HIERARCHICAL UPSCALING AND MODEL REDUCTION TECHNIQUES FOR MULTISCALE DUAL-CONTINUUM SYSTEMS

A Dissertation

by

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ABSTRACT

Simulation in media with multiple interacting continua is often challenging due to distinct properties of the continua, multiple scales and high contrast. Thus, some type of model reduction is required. One of the approaches is a multi-continuum technique, where every process in each continuum is modeled separately and an interaction term is added. Direct numerical simulation in multiscale multi-continuum media is very expensive as it requires a large number of degrees of freedom to completely resolve the micro-scale variation. In this work, we present efficient upscaling and model reduction methods for multiscale dual-continuum systems.

We first consider the numerical homogenization of a multiscale dual-continuum system where the interaction terms between the continua are scaled as $O(1/\epsilon^2)$ where ϵ is the microscopic scale. Computing the effective coefficients of the homogenized equations can be expensive because one needs to solve local cell problems for a large number of macroscopic points. We develop a hierarchical approach for solving these cell problems at a dense network of macroscopic points with an essentially optimal computation cost. The method employs the fact that neighboring representative volume elements (RVEs) share similar features; and effective properties of the neighboring RVEs are close to each other. The hierarchical approach reduces computation cost by using different levels of resolution for cell problems at different macroscopic points. Solutions of the cell problems which are solved with a higher level of resolution are employed to correct the solutions at neighboring macroscopic points that are computed by approximation spaces with a lower level of resolution.

We then consider the case where the interaction terms of the dual-continuum system are scaled as $O(1/\epsilon)$. We derive the homogenized problem that is a dual-continuum sys-

tem which contains features that are not in the original two scale problem. In particular, the homogenized dual-continuum system contains extra convection terms and negative interaction coefficients while the interaction coefficient between the continua in the original two scale system obtains both positive and negative values. We prove rigorously the homogenization convergence and homogenization convergence rate. Homogenization of dual-continuum system of this type has not been considered before. We present the numerical examples for computing effective coefficients using hierarchical finite element methods.

We assume the above mentioned homogenized equation still possess some degree of multiscale and high contrast features caused by channels in the media. This motivates us to develop the generalized multiscale finite element method (GMsFEM) for an upscaled multiscale dual-continuum equations with general convection and interaction terms. GMsFEM systematically generates either uncoupled or coupled multiscale basis, via establishing local snapshots and spectral decomposition in the snapshot space. Then the global problem is solved in the constructed multiscale space with a reduced dimensional structure. Convergence analysis of the proposed GMsFEM is accompanied with the numerical results, which support the theoretical results.

DEDICATION

To my wife, parents, parents-in-law, brother and grandmother.

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1. INTRODUCTION

Fluid flow simulation was early known to be based on the concept of porous medium as a single continuum. However, in nature, a porous medium (as stratum or fissured rock) may possess some degree of fracturing. This hence motivated the notion of dual continua, or more generally, multi-continua (see [1], for instance), thanks to mean characteristics (porosity, permeability, pressure, ...) of the media and flow. For example (see [1]), a dual-continuum background can consist of a matrix (first continuum) and a system of naturally connected fractures (second continuum). In such heterogeneous media, the simulation of flow is hard, mainly because of the distinct properties of continua, multiple scales and high contrast and it requires some model reduction techniques. In multi-continuum approach, [1, 2, 3, 4, 5, 6] the equations for each continuum are written separately with some interaction terms (exchange terms) that represent interrelations between the continua.

In many scientific problems involving multi-continuum media, each continuum possesses multiscale feature. Direct numerical simulation in multiscale multi-continuum media is very expensive as it requires a large number of degrees of freedom to completely resolve the fine-scale features. For this reason, some type of upscaling or multiscale method is needed to average or capture the micro-scale effect on the macro-scales. In this work, we study efficient upscaling and model reduction methods for multiscale dual-continuum systems.

When there is a scale separation and the coefficients of the multiscale equations are periodic or locally periodic, the equations can be approximated by the corresponding homogenized equations whose coefficients do not vary rapidly. The theory of homogenization has a long and successful history. We mention only those now classical references Bensoussan et al. [7], Bakhvalov and Panasenko [8] and Jikov et al. [9]. However, for multiscale multi-continuum systems, there has been very little literature. The homogenization of these systems can result in very interesting effective phenomena that are not often seen in homogenization literature.

Homogenization of multiscale multicontinuum systems have not been paid much attention. We contribute in this work the first rigorous results on homogenization of a two scale two continuum system where the interaction between the continua is scaled as $1/\epsilon^2$ where ϵ represents the microscopic scale of the medium. We derive the homogenized problem from the two-scale asymptotic expansion [7, 8, 9]. We show that for this scale of the interaction term, we obtain the same limit for both continua. Other scaling regimes of this term give rise to other limiting behaviours which will be studied in our forthcoming publications. The effective coefficients of the homogenized equation are established via the solutions of cell problems which are systems of equations of a similar form as the two continuum system. Since the two scale coefficients depend on both macro- and micro-scale variables, a different set of cell equations needs to be solved at each macroscopic point. The number of equations to be solved is thus very large. Solving them using the same fine mesh at every macroscopic point is extremely expensive.

One of the main contributions of the dissertation is the development of a hierarchical approach to solve these cell problems to obtain the effective coefficients for the multi-continuum system for a large number of macroscopic points, using an optimal number of degrees of freedom, without sacrificing the accuracy. It solves cell problems for a dense hierarchical network of macroscopic points with different levels of resolution. The problems at those points belonging to a lower level in the hierarchy are solved with a higher level of accuracy. For the solution at a macroscopic point at a higher level in the hierarchy which is obtained with a lower level of accuracy, we use solutions at nearby macroscopic points that are solved with a higher level of accuracy to correct the error. We show that this hierarchical FE approach obtains the same level of accuracy at every macroscopic point as that obtained when every cell problem is solved with the highest level of resolution (we will refer to this as the full reference solve below), but uses only an essentially optimal number of degrees of freedom that is equal to that required to solve only one cell problem at the finest level of resolution (apart from a possible logarithmic factor).

In the second part of the dissertation, we study the homogenization of the two-scale dual-continuum system with the interaction terms scaled as $\mathcal{O}(\frac{1}{\epsilon})$ where ϵ represents the micro-scale. We show that the homogenized equation for this case is very interesting and complicated. The homogenized dual-continuum system consists of convection continuum interacting terms which do not appear in the original two-scale system. Furthermore, the homogenized dual-continuum system has negative interaction coefficients while the interaction coefficients of the original two scale system can have both positive and negative values. The phenomenon of the convection term in the homogenization limit has been discovered before, e.g. in Allaire and Piatnitski [10] when homogenization of a reaction-diffusion equation with a large reaction term is studied. However, in [10], this term is due to the dependence on the ϵ^2 microscopic time scale. In this work, we have a convection term in the homogenized equation where the original multiscale equation is without this microscopic time scale.

We provide a rigorous proof of homogenization convergence. The proof is new and difficult because of the $\frac{1}{\epsilon}$-scale of the interaction terms in our system and the complicated homogenization limit. We also derive a homogenization error under regularity conditions for the solutions to the cell problems and the homogenized equation. Such a homogenization error has never been derived for multiscale multi-continuum systems before.

Although the homogenization technique is effective for simulations in media with scale separation or periodic structure, it is limited to the problems where the media of interest locally has a few important modes. In order to overcome the limits of homogenization technique as well as integrate the heterogeneity of the multicontinua and reduce the computational cost, based on the multiscale finite element method (MsFEM) as in [11, 12], the generalized multiscale finite element method (GMsFEM) was developed ([13]). This method allows one to systematically construct multiple multiscale basis functions, by adding new degrees of freedom (basis functions) in each coarse block. These new basis functions are calculated by constructing the local snapshots and performing local spectral decomposition in the snapshot space. That is, the producing eigenfunctions can convey the local characteristics to the global ones, via the multiscale basis functions in coarse grid.

The generalized multiscale finite element methods (GMsFEM) ([13]) has been successfully applied to a number of multiscale multi-continuum problems. A recent example is about shale gas transport in dual-continuum background consisting of organic and inorganic materials ([14]). In this spirit, a third continuum can be added to dual continua as an extension (see [15], for instance). More generally, flow simulation in heterogeneously varying multicontinua was investigated (see [6, 16, 17], for instance). Additionally, there are various and active studies on new model reduction techniques including constraint energy minimizing GMsFEM (CEM-GMsFEM) ([18, 19, 20]) and related numerical methods for multi-continuum systems ([21]) including non-local multi-continuum method (NLMC) ([22, 23, 24, 25]). These methods construct multiscale basis functions solving well-designed local constrained energy minimization problems. The basis functions of reduced system are related to the solutions on each continuum in each coarse elements and these approaches effectively handle high-contrast as well as multiscale features in multi-continuum media.

Herein, we develop the GMsFEM for an upscaled multiscale multi-continuum system. We consider the special case where the multiple continua occur at many scales. Starting from a microscopic scale, the multi-continua are upscaled via homogenization, to reach an intermediate scale. At this stage, the multicontinua still possess some degree of multiscale. Hence, they are then simulated by the GMsFEM, to arrive at coarse-grid (macroscopic) level. More specifically, being motivated my the homogenized equation derived in the second part of the dissertation, we develop the GMsFEM for a multiscale upscaled dual-continuum system with general convection and interaction terms. The GMsFEM has never been utilized for this type of upscaled dual-continuum equations.

The dissertation is organized as follows. In Chapter 2, we study the homogenization and the hierarchical finite element algorithm for a multiscale dual-continuum system where the interaction between the continua is scaled as $\frac{1}{\epsilon^2}$. We outline the hierarchical finite element algorithm for solving the cell problems at all macroscopic points. We present a rigorous error estimates and the numerical results. We rigorously prove the homogenization convergence. In Chapter 3, We study the homogenization of a multiscale dual-continuum system where the interaction term is scaled as $\frac{1}{\epsilon}$. We perform the two scale asymptotic expansion to derive the homogenized dual-continuum systems. We then state the main results on the convergence of the solution of the multiscale multi-continuum system to the solution of the homogenized multi-continuum system. We derive a corrector and prove a homogenization error estimate. In Chapter 4, We utilize the GMsFEM for an upscaled dual-continuum system derived in Chapter 3. We provide an overview of the uncoupled and coupled GMsFEM. We derive convergence analysis, for both uncoupled and coupled GMsFEM. Then we present the numerical results. Finally, the appendix in the end of the dissertation contains the proofs of the existence and uniqueness of solutions to both the original two scale system and the homogenized equations in Chapter 3.

Throughout the dissertation, by ∇, we denote the gradient with respect to x of a function that depends only on the variable x, or the variables x and t. By ∇_x, we denote the partial gradient with respect to x of a function that depends on x, t and also other variables. Repeated indices indicate summation. The notation $\#$ denotes spaces of periodic functions.

2. HIERARCHICAL MULTISCALE FINITE ELEMENT METHOD FOR MULTI-CONTINUUM MEDIA

* The hierarchical finite element method has been developed to solve the cell problems and compute the effective coefficients for multiscale equations. In [26], the method was developed for the effective coefficients of deterministic two-scale Stokes-Darcy systems in a slowly varying porous medium. In [27], they use the hierarchical algorithm for a two-scale ergodic random homogenization problem without assuming microscopic periodicity. In this chapter, we follow the framework of these papers, but we utilize the hierarchical approach to compute homogenization coefficients for a two-scale dual-continuum system where the interaction terms are scaled as $\mathcal{O}(\frac{1}{\epsilon^2})$. The interaction terms give the interesting cell problems (2.4) in the form of a system of coupled equations.

2.1 Problem formulation

2.1.1 Homogenization of multi-continuum systems

In multi-continuum approaches, equations for each continuum are written separately. We denote by u_i the solution for ith continuum. In the general case where each continuum interacts with every other continuum, we have the following system of equations introduced in [6]

$$C_{ii}^\epsilon(x)\frac{\partial u_i^\epsilon(t,x)}{\partial t} = \text{div}(\kappa_i^\epsilon(x)\nabla u_i^\epsilon(t,x)) + Q_i^\epsilon(u_1^\epsilon(t,x),...,u_N^\epsilon(t,x)) + q_i, \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^d$ is a domain ($d = 2,3$), κ_i^ϵ are the multiscale permeability and C_{ii}^ϵ are the multiscale porosities, q_i are the source terms, and the functions Q_i^ϵ of $(u_1,...,u_N)$ are

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exchange terms (see [1, 2, 3, 4, 5]) that describe the interaction of continua; ϵ represents the microscopic scale of the local variation.

In this chapter, we consider a dual-continuum system. Let Y be the unit cube in $\mathbb{R}^d$. Let $\mathcal{C}_{ii}(x,y)$, $\kappa_i(x,y)$ $(i=1,2)$ be continuous functions on $\Omega \times Y$ which are Y-periodic with respect to y and q be a function in $L^2(\Omega)$. We assume further that there is a constant $c > 0$ such that for all $x \in \Omega, y \in Y$

$$\mathcal{C}_{ii}(x,y) \geq c,\ \kappa_i(x,y) \geq c,\ Q(x,y) \geq c. \tag{2.1}$$

We define the two scale coefficients as

$$\mathcal{C}^\epsilon_{ii}(x) = \mathcal{C}_{ii}(x, \frac{x}{\epsilon}),\ \ \kappa^\epsilon_i(x) = \kappa_i(x, \frac{x}{\epsilon}),\ \ Q^\epsilon(x) = Q(x, \frac{x}{\epsilon}).$$

We consider in this work the case where the interaction terms are scaled as $O(1/\epsilon^2)$; this case has the most interesting cell problems in the form of a coupled system. We consider the multiscale dual-continuum system

$$\begin{aligned}
\mathcal{C}^\epsilon_{11}(x)\frac{\partial u^\epsilon_1(t,x)}{\partial t} &= \text{div}(\kappa^\epsilon_1(x)\nabla u^\epsilon_1(t,x)) + \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_2(t,x) - u^\epsilon_1(t,x)) + q,\\
\mathcal{C}^\epsilon_{22}(x)\frac{\partial u^\epsilon_2(t,x)}{\partial t} &= \text{div}(\kappa^\epsilon_2(x)\nabla u^\epsilon_2(t,x)) + \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_1(t,x) - u^\epsilon_2(t,x)) + q,
\end{aligned} \tag{2.2}$$

with the Dirichlet boundary condition $u^\epsilon_1(t,x) = u^\epsilon_2(t,x) = 0$ for $x \in \partial\Omega$, and with the initial condition $u^\epsilon_1(0,x) = g_1$, $u^\epsilon_2(0,x) = g_2$ where g_1 and g_2 are in $L^2(\Omega)$. We consider the following two-scale asymptotic expansion of u^ϵ_1 and u^ϵ_2.

$$\begin{aligned}
u^\epsilon_1(t,x) &= u_{10}(t,x,\frac{x}{\epsilon}) + \epsilon u_{11}(t,x,\frac{x}{\epsilon}) + \cdots\ ,\\
u^\epsilon_2(t,x) &= u_{20}(t,x,\frac{x}{\epsilon}) + \epsilon u_{21}(t,x,\frac{x}{\epsilon}) + \cdots\ ,
\end{aligned}$$

where the functions $u_{1i}(t,x,y)$ and $u_{2i}(t,x,y)$ are periodic with respect to y. Performing the two-scale asymptotic expansion, from (2.2) we obtain

$$
\begin{aligned}
\mathcal{C}_{11}\frac{\partial(u_{10}+\epsilon u_{11}+\cdots)}{\partial t} &= (\operatorname{div}_x+\frac{1}{\epsilon}\operatorname{div}_y)(\kappa_1(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{10}+\epsilon u_{11}+\cdots)) \\
&\quad+\frac{1}{\epsilon^2}Q(u_{20}+\epsilon u_{21}-u_{10}-\epsilon u_{11}+\cdots)+q, \\
\mathcal{C}_{22}\frac{\partial(u_{20}+\epsilon u_{21}+\cdots)}{\partial t} &= (\operatorname{div}_x+\frac{1}{\epsilon}\operatorname{div}_y)(\kappa_2(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{20}+\epsilon u_{21}+\cdots)) \\
&\quad+\frac{1}{\epsilon^2}Q(u_{10}+\epsilon u_{11}-u_{20}-\epsilon u_{21}+\cdots)+q,
\end{aligned}
\tag{2.3}
$$

For the $O(\epsilon^{-2})$ terms, we obtain,

$$
\begin{aligned}
\operatorname{div}_y(\kappa_1(x,y)\nabla_y u_{10}(t,x,y))+Q(x,y)(u_{20}(t,x,y)-u_{10}(t,x,y))=0 \\
\operatorname{div}_y(\kappa_2(x,y)\nabla_y u_{20}(t,x,y))+Q(x,y)(u_{10}(t,x,y)-u_{20}(t,x,y))=0.
\end{aligned}
$$

From this, we have

$$
\begin{aligned}
-\int_Y \kappa_1\nabla_y u_{10}\cdot\nabla_y u_{10}dy+\int_Y Q(u_{20}-u_{10})u_{10}dy=0 \\
-\int_Y \kappa_2\nabla_y u_{20}\cdot\nabla_y u_{20}dy+\int_Y Q(u_{10}-u_{20})u_{20}dy=0
\end{aligned}
$$

Adding these two equations, we obtain

$$
\int_Y \kappa_1\nabla_y u_{10}\cdot\nabla_y u_{10}dy+\int_Y \kappa_2\nabla_y u_{20}\cdot\nabla_y u_{20}dy+\int_Y Q(u_{20}-u_{10})^2dy=0.
$$

This implies $\nabla_y u_{10}=0$, $\nabla_y u_{20}=0$. i.e. u_{10} and u_{20} are independent of y, and $u_{10}(t,x)=u_{20}(t,x)=u_0(t,x)$ as $Q(x,y)>c>0\ \forall\, x\in\Omega, y\in Y$. For the $O(\epsilon^{-1})$ terms in (2.3), we have,

$$\mathrm{div}_x(\kappa_1\nabla_y u_{10}) + \mathrm{div}_y(\kappa_1\nabla u_{10}) + \mathrm{div}_y(\kappa_1\nabla_y u_{11}) + Q(u_{21} - u_{11}) = 0$$
$$\mathrm{div}_x(\kappa_2\nabla_y u_{20}) + \mathrm{div}_y(\kappa_2\nabla u_{20}) + \mathrm{div}_y(\kappa_2\nabla_y u_{21}) + Q(u_{11} - u_{21}) = 0.$$

Since u_{10} and u_{20} are independent of y, we have

$$\mathrm{div}_y(\kappa_1\nabla_y u_{11}) + Q(u_{21} - u_{11}) = -\,\mathrm{div}_y(\kappa_1\nabla u_0)$$
$$\mathrm{div}_y(\kappa_2\nabla_y u_{21}) + Q(u_{11} - u_{21}) = -\,\mathrm{div}_y(\kappa_2\nabla u_0)$$

Thus $u_{11} = \frac{\partial u_0}{\partial x_i} N_1^i$ and $u_{21} = \frac{\partial u_0}{\partial x_i} N_2^i$ where $N_1^i(x,\cdot) \in H^1_\#(Y)/\mathbb{R}$, and $N_2^i(x,\cdot) \in H^1_\#(Y)/\mathbb{R}$ are solutions of the cell problem

$$\begin{aligned} \mathrm{div}_y(\kappa_1(x,y)(e^i + \nabla_y N_1^i)) + Q(x,y)(N_2^i - N_1^i) = 0 \\ \mathrm{div}_y(\kappa_2(x,y)(e^i + \nabla_y N_2^i)) + Q(x,y)(N_1^i - N_2^i) = 0, \end{aligned} \tag{2.4}$$

where e^i is the ith unit vector in the standard basis of $\mathbb{R}^d$. For the $O(\epsilon^0)$ terms in (2.3), integrating over Y, one has

$$\begin{aligned}
&\int_Y \mathcal{C}_{11}\frac{\partial u_0}{\partial t}dy \\
&= \int_Y \mathrm{div}_x(\kappa_1\nabla u_0)dy + \int_Y \mathrm{div}_x(\kappa_1\nabla_y u_{11})dy + \int_Y Q(u_{22} - u_{12})dy + \int_Y q dy, \\
&\int_Y \mathcal{C}_{22}\frac{\partial u_0}{\partial t}dy \\
&= \int_Y \mathrm{div}_x(\kappa_2\nabla u_0)dy + \int_Y \mathrm{div}_x(\kappa_2\nabla_y u_{21})dy + \int_Y Q(u_{12} - u_{22})dy + \int_Y q dy.
\end{aligned}$$

Adding these two equations, one obtains the homogenized equation

$$\left(\int_Y \mathcal{C}_{11}dy + \int_Y \mathcal{C}_{22}dy\right)\frac{\partial u_0}{\partial t} = \mathrm{div}(\kappa_1^*\nabla u_0) + \mathrm{div}(\kappa_2^*\nabla u_0) + \int_Y 2q dy \qquad \text{in } \Omega \tag{2.5}$$

where the x-dependent permeabilities are defined as

$$\kappa_{1ij}^*(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i})dy, \;\; \kappa_{2ij}^*(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i})dy \tag{2.6}$$

We will show later that the matrix $\kappa_{1ij}^*(x) + \kappa_{2ij}^*(x)$ is symmetric and positive definite. We will also show that the initial condition for u_0 is

$$u_0(0,x) = \frac{\langle C_{11}\rangle g_1(x) + \langle C_{22}\rangle g_2(x)}{\langle C_{11}\rangle + \langle C_{22}\rangle} \tag{2.7}$$

where $\langle C_{ii}\rangle = \int_Y C_{ii}(y)dy$ for $i = 1, 2$. Equation (2.5) together with initial condtion (2.7) has a unique solution (see, e.g., [28]).

2.1.2 Uniqueness of solution to the cell problem

We write the system (2.4) in the variational form as

$$\begin{aligned}
&\int_Y \kappa_1(x,y)\nabla_y N_1^i(x,y)\cdot\nabla_y\phi_1(y)dy - \int_Y Q(x,y)(N_2^i - N_1^i)\phi_1(y)dy \\
&\qquad\qquad\qquad\qquad = -\int_Y \kappa_1(x,y)e^i\cdot\nabla_y\phi_1(y)dy \\
&\int_Y \kappa_2(x,y)\nabla_y N_2^i(x,y)\cdot\nabla_y\phi_2(y)dy - \int_Y Q(x,y)(N_1^i - N_2^i)\phi_2(y)dy \\
&\qquad\qquad\qquad\qquad = -\int_Y \kappa_2(x,y)e^i\cdot\nabla_y\phi_2(y)dy
\end{aligned} \tag{2.8}$$

where $\phi_1, \phi_2 \in H^1_\#(Y)$. Let W be the space $H^1_\#(Y) \times H^1_\#(Y)/(c,c), c \in \mathbb{R}$. The space W is equipped with the norm

$$|||(\phi_1,\phi_2)||| = \|\nabla_y\phi_1\|_{L^2(Y)} + \|\nabla_y\phi_2\|_{L^2(Y)} + \|\phi_1 - \phi_2\|_{L^2(Y)}.$$

For $x \in \Omega$, we define the bilinear form $B(x;\cdot,\cdot) : W \times W \to \mathbb{R}$ as

$$\begin{aligned}
&B(x;(\phi_1,\phi_2),(\psi_1,\psi_2))\\
&= \int_Y \kappa_1(x,y)\nabla_y\phi_1(y)\cdot\nabla_y\psi_1(y)dy + \int_Y \kappa_2(x,y)\nabla_y\phi_2(y)\cdot\nabla_y\psi_2(y)dy\\
&+ \int_Y Q(x,y)(\phi_1(x,y)-\phi_2(x,y))(\psi_1(x,y)-\psi_2(x,y))dy
\end{aligned}$$

for $(\phi_1,\phi_2) \in W$ and $(\psi_1,\psi_2) \in W$. From (2.1), we deduce that the bilinear form B is uniformly coercive and bounded with respect to $x \in \Omega$, i.e. there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}
B(x;(\phi_1,\phi_2),(\phi_1,\phi_2)) &\geq c_1|||(\phi_1,\phi_2)|||^2,\\
B(x;(\phi_1,\phi_2),(\psi_1,\psi_2)) &\leq c_2|||(\phi_1,\phi_2)|||\cdot|||(\psi_1,\psi_2)|||,
\end{aligned} \tag{2.9}$$

for all $(\phi_1,\phi_2) \in W$ and $(\psi_1,\psi_2) \in W$. Adding the two equations in (2.8), we obtain

$$B(x;(N_1^i,N_2^i),(\phi_1,\phi_2)) = -\int_Y \kappa_1(x,y)e^i\cdot\nabla_y\phi_1(y)dy - \int_Y \kappa_2(x,y)e^i\cdot\nabla_y\phi_2(y)dy.$$

Theorem 2.1.1. *Problem (2.8) has a unique solution* $(N_1^i,N_2^i) \in W$.

Proof. The conclusion follows from the boundedness and coerciveness of the bilinear form B and the Lax-Milgram lemma. □

2.2 Hierarchical finite element algorithm

Computing effective coefficients $\kappa_i^*(x)$ requires the solutions of the cell problems (2.4) at many macroscopic points which can be very expensive. We develop in this section the hierarchical FE method which computes the solution of the cell problems at a dense network of macroscopic points using only an essentially optimal number of degrees of freedom which is equal to that for solving one cell problem (apart from a multiplying logarithmic factor). We assume that the coefficients are sufficiently smooth with respect to the macroscopic variable x. We make the following assumption.

Assumption 2.2.1. *There is a constant $C > 0$ such that for all x, $x' \in \Omega$,*

$$\|\kappa_1(x,\cdot) - \kappa_1(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|,\ \|\kappa_2(x,\cdot) - \kappa_2(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|,$$
$$\textit{and } \|Q(x,\cdot) - Q(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|.$$

Remark. The main necessary condition for our proposed method to work is that the two scale coefficients possess Lipschitz (or Holder) smoothness with respect to the macroscopic variable. However, this assumption is reasonable as the macroscopic properties of the media normally vary smoothly.

2.2.1 Overview of hierarchical algorithm

We develop an efficient hierarchical finite element algorithm to solve the coupled cell problem (2.4) numerically and to approximate the effective properties $\kappa_i^*(x)$ in (2.6) for a dense network of macroscopic points $x \in \Omega$. We follow the algorithm introduced in [26].

We outline the algorithm as follows.

Step 1 : Build nested finite element spaces. We employ Galerkin FE to obtain an approximation of the solution $(N_1^i, N_2^i) \in W$ of (2.4) for each macroscopic point $x \in \Omega$ using FE spaces of different levels of resolution. We assume that there exists a hierarchy of FE spaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_L \subset H^1_\#(Y)$ where the integer index L denotes the resolution level. We assume further the following approximation properties: for $w \in H^2_\#(Y)$,

$$\inf_{\phi \in \mathcal{V}_{L-l}} \|\nabla_y (w - \phi)\|_{L^2(Y)} + 2^{L-l}\|w - \phi\|_{L^2(Y)} \le C 2^{-L+l}\|w\|_{H^2(Y)}, \tag{2.10}$$

where the constant C is independent of L and l.

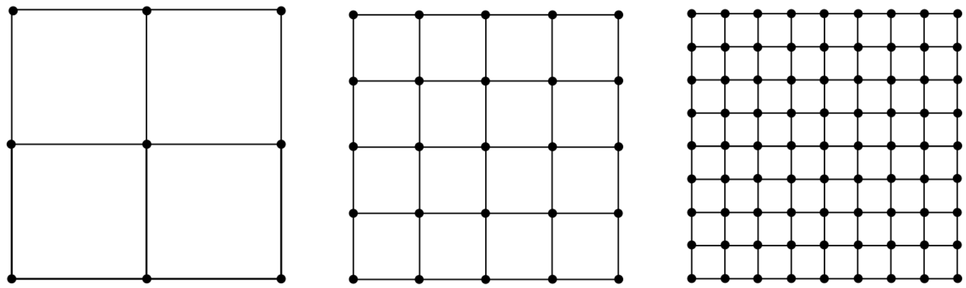
Step 2 : Build a hierarchy of macrogrids. We solve the cell equations at different macroscopic points $x \in \Omega$ with different levels of accuracy. We use the solutions solved with a higher accuracy level to correct the solutions obtained with a lower accuracy level. We achieve this by solving the cell problems at different macroscopic points using different FE spaces in the hierarchy in Step 1. This can be done by constructing a hierarchy of macro-grid points. We construct a nested macro-grid, $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_L \subset \Omega$ as follows. First, we build an initial grid $\mathcal{T}_0$ with a proper grid spacing H, the maximal distance between neighboring nodes. We then inductively construct $\mathcal{T}_l$, a refinement of $\mathcal{T}_{l-1}$, with grid spacing $H2^{-l}$. Then, we define the hierarchy of macro-grids, $\{S_0, S_1, \cdots, S_L\}$ as $S_0 = \mathcal{T}_0$, $S_1 = \mathcal{T}_1 \backslash S_0$, and for each $l > 1$, we have

$$S_l = \mathcal{T}_l \Big\backslash \Big(\bigcup_{k<l} S_k \Big).$$

We call the nodes in the lowest level grid S_0 the anchor points. In this way, we obtain a dense hierarchy of the macro-grids. That is, each point $x \in S_l$ has at least one point from one of the previous levels, $x' \in \bigcup_{k<l} S_k$ such that dist$(x, x') < O(H2^{-l})$. Figures 2.1 and 2.2 show an example of 3-level hierarchy of macrogrids $\mathcal{T}_l$, S_l, $l = 1, 2, 3$, constructed in $\Omega = [0, 1]^2$.

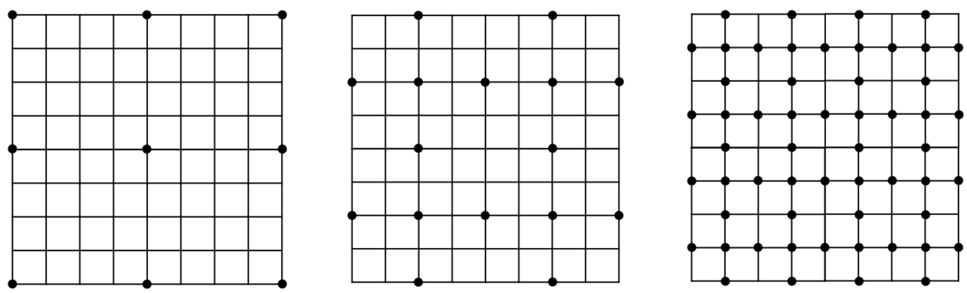
Step 3 : Calculating the correction term. We relate the nested FE spaces and the hierarchy of macrogrids for our algorithm. We first solve the cell problems at anchor points using the standard Galerkin FE with FE space $\mathcal{V}_L$. That is, for the points in the coarsest macro-grid S_0, we solve the cell problems with the finest mesh. More precisely, we find $\bar{N}_1^i(x, \cdot), \bar{N}_2^i(x, \cdot) \in \mathcal{V}_L$, such that

$$B(x; (\bar{N}_1^i, \bar{N}_2^i), (\phi_1, \phi_2)) = -\int_Y \kappa_1(x, y) e^i \cdot \nabla_y \phi_1(y) dy - \int_Y \kappa_2(x, y) e^i \cdot \nabla_y \phi_2(y) dy$$



(a) $\mathcal{T}_0$ (b) $\mathcal{T}_1$ (c) $\mathcal{T}_2$

Figure 2.1: 3-level nested macrogrids



(a) S_0 (b) S_1 (c) S_2

Figure 2.2: 3-level hierarchy of macrogrids

for all $\phi_1, \phi_2 \in \mathcal{V}_L$. Proceeding inductively, for $x \in S_l$ ($l = 1, \cdots, L$), we choose the points $\{x_1, x_2, \cdots, x_n\} \in (\bigcup_{l'<l} S_{l'})$ so that the distance between x and each point in $\{x_1, x_2, \cdots, x_n\}$ is $O(H2^{-l})$. This is possible from the assumption for the hierarchy of macroscopic points above. We define the l-th macro-grid interpolation by

$$I_l^x(N_k^i) = \sum_{j=1}^{n} c_j N_k^i(x_j, \cdot),$$

where the coefficients c_j satisfy $\sum_{j=1}^n c_j = 1$ ($k = 1, 2$). We refer to the l-th macro-grid interpolation of Galerkin approximations as $I_l^x(\bar{N}_k^i) = \sum_{j=1}^n c_j \bar{N}_k^i(x_j, \cdot)$. We solve the following problem: Find $\bar{N_1^i}^c(x,\cdot), \bar{N_2^i}^c(x,\cdot) \in \mathcal{V}_{L-l}$ such as

$$
\begin{aligned}
&B(x; (\bar{N_1^i}^c, \bar{N_2^i}^c), (\phi_1, \phi_2)) \\
&= -\sum_{j=1}^{n} c_j \int_Y (\kappa_1(x,y) - \kappa_1(x_j,y)) \nabla_y \bar{N_1^i}(x_j,y) \cdot \nabla_y \phi_1(y) dy \\
&- \sum_{j=1}^{n} c_j \int_Y (\kappa_1(x,y) - \kappa_1(x_j,y)) e^i \cdot \nabla_y \phi_1(y) dy \\
&- \sum_{j=1}^{n} c_j \int_Y (\kappa_2(x,y) - \kappa_2(x_j,y)) \nabla_y \bar{N_2^i}(x_j,y) \cdot \nabla_y \phi_2(y) dy \\
&- \sum_{j=1}^{n} c_j \int_Y (\kappa_2(x,y) - \kappa_2(x_j,y)) e^i \cdot \nabla_y \phi_2(y) dy \\
&+ \sum_{j=1}^{n} c_j \int_Y (Q(x_j,y) - Q(x,y))(\bar{N_1^i}(x_j,y) - \bar{N_2^i}(x_j,y))(\phi_1(y) - \phi_2(y)) dy,
\end{aligned}
\tag{2.11}
$$

for all $\phi_1,\ \phi_2 \in \mathcal{V}_{L-l}$. Note that right-hand side data is all known since we have already computed $\{\bar{N_k^i}(x_j,\cdot)\}_{j=1}^n$ inductively using finer mesh spaces at macro-grid points in $(\bigcup_{l'<l} S_{l'})$. We let

$$
\bar{N_k^i}(x,\cdot) = \bar{N_k^i}^c(x,\cdot) + I_l^x(\bar{N_k^i}), \tag{2.12}
$$

be the FE approximation for $N_k^i(x,\cdot)$ where $k = 1, 2$. A main goal of this chapter is to prove that the approximation (2.12) for $N_k^i(x,\cdot)$ has the same order of accuracy compared to the approximation we obtain by solving (2.8) using the finest FE space $\mathcal{V}_L$ at all macroscopic points. We also prove that we reduce the computation cost with the approximation (2.12) to the optimal level.

Remark. In the following, for simplicity, we use a simple 1-point interpolation to compute the correction term $(\bar{N_1^i}^c, \bar{N_2^i}^c)$. More precisely, for $x \in S_l$ we choose $x' \in (\bigcup_{l'<l} S_{l'})$

such that $\mathrm{dist}(x, x') < O(H2^{-l})$. We let

$$I_l^x(\bar{N}_k^i) = \bar{N}_k^i(x', \cdot), \ \ k = 1, 2$$

be the macro-grid interpolation. The FE approximation is

$$\bar{N}_k^i(x, \cdot) = \bar{N_k^i}^c(x, \cdot) + \bar{N}_k^i(x', \cdot), \ \ k = 1, 2.$$

Remark. Note that as the level l goes higher, we use coarser FE spaces for the corresponding finer macro grids. This balance guarantees that although we use coarser FE spaces, the FE error is still optimal, but with much less computation cost.

2.2.2 Error estimates

We require that the coefficients κ_i and Q satisfy Assumption 2.2.1 and (2.1). We prove that the hierarchical method achieves the same order of accuracy as the full solve. For simplicity, we consider 1-point interpolation for our proof; the proof for the general case is similar.

Lemma 2.2.1. *There exists a positive number C such that $|||(N_1^i(x, \cdot), N_2^i(x, \cdot))||| \leq C$ for all $x \in \Omega$.*

Proof. From (2.8), we obtain

$$\begin{aligned} &B(x; (N_1^i(x, \cdot), N_2^i(x, \cdot)), (N_1^i(x, \cdot), N_2^i(x, \cdot))) \\ &= -\int_Y \kappa_1(x, y) e^i \cdot \nabla_y N_1^i(x, y) dy - \int_Y \kappa_2(x, y) e^i \cdot \nabla_y N_2^i(x, y) dy. \end{aligned}$$

Using the uniform coercivity of the bilinear form $B(x;\cdot,\cdot)$ with respect to x, we get

$$C|||(N_1^i(x,\cdot),N_2^i(x,\cdot))||| \leq (||\nabla_y N_1^i(x,\cdot)||_{L^2(Y)} + ||\nabla_y N_2^i(x,\cdot)||_{L^2(Y)})$$

for $C > 0$. From this we get the conclusion. □

Let $N_k^{ic}(x,\cdot) = N_k^i(x,\cdot) - N_k^i(x',\cdot)$. We have that $(N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot)) \in W$ satisfies

$$\begin{aligned}
&B(x;(N_1^{ic},N_2^{ic}),(\phi_1,\phi_2)) \\
&= -\int_Y (\kappa_1(x,y)-\kappa_1(x',y))\nabla_y N_1^i(x',y)\cdot\nabla_y\phi_1(y)dy \qquad (2.13)\\
&-\int_Y (\kappa_1(x,y)-\kappa_1(x',y))e^i\cdot\nabla_y\phi_1(y)dy \\
&-\int_Y (\kappa_2(x,y)-\kappa_2(x',y))\nabla_y N_2^i(x',y)\cdot\nabla_y\phi_2(y)dy \qquad (2.14)\\
&-\int_Y (\kappa_2(x,y)-\kappa_2(x',y))e^i\cdot\nabla_y\phi_2(y)dy \\
&+\int_Y (Q(x',y)-Q(x,y))(N_1^i(x',y)-N_2^i(x',y))(\phi_1(y)-\phi_2(y))dy \qquad (2.15)
\end{aligned}$$

$\forall\,(\phi_1,\phi_2) \in W$.

Proposition 2.2.2. *There exists $C > 0$ such that*

$$|||(N_1^{ic}(x,\cdot),N_2^{ic}(x,\cdot)))||| \leq C|x-x'| \qquad (2.16)$$

for $x \in \mathcal{T}_L$.

Proof. From (2.15), for $(\phi_1, \phi_2) = (N_1^{ic}(x, \cdot), N_2^{ic}(x, \cdot)) \in W$ we have

$$
\begin{aligned}
&B(x; (N_1^{ic}, N_2^{ic}), (N_1^{ic}, N_2^{ic})) \\
&= -\int_Y (\kappa_1(x, y) - \kappa_1(x', y))\nabla_y N_1^i(x', y) \cdot \nabla_y N_1^{ic}(x, y)dy \\
&- \int_Y (\kappa_1(x, y) - \kappa_1(x', y)e^i \cdot \nabla_y N_1^{ic}(x, y)dy \\
&- \int_Y (\kappa_2(x, y) - \kappa_2(x', y))\nabla_y N_2^i(x', y) \cdot \nabla_y N_2^{ic}(x, y)dy \\
&- \int_Y (\kappa_2(x, y) - \kappa_2(x', y))e^i \cdot \nabla_y N_2^{ic}(x, y)dy \\
&+ \int_Y (Q(x', y) - Q(x, y))(N_1^i(x', y) - N_2^i(x', y))(N_1^{ic}(x, y) - N_2^{ic}(x, y))dy.
\end{aligned}
$$

As $\nabla_y N_1^i(x', \cdot)$ and $\nabla_y N_2^i(x', \cdot)$ are uniformly bounded in $L^2(Y)$ with respect to $x \in \Omega$ by Lemma 2.2.1. From Assumption 2.2.1we have

$$
\begin{aligned}
&|||(N_1^{ic}(x, \cdot), N_2^{ic}(x, \cdot))|||^2 \\
&\leq C|x - x'|(||\nabla_y N_1^{ic}(x, \cdot)||_{L^2(Y)} + ||\nabla_y N_2^{ic}(x, \cdot)||_{L^2(Y)} + ||N_2^{ic}(x, \cdot) - N_1^{ic}(x, \cdot)||_{L^2(Y)}).
\end{aligned}
$$

Thus

$$
|||(N_1^{ic}(x, \cdot), N_2^{ic}(x, \cdot))||| \leq C|x - x'| \tag{2.17}
$$

where the constant C is independent of x. □

Lemma 2.2.3. *There is a positive constant C such that*

$$
||\Delta_y N_1^i(x, \cdot)||_{L^2(Y)} + ||\Delta_y N_2^i(x, \cdot)||_{L^2(Y)} \leq C \tag{2.18}
$$

for all $x \in \Omega$.

Proof. We rewrite cell problem (2.4) as

$$\kappa_1 \Delta_y N_1^i + \nabla_y \kappa_1 \cdot \nabla_y N_1^i + \mathrm{div}_y(\kappa_1 e^i) + Q(x,y)(N_2^i - N_1^i) = 0$$

$$\kappa_2 \Delta_y N_2^i + \nabla_y \kappa_2 \cdot \nabla_y N_2^i + \mathrm{div}_y(\kappa_2 e^i) + Q(x,y)(N_1^i - N_2^i) = 0.$$

Rearranging these equations, we have,

$$\Delta_y N_1^i = -\frac{1}{\kappa_1}(\nabla_y \kappa_1 \cdot \nabla_y N_1^i + \mathrm{div}_y(\kappa_1 e^i) + Q(x,y)(N_2^i - N_1^i))$$

$$\Delta_y N_2^i = -\frac{1}{\kappa_2}(\nabla_y \kappa_2 \cdot \nabla_y N_2^i + \mathrm{div}_y(\kappa_2 e^i) + Q(x,y)(N_1^i - N_2^i)).$$

By the uniform boundedness of $|||(N_1^i(x,\cdot), N_2^i(x,\cdot))|||$ with respect to x and Lemma 2.2.1, we deduce that $||\Delta_y N_1^i(x,\cdot)||_{L^2(Y)}$ and $||\Delta_y N_2^i(x,\cdot)||_{L^2(Y)}$ are uniformly bounded for all $x \in \Omega$. □

Lemma 2.2.4. *There exists a positive constant C such that*

$$||\Delta_y N_1^{i^c}(x,\cdot)||_{L^2(Y)} \leq C|x - x'|, \;\; ||\Delta_y N_2^{i^c}(x,\cdot)||_{L^2(Y)} < C|x - x'|$$

for all $x \in \mathcal{T}_L$.

Proof. From (2.15), we have

$$\begin{aligned}
&\kappa_1(x,y)\Delta_y N_1^{i^c}(x,y) + \nabla_y \kappa_1(x,y) \cdot \nabla_y N_1^{i^c}(x,y) = -Q(x,y)(N_2^{i^c}(x,y) - N_1^{i^c}(x,y)) \\
&- \nabla_y(\kappa_1(x,y) - \kappa_1(x',y)) \cdot \nabla_y N_1^i(x',y) - (\kappa_1(x,y) - \kappa_1(x',y))\Delta_y N_1^i(x',y) \\
&- \mathrm{div}_y(\kappa_1(x,y) - \kappa_1(x',y)e^i) + (Q(x',y) - Q(x,y))(N_2^i(x',y) - N_1^i(x',y)),
\end{aligned}$$

$$
\begin{aligned}
&\kappa_2(x,y)\Delta_y N_2^{ic}(x,y) + \nabla_y \kappa_2(x,y)\cdot\nabla_y N_2^{ic}(x,y) = -Q(x,y)(N_1^{ic}(x,y) - N_2^{ic}(x,y)) \\
&- \nabla_y(\kappa_2(x,y) - \kappa_2(x',y))\cdot\nabla_y N_2^i(x',y) - (\kappa_2(x,y) - \kappa_2(x',y))\Delta_y N_2^i(x',y) \\
&- \mathrm{div}_y(\kappa_2(x,y) - \kappa_2(x',y)e^i) + (Q(x',y) - Q(x,y))(N_1^i(x',y) - N_2^i(x',y)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta_y N_1^{ic}(x,y) = \frac{1}{\kappa_1}\{&-\nabla_y\kappa_1(x,y)\cdot\nabla_y N_1^{ic}(x,y) - Q(x,y)(N_2^{ic}(x,y) - N_1^{ic}(x,y)) \\
&- \nabla_y(\kappa_1(x,y) - \kappa_1(x',y))\cdot\nabla_y N_1^i(x',y) - (\kappa_1(x,y) - \kappa_1(x',y))\Delta_y N_1^i(x',y) \\
&- \mathrm{div}_y(\kappa_1(x,y) - \kappa_1(x',y)e^i) + (Q(x',y) - Q(x,y))(N_2^i(x',y) - N_1^i(x',y))\},
\end{aligned}
$$

$$
\begin{aligned}
\Delta_y N_2^{ic}(x,y) = \frac{1}{\kappa_2}\{&-\nabla_y\kappa_2(x,y)\cdot\nabla_y N_2^{ic}(x,y) - Q(x,y)(N_1^{ic}(x,y) - N_2^{ic}(x,y)) \\
&- \nabla_y(\kappa_2(x,y) - \kappa_2(x',y))\cdot\nabla_y N_2^i(x',y) - (\kappa_2(x,y) - \kappa_2(x',y))\Delta_y N_2^i(x',y) \\
&- \mathrm{div}_y(\kappa_2(x,y) - \kappa_2(x',y)e^i) + (Q(x',y) - Q(x,y))(N_1^i(x',y) - N_2^i(x',y))\}.
\end{aligned}
$$

From Lemma 2.2.1 and Proposition 2.2.2, we have

$$
||\Delta_y N_1^{ic}(x,\cdot)||_{L^2(Y)}, ||\Delta_y N_2^{ic}(x,\cdot)||_{L^2(Y)} \le C|x - x'|.
$$

for some constant $C > 0$. □

We choose $(N_1^{ic}, N_2^{ic}) \in W$ such that

$$
\int_Y (N_1^{ic} + N_2^{ic})dy = 0.
$$

We then have

Lemma 2.2.5. *There is a positive constant C such that $||N_1^{ic}(x,\cdot)||_{L^2(Y)} \le C|x - x'|$ and $||N_2^{ic}(x,\cdot)||_{L^2(Y)} \le C|x - x'|$ for all $x \in \mathcal{T}_L$.*

Proof. We note that

$$2(||N_1^{ic}||^2_{L^2(Y)} + ||N_2^{ic}||^2_{L^2(Y)}) = ||N_1^{ic} + N_2^{ic}||^2_{L^2(Y)} + ||N_1^{ic} - N_2^{ic}||^2_{L^2(Y)}. \tag{2.19}$$

Since $\int_Y (N_1^{ic} + N_2^{ic}) dy = 0$, by Poincare inequality, and (2.17), the following inequalities hold.

$$||N_1^{ic} + N_2^{ic}||_{L^2(Y)} \leq C||\nabla_y(N_1^{ic} + N_2^{ic})||_{L^2(Y)} \leq C(||\nabla_y N_1^{ic}||_{L^2(Y)} + ||\nabla_y N_2^{ic}||_{L^2(Y)})$$
$$\leq C|x - x'|$$

And then by (2.19),

$$2(||N_1^{ic}||^2_{L^2(Y)} + ||N_2^{ic}||^2_{L^2(Y)}) \leq C|x - x'|^2.$$

□

Proposition 2.2.6. *There is a constant $C > 0$ such that $||N_1^{ic}||_{H^2(Y)} \leq C|x - x'|$ and $||N_2^{ic}||_{H^2(Y)} \leq C|x - x'|$ for all $x \in \mathcal{T}_L$.*

Proof. Let $\omega \subset \mathbb{R}^d$ be a domain such that $Y \subset \omega$. Let $\phi \in \mathcal{C}_0^\infty(\omega)$ be such that $\phi = 1$ in Y. We have

$$\Delta_y(\phi N_1^{ic}) = \Delta_y \phi N_1^{ic} + 2\nabla\phi \cdot \nabla N_1^{ic} + \phi \Delta_y N_1^{ic}.$$

Since $\phi N_1^{ic} = 0$ on $\partial\omega$, applying elliptic regularity, we have

$$||N_1^{ic}||_{H^2(Y)} \leq ||\phi N_1^{ic}||_{H^2(\omega)} \leq ||\Delta_y \phi N_1^{ic} + 2\nabla_y \phi \cdot \nabla_y N_1^{ic} + \phi \Delta_y N_1^{ic}||_{L^2(\omega)}. \tag{2.20}$$

By Proposition 2.2.2, Lemmas 2.2.4 and 2.2.5, and the Y-periodicity of N_1^{ic},

$$||N_1^{ic}(x,\cdot)||_{L^2(\omega)} \le C|x-x'|, ||\nabla_y N_1^{ic}(x,\cdot)||_{L^2(\omega)} \le C|x-x'|, ||\Delta_y N_1^{ic}(x,\cdot)||_{L^2(\omega)} \le C|x-x'|$$

for all $x \in \mathcal{T}_L$. Then from (2.20), $||N_1^{ic}||_{H^2(Y)} \le C|x-x'|$. Similarly, $||N_2^{ic}||_{H^2(Y)} \le C|x-x'|$ for $C > 0$. □

We consider the problem: Find $\bar{\bar{N}}_1^{ic}(x,y) \in \mathcal{V}_{L-l}$ and $\bar{\bar{N}}_2^{ic}(x,y) \in \mathcal{V}_{L-l}$ such that

$$\begin{aligned}
&B(x;(\bar{\bar{N}}_1^{ic},\bar{\bar{N}}_2^{ic}),(\phi_1,\phi_2)) \\
&= -\int_Y (\kappa_1(x,y)-\kappa_1(x',y))\nabla_y N_1^i(x',y)\cdot\nabla_y\phi_1(y)dy \qquad (2.21)\\
&-\int_Y (\kappa_1(x,y)-\kappa_1(x',y))e^i\cdot\nabla_y\phi_1(y)dy \\
&-\int_Y (\kappa_2(x,y)-\kappa_2(x',y))\nabla_y N_2^i(x',y)\cdot\nabla_y\phi_2(y)dy \qquad (2.22)\\
&-\int_Y (\kappa_2(x,y)-\kappa_2(x',y))e^i\cdot\nabla_y\phi_2(y)dy \\
&+\int_Y (Q(x',y)-Q(x,y))(N_1^i(x',y)-N_2^i(x',y))(\phi_1(y)-\phi_2(y))dy, \qquad (2.23)
\end{aligned}$$

for all $\phi_1 \in \mathcal{V}_{L-l}$ and $\phi_2 \in \mathcal{V}_{L-l}$. This is the FE approximation of (2.15). We then have the following result.

Lemma 2.2.7. *There is a positive constant C^0 such that*

$$|||(N_1^{ic}(x,\cdot)-\bar{\bar{N}}_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot)-\bar{\bar{N}}_2^{ic}(x,\cdot))||| \le C^0 2^{-L}.$$

Proof. It follows from Cea's Lemma, Proposition 2.2.6 and (2.10) that

$$|||(N_1^{i^c} - \bar{\bar{N}}_1^{i^c}, N_2^{i^c} - \bar{\bar{N}}_2^{i^c})||| \le C2^{-(L-l)}(\|N_1^{i^c}\|_{H^2(Y)} + \|N_2^{i^c}\|_{H^2(Y)})$$
$$\le C2^{-(L-l)}|x - x'| \le C^0 2^{-L}.$$

□

Proposition 2.2.8. *There is a constant $c_l > 0$ which only depends on the level S_l of $x \in \mathcal{T}_L$ such that*

$$|||(\bar{N}_1^i(x,\cdot) - N_1^i(x,\cdot), \bar{N}_2^i(x,\cdot) - N_2^i(x,\cdot))||| \le c_l 2^{-L}.$$

Proof. We will prove the proposition by induction. The conclusion holds for $l = 0$. We assume that for all $x' \in S_{l'}$ where $l' \le l - 1$.

$$|||(\bar{N}_1^i(x',\cdot) - N_1^i(x',\cdot), \bar{N}_2^i(x',\cdot) - N_2^i(x',\cdot))||| \le c_{l-1} 2^{-L}. \tag{2.24}$$

From (2.11) and (2.23), we have

$$B(x; (\bar{N}_1^{i^c}(x,\cdot) - \bar{\bar{N}}_1^{i^c}(x,\cdot), \bar{N}_2^{i^c}(x,\cdot) - \bar{\bar{N}}_2^{i^c}(x,\cdot)), (\phi_1, \phi_2))$$
$$= -\int_Y (\kappa_1(x,y) - \kappa_1(x',y))\nabla_y(\bar{N}_1^i(x',y) - N_1^i(x',y)) \cdot \nabla_y \phi_1(y) dy$$
$$- \int_Y (\kappa_2(x,y) - \kappa_2(x',y))\nabla_y(\bar{N}_2^i(x',y) - N_2^i(x',y)) \cdot \nabla_y \phi_2(y) dy$$
$$+ \int_Y (Q(x',y) - Q(x,y))((\bar{N}_1^i(x',y) - \bar{N}_2^i(x',y)) - (N_1^i(x',y) - N_2^i(x',y)))$$
$$\cdot(\phi_1(y) - \phi_2(y))dy$$

for all $\phi_1 \in \mathcal{V}_{L-l}$ and $\phi_2 \in \mathcal{V}_{L-l}$. From Assumption 2.2.1 and the induction hypothesis, we have

$$|||(\bar{N}_1^{i^c}(x,\cdot) - \bar{\bar{N}}_1^{i^c}(x,\cdot), \bar{N}_2^{i^c}(x,\cdot) - \bar{\bar{N}}_2^{i^c}(x,\cdot))||| \leq \gamma c_{l-1} 2^{-L-l}. \tag{2.25}$$

where $\gamma > 0$ is independent of x and l. By Lemma 2.2.7 and (2.25),

$$\begin{aligned} &|||(N_1^{i^c}(x,\cdot) - \bar{N}_1^{i^c}(x,\cdot), N_2^{i^c}(x,\cdot) - \bar{N}_2^{i^c}(x,\cdot))||| \\ \leq &|||(N_1^{i^c}(x,\cdot) - \bar{\bar{N}}_1^{i^c}(x,\cdot), N_2^{i^c}(x,\cdot) - \bar{\bar{N}}_2^{i^c}(x,\cdot))||| \\ +&|||(\bar{N}_1^{i^c}(x,\cdot) - \bar{\bar{N}}_1^{i^c}(x,\cdot), \bar{N}_2^{i^c}(x,\cdot) - \bar{\bar{N}}_2^{i^c}(x,\cdot))||| \\ \leq &C^0 2^{-L} + \gamma c_{l-1} 2^{-L-l}. \end{aligned} \tag{2.26}$$

Using $\bar{N}_k^i(x,y) = \bar{N}_k^{i^c}(x,y) + \bar{N}_k^i(x',y)$, We have

$$|||(N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot))||| \leq c_l 2^{-L},$$

where

$$c_l = \gamma c_{l-1} 2^{-l} + c_{l-1} + C^0. \tag{2.27}$$

□

Theorem 2.2.9. *Under Assumption 2.2.1 and the uniform boundedness of $\kappa_i(x,y)$ and $Q(x,y)$, there is a positive constant C_* which depends only on the functions κ_1, κ_2 and Q so that,*

$$|||(N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot))||| \leq C_* l 2^{-L} \tag{2.28}$$

for $x \in S_l$.

Proof. We let $\bar{l}$ be an integer independent of L such that $l2^{-l} < \frac{1}{2\gamma}$ for $l > \bar{l}$. And let

$$C_* = \max\left\{ \max_{0\le l\le \bar{l}} \left\{\frac{c_l}{l}\right\}, 2C^0 \right\}, \tag{2.29}$$

where C^0 and c_l are the constants in Lemma 2.2.7 and Proposition 2.2.8. Now we prove

$$|||(N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot))||| \le C_* l 2^{-L} \tag{2.30}$$

by induction. From (2.29), this holds for all $l \le \bar{l}$. Suppose that (2.30) holds for all $l' \le l$. Then from (2.27), we obtain

$$c_l \le ((l-1)C_* + \frac{1}{2\gamma}\gamma C_* + \frac{C_*}{2}) = C_* l. \tag{2.31}$$

□

Theorem 2.2.10. *The total number of degrees of freedom required to solve (2.8) for all points in $S_0, S_1, \cdots, S_L$ is $\mathcal{O}((L+1)2^{dL})$ for the hierarchical solve while it is $\mathcal{O}((2^{dL})^2)$ in the full solve where cell problems are solved with the finest mesh level at all macrogrid points.*

Proof. Since the number of macroscopic points in S_l is $\mathcal{O}(2^{dl})$, and the space $\mathcal{V}_{L-l}$ is of dimension $\mathcal{O}(2^{d(L-l)})$, the total number of degrees of freedom for solving (2.8) for all points in S_l is $\mathcal{O}(2^{dl})\mathcal{O}(2^{d(L-l)}) = \mathcal{O}(2^{dL})$. Therefore, the total number of degrees of freedom required to solve (2.8) for all points in $S_0, S_1, \cdots, S_L$ is $\mathcal{O}((L+1)2^{dL})$. □

2.3 Numerical example

In this section, we apply the hierarchical finite element algorithm to a numerical example for computing the effective coefficients of a multiscale multi-continuum system at a dense network of macrogrid points. To show the accuracy of the algorithm, we compare the results to the approximations to the effective coefficients obtained from solving the cell problems using the finest meshes at all macroscopic points.

2.3.1 Numerical Implementation

We let $\Omega = [0,1]^2$ be the macroscopic domain and $Y = [0,1]^2$ be the unit cell. We consider the locally periodic coefficients

$$\kappa_1(x_1, y_1, y_2) = (2 - ax_1)\cos(2\pi y_1)\sin(2\pi y_2) + 3$$
$$\kappa_2(x_1, y_1, y_2) = (2 - ax_1)\sin(2\pi y_1)\cos(2\pi y_2) + 3$$
$$Q(x_1, y_1, y_2) = (1 + ax_1)\sin(2\pi y_1)\sin(2\pi y_2) + 3$$

where the constant a is chosen below. We use 4 square meshes in $[0,1]^2$ to construct a nested sequence of FE spaces, $\{\mathcal{V}_{3-l}\}_{l=0}^{3}$ so that the mesh size of each space is $h_l = 2^l \cdot 2^{-4}$ for $l = 0,1,2,3$. Since κ_1, κ_2 and Q are independent of x_2, we only consider 1-dimensional macrogrids in $[0,1]$. The nested macrogrids $\{\mathcal{T}_l\}_{l=0}^{L} \subset [0,1]$ and the subsequent macrogrid hierarchy, $\{S_l\}_{l=0}^{3}$ are constructed as follows. We first let $\mathcal{T}_0 = S_0 = \{0, \frac{1}{2}, 1\}$. Considering that our macrogrids have grid spacing $H2^{-l}$ for $l = 0,1,2,3$, where $H = \frac{1}{2}$ in this case, we have following hierarchy of macrogrids.

$$S_0 = \{0, \frac{1}{2}, 1\},\ S_1 = \{\frac{1}{4}, \frac{3}{4}\},\ S_2 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\},\ S_3 = \{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}\}$$

Figure 2.3 indicates how these macrogrids and the approximation spaces are related in numerical implementation.

We implement the algorithm as follows. For $x' \in S_0 = \{0, \frac{1}{2}, 1\}$, we solve (2.8) for $\bar{N_1^i}(x', \cdot)$, $\bar{N_2^i}(x', \cdot) \in \mathcal{V}_3$, for all $\phi_1,\ \phi_2 \in \mathcal{V}_3$ by the standard Galerkin FEM. We then use a simple 1-point interpolation to compute the correction terms. That is, for $x \in S_l$ we

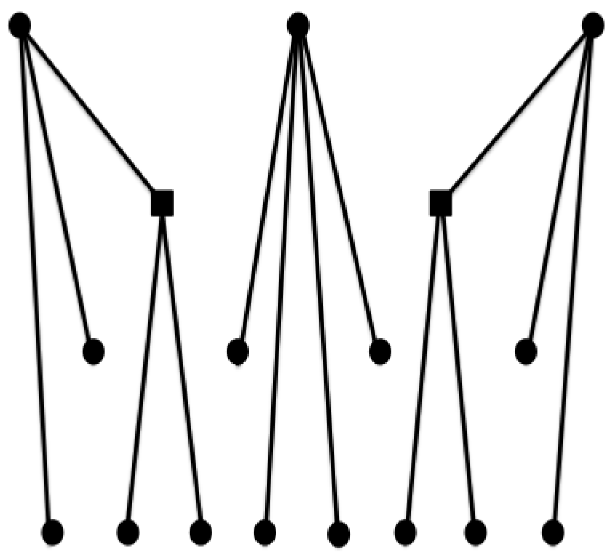


Figure 2.3: The hierarchy of one dimensional macrogrids and corresponding mesh size of FE spaces for 1-pt interpolation method. The lines indicates correction relations. The squares indicate the points at which the solutions are corrected with the lower level solutions and used once more to correct upper level solutions.

choose $x' \in (\bigcup_{k<l} S_k)$ such that $|x'-x| \leq 2^{-l}$. We let the lth macrogrid interpolation be

$$I_l^x(\bar{N}_k^i) = \bar{N}_k^i(x', \cdot),\ (k=1,2).$$

We find $\bar{N_1^i}^c(x,y)$ and $\bar{N_2^i}^c(x,y)$ in $\mathcal{V}_{L-l}$ such that

$$\begin{aligned}
&\int_Y \kappa_1(x,y)\nabla_y \bar{N_1^i}^c(x,y)\cdot\nabla_y\phi_1(y)dy - \int_Y Q(x,y)(\bar{N_2^i}^c(x,y) - \bar{N_1^i}^c(x,y))\phi_1(y)dy \\
&= -\int_Y \kappa_1(x,y)\nabla_y \bar{N}_1^i(x',y)\cdot\nabla_y\phi_1(y)dy - \int_Y \kappa_1(x,y)e^i\cdot\nabla_y\phi_1(y)dy \\
&\quad + \int_Y Q(x,y)(\bar{N}_2^i(x',y) - \bar{N}_1^i(x',y))\phi_1(y)dy,
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
&\int_Y \kappa_2(x,y)\nabla_y \bar{N_2^i}^c(x,y)\cdot\nabla_y\phi_2(y)dy - \int_Y Q(x,y)(\bar{N_1^i}^c(x,y) - \bar{N_2^i}^c(x,y))\phi_2(y)dy \\
&= -\int_Y \kappa_2(x,y)\nabla_y \bar{N}_2^i(x',y)\cdot\nabla_y\phi_2(y)dy - \int_Y \kappa_2(x,y)e^i\cdot\nabla_y\phi_2(y)dy \\
&\quad + \int_Y Q(x,y)(\bar{N}_1^i(x',y) - \bar{N}_2^i(x',y))\phi_2(y)dy,
\end{aligned} \tag{2.33}$$

for $\forall \phi_1, \phi_2 \in \mathcal{V}_{L-l}$. We let

$$\bar{N}_k^i(x,\cdot) = \bar{N}_k^i(x',\cdot) + \bar{N_k^i}^c(x,\cdot),\ (k=1,2)$$

be the approximation to $N_k^i(x,\cdot)$. We continue inductively. For example, for $x=\frac{1}{2}\in S_0$, we compute $\bar{N}_1^i(\frac{1}{2},\cdot)$, $\bar{N}_2^i(\frac{1}{2},\cdot)$ using the standard Galerkin FEM. Then for $\frac{3}{8}\in S_1$, we find the correction terms $\bar{N_1^i}^c(\frac{3}{8},\cdot)$, $\bar{N_2^i}^c(\frac{3}{8},\cdot)\in\mathcal{V}_{L-1}$ that satisfy (2.32) and (2.33), where

$x' = \frac{1}{2}$. And we let the solutions at $x = \frac{3}{8}$ be

$$\bar{N}_k^i(\frac{3}{8}, y) = \bar{N}_k^i(\frac{1}{2}, y) + \bar{N}_k^{i^c}(\frac{3}{8}, y),\ (k = 1, 2).$$

We continue this procedure based on Figure 2.3.

Tables 3.1 and 2.2 indicate κ^*_{111} and κ^*_{211} obtained by both the hierarchical solve and the full solve where the finest mesh is used for all cell problems, at each x_1 and the relative errors between them, where relative errors are calculated by $\frac{100|\kappa^*_{full} - \kappa^*_{hier}|}{\kappa^*_{full}}$ with obvious notations for $a = 1$ and $a = 0.1$ respectively. The results show clearly that the effective coefficients obtained from the hierarchical algorithm are very closed to the reference effective coefficients. We can see from the tables that relatively large errors occur at the highest level macroscopic points where more than one layer of corrections is performed, i.e. the corrector itself is corrected by the solution at a macroscopic point belonging to a lower level. We note that the error for the case $a = 0.1$ is much smaller as the change of κ_i in x is much smaller. That is, large Lipschitz constants in Assumption 2.2.1 tend to result in large errors. The results in Tables 3.1 and 2.2 are obtained when only one corrector point is employed. If we use more corrector points, the error can be reduced significantly. In Table 2.3 we show the relative errors, in comparison to the coefficients obtained from the full solve where the finest mesh is used for all the cell problems, for the effective coefficients obtained from the hierarchical solve for the two cases where one-point and two-point interpolations are used. The table shows that the result can be improved by employing two-point interpolation.

x_1	$\kappa^*_{111}(x_1)$			$\kappa^*_{211}(x_1)$		
	Full	Hierarchical	Errors (%)	Full	Hierarchical	Errors (%)
0	2.8211	2.8211	0.0000	2.8304	2.8304	0.0000
$\frac{1}{16}$	2.8333	2.8267	0.2312	2.8413	2.8397	0.0582
$\frac{1}{8}$	2.8448	2.8408	0.1414	2.8518	2.8491	0.0968
$\frac{3}{16}$	2.8559	2.8593	0.1184	2.8619	2.8624	0.0159
$\frac{1}{4}$	2.8664	2.8641	0.0803	2.8716	2.8707	0.0322
$\frac{5}{16}$	2.8765	2.8690	0.2605	2.8809	2.8787	0.0764
$\frac{3}{8}$	2.8860	2.8887	0.0933	2.8898	2.8919	0.0712
$\frac{7}{16}$	2.8952	2.8998	0.1608	2.8983	2.8995	0.0390
$\frac{1}{2}$	2.9038	2.9038	0.0000	2.9065	2.9065	0.0000
$\frac{9}{16}$	2.9120	2.9078	0.1450	2.9143	2.9133	0.0349
$\frac{5}{8}$	2.9199	2.9178	0.0706	2.9217	2.9201	0.0564
$\frac{11}{16}$	2.9273	2.9319	0.1572	2.9288	2.9303	0.0496
$\frac{3}{4}$	2.9343	2.9351	0.0288	2.9355	2.9361	0.0180
$\frac{13}{16}$	2.9409	2.9383	0.0857	2.9419	2.9416	0.0093
$\frac{7}{8}$	2.9471	2.9485	0.0476	2.9479	2.9492	0.0414
$\frac{15}{16}$	2.9530	2.9558	0.0979	2.9536	2.9543	0.0229
1	2.9584	2.9584	0.0000	2.9598	2.9590	0.0000

Table 2.1: a = 1, the effective coefficients $\kappa^*_{111}(x_1)$ and $\kappa^*_{211}(x_1)$ computed by full mesh reference and hierarchical solve along with percentage relative errors between those.

2.4 Proof of homogenization convergence

In this section, we prove rigorously the homogenization convergence, i.e. the convergence of the solution of the two-scale equation (2.2) to the solution of the homogenized equation (2.5). Throughout this section, we denote the spaces $L^2(\Omega)$ and $H^1_0(\Omega)$ as H and V respectively. We recall the two-scale multi-continuum system

$$\mathcal{C}^\epsilon_{11}\frac{\partial u^\epsilon_1(t,x)}{\partial t} - \text{div}(\kappa^\epsilon_1(x)\nabla u^\epsilon_1(t,x)) - \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_2(t,x) - u^\epsilon_1(t,x)) = q, \tag{2.34}$$

$$\mathcal{C}^\epsilon_{22}\frac{\partial u^\epsilon_2(t,x)}{\partial t} - \text{div}(\kappa^\epsilon_2(x)\nabla u^\epsilon_2(t,x)) - \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_1(t,x) - u^\epsilon_2(t,x)) = q. \tag{2.35}$$

x_1	$\kappa^*_{111}(x_1)$			$\kappa^*_{211}(x_1)$		
	Full	Hierarchical	Errors (%)	Full	Hierarchical	Errors (%)
0	2.8210	2.8211	0.0000	2.8304	2.8304	0.0000
$\frac{1}{16}$	2.8224	2.8217	0.0241	2.8315	2.8314	0.0061
$\frac{1}{8}$	2.8236	2.8231	0.0161	2.8326	2.8323	0.0107
$\frac{3}{16}$	2.8248	2.8252	0.0125	2.8337	2.8338	0.0020
$\frac{1}{4}$	2.8261	2.8257	0.0112	2.8348	2.8347	0.0040
$\frac{5}{16}$	2.8273	2.8263	0.0347	2.8359	2.8356	0.0099
$\frac{3}{8}$	2.8285	2.8289	0.0154	2.8370	2.8373	0.0103
$\frac{7}{16}$	2.8297	2.8303	0.0232	2.8381	2.8383	0.0059
$\frac{1}{2}$	2.8309	2.8309	0.0000	2.8392	2.8392	0.0000
$\frac{9}{16}$	2.8321	2.8314	0.0230	2.8403	2.8401	0.0058
$\frac{5}{8}$	2.8333	2.8328	0.0150	2.8413	2.8410	0.0101
$\frac{11}{16}$	2.8345	2.8354	0.0327	2.8424	2.8427	0.0095
$\frac{3}{4}$	2.8356	2.8359	0.0100	2.8435	2.8436	0.0037
$\frac{13}{16}$	2.8368	2.8364	0.0125	2.8445	2.8445	0.0019
$\frac{7}{8}$	2.8380	2.8384	0.0144	2.8456	2.8459	0.0098
$\frac{15}{16}$	2.8391	2.8398	0.0222	2.8466	2.8468	0.0056
1	2.8403	2.8403	0.0000	2.8477	2.8477	0.0000

Table 2.2: a = .1, the effective coefficients $\kappa^*_{111}(x_1)$ and $\kappa^*_{211}(x_1)$ computed by full mesh reference and hierarchical solve along with percentage relative errors between those.

1-pt interpolation

x_1	Relative Errors (%)	
	κ^*_{111}	κ^*_{211}
$\frac{1}{16}$	0.2312	0.0582
$\frac{1}{8}$	0.1414	0.0968
$\frac{3}{16}$	0.1184	0.0159
$\frac{1}{4}$	0.0803	0.0322
$\frac{5}{16}$	0.2605	0.0764
$\frac{3}{8}$	0.0933	0.0712
$\frac{7}{16}$	0.1608	0.0390
$\frac{9}{16}$	0.1450	0.0349
$\frac{5}{8}$	0.0706	0.0564
$\frac{11}{16}$	0.1572	0.0496
$\frac{3}{4}$	0.0288	0.0180
$\frac{13}{16}$	0.0857	0.0093
$\frac{7}{8}$	0.0476	0.0414
$\frac{15}{16}$	0.0979	0.0229

2-pt interpolation

x_1	Relative Errors (%)	
	κ^*_{111}	κ^*_{211}
$\frac{1}{16}$	0.0072	0.0022
$\frac{1}{8}$	0.0091	0.0030
$\frac{3}{16}$	0.099	0.0026
$\frac{1}{4}$	0.0068	0.0013
$\frac{5}{16}$	0.0080	0.0021
$\frac{3}{8}$	0.0061	0.0020
$\frac{7}{16}$	0.0042	0.0013
$\frac{9}{16}$	0.0026	0.0008
$\frac{5}{8}$	0.0031	0.0011
$\frac{11}{16}$	0.0033	0.0009
$\frac{3}{4}$	0.0021	0.0004
$\frac{13}{16}$	0.0026	0.0007
$\frac{7}{8}$	0.0020	0.0007
$\frac{15}{16}$	0.0014	0.0004

Table 2.3: Percentage relative errors between full mesh reference solve and hierarchical solve when a = 1.

We have the following theorem.

Lemma 2.4.1. *The solution* $(u^{\epsilon}{}_1, u^{\epsilon}{}_2)$ *of (2.34) and (2.35) are uniformly bounded in* $L^{\infty}(0,T;H)$ *and* $L^2(0,T;V)$.

Proof. Multiplying ϕ_1 and $\phi_2 \in V$ to (2.34) and (2.35) respectively and integrating over Ω, one has

$$\begin{aligned} \int_{\Omega} \mathcal{C}^{\epsilon}_{11} \frac{\partial u^{\epsilon}_1}{\partial t} \phi_1 dx + \int_{\Omega} \kappa^{\epsilon}_1 \nabla u^{\epsilon}_1 \cdot \nabla \phi_1 dx - \int_{\Omega} \frac{1}{\epsilon^2} Q^{\epsilon}(u^{\epsilon}_2 - u^{\epsilon}_1)\phi_1 dx &= \int_{\Omega} q\phi_1 dx, \\ \int_{\Omega} \mathcal{C}^{\epsilon}_{22} \frac{\partial u^{\epsilon}_2}{\partial t} \phi_2 dx + \int_{\Omega} \kappa^{\epsilon}_2 \nabla u^{\epsilon}_2 \cdot \nabla \phi_2 dx - \int_{\Omega} \frac{1}{\epsilon^2} Q^{\epsilon}(u^{\epsilon}_1 - u^{\epsilon}_2)\phi_2 dx &= \int_{\Omega} q\phi_2 dx. \end{aligned} \tag{2.36}$$

Summing these equations, we get

$$\begin{aligned} \int_{\Omega} \mathcal{C}^{\epsilon}_{11} \frac{\partial u^{\epsilon}_1(t)}{\partial t} \phi_1 dx + \int_{\Omega} \kappa^{\epsilon}_1 \nabla u^{\epsilon}_1(t) \cdot \nabla \phi_1 dx - \int_{\Omega} \frac{1}{\epsilon^2} Q^{\epsilon}(u^{\epsilon}_2(t) - u^{\epsilon}_1(t))\phi_1 dx \\ + \int_{\Omega} \mathcal{C}^{\epsilon}_{22} \frac{\partial u^{\epsilon}_2}{\partial t}(t) \phi_2 dx + \int_{\Omega} \kappa^{\epsilon}_2 \nabla u^{\epsilon}_2(t) \cdot \nabla \phi_2 dx - \int_{\Omega} \frac{1}{\epsilon^2} Q^{\epsilon}(u^{\epsilon}_1(t) - u^{\epsilon}_2(t))\phi_2 dx \\ = \int_{Q} q(t)\phi_1 dx + \int_{\Omega} q(t)\phi_2 dx \end{aligned} \tag{2.37}$$

$\forall \phi_1, \phi_2 \in V$. Substituting u^{ϵ}_1 and u^{ϵ}_2 into ϕ_1 and ϕ_2 in (2.37) respectively, we have

$$\begin{aligned} &\int_{\Omega} \mathcal{C}^{\epsilon}_{11} \frac{\partial u^{\epsilon}_1(t)}{\partial t} u^{\epsilon}_1(t) dx + \int_{\Omega} \mathcal{C}^{\epsilon}_{22} \frac{\partial u^{\epsilon}_2(t)}{\partial t} u^{\epsilon}_2(t) dx + \int_{\Omega} \kappa^{\epsilon}_1 \nabla u^{\epsilon}_1(t) \cdot \nabla u^{\epsilon}_1(t) dx \\ &+ \int_{\Omega} \kappa^{\epsilon}_2 \nabla u^{\epsilon}_2(t) \cdot \nabla u^{\epsilon}_2(t) dx + \frac{1}{\epsilon^2} \int_{\Omega} Q^{\epsilon}(u^{\epsilon}_2(t) - u^{\epsilon}_1(t))^2 dx = \int_{\Omega} q u^{\epsilon}_1(t) + \int_{\Omega} q u^{\epsilon}_2(t) dx. \end{aligned}$$

Integrating this equation over $(0, \tau)$, we get

$$\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon |u_1^\epsilon(\tau,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon |u_2^\epsilon(\tau,x)|^2 dx + \int_0^\tau \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla u_1^\epsilon dx dt$$
$$+ \int_0^\tau \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla u_2^\epsilon dx dt + \frac{1}{\epsilon^2}\int_0^\tau \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon)^2 dx dt$$
$$= \int_0^\tau \int_\Omega q u_1^\epsilon dx dt + \int_0^\tau \int_\Omega q u_2^\epsilon dx dt + \frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon |u_1^\epsilon(0,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon |u_2^\epsilon(0,x)|^2 dx. \tag{2.38}$$

Therefore,

$$\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon |u_1^\epsilon(\tau,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon |u_2^\epsilon(\tau,x)|^2 dx$$
$$+ \int_0^\tau \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla u_1^\epsilon dx dt + \int_0^\tau \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla u_2^\epsilon dx dt$$
$$\leq \frac{\varepsilon}{2}\int_0^\tau \int_\Omega |q|^2 dx dt + \frac{1}{2\varepsilon}\int_0^\tau \int_\Omega |u_1^\epsilon|^2 dx dt + \frac{\varepsilon}{2}\int_0^\tau \int_\Omega |q|^2 dx dt$$
$$+ \frac{1}{2\varepsilon}\int_0^\tau \int_\Omega |u_2^\epsilon|^2 dx dt + \int_\Omega |\mathcal{C}_{11}^\epsilon| |u_1^\epsilon(0,x)|^2 dx + \int_\Omega |\mathcal{C}_{22}^\epsilon| |u_2^\epsilon(0,x)|^2 dx.$$

Using the uniform boundedness from below of C_{11}^ϵ and C_{22}^ϵ, we have

$$c\|u_1^\epsilon(\tau,\cdot)\|_H^2 + c\|u_2^\epsilon(\tau,\cdot)\|_H^2 + \int_0^\tau \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla u_1^\epsilon dx dt + \int_0^\tau \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla u_2^\epsilon dx dt$$
$$\leq c_1 \frac{\varepsilon}{2} + \frac{1}{2\varepsilon}\int_0^T \|u_1^\epsilon(t,\cdot)\|_H^2 dt + \frac{1}{2\varepsilon}\int_0^T \|u_2^\epsilon(t,\cdot)\|_H^2 dt.$$

Choosing ε sufficiently large, we deduce that u_1^ϵ and u_2^ϵ are uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$. □

Note that because of the 5th term of equation (2.38), $\lim_{\epsilon\to 0} u_1^\epsilon = \lim_{\epsilon \to 0} u_2^\epsilon$. Thus, there exist subsequences of u_1^ϵ and u_2^ϵ, which we still denote by u_1^ϵ and u_2^ϵ , and u_0 such that

$$u_1^\epsilon, u_2^\epsilon \rightharpoonup u_0 \text{ in } L^2(0,T;V).$$

Recall that $(N_1^i, N_2^i) \in W$ is the solution of cell problem.

$$
\begin{aligned}
&\text{div}_y(\kappa_1(x,y)(e^i + \nabla_y N_1^i(x,y))) + Q(x,y)(N_2^i(x,y) - N_1^i(x,y)) = 0 \\
&\text{div}_y(\kappa_2(x,y)(e^i + \nabla_y N_2^i(x,y))) + Q(x,y)(N_1^i(x,y) - N_2^i(x,y)) = 0.
\end{aligned}
\tag{2.39}
$$

We assume that N_1^i and N_2^i are sufficiently smooth with respect to both x and y. Let $\omega_1(x) = \frac{x_i}{\epsilon} + N_1^i(x, \frac{x}{\epsilon})$ and $\omega_2(x) = \frac{x_i}{\epsilon} + N_2^i(x, \frac{x}{\epsilon})$. We define ω_1^ϵ and ω_2^ϵ as

$$
\omega_1^\epsilon(x) = \epsilon\omega_1(x, \frac{x}{\epsilon}), \;\; \omega_2^\epsilon(x) = \epsilon\omega_2(x, \frac{x}{\epsilon}).
$$

Assuming that κ_1, κ_2, N_1^i and N_2^i are sufficiently smooth, for all $\psi_1, \psi_2 \in V$ we have

$$
\begin{aligned}
&-\int_\Omega \text{div}(\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x))\psi_1(x)dx - \frac{1}{\epsilon^2}\int_\Omega Q^\epsilon(x)(\omega_2^\epsilon(x) - \omega_1^\epsilon(x))\psi_1(x)dx \\
=&-\frac{1}{\epsilon}\int_\Omega \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})))\psi_1(x)dx \\
&-\frac{1}{\epsilon}\int_\Omega Q(x,\frac{x}{\epsilon})(N_2^i(x,\frac{x}{\epsilon}) - N_1^i(x,\frac{x}{\epsilon}))\psi_1(x)dx \\
&-\epsilon\int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_x(N_1^i(x,\frac{x}{\epsilon})))\psi_1(x)dx \\
&-\int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})))\psi_1(x)dx \\
&-\int_\Omega \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))\psi_1(x)dx \\
=&-\epsilon\int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))\psi_1(x)dx \\
&-\int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})))\psi_1(x)dx \\
&-\int_\Omega \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))\psi_1(x)dx
\end{aligned}
\tag{2.40}
$$

and

$$
\begin{aligned}
&-\int_\Omega \mathrm{div}(\kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x))\psi_2(x)dx - \frac{1}{\epsilon^2}\int_\Omega Q^\epsilon(x)(\omega_1^\epsilon(x)-\omega_2^\epsilon(x))\psi_2(x)dx \\
=&-\frac{1}{\epsilon}\int_\Omega \mathrm{div}_y(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})))\psi_2(x)dx \\
&-\frac{1}{\epsilon}\int_\Omega Q(x,y)(N_1^i(x,\frac{x}{\epsilon})-N_2^i(x,\frac{x}{\epsilon}))\psi_2(x)dx \\
&-\epsilon\int_\Omega \mathrm{div}_x(\kappa_2(x,\frac{x}{\epsilon})\nabla_x(N_2^i(x,\frac{x}{\epsilon})))\psi_2(x)dx \\
&-\int_\Omega \mathrm{div}_x(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})))\psi_2(x)dx \\
&-\int_\Omega \mathrm{div}_y(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon}))\psi_2(x)dx \\
=&-\epsilon\int_\Omega \mathrm{div}_x(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon}))\psi_2(x)dx \\
&-\int_\Omega \mathrm{div}_x(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})))\psi_2(x)dx \\
&-\int_\Omega \mathrm{div}_y(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon}))\psi_2(x)dx
\end{aligned}
\tag{2.41}
$$

due to (2.39). Let $\phi_1(x)=\phi(x)\omega_1^\epsilon(x)$, $\phi_2(x)=\phi(x)\omega_2^\epsilon(x)$ where $\phi\in\mathcal{C}_0^\infty(\Omega)$ in (2.36), we have

$$
\begin{aligned}
\int_\Omega \mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\omega_1^\epsilon dx + \int_\Omega \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t}\phi\omega_2^\epsilon dx + \int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla(\phi\omega_1^\epsilon)dx + \int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla(\phi\omega_2^\epsilon)dx \\
+\int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u_1^\epsilon-u_2^\epsilon)(\omega_1^\epsilon-\omega_2^\epsilon)\phi dx = \int_\Omega q\phi\omega_1^\epsilon dx + \int_\Omega q\phi\omega_2^\epsilon dx.
\end{aligned}
\tag{2.42}
$$

Let $\psi_1(x)$ and $\psi_2(x)$ in (2.40) and (2.41) be ϕu_1^ϵ and ϕu_2^ϵ respectively. We have

$$\begin{aligned}
&\int_\Omega \kappa_1^\epsilon \nabla \omega_1^\epsilon \cdot \nabla(\phi u_1^\epsilon) dx + \int_\Omega \kappa_2^\epsilon \nabla \omega_2^\epsilon \cdot \nabla(\phi u_2^\epsilon) dx + \int_\Omega \frac{1}{\epsilon^2} Q^\epsilon (\omega_1^\epsilon - \omega_2^\epsilon)(u_1^\epsilon - u_2^\epsilon) \phi dx \\
&= -\epsilon \int_\Omega \mathrm{div}_x (\kappa_1(x, \frac{x}{\epsilon}) \nabla_x N_1^i(x, \frac{x}{\epsilon})) \phi u_1^\epsilon dx \\
&- \int_\Omega \mathrm{div}_x (\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon}))) \phi u_1^\epsilon dx \\
&- \int_\Omega \mathrm{div}_y (\kappa_1(x, \frac{x}{\epsilon}) \nabla_x N_1^i(x, \frac{x}{\epsilon})) \phi u_1^\epsilon dx - \epsilon \int_\Omega \mathrm{div}_x (\kappa_2(x, \frac{x}{\epsilon}) \nabla_x N_2^i(x, \frac{x}{\epsilon})) \phi u_2^\epsilon dx \\
&- \int_\Omega \mathrm{div}_x (\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon}))) \phi u_2^\epsilon dx - \int_\Omega \mathrm{div}_y (\kappa_2(x, \frac{x}{\epsilon}) \nabla_x N_2^i(x, \frac{x}{\epsilon})) \phi u_2^\epsilon dx.
\end{aligned} \tag{2.43}$$

Let $\quad \in C_0^\infty(0,T)$. We multiply (2.42) and (2.43) by ψ and intergrate over $(0,T)$ with respect to t. After subtracting the resulting equations by each other, we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt + \int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon\cdot\nabla\phi\omega_1^\epsilon\psi dxdt \\
&-\int_0^T\int_\Omega \kappa_1^\epsilon\nabla\omega_1^\epsilon\cdot\nabla\phi u_1^\epsilon\psi dxdt + \int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t}\phi\psi\omega_2^\epsilon dxdt \\
&+\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla\phi\omega_2^\epsilon\psi dxdt - \int_0^T\int_\Omega \kappa_2^\epsilon\nabla\omega_2^\epsilon\cdot\nabla\phi u_2^\epsilon\psi dxdt \\
=&\int_0^T\int_\Omega q\phi\omega_1^\epsilon\psi dxdt + \int_0^T\int_\Omega q\phi\omega_2^\epsilon\psi dxdt \\
&+\epsilon\int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_1^i(\cdot,\frac{\cdot}{\epsilon})\phi u_1^\epsilon\psi dxdt \\
&+\int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_1^i(\cdot,\cdot\epsilon)))\phi u_1^\epsilon\psi dxdt \\
&+\int_0^T\int_\Omega \mathrm{div}_y(\kappa_1(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_1^i(\cdot,\frac{\cdot}{\epsilon})\phi u_1^\epsilon\psi dxdt \\
&+\epsilon\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_2^i(\cdot,\frac{\cdot}{\epsilon}))\phi u_2^\epsilon\psi dxdt \\
&+\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_2^i(\cdot,\frac{\cdot}{\epsilon})))\phi u_2^\epsilon\psi dxdt \\
&+\int_0^T\int_\Omega \mathrm{div}_y(\kappa_2(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_2^i(\cdot,\frac{\cdot}{\epsilon})\phi u_2^\epsilon\psi dxdt.
\end{aligned}
\tag{2.44}
$$

We have the following lemma.

Lemma 2.4.2. *The functions* $\int_0^T \quad (t)u_1^\epsilon(x,t)dt$ *and* $\int_0^T \quad (t)u_2^\epsilon(x,t)dt$ *converge strongly in* H *to* $\int_0^T \psi(t)u_0(x,t)dt$*.*

Proof This is the standard result in Jikov et al. [9]. As u_1^ϵ is uniformly bounded in $L^2(0,T;V)$, $\int_0^T \quad (t)u_1^\epsilon(x,t)dt$ is uniformly bounded in V when $\epsilon\to 0$. Thus we can extract a subsequence which converges weakly in V and strongly in H. As for all $\phi\in C_0^\infty(\Omega)$,

$$\int_\Omega\int_0^T \quad (t)u_1^\epsilon(x,t)\phi(x)dtdx \to \int_\Omega\int_0^T \psi(t)u_0(x,t)\phi(x)dtdx,$$

the limit is $\int_0^T \psi(t)u_0(x,t)dt$. $\square$

We have

$$\int_0^T \int_\Omega C_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt = -\int_\Omega C_{11}^\epsilon \left(\int_0^T u_1^\epsilon \frac{\partial \psi}{\partial t}dt\right)\phi\omega_1^\epsilon dx.$$

As C_{11}^ϵ converges weakly to $\int_Y C_{11}(x,y)dy$ in H, $\int_0^T u_1^\epsilon \frac{\partial\psi}{\partial t}dt$ converges weakly to $\int_0^T u_0 \frac{\partial\psi}{\partial t}dt$ in V, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T \int_\Omega C_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt &= -\int_0^T \int_\Omega \left(\int_Y C_{11}(x,y)dy\right) u_0 \frac{\partial\psi}{\partial t}\phi x_i dxdt \\ &= \int_0^T \int_\Omega \left(\int_Y C_{11}(x,y)dy\right)\frac{\partial u_0}{\partial t}\psi\phi x_i dxdt.\end{aligned}$$

We note that

$$\begin{aligned}\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x) &= \kappa_1(x,\frac{x}{\epsilon})\big((e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})) + \epsilon\nabla_x N_1^i(x,\frac{x}{\epsilon})\big),\\ \kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x) &= \kappa_2(x,\frac{x}{\epsilon})\big((e^i + \nabla_y N_2^i(x,\frac{x}{\epsilon})) + \epsilon\nabla_x N_2^i(x,\frac{x}{\epsilon})\big).\end{aligned}$$

Also, note that due to Y-periodicity of κ and N^i, we have

$$\begin{aligned}\kappa_1(x,\frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})) &\rightharpoonup \int_Y \kappa_1(x,y)(e^i + \nabla_y N_1^i(x,y))dy\\ \kappa_2(x,\frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x,\frac{x}{\epsilon})) &\rightharpoonup \int_Y \kappa_2(x,y)(e^i + \nabla_y N_2^i(x,y))dy \ \text{ in } \ H.\end{aligned}$$

We observe that $x_i + \epsilon N_1^i \to x_i$ strongly in H and $\int_\Omega q\phi\omega_k^\epsilon dx \to \int_\Omega q\phi x_i dx$ since $\omega_k^\epsilon\phi \rightharpoonup$

$x_i\phi$ in H. Passing to the limit in (2.44), we obtain from Lemma 2.4.2,

$$\int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{11} dy \right) \frac{\partial u_0}{\partial t} \phi\psi x_i dxdt + \lim_{\epsilon\to 0} \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla\phi\psi x_i dxdt$$
$$- \int_0^T \int_\Omega \int_Y \kappa_1 (e^i + \nabla_y N_1^i) dy \cdot \nabla\phi\psi u_0 dxdt + \int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{22} dy \right) \frac{\partial u_0}{\partial t} \phi\psi x_i dxdt$$
$$+ \lim_{\epsilon\to 0} \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla\phi\psi x_i dxdt - \int_0^T \int_\Omega \int_Y \kappa_2 (e^i + \nabla_y N_2^i) dy \cdot \nabla\phi\psi u_0 dxdt$$
$$= 2 \int_0^T \int_\Omega q\phi\psi x_i dxdt + \lim_{\epsilon\to 0} \int_0^T \int_\Omega \mathrm{div}_x (\kappa_1 (\cdot, \frac{\cdot}{\epsilon})(e^i + \nabla_y N_1^i (\cdot, \frac{\cdot}{\epsilon}))) \phi\psi u_1^\epsilon dxdt$$
$$+ \lim_{\epsilon\to 0} \int_0^T \int_\Omega \mathrm{div}_x (\kappa_2 (\cdot, \frac{\cdot}{\epsilon})(e^i + \nabla_y N_2^i (\cdot, \frac{\cdot}{\epsilon}))) \phi\psi u_2^\epsilon dxdt. \tag{2.45}$$

Let ϕ_1 and ϕ_2 in (2.36) be ϕx_i where $\phi \in C_0^\infty(\Omega)$. Adding the two equations, we have

$$\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} \phi\psi x_i dxdt + \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t} \phi\psi x_i dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla(\phi x_i) \psi dxdt$$
$$+ \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla(\phi x_i) \psi dxdt = 2 \int_0^T \int_\Omega q\phi\psi x_i dxdt.$$

Passing to the limit, we obtain

$$\int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{11} dy \right) \frac{\partial u_0}{\partial t} \phi\psi x_i dxdt + \int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{22} dy \right) \frac{\partial u_0}{\partial t} \phi\psi x_i dxdt$$
$$+ \lim_{\epsilon\to 0} \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \nabla \cdot (\phi x_i) \psi dxdt + \lim_{\epsilon\to 0} \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla(\phi x_i) \psi dxdt \tag{2.46}$$
$$= 2 \int_0^T \int_\Omega q(\phi x_i) \psi dxdt.$$

Using (2.45) and (2.46), one obtains

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi\psi dx dt + \lim_{\epsilon\to 0}\int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot e^i \phi\psi dx dt \\
&= -\int_0^T \int_\Omega \left(\int_Y \kappa_1(e^i + \nabla_y N_1^i) dy\right) \cdot \nabla\phi\psi u_0 dx dt \\
&\quad - \int_0^T \int_\Omega \left(\int_Y \kappa_2(e^i + \nabla_y N_2^i) dy\right) \cdot \nabla\phi\psi u_0 dx dt \\
&\quad - \lim_{\epsilon\to 0}\int_0^T \int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon})))\phi\psi u_1^\epsilon dx dt \\
&\quad - \lim_{\epsilon\to 0}\int_0^T \int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon})))\phi\psi u_2^\epsilon dx dt.
\end{aligned}$$

Since κ_1, κ_2, N_1^i and N_2^i are independent of t, by Lemma 2.4.2, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi\psi dx dt + \lim_{\epsilon\to 0}\int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot e^i \phi\psi dx dt \\
&\qquad = -\int_0^T \int_\Omega \left(\int_Y \kappa_1(e^i + \nabla_y N_1^i) dy\right) \cdot \nabla\phi\psi u_0 dx dt \\
&\qquad\quad - \int_0^T \int_\Omega \left(\int_Y \kappa_2(e^i + \nabla_y N_2^i) dy\right) \cdot \nabla\phi\psi u_0 dx dt \\
&\qquad\quad - \int_0^T \int_\Omega \mathrm{div} \left(\int_Y \kappa_1(e^i + \nabla_y N_1^i) dy\right) \phi\psi u_0 dx dt \\
&\qquad\quad - \int_0^T \int_\Omega \mathrm{div} \left(\int_Y \kappa_2(e^i + \nabla_y N_2^i) dy\right) \phi\psi u_0 dx dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T \left(\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi dx\right) \psi dt + \lim_{\epsilon\to 0}\int_0^T \left(\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot e^i \phi dx\right) \psi dt \\
&= \int_0^T \left(\int_\Omega \left(\int_Y \kappa_1(e^i + \nabla_y N_1^i) dy\right) \cdot \nabla u_0 \phi dx\right) \psi dt \\
&+ \int_0^T \left(\int_\Omega \left(\int_Y \kappa_2(e^i + \nabla_y N_2^i) dy\right) \cdot \nabla u_0 \phi dx\right) \psi dt
\end{aligned}$$

From this, we deduce

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left[\int_0^T\int_\Omega \kappa_1^\epsilon(x)\nabla u_1^\epsilon(x)\cdot\nabla\phi\psi dxdt+\int_0^T\int_\Omega \kappa_2^\epsilon(x)\nabla u_2^\epsilon(x)\cdot\nabla\phi\psi dxdt\right]\\
&=\lim_{\epsilon\to 0}\left[\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot e^i\frac{\partial\phi}{\partial x_i}\psi dxdt+\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot e^i\frac{\partial\phi}{\partial x_i}\psi dxdt\right]\\
&=\int_0^T\int_\Omega\left(\int_Y \kappa_1(x,y)(\delta_{ij}+\frac{\partial N_1^i(x,y)}{\partial y_j})dy\right)\frac{\partial u_0}{\partial x_j}(x)\frac{\partial\phi}{\partial x_i}\psi dxdt\\
&+\int_0^T\int_\Omega\left(\int_Y \kappa_2(x,y)(\delta_{ij}+\frac{\partial N_2^i(x,y)}{\partial y_j})dy\right)\frac{\partial u_0}{\partial x_j}(x)\frac{\partial\phi}{\partial x_i}\psi dxdt
\end{aligned}
\tag{2.47}
$$

For consistency with formula (2.6), we note the following result.

Lemma 2.4.3. $\int_Y \kappa_1\frac{\partial N_1^j(x,y)}{\partial y_i}dy+\int_Y \kappa_2\frac{\partial N_2^j(x,y)}{\partial y_i}dy=\int_Y \kappa_1\frac{\partial N_1^i(x,y)}{\partial y_j}dy+\int_Y \kappa_2\frac{\partial N_2^i(x,y)}{\partial y_j}dy$

Proof. From the cell problem, we have

$$
\begin{aligned}
&\int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_1^j dy+\int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy\\
&+\int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy+\int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_2^j dy=0\\
&\int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_1^i dy+\int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy\\
&+\int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy+\int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_2^i dy=0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_Y \kappa_1\frac{\partial N_1^j}{\partial y_i}dy+\int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy\\
&+\int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy+\int_Y \kappa_2\frac{\partial N_2^j}{\partial y_i}dy\\
&=\int_Y \kappa_1\frac{\partial N_1^i}{\partial y_j}dy+\int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy\\
&+\int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy+\int_Y \kappa_2\frac{\partial N_2^i}{\partial y_j}dy.
\end{aligned}
\tag{2.48}
$$

Now we show

$$\int_Y \kappa_2(e^i + \nabla_y N_2^i) \cdot \nabla_y N_1^j dy + \int_Y \kappa_1(e^i + \nabla_y N_1^i) \cdot \nabla_y N_2^j dy$$
$$= \int_Y \kappa_2(e^j + \nabla_y N_2^j) \cdot \nabla_y N_1^i dy + \int_Y \kappa_1(e^j + \nabla_y N_1^j) \cdot \nabla_y N_2^i dy.$$

From the cell problem, we know that

$$\begin{aligned}
&\int_Y \kappa_2(e^i + \nabla_y N_2^i) \cdot \nabla_y N_1^j dy + \int_Y \kappa_1(e^i + \nabla_y N_1^i) \cdot \nabla_y N_2^j dy \\
&= \int_Y Q(N_1^i - N_2^i)N_1^j + Q(N_2^i - N_1^i)N_2^j dy \\
&= \int_Y Q(N_1^i N_1^j - N_2^i N_1^j + N_2^i N_2^j - N_1^i N_2^j) dy \qquad (2.49) \\
&= \int_Y Q(N_1^j - N_2^j)N_1^i + Q(N_2^j - N_1^j)N_2^i dy \\
&= \int_Y \kappa_2(e^j + \nabla_y N_2^j) \cdot \nabla_y N_1^i dy + \int_Y \kappa_1(e^j + \nabla_y N_1^j) \cdot \nabla_y N_2^i dy.
\end{aligned}$$

Thus, by (2.48) and (2.49), we have the result. □

Theorem 2.4.4. *Assume that the solution N_1^i and N_2^i of cell problem* (2.8) *belong to $C^2(\bar{\Omega}, C^2(\bar{Y}))$ and the coefficients κ_1 and κ_2 belong to $C^1(\bar{\Omega}, C^1(\bar{Y}))$. The limit function u_0 of the sequences u_1^ϵ, u_2^ϵ is the unique solution of the homogenized equation (2.5) with the initial condition* (2.7).

Proof. Note that from the equation (2.34), we obtain

$$\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} \phi dx \psi dt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla \phi dx \psi dt + \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t} \phi dx \psi dt$$
$$+ \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla \phi dx \psi dt = 2 \int_0^T \int_\Omega q \phi dx \psi dt.$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega)$ and $\psi \in C_0^\infty((0,T))$. Passing to the limit, from (2.47), Lemmas 2.4.2

and 2.4.3, we have

$$\int_0^T \int_\Omega \{(\int_Y \mathcal{C}_{11} dy) + (\int_Y \mathcal{C}_{22} dy)\} \frac{\partial u_0}{\partial t} \phi dx \psi dt$$
$$= \int_0^T \int_\Omega \operatorname{div}(\kappa_1^* \nabla u_0) \phi dx \psi dt + \int_0^T \int_\Omega \operatorname{div}(\kappa_2^* \nabla u_0) \phi dx \psi dt + \int_0^T \int_\Omega 2q \phi dx \psi dt,$$

where

$$\kappa_{1ij}^*(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i}) dy$$
$$\kappa_{2ij}^*(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i}) dy.$$

We now show the initial condition. Adding (2.34) and (2.35), we have

$$\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t} - \operatorname{div}(\kappa_1^\epsilon \nabla u_1^\epsilon) - \operatorname{div}(\kappa_2^\epsilon \nabla u_2^\epsilon) = 2q.$$

As u_1^ϵ and u_2^ϵ are bounded in $L^2(0,T;V)$, we deduce that $\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}$ is bounded in $L^2(0,T;V')$. Let $\psi(t,x) \in \mathcal{C}_0^\infty(0,T;V)$, i.e. $\psi(0,x) = \psi(T,x) = 0$. We have

$$\int_0^T \int_\Omega \big(\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\big) \psi dx dt = -\int_0^T \int_\Omega \big(\mathcal{C}_{11}^\epsilon u_1^\epsilon + \mathcal{C}_{22}^\epsilon u_2^\epsilon\big) \frac{\partial \psi}{\partial t} dx dt$$
$$\to -\int_0^T \int_\Omega \big(\langle \mathcal{C}_{11} \rangle + \langle \mathcal{C}_{22} \rangle\big) u_0 \frac{\partial \psi}{\partial t} dx dt = \int_0^T \int_\Omega \big(\langle \mathcal{C}_{11} \rangle + \langle \mathcal{C}_{22} \rangle\big) \frac{\partial u_0}{\partial t} \psi dx dt.$$

This shows that the weak limit of $\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}$ in $L^2(0,T;V')$ is $\big(\langle \mathcal{C}_{11} \rangle + \langle \mathcal{C}_{22} \rangle\big) \frac{\partial u_0}{\partial t}$. Now we choose $\psi \in \mathcal{C}^\infty(0,T;V)$ so that $\psi(T,x) = 0$. Then

$$\int_0^T \int_\Omega \big(\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\big)\psi dxdt$$
$$= -\int_0^T \int_\Omega \big(\mathcal{C}_{11}^\epsilon u_1^\epsilon + \mathcal{C}_{22}^\epsilon u_2^\epsilon\big)\frac{\partial \psi}{\partial t} dxdt + \int_\Omega \big(\mathcal{C}_{11}^\epsilon u_1^\epsilon(0,x) + \mathcal{C}_{22}^\epsilon u_2^\epsilon(0,x)\big)\psi(0,x)dx$$
$$\to -\int_0^T \int_\Omega \big(\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle\big)u_0 \frac{\partial \psi}{\partial t} dxdt + \int_\Omega \big(\langle \mathcal{C}_{11}\rangle g_1 + \langle \mathcal{C}_{22}\rangle g_2\big)\psi(0,x)dx.$$

On the other hand

$$\int_0^T \int_\Omega \big(\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\big)\psi dxdt \to \int_0^T \int_\Omega \big(\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle\big)\frac{\partial u_0}{\partial t}\psi dxdt.$$
$$= -\int_0^T \int_\Omega \big(\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle\big)u_0 \frac{\partial \psi}{\partial t} dxdt + \int_\Omega \big(\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle\big)u_0(0,x)\psi(0,x)dx.$$

This shows that $\big(\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle\big)u_0(0,x) = \langle \mathcal{C}_{11}\rangle g_1(x) + \langle \mathcal{C}_{22}\rangle g_2(x)$. i.e. the initial condition of u_0 is

$$u_0(0,x) = \frac{\langle \mathcal{C}_{11}\rangle g_1(x) + \langle \mathcal{C}_{22}\rangle g_2(x)}{\langle \mathcal{C}_{11}\rangle + \langle \mathcal{C}_{22}\rangle} \tag{2.50}$$

□

3. HOMOGENIZATION OF A MULTISCALE MULTI-CONTINUUM SYSTEM

In this chapter, we study the homogenization of a multiscale dual-continuum system where the interaction terms are scaled as $\mathcal{O}(\frac{1}{\epsilon})$. We first derive an interesting homogenization equation using two-scale asymptotic expansion. Then we provide a rigorous proof of the homogenization convergence and convergence rate which are stated in Theorems 3.1.1 and 3.1.2.

3.1 Problem formulation

3.1.1 Two scale multicontinuum problem

Let Ω be a bounded domain in $\mathbb{R}^d$. Let Y be a unit cube in $\mathbb{R}^d$. Let $Q(x,y)$, $\mathcal{C}_{ii}(x,y)$ and $\kappa_i(x,y)$ $(i=1,2)$ be continuous functions on $\Omega\times Y$ which are Y-periodic with respect to y. We assume that

$$\int_Y Q(x,y)dy = 0. \tag{3.1}$$

Let $T>0$. Let q be a function in $L^2((0,T)\times\Omega)$. Let $\epsilon>0$ be a small quantity that represents the microscopic scale the coefficients depend on. We define the two scale coefficients as

$$\mathcal{C}_{ii}^{\epsilon}(x) = \mathcal{C}_{ii}(x,\frac{x}{\epsilon}),\ \kappa_i^{\epsilon}(x) = \kappa_i(x,\frac{x}{\epsilon}),\ \ i=1,2,\ \text{ and }\ Q^{\epsilon}(x) = Q(x,\frac{x}{\epsilon}). \tag{3.2}$$

Let H denote the space $L^2(\Omega)$ and V denote the space $H_0^1(\Omega)$. We consider the following dual-continuum system.

$$\begin{aligned} \mathcal{C}_{11}^{\epsilon}(x)\frac{\partial u_1^{\epsilon}(t,x)}{\partial t} &= \text{div}(\kappa_1^{\epsilon}(x)\nabla u_1^{\epsilon}(t,x)) + \frac{1}{\epsilon}Q^{\epsilon}(x)(u_2^{\epsilon}(t,x) - u_1^{\epsilon}(t,x)) + q, \;\; x \in \Omega, \\ \mathcal{C}_{22}^{\epsilon}(x)\frac{\partial u_2^{\epsilon}(t,x)}{\partial t} &= \text{div}(\kappa_2^{\epsilon}(x)\nabla u_2^{\epsilon}(t,x)) + \frac{1}{\epsilon}Q^{\epsilon}(x)(u_1^{\epsilon}(t,x) - u_2^{\epsilon}(t,x)) + q, \;\; x \in \Omega, \end{aligned} \tag{3.3}$$

with the Dirichlet boundary condition $u_1^{\epsilon}(t,x) = u_2^{\epsilon}(t,x) = 0$ for $x \in \partial\Omega$, and with the initial condition $u_1^{\epsilon}(0,x) = g_1(x)$, $u_2^{\epsilon}(0,x) = g_2(x)$ where g_1 and g_2 are in H. We assume there exist positive constants $\underline{C}$, $\underline{\kappa}$ such that

$$\mathcal{C}_{ii}(x,y) \geq \underline{C}, \; \kappa_i(x,y) \geq \underline{\kappa}. \tag{3.4}$$

Remark. Throughout this chapter, we assume $\int_Y Q(x,y)dy = 0$. This assumption makes the presentation clearer and more concise. However, in realistic models, we have $Q(x,y) > 0$. One can handle this case under following assumption: $Q(x,y) = \frac{1}{\epsilon}Q^*(x,y) + \bar{Q}(x)$, where

$$Q^*(x,y) = Q(x,y) - \int_Y Q(x,y)dy \;\text{ and }\; \bar{Q}(x) = \int_Y Q(x,y)dy. \tag{3.5}$$

Obviously, now we have $\int_Y Q^*(x,y)dy = 0$. The term $\bar{Q}(x)$ will contribute to the homogenized equation and $Q^*(x,y)$ will play the role of $Q(x,y)$ in this chapter.

In the weak form, equations (3.3) are of the form

$$\begin{aligned} \int_{\Omega} \mathcal{C}_{11}^{\epsilon}\frac{\partial u_1^{\epsilon}}{\partial t}\phi_1 dx + \int_{\Omega} \kappa_1^{\epsilon}\nabla u_1^{\epsilon}\cdot\nabla\phi_1 dx - \frac{1}{\epsilon}\int_{\Omega} Q^{\epsilon}(u_2^{\epsilon} - u_1^{\epsilon})\phi_1 dx = \int_{\Omega} q\phi_1 dx, \\ \int_{\Omega} \mathcal{C}_{22}^{\epsilon}\frac{\partial u_2^{\epsilon}}{\partial t}\phi_2 dx + \int_{\Omega} \kappa_2^{\epsilon}\nabla u_2^{\epsilon}\cdot\nabla\phi_2 dx - \frac{1}{\epsilon}\int_{\Omega} Q^{\epsilon}(u_1^{\epsilon} - u_2^{\epsilon})\phi_2 dx = \int_{\Omega} q\phi_2 dx. \end{aligned} \tag{3.6}$$

for all ϕ_1 and ϕ_2 in $C_0^\infty(\Omega)$. We use $L^2(0,T;X)$, $L^\infty(0,T;X)$ to represent the Bochner space with the norm

$$\|v\|_{L^2(0,T;X)} := \left(\int_0^T \|v\|_X^2 \, \mathrm{d}t \right)^{1/2},$$

$$\|v\|_{L^\infty(0,T;X)} := \sup_{0\le t\le T} \|v\|_X,$$

where $(X, \|\cdot\|_X)$ is a Banach space. Also, we define

$$H^1(0,T;X) := \left\{ v \in L^2(0,T;X) \, : \, \partial_t v \in L^2(0,T;X) \right\} .$$

We will prove in the appendix that system (3.3) has a unique solution $(u_1^\epsilon, u_2^\epsilon) \in L^2(0,T;V) \cap H^1(0,T;V') \times L^2(0,T;V) \cap H^1(0,T;V')$ which satisfies

$$||u_1^\epsilon||_{L^2(0,T;V)} + ||u_1^\epsilon||_{H^1(0,T;V')} + ||u_2^\epsilon||_{L^2(0,T;V)} + ||u_2^\epsilon||_{H^1(0,T;V')} \le C \tag{3.7}$$

for a constant $C > 0$ independent of ϵ.

3.1.2 Homogenization of multi-continuum system

We study homogenization of this multi-continuum system by using the standard two scale asymptotic expansion. We consider the two scale asymptotic expansion of u_1^ϵ and u_2^ϵ

$$\begin{aligned} u_1^\epsilon(t,x) &= u_{10}(t,x,\frac{x}{\epsilon}) + \epsilon u_{11}(t,x,\frac{x}{\epsilon}) + \cdots \\ u_2^\epsilon(t,x) &= u_{20}(t,x,\frac{x}{\epsilon}) + \epsilon u_{21}(t,x,\frac{x}{\epsilon}) + \cdots , \end{aligned} \tag{3.8}$$

where the functions $u_{1j}(t,x,y)$ and $u_{2j}(t,x,y)$ are Y-periodic with respect to y. From (3.3), we have

$$
\begin{aligned}
&\mathcal{C}_{11}\frac{\partial(u_{10}+\epsilon u_{11}+\cdots)}{\partial t}\\
&=(\mathrm{div}_x+\frac{1}{\epsilon}\,\mathrm{div}_y)(\kappa_1(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{10}+\epsilon u_{11}+\cdots))\\
&\quad+\frac{1}{\epsilon}Q(u_{20}+\epsilon u_{21}-u_{10}-\epsilon u_{11}+\cdots)+q,\\
&\mathcal{C}_{22}\frac{\partial(u_{20}+\epsilon u_{21}+\cdots)}{\partial t}\\
&=(\mathrm{div}_x+\frac{1}{\epsilon}\,\mathrm{div}_y)(\kappa_2(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{20}+\epsilon u_{21}+\cdots))\\
&\quad+\frac{1}{\epsilon}Q(u_{10}+\epsilon u_{11}-u_{20}-\epsilon u_{21}+\cdots)+q.
\end{aligned}
\tag{3.9}
$$

Collecting the ϵ^{-2} terms, we obtain

$$
\begin{aligned}
&\mathrm{div}_y(\kappa_1(x,y)\nabla_y u_{10}(t,x,y))=0\\
&\mathrm{div}_y(\kappa_2(x,y)\nabla_y u_{20}(t,x,y))=0.
\end{aligned}
\tag{3.10}
$$

From this, we deduce u_{10} and u_{20} are independent of y. Collecting the ϵ^{-1} terms we obtain

$$
\begin{aligned}
&\mathrm{div}_y(\kappa_1\nabla u_{10})+\mathrm{div}_y(\kappa_1\nabla_y u_{11})+Q(u_{20}-u_{10})=0\\
&\mathrm{div}_y(\kappa_2\nabla u_{20})+\mathrm{div}_y(\kappa_2\nabla_y u_{21})+Q(u_{10}-u_{20})=0.
\end{aligned}
\tag{3.11}
$$

Therefore,

$$
\begin{aligned}
u_{11}(t,x,y)&=\sum_{i=1}^{d}N_1^i(x,y)\frac{\partial u_{10}(t,x)}{\partial x_i}+M_1(x,y)(u_{20}(t,x)-u_{10}(t,x))\\
u_{21}(t,x,y)&=\sum_{i=1}^{d}N_2^i(x,y)\frac{\partial u_{20}(t,x)}{\partial x_i}+M_2(x,y)(u_{10}(t,x)-u_{20}(t,x)),
\end{aligned}
\tag{3.12}
$$

where $N_1^i(x,y)$, $N_2^i(x,y)$ $(i=1,\ldots,d)$, $M_1(x,y)$ and $M_2(x,y)$, as functions of y are the solutions of the following cell problems respectively.

$$\begin{aligned}
&\mathrm{div}_y(\kappa_1(x,y)(e^i+\nabla_y N_1^i(x,y)))=0\\
&\mathrm{div}_y(\kappa_1(x,y)\nabla_y M_1(x,y))+Q(x,y)=0\\
&\mathrm{div}_y(\kappa_2(x,y)(e^i+\nabla_y N_2^i(x,y)))=0\\
&\mathrm{div}_y(\kappa_2(x,y)\nabla_y M_2(x,y))+Q(x,y)=0
\end{aligned} \tag{3.13}$$

with the periodic boundary condition, where e^i is the ith standard basis vector of $\mathbb{R}^d$. Problems (3.13 (a),(c)) have a unique solution in $H^1_\#(Y)/\mathbb{R}$; problems (3.13 (b),(d)) have a unique solution since $\int_Y Q(x,y)dy=0$. Collecting the ϵ^0 terms, we have,

$$\begin{aligned}
&\mathcal{C}_{11}\frac{\partial u_{10}}{\partial t}=\mathrm{div}_x(\kappa_1\nabla u_{10})+\mathrm{div}_y(\kappa_1\nabla_x u_{11})\\
&+\mathrm{div}_x(\kappa_1\nabla_y u_{11})+\mathrm{div}_y(\kappa_1\nabla_y u_{12})+Q(u_{21}-u_{11})+q\\
&\mathcal{C}_{22}\frac{\partial u_{20}}{\partial t}=\mathrm{div}_x(\kappa_2\nabla u_{20})+\mathrm{div}_y(\kappa_2\nabla_x u_{21})\\
&+\mathrm{div}_x(\kappa_2\nabla_y u_{21})+\mathrm{div}_y(\kappa_2\nabla_y u_{22})+Q(u_{11}-u_{21})+q.
\end{aligned} \tag{3.14}$$

Integrating with respect to y over Y and using (3.12), we have

$$\begin{aligned}
&\left(\int_Y \mathcal{C}_{11}dy\right)\frac{\partial u_{10}}{\partial t}=\mathrm{div}(\kappa_1^*\nabla u_{10})+\mathrm{div}\left((\int_Y \kappa_1\nabla_y M_1 dy)(u_{20}-u_{10})\right)\\
&+\left((\int_Y QN_2^i dy)\frac{\partial u_{20}}{\partial x_i}-(\int_Y QN_1^i dy)\frac{\partial u_{10}}{\partial x_i}\right)-\int_Y Q(M_1+M_2)dy(u_{20}-u_{10})+q\\
&\left(\int_Y \mathcal{C}_{22}dy\right)\frac{\partial u_{20}}{\partial t}=\mathrm{div}(\kappa_2^*\nabla u_{20})+\mathrm{div}\left((\int_Y \kappa_2\nabla_y M_2 dy)(u_{10}-u_{20})\right)\\
&+\left((\int_Y QN_1^i dy)\frac{\partial u_{10}}{\partial x_i}-(\int_Y QN_2^i dy)\frac{\partial u_{20}}{\partial x_i}\right)-\int_Y Q(M_1+M_2)dy(u_{10}-u_{20})+q,
\end{aligned} \tag{3.15}$$

where

$$\kappa^*_{1ij}(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i})dy$$
$$\kappa^*_{2ij}(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i})dy. \tag{3.16}$$

We note that $\kappa^*_{1ij}(x)$ and $\kappa^*_{2ij}(x)$ are standard homogenized coefficients for elliptic problems [7]. They are symmetric and positive definite ([7]). We will show in Section 3.2 that the initial conditions for u_{10}, u_{20} are

$$u_{10}(0,x) = g_1(x),\ u_{20}(0,x) = g_2(x). \tag{3.17}$$

In the appendix, we show that the homogenized problem (3.15) with these initial conditions has a unique solution.

Remark. The case where the continuum interacting term is scaled as $1/\epsilon$ considered in this chapter has the most interesting homogenization limit, in comparison to other scalings, e.g. the $1/\epsilon^2$ scale case considered in [29]. It can be shown that the continuum interacting coefficient $-\int_Y Q(M_1+M_2)dy$ in (3.15) is always negative while the interaction coefficient $\frac{1}{\epsilon}Q$ in the two-scale problem can be both positive and negative due to Assumption (3.1). The homogenized equation (3.15) has convection terms, which is different from the original equation (3.3).

We have the following homogenization results.

Theorem 3.1.1. *Assume that the solution N_1^i and N_2^i ($i = 1,\dots,d$) of cell problem (3.13 (a),(c)) belong to $C^2(\bar{\Omega}, C^2(\bar{Y}))$ and the coefficients κ_1 and κ_2 belong to $C^1(\bar{\Omega}, C^1(\bar{Y}))$. The sequence $(u_1^\epsilon, u_2^\epsilon)$ of the solutions to (3.3) converges weakly to (u_{10}, u_{20}) in $L^2(0,T;V)\times L^2(0,T;V)$, where (u_{10}, u_{20}) is the solution of the homogenized equations* (3.15) *with initial conditions* (3.17).

Since $\int_Y Q(x,y)dy = 0$, there is a vector function $\mathcal{Q}(x,y)$ which is periodic with respect to y such that $Q(x,y) = \text{div}_y\, \mathcal{Q}(x,y)$ (see [9], section 1.1). We have the following result on homogenization convergence rate.

Theorem 3.1.2. *Assume $\kappa_1, \kappa_2 \in C^1(\bar{\Omega}; C(\bar{Y}))$, $u_{10},\ u_{20} \in C([0,T]; C^2(\bar{\Omega})) \cap C^1([0,T]; C^1(\bar{\Omega}))$, $N_k^i, M_k \in C^1(\bar{\Omega}, C^1(\bar{Y}))$, ($i = 1, \ldots, d$, $k = 1, 2$), $\mathcal{Q} \in C^2(\bar{\Omega}; C^1(\bar{Y}))^2$. Then we have*

$$||\nabla u_1^\epsilon - \nabla u_{10} - \nabla_y u_{11}(\cdot, \cdot, \frac{\cdot}{\epsilon})||_{L^2(0,T;H)} + ||\nabla u_2^\epsilon - \nabla u_{20} - \nabla_y u_{21}(\cdot, \cdot, \frac{\cdot}{\epsilon})||_{L^2(0,T;H)} \leq c\epsilon^{\frac{1}{2}} \tag{3.18}$$

where the constant c is independent of ϵ.

We prove Theorems 3.1.1 and 3.1.2 in Sections 3 and 4 respectively.

3.2 Proof of homogenization convergence

In this section, we prove Theorem 3.1.1 on homogenization convergence for the solution of the two scale multi-continuum system (3.3). From (3.7), there exists a subsequence of $(u_1^\epsilon, u_2^\epsilon)$, which we still denote by $(u_1^\epsilon, u_2^\epsilon)$, u_{10} and u_{20} such that

$$u_1^\epsilon \rightharpoonup u_{10},\ u_2^\epsilon \rightharpoonup u_{20} \ \text{ in } \ L^2(0,T;V). \tag{3.19}$$

We show that (u_{10}, u_{20}) satisfies the homogenized problem (3.15). Recall N_1^i, N_2^i, M_1 and M_2 in $H^1_\#(Y)$ as functions of y are the solutions of the cell problems (3.13). Fixing $i = 1, \ldots, d$, we consider

$$\omega_1^\epsilon(x) = x_i + \epsilon N_1^i(x, \frac{x}{\epsilon}) \ \text{ and } \ \omega_2^\epsilon(x) = x_i + \epsilon N_2^i(x, \frac{x}{\epsilon}). \tag{3.20}$$

Under regularity conditions for κ_1, κ_2, N_1^i and N_2^i, we have

$$\begin{aligned}
&- \operatorname{div}(\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x)) \\
&= -\frac{1}{\epsilon}\text{div}_y(\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon}))) - \epsilon\,\text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon})) \\
&- \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon}))) - \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&- \operatorname{div}(\kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x)) \\
&= -\frac{1}{\epsilon}\text{div}_y(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon}))) - \epsilon\,\text{div}_x(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon})) \\
&- \text{div}_x(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon}))) - \text{div}_y(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon})).
\end{aligned} \tag{3.22}$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega)$. From (3.6), we have

$$\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\omega_1^\epsilon dx + \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla(\phi\omega_1^\epsilon)dx - \int_\Omega \frac{1}{\epsilon}Q^\epsilon(u_2^\epsilon - u_1^\epsilon)\phi\omega_1^\epsilon dx = \int_\Omega q\phi\omega_1^\epsilon dx, \tag{3.23}$$

and

$$\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\phi\omega_2^\epsilon dx + \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla(\phi\omega_2^\epsilon)dx - \int_\Omega \frac{1}{\epsilon}Q^\epsilon(u_1^\epsilon - u_2^\epsilon)\phi\omega_2^\epsilon dx = \int_\Omega q\phi\omega_2^\epsilon dx. \tag{3.24}$$

Multiplying (3.21) and (3.22) by ϕu_1^ϵ and ϕu_2^ϵ respectively and integrate over Ω we have

$$\begin{aligned}
&\int_\Omega \kappa_1^\epsilon \nabla\omega_1^\epsilon \cdot \nabla(\phi u_1^\epsilon)dx = -\epsilon\int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))\phi u_1^\epsilon dx \\
&- \int_\Omega \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon})))\phi u_1^\epsilon dx - \int_\Omega \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon}))\phi u_1^\epsilon dx,
\end{aligned} \tag{3.25}$$

and

$$\int_\Omega \kappa_2^\epsilon \nabla\omega_2^\epsilon \cdot \nabla(\phi u_2^\epsilon)dx = -\epsilon\int_\Omega \text{div}_x(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon}))\phi u_2^\epsilon dx$$
$$-\int_\Omega \text{div}_x(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})))\phi u_2^\epsilon dx - \int_\Omega \text{div}_y(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon}))\phi u_2^\epsilon dx. \tag{3.26}$$

Let $\in C_0^\infty(0,T)$. Subtracting (3.25), (3.26) from (3.23) and (3.24) respectively, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt + \int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon\cdot\nabla\phi\omega_1^\epsilon\psi dxdt\\
&-\int_0^T\int_\Omega \frac{1}{\epsilon}Q^\epsilon(u_2^\epsilon-u_1^\epsilon)\phi\omega_1^\epsilon\psi dxdt - \int_0^T\int_\Omega \kappa_1^\epsilon\nabla\omega_1^\epsilon\cdot(\nabla\phi u_1^\epsilon)\psi dxdt\\
&=\int_0^T\int_\Omega q\phi\omega_1^\epsilon\psi dxdt + \epsilon\int_0^T\int_\Omega \text{div}_x\left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon})\right)\phi u_1^\epsilon\psi dxdt\\
&+\int_0^T\int_\Omega \text{div}_x\left(\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon}))\right)\phi u_1^\epsilon\psi dxdt\\
&+\int_0^T\int_\Omega \text{div}_y\left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x N_1^i(x,\frac{x}{\epsilon})\right)\phi u_1^\epsilon\psi dxdt
\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\phi\psi\omega_2^\epsilon dxdt + \int_0^T\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon\cdot\nabla\phi\omega_2^\epsilon\psi dxdt\\
&-\int_0^T\int_\Omega \frac{1}{\epsilon}Q^\epsilon(u_1^\epsilon-u_2^\epsilon)\phi\omega_2^\epsilon\psi dxdt - \int_0^T\int_\Omega \kappa_2^\epsilon\nabla\omega_2^\epsilon\cdot(\nabla\phi u_2^\epsilon)\psi dxdt\\
&=\int_0^T\int_\Omega q\phi\omega_2^\epsilon\psi dxdt + \epsilon\int_0^T\int_\Omega \text{div}_x\left(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon})\right)\phi u_2^\epsilon\psi dxdt\\
&+\int_0^T\int_\Omega \text{div}_x\left(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon}))\right)\phi u_2^\epsilon\psi dxdt\\
&+\int_0^T\int_\Omega \text{div}_y\left(\kappa_2(x,\frac{x}{\epsilon})\nabla_x N_2^i(x,\frac{x}{\epsilon})\right)\phi u_2^\epsilon\psi dxdt.
\end{aligned}\tag{3.28}$$

We have the following lemma.

Lemma 3.2.1. *The functions* $\int_0^T \quad (t)u_1^\epsilon(t,x)dt$ *and* $\int_0^T \quad (t)u_2^\epsilon(t,x)dt$ *converge strongly in* H *to* $\int_0^T \psi(t)u_{10}(t,x)dt$ *and* $\int_0^T \psi(t)u_{20}(t,x)dt$ *respectively, for* $\psi \in C_0^\infty(0,T)$.

Proof This is the standard result in Jikov et al. [9]. As u_1^ϵ is uniformly bounded in $L^2(0,T;V)$, we have that $\int_0^T \quad (t)u_1^\epsilon(t,x)dt$ is uniformly bounded in V. Thus we can extract a subsequence which converges weakly in V and strongly in H. As for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \int_0^T \quad (t)u_1^\epsilon(t,x)\phi(x)dtdx \to \int_\Omega \int_0^T \psi(t)u_{10}(t,x)\phi(x)dtdx,$$

the limit is $\int_\Omega \psi(t)u_{10}(t,x)dt$. □

We have

$$\int_0^T \int_\Omega C_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt = -\int_\Omega C_{11}^\epsilon \left(\int_0^T u_1^\epsilon \frac{\partial \psi}{\partial t}dt\right)\phi\omega_1^\epsilon dx.$$

Note that C_{11}^ϵ converges weakly to $\int_Y C_{11}(x,y)dy$ in H, and $\int_0^T u_1^\epsilon \frac{\partial\psi}{\partial t}dt$ converges strongly to $\int_0^T u_{10}\frac{\partial\psi}{\partial t}dt$ in H. Thus

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T \int_\Omega C_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt &= -\int_0^T \int_\Omega \left(\int_Y C_{11}(x,y)dy\right) u_{10}\frac{\partial\psi}{\partial t}\phi x_i dxdt \\ &= \int_0^T \int_\Omega \left(\int_Y C_{11}(x,y)dy\right)\frac{\partial u_{10}}{\partial t}\psi\phi x_i dxdt.\end{aligned}$$

Note that we have

$$\begin{aligned}\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x) &= \kappa_1(x,\frac{x}{\epsilon})\big((e^i + \nabla_y N_1^i(x,\frac{x}{\epsilon})) + \epsilon\nabla_x N_1^i(x,\frac{x}{\epsilon})\big),\\ \kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x) &= \kappa_2(x,\frac{x}{\epsilon})\big((e^i + \nabla_y N_2^i(x,\frac{x}{\epsilon})) + \epsilon\nabla_x N_2^i(x,\frac{x}{\epsilon})\big).\end{aligned} \tag{3.29}$$

Also, note that due to Y-periodicity of κ and N^i, we have

$$\begin{aligned}
&\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon})) \rightharpoonup \int_Y \kappa_1(x,y)(e^i+\nabla_y N_1^i(x,y))dy,\\
&\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})) \rightharpoonup \int_Y \kappa_2(x,y)(e^i+\nabla_y N_2^i(x,y))dy \text{ in } H.
\end{aligned} \tag{3.30}$$

Passing to the limit in (3.27), (3.28), we obtain from Lemma 3.2.1,

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\int_Y \mathcal{C}_{11}dy\right)\frac{\partial u_{10}}{\partial t}\phi\psi x_i dxdt - \int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla\phi\psi u_{10}dxdt\\
&\quad+\lim_{\epsilon\to 0}\left(\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla\phi\psi\omega_1^\epsilon dxdt - \frac{1}{\epsilon}\int_0^T\int_\Omega Q^\epsilon(u_2^\epsilon-u_1^\epsilon)\phi\omega_1^\epsilon\psi dxdt\right)\\
&\quad=\int_0^T\int_\Omega q\phi x_i\psi dxdt + \int_0^T\int_\Omega \operatorname{div}\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\phi u_{10}\psi dxdt
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\int_Y \mathcal{C}_{22}dy\right)\frac{\partial u_{20}}{\partial t}\phi\psi x_i dxdt - \int_0^T\int_\Omega\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\cdot\nabla\phi\psi u_{20}dxdt\\
&\quad+\lim_{\epsilon\to 0}\left(\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla\phi\psi\omega_2^\epsilon dxdt - \frac{1}{\epsilon}\int_0^T\int_\Omega Q^\epsilon(u_1^\epsilon-u_2^\epsilon)\phi\omega_2^\epsilon\psi dxdt\right)\\
&\quad=\int_0^T\int_\Omega q\phi x_i\psi dxdt + \int_0^T\int_\Omega \operatorname{div}\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\phi u_{20}\psi dxdt.
\end{aligned} \tag{3.32}$$

Letting ϕ_1 and ϕ_2 in (3.6) be ϕx_i for $\phi\in C_0^\infty(\Omega)$, we get

$$\begin{aligned}
&\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\psi x_i dxdt + \int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla(\phi x_i)\psi dxdt\\
&-\frac{1}{\epsilon}\int_0^T\int_\Omega Q^\epsilon(u_2^\epsilon-u_1^\epsilon)\phi\psi x_i dxdt = \int_0^T\int_\Omega q\phi\psi x_i dxdt.
\end{aligned} \tag{3.33}$$

Passing to the limit when $\epsilon \to 0$, we obtain

$$\int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{11} dy \right) \frac{\partial u_{10}}{\partial t} \phi \psi x_i dx dt$$
$$+ \lim_{\epsilon \to 0} \left(\int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla(\phi x_i) \psi dx dt - \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon) \phi \psi x_i dx dt \right) \tag{3.34}$$
$$= \int_0^T \int_\Omega q \phi x_i \psi dx dt.$$

Subtracting (3.34) from (3.31), one obtains

$$- \int_0^T \int_\Omega \left(\int_Y \kappa_1 (e^i + \nabla_y N_1^i) dy \right) \cdot \nabla \phi \psi u_{10} dx dt$$
$$- \lim_{\epsilon \to 0} \left(\int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi \psi dx dt + \int_0^T \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon) N_1^i (x, \frac{x}{\epsilon}) \phi \psi dx dt \right) \tag{3.35}$$
$$= - \int_0^T \int_\Omega \left(\int_Y \kappa_1 (e^i + \nabla_y N_1^i) dy \right) \cdot \nabla (u_{10} \phi) \psi dx dt.$$

Using Lemma 3.2.1, we get

$$- \int_0^T \int_\Omega \left(\int_Y \kappa_1 (e^i + \nabla_y N_1^i) dy \right) \cdot \nabla \phi \psi u_{10} dx dt$$
$$- \lim_{\epsilon \to 0} \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi \psi dx dt - \int_0^T \int_\Omega \left(\int_Y Q N_1^i dy \right) (u_{20} - u_{10}) \phi \psi dx dt \tag{3.36}$$
$$= - \int_0^T \int_\Omega \int_Y \kappa_1 (e^i + \nabla_y N_1^i) dy \cdot \nabla (u_{10} \phi) \psi dx dt.$$

From this, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot e^i \phi\psi dx dt \\
&= -\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla\phi\psi u_{10}dxdt \\
&\quad+\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla(u_{10}\phi)\psi dxdt \\
&\quad-\int_0^T\int_\Omega\left(\int_Y QN_1^i dy\right)(u_{20}-u_{10})\phi\psi dxdt \\
&\quad=\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla u_{10}\phi\psi dxdt \\
&\quad+\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1\cdot e^i dy\right)(u_{20}-u_{10})\phi\psi dxdt,
\end{aligned} \tag{3.37}$$

where we use (3.13 (a),(b)) for the last term of (3.37). Similarly,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot e^i \phi\psi dx dt \\
&=\int_0^T\int_\Omega\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\cdot\nabla u_{20}\phi dx\psi dt \\
&+\int_0^T\int_\Omega\left(\int_Y \kappa_2\nabla_y M_2\cdot e^i dy\right)(u_{10}-u_{20})\phi\psi dxdt.
\end{aligned} \tag{3.38}$$

From (3.37), one obtains

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla\phi\psi dx dt \\
&=\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla u_{10}\frac{\partial\phi}{\partial x_i}\psi dxdt \\
&+\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1\cdot e^i dy\right)(u_{20}-u_{10})\frac{\partial\phi}{\partial x_i}\psi dxdt \\
&=\int_0^T\int_\Omega \kappa_1^*\nabla u_{10}\cdot\nabla\phi\psi dxdt+\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1 dy\right)\cdot\nabla\phi\psi(u_{20}-u_{10})dxdt,
\end{aligned} \tag{3.39}$$

where we have used the standard result on the symmetry of the homogenized coefficient κ_1^* defined in (3.16) (see, e.g., [7]). Similarly, we deduce

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla\phi\psi dxdt$$
$$= \int_0^T\int_\Omega \kappa_2^* \nabla u_{20}\cdot\nabla\phi\psi dxdt + \int_0^T\int_\Omega \left(\int_Y \kappa_2\nabla_y M_2 dy\right)\cdot\nabla\phi\psi(u_{10}-u_{20})dxdt, \tag{3.40}$$

where κ_2^* is defined in (3.16). We define γ_1^ϵ and γ_2^ϵ as

$$\gamma_1^\epsilon(x) = \epsilon M_1(x,\frac{x}{\epsilon}),\ \gamma_2^\epsilon(x) = \epsilon M_2(x,\frac{x}{\epsilon}). \tag{3.41}$$

Under the smoothness conditions for κ_1, M_1, we have

$$-\text{div}(\kappa_1^\epsilon(x)\nabla\gamma_1^\epsilon(x)) = -\frac{1}{\epsilon}\text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_y M_1(x,\frac{x}{\epsilon})) - \epsilon\,\text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon}))$$
$$- \text{div}_x(\kappa_1(x,\frac{x}{\epsilon})\nabla_y M_1(x,\frac{x}{\epsilon})) - \text{div}_y(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon})). \tag{3.42}$$

Letting $\phi_1(x) = \phi(x)\gamma_1^\epsilon(x)$ where $\phi\in\mathcal{C}_0^\infty(\Omega)$ in (3.6), we obtain

$$\int_\Omega \mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\gamma_1^\epsilon dx + \int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla(\phi\gamma_1^\epsilon)dx - \int_\Omega\frac{1}{\epsilon}Q^\epsilon(u_2^\epsilon - u_1^\epsilon)\phi\gamma_1^\epsilon dx = \int_\Omega q\phi\gamma_1^\epsilon dx. \tag{3.43}$$

Let $\psi \in C_0^\infty((0,T))$. From (3.42) we have

$$\begin{aligned}\int_0^T \int_\Omega \mathrm{div}(\kappa_1^\epsilon \nabla \gamma_1^\epsilon)\phi(x) u_1^\epsilon \psi(t) dxdt = -\frac{1}{\epsilon}\int_0^T \int_\Omega Q(x,\frac{x}{\epsilon})\phi(x) u_1^\epsilon \psi(t) dxdt \\ +\epsilon \int_0^T \int_\Omega \mathrm{div}_x \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon})\right)\phi(x) u_1^\epsilon \psi(t) dxdt \\ +\int_0^T \int_\Omega \mathrm{div}_x \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_y M_1(x,\frac{x}{\epsilon})\right)\phi(x) u_1^\epsilon \psi(t) dxdt \\ +\int_0^T \int_\Omega \mathrm{div}_y \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon})\right)\phi(x) u_1^\epsilon \psi(t) dxdt.\end{aligned} \tag{3.44}$$

Adding (3.43) and (3.44), we have

$$\begin{aligned}\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\gamma_1^\epsilon \psi dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla(\phi\gamma_1^\epsilon)\psi dxdt \\ -\int_0^T \int_\Omega \frac{1}{\epsilon} Q^\epsilon (u_2^\epsilon - u_1^\epsilon)\phi\gamma_1^\epsilon \psi dxdt + \int_0^T \int_\Omega \mathrm{div}(\kappa_1^\epsilon \nabla\gamma_1^\epsilon)\phi u_1^\epsilon \psi dxdt \\ = \int_0^T \int_\Omega q\phi\gamma_1^\epsilon \psi dxdt - \frac{1}{\epsilon}\int_0^T \int_\Omega Q(x,\frac{x}{\epsilon})\phi u_1^\epsilon \psi dxdt \\ +\epsilon \int_0^T \int_\Omega \mathrm{div}_x \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon})\right)\phi u_1^\epsilon \psi dxdt \\ +\int_0^T \int_\Omega \mathrm{div}_x \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_y M_1(x,\frac{x}{\epsilon})\right)\phi u_1^\epsilon \psi dxdt \\ +\int_0^T \int_\Omega \mathrm{div}_y \left(\kappa_1(x,\frac{x}{\epsilon})\nabla_x M_1(x,\frac{x}{\epsilon})\right)\phi u_1^\epsilon \psi dxdt.\end{aligned} \tag{3.45}$$

We note that on the left hand side of (3.45),

$$\begin{aligned}\int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla(\phi\gamma_1^\epsilon)\psi dxdt + \int_0^T \int_\Omega \mathrm{div}(\kappa_1^\epsilon \nabla\gamma_1^\epsilon)\phi u_1^\epsilon \psi dxdt = \\ \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla\phi\gamma_1^\epsilon \psi dxdt - \int_0^T \int_\Omega \kappa_1^\epsilon \nabla\gamma_1^\epsilon \cdot \nabla\phi u_1^\epsilon dxdt.\end{aligned}$$

Passing (3.45) to the limit, using Lemma 3.2.1, one obtains

$$
\begin{aligned}
&-\int_0^T\int_\Omega\left(\int_Y \kappa_1(x,y)\nabla_y M_1(x.y)dy\right)\cdot\nabla\phi(x)u_{10}\psi(t)dxdt\\
&-\int_0^T\int_\Omega\left(\int_Y Q(x,y)M_1(x,y)dy\right)(u_{20}-u_{10})\phi(x)\psi(t)dxdt\\
=&-\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_\Omega Q(x,\frac{x}{\epsilon})\phi(x)u_1^\epsilon\psi(t)dxdt\\
&+\int_0^T\int_\Omega \operatorname{div}\left(\int_Y \kappa_1(x,y)\nabla_y M_1(x,y)dy\right)\phi(x)u_{10}\psi(t)dxdt\\
&+\int_0^T\int_\Omega\left(\int_Y \operatorname{div}_y(\kappa_1(x,y)\nabla_x M_1(x,y))dy\right)\phi(x)u_{10}\psi(t)dxdt.
\end{aligned}
\tag{3.46}
$$

Due to periodicity, the last term on the right hand side equals 0. We thus have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_\Omega Q(x,\frac{x}{\epsilon})u_1^\epsilon\phi(x)\psi(t)dxdt=\int_0^T\int_\Omega\left(\int_Y QM_1dy\right)(u_{20}-u_{10})\phi\psi dxdt\\
&-\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1dy\right)\cdot\nabla(\phi u_{10})\psi dxdt+\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1dy\right)\cdot\nabla\phi u_{10}\psi dxdt\\
&=\int_0^T\int_\Omega\left(\int_Y QM_1dy\right)(u_{20}-u_{10})\phi\psi dxdt-\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1dy\right)\cdot\nabla u_{10}\phi\psi dxdt.
\end{aligned}
\tag{3.47}
$$

Similarly, we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_\Omega Q(x,\frac{x}{\epsilon})u_2^\epsilon\phi(x)\psi(t)dxdt\\
&=\int_0^T\int_\Omega\left(\int_Y QM_2dy\right)(u_{10}-u_{20})\phi\psi dxdt-\int_0^T\int_\Omega\left(\int_Y \kappa_2\nabla_y M_2dy\right)\cdot\nabla u_{20}\phi\psi dxdt.
\end{aligned}
\tag{3.48}
$$

Thus

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_\Omega Q(x,\frac{x}{\epsilon})(u_2^\epsilon-u_1^\epsilon)\phi(x)\psi(t)dxdt\\
=&-\int_0^T\int_\Omega\left(\int_Y Q(M_1+M_2)dy\right)(u_{20}-u_{10})\phi\psi dxdt\\
&-\int_0^T\int_\Omega\left(\int_Y \kappa_2\nabla_y M_2dy\right)\cdot\nabla u_{20}\phi\psi dxdt+\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y M_1dy\right)\cdot\nabla u_{10}\phi\psi dxdt.\\
=&-\int_0^T\int_\Omega\left(\int_Y Q(M_1+M_2)dy\right)(u_{20}-u_{10})\phi(x)\psi(t)dxdt\\
&-\int_0^T\int_\Omega\left(\int_Y \kappa_2 e^i\cdot\nabla_y M_2dy\right)\frac{\partial u_{20}}{\partial x_i}\phi dx\psi dt\\
&+\int_0^T\int_\Omega\left(\int_Y \kappa_1 e^i\cdot\nabla_y M_1dy\right)\frac{\partial u_{10}}{\partial x_i}\phi dx\psi dt\\
=&-\int_0^T\int_\Omega\left(\int_Y Q(M_1+M_2)dy\right)(u_{20}-u_{10})\phi\psi dxdt\\
&+\int_0^T\int_\Omega\left(\int_Y \kappa_2\nabla_y N_2^i\cdot\nabla_y M_2dy\right)\frac{\partial u_{20}}{\partial x_i}\phi\psi dx\psi dt\\
&-\int_0^T\int_\Omega\left(\int_Y \kappa_1\nabla_y N_1^i\cdot\nabla_y M_1dy\right)\frac{\partial u_{10}}{\partial x_i}\phi\psi dxdt
\end{aligned}
\tag{3.49}
$$

where we have used cell problems (3.13 (a),(c)). Using cell problems (3.13 (b),(d)), we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\int_\Omega Q(x,\frac{x}{\epsilon})(u_2^\epsilon-u_1^\epsilon)\phi\psi dxdt\\
&\qquad=-\int_0^T\int_\Omega\left(\int_Y Q(M_1+M_2)dy\right)(u_{20}-u_{10})\phi\psi dxdt\\
&\qquad+\int_0^T\int_\Omega\left(\int_Y QN_2^idy\right)\frac{\partial u_{20}}{\partial x_i}\phi\psi dxdt-\int_0^T\int_\Omega\left(\int_Y QN_1^idy\right)\frac{\partial u_{10}}{\partial x_i}\phi\psi dxdt.
\end{aligned}
\tag{3.50}
$$

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1

From (3.6)

$$\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla\phi\psi dxdt - \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon)\phi\psi dxdt$$
$$= \int_0^T \int_\Omega q\phi\psi dxdt. \tag{3.51}$$

for all $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty((0,T))$. Passing to the limit, from (3.39), (3.50), Lemma 3.2.1, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{11} dy \frac{\partial u_{10}}{\partial t}\phi\psi dxdt \\
&\quad + \int_0^T \int_\Omega \kappa_1^* \nabla u_{10} \cdot \nabla\phi\psi dxdt + \int_0^T \int_\Omega \left(\int_Y \kappa_1 \nabla_y M_1 dy\right) \cdot \nabla\phi (u_{20} - u_{10})\psi dxdt \\
&\quad - \int_0^T \int_\Omega \left(\left(\int_Y QN_2^i dy\right) \frac{\partial u_{20}}{\partial x_i} - \left(\int_Y QN_1^i dy\right) \frac{\partial u_{10}}{\partial x_i}\right) \phi\psi dxdt \\
&\quad + \int_0^T \int_\Omega \left(\int_Y Q(M_1 + M_2) dy\right) (u_{20} - u_{10})\phi\psi dxdt = \int_0^T \int_\Omega q\phi\psi dxdt.
\end{aligned} \tag{3.52}$$

Similarly, we derive

$$\begin{aligned}
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{22} dy \frac{\partial u_{20}}{\partial t}\phi dx\psi dt \\
&\quad + \int_0^T \int_\Omega \kappa_2^* \nabla u_{20} \cdot \nabla\phi\psi dxdt + \int_0^T \int_\Omega \left(\int_Y \kappa_2 \nabla_y M_2 dy\right) \cdot \nabla\phi (u_{10} - u_{20})\psi dxdt \\
&\quad - \int_0^T \int_\Omega \left(\left(\int_Y QN_1^i dy\right) \frac{\partial u_{10}}{\partial x_i} - \left(\int_Y QN_2^i dy\right) \frac{\partial u_{20}}{\partial x_i}\right) \phi\psi dxdt \\
&\quad + \int_0^T \int_\Omega \left(\int_Y Q(M_1 + M_2) dy\right) (u_{10} - u_{20})\phi\psi dxdt. = \int_0^T \int_\Omega q\phi\psi dxdt,
\end{aligned} \tag{3.53}$$

where

$$\kappa^*_{1ij}(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i})dy, \ \ \kappa^*_{2ij}(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i})dy. \tag{3.54}$$

We now prove the initial condition of $u_{10},\ u_{20}$. From (3.3) and Theorem A.0.1, we deduce that $\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}$ is bounded in $L^2(0,T;V')$. Similarly, $\mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}$ is bounded in $L^2(0,T;V')$. Let $\psi(t,x) \in \mathcal{C}_0^\infty(0,T;V)$, i.e. $\psi(0,x) = \psi(T,x) = 0$. As $\epsilon \to 0$, we have

$$\begin{aligned} \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\psi dxdt = -\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon \frac{\partial \psi}{\partial t} dxdt \to -\int_0^T \int_\Omega \langle \mathcal{C}_{11}\rangle u_{10}\frac{\partial \psi}{\partial t}dxdt \\ = \int_0^T \int_\Omega \langle \mathcal{C}_{11}\rangle \frac{\partial u_{10}}{\partial t}\psi dxdt \end{aligned} \tag{3.55}$$

where $\langle\cdot\rangle$ denotes the integral average over Y. Note that we used Lemma 3.2.1. This shows that the weak limit of $\mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}$ in $L^2(0,T;V')$ is $\langle \mathcal{C}_{11}\rangle \frac{\partial u_{10}}{\partial t}$. Now we choose $\psi \in \mathcal{C}^\infty(0,T;V)$ so that $\psi(T,x) = 0$. Then

$$\begin{aligned} \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\psi dxdt = -\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon \frac{\partial \psi}{\partial t}dxdt + \int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon(0,x)\psi(0,x)dx \\ \to -\int_0^T \int_\Omega \langle \mathcal{C}_{11}\rangle u_{10}\frac{\partial \psi}{\partial t}dxdt + \int_\Omega \langle \mathcal{C}_{11}\rangle g_1 \psi(0,x)dx. \end{aligned} \tag{3.56}$$

On the other hand

$$\begin{aligned} &\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\psi dxdt \to \int_0^T \int_\Omega \langle \mathcal{C}_{11}\rangle \frac{\partial u_{10}}{\partial t}\psi dxdt \\ &= -\int_0^T \int_\Omega \langle \mathcal{C}_{11}\rangle u_{10}\frac{\partial \psi}{\partial t}dxdt + \int_\Omega \langle \mathcal{C}_{11}\rangle u_{10}(0,x)\psi(0,x)dx. \end{aligned} \tag{3.57}$$

This shows that $\langle \mathcal{C}_{11}\rangle u_{10}(0,x) = \langle \mathcal{C}_{11}\rangle g_1(x)$. i.e. the initial condition of u_{10} is $u_{10}(0,x) = g_1(x)$. Similarly, we have initial condition $u_{20}(0,x) = g_2(x)$.

3.3 Homogenization error

We prove Theorem 3.1.2 in this section. Let

$$u^\epsilon_{11}(t,x) = u_{10}(t,x) + \epsilon u_{11}(t,x,\frac{x}{\epsilon}),\ u^\epsilon_{21}(t,x) = u_{20}(t,x) + \epsilon u_{21}(t,x,\frac{x}{\epsilon}). \tag{3.58}$$

Using (3.12) we have

$$\begin{aligned}
&\text{div}(\kappa^\epsilon_1(x)\nabla u^\epsilon_{11}(t,x)) + \frac{1}{\epsilon}Q^\epsilon(x)(u^\epsilon_{21}(t,x) - u^\epsilon_{11}(t,x))\\
&= \text{div}(\kappa^\epsilon_1\nabla u_{10}) + \epsilon\,\text{div}(\kappa^\epsilon_1\nabla_x u_{11}) + \text{div}(\kappa^\epsilon_1\nabla_y N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i})\\
&\quad + \text{div}(\kappa^\epsilon_1\nabla_y M_1(x,\frac{x}{\epsilon})(u_{20} - u_{10})) + \frac{1}{\epsilon}Q^\epsilon(u_{20} - u_{10})\\
&\quad + Q^\epsilon(N^i_2(x,\frac{x}{\epsilon})\frac{\partial u_{20}}{\partial x_i} - N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i}) + Q^\epsilon(M_2(x,\frac{x}{\epsilon}) + M_1(x,\frac{x}{\epsilon}))(u_{10} - u_{20})\\
&= \text{div}(\kappa^\epsilon_1\nabla u_{10}) + \epsilon\,\text{div}(\kappa^\epsilon_1\nabla_x u_{11}) + \text{div}(\kappa^\epsilon_1\nabla_y N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i})\\
&\quad + \text{div}(\kappa^\epsilon_1\nabla_y M_1(x,\frac{x}{\epsilon})(u_{20} - u_{10})) + \text{div}(\mathcal{Q}(x,\frac{x}{\epsilon})(u_{20} - u_{10}))\\
&\quad - \text{div}_x(\mathcal{Q}(x,\frac{x}{\epsilon})(u_{20} - u_{10})) + Q(x,\frac{x}{\epsilon})(N^i_2(x,\frac{x}{\epsilon})\frac{\partial u_{20}}{\partial x_i} - N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i})\\
&\quad + Q(x,\frac{x}{\epsilon})(M_2(x,\frac{x}{\epsilon}) + M_1(x,\frac{x}{\epsilon}))(u_{10} - u_{20})\\
&= \text{div}(\kappa^\epsilon_1\nabla u_{10}) + \epsilon\,\text{div}(\kappa^\epsilon_1\nabla_x u_{11}) + \text{div}(\kappa^\epsilon_1\nabla_y N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i})\\
&\quad + \text{div}(\kappa^\epsilon_1\nabla_y M_1(x,\frac{x}{\epsilon})(u_{20} - u_{10})) + \text{div}(\mathcal{Q}(x,\frac{x}{\epsilon})(u_{20} - u_{10}))\\
&\quad - \text{div}_x(\mathcal{Q}(x,\frac{x}{\epsilon})(u_{20} - u_{10})) - \text{div}(\int_Y \mathcal{Q}(x,y)dy(u_{20} - u_{10}))\\
&\quad + \text{div}(\int_Y \mathcal{Q}(x,y)dy(u_{20} - u_{10})) + Q(x,\frac{x}{\epsilon})(N^i_2(x,\frac{x}{\epsilon})\frac{\partial u_{20}}{\partial x_i} - N^i_1(x,\frac{x}{\epsilon})\frac{\partial u_{10}}{\partial x_i})\\
&\quad + Q(x,\frac{x}{\epsilon})(M_2(x,\frac{x}{\epsilon}) + M_1(x,\frac{x}{\epsilon}))(u_{10} - u_{20}).
\end{aligned} \tag{3.59}$$

We let $F(t,x,y)$ be defined as

$$
\begin{aligned}
&F(t,x,y) = \kappa_1(x,y)\nabla u_{10}(t,x) + \kappa_1(x,y)\nabla_y N_1^i(x,y)\frac{\partial u_{10}(t,x)}{\partial x_i} \\
&+ \kappa_1(x,y)\nabla_y M_1(x,y)(u_{20}(t,x) - u_{10}(t,x)) \\
&+ \mathcal{Q}(x,y)(u_{20}(t,x) - u_{10}(t,x)) - \int_Y \mathcal{Q}(x,y)dy(u_{20}(t,x) - u_{10}(t,x)) \\
&- \left(\int_Y \kappa_1(x,y)dy\nabla u_{10}(t,x) + \int_Y \kappa_1(x,y)\nabla_y N_1^i(x,y)dy\frac{\partial u_{10}(t,x)}{\partial x_i}\right. \\
&\left.+ \int_Y \kappa_1(x,y)\nabla_y M_1(x,y)dy(u_{20}(t,x) - u_{10}(t,x))\right).
\end{aligned}
\tag{3.60}
$$

We let

$$
\begin{aligned}
&G(t,x,y) = -\operatorname{div}_x(\mathcal{Q}(x,y)(u_{20}-u_{10})) + \operatorname{div}\big(\int_Y \mathcal{Q}(x,y)dy(u_{20}-u_{10})\big) \\
&+ Q(x,y)\big(N_2^i(x,y)\frac{\partial u_{20}}{\partial x_i} - N_1^i(x,y)\frac{\partial u_{10}}{\partial x_i}\big) + Q(x,y)(M_2(x,y)+M_1(x,u))(u_{10}-u_{20}) \\
&- \left(\int_Y Q(x,y)N_2^i(x,y)dy\frac{\partial u_{20}}{\partial x_i} - \int_Y Q(x,y)N_1^i(x,y)dy\frac{\partial u_{10}}{\partial x_i}\right) \\
&- \int_Y Q(x,y)(M_2(x,y)+M_1(x,y))(u_{10}-u_{20})dy.
\end{aligned}
\tag{3.61}
$$

Note that from (3.13), we deduce $\operatorname{div}_y F(t,x,y) = 0$. Further, we have $\int_Y F_i(t,x,y)dy = 0$, $i = 1,\dots,d$. From the hypothesis of the theorem, $F_i(t,x,y) \in C(0,T;C^1(\bar{\Omega};C(\bar{Y})))$. Thus, from p.6 of [9], there are functions $\alpha_{ij}(t,x,y) \in C(0,T;C^1(\bar{\Omega};C^1(\bar{Y})))$ such that

$$
\alpha_{ij} = -\alpha_{ji} \text{ and } F_i(t,x,y) = \frac{\partial}{\partial y_j}\alpha_{ij}(t,x,y), \tag{3.62}
$$

for $i,j = 1,\dots,d$. From this, we have

$$
F_i(t,x,\frac{x}{\epsilon}) = \epsilon\frac{d}{dx_j}\alpha_{ij}(t,x,\frac{x}{\epsilon}) - \epsilon\frac{\partial}{\partial x_j}\alpha_{ij}(t,x,\frac{x}{\epsilon}), \tag{3.63}
$$

where $\frac{d}{dx_j}$ is the total partial derivative with respect to x_j of a function of t and x. Then for any $\phi(x) \in V$, we have

$$\begin{aligned}
&\int_\Omega F_i(t,x,\frac{x}{\epsilon})\frac{\partial}{\partial x_i}\phi(x)dx \\
&= \int_\Omega \big(\epsilon\frac{d}{dx_j}\alpha_{ij}(t,x,\frac{x}{\epsilon}) - \epsilon\frac{\partial}{\partial x_j}\alpha_{ij}(t,x,\frac{x}{\epsilon})\big)\frac{\partial}{\partial x_i}\phi(x)dx \\
&= -\epsilon\int_\Omega \alpha_{ij}(t,x,\frac{x}{\epsilon})\frac{\partial^2\phi(x)}{\partial x_j\partial x_i}dx - \epsilon\int_\Omega \frac{\partial}{\partial x_j}\alpha_{ij}(t,x,\frac{x}{\epsilon})\frac{\partial}{\partial x_i}\phi(x)dx \\
&= -\epsilon\int_\Omega \frac{\partial}{\partial x_j}\alpha_{ij}(t,x,\frac{x}{\epsilon})\frac{\partial}{\partial x_i}\phi(x)dx.
\end{aligned} \tag{3.64}$$

As $\int_Y G(t,x,y)dy = 0$, there exists a vector function $\mathcal{G} \in C(0,T;C^1(\bar{\Omega};C^1(\bar{Y})))$ which is Y-periodic with respect to y such that $\mathrm{div}_y\,\mathcal{G} = G$. Thus for any $\phi(x) \in V$, we have

$$\begin{aligned}
\int_\Omega G(t,x,\frac{x}{\epsilon})\phi dx &= \int_\Omega \mathrm{div}_y\,\mathcal{G}(t,x,\frac{x}{\epsilon})\phi dx \\
&= \epsilon\int_\Omega \mathrm{div}\,\mathcal{G}(t,x,\frac{x}{\epsilon})\phi dx - \epsilon\int_\Omega \mathrm{div}_x\,\mathcal{G}(t,x,\frac{x}{\epsilon})\phi dx \\
&= -\epsilon\int_\Omega \mathcal{G}(t,x,\frac{x}{\epsilon})\cdot\nabla\phi dx - \epsilon\int_\Omega \mathrm{div}_x\,\mathcal{G}(t,x,\frac{x}{\epsilon})\phi dx.
\end{aligned} \tag{3.65}$$

From (3.58), we have

$$\left\|\mathcal{C}_{11}^\epsilon\frac{\partial u_{11}^\epsilon}{\partial t}(t) - \int_Y \mathcal{C}_{11}dy\frac{\partial u_{10}}{\partial t}(t)\right\|_{V'} \leq c\epsilon \tag{3.66}$$

where c is independent of t. From (3.59), (3.64), (3.65) and (3.66), we have

$$\Big\|\Big(\mathcal{C}_{11}^\epsilon \frac{\partial u_{11}^\epsilon}{\partial t}(t) - \mathbf{div}(\kappa_1^\epsilon(x)\nabla u_{11}^\epsilon(t)) - \frac{1}{\epsilon}Q^\epsilon(u_{21}^\epsilon(t) - u_{11}^\epsilon(t))\Big)$$
$$-\Big(\int_Y \mathcal{C}_{11}dy\frac{\partial u_{10}}{\partial t}(t) - \mathrm{div}(\kappa_1^*\nabla u_{10}(t)) - \mathrm{div}\big(\int_Y \kappa_1\nabla_y M_1 dy(u_{20}(t) - u_{10}(t))\big)$$
$$-\big(\int_Y QN_2^i dy\frac{\partial u_{20}}{\partial x_i}(t) - \int_Y QN_1^i dy\frac{\partial u_{10}}{\partial x_i}(t)\big) + \int_Y Q(M_1 + M_2)dy(u_{20}(t) - u_{10}(t))\Big)\Big\|_{V'}$$
$$\leq c\epsilon. \tag{3.67}$$

Let $\tau^\epsilon \in \mathcal{D}(\Omega)$ be such that

$$\tau^\epsilon(x) = 0 \quad \text{if} \;\; d(x, \partial\Omega) \leq \epsilon, \;\; \tau^\epsilon(x) = 1 \quad \text{if} \;\; d(x, \partial\Omega) \geq 2\epsilon, \;\; \epsilon|\nabla\tau^\epsilon(x)| \leq C,$$

where C is independent of ϵ. We define the functions

$$\omega_{11}^\epsilon(t, x) = u_{10}(t, x) + \epsilon\tau^\epsilon(x)u_{11}(t, x, \frac{x}{\epsilon}), \;\; \omega_{21}^\epsilon(t, x) = u_{20}(t, x) + \epsilon\tau^\epsilon(x)u_{21}(t, x, \frac{x}{\epsilon}). \tag{3.68}$$

Using the smoothness asumptions of the theorem, we have

$$\begin{aligned}
&\nabla(u_{11}^\epsilon(t, x) - \omega_{11}^\epsilon(t, x)) \\
&= -\epsilon\nabla\tau^\epsilon(x)u_{11}(t, x, \frac{x}{\epsilon}) + \epsilon(1 - \tau^\epsilon(x))\nabla_x u_{11}(t, x, \frac{x}{\epsilon}) + (1 - \tau^\epsilon(x))\nabla_y u_{11}(t, x, \frac{x}{\epsilon}), \\
&\nabla(u_{21}^\epsilon(t, x) - \omega_{21}^\epsilon(t, x)) \\
&= -\epsilon\nabla\tau^\epsilon(x)u_{21}(t, x, \frac{x}{\epsilon}) + \epsilon(1 - \tau^\epsilon(x))\nabla_x u_{21}(t, x, \frac{x}{\epsilon}) + (1 - \tau^\epsilon(x))\nabla_y u_{21}(t, x, \frac{x}{\epsilon}).
\end{aligned} \tag{3.69}$$

It follows from (3.69) that

$$\|u_{11}^{\epsilon}(t) - \omega_{11}^{\epsilon}(t)\|_{H^1(\Omega)} \leq c\epsilon^{\frac{1}{2}}, \;\; \|u_{21}^{\epsilon}(t) - \omega_{21}^{\epsilon}(t)\|_{H^1(\Omega)} \leq c\epsilon^{\frac{1}{2}} \tag{3.70}$$

where the constant c is independent of t. From (3.70), we have

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^{\epsilon}(x)\frac{\partial(u_{11}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x))}{\partial t}\phi(x)dx + \int_\Omega \kappa_1^{\epsilon}(x)\nabla(u_{11}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x)) \cdot \nabla\phi(x)dx \\
&- \int_\Omega \mathcal{Q}(x,\frac{x}{\epsilon})((u_{11}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x)) - (u_{21}^{\epsilon}(t,x) - \omega_{21}^{\epsilon}(t,x)))\nabla\phi(x)dx \\
&- \int_\Omega \mathcal{Q}(x,\frac{x}{\epsilon})((\nabla u_{11}^{\epsilon}(t,x) - \nabla\omega_{11}^{\epsilon}(t,x)) - (\nabla u_{21}^{\epsilon}(t,x) - \nabla\omega_{21}^{\epsilon}(t,x)))\phi(x)dx \\
&- \int_\Omega \operatorname{div}_x \mathcal{Q}(x,\frac{x}{\epsilon})((u_{11}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x)) - (u_{21}^{\epsilon}(t,x) - \omega_{21}^{\epsilon}(t,x)))\phi(x)dx \\
&\leq c\Bigg(\left\|\frac{\partial(u_{11}^{\epsilon}(t) - \omega_{11}^{\epsilon}(t))}{\partial t}\right\|_{H^1(\Omega)} + \|u_{11}^{\epsilon}(t) - \omega_{11}^{\epsilon}(t)\|_{H^1(\Omega)} \\
&+ \|u_{21}^{\epsilon}(t) - \omega_{21}^{\epsilon}(t)\|_{H^1(\Omega)}\Bigg) \|\phi\|_V \leq c\epsilon^{\frac{1}{2}} \|\phi\|_V
\end{aligned} \tag{3.71}$$

for all $\phi \in V$, where $c > 0$ is independent of t. Then we obtain

$$\begin{aligned}
&\Bigg\|\left(\mathcal{C}_{11}^{\epsilon}(x)\frac{\partial u_{11}^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_1^{\epsilon}(x)\nabla u_{11}^{\epsilon}(t,x)) - \frac{1}{\epsilon}Q^{\epsilon}(x)(u_{21}^{\epsilon}(t,x) - u_{11}^{\epsilon}(t,x))\right) \\
&- \left(\mathcal{C}_{11}^{\epsilon}(x)\frac{\partial \omega_{11}^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_1^{\epsilon}(x)\nabla \omega_{11}^{\epsilon}(t,x)) - \frac{1}{\epsilon}Q^{\epsilon}(x)(\omega_{21}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x))\right)\Bigg\|_{V'} \\
&\leq c\epsilon^{\frac{1}{2}}.
\end{aligned} \tag{3.72}$$

From (3.3), (3.15), (3.67) and (3.72), we obtain

$$\Bigg\| \left(\mathcal{C}_{11}^{\epsilon}(x) \frac{\partial u_1^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_1^{\epsilon}(x)\nabla u_1^{\epsilon}(t,x)) - \frac{1}{\epsilon} Q^{\epsilon}(x)(u_2^{\epsilon}(t,x) - u_1^{\epsilon}(t,x)) \right)$$
$$- \left(\mathcal{C}_{11}^{\epsilon}(x) \frac{\partial \omega_{11}^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_1^{\epsilon}(x)\nabla \omega_{11}^{\epsilon}(t,x)) - \frac{1}{\epsilon} Q^{\epsilon}(x)(\omega_{21}^{\epsilon}(t,x) - \omega_{11}^{\epsilon}(t,x)) \right) \Bigg\|_{V'}$$
$$\leq c\epsilon^{\frac{1}{2}} \tag{3.73}$$

where c is independent of t. Similarly,

$$\Bigg\| \left(\mathcal{C}_{22}^{\epsilon}(x) \frac{\partial u_2^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_2^{\epsilon}(x)\nabla u_2^{\epsilon}(t,x)) - \frac{1}{\epsilon} Q^{\epsilon}(x)(u_1^{\epsilon}(t,x) - u_2^{\epsilon}(t,x)) \right)$$
$$- \left(\mathcal{C}_{22}^{\epsilon}(x) \frac{\partial \omega_{21}^{\epsilon}(t,x)}{\partial t} - \text{div}(\kappa_2^{\epsilon}(x)\nabla \omega_{21}^{\epsilon}(t,x)) - \frac{1}{\epsilon} Q^{\epsilon}(x)(\omega_{11}^{\epsilon}(t,x) - \omega_{21}^{\epsilon}(t,x)) \right) \Bigg\|_{V'}$$
$$\leq c\epsilon^{\frac{1}{2}}. \tag{3.74}$$

Let $\lambda > 0$. Let $\hat{u}_i^{\epsilon}(t,x) = u_i^{\epsilon}(t,x)e^{-\lambda t},\ \hat{\omega}_{i1}^{\epsilon}(t,x) = \omega_{i1}^{\epsilon}(t,x)e^{-\lambda t}$ for $i = 1, 2$. From (3.73) and (3.74), we deduce

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial}{\partial t}(u_1^\epsilon - \omega_{11}^\epsilon)(\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon)dx + \int_\Omega \kappa_1^\epsilon \big(\nabla u_1^\epsilon - \nabla \omega_{11}^\epsilon\big) \cdot (\nabla \hat{u}_1^\epsilon - \nabla \hat{\omega}_{11}^\epsilon)dx \\
&+ \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial}{\partial t}(u_2^\epsilon - \omega_{21}^\epsilon)(\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)dx + \int_\Omega \kappa_2^\epsilon \big(\nabla u_2^\epsilon - \nabla \omega_{21}^\epsilon\big) \cdot (\nabla \hat{u}_2^\epsilon - \nabla \hat{\omega}_{21}^\epsilon)dx \\
&- \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \big((\nabla u_1^\epsilon - \nabla \omega_{11}^\epsilon) - (\nabla u_2^\epsilon - \nabla \omega_{21}^\epsilon)\big)\big((\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon) - (\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)\big)dx \\
&- \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \big((\nabla \hat{u}_1^\epsilon - \nabla \hat{\omega}_{11}^\epsilon) - (\nabla \hat{u}_2^\epsilon - \nabla \hat{\omega}_{21}^\epsilon)\big)\big((u_1^\epsilon - \omega_{11}^\epsilon) - (u_2^\epsilon - \omega_{21}^\epsilon)\big)dx \\
&- \int_\Omega \mathrm{div}_x\, \mathcal{Q}(x, \frac{x}{\epsilon})\big((u_1^\epsilon - \omega_{11}^\epsilon) - (u_2^\epsilon - \omega_{21}^\epsilon)\big)\big((\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon) - (\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)\big)dx \\
&\leq c\epsilon^{\frac{1}{2}}\big(||\hat{u}_1^\epsilon(t) - \hat{\omega}_{11}^\epsilon(t)||_V + ||\hat{u}_2^\epsilon(t) - \hat{\omega}_{21}^\epsilon(t)||_V\big).
\end{aligned} \tag{3.75}$$

As $u_i^\epsilon(t,x) = \hat{u}_i^\epsilon(t,x)e^{\lambda t},\ \omega_{i1}^\epsilon(t,x) = \hat{\omega}_{i1}^\epsilon(t,x)e^{\lambda t}$

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial}{\partial t}(\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon)(\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon)dx + \lambda \int_\Omega \mathcal{C}_{11}^\epsilon (\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon)^2 dx + \int_\Omega \kappa_1^\epsilon |\nabla \hat{u}_1^\epsilon - \nabla \hat{\omega}_{11}^\epsilon|^2 dx \\
&+ \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial}{\partial t}(\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)(\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)dx + \lambda \int_\Omega \mathcal{C}_{22}^\epsilon (\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)^2 dx + \int_\Omega \kappa_2^\epsilon |\nabla \hat{u}_2^\epsilon - \nabla \hat{\omega}_{21}^\epsilon|^2 dx \\
&- 2\int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \big((\nabla \hat{u}_1^\epsilon - \nabla \hat{\omega}_{11}^\epsilon) - (\nabla \hat{u}_2^\epsilon - \nabla \hat{\omega}_{21}^\epsilon)\big)\big((\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon) - (\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon))\big)dx \\
&- \int_\Omega \mathrm{div}_x\, \mathcal{Q}(x, \frac{x}{\epsilon})\big((\hat{u}_1^\epsilon - \hat{\omega}_{11}^\epsilon) - (\hat{u}_2^\epsilon - \hat{\omega}_{21}^\epsilon)\big)^2 dx \\
&\leq ce^{-\lambda t}\epsilon^{\frac{1}{2}}\big(||\hat{u}_1^\epsilon(t) - \hat{\omega}_{11}^\epsilon(t)||_V + ||\hat{u}_2^\epsilon(t) - \hat{\omega}_{21}^\epsilon(t)||_V\big).
\end{aligned} \tag{3.76}$$

Integrating over $[0,T]$ we get

$$\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon(T)-\hat{\omega}_{11}^\epsilon(T))^2dxdt+\lambda\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon-\hat{\omega}_{11}^\epsilon)^2dxdt$$
$$+\int_0^T\int_\Omega \kappa_1^\epsilon|\nabla\hat{u}_1^\epsilon-\nabla\hat{\omega}_{11}^\epsilon|^2dxdt+\frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon(T)-\hat{\omega}_{21}^\epsilon(T))^2dxdt$$
$$+\lambda\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon-\hat{\omega}_{21}^\epsilon)^2dxdt+\int_0^T\int_\Omega \kappa_2^\epsilon|\nabla\hat{u}_2^\epsilon-\nabla\hat{\omega}_{21}^\epsilon|^2dxdt$$
$$-2\int_0^T\int_\Omega \mathcal{Q}(x,\frac{x}{\epsilon})\cdot\big((\nabla\hat{u}_1^\epsilon-\nabla\hat{\omega}_{11}^\epsilon)-(\nabla\hat{u}_2^\epsilon-\nabla\hat{\omega}_{21}^\epsilon)\big)\big((\hat{u}_1^\epsilon-\hat{\omega}_{11}^\epsilon)-(\hat{u}_2^\epsilon-\hat{\omega}_{21}^\epsilon)\big)dxdt$$
$$-\int_0^T\int_\Omega \operatorname{div}_x\mathcal{Q}(x,\frac{x}{\epsilon})\big((\hat{u}_1^\epsilon-\hat{\omega}_{11}^\epsilon)-(\hat{u}_2^\epsilon-\hat{\omega}_{21}^\epsilon)\big)^2dxdt$$
$$\leq c\epsilon^{\frac{1}{2}}\left(\int_0^T||\hat{u}_1^\epsilon(t)-\hat{\omega}_{11}^\epsilon(t)||_V+||\hat{u}_2^\epsilon(t)-\hat{\omega}_{21}^\epsilon(t)||_V dt\right)$$
$$+\frac{1}{2}\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon(0)-\hat{\omega}_{11}^\epsilon(0))^2dxdt+\frac{1}{2}\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon(0)-\hat{\omega}_{21}^\epsilon(0))^2dxdt \tag{3.77}$$

By Cauchy Schwartz and Young's inequalities, using the boundedness of $\mathcal{Q}(x,y)$, one obtains

$$\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon(T)-\hat{\omega}_{11}^\epsilon(T))^2dxdt+\lambda\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon-\hat{\omega}_{11}^\epsilon)^2dxdt$$
$$+\int_0^T\int_\Omega \kappa_1^\epsilon|\nabla\hat{u}_1^\epsilon-\nabla\hat{\omega}_{11}^\epsilon|^2dxdt\frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon(T)-\hat{\omega}_{21}^\epsilon(T))^2dxdt$$
$$+\lambda\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon-\hat{\omega}_{21}^\epsilon)^2dxdt+\int_0^T\int_\Omega \kappa_2^\epsilon|\nabla\hat{u}_2^\epsilon-\nabla\hat{\omega}_{21}^\epsilon|^2dxdt$$
$$-\frac{1}{2\varepsilon}(||\nabla\hat{u}_1^\epsilon(t)-\nabla\hat{\omega}_{11}^\epsilon(t)||^2_{L^2(0,T;H)}+||\nabla\hat{u}_2^\epsilon(t)-\nabla\hat{\omega}_{21}^\epsilon(t)||^2_{L^2(0,T;H)})$$
$$-\frac{\varepsilon}{2}(||\hat{u}_1^\epsilon(t)-\hat{\omega}_{11}^\epsilon(t)||^2_{L^2(0,T;H)}+||\hat{u}_2^\epsilon(t)-\hat{\omega}_{21}^\epsilon(t)||^2_{L^2(0,T;H)})$$
$$-c_1(||\hat{u}_1^\epsilon(t)-\hat{\omega}_{11}^\epsilon(t)||^2_{L^2(0,T;H)}+||\hat{u}_2^\epsilon(t)-\hat{\omega}_{21}^\epsilon(t)||^2_{L^2(0,T;H)})$$
$$\leq c\epsilon^{\frac{1}{2}}\bigg(\big(\int_0^T||\hat{u}_1^\epsilon-\hat{\omega}_{11}^\epsilon||_V^2dt\big)^{\frac{1}{2}}+\big(\int_0^T||\hat{u}_2^\epsilon(t)-\hat{\omega}_{21}^\epsilon(t)||_V^2dt\big)^{\frac{1}{2}}\bigg)$$
$$+\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon(\hat{u}_1^\epsilon(0)-\hat{\omega}_{11}^\epsilon(0))^2dx+\frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon(\hat{u}_2^\epsilon(0)-\hat{\omega}_{21}^\epsilon(0))^2dx \tag{3.78}$$

where the constant $\frac{1}{2\varepsilon} > 0$ can be chosen to be smaller than $\underline{\kappa}$ in (3.4). Choosing λ large enough, we obtain from (3.4),

$$\begin{aligned}
&c_2(||\hat{u}_1^\epsilon(T) - \hat{\omega}_{11}^\epsilon(T)||_H^2 + c||\hat{u}_2^\epsilon(T) - \hat{\omega}_{21}^\epsilon(T)||_H^2 \\
&+ ||\nabla\hat{u}_1^\epsilon - \nabla\hat{\omega}_{11}^\epsilon||_{L^2(0,T;H)}^2 + ||\nabla\hat{u}_2^\epsilon - \nabla\hat{\omega}_{21}^\epsilon||_{L^2(0,T;H)}^2) \\
&\leq c_3(\epsilon^{\frac{1}{2}}||\nabla\hat{u}_1^\epsilon - \nabla\hat{\omega}_{11}^\epsilon||_{L^2(0,T;H)} + \epsilon^{\frac{1}{2}}||\nabla\hat{u}_2^\epsilon - \nabla\hat{\omega}_{21}^\epsilon||_{L^2(0,T;H)} \\
&+ ||\hat{u}_1^\epsilon(0) - \hat{\omega}_{11}^\epsilon(0)||_H^2 + ||\hat{u}_2^\epsilon(0) - \hat{\omega}_{21}^\epsilon(0)||_H^2).
\end{aligned} \tag{3.79}$$

Thus

$$\begin{aligned}
&||u_1^\epsilon(T) - \omega_{11}^\epsilon(T)||_H^2 + ||u_2^\epsilon(T) - \omega_{21}^\epsilon(T)||_H^2 \\
&+ ||\nabla u_1^\epsilon - \nabla\omega_{11}^\epsilon||_{L^2(0,T;H)}^2 + ||\nabla u_2^\epsilon - \nabla\omega_{21}^\epsilon||_{L^2(0,T;H)}^2 \\
&\leq \frac{c_3}{c_2}(\epsilon^{\frac{1}{2}}||\nabla u_1^\epsilon - \nabla\omega_{11}^\epsilon||_{L^2(0,T;H)} + \epsilon^{\frac{1}{2}}||\nabla u_2^\epsilon - \nabla\omega_{21}^\epsilon||_{L^2(0,T;H)} \\
&+ ||u_1^\epsilon(0) - \omega_{11}^\epsilon(0)||_H^2 + ||u_2^\epsilon(0) - \omega_{21}^\epsilon(0)||_H^2).
\end{aligned} \tag{3.80}$$

Since $u_i^\epsilon(0) = u_{i0}(0) = g_i(x)$, we deduce that

$$u_i^\epsilon(0) - \omega_{i1}^\epsilon(0) = u_i^\epsilon(0) - u_{i0}(0) - \epsilon\tau u_{i1}(0, x, \frac{x}{\epsilon}) = -\epsilon\tau u_{i1}(0, x, \frac{x}{\epsilon}). \tag{3.81}$$

As $u_{i1}(t, x, y) \in C([0, T] \times \bar{\Omega} \times \bar{Y})$, we have $||u_i^\epsilon(0) - \omega_{i1}^\epsilon(0)||_H \leq c\epsilon$. From this we obtain

$$||\nabla u_1^\epsilon - \nabla\omega_{11}^\epsilon||_{L^2(0,T;H)} + ||\nabla u_2^\epsilon - \nabla\omega_{21}^\epsilon||_{L^2(0,T;H)} \leq c\epsilon^{\frac{1}{2}}. \tag{3.82}$$

From (3.70), we have

$$||\nabla u_1^\epsilon - \nabla u_{11}^\epsilon||_{L^2(0,T;H)} + ||\nabla u_2^\epsilon - \nabla u_{21}^\epsilon||_{L^2(0,T;H)} \leq c\epsilon^{\frac{1}{2}}. \tag{3.83}$$

The conclusion follows. $\square$

3.4 Numerical example

In this section, we apply hierarchical finite element algorithm to a numerical example for a multi-continuum system with highly oscillatory coefficients. We will utilize our algorithm to numerically approximate the effective coefficients for macroscopic points. To show the accuracy of the algorithm, we will compare the results to the approximations to the effective coefficients obtained from full reference solve that uses the finest meshes at all macroscopic points.

3.4.1 Numerical Implementation

We let $\Omega = [0,1]^2$ be the macroscopic domain and $Y = [0,1]^2$ be the unit cell. We suppose that we are given

$$\begin{aligned} \kappa_1(x_1, y_1, y_2) &= (2 - ax_1)\cos(2\pi y_1)\sin(2\pi y_2) + 3 \\ \kappa_2(x_1, y_1, y_2) &= (2 - ax_1)\sin(2\pi y_1)\cos(2\pi y_2) + 3 \\ Q(x_1, y_1, y_2) &= (1 + ax_1)\sin(2\pi y_1)\sin(2\pi y_2) \end{aligned} \tag{3.84}$$

We use 4 square meshes in $[0,1]^2$ to construct a nested sequence of FE spaces, $\{\mathcal{V}_{3-l}\}_{l=0}^{3}$ so that the mesh size of each space is $h_l = 2^l(2^{-4})$ for $l = 0, 1, 2, 3$. Since κ_1, κ_2 and Q are independent of x_2, we only consider 1-dimensional macrogrids in $[0,1]$. We develop the nested macrogrids $\{\mathcal{T}_l\}_{l=0}^{L} \subset [0,1]$ and the subsequent macrogrid hierarchy, $\{S_l\}_{l=0}^{3}$. We first let $\mathcal{T}_0 = S_0 = \{0, \frac{1}{2}, 1\}$. Considering that our macrogrids have grid spacing $H2^{-l}$ for $l = 0, 1, 2, 3$, where $H = \frac{1}{2}$ in this case, we have following hierarchy of macrogrids.

$$S_0 = \{0, \frac{1}{2}, 1\},\ S_1 = \{\frac{1}{4}, \frac{3}{4}\},\ S_2 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\},\ S_3 = \{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}\} \tag{3.85}$$

1-pt interpolation

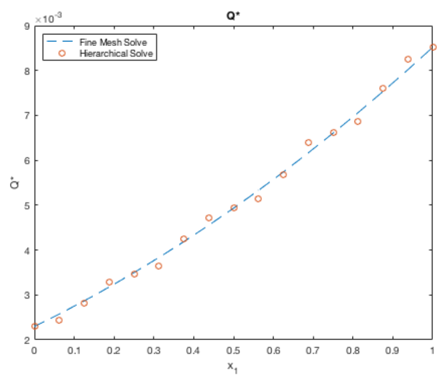
x_1	Relative Errors (%)	
	κ^*_{111}	Q^*
$\frac{1}{16}$	0.2356	5.2251
$\frac{1}{8}$	0.1469	1.9858
$\frac{3}{16}$	0.1172	3.8566
$\frac{1}{4}$	0.0849	0.7521
$\frac{5}{16}$	0.2680	4.9487
$\frac{3}{8}$	0.0965	1.4295
$\frac{7}{16}$	0.1633	3.8556
$\frac{9}{16}$	0.1473	3.5702
$\frac{5}{8}$	0.0730	1.1951
$\frac{11}{16}$	0.1605	3.6592
$\frac{3}{4}$	0.0302	0.3319
$\frac{13}{16}$	0.0860	2.7865
$\frac{7}{8}$	0.0490	0.9929
$\frac{15}{16}$	0.0992	2.9284

2-pt interpolation

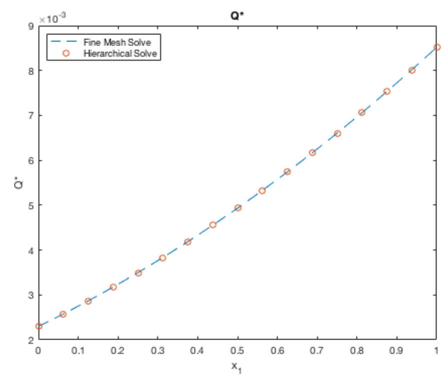
x_1	Relative Errors (%)	
	κ^*_{111}	Q^*
$\frac{1}{16}$	0.0076	0.0710
$\frac{1}{8}$	0.0097	0.0991
$\frac{3}{16}$	0.0105	0.1016
$\frac{1}{4}$	0.0073	0.0782
$\frac{5}{16}$	0.0085	0.0808
$\frac{3}{8}$	0.0066	0.0651
$\frac{7}{16}$	0.0044	0.0376
$\frac{9}{16}$	0.0027	0.0221
$\frac{5}{8}$	0.0033	0.0353
$\frac{11}{16}$	0.0036	0.0348
$\frac{3}{4}$	0.0023	0.0307
$\frac{13}{16}$	0.0028	0.0267
$\frac{7}{8}$	0.0021	0.0224
$\frac{15}{16}$	0.0015	0.0098

Table 3.1: 1-pt and 2-pt interpolation : Percentage relative errors between full mesh reference solve and hierarchical solve when a = 1.

Figure 2.3 indicates how these macrogrids and the approximation spaces are related in numerical implementation. Table 3.1, 2.2 indicate κ^*_{111} and Q^* obtained by both hierarchical and full solve at each x_1 and relative errors between them, where $Q^* = \int_Y Q(M_1 + M_2)\mathrm{d}y$ and relative errors are calculated by $\frac{100|\kappa^*_{full} - \kappa^*_{hier}|}{\kappa^*_{full}}$ with obvious notations.



(a) Q^{*}, $a = 1$, 1 point-interpolation



(b) Q^{*}, $a = 1$, 2 point-interpolation

Figure 3.1: hierarchical solve "-o-" vs. full mesh solve "–"

4. MULTISCALE SIMULATION FOR UPSCALED MULTI-CONTINUUM FLOWS

* The generalized multiscale finite element methods (GMsFEM) ([30],[13]) is a multiscale model reduction technique for solving problems with multiscale and high contrast. The GMsFEM uses a coarse grid and constructs a local reduced-order model for each coarse region. The main idea is to systematically select important degrees of freedom for the solution in each coarse block. This is achieved by constructing local snapshot spaces and selecting multiscale basis functions. Snapshot spaces are constructed by solving local problems subject to proper boundary conditions. The multiscale basis functions are obtained by well-designed local spectral problems. In this chapter we utilize the GMsFEM for solving the upscaled multiscale dual-continuum system derived in Chapter 3.

4.1 Function spaces

Let Ω be our computational domain in $\mathbb{R}^2$. The spaces of functions, vector fields in $\mathbb{R}^2$, and 2×2 matrix fields defined over Ω are respectively denoted by italic capitals (e.g., $L^2(\Omega)$), boldface Roman capitals (e.g., $\boldsymbol{V}$), and special Roman capitals (e.g., $\mathbb{S}$).

Consider the space $V := H_0^1(\Omega) = W_0^{1,2}(\Omega)$. Its dual space (also called the adjoint space), which consists of continuous linear functionals on $H_0^1(\Omega)$, is denoted by $H^{-1}(\Omega)$, and the value of a functional $f \in H^{-1}(\Omega)$ at a point $v \in H_0^1(\Omega)$ is denoted by the inner product $\langle f, v \rangle$.

The Sobolev norm $\| \cdot \|_{W_0^{1,2}(\Omega)}$ is of the form

$$\|v\|_{W_0^{1,2}(\Omega)} = \left(\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{\boldsymbol{L}^2(\Omega)}^2 \right)^{\frac{1}{2}} .$$

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Here, $\|\nabla v\|_{\boldsymbol{L}^2(\Omega)} := \||\nabla v|\|_{\boldsymbol{L}^2(\Omega)}$, where $|\nabla v|$ denotes the Euclidean norm of the 2-component vector-valued function ∇v; and for $\boldsymbol{v} = (v_1, v_2)$, $\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla \boldsymbol{v}|\|_{\mathbb{L}^2(\Omega)}$, where $|\nabla \boldsymbol{v}|$ denotes the Frobenius norm of the 2×2 matrix $\nabla \boldsymbol{v}$. We recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ is defined by $|\boldsymbol{X}|^2 := \boldsymbol{X} \cdot \boldsymbol{X} = \mathrm{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$.

The dual norm to $\|\cdot\|_{H_0^1(\Omega)}$ is $\|\cdot\|_{H^{-1}(\Omega)}$, that is,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|\langle f, v\rangle|}{\|v\|_{H_0^1(\Omega)}} .$$

For every $1 \le r < \infty$, we use $\boldsymbol{L}^r(0,T;\boldsymbol{X})$ to represent the Bochner space with the norm

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^r(0,T;\boldsymbol{X})} := \left(\int_0^T \|\boldsymbol{w}\|_{\boldsymbol{X}}^r \,\mathrm{d}t\right)^{1/r} < +\infty ,$$

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{X})} := \sup_{0 \le t \le T} \|\boldsymbol{w}\|_{\boldsymbol{X}} < +\infty ,$$

where $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$ is a Banach space. Also, we define

$$\boldsymbol{H}^1(0,T;\boldsymbol{X}) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{X}) \,:\, \partial_t \boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{X})\right\} .$$

To shorten notation, we denote the space for $\boldsymbol{u}(\cdot,t) = (u_1(\cdot,t), u_2(\cdot,t))$ by $\boldsymbol{V} = V \times V = H_0^1(\Omega) \times H_0^1(\Omega)$, where $t \in [0,T], T > 0$.

4.2 Problem formulation

In [29, 31], Park and Hoang have studied homogenization of multi-continuum systems (see [1, 2, 3, 4, 5, 6], for instance). Specially, in [31], homogenization was developed for a two-scale dual-continuum system

$$\begin{aligned}
\mathcal{C}_{11}^\epsilon(\boldsymbol{x})\frac{\partial u_1^\epsilon(\boldsymbol{x},t)}{\partial t} &= \mathrm{div}(\kappa_1^\epsilon(\boldsymbol{x})\nabla u_1^\epsilon(\boldsymbol{x},t)) + \frac{1}{\epsilon}Q^\epsilon(\boldsymbol{x})(u_2^\epsilon(\boldsymbol{x},t) - u_1^\epsilon(\boldsymbol{x},t)) + f_1 , \\
\mathcal{C}_{22}^\epsilon(\boldsymbol{x})\frac{\partial u_2^\epsilon(\boldsymbol{x},t)}{\partial t} &= \mathrm{div}(\kappa_2^\epsilon(\boldsymbol{x})\nabla u_2^\epsilon(\boldsymbol{x},t)) + \frac{1}{\epsilon}Q^\epsilon(\boldsymbol{x})(u_1^\epsilon(\boldsymbol{x},t) - u_2^\epsilon(\boldsymbol{x},t)) + f_2 ,
\end{aligned} \tag{4.1}$$

where $\boldsymbol{x} \in \Omega \subset \mathbb{R}^2$, $f_1, f_2 \in L^2(\Omega)$, ϵ represents the microscopic scale of the local variation, and the interaction terms are scaled as $\mathcal{O}(\epsilon^{-1})$. Let Y be a unit cube in $\mathbb{R}^2$. The coefficients $\mathcal{C}_{ii}^\epsilon$, κ_i^ϵ and Q^ϵ are defined as

$$\mathcal{C}_{ii}^\epsilon(\boldsymbol{x}) = \mathcal{C}_{ii}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}\right), \ \kappa_i^\epsilon(\boldsymbol{x}) = \kappa_i\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}\right) \ \text{and} \ Q^\epsilon(\boldsymbol{x}) = Q\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}\right), \quad i = 1, 2, \tag{4.2}$$

where $\mathcal{C}_{ii}(\boldsymbol{x}, \boldsymbol{y})$, $\kappa_i(\boldsymbol{x}, \boldsymbol{y})$ and $Q(\boldsymbol{x}, \boldsymbol{y})$ are Y-periodic functions from $\Omega \times Y$. The following homogenized equations of the system (4.1) were derived in [31]:

$$\begin{aligned}
\left(\int_Y \mathcal{C}_{11}\,\mathrm{d}y\right)\frac{\partial u_{1,0}}{\partial t} &= \operatorname{div}(\kappa_1^*\nabla u_{1,0}) + \operatorname{div}\left[\left(\int_Y \kappa_1 \nabla_y M_1\,\mathrm{d}y\right)(u_{2,0} - u_{1,0})\right] \\
&\quad + \sum_{i=1}^{2}\left[\left(\int_Y QN_2^i\,\mathrm{d}y\right)\frac{\partial u_{2,0}}{\partial x_i} - \left(\int_Y QN_1^i\,\mathrm{d}y\right)\frac{\partial u_{1,0}}{\partial x_i}\right] \\
&\quad - \left(\int_Y Q(M_1 + M_2)\,\mathrm{d}y\right)(u_{2,0} - u_{1,0}) + f_1\,, \\
\left(\int_Y \mathcal{C}_{22}\mathrm{d}y\right)\frac{\partial u_{2,0}}{\partial t} &= \operatorname{div}(\kappa_2^*\nabla u_{2,0}) + \operatorname{div}\left[\left(\int_Y \kappa_2 \nabla_y M_2\,\mathrm{d}y\right)(u_{1,0} - u_{2,0})\right] \\
&\quad + \sum_{i=1}^{2}\left[\left(\int_Y QN_1^i\,\mathrm{d}y\right)\frac{\partial u_{1,0}}{\partial x_i} - \left(\int_Y QN_2^i\,\mathrm{d}y\right)\frac{\partial u_{2,0}}{\partial x_i}\right] \\
&\quad - \left(\int_Y Q(M_1 + M_2)\,\mathrm{d}y\right)(u_{1,0} - u_{2,0}) + f_2\,,
\end{aligned} \tag{4.3}$$

where κ_1^* and κ_2^* are symmetric and positive definite, f_1 and f_2 are in $L^2(\Omega)$. The coefficients $\int_Y \kappa_i \nabla_y M_i\,\mathrm{d}y$ and $\int_Y QN_j^i\,\mathrm{d}y$ (where $i, j = 1, 2$) can be either positive or negative, and $\left(-\int_Y Q(M_1 + M_2)\,\mathrm{d}y\right)$ is uniformly negative in Ω. These homogenized equations still possess some degree of multiscale. This motivates our research (herein) on numerical multiscale simulation for a dual-continuum system with general convection and reaction terms:

$$
\begin{aligned}
&\mathcal{C}_{11}(\boldsymbol{x})\frac{\partial u_1(\boldsymbol{x},t)}{\partial t} - \operatorname{div}(\kappa_1(\boldsymbol{x})\nabla u_1(\boldsymbol{x},t)) + \boldsymbol{b}_1(\boldsymbol{x})\cdot\nabla(u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)) \\
&\quad + \ Q_1(\boldsymbol{x})(u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)) = f_1(\boldsymbol{x})\,, \\
&\mathcal{C}_{22}(\boldsymbol{x})\frac{\partial u_2(\boldsymbol{x},t)}{\partial t} - \operatorname{div}(\kappa_2(\boldsymbol{x})\nabla u_2(\boldsymbol{x},t)) + \boldsymbol{b}_2(\boldsymbol{x})\cdot\nabla(u_2(\boldsymbol{x},t)-u_1(\boldsymbol{x},t)) \\
&\quad + \ Q_2(\boldsymbol{x})(u_2(\boldsymbol{x},t)-u_1(\boldsymbol{x},t)) = f_2(\boldsymbol{x})\,,
\end{aligned} \tag{4.4}
$$

in $\Omega\times(0,T)$, with the Dirichlet boundary condition $u_1(\boldsymbol{x}) = u_2(\boldsymbol{x}) = 0$ on $\partial\Omega\times(0,T)$, and with suitable initial conditions (when $t = 0, T$), given $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) \in \boldsymbol{L}^2(\Omega)$. We will show later that (4.4) has a unique solution under certain conditions. One of the main difficulties as well as contributions of this work is that in (4.4), we use different Q_1 and Q_2 rather than the same Q in (4.3). Note that $C_0^\infty(\Omega)$ is dense in V. The variational form of (4.4) is as follows: Find $\boldsymbol{u} = (u_1, u_2) \in \boldsymbol{V}$ such that

$$
\begin{aligned}
&\int_\Omega \mathcal{C}_{11}\frac{\partial u_1}{\partial t}\phi_1\,\mathrm{d}x + \int_\Omega \kappa_1(\boldsymbol{x})\nabla u_1\cdot\nabla\phi_1\,\mathrm{d}x + \int_\Omega \boldsymbol{b}_1(\boldsymbol{x})\cdot\nabla(u_1-u_2)\phi_1\,\mathrm{d}x \\
&+ \int_\Omega Q_1(\boldsymbol{x})(u_1-u_2)\phi_1\,\mathrm{d}x = \int_\Omega f_1\phi_1\,\mathrm{d}x\,, \\
&\int_\Omega \mathcal{C}_{22}\frac{\partial u_2}{\partial t}\phi_2\,\mathrm{d}x + \int_\Omega \kappa_2(\boldsymbol{x})\nabla u_2\cdot\nabla\phi_2\,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2(\boldsymbol{x})\cdot\nabla(u_2-u_1)\phi_2\,\mathrm{d}x \\
&+ \int_\Omega Q_2(\boldsymbol{x})(u_2-u_1)\phi_2\,\mathrm{d}x = \int_\Omega f_2\phi_2\,\mathrm{d}x\,,
\end{aligned} \tag{4.5}
$$

for all $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \boldsymbol{V}$, for a.e. $t \in (0,T)$. Before studying this problem, we first consider the following interesting static dual-continuum system:

$$
\begin{aligned}
&-\operatorname{div}(\kappa_1(\boldsymbol{x})\nabla u_1(\boldsymbol{x})) + \boldsymbol{b}_1(\boldsymbol{x})\cdot\nabla(u_1(\boldsymbol{x})-u_2(\boldsymbol{x})) + Q_1(\boldsymbol{x})(u_1(\boldsymbol{x})-u_2(\boldsymbol{x})) = f_1(\boldsymbol{x})\,, \\
&-\operatorname{div}(\kappa_2(\boldsymbol{x})\nabla u_2(\boldsymbol{x})) + \boldsymbol{b}_2(\boldsymbol{x})\cdot\nabla(u_2(\boldsymbol{x})-u_1(\boldsymbol{x})) + Q_2(\boldsymbol{x})(u_2(\boldsymbol{x})-u_1(\boldsymbol{x})) = f_2(\boldsymbol{x})\,,
\end{aligned} \tag{4.6}
$$

in Ω, with the Dirichlet boundary condition $u_1(\boldsymbol{x}) = u_2(\boldsymbol{x}) = 0$ on $\partial\Omega$, where $\kappa_1(\boldsymbol{x})$ and $\kappa_2(\boldsymbol{x})$ are permeability coefficients in high contrast media, provided $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) \in \boldsymbol{L}^2(\Omega)$.

For later use, we define

$$\boldsymbol{b}_s = \frac{\boldsymbol{b}_1 + \boldsymbol{b}_2}{2}\,, \quad \boldsymbol{b}_a = \frac{\boldsymbol{b}_1 - \boldsymbol{b}_2}{2}\,, \quad Q_s = \frac{Q_1 + Q_2}{2}\,, \quad Q_a = \frac{Q_1 - Q_2}{2}\,, \tag{4.7}$$

in variable $\boldsymbol{x}$.

Throughout this section, we assume the following.

Assumption 4.2.1. *There are some positive constants $\bar{\mathcal{C}}, \underline{\mathcal{C}}$, $\bar{b}$, $\bar{Q}$ and $\bar{\kappa}, \underline{\kappa}$ such that $\bar{\mathcal{C}} \geq \mathcal{C}_{ii} \geq \underline{\mathcal{C}}$, $|\boldsymbol{b}_i| \leq \bar{b}$, $|Q_i| \leq \bar{Q}$, $\bar{\kappa} \geq \kappa_i \geq \underline{\kappa}$ $(i = 1, 2)$, and we further assume that $1 > \bar{b}/\sqrt{\underline{\kappa}}$, $|\boldsymbol{b}_s| \gg |\boldsymbol{b}_a|$ and $|Q_s| \gg |Q_a|$.*

The system (4.6) can be written in the variational form

$$\begin{aligned}
&\int_\Omega \kappa_1(\boldsymbol{x}) \nabla u_1(\boldsymbol{x}) \cdot \nabla \phi_1(\boldsymbol{x})\, \mathrm{d}x + \int \boldsymbol{b}_1 \cdot \nabla (u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})) \phi_1(\boldsymbol{x})\, \mathrm{d}x \\
&+ \int_\Omega Q_1(\boldsymbol{x})(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x}))\phi_1(\boldsymbol{x})\, \mathrm{d}x = \int_\Omega f_1(\boldsymbol{x})\phi_1(\boldsymbol{x})\, \mathrm{d}x\,, \\
&\int_\Omega \kappa_2(\boldsymbol{x}) \nabla u_2(\boldsymbol{x}) \cdot \nabla \phi_2(\boldsymbol{x})\, \mathrm{d}x + \int_\Omega \boldsymbol{b}_2 \cdot \nabla (u_2(\boldsymbol{x}) - u_1(\boldsymbol{x})) \phi_2(\boldsymbol{x})\, \mathrm{d}x \\
&+ \int_\Omega Q_2(\boldsymbol{x})(u_2(\boldsymbol{x}) - u_1(\boldsymbol{x}))\phi_2(\boldsymbol{x})\, \mathrm{d}x = \int_\Omega f_2(\boldsymbol{x})\phi_2(\boldsymbol{x})\, \mathrm{d}x\,,
\end{aligned} \tag{4.8}$$

for all $\phi_1(\boldsymbol{x}),\ \phi_2(\boldsymbol{x}) \in V$. We define a norm $||\cdot||_a$ on the space $\boldsymbol{V}$ as

$$||(u_1, u_2)||_a = \left(\left\| \kappa_1^{\frac{1}{2}} \nabla u_1 \right\|_{L^2(\Omega)}^2 + \left\| \kappa_2^{\frac{1}{2}} \nabla u_2 \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} . \tag{4.9}$$

We define a bilinear form $b(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \longrightarrow \mathbb{R}$ as

$$\begin{aligned} b((u_1,u_2),(v_1,v_2)) = &\int_\Omega \kappa_1 \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x + \int_\Omega \kappa_2 \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x \\ &+ \int_\Omega \boldsymbol{b}_1 \cdot \nabla(u_1 - u_2) v_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2 \cdot \nabla(u_2 - u_1) v_2 \,\mathrm{d}x \\ &+ \int_\Omega Q_1 (u_1 - u_2) v_1 \,\mathrm{d}x + \int_\Omega Q_2 (u_2 - u_1) v_2 \,\mathrm{d}x \,. \end{aligned} \tag{4.10}$$

4.2.1 Existence and uniqueness of weak solutions

In this section, we will show that each of the systems (4.8) and (4.5) has a unique solution under certain conditions.

Lemma 4.2.1. *Under Assumption 4.2.1, there are some positive constants K, α and C_b such that for all $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}$, we have*

$$b((u_1,u_2),(v_1,v_2)) \le C_b ||\boldsymbol{u}||_a \, ||\boldsymbol{v}||_a \,, \tag{4.11}$$

$$b((u_1,u_2),(u_1,u_2)) + K||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} \ge \alpha ||\boldsymbol{u}||_a^2 \,. \tag{4.12}$$

Proof. First, we prove (4.11). Note that

$$\begin{aligned} b((u_1,u_2),(v_1,v_2)) \le &\sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} \left\| \kappa_i^{\frac{1}{2}} \nabla v_i \right\|_{L^2(\Omega)} \\ + \frac{\bar{b}}{\sqrt{\underline{\kappa}}} \sum_{i=1}^{2}\sum_{j=1}^{2} &\left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} ||v_j||_{L^2(\Omega)} + \bar{Q} \sum_{i=1}^{2}\sum_{j=1}^{2} ||u_i||_{L^2(\Omega)} ||v_j||_{L^2(\Omega)} \,. \end{aligned} \tag{4.13}$$

By the Poincaré inequality, there exits a positive constant $C_p(\Omega)$ such that

$$||v_i||_{L^2(\Omega)} \le C_p ||\nabla v_i||_{L^2(\Omega)} \le \frac{C_p}{\sqrt{\underline{\kappa}}} \left\| \kappa_i^{\frac{1}{2}} \nabla v_i \right\|_{L^2(\Omega)} \,, \tag{4.14}$$

for all $v_i \in H_0^1(\Omega),\ i = 1, 2$. Thus, we get

$$\begin{aligned}
b((u_1,u_2),(v_1,v_2)) &\le \sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} \left\| \kappa_i^{\frac{1}{2}} \nabla v_i \right\|_{L^2(\Omega)} \\
&\quad + \frac{\bar{b} C_p}{\underline{\kappa}} \sum_{i=1}^{2} \sum_{j=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} \left\| \kappa_j^{\frac{1}{2}} \nabla v_j \right\|_{L^2(\Omega)} \\
&\quad + \frac{\bar{Q} {C_p}^2}{\underline{\kappa}} \sum_{i=1}^{2} \sum_{j=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} \left\| \kappa_j^{\frac{1}{2}} \nabla v_j \right\|_{L^2(\Omega)} \\
&\le \left(\left(1 + \frac{2\bar{b} C_p}{\underline{\kappa}} + \frac{2\bar{Q} {C_p}^2}{\underline{\kappa}} \right) \sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\left(1 + \frac{2\bar{b} C_p}{\underline{\kappa}} + \frac{2\bar{Q} {C_p}^2}{\underline{\kappa}} \right) \sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla v_i \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.15}$$

From (4.15), we obtain the boundedness of $b(\cdot, \cdot)$ as in (4.11).

To prove (4.12), we first note that

$$
\begin{aligned}
b((u_1,u_2),(u_1,u_2)) &= \int_\Omega \kappa_1 \nabla u_1 \cdot \nabla u_1 \,\mathrm{d}x + \int_\Omega \kappa_2 \nabla u_2 \cdot \nabla u_2 \,\mathrm{d}x \\
&\quad + \int_\Omega \boldsymbol{b}_1 \cdot \nabla (u_1 - u_2) u_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2 \cdot \nabla (u_2 - u_1) u_2 \,\mathrm{d}x \\
&\quad + \int_\Omega Q_1 (u_1 - u_2) u_1 \,\mathrm{d}x + \int_\Omega Q_2 (u_2 - u_1) u_2 \,\mathrm{d}x \\
&\geq \sum_{i=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 - \frac{\bar{b}}{\sqrt{\underline{\kappa}}} \sum_{i=1}^2 \sum_{j=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)} ||u_j||_{L^2(\Omega)} \\
&\quad - \bar{Q} \sum_{i=1}^2 \sum_{j=1}^2 ||u_i||_{L^2(\Omega)} ||u_j||_{L^2(\Omega)} \\
&\geq \sum_{i=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 - \frac{\bar{b}}{2\sqrt{\underline{\kappa}}} \sum_{i=1}^2 \sum_{j=1}^2 \left(\left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 + ||u_j||_{L^2(\Omega)}^2 \right) \\
&\quad - \frac{\bar{Q}}{2} \sum_{i=1}^2 \sum_{j=1}^2 \left(||u_i||_{L^2(\Omega)}^2 + ||u_j||_{L^2(\Omega)}^2 \right) \\
&= \sum_{i=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 - \frac{\bar{b}}{\sqrt{\underline{\kappa}}} \sum_{i=1}^2 \left(\left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 + ||u_i||_{L^2(\Omega)}^2 \right) \\
&\quad - 2\bar{Q} \sum_{i=1}^2 ||u_i||_{L^2(\Omega)}^2 \\
&= \left(1 - \frac{\bar{b}}{\sqrt{\underline{\kappa}}} \right) \sum_{i=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 - \left(\frac{\bar{b}}{\sqrt{\underline{\kappa}}} + 2\bar{Q} \right) \sum_{i=1}^2 ||u_i||_{L^2(\Omega)}^2 .
\end{aligned}
\tag{4.16}
$$

Thus, we deduce that

$$
b((u_1,u_2),(u_1,u_2)) + K \sum_{i=1}^2 ||u_i||_{L^2(\Omega)}^2 \geq \alpha \sum_{i=1}^2 \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 , \tag{4.17}
$$

where $K = \dfrac{\bar{b}}{\sqrt{\underline{\kappa}}} + 2\bar{Q}$ and $1 - \dfrac{\bar{b}}{\sqrt{\underline{\kappa}}} \geq \alpha > 0$ by Assumption 4.2.1. Hence, (4.12) holds. □

The following assumption is made for later use.

Assumption 4.2.2. *We assume that* $\alpha > \dfrac{K\, C_p}{\sqrt{\underline{\kappa}}}$ *, where* C_p*,* K *and* α *are from the proof of Lemma 4.2.1.*

We now present the main results of this section under Assumptions 4.2.1 and 4.2.2.

Lemma 4.2.2. *Under Assumption 4.2.1 and 4.2.2, we have*

$$b((u_1, u_2), (u_1, u_2)) \geq C_c ||\boldsymbol{u}||_a^2 \,, \tag{4.18}$$

for some constant $C_c > 0$*.*

Proof. From (4.17) in the proof of Lemma 4.2.1 and the Poincaré inequality (4.14), we obtain

$$b((u_1, u_2), (u_1, u_2)) + \frac{K\, C_p}{\sqrt{\underline{\kappa}}} \sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 \geq \alpha \sum_{i=1}^{2} \left\| \kappa_i^{\frac{1}{2}} \nabla u_i \right\|_{L^2(\Omega)}^2 \,. \tag{4.19}$$

Then, it follows that

$$b((u_1, u_2), (u_1, u_2)) \geq C_c ||\boldsymbol{u}||_a^2 \,, \tag{4.20}$$

where $C_c = \alpha - \dfrac{K\, C_p}{\sqrt{\underline{\kappa}}} > 0$ by Assumption 4.2.2. □

Theorem 4.2.3. *Under Assumption 4.2.1 and 4.2.2, we have a unique solution of the problem (4.8) with respect to* $|| \cdot ||_a$*.*

Proof. The theorem directly results from Lemmas 4.2.1, 4.2.2 and the Lax-Milgram Theorem. □

Also for later use, note that under Assumption 4.2.1 and 4.2.2, the following assumptions are satisfied.

Assumption 4.2.3. *There exist constants C_1, $C_2 > 0$ such that*

$$
\begin{aligned}
&b((u_1,u_2),(v_1,v_2)) \leq C_1 ||\boldsymbol{u}||_a \, ||\boldsymbol{v}||_a, \\
&b((u_1,u_2),(u_1,u_2)) \geq C_2 ||\boldsymbol{u}||_a^2 \, ,
\end{aligned}
\tag{4.21}
$$

for all $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}$.

Theorem 4.2.4. *Under Assumption 4.2.1, the problem (4.5) has a unique solution.*

Proof. We refer to [28, 31] and Lemma 4.2.1 for the proof. □

4.2.2 Fine-scale finite element discretization

We provide finite element approximation of the solutions to (4.8) and (4.5). Let $\boldsymbol{V}_h = V_h^1 \times V_h^2 = V_h \times V_h \, (\subset \boldsymbol{V})$, a Cartesian product space, be the first-order Galerkin finite element basis space, with respect to the fine grid $\mathcal{T}_h$. That is, in this chapter, $V_h^i = V_h$ is a conforming finite element space of each continuum i (for $i = 1, 2$) on $\mathcal{T}_h$.

We first consider the proposed static case (4.6), that is, solving the following problem for $\boldsymbol{u}_h = (u_{h,1}, u_{h,2}) \, (\in \boldsymbol{V}_h)$:

$$
\begin{aligned}
&\int_\Omega \kappa_1(\boldsymbol{x}) \nabla u_{h,1}(\boldsymbol{x}) \cdot \nabla \phi_1(\boldsymbol{x}) \, \mathrm{d}x + \int \boldsymbol{b}_1(\boldsymbol{x}) \cdot \nabla (u_{h,1}(\boldsymbol{x}) - u_{h,2}(\boldsymbol{x})) \phi_1(\boldsymbol{x}) \, \mathrm{d}x \\
&\quad + \int_\Omega Q_1(\boldsymbol{x}) (u_{h,1}(\boldsymbol{x}) - u_{h,2}(\boldsymbol{x})) \phi_1(\boldsymbol{x}) \, \mathrm{d}x = \int_\Omega f_1(\boldsymbol{x}) \phi_1(\boldsymbol{x}) \, \mathrm{d}x \, , \\
&\int_\Omega \kappa_2(\boldsymbol{x}) \nabla u_{h,2}(\boldsymbol{x}) \cdot \nabla \phi_2(\boldsymbol{x}) \, \mathrm{d}x + \int_\Omega \boldsymbol{b}_2(\boldsymbol{x}) \cdot \nabla (u_{h,2}(\boldsymbol{x}) - u_{h,1}(\boldsymbol{x})) \phi_2(\boldsymbol{x}) \, \mathrm{d}x \\
&\quad + \int_\Omega Q_2(\boldsymbol{x}) (u_{h,2}(\boldsymbol{x}) - u_{h,1}(\boldsymbol{x})) \phi_2(\boldsymbol{x}) \, \mathrm{d}x = \int_\Omega f_2(\boldsymbol{x}) \phi_2(\boldsymbol{x}) \, \mathrm{d}x \, ,
\end{aligned}
\tag{4.22}
$$

for all $(\phi_1, \phi_2) \in \boldsymbol{V}_h$.

Lemma 4.2.5. *Assuming $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, we have*

$$
\inf_{\boldsymbol{v} \in \boldsymbol{V}_h} ||\boldsymbol{u} - \boldsymbol{v}||_a \leq C_A(\bar{\kappa}) h \, \|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \, ,
$$

where $\bar{\kappa} \geq \kappa_i$ (as in Assumption 4.2.1 for $i = 1, 2$).

Proof. The proof is quite standard by the defintion (4.9) of norm $\|\cdot\|_a$ and the Bramble-Hilbert Lemma. □

Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{L}^2(\Omega)} := \int_\Omega u_1 v_1 \,\mathrm{d}x + \int_\Omega u_2 v_2 \,\mathrm{d}x$, where $\boldsymbol{u} = (u_1, u_2),\ \boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}$. We consider the adjoint problem of (4.8) : Find $\boldsymbol{w} \in \boldsymbol{V}$ that satisfies

$$b(\boldsymbol{v}, \boldsymbol{w}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{L}^2(\Omega)}, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}\,. \tag{4.23}$$

Theorem 4.2.6. *Assume that each of the problem (4.8) and its corresponding adjoint problem has a unique solution in $\boldsymbol{V}$. We further assume that the solution $\boldsymbol{w} = (w_1, w_2) \in \boldsymbol{V}$ of the above adjoint problem (4.23) satisfies*

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)} \leq C_R ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}\,, \tag{4.24}$$

for all $\boldsymbol{f} = (f_1, f_2) \in \boldsymbol{L}^2(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{V}$ be the solution of (4.8). Then, there are positive constants h_0 and C such that for all $h \leq h_0$, the problem (4.22) has a unique solution $\boldsymbol{u}_h = (u_{h,1}, u_{h,2}) \in \boldsymbol{V}_h$ that satisfies

$$||\boldsymbol{u} - \boldsymbol{u}_h||_a \leq C \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} ||\boldsymbol{u} - \boldsymbol{v}||_a\,, \tag{4.25}$$

where we may take $C = 2C_b/\alpha$, with C_b and α from Lemma 4.2.1.

Proof. The Theorem is proved based on the procedure in [32]. From Lemma 4.2.1, we get

$$\alpha||\boldsymbol{u} - \boldsymbol{u}_h||_a^2 \leq b(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) + K||\boldsymbol{u} - \boldsymbol{u}_h||_{\boldsymbol{L}^2(\Omega)}^2\,, \tag{4.26}$$

where K and α are as in the proof of Lemma 4.2.1. From (4.22), for any $\boldsymbol{v} \in \boldsymbol{V}_h$, we always have $b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})=0$. Thus,

$$
\begin{aligned}
& b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)+K||\boldsymbol{u}-\boldsymbol{u}_h||^2_{\boldsymbol{L}^2(\Omega)} \\
& = b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{v})+K||\boldsymbol{u}-\boldsymbol{u}_h||^2_{\boldsymbol{L}^2(\Omega)} \\
& \leq C_b||\boldsymbol{u}-\boldsymbol{u}_h||_a\,||\boldsymbol{u}-\boldsymbol{v}||_a+K||\boldsymbol{u}-\boldsymbol{u}_h||^2_{\boldsymbol{L}^2(\Omega)}\,,
\end{aligned}
\tag{4.27}
$$

where the last inequality follows from (4.11). Let $\boldsymbol{w}\in\boldsymbol{V}$ be the solution to the problem (4.23) with $\boldsymbol{f}=\boldsymbol{u}-\boldsymbol{u}_h$, that is, $b(\boldsymbol{v},\boldsymbol{w})=\langle\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}\rangle_{\boldsymbol{L}^2(\Omega)}$ for all $\boldsymbol{v}\in\boldsymbol{V}$. Then, for any $\boldsymbol{w}_h\in\boldsymbol{V}_h$, we obtain

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2(\Omega)} &= \langle\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h\rangle_{\boldsymbol{L}^2(\Omega)}=b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})=b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w}-\boldsymbol{w}_h) \\
&\leq C_b||\boldsymbol{u}-\boldsymbol{u}_h||_a\,||\boldsymbol{w}-\boldsymbol{w}_h||_a\,.
\end{aligned}
\tag{4.28}
$$

By Lemma 4.2.5 for $||\boldsymbol{w}-\boldsymbol{w}_h||_a$, (4.28) becomes

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2(\Omega)} &\leq C_bC_Ah||\boldsymbol{u}-\boldsymbol{u}_h||_a\,\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)} \\
&\leq C_bC_AC_Rh||\boldsymbol{u}-\boldsymbol{u}_h||_a\,||\boldsymbol{u}-\boldsymbol{u}_h||_{\boldsymbol{L}^2(\Omega)}\,,
\end{aligned}
\tag{4.29}
$$

where the last inequality follows from assumption (4.24). Simplifying (4.29), we get

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C_bC_AC_Rh||\boldsymbol{u}-\boldsymbol{u}_h||_a\,. \tag{4.30}
$$

From this inequality and (4.27), we derive from (4.26) that

$$
\alpha||\boldsymbol{u}-\boldsymbol{u}_h||^2_a \leq C_b||\boldsymbol{u}-\boldsymbol{u}_h||_a\,||\boldsymbol{u}-\boldsymbol{v}||_a+K(C_bC_AC_Rh)^2||\boldsymbol{u}-\boldsymbol{u}_h||^2_a\,. \tag{4.31}
$$

For $h \leq h_0$, where $h_0 = \dfrac{\sqrt{\alpha}}{\sqrt{2K}C_b C_A C_R}$, we obtain

$$||\boldsymbol{u} - \boldsymbol{u}_h||_a \leq \frac{2C_b}{\alpha}||\boldsymbol{u} - \boldsymbol{v}||_a \,, \tag{4.32}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_h$, and the desired result (4.25) follows. The proof of uniqueness of the solution to (4.22) is quite straightforward ([32]). □

We now investigate the dynamic case, that is, the variational problem (4.5) of (4.4) for $\boldsymbol{u}_h = (u_{h,1}, u_{h,2}) \in \boldsymbol{V}_h$:

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}\frac{\partial u_{h,1}}{\partial t}\phi_1 \,\mathrm{d}x + \int_\Omega \kappa_1(\boldsymbol{x})\nabla u_{h,1}\cdot\nabla\phi_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_1(\boldsymbol{x})\cdot\nabla(u_{h,1} - u_{h,2})\phi_1 \,\mathrm{d}x \\
&\quad + \int_\Omega Q_1(\boldsymbol{x})(u_{h,1} - u_{h,2})\phi_1 \,\mathrm{d}x = \int_\Omega f_1\phi_1 \,\mathrm{d}x \,, \\
&\int_\Omega \mathcal{C}_{22}\frac{\partial u_{h,2}}{\partial t}\phi_2 \,\mathrm{d}x + \int_\Omega \kappa_2(\boldsymbol{x})\nabla u_{h,2}\cdot\nabla\phi_2 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2(\boldsymbol{x})\cdot\nabla(u_{h,2} - u_{h,1})\phi_2 \,\mathrm{d}x \\
&\quad + \int_\Omega Q_2(\boldsymbol{x})(u_{h,2} - u_{h,1})\phi_2 \,\mathrm{d}x = \int_\Omega f_2\phi_2 \,\mathrm{d}x \,,
\end{aligned} \tag{4.33}$$

for all $(\phi_1, \phi_2) \in \boldsymbol{V}_h$ and a.e. $t \in (0, T)$. We define the following bilinear forms in $\boldsymbol{V} \times \boldsymbol{V}$:

$$\begin{aligned}
c((u_1, u_2), (v_1, v_2)) &= \int_\Omega \mathcal{C}_{11}u_1 v_1 \,\mathrm{d}x + \int_\Omega \mathcal{C}_{22}u_2 v_2 \,\mathrm{d}x \,, \\
a((u_1, u_2), (v_1, v_2)) &= \int_\Omega \kappa_1\nabla u_1\cdot\nabla v_1 \,\mathrm{d}x + \int_\Omega \kappa_2\nabla u_2\cdot\nabla v_2 \,\mathrm{d}x \,.
\end{aligned} \tag{4.34}$$

Let us hence define the norms $||\boldsymbol{u}||_c^2 = c(\boldsymbol{u}, \boldsymbol{u}) = \langle\boldsymbol{u}, \boldsymbol{u}\rangle_c$ and $||\boldsymbol{u}||_a^2 = a(\boldsymbol{u}, \boldsymbol{u}) = \langle\boldsymbol{u}, \boldsymbol{u}\rangle_a$.

Lemma 4.2.7. *Under Assumption 4.2.3, we have*

$$
\begin{aligned}
&||\boldsymbol{u}(\cdot,T)-\boldsymbol{u}_h(\cdot,T)||_c^2+\int_0^T||\boldsymbol{u}-\boldsymbol{u}_h||_a^2\,\mathrm{d}t\\
&\le C\inf_{\boldsymbol{w}\in\boldsymbol{V}_h}\left(\int_0^T\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2\mathrm{d}t+\int_0^T\|\boldsymbol{w}-\boldsymbol{u}\|_a^2\,\mathrm{d}t+||\boldsymbol{w}(\cdot,0)-\boldsymbol{u}(\cdot,0)||_c^2\right),
\end{aligned}
\tag{4.35}
$$

where $\mathbf{u}$ *and* $\mathbf{u}_h$ *satisfy (4.5) and (4.33), respectively.*

Proof. The proof is based on [15, 6]. From the systems (4.5), (4.33), c as in (4.34) and b as in (4.10), we get

$$
c\left(\frac{\partial(\boldsymbol{u}-\boldsymbol{u}_h)}{\partial t},\boldsymbol{v}\right)+b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})=0\,,
\tag{4.36}
$$

for all $\boldsymbol{v}\in\boldsymbol{V}_h$.

Given $\boldsymbol{w}\in\boldsymbol{V}_h$, let $\boldsymbol{v}=\boldsymbol{w}-\boldsymbol{u}_h\in\boldsymbol{V}_h$. For the constants $C_1, C_2>0$ in Assumption 4.2.3, from (4.36), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}-\boldsymbol{u}_h||_c^2+C_2\,\|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2\\
&=c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_h)}{\partial t},\boldsymbol{w}-\boldsymbol{u}_h\right)+C_2\,\|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2\\
&\le c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_h)}{\partial t},\boldsymbol{w}-\boldsymbol{u}_h\right)+b(\boldsymbol{w}-\boldsymbol{u}_h,\boldsymbol{w}-\boldsymbol{u}_h)\\
&=c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t},\boldsymbol{w}-\boldsymbol{u}_h\right)+b(\boldsymbol{w}-\boldsymbol{u},\boldsymbol{w}-\boldsymbol{u}_h)\\
&\le\left|c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t},\boldsymbol{w}-\boldsymbol{u}_h\right)\right|+C_1\|\boldsymbol{w}-\boldsymbol{u}\|_a\|\boldsymbol{w}-\boldsymbol{u}_h\|_a\\
&\le\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c\|\boldsymbol{w}-\boldsymbol{u}_h\|_c+C_1\|\boldsymbol{w}-\boldsymbol{u}\|_a\|\boldsymbol{w}-\boldsymbol{u}_h\|_a\,,
\end{aligned}
\tag{4.37}
$$

where the last inequality follows from the Cauchy-Schwarz inequality.

Applying Young's inequality for the right hand side of the last inequality of (4.37), we get

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,||\boldsymbol{w}-\boldsymbol{u}_h||_c^2 + C_2\ \|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2 \\
&\leq \frac{1}{2}\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 + \frac{1}{2}\,\|\boldsymbol{w}-\boldsymbol{u}_h\|_c^2 + \frac{C_1^2}{3C_2}\,\|\boldsymbol{w}-\boldsymbol{u}\|_a^2 + \frac{3C_2}{4}\,\|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2\ .
\end{aligned} \tag{4.38}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,||\boldsymbol{w}-\boldsymbol{u}_h||_c^2 - \frac{1}{2}\,\|\boldsymbol{w}-\boldsymbol{u}_h\|_c^2 + \frac{C_2}{4}\ \|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2 \\
&\leq \frac{1}{2}\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 + \frac{C_1^2}{3C_2}\,\|\boldsymbol{w}-\boldsymbol{u}\|_a^2\ .
\end{aligned} \tag{4.39}$$

Multiplying both sides of (4.39) by multiplicative integrating factor $e^{\int(-1)\,\mathrm{d}t} = e^{-t}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\left(\frac{d}{dt}\,||\boldsymbol{w}-\boldsymbol{u}_h||_c^2\right)e^{-t} - e^{-t}\ \|\boldsymbol{w}-\boldsymbol{u}_h\|_c^2\right) + e^{-t}\,\frac{C_2}{4}\ \|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2 \\
&\leq e^{-t}\ \left(\frac{1}{2}\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 + \frac{C_1^2}{3C_2}\,\|\boldsymbol{w}-\boldsymbol{u}\|_a^2\right)\ .
\end{aligned} \tag{4.40}$$

Taking $\displaystyle\int_0^T \cdot\ \mathrm{d}t$ both sides of (4.40), we get

$$\begin{aligned}
&\frac{1}{2}\,||\boldsymbol{w}(\cdot,T)-\boldsymbol{u}_h(\cdot,T)||_c^2\ e^{-T} + \int_0^T e^{-t}\ \frac{C_2}{4}\ \|\boldsymbol{w}-\boldsymbol{u}_h\|_a^2\ \mathrm{d}t \\
&\leq \frac{1}{2}\,||\boldsymbol{w}(\cdot,0)-\boldsymbol{u}_h(\cdot,0)||_c^2 + \int_0^T e^{-t}\ \left(\frac{1}{2}\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 + \frac{C_1^2}{3C_2}\,\|\boldsymbol{w}-\boldsymbol{u}\|_a^2\right)\,\mathrm{d}t\,.
\end{aligned} \tag{4.41}$$

Note that $e^{-T} \leq e^{-t} \leq 1,\ \forall t \in [0, T]$. Let

$$M = \frac{\max\left\{\frac{1}{2}, \frac{C_1^2}{3C_2}\right\}}{\min\left\{\frac{e^{-T}}{2}, \frac{e^{-T}C_2}{4}\right\}}.$$

Therefore,

$$\begin{aligned} &||\boldsymbol{w}(\cdot, T) - \boldsymbol{u}_h(\cdot, T)||_c^2 + \int_0^T \|\boldsymbol{w} - \boldsymbol{u}_h\|_a^2 \, \mathrm{d}t \\ &\leq M \left(\int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t + \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \, \mathrm{d}t + ||\boldsymbol{w}(\cdot, 0) - \boldsymbol{u}_h(\cdot, 0)||_c^2 \right). \end{aligned} \tag{4.42}$$

We define the initial value $\boldsymbol{u}_h(\cdot, 0)$ such that $c(\boldsymbol{u}(\cdot, 0), \boldsymbol{v}) = c(\boldsymbol{u}_h(\cdot, 0), \boldsymbol{v})$, so $||\boldsymbol{u}(\cdot, 0) - \boldsymbol{u}_h(\cdot, 0)||_c = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}$. By triangle inequality, we thus have

$$||\boldsymbol{w}(\cdot, 0) - \boldsymbol{u}_h(\cdot, 0)||_c \leq ||\boldsymbol{w}(\cdot, 0) - \boldsymbol{u}(\cdot, 0)||_c\,. \tag{4.43}$$

From (4.42) and (4.43), we obtain

$$\begin{aligned} &||\boldsymbol{u}(\cdot, T) - \boldsymbol{u}_h(\cdot, T)||_c^2 + \int_0^T \|\boldsymbol{u} - \boldsymbol{u}_h\|_a^2 \, \mathrm{d}t \\ &\leq 2(||\boldsymbol{w}(\cdot, T) - \boldsymbol{u}_h(\cdot, T)||_c^2 + ||\boldsymbol{w}(\cdot, T) - \boldsymbol{u}(\cdot, T)||_c^2 \\ &+ \int_0^T \|\boldsymbol{w} - \boldsymbol{u}_h\|_a^2 \, \mathrm{d}t + \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \, \mathrm{d}t) \\ &\leq 2M \int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t + 2M \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \, \mathrm{d}t + 2||\boldsymbol{w}(\cdot, T) - \boldsymbol{u}(\cdot, T)||_c^2 \\ &+ 2 \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \, \mathrm{d}t + 2M||\boldsymbol{w}(\cdot, 0) - \boldsymbol{u}(\cdot, 0)||_c^2\,. \end{aligned} \tag{4.44}$$

To simplify the above inequality, we note that

$$\left\| \int_0^T \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\, \mathrm{d}t \right\|_c^2 \le T \int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t\,. \tag{4.45}$$

Indeed, let

$$\boldsymbol{z} = \boldsymbol{z}(\cdot) = (\boldsymbol{w}(\cdot,T) - \boldsymbol{u}(\cdot,T)) - (\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0)) = \int_0^T \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\, \mathrm{d}t\,. \tag{4.46}$$

Then,

$$\begin{aligned} \|\boldsymbol{z}\|_c^2 = \langle \boldsymbol{z}, \boldsymbol{z} \rangle_c &= \left\langle \boldsymbol{z}\,,\ \int_0^T \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\, \mathrm{d}t \right\rangle_c = \int_0^T \left\langle \boldsymbol{z}\,,\ \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\rangle_c \mathrm{d}t \\ &\le \int_0^T \|\boldsymbol{z}\|_c \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c \mathrm{d}t = \|\boldsymbol{z}\|_c \int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c \mathrm{d}t\,. \end{aligned}$$

Thus,

$$\|\boldsymbol{z}\|_c \le \int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c \mathrm{d}t\,. \tag{4.47}$$

Now, by Hölder's inequality for the right hand side of (4.47), we get

$$\|\boldsymbol{z}\|_c^2 \le \left(\int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c \cdot 1\, \mathrm{d}t \right)^2 \le T \left(\int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t \right),$$

which is (4.45).

Therefore, from (4.46), we get

$$\begin{aligned} ||\boldsymbol{w}(\cdot,T) - \boldsymbol{u}(\cdot,T)||_c^2 &= \|\boldsymbol{z} + (\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0))\|_c^2 \\ &\le 2\|\boldsymbol{z}\|_c^2 + 2||\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0)||_c^2 \\ &\le 2T \int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t + 2||\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0)||_c^2\,. \end{aligned}$$

Finally, there exists $C > 0$ such that (4.44) becomes

$$
\begin{aligned}
&||\boldsymbol{u}(\cdot,T) - \boldsymbol{u}_h(\cdot,T)||_c^2 + \int_0^T \|\boldsymbol{u} - \boldsymbol{u}_h\|_a^2 \,\mathrm{d}t \\
&\leq C\left(\int_0^T \left\|\frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial t}\right\|_c^2 \mathrm{d}t + \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \,\mathrm{d}t + ||\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0)||_c^2\right),
\end{aligned}
\tag{4.48}
$$

and (4.35) follows. □

Let us define additional bilinear forms before proceeding to the next section. For $\boldsymbol{u} = (u_1, u_2) \in \boldsymbol{V}$, using notation from (4.7), the problem (4.5) can be written as

$$
\begin{aligned}
&\int_\Omega \mathcal{C}_{11}\frac{\partial u_1}{\partial t} v_1 \,\mathrm{d}x + \int_\Omega \kappa_1 \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x \\
&+ \int_\Omega \boldsymbol{b}_s \cdot \nabla(u_1 - u_2) v_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_a \cdot \nabla(u_1 - u_2) v_1 \,\mathrm{d}x \\
&+ \int_\Omega Q_s(u_1 - u_2) v_1 \,\mathrm{d}x + \int_\Omega Q_a(u_1 - u_2) v_1 \,\mathrm{d}x = \int_\Omega f_1 v_1 \,\mathrm{d}x \,, \\
&\int_\Omega \mathcal{C}_{22}\frac{\partial u_2}{\partial t} v_2 \,\mathrm{d}x + \int_\Omega \kappa_2 \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x \\
&+ \int_\Omega \boldsymbol{b}_s \cdot \nabla(u_2 - u_1) v_2 \,\mathrm{d}x - \int_\Omega \boldsymbol{b}_a \cdot \nabla(u_2 - u_1) v_2 \,\mathrm{d}x \\
&+ \int_\Omega Q_s(u_2 - u_1) v_2 \,\mathrm{d}x - \int_\Omega Q_a(u_2 - u_1) v_2 \,\mathrm{d}x = \int_\Omega f_2 v_2 \,\mathrm{d}x \,.
\end{aligned}
\tag{4.49}
$$

Also, we define the following bilinear forms in $\boldsymbol{V} \times \boldsymbol{V}$:

$$\begin{aligned}
\beta((u_1,u_2),(v_1,v_2)) &= \int_\Omega \boldsymbol{b}_1 \cdot \nabla(u_1-u_2)v_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2 \cdot \nabla(u_2-u_1)v_2 \,\mathrm{d}x\,, \\
q((u_1,u_2),(v_1,v_2)) &= \int_\Omega Q_1(u_1-u_2)v_1 \,\mathrm{d}x + \int_\Omega Q_2(u_2-u_1)v_2 \,\mathrm{d}x\,, \\
q_s((u_1,u_2),(v_1,v_2)) &= \int_\Omega Q_s(u_1-u_2)v_1 \,\mathrm{d}x + \int_\Omega Q_s(u_2-u_1)v_2 \,\mathrm{d}x\,, \\
q_a((u_1,u_2),(v_1,v_2)) &= \int_\Omega Q_a(u_1-u_2)v_1 \,\mathrm{d}x - \int_\Omega Q_a(u_2-u_1)v_2 \,\mathrm{d}x\,, \\
a_{Q_s}((u_1,u_2),(v_1,v_2)) &= a((u_1,u_2),(v_1,v_2)) + q_s((u_1,u_2),(v_1,v_2))\,, \\
b((u_1,u_2),(v_1,v_2)) &= a((u_1,u_2),(v_1,v_2)) + \beta((u_1,u_2),(v_1,v_2)) + q((u_1,u_2),(v_1,v_2))\,.
\end{aligned} \tag{4.50}$$

Here,

$$\begin{aligned}
a_i^{(j)}(u_i,v_i) &= \int_{\omega_j} \kappa_i \nabla u_i \cdot \nabla v_i \,\mathrm{d}x\,, \\
a^{(j)}((u_1,u_2),(v_1,v_2)) &= a_1^{(j)}(u_1,v_1) + a_2^{(j)}(u_2,v_2)\,, \\
a_{Q_s}^{(j)}((u_1,u_2),(v_1,v_2)) &= a^{(j)}((u_1,u_2),(v_1,v_2)) + q_s^{(j)}((u_1,u_2),(v_1,v_2)),
\end{aligned} \tag{4.51}$$

where $u_1, u_2, v_1, v_2 \in H_0^1(\omega_j) = V(\omega_j)$. Note that $q_s(\boldsymbol{u},\boldsymbol{v}) = q_s(\boldsymbol{v},\boldsymbol{u})$. We define the norm $||\boldsymbol{u}||_{a_{Q_s}} = a_{Q_s}(\boldsymbol{u},\boldsymbol{u})$.

4.3 Overview of the GMsFEM

We refer the readers to [13] for the details of the GMsFEM, and [33, 6] for a brief overview of the GMsFEM. Broadly speaking, solving Eq. (4.6) on a fine grid using the standard FEM method is very expensive (due to heterogeneous coefficients). If we use coarse grid with the FEM, the solution is not accurate because of the loss of some important local information. Thus, we utilize the GMsFEM, where local problems are solved in each coarse neighborhood, to systematically construct multiscale basis functions containing local heterogenity information. More specifically, by first solving local snapshot and local eigenvalue problems, we then deduce a so-called multiscale space as global offline space $\boldsymbol{V}_{\text{ms}}$ (consisting of multiscale basis functions). Hence, for all $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}_{\text{ms}}$, the GMsFEM solution $\boldsymbol{u}_{\text{ms}} = (u_{\text{ms},1}, u_{\text{ms},2})$ ($\in \boldsymbol{V}_{\text{ms}}$) is defined via the following system:

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11} \frac{\partial u_{\text{ms},1}}{\partial t} v_1 \,\mathrm{d}x + \int_\Omega \kappa_1(\boldsymbol{x}) \nabla u_{\text{ms},1} \cdot \nabla v_1 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_1(\boldsymbol{x}) \cdot \nabla (u_{\text{ms},1} - u_{\text{ms},2}) v_1 \,\mathrm{d}x \\
&+ \int_\Omega Q_1(\boldsymbol{x})(u_{\text{ms},1} - u_{\text{ms},2}) v_1 \,\mathrm{d}x = \int_\Omega f_1 v_1 \,\mathrm{d}x \,, \\
&\int_\Omega \mathcal{C}_{22} \frac{\partial u_{\text{ms},2}}{\partial t} v_2 \,\mathrm{d}x + \int_\Omega \kappa_2(\boldsymbol{x}) \nabla u_{\text{ms},2} \cdot \nabla v_2 \,\mathrm{d}x + \int_\Omega \boldsymbol{b}_2(\boldsymbol{x}) \cdot \nabla (u_{\text{ms},2} - u_{\text{ms},1}) v_2 \,\mathrm{d}x \\
&+ \int_\Omega Q_2(\boldsymbol{x})(u_{\text{ms},2} - u_{\text{ms},1}) v_2 \,\mathrm{d}x = \int_\Omega f_2 v_2 \,\mathrm{d}x \,.
\end{aligned} \tag{4.52}$$

4.3.1 Coarse and fine grids

First, let $\mathcal{T}^H$ be a coarse grid, with grid size H. In $\mathcal{T}^H$, each coarse block can be denoted by K_i. A refinement of $\mathcal{T}^H$ is called a fine grid $\mathcal{T}_h$, with grid size h ($\ll H$). We denote by N the total number of coarse blocks, and N_v the total number of interior vertices of $\mathcal{T}^H$. Let $\{\boldsymbol{x}_i\}_{i=1}^N$ be the set of all vertices in $\mathcal{T}^H$. The jth coarse neighborhood is defined by

$$\omega_j = \bigcup \{K_i \in \mathcal{T}^H : \boldsymbol{x}_j \in \overline{K_i}\}. \tag{4.53}$$

Next, we will present the definitions of the uncoupled multiscale basis functions (uncoupled GMsFEM) and the coupled multiscale basis functions (coupled GMsFEM). For each case, based on the above general procedure, we first generate a local snapshot space for each coarse neighborhood ω_j, then solve an appropriate local spectral problem defined on the snapshot space, to establish a multiscale (offline) space. There are several choices of snapshot spaces (see [13, 33], for instance). In this work, for each case, its snapshot space is a set of **harmonic basis functions** (to be specified in the next subsections), which are solutions for the corresponding harmonic extension problem. Note that the snapshot functions and the basis functions are time-independent.

4.3.2 Uncoupled GMsFEM

As in [33], let $V_h^i(\omega_j) = V_h(\omega_j)$ be a fine-scale FEM space, which is the restriction in ω_j the conforming space $V_h^i = V_h$ (introduced in Section 4.2.2), for the ith continuum ($i = 1, 2$). Let $J_h(\omega_j)$ be the set of all nodes of the fine grid $\mathcal{T}_h$ belonging to $\partial\omega_j$. We denote by J_j the cardinality of $J_h(\omega_j)$.

For the case of uncoupled GMsFEM, multiscale basis functions will be established for each ith continuum separately, by taking into account only the permeability κ_i and neglecting the transfer functions.

More specifically, on each coarse neighborhood ω_j, for each ith continuum, we first find the kth snapshot function $\phi_{k,i}^{(j),\text{snap}} \in V_h(\omega_j)$ such that

$$\begin{aligned} -\operatorname{div}(\kappa_i \nabla \phi_{k,i}^{(j),\text{snap}}) &= 0 \quad \text{in } \omega_j , \\ \phi_{k,i}^{(j),\text{snap}} &= \delta_{k,i} \quad \text{on } \partial\omega_j , \end{aligned} \tag{4.54}$$

where $\delta_{k,i}$ is a discrete delta function such that

$$\delta_{k,i}(\boldsymbol{x}_l^j) = \begin{cases} 1 & l = k , \\ 0 & l \neq k , \end{cases}$$

for all $\boldsymbol{x}_l^j$ in $J_h(\omega_j)$, $1 \le k \le J_j$. The solutions of this problem (4.54) are called harmonic basis functions. Then, the local snapshot space on ω_j for the ith continuum is defined as

$$V_{\text{snap}}^i(\omega_j) = \text{span}\{\phi_{k,i}^{(j),\text{snap}} \,\big|\, 1 \le k \le J_j\} , \tag{4.55}$$

where J_j is the cardinality of $J_h(\omega_j)$ as above.

To construct local multiscale basis functions on ω_j corresponding to the ith continuum ($i = 1, 2$), we now solve local spectral problems: Find the eigenfunctions $\;^{(j)}_{k,i} \in V^i_{\text{snap}}(\omega_j)$ and eigenvalues $\lambda^{(j)}_{k,i} \in \mathbb{R}$ such that

$$a^{(j)}_i(\psi^{(j)}_{k,i}, v_i) = \lambda^{(j)}_{k,i} s^{(j)}_i(\psi^{(j)}_{k,i}, v_i)\,, \tag{4.56}$$

for all v_i in $V^i_{\text{snap}}(\omega_j)$, where $s^{(j)}_i$ is defined as follows ([33, 6]):

$$s^{(j)}_i(u_i, v_i) = \int_{\omega_j} \kappa_i \left(\sum_{j=1}^{N_v} |\nabla \chi_{j,i}|^2 \right) u_i v_i \,\mathrm{d}x\,, \tag{4.57}$$

where each $\chi_{j,i}$ is a standard multiscale finite element basis function for the coarse node $\boldsymbol{x}_j$ (that is, with linear boundary conditions for cell problems) in the ith continuum, and $\{\chi_{j,i}\}_{j=1}^{N_v}$ is a set of partition of unity functions (for coarse grid) supported in the intersection of ω_j and the ith continuum. More specifically, based on [34],

$$\begin{aligned} -\operatorname{div}(\kappa_i \nabla \chi_{j,i}) = 0 \;\text{ in } K \in \omega_j\,, \\ \chi_{j,i} = \chi^0_{j,i} \;\text{ on }\; \partial K\,, \quad \forall K \in \omega_j\,, \end{aligned} \tag{4.58}$$

where each $\chi^0_{j,i}$ is a standard linear (and continuous) partition of unity function, and note that $\chi^0_{j,i} = 0$ on $\partial\omega_j$.

After sorting the eigenvalues $\lambda^{(j)}_{k,i}$ (for $k = 1, 2, \cdots$) from (4.56) in ascending order, we choose the first corresponding L_j eigenfunctions from (4.56), and still denote them by $\;^{(j)}_{1,i}, \cdots, \psi^{(j)}_{L_j,i}$. At the last step, the kth multiscale basis function for the ith continuum on the coarse neighborhood ω_j is defined by

$$\;^{(j),\text{ms}}_{k,i} = \chi_{j,i} \psi^{(j)}_{k,i}\,, \tag{4.59}$$

where $1 \leq k \leq L_j$, and $\{\chi_{j,i}\}_{j=1}^{N_v}$ is from (4.58).

We define the local auxiliary offline multiscale space $V_{\text{ms}}^i(\omega_j)$ for the coarse neighborhood ω_j corresponding to the ith continuum, using the first L_j multiscale basis functions as follows:

$$V_{\text{ms}}^i(\omega_j) = \text{span}\left\{ \ {}_{k,i}^{(j),\text{ms}} \,\middle|\, 1 \leq k \leq L_j \right\} . \tag{4.60}$$

Then, the global offline space for the ith continuum is

$$V_{\text{ms}}^i = \sum_{j=1}^{N_v} V_{\text{ms}}^i(\omega_j) = \text{span}\left\{ \ {}_{k,i}^{(j),\text{ms}} \,\middle|\, 1 \leq j \leq N_v\,, 1 \leq k \leq L_j \right\} .$$

The multiscale space $\boldsymbol{V}_{\text{ms}}$ can be taken as the global offline space: $\boldsymbol{V}_{\text{ms}} = V_{\text{ms}}^1 \times V_{\text{ms}}^2$.

4.3.3 Coupled GMsFEM

In the coupled GMsFEM, the multiscale basis functions will be created by first solving a coupled problem for snapshot space, then applying a spectral decomposition.

Note that for the case of coupled GMsFEM, the interaction terms Q_1 and Q_2 from (4.5) will be taken into account. For eigenvalue problem, the operator should be symmetric. Therefore, we wish to only consider the dominant symmetric part Q_s (of Q_1 and Q_2) and ignore Q_a from (4.49), which is equivalent to (4.5). In order to do so, we will utilize Assumption 4.2.1 (that is, $|\boldsymbol{b}_s| \gg |\boldsymbol{b}_a|$ and $|Q_s| \gg |Q_a|$) and Lemma 4.4.8 in Section 4.4.

More specifically, we find the snapshot functions $\boldsymbol{\phi}_{k,r}^{(j),\text{snap}} = \left(\phi_{k,1,r}^{(j),\text{snap}}, \phi_{k,2,r}^{(j),\text{snap}}\right)$ in $\boldsymbol{V}_h(\omega_j) = V_h(\omega_j) \times V_h(\omega_j)$ (the spaces are from Subsections 4.2.2 and 4.3.2) such that

$$\begin{aligned} -\operatorname{div}\left(\kappa_1 \nabla \phi_{k,1,r}^{(j),\text{snap}}\right) + Q_s\left(\phi_{k,1,r}^{(j),\text{snap}} - \phi_{k,2,r}^{(j),\text{snap}}\right) &= 0 \quad \text{in } \omega_j, \\ -\operatorname{div}\left(\kappa_2 \nabla \phi_{k,2,r}^{(j),\text{snap}}\right) + Q_s\left(\phi_{k,2,r}^{(j),\text{snap}} - \phi_{k,1,r}^{(j),\text{snap}}\right) &= 0 \quad \text{in } \omega_j, \\ \boldsymbol{\phi}_{k,r}^{(j),\text{snap}} &= \boldsymbol{\delta}_{k,r} \quad \text{on } \partial\omega_j\,, \end{aligned} \tag{4.61}$$

where each $\boldsymbol{\delta}_{k,r}$ is defined as

$$\boldsymbol{\delta}_{k,r}(\boldsymbol{x}_l) = \delta_k(\boldsymbol{x}_l)\boldsymbol{e}_r, \;\; r = 1, 2\,, \tag{4.62}$$

in which $\{\boldsymbol{e}_r \,|\, r = 1, 2\}$ is a standard basis in $\mathbb{R}^2$, $1 \leq k \leq J_j$. The solutions of this problem (4.61) are called harmonic basis functions. Then, the local snapshot space is defined as

$$\boldsymbol{V}_{\text{snap}}(\omega_j) = \text{span}\left\{\boldsymbol{\phi}_{k,r}^{(j),\text{snap}} \,\middle|\, 1 \leq k \leq J_j,\; 1 \leq r \leq 2\right\}\,. \tag{4.63}$$

Next, local eigenvalue problems are solved, to construct local multiscale basis functions. That is, we find the eigenfunctions $\boldsymbol{\psi}_k^{(j)} = \left(\;\;{}^{(j)}_{k,1}, \psi^{(j)}_{k,2}\right) \in \boldsymbol{V}_{\text{snap}}(\omega_j)$ and eigenvalues $\lambda_k^{(j)} \in \mathbb{R}$ such that

$$a_{Q_s}^{(j)}\left(\boldsymbol{\psi}_k^{(j)}, \boldsymbol{v}\right) = \lambda_k^{(j)} s^{(j)}\left(\boldsymbol{\psi}_k^{(j)}, \boldsymbol{v}\right), \tag{4.64}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_{\text{snap}}(\omega_j)$, where $s^{(j)}$ is defined as follows ([33, 6]):

$$s^{(j)}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{2} s_i^{(j)}(u_i, v_i) = \sum_{i=1}^{2} \int_{\omega_j} \kappa_i \left(\sum_{j=1}^{N_v} |\nabla \chi_{j,i}|^2\right) u_i v_i \,\mathrm{d}x\,, \tag{4.65}$$

in which $\{\chi_{j,i}\}_{j=1}^{N_v}$ is from (4.58).

After arranging the eigenvalues $\lambda_k^{(j)}$ (for $k = 1, 2, \cdots$) from (4.64) in ascending order, we take the first corresponding L_j eigenfunctions from (4.64), and still denote them by $\boldsymbol{\psi}_1^{(j)}, \cdots, \boldsymbol{\psi}_{L_j}^{(j)}$. At the final step, we define the kth multiscale basis functions for the coarse region ω_j by

$$\boldsymbol{\psi}_k^{(j),\text{ms}} = (\chi_{j,1} \;\; {}^{(j)}_{k,1}, \chi_{j,2} \;\; {}^{(j)}_{k,2})\,, \tag{4.66}$$

where $1 \leq k \leq L_j$, and $\{\chi_{j,i}\}_{j=1}^{N_v}$ is from (4.58).

The local auxiliary offline multiscale space $\boldsymbol{V}_{\text{ms}}(\omega_j)$ is defined by using the first L_j

multiscale basis functions as follows:

$$\boldsymbol{V}_{\text{ms}}(\omega_j) = \text{span}\left\{\boldsymbol{\psi}_k^{(j),\text{ms}} \,\middle|\, 1 \leq k \leq L_j\right\}. \tag{4.67}$$

Then, the multiscale space $\boldsymbol{V}_{\text{ms}}$ can be taken as the global offline space:

$$\boldsymbol{V}_{\text{ms}} = \sum_{j=1}^{N_v} \boldsymbol{V}_{\text{ms}}(\omega_j) = \text{span}\left\{\boldsymbol{\psi}_k^{(j),\text{ms}} \,\middle|\, 1 \leq j \leq N_v\,, 1 \leq k \leq L_j\right\}.$$

4.4 Convergence Analysis (GMsFEM)

In this section, we show convergence analysis for both uncoupled and coupled GMsFEM. First, best (a-priori) error estimate is provided, for our semi-discrete problem. We will compare the difference between the reference weak solution $\boldsymbol{u} \in \boldsymbol{V}$ defined in (4.5) and the multiscale solution $\boldsymbol{u}_{\text{ms}} \in \boldsymbol{V}_{\text{ms}}$ defined in (4.52), by using the projection error of $\boldsymbol{u}$ onto $\boldsymbol{V}_{\text{ms}}$ in various norms.

Lemma 4.4.1. *Under Assumption 4.2.3, for $\boldsymbol{u}$ and $\boldsymbol{u}_{\text{ms}}$ defined in (4.5) and (4.52), respectively, where $\boldsymbol{V}_{\text{ms}}$ is constructed via the uncoupled GMsFEM, we have the following result:*

$$\begin{aligned}
&||\boldsymbol{u}(\cdot,T) - \boldsymbol{u}_{\text{ms}}(\cdot,T)||_c^2 + \int_0^T ||\boldsymbol{u} - \boldsymbol{u}_{\text{ms}}||_a^2 \,\mathrm{d}t \\
&\leq C \inf_{w \in \boldsymbol{V}_{\text{ms}}} \left(\int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial t} \right\|_c^2 \mathrm{d}t + \int_0^T \|\boldsymbol{w} - \boldsymbol{u}\|_a^2 \,\mathrm{d}t + ||\boldsymbol{w}(\cdot,0) - \boldsymbol{u}(\cdot,0)||_c^2 \right).
\end{aligned} \tag{4.68}$$

Proof. The proof is similar to that of Lemma 4.2.7. □

In the spirit of this Lemma, based on [6], to complete the convergence proof for our proposed approach, we will find an appropriate function $\boldsymbol{w}$ in the multiscale space $\boldsymbol{V}_{\text{ms}}$, then estimate the error $\boldsymbol{w} - \boldsymbol{u}$ (the so-called projection error of $\boldsymbol{u}$ onto $\boldsymbol{V}_{\text{ms}}$) in various norms on the right hand side of (4.68). More specifically, we will define an approximation $\boldsymbol{u}_{\text{snap}} \in \boldsymbol{V}_{\text{snap}}$ (called snapshot projection) of $\boldsymbol{u}$ in the snapshot space (which is the set of all snapshot functions). We can express $\boldsymbol{w} - \boldsymbol{u} = \boldsymbol{w} - \boldsymbol{u}_{\text{snap}} + \boldsymbol{u}_{\text{snap}} - \boldsymbol{u}$, where the last term $\boldsymbol{u}_{\text{snap}} - \boldsymbol{u}$ corresponds to an irreducible error of our method, and can be assumed to be very small by utilizing a large enough collection of snapshot functions. It hence suffices to only estimate $\boldsymbol{w} - \boldsymbol{u}_{\text{snap}}$ by choosing a suitable function $\boldsymbol{w} \in \boldsymbol{V}_{\text{ms}}$.

We will define $\boldsymbol{w} \in \boldsymbol{V}_{\text{ms}}$ as the projection of $\boldsymbol{u}_{\text{snap}}$ onto the multiscale space $\boldsymbol{V}_{\text{ms}}$. In particular, first, in the case of uncoupled GMsFEM, the snapshot projection $\boldsymbol{u}_{\text{snap}}$ (in $\boldsymbol{V}_{\text{snap}}$) of $\boldsymbol{u}$ can be represented by the set of $\;^{(j)}_{k,i}(\boldsymbol{x})$ from (4.56) as follows:

$$\boldsymbol{u}_{\text{snap}}(\boldsymbol{x}, t) = (u_{\text{snap},1}, u_{\text{snap},2})\,, \quad u_{\text{snap},i} = \sum_{j=1}^{N_v} \sum_{k} d^{(j)}_{k,i}(t)\, \chi_{j,i}(\boldsymbol{x})\, \psi^{(j)}_{k,i}(\boldsymbol{x})\,. \tag{4.69}$$

We define the local component of $u^{(j)}_{\text{snap},i}$ by

$$u^{(j)}_{\text{snap},i}(\boldsymbol{x}, t) = \sum_{k} d^{(j)}_{k,i}(t)\, \psi^{(j)}_{k,i}(\boldsymbol{x})\,, \quad \text{with } u^{(j)}_{\text{snap},i}|_{\partial\omega_j} = u_i|_{\partial\omega_j}\,. \tag{4.70}$$

Then, the projection $\boldsymbol{w}$ of $\boldsymbol{u}_{\text{snap}}$ in the multiscale space $\boldsymbol{V}_{\text{ms}}$ is defined as

$$\boldsymbol{w}(\boldsymbol{x}, t) = (w_1, w_2)\,, \quad w_i = \sum_{j=1}^{N_v} \sum_{k=1}^{L_j} d^{(j)}_{k,i}(t)\, \psi^{(j),\text{ms}}_{k,i}(\boldsymbol{x}) = \sum_{j=1}^{N_v} \sum_{k=1}^{L_j} d^{(j)}_{k,i}(t)\, \chi_{j,i}(\boldsymbol{x})\, \psi^{(j)}_{k,i}(\boldsymbol{x})\,, \tag{4.71}$$

where the collection of local multiscale basis functions $\left\{ \psi^{(j),\text{ms}}_{k,i}(\boldsymbol{x}) \,\middle|\, 1 \le k \le L_j \right\}$ is from (4.59).

Second, in the case of coupled GMsFEM, the snapshot projection $\boldsymbol{u}_{\text{snap}}$ (in $\boldsymbol{V}_{\text{snap}}$) of $\boldsymbol{u}$

can be represented by the set of $\boldsymbol{\psi}_k^{(j)}(\boldsymbol{x}) = \Big(\ {}^{(j)}_{k,1}(\boldsymbol{x}), \psi^{(j)}_{k,2}(\boldsymbol{x})\Big)$ from (4.64) as follows:

$$\boldsymbol{u}_{\text{snap}}(\boldsymbol{x}, t) = (u_{\text{snap},1}, u_{\text{snap},2})\,, \quad u_{\text{snap},i} = \sum_{j=1}^{N_v} \sum_{k} d_{k,i}^{(j)}(t)\, \chi_{j,i}(\boldsymbol{x})\, \psi_{k,i}^{(j)}(\boldsymbol{x})\,. \tag{4.72}$$

We define the local component of $u_{\text{snap},i}^{(j)}$ by

$$u_{\text{snap},i}^{(j)}(\boldsymbol{x}, t) = \sum_{k} d_{k,i}^{(j)}(t)\, \psi_{k,i}^{(j)}(\boldsymbol{x})\,, \quad \text{with } u_{\text{snap},i}^{(j)}|_{\partial\omega_j} = u_i|_{\partial\omega_j}\,. \tag{4.73}$$

Then, the projection $\boldsymbol{w}$ of $\boldsymbol{u}_{\text{snap}}$ in the multiscale space $\boldsymbol{V}_{\text{ms}}$ is defined as

$$\boldsymbol{w}(\boldsymbol{x}, t) = (w_1, w_2)\,, \quad w_i = \sum_{j=1}^{N_v} \sum_{k=1}^{L_j} d_{k,i}^{(j)}(t)\, \psi_{k,i}^{(j),\text{ms}}(\boldsymbol{x}) = \sum_{j=1}^{N_v} \sum_{k=1}^{L_j} d_{k,i}^{(j)}(t)\, \chi_{j,i}(\boldsymbol{x})\, \psi_{k,i}^{(j)}(\boldsymbol{x})\,, \tag{4.74}$$

where the collection of local multiscale basis functions $\left\{\boldsymbol{\psi}_k^{(j),\text{ms}}(\boldsymbol{x}) \,\middle|\, 1 \le k \le L_j\right\}$ is from (4.66).

Now, we present the main results of this section.

4.4.1 Uncoupled GMsFEM

Convergence analysis is presented for the uncoupled GMsFEM. We will compare the difference between the reference weak solution $\boldsymbol{u}$ defined in (4.5) and the multiscale solution $\boldsymbol{u}_{\text{ms}}$ defined in (4.52) from the uncoupled GMsFEM.

Lemma 4.4.2. *For the uncoupled GMsFEM, if* $\mathbf{u}$ *in (4.5) satisfies*

$$\int_{\omega_j} \kappa_1 \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x + \int_{\omega_j} \kappa_2 \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x = \int_{\omega_j} f_1 v_1 \,\mathrm{d}x + \int_{\Omega_j} f_2 v_2 \,\mathrm{d}x\,, \tag{4.75}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\omega_j)$, then we have

$$\begin{aligned} &\int_{\omega_j} \kappa_1 \chi_{j,1}^2 |\nabla u_1|^2 \,\mathrm{d}x + \int_{\omega_j} \kappa_2 \chi_{j,2}^2 |\nabla u_2|^2 \,\mathrm{d}x \\ &\leq C \sum_{i=1}^{2} \left(\int_{\omega_j} \frac{\chi_{j,i}^4}{\kappa_i |\nabla \chi_{j,i}|^2} f_i^2 \,\mathrm{d}x + \int_{\omega_j} \kappa_i |\nabla \chi_{j,i}|^2 u_i^2 \,\mathrm{d}x \right). \end{aligned} \tag{4.76}$$

Proof. We base on [6] for the proof. Take $v_i = (\chi_{j,i}^2) u_i$ (for $i = 1, 2$), we obtain

$$\sum_{i=1}^{2} \int_{\omega_j} \kappa_i (\nabla u_i) \cdot \nabla (\chi_{j,i}^2 u_i) \,\mathrm{d}x = \sum_{i=1}^{2} \int_{\omega_j} f_i (\chi_{j,i}^2) u_i \,\mathrm{d}x \,.$$

This leads to

$$\begin{aligned} \sum_{i=1}^{2} \int_{\omega_j} \kappa_i \chi_{j,i}^2 |\nabla u_i|^2 \,\mathrm{d}x &= \sum_{i=1}^{2} \int_{\omega_j} f_i \frac{\chi_{j,i}^2}{(\nabla \chi_{j,i}) \sqrt{\kappa_i}} \sqrt{\kappa_i} u_i \nabla \chi_{j,i} \,\mathrm{d}x \\ &\quad - 2 \sum_{i=1}^{2} \int_{\omega_j} \kappa_i (\chi_{j,i}) (\nabla u_i) \cdot (\nabla \chi_{j,i}) u_i \,\mathrm{d}x \\ &\leq \frac{\epsilon}{2} \sum_{i=1}^{2} \int_{\omega_j} f_i^2 \frac{\chi_{j,i}^4}{|\nabla \chi_{j,i}|^2 \kappa_i} \,\mathrm{d}x + \frac{1}{2\epsilon} \sum_{i=1}^{2} \int_{\omega_j} \kappa_i (u_i \nabla \chi_{j,i})^2 \,\mathrm{d}x \\ &\quad + \epsilon \sum_{i=1}^{2} \int_{\omega_j} \kappa_i \chi_{j,i}^2 |\nabla u_i|^2 \,\mathrm{d}x + \frac{1}{\epsilon} \sum_{i=1}^{2} \int_{\omega_j} \kappa_i (u_i \nabla \chi_{j,i})^2 \,\mathrm{d}x \,, \end{aligned}$$

where the last inequality follows from Young's inequality. Let $\epsilon = 1/2$, and move the third term on the right hand side to the left hand side of the above inequality. Then, for some constant $C > 0$, the desired inequality (4.76) holds. □

We finally have the following error estimate.

Theorem 4.4.3. *Let $\boldsymbol{u}$ be the solution of (4.5), $\boldsymbol{u}_{\text{snap}}$ and $\boldsymbol{w}$ be defined in (4.69) and (4.71),* *respectively. Then, we obtain the following result:*

$$
\begin{aligned}
&\int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u}_{\text{snap}})}{\partial t} \right\|_c^2 \mathrm{d}t + \int_0^T ||\boldsymbol{w} - \boldsymbol{u}_{\text{snap}}||_a^2 \, \mathrm{d}t + ||\boldsymbol{w}(\cdot,0) - \boldsymbol{u}_{\text{snap}}(\cdot,0)||_c^2 \\
&\leq \frac{C}{\Lambda_1} \left(\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_a^2 \mathrm{d}t + \int_0^T ||\boldsymbol{u}||_a^2 \, \mathrm{d}t + ||\boldsymbol{u}(\cdot,0)||_a^2 \right),
\end{aligned}
\tag{4.77}
$$

where $\Lambda_1 = \min_{j,i}\{\lambda^{(j)}_{L_j+1,i}\}$.

Proof. We base on [6, 15] for the proof of this Theorem. That is, our proof follows from Lemmas 4.4.11, 4.4.12 and 4.4.13 at the end of this section.

□

4.4.2 Coupled GMsFEM

Convergence analysis is provided for the coupled GMsFEM. We will compare the difference between the reference weak solution $\boldsymbol{u}$ defined in (4.5) and the multiscale solution $\boldsymbol{u}_{\text{ms}}$ defined in (4.52) from the coupled GMsFEM.

We will utilize the notation from (4.50) and (4.51). Assume that there is some positive constant $\overline{Q_s}$ such that $|Q_s| \leq \overline{Q_s}$. Then, it is easy to show that

$$
\left(1 - \frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right) a(\boldsymbol{u},\boldsymbol{u}) \leq a_{Q_s}(\boldsymbol{u},\boldsymbol{u}) \leq \left(1 + \frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right) a(\boldsymbol{u},\boldsymbol{u}), \tag{4.78}
$$

where $C_p(\Omega)$ is from (4.14). We now have the following lemma.

Lemma 4.4.4. *Assume* $\left(1 - \frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right) > 0$*. Then, there exist constants* $m_1, m_2 > 0$ *such that*

$$
m_1 \, a(\boldsymbol{u},\boldsymbol{u}) \leq a_{Q_s}(\boldsymbol{u},\boldsymbol{u}) \leq m_2 \, a(\boldsymbol{u},\boldsymbol{u}). \tag{4.79}
$$

Throughout this section, we always assume that $\left(1 - \frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right) > 0$ holds. Recall that $a_{Q_s}(\boldsymbol{u},\boldsymbol{v}) = a_{Q_s}(\boldsymbol{v},\boldsymbol{u})$.

Lemma 4.4.5. *Let K, α and C_b be defined as in Lemma 4.2.1 and its proof.*

$$
\begin{aligned}
&b((u_1,u_2),(v_1,v_2)) \leq C_b\left(1-\frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right)^{-1}||\boldsymbol{u}||_{a_{Q_s}}\,||\boldsymbol{v}||_{a_{Q_s}}, \\
&b((u_1,u_2),(u_1,u_2)) + K||\boldsymbol{u}||^2_{L^2(\Omega)} \geq \alpha\left(1+\frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right)^{-1}||\boldsymbol{u}||^2_{a_{Q_s}},
\end{aligned}
\tag{4.80}
$$

for all (u_1,u_2), $(v_1,v_2) \in \boldsymbol{V}$.

Proof. The result follows from Lemma 4.2.1 and (4.78). □

The following assumption is for later theorem.

Assumption 4.4.1. *We assume that $\alpha\left(1+\frac{2\overline{Q_s}C_p^2}{\underline{\kappa}}\right)^{-1} > \frac{K\,C_p}{\sqrt{\underline{\kappa}}}$ where K, α and C_p are from the proof of Lemma 4.2.1.*

Theorem 4.4.6. *Under Assumptions 4.2.1 and 4.4.1, we have a unique solution of the problem (4.8) with respect to $||\cdot||_{a_{Q_s}}$.*

Proof. The result follows from Lemma 4.4.5, the Poincaré inequality and the Lax-Milgram Theorem. □

Under Assumptions 4.2.1 and 4.4.1, the following assumptions are satisfied.

Assumption 4.4.2. *There exists constants D_1, $D_2 > 0$ such that*

$$
\begin{aligned}
&b((u_1,u_2),(v_1,v_2)) \leq D_1||\boldsymbol{u}||_{a_{Q_s}}\,||\boldsymbol{v}||_{a_{Q_s}}, \\
&b((v_1,v_2),(v_1,v_2)) \geq D_2||\boldsymbol{v}||^2_{a_{Q_s}},
\end{aligned}
\tag{4.81}
$$

for all $\boldsymbol{u} = (u_1,u_2)$, $\boldsymbol{v} = (v_1,v_2) \in \boldsymbol{V}$.

Lemma 4.4.7. *Under Assumption 4.4.2, in the coupled GMsFEM, for* $\boldsymbol{u}$ *and* $\boldsymbol{u}_{\text{ms}}$ *respectively defined in (4.5) and (4.52), we have the following result:*

$$\begin{aligned}
&\|\boldsymbol{u}(\cdot,T)-\boldsymbol{u}_{\text{ms}}(\cdot,T)\|_c^2+\int_0^T \|\boldsymbol{u}-\boldsymbol{u}_{\text{ms}}\|_{a_{Q_s}}^2 \,\mathrm{d}t \\
&\le C \inf_{\boldsymbol{w}\in \boldsymbol{V}_{\text{ms}}} \left(\int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 \mathrm{d}t + \int_0^T ||\boldsymbol{w}-\boldsymbol{u}||_{a_{Q_s}}^2 \,\mathrm{d}t + ||\boldsymbol{w}(\cdot,0)-\boldsymbol{u}(\cdot,0)||_c^2 \right).
\end{aligned} \tag{4.82}$$

Proof. The proof is similar to that of Lemma 4.2.7. □

We hence obtain the following convergence result, under weaker condition on the bilinear form b.

Lemma 4.4.8. *Assume that there exist positive constants* $\overline{Q_a}$, D_1 *and* D_2 *such that* $|Q_a| \le \overline{Q_a}$ *and*

$$D_2||\boldsymbol{v}||_{a_{Q_s}}^2 \le b((v_1,v_2),(v_1,v_2)) \le D_1||\boldsymbol{v}||_{a_{Q_s}}^2 , \tag{4.83}$$

for all $\boldsymbol{v}=(v_1,v_2)\in \boldsymbol{V}$. *For* $\boldsymbol{u}$ *and* $\boldsymbol{u}_{\text{ms}}$ *respectively defined in (4.5) and (4.52) from the coupled GMsFEM, the following result holds:*

$$\begin{aligned}
&||\boldsymbol{u}(\cdot,T)-\boldsymbol{u}_{\text{ms}}(\cdot,T)||_c^2+\int_0^T \|\boldsymbol{u}-\boldsymbol{u}_{\text{ms}}\|_{a_{Q_s}}^2 \,\mathrm{d}t \\
&\le C \inf_{w\in \boldsymbol{V}_{\text{ms}}} \Bigg(\int_0^T \left\| \frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c^2 \mathrm{d}t + \int_0^T \|\boldsymbol{w}-\boldsymbol{u}\|_{a_{Q_s}}^2 \,\mathrm{d}t \\
&+ \bar{b}\int_0^T ||\nabla \boldsymbol{w}-\nabla\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^2 \,\mathrm{d}t \\
&+ \overline{Q_a}\int_0^T ||\boldsymbol{w}-\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^2 \,\mathrm{d}t + ||\boldsymbol{w}(\cdot,0)-\boldsymbol{u}_{\text{ms}}(\cdot,0)||_c^2 \Bigg),
\end{aligned} \tag{4.84}$$

where $\bar{b}$ *is from Assumption 4.2.1.*

Note that the constant C in this Lemma can be different from the one in Lemma 4.4.7.

Proof. Recall that for all $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}_{\text{ms}}$, from (4.5) and (4.52), we have

$$c\left(\frac{\partial(\boldsymbol{u} - \boldsymbol{u}_{\text{ms}})}{\partial t}, \boldsymbol{v}\right) + b(\boldsymbol{u} - \boldsymbol{u}_{\text{ms}}, \boldsymbol{v}) = 0\,. \tag{4.85}$$

Given $\boldsymbol{w} \in \boldsymbol{V}_{\text{ms}}$, we let $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}_{\text{ms}} \in \boldsymbol{V}_{\text{ms}}$. Using notation from (4.50), and Young's inequality, we note that

$$\begin{aligned}\beta(\boldsymbol{w} - \boldsymbol{u}, \boldsymbol{w} - \boldsymbol{u}_{\text{ms}}) &= \beta((w_1 - u_1, w_2 - u_2), (w_1 - u_{\text{ms},1}, w_2 - u_{\text{ms},2})) \\ &= \int_\Omega \boldsymbol{b}_1 \cdot \nabla((w_1 - u_1) - (w_2 - u_2))(w_1 - u_{\text{ms},1})\,\mathrm{d}x \\ &\quad + \int_\Omega \boldsymbol{b}_2 \cdot \nabla((w_2 - u_2) - (w_1 - u_1))(w_2 - u_{\text{ms},2})\,\mathrm{d}x \\ &\leq \frac{1}{c_1}\bar{b}||\nabla \boldsymbol{w} - \nabla \boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + \frac{c_1}{2}||\boldsymbol{w} - \boldsymbol{u}_{\text{ms}}||^2_{\boldsymbol{L}^2(\Omega)}\,,\end{aligned} \tag{4.86}$$

for some $c_1 > 0$. Also,

$$q_a(\boldsymbol{w} - \boldsymbol{u}, \boldsymbol{w} - \boldsymbol{u}_{\text{ms}}) \leq \frac{1}{d_1}\overline{Q_a}\;||\boldsymbol{w} - \boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + \frac{d_1}{2}||\boldsymbol{w} - \boldsymbol{u}_{\text{ms}}||^2_{\boldsymbol{L}^2(\Omega)}\,, \tag{4.87}$$

for some $d_1 > 0$. Hence, for D_2 from (4.83), utilizing (4.85), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_c^2 + D_2||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_{a_{Q_s}}^2 \\
&= c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}})}{\partial t}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\right) + D_2||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_{a_{Q_s}} \\
&\leq c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}})}{\partial t}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\right) + b(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&= c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}})}{\partial t}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\right) + a(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) + \beta(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&+ q(\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&= c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\right) + a_{Q_s}(\boldsymbol{w}-\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) + \beta(\boldsymbol{w}-\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&+ q_a(\boldsymbol{w}-\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&\leq \left|c\left(\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\right)\right| + \|\boldsymbol{w}-\boldsymbol{u}\|_{a_{Q_s}} \|\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\|_{a_{Q_s}} + \beta(\boldsymbol{w}-\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&+ q_a(\boldsymbol{w}-\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}_{\text{ms}}) \\
&\leq \left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u})}{\partial t}\right\|_c \|\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\|_c + \|\boldsymbol{w}-\boldsymbol{u}\|_{a_{Q_s}} \|\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\|_{a_{Q_s}} \\
&+ \frac{1}{c_1}\bar{b}||\nabla\boldsymbol{w}-\nabla\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^2 + \frac{c_1}{2}||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_{\boldsymbol{L}^2(\Omega)}^2 \\
&+ \frac{1}{d_1}\overline{Q_a}\,||\boldsymbol{w}-\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^2 + \frac{d_1}{2}||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_{\boldsymbol{L}^2(\Omega)}^2,
\end{aligned}
\tag{4.88}
$$

where the last inequality follows from (4.86) and (4.87).

From the Poincaré inequality (4.14), there exists $C_p, D > 0$ such that $\|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)}^2 \leq C_p^2\|\nabla\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)}^2 \leq D\|\boldsymbol{z}\|_{a_{Q_s}}^2, \forall \boldsymbol{z} \in \boldsymbol{V} \subset \boldsymbol{L}^2(\Omega)$. Thus, in the last inequality of (4.88),

$$\frac{c_1+d_1}{2}||\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}||_{\boldsymbol{L}^2(\Omega)}^2 \leq \frac{D(c_1+d_1)}{2}\|\boldsymbol{w}-\boldsymbol{u}_{\text{ms}}\|_{a_{Q_s}}^2.$$

We define the initial value $\boldsymbol{u}_{\text{ms}}(\cdot,0)$ such that $c(\boldsymbol{u}(\cdot,0),\boldsymbol{v}) = c(\boldsymbol{u}_{\text{ms}}(\cdot,0),\boldsymbol{v})$, so $||\boldsymbol{u}(\cdot,0)-\boldsymbol{u}_{\text{ms}}(\cdot,0)||_c = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}$. Then, the rest of the proof is similar to that of

Lemma 4.2.7. □

Lemma 4.4.9. *For the coupled GMsFEM, if* $\mathbf{u}$ *from (4.5) satisfies*

$$\int_{\omega_j} \kappa_1 \nabla u_1 \cdot \nabla v_1 \,\mathrm{d}x + \int_{\omega_j} \kappa_2 \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x + q_s(\boldsymbol{u}, \boldsymbol{v}) = \int_{\omega_j} f_1 v_1 \,\mathrm{d}x + \int_{\omega_j} f_2 v_2 \,\mathrm{d}x \,, \quad (4.89)$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\omega_j)$*, we have*

$$\begin{aligned} &\int_{\omega_j} \kappa_1 \chi_j^2 |\nabla u_1|^2 \,\mathrm{d}x + \int_{\omega_j} \kappa_2 \chi_j^2 |\nabla u_2|^2 \,\mathrm{d}x + q_s((u_1, u_2), (\chi_{j,1}^2 \, u_1, \chi_{j,2}^2 \, u_2)) \\ &\leq C \sum_{i=1}^{2} \bigg(\int_{\omega_j} \frac{\chi_j^4}{\kappa_i |\nabla \chi_j|^2} f_i^2 \,\mathrm{d}x + \int_{\omega_j} \kappa_i |\nabla \chi_j|^2 u_i^2 \,\mathrm{d}x \bigg). \end{aligned} \quad (4.90)$$

Proof. The proof of this Lemma readily follows from that of Lemma 4.4.2 and thanks to [6]. □

Theorem 4.4.10. *Let* $\mathbf{u}$ *be the solution of (4.5),* $\boldsymbol{u}_{\text{snap}}$ *and* $\boldsymbol{w}$ *be defined in (4.72) and (4.74), respectively. Then, we have the following estimate:*

$$\begin{aligned} &\int_0^T \left\| \frac{\partial (\boldsymbol{w} - \boldsymbol{u}_{\text{snap}})}{\partial t} \right\|_c^2 \mathrm{d}t + \int_0^T ||\boldsymbol{w} - \boldsymbol{u}_{\text{snap}}||_{a_{Q_s}}^2 \,\mathrm{d}t + ||\boldsymbol{w}(\cdot, 0) - \boldsymbol{u}_{\text{snap}}(\cdot, 0)||_c^2 \\ &\leq \frac{C}{\Lambda_2} \left(\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_{a_{Q_s}}^2 \mathrm{d}t + \int_0^T ||\boldsymbol{u}||_{a_{Q_s}}^2 \,\mathrm{d}t + ||\boldsymbol{u}(\cdot, 0)||_{a_{Q_s}}^2 \right) \,, \end{aligned} \quad (4.91)$$

where $\Lambda_2 = \min_j \{\lambda_{L_j+1}^{(j)}\}$.

Proof. Following the proof in [6, 15], our proof is derived from Lemmas 4.4.11, 4.4.12 and 4.4.13. □

4.4.3 Lemmas for the main convergence results

In this part, we provide and prove some Lemmas that Theorems 4.4.3 and 4.4.10 directly follow from.

Lemma 4.4.11. *Let $\boldsymbol{u}$, $\boldsymbol{u}_{\text{snap}}$, $\boldsymbol{w}$, Λ_1 and Λ_2 be defined in Theorems 4.4.3 and 4.4.10. For the uncoupled GMsFEM, we have*

$$\int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u}_{\text{snap}})}{\partial t} \right\|_c^2 \mathrm{d}t \leq \frac{C}{\Lambda_1} \int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_a^2 \mathrm{d}t\,. \tag{4.92}$$

For the coupled GMsFEM, we have

$$\int_0^T \left\| \frac{\partial(\boldsymbol{w} - \boldsymbol{u}_{\text{snap}})}{\partial t} \right\|_c^2 \mathrm{d}t \leq \frac{C}{\Lambda_2} \int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|_{a_{Q_s}}^2 \mathrm{d}t\,. \tag{4.93}$$

Proof. Based on [6, 15], we will first derive the proof for the case of uncoupled GMsFEM. Note that

$$\begin{aligned}
&\left\| \frac{\partial \boldsymbol{w}}{\partial t} - \frac{\partial \boldsymbol{u}_{\text{snap}}}{\partial t} \right\|_c^2 \\
&= \sum_{i=1}^{2} \int_\Omega \mathcal{C}_{ii} \left(\frac{\partial w_i}{\partial t} - \frac{\partial u_{\text{snap},i}}{\partial t} \right)^2 \mathrm{d}x \\
&= \sum_{i=1}^{2} \int_\Omega \mathcal{C}_{ii} \left(\sum_{j=1}^{N_v} \sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \chi_{j,i}(\boldsymbol{x}) \psi_{k,i}^{(j)}(\boldsymbol{x}) \right)^2 \mathrm{d}x \\
&\leq C \sum_{i=1}^{2} \sum_{j=1}^{N_v} \int_{\omega_j} \kappa_i \left(\sum_{j=1}^{N_v} |\nabla \chi_{j,i}|^2 \right) \left(\sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \quad {}_{k,i}^{(j)}(\boldsymbol{x}) \right)^2 \mathrm{d}x \\
&= C \sum_{i=1}^{2} \sum_{j=1}^{N_v} s_i^{(j)} \left(\sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \quad {}_{k,i}^{(j)}(\boldsymbol{x}), \sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \quad {}_{k,i}^{(j)}(\boldsymbol{x}) \right).
\end{aligned} \tag{4.94}$$

By the spectral problem (4.56) and the orthogonality of eigenfunctions $\{\psi_{k,i}^{(j)}(\boldsymbol{x})\}_k$, we have

$$
\begin{aligned}
& s_i^{(j)} \left(\sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}), \sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}) \right) \\
& \leq \frac{1}{\lambda_{L_j+1,i}^{(j)}} a_i^{(j)} \left(\sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}), \sum_{k>L_j} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}) \right) \\
& \leq \frac{1}{\lambda_{L_j+1,i}^{(j)}} a_i^{(j)} \left(\sum_{k} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}), \sum_{k} \frac{\partial d_{k,i}^{(j)}(t)}{\partial t} \; {}_{k,i}^{(j)}(\boldsymbol{x}) \right) \\
& = \frac{1}{\lambda_{L_j+1,i}^{j}} a_i^{(j)} \left(\frac{\partial u_{\text{snap},i}^{(j)}}{\partial t}, \frac{\partial u_{\text{snap},i}^{(j)}}{\partial t} \right) .
\end{aligned}
\tag{4.95}
$$

Therefore, (4.94) becomes

$$
\left\| \frac{\partial \boldsymbol{w}}{\partial t} - \frac{\partial \boldsymbol{u}_{\text{snap}}}{\partial t} \right\|_c^2 \leq C \sum_{i=1}^{2} \sum_{j=1}^{N_v} \frac{1}{\lambda_{L_j+1,i}^{j}} a_i^{(j)} \left(\frac{\partial u_{\text{snap},i}^{(j)}}{\partial t}, \frac{\partial u_{\text{snap},i}^{(j)}}{\partial t} \right) . \tag{4.96}
$$

Since $u_{\text{snap},i}^{(j)}$ is the projection of u_i in each ω_j by the definition (4.69), it follows that

$$
a_i^{(j)}(u_i, v_i) = a_i^{(j)} \left(u_{\text{snap},i}^{(j)}, v_i \right) \, , \quad \forall v_i \in V_{\text{snap}}^i(\omega_j) \, .
$$

More specifically, let $v_i = u_{\text{snap},i}^{(j)}$, we have

$$
a_i^{(j)} \left(u_{\text{snap},i}^{(j)}, u_{\text{snap},i}^{(j)} \right) = a_i^{(j)} \left(u_i, u_{\text{snap},i}^{(j)} \right) \, ,
$$

$$
\left\| u_{\text{snap},i}^{(j)} \right\|_{a_i^{(j)}}^2 \leq \| u_i \|_{a_i^{(j)}} \left\| u_{\text{snap},i}^{(j)} \right\|_{a_i^{(j)}} \, .
$$

Hence,

$$
a_i^{(j)} \left(u_{\text{snap},i}^{(j)}, u_{\text{snap},i}^{(j)} \right) \leq a_i^{(j)}(u_i, u_i) \, . \tag{4.97}
$$

Similarly,

$$a_i^{(j)}\left(\frac{\partial u_{\text{snap},i}^{(j)}}{\partial t}, \frac{\partial u_{\text{snap},i}^{(j)}}{\partial t}\right) \leq a_i^{(j)}\left(\frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial t}\right).$$

Thus, from (4.96), we get

$$\begin{aligned}
\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_{\text{snap}})}{\partial t}\right\|_c^2 &\leq C\sum_{i=1}^{2}\sum_{j=1}^{N_v}\frac{1}{\lambda_{L_j+1,i}^{(j)}}a_i^{(j)}\left(\frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial t}\right) \\
&\leq \frac{C}{\min\limits_{j,i}\{\lambda_{L_j+1,i}^{(j)}\}}a\left(\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial \boldsymbol{u}}{\partial t}\right) \\
&= \frac{C}{\min\limits_{j,i}\{\lambda_{L_j+1,i}^{(j)}\}}\left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_a^2.
\end{aligned} \tag{4.98}$$

For the case of coupled GMsFEM, recall that $s^{(j)}(\cdot,\cdot) = \sum_{i=1}^{2} s_i^{(j)}(\cdot,\cdot)$. Applying the same arguments, we get

$$\left\|\frac{\partial(\boldsymbol{w}-\boldsymbol{u}_{\text{snap}})}{\partial t}\right\|_c^2 \leq C\sum_{i=1}^{2}\sum_{j=1}^{N_v} s_i^{(j)}\left(\sum_{k>L_j}\frac{\partial d_{k,i}^{(j)}(t)}{\partial t}\ {}_{k,i}^{(j)}(\boldsymbol{x}), \sum_{k>L_j}\frac{\partial d_{k,i}^{(j)}(t)}{\partial t}\ {}_{k,i}^{(j)}(\boldsymbol{x})\right).$$

□

Lemma 4.4.12. *Let $\boldsymbol{u}$, $\boldsymbol{u}_{\text{snap}}$, $\boldsymbol{w}$, Λ_1 and Λ_2 be defined in Theorems 4.4.3 and 4.4.10. For the uncoupled GMsFEM, we have*

$$\int_0^T \|\boldsymbol{w}-\boldsymbol{u}_{\text{snap}}\|_a^2 \,\mathrm{d}t \leq \frac{C}{\Lambda_1}\int_0^T \|\boldsymbol{u}\|_a^2 \,\mathrm{d}t. \tag{4.99}$$

For the coupled GMsFEM, we have

$$\int_0^T \|\boldsymbol{w}-\boldsymbol{u}_{\text{snap}}\|_{a_{Q_s}}^2 \,\mathrm{d}t \leq \frac{C}{\Lambda_2}\int_0^T \|\boldsymbol{u}\|_{a_{Q_s}}^2 \,\mathrm{d}t. \tag{4.100}$$

Proof. This Lemma's proof is based on [6, 15].

For the case of uncoupled GMsFEM, we define

$$e_i^{(j)} = \sum_{k>L_j} d_{k,i}^{(j)}(t)\, \psi_{k,i}^{(j)}(\boldsymbol{x})\,.$$

By (4.71) and (4.69), we have

$$\begin{aligned}
&\|\boldsymbol{w} - \boldsymbol{u}_{\text{snap}}\|_a^2 \\
&= \sum_{i=1}^{2} \int_\Omega \kappa_i \left|\nabla(w_i - u_{\text{snap},i})\right|^2 \, \mathrm{d}x \\
&= \sum_{i=1}^{2} \int_\Omega \kappa_i \left|\sum_{j=1}^{N_v} \sum_{k>L_j} \nabla \left(d_{k,i}^{(j)}(t)\, \chi_{j,i} \quad {}^{(j)}_{k,i}\right)\right|^2 \, \mathrm{d}x \\
&\leq N_v \sum_{i=1}^{2} \sum_{j=1}^{N_v} \int_{\omega_j} \kappa_i \left|\nabla\left(\chi_{j,i} e_i^{(j)}\right)\right|^2 \, \mathrm{d}x \\
&\leq 2N_v \sum_{i=1}^{2} \sum_{j=1}^{N_v} \left(\int_{\omega_j} \kappa_i \left|\nabla \chi_{j,i}\right|^2 \left|e_i^{(j)}\right|^2 \, \mathrm{d}x + \int_{\omega_j} \kappa_i \left|\chi_{j,i}\right|^2 \left|\nabla e_i^{(j)}\right|^2 \, \mathrm{d}x \right).
\end{aligned} \tag{4.101}$$

Note that

$$\int_{\omega_j} \kappa_i \left|\nabla \chi_{j,i}\right|^2 \left|e_i^{(j)}\right|^2 \, \mathrm{d}x \leq \int_{\omega_j} \kappa_i \left(\sum_{j=1}^{N_v} \left|\nabla \chi_{j,i}\right|^2\right) \left|e_i^{(j)}\right|^2 \, \mathrm{d}x = s_i^{(j)}\left(e_i^{(j)}, e_i^{(j)}\right).$$

From this, Lemma 4.4.2 and (4.54), there exists some positive constant D_3 such that

$$\int_{\omega_j} \kappa_i \left|\chi_{j,i}\right|^2 \left|\nabla e_i^{(j)}\right|^2 \, \mathrm{d}x \leq D_3 \int_{\omega_j} \kappa_i \left|\nabla \chi_{j,i}\right|^2 \left|e_i^{(j)}\right|^2 \leq D_3 s_i^{(j)}\left(e_i^{(j)}, e_i^{(j)}\right).$$

Therefore,

$$\|\boldsymbol{w} - \boldsymbol{u}_{\text{snap}}\|_a^2 \leq D_4 \sum_{i=1}^{2} \sum_{j=1}^{N_v} s_i^{(j)}\left(e_i^{(j)}, e_i^{(j)}\right).$$

Finally, based on bilinearity of $a_i^{(j)}$ and $s_i^{(j)}$ as well as the orthogonality of $\{\psi_{k,i}^j\}_k$, and the definition of the eigenprojection, for the case of uncoupled GMsFEM, as in (4.97), we get

$$s_i^{(j)}\left(e_i^{(j)}, e_i^{(j)}\right) \le \frac{1}{\lambda_{L_j+1,i}^{(j)}} a_i^{(j)}\left(e_i^{(j)}, e_i^{(j)}\right) \le \frac{1}{\lambda_{L_j+1,i}^{(j)}} a_i^{(j)}\left(u_{\text{snap},i}^{(j)}, u_{\text{snap},i}^{(j)}\right)$$
$$\le \frac{1}{\lambda_{L_j+1,i}^{(j)}} a_i^{(j)}\left(u_i, u_i\right) .$$

Hence, the desired result (4.99) follows.

For the case of coupled GMsFEM, similar arguments are applied for

$$\boldsymbol{e}^{(j)} = \sum_{k>L_j} d_k^{(j)}(t)\, \boldsymbol{\psi}_k^{(j)}(\boldsymbol{x}) .$$

□

Lemma 4.4.13. *Let $\boldsymbol{u}$, $\boldsymbol{u}_{\text{snap}}$, $\boldsymbol{w}$, Λ_1 and Λ_2 be defined in Theorems 4.4.3 and 4.4.10. For the uncoupled GMsFEM, we have*

$$\|\boldsymbol{w}(\cdot,0) - \boldsymbol{u}_{\text{snap}}(\cdot,0)\|_c^2 \le \frac{C}{\Lambda_1} \|\boldsymbol{u}(\cdot,0)\|_a^2 . \tag{4.102}$$

For the coupled GMsFEM, we have

$$\|\boldsymbol{w}(\cdot,0) - \boldsymbol{u}_{\text{snap}}(\cdot,0)\|_c^2 \le \frac{C}{\Lambda_2} \|\boldsymbol{u}(\cdot,0)\|_{a_{Q_s}}^2 . \tag{4.103}$$

Proof. For the case of uncoupled GMsFEM, as in Lemma 4.4.12, we let

$$e_{0,i}^{(j)} = \sum_{k>L_j} d_{k,i}^{(j)}(0)\psi_{k,i}^{(j)}(\boldsymbol{x}) .$$

Then, following the proof of Lemma 4.4.12, we get

$$\begin{aligned}
&\|\boldsymbol{w}(\cdot,0)-\boldsymbol{u}_{\text{snap}}(\cdot,0)\|_c^2\\
&=\sum_{i=1}^{2}\int_{\Omega}\mathcal{C}_{ii}\left|u_{\text{snap},i}(\cdot,0)-w_i(\cdot,0)\right|^2\,\mathrm{d}x\\
&=\sum_{i=1}^{2}\int_{\Omega}\mathcal{C}_{ii}\left|\sum_{j=1}^{N_v}\chi_{j,i}\,e_{0,i}^{(j)}\right|^2\,\mathrm{d}x\\
&=\left\|\sum_{j=1}^{N_v}\chi_{j,i}\,e_0^{(j)}\right\|_c^2\\
&\leq D_6\sum_{j=1}^{N_v}\left\|e_0^{(j)}\right\|_c^2\\
&\leq D_6\,D_7\frac{1}{\Lambda_1}\sum_{i=1}^{2}\sum_{j=1}^{N_v}a_i^{(j)}\left(e_{0,i}^{(j)},e_{0,i}^{(j)}\right)\\
&\leq D_6\,D_7\frac{1}{\Lambda_1}\sum_{i=1}^{2}\sum_{j=1}^{N_v}a_i^{(j)}\left(u_{\text{snap},i}^{(j)}(\cdot,0),u_{\text{snap},i}^{(j)}(\cdot,0)\right)\\
&\leq D_6\,D_7\frac{1}{\Lambda_1}\sum_{i=1}^{2}\sum_{j=1}^{N_v}a_i^{(j)}\left(u_i(\cdot,0),u_i(\cdot,0)\right)\\
&\leq\frac{C}{\Lambda_1}\|\boldsymbol{u}(\cdot,0)\|_a^2\,.
\end{aligned}\tag{4.104}$$

For the case of coupled GMsFEM, similar arguments are utilized for

$$\boldsymbol{e}_0^{(j)}=\sum_{k>L_j}d_k^{(j)}(0)\boldsymbol{\psi}_k^{(j)}(\boldsymbol{x})\,.$$

□

4.5 Numerical results

In this section, we present numerical results for both coupled and uncoupled GMsFEM.

Let $\Omega = [0,1]^2$, and consider the following problem:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}(\boldsymbol{x},t) - \operatorname{div}(\kappa_1(\boldsymbol{x})\nabla u_1(\boldsymbol{x},t)) + \boldsymbol{b}_1(\boldsymbol{x})\cdot\nabla(u_1(\boldsymbol{x},t) - u_2(\boldsymbol{x},t)) \\
&\quad + Q_1(u_1(\boldsymbol{x},t) - u_2(\boldsymbol{x},t)) = 1\,, \\
&\frac{\partial u_2}{\partial t}(\boldsymbol{x},t) - \operatorname{div}(\kappa_2(\boldsymbol{x})\nabla u_2(\boldsymbol{x},t)) + \boldsymbol{b}_2(\boldsymbol{x})\cdot\nabla(u_2(\boldsymbol{x},t) - u_1(\boldsymbol{x},t)) \\
&\quad + Q_2(u_2(\boldsymbol{x},t) - u_1(\boldsymbol{x},t)) = 1,
\end{aligned} \tag{4.105}$$

where we let

$$\begin{aligned}
\boldsymbol{b}_1(\boldsymbol{x}) &= 10\,((1-\cos(2\pi x_1))\sin(2\pi x_2), -\sin(2\pi x_1)(1-\cos(2\pi x_2))), \\
\boldsymbol{b}_2(\boldsymbol{x}) &= 10\,(-\sin(2\pi x_1)(1-\cos(2\pi x_2)), (1-\cos(2\pi x_1))\sin(2\pi x_2)).
\end{aligned} \tag{4.106}$$

The above equations represent a model problem where we deal with fluid flow through two continua. First, we note that the model is of **mathematical interest** as it is for a system of equations with multiscale coefficients. The model problem can be derived from upscaling of highly heterogeneous media using Representative Volume Element (RVE) approach described in [35]. In that paper, the authors use sub RVE scale to formulate a multi-continuum model at fine-grid scale, which is further upscaled. In this work, we assume that the resulting multi-continuum model can have highly heterogeneous coefficients. For example, with respect to the second continuum, the first continuum has much higher permeability fields in channels, which are much larger compared to RVE scales. In general, our proposed approach can handle any variations of permeability fields κ_1 and κ_2. In this work, we consider some model problems. In the future, we plan to study more realistic examples.

Figs. 4.1a and 4.1b indicate that the high-contrast permeability coefficients κ_1 and κ_2 are used. We compare the fine-scale solutions with the multiscale ones, by computing relative errors in weighted L^2 norm and H^1 semi-norm. In particular, we use

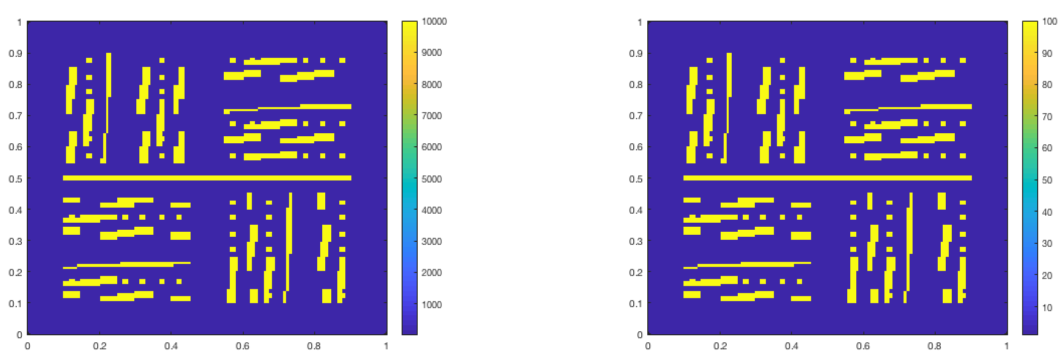
$$\begin{aligned} &100\,||u_{\text{ms},i} - u_{h,i}||_{L^2_{a_i}} / ||u_{h,i}||_{L^2_{a_i}}\,, \\ &100\,\|u_{\text{ms},i} - u_{h,i}\|_{H^1_{a_i}} / \|u_{h,i}\|_{H^1_{a_i}}\,, \end{aligned} \tag{4.107}$$

where

$$||u_i||_{L^2_{a_i}} = \int_\Omega \kappa_i u_i^2 \,\mathrm{d}x\,, \qquad \|u_i\|_{H^1_{a_i}} = \int_\Omega \kappa_i |\nabla u_i|^2 \,\mathrm{d}x\,,$$

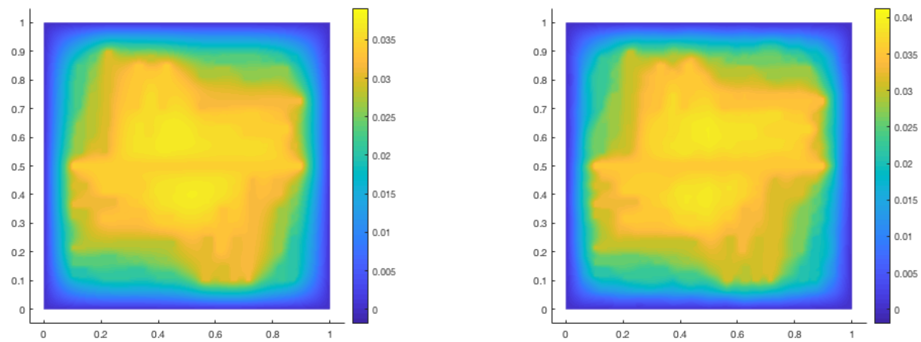
(for $i = 1, 2$).

We denote by DOF_{fine} the number of degrees of freedom (basis functions) for fine-scale FEM. Tables 4.1, 4.2, 4.3 and 4.4 represent the errors obtained from the coupled and uncoupled GMsFEM with various Q_1 and Q_2 (see Figs. 4.2a and 4.2b). From Tables 4.1 and 4.2, we observe that the coupled GMsFEM has higher accuracy compared with the uncoupled GMsFEM, when Q_1 and Q_2 are large and positive. Tables 4.3 and 4.4 show that both of the coupled and uncoupled GMsFEM still have good convergence with some negative Q_1 and Q_2. A fine-scale reference solution u_1 (obtained from the FEM) is plotted Fig. 4.3a, while Fig. 4.3b represents solution u_1 obtained from the GMsFEM.



(a) $\kappa_1(\boldsymbol{x})$. The value in each channel is 10^4. (b) $\kappa_2(\boldsymbol{x})$. The value in each channel is 100.

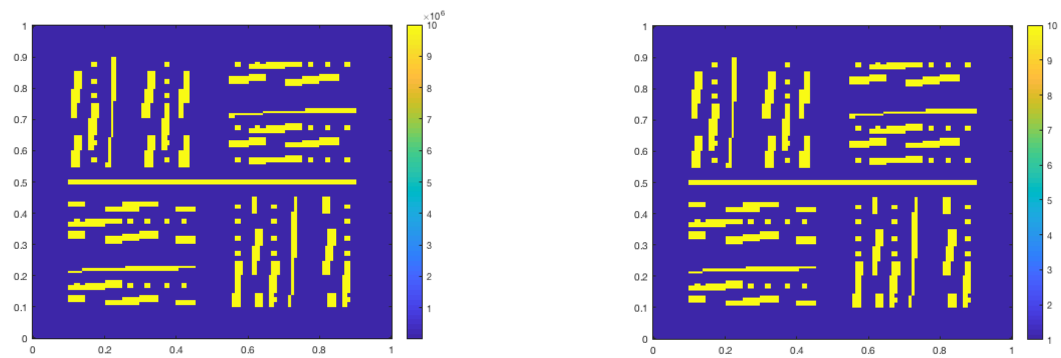
Figure 4.1: Permeability coefficients κ_1 and κ_2 for numerical implementation.



(a) FEM, u_1 as a fine-scale reference solution, $DOF_{\text{fine}} = 32768$.

(b) Coupled GMsFEM, u_1 as a coarse-scale solution, $\dim(\boldsymbol{V}_{\text{ms}}) = 2700$.

Figure 4.3: Solutions using the FEM and Coupled GMsFEM.



(a) $\hat{Q}(\boldsymbol{x})$. The value in each channel is 10^7.

(b) $\tilde{Q}(\boldsymbol{x})$. The value in each channel is 10.

Figure 4.2: Interaction coefficients Q_1 and Q_2 for numerical implementation.

dim($\boldsymbol{V}_{\text{ms}}$)	u_1		u_2	
	H^1 Errors(%)	L^2 Errors(%)	H^1 Errors(%)	L^2 Errors(%)
1800	11.619	1.162	10.246	1.173
2700	6.994	0.449	6.811	0.456
3600	6.129	0.335	5.832	0.340
4500	5.214	0.223	4.768	0.228
5400	3.726	0.117	3.532	0.120
7200	2.253	0.045	2.186	0.047

Table 4.1: Coupled GMsFEM, $Q_1 = Q_2 = \hat{Q}$, DOF_{fine} = 32768.

dim($\boldsymbol{V}_{\text{ms}}$)	u_1		u_2	
	H^1 Errors(%)	L^2 Errors(%)	H^1 Errors(%)	L^2 Errors(%)
1800	16.170	2.987	17.450	2.998
2700	8.213	1.020	9.976	1.026
3600	6.630	0.756	8.637	0.760
4500	5.554	0.544	7.490	0.547
5400	4.717	0.435	6.776	0.438
7200	2.712	0.237	5.065	0.239

Table 4.2: Uncoupled GMsFEM, $Q_1 = Q_2 = \hat{Q}$, DOF_{fine} = 32768.

dim($\boldsymbol{V}_{\text{ms}}$)	u_1		u_2	
	H^1 Errors(%)	L^2 Errors(%)	H^1 Errors(%)	L^2 Errors(%)
1800	16.051	2.250	17.558	2.547
2700	8.232	0.571	7.957	0.567
3600	6.621	0.375	6.579	0.381
4500	5.567	0.255	5.374	0.252
5400	4.729	0.195	4.578	0.179
7200	2.696	0.064	2.628	0.061

Table 4.3: Coupled GMsFEM, $Q_1 = -10\tilde{Q}, Q_2 = -\tilde{Q}$, DOF_{fine} = 32768.

$\dim(\boldsymbol{V}_{\text{ms}})$	u_1		u_2	
	H^1 Errors(%)	L^2 Errors(%)	H^1 Errors(%)	L^2 Errors(%)
1800	16.233	2.314	15.873	2.266
2700	8.213	0.581	7.951	0.566
3600	6.620	0.377	6.54	0.381
4500	5.563	0.258	5.371	0.252
5400	4.733	0.196	4.558	0.180
7200	2.693	0.064	2.626	0.061

Table 4.4: Uncoupled GMsFEM, $Q_1 = -10\tilde{Q}, Q_2 = -\tilde{Q}$, DOF_{fine} = 32768.

5. CONCLUSIONS

In this dissertation, we developed an efficient algorithm for computing the effective coefficients of a coupled multiscale multi-continuum system where the interaction terms are scaled as $O(\frac{1}{\epsilon^2})$. We solved the cell problems using hierarchical finite element algorithm and used the solutions to compute the effective coefficients. To establish the hierarchical FE algorithm, we first constructed a dense hierarchy of macrogrids and the corresponding nested FE spaces. Based on the hierarchy, we solve the cell problems using different resolution FE spaces at different macroscopic points. We use solutions solved with a higher level of accuracy to correct solutions obtained with a lower level of accuracy at nearby macroscopic points. We rigorously showed that this hierarchical FE method achieves the same order of accuracy as the reference full solve where cell problems at every macroscopic point are solved with the highest level of accuracy, at a significantly reduced computation cost, using an essentially optimal number of degrees of freedom. The algorithm was implemented on macroscopic points in a one dimensional domain. The numerical results strongly support the error estimates.

We analyzed the homogenization of a two-scale dual-continuum system. The coupled exchange terms are scaled as $\mathcal{O}(\frac{1}{\epsilon})$. This scale gives an interesting homogenization limit which contains convection, coupled reaction terms with negative interaction coefficients while the original two scale system does not contain these features. We proved rigorously the homogenization convergence. We proved rigorously also the homogenization convergence rate. These proofs of homogenization convergence and error are significantly more complicated than those for the scaling $O(\frac{1}{\epsilon^2})$ considered in Chapter 1 due to the complicated form of the homogenized equation. The effective coefficients are approximated implementing hierarchical algorithm introduced in Chapter 2.

We proposed a generalized multiscale finite element method (GMsFEM), to speedily and effectively solve an upscaled multiscale dual-continuum system motivated by the homogenized equation derived in Chapter 3. The GMsFEM systematically produces either uncoupled or coupled multiscale basis functions (called uncoupled or coupled GMsFEM, respectively). That is, multiscale basis functions are constructed for separately for each solution (uncouple GMsFEM), or jointly for the system (coupled GMsFEM). Our numerical results show that the combination of the GMsFEM and dual-continuum approach is able to compute solutions with high efficiency and accuracy, which are even higher when the coupled multiscale basis functions are applied. In a future contribution, we will extend this strategy to a dual-continuum system of homogenized nonlinear equations.

REFERENCES

[1] G. Barenblatt, I. P. Zheltov, and I. Kochina, “Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata],” *Journal of applied mathematics and mechanics*, vol. 24, no. 5, pp. 1286–1303, 1960.

[2] J. Warren, P. J. Root, *et al.*, “The behavior of naturally fractured reservoirs,” *Society of Petroleum Engineers Journal*, vol. 3, no. 03, pp. 245–255, 1963.

[3] H. Kazemi, L. Merrill Jr, K. Porterfield, P. Zeman, *et al.*, “Numerical simulation of water-oil flow in naturally fractured reservoirs,” *Society of Petroleum Engineers Journal*, vol. 16, no. 06, pp. 317–326, 1976.

[4] Y.-S. Wu, K. Pruess, *et al.*, “A multiple-porosity method for simulation of naturally fractured petroleum reservoirs,” *SPE Reservoir Engineering*, vol. 3, no. 01, pp. 327–336, 1988.

[5] K. Pruess and T. Narasimhan, “On fluid reserves and the production of superheated steam from fractured, vapor-dominated geothermal reservoirs,” *Journal of Geophysical Research: Solid Earth*, vol. 87, no. B11, pp. 9329–9339, 1982.

[6] E. T. Chung, Y. Efendiev, W. T. Leung, and M. Vasilyeva, “Coupling of multiscale and multi-continuum approaches,” *GEM - International Journal on Geomathematics*, vol. 8, pp. 9–41, Apr 2017.

[7] G. Papanicolau, A. Bensoussan, and J.-L. Lions, *Asymptotic analysis for periodic structures*. Elsevier, 1978.

[8] N. Bakhvalov and G. Panasenko, *Homogenisation averaging processes in periodic media*. Springer, 1989.

[9] V. Jikov, S. Kozlov, and O. Oleinik, *Homogenization of differential operators and integral functionals*. Springer Science & Business Media, 2012.

[10] G. Allaire and A. Piatnitski, "Homogenization of nonlinear reaction-diffusion equation with a large reaction term," *Annali dell'Universita di Ferrara*, vol. 56, no. 1, pp. 141–161, 2010.

[11] Y. Efendiev and T. Hou, *Multiscale Finite Element Methods: Theory and Applications*. Springer, 2009.

[12] Y. Efendiev, T. Hou, and V. Ginting, "Multiscale finite element methods for nonlinear problems and their applications," *Commun. Math. Sci.*, vol. 2, no. 4, pp. 553–589, 2004.

[13] Y. Efendiev, J. Galvis, and T. Y. Hou, "Generalized multiscale finite element methods (GMsFEM)," *J. Comput. Phys.*, vol. 251, pp. 116–135, Oct. 2013.

[14] I. Y. Akkutlu, Y. Efendiev, M. Vasilyeva, and Y. Wang, "Multiscale model reduction for shale gas transport in a coupled discrete fracture and dual-continuum porous media," *Journal of Natural Gas Science and Engineering*, vol. 48, pp. 65 – 76, 2017. Multiscale and Multiphysics Techniques and their Applications in Unconventional Gas Reservoirs.

[15] M. Wang, S. W. Cheung, E. T. Chung, M. Vasilyeva, and Y. Wang, "Generalized multiscale multicontinuum model for fractured vuggy carbonate reservoirs," *Journal of Computational and Applied Mathematics*, vol. 366, p. 112370, 2020.

[16] D. Spiridonov, M. Vasilyeva, and E. T. Chung, "Generalized multiscale finite element method for multicontinua unsaturated flow problems in fractured porous media," *Journal of Computational and Applied Mathematics*, vol. 370, p. 112594, 2020.

[17] A. Tyrylgin, M. Vasilyeva, D. Spiridonov, and E. T. Chung, "Generalized multiscale finite element method for the poroelasticity problem in multicontinuum media," *Journal of Computational and Applied Mathematics*, vol. 374, p. 112783, 2020.

[18] E. T. Chung, Y. Efendiev, and W. T. Leung, "Constraint energy minimizing generalized multiscale finite element method," *Computer Methods in Applied Mechanics and Engineering*, vol. 339, pp. 298 – 319, 2018.

[19] E. T. Chung, Y. Efendiev, and W. T. Leung, "Fast online generalized multiscale finite element method using constraint energy minimization," *Journal of Computational Physics*, vol. 355, pp. 450 – 463, 2018.

[20] S. Fu, E. Chung, and T. Mai, "Constraint energy minimizing generalized multiscale finite element method for nonlinear poroelasticity and elasticity," *arXiv e-prints*, p. arXiv:1909.13267, Sep 2019.

[21] S. W. Cheung, E. T. Chung, Y. Efendiev, W. Tat Leung, and M. Vasilyeva, "Constraint Energy Minimizing Generalized Multiscale Finite Element Method for dual continuum model," *arXiv e-prints*, p. arXiv:1807.10955, Jul 2018.

[22] M. Vasilyeva, E. T. Chung, S. W. Cheung, Y. Wang, and G. Prokopev, "Nonlocal multicontinua upscaling for multicontinua flow problems in fractured porous media," *Journal of Computational and Applied Mathematics*, vol. 355, pp. 258–267, 2019.

[23] E. T. Chung, Y. Efendiev, W. T. Leung, M. Vasilyeva, and Y. Wang, "Non-local multi-continua upscaling for flows in heterogeneous fractured media," *Journal of Computational Physics*, vol. 372, pp. 22–34, 2018.

[24] M. Vasilyeva, E. T. Chung, Y. Efendiev, and A. Tyrylgin, "A three-level multi-continua upscaling method for flow problems in fractured porous media," *arXiv e-prints*, p. arXiv:1810.01581, Oct 2018.

[25] M. Vasilyeva, M. Babaei, E. T. Chung, and V. Alekseev, "Upscaling of the single-phase flow and heat transport in fractured geothermal reservoirs using nonlocal multicontinuum method," *Computational Geosciences*, pp. 1–15, 2019.

[26] D. L. Brown, Y. Efendiev, and V. H. Hoang, "An efficient hierarchical multiscale finite element method for stokes equations in slowly varying media," *Multiscale Modeling & Simulation*, vol. 11, no. 1, pp. 30–58, 2013.

[27] D. L. Brown and V. H. Hoang, "A hierarchical finite element monte carlo method for stochastic two-scale elliptic equations," *Journal of Computational and Applied Mathematics*, vol. 323, pp. 16–35, 2017.

[28] J. Wloka, "Partial differential equations. 1987," *Cambridge University*.

[29] J. S. R. Park and V. H. Hoang, "Hierarchical multiscale finite element method for multi-continuum media," *Journal of Computational and Applied Mathematics*, vol. 369, 2019.

[30] E. Chung, Y. Efendiev, and T. Y. Hou, "Adaptive multiscale model reduction with generalized multiscale finite element methods," *Journal of Computational Physics*, vol. 320, pp. 69–95, 2016.

[31] J. S. R. Park and V. H. Hoang, "Homogenization of a multiscale multi-continuum system," *arXiv e-prints*, p. arXiv:1910.07098, Oct 2019.

[32] S. Brenner and R. Scott, *The mathematical theory of finite element methods*, vol. 15. Springer Science & Business Media, 2007.

[33] S. Fu, E. Chung, and T. Mai, "Generalized multiscale finite element method for a strain-limiting nonlinear elasticity model," *Journal of Computational and Applied Mathematics*, vol. 359, pp. 153 – 165, 2019.

[34] Y. Efendiev, J. Galvis, and X.-H. Wu, "Multiscale finite element methods for high-contrast problems using local spectral basis functions," *Journal of Computational Physics*, vol. 230, no. 4, pp. 937 – 955, 2011.

[35] E. T. Chung, Y. Efendiev, W. T. Leung, and M. Vasilyeva, "Nonlocal multicontinua with Representative Volume Elements. Bridging separable and non-separable scales," *arXiv e-prints*, p. arXiv:2001.07988, Jan 2020.

[36] V. H. Hoang and C. Schwab, "High-dimensional finite elements for elliptic problems with multiple scales," *Multiscale Modeling & Simulation*, vol. 3, no. 1, pp. 168–194, 2005.

[37] L. Tartar, *The general theory of homogenization*, vol. 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2009. A personalized introduction.

[38] M. E. Jarroudi, "Homogenization of a nonlinear elastic fibre-reinforced composite: A second gradient nonlinear elastic material," *Journal of Mathematical Analysis and Applications*, vol. 403, no. 2, pp. 487 – 505, 2013.

[39] S. W. Cheung, E. T. Chung, and W. T. Leung, "Constraint energy minimizing generalized multiscale discontinuous galerkin method," *arXiv preprint arXiv:1909.12461*, 2019.

[40] I. Akkutlu, Y. Efendiev, and M. Vasilyeva, "Multiscale model reduction for shale gas transport in fractured media," *Computational Geosciences*, pp. 1–21, 2015.

[41] Y.-S. Wu, G. Qin, R. Ewing, Y. Efendiev, Z. Kang, and Y. Ren, "A multiple-continuum approach for modeling multiphase flow in naturally fractured vuggy petroleum reservoirs," *SPE-104173, presented at the 2006 SPE International Oil & Gas Conference and Exhibition in China held in Beijing, China*.

[42] K. Gao, S. Fu, R. L. Gibson, E. T. Chung, and Y. Efendiev, "Generalized multiscale finite-element method (gmsfem) for elastic wave propagation in heterogeneous, anisotropic media," *Journal of Computational Physics*, vol. 295, pp. 161–188, 2015.

[43] Y. Efendiev and A. Pankov, "Numerical homogenization and correctors for nonlinear elliptic equations," *SIAM J. Appl. Math.*, vol. 65, no. 1, pp. 43–68, 2004.

[44] W. C. Tan and V. H. Hoang, "High dimensional finite element method for multiscale nonlinear monotone parabolic equations," *Journal of Computational and Applied Mathematics*, vol. 345, pp. 471–500, 2019.

[45] Y. Efendiev, J. Galvis, G. Li, and M. Presho, "Generalized multiscale finite element methods. Oversampling strategies," *International Journal for Multiscale Computational Engineering*, vol. 12, no. 6, pp. 465–484, 2014.

[46] S. Fu and E. T. Chung, "Constraint energy minimizing generalized multiscale finite element method for high-contrast linear elasticity problem," *arXiv e-prints*, p. arXiv:1809.03726, Sep 2018.

[47] V. M. Calo, Y. Efendiev, J. Galvis, and G. Li, "Randomized oversampling for generalized multiscale finite element methods," *Multiscale Modeling & Simulation*, vol. 14, no. 1, pp. 482–501, 2016.

[48] E. Chung, Y. Efendiev, K. Shi, and S. Ye, "A multiscale model reduction method for nonlinear monotone elliptic equations in heterogeneous media," *Networks & Heterogeneous Media*, vol. 12, no. 4, pp. 619–642, 2017.

[49] N. Zhang, Y. Wang, Q. Sun, and Y. Wang, "Multiscale mass transfer coupling of triple-continuum and discrete fractures for flow simulation in fractured vuggy porous media," *International Journal of Heat and Mass Transfer*, vol. 116, pp. 484 – 495, 2018.

[50] Y.-S. Wu, Y. Di, Z. Kang, and P. Fakcharoenphol, “A multiple-continuum model for simulating single-phase and multiphase flow in naturally fractured vuggy reservoirs,” *Journal of Petroleum Science and Engineering*, vol. 78, no. 1, pp. 13–22, 2011.

[51] K. Gao, E. T. Chung, R. L. Gibson Jr, S. Fu, and Y. Efendiev, “A numerical homogenization method for heterogeneous, anisotropic elastic media based on multiscale theory,” *Geophysics*, vol. 80, no. 4, pp. D385–D401, 2015.

[52] Y. Efendiev and A. Pankov, “On homogenization of almost periodic nonlinear parabolic operators,” *Int. J. Evol. Equ.*, vol. 1, no. 3, pp. 203–209, 2005.

[53] A.-M. Matache and C. Schwab, “Two-scale fem for homogenization problems,” *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 36, no. 04, pp. 537–572, 2002.

[54] Y. Cho, R. L. Gibson Jr, M. Vasilyeva, and Y. Efendiev, “Generalized multiscale finite elements for simulation of elastic-wave propagation in fractured media,” *Geophysics*, vol. 83, no. 1, pp. WA9–WA20, 2017.

[55] E. T. Chung, Y. Efendiev, and W. T. Leung, “Generalized multiscale finite element methods for wave propagation in heterogeneous media,” *Multiscale Modeling & Simulation*, vol. 12, no. 4, pp. 1691–1721, 2014.

[56] J. S. R. Park, S. W. Cheung, T. Mai, and V. H. Hoang, “Multiscale simulations for upscaled multi-continuum flows,” *Journal of Computational and Applied Mathematics*, p. 112782, 2020.

[57] G. Allaire, “Homogenization and two-scale convergence,” *SIAM Journal on Mathematical Analysis*, vol. 23, no. 6, pp. 1482–1518, 1992.

[58] Y. Efendiev and A. Pankov, “Numerical homogenization of monotone elliptic operators,” *Multiscale Modeling & Simulation*, vol. 2, no. 1, pp. 62–79, 2003.

[59] V. M. Calo, E. T. Chung, Y. Efendiev, and W. T. Leung, "Multiscale stabilization for convection-dominated diffusion in heterogeneous media," *Computer Methods in Applied Mechanics and Engineering*, vol. 304, pp. 359–377, 2016.

[60] P. Henning and M. Ohlberger, "The heterogeneous multiscale finite element method for elliptic homogenization problems in perforated domains," *Numerische Mathematik*, vol. 113, no. 4, pp. 601–629, 2009.

[61] S. Boyaval, "Reduced-basis approach for homogenization beyond periodic setting," *SIAM J. Multiscale Modeling and Simulation*, vol. 7, no. 1, pp. 466–494, 2008.

[62] G. Allaire and R. Brizzi, "A multiscale finite element method for numerical homogenization," *SIAM J. Multiscale Modeling and Simulation*, vol. 4, no. 3, pp. 790–812, 2005.

[63] Y. Efendiev, J. Galvis, R. Lazarov, M. Moon, and M. Sarkis, "Generalized multiscale finite element method. Symmetric interior penalty coupling," *Journal of Computational Physics*, vol. 255, pp. 1–15, 2013.

[64] T. Arbogast and H. L. Lehr, "Homogenization of a Darcy-Stokes system modeling vuggy porous media," *Comput. Geosci.*, vol. 10, no. 3, pp. 291–302, 2006.

[65] E. Chung, Y. Efendiev, and G. Li, "An adaptive GMsFEM for high-contrast flow problems," *Journal of Computational Physics*, vol. 273, pp. 54–76, 2014.

[66] H. Y. Chan, E. Chung, and Y. Efendiev, "Adaptive mixed gmsfem for flows in heterogeneous media," *Numerical Mathematics: Theory, Methods and Applications*, vol. 9, no. 4, pp. 497–527, 2016.

[67] E. T. Chung, W. T. Leung, and S. Pollock, "Goal-oriented adaptivity for gmsfem," *Journal of Computational and Applied Mathematics*, vol. 296, pp. 625–637, 2016.

[68] E. T. Chung, W. T. Leung, and M. Vasilyeva, "Mixed gmsfem for second order elliptic problem in perforated domains," *Journal of Computational and Applied Mathematics*, vol. 304, pp. 84–99, 2016.

[69] L. J. F. Chen, E. Chung, "Least squares GMsFEM," 2015. in preparation.

[70] J. Fish and R. Fan, "Mathematical homogenization of nonperiodic heterogeneous media subjected to large deformation transient loading," *International Journal for numerical methods in engineering*, vol. 76, no. 7, pp. 1044–1064, 2008.

[71] E. Skjetne and J.-L. Auriault, "Homogenization of wall-slip gas flow through porous media," *Transport in porous Media*, vol. 36, no. 3, pp. 293–306, 1999.

[72] M. Wang, C. Wei, H. Song, Y. Efendiev, Y. Wang, *et al.*, "Generalized multiscale coupling of triple-continuum model and discrete fracture network for carbonate reservoir simulation," in *SPE Annual Technical Conference and Exhibition*, Society of Petroleum Engineers, 2017.

[73] E. T. Chung, Y. Efendiev, and Y. Li, "Space-time gmsfem for transport equations," *GEM-International Journal on Geomathematics*, vol. 9, no. 2, pp. 265–292, 2018.

[74] W. T. Leung, E. T. Chung, Y. Efendiev, M. Vasilyeva, M. Wheeler, *et al.*, "Nonlinear upscaling of two-phase flow using non-local multi-continuum approach," in *SPE Reservoir Simulation Conference*, Society of Petroleum Engineers, 2019.

[75] E. T. Chung, Y. Efendiev, B. Jin, W. T. Leung, and M. Vasilyeva, "Generalized multiscale inversion for heterogeneous problems," *arXiv preprint arXiv:1707.08194*, 2017.

[76] E. T. Chung, Y. Efendiev, W. T. Leung, and M. Wheeler, "Nonlinear nonlocal multicontinua upscaling framework and its applications," *International Journal for Multiscale Computational Engineering*, vol. 16, no. 5, 2018.

[77] V. H. Hoang, "Sparse finite element method for periodic multiscale nonlinear monotone problems," *Multiscale Modeling & Simulation*, vol. 7, no. 3, pp. 1042–1072, 2009.

[78] V. H. Hoang, "Homogenization of the stokes flow with small viscosity in a non-periodic porous medium," *Rend. Mat. Appl.(7)*, vol. 22, no. 223-248, p. 2002, 2003.

[79] V. T. Chu, *Multiscale maxwell equations: homogenization and high dimensional finite element method*. PhD thesis, 2016.

APPENDIX A

EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

In this appendix, we present the proof of the existence and uniqueness of a weak solution of (3.3) and (3.15). We recall that the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by H and V respectively. We first note that for $u \in L^2(0,T;V) \cap H^1(0,T;V')$, we have $u \in C(0,T;H)$ ([28], p.394). This will make sense of the initial conditions (A.2). Considering the fact that $C_0^\infty(\Omega)$ is dense in V and (3.6), we have the variational problem : Find $u_1^\epsilon, u_2^\epsilon \in L^2(0,T;V)$ such that $\frac{\partial u_1^\epsilon}{\partial t}, \frac{\partial u_2^\epsilon}{\partial t} \in L^2(0,T;V')$ and

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon(x)\frac{\partial u_1^\epsilon(t,x)}{\partial t}\phi_1(t,x)dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon(x)\nabla u_1^\epsilon(t,x)\cdot\nabla\phi_1(t,x)dxdt \\
&\quad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon(x)(u_1^\epsilon(t,x) - u_2^\epsilon(t,x))\phi_1(t,x)dxdt = \int_0^T \int_\Omega q\phi_1(t,x)dxdt \\
&\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon(x)\frac{\partial u_2^\epsilon(t,x)}{\partial t}\phi_2(t,x)dxdt + \int_0^T \int_\Omega \kappa_2^\epsilon(x)\nabla u_2^\epsilon(t,x)\cdot\nabla\phi_2(t,x)dxdt \\
&\quad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon(x)(u_2^\epsilon(t,x) - u_1^\epsilon(t,x))\phi_2(t,x)dxdt = \int_0^T \int_\Omega q\phi_2(t,x)dxdt
\end{aligned} \tag{A.1}$$

for all $\phi_1, \phi_2 \in L^2(0,T;V)$, and

$$u_1^\epsilon(0,x) = g_1(x) \in H \;\text{ and }\; u_2^\epsilon(0,x) = g_2(x) \in H. \tag{A.2}$$

Note that as $u_1^\epsilon, u_2^\epsilon \in C(0,T;H)$, it is possible to use the pointwise values of them in H at $t = 0$ to impose the initial conditions.

Let W be the space $V \times V$. We define a bilinear form $a : W \times W \to \mathbb{R}$ as

$$
\begin{aligned}
&a((u_1(t), u_2(t)), (v_1(t), v_2(t))) \\
&= \int_\Omega \kappa_1^\epsilon \nabla u_1(t) \cdot \nabla v_1(t) dx + \int_\Omega \kappa_2^\epsilon \nabla u_2(t) \cdot \nabla v_2(t) dx + \frac{1}{\epsilon} \int_\Omega Q^\epsilon (u_1(t) - u_2(t)) v_1(t) dx \\
&+ \frac{1}{\epsilon} \int_\Omega Q^\epsilon (u_2(t) - u_1(t)) v_2(t) dx.
\end{aligned}
\tag{A.3}
$$

Recall that since $\int_Y Q(x,y)dy = 0$, there is a vector function $\mathcal{Q}(x,y)$ which is periodic with respect to y such that $Q(x,y) = \operatorname{div}_y \mathcal{Q}(x,y)$. We have the following theorem.

Theorem A.0.1. *Assume that the vector function $\mathcal{Q}(x,y)$ is in $C^1(\bar{\Omega}; C^1(\bar{Y}))^2$. Then the sequences u_1^ϵ and u_2^ϵ satisfying (3.3) are uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$.*

Proof. As $\operatorname{div} \mathcal{Q}(x, \frac{x}{\epsilon}) = \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon}) + \frac{1}{\epsilon} \operatorname{div}_y \mathcal{Q}(x, \frac{x}{\epsilon})$, $Q(x, \frac{x}{\epsilon}) = \epsilon \operatorname{div} \mathcal{Q}(x, \frac{x}{\epsilon}) - \epsilon \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})$. Note that

$$
\begin{aligned}
&\int_\Omega Q^\epsilon(x)(u_2^\epsilon - u_1^\epsilon)\phi_1 dx = -\epsilon \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla (u_2^\epsilon - u_1^\epsilon)\phi_1 dx \\
&- \epsilon \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla \phi_1 (u_2^\epsilon - u_1^\epsilon) dx - \epsilon \int_\Omega \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})(u_2^\epsilon - u_1^\epsilon)\phi_1 dx,
\end{aligned}
\tag{A.4}
$$

for all ϕ_1, $\phi_2 \in \mathcal{C}_0^\infty(\Omega)$. Thus, from (3.6), we have

$$
\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} \phi_1 dx dt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla \phi_1 dx dt \\
&= - \int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla (u_2^\epsilon - u_1^\epsilon)\phi_1 dx dt - \int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla \phi_1 (u_2^\epsilon - u_1^\epsilon) dx dt \\
&- \int_0^T \int_\Omega \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})(u_2^\epsilon - u_1^\epsilon)\phi_1 dx dt + \int_0^T \int_\Omega q \phi_1 dx dt.
\end{aligned}
\tag{A.5}
$$

We let $\hat{u}_1^\epsilon = u_1^\epsilon e^{-\lambda t}$, $\hat{u}_2^\epsilon = u_2^\epsilon e^{-\lambda t}$ and $\check{\phi}_1 = \phi_1 e^{\lambda t}$. Then,

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial \hat{u}_1^\epsilon}{\partial t} \check{\phi}_1 dxdt + \lambda \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \hat{u}_1^\epsilon \check{\phi}_1 dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla \hat{u}_1^\epsilon \cdot \nabla \check{\phi}_1 dxdt \\
&= -\int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla(\hat{u}_2^\epsilon - \hat{u}_1^\epsilon) \check{\phi}_1 dxdt - \int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla \check{\phi}_1 (\hat{u}_2^\epsilon - \hat{u}_1^\epsilon) dxdt \\
&- \int_0^T \int_\Omega \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})(\hat{u}_2^\epsilon - \hat{u}_1^\epsilon) \check{\phi}_1 dxdt + \int_0^T \int_\Omega q \check{\phi}_1 e^{-\lambda t} dxdt.
\end{aligned} \tag{A.6}$$

Similarly, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial \hat{u}_2^\epsilon}{\partial t} \check{\phi}_2 dxdt + \lambda \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \hat{u}_2^\epsilon \check{\phi}_2 dxdt + \int_0^T \int_\Omega \kappa_2^\epsilon \nabla \hat{u}_2^\epsilon \cdot \nabla \check{\phi}_2 dxdt \\
&= -\int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla(\hat{u}_1^\epsilon - \hat{u}_2^\epsilon) \check{\phi}_2 dxdt - \int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla \check{\phi}_2 (\hat{u}_1^\epsilon - \hat{u}_2^\epsilon) dxdt \\
&- \int_0^T \int_\Omega \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})(\hat{u}_1^\epsilon - \hat{u}_2^\epsilon) \check{\phi}_2 dxdt + \int_0^T \int_\Omega q \check{\phi}_2 e^{-\lambda t} dxdt.
\end{aligned} \tag{A.7}$$

Let $\check{\phi}_1 = \hat{u}_1^\epsilon$, $\check{\phi}_2 = \hat{u}_2^\epsilon$. Taking the sum of the above two equations we get

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial \hat{u}_1^\epsilon}{\partial t} \hat{u}_1^\epsilon dxdt + \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial \hat{u}_2^\epsilon}{\partial t} \hat{u}_2^\epsilon dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla \hat{u}_1^\epsilon \cdot \nabla \hat{u}_1^\epsilon dxdt \\
&+ \int_0^T \int_\Omega \kappa_2^\epsilon \nabla \hat{u}_2^\epsilon \cdot \nabla \hat{u}_2^\epsilon dxdt + \lambda \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon (\hat{u}_1^\epsilon)^2 dxdt + \lambda \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon (\hat{u}_2^\epsilon)^2 dxdt \\
&= 2 \int_0^T \int_\Omega \mathcal{Q}(x, \frac{x}{\epsilon}) \cdot \nabla(\hat{u}_1^\epsilon - \hat{u}_2^\epsilon)(\hat{u}_1^\epsilon - \hat{u}_2^\epsilon) dxdt + \int_0^T \int_\Omega \operatorname{div}_x \mathcal{Q}(x, \frac{x}{\epsilon})(\hat{u}_1^\epsilon - \hat{u}_2^\epsilon)^2 dxdt \\
&+ \int_0^T \int_\Omega q e^{-\lambda t}(\hat{u}_1^\epsilon + \hat{u}_2^\epsilon) dxdt.
\end{aligned} \tag{A.8}$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{11}|\hat{u}^\epsilon_1(T,x)|^2dx + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{22}|\hat{u}^\epsilon_2(T,x)|^2dx + \int_0^T\int_\Omega \kappa^\epsilon_1\nabla\hat{u}^\epsilon_1\cdot\nabla\hat{u}^\epsilon_1 dxdt \\
&+\int_0^T\int_\Omega \kappa^\epsilon_2\nabla\hat{u}^\epsilon_2\cdot\nabla\hat{u}^\epsilon_2 dxdt + \lambda\int_0^T\int_\Omega \mathcal{C}^\epsilon_{11}(\hat{u}^\epsilon_1)^2dxdt + \lambda\int_0^T\int_\Omega \mathcal{C}^\epsilon_{22}(\hat{u}^\epsilon_2)^2dxdt \\
&= 2\int_0^T\int_\Omega \mathcal{Q}(x,\frac{x}{\epsilon})\cdot\nabla(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)dxdt + \int_0^T\int_\Omega \mathrm{div}_x\,\mathcal{Q}(x,\frac{x}{\epsilon})(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)^2dxdt \\
&+\int_0^T\int_\Omega qe^{-\lambda t}(\hat{u}^\epsilon_1+\hat{u}^\epsilon_2)dxdt + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{11}|\hat{u}^\epsilon_1(0,x)|^2dx + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{22}|\hat{u}^\epsilon_2(0,x)|^2dx.
\end{aligned}
\tag{A.9}
$$

Since $\mathcal{Q}(x,y)\in C^1(\bar{\Omega};C^1(\bar{Y}))^2$, using Cauchy Schwartz and Young's inequalities, we have

$$
\begin{aligned}
&2\int_0^T\int_\Omega \mathcal{Q}(x,\frac{x}{\epsilon})\cdot\nabla(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)dxdt \\
&\qquad\le \frac{1}{\varepsilon}(||\hat{u}^\epsilon_1||^2_{L^2(0,T;V)} + ||\hat{u}^\epsilon_2||^2_{L^2(0,T;V)}) + \varepsilon(||\hat{u}^\epsilon_1||^2_{L^2(0,T;H)} + ||\hat{u}^\epsilon_2||^2_{L^2(0,T;H)}).
\end{aligned}
\tag{A.10}
$$

Similarly,

$$
\int_0^T\int_\Omega \mathrm{div}_x\,\mathcal{Q}(x,\frac{x}{\epsilon})(\hat{u}^\epsilon_1-\hat{u}^\epsilon_2)^2dxdt \le c_2(||\hat{u}^\epsilon_1||^2_{L^2(0,T;H)} + ||\hat{u}^\epsilon_2||^2_{L^2(0,T;H)}) \tag{A.11}
$$

and

$$
\int_0^T\int_\Omega qe^{-\lambda t}(\hat{u}^\epsilon_1+\hat{u}^\epsilon_2)dxdt \le c_3 + c_4(||\hat{u}^\epsilon_1||^2_{L^2(0,T;H)} + ||\hat{u}^\epsilon_2||^2_{L^2(0,T;H)}). \tag{A.12}
$$

Choosing ε and λ sufficiently large, we deduce $\hat{u}^\epsilon_1$ and $\hat{u}^\epsilon_2$, thus, u^ϵ_1 and u^ϵ_2 are uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$. □

Lemma A.0.2. *Assume $Q(x,y)\in L^\infty(\Omega\times Y)$ and $\kappa_i(x,y)\in L^\infty(\Omega\times Y)$. There exists*

$C > 0$ *such that*

$$a((u_1, u_2), (v_1, v_2)) \leq C\big(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2\big)^{\frac{1}{2}} \cdot \big(||\nabla v_1||_H^2 + ||\nabla v_2||_H^2\big)^{\frac{1}{2}} \qquad \text{(A.13)}$$

for $(u_1, u_2), (v_1, v_2) \in W$. *And there exists* $k \geq 0$ *such that*

$$a((\phi_1, \phi_2), (\phi_1, \phi_2)) + k||\phi_1||_H^2 + k||\phi_2||_H^2 \geq \alpha(||\nabla \phi_1||_H^2 + ||\nabla \phi_2||_H^2), \qquad \text{(A.14)}$$

for all $\phi_1, \phi_2 \in V$. *Here,* C *and* k *depend on* ϵ.

Proof. It is not difficult to show (A.13). Since $Q \in C(\bar{\Omega} \times \bar{Y})$, we have

$$\begin{aligned}
&a((u_1, u_2), (u_1, u_2)) \\
&= \int_\Omega \kappa_1^\epsilon \nabla u_1 \cdot \nabla u_1 dx + \int_\Omega \kappa_2^\epsilon \nabla u_2 \cdot \nabla u_2 dx + \frac{1}{\epsilon} \int_\Omega Q^\epsilon (u_1 - u_2)^2 dx \qquad \text{(A.15)} \\
&\geq \underline{\kappa}(\|\nabla u_1\|_H^2 + \|\nabla u_2\|_H^2) - k(\|u_1\|_H^2 + \|u_2\|_H^2)
\end{aligned}$$

for some $k > 0$ depending on ϵ. The last inequality follows from Cauchy-Schwarz and Young's inequalities. □

Theorem A.0.3. *Let* $Q(x, y)$, $\mathcal{C}_{ii}(x, y)$ *and* $\kappa_i(x, y)$ *(*$i = 1, 2$*) be in* $C(\bar{\Omega} \times \bar{Y})$. *There exists a unique solution for problem (A.1).*

Proof. We follow the standard proof for parabolic equations in [28]. We note that $u_1^\epsilon, u_2^\epsilon$ are weak solutions of (3.3) if for almost all $t \in [0, T]$

$$\begin{aligned}
\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon(t)}{\partial t} \phi_1 dx + \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon(t) \cdot \nabla \phi_1 dx + \frac{1}{\epsilon} \int_\Omega Q^\epsilon (u_1^\epsilon(t) - u_2^\epsilon(t)) \phi_1 dx = \int_\Omega q \phi_1 dx \\
\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon(t)}{\partial t} \phi_2 dx + \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon(t) \cdot \nabla \phi_2 dx + \frac{1}{\epsilon} \int_\Omega Q^\epsilon (u_2^\epsilon(t) - u_1^\epsilon(t)) \phi_2 dx = \int_\Omega q \phi_2 dx
\end{aligned} \qquad \text{(A.16)}$$

for all $\phi_1, \phi_2 \in V$, $u_1^\epsilon(0,x) = g_1(x) \in H$ and $u_2^\epsilon(0,x) = g_2(x) \in H$. Let $\{\omega_k\}$ be an orthogonal basis of V and an orthonormal basis of H. For fixed integer $m > 0$, we consider functions

$$u_{1m}^\epsilon(t) = \sum_{k=1}^{m} d_{1m}^k(t)\omega_k, \;\; u_{2m}^\epsilon(t) = \sum_{k=1}^{m} d_{2m}^k(t)\omega_k, \tag{A.17}$$

where the coefficients d_{1m}^k, d_{2m}^k satisfy

$$d_{1m}^k(0) = \int_\Omega g_1\omega_k dx, \;\; d_{2m}^k(0) = \int_\Omega g_2\omega_k dx \tag{A.18}$$

and

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_{1m}^\epsilon(t)}{\partial t}\omega_{k_1} dx + \int_\Omega \kappa_1^\epsilon \nabla u_{1m}^\epsilon(t) \cdot \nabla\omega_{k_1} dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon(u_{1m}^\epsilon(t) - u_{2m}^\epsilon(t))\omega_{k_1} dx \\
&\qquad = \int_\Omega q\omega_{k_1} dx \\
&\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_{2m}^\epsilon(t)}{\partial t}\omega_{k_2} dx + \int_\Omega \kappa_2^\epsilon \nabla u_{2m}^\epsilon(t) \cdot \nabla\omega_{k_2} dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon(u_{2m}^\epsilon(t) - u_{1m}^\epsilon(t))\omega_{k_2} dx \\
&\qquad = \int_\Omega q\omega_{k_2} dx
\end{aligned} \tag{A.19}$$

a.e. on [0,T], where $k_1, k_2 = 1, 2, \ldots, m$. This problem can be written as a system of ODEs

$$\begin{aligned}
&\sum_{l=1}^{m}[M_1]_{kl}\frac{d}{dt}d_{1m}^k(t) + \sum_{l=1}^{m}[A_1 + M_Q]_{kl}d_{1m}^l(t) - \sum_{l=1}^{m}[M_Q]_{kl}d_{2m}^l(t) = \int q\omega_k dx \\
&\sum_{l=1}^{m}[M_2]_{kl}\frac{d}{dt}d_{2m}^k(t) + \sum_{l=1}^{m}[A_2 + M_Q]_{kl}d_{2m}^l(t) - \sum_{l=1}^{m}[M_Q]_{kl}d_{1m}^l(t) = \int a\omega_k dx
\end{aligned} \tag{A.20}$$

for $k = 1, 2, \ldots, m$, where

$$[M_i]_{kl} = \int_\Omega C_{ii}^\epsilon \omega_k \omega_l dx,\ [M_Q]_{kl} = \frac{1}{\epsilon}\int_\Omega Q^\epsilon \omega_k \omega_l dx,\ [A_i]_{kl} = \int_\Omega \kappa_i^\epsilon \nabla\omega_k \cdot \nabla\omega_l dx. \quad \text{(A.21)}$$

Since M_1 and M_2 are positive definite and symmetric Gram matrices, they are invertible. Hence, (A.20) has unique solutions.

It can be shown that $u_{1m}^\epsilon, u_{2m}^\epsilon$ are uniformly bounded in both $L^2(0,T;V),\ L^\infty(0,T;H)$ and $\mathcal{C}_{11}^\epsilon \frac{\partial u_{1m}^\epsilon}{\partial t}, \mathcal{C}_{22}^\epsilon \frac{\partial u_{2m}^\epsilon}{\partial t}$ are uniformly bounded in $L^2(0,T;V')$ for all m. The proof is similar to that of A.0.1 and 3.1.1.

From these results, we deduce that there exist functions $u_1^\epsilon, u_2^\epsilon, \eta_1^\epsilon, \eta_2^\epsilon$ such that

$$u_{im}^\epsilon \rightharpoonup u_i^\epsilon \text{ in } L^2(0,T;V), \quad \mathcal{C}_{ii}^\epsilon \frac{\partial u_{im}^\epsilon}{\partial t} \rightharpoonup \eta_i^\epsilon \text{ in } L^2(0,T;V'), \ \ i = 1, 2. \quad \text{(A.22)}$$

Let $\psi_1(t), \psi_2(t) \in C^1[0,T]$ with $\psi_1(T) = \psi_2(T) = 0$. Let $\phi_{1k} = \psi_1\omega_k$, $\phi_{2k} = \psi_2\omega_k$. From (A.19) we get

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_{1m}^\epsilon}{\partial t}\phi_{1k} dxdt + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_{1m}^\epsilon \cdot \nabla\phi_{1k} dxdt \\
&\qquad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon (u_{1m}^\epsilon - u_{2m}^\epsilon)\phi_{1k} dxdt = \int_0^T \int_\Omega q\phi_{1k} dxdt \\
&\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_{2m}^\epsilon}{\partial t}\phi_{2k} dxdt + \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_{2m}^\epsilon \cdot \nabla\phi_{2k} dxdt \\
&\qquad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon (u_{2m}^\epsilon - u_{1m}^\epsilon)\phi_{2k} dxdt = \int_0^T \int_\Omega q\phi_{2k} dxdt.
\end{aligned} \quad \text{(A.23)}$$

Since $\phi(T) = 0$, integrating by parts we obtain

$$\begin{aligned}
-\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon u_{1m}^\epsilon \frac{\partial \phi_{1k}}{\partial t} dxdt &+ \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_{1m}^\epsilon \cdot \nabla \phi_{1k} dxdt \\
&+ \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon (u_{1m}^\epsilon - u_{2m}^\epsilon) \phi_{1k} dxdt \\
&= \int_0^T \int_\Omega q \phi_{1k} dxdt + \int_\Omega \mathcal{C}_{11}^\epsilon u_{1m}^\epsilon(0) \phi_{1k}(0) dx \\
-\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon u_{2m}^\epsilon \frac{\partial \phi_{2k}}{\partial t} dxdt &+ \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_{2m}^\epsilon \cdot \nabla \phi_{2k} dxdt \\
&+ \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon (u_{2m}^\epsilon - u_{1m}^\epsilon) \phi_{2k} dxdt \\
&= \int_0^T \int_\Omega q \phi_{2k} dxdt + \int_\Omega \mathcal{C}_{22}^\epsilon u_{2m}^\epsilon(0) \phi_{2k}(0) dx.
\end{aligned} \tag{A.24}$$

Note that $u_{1m}^\epsilon(0) \to g_1$, $u_{2m}^\epsilon(0) \to g_2$ in H as $m \to \infty$. Passing to the limit, $m \to \infty$, we obtain

$$\begin{aligned}
-\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon \frac{\partial \phi_{1k}}{\partial t} dxdt &+ \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla \phi_{1k} dxdt \\
&+ \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon (u_1^\epsilon - u_2^\epsilon) \phi_{1k} dxdt \\
&= \int_0^T \int_\Omega q \phi_{1k} dxdt + \int_\Omega \mathcal{C}_{11}^\epsilon g_1 \phi_{1k}(0) dx, \\
-\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon u_2^\epsilon \frac{\partial \phi_{2k}}{\partial t} dxdt &+ \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla \phi_{2k} dxdt \\
&+ \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon) \phi_{2k} dxdt \\
&= \int_0^T \int_\Omega q \phi_{2k} dxdt + \int_\Omega \mathcal{C}_{22}^\epsilon g_2 \phi_{2k}(0) dx.
\end{aligned} \tag{A.25}$$

We partially integrate the first terms of the equations in (A.25) and obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t} \phi_{1k} dxdt + \int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon(0)\phi_{1k}(0)dx + \int_0^T \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon \cdot \nabla \phi_{1k} dxdt \\
&\quad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon (u_1^\epsilon - u_2^\epsilon)\phi_{1k} dxdt = \int_0^T \int_\Omega q\phi_{1k} dxdt + \int_\Omega \mathcal{C}_{11}^\epsilon g_1 \phi_{1k}(0)dx, \\
&\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t} \phi_{2k} dxdt + \int_\Omega \mathcal{C}_{22}^\epsilon u_2^\epsilon(0)\phi_{2k}(0)dx + \int_0^T \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon \cdot \nabla \phi_{2k} dxdt \\
&\quad + \frac{1}{\epsilon}\int_0^T \int_\Omega Q^\epsilon (u_2^\epsilon - u_1^\epsilon)\phi_{2k} dxdt = \int_0^T \int_\Omega q\phi_{2k} dxdt + \int_\Omega \mathcal{C}_{22}^\epsilon g_2 \phi_{2k}(0)dx.
\end{aligned} \tag{A.26}$$

As this holds for all $\psi_1, \psi_2 \in \mathcal{D}((0,T))$, it follows that

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon(t)}{\partial t}\omega_k dx + \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon(t) \cdot \nabla \omega_k dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon (u_1^\epsilon(t) - u_2^\epsilon(t))\omega_k dx = \int_\Omega q\omega_k dx, \\
&\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon(t)}{\partial t}\omega_k dx + \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon(t) \cdot \nabla \omega_k dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon (u_2^\epsilon(t) - u_1^\epsilon(t))\omega_k dx = \int_\Omega q\omega_k dx
\end{aligned} \tag{A.27}$$

a.e. on [0,T], and

$$\int_\Omega \mathcal{C}_{11}^\epsilon u_1^\epsilon(0)\omega_k(0)dx = \int_\Omega \mathcal{C}_{11}^\epsilon g_1 \omega_k(0)dx, \ \int_\Omega \mathcal{C}_{22}^\epsilon u_2^\epsilon(0)\omega_k(0)dx = \int_\Omega \mathcal{C}_{22}^\epsilon g_2 \omega_k(0)dx \tag{A.28}$$

for all k. Thus, from (A.27) and (A.28), we deduce

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon(t)}{\partial t}\phi_1 dx + \int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon(t) \cdot \nabla \phi_1 dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon (u_1^\epsilon(t) - u_2^\epsilon(t))\phi_1 dx = \int_\Omega q\phi_1 dx, \\
&\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon(t)}{\partial t}\phi_2 dx + \int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon(t) \cdot \nabla \phi_2 dx + \frac{1}{\epsilon}\int_\Omega Q^\epsilon (u_2^\epsilon(t) - u_1^\epsilon(t))\phi_2 dx = \int_\Omega q\phi_2 dx
\end{aligned} \tag{A.29}$$

a.e. on [0,T], for all $\phi_1, \phi_2 \in V$ and $\mathcal{C}_{ii}^\epsilon u_i^\epsilon(0) = \mathcal{C}_{ii}^\epsilon g_i$, hence, $u_i^\epsilon(0) = g_i$. Thus, $u_1^\epsilon, u_2^\epsilon$ are solutions of (A.1). We now show the uniqueness of the solutions. Assume $u_1^\epsilon, u_2^\epsilon$, $v_1^\epsilon, v_2^\epsilon$

are two solution sets of (A.1). We let $u_1^\epsilon - v_1^\epsilon = \delta_1$, $u_2^\epsilon - v_2^\epsilon = \delta_2$. Then from (A.1), we get

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon(x) \frac{\partial \delta_1(t,x)}{\partial t} \phi_1(t,x) dx dt + \int_0^T \int_\Omega \kappa_1^\epsilon(x) \nabla \delta_1(t,x) \cdot \nabla \phi_1(t,x) dx dt \\
&\qquad + \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon(x)(\delta_1(t,x) - \delta_2(t,x)) \phi_1(t,x) dx dt = 0, \\
&\int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon(x) \frac{\partial \delta_2(t,x)}{\partial t} \phi_2(t,x) dx dt + \int_0^T \int_\Omega \kappa_2^\epsilon(x) \nabla \delta_2(t,x) \cdot \nabla \phi_2(t,x) dx dt \\
&\qquad + \frac{1}{\epsilon} \int_0^T \int_\Omega Q^\epsilon(x)(\delta_2(t,x) - \delta_1(t,x)) \phi_2(t,x) dx dt = 0
\end{aligned} \tag{A.30}$$

for all $\phi_1, \phi_2 \in L^2(0,T;V)$. Letting $\hat{\delta}_1(t) = \delta_1(t)e^{-\lambda t}$, $\hat{\delta}_2(t) = \delta_2(t)e^{-\lambda t}$, $\check{\phi}_1 = \phi_1 e^{\lambda t}$ and $\check{\phi}_2 = \phi_2 e^{\lambda t}$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial \hat{\delta}_1}{\partial t} \check{\phi}_1 dx dt + \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial \hat{\delta}_2}{\partial t} \check{\phi}_2 dx dt + \lambda \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon \hat{\delta}_1 \check{\phi}_1 dx dt \\
&\quad + \lambda \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon \hat{\delta}_2 \check{\phi}_2 dx dt + \int_0^T a((\hat{\delta}_1(t), \hat{\delta}_2(t)), (\check{\phi}_1(t), \check{\phi}_2(t))) dt = 0.
\end{aligned} \tag{A.31}$$

Letting $\check{\phi}_i = \hat{\delta}_i$, we have

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \mathcal{C}_{11}^\epsilon |\hat{\delta}_1(T)|^2 dx + \frac{1}{2} \int_\Omega \mathcal{C}_{22}^\epsilon |\hat{\delta}_2(T)|^2 dx + \lambda \int_0^T \int_\Omega \mathcal{C}_{11}^\epsilon |\hat{\delta}_1|^2 dx dt \\
&+ \lambda \int_0^T \int_\Omega \mathcal{C}_{22}^\epsilon |\hat{\delta}_2|^2 dx dt + \int_0^T a((\hat{\delta}_1(t), \hat{\delta}_2(t)), (\hat{\delta}_1(t), \hat{\delta}_2(t))) dt = 0.
\end{aligned} \tag{A.32}$$

Note that $\hat{\delta}_i(0) = 0$. By Lemma A.0.2, choosing sufficiently large λ, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon|\hat{\delta}_1(T)|^2dx + \frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon|\hat{\delta}_2(T)|^2dx + \alpha\int_0^T(||\nabla\hat{\delta}_1(t)||_H^2 + ||\nabla\hat{\delta}_2(t)||_H^2)dt \\
&\qquad\leq \frac{1}{2}\int_\Omega \mathcal{C}_{11}^\epsilon|\hat{\delta}_1(T)|^2dx + \frac{1}{2}\int_\Omega \mathcal{C}_{22}^\epsilon|\hat{\delta}_2(T)|^2dx + \lambda\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon|\hat{\delta}_1|^2dxdt \qquad \text{(A.33)} \\
&\qquad+\lambda\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon|\hat{\delta}_2|^2dxdt + \int_0^T a((\hat{\delta}_1(t),\hat{\delta}_2(t)),(\hat{\delta}_1(t),\hat{\delta}_2(t)))dt = 0.
\end{aligned}$$

This implies $\hat{\delta}_1 = \hat{\delta}_2 = 0$, thus, $\delta_1 = \delta_2 = 0$ a.e. on $[0,T]\times\Omega$. We deduce $u_1^\epsilon = v_1^\epsilon,\ u_2^\epsilon = v_2^\epsilon$. □

Now we show the uniqueness of a solution of the homogenized system (3.15). The homogenized problem (3.15) can be written in variational form. We find $u_{10}, u_{20} \in L^2(0,T;V)$ such that $\frac{\partial u_{10}}{\partial t}, \frac{\partial u_{20}}{\partial t} \in L^2(0,T;V')$ satisfying

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\int_Y \mathcal{C}_{11}dy\right)\frac{\partial u_{10}}{\partial t}\phi_1dxdt + \int_0^T\int_\Omega \kappa_1^*\nabla u_{10}\cdot\nabla\phi_1dxdt \\
&+\int_0^T\int_\Omega(\int_Y \kappa_1\nabla_y M_1dy)\cdot\nabla\phi_1(u_{20}-u_{10})dxdt \\
&+\int_0^T\int_\Omega[(\int_Y QN_1^idy)\frac{\partial u_{10}}{\partial x_i} - (\int_Y QN_2^idy)\frac{\partial u_{20}}{\partial x_i}]\phi_1dxdt \qquad \text{(A.34)} \\
&-\int_0^T\int_\Omega(\int_Y Q(M_1+M_2)dy)(u_{10}-u_{20})\phi_1dxdt \\
&=\int_0^T\int_\Omega q\phi_1dxdt,
\end{aligned}$$

$$
\begin{aligned}
&\int_0^T \int_\Omega \left(\int_Y \mathcal{C}_{22} dy\right) \frac{\partial u_{20}}{\partial t}\phi_2 dxdt + \int_0^T \int_\Omega \kappa_2^* \nabla u_{20}\cdot\nabla\phi_2 dxdt \\
&+ \int_0^T \int_\Omega (\int_Y \kappa_2 \nabla_y M_2 dy)\cdot\nabla\phi_2 (u_{10}-u_{20}) dxdt \\
&+ \int_0^T \int_\Omega [(\int_Y QN_2^i dy)\frac{\partial u_{20}}{\partial x_i} - (\int_Y QN_1^i dy)\frac{\partial u_{10}}{\partial x_i}]\phi_2 dxdt \\
&- \int_0^T \int_\Omega (\int_Y Q(M_1+M_2)dy)(u_{20}-u_{10})\phi_2 dxdt \\
&= \int_0^T \int_\Omega q\phi_2 dxdt,
\end{aligned} \tag{A.35}
$$

for all $\phi_1, \phi_2 \in L^2(0,T;V)$. We define the bilinear form $b: W\times W \to \mathbb{R}$ by

$$
\begin{aligned}
&b((u_{10}(t),u_{20}(t)),(\phi_1(t),\phi_2(t))) = \\
&\int_\Omega \kappa_1^*\nabla u_{10}\cdot\nabla\phi_1 dx + \int_\Omega (\int_Y \kappa_1\nabla_y M_1 dy)\cdot\nabla\phi_1(u_{20}-u_{10})dx \\
&+ \int_\Omega [(\int_Y QN_1^i dy)\frac{\partial u_{10}}{\partial x_i} - (\int_Y QN_2^i dy)\frac{\partial u_{20}}{\partial x_i}]\phi_1 dx \\
&- \int_\Omega (\int_Y Q(M_1+M_2)dy)(u_{10}-u_{20})\phi_1 dx \\
&+ \int_\Omega \kappa_2^*\nabla u_{20}\cdot\nabla\phi_2 dx + \int_\Omega (\int_Y \kappa_2\nabla_y M_2 dy)\cdot\nabla\phi_2(u_{10}-u_{20})dx \\
&+ \int_\Omega [(\int_Y QN_2^i dy)\frac{\partial u_{20}}{\partial x_i} - (\int_Y QN_1^i dy)\frac{\partial u_{10}}{\partial x_i}]\phi_2 dx \\
&- \int_\Omega (\int_Y Q(M_1+M_2)dy)(u_{20}-u_{10})\phi_2 dx.
\end{aligned} \tag{A.36}
$$

Lemma A.0.4. *Assume $Q, \kappa_j \in C(\bar{\Omega}; C(\bar{Y}))$, $N_j^i,\ M_j \in C(\bar{\Omega}; C^1(\bar{Y}))$ for $j = 1,\ 2$. There exists $C > 0$ such that*

$$
b((u_1,u_2),(v_1,v_2)) \le C\big(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2\big)^{\frac{1}{2}} \cdot \big(||\nabla v_1||_H^2 + ||\nabla v_2||_H^2\big)^{\frac{1}{2}} \tag{A.37}
$$

for $(u_1,u_2),(v_1,v_2) \in W$. There exists $k \ge 0$ such that

$$
b((u_1,u_2),(u_1,u_2)) + k||u_1||_H^2 + k||u_2||_H^2 \ge \alpha(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2) \tag{A.38}
$$

for all $u_1, u_2 \in V$, for a constant $\alpha > 0$.

Proof. We first show (A.37). We have

$$\begin{aligned}
& b((u_1, u_2), (v_1, v_2)) \\
\leq & c_1(||\nabla u_1||_H \cdot ||\nabla v_1||_H + ||\nabla v_1||_H \cdot ||u_1||_H + ||\nabla v_1||_H \cdot ||u_2||_H + ||\nabla u_1||_H \cdot ||v_1||_H \\
& + ||\nabla u_2||_H \cdot ||v_1||_H + ||u_1||_H \cdot ||v_1||_H + ||u_2||_H \cdot ||v_1||_H + ||\nabla u_2||_H \cdot ||\nabla v_2||_H \\
& + ||\nabla v_2||_H \cdot ||u_2||_H + ||\nabla v_2||_H \cdot ||u_1||_H + ||\nabla u_2||_H \cdot ||v_2||_H + ||\nabla u_1||_H \cdot ||v_2||_H \\
& + ||u_2||_H \cdot ||v_2||_H + ||u_2||_H \cdot ||v_2||_H) \\
\leq & 4c_1\big(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2 + ||u_1||_H^2 + ||u_2||_H^2\big)^{\frac{1}{2}} \\
& \cdot \big(||\nabla v_1||_H^2 + ||\nabla v_2||_H^2 + ||v_1||_H^2 + ||v_2||_H^2\big)^{\frac{1}{2}} \\
\leq & C\big(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2\big)^{\frac{1}{2}} \cdot \big(||\nabla v_1||_H^2 + ||\nabla v_2||_H^2\big)^{\frac{1}{2}}.
\end{aligned} \tag{A.39}$$

The last inequality follows from Poincare inequality. We now prove (A.38). As κ_1^* and κ_2^* are positive definite, we have

$$\begin{aligned}
& b((u_1, u_2), (u_1, u_2)) \\
& \geq c_1(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2) - c_2(||\nabla u_1||_H \cdot ||u_1||_H + ||\nabla u_1||_H \cdot ||u_2||_H \\
& + ||\nabla u_1||_H \cdot ||u_1||_H + ||\nabla u_2||_H \cdot ||u_1||_H + ||u_1||_H \cdot ||u_1||_H + ||u_2||_H \cdot ||u_1||_H \\
& + ||\nabla u_2||_H \cdot ||u_2||_H + ||\nabla u_2||_H \cdot ||u_1||_H + ||\nabla u_2||_H \cdot ||u_2||_H + ||\nabla u_1||_H \cdot ||u_2||_H \\
& + ||u_2||_H \cdot ||u_2||_H + ||u_2||_H \cdot ||u_2||_H) \\
& \geq c_1(||\nabla u_1||_H^2 + ||\nabla u_2||_H^2) - (\varepsilon_1||\nabla u_1||_H^2 + \delta_1||u_1||_H^2 + \varepsilon_2||\nabla u_2||_H^2 + \delta_2||u_2||_H^2).
\end{aligned} \tag{A.40}$$

The last inequality follows from Young's inequality, $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, valid for all $\varepsilon > 0$.

Thus, we can choose $\varepsilon_1,\ \varepsilon_2$ small enough and get the conclusion. $\square$

Theorem A.0.5. *Assume* $Q, \kappa_j \in C(\bar{\Omega}; C(\bar{Y}))$, $N_j^i,\ M_j \in C(\bar{\Omega}; C^1(\bar{Y}))$ *for* $j = 1,\ 2$. *There exists a unique solution for problem (A.34).*

Proof. The existence follows from Theorem 3.1.1. We only prove the uniqueness. Assume (u_{10}, u_{20}), (v_{10}, v_{20}) are two solutions of (A.34). We let $u_{10} - v_{10} = \delta_1$, $u_{20} - v_{20} = \delta_2$. From (A.34), we obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{11} dy \frac{\partial \delta_1}{\partial t} \phi_1 dx dt + \int_0^T \int_\Omega \kappa_1^* \nabla \delta_1 \cdot \nabla \phi_1 dx dt \\
&+ \int_0^T \int_\Omega (\int_Y \kappa_1 \nabla_y M_1 dy) \cdot \nabla \phi_1 (\delta_2 - \delta_1) dx dt \\
&+ \int_0^T \int_\Omega [(\int_Y Q N_1^i dy) \frac{\partial \delta_1}{\partial x_i} - (\int_Y Q N_2^i dy) \frac{\partial \delta_2}{\partial x_i}] \phi_1 dx dt \\
&- \int_0^T \int_\Omega (\int_Y Q(M_1 + M_2) dy)(\delta_1 - \delta_2) \phi_1 dx dt = 0, \\
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{22} dy \frac{\partial \delta_2}{\partial t} \phi_2 dx dt + \int_0^T \int_\Omega \kappa_2^* \nabla \delta_2 \cdot \nabla \phi_2 dx dt \\
&+ \int_0^T \int_\Omega (\int_Y \kappa_2 \nabla_y M_2 dy) \cdot \nabla \phi_2 (\delta_1 - \delta_2) dx dt \\
&+ \int_0^T \int_\Omega [(\int_Y Q N_2^i dy) \frac{\partial \delta_2}{\partial x_i} - (\int_Y Q N_1^i dy) \frac{\partial \delta_1}{\partial x_i}] \phi_2 dx dt \\
&- \int_0^T \int_\Omega (\int_Y Q(M_1 + M_2) dy)(\delta_2 - \delta_1) \phi_2 dx dt = 0
\end{aligned} \tag{A.41}
$$

for all $\phi_1, \phi_2 \in L^2(0, T; V)$. Let $\hat{\delta}_1(t) = \delta_1(t) e^{-\lambda t}$, $\hat{\delta}_2(t) = \delta_2(t) e^{-\lambda t}$, $\check{\phi}_1 = \phi_1 e^{\lambda t}$ and $\check{\phi}_2 = \phi_2 e^{\lambda t}$. We have

$$
\begin{aligned}
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{11} dy \frac{\partial \hat{\delta}_1}{\partial t} \check{\phi}_1 dx dt + \lambda \int_0^T \int_\Omega \int_Y \mathcal{C}_{11} dy \hat{\delta}_1 \check{\phi}_1 dx dt \\
&+ \int_0^T \int_\Omega \kappa_1^* \nabla \hat{\delta}_1 \cdot \nabla \check{\phi}_1 dx dt + \int_0^T \int_\Omega (\int_Y \kappa_1 \nabla_y M_1 dy) \cdot \nabla \check{\phi}_1 (\hat{\delta}_2 - \hat{\delta}_1) dx dt \\
&+ \int_0^T \int_\Omega [(\int_Y Q N_1^i dy) \frac{\partial \hat{\delta}_1}{\partial x_i} - (\int_Y Q N_2^i dy) \frac{\partial \hat{\delta}_2}{\partial x_i}] \check{\phi}_1 dx dt \\
&- \int_0^T \int_\Omega (\int_Y Q(M_1 + M_2) dy)(\hat{\delta}_1 - \hat{\delta}_2) \check{\phi}_1 dx dt = 0, \\
&\int_0^T \int_\Omega \int_Y \mathcal{C}_{22} dy \frac{\partial \hat{\delta}_2}{\partial t} \check{\phi}_2 dx dt + \lambda \int_0^T \int_\Omega \int_Y \mathcal{C}_{22} dy \hat{\delta}_2 \check{\phi}_2 dx dt \\
&+ \int_0^T \int_\Omega \kappa_2^* \nabla \hat{\delta}_2 \cdot \nabla \check{\phi}_2 dx dt + \int_0^T \int_\Omega (\int_Y \kappa_2 \nabla_y M_2 dy) \cdot \nabla \check{\phi}_2 (\hat{\delta}_1 - \hat{\delta}_2) dx dt \\
&+ \int_0^T \int_\Omega [(\int_Y Q N_2^i dy) \frac{\partial \hat{\delta}_2}{\partial x_i} - (\int_Y Q N_1^i dy) \frac{\partial \hat{\delta}_1}{\partial x_i}] \check{\phi}_2 dx dt \\
&- \int_0^T \int_\Omega (\int_Y Q(M_1 + M_2) dy)(\hat{\delta}_2 - \hat{\delta}_1) \check{\phi}_2 dx dt = 0.
\end{aligned}
\tag{A.42}
$$

We let $\check{\phi}_i = \hat{\delta}_i$. Since $\hat{\delta}_i(0) = 0$, adding above 2 equations, we have

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \int_Y \mathcal{C}_{11} dy |\hat{\delta}_1(T)|^2 dx + \frac{1}{2} \int_\Omega \int_Y \mathcal{C}_{22} dy |\hat{\delta}_2(T)|^2 dx \\
&+ \lambda \int_0^T \int_\Omega \int_Y \mathcal{C}_{11} dy |\hat{\delta}_1|^2 dx dt + \lambda \int_0^T \int_\Omega \int_Y \mathcal{C}_{22} dy |\hat{\delta}_2|^2 dx dt \\
&+ \int_0^T b((\hat{\delta}_1(t), \hat{\delta}_2(t)), (\hat{\delta}_1(t), \hat{\delta}_2(t))) dt = 0.
\end{aligned}
\tag{A.43}
$$

Choosing λ large enough, we have

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \int_Y \mathcal{C}_{11} dy |\hat{\delta}_1(T)|^2 dx + \frac{1}{2} \int_\Omega \int_Y \mathcal{C}_{22} dy |\hat{\delta}_2(T)|^2 dx \\
&\qquad + \alpha \int_0^T (||\nabla \hat{\delta}_1(t)||_H^2 + ||\nabla \hat{\delta}_2(t)||_H^2) dt = 0,
\end{aligned}
\tag{A.44}
$$

by Lemma A.0.4. We deduce $\hat{\delta}_1 = \hat{\delta}_2 = 0$ thus $\delta_1 = \delta_2 = 0$. We have $u_{10} = v_{10}$,

$u_{20} = v_{20}$. □