1}$ be the Zariski closure of the subgroup inside $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ generated by all $\overline{\gamma}_0$ with γ_0 running over all elements of $\prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$. Then, $\overline{Q}_{1,1} \cong \prod_{i=1}^{s}(\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$. Since $\mathrm{Lie}(\cdot)$ is a left exact functor, when we restrict $\mathrm{d}\overline{\boldsymbol{\pi}}_1$ to $\mathrm{Lie}(\overline{Q}_{1,1})$, we obtain a surjection onto $\prod_{i=1}^{s}(\mathrm{Mat}_{r/s}(\overline{\mathbf{k}})_i)$. Thus, $\mathrm{d}\overline{\boldsymbol{\pi}}_1$ is surjective and so Q_1 is smooth over $\mathbf{k}$.

Now let $n=2$. We consider the following short exact sequence of linear algebraic groups

$$1 \to \overline{Q}_2 \to \overline{\Gamma}_{\mathrm{P}_2 M_\rho} \xrightarrow{\overline{\boldsymbol{\pi}}_2} \overline{\Gamma}_{\mathrm{P}_1 M_\rho} \to 1.$$

Since Q_1 is smooth over $\mathbf{k}$, by Theorem 3 and (4.29) the defining ideal of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_0, \mathbf{Y}_1, 1/\det \mathbf{Y}_0]$ generated by the entries of

$$d_{t,2}[\mathbf{B} \cdot ((\mathrm{A}^{-1})^\intercal \otimes \mathrm{A})] \cdot \mathbf{vec}\left(\begin{smallmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_0 \end{smallmatrix} \right). \tag{4.33}$$

Recall that $\mathbf{Y}_0 = ((Y_0)_{i,j})$ and $\mathbf{Y}_1 = ((Y_1)_{i,j})$. Since the defining ideal of $\prod_{i=1}^s (\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_0, 1/\det \mathbf{Y}_0]$ generated by the entries of (4.31), we see that the defining ideal of $\prod_{i=1}^s (\mathrm{Mat}_{r/s}(\overline{\mathbf{k}}))_i$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_1]$ generated by the entries of

$$(\mathbf{B} \cdot ((\mathrm{A}^{-1})^\intercal \otimes \mathrm{A})) \cdot \mathbf{vec}(\mathbf{Y}_1). \tag{4.34}$$

Therefore, using (4.32) we see that for all $\gamma_0 \in \prod\limits_{i=1}^s (\mathrm{GL}_{r/s} / \overline{\mathbf{k}})_i$ and $\gamma_1 \in \prod\limits_{i=1}^s (\mathrm{Mat}_{r/s} / \overline{\mathbf{k}})_i$, we have

$$d_{t,2}[\mathbf{B} \cdot ((\mathrm{A}^{-1})^\intercal \otimes \mathrm{A})] \cdot \mathbf{vec} \begin{pmatrix} \gamma_1 \\ \gamma_0 \end{pmatrix} = \mathbf{0}.$$

Thus, by (4.33) we have

$$\overline{\Gamma}_{\mathrm{P}_1 M_\rho} = \left\{ \left(\begin{array}{c|c} \gamma_0 & \gamma_1 \\ \hline \mathbf{0} & \gamma_0 \end{array} \right) \, : \, \gamma_0 \in \prod_{i=1}^s (\mathrm{GL}_{r/s} / \overline{\mathbf{k}})_i, \; \gamma_1 \in \prod_{i=1}^s (\mathrm{Mat}_{r/s} / \overline{\mathbf{k}})_i \right\}. \tag{4.35}$$

By (4.35), the defining ideal of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ is the ideal in $\overline{\mathbf{k}}[\mathbf{Y}_0, \mathbf{Y}_1, 1/\det \mathbf{Y}_0]$ generated by

$$\{(Y_0)_{i,j}, (Y_1)_{i,j} \mid (i,j) \neq (ur/s + v_1, ur/s + v_2), u = 0, \ldots, s-1 \text{ and } v_1, v_2 = 1, \ldots, r/s\}. \tag{4.36}$$

By (4.33) and (4.36), we see that the entries of $d_{t,2}[\mathbf{B} \cdot ((A^{-1})^\intercal \otimes A)]$ that give relations among

$$\{(Y_0)_{i,j}, (Y_1)_{i,j} \mid (i,j) = (ur/s + v_1, ur/s + v_2), u = 0, \ldots, s-1 \text{ and } v_1, v_2 = 1, \ldots, r/s\}$$

are all zero. Therefore, the hyperderivatives of these entries are also all zero. Using this and (4.36), we see that for any $\overline{\gamma}_{0,1} = \left(\begin{smallmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{smallmatrix}\right) \in \overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}})$,

$$\partial_t^2 \left(\mathbf{B} \cdot ((\mathrm{A}^{-1})^\mathsf{T} \otimes \mathrm{A})\right) \cdot \gamma_0 = \mathbf{0}; \;\; \partial_t^1 \left(\mathbf{B} \cdot ((\mathrm{A}^{-1})^\mathsf{T} \otimes \mathrm{A})\right) \cdot \gamma_1 = \mathbf{0},$$

and so by (4.29) for $n = 2$, we see that

$$\overline{\gamma}_1 = \left(\begin{array}{c|c|c} \gamma_0 & \gamma_1 & \mathbf{0} \\ \hline \mathbf{0} & \gamma_0 & \gamma_1 \\ \hline \mathbf{0} & \mathbf{0} & \gamma_0 \end{array}\right) \in \overline{\Gamma}_{\mathrm{P}_2 M_\rho}(\overline{\mathbf{k}})$$

is a pre-image of $\overline{\gamma}_{0,1}$ under the map $\overline{\boldsymbol{\pi}}_2$. Let $\overline{Q}_{2,1}$ be the Zariski closure of the subgroup inside $\overline{\Gamma}_{\mathrm{P}_2 M_\rho}$ generated by all $\overline{\gamma}_1$ with γ_0 and γ_1 respectively running over all elements of $\prod_{i=1}^s (\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i$ and $\prod_{i=1}^s (\mathrm{Mat}_{r/s}(\overline{\mathbf{k}}))_i$, that is, $\overline{\gamma}_{0,1}$ running over all elements of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}})$. Then, $\overline{Q}_{2,1} \cong \overline{\Gamma}_{\mathrm{P}_1 M_\rho}$. Since $\mathrm{Lie}(\cdot)$ is a left exact functor, when we restrict $\mathrm{d}\overline{\boldsymbol{\pi}}_2$ to $\mathrm{Lie}(\overline{Q}_{2,1})$, we obtain a surjection onto $\mathrm{Lie}\,\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$.

For the general n case, after having proven that Q_{n-1} is smooth over $\mathbf{k}$, it follows by Theorem 3 and by applying the same methods used to determine (4.35) that

$$\overline{\Gamma}_{\mathrm{P}_{n-1} M_\rho} = \left\{ \left(\begin{array}{c|c|c|c} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \\ \hline & \gamma_0 & \ddots & \vdots \\ \hline & & \ddots & \gamma_1 \\ \hline & & & \gamma_0 \end{array}\right) : \begin{array}{c} \gamma_0 \in \displaystyle\prod_{i=1}^s (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i, \gamma_j \in \prod_{i=1}^s (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i, \\ \forall\, j = 1, \ldots, n-1 \end{array} \right\}. \quad (4.37)$$

Similar to $\overline{Q}_{1,1}$ and $\overline{Q}_{2,1}$, we construct $\overline{Q}_{n,1}$, the Zariski closure of the subgroup inside $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$

generated by

$$\left\{\left(\begin{array}{c|c|c|c|c} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} & \mathbf{0} \\ \hline & \gamma_0 & \ddots & \ddots & \gamma_{n-1} \\ \hline & & \ddots & \ddots & \vdots \\ \hline & & & \ddots & \gamma_1 \\ \hline & & & & \gamma_0 \end{array}\right) : \begin{array}{c} \gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i, \gamma_j \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i, \\ \forall j = 1, \ldots, n-1 \end{array}\right\} \subseteq \overline{\Gamma}_{\mathrm{P}_n M_\rho}. \tag{4.38}$$

Then, $\overline{Q}_{n,1} \cong \overline{\Gamma}_{\mathrm{P}_{n-1}M_\rho}$. Since $\mathrm{Lie}(\cdot)$ is a left exact functor, restricting $\mathrm{d}\overline{\boldsymbol{\pi}}_n$ to $\mathrm{Lie}\,\overline{Q}_{n,1}$ gives a surjection onto $\mathrm{Lie}\,\overline{\Gamma}_{\mathrm{P}_{n-1}M_\rho}$. The details are similar to the $n = 2$ case and so, we leave the task of constructing $\overline{Q}_{n,1}$ to the reader. □

4.5 Algebraic Independence of periods and quasi-periods

The following result proves Theorem 2.

Theorem 12. *Fix $n \geq 1$. Let ρ be a Drinfeld $\mathbf{A}$-module of rank r defined over k^{sep} and $\mathrm{P}_n\rho$ be its associated n-th prolongation t-module. Suppose that K_ρ is separable over k. Let M_ρ and P_nM_ρ be the t-motives corresponding to ρ and $\mathrm{P}_n\rho$ respectively. Then,* $\mathrm{tr.deg}_{\overline{k}}\,\overline{k}(\Psi_{\mathrm{P}_n\rho}(\theta)) = (n+1)\cdot r^2/s$, *where $s = [K_\rho : k]$. In particular,*

$$\mathrm{tr.deg}_{\overline{k}}\ \overline{k}\left(\bigcup_{s=1}^{n}\bigcup_{i=1}^{r-1}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), \partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j))\}\right) = (n+1)\cdot r^2/s.$$

Proof. By Theorem 7, we have

$$\overline{k}(\Psi_{\mathrm{P}_n\rho}(\theta)) = \overline{k}\left(\bigcup_{s=1}^{n}\bigcup_{i=1}^{r-1}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), \partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j))\}\right). \tag{4.39}$$

Moreover, by Theorem 6 we have

$$\dim \Gamma_{\mathrm{P}_nM_\rho} = \mathrm{tr.deg}_{\overline{k}}\,\overline{k}(\Psi_{\mathrm{P}_n\rho}(\theta)).$$

Then, the result follows from Corollary 3 and Lemma 3. $\square$

5. HYPERDERIVATIVES OF LOGARITHMS AND QUASI-LOGARITHMS

In this chapter, we fix a Drinfeld $\mathbf{A}$-module ρ of rank r defined over k^{sep} and an $\mathbf{A}$-basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ_ρ as in §3.1. We let M_ρ be the t-motive associated to ρ along with a fixed $\overline{k}(t)$-basis $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$, multiplication by σ given by Φ_ρ as in (3.2), and rigid analytic trivialization Ψ_ρ as in (3.10). Also for each $n \geq 1$, let $\mathrm{P}_n M_\rho$ be the t-motive corresponding to the n-th prolongation $\mathrm{P}_n\rho$ of ρ as in §3.1. Note that the 0-th prolongation t-motive $\mathrm{P}_0 M_\rho$ is simply M_ρ via the map $D_0 m \mapsto m$ for all $m \in M_\rho$. Then, a $\overline{k}(t)$-basis of $\mathrm{P}_n M_\rho$ is $\boldsymbol{D}_n\boldsymbol{m} \in \mathrm{Mat}_{(n+1)r\times 1}(\mathrm{P}_n M_\rho)$ (see (2.3)) such that multiplication by σ is given by $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$ (see (2.4)) with rigid analytic trivialization $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$ (see (2.5)). We also set $\mathbf{K}_\rho := \mathrm{End}_{\mathcal{T}}(M_\rho)$ as in (4.24) and let K_ρ denote the field of fractions of $\mathrm{End}(\rho)$.

In what follows, we adapt the methods of Chang and Papanikolas (see [12, §5]).

5.1 t-motives and quasi-logarithms

Given $u \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u) = \alpha \in k^{\mathrm{sep}}$, we set $f_u(t)$ to be the Anderson generating function of ρ with respect to u as in (3.5). Then, for $n \geq 1$ we see that by (4.3) the Anderson generating function of $\mathrm{P}_n\rho$ with respect to $\boldsymbol{u}_n = [u, 0, \ldots, 0]^{\mathsf{T}} \in \mathbb{K}^{n+1}$ is $\mathcal{G}_{u,1}(t) = [f_u(t), \partial_t^1(f_u(t)), \ldots, \partial_t^n(f_u(t))]^{\mathsf{T}}$. Moreover, by (3.4) we obtain

$$\mathrm{Exp}_{\mathrm{P}_n\rho}(\boldsymbol{u}_n) = [\mathrm{Exp}_\rho(u), 0 \ldots, 0]^{\mathsf{T}} = [\alpha, 0 \ldots, 0]^{\mathsf{T}} \in (k^{\mathrm{sep}})^{n+1}.$$

We define

$$\mathbf{s}_\alpha := \begin{pmatrix} -(t-\theta)f_u(t) - \alpha \\ -(\kappa_2^{(-1)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-1)} f_u^{(r-2)}(t) + \kappa_r^{(-1)} f_u^{(r-1)}(t)) \\ -(\kappa_3^{(-2)} f_u^{(1)}(t) + \cdots + \kappa_{r-1}^{(-2)} f_u^{(r-3)}(t) + \kappa_r^{(-2)} f_u^{(r-2)}(t)) \\ \vdots \\ -\kappa_r^{(-r+1)} f_u^{(1)}(t) \end{pmatrix}^{\mathsf{T}} \in \mathrm{Mat}_{1\times r}(\mathbb{T}),$$

as in [12, §4.2] and let $\mathbf{h}_{\alpha,n} := (\alpha, 0, \ldots, 0) \in \mathrm{Mat}_{1\times(n+1)r}(k^{\mathrm{sep}})$. Then, we define the pre-t-motive $Y_{\alpha,n}$ of dimension $(n+1)r+1$ over $\overline{k}(t)$ such that multiplication by σ is given by $\Phi_{\alpha,n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{h}_{\alpha,n} & 1 \end{pmatrix}$. If we set $\mathbf{g}_{\alpha,n} := (\mathbf{s}_\alpha, \partial_t^1(\mathbf{s}_\alpha), \ldots, \partial_t^n(\mathbf{s}_\alpha))$, then we obtain the difference equation $\mathbf{g}_{\alpha,n}^{(-1)}\Phi_{\mathrm{P}_n\rho} = \mathbf{g}_{\alpha,n} + \mathbf{h}_{\alpha,n}$, and so we set $\Psi_{\alpha,n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{g}_{\alpha,n}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix}$ to obtain $\Psi_{\alpha,n}^{(-1)} = \Phi_{\alpha,n}\Psi_{\alpha,n}$. Thus, $Y_{\alpha,n}$ is rigid analytically trivial.

Proposition 4. *Let $u \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u) = \alpha \in k^{\mathrm{sep}}$. The rigid analytically trivial pre-t-motive $Y_{\alpha,n}$ is a t-motive.*

Proof. To prove that $Y_{\alpha,n}$ is a t-motive, we follow the arguments of the proof of [34, Prop. 6.1.3]. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $\mathbf{A}$-finite dual t-motives such that multiplication by σ on $\overline{k}[t]$-bases are represented by Φ_1 amd Φ_2 respectively. Then, multiplication by σ on a suitable $\overline{k}[t]$-basis of the tensor product $\mathcal{M}_1 \otimes_{\overline{k}(t)} \mathcal{M}_2$ is represented by the Kronecker product $\Phi_1 \otimes \Phi_2$ (see [34, §3.2.5]). Let $\mathcal{C}$ be the $\mathbf{A}$-finite dual t-motive associated to the Carlitz module $\mathfrak{C}$ (rank 1 Drinfeld $\mathbf{A}$-module) uniquely determined by $\mathfrak{C}_t = \theta + \tau$ (see [34, §3.4.3]. Let $C := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{C}$ be the pre-t-motive associated to $\mathcal{C}$.

We claim that the pre-t-motive $C \otimes_{\overline{k}(t)} Y_{\alpha,n}$ is in the essential image of the functor $\mathcal{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$ of [34, Thm. 3.4.9] (see §2.2). By the definition of the category $\mathcal{T}$ in [34, §3.4.10] (see §2.2), it follows that $Y_{\alpha,n}$ is a t-motive.

Let $\mathcal{N} := \mathrm{Mat}_{1\times(n+1)r+1}(\overline{k}[t])$ and let $\boldsymbol{e} := [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{(n+1)r+1}]^{\mathsf{T}}$ be its standard $\overline{k}[t]$-basis. We give $\mathcal{N}$ a left $\overline{k}[t,\sigma]$-module structure by setting

$$\sigma\boldsymbol{e} = (t-\theta)\Phi_{\alpha,n}\boldsymbol{e}.$$

We obtain the following short exact sequence of $\overline{k}[t,\sigma]$-modules:

$$0 \to \mathcal{C} \otimes_{\overline{k}[t]} \mathrm{P}_n\mathcal{M}_\rho \to \mathcal{N} \to \mathcal{C} \to 0. \tag{5.1}$$

Since $\mathcal{C}$ and $\mathcal{C} \otimes_{\overline{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ are finitely generated left $\overline{k}[\sigma]$-modules, it follows from [2, Prop. 4.3.2] that $\mathcal{N}$ is free and finitely generated as a left $\overline{k}[\sigma]$-module. Since $\mathcal{C} \otimes_{\overline{k}[t]} \mathrm{P}_n\mathcal{M}_\rho$ is an $\mathbf{A}$-finite dual t-motive, we have $(t-\theta)^u(\mathcal{C} \otimes_{\overline{k}[t]} \mathrm{P}_n\mathcal{M}_\rho) \subseteq \sigma(\mathcal{C} \otimes_{\overline{k}[t]} \mathrm{P}_n\mathcal{M}_\rho)$ for $u \in \mathbb{N}$ sufficiently large. Moreover, $(t-\theta)\mathcal{C} = \sigma\mathcal{C}$ and so, by (5.1) we obtain $(t-\theta)^v\mathcal{N} \subseteq \sigma\mathcal{N}$ for $v \in \mathbb{N}$ sufficiently large. Thus, by [34, §3.4.1] (see §2.2) we see that $\mathcal{N}$ is an $\mathbf{A}$-finite dual t-motive. This proves our claim. $\square$

5.2 Non-triviality in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$

We continue with the t-motive $Y_{\alpha,n}$ from the previous section. Recall the trivial object $\mathbf{1}$ of $\mathcal{R}$ (see §2.2). Note that $Y_{\alpha,n}$ represents a class in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Suppose $e \in \mathrm{End}_{\mathcal{T}}(M_\rho)$ and let $\mathrm{E} \in \mathrm{Mat}_r(\overline{k}(t))$ such that $e(\boldsymbol{m}) = \mathrm{E}\boldsymbol{m}$. If we set

$$\mathbf{E} := \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathrm{E} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\overline{k}(t)), \tag{5.2}$$

then one checks easily that $\mathbf{E}$ represents an element $\mathbf{e}$ of $\mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$. For classes Y_1 and Y_2 in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, if multiplication by σ on suitable $\overline{k}(t)$-bases are represented by $\left(\begin{smallmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{v}_1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{v}_2 & 1 \end{smallmatrix}\right)$ respectively, then their Baer sum in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$ is achieved by the matrix $\left(\begin{smallmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{v}_2+\mathbf{v}_2 & 1 \end{smallmatrix}\right)$. Moreover, we see that multiplication by σ on a $\overline{k}(t)$-basis of the pushout $\mathbf{e}_*Y_1$ is represented by $\left(\begin{smallmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \mathbf{v}_1\mathbf{E} & 1 \end{smallmatrix}\right)$.

Theorem 13. *Suppose $u_1, \dots, u_w \in \mathbb{K}$ such that $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each i. For $n \geq 1$, we let $Y_{i,n} := Y_{\alpha_i,n}$ be as above. Suppose that $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w) = r/s + w$. Then, for $e_1, \dots, e_w \in \mathbf{K}_\rho$, not all zero, $S := \mathbf{e}_{1*}Y_{1,n} + \cdots + \mathbf{e}_{w*}Y_{w,n}$ is non-trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$, where each $\mathbf{e}_i \in \mathrm{End}_{\mathcal{T}}(\mathrm{P}_n M_\rho)$ corresponds to e_i as in* (5.2)*.*

Proof. We adapt the ideas of the proof of [12, Thm. 4.2.2]. For each i, we let $\mathbf{h}_{i,n} := \mathbf{h}_{\alpha_i,n}$ and $\mathbf{g}_{i,n} := \mathbf{g}_{\alpha_i,n}$. Fix $E_i \in \mathrm{Mat}_r(\overline{k}(t))$ so that $e_i(\boldsymbol{m}) = E_i\boldsymbol{m}$ for each i. Then $\mathbf{e}_i(\boldsymbol{D}_n\boldsymbol{m}) = \mathbf{E}_i \cdot \boldsymbol{D}_n\boldsymbol{m}$, where $\mathbf{E}_i$ is as in (5.2). By choosing an appropriate $\overline{k}(t)$-basis $\mathbf{s}$ for S, multiplication by σ on $\mathbf{s}$ is represented by

$$\Phi_S := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^w \mathbf{h}_{i,n}\mathbf{E}_i & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\overline{k}(t)),$$

and a corresponding rigid analytic trivialization is represented by

$$\Psi_S := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & \mathbf{0} \\ \sum_{i=1}^{w} \mathbf{g}_{i,n}\mathbf{E}_i\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\mathbb{L}).$$

Suppose on the contrary that S is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho)$. Then, there exists a $\overline{k}(t)$-basis $\mathbf{s}'$ of S such that $\sigma\mathbf{s}' = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\mathbf{s}'$, where $\Phi_{\mathrm{P}_n\rho} \oplus (1)$ is the block diagonal matrix with $\Phi_{\mathrm{P}_n\rho}$ and 1 in the diagonal blocks and all other entries are zero. If we let $\gamma = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \gamma_0 \cdots \gamma_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\overline{k}(t))$, where $\gamma_i := (\gamma_{i1}, \dots, \gamma_{ir})$ for each i be the matrix such that $\mathbf{s}' := \gamma\mathbf{s}$, then we obtain

$$\gamma^{(-1)}\Phi_S = (\Phi_{\mathrm{P}_n\rho} \oplus (1))\gamma. \tag{5.3}$$

Note from [34, Proof of Prop. 3.4.5] that all denominators of entries of γ are in $\mathbb{F}_q[t]$ and so in particular, each γ_{ij} is regular at $t = \theta, \theta^q, \theta^{q^2}, \dots$. Using $\Phi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Phi_\rho]$, the $((n+1)r + 1, (n-j) \cdot r + 1)$-th entry of (5.3) for each $j = 1, \dots, n$ is

$$\sum_{h=0}^{n-j} \gamma_{h,r}^{(-1)} \partial_t^{n-j-h} \left((t-\theta)/\kappa_r^{(-r)}\right) = \gamma_{n-j,1},$$

and the $((n+1)r + 1, nr + 1)$-th entry is

$$\sum_{j=0}^{n} \gamma_{n-j,r}^{(-1)} \partial_t^{j} \left((t-\theta)/\kappa_r^{(-r)}\right) + \sum_{i=1}^{w} \alpha_i (E_i)_{11} = \gamma_{n,1}.$$

For each $j = 0, 1, \dots, n$, applying $(-1)^j\partial_t^j(\cdot)$ to each $((n+1)r + 1, (n-j) \cdot r + 1)$-th entry and then adding them, we obtain (by applying the product rule of hyperderivatives and using the property $\partial_t^v\partial_t^w(f(t)) = \binom{v+w}{v}\partial_t^{v+w}(f(t))$)

$$\sum_{j=0}^{n} (-1)^j \partial_t^j (\gamma_{n-j,r})^{(-1)} \left((t-\theta)/\kappa_r^{(-r)}\right) + \sum_{i=1}^{w} \alpha_i (E_i)_{11} = \sum_{j=0}^{n} (-1)^j \partial_t^j (\gamma_{n-j,1}). \tag{5.4}$$

Specializing both sides of this equation at $t = \theta$, we obtain

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) = \sum_{i=1}^{w} \alpha_i (E_i)_{11}(\theta). \tag{5.5}$$

Moreover, (5.3) implies that $(\gamma\Psi_S)^{(-1)} = (\Phi_{\mathrm{P}_n\rho} \oplus (1))(\gamma\Psi_S)$ and so by [34, §4.1.6], for some $\delta = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & \mathbf{0} \\ \delta_0 \ldots \delta_n & 1 \end{pmatrix} \in \mathrm{GL}_{(n+1)r+1}(\mathbf{k})$ where $\delta_i := (\delta_{i1}, \ldots, \delta_{ir})$ for each i, we have

$$\gamma\Psi_S = (\Psi_{\mathrm{P}_n\rho} \oplus (1))\delta. \tag{5.6}$$

Since $\Psi_{\mathrm{P}_n\rho} = d_{t,n+1}[\Psi_\rho]$, by applying to (5.6) the same methods applied on (5.3) to obtain (5.4), it follows that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j}) + \sum_{i=1}^{w} \mathbf{s}_i E_i = \sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j})\Psi_\rho^{-1}, \tag{5.7}$$

where for each i and j, we set $\partial_t^j(\gamma_i) := (\partial_t^j(\gamma_{i1}), \ldots, \partial_t^j(\gamma_{ir}))$. Since for each i the first entry of $\mathbf{s}_i(\theta)$ is $u_i - \alpha_i$, using [12, Prop. 4.1.1(b)] and specializing both sides of (5.7) at $t = \theta$, we see that

$$\sum_{j=0}^{n}(-1)^j \partial_t^j(\gamma_{n-j,1})(\theta) + \sum_{i=1}^{w}(u_i - \alpha_i)(E_i)_{11}(\theta) = -\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m,$$

and so from (5.5) we have

$$\sum_{m=1}^{r}\sum_{j=0}^{n}(-1)^j \partial_t^j(\delta_{n-j,m})(\theta)\lambda_m + \sum_{i=1}^{w}(E_i)_{11}(\theta)u_i = 0.$$

Since $e_1, \ldots, e_w$ are not all zero, E_i is nonzero for some i. Moreover, by Proposition 2 we see that $\mathbf{K}_\rho \cong K_\rho$ and so E_i is invertible. By [12, Prop. 4.1.1(b),(c)] we get $(E_i)_{11}(\theta) \in K_\rho^\times$ and thus we get a contradiction. □

5.3 Construction of the t-motives Y and N

In this section, we construct a t-motive that is suitable for the investigation of the hyperderivatives of logarithms and quasi-logarithms of the Drinfeld $\mathbf{A}$-module ρ, and the study of its Galois group. Suppose that we have $u_1, \ldots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each i. We let $\mathbf{h}_{\alpha_i} := \mathbf{h}_{\alpha_i,n}, \mathbf{g}_{\alpha_i} := \mathbf{g}_{\alpha_i,n}, Y_{i,n} := Y_{\alpha_i,n}, \Phi_{i,n} := \Phi_{\alpha_i,n}$ and $\Psi_{i,n} := \Psi_{\alpha_i,n}$ for each $n \geq 1$. The matrix $\Psi_n := \oplus_{i=1}^w \Psi_{i,n}$ gives the rigid analytic trivialization for $Y_n := \oplus_{i=1}^w Y_{i,n}$.

Define the t-motive N_n such that multiplication by σ on a $\overline{k}(t)$-basis is given by $\Phi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\overline{k}(t))$ along with rigid analytic trivialization $\Psi_{N_n} \in \mathrm{GL}_{(n+1)rw+1}(\mathbb{T})$ such that

$$\Phi_{N_n} := \begin{pmatrix} \Phi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Phi_{\mathrm{P}_n\rho} & \\ \mathbf{h}_{\alpha_1} & \ldots & \mathbf{h}_{\alpha_w} & 1 \end{pmatrix}, \quad \text{and} \quad \Psi_{N_n} := \begin{pmatrix} \Psi_{\mathrm{P}_n\rho} & & & \\ & \ddots & & \\ & & \Psi_{\mathrm{P}_n\rho} & \\ \mathbf{g}_{\alpha_1}\Psi_{\mathrm{P}_n\rho} & \ldots & \mathbf{g}_{\alpha_w}\Psi_{\mathrm{P}_n\rho} & 1 \end{pmatrix}.$$

Similar to the $n = 0$ case (see [12, §5.1]), N_n is an extension of $\mathbf{1}$ by $(\mathrm{P}_n M_\rho)^w$ which is a pullback of the surjective map $Y_n \twoheadrightarrow \mathbf{1}^w$ and the diagonal map $\mathbf{1} \to \mathbf{1}^w$. Thus, the two t-motives Y_n and N_n generate the same Tannakian subcategory of $\mathcal{T}$ and hence the Galois groups Γ_{Y_n} and Γ_{N_n} are isomorphic. For any $\mathbf{k}$-algebra R, an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form

$$\nu = \begin{pmatrix} \mu & & & \\ & \ddots & & \\ & & \mu & \\ \mathbf{v}_1 & \ldots & \mathbf{v}_w & 1 \end{pmatrix},$$

where $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and for each i, we have $\mathbf{v}_i = (\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,n+1})$ such that $\mathbf{v}_{i,h} \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$, for each $h = 0, \ldots, n$. Since $(\mathrm{P}_n M_\rho)^w$ is a sub-t-motive of N_n, we have the following short exact sequence of affine group schemes over $\mathbf{k}$,

$$1 \to X_n \to \Gamma_{N_n} \xrightarrow{\pi_n} \Gamma_{\mathrm{P}_n M_\rho} \to 1, \tag{5.8}$$

where $\pi_n^{(\mathrm{R})} : \Gamma_{N_n}(\mathrm{R}) \to \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is the map $\nu \mapsto \mu$ (cf. [12, p.138]). It can be checked directly that via conjugation, (5.8) gives an action of any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on

$$\mathbf{v} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \mathbf{u}_1 & \dots & \mathbf{u}_w & 1 \end{pmatrix} \in X_n(\mathrm{R})$$

given by

$$\nu \mathbf{v} \nu^{-1} = \begin{pmatrix} \mathrm{Id}_{(n+1)r} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{(n+1)r} & \\ \mathbf{u}_1 \mu^{-1} & \dots & \mathbf{u}_w \mu^{-1} & 1 \end{pmatrix}. \tag{5.9}$$

Lemma 4. *Let $n \geq 1$. If K_ρ is separable over k, then X_n in* (5.8) *is* $\mathbf{k}$*-smooth.*

Remark 3. Similar to what was said in Remark 2, Lemma 3 and this lemma are the reasons for the separability hypothesis in Theorem 4. However, suppose to the contrary that K_ρ is not separable over k but the hyperdifferential operator ∂_θ^j can be extended to K_ρ. In this case, if for $n \geq 1$ each Q_n and X_n are $\mathbf{k}$-smooth, then Theorem 4 holds for the Drinfeld $\mathbf{A}$-module ρ defined over k^{sep}.

Proof of Lemma 4. Similar to Lemma 3, we adapt the ideas of the proof of [11, Prop. 4.1.2] and the proof of a lemma from a preliminary version of [12] (Lemma 5.1.3: arXiv:1005.5120v1). To prove this, by [37, Cor. 12.1.3] it suffices to show that for $n \geq 1$, the induced tangent map $\mathrm{d}\pi_n$ at the identity is surjective onto $\operatorname{Lie} \Gamma_{\mathrm{P}_n M_\rho}$. We prove this for $w = 1$. The argument used in this case can be applied in a straightforward manner to prove the arbitrary w case, which we leave to the reader. First suppose $n = 1$. Let $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ and A be as in Lemma 3, that is, conjugation by $d_{t,2}[\mathrm{A}] \in \mathrm{GL}_{2r}(\mathbf{k}^{\mathrm{sep}})$ on $\Gamma_{\mathrm{P}_1 M_\rho}$ gives (see (4.35))

$$\overline{\Gamma}_{\mathrm{P}_1 M_\rho} = \left\{ \left(\begin{array}{c|c} \gamma_0 & \gamma_1 \\ \hline \mathbf{0} & \gamma_0 \end{array} \right) : \gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s} / \overline{\mathbf{k}})_i,\ \gamma_1 \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s} / \overline{\mathbf{k}})_i \right\}. \tag{5.10}$$

Let $(\oplus_{i=1}^{w} d_{t,2}[\mathrm{A}]) \oplus (1) \in \mathrm{GL}_{2rw+1}(\mathbf{k}^{\mathrm{sep}})$ be the block diagonal matrix such that $d_{t,2}[\mathrm{A}]$ is in the first w diagonal blocks and 1 in the last diagonal and all other entries are zero. Then, via conjugation by $(\oplus_{i=1}^{w} d_{t,2}[\mathrm{A}]) \oplus (1)$ on Γ_{N_1} we obtain $\overline{\Gamma}_{N_1}$ such that we have an isomorphism $\Gamma_{N_1} \times_{\mathbf{k}} \overline{\mathbf{k}} \cong \overline{\Gamma}_{N_1}$. Moreover, $\overline{\Gamma}_{N_1}$ is an algebraic subgroup of $\mathrm{GL}_{2rw+1}/\overline{\mathbf{k}}$ such that $\overline{\pi}_1 : \overline{\Gamma}_{N_1} \to \overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ induced by π_1 is surjective. Thus, we are reduced to proving that the induced tangent map $\mathrm{d}\overline{\pi}_1 : \mathrm{Lie}\,\overline{\Gamma}_{N_1} \to \mathrm{Lie}\,\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$ is surjective. Let $w = 1$ and consider the short exact sequence of linear algebraic groups

$$1 \to \overline{X}_1 \to \overline{\Gamma}_{N_1} \xrightarrow{\overline{\pi}_1} \overline{\Gamma}_{\mathrm{P}_1 M_\rho} \to 1. \tag{5.11}$$

From $\overline{\pi}_1$, we see that $\overline{X}_1$ is contained in the $2r$-dimensional additive group

$$G := \left\{ \begin{pmatrix} \mathrm{Id}_{r/s} & & & \\ & \ddots & & \\ & & \mathrm{Id}_{r/s} & \\ \mathbf{v}_1 & \dots & \mathbf{v}_{2s} & 1 \end{pmatrix} : \mathbf{v}_i \in \mathbb{G}_a^{r/s} \right\},$$

where we call $\mathbf{v}_1, \dots, \mathbf{v}_{2s}$ the coordinates of G. We see that via conjugation, $\overline{X}_1(\overline{\mathbf{k}})$ has a $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}})$-module structure coming from (5.20) (see (5.9)). Using (5.10) and this module structure, one checks easily that there is a natural decomposition $\overline{X}_1(\overline{\mathbf{k}}) = \prod_i^{2s} W_i$ such that each W_i is either zero or $\overline{\mathbf{k}}^{r/s}$. Fix any $1 \leq i \leq s$. For any $\gamma_i \in \mathrm{GL}_{r/s}(\overline{\mathbf{k}})$, we let

$$\overline{\gamma}_i = \left(\begin{array}{ccccccccccccc} \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & & & & & & \\ & \ddots & & & & & & \ddots & & & & & \\ & & \gamma_i & & & & & & \mathbf{0} & & & & \\ & & & \ddots & & & & & & \ddots & & & \\ & & & & \mathrm{Id}_{r/s} & & & & & & \mathbf{0} & & \\ \hline & & & & & & \mathrm{Id}_{r/s} & & & & & & \\ & & & & & & & \ddots & & & & & \\ & & & & & & & & \gamma_i & & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & \mathrm{Id}_{r/s} & & \\ \mathbf{u}_1 & \dots & \mathbf{u}_i & \dots & \mathbf{u}_s & & \mathbf{u}_{s+1} & \dots & \mathbf{u}_{s+i} & \dots & \mathbf{u}_{2s} & & 1 \end{array} \right) \in \overline{\Gamma}_{N_1}(\overline{\mathbf{k}})$$

be an arbitrary element, which by (5.10) and (5.11) is a pre-image of the matrix formed by the upper left $2r \times 2r$ square of $\overline{\gamma_i}$ under the map $\overline{\pi}_1$. For each $j \neq i$ with $1 \leq j \leq s$, we claim that if $\mathbf{u}_j \neq 0$ and $\mathbf{u}_{s+j} \neq 0$, then $W_j = W_{s+j} = \overline{\mathbf{k}}^{r/s}$. To prove this claim, assuming that $\mathbf{u}_j \neq 0$ and $\mathbf{u}_{s+j} \neq 0$ we pick $\delta_j \in \mathrm{GL}_{r/s}(\overline{\mathbf{k}})$ so that $\mathbf{u}_j \delta_j - \mathbf{u}_j \neq 0$ and $\mathbf{u}_{s+j}\delta_j - \mathbf{u}_{s+j} \neq 0$, and let $\overline{\delta}_j \in \overline{\Gamma}_{N_1}(\overline{\mathbf{k}})$ be such that

$$\overline{\pi}_1(\overline{\delta}_j) = \left(\begin{array}{ccccc} \mathrm{Id}_{r/s} & & & \mathbf{0} & \\ & \ddots & & & \ddots \\ & \delta_j & & & \mathbf{0} \\ & & \ddots & & & \ddots \\ & & \mathrm{Id}_{r/s} & & & \mathbf{0} \\ \hline & & & \mathrm{Id}_{r/s} & & \\ & & & & \ddots & \\ & & & & \delta_j & \\ & & & & & \ddots \\ & & & & & \mathrm{Id}_{r/s} \end{array}\right) \in \overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}}).$$

Then one checks directly that $\overline{\delta}_j^{-1}\overline{\gamma}_i\overline{\delta}_j\overline{\gamma}_i^{-1}$ is an element of $\overline{X}_1(\overline{\mathbf{k}})$ and its $\mathbf{v}_j$ and $\mathbf{v}_{s+j}$ coordinate vectors respectively are $\mathbf{u}_j\delta_j - \mathbf{u}_j$ and $\mathbf{u}_{s+j}\delta_j - \mathbf{u}_{s+j}$, and so it follows that $W_j = W_{s+j} = \overline{\mathbf{k}}^{r/s}$. Therefore, multiplying $\overline{\gamma}_i$ by a suitable element of $\overline{X}_1(\overline{\mathbf{k}})$ we get an element of the form

$$\overline{\gamma}_i' = \left(\begin{array}{ccccccccccccc} \mathrm{Id}_{r/s} & & & & & 0 & & & & & & & \\ & \ddots & & & & & \ddots & & & & & & \\ & & \gamma_i & & & & & \mathbf{0} & & & & & \\ & & & \ddots & & & & & \ddots & & & & \\ & & & & \mathrm{Id}_{r/s} & & & & & 0 & & & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & & & \\ & & & & & & \ddots & & & & & & \\ & & & & & & & \gamma_i & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & & & \\ \mathbf{0} & \dots & \mathbf{u}_i & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{u}_{s+i} & \dots & \mathbf{0} & 1 & & \end{array}\right) \in \overline{\Gamma}_{N_1}(\overline{\mathbf{k}}).$$

For any $\mathbf{b}_i \in \mathrm{Mat}_{r/s}(\overline{\mathbf{k}})$, by using a similar method as above where we take an element of the form $\overline{\delta}_j$, we obtain an element of the form

$$\overline{\mathbf{b}}_i' = \left(\begin{array}{ccccccccccccc} \mathrm{Id}_{r/s} & & & & & 0 & & & & & & & \\ & \ddots & & & & & \ddots & & & & & & \\ & & \mathrm{Id}_{r/s} & & & & & \mathbf{b}_i & & & & & \\ & & & \ddots & & & & & \ddots & & & & \\ & & & & \mathrm{Id}_{r/s} & & & & & 0 & & & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & & & \\ & & & & & & \ddots & & & & & & \\ & & & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & & & \\ \mathbf{0} & \dots & \mathbf{w}_i & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{w}_{s+i} & \dots & \mathbf{0} & 1 & & \end{array}\right) \in \overline{\Gamma}_{N_1}(\overline{\mathbf{k}}),$$

which is a pre-image of the matrix formed by the upper left $2r \times 2r$ square of $\overline{\mathbf{b}}_i'$ under the map $\overline{\pi}_1$. Let $\overline{X}_{1,i}$ be the Zariski closure inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all $\overline{\gamma}_i'$ with γ_i running over all elements of $\mathrm{GL}_{r/s}(\overline{\mathbf{k}})$ and $\mathbf{b}_i'$ with $\mathbf{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\overline{\mathbf{k}})$. Let

$$\left(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}}\right)_i := \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 \\ \mathbf{0} & \gamma_0 \end{pmatrix} : \gamma_0 \in (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i;\ \gamma_1 \in (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i \right\},$$

where we set $(\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i$ and $(\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i$ to be the canonical embeddings of $\mathrm{GL}_{r/s}/\overline{\mathbf{k}}$ and $\mathrm{Mat}_{r/s}/\overline{\mathbf{k}}$ into the i-th diagonal block matrix of $\mathrm{GL}_r/\overline{\mathbf{k}}$ and $\mathrm{Mat}_r/\overline{\mathbf{k}}$ respectively. Note that $\dim \overline{X}_{1,i} \leq 2r^2/s^2 + 2r/s$. First suppose that $\dim \overline{X}_{1,i} = 2r^2/s^2 + 2r/s$. Then, we could simply take $\overline{\gamma}_i'$ and $\overline{\mathbf{b}}_i'$ so that $\mathbf{u}_i, \mathbf{u}_{s+1}, \mathbf{w}_i$ and $\mathbf{w}_{s+i}$ are zero. Taking the Zariski closure $\overline{X}_{1,0,i}$ inside $\overline{\Gamma}_{N_1}$ of the subgroup generated by all such $\overline{\gamma}_i'$ and $\mathbf{b}_i'$ with γ_i running over all elements of $\mathrm{GL}_{r/s}(\overline{\mathbf{k}})$ and $\mathbf{b}_i'$ with $\mathbf{b}_i$ running over all elements of $\mathrm{Mat}_{r/s}(\overline{\mathbf{k}})$, we obtain

$$\overline{X}_{1,0,i} = \left\{ \begin{pmatrix} \nu_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \nu_i \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}}))_i \right\}. \tag{5.12}$$

Restricting $\mathrm{d}\overline{\pi}_1$ to $\mathrm{Lie}\,\overline{X}_{1,0,i}$, we obtain a surjection onto $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}})_i$. As we vary all $1 \leq i \leq s$, the surjection of $\mathrm{d}\overline{\pi}_1$ follows.

Next, suppose that $\dim X_{1,i} < 2r^2/s^2 + 2r/s$. Then, via $\overline{\pi}_1$ we have a short exact sequence,

$$1 \to Q_{1,i} \to \overline{X}_{1,i} \xrightarrow{\overline{\pi}_{1,i}} (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}})_i \to 1,$$

where $Q_{1,i}$ is contained in an additive subgroup of G whose $\mathbf{v}_j$ coordinate vector is zero for all $j \neq i, s+i$ and $\dim Q_{1,i} < 2r/s$.

We first show that $\dim Q_{1,i} = 0$. We follow the argument of the proof of [11, Lem. 4.1.1].

Suppose $\dim Q_{1,i} = m$, where $1 \leq m < 2r/s$. Note that $Q_{1,i}$ is a vector group. We claim that

$$Q_{1,i} \subseteq \left\{ \left(\begin{array}{ccccccccccc} \mathrm{Id}_{r/s} & & & & & \mathbf{0} & & & & & \\ & \ddots & & & & & \ddots & & & & \\ & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} & & & \\ & & & \ddots & & & & & \ddots & & \\ & & & & \mathrm{Id}_{r/s} & & & & & \mathbf{0} & \\ \hline & & & & & \mathrm{Id}_{r/s} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \mathrm{Id}_{r/s} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \mathrm{Id}_{r/s} & \\ \mathbf{0} & \dots & \mathbf{v}_i & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{v}_{s+i} & \dots & \mathbf{0} & 1 \end{array} \right) : \begin{array}{l} \mathbf{v}_i, \mathbf{v}_{s+i} \in \mathbb{G}_a^{r/s}, \text{ and} \\ \text{if } \mathbf{v}_i = (\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,r/s}), \text{ then} \\ \mathbf{v}_{i,u} = 0 \text{ for some } u \in \left\{1, \dots, \dfrac{r}{s}\right\} \end{array} \right\}.$$

If $\mathbf{v}_{i,j} \neq 0$ for all $j \in \{1, \dots, r/s\}$, then there exists $\overline{\mathbf{k}}$-linearly independent elements $\mu_1, \dots, \mu_m \in Q_{1,i}(\overline{\mathbf{k}})$ such that all the entries of μ_1 in the $\mathbf{v}_i$ coordinate vector are non-zero. For $a \in \overline{\mathbf{k}}$ such that $a \neq 0, 1$, pick $\eta \in \overline{X}_{1,i}(\overline{\mathbf{k}})$ such that

$$\overline{\pi}_{1,i}(\eta) = \left(\begin{array}{cccccccc} \mathrm{Id}_{r/s} & & & & \mathbf{0} & & & \\ & \ddots & & & & \ddots & & \\ & & \mathfrak{a} & & & & \mathbf{0} & \\ & & & \ddots & & & & \ddots \\ & & & \mathrm{Id}_{r/s} & & & & \mathbf{0} \\ \hline & & & & \mathrm{Id}_{r/s} & & & \\ & & & & & \ddots & & \\ & & & & & & \mathfrak{a} & \\ & & & & & & & \ddots \\ & & & & & & & \mathrm{Id}_{r/s} \end{array} \right) \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}}))_i,$$

where

$$\mathfrak{a} := \left(\begin{array}{cccccc} a & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ \hline & & & & a & 1 \\ & & & & & \ddots \\ & & & & & 1 \end{array} \right).$$

Then, one checks directly that $\eta^{-1}\mu_1\eta, \mu_1, \mu_2, \dots, \mu_m$ are $\overline{\mathbf{k}}$-linearly independent in $Q_{1,i}(\overline{\mathbf{k}})$, which contradicts $\dim Q_{1,i} = m$. This proves our claim. Thus, $\mathbf{v}_{i,u} = 0$ for some $u \in \{1, \dots, r/s\}$.

Now, since $m \neq 0$, at least one of $\mathbf{v}_{i,j}$, $j \in \{1, \dots, r/s\}$ is non-zero, say $\mathbf{v}_{i,v}$. Let $\mathbf{P}_{u,v}$ be the permutation matrix obtained by switching the $((i-1)r/s + u)$-th column and the $((i-1)r/s + v)$-

column of the $r \times r$ identity matrix. Pick $\gamma \in \overline{X}_{1,i}(\overline{\mathbf{k}})$ such that

$$\overline{\pi}_{1,i}(\gamma) = \begin{pmatrix} \mathbf{P}_{u,v} & \\ & \mathbf{P}_{u,v} \end{pmatrix} \in (\overline{\Gamma}_{\mathrm{P}_1 M_\rho}(\overline{\mathbf{k}}))_i.$$

Then, since $\gamma^{-1} Q_{1,i} \gamma \subseteq Q_{1,i}$, we get a contradiction to $\mathbf{v}_{i,u} = 0$. Therefore, $\dim Q_{1,i} = 0$.

Now, we claim that $\mathrm{d}\overline{\pi}_{1,i} : \mathrm{Lie}\, \overline{X}_{1,i} \to \mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}})_i$ is surjective. As we vary all $1 \leq i \leq s$, the surjection of $\mathrm{d}\overline{\pi}_1$ follows. To prove that $\mathrm{d}\overline{\pi}_{1,i}$ is surjective, we follow the argument of the proof of [11, Prop. 4.1.2]. We let the coordinates of $\overline{X}_{1,i}$ be as follows:

$$\mathbf{Z}_1 := \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \mathbf{0} \\ & \mathcal{Z}_0 & \mathbf{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & 1 \end{pmatrix}, \tag{5.13}$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/s} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{Z}_1 = \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & (Z_1) & & \\ & & & \ddots & \\ & & & & \mathbf{0} \end{pmatrix},$$

such that (Z_0) and (Z_1) are the coordinates of $\mathrm{GL}_{r/s}$ and $\mathrm{Mat}_{r/s}$ respectively, and for each $h = 0, 1$, we define (Z_h) to be the $r/s \times r/s$ block $((Z_h)_{a,b})$ for $1 \leq a, b \leq r/s$. Moreover, $\mathcal{W}_h := (0, \dots, 0, (W_h), 0, \dots 0)$, where we set $(W_h) := (W_{h,1}, \dots, W_{h,r/s})$ for each $h = 0, 1$. For $1 \leq u, v \leq r/s$, we define the following one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_1 M_\rho}$:

$$T_{uv} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \mathbf{0} \\ \mathbf{0} & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{uv} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathcal{C}_{uv} \\ \mathbf{0} & \mathrm{Id}_r \end{pmatrix} \right\}, \tag{5.14}$$

where we set

$$\mathcal{B}_{uv} := \begin{pmatrix} \mathrm{Id}_{r/s} & & & & \\ & \ddots & & & \\ & & B_{uv} & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{C}_{uv} := \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & C_{uv} & & \\ & & & \ddots & \\ & & & & \mathbf{0} \end{pmatrix} \tag{5.15}$$

such that

$$B_{vv} := \left\{ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & * & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \right\}, \quad B_{uv} := \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\}, \tag{5.16}$$

and

$$C_{uv} := \left\{ \begin{pmatrix} 0 & & & \\ & \ddots & * & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \right\}, \tag{5.17}$$

where $*$ in B_{uv} and C_{uv} are in the (u, v)-coordinates. Note that the Lie algebras of the $2 \cdot r^2/s^2$ algebraic groups T_{uv} and U_{uv} span $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}})_i$. We construct one dimension algebraic subgroups T'_{uv} and U'_{uv} of $\overline{X}_{1,i}$ so that $T'_{uv} \cong T_{uv}$ and $U'_{uv} \cong U_{uv}$. Then, since $\mathrm{Lie}(\cdot)$ is a left exact functor, it follows that $\mathrm{Lie}\, T'_{uv} \cong \mathrm{Lie}\, T_{uv}$ and $\mathrm{Lie}\, U'_{uv} \cong \mathrm{Lie}\, U_{uv}$, and so $\mathrm{d}\overline{\pi}_{1,i}$ is surjective. Since $Q_{1,i}$ is a zero dimensional vector group, $\overline{\pi}_{1,i}$ is injective on points and so it follows by checking directly that

- for $w \neq v$, all $W_{0,w}$ and $W_{1,w}$ coordinates of $\overline{\pi}_{1,i}^{-1}(T_{uv})$ are zero;
- all (W_0) coordinates of $\overline{\pi}_{1,i}^{-1}(U_{uv})$ are zero, and for $w \neq v$, all $W_{1,w}$ coordinates of $\overline{\pi}_{1,i}^{-1}(U_{uv})$

are zero.

To construct T'_{vv}, we let $a_v \in \overline{\mathbf{k}}^{\times} \setminus \overline{\mathbb{F}_q}^{\times}$ and pick element $\gamma_{1,v} \in \overline{X}_{1,i}(\overline{\mathbf{k}})$ so that

$$\overline{\pi}_{\ell,i}(\gamma_{1,v}) = \begin{pmatrix} \mathfrak{a}_v & \\ & \mathfrak{a}_v \end{pmatrix}, \text{ where } \mathfrak{a}_v = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a_v & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in (\mathrm{GL}_{r/s}(\overline{\mathbf{k}}))_i, \tag{5.18}$$

such that a_v is in the $(i \cdot r/s + v)$-th diagonal entry of $\mathfrak{a}_v$. For $1 \leq v \leq r/s$, we let $c_{0,v}$ and $c_{1,v}$ respectively be the $(2r+1, (i-1) \cdot r/s + v)$-th and the $(2r+1, (r + (i-1) \cdot r/s) + v)$-th the entry of $\gamma_{1,v}$. Let T'_{vv} be the Zariski closure of the subgroup of $\overline{X}_{1,i}$ generated by $\gamma_{1,v}$, for each $v = 1, \ldots, r/s$. Then, one checks directly that the defining equations of the one dimensional subgroup T'_{vv} of $\overline{X}_{1,i}$ can be written as follows:

$$\begin{cases} (a_v - 1)W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, \ \ \forall 1 \leq v \leq r^2/s, \\ (Z_0)_{w,w} = 1 \ \ \forall w \neq v, \ \ 1 \leq v \leq r^2/s, \\ (Z_1)_{u,v} = 0 \ \ \forall 1 \leq u, v \leq r/s, \\ W_{h,w} = 0 \ \ \forall w \neq v; \ \ h = 0, 1, \ \ 1 \leq v \leq r^2/s, \\ W_{0,v} \cdot c_{1,v} - W_{1,v} \cdot c_{0,v} = 0 \ \ \forall 1 \leq v \leq r^2/s. \end{cases}$$

Then, we see that $T'_{vv} \cong T_{vv}$ via $\overline{\pi}_{1,i}$. Similarly, we use the methods used for T'_{vv} to construct U'_{vv} such that $U'_{vv} \cong U_{vv}$ for all $v = 1, \ldots, r/s$. To construct T'_{uv} when $u \neq v$, we let $b_{u,v} \in T_{uv}(\mathbf{k})$ be a $\mathbf{k}$-rational basis for the one dimensional vector group T_{uv} and pick $b'_{u,v} \in \overline{X}_{1,i}(\overline{\mathbf{k}})$ so that $\overline{\boldsymbol{\pi}}_{1,i}(b'_{u,v}) = b_{u,v}$. We define T'_{uv} to be the one dimensional vector group in $\overline{X}_{1,i}$ via the conjugations

$$\eta_v^{-1} b'_{uv} \eta_v, \quad \text{for } \eta_v \in T'_{vv}, \ \ v = 1, \ldots, r/s.$$

Then, we obtain $T'_{uv} \cong T_{uv}$ via $\overline{\pi}_{1,i}$. Similarly, to construct U'_{uv} for $u \neq v$, we let $d_{u,v} \in U_{uv}(\mathbf{k})$ be a $\mathbf{k}$-rational basis for the one dimensional vector group U_{uv} and pick $d'_{u,v} \in \overline{X}_{1,i}(\overline{\mathbf{k}})$ so that

$\overline{\pi}_{1,i}(d'_{u,v}) = d_{u,v}$. We define U'_{uv} to be the one dimensional vector group in $\overline{X}_{1,i}$ via the conjugations

$$\eta_v^{-1} d'_{u,v} \eta_v, \quad \text{for } \eta_v \in T'_{vv},\ \ v = 1, \ldots, r/s.$$

Then, we obtain $U'_{uv} \cong U_{uv}$ via $\overline{\pi}_{1,i}$. This proves our claim.

Now suppose $n > 1$. We follow the methods used for $n = 1$ to prove that the induced tangent map $\mathrm{d}\pi_n$ at the identity is surjective onto $\operatorname{Lie}\Gamma_{\mathrm{P}_n M_\rho}$. Recall $\overline{\Gamma}_{\mathrm{P}_n M_\rho}$ and A from Lemma 3, where conjugation by $d_{t,n+1}[\mathrm{A}]$ on $\Gamma_{\mathrm{P}_n M_\rho}$ gives (see (4.37))

$$\overline{\Gamma}_{\mathrm{P}_n M_\rho} = \left\{ \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ & \gamma_0 & \ddots & \vdots \\ & & \ddots & \gamma_1 \\ & & & \gamma_0 \end{pmatrix} : \gamma_0 \in \prod_{i=1}^{s} (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i,\ \gamma_j \in \prod_{i=1}^{s} (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i, j = 1, \ldots, n \right\}. \tag{5.19}$$

Let $(\oplus_{i=1}^{w} d_{t,n+1}[\mathrm{A}]) \oplus (1) \in \mathrm{GL}_{(n+1)rw+1}(\mathbf{k}^{\mathrm{sep}})$ be the block diagonal matrix such that $d_{t,n+1}[\mathrm{A}]$ is in the first w diagonal blocks and 1 in the last diagonal and all other entries are zero. Then, via conjugation by $(\oplus_{i=1}^{w} d_{t,n+1}[\mathrm{A}]) \oplus (1)$ on Γ_{N_n} we obtain $\overline{\Gamma}_{N_n}$ such that we have an isomorphism $\Gamma_{N_n} \times_{\mathbf{k}} \overline{\mathbf{k}} \cong \overline{\Gamma}_{N_n}$. Moreover, $\overline{\Gamma}_{N_n}$ is an algebraic subgroup of $\mathrm{GL}_{(n+1)rw+1}/\overline{\mathbf{k}}$ such that $\overline{\pi}_1 : \overline{\Gamma}_{N_n} \to \overline{\Gamma}_{\mathrm{P}_n M_\rho}$ induced by π_n is surjective. Thus, we are reduced to proving that the induced tangent map $\mathrm{d}\overline{\pi}_n : \operatorname{Lie}\overline{\Gamma}_{N_n} \to \operatorname{Lie}\overline{\Gamma}_{\mathrm{P}_n M_\rho}$ is surjective. Let $w = 1$ and consider the short exact sequence of linear algebraic groups

$$1 \to \overline{X}_n \to \overline{\Gamma}_{N_n} \xrightarrow{\overline{\pi}_n} \overline{\Gamma}_{\mathrm{P}_n M_\rho} \to 1. \tag{5.20}$$

Fix $1 \leq i \leq s$. We follow the methods used for the construction of $\overline{X}_{1,i}$ above to construct $\overline{X}_{n,i}$, the Zariski closure inside $\overline{\Gamma}_{N_n}$ of the subgroup generated by suitably chosen elements of $\overline{\Gamma}_{N_n}$ such that $\overline{X}_{n,i}$ is contained in the $nr^2/s^2 + nr^2/s$ dimensional group,

$$H_{n,i} := \left\{ \left(\begin{array}{c|c|c|c|c} \eta_0 & \eta_1 & \dots & \eta_n & \mathbf{0} \\ \hline & \eta_0 & \ddots & \vdots & \vdots \\ \hline & & \ddots & \eta_1 & \vdots \\ \hline & & & \eta_0 & \mathbf{0} \\ \hline \mathbf{s}_0 & \mathbf{s}_1 & \dots & \mathbf{s}_n & 1 \end{array} \right) : \begin{array}{l} \eta_0 \in (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i,\ \eta_j \in (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i, j = 1, \dots, n \\ \mathbf{s}_h = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{s}_{h,i}, \mathbf{0}, \dots, \mathbf{0}), \quad \mathbf{s}_{h,i} \in \mathbb{G}_a^{r/s} \\ \text{for each } h = 0, \dots, n \end{array} \right\}. \tag{5.21}$$

Let

$$(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\overline{\mathbf{k}})_i := \left\{ \left(\begin{array}{c|c|c|c} \gamma_0 & \gamma_1 & \dots & \gamma_n \\ \hline & \gamma_0 & \ddots & \vdots \\ \hline & & \ddots & \gamma_1 \\ \hline & & & \gamma_0 \end{array} \right) : \begin{array}{l} \gamma_0 \in (\mathrm{GL}_{r/s}/\overline{\mathbf{k}})_i,\ \gamma_j \in (\mathrm{Mat}_{r/s}/\overline{\mathbf{k}})_i, \\ \text{where } j = 1, \dots, n \end{array} \right\}.$$

Note that $\dim \overline{X}_{n,i} \leq \dim H_{n,i} = nr^2/s^2 + nr/s$. If $\dim \overline{X}_{n,i} = nr^2/s^2 + nr/s$, similar to $\overline{X}_{1,0,i}$ as in (5.12) we simply construct

$$\overline{X}_{1,0,i} = \left\{ \begin{pmatrix} \vartheta_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : \vartheta_i \in (\Gamma_{\mathrm{P}_n M_\rho}(\overline{\mathbf{k}}))_i \right\},$$

and restrict $\mathrm{d}\overline{\pi}_n$ to $\mathrm{Lie}\,\overline{X}_{1,0,i}$ to obtain a surjection onto $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_1 M_\rho}/\overline{\mathbf{k}})_i$. As we vary all $1 \leq i \leq s$, the surjection of $\mathrm{d}\overline{\pi}_n$ follows.

Next, suppose $\dim \overline{X}_{n,i} < nr^2/s^2 + nr/s$. Then, via $\overline{\pi}_n$ we have a short exact sequence,

$$1 \to Q_{n,i} \to \overline{X}_{n,i} \xrightarrow{\overline{\pi}_{n,i}} (\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\overline{\mathbf{k}})_i \to 1.$$

The methods used above to prove $\dim Q_{1,i} = 0$ can be applied in a straightforward manner to prove $\dim Q_{n,i} = 0$, which we leave to the reader. Similar to the coordinates $\mathbf{Z}_1$ of $\overline{X}_{1,i}$ in (5.13), we let the coordinates of $\overline{X}_{n,i}$ be as follows:

$$\mathbf{Z}_n = \begin{pmatrix} \mathcal{Z}_0 & \mathcal{Z}_1 & \dots & \mathcal{Z}_n & \mathbf{0} \\ & \mathcal{Z}_0 & \ddots & \vdots & \vdots \\ & & \ddots & \mathcal{Z}_1 & \vdots \\ & & & \mathcal{Z}_0 & \mathbf{0} \\ \mathcal{W}_0 & \mathcal{W}_1 & \dots & \mathcal{W}_n & 1 \end{pmatrix},$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} \mathrm{Id}_{r/s} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathrm{Id}_{r/s} \end{pmatrix}, \quad \mathcal{Z}_j = \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & (Z_0) & & \\ & & & \ddots & \\ & & & & \mathbf{0} \end{pmatrix},$$

for each $j = 1, \dots, n$ such that (Z_0) is as in (5.13) and (Z_j) is the $r/s \times r/s$ block $((Z_j)_{a,b})$ for $1 \leq a, b \leq r/s$. Moreover, $\mathcal{W}_h := (0, \dots, 0, (W_h), 0, \dots 0)$, where we set $(W_h) := (W_{h,1}, \dots, W_{h,r/s})$ for each $h = 0, \dots, n$. Similar to (5.14), we construct one-dimensional subgroups of $\overline{\Gamma}_{\mathrm{P}_\ell M_\rho}$:

$$T_{0,u,v} := \left\{ \begin{pmatrix} \mathcal{B}_{uv} & \mathbf{0} & \dots & \mathbf{0} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \mathcal{B}_{uv} \end{pmatrix} \right\}, \quad U_{i,u,v} := \left\{ \begin{pmatrix} \mathrm{Id}_r & \mathbf{0} & \dots & \mathcal{C}_{uv} & \dots & \mathbf{0} \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \mathcal{C}_{uv} \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \mathbf{0} \\ & & & & & \mathrm{Id}_r \end{pmatrix} \right\},$$

such that $\mathcal{B}_{uv}$ and $\mathcal{C}_{uv}$ are as in (5.15), and $\mathcal{C}_{uv}$ is in the i-th diagonal block of $U_{i,u,v}$. Similar to the $n = 1$ case, note that the Lie algebras of the $n \cdot r^2/s^2$ algebraic groups $T_{0,u,v}$ and $U_{i,u,v}$ span $\mathrm{Lie}(\overline{\Gamma}_{\mathrm{P}_n M_\rho}/\overline{\mathbf{k}})_i$. We construct one dimension algebraic subgroups $T'_{0,u,v}$ and $U'_{i,u,v}$ of $\overline{X}_{n,i}$ so that $T'_{0,u,v} \cong T_{0,u,v}$ and $U'_{i,u,v} \cong U_{i,u,v}$. Then, since $\mathrm{Lie}(\cdot)$ is a left exact functor, it follows that $\mathrm{Lie}\, T'_{0,u,v} \cong \mathrm{Lie}\, T_{0,u,v}$ and $\mathrm{Lie}\, U'_{i,u,v} \cong \mathrm{Lie}\, U_{i,u,v}$, and so $\mathrm{d}\overline{\pi}_{n,i}$ is surjective. Since $Q_{n,i}$ is a zero dimensional vector group, $\overline{\pi}_{n,i}$ is injective on points and so it follows by checking directly that

- for $w \neq v$ and $h = 0, \ldots, n$, all $W_{h,w}$ coordinates of $\overline{\pi}_{n,i}^{-1}(T_{0,u,v})$ are zero;
- all (W_0) coordinates of $\overline{\pi}_{n,i}^{-1}(U_{i,u,v})$ are zero, and for $w \neq v$ and $j = 1, \ldots, n$, all $W_{j,w}$ coordinates of $\overline{\pi}_{n,i}^{-1}(U_{i,u,v})$ are zero.

To construct $T'_{0,v,v}$, we let $a_v \in \overline{\mathbf{k}}^{\times} \setminus \overline{\mathbb{F}_q}^{\times}$ and pick elements $\gamma_{n,v} \in \overline{X}_{n,i}(\overline{\mathbf{k}})$ so that

$$\overline{\pi}_{n,i}(\gamma_{n,v}) = \begin{pmatrix} \mathfrak{a}_v & & \\ & \ddots & \\ & & \mathfrak{a}_v \end{pmatrix},$$

where $\mathfrak{a}_v$ is as in (5.18). For $1 \leq v \leq r/s$ and $h = 0, \ldots, n$, we let $c_{h,v}$ be the $(nr+1, hr+(i-1)\cdot r/s+v)$-th the entry of $\gamma_{n,v}$. Let $T'_{0,v,v}$ be the Zariski closure of the subgroup of $\overline{X}_{n,i}$ generated by $\gamma_{n,v}$. Then, one checks directly that the defining equations of the one dimensional subgroup $T'_{0,v,v}$ of $\overline{X}_{n,i}$ can be written as follows:

$$\begin{cases} (a_v - 1)W_{0,v} - c_{0,v}((Z_0)_{v,v} - 1) = 0, \ \ \forall 1 \leq v \leq r^2/s, \\ (Z_0)_{w,w} = 1 \ \ \forall w \neq v, \ \ 1 \leq v \leq r^2/s, \\ (Z_j)_{u,v} = 0 \ \ \forall\, j = 1, \ldots, \ell, \ \ 1 \leq u, v \leq r/s, \\ W_{h,w} = 0 \ \ \forall\, w \neq v; \ \ h = 0, \ldots, \ell, \ \ 1 \leq v \leq r^2/s, \\ W_{h_1,v} \cdot c_{h_2,v} - W_{h_2,v} \cdot c_{h_1,v} = 0 \ \ \forall\, h_1, h_2 \in \{0, \ldots, \ell\}, \ \ 1 \leq v \leq r^2/s. \end{cases}$$

Then, we see that $T'_{0,v,v} \cong T_{0,v,v}$ via $\overline{\pi}_{n,i}$. Similarly, we use the methods used for $T'_{0,v,,v}$ to construct $U'_{i,v,v}$ such that $U'_{i,v,v} \cong U_{i,v,v}$ for all $v = 1, \ldots, r/s$. To construct $T'_{0,u,v}$ and $U_{i,u,v}$ for $u \neq v$ such that $T'_{0,u,v} \cong T_{0,u,v}$ and $U'_{i,u,v} \cong U_{i,u,v}$, we use conjugation where the arguments are essentially the same as the arguments used to construct T'_{uv} and U'_{uv} in the $n = 1$ case, and so we omit the details. $\square$

5.4 Algebraic independence of logarithms and quasi-logarithms

In this section, we prove Theorem 4 (restated as Theorem 14) and Corollary 1. Recall the short exact sequence (5.8). As was shown for the $n = 0$ case (Drinfeld module case) in [12, §5.1], we will show that X_n can be identified with a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^B)^w$. Let $\mathbf{n} \in \mathrm{Mat}_{((n+1)rw+1)\times 1}(N_n)$ be the $\overline{k}(t)$-basis of N_n such that $\sigma \mathbf{n} = \Phi_{N_n}\mathbf{n}$. Recall that the entries of $\Psi_{N_n}^{-1}\mathbf{n}$ form a $\mathbf{k}$-basis of N_n^B. If we write $\mathbf{n} = [\mathbf{n}_1, \ldots, \mathbf{n}_w, y]^\mathsf{T}$ where each $\mathbf{n}_i \in \mathrm{Mat}_{(n+1)r\times 1}(N_n)$, then $[\mathbf{n}_1, \ldots, \mathbf{n}_w]^\mathsf{T}$ is a $\overline{k}(t)$-basis of $(\mathrm{P}_n M_\rho)^w$ and the entries of $\mathbf{u} := [\Psi_{\mathrm{P}_n M_\rho}^{-1}\mathbf{n}_1, \ldots, \Psi_{\mathrm{P}_n M_\rho}^{-1}\mathbf{n}_w]^\mathsf{T}$ form a $\mathbf{k}$-basis of $((\mathrm{P}_n M_\rho)^B)^w$. Given any $\mathbf{k}$-algebra R, we recall the action of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ on $\mathrm{R} \otimes_{\mathbf{k}} ((\mathrm{P}_n M_\rho)^B)^w$ from [34, §4.5] (see also (4.13)) as follows: for any $\mu \in \Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ and any $\mathbf{v}_h \in \mathrm{Mat}_{1\times(n+1)r}(\mathrm{R})$, $0 \le h \le n$, the action of μ on $(\mathbf{v}_1, \ldots, \mathbf{v}_w)\cdot \mathbf{u} \in R \otimes_{\mathbf{k}} ((\mathrm{P}_n M_\rho)^B)^w$ is

$$(\mathbf{v}_1, \ldots, \mathbf{v}_w)\cdot \mathbf{u} \mapsto (\mathbf{v}_1\mu^{-1}, \ldots, \mathbf{v}_w\mu^{-1})\cdot \mathbf{u}. \tag{5.22}$$

Thus, by (5.9) the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on $((\mathrm{P}_n M_\rho)^B)^w$ is compatible with the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on X_n. Then, when we regard $((\mathrm{P}_n M_\rho)^B)^w$ as a vector group over $\mathbf{k}$, by Lemma 4 we get the desired result.

Since X_n is a $\Gamma_{\mathrm{P}_n M_\rho}$-submodule of $((\mathrm{P}_n M_\rho)^w)^B$, by the equivalence of categories $\mathcal{T}_{\mathrm{P}_n M_\rho} \approx \mathbf{Rep}(\Gamma_{\mathrm{P}_n M_\rho}, \mathbf{k})$, there exists a sub-$t$-motive V_n of $(\mathrm{P}_n M_\rho)^w$ such that

$$X_n \cong V_n^B. \tag{5.23}$$

By (4.12), we see that for any $n \ge 1$ and $0 \le j \le n-1$ we obtain a short exact sequence of t-motives

$$0 \to (\mathrm{P}_j M_\rho)^w \xrightarrow{\iota} (\mathrm{P}_n M_\rho)^w \xrightarrow{\mathbf{pr}_w} (\mathrm{P}_{n-j-1} M_\rho)^w \to 0. \tag{5.24}$$

Lemma 5. *For $n \ge 1$, let V_n be as in* (5.23). *Then, for $0 \le j \le n-1$ there is a surjective map of t-motives $\overline{\mathbf{pr}}_w : V_n \to V_{n-j-1}$ via the map $\mathbf{pr}_w$ in* (5.24).

Proof. We prove the result for $w = 1$. The following argument for $w = 1$ can be applied in a straightforward manner to prove the arbitrary w case, which we leave to the reader. Let $w = 1$. For $n \geq 1$, recall from (2.3) that if $\boldsymbol{m} \in \mathrm{Mat}_{r\times 1}(M_\rho)$ is a $\overline{k}(t)$-basis of M_ρ, then $\boldsymbol{D}_n\boldsymbol{m}$ forms a $\overline{k}(t)$-basis of P_nM_ρ. Let $[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal$ be a $\overline{k}(t)$-basis of N_n. Then, $\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal$ forms a $\mathbb{F}_q(t)$-basis of N_n^B. By construction, P_jM_ρ is a sub-t-motive of N_n for each $j \leq n$ and we have a short exact sequence of t-motives

$$0 \to \mathrm{P}_jM_\rho \xrightarrow{\iota} N_n \xrightarrow{\mathbf{pr}} N_{n-j-1} \to 0, \tag{5.25}$$

where $\mathbf{pr}(D_hm) := D_{h-j-1}m$ for $h > j$, $\mathbf{pr}(D_hm) := 0$ for $h \leq j$ and $m \in M_\rho$, and $\mathbf{pr}(x) = x$ for $x \in Y_n/\mathrm{P}_nM_\rho$. Therefore, as t-motives

$$N_n/\mathrm{P}_jM_\rho \cong N_{n-j-1},$$

and so we have a surjective map of affine group schemes $\Gamma_{N_n} \twoheadrightarrow \Gamma_{N_{n-j-1}}$. We now determine this surjective map. For any $\mathbf{k}$-algebra R, we recall the action of $\Gamma_{N_n}(\mathrm{R})$ on $\mathrm{R} \otimes_{\mathbf{k}} (N_n)^B$ from [34, §4.5] as follows: for any $\nu_n \in \Gamma_{N_n}(\mathrm{R})$, $b \in \mathrm{R}$ and $a_h \in \mathrm{Mat}_{1\times r}(\mathrm{R})$ where $0 \leq h \leq n$, the action of ν_n on $(a_0, \ldots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal \in R \otimes_{\mathbf{k}} (N_n)^B$ is

$$(a_0, \ldots, a_n, b) \cdot \Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal \mapsto (a_0, \ldots, a_n, b) \cdot \nu_n^{-1}\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal \tag{5.26}$$

By the definition of Ψ_{N_n}, we see that $\Psi_{N_n}^{-1}[\boldsymbol{D}_n\boldsymbol{m}^\intercal, y]^\intercal = [\boldsymbol{D}_n(\Psi_\rho^{-1}\boldsymbol{m})^\intercal, D_n(-\mathbf{s}_{\alpha_1}\boldsymbol{m}) + y]^\intercal$. We restrict the action of ν_n to $\mathrm{R} \otimes_{\mathbf{k}} (N_{n-j-1}M_\rho)^B$ via the map $\mathbf{pr}$ in (5.25). Note that an element of $\Gamma_{N_n}(\mathrm{R})$ is of the form $\left(\begin{smallmatrix} \mu_n & 0 \\ \mathbf{w}_n & 1 \end{smallmatrix}\right)$, where $\mu_n \in \Gamma_{\mathrm{P}_nM_\rho}(\mathrm{R})$ and $\mathbf{w}_n = (w_0, \ldots, w_n)$ such that each $w_h \in \mathbb{G}_a^r(\mathrm{R}) = \mathrm{Mat}_{1\times r}(\mathrm{R})$. Through $\mathbf{pr}$, we see that ν_n leaves $(N_{n-j-1})^B$ invariant and so for

$\nu_n = \left(\begin{smallmatrix} \mu_n & 0 \\ \mathbf{w}_n & 1 \end{smallmatrix}\right) \in \Gamma_{N_n}(\mathrm{R})$, we obtain

$$\nu_{n-j-1} = \begin{pmatrix} \mu_{n-j-1} & 0 \\ \mathbf{w}_{n-j-1} & 1 \end{pmatrix} \in \Gamma_{N_{n-j-1}}(\mathrm{R}),$$

where μ_{n-j-1} is the matrix formed by the $r(n-j)\times r(n-j)$ upper-left square of μ_n and $\mathbf{w}_{n-j-1} = (w_0, \ldots, w_{n-j-1})$. Note that by Theorem 9, we have $\mu_{n-j-1} \in \Gamma_{\mathrm{P}_{n-j-1}M_\rho}(\mathrm{R})$. Thus, the surjective map $\Gamma_{N_n} \to \Gamma_{N_{n-j-1}}$ is given by

$$\nu_n \mapsto \nu_{n-j-1}, \tag{5.27}$$

(cf. [11, proof of Prop. 3.1.2]). Since X_n and X_{n-j-1} are $\mathbf{k}$-smooth by Lemma 4, this map gives a surjective map of group schemes $X_n \to X_{n-j-1}$. By (5.23), this corresponds to a map of representations of $\Gamma_{\mathrm{P}_nM_\rho}$ over $\mathbf{k}$, $\overline{\mathbf{pr}}_w^B : V_n^B \to V_{n-j-1}^B$ via the map $\mathbf{pr}_w^B$, where $\mathbf{pr}_w$ is as in (5.24). By the equivalence of categories $\mathcal{T}_{\mathrm{P}_nM_\rho} \approx \mathbf{Rep}(\Gamma_{\mathrm{P}_nM_\rho}, \mathbf{k})$, we get the required conclusion. $\square$

Theorem 14. *Let ρ be a Drinfeld $\mathbf{A}$-module defined over k^{sep}. Suppose that K_ρ is separable over k. Let $u_1, \ldots, u_w \in \mathbb{K}$ with $\mathrm{Exp}_\rho(u_i) = \alpha_i \in k^{\mathrm{sep}}$ for each i and suppose that $\dim_{K_\rho} \mathrm{Span}_{K_\rho}(\lambda_1, \ldots, \lambda_r, u_1, \ldots, u_w) = r/s + w$. For $n \geq 1$, let N_n and Ψ_{N_n} be defined as in §5.3, and for each $i = 1, \ldots, w$, let $Y_{i,n} := Y_{u_i,n}$ be defined as in §5.2. Then, $\dim \Gamma_{N_n} = (n+1)\cdot r(r/s + w)$. In particular,*

$$\operatorname{tr.deg}_{\overline{k}} \overline{k}\Bigg(\bigcup_{s=0}^{n} \bigcup_{i=1}^{r-1} \bigcup_{m=1}^{w} \bigcup_{j=1}^{r} \{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m)\} \Bigg) = (n+1)\cdot(r^2/s + rw).$$

Proof. From the construction of Ψ_{N_n}, by Theorem 7 we have

$$\overline{k}(\Psi_{N_n}(\theta)) = \overline{k}\Bigg(\bigcup_{s=0}^{n} \bigcup_{i=1}^{r-1} \bigcup_{m=1}^{w} \bigcup_{j=1}^{r} \{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m)\} \Bigg),$$

and by Theorem 6 and Theorem 12, we have

$$\dim \Gamma_{N_n} = \operatorname{tr.deg}_{\overline{k}} \overline{k}(\Psi_n(\theta)) \leq (n+1)\frac{r^2}{s} + (n+1)rw. \tag{5.28}$$

Thus, we need to prove that $\dim X_n = (n+1)rw$, where X_n is as in (5.8). By (5.23) it suffices to show that $V_n^B \cong ((\mathrm{P}_n M_\rho)^w)^B$. To prove this, we adapt the arguments of the proof of [12, Thm. 5.1.5] (see also [22, Lem. 1.2]).

Note from (5.24) that for $n \geq 1$ we have a short exact sequence of t-motives

$$0 \to (\mathrm{P}_0 M_\rho)^w \xrightarrow{\iota} (\mathrm{P}_n M_\rho)^w \xrightarrow{\mathbf{pr}_{n,w}} (\mathrm{P}_{n-1} M_\rho)^w \to 0.$$

By Lemma 5, there is a surjective map $\overline{\mathbf{pr}}_{n,w} : V_n \to V_{n-1}$ via $\mathbf{pr}_{n,w}$. Then $\ker(\overline{\mathbf{pr}}_{n,w})$ is a sub-t-motive of M_ρ^w.

Suppose for now that $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$. We claim that the extension N_n/V_n is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_n M_\rho/V_n)$. Since $X_n \cong V_n^B$, we see that Γ_{N_n} acts on N_n^B/V_n^B through $\Gamma_{N_n}/X_n \cong \Gamma_{\mathrm{P}_n M_\rho}$ via (5.8). Since $\overline{\mathbf{pr}}_{n,w}$ is surjective onto $(\mathrm{P}_{n-1} M_\rho)^w$, by (5.25) we see that $N_n^B/V_n^B \cong N_0^B/(\ker \overline{\mathbf{pr}}_{n,w})^B$. Recall that for any $\mathbf{k}$-algebra R, an element of $\Gamma_{\mathrm{P}_n M_\rho}(\mathrm{R})$ is of the form (4.22) such that γ is an element of $\Gamma_{M_\rho}(\mathrm{R})$. Then, (5.26) shows the action of $\Gamma_{\mathrm{P}_n M_\rho}$ on N_n^B/V_n^B is the same as the action of Γ_{M_ρ} on it. It follows that N_n^B/V_n^B is an extension of $\mathbf{k}$ by $((\mathrm{P}_n M_\rho)^w)^B/V_n^B$ in $\mathbf{Rep}(\Gamma_{M_\rho}, \mathbf{k})$. By [12, Cor. 3.5.7] and the equivalence of category $\mathcal{T}_{M_\rho} \approx \mathbf{Rep}(\Gamma_{M_\rho}, \mathbf{k})$, we get the required conclusion of the claim.

Now, we prove the main result by induction. For the base case $n = 1$, suppose on the contrary that $V_1^B \subsetneq ((\mathrm{P}_1 M_\rho)^w)^B$. From [12, Thm. 5.1.5], we have $M_\rho^w \cong V_0$ and so, since $M_\rho^w \cong (\mathrm{P}_0 M_\rho)^w$ we have $\ker(\overline{\mathbf{pr}}_{1,w}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [12, Cor. 3.3.3] and $\ker(\overline{\mathbf{pr}}_{1,w})$ is a sub-t-motive of M_ρ^w, there exists a non-trivial morphism $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{1,w}) \subseteq \ker \phi_1$. Moreover, the morphism ϕ_1 factors through the map $M_\rho^w/\ker(\overline{\mathbf{pr}}_{1,w}) \to M_\rho^w/(\ker \phi_1)$:

$$\begin{tikzcd} M_\rho^w \arrow[d] \arrow[dr, "\phi_1"] & \\ M_\rho^w/\ker(\overline{\mathbf{pr}}_{1,w}) \arrow[r] & M_\rho^w/(\ker \phi_1) \cong M_\rho. \end{tikzcd}$$

Since $\phi_1 \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, there exist $e_{i,1} \in \mathbf{K}_\rho$ not all zero such that $\phi_1(n_1, \ldots, n_w) = \sum_{i=1}^w e_{i,1}(n_i)$. Suppose that $E_{i,1} \in \mathrm{Mat}_r(\overline{k}(t))$ satisfies $e_{i,1}(\boldsymbol{m}) = E_{i,1}\boldsymbol{m}$. Recall from §2.5 that

$\boldsymbol{D}_1\boldsymbol{m}$ forms a $\overline{k}(t)$-basis of $\mathrm{P}_1 M_\rho$. Set

$$\mathbf{E}_{i,1} := \begin{pmatrix} \mathbf{0} & E_{i,1} \\ & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{2r}(\overline{k}(t)).$$

Then, by (5.2) there exists $\mathbf{e}_{i,1} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_1 M)^w)$ such that

$$\mathbf{e}_{i,1}(\boldsymbol{D}_1\boldsymbol{m}) = \mathbf{E}_{i,1}\boldsymbol{D}_1\boldsymbol{m}.$$

Let $\psi_1 \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_1 M_\rho)^w, \mathrm{P}_1 M_\rho)$ such that $\psi_1(D_j n_1, \ldots, D_j n_w) = \sum_{i=1}^w \mathbf{e}_{i,1}(D_j n_i)$ for each $j = 0, 1$. We see that $\ker\psi_1 / M_\rho^w \cong \ker\phi_1$ and $\mathrm{P}_1 M_\rho^w / \ker\psi_1 \cong M_\rho^w / \ker\phi_1 \cong M_\rho$. Then the pushout $\psi_{1*}N_1 := \mathbf{e}_{1,1*}Y_{1,1} + \cdots + \mathbf{e}_{w,1*}Y_{w,1}$ is a quotient of N_1/V_1. By using the claim above, it follows that $\psi_{1*}N_1$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_1 M_\rho)$. However by Theorem 13, this is a contradiction.

Now suppose that we have shown the result for $n-1$, that is, $V_{n-1} \cong (\mathrm{P}_{n-1} M_\rho)^w$. Suppose that $V_n^B \subsetneq ((\mathrm{P}_n M_\rho)^w)^B$. Then, $\ker(\overline{\mathbf{pr}}_{n,w}) \subsetneq M_\rho^w$. Since M_ρ^w is completely reducible in $\mathcal{T}_{M_\rho}$ by [12, Cor. 3.3.3] and $\ker(\overline{\mathbf{pr}}_{n,w})$ is a sub-t-motive of M_ρ^w, there exists a non-trivial morphism $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$ so that $\ker(\overline{\mathbf{pr}}_{n,w}) \subseteq \ker\phi_n$. Moreover, the morphism ϕ_n factors through the map $M_\rho^w / \ker(\overline{\mathbf{pr}}_{n,w}) \to M_\rho^w / (\ker\phi_n)$:

$$\begin{tikzcd} M_\rho^w \arrow[d] \arrow[dr, "\phi_n"] & \\ M_\rho^w / \ker(\overline{\mathbf{pr}}_{n,w}) \arrow[r] & M_\rho^w / (\ker\phi_n) \cong M_\rho. \end{tikzcd}$$

Since $\phi_n \in \mathrm{Hom}_{\mathcal{T}}(M_\rho^w, M_\rho)$, we can write $\phi_n(n_1, \ldots, n_w) = \sum_{i=1}^w e_{i,n}(n_i)$ for some $e_{1,n}, \ldots, e_{w,n}$ $\in \mathbf{K}_\rho$ not all zero. Suppose that $e_{i,n}(\boldsymbol{m}) = E_{i,n}\boldsymbol{m}$ where $E_{i,n} \in \mathrm{Mat}_r(\overline{k}(t))$. Recall from §2.5

that $\boldsymbol{D}_n\boldsymbol{m}$ forms a $\overline{k}(t)$-basis of P_nM_ρ. Set

$$\mathbf{E}_{i,n} := \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & E_{i,n} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{0} \end{pmatrix} \in \mathrm{Mat}_{(n+1)r}(\overline{k}(t)).$$

Then, by (5.2) there exists $\mathbf{e}_{i,n} \in \mathrm{End}_{\mathcal{T}}((\mathrm{P}_nM)^w)$ such that

$$\mathbf{e}_{i,n}(\boldsymbol{D}_1\boldsymbol{m}) = \mathbf{E}_{i,n}\boldsymbol{D}_1\boldsymbol{m}.$$

Let $\psi_n \in \mathrm{Hom}_{\mathcal{T}}((\mathrm{P}_nM_\rho)^w, \mathrm{P}_nM_\rho)$ such that $\psi_1(D_jn_1, \dots, D_jn_w) = \sum_{i=1}^{w} \mathbf{e}_{i,1}(D_jn_i)$ for each $j = 0, \dots, n$. Similar to the base case, we see that $\ker\psi_n/(\mathrm{P}_{n-1}M_\rho)^w \cong \ker\phi_n$ and that $\mathrm{P}_nM_\rho^w/\ker\psi_n \cong M_\rho^w/\ker\phi_1 \cong M_\rho$. Then the pushout $\psi_{n*}N_n := \mathbf{e}_{1,n*}Y_{1,n} + \dots + \mathbf{e}_{w,n*}Y_{w,n}$ is a quotient of N_n/V_n. By using the claim above, it follows that $\psi_{n*}N_n$ is trivial in $\mathbf{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathrm{P}_nM_\rho)$. But by Theorem 13, this is a contradiction. □

Proof of Corollary 1. We adapt the ideas of the proof of [11, Thm. 4.3.3] and [12, Cor. 5.1.6]. We define $W := \mathrm{Span}_{K_\rho}(\lambda_1, \dots, \lambda_r, u_1, \dots, u_w)$ and let $\{\eta_1, \dots, \eta_\alpha\}$ be a K_ρ-basis of W. Clearly, $r/s \le \alpha \le r/s + w$. Since the quasi-periodic functions F_δ are linear in δ and satisfy the difference equation (1.2), we have

$$\overline{k}\left(\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\lambda_j, F_{\tau^i}(\lambda_j), u_m, F_{\tau^i}(u_m)\}\right) = \overline{k}\left(\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\left\{F_{\delta_j}(\eta_m)\right\}\right).$$

Moreover, for any $i_1, i_2 \in \{1, \dots, r\}$, $j_1, j_2 \in \{1, \dots, \alpha\}$, $s \in \{0, \dots, n\}$ and $v_1, v_2 \in K_\rho$, by the product rule of hyperderivatives we obtain

$$\partial_\theta^s\left(v_1F_{\delta_{i_1}}(\eta_{j_1}) + v_2F_{\delta_{i_2}}(\eta_{j_2})\right) = \sum_{h=0}^{s}\left(\partial_\theta^{s-h}(v_1)\partial_\theta^h\left(F_{\delta_{i_1}}(\eta_{j_1})\right) + \partial_\theta^{s-h}(v_2)\partial_\theta^h\left(F_{\delta_{i_1}}(\eta_{j_2})\right)\right).$$

Thus,

$$\overline{k}\left(\bigcup_{s=0}^{n}\bigcup_{i=1}^{r-1}\bigcup_{m=1}^{w}\bigcup_{j=1}^{r}\{\partial_\theta^s(\lambda_j), \partial_\theta^s(F_{\tau^i}(\lambda_j)), \partial_\theta^s(u_m), \partial_\theta^s(F_{\tau^i}(u_m))\}\right)$$
$$= \overline{k}\left(\bigcup_{s=0}^{n}\bigcup_{j=1}^{r}\bigcup_{m=1}^{\alpha}\left\{\partial_\theta^s(F_{\delta_j}(\eta_m))\right\}\right).$$

Then, the result follows from Theorem 14. $\square$

6. CONCLUSION

In this dissertation, we determined all algebraic independence results among hyperderivatives of periods, quasi-periods, logarithms and quasi-logarithms of Drinfeld $\mathbf{A}$-modules. A natural next step would be to investigate the transcendence of the entries of the periods and logarithms of a general uniformizable Anderson t-module. Let ϕ be a uniformizable Anderson t-module of dimension d and rank r defined over $\overline{k}$. Similar to the case of the Drinfeld module ρ above, we define its dual t-motive $\mathcal{M}_\phi$ by setting $\mathcal{M}_\phi := \mathrm{Mat}_{1\times d}(\overline{k}[\sigma])$. If $\mathcal{M}_\phi$ is $\mathbf{A}$-finite, then $r = \mathrm{rank}_{\overline{k}[t]} \mathcal{M}_\phi$ is the rank of $\mathcal{M}_\phi$.

The *t-motive* (in the sense of [1]) $\mathcal{N}_\phi$ of ϕ is defined by setting $\mathcal{N}_\phi = \mathrm{Mat}_{1\times d}(\overline{k}[\tau])$. We let $t{\cdot}m := m\phi_t, \forall\, m \in \mathcal{M}_\phi$ and thus we give $\mathcal{N}_\phi$ a unique structure of a left $\overline{k}[t,\tau]$-module. Moreover, for any $m \in \mathcal{N}_\phi$, we have that $(t-\theta)^d \cdot m \in \tau\mathcal{M}_\phi$. If in addition $\mathcal{N}_\phi$ is free and finitely generated as a left $\overline{k}[t]$-module, ϕ is called an *abelian t-module*.

Similar to the case of the Drinfeld modules, suppose that $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r\}$ forms an $\mathbf{A}$-basis of the period lattice Λ_ϕ and $\{\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_r\}$ is a $\overline{k}$-basis of $\mathrm{H}^1_{\mathrm{DR}}(\phi)$, the de Rham module of a uniformizable, abelian, and $\mathbf{A}$-finite t-module ϕ. In [32], Papanikolas and the author investigated how to use rigid analytic trivialization to study periods, quasi-periods, logarithms and quasi-logarithms of the abelian Anderson t-module ϕ. Using Yu's sub-t-module theorem [44], the following result was proved.

Proposition 5 (Namoijam-Papanikolas, [32])**.** *Let $\phi : \mathbf{A} \to \mathrm{Mat}_d(\overline{k}[\tau])$ be a uniformizable abelian t-module defined over $\overline{k}$, and assume further that ϕ is simple (ϕ has no proper non-trivial sub-t-modules). Then,* $\dim_{\overline{k}} \mathrm{Span}_{\overline{k}}\big(\mathrm{F}_{\boldsymbol{\delta}_i}(\boldsymbol{\lambda}_j) : 1 \le i, j \le r\big) = r^2/s$, *where* $s = [\mathrm{End}(\phi) : \mathbf{A}]$.

Even studying the Galois groups associated to simple, uniformizable, abelian, and $\mathbf{A}$-finite Anderson t-modules and *strictly pure* Anderson t-modules (see [24, § 5.2], [32]) to determine the transcendence of their periods and quasi-periods would be interesting. The difficulty that arises in the application of methods similar to the ones used in this dissertation or [12] to find the Galois group of the t-motives associated to Anderson t-modules of dimension greater that 1 is that the Zariski closure of the image of a map similar to the one in Theorem 8 need not be open. Therefore, we only get containment of this Zariski closure in our Galois group. We plan to investigate this in future research.

Lemma 6 (Namoijam-Papanikolas, [32]). *Suppose that $\phi : \mathbf{A} \to \mathrm{Mat}_d(k_\infty^{\mathrm{sep}}[\tau])$ is an abelian t-module defined over k_∞^{sep} and that $\boldsymbol{\delta}$ is a ϕ-biderivation also defined over k_∞^{sep}. For any $\boldsymbol{x} \in \mathbb{C}_\infty^d$ such that $\mathrm{Exp}_\phi(\boldsymbol{x}) \in (k_\infty^{\mathrm{sep}})^d$, we have $\boldsymbol{x} \in (k_\infty^{\mathrm{sep}})^d$ and $\mathrm{F}_{\boldsymbol{\delta}}(\boldsymbol{x}) \in k_\infty^{\mathrm{sep}}$.*

By this lemma, we can consider the hyperderivatives of the periods and logarithms of a uniformizable, abelian and $\mathbf{A}$-finite Anderson t-module ϕ defined over k^{sep}. Moreover, we determined that the evaluation of the rigid analytic trivialization matrix of the prolongations of ϕ yields all its periods, quasi-periods, logarithms, quasi-logarithms, and their hyperderivatives. Therefore, we intend to study prolongations of Anderson t-modules of dimension greater than 1 by adapting the methods used in this dissertation.

Thakur [38] defined the multiple zeta values

$$\zeta_A(\mathrm{s}) = \sum_{\deg(a_1) > \cdots > \deg(a_r)} \frac{1}{a_1^{s_1} \ldots a_r^{s_r}},$$

where each $a_i \in A_+$, and $\mathrm{s} = (s_1, \ldots, s_r) \in \mathbb{N}$. Let $L_0 := 1$ and for $i \geq 1$ define $L_i := \prod_{j=1}^{i}(\theta - \theta^{q^j})$. Chang [8] introduced the Carlitz multiple polylogarithm (CMPL)

$$\mathbf{Li}_{\mathrm{s}}(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \ldots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \ldots L_{i_r}^{s_r}},$$

where $\mathrm{s} = (s_1, \ldots, s_r) \in \mathbb{N}$. Chang showed that $\zeta_A(\mathrm{s})$ can be expressed as a k-linear combination of CMPL's and further showed that each monomial of CMPL's at algebraic points is transcendental over k. The Carlitz multiple star polylogarithm (CMSPL) is defined as

$$\mathbf{Li}^*_{\mathrm{s}}(z_1, \ldots, z_r) \ := \ \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \ldots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \ldots L_{i_r}^{s_r}}.$$

Chang and Mishiba [10] introduced CMSPL's and used them to construct an Anderson t-module such that multiple zeta values appear in some form as a certain coordinate of a logarithm at an algebraic point. Chang, Green and Mishiba [9] gave explicit formula for the coordinates of this logarithm and hyperderivatives of the CMSPL's appear in some form.

Since we were successful in the investigation of hyperderivatives of periods and logarithms of Drinfeld $\mathbf{A}$-modules, we are hopeful that methods used in this dissertation can be adapted for the study of transcendence of these CMPL's and CMSPL's, and their hyperderivatives. The $\overline{k}$-linear independence of the Carlitz zeta values $\zeta_C(s) = \sum_{a \in A_+} \frac{1}{a^n}$ was established by Yu [43], [44], and their algebraic relations over $\overline{k}$ were completely proved by Chang and Yu [13]. Transcendence and algebraic relations among CMPL's have been studied by Chang [8] and Mishiba ([31], [30]).

REFERENCES

[1] G. W. Anderson, *t-motives*, Duke Math. J. **53** (1986), no. 2, 457–502.

[2] G. W. Anderson, W. D. Brownawell, and M. A. Papanikolas, *Determination of the algebraic relations among special Γ-values in positive characteristic*, Ann. of Math. **160** (2004), no. 2, 237–313.

[3] W. D. Brownawell, *Linear independence and divided derivatives of a Drinfeld module. I.*, Number Theory in Progress, Zakopane-Kościelisko **Vol. 1** (1997), 47–61.

[4] ______, *Minimal extensions of algebraic groups and linear independence*, J. Number Theory **90** (2001), no. 2, 239–254.

[5] W. D. Brownawell and L. Denis, *Linear independence and divided derivatives of a Drinfeld module. II.*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1581–1593.

[6] W. D. Brownawell and M. A. Papanikolas, *Linear independence of Gamma-values in positive characteristic*, J. Reine Angew. Math. **549** (2002), 91–148.

[7] ______, *A rapid introduction to Drinfeld modules, t-modules, and t-motives*, t-Motives: Hodge Structures, transcendence, and other motivic aspects, BIRS, Banff, Canada (to appear) (2009).

[8] C.-Y. Chang, *Linear independence of monomials of multizeta values in positive characteristic*, Compositio Math. **150** (2014), 1789–1808.

[9] C.-Y. Chang, N. Green, and Y. Mishiba, *Taylor coefficients of t-motivic multiple zeta values and explicit formulae*, arXiv:1902.06879 (2019).

[10] C.-Y. Chang and Y. Mishiba, *Logarithmic interpretation of multiple zeta values in positive characteristic*, arXiv:1710.10849 (2017).

[11] C.-Y. Chang and M. A. Papanikolas, *Algebraic relations among periods and logarithms of rank 2 Drinfeld modules*, Amer. J. Math. **133** (2011), no. 2, 359–391.

[12] ______, *Algebraic independence of periods and logarithms of Drinfeld modules. with an appendix by B. Conrad*, J. Amer. Math. Soc. **25** (2012), no. 1, 123–150.

[13] C.-Y. Chang and J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216** (2007), 321–345.

[14] K. Conrad, *The digit principle*, J. Number Theory **84** (2000), no. 2, 230–257.

[15] L. Denis, *Transcendence et dérivées de l'exponentielle de Carlitz*, Séminaire de Théorie des Nombres, (Paris, 1991-92), Birkhäuse, Boston, MA (1993), 1–21.

[16] ______, *Dérivées d'un module de Drinfeld et transcendance*, Duke Math. J. **80** (1995), no. 1, 1–13.

[17] ______, *Indépendance algébrique des dérivées d'une période du module de Carlitz*, J. Austral. Math. Soc. Ser. A **69** (2000), no. 1, 8–18.

[18] V. G. Drinfeld, *Elliptic modules*, Math. Sb. (N.S.) **94** (1974), 594–627.

[19] B. Farb and R. K. Dennis, *Noncommutative algebra*, Springer-Verlag, New York, 1993.

[20] E.-U. Gekeler, *On the de Rham isomorphism for Drinfeld modules*, J. Reine Angew. Math. **401** (1989), 188–208.

[21] N. Green, *Tensor powers of rank* 1 *Drinfeld modules and periods*, J. Number Theory (to appear), arXiv:1706.1706.03854 (2017).

[22] C. Hardouin, *Unipotent radical of Tannakian Galois groups in positive characteristic*, Arithmetic and Galois theories of differential equations, Seminars and Congresses **23** (2011), 223–239.

[23] ______, *Galoisian approach to differential transcendence*, in "Galois Theories of Linear Difference Equations: An Introduction" with J. Sauloy and M. F. Singer, Mathematical Surveys and Monograph of the AMS, 2016.

[24] U. Hartl and A.-K. Juschka, *Pink's theory of Hodge structures and the Hodge conjecture over function fields*, t-motives: Hodge structures, transcendence and other motivic aspects, European Mathematical Society (EMS), Zurich, (to appear) (2016).

[25] K. Iima and Y. Yoshino, *Gröbner bases for the polynomial ring with infinite variables and their applications*, Communications in Algebra **37:10** (2009), 3424–3437.

[26] S. Jeong, *Calculus in positive characteristic p*, J. Number Theory **131** (2011), no. 6, 1089–1104.

[27] A. Maurischat, *Periods of t-modules as special values*, J. Number Theory, David Goss Memorial Issue. (2018).

[28] ______, *Prolongations of t-motives and algebraic independence of periods*, Doc. Math. **23** (2018), 815–838.

[29] ______, *Anderson t-modules with thin t-adic Galois representations*, arXiv:1907.05144 (2019).

[30] Y. Mishiba, *On algebraic independence of certain multizeta values in characteristic p*, J. Number Theory **173** (2017), 512–528.

[31] ______, *Algebraic independence of the Carlitz period and the positive characteristic multizeta values at n and (n, n)*, Proc. Amer. Math. Soc. **143** (2019), no. 9, 3753–3763.

[32] C. Namoijam and M. A. Papanikolas, *Hyperderivatives of periods and quasi-periods for Anderson t-modules*, Preprint (2020).

[33] K. Okugawa, *Differential algebra of nonzero characteristic*, Kinokuniya Co. Ltd., Tokyo (1987).

[34] M. A. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. **171** (2008), no. 1, 123–174.

[35] F. Pellarin, *Aspects de l'indépendance algébrique en caractéristique non nulle*, Astérisque (2008), no. 973, viii, 205–242.

[36] R. Pink, *The Mumford-Tate conjecture for Drinfeld modules*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 3, 393–425.

[37] T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Birkhäuser, Boston, 1998.

[38] D. S. Thakur, *Function Field Arithmetic*, World Scientific Publishing, River Edge, NJ, 2004.

[39] A. Thiery, *Indépendance algébrique des périodes et quasi-périodes dun module de Drinfeld*, The arithmetic of function fields (Columbus, OH, 1991), de Gruyter, Berlin (1992), 265–284.

[40] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Springer, New York, 1979.

[41] J. Yu, *Transcendence and Drinfeld modules*, Invent. Math. **83** (1986), 507–517.

[42] ———, *On periods and quasi-periods of Drinfeld modules*, Compositio Math. **74** (1990), no. 3, 235–245.

[43] ———, *Transcendence and special zeta values in characteristic* p, Ann. of Math. **134** (1991), no. 1, 1–23.

[44] ———, *Analytic homomorphisms into Drinfeld modules*, Ann. of Math. **145** (1997), no. 2, 215–233.

APPENDIX A

DIFFERENTIAL ALGEBRAIC GEOMETRY

We present a few topics from differential algebraic geometry in positive characteristic [33] (cf. [23] for characteristic zero). For the most part, we follow the terminology of [23]. Even though the proofs of most of the results presented here are covered in [33], we present them here nevertheless for completeness.

A.1 Differential algebra in positive characteristic

Let R be a commutative ring with unity of characteristic $p > 0$. A *differential ring* or *∂-ring* is a pair (R, ∂), where ∂ represents a sequence of additive maps $\partial^j : R \to R$ that satisfy

1. $\partial^0(a) = a$,
2. $\partial^j(a + b) = \partial^j(a) + \partial^j(b)$,
3. $\partial^j(ab) = \sum_{i=0}^{j} \partial^i(a)\partial_\theta^{j-i}(b)$,
4. $\partial^k\partial^j(a) = \binom{k+j}{j}\partial^{k+j}(a)$,

for all $a, b \in R$ and $j, k \geq 0$. If R is a field, then we say that (R, ∂) is a *differential field* or a *∂-field*. When the context is clear, we shall write R instead of (R, ∂). Moreover, a *∂-morphism* between two ∂-rings R and S is a morphism of rings that commute with ∂. For a ∂-ring R, if we let $\mathfrak{I} \subseteq R$ be an ideal, then $\mathfrak{I}$ is called a *∂-ideal* if $\partial^j(\mathfrak{I}) \subseteq \mathfrak{I}$ for all $j \geq 1$. If, in addition, $\mathfrak{I}$ is a radical (respectively prime) ideal of the ∂-ring R regarded as a ring, then we say that $\mathfrak{I}$ is a *radical* (respectively *prime) ∂-ideal* of the ∂-ring R. For a set $\Sigma \subseteq R$, the intersection of all ∂-ideals containing Σ is a ∂-ideal of R, which we denote by $\mathfrak{D}(\Sigma)$ and it is the smallest ∂-ideal of R containing Σ. We see that $\mathfrak{D}(\Sigma)$ is the ideal, generated $\{\partial^j(a) \mid j \geq 0, a \in \Sigma\}$, of the ∂-ring R regarded as a ring. We denote by $\mathfrak{R}(\mathfrak{D}(\Sigma))$ or $\mathfrak{R}(\Sigma)$ the radical of $\mathfrak{D}(\Sigma)$ in the ∂-ring R.

Proposition 6 (Okugawa [33, p.45, Thm. 5])**.** *Let R be a ∂-ring of characteristic $p > 0$ and let $\mathfrak{I} \subseteq R$ be a ∂-ideal of R. Then, the radical $\mathfrak{R}(\mathfrak{I})$ is a ∂-ideal of R.*

Proof. It suffices to prove that $\partial^j(\mathfrak{R}(\mathfrak{I})) \subseteq \mathfrak{R}(\mathfrak{I})$ for all $j \geq 1$. Let $a \in \mathfrak{R}(\mathfrak{I})$. Then $a^n \in \mathfrak{I}$ for some $n \geq 1$. For a sufficiently large $e \geq 1$, we see that

$$a^m \cdot a^n = a^{p^e} \in \mathfrak{I}$$

for some $m \in \mathbb{N}$. Since ∂ satisfies the same properties as hyperderivatives (see §2.4), by Proposition 1(b), for all $j \in \mathbb{N}$ we see that

$$\partial^{jp^e}(a^{p^e}) = (\partial^j(a))^{p^e}.$$

Since $\mathfrak{I}$ is a ∂-ideal of R, we have $\partial^{jp^e}(a^{p^e}) \in \mathfrak{I}$ for all $j \geq 1$. Thus, $(\partial^j(a))^{p^e} \in \mathfrak{I}$ and so $\partial^j(a) \in \mathfrak{R}(\mathfrak{I})$. □

Remark 4. The proof of Proposition 6 does not work in characteristic 0. See [23, Prop. 2.19] for characteristic 0.

Theorem 15 (cf. [33, p.63 Thm. 1] and [23, Lem. 2.22])**.** *Let R be a ∂-ring of characteristic $p > 0$ and let $\mathfrak{I}$ is a proper ∂-ideal of R. If S is a multiplicative subset of R such that $S \cap \mathfrak{I} = \emptyset$, then there exists a prime ∂-ideal $\mathbf{p}$ of R such that $\mathfrak{I} \subseteq \mathbf{p}$ but $S \cap \mathbf{p} = \emptyset$.*

Proof. Consider the set of all ∂-ideals of R that do not intersect S but contain $\mathfrak{I}$, ordered by inclusion. Clearly, this set is not empty since it contains $\mathfrak{I}$. By Zorn's Lemma, this set has a maximal element $\mathbf{p}$. We claim that $\mathbf{p}$ is a prime ideal of R. Suppose there exist elements $a, b \in R$ such that $a \notin \mathbf{p}, b \notin \mathbf{p}$ and $ab \in \mathbf{p}$. By the maximal property of $\mathbf{p}$, we see that $\mathfrak{D}(\mathbf{p}, a) \cap S \neq \emptyset$ and $\mathfrak{D}(\mathbf{p}, b) \cap S \neq \emptyset$. Let $s_1 \in \mathfrak{D}(\mathbf{p}, a) \cap S$ and $s_2 \in \mathfrak{D}(\mathbf{p}, b) \cap S$. Then, $s_1 \cdot s_2 \in \mathfrak{D}(\mathbf{p}, ab) \subset \mathbf{p}$, which is a contradiction since S is a multiplicative set. □

A.2 Kolchin topology in positive characteristic

The ∂-*polynomial ring* denoted by $R\{y_1, \ldots, y_m\}$ in the ∂-variables $(y_1, \ldots, y_m)$ is the polynomial ring over R in the variables $\partial^j(y_i),\ j \geq 1,\ i = 1, \ldots, m$ made into a ∂-ring by setting

(a) $\partial^j(a) := \partial^j(a)$ for $a \in R$,

(b) $\partial^k(\partial^j(y_i)) := \binom{k+j}{j}\partial^{k+j}(y_i)$, $k \geq 1$.

Here $y_1, \ldots, y_m$ are called ∂-indeterminates.

We can now define the Kolchin topology in positive characteristic. Let K be a ∂-field. A ∂-*extension field of* K is a ∂-field L which is an extension field of the ∂-field K. Let $\overline{K}$ be an algebraic closure of the field K and K^{sep} be the separable closure of K in $\overline{K}$.

Proposition 7. *There is a unique extension of $\partial^j : K \to K$ to $\partial^j : K^{\mathrm{sep}} \to K^{\mathrm{sep}}$, which satisfy all the rules of ∂.*

Proof. The proof follows the same argument as that for hyperderivatives. See [14, Thm. 5]. $\square$

Let $a \in \overline{K} \setminus K^{\mathrm{sep}}$. We say that ∂ can be *extended to* a if ∂ can be extended to some extension field of K^{sep} that contains a. The largest extension field $\overline{K}^{\partial}$ of K^{sep} in $\overline{K}$ that has an extension of ∂ is called the ∂-*closure* of K in $\overline{K}$.

Let $S \subseteq K\{y_1, \ldots, y_m\}$ be a set of ∂-polynomials. The *zero set* of S is defined as

$$\mathfrak{Z}(S) := \{(a_1, \ldots, a_m) \in (\overline{K}^{\partial})^m \mid f(a_1, \ldots, a_m) = 0,\ \forall f \in S\}.$$

Proposition 8 (cf. [23, Prop. 3.2])**.** *For $i \geq 0$, let $S, T, S_i \subset K\{y_1, \ldots, y_m\}$. We have the following properties.*

1. $\mathfrak{Z}(0) = (\overline{K}^{\partial})^m$ *and* $\mathfrak{Z}(R) = \emptyset$.

2. $S \subset T$ *implies that* $\mathfrak{Z}(T) \subset \mathfrak{Z}(S)$.

3. $\mathfrak{Z}(S) = \mathfrak{Z}(\mathfrak{D}(S)) = \mathfrak{Z}(\mathfrak{R}(S))$.

4. $\mathfrak{Z}(\cup S_i) = \mathfrak{Z}(\sum \mathfrak{D}(S_i)) = \cap \mathfrak{Z}(S_i)$.

5. $\mathfrak{Z}(\mathfrak{D}(S) \cap \mathfrak{D}(T)) = \mathfrak{Z}(\mathfrak{D}(S)\mathfrak{D}(T)) = \mathfrak{Z}(S) \cup \mathfrak{Z}(T)$.

Proof. The proofs of the assertions follow the same line of argument as that for the Zariski topology. □

A set $X \subseteq (\overline{K}^{\partial})^m$ is said to be *K-∂-closed* if there exists a subset $S \subseteq K\{y_1, \ldots, y_m\}$ such that $X = \mathfrak{Z}(S)$. If we set

$$\mathfrak{I}(X) := \{P \in K\{y_1, \ldots, y_m\} \mid P(a_1, \ldots, a_m) = 0 \ \ \forall\, (a_1, \ldots, a_m) \in X\},$$

then $\mathfrak{I}(X)$ is a radical ∂-ideal in R, and we call it the *defining K-∂-ideal* of X.

If X is not K-∂-closed, then its *K-∂-closure* $\overline{X}^{\partial}$ is the smallest K-∂-closed subset of $(\overline{K}^{\partial})^m$ containing X, that is, $\mathfrak{Z}(\mathfrak{I}(X))$.

Proposition 9 (cf. [23, Prop. 3.8]). *Let $X_1, X_2 \subseteq (\overline{K}^{\partial})^m$. Then,*

1. *If $X_1 \subseteq X_2$, then $\mathfrak{I}(X_2) \subseteq \mathfrak{I}(X_1)$,*

2. $\mathfrak{I}(X_1 \cup X_2) = \mathfrak{I}(X_1) \cap \mathfrak{I}(X_2)$.

Proof. The proofs of the assertions follow the same line of argument as that for the Zariski topology. □

Theorem 16 (Okugawa [33, p.71 Thm. 2]). *Let $\mathbf{p}$ be a prime ∂-ideal of the polynomial ring $K\{y_1, \ldots, y_m\}$. There exists a zero $\xi \in (\overline{K}^{\partial})^m$ of $\mathbf{p}$ such that*

$$\mathbf{p} = \{f \in K\{y_1, \ldots, y_m\} \mid f(\xi) = 0\}.$$

Proof. Let $K\{y_1, \ldots, y_m\}/\mathbf{p}$ be the residue ring of $K\{y_1, \ldots, y_m\}$ mod $\mathbf{p}$, and φ the canonical ∂-morphism of $K\{y_1, \ldots, y_m\}$ onto $K\{y_1, \ldots, y_m\}/\mathbf{p}$. Since φ induces a ∂-isomorphism of K onto $\varphi(K)$, we identify each $a \in K$ with $\varphi(a)$. Then the ∂-field of quotients $Q(K\{y_1, \ldots, y_m\}/\mathbf{p})$

of $K\{y_1, \ldots, y_m\}/\mathbf{p}$ is a ∂-extension of K. If we set $\xi = (\xi_1, \ldots, \xi_m)$ where each $\xi_i := \varphi(y_i)$, then we get the desired result. $\square$

Proposition 10 (Okugawa [33, p.72 Thm.4])**.** *Let $S \subseteq K\{y_1, \ldots, y_m\}$ be a set of ∂-polynomials. If $\mathfrak{Z}(S) = \emptyset$, then $\mathfrak{R}(S) = K\{y_1, \ldots, y_m\}$.*

Proof. Assume that $\mathfrak{R}(S) \neq K\{y_1, \ldots, y_m\}$. By Theorem 15 there is a prime differential ideal $\mathbf{p}$ such that $\mathfrak{R}(S) \subseteq \mathbf{p}$. Then, the zero ξ of $\mathbf{p}$ from Theorem 16 is such that $\xi \in \mathfrak{Z}(S)$, which is a contradiction to the hypothesis. $\square$

The differential algebraic geometry analogue of Hilbert's Nullstellensatz is as follows.

Theorem 17 (Okugawa [33, p.72, Cor. to Thms. 3 and 4])**.** *For the ∂-field K, let $K\{y_1, ..., y_m\}$ be a ∂-polynomial ring and let $S \subseteq K\{y_1, \ldots, y_m\}$. Then*

$$\mathfrak{R}(S) = \mathfrak{I}(\mathfrak{Z}(S)).$$

Proof. It is clear that $\mathfrak{R}(S) \subseteq \mathfrak{I}(\mathfrak{Z}(S))$. Therefore, it suffices to show that if $g \in \mathfrak{I}(\mathfrak{Z}(S))$, then $g \in \mathfrak{R}(S)$. Let y_{m+1} be a ∂-indeterminate over $K\{y_1, \ldots, y_m\}$. Consider the set

$$S' := \{f, 1 - y_{m+1}g \mid f \in S\} \subseteq K\{y_1, \ldots, y_m, y_{m+1}\},$$

where $g \in \mathfrak{I}(\mathfrak{Z}(S))$. Since g vanishes at every zero of S, the set S' has no zeros. Thus by Proposition 10, the ideal $\mathfrak{D}(S')$ of the ∂-ring $K\{y_1, \ldots, y_m, y_{m+1}\}$ contains 1 and so

$$1 = \sum_{\substack{L_i \in S \\ j \geq 1}} Q_{i,\ell_j} \cdot \partial^{\ell_j}(L_i) + \sum_{h \geq 1} Q_{\ell_h} \partial^{\ell_h}(1 - y_{m+1}g), \tag{A.1}$$

where each $Q_{i,\ell_j}, Q_{\ell_h} \in K\{y_1, \ldots, y_m, y_{m+1}\}$ and $\ell_j, \ell_h \in \mathbb{Z}_{\geq 0}$. Note that the right hand side of (A.1) has finitely many terms. Since ∂ satisfies the same properties as hyperderivatives (see §2.4), by Proposition 1(c) we see that for $u \geq 1$

$$\partial^u \left(\frac{1}{g}\right) = \frac{B}{g^v}, \tag{A.2}$$

for some $B \in K\{y_1, \ldots, y_m\}$ and $v \in \mathbb{N}$. Substituting y_{m+1} for $1/g$, we have $1 - y_{m+1}g = 0$ and so by (A.2), we obtain

$$1 = \sum_{\substack{L_i \in S \\ j \geq 1}} \frac{B_{i,\ell_j}}{g^{v_{i,\ell_j}}} \cdot \partial^{\ell_j}(L_i),$$

where each $B_{i,\ell_j} \in K\{y_1, \ldots, y_m\}$ and $v_{i,\ell_j} \in \mathbb{N}$. Multiplying both sides by a sufficiently large power w of g to clear denominators, we see that $g^w \in \mathfrak{D}(S)$ and so $g \in \mathfrak{R}(S)$. $\square$

Given a K-∂-closed set $X \subseteq ((\overline{K}^{\partial})^m, \partial)$, we consider the Zariski closure $\overline{X}^Z \subseteq \overline{K}^m$ of X, the closure of X as a subset of $(\overline{K}^{\partial})^m$ equipped with the Zariski topology. Let $S \subseteq K[y_1, \ldots, y_m]$ be a set of polynomials. The *zero set* of S is defined as

$$\mathcal{Z}(S) := \{(a_1, \ldots, a_m) \in \overline{K}^m \mid f(a_1, \ldots, a_m) = 0, \, \forall f \in S\}.$$

Lemma 7 (cf. [23, Lem. 3.42])**.** *Let $X \subseteq (\overline{K}^{\partial})^m$ be a K-∂-closed set and let its defining K-∂-ideal be $\mathfrak{I}(X) \subseteq K\{y_1, \ldots, y_m\}$. Also, let $K[y_1, \ldots, y_m]$ be the polynomial ring in the variables $y_1, \ldots, y_m$ over the field K. Then its Zariski closure is the set*

$$\overline{X}^Z = \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \ldots, y_m]),$$

where $\mathfrak{I}(X) \cap K[y_1, \ldots, y_m] \subseteq K[y_1, \ldots, y_m]$.

Proof. We follow the outline of the proof of [23, Lem. 3.42]. Since $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \ldots, y_m])$ is Zariski closed, it is straightforward to see that

$$X \subseteq \overline{X}^Z \subseteq \mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \ldots, y_m]).$$

Conversely, if $S \subseteq K[y_1, \ldots, y_m] \subseteq K\{y_1, \ldots, y_m\}$ is such that $X \subseteq \mathfrak{Z}(S)$, then by Theorem 17 we have $\mathfrak{R}(S) \subseteq \mathfrak{I}(X)$. This implies that $S \subseteq \mathfrak{R}(S) \cap K[y_1, \ldots, y_m] \subseteq \mathfrak{I}(X) \cap K[y_1, \ldots, y_m]$. Thus, $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \ldots, y_m]) \subseteq \mathcal{Z}(S)$. Since S was chosen arbitrarily, we see that $\mathcal{Z}(\mathfrak{I}(X) \cap K[y_1, \ldots, y_m]) \subseteq \overline{X}^Z$. □

If $f \in K\{y_1, \ldots, y_m\}$ is a linear combination over the ∂-field K of 1 and elements of the set $\{\partial^j(y_i) \mid j \geq 0, i = 1, \ldots, m\}$, then we say that f is a *linear ∂-polynomial* in $K\{y_1, \ldots, y_m\}$. Moreover if the coefficient of 1 is 0, then we say that f is a *linear homogeneous ∂-polynomial*.

Proposition 11 (Okugawa [33, p.74 Thm. 5])**.** *Let $S \subseteq K\{y_1, \ldots, y_m\}$ be a set of linear ∂-polynomials, then*

$$\mathfrak{R}(S) = \mathfrak{D}(S).$$

Proof. It suffices to show that $\mathfrak{D}(S)$ is a prime ideal of the ∂-ring $K\{y_1, \ldots, y_m\}$ regarded as a ring. By definition $\mathfrak{D}(S)$ is generated, as an ideal of the ring $K\{y_1, \ldots, y_m\}$, by $\{\partial^j(L_i) \mid i, j \geq 0, L_i \in S\}$. Suppose that $f, g \notin \mathfrak{D}(S)$ such that $fg \in \mathfrak{D}(S)$. Then,

$$fg = \sum_{\substack{L_i \in S \\ j \geq 1}} h_{i,\ell_j} \partial^{\ell_j}(L_i),$$

where $\ell_j \in \mathbb{N}$, and $h_{i,\ell_j} \in K\{y_1, \ldots, y_m\}$, and all but finitely many h_{i,ℓ_j} are zero. We see that fg is a polynomial in a finite subset of the variables $\{\partial^j(y_i) \mid j \geq 0,\ i = 1, \ldots, m\}$ over the ∂-field K regarded as a field. Let us denote this subset of variables by $\{x_1, \ldots, x_n\}$ for some $n \geq 1$. Then, $L = (\{\partial^{\ell_j}(L_i)\})$ is an ideal in $K[x_1, \ldots, x_n]$ such that $f, g \notin L$ and $fg \in L$ and so, L is not a prime ideal. However, for a polynomial ring in finitely many indeterminates, ideals generated by linear polynomials are prime ideals and thus, we obtain a contradiction. □

The reader is directed to [33] for a detailed account of different algebra in positive characteristic and [23] for a detailed account of differential algebraic geometry in characteristic zero.