SOME HOMOLOGICAL PROPERTIES OF LATTICE IDEALS

A Dissertation

by

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ABSTRACT

The interplay of algebra and combinatorics is fruitful in both fields: combinatorics provides algebraic structures with tractable realizations, while algebra underpins combinatorial objects with a rigorous framework. Pioneered by Hochster and Stanley, interest in combinatorial commutative algebra has grown rapidly, often including techniques from simplicial topology and convex geometry.

This thesis presents two main results that combine commutative algebra and combinatorics. The first result considers the Cohen–Macaulayness of a lattice ideal and its associated toric ideal. Despite the deep algebraic connection between these two ideals, we produce infinitely many examples, in every codimension, of pairs where one of these ideals is Cohen–Macaulay but the other is not.

The second result describes the free resolution of the ground field over the quotient ring by a specific type of lattice ideal, that defining a rational normal 2-scroll. This chapter also includes a computation of the Betti numbers of the ground field when resolved over the ring coming from an arbitrary rational normal k-scroll.

DEDICATION

To my grandparents: Alicja & Kazimierz and Jadwiga & Marian. May my life and work honor yours.

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There is an idiom used in Polish „trafiło się jak ślepej kurze ziarno” which roughly translates to “chanced upon like a blind hen finds a kernel." The phrase is used to describe when someone, either unwittingly or undeservedly, stumbles upon a lucky find. Five years ago, when I chose Professor Laura Matusevich as an advisor, I could not have known how truly fortunate I was in my decision, nor am I yet sure that I deserve such a wonderful advisor. Laura has been a guide, a mentor, a role model, and an enduring source of support and patience as I made my way through this program. She has helped me grow in every aspect: in study and research, in writing and presentation, in professional development and mathematical confidence. Any achievements I have made at Texas A&M and all opportunities I have in the future are in large part due to her, and I am forever grateful.

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

LIST OF TABLES

TABLE Page

1. INTRODUCTION AND BACKGROUND

1.1 Introduction

Combinatorial commutative algebra uses combinatorial tools to answer questions in algebra. Often, algebraic questions are broad and abstract, and a combinatorial realization gives a tangible way to study algebraic objects and properties.

One central object of study in commutative algebra is the *minimal free resolution* of a module. We give a rigorous definition in Definition 1.2.12, but provide an overview and two introductory examples here. For a module M with generators $m_1, \dots, m_r$ over a ring A, we can ask what the relations among these generators are. We can write this down explicitly by mapping a free module F_0 of rank r onto M that takes the ith basis element of F to $m_i \in M$. The relations among the generators of M, called the *syzygies*, are given by the kernel K_0 of our surjective map $f_0 : F_0 \to M$.

This kernel K_0 is in its own right an A-module, with generators $k_1, \dots, k_s$. These generators have relations among each other, so we can map a rank s free module F_1 onto K_0 in F_0 by taking the ith basis element of F_1 to k_i. The relations among the generators of K_0 are given in the kernel K_1 of $f_1 : F_1 \to F_0$.

The kernel K_1 of the map $F_1 \to F_0$ is also an A-module, generated by some $\ell_1, \dots, \ell_t$. To find the relations among the ℓ_i we map an A-module F_2 of rank t onto K_1 in F_1 by taking the ith basis element of F_2 to ℓ_i. The relations among the generators of K_1 are given in the kernel K_2 of $f_2 : F_2 \to F_1$.

Now, rinse and repeat: this process can be continued! At each step, we uncover relations among the generators of the module we map onto, which themselves represent relations of a module we map onto, which themselves represent relations... And so on.

We can see that this is an *exact* sequence, that is, the image of each map is the kernel of the next. Furthermore, if we fix a basis for our free modules, these relations are then encoded as the columns of the matrices we use to represent the maps. We will give both a schematic and a concrete example.

$$\cdots \longrightarrow F_2 \xrightarrow{\begin{bmatrix}\text{relations among the}\\ \text{relations among the}\\ \text{generators of } M\end{bmatrix}} F_1 \xrightarrow{\begin{bmatrix}\text{relations among the}\\ \text{generators of } M\end{bmatrix}} F_0 \xrightarrow{\begin{bmatrix}\text{a system of}\\ \text{generators of } M\end{bmatrix}} M$$

This process, so far loosely defined, benefits from an imposition of *minimality*; we do not want to repeat generators or relations. Unfortunately, for a general module, the cardinality of a minimal generating set is not well-defined. Take, for example, the ideal $\langle 4, 6\rangle = \langle 2\rangle$ in the integers, but the generators 4 and 6 are both necessary to generate the ideal. However, over a ring with a unique maximal ideal $\mathfrak{m}$, the cardinality of a minimal set of generators *is* well-defined. A strange-looking condition, that each entry of the matrix representing the ∂_i is contained in $\mathfrak{m}$, ensures that we simultaneously minimize the ranks of the F_i, making the idea of a *minimal free resolution* precise. Even more miraculously, this resolution turns out to be unique up to chain isomorphism, a true invariant of our module M.

Example 1.1.1. Take the module $I = \langle xy - z^2, z^2w, w^3\rangle$ in $S = \mathbb{C}[x, y, z, w]$. Because I has 3 minimal generators, our first free module F_0 will be S^3. The map $f_0 : F_0 \to I$ is

$$f_0 = \begin{bmatrix} xy - z^2 & z^2w & w^3 \end{bmatrix}.$$

If we use m_1, m_2, m_3 to denote $xy - z^2$, z^2w, and w^3 respectively, then the minimal relations among the generators of I are

$$\begin{aligned}
0 &= -z^2w\, m_1 && +(xy - z^2)\, m_2 && \\
0 &= -w^3\, m_1 && -w^2\, m_2 && +xy\, m_3 \\
0 &= && -w^2\, m_2 && +z^2\, m_3
\end{aligned}$$

All other relations among the m_i are combinations of those written above. Therefore, if we use

e_1, e_2, e_3 as the basis for $F_0 = S^3$, then the kernel $K_0 = \ker(f_0)$ is generated by

$$\begin{aligned} k_1 &= -z^2 w\, e_1 & +(xy - z^2)\, e_2 & \\ k_2 &= -w^3\, e_1 & -w^2\, e_2 & \qquad +xy\, e_3 \\ k_3 &= & -w^2\, e_2 & \qquad +z^2\, e_3 \end{aligned}$$

This kernel K_0 is generated by 3 elements, so $F_1 = S^3$. The map $f_1 : S^3 \to S^3$ takes the basis of F_1 and maps it onto K_0, so

$$f_1 = \begin{bmatrix} -z^2 w & -w^3 & 0 \\ (xy - z^2) & -w^2 & -w^2 \\ 0 & xy & z^2 \end{bmatrix}.$$

We can see that the columns of f_1 each correspond to a relation among the generators of I.

The generators of the kernel $K_1 = \ker(f_1)$ have only one relation

$$0 = w^2\, k_1 - z^2\, k_2 + xy\, k_3.$$

Therefore, if we use $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for the basis of $F_1 = S^3$, a basis for the kernel $K_1 = \ker(f_1)$ is given by

$$\ell_1 = w^2\, \varepsilon_1 - z^2\, \varepsilon_2 + xy\, \varepsilon_3.$$

This kernel K_1 is generated by one element, so $F_2 = S^1$ and the map $f_2 : S \to S^3$ is

$$f_2 = \begin{bmatrix} w^2 \\ -z^2 \\ xy \end{bmatrix}.$$

The kernel of f_2 is zero, i.e. ℓ_1 has no non-trivial relations. Therefore F_3 and all subsequent F_i are 0.

Sewing this all together, the minimal free resolution of I as an S-module is

$$0 \to S \xrightarrow{f_2} S^3 \xrightarrow{f_1} S^3 \xrightarrow{f_0} I \to 0.$$

The zero at the end is often added so that every spot in our sequence is exact, that is, the kernel of each map is the image of the next.

Example 1.1.1 is finite in the sense that $F_i = 0$ for $i \geq 3$. The fact that this iterative mapping onto kernels peters out is a priori not guaranteed by the construction, and is, in fact, rarely the case. It is only because I is a module over the polynomial ring S that this termination occurs. For modules over general rings, resolutions are typically infinite, even in very simplistic examples.

Example 1.1.2. Let $A = \Bbbk[x]/\langle x^2 \rangle$. The minimal free resolution of $\Bbbk = A/\langle x \rangle$ as an A-module is

$$\cdots \xrightarrow{\cdot x} A \xrightarrow{\cdot x} A \xrightarrow{\cdot x} A \xrightarrow{\cdot x} A \to \Bbbk \to 0.$$

This infinite length, along with the frequent failure of techniques used for S-modules, makes finding explicit minimal free resolutions over more general rings much more difficult. Chapter 3, however, presents explicitly the minimal free resolution for the ground field $\Bbbk$ over a family of rings coming from rational normal scrolls.

Besides giving a full presentation of the module and its relations, many of algebraic properties can be understood from a minimal free resolution. Chapter 2 focuses on the Cohen–Macaulay property, while Chapter 3 focuses on the Koszul property, both defined in subsection 1.2.3. The Cohen–Macaulay property is often used as a baseline algebraic "niceness" qualifier; in the words of Hochster, "life is really worth living" in a Cohen–Macaulay ring [Hoc78, p.887].

The context that we explore these properties is that of *lattice* and *toric ideals*, defined in subsection 1.2.2. These binomial ideals have deep combinatorial properties, often allowing us to understand algebraic aspects like the Cohen–Macaulay property from an associated combinatorial object.

We will give two examples previewing the constructions from Chapters 2 and 3.

1.1.1 Preview of Chapter 2

Consider the rank 2 lattice $\mathscr{L}$ in $\mathbb{Z}^4$ generated by the columns of the matrix

$$\begin{bmatrix} 14 & 1 \\ 0 & -5 \\ -16 & 1 \\ 2 & 3 \end{bmatrix}.$$

To a lattice in $\mathbb{Z}^n$, one can associate a binomial *lattice ideal* $I_{\mathscr{L}} = \langle x^a - x^b \,|\, a - b \in \mathscr{L} \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$. For our lattice $\mathscr{L}$, the lattice ideal is simply

$$I_{\mathscr{L}} = \langle x_1^{14} x_4 - x_3^{16}, x_1 x_3 x_4^3 - x_2^5 \rangle.$$

Note that there are some "holes" in our lattice, that is, there are elements u in $\mathscr{L}$ where not all rational multiples of u are in $\mathscr{L}$, despite having integer entries. For example, $(7, 0, -8, 1)$ is not in $\mathscr{L}$, though $2(7, 0, -8, 1) = (14, 0, -16, 2)$ is in $\mathscr{L}$. With this in mind, we define the *saturation* of $\mathscr{L}$, $\mathscr{L}^{\text{sat}} = \{u \,|\, mu \in \mathscr{L} \text{ for some } m \in \mathbb{N}\}$. This saturated lattice is generated by the columns of the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & -3 \\ -2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The corresponding lattice ideal for $\mathscr{L}^{\text{sat}}$ is

$$I_{\mathscr{L}^{\text{sat}}} = \langle x_1 x_3 x_4^3 - x_2^5, x_1 x_2^2 - x_3^2 x_4, x_1^2 x_4^2 - x_2^3 x_3, x_1^3 x_4 - x_2 x_3^3, x_1^4 x_2 - x_3^5 \rangle.$$

The ideal $I_{\mathscr{L}^{\text{sat}}}$ is prime, while $I_{\mathscr{L}}$ is not. In fact, $I_{\mathscr{L}^{\text{sat}}}$ is a minimal prime of $I_{\mathscr{L}}$, so there is a strong algebraic connection between the two. One can check, perhaps in the computer algebra system `Macaulay2` [GS], that $I_{\mathscr{L}}$ is Cohen–Macaulay, while $I_{\mathscr{L}^{\text{sat}}}$ is not.

However, this is not always the case. Consider the lattice $\mathscr{L}$ generated by the columns of

$$\begin{bmatrix} -1 & 6 \\ 1 & 7 \\ 12 & -7 \\ -12 & -6 \end{bmatrix}.$$

The saturation of $\mathscr{L}$, the lattice $\mathscr{L}^{\text{sat}}$, is generated by the columns of

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ -2 & 1 \\ 0 & -6 \end{bmatrix}.$$

The corresponding lattice ideals are

$$I_{\mathscr{L}} = \langle x_1^6x_2^7 - x_3^7x_4^6, x_1^7x_2^6x_4^6 - x_3^19, x_2x_3^{12} - x_1x_4^{12}, x_1^5x_2^8x_3^5 - x_4^{18}\rangle$$

and

$$I_{\mathscr{L}^{\text{sat}}} = \langle x_1x_2 - x_3^2, x_1^2x_2^3x_3 - x_4^6\rangle.$$

One can check that $I_{\mathscr{L}^{\text{sat}}}$ is prime and Cohen–Macaulay, while $I_{\mathscr{L}}$ is neither. This pair of pairs show that there is no way to deduce the whether one of $I_{\mathscr{L}}$, $I_{\mathscr{L}^{\text{sat}}}$ is Cohen–Macaulay just by checking if the other is Cohen–Macaulay, at least when our lattice has rank 2. Chapter 2 gives a method for constructing such pairs $\mathscr{L}$ and $\mathscr{L}^{\text{sat}}$ of rank 2, then building on those small examples to create examples of every rank. The methods invoke the combinatorics of lattice ideals to give criteria for being Cohen–Macaulay.

1.1.2 Preview of Chapter 3

Minimal free resolutions have enjoyed a rich history of study. The most common setting is resolutions over the polynomial ring. Though still not entirely understood, this is the most tractable environment, not least because Theorem 1.2.21 guarantees that a minimal free resolution over the polynomial ring must have $F_i = 0$ for large enough i.

Minimal free resolutions over other rings, even quotient rings of the polynomial ring, fail to satisfy Hilbert's Syzygy Theorem miserably, and, consequently, the resolutions do not terminate. This is coupled with an often exponential growth in the ranks of each of the free modules, making these resolutions almost impossible to wrangle.

However, in the case when the ring is a quotient of the polynomial ring by a specific type of lattice ideal, one arising from a rational normal 2-scroll, the resolution is presented in Chapter 3. The resolution is shockingly structured, allowing us to write it out explicitly.

We illustrate our results in an example. Consider $R = \Bbbk[x_1, \ldots, x_6]/I$, where I is the ideal of 2×2 minors of the matrix

$$\left[\begin{array}{cc|cc} x_1 & x_2 & x_4 & x_5 \\ x_2 & x_3 & x_5 & x_6 \end{array}\right].$$

We can also check that I is the lattice ideal corresponding to the saturated lattice in $\mathbb{Z}^6$ generated by the columns of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 2 \\ 1 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The ideal I gives the defining equations of the rational normal scroll $\mathcal{S}(2,2)$, an algebraic variety defined fully in Definition 3.0.1. In this case, the minimal free resolution of the ground field $\Bbbk$ over

R is

$$\cdots \to R^{64\cdot 3^{i-3}} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_4} R^{64} \xrightarrow{\partial_3} R^{21} \xrightarrow{\partial_2} R^{6} \xrightarrow{[x_1\ x_2\ \cdots\ x_6]} R \to \Bbbk \to 0$$

The matrices giving the differentials ∂_i are highly structured. Throughout this work, we adopt the following notations: $\mathbf{0}^{p\times q}$ denotes a zero matrix of size $p \times q$; where it causes no confusion, zero blocks or entries of a matrix are indicated by $\mathbf{0}$ or simply left empty; $\mathbb{1}_\ell$ is the $\ell \times \ell$ identity matrix; direct sum of matrices denotes concatenation of blocks along the main diagonal (with off-diagonalblocks equal to zero). With these conventions,

$$\partial_2 = \left[\begin{array}{cccc|ccccc}
\varphi_0 & & & & x_4 & & & & \\
 & \varphi_0 & & & & x_4 & & & \\
 & & & & & & x_4 & x_5 & x_6 \\
\hline
 & & \varphi_0 & & -x_1 & -x_2 & -x_3 & & \\
 & & & \varphi_0 & & & & -x_3 & \\
 & & & & & & & & -x_3
\end{array}\right]$$

where $\varphi_0 = \left[\begin{array}{cc|cc} x_2 & x_3 & x_5 & x_6 \\ -x_1 & -x_2 & -x_4 & -x_5 \end{array}\right]$;

$$\partial_3 = \left[\begin{array}{c|cccc}
\varphi_1^{\oplus 4} & \multicolumn{2}{c}{x_4 \cdot \mathbb{1}_8} & \multicolumn{2}{c}{} \\
 & \multicolumn{2}{c}{} & \multicolumn{2}{c}{-x_3 \cdot \mathbb{1}_8} \\
\hline
 & -\varphi_0 & & & \\
 & & -\varphi_0 & & \\
 & & & -\varphi_0 & \\
 & & & & -\varphi_0
\end{array}\right]$$

where

$$\varphi_1 = \left[\begin{array}{cccc|cccc|cccc}
x_2 & x_3 & x_5 & x_6 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & -x_2 & -x_4 & -x_5 & 0 & x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & 0 & x_2 & x_3 & x_5 & x_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_1 & -x_2 & -x_4 & -x_5
\end{array}\right];\text{ and for } i \geq 4,$$

$$
\partial_i = \left[\begin{array}{c|cc}
\varphi_{i-2}^{\oplus 4} & x_4 \cdot \mathbb{1}_{8\cdot 3^{i-3}} & \\
\cline{2-3}
 & & -x_3 \cdot \mathbb{1}_{8\cdot 3^{i-3}} \\
\hline
 & \multicolumn{2}{c}{-\varphi_{i-3}^{\oplus 4}}
\end{array}\right]
$$

where

$$
\varphi_2 = \left[\begin{array}{c|cccccccccccc|c}
\varphi_1 & x_1 & x_2 & x_4 & x_5 & & & & & & & & & \\
 & 0 & 0 & 0 & 0 & & & & & & & & & \\
\hline
 & -x_2 & -x_3 & -x_5 & -x_6 & & & & & & & & & \\
 & x_1 & x_2 & x_4 & x_5 & -x_2 & -x_3 & -x_5 & -x_6 & & & & & \\
 & & & & & x_1 & x_2 & x_4 & x_5 & -x_2 & -x_3 & -x_5 & -x_6 & \\
 & & & & & & & & & x_1 & x_2 & x_4 & x_5 & \\
\hline
 & & & & & & & & & 0 & 0 & 0 & 0 & \\
 & & & & & & & & & -x_2 & -x_3 & -x_5 & -x_6 & \varphi_1
\end{array}\right]
$$

and $\varphi_i = \varphi_{i-1} \bigoplus \varphi_{i-2}^{\oplus 3} \bigoplus \varphi_{i-1}$ for $i \geq 3$.

Chapter 3 gives the general construction for these resolutions, as well as a computation for the ranks of the free modules in a slightly more general case, where the ambient ring is the quotient by the defining ideal of a rational normal k-scroll.

1.2 Background

Throughout this work, assume all rings are commutative rings with unity. We adopt the convention that $\Bbbk$ is a field and $S = \Bbbk[x_1, \ldots, x_n]$ is the polynomial ring over $\Bbbk$ with n indeterminates. In this setting, we will use multinomial notation, that is, for an integer vector $u \in \mathbb{N}^n$, we use x^u to denote the monomial $x^{u_1}x^{u_2}\cdots x^{u_N}$. We include 0 in the natural numbers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and use $[n]$ to denote the set $\{1, \ldots, n\}$. Foundational material on modules can be found in [Hun80] and [DF04], while the material on Gröbner bases can be found in [CLO15].

1.2.1 Gradings and Free Resolutions

Definition 1.2.1. Let G be an abelian monoid. If we can decompose a ring A into abelian groups in the following way

$$A = \bigoplus_{g\in G} A_g \quad \text{satisfying } A_gA_h \subseteq A_{g+h} \text{ for all } g, h \in G,$$

we say A is a *graded*, or *G-graded*, ring. If an element $a \in A_g$, we say a is *homogeneous of degree* a and write $\deg_G(a) = g$, or just $\deg(a) = g$ if G is clear from context. If I is an ideal of A generated by homogeneous elements, we call I a *homogeneous* ideal.

Though this definition is for any G, the most commonly seen examples use the familiar monoids $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}^n$, or $\mathbb{Z}^n$.

Remark. Often multiple gradings can exist for the same ring. The polynomial ring S in n variables can be graded both by $\mathbb{N}$ and by $\mathbb{N}^n$. If you set S_d to be the $\Bbbk$-vector space of homogeneous degree d polynomials, i.e. consider $\deg(x_i) = 1$, this gives an $\mathbb{N}$-grading. If you set S_u to be the $\Bbbk$-vector space spanned by the monomial x^u, i.e. consider $\deg(x_i) = e_i$ where e_i is the length n vector with a 1 in the ith spot and 0's everywhere else, this gives an $\mathbb{N}^n$-grading.

A graded ring may mimic the behavior of a local ring in the following sense.

Definition 1.2.2. An $\mathbb{N}$-graded ring A is called *graded local* if A has a unique homogeneous maximal ideal. As in the notation for a local ring, we call this maximal ideal $\mathfrak{m}$.

In particular, the $\mathbb{N}$-graded polynomial ring is graded local. Throughout this work, definitions and results stated for local rings hold for graded local rings (and in particular S).

There is a corresponding notion of a graded module when working with graded rings.

Definition 1.2.3. Let $A = \bigoplus_{g \in G} A_g$ be a graded ring. If we can decompose an R-module M in the following way

$$M = \bigoplus_{g \in G} M_g \quad \text{satisfying } A_g \cdot M_h \subseteq M_{g+h} \text{ for all } g, h \in G,$$

we say M is a *graded module*. The M_g are called the *graded components*, or *graded pieces*, of M.

These gradings extend to maps between graded modules.

Definition 1.2.4. Let A be a G-graded ring. An A-module homomorphism $f : M \to N$ is called a *graded map* (or *homogeneous map*) of *degree* g if $f(M_h) \subseteq N_{g+h}$.

Given a map between graded modules, it is often convenient if it preserves degree. We can force this by introducing a shift to the graded components of M by some $g \in G$, which we denote by $M(g)$.

Definition 1.2.5. Let M be a graded module. The module $M(g)$, called the g^{th} *twist of* M, is the graded module that is isomorphic to M and has graded components $M(g)_h = M_{g+h}$.

Example 1.2.6. Consider the ring $\mathbb{C}[x]$ as graded by $\mathbb{N}$ and the map of graded $\mathbb{C}[x]$-modules $\mathbb{C}[x] \to \mathbb{C}[x]$ that takes f to $x^2 f$. This map takes elements in $\mathbb{C}[x]_i$ to elements in $\mathbb{C}[x]_{i+2}$, an unfortunate consequence of multiplying by a degree 2 element. If we would like the map to go between components of the same degree, we can introduce an articial twist and consider it as a map $\mathbb{C}[x](-2) \to \mathbb{C}[x]$. Now the map takes the ith graded component $\mathbb{C}[x](-2)_i = \mathbb{C}[x]_{i-2}$ to the ith graded component $\mathbb{C}[x]_i$.

With these twists, we can define the graded analogue of free modules.

Definition 1.2.7. Let A be a graded ring. An A-module M is a *graded free module* if M is the direct sum of modules $A(g)$ for various $g \in G$.

With this language of grading in hand, we take a moment to discuss gradings in the polynomial setting.

The most common gradings of S are by $\mathbb{N}$ or by $\mathbb{N}^n$, as discussed in Remark 1.2.1. These gradings, besides being extremely natural, offer many desirable qualities that may not carry over to an arbitrary grading.

Each grading of S by an abelian group G comes with its degree map, which can be considered as a map $\deg : \mathbb{Z}^n \to G$ where $\deg(e_i) = \deg_G(x_i)$. We will call the kernel of this map K.

Theorem 1.2.8. *[MS05, Theorem 8.6] The following conditions are equivalent for a polynomial ring S graded by an abelian group G.*

1. *The only polynomials of degree 0 are constants, i.e. $S_0 = \mathbb{k}$.*

2. *For all $g \in G$, the $\mathbb{k}$-vector space S_g is finite dimensional.*

3. *For all finitely generated graded S-modules M and all $g \in G$, the $\mathbb{k}$-vector space M_g is finite dimensional.*

4. *The only nonnegative element in K is 0, i.e. $K \cap \mathbb{N}^n = 0$.*

Definition 1.2.9. If any of the equivalent conditions of Definition 1.2.8 hold for a torsion-free abelian group G, then we call the grading of S by G *positive*.

With a decomposition of a modules into its graded components, we may be interested in the relative sizes of the graded components of M. Though the following definitions can be extended to all positive gradings of S (and indeed to other graded rings), we will limit ourselves to the most basic context, that when S is graded by $\mathbb{N}$.

Definition 1.2.10. Let M be a finitely generated graded module over S considered with the $\mathbb{N}$ grading. For a nonnegative integer d, the *Hilbert function of* M

$$H_M : \mathbb{N} \to \mathbb{N} \tag{1.1}$$

$$d \mapsto \dim_{\Bbbk} M_d \tag{1.2}$$

returns the dimension of the graded component M_d as a $\Bbbk$-vector space. If we arrange this data in a formal power series, we call it the *Hilbert series of* M, denoted $\mathrm{Hilb}_M(t) = \sum\limits_{d\geq 0} H_M(d)t^d$.

Example 1.2.11. Take $M = S$. Then $H_S(d)$ is the number of monomials in n variables of degree d. A quick combinatorial computation shows that this is $\binom{n+d-1}{d-1}$. Therefore the Hilbert series is

$$\mathrm{Hilb}_M(t) = \sum_{d\geq 0} \binom{n+d-1}{d-1} t^d = \frac{1}{(1-t)^n}.$$

Though the Hilbert series captures much of the structure of the module and its graded components, it is difficult to see how these pieces fit together. An alternate display of the module structure is given by a minimal free resolution, sketched out in the introduction.

Definition 1.2.12. For any ring A and A-module M, a *projective resolution* of M is a chain complex

$$\mathcal{F}_\bullet : \cdots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0,$$

where each F_i is a projective module and $\mathcal{F}_\bullet$ is *exact*, that is, $\ker \partial_i = \operatorname{Im} \partial_{i-1}$ for every $i \geq 1$ and $\operatorname{Im} \partial_0 = M$. If each F_i is, in fact, free, we call $\mathcal{F}_\bullet$ a *free resolution*.

In the case where $(A, \mathfrak{m})$ is a (graded) local ring, the idea of projective and free modules coincide. We say $\mathcal{F}_\bullet$ is *minimal* if $\partial_k F_k \subseteq \mathfrak{m} F_{k-1}$. In the case where A is a graded ring, we say $\mathcal{F}_\bullet$ is *graded* if the F_i are graded free modules, and the ∂_i are homogoneous maps of degree 0.

There are certainly many different projective or free resolutions of a given module M. For a given resolution $\mathcal{F}_\bullet$, we can describe its length, and in turn use this to define an invariant of our module M that roughly captures "how far" a module is from being projective.

Definition 1.2.13. The *length* of a resolution $\mathcal{F}_\bullet$ is $\sup\{i \mid F_i \neq 0\}$. For a module M, the projective dimension of M is

$$\mathrm{pd}_A(M) \coloneqq \min\{r \mid r = \text{ length of a projective resolution of } M\}.$$

If M has no finite-length projective resolutions, we say $\mathrm{pd}_A(M) = \infty$.

The criterion for minimality simply means that any matrix representing the ∂_i has entries in $\mathfrak{m}$. The choice of the word "minimal" for this definition comes from the following fact.

Lemma 1.2.14. *A free resolution $\mathcal{F}_\bullet$ is minimal if and only if, for each i, a basis of F_{i-1} maps onto a minimal set of generators for* $\operatorname{coker} \partial_i = F_{i-1}/\operatorname{Im} \partial_i$.

The proof utilizes Nakayama's Lemma (see, for example, [Eis95, Lemma 19.4]).

While there exist many free resolutions of M, minimal free resolutions *are* unique to M, up to chain isomorphism. The uniqueness of the minimal free resolution allows us to define the following invariant.

Definition 1.2.15. For a minimal free resolution $\mathcal{F}_\bullet$ of M, the *Betti numbers* $\beta_i^A(M)$ of M are the ranks of the free modules F_i. In the case where $\mathcal{F}_\bullet$ is a graded minimal free resolution, then we can write each of the free modules as $F_i = \bigoplus\limits_{g \in G} A(-g)^{\beta_{i,g}}$. We call the $\beta_{i,g}$ the *graded Betti numbers*.

Note that $\beta_i(M) = \sum\limits_{g \in G} \beta_{i,g}(M)$. Furthermore, $\mathrm{pd}_A(M) = \max\{r \mid \beta_r \neq 0\}$.

Example 1.2.16. Let $I = \langle xy - z^2, xw + zw, z^4 \rangle \subseteq \Bbbk[x, y, z, w] = S$. Then S/I has the minimal graded free resolution

$$\mathcal{F}_\bullet : 0 \to S^2(-7) \xrightarrow{\partial_3} \begin{matrix} S(-4) \\ \oplus \\ S^3(-6) \end{matrix} \xrightarrow{\partial_2} \begin{matrix} S^2(-2) \\ \oplus \\ S(-4) \end{matrix} \xrightarrow{\partial_1} S \xrightarrow{\partial_0} S/I \to 0,$$

where ∂_0 is the usual quotient map and the remaining maps are

$$\partial_1 = \begin{bmatrix} xy - z^2 & xw + zw & z^4 \end{bmatrix};$$

$$\partial_2 = \begin{bmatrix} -xw - zw & -z^4 & 0 & z^3w \\ xy - z^2 & 0 & -z^4 & -yz^3 \\ 0 & xy - z^2 & xw + zw & yw + zw \end{bmatrix};$$

$$\text{and } \partial_3 = \begin{bmatrix} 0 & z^3 \\ -w & 0 \\ y & -y - z \\ -z & x + z \end{bmatrix}.$$

From the resolution, we see that the graded Betti numbers are

$$\beta_{0,0} = 1;$$
$$\beta_{1,2} = 2 \text{ and } \beta_{1,4} = 1;$$
$$\beta_{2,4} = 1 \text{ and } \beta_{2,6} = 3;$$
$$\text{and } \beta_{3,7} = 2$$

The coarser Betti numbers are simply

$$\beta_0 = 1; \quad \beta_1 = 3; \quad \beta_2 = 4; \quad \text{and} \quad \beta_3 = 2.$$

The Hilbert series and minimal free resolution of an S-module are not independent of each other, and, in fact, fit together in the following way.

Theorem 1.2.17. *[EH12, Proposition 4.27] For a finitely generated S-module M considered with the $\mathbb{N}$-grading, the Hilbert series of M and the graded Betti numbers of M are related by the* *following equality:*

$$\mathrm{Hilb}_M(t) = \frac{Q(t)}{(1-t)^n} \;\; \textit{with} \;\; Q(t) = \sum_{i\geq 0}(-1)^i \left(\sum_{j\geq 0} \beta_{i,j} t^j\right).$$

Example 1.2.18. We demonstrate with the ideal from Example 1.2.16. The Hilbert series of S/I in this case is

$$\mathrm{Hilb}_{S/I}(t) = \frac{1-(2t^2+t^4)+(t^4+3t^6)-(2t^7)}{(1-t)^4},$$

which simplifies to

$$\mathrm{Hilb}_{S/I}(t) = \frac{1-2t^2+3t^6-2t^7}{(1-t)^4}.$$

Both the Hilbert series and Betti numbers of an ideal $I \subseteq S$ are closely tied to the same invariants for its initial ideal $\mathrm{in}_\prec(I)$ under some monomial order $\prec$. The Hilbert function are preserved under initial degeneration, while the Betti numbers for the initial ideal serve as an upper bound on the Betti numbers of the ideal. These statements are given precise formulation in the following two propositions.

Proposition 1.2.19. *(cf. [EH12, Proposition 4.29]) The Hilbert series of S/I is the same as the Hilbert series of $S/\mathrm{in}_\prec(I)$, that is,*

$$\mathrm{Hilb}_{S/I}(t) = \mathrm{Hilb}_{S/\mathrm{in}_\prec(I)}(t)$$

This follows from the fact that the Hilbert functions of both quotient rings are the same: the monomials that form a basis for a given degree are the same in both S/I and $S/\mathrm{in}_\prec(I)$.

Proposition 1.2.20. *(cf. [EH12, Theorem 6.8] Let $I \subseteq J \subseteq S$ be homogeneous ideals in S. Then*

$$\beta_{i,j}^{S/I}(S/J) \leq \beta_{i,j}^{S/\mathrm{in}\, I}(S/\mathrm{in}\, J).$$

In particular, if $I = 0$, then

$$\beta_{i,j}^{S}(S/J) \leq \beta_{i,j}^{S}(S/\mathrm{in}\, J).$$

The bare definitions of minimal free resolutions do not imply any sort of controlled behavior, not in the Betti numbers and not in the differentials ∂_i. However, in the particular case that we consider modules over the polynomial ring S, the following theorem ensures a particular finiteness.

Theorem 1.2.21 (Hilbert's Syzygy Theorem)**.** *Every finitely generated module M over S has a free resolution where $F_i = 0$ for $i > n$.*

Hilbert's Syzygy Theorem is very far from true for general rings, as seen in Example 1.1.2.

Because they directly reflect a module's internal structure, the innards of a minimal free resolutions are as varied and complicated as the modules they resolve. Free resolutions over polynomial rings have been the focus of intense study: there are various approaches to producing minimal free resolutions over S that vary from the combinatorial to the computational. Over more general rings, however, free resolutions are typically infinite, and are consequently harder to work with. On top of their infinite length, Betti numbers become very large; for example, a result of Avramov [Avr98] guarantees that, if R is not a complete intersection (see Definition 1.2.44), the Betti numbers of the field $\Bbbk$ as an R-module grow exponentially. Despite these challenges, a particular family of infinite minimal free resolutions is presented Chapter 3.

1.2.2 Lattice and Toric Ideals

Definition 1.2.22. Let $\mathscr{L} \leq \mathbb{Z}^n$ be a lattice, that is, a subgroup of $\mathbb{Z}^n$. The *lattice ideal* corresponding to $\mathscr{L}$ is

$$I_{\mathscr{L}} := \langle x^a - x^b \mid a, b \in \mathbb{N}^n \text{ and } a - b \in \mathscr{L} \rangle \subset \Bbbk[x].$$

For $u \in \mathbb{Z}^n$, write $u = u_+ - u_-$, where $u_+, u_- \in \mathbb{N}^n$ are defined via $(u_+)_i = \begin{cases} u_i & u_i \geq 0 \\ 0 & u_i < 0 \end{cases}$.
It is not hard to check that

$$I_{\mathscr{L}} = \langle x^{u_+} - x^{u_-} \mid u \in \mathscr{L} \rangle.$$

For $B \in \mathbb{Z}^{n\times m}$ of full rank m, we set $\mathbb{Z}B$ to be the sublattice of $\mathbb{Z}^n$ generated over $\mathbb{Z}$ by the columns of B. By construction, $\mathbb{Z}B$ is a rank m lattice. Without loss of generality, we may assume that B contains no zero rows, for if a row of B is zero, the lattice $\mathbb{Z}B$ can be naturally embedded in $\mathbb{Z}^{n-1}$. This gives the following algebraic implication for $I_{\mathscr{L}}$.

Theorem 1.2.23. *[ES96, Corollary 2.2] The codimension of a lattice ideal $I_{\mathscr{L}}$ equals the rank of $\mathscr{L}$.*

We assume throughout that we work with a *positive* lattice $\mathscr{L}$, meaning that $\mathscr{L} \cap \mathbb{N}^n = \{0\}$. This positivity condition ensures that $I_{\mathscr{L}}$ is homogeneous with respect to some $\mathbb{Z}$-grading of $\Bbbk[x]$ for which the values $\deg(x_i)$ are positive integers. This places certain stipulations on the matrix B.

Definition 1.2.24. A matrix B is *mixed* if every column contains a strictly positive and strictly negative entry.

Remark. For a positive lattice $\mathbb{Z}B$, the matrix B must be *mixed*.

Definition 1.2.25. The *saturation* of a lattice $\mathscr{L} \subset \mathbb{Z}^n$ is $\mathscr{L}^{\text{sat}} := \{u \in \mathbb{Z}^n \mid m{\cdot}u \in \mathscr{L} \text{ for some } m \in \mathbb{N}\}$. We say that $\mathscr{L}$ is *saturated* if $\mathscr{L} = \mathscr{L}^{\text{sat}}$.

By [ES96, Corollary 2.2], if $\Bbbk$ is algebraically closed, then $I_{\mathscr{L}}$ is prime if and only if $\mathscr{L}$ is saturated. (Lattice ideals corresponding to saturated lattices are always prime, it is the converse of this assertion that requires the base field assumption.) We remark that a lattice $\mathscr{L}$ is positive if and only if $\mathscr{L}^{\text{sat}}$ is positive.

Remark. If a lattice $\mathscr{L}$ is given as the $\mathbb{Z}$-span of some $B \in \mathbb{Z}^{n\times m}$ of full rank m, then $\mathscr{L}$ is saturated if and only if the gcd of all $m \times m$ minors of B is 1 (see [FS96]).

Lattice ideals corresponding to saturated lattices are known as *toric ideals*. Toric ideals have the following equivalent definition.

Definition 1.2.26. Given a collection $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{N}^d$, consider it as both a configuration of n points in $\mathbb{R}^d$ and the $d \times n$ matrix with ith column α_i. To avoid pathologies, assume $d \leq n$ and $\operatorname{rank} \mathcal{A} = d$. This configuration $\mathcal{A}$ induces a map between polynomial rings:

$$\varphi_{\mathcal{A}} \colon \Bbbk[x_1, \ldots, x_n] \to \Bbbk[t_1, \ldots, t_d]$$
$$x_i \mapsto \mathbf{t}^{\alpha_i}$$

If we extend this map, we see it takes a monomial $\mathbf{x}^u$ to $\varphi_{\mathcal{A}}(\mathbf{x}^u) = \mathbf{t}^{\mathcal{A}u}$. The *toric ideal* $I_{\mathcal{A}}$ associated to A is the kernel of $\varphi_{\mathcal{A}}$. The *toric ring*, or *semigroup ring*, R associated to $\mathcal{A}$ is the image of the map $\varphi_{\mathcal{A}}$, which may also be viewed as

$$R = \Bbbk[\mathbf{t}^{\alpha_1}, \mathbf{t}^{\alpha_2}, \ldots, \mathbf{t}^{\alpha_n}] \cong \Bbbk[x_1, \ldots, x_n]/I_{\mathcal{A}}.$$

The semigroup in question is the semigroup $\mathbb{N}\mathcal{A}$. This formulation of a toric ideal makes it clear that the ideal $I_{\mathcal{A}}$ is prime, as the quotient by $I_{\mathcal{A}}$ is an integral domain. The ideal $I_{\mathcal{A}}$ is a prime binomial ideal, and, in particular, can be written as $I_{\mathcal{A}} = \langle x^u - x^v \mid \mathcal{A}u = \mathcal{A}v \rangle$. From this expression, we can see that a toric ideal equivalent to a saturated lattice ideal: in particular, $I_{\mathscr{L}} = I_{\mathcal{A}}$ when $\mathscr{L} = \ker \mathcal{A}$.

Because $I_{\mathcal{A}} = \langle x^u - x^v \mid \mathcal{A}u = \mathcal{A}v \rangle$, we can see that $I_{\mathcal{A}}$ is always homogeneous with respect to the grading $\deg(x_i) = \alpha_i$. The ideal $I_{\mathcal{A}}$ is also homogeneous with respect to the usual grading $\deg(x_i) = 1$ if the α_i lie on a common affine hyperplane in $\mathbb{R}^d$ [CLS11, Theorem 2.1.4], in which case $I_{\mathcal{A}}$ defines a projective variety. Toric varieties are a well-loved object in algebraic geometry and an active research area both in their own right and in conjunction with many other mathematical fields such as polytopes and polyhedra, combinatorics, commutative algebra, symplectic geometry, and topology; various texts in the last forty years describe these varieties [CLS11, Ewa96, Ful93, Mic18, Oda88, Stu96] as a cohesive subject, though interest in these objects from individual angles predates this perspective. The discrete geometry of the configuration $\mathcal{A}$ and the poset structure of the semigroup $\mathbb{N}\mathcal{A}$ often inform the algebraic properties of $S/I_{\mathcal{A}}$, giving multiple access points for understanding these rings.

Example 1.2.27. Take $n = 4$ and consider

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The map from Definition 1.2.26 is the map $\varphi_{\mathcal{A}} : \Bbbk[x_1, \ldots, x_4] \to \Bbbk[t_1, t_2]$ where

$$x_1 \mapsto t_1; \quad x_2 \mapsto t_1 t_2; \quad x_3 \mapsto t_1 t_2^2; \quad \text{and} \quad x_4 \mapsto t_1 t_2^3.$$

The corresponding toric ideal is

$$I_{\mathcal{A}} = \langle x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3 \rangle.$$

The corresponding lattice ideal comes from the rank 2 lattice $\mathscr{L} \leq \mathbb{Z}^4$

$$\ker \mathcal{A} = \mathscr{L} = \mathbb{Z} \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Because $\mathscr{L}$ is saturated, the lattice ideal and the toric ideal are equal, i.e. $I_{\mathscr{L}} = I_{\mathcal{A}}$.

We can spot the two columns of B appearing in the exponents for the first two generators of $I_{\mathscr{L}}$. Note that the set $\{x^{u_+} - x^{u_-} \mid u \text{ is a column of } B\}$ is not sufficient to generate the entire ideal, as $\langle x_1x_3 - x_2^2, x_2x_4 - x_3^2 \rangle \subsetneq I_{\mathscr{L}}$.

There is a strong relationship between the lattice ideals $I_{\mathscr{L}}$ and $I_{\mathscr{L}^{\text{sat}}}$, namely, $I_{\mathscr{L}^{\text{sat}}}$ is a minimal prime of $I_{\mathscr{L}}$. Furthermore, if $\Bbbk$ is algebraically closed, every associated prime of $I_{\mathscr{L}}$ is isomorphic to $I_{\mathscr{L}^{\text{sat}}}$ by a rescaling of the variables [ES96, Corollary 2.2]. On the other hand, toric ideals are better understood than lattice ideals. For instance, there is a combinatorial/topological criterion [TH86] to decide when a quotient by a toric ideal is Cohen–Macaulay (see Definition 1.2.40), but there is no such characterization of the Cohen–Macaulay property for lattice ideals currently available in the literature, beyond certain special cases [PS98b, PS98a, Sab11, Eto99]. At the root of most of these results is a topological method for computing the graded Betti numbers of a lattice ideal as the ranks of the homology groups of certain simplicial complexes (see Lemma 2.2.2); however, these simplicial complexes are not easily controlled in general.

1.2.3 Algebraic Properties of Ideals and Resolutions

Many algebraic properties have interpretations from the perspective of minimal free resolutions. This section will, in particular, define what it means for a ring to be *Cohen–Macaulay* or *Koszul*, and formulate these definitions in terms of minimal free resolutions. These definitions can be found in greater detail in [Eis95], [Mat89], [BH93], and [EH12].

The Cohen–Macualay property occupies a central place in commutative algebra. Cohen–Macaulay rings form a broad class of rings that includes various interesting examples, but remains fairly well-understood. In the "onion" of desirable algebraic properties, the Cohen–Macaulay property is the outer skin.

Definition 1.2.28. For a Noetherian ring A and an A-module M, a sequence of elements $(a_1, \ldots, a_r)$ in A is called a *regular sequence* for M (or an M-sequence) if a_i is a non-zero divisor on $M/\langle a_1, \ldots, a_{i-1}\rangle M$ for each i and $M/\langle a_1, \ldots, a_r\rangle M$ is not 0.

Definition 1.2.29. If $IM \neq M$, the *depth* of the ideal I on the module M is the maximum length of a maximal M-sequence of elements in I. For this, we write $\operatorname{depth}(I, M)$. If $(A, \mathfrak{m})$ is a local ring, we use $\operatorname{depth}(M)$ to denote the maximum length of a maximal M-sequence of elements in $\mathfrak{m}$, that is, $\operatorname{depth}(M) = \operatorname{depth}(\mathfrak{m}, M)$. In particular, if we consider A as a module over itself, we write $\operatorname{depth}(A) = \operatorname{depth}(\mathfrak{m}, A)$.

One can prove that these M-sequences all have the same length (see [Eis95, Theorem 17.4]), so depth is a well-defined invariant.

Definition 1.2.30. The *Krull dimension* of a ring A (often simply called the *dimension*), denoted $\dim A$, is the supremum of lengths r of chains of prime ideals $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_r$ in A.

The *Krull dimension* of an ideal $I \subsetneq R$, denoted $\dim I$, is the Krull dimension of A/I.

Definition 1.2.31. The *codimension* of a prime ideal $I \subsetneq A$, denoted $\operatorname{codim}(I)$, or often called the *height* of I, is the supremum of lengths r of chains of prime ideals $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_r = I$. In the case that I is not prime, then the codimension of I is the minimum of the codimensions of the primes containing I.

Example 1.2.32. The dimension of the polynomial ring S in n variables has dimension $\dim S = n$. One example of a length n chain of primes in S is

$$\langle 0 \rangle \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \ldots \subsetneq \langle x_1, \ldots, x_n \rangle.$$

Krull dimension can also be defined in terms of the Hilbert series, in particular when we are working with modules over the $\mathbb{N}$-graded polynomial ring S.

Proposition 1.2.33. *[BH93, Corollary 4.4.14] Let M be a graded S-module with Hilbert series written as $\operatorname{Hilb}_M(t) = \dfrac{p(t)}{(1-t)^d}$ with $p(1) \neq 0$. The Krull dimension of M is equal to d.*

Example 1.2.34. The polynomial ring S has Hilbert series $H_S(t) = \frac{1}{(1-t)^n}$. Therefore S has Krull dimension n.

Example 1.2.35. Take M to be S/I as in Example 1.2.18. The Hilbert series of S/I is

$$\operatorname{Hilb}_{S/I}(t) = \frac{1 - 2t^2 + 3t^6 - 2t^7}{(1-t)^4}.$$

Both the numerator and denominator are divisible by $(1-t)^2$, so we can further reduce this to write the Hilbert series of S/I as

$$\operatorname{Hilb}_{S/I}(t) = \frac{-1 + 2t + t^2 + t^4 - 2t^5}{(1-t)^2}.$$

From this we can see that the Krull dimension of S/I is 2.

Though intuitive, it is not quite correct to simply assume that $\operatorname{codim} I = \dim A - \dim I$. However, this will be true in the cases considered in this work.

Remark. If A is a domain that is finitely generated over a field and $I \subsetneq A$ is an ideal, then it is true that $\operatorname{codim} I = \dim A - \dim I$. In particular, for ideals in the polynomial ring S in n variables, the codimension is $\operatorname{codim} I = n - \dim I$. See the introduction of Chapter 9 (page 226) in [Eis95] for a discussion of this fact and an example where the equality does not hold.

The definition for Krull dimension and codimension extend to A-modules via a special ideal connected to M.

Definition 1.2.36. The annihilator of M in A is the ideal $\operatorname{Ann}_A(M) := \{a \in A \mid aM = 0\}$. If the ring A is clear from context, we simply write $\operatorname{Ann}(M)$.

Definition 1.2.37. The Krull dimension of an A-module M is defined as the dimension of the quotient of A by $\operatorname{Ann}(M)$.

This definition of dimension does introduce an awkward ambiguity in the case when M is an ideal in A, and these two numbers can be quite different. For example, if A is a domain, then $\operatorname{Ann}(M) = 0$, so $\dim(M) = \dim(A/0) = \dim A$, whereas $\dim A/I$ could be many things. We establish the convention that, when we write $\dim I$, we always mean $\dim A/I$, not $\dim A/\operatorname{Ann}(I)$.

Both depth and codimension seem to give some notion of algebro-geometric "size." These notions coincide when our object is Cohen–Macaulay.

Definition 1.2.38. Let A be a Noetherian ring such that $\operatorname{depth}(\mathfrak{m}) = \operatorname{codim}(\mathfrak{m})$ for every maximal ideal in A.

This definition can be restated in terms of localizations.

Proposition 1.2.39. *[Eis95, Proposition 18.8] A Noetherian ring A is Cohen–Macaulay if and only if the localization $A_{\mathfrak{m}}$ is Cohen–Macaulay for every maximal ideal $\mathfrak{m}$ in A.*

When A is a local ring, this criterion reduces to a single check and can be extended to A-modules. The following is the most commonly seen definition of Cohen–Macaulay.

Definition 1.2.40. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated A-module. We call M *Cohen–Macaulay* if the equality $\operatorname{depth}(M) = \dim(M)$ holds. In particular, the ring A Cohen–Macaulay if it is Cohen–Macaulay as a module over itself. We call an ideal I Cohen–Macaulay if A/I is Cohen–Macaulay as an A-module.

Despite its terse and algebraic definition, the Cohen–Macaulay property has many equivalent formulations, particularly when considering modules over a polynomial ring S. However, in the case that A is a quotient of a Noetherian local ring, the Auslander–Buchsbaum formula gives a characterization of A in terms of its projective dimension and the depth of its parent ring. The formula is, in fact, true for more general modules over a Noetherian local ring, but this formulation shall suffice for our purposes.

Theorem 1.2.41. *(Auslander–Buchsbaum formula) If $A = Q/I$, where Q is a Noetherian local ring, then* $\operatorname{depth}_Q(A) = \operatorname{depth}(Q) - \operatorname{pd}_Q(A)$.

In particular, we can use this formula when A is a quotient of the polynomial ring by an ideal. Therefore, we have a string of equalities relating the various depths and various dimensions of quotient rings of S that we can use to reformulate the criterion for a ring S/I to be Cohen–Macaulay. (We must also assume that S has a positive $\mathbb{N}$-grading; this will be the case in all subsequent material.)

Proposition 1.2.42. *A quotient ring $R = S/I$ is a Cohen–Macaulay S-module if* $\operatorname{pd}_S(R) = \operatorname{codim} I$.

Proof. By Theorem 1.2.41, $\operatorname{pd}_S(R) = \operatorname{depth} S - \operatorname{depth} R$. By the hypothesis, then $\operatorname{codim} I = \operatorname{depth} S - \operatorname{depth} R$. Because S is a domain, $\operatorname{codim} I = n - \dim R$. We also know that $\operatorname{depth} S = n$. Therefore $n - \dim R = n - \operatorname{depth} R$, and so $\dim R = \operatorname{depth} R$, and thus R is Cohen–Macaulay. $\square$

Example 1.2.43. The S-module S/I given in Example 1.2.16 is *not* Cohen–Macaulay, as its codimension is 2, but its minimal free resolution has length 3.

Though there are many examples of rings that are Cohen–Macaulay, we will only pause to identify two: complete intersections and normal semigroup rings.

A general description of complete intersections would take us farther afield than necessary, so we shall limit our scope to quotient rings of the polynomial ring.

Definition 1.2.44. An S-module R is a *complete intersection* if it can be written as $R = S/I$ where I is generated by a regular sequence in S. If I is an ideal in S generated by a regular S-sequence, we may call I itself a complete intersection.

The following is true for all complete intersections.

Theorem 1.2.45. *[BH93, Section 2.3] If R is a complete intersection, then R is Cohen–Macaulay.*

Chapter 2 explores the relationship between lattice and toric ideals and when they are (or are not) Cohen–Macaulay. This chapter will use heavily that complete intersections are Cohen–Macaulay.

The characterization for semigroup rings to be Cohen–Macaulay is combinatorial and depends on the normality of the underlying semigroup $\mathbb{N}\mathcal{A}$.

Definition 1.2.46. A semigroup $\mathbb{N}\mathcal{A}$ is *normal* if $\mathbb{N}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}$.

Morally, this means that our semigroup does not have gaps. Though unassuming, this property turns out to be exactly what we need.

Theorem 1.2.47. *[Hoc72, Theorem 1] Let $\mathbb{N}\mathcal{A}$ be a normal semigroup. Then the semigroup ring associated to $\mathcal{A}$ is Cohen–Macaulay.*

Theorem 1.2.47 is the criterion we will use in Chapter 3 to determine if a semigroup ring is Cohen–Macaulay.

Until this point, the definitions proposed have focused on regular sequences and prime ideals, eventually culminating in a definition of the Cohen–Macaulay property that can be associated to the length of a minimal free resolution by Proposition 1.2.42. We might also be interested in algebraic properties that can be understood by considering the maps in a minimal free resolutions. We now turn our attention to the Koszul property. Though we will be focusing on the commutative case, the definition of Koszul extends to non-commutative algebras.

Let $A = \bigoplus_{i \geq 0} A_i$ be a standard graded $\Bbbk$-algebra, and let $\beta_i^A(\Bbbk)$ be the ith Betti number of $\Bbbk$ as an A-module.

Definition 1.2.48. The *Poincaré series* $P_A(t)$ is the formal power series with $\beta_i^A(\Bbbk)$ as the coefficient of t^i, i.e. $P_A(t) = \sum_{i \geq 0} \beta_i^A(\Bbbk) \cdot t^i$.

When A is a *Koszul* ring, there is a strong relationship between the Poincaré series and the Hilbert series. The following result can be taken as a definition.

Theorem 1.2.49. *(cf. [Frö99, Definition-Theorem 1]) A graded algebra A is Koszul if and only if the following equivalent conditions are satisfied.*

1. *The minimal graded A-resolution of $\Bbbk$ is linear, that is, the degree of each map in the resolution is 1.*
2. *The Hilbert series and the Poincaré series of A satisfy the equality* $\operatorname{Hilb}_A(-t)P_A(t) = 1$.

An equivalent formulation for condition 1 of Theorem 1.2.49 is that $\beta_{i,j}^A(\Bbbk) = 0$ when $i \neq j$.

Remark. A Koszul algebra must be defined by exactly quadratic relations.

This comes from the minimal graded resolution of $\Bbbk$. Let us denote this resolution

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \Bbbk \to 0.$$

The columns of ∂_0 are the minimal generators of A and the columns of ∂_1 are the relations on the generators. Because all entries in both maps are degree 1, their composition $\partial_0\partial_1$ is quadratic, and it is precisely in this composition that we have the defining relations among the minimal generators.

Koszul algebras are independently interesting to many mathematicians, boasting a variety of introductions and surveys [CDNR13, Con14, Frö99, Pri70, PP05]. Though well-studied, there are many open questions about their characterization. One such conjecture about the Koszul property of toric rings is given below.

Conjecture 1.2.50. *(Bøgvad's Conjecture [MP15, Conjecture 8.19] [Stu96, Conjecture 13.9]) The toric ring of a smooth projectively normal toric variety is Koszul.*

The Koszul property has received much attention in combinatorial settings. An early result [Frö75] states that if $R = \Bbbk[x_1, \ldots, x_n]/I$, where I is generated by monomials of degree two, then R is Koszul. By Proposition 1.2.20 and Condition 1 of Theorem 1.2.49, if $R = \Bbbk[x_1, \ldots, x_n]/J$ where J has a quadratic initial ideal, then R is Koszul. For semigroup rings, a characterization of the Koszul property is an open problem, see [Pee07] for a survey of known results on resolutions over semigroup rings.

In many cases of rings that are known to be Koszul, the resolution of the residue field is not explicitly known. For semigroup rings, we are aware only of resolutions over the rings associated to rational normal curves [GHP08]. In fact, [GHP08] gives the minimal free resolution for any monomial ideal in this case. In a similar vein, Section 3 discusses the resolution of the field $\Bbbk$ over a generalization of the rational normal, the rational normal 2-scroll.

1.2.4 Simplicial Complexes

Though simplicial complexes are combinatorial and topological by nature, there are many constructions that use them to glean information about algebraic objects. The material from this subsection can all be found in [Sta96]. [MS05, Section 1] also gives an overview of this material.

Definition 1.2.51. A simplicial complex Δ on vertices $[n]$ is a collection of subsets of $[n]$ that is closed under taking subsets, that is, if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. The elements of Δ are called *faces* and the dimension of a face F is $\dim F = |F| + 1$. In this case, F is called an i*-face* of Δ. The dimension of Δ is $\max\{\dim F \mid F \in \Delta\}$.

Because Δ is closed under taking subsets, it is often expressed by a list of its maximal faces. It is often convenient to represent a simplicial complex graphically and study the underlying topology of this realization.

Example 1.2.52. The set

$$\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{4,5\}, \{2,3,4\}\}$$

is a simplicial complex. It is easier to simply record the maximal faces (with respect to inclusion), so we can write $\Delta = \{\{1,2\}, \{1,3\}, \{2,3,4\}, \{4,5\}\}$.

The simplicial complex Δ has dimension 2. Figure 1.1 gives the pictorial representation of Δ.

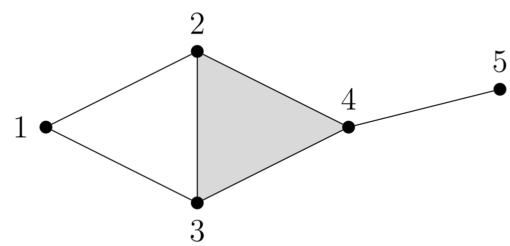

Figure 1.1: The simplicial complex Δ from Example 1.2.52.

One assignment of a simplicial complex to an algebraic structure comes from Stanley–Reisner theory. In particular, this theory provides a correspondence between simplicial complexes and squarefree monomial ideals.

Definition 1.2.53. The Stanley–Reisner ideal of a simplicial complex Δ is the squarefree monomial ideal

$$I_\Delta = \langle x^F \mid F \notin \Delta \rangle$$

generated by the *non-faces* of Δ. The quotient ring S/I_Δ is called the *Stanley–Reisner ring* of Δ and is often denoted $\Bbbk[\Delta]$.

This correspondence is, in fact, a bijection.

Example 1.2.54. For the simplicial complex Δ from Example 1.2.52, the corresponding Stanley–

Reisner ideal is

$$I_\Delta = \langle x_1x_4,\, x_1x_4,\, x_2x_5,\, x_3x_5,\, x_1x_2x_3\rangle.$$

Note that it suffices to take the minimal non-faces of Δ to generate I_Δ.

While this correspondence reveals many algebraic aspects of $\Bbbk[\Delta]$, the one used in Section 3 is the formulation of the Hilbert function of $\Bbbk[\Delta]$ from the combinatorial data of Δ.

Theorem 1.2.55. *The Hilbert series of a Stanley–Reisner ring $\Bbbk[\Delta]$ of dimension k can be given in terms of the face numbers of the corresponding simplicial complex Δ, that is,*

$$\mathrm{Hilb}_{\Bbbk[\Delta]}(t) = \frac{1}{(1-t)^{k+1}} \sum_{d=0}^{k+1} f_{d-1} t^d (1-t)^{k+1-d},$$

where f_d is the number of d-dimensional faces of Δ.

Example 1.2.56. For the Stanley–Reisner ring $\Bbbk[\Delta]$, where Δ is the 2-dimensional simplicial complex Δ from Example 1.2.52, the f_d are

$$f_{-1} = 1;\; f_0 = 5;\; f_1 = 6; \text{ and } f_2 = 1.$$

Therefore the Hilbert function of $\Bbbk[\Delta] = S/I_\Delta$ is

$$\mathrm{Hilb}_{\Bbbk[\Delta]}(t) = \frac{1}{(1-t)^3}[1(1-t)^3 + 5t(1-t)^2 + 6t^2(1-t) + 1t^3].$$

After simplifying, we see that

$$\mathrm{Hilb}_{\Bbbk[\Delta]}(t) = \frac{1+2t-t^2-t^3}{(1-t)^3}.$$

Recall that, from Proposition 1.2.19, the Hilbert series of an ideal and its initial ideal are the same, so if we are examining an ideal I whose initial ideal is squarefree, we can obtain the Hilbert series of S/I from the Hilbert series of $S/\operatorname{in}_\prec(I)$, which in turn can be obtained from its simplicial realization.

Remark. If the squarefree monomial ideal I_Δ is generated by monomials of degree 2, then $\Bbbk[\Delta]$ is a Koszul ring by [Frö75]. In terms of the simplicial complex Δ, this implies, for every $T \subseteq [n]$, if every pair of elements from T is in Δ, then T is also in Δ. Such a simplicial complex is often called a *flag complex*, or a *clique complex*.

We finish by considering two constructions that are used to make arbitrary rank examples in Chapter 2: the join and suspension of simplicial complexes.

Definition 1.2.57. The *join* of two simplicial complexes Δ and Δ' is the simplicial complex on the union of their vertices that has faces $F \cup F'$ where F is a face of Δ and F' is a face of Δ'. If Δ' is the zero-dimensional simplicial complex with two vertices, then $\Delta \star \Delta'$ is the *suspension* of Δ, denoted $\Sigma\Delta$.

In terms of reduced homology, it is known that

$$\tilde{H}_{j+1}(\Sigma\Delta; \Bbbk) \cong \tilde{H}_j(\Delta; \Bbbk) \qquad \text{for } j \geq -1. \tag{1.3}$$

Example 1.2.58. Let $\Delta = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ and $\Delta' = \{\{5,6\}\}$. Then the join of Δ and Δ' is

$$\Delta \star \Delta' = \{\{1,2,5,6\},\{2,3,5,6\},\{3,4,5,6\},\{1,4,5,6\}\}$$

and, if we call the two new vertices 7 and 8, the suspension of Δ is

$$\Sigma\Delta = \{\{1,2,7\},\{2,3,7\},\{3,4,7\},\{1,4,7\},\{1,2,8\},\{2,3,8\},\{3,4,8\},\{1,4,8\}\}.$$

Pictures for Δ, Δ', $\Delta \star \Delta'$, and $\Sigma\Delta$ are given in Figure 1.2. Note that $\Sigma\Delta$ has hollow interior, whereas $\Delta \star \Delta'$ is solid.

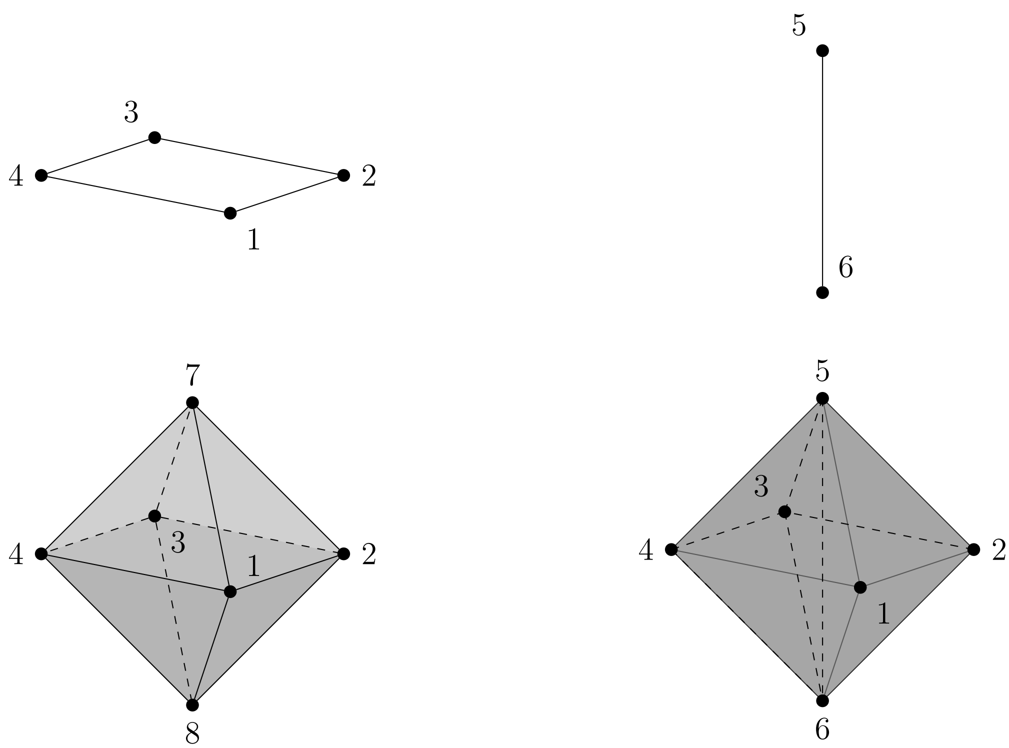

Figure 1.2: Clockwise from the upper left: the simplicial complexes Δ, Δ', $\Delta \star \Delta'$, and $\Sigma\Delta$ from Example 1.2.58.

2. COUNTEREXAMPLES FOR COHEN–MACAULAYNESS OF LATTICE IDEALS *

Let $n > 2$ be an integer.

In this chapter, we construct, for each codimension $m \geq 2$, infinitely many matrices B for which one of $I_{\mathbb{Z}B}$, $I_{\mathbb{Z}B^{\mathrm{sat}}}$ is Cohen–Macaulay (in fact, a complete intersection), but the other one is not. This means that the most obvious place to look for a criterion for the Cohen–Macaulay property for lattice ideals, namely the associated toric ideals, does not directly yield positive results.

2.1 Lattice Ideals in Codimension 2

In this section we study lattice ideals in codimension 2. We recall results from [PS98b] that characterize when such ideals are (not) Cohen–Macaulay, and use them to construct examples.

Let $\mathscr{L} \subset \mathbb{Z}^n$ be a rank m lattice, and let $B = [b_{ij}]$ be an integer $n \times m$ matrix whose columns are a $\mathbb{Z}$-basis of $\mathscr{L}$.

Recall that an integer matrix is *mixed* if every column contains a strictly positive and a strictly negative entry. We emphasize that matrices B corresponding to $\mathbb{Z}$-generators of positive lattices are mixed. A particular type of mixed matrix is described below, as well as its algebraic reflection in lattice ideals.

Definition 2.1.1. An integer matrix is *dominating* if it contains no square mixed submatrices.

Theorem 2.1.2. *[FS96, Theorem 2.9] Let $\mathscr{L} \subset \mathbb{Z}^n$ be a rank m lattice. The lattice ideal $I_{\mathscr{L}}$ is a complete intersection if and only if $\mathscr{L} = \mathbb{Z}B$ for some dominating matrix B. In this case, $I_{\mathscr{L}} = \langle x^{u_+} - x^{u_-} \mid u \text{ is a column of } B\rangle$.*

This theorem can be restated combinatorially using the following gadget.

Definition 2.1.3. If $\mathscr{L} = \mathbb{Z}B$, use b_i to denote the ith row of B. The collection $G_B := \{b_1, \ldots, b_n\} \subseteq \mathbb{Z}^m$ is called a *Gale diagram* of $\mathscr{L}$.

*Parts of this chapter are reprinted with permission from [Lyu88]: "Counterexamples for Cohen–Macaulayness of Lattice Ideals" by Laura Felicia Matusevich and Aleksandra Sobieska, 2018. *Communications in Algebra*, 47:6, 2494-2502.

Any $\mathbb{Z}$-basis for $\mathscr{L}$ yields a Gale diagram, which means that Gale diagrams are unique up to the action of $\mathrm{GL}_m(\mathbb{Z})$. This elementary combinatorial object gives some insight to the nature of $I_{\mathscr{L}}$: in the codimension 2 case, Gale diagrams can be used to restate Theorem 2.1.2 and also to give a characterization for when a lattice ideal is Cohen–Macaulay.

Definition 2.1.4. A Gale diagram G_B is *imbalanced* if $b_{i1} = 0$ or $b_{i2} \geq 0$ for each $i = 1, \ldots, n$.

Theorem 2.1.5. *[PS98b, Lemma 3.1, Proposition 4.1] Let $\mathscr{L} \subset \mathbb{Z}^n$ be a rank 2 lattice.*

1. *The lattice ideal $I_{\mathscr{L}}$ is a complete intersection if and only if $\mathscr{L}$ has an imbalanced Gale diagram G_B. In this case, $I_{\mathscr{L}} = \langle x^{u_+} - x^{u_-} \mid u$ is a column of $B\rangle$.*
2. *The lattice ideal $I_{\mathscr{L}}$ is **not** Cohen–Macaulay if and only if it has a Gale diagram G_B which intersects each of the four open quadrants of $\mathbb{R}^2$.*

Because stretching or skewing a lattice $\mathbb{Z}B$ corresponds to multiplying B by a nonsingular 2×2 integer matrix, it is natural to wonder how such an action transforms $G_{\mathbb{Z}B}$ and how this is reflected in the corresponding lattice ideal. We illustrate below how multiplication of B by a nonsingular 2×2 integer matrix can change a non-Cohen–Macaulay $I_{\mathbb{Z}B}$ to a complete intersection and vice-versa.

Proposition 2.1.6. *For any 4×2 matrix B such that G_B touches all four open quadrants of $\mathbb{R}^2$ (so that $I_{\mathbb{Z}B}$ is not Cohen–Macaulay), there exists a nonsingular 2×2 integer matrix M such that G_{BM} is imbalanced (so that $I_{\mathbb{Z}BM}$ is a complete intersection).*

Proof. Let $b_1, \ldots, b_4$ be the rows of $B = [b_{ij}]$ and assume that b_i lies in the ith open quadrant of $\mathbb{R}^2$.

If both sets $\{b_1, b_3\}$, $\{b_2, b_4\}$ are linearly dependent, let $M = \left[\begin{smallmatrix} -b_{12} & -b_{22} \\ b_{11} & b_{21} \end{smallmatrix}\right]$. Since b_1 and b_2 are linearly independent, $\det(M) \neq 0$. But every row of BM contains a zero entry, so the corresponding Gale diagram G_{BM} is imbalanced.

Now assume that $\{b_1, b_3\}$ are linearly independent. Then the cone of nonnegative combinations of b_1 and b_3 contains either the second or the fourth quadrant of $\mathbb{R}^2$. Suppose that it contains the

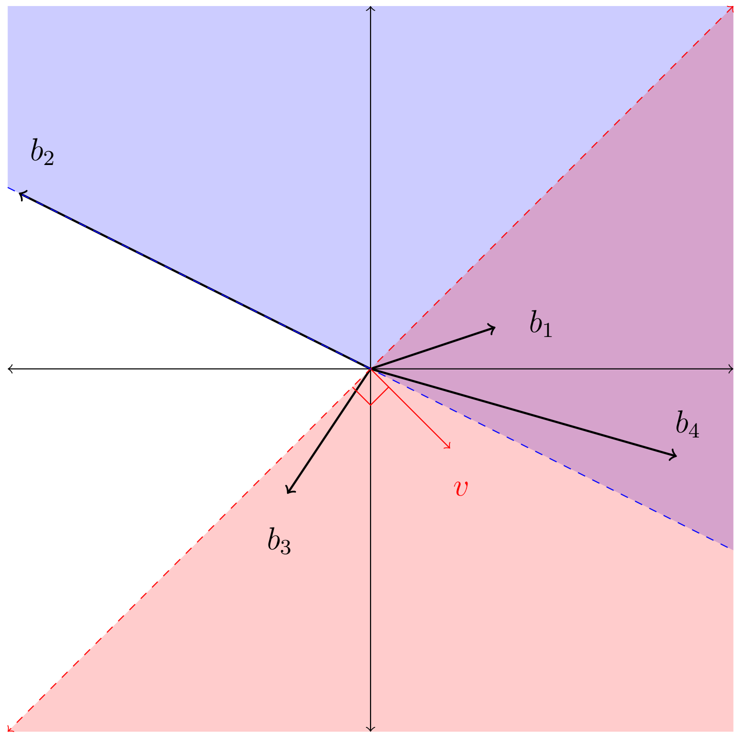

Figure 2.1: A non-Cohen–Macaulay Gale diagram with new quadrants shaded.

fourth quadrant. Then we can find $v = (v_1, v_2) \in \mathbb{Z}^2$ lying in the fourth quadrant of $\mathbb{R}^2$ such that the angles between v and b_1, and between v and b_3 are both less than $\pi/2$, so that $b_1 \cdot v \geq 0$ and $b_3 \cdot v \geq 0$. Since they lie in the same quadrant, $b_4 \cdot v \geq 0$. Now let $M = \left[\begin{smallmatrix} b_{22} & v_1 \\ -b_{21} & v_2 \end{smallmatrix}\right]$. Since v lies in the fourth quadrant, and $(b_{22}, -b_{21})$ lies in the first, they are linearly independent, and so $\det(M) \neq 0$. By construction, G_{BM} is imbalanced. See Figure 2.1 for a pictorial illustration of this argument. $\square$

Example 2.1.7. To illustrate Proposition 2.1.6, consider the following example, shown pictorially in Figure 2.1. Take

$$B = \begin{bmatrix} 3 & 1 \\ -8 & 4 \\ -2 & -3 \\ 7 & -2 \end{bmatrix}.$$

The Gale diagram shows that the corresponding lattice ideal $I_{\mathbb{Z}B}$ is non-Cohen–Macaulay by Theorem 2.1.5. By the construction in Proposition 2.1.6, we can take $v = (2, -2)$ and $M = \begin{bmatrix} 4 & 2 \\ 8 & -2 \end{bmatrix}$, so that

$$BM = \begin{bmatrix} 20 & 4 \\ 0 & -24 \\ -32 & 2 \\ 12 & 18 \end{bmatrix}.$$

The Gale diagram of BM is imbalanced, so the ideal $I_{\mathbb{Z}BM}$ is a complete intersection.

In the next proposition, we determine which imbalanced Gale diagrams can be transformed into Gale diagrams of non-Cohen–Macaulay lattice ideals.

Definition 2.1.8. If $b \in \mathbb{R}^2 \setminus \{0\}$, the *ray* spanned by b is defined to be $\{tb \mid t \in \mathbb{R}, t \geq 0\}$.

If $B \in \mathbb{Z}^{n \times 2}$, we consider the collection of rays spanned by the rows of B, and associate this collection to the Gale diagram G_B. Since we assume all rows of B are nonzero, none of these rays is a point.

Proposition 2.1.9. *Let B be an $n \times 2$-matrix such that G_B is imbalanced. There exists a nonsingular 2×2 integer matrix M such that G_{BM} meets the four open quadrants of $\mathbb{R}^2$ (which implies that $I_{\mathbb{Z}BM}$ is not Cohen–Macaulay) if and only if G_B spans more than three rays.*

Proof. Note that if M is a nonsingular 2×2 integer matrix, then G_B and G_{BM} span the same number of rays. If G_{BM} meets the four open quadrants of $\mathbb{R}^2$, it spans at least four rays, and therefore, so does G_B.

Now assume that G_B spans more than three rays. We first consider the case where G_B is contained in the coordinate axes. Then G_B must span all four half axes. Using $M = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, we see that G_{BM} meets the four open quadrants of $\mathbb{R}^2$.

Now consider the case where our imbalanced G_B is not contained in the coordinate axes (and spans at least four rays). Then, since $\mathbb{Z}B \cap \mathbb{N}^n = 0$, we have that both the first and second open quadrants of $\mathbb{R}^2$ contain elements of G_B, and there is some b_j where $b_{j1} = 0$ and $b_{j2} < 0$.

Consider the rightmost and leftmost elements of G_B, that is, the vectors with the smallest positive and largest negative slopes. Denote them $b_a = (b_{a1}, b_{a2})$ and $b_d = (b_{d1}, b_{d2})$ respectively. Alternatively, we can characterize this by

$$\arccos(b_{a1}/|b_a|) < \arccos(b_{i1}/|b_i|) < \arccos(b_{d1}/|b_d|)$$

for all i such that b_i lies in the upper half plane.

Since G_B spans at least four rays, there exists $b_c = (b_{c1}, b_{c2}) \in G_B$ between b_a and b_d, where we formalize "between" to mean that $\arccos(b_{a1}/|b_a|) < \arccos(b_{c1}/|b_c|) < \arccos(b_{d1}/|b_d|)$. As this is moderately unappealing, the reader may choose to simply visualize starting at the positive x-axis and sweeping counter-clockwise, and declaring a vector between two others if they encounter it after the first vector but before the second.

Now choose $s = (s_1, s_2) \in \mathbb{Z}^2$ between b_a and b_c and $t = (t_1, t_2) \in \mathbb{Z}^2$ between b_c and b_d. We may assume that $s_1 > 0$ and $t_1 < 0$. Note that $s_2, t_2 > 0$.

The matrix

$$M = \begin{bmatrix} -s_2 & t_2 \\ s_1 & -t_1 \end{bmatrix}$$

is nonsingular, since s and t are linearly independent (they belong to adjacent quadrants of $\mathbb{R}^2$).

As stated previously, G_B contains at least one element whose first entry is zero, and whose second entry is negative. Our construction yields the following sign pattern:

$$\begin{bmatrix} b_{a1} & b_{a2} \\ b_{c1} & b_{c2} \\ b_{d1} & b_{d2} \\ 0 & b_{j2} \end{bmatrix} M = \begin{bmatrix} - & + \\ + & + \\ + & - \\ - & - \end{bmatrix}.$$

We conclude that the Gale diagram G_{BM} has an arrangement of vectors in all four open quadrants of $\mathbb{R}^2$, so $I_{\mathbb{Z}BM}$ is not Cohen–Macaulay. Figure 2.2 illustrates this argument. $\square$

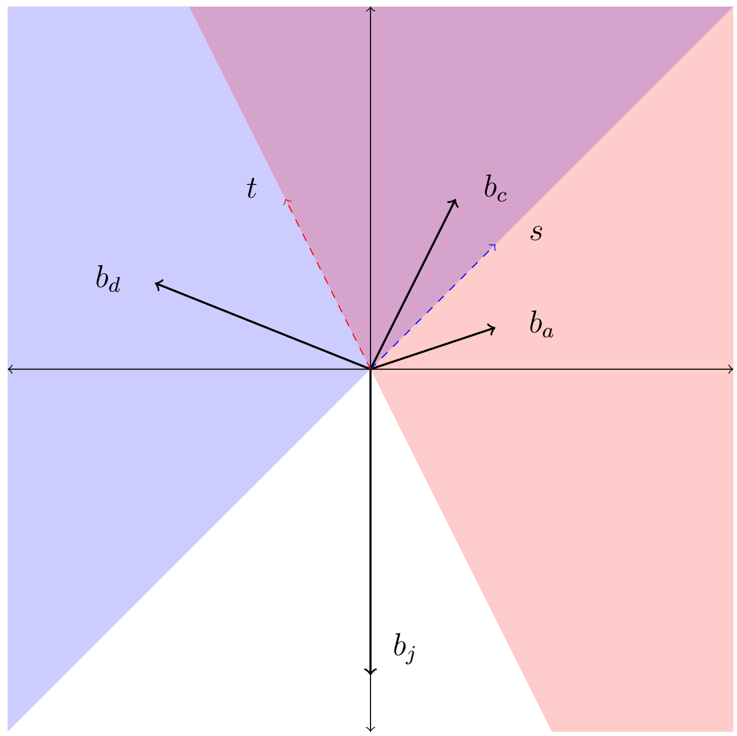

Figure 2.2: An imbalanced Gale diagram with new quadrants shaded. (Reprinted with permission from [MS19]).

Example 2.1.10. To illustrate Proposition 2.1.9, consider the following example, shown pictorially in Figure 2.2. Take

$$B = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -5 & 2 \\ 0 & -7 \end{bmatrix}.$$

The Gale diagram shows that the corresponding lattice ideal $I_{\mathbb{Z}}B$ is a complete intersection by Theorem 2.1.5. Following the construction in Proposition 2.1.9, we can choose $s = (3, 3)$ and

$t = (-2, 4)$, so $M = \begin{bmatrix} -3 & 4 \\ 3 & 2 \end{bmatrix}$ and

$$BM = \begin{bmatrix} -6 & 14 \\ 6 & 16 \\ 21 & -16 \\ -21 & -14 \end{bmatrix}.$$

The Gale diagram of BM touches all open quadrants of $\mathbb{R}^2$, so the ideal $I_{\mathbb{Z}BM}$ is not Cohen–Macaulay.

The main result in this section brings together Propositions 2.1.6 and 2.1.9.

Theorem 2.1.11. *There are infinitely many examples of rank* 2 *non-saturated lattices* $\mathscr{L}$ *such that one of* $I_{\mathscr{L}}$, $I_{\mathscr{L}^{\text{sat}}}$ *is a complete intersection, and the other one is not Cohen–Macaulay.*

Proof. Note that there are infinitely many 4×2 integer matrices B with Gale diagram meeting the four open quadrants of $\mathbb{R}^2$ and whose columns span a saturated lattice. Let M as in Proposition 2.1.6. Define $\mathscr{L} \subset \mathbb{Z}^4$ to be the lattice spanned by the columns BM, so that $\mathscr{L}^{\text{sat}}$ is the lattice spanned by the columns of B. We see that $I_{\mathscr{L}}$ is a complete intersection, while $I_{\mathscr{L}^{\text{sat}}}$ is not Cohen–Macaulay.

Similarly, there are infinitely many $B \in \mathbb{Z}^{n\times 2}$ whose columns span a saturated lattice, and whose Gale diagram is imbalanced and spans at least four rays. With M as in Proposition 2.1.9, and $\mathscr{L} \subset \mathbb{Z}^n$ the lattice spanned by the columns of BM, we have that $I_{\mathscr{L}}$ is not Cohen–Macaulay, and $I_{\mathscr{L}^{\text{sat}}}$ is a complete intersection. $\square$

Example 2.1.12. We can check with the criterion from Remark 1.2.2 that the lattices $\mathbb{Z}B$ in examples 2.1.7 and 2.1.10 are both saturated. The constructed lattice ideals $I_{\mathbb{Z}BM}$ have corresponding toric ideal $I_{\mathbb{Z}BM^{\text{sat}}} = I_{\mathbb{Z}B^{\text{sat}}} = I_{\mathbb{Z}B}$. Therefore these are examples where the lattice ideal from $\mathbb{Z}BM$ is a complete intersection but the toric ideal from $\mathbb{Z}B$ is not Cohen-Macaulay and vice versa.

2.2 Lattice Ideals in Codimension ≥ 3

The goal of this section is to generalize Theorem 2.1.11 to higher ranks. We begin by reviewing relevant results on Betti numbers of lattice ideals.

Let $B \in \mathbb{Z}^{n\times m}$ of full rank m, and consider $R = S/I_{\mathbb{Z}B}$, $\Gamma = \mathbb{Z}^n/\mathbb{Z}B$.

Definition 2.2.1. The congruence classes of $\mathbb{Z}^n$ modulo Γ are called *fibers*.

The polynomial ring S is Γ-graded by setting $\deg(x^u)$ to be the fiber of Γ containing u. The ideal $I_{\mathbb{Z}B}$ is homogeneous with respect to this grading, and moreover the quotient R is *finely graded*, meaning that its graded pieces have dimension at most one. If C is a fiber of Γ, we denote the corresponding graded piece of R by R_C. This grading gives a decomposition $\operatorname{Tor}_j^S(R,\Bbbk) = \bigoplus_{C\in\Gamma} \operatorname{Tor}_j^S(R,\Bbbk)_C$. The multi-graded Betti number of R in degree C is $\beta_{j,C} = \beta_{j,C}(R) = \dim_{\Bbbk} \operatorname{Tor}_j^S(R,\Bbbk)_C$.

The Betti number $\beta_{j,C}$ can be computed using simplicial homology. For each fiber C of Γ, define a simplicial complex Δ_C on $[n]$, where $F \subseteq [n]$ is a face of Δ_C if and only if C contains a nonnegative vector a whose support contains F.

Lemma 2.2.2. *[AH96, Lemma 4.1], [PS98b, Lemma 2.1] The multigraded Betti number $\beta_{j+1,C}(R)$ equals the rank of the j-th reduced homology group $\tilde{H}_j(\Delta_C;\Bbbk)$ of the simplicial complex Δ_C.*

We recall from Proposition 1.2.42 that R is Cohen–Macaulay if and only if $\beta_{j,C}(R) = 0$ whenever $j > \operatorname{codim}_S(R)$.

Since the simplicial complex Δ_C is defined using the nonnegative elements of the fiber C, we consider only those elements when working with specific fibers.

Example 2.2.3. Consider the lattice $\mathbb{Z}B \subset \mathbb{Z}^4$ where $B = \begin{bmatrix} 2 & 3 & -1 & -4 \\ -1 & 3 & 5 & -7 \end{bmatrix}^T$. For the fiber $C = \{\ (2,\ 6,\ 5,\ 0), (3,\ 3,\ 0,\ 7), (0,\ 3,\ 6,\ 4), (1,\ 0,\ 1,\ 11)\ \}$, the simplicial complex Δ_C is a hollow tetrahedron. By Lemma 2.2.2, we see that $\beta_{3,C}(R) = \operatorname{rank}_{\Bbbk} \tilde{H}_2(\Delta_C;\Bbbk) = 1$. Since R has codimension 2, we conclude that R is not Cohen–Macaulay.

In the previous example, the high syzygy constructed occurs in the fiber C containing $(B_1)_+ + (B_2)_+$, where B_1 and B_2 are the columns of B. This turns out to be the case in general.

Proposition 2.2.4. *Let B be an $n \times 2$ integer matrix of rank 2 with columns B_1 and B_2 such that G_B meets the four open quadrants of $\mathbb{R}^2$. Let $C_\circ$ be the fiber of $\mathbb{Z}^n/\mathbb{Z}B$ containing $(B_1)_+ + (B_2)_+$. Then $\beta_{3,C_\circ}(S/I_{\mathbb{Z}B}) > 0$.*

Proof. Since G_B meets the four open quadrants of $\mathbb{R}^2$, we may assume that for $1 \leq i \leq 4$, the ith row of B meets the ith quadrant of $\mathbb{R}^2$. For convenience, we write the submatrix of B consisting of its first four rows as:

$$\begin{bmatrix} t_1 & t_2 \\ y_1 & y_2 \\ z_1 & z_2 \\ w_1 & w_2 \end{bmatrix} \text{ with sign pattern } \begin{bmatrix} + & + \\ + & - \\ - & + \\ - & - \end{bmatrix}.$$

An element of $C_\circ$, different from $(B_1)_+ + (B_2)_+$, whose entries are all nonnegative can be written as $B\begin{bmatrix} u \\ v \end{bmatrix} + (B_1)_+ + (B_2)_+$, where $0 \neq \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{Z}^2$ is such that

$$B\begin{bmatrix} u \\ v \end{bmatrix} + (B_1)_+ + (B_2)_+ \geq 0 \text{ coordinatewise.} \tag{2.1}$$

In particular, restricting to the first four rows of B, we have that

$$(u+1)t_1 + (v+1)t_2 \geq 0 \tag{2.2}$$

$$(u+1)y_1 + vy_2 \geq 0 \tag{2.3}$$

$$uz_1 + (v+1)z_2 \geq 0 \tag{2.4}$$

$$uw_1 + vw_2 \geq 0 \tag{2.5}$$

Because both $w_1, w_2 < 0$ and either u or v is nonzero, equation (2.5) implies that at least one of

$u, v < 0$. Suppose then that $u < 0$. Since $y_1 > 0$, we have $(u+1)y_1 \leq 0$, so $(u+1)y_1 + vy_2 \leq vy_2$. By (2.3), we have $vy_2 \geq 0$. As $y_2 < 0$, we see that $v \leq 0$. If $v = 0$, equation (2.3) reduces to $(u+1)y_1 \geq 0$, so $(u+1)y_1 = 0$, which means that $u = -1$. Otherwise, $v < 0$, but then both $(u+1)t_1, (v+1)t_2 \leq 0$, so by (2.2), $(u+1)t_1 = (v+1)t_2 = 0$, and therefore $u = v = -1$.

The case where $v < 0$ is similar, leading to the possibilities $u = 0, v = -1$, and $u = v = -1$.

We conclude that the only coordinatewise nonnegative elements of $C_\circ$ other than $(B_1)_+ + (B_2)_+$ are $(B_1)_+ + (B_2)_-$, $(B_1)_- + (B_2)_+$, and $(B_1)_- + (B_2)_-$, obtained by taking $\begin{bmatrix} u \\ v \end{bmatrix}$ equal to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$, respectively. Consequently, the maximal faces of $\Delta_{C_\circ}$ are

$$\{i \mid b_{i1} > 0 \text{ or } b_{i2} > 0\} \supseteq \{1,2,3\}, \qquad \{i \mid b_{i1} > 0 \text{ or } b_{i2} < 0\} \supseteq \{1,2,4\},$$
$$\{i \mid b_{i1} < 0 \text{ or } b_{i2} > 0\} \supseteq \{1,3,4\}, \qquad \{i \mid b_{i1} < 0 \text{ or } b_{i2} < 0\} \supseteq \{2,3,4\}.$$

We see that $\Delta_{C_\circ}$ has a hollow tetrahedron as a deformation retract, and hence $\beta_{3,C_\circ}(\Bbbk[x]/I_{\mathbb{Z}B}) > 0$. $\square$

We wish to construct examples of lattice ideals in codimension greater than 2 generalizing Theorem 2.1.11. We do this using block matrices.

Suppose G is an $n \times m$ matrix of full rank of the form

$$G = \left[\begin{array}{c|c|c|c} G_1 & 0 & 0 & 0 \\ \hline 0 & G_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & G_r \end{array}\right],$$

where each G_i is an $n_i \times m_i$-matrix and the columns of G form a basis for the lattice $\mathbb{Z}G$.

Because of the block structure of G, $\mathbb{Z}G = \mathbb{Z}G_1 \oplus \mathbb{Z}G_2 \oplus \cdots \oplus \mathbb{Z}G_r$. Any fiber C of $\mathbb{Z}^n/\mathbb{Z}G$ has the form $C = C_1 \times \cdots \times C_r$, where C_i is a fiber of $\mathbb{Z}^{n_i}/\mathbb{Z}G_i$. In particular, C has coordinatewise positive (resp. nonnegative) elements of the form $(\alpha_1, \ldots, \alpha_r)$, where α_i is a coordinatewise positive (resp. nonnegative) element of the fiber $C_i \in \mathbb{Z}^{n_i}/\mathbb{Z}G_i$. The resulting complex of supports Δ_C is then $\Delta_{C_1} \star \Delta_{C_2} \star \cdots \star \Delta_{C_r}$, where $\star$ denotes the join of two simplicial complexes, as in Definition 1.2.57.

Using block matrices, it is easy to construct non-Cohen–Macaulay lattice ideals of any codimension $m \geq 2$.

Proposition 2.2.5. *Let $B_\circ \in \mathbb{Z}^{n\times 2}$ such that $G_{B_\circ}$ meets the four open quadrants of $\mathbb{R}^2$ and let and $H = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{Z}^{2\times 1}$. For $m > 2$, let*

$$B = \left[\begin{array}{c|c|c|c} B_\circ & 0 & 0 & 0 \\ \hline 0 & H & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & H \end{array}\right] \in \mathbb{Z}^{(n+2(m-2))\times m}.$$

Note that $\mathbb{Z}B$ is a positive lattice if and only if $\mathbb{Z}B_\circ$ is a positive lattice. Set $\Gamma = \mathbb{Z}^{n+2(m-2)}/\mathbb{Z}B$. If C is the fiber of Γ containing $(B_1)_+ + (B_2)_+ + \cdots + (B_m)_+$, then

$$\beta_{m+1,C}(\Bbbk[x_1, \ldots, x_{n+2(m-2)}]/I_{\mathbb{Z}B}) = 1.$$

Consequently $I_{\mathbb{Z}B}$ is not Cohen–Macaulay.

Proof. Let $C_\circ$ be the fiber in $\mathbb{Z}^n/\mathbb{Z}B_\circ$ containing $((B_\circ)_1)_+ + ((B_\circ)_2)_+$, and $\Delta_\circ = \Delta_{C_\circ}$. By Lemma 2.2.2 and Proposition 2.2.4, $\tilde{H}_2(\Delta_\circ; \Bbbk)$ has rank at least 1.

Let C' be the fiber in $\mathbb{Z}^2/\mathbb{Z}H$ containing $(1, 0)$. We see that $C' = \{(1, 0), (0, 1)\}$, and therefore $\Delta' := \Delta_{C'}$ is the zero dimensional simplicial complex with two vertices.

Then $\Delta_C = \Delta_\circ \star \Delta' \star \cdots \star \Delta'$ is a sequence of suspensions of $\Delta_\circ$. Repeatedly applying (1.3), and using that $\tilde{H}_2(\Delta_\circ; \Bbbk)$ has rank at least 1, we see that $\tilde{H}_m(\Delta_C; \Bbbk)$ has rank at least 1, and therefore

$$\beta_{m+1,C}(\Bbbk[x_1, \ldots, x_{n+2(m-2)}]/I_{\mathbb{Z}B}) \geq 1.$$

$\square$

The property of being a complete intersection also behaves well with respect to block matrices for lattice ideals. This is stated precisely as follows.

Lemma 2.2.6. *Let $B^{(1)} \in \mathbb{Z}^{n_1 \times m_1}$ and $B^{(2)} \in \mathbb{Z}^{n_2 \times m_2}$ have rank m_1 and m_2 respectively. Let $B = \left[\begin{array}{c|c} B^{(1)} & 0 \\ \hline 0 & B^{(2)} \end{array}\right] \in \mathbb{Z}^{(n_1+n_2)\times(m_1+m_2)}$. The binomials corresponding to the columns of B generate $I_{\mathbb{Z}B}$ if and only if the binomials corresponding to the columns of $B^{(i)}$ generate $I_{\mathbb{Z}B^{(i)}}$ for $i = 1, 2$.*

Proof. For ease in the notation, we consider

$$I_{\mathbb{Z}B^{(1)}} \subset \Bbbk[x_1, \ldots, x_{n_1}] = \Bbbk[x] \text{ and } I_{\mathbb{Z}B^{(2)}} \subset \Bbbk[y_1, \ldots, y_{n_2}] = \Bbbk[y]$$

so that $I_{\mathbb{Z}B} \subset \Bbbk[x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}] = k[x, y]$. We also consider $\Bbbk[x], \Bbbk[y] \subset \Bbbk[x, y]$.

The lattice ideal $I_{\mathbb{Z}B}$ is generated by binomials $x^{u_+}y^{v_+} - x^{u_-}y^{v_-}$ for $u \in \mathbb{Z}B^{(1)} \subset \mathbb{Z}^{n_1}$ and $v \in \mathbb{Z}B^{(2)} \subset \mathbb{Z}^{n_2}$. Note that $x^{u_+}y^{v_+} - x^{u_-}y^{v_-} = x^{u_+}(y^{v_+} - y^{v_-}) + y^{v_-}(x^{u_+} - x^{u_-})$. This implies that $I_{\mathbb{Z}B} = \Bbbk[x, y] \cdot I_{\mathbb{Z}B^{(1)}} + \Bbbk[x, y] \cdot I_{\mathbb{Z}B^{(2)}}$. Moreover, $I_{\mathbb{Z}B^{(1)}} = I_{\mathbb{Z}B} \cap \Bbbk[x]$ and $I_{\mathbb{Z}B^{(2)}} = I_{\mathbb{Z}B} \cap \Bbbk[y]$. Our statement follows from these relationships. $\square$

We now come to the main result in this article.

Theorem 2.2.7. *Let $m \geq 2$ be an integer. There are infinitely many examples of rank m non-saturated lattices $\mathscr{L}$ such that one of $I_{\mathscr{L}}$, $I_{\mathscr{L}^{sat}}$ is a complete intersection, and the other one is not Cohen–Macaulay.*

Proof. When $m = 2$, this is Theorem 2.1.11. Let $B_\circ \in \mathbb{Z}^{n \times 2}$ and $H = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{Z}^{2\times 1}$. Suppose now $m > 2$ and consider the matrix

$$B = \left[\begin{array}{c|c|c|c} B_\circ & 0 & 0 & 0 \\ \hline 0 & H & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & H \end{array}\right] \in \mathbb{Z}^{(n+2(m-2))\times m},$$

Let $B_\circ \in \mathbb{Z}^{4\times 2}$ whose Gale diagram meets the four open quadrants of $\mathbb{R}^2$, and whose columns span a saturated lattice (there are infinitely many such matrices). Let $M_\circ$ be as in Proposition 2.1.6, and consider the nonsingular matrix

$$M = \left[\begin{array}{c|c|c|c} M_\circ & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \in \mathbb{Z}^{m\times m}. \tag{2.6}$$

Then we have

$$BM = \left[\begin{array}{c|c|c|c} B_\circ M_\circ & 0 & 0 & 0 \\ \hline 0 & H & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & H \end{array}\right] \in \mathbb{Z}^{(n+2(m-2))\times m}.$$

If $\mathscr{L}$ is the lattice spanned by the columns of BM, then $\mathscr{L}^{\text{sat}}$ is the lattice spanned by the columns of B. By Lemma 2.2.6, $I_{\mathscr{L}}$ is a complete intersection, and by Proposition 2.2.5, $I_{\mathscr{L}^{\text{sat}}}$ is not Cohen–Macaulay.

Now let $B_\circ \in \mathbb{Z}^{n\times 2}$ whose columns span a saturated lattice, and whose Gale diagram is imbalanced and spans at least four rays (there are infinitely many such matrices). Let $M_\circ$ be as in Proposition 2.1.9, and construct M using (2.6). If $\mathscr{L}$ is the lattice spanned by the columns of BM, then $\mathscr{L}^{\text{sat}}$ is the lattice spanned by the columns of B. By Proposition 2.2.5 $I_{\mathscr{L}}$ is not Cohen–Macaulay, and by Lemma 2.2.6 $I_{\mathscr{L}^{\text{sat}}}$ is a complete intersection. $\square$

3. FREE RESOLUTIONS OVER SCROLLS *

In this chapter, we consider the next class of examples after rational normal curves, namely rational normal scrolls. We compute the Betti numbers of the residue field (Theorem 3.1.1), and for 2-scrolls, we give its minimal free resolution (Theorem 3.2.2).

Notation: We work in $n = \sum_{i=1}^{k} m_i$ variables, and denote the polynomial ring by $S = \Bbbk[x_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq m_i]$.

Definition 3.0.1. The *rational normal k-scroll* $\mathcal{S}(m_1 - 1, m_2 - 1, \ldots, m_k - 1)$ is the variety in $\mathbb{P}^{n-1}$ defined by the ideal $I_2(M)$ of 2×2 minors of the $2 \times (n - k)$ matrix

$$M = \left[\begin{array}{ccc|ccc|c|ccc} x_{1,1} & \ldots & x_{1,m_1-1} & x_{2,1} & \ldots & x_{2,m_2-1} & \ldots\ldots & x_{k,1} & \ldots & x_{k,m_k-1} \\ x_{1,2} & \ldots & x_{1,m_1} & x_{2,2} & \ldots & x_{2,m_2} & \ldots\ldots & x_{k,2} & \ldots & x_{k,m_k} \end{array}\right]. \tag{3.1}$$

Throughout, we often forego writing “rational normal" and call $\mathcal{S}(m_1 - 1, m_2 - 1, \ldots, m_k - 1)$ a k-scroll and $\mathcal{S}(m - 1, n - m - 1)$ a scroll.

When $k = 1$, $\mathcal{S}(n - 1)$ is a *rational normal curve*, that is, the variety defined by 2×2 the minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_{n-1} \\ x_2 & x_3 & \ldots & x_n \end{bmatrix}.$$

As we mentioned in the introduction, the rings that are studied in this chapter are Koszul.

Theorem 3.0.2. *For M as in* (3.1)*, $R = S/I_2(M)$ is a Koszul ring.*

Proof. By [BHV94, Theorem 2.2], a sufficient condition for a quotient $\Bbbk[x_1, \ldots, x_n]/I$ to be Koszul is the existence of a homogeneous quadratic Gröbner basis for I. It follows that R is Koszul, since the 2×2 minors of M form a Gröbner basis for $I_2(M)$ with respect to a reverse lexicographic ordering (see [KPU09, Lemma 2.2]). $\square$

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Furthermore, rational normal k-scrolls are projective toric varieties. Therefore their defining ideal $I_2(M)$ can be realized as a toric ideal $I_{\mathcal{A}}$ [Pet08, Lemma 2.1], where $\mathcal{A}$ is the $(k+1)\times n$ matrix

$$\mathcal{A} = \left[\begin{array}{cccc|cccc|c|cccc} 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0\cdots\cdots 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0\cdots\cdots 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & & & & & & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0\cdots\cdots 0 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & m_1-1 & 0 & 1 & \cdots & m_2-1 & \cdots\cdots & 0 & 1 & \cdots & m_k-1 \end{array}\right], \tag{3.2}$$

so that $R = S/I_2(M)$ is a semigroup ring.

Example 3.0.3. The rational normal 2-scroll $\mathcal{S}(3,2)$ can be defined as the ideal of minors $I_2(M)$ of

$$M = \left[\begin{array}{ccc|cc} x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} \\ x_{1,2} & x_{1,3} & x_{1,4} & x_{2,2} & x_{2,3} \end{array}\right]$$

and as the toric ideal $I_{\mathcal{A}}$ defined by

$$\mathcal{A} = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 \end{array}\right].$$

By either approach, the ideal defining $\mathcal{S}(3,2)$ is

$$\langle x_{1,1}x_{1,3} - x_{1,2}^2, \quad x_{1,1}x_{1,4} - x_{1,2}x_{1,3}, \quad x_{1,1}x_{2,2} - x_{1,2}x_{2,1}, \quad x_{1,1}x_{2,3} - x_{1,2}x_{2,2} \quad x_{1,2}x_{1,4} - x_{1,3}^2,$$
$$x_{1,2}x_{2,2} - x_{1,3}x_{2,1}, \quad x_{1,2}x_{2,3} - x_{1,3}x_{2,2} \quad x_{1,3}x_{2,2} - x_{1,4}x_{2,1} \quad x_{1,3}x_{2,3} - x_{1,4}x_{2,2} \quad x_{2,1}x_{2,3} - x_{2,2}^2\rangle.$$

Throughout this chapter, we will occasionally commit the minor sin of identifying a scroll $\mathcal{S}$ with its defining ideal $I_2(M)$ with its coordinate ring R.

3.1 Betti Numbers of $\Bbbk$ over k-scrolls

Our first main theorem gives the Betti numbers of the field $\Bbbk$ over $R = S/I_2(M)$, where M is as in 3.1.

Theorem 3.1.1. *Let $I_2(M)$ define the rational normal k-scroll $\mathcal{S}(m_1 - 1, \ldots, m_k - 1)$. If $R = S/I_2(M)$, then the ith Betti number of $\Bbbk$ as an R-module is*

$$\beta_i^R(\Bbbk) = \sum_{j=0}^{i} \binom{k+1}{j} (n-k-1)^{i-j}$$

In particular $\beta_{k+r}^R(\Bbbk) = (n-k-1)^{r-1}(n-k)^{k+1}$ for $r \geq 1$.

Because R is Koszul, Theorem 1.2.49 implies that we can obtain the Poincaré series of R by inverting its Hilbert series. Since Hilbert series are preserved under Gröbner degeneration, it is enough to compute the Hilbert series of $S/\operatorname{in}_\prec(I_2(M))$ for $\prec$ a monomial order in S. This task is easiest if we are fortunate enough that our ideal has a squarefree initial ideal. The next result states that this is indeed the case for scrolls.

Theorem 3.1.2. *Let $\prec$ be the lexicographic monomial order on S given by $x_{1,1} \succ x_{1,2} \succ \ldots \succ x_{1,m_1} \succ x_{2,1} \succ \ldots \succ x_{k,m_k}$, then*

$$\operatorname{in}_\prec(I_2(M)) = \langle x_{i,j}x_{i,\ell} \mid |j-\ell| \geq 2 \rangle + \langle x_{i,j}x_{r,s} \mid 1 \leq i < r \leq k, 1 \leq j < m_i, 1 < s \leq m_r \rangle, \tag{3.3}$$

that is, $\operatorname{in}_\prec(I_2(M))$ is generated by the products of variables on the main diagonals of M. In particular, $\operatorname{in}_\prec(I_2(M))$ is a squarefree monomial ideal.

Example 3.1.3. The initial ideal for $I_2(M)$ as in Example 3.0.3 is $\operatorname{in}_\prec(I_2(M)) = \langle x_{1,1}x_{1,3}, x_{1,1}x_{1,4}, x_{1,1}x_{2,2}, x_{1,1}x_{2,3},\ x_{1,2}x_{1,4}, x_{1,2}x_{2,2}, x_{1,2}x_{2,3},\ x_{1,3}x_{2,2}, x_{1,3}x_{2,3},\ x_{2,1}x_{2,3} \rangle$.

Denote by D the ideal on the right hand side of (3.3). To prove Theorem 3.1.2, we begin by pinpointing which monomials are *not* in D.

Lemma 3.1.4. *Suppose $x^u \notin D$.*

a) If there exists i such that x^u contains two variables with first index i with nonzero exponents, then u is of the form

$$u = (0 \dots 0\, a_1 | 0 \dots 0\, a_2 | \dots | 0 \dots 0\, a_{i-1} | 0 \dots 0\, a_{i,\ell}\, a_{i,\ell+1}\, 0 \dots 0 | a_{i+1}\, 0 \dots 0 | \dots | a_k\, 0 \dots 0)$$

b) Otherwise, u is of the form

$$u = (0 \dots 0\, a_1 | 0 \dots 0\, a_2 | \dots | 0 \dots 0\, a_{i-1} | a_i\, 0 \dots 0 | a_{i+1}\, 0 \dots 0 | \dots | a_k\, 0 \dots 0)$$

Proof. The lemma follows from these observations.

i) If x^u contains the variables $x_{i,j}, x_{i,\ell}$ with $j < \ell$ both with nonzero exponent, then $\ell = j + 1$. Consequently, x^u cannot contain 3 variables from the same block with nonzero exponent.

ii) If x^u contains variables $x_{i,j}, x_{r,s}$ with $i < r$ and $j < m_i$, both with nonzero exponent, then $s = 1$.

□

The following result is used to show that D is equal to $\mathrm{in}_\prec I_2(M)$.

Proposition 3.1.5. *Let $\mathcal{A}$ be as in* (3.2) *(so that $I_2(M) = I_\mathcal{A}$). If $x^u \notin D$, $x^u \succ x^v$, and $\mathcal{A}u = \mathcal{A}v$, then $u = v$.*

Proof. In Lemma 3.1.4, case b) is a special case of a) where $\ell = 1$ and $a_{i,\ell+1} = 0$, so we may assume u satisfies case a). We also assume $u \neq 0$, and write $v = (b_{1,1}, b_{1,2}, \dots, b_{k,m_k})$.

Suppose $a_1 \neq 0$. Since $x^u \succ x^v$, the monomial x^v cannot contain any variable greater than x_{1,m_1}. Then, as $\mathcal{A}u = \mathcal{A}v$, x^u and x^v must contain the same power of x_{1,m_1}. The same argument implies that x^u and x^v contain the same powers of all variables up to and including $x_{i-1,m_{i-1}}$.

Now again, since $x^u \succ x^v$ lexicographically, $a_{i,\ell} \geq b_{i,\ell}$ and $b_{i,0} = \ldots = b_{i,\ell-1} = 0$. As $\mathcal{A}u = \mathcal{A}v$, we have $a_{i,\ell} + a_{i,\ell+1} = b_{i,\ell} + b_{i,\ell+1} + \ldots + b_{i,m_i}$. But if $a_{i,\ell} > b_{i,\ell}$, then $(\mathcal{A}u)_{k+1} > (\mathcal{A}v)_{k+1}$. This implies that $a_{i,\ell} = b_{i,\ell}$, and similarly $a_{i,\ell+1} = b_{i,\ell+1}$, so $b_{i,t} = 0$ for $t \geq \ell + 2$.

To finish the proof, note that $(\mathcal{A}u)_{k+1} = (m_1-1)a_1+\ldots+(m_{i-1}-1)a_{i-1}+(\ell-1)a_{i,\ell}+\ell a_{i,\ell+1} = (m_1 - 1)b_{1,m_1} + \ldots + (m_{i-1} - 1)b_{i-1,m_{i-1}} + (\ell - 1)b_{i,\ell} + \ell b_{i,\ell+1}$. Because $\mathcal{A}u = \mathcal{A}v$, this implies that $b_{j,t} = 0$ for $j > i$ and $t > 1$. Again, using $\mathcal{A}u = \mathcal{A}v$, we conclude that $b_{j,1} = a_j$ for all $j > i$. $\square$

We are ready to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Since $I_2(M)$ is $\mathcal{A}$-homogeneous, its initial ideal is generated by the initial forms of $\mathcal{A}$-homogeneous elements of $I_2(M)$. If $P \in I_2(M)$ is $\mathcal{A}$-homogeneous and $\mathrm{in}_\prec P \notin D$, then P has one term by Proposition 3.1.5. But since $I_2(M)$ is a toric ideal, it contains no monomials, so that such a P cannot belong to $I_2(M)$. We conclude that if P is $\mathcal{A}$-homogeneous and $P \in I_2(M)$, then $\mathrm{in}_\prec P \in D$. $\square$

With a squarefree initial ideal in hand, we now turn to Stanley–Reisner theory. Let Δ be the simplicial complex on the vertex set $\{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq m_i\}$ whose Stanley–Reisner ideal is $D = \mathrm{in}_\prec I_2(M)$. By definition, this means that D is generated by monomials whose index sets correspond to nonfaces of Δ. It follows from Lemma 3.1.4 that Δ is the simplicial complex whose maximal faces are

$$\{(1,m_1),(2,m_2),\ldots,(i,m_{i-1}),(i,j),(i,j+1),(i+1,1),\ldots,(k,1)\} \qquad \text{for } 1 \leq i \leq k,\ 1 \leq j \leq m_i - 1, \tag{3.4}$$

in particular, Δ is pure of dimension k. Figure 3.1 illustrates this simplicial complex in an example.

Recall from Theorem 1.2.55 that the Hilbert series of a Stanley–Reisner ring can be given in terms of the face numbers of the corresponding simplicial complex by

$$\mathrm{Hilb}_{S/D}(t) = \frac{1}{(1-t)^{k+1}} \sum_{d=0}^{k+1} f_{d-1} t^d (1-t)^{k+1-d},$$

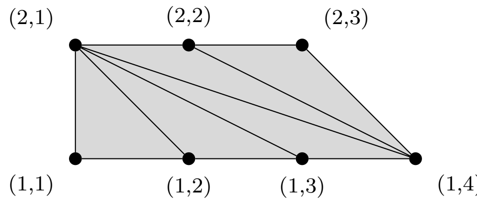

Figure 3.1: The simplicial complex Δ for $\mathcal{S}(3,2)$.
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where f_d is the number of d-dimensional faces of Δ. We now compute these face numbers for our simplicial complex Δ.

Proposition 3.1.6. *If Δ is the simplicial complex whose Stanley–Reisner ideal is D, then $f_d = \binom{k}{d}n - d\binom{k+1}{d+1}$ for $d \geq -1$. In particular, the face numbers of Δ depend only on k and n, and not on $m_1, \ldots, m_k$.*

Proof. We prove this by induction on k. Note that, by construction, $f_0 = n$, regardless of the value of k.

If $k = 1$, Δ has $f_1 = n-1$ one-dimensional faces, namely $\{(1,i),(1,i+1)\}$ for $i = 1,\ldots,n-1$ (cf [PRS98, Theorem 3.9]).

For the inductive step, let Δ be the complex associated to $\mathcal{S}(m_1-1,\ldots,m_k-1)$ and Δ' be the complex associated to $\mathcal{S}(m_1-1,\ldots,m_{k+1}-1)$. The complex Δ is naturally a subcomplex of Δ'. We assume that $f_d(\Delta) = \binom{k}{d}(m_1+\ldots+m_k) - d\binom{k+1}{d+1}$. Using the description of the facets of Δ from (3.4) (and the corresponding description for the facets of Δ') we see that the d-dimensional faces of Δ' are:

- $f_d(\Delta)$ d-dimensional faces of Δ,
- $f_{d-1}(\Delta)$ faces of the form $\tau \cup \{(k+1,1)\}$, where τ is a $(d-1)$-dimensional face of Δ,
- $\binom{k}{d}(m_{k+1}-1)$ faces with d vertices from the set $\{(i,m_i) \mid 1 \leq i \leq k\}$ and one vertex from $\{(k+1,j) \mid 2 \leq j \leq m_{k+1}\}$, and
- $\binom{k}{d-1}(m_{k+1}-1)$ faces with $d-1$ vertices from $\{(i,m_i) \mid 1 \leq i \leq k\}$ union an element of $\{\{(k+1,j),(k+1,j+1)\} \mid 1 \leq j \leq m_{k+1}-1\}$.

Adding these together and applying the inductive hypothesis yields $f_d(\Delta') = \binom{k+1}{d}(m_1 + \ldots + m_{k+1}) - d\binom{k+2}{d+2}$, as we wanted. $\square$

The following result gives the Hilbert series of R; the proof is a straightforward, if hefty, bullying of binomial coefficients.

Proposition 3.1.7. *The Hilbert series of R is* $\mathrm{Hilb}_R(t) = \mathrm{Hilb}_{S/D}(t) = \frac{1+(n-k-1)t}{(1-t)^{k+1}}$ $\square$

Remark. For those familiar with the machinery, there is alternate way to compute the Hilbert series of R. It is known that the Eagon-Northcott complex is a minimal free resolution for arbitrary k-scrolls (see [CJ97] or [Eis05, Corollary A2.62]), which reveals that the Betti numbers of $S/I_2(M)$ over S are $\beta_{0,0} = 1$ and $\beta_{i,i+1} = i\binom{n-k}{i+1}$ for $i \geq 1$. The Hilbert series of R must then be

$$\mathrm{Hilb}_R(t) = \frac{\sum_{i=0}^{n-k}(-1)^{i-1}(i-1)\binom{n-k}{i}t^i}{(1-t)^n},$$

which simplifies to the same Hilbert series as that in Proposition 3.1.7.

We are finally ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Since R is a Koszul ring, it follows from Proposition 3.1.7 that the Poincaré series of R is

$$P_R(t) = \frac{1}{\mathrm{Hilb}_R(-t)} = \frac{(1+t)^{k+1}}{1-(n-k-1)t} = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\binom{k+1}{j}(n-k-1)^{i-j}\right]t^i.$$

For the last equality, we use $(1+t)^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}t^i$ and $\dfrac{1}{1-(n-k-1)t} = \sum_{i=0}^{\infty}(n-k-1)^i t^i$. We conclude that $\beta_i^R(\Bbbk) = \sum_{j=0}^{i}\binom{k+1}{j}(n-k-1)^{i-j}$. The special formula for $\beta_{k+r}(\Bbbk)$ follows from the simplification of this sum when $\binom{k+1}{j}$ becomes 0. $\square$

3.2 The Resolution of $\Bbbk$ for $k = 2$

One of the difficulties when dealing with infinite free resolutions and unbounded Betti numbers is to give an explicit presentation for the differentials. In the case $k = 2$, the combinatorics of the ring R ensure a strong block structure that makes giving explicit matrices achievable.

Notation. In the case $k = 2$, we write $\mathcal{S}(m-1, n-m-1)$ instead of $\mathcal{S}(m_1 - 1, m_2 - 1)$, and forego double indexing to replace $x_{1,j}$ by x_j and $x_{2,j}$ by x_{m+j}. Finally, we denote $p = n - m$.

With this new notation, the matrix (3.1) is replaced by the $2 \times (n-2)$ matrix

$$M = \left[\begin{array}{cccc|cccc} x_1 & x_2 & \dots & x_{m-1} & x_{m+1} & x_{m+2} & \dots & x_{n-1} \\ x_2 & x_3 & \dots & x_m & x_{m+2} & x_{m+3} & \dots & x_n \end{array}\right],$$

and the ideal $I_2(M)$ is the toric ideal $I_{\mathcal{A}}$ associated to the $3 \times n$ matrix

$$\mathcal{A} = \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & m-1 & 0 & 1 & \dots & p-1 \end{array}\right].$$

Example 3.2.1. This abbreviated notation rewrites the matrix M from Example 3.0.3 as

$$M = \left[\begin{array}{ccc|cc} x_1 & x_2 & x_3 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_6 & x_7 \end{array}\right].$$

Our ultimate goal is to construct the minimal free resolution of $\Bbbk$ as an R-module. Our point of departure is the short exact sequence

$$0 \to \langle x_1, \dots, x_m \rangle \cap \langle x_{m+1}, \dots, x_n \rangle \to \langle x_1, \dots, x_m \rangle \oplus \langle x_{m+1}, \dots, x_n \rangle \to \langle x_1, \dots, x_n \rangle \to 0. \tag{3.5}$$

We construct free resolutions $(\mathcal{F}_\bullet(I_1), \partial_{I_1,i})$, $(\mathcal{F}_\bullet(I_2), \partial_{I_2,i})$, and $(\mathcal{F}_\bullet, \partial_{J,i})$ of the ideals $I_1 = \langle x_1, \dots, x_m \rangle$, $I_2 = \langle x_{m+1}, \dots, x_n \rangle$ and $J = I_1 \cap I_2$ respectively. We then combine these resolutions via mapping cone to make a resolution of $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. Augmenting the resolution of $\mathfrak{m}$ to be a resolution of $\Bbbk = R/\mathfrak{m}$ results in a shift of one step, and minimality is assured by the previous

Betti number computations. We obtain the resolution

$$\mathcal{F}_\bullet : \cdots \xrightarrow{\partial_6} R^{(n-2)^3(n-3)^2} \xrightarrow{\partial_5} R^{(n-2)^3(n-3)} \xrightarrow{\partial_4} R^{(n-2)^3} \xrightarrow{\partial_3} R^{n^2-3n+3} \xrightarrow{\partial_2} R^n \xrightarrow{\partial_1} R \xrightarrow{\partial_0} \Bbbk \to 0.$$

3.2.1 The Differentials of $\mathcal{F}_\bullet$

Our first objective is to explicitly describe the differentials ∂_i of $\mathcal{F}_\bullet$. These differentials are induced by a mapping cone. More precisely,

$$\partial_1 = [x_1\ x_2\ \cdots\ x_n],\ \partial_2 = \left[\begin{array}{cc|c} \partial_{I_1,1} & & \\ \cline{1-2} & \partial_{I_2,1} & \alpha_0 \end{array}\right],\ \partial_{i+1} = \left[\begin{array}{cc|c} \partial_{I_1,i} & & \\ \cline{1-2} & \partial_{I_2,i} & \alpha_{i-1} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & -\partial_{J,i-1} \end{array}\right] \text{ for all } i \geq 2.$$

The maps α are the chain maps from $\mathcal{F}_\bullet(J)$ to $\mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$, which are:

$$\alpha_0 = \left[\begin{array}{cccccccc} x_{m+1} & 0 & \dots & 0 & & & & \\ 0 & x_{m+1} & \dots & 0 & & \mathbf{0} & & \\ 0 & \ddots & \ddots & 0 & & & & \\ 0 & 0 & \dots & x_{m+1} & x_{m+2} & x_{m+3} & \dots & x_n \\ \hline -x_1 & -x_2 & \dots & -x_m & 0 & 0 & \cdots & 0 \\ & & & & -x_m & 0 & \cdots & 0 \\ & \mathbf{0} & & & 0 & -x_m & \cdots & 0 \\ & & & & 0 & \ddots & \ddots & 0 \\ & & & & 0 & 0 & \cdots & -x_m \end{array}\right] \in R^{n\times(n-1)}$$

$$\alpha_i = \begin{bmatrix} x_{m+1} \cdot \mathbb{1}_{(m-1)(n-2)(n-3)^{i-1}} & 0 \\ 0 & -x_m \cdot \mathbb{1}_{(p-1)(n-2)(n-3)^{i-1}} \end{bmatrix} \in R^{(n-2)^2(n-3)^{i-1}\times(n-2)^2(n-3)^{i-1}}$$

The constituent resolutions $\mathcal{F}_\bullet(I_1)$, $\mathcal{F}_\bullet(I_2)$, and $\mathcal{F}_\bullet(J)$ have highly structured differentials, the building blocks of which are now given:

$$\varphi_0 = \begin{bmatrix} x_2 & x_3 & \dots & x_m & x_{m+2} & \dots & x_n \\ -x_1 & -x_2 & \dots & -x_{m-1} & -x_{m+1} & \dots & -x_{n-1} \end{bmatrix} \in R^{2\times(n-2)}$$

$$\Phi_d = \begin{bmatrix} \varphi_0 & \mathbf{0}^{1\times(n-2)} & \dots & \dots & \dots & \mathbf{0}^{1\times(n-2)} \\ \mathbf{0}^{1\times(n-2)} & \varphi_0 & \mathbf{0}^{1\times(n-2)} & \dots & \dots & \mathbf{0}^{1\times(n-2)} \\ \vdots & \mathbf{0}^{1\times(n-2)} & \varphi_0 & \mathbf{0}^{1\times(n-2)} & \dots & \mathbf{0}^{1\times(n-2)} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}^{1\times(n-2)} & \dots & \dots & \dots & \mathbf{0}^{1\times(n-2)} & \varphi_0 \end{bmatrix} \in R^{d\times(d-1)(n-2)}$$

Note that Φ_d is very sparse and consists of block components, but is not a block diagonal matrix. The structure of Φ_d is illustrated in Figure 3.2, with gray squares denoting non-zero entries, and empty squares denoting 0. These nonzero φ_0-blocks appear $d-1$ times.

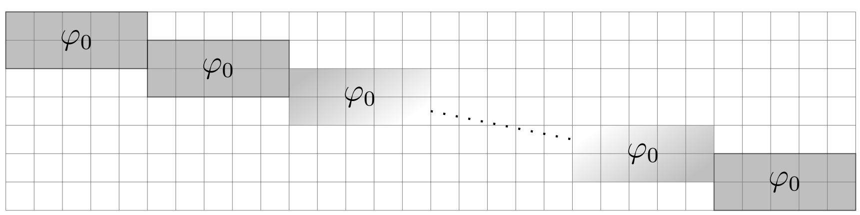

Figure 3.2: The structure of Φ_d.
Reprinted with permission from [MS20].

We denote by $u_i \in R^{(m-2)(n-2)\times(n-2)}$ and $v_i \in R^{(p-2)(n-2)\times(n-2)}$ the following matrices, which are almost entirely composed of zeros save for a single row that equals the first row or second row of M, respectively. More precisely,

$$u_i = \begin{bmatrix} & & & \mathbf{0}^{i(n-2)+m-1\times n-2} & & & \\ x_1 & \dots & x_{m-1} & x_{m+1} & \dots & x_{n-1} \\ & & & \mathbf{0}^{((m-i-2)(n-2)-m)\times n-2} & & & \end{bmatrix} \quad v_i = \begin{bmatrix} & & & \mathbf{0}^{i(n-2)+m-2\times n-2} & & & \\ -x_2 & \dots & -x_m & -x_{m+2} & \dots & -x_n \\ & & & \mathbf{0}^{(((p-i-2)(n-2))-m+1)\times n-2} & & & \end{bmatrix}.$$

Despite the length of the exponents, these matrices are simple: u_i is the $(m-2)(n-2)\times(n-2)$ matrix with the top row of M in the $(i(n-2)+m)$-th row, and v_i is the $(p-2)(n-2)\times(n-2)$ matrix with the negative of the bottom row of M in the $(i(n-2)+m-1)$-st row.

Finally, we introduce the following notation:

$$
\varphi_1 = \left[\begin{array}{c|cccccccc|c}
 & x_{m+1} & 0 & \dots & 0 & & & & & \\
 & 0 & x_{m+1} & \dots & 0 & & & & & \\
\Phi_{m-1} & 0 & \ddots & \ddots & 0 & & \mathbf{0} & & & \mathbf{0} \\
 & 0 & 0 & \dots & x_{m+1} & x_{m+2} & x_{m+3} & \dots & x_n & \\
\hline
 & -x_1 & -x_2 & \dots & -x_{m-1} & -x_m & 0 & \cdots & 0 & \\
 & & & & & 0 & -x_m & \cdots & 0 & \\
\mathbf{0} & & & \mathbf{0} & & 0 & \ddots & \ddots & 0 & \Phi_{p-1} \\
 & & & & & 0 & 0 & \cdots & -x_m &
\end{array}\right]
$$

$$
\varphi_2 = \left[\begin{array}{c|ccccccccc|c}
\bigoplus_{m-2}\varphi_1 & u_0 & u_1 & \cdots & u_{m-3} & \mathbf{0} & & \mathbf{0} & & & \mathbf{0} \\
\hline
\mathbf{0} & & & & & -\Phi_{n-2} & & & & & \mathbf{0} \\
\hline
\mathbf{0} & & \mathbf{0} & & & \mathbf{0} & v_0 & v_1 & \cdots & v_{p-3} & \bigoplus_{p-2}\varphi_1
\end{array}\right]
$$

$$
\varphi_i = \varphi_{i-1}^{\oplus(m-2)} \bigoplus \varphi_{i-2}^{\oplus(n-3)} \bigoplus \varphi_{i-1}^{\oplus(p-2)} \text{ for } i \geq 3
$$

Note that φ_1 is in $R^{(n-2)\times(n-2)(n-3)}$, φ_2 is in $R^{(n-2)(n-3)\times(n-2)(n-3)^2}$, and φ_i is in $R^{(n-2)(n-3)^{i-1}\times(n-2)(n-3)^i}$.

The presentation for φ_2 is perhaps deceiving; the brunt of the matrix is a direct sum of φ_1's. It is only (most of) the middle $(n-3)(n-2)$ columns that have additional entries above or below the middle $-\Phi_{n-2}$.

Using these φ's, we construct resolutions of J, I_1 and I_2. The ideal J has resolution over R as shown below:

$$
\mathcal{F}_\bullet(J) : \cdots \xrightarrow{\partial_{J,4}} R^{(n-2)^2(n-3)^2} \xrightarrow{\partial_{J,3}} R^{(n-2)^2(n-3)} \xrightarrow{\partial_{J,2}} R^{(n-2)^2} \xrightarrow{\partial_{J,1}} R^{n-1} \xrightarrow{\partial_{J,0}} J \to 0
$$

where

$$\partial_{J,0} = \begin{bmatrix} x_1x_{m+1} & x_2x_{m+1} & \dots & x_mx_{m+1} & x_mx_{m+2} & x_mx_{m+3} & \dots & x_mx_n \end{bmatrix} \in R^{1\times(n-1)}$$

$$\partial_{J,1} = \Phi_{n-1} \in R^{(n-1)\times(n-2)^2}$$

$$\partial_{J,i} = \varphi_{i-1}^{\oplus(n-2)} \in R^{(n-2)^2(n-3)^{i-2}\times(n-2)^2(n-3)^{i-1}} \text{ for } i \geq 2$$

The ideal I_1 has resolution over R as shown below:

$$\mathcal{F}_\bullet(I_1) : \cdots \to R^{(m-1)(n-2)(n-3)^2} \xrightarrow{\partial_{I_1,3}} R^{(m-1)(n-2)(n-3)} \xrightarrow{\partial_{I_1,2}} R^{(m-1)(n-2)} \xrightarrow{\partial_{I_1,1}} R^m \xrightarrow{\partial_{I_1,0}} I_1 \to 0$$

where

$$\partial_{I_1,0} = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix} \in R^{1\times m}\,; \qquad \partial_{I_1,1} = \Phi_m \in R^{m\times(m-1)(n-2)}$$

$$\partial_{I_1,i} = \varphi_{i-1}^{\oplus(m-1)} \in R^{(m-1)(n-2)(n-3)^{i-2}\times(m-1)(n-2)(n-3)^{i-1}} \text{ for } i \geq 2$$

The ideal I_2 has resolution over R as shown below:

$$\mathcal{F}_\bullet(I_2) : \cdots \to R^{(p-1)(n-2)(n-3)^2} \xrightarrow{\partial_{I_2,3}} R^{(p-1)(n-2)(n-3)} \xrightarrow{\partial_{I_2,2}} R^{(p-1)(n-2)} \xrightarrow{\partial_{I_2,1}} R^p \xrightarrow{\partial_{I_2,0}} I_2 \to 0$$

where

$$\partial_{I_2,0} = \begin{bmatrix} x_{m+1} & x_{m+2} & \dots & x_n \end{bmatrix} \in R^{1\times p}\,; \qquad \partial_{I_2,1} = \Phi_p \in R^{(p)\times(p-1)(n-2)}$$

$$\partial_{I_2,i} = \varphi_{i-1}^{\oplus(p-1)} \in R^{(p-1)(n-2)(n-3)^{i-2}\times(p-1)(n-2)(n-3)^{i-1}} \text{ for } i \geq 2$$

Our main result is as follows.

Theorem 3.2.2. *$\mathcal{F}_\bullet$ constructed above is the minimal free resolution of $\Bbbk$ over R.*

3.2.2 Outline of the proof of Theorem 3.2.2

The remainder of this section is devoted to showing that $\mathcal{F}_\bullet$ is indeed the minimal free resolution of $\Bbbk$ over R. We now lay out the steps in this proof.

Most of the work goes to showing that $\mathcal{F}_\bullet(I_1)$, $\mathcal{F}_\bullet(I_2)$ and $\mathcal{F}_\bullet(J)$ are free resolutions of I_1, I_2

and J respectively. The matrices considered in these three cases have very similar structure, and the details in proving exactness are virtually identical. Thus, we give only the proof that $\mathcal{F}_\bullet(J)$ is a resolution. Exactness of $\mathcal{F}_\bullet(J)$ is shown in Subsection 3.2.3, using ideas from [**?**].

What remains is to provide the map of complexes $\alpha : \mathcal{F}_\bullet(J) \to \mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$ lifting the inclusion $J \to I_1 \oplus I_2$ from the short exact sequence (3.5). This is done in Subsection 3.2.4.

Once α is constructed, the mapping cone procedure ensures that $\mathcal{F}_\bullet$ is exact, and thus a free resolution of $\Bbbk$. That it is the *minimal* free resolution of $\Bbbk$ follows by inspection, or by Theorem 3.1.1.

3.2.3 $\mathcal{F}_\bullet(J)$ is exact

We need generators for $J = \langle x_1, \dots, x_m \rangle \cap \langle x_{m+1}, \dots, x_n \rangle$. Clearly,

$$J = \left\langle \begin{array}{cccc} x_1x_{m+1}, & x_1x_{m+2}, & \dots, & x_1x_n, \\ x_2x_{m+1}, & x_2x_{m+2}, & \dots, & x_2x_n, \\ \vdots & \vdots & \vdots & \vdots \\ x_mx_{m+1} & x_mx_{m+2}, & \dots, & x_mx_n \end{array} \right\rangle.$$

However, many of these monomials are equal in R; in fact, $x_ix_j = x_kx_\ell$ if $i + j = k + \ell$, as long as $j \neq m$ and $k \neq m+1$. This means that, in the above arrangement, all monomials on the same skew-diagonal are the same, for example, $x_3x_{m+1} = x_2x_{m+2} = x_1x_{m+3}$. Consequently,

$$J = \langle x_1x_{m+1}, x_2x_{m+1}, \dots, x_mx_{m+1}, x_mx_{m+2}, x_mx_{m+3}, \dots, x_mx_n \rangle.$$

We start by checking that we are working with complexes.

Proposition 3.2.3. *$\mathcal{F}_\bullet(J)$, $\mathcal{F}_\bullet(I_1)$ and $\mathcal{F}_\bullet(I_2)$ are complexes.*

Proof. This is a straightforward, if tedious, calculation. A key observation is that $\varphi_i \circ \varphi_{i+1} = 0$ for all i. This follows, as each of these compositions has entries that are either 0 or binomials in $I_2(M)$. Given the direct sum structure of the differentials, this is enough to show our proposed differentials compose to zero. $\square$

Our next goal is to show that $\mathcal{F}_\bullet(J)$ is exact. (The same argument, with minor modifications, shows the same for $\mathcal{F}_\bullet(I_1)$ and $\mathcal{F}_\bullet(I_2)$.) We need some notation.

Definition 3.2.4. Let A be a Noetherian commutative ring. Let $f : F \to G$ be a map of free A-modules, which is represented by a matrix with entries in A. The *rank* of f is the size of the largest nonvanishing minor of this matrix. If f has rank r, we use $I(f)$ to denote the ideal generated by the $r \times r$ minors of (the matrix representing) f.

The following results are used to prove exactness.

Lemma 3.2.5. *[Eis95, Lemma 20.10] Let A be a commutative Noetherian ring. A complex $F \xrightarrow{f} G \xrightarrow{g} H$ of free A-modules with $I(f) = I(g) = A$ is exact iff $\operatorname{rank} f + \operatorname{rank} g = \operatorname{rank} G$.*

Lemma 3.2.6 (Sylvester's Rank Inequality)**.** *If U and V are matrices with entries in a field, where U is $r \times s$ and V is $s \times t$, then*

$$\operatorname{rank} U + \operatorname{rank} V - s \leq \operatorname{rank} UV.$$

By Lemma 3.2.5, it is important to know the ranks of the differentials of $\mathcal{F}_\bullet(J)$. Due to the block structure, we must first address the matrices φ_i.

Proposition 3.2.7. *The rank of Φ_d is $d-1$ for all $d \geq 2$ and the rank φ_i is $(n-3)^i$ for all $i \geq 0$.*

Proof. Because R is a domain, dependences among rows of a matrix over R can be read off from the vanishing of minors. In fact, the rank of a matrix over R equals the rank of that matrix over the field of fractions of R. In this proof, we work over the field of fractions of R, which gives us access to Lemma 3.2.6.

It is clear that $\operatorname{rank} \varphi_0 = 1$, as all 2×2 minors of φ_0 are exactly the same as the minors of M, which belong to $I_2(M)$.

Next we must show that $\operatorname{rank} \varphi_1 = n-3$. We know that the rank of φ_1 is at least $n-3$, as the minor of size $n-3$ corresponding to rows $2, 3, \ldots, n-2$ and columns $1, 1+(n-2), 1+$

$2(n-2), \ldots, 1+(n-4)(n-2)$ equals $(-1)^{n-3}x_1 \neq 0$. On the other hand, by Lemma 3.2.6, $\operatorname{rank} \varphi_0 + \operatorname{rank} \varphi_1 - (n-2) \leq \operatorname{rank}(\varphi_0 \circ \varphi_1) = 0$, so $\operatorname{rank} \varphi_1 \leq n-3$. Consequently $\operatorname{rank} \varphi_1 = n-3$.

To compute $\operatorname{rank} \varphi_2$, we consider the minor of size $(n-3)^2$ corresponding to rows $\{s+t(n-2) \mid 2 \leq s \leq n-2, 0 \leq t \leq n-4\}$ and columns $1, (n-2)+1, 2(n-2)+1, \ldots, ((n-3)^2-1)(n-2)+1$ which equals $x_1^{(n-3)^2} \neq 0$, so that $\operatorname{rank} \varphi_2 \geq (n-3)^2$. Again by Lemma 3.2.6 and because $\varphi_1 \circ \varphi_2 = 0$, we know that $\operatorname{rank} \varphi_2 \leq (n-2)(n-3) - \operatorname{rank} \varphi_1 = (n-2)(n-3)-(n-3) = (n-3)^2$. We conclude that $\operatorname{rank} \varphi_2 = (n-3)^2$.

The rank computations for the remaining maps φ_i follow easily from the block structure: $\operatorname{rank} \varphi_i = (m-2)(n-3)^{i-1} + (n-3)(n-3)^{i-2} + (p-2)(n-3)^{i-1} = (n-3)^i$ for any $i \geq 2$.

In the case of Φ_d, we consider the minor corresponding to rows $2, 3, \ldots, d$ and columns $1, 1+(n-2), 1+2(n-2), \ldots, 1+(d-2)(n-2)$ which equals $(-1)^{d-1}x_1^{d-1}$, so that $\operatorname{rank} \Phi_d \geq d-1$. On the other hand, because $\varphi_0 \circ \varphi_1 = 0$, $\Phi_d \circ \bigoplus_{d-1} \varphi_1 = 0$. Therefore $\operatorname{rank} \Phi_d \leq dp-1)(n-2) - (d-1) \operatorname{rank} \varphi_1 = d-1$, and in fact $\operatorname{rank} \Phi_d = d-1$. □

The ranks of the differentials of $\mathcal{F}_\bullet(J)$ can be computed directly from Proposition 3.2.7.

Corollary 3.2.8. *The ranks of the $\partial_{J,i}$ are:*

1. $\operatorname{rank} \partial_{J,1} = n-2$
2. $\operatorname{rank} \partial_{J,2} = (n-2)(n-3)$
3. $\operatorname{rank} \partial_{J,i} = (n-2)(n-3)^{i-1}$ *for all* $i \geq 2$ □

In order to apply Lemma 3.2.5, we need more information regarding the ideals of maximal nonvanishing minors of the matrices involved.

Proposition 3.2.9. *For $i \in [n]$, we have $x_i^{p-1} \in I(\Phi_p)$ and $x_i^{(n-3)^j} \in I(\varphi_j)$ for all $j \geq 0$.*

Proof. Because we are considering the ideals generated by the minors, we can ignore signs in our computations. It is clear that $x_i \in I(\varphi_0)$ for all $i \in [n]$. To see that any $x_i^{n-3} \in I(\varphi_1)$ for

$x_i^{n-3} \in I(\varphi_1)$	rows $r_{i,1}$	columns $c_{i,1}$
$1 \leq i \leq m-1$	$2, 3, \ldots, n-2$	$i + j(n-2)$ for $0 \leq j \leq n-4$
$i = m$	$1, 2, \ldots, m-2,$ $m, \ldots, n-2$	$(m-1) + j(n-2)$ for $0 \leq j \leq m-3$ $\ell + (m-2)(n-2)$ for $m \leq \ell \leq n-2$
$i = m+1$	$1, 2, \ldots, m-1,$ $m+1, \ldots, n-2$	$\ell + (m-2)(n-2)$ for $1 \leq \ell \leq m-1$ $m + j(n-2)$ for $m-1 \leq j \leq n-4$
$m+2 \leq i \leq n$	$1, 2, \ldots, n-3$	$i - 2 + j(n-2)$ for $0 \leq j \leq n-4$

Table 3.1: Rows and columns giving variable powers in minors of φ_1

$x_i^{(n-3)^2} \in I(\varphi_2)$	rows $r_{i,2}$	columns $c_{i,2}$
$1 \leq i \leq m-1$	$s+t(n-2)$ for $s \in r_{i,1}$ and $0 \leq t \leq n-4$	$i + j(n-2)$ for $0 \leq j \leq (n-4)(n-2)$
$i = m$	$s+t(n-2)$ for $s \in r_{i,1}$ and $0 \leq t \leq n-4$ $\backslash (m-1)(n-2)$ $\cup (m-2)(n-1)+1$	$a+j(n-2)(n-3)$ for $a \in c_{i,1}$ and $0 \leq j \leq (n-4)$ but $j \neq m-2$ $(m-1) + \ell(n-2) + (m-2)(n-2)(n-3)$ for $0 \leq \ell \leq n-4$
$i = m+1$	$s+t(n-2)$ for $s \in r_{i,1}$ and $0 \leq t \leq n-4$ $\backslash (m-2)(n-2)+1$ $\cup\, m+(m-2)(n-2)$	$a+j(n-2)(n-3)$ for $a \in c_{i,1}$ and $0 \leq j \leq (n-4)$ but $j \neq m-2$ $m + \ell(n-2) + (m-2)(n-2)(n-3)$ for $0 \leq$ $\ell \leq n-4$
$m+2 \leq i \leq n$	$s+t(n-2)$ for $s \in r_{i,1}$ and $0 \leq t \leq n-4$	$(i-2) + j(n-2)$ for $0 \leq j \leq (n-4)(n-2)$

Table 3.2: Rows and columns giving variable powers in minors of φ_2

any $i \in [n]$, we can consider the minors corresponding to the rows $r_{i,1}$ and columns $c_{i,1}$ listed in Table 3.1.

The proposed submatrices of φ_1 whose rows and columns listed above are strictly triangular, so the minors are easily computed.

We can give similar recipes for the appropriate minors in φ_2, provided in Table 3.2.

The block structures of the successive φ_j's combined with the previous two statements is enough to see that $x_i^{(n-3)^j} \in I(\varphi_j)$. Finally, use the minors whose columns and rows are given in Table 3.3 to obtain $x_i^{d-1} \in I(\Phi_d)$.

□

$x_i^{d-1} \in I(\Phi_d)$	rows	columns
$1 \leq i \leq m-1$	$2, \ldots, d$	$i + j(n-2)$ for $0 \leq j \leq d-2$
$i = m$	$1, \ldots, d-1$	$m - 1 + j(n-2)$ for $0 \leq j \leq d-2$
$i = m+1$	$2, \ldots, d$	$m + j(n-2)$ for $0 \leq j \leq d-2$
$m+2 \leq i \leq n$	$1, \ldots, d-1$	$i - 2 + j(n-2)$ for $0 \leq j \leq d-2$

Table 3.3: Rows and columns giving variable powers in minors of Φ_d

We are now ready to give the main result in this subsection.

Theorem 3.2.10. *The complexes $\mathcal{F}_\bullet(J)$, $\mathcal{F}_\bullet(I_1)$ and $\mathcal{F}_\bullet(I_2)$ are exact.*

Proof. We only provide details for $\mathcal{F}_\bullet(J)$. We show that we have exactness after localizing at any prime ideal of R, from which exactness over R follows. If $\mathfrak{q}$ is any prime ideal in R, we denote by $\partial_{J,i,\mathfrak{q}}$ the localized map induced by $\partial_{J,i}$. The (unique) graded maximal ideal of R is $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$.

Corollary 3.2.8 provides the ranks of the maps $\partial_{J,i}$ over R. Because R is a domain, $I(\partial_{J,i})$ contains exclusively non-zero divisors for all i. This means that, when localizing, the rank of $\partial_{J,i}$ does not change. Furthermore, localization at any prime ideal $\mathfrak{q} \neq \mathfrak{m}$ yields $I(\partial_{J,i,\mathfrak{q}}) = R_\mathfrak{q}$, because each $I(\partial_{J,i})$ contains some power of every x_ℓ, by Proposition 3.2.9. By Lemma 3.2.5, this proves that $\mathcal{F}_\bullet(J)$ is exact after localization at any prime ideal $\mathfrak{q} \neq \mathfrak{m}$.

We conclude that, if $\mathcal{F}_\bullet(J)$ has a nonzero homology module, it is only supported at the graded maximal ideal $\mathfrak{m}$, and therefore has depth 0. Our goal now is to derive a contradiction.

We localize at $\mathfrak{m}$, and use F_i to denote the free $R_\mathfrak{m}$-modules appearing in the localization of $\mathcal{F}_\bullet(J)$. Use $B_i \subseteq C_i \subseteq F_i$ to denote the i-cycles and i-boundaries, and $H_i = C_i/B_i$. The ring R is a semigroup ring corresponding to a saturated (normal) semigroup, and is therefore Cohen–Macaulay by Hochster's theorem. Since $\dim R = 3$, it follows that $R_\mathfrak{m}$ has depth 3. Consequently all the free modules over $R_\mathfrak{m}$ also have depth 3, in particular the F_i. Any submodules of the free modules F_i must have depth at least 3, so we have $\operatorname{depth}(C_i) \geq 3$ and $\operatorname{depth}(B_i) \geq 3$. From the exact sequence

$$0 \to B_i \to C_i \to H_i \to 0$$

it follows that $\operatorname{depth}(H_i) \geq \min\{\operatorname{depth}(C_i), \operatorname{depth}(B_i) - 1\}$ (see [**?**, Corollary 18.6.a]), so that $\operatorname{depth}(H_i) \geq 2$. This contradicts that $\operatorname{depth}(H_i) = 0$.

Therefore localizations of $\mathcal{F}_\bullet(J)$ at all prime ideals (now including $\mathfrak{m}$) are exact, and consequently $\mathcal{F}_\bullet$ is exact. □

3.2.4 The Mapping Cone

We recall that we have an exact sequence

$$0 \to J \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} I_1 \oplus I_2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} \mathfrak{m} \to 0.$$

The relevant result for us is that, if we have resolutions of J and $I_1 \oplus I_2$, the inclusion $J \to I_1 \oplus I_2$ can be lifted to a map of complexes between the corresponding resolutions, and the associated mapping cone is a resolution of $\mathfrak{m}$. The definition of the mapping cone of a map of complexes is given below; we refer to the appendix of [**?**] for more information.

Definition 3.2.11. If $\alpha : F \to G$ is a map of complexes, and we write f, g for the differentials of F and G respectively, then the *mapping cone* $M(\alpha)$ of α is the complex such that $M(\alpha)_{i+1} = F_i \oplus G_{i+1}$ where the differential ∂_{i+1} is shown:

$$\begin{tikzcd} F_i \arrow[r, "-f_i"] \arrow[dr, "\alpha_i"] & F_{i-1} \\ \bigoplus & \bigoplus \\ G_{i+1} \arrow[r, "g_{i+1}"] & G_i \end{tikzcd}$$

that is, $\partial_{i+1}(a, b) = (-f_i(a), g_{i+1}(b) + \alpha_i(a))$.

We now construct the map of complexes that lifts the inclusion $J \to I_1 \oplus I_2$.

Proposition 3.2.12. *The map of complexes $\alpha : \mathcal{F}_\bullet(J) \to \mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$ is given by*

$$\alpha_0 = \left[\begin{array}{cccccccc} x_{m+1} & 0 & \dots & 0 & & & & \\ 0 & x_{m+1} & \dots & 0 & & \mathbf{0} & & \\ 0 & \ddots & \ddots & 0 & & & & \\ 0 & 0 & \dots & x_{m+1} & x_{m+2} & x_{m+3} & \dots & x_n \\ \hline -x_1 & -x_2 & \dots & -x_m & 0 & 0 & \cdots & 0 \\ & & & & -x_m & 0 & \cdots & 0 \\ & \mathbf{0} & & & 0 & -x_m & \cdots & 0 \\ & & & & 0 & \ddots & \ddots & 0 \\ & & & & 0 & 0 & \cdots & -x_m \end{array}\right] \in R^{n\times(n-1)}$$

$$\alpha_i = \begin{bmatrix} x_{m+1} * \mathbb{1}_{(m-1)(n-2)(n-3)^{i-1}} & 0 \\ 0 & -x_m * \mathbb{1}_{(p-1)(n-2)(n-3)^{i-1}} \end{bmatrix} \in R^{(n-2)^2(n-3)^{i-1}\times(n-2)^2(n-3)^{i-1}} \mathit{for}\, i \geq 1$$

Proof. We first check that $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \partial_{J,0} = (\partial_{I_1,0} \oplus \partial_{I_2,0})\alpha_0$. We compute both sides explicitly:

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \partial_{J,0} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} x_1x_{m+1} & x_2x_{m+1} & \dots & x_mx_{m+1} & x_mx_{m+2} & x_mx_{m+3} & \dots & x_mx_n \end{bmatrix} \\ &= \begin{bmatrix} x_1x_{m+1} & x_2x_{m+1} & \dots & x_mx_{m+1} & x_mx_{m+2} & x_mx_{m+3} & \dots & x_mx_n \\ -x_1x_{m+1} & -x_2x_{m+1} & \dots & -x_mx_{m+1} & -x_mx_{m+2} & -x_mx_{m+3} & \dots & -x_mx_n \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
&(\partial_{I_1,0} \oplus \partial_{I_2,0})\alpha_0 = \\
&= \begin{bmatrix} x_1 & \dots & x_m & 0 & \dots & 0 \\ 0 & \dots & 0 & x_{m+1} & \dots & x_n \end{bmatrix}
\left[\begin{array}{cccccccc}
x_{m+1} & 0 & \dots & 0 & & & & \\
0 & x_{m+1} & \dots & 0 & & \mathbf{0} & & \\
0 & \ddots & \ddots & 0 & & & & \\
0 & 0 & \dots & x_{m+1} & x_{m+2} & x_{m+3} & \dots & x_n \\
\hline
-x_1 & -x_2 & \dots & -x_m & 0 & 0 & \cdots & 0 \\
 & & & & -x_m & 0 & \cdots & 0 \\
 & \mathbf{0} & & & 0 & -x_m & \cdots & 0 \\
 & & & & 0 & \ddots & \ddots & 0 \\
 & & & & 0 & 0 & \cdots & -x_m
\end{array}\right] \\
&= \begin{bmatrix} x_1x_{m+1} & x_2x_{m+1} & \dots & x_mx_{m+1} & x_mx_{m+2} & x_mx_{m+3} & \dots & x_mx_n \\ -x_1x_{m+1} & -x_2x_{m+1} & \dots & -x_mx_{m+1} & -x_mx_{m+2} & -x_mx_{m+3} & \dots & -x_mx_n \end{bmatrix}.
\end{aligned}
$$

Now we can check if $\alpha_0\partial_{J,1} = (\partial_{I_1,1} \oplus \partial_{I_2,1})\alpha_1$. Without further ado:

$$
\begin{aligned}
\alpha_0\partial_{J,1} &= \\
&= \left[\begin{array}{cccccccc}
x_{m+1} & 0 & \dots & 0 & & & & \\
0 & x_{m+1} & \dots & 0 & & \mathbf{0} & & \\
0 & \ddots & \ddots & 0 & & & & \\
0 & 0 & \dots & x_{m+1} & x_{m+2} & x_{m+3} & \dots & x_n \\
\hline
-x_1 & -x_2 & \dots & -x_m & 0 & 0 & \cdots & 0 \\
 & & & & -x_m & 0 & \cdots & 0 \\
 & \mathbf{0} & & & 0 & -x_m & \cdots & 0 \\
 & & & & 0 & \ddots & \ddots & 0 \\
 & & & & 0 & 0 & \cdots & -x_m
\end{array}\right] \Phi_{n-1} \\
&= x_{m+1}\Phi_m \oplus (-x_m)\Phi_p
\end{aligned}
$$

where $x_{m+1}\varphi_0$ appears $m-1$ times and $-x_m\varphi_0$ appears $p-1$ times. However, because of the diagonal structure of α_1, this is clearly the same as $(\Phi_m \oplus \Phi_p)\alpha_1 = (\partial_{I_1,1} \oplus \partial_{I_2,1})\alpha_1$.

For remaining $i \geq 2$, $\partial_{J,i} = \partial_{I_1,i} \oplus \partial_{I_2,i}$, and all the α_i are diagonal matrices, so the products are easily verified to be equal. □

Proof of Theorem 3.2.2. Since $\mathcal{F}_\bullet(J)$ is a resolution of J and $\mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$ resolves $I_1 \oplus I_2$, the mapping cone of $\alpha : \mathcal{F}_\bullet(J) \to \mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$ is a resolution of $\mathfrak{m}$. Augmenting the resolution to be a resolution of $\Bbbk = R/\mathfrak{m}$ results in a shift of one step, and so we finally have the resolution $\mathcal{F}_\bullet$. Comparing the rank of the free modules in each step to the Betti numbers computed in Theorem 3.1.1, we conclude that $\mathcal{F}_\bullet$ is not only exact, but minimal. □

4. SUMMARY AND CONCLUSIONS

Both chapters of this dissertation come to ultimately algebraic conclusions via combinatorial tools. Chapter 2 uses the combinatorics of Gale diagrams and simplicial complexes to understand homological properties of lattice ideals. Chapter 3 focuses on a specifically combinatorial ring, the coordinate ring of rational normal scrolls, and leverages its internal combinatorics to obtain an algebraic resolution.

The flavor of the results in Chapters 2 and 3 are somewhat antithetical. Chapter 2 proves that, despite a deep algebraic connection between the two ideals, the Cohen–Macaulayness of lattice ideals and their corresponding toric ideals have no bearing on one another, a strongly negative result. In particular, one cannot check for Cohen–Macaulayness of one by checking Cohen–Macaulayness of the other. This was somewhat surprising, as the well-behaved nature of the toric ideal led us to expect that, if a lattice ideal was Cohen–Macaulay, the corresponding lattice idea would be Cohen–Macaulay as well.

This erroneous hypothesis was supported by early computations in the programming language `Macaulay2` [GS], which created 4×2 integer matrices B and tested the ideals $I_{\mathbb{Z}B}$ and $I_{\mathbb{Z}B^{\text{sat}}}$ for Cohen–Macaulayness. Most pairs were either both Cohen–Macaulay or neither ideal was, and in cases where the two differed, the toric ideal was Cohen–Macaulay. However, after hundreds of automated examples were checked, a pair of ideals emerged where the toric ideal was not Cohen–Macaulay and the lattice ideal was. Despite this surprise, the goal was now clear, and, using Gale diagrams, we were able to create and extend these counter-examples.

On the other hand, Chapter 3 tackles an infinite minimal free resolution, something that can rarely be shown explicitly, and shows that completely characterizing the resolution is not only feasible, but very finitely feasible. Despite the infinite nature of the resolution, the Betti numbers and the differentials show a replicating structure that allows us to characterize them completely. Furthermore, the differentials are fairly sparse and can be broken down into block components. These blocks mimic the two-fold structure of the rational normal 2-scroll. However, it is not enough to simply paste two of the resolutions from [GHP08] for the rational normal curve together, as the interaction between their equations must also make an appearance in the differentials. Unlike Chapter 2, the results in Chapter 3 indicate that a deeper study of these algebraic objects may prove fruitful.

Chapters 2 and 3 do share a similarity in their constructive nature. This explicit realization of abstract algebraic objects is available to us because of the underlying combinatorics of both cases, and is often not shared in more general algebraic results. Lattice ideals are at the heart of both chapters and their connection to simplicial and convex geometry ensures that the often elusive homological properties of a general ideal can be fully described.

4.1 Further Study

The nature of Chapter 2 shows that the Cohen–Macaulayness of a lattice ideal and the corresponding toric ideal are unrelated, but one may wish to investigate other algebraic properties, such as regularity and normality. It is possible that there is an implication in these cases. However, showing again that these characteristics may fail to be related would be interesting in its own right. One may also wish to limit oneself to a particular subclass of lattice ideals, perhaps those that are Gorenstein, and see if an implication holds in this setting.

Chapter 3 inspires courage for tackling more free resolutions in the toric case. The obvious generalization is finding the minimal free resolution of the rational normal k-scroll. Though resolutions of $\Bbbk$ over two of the classical examples of semigroup rings – those from rational normal curves [GHP08] and rational normal 2-scrolls [MS20] – are known, I still seek an explicit resolution of $\Bbbk$ for semigroup rings from a general k-scroll. The stabilization of the Betti numbers and repetitive nature of the differentials known from the finished cases are encouraging, but working with examples for the k-scroll offers challenges: exponential growth in n of the Betti numbers means that even the most atomic bits of the differentials are massive after just a few steps. This is compounded by the need for larger n as k increases. Nonetheless, the highly structured nature of the two known cases gives courage for computing the general case.

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