STATISTICAL PHYSICS MODELS GOVERNED BY DIFFUSION

A Dissertation

by

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ABSTRACT

In this article we consider two probability models: stationary diffusion limited aggregation (SDLA) and finitary random interlacements (FRI). SDLA is a stochastic process on the upper half planar lattice, growing from an infinite line, with local growth rate proportional to stationary harmonic measure. We first prove that stationary harmonic measure of an infinite set in the upper planar lattice can be represented as the proper scaling limit of the classical harmonic measure of truncations of the infinite set. Then we construct an infinite SDLA that is ergodic with respect to left-right integer translation. For FRI, we prove a phase transition in the connectivity of FRI $\mathcal{FI}^{u,T}$ on $\mathbb{Z}^d$ with respect to the average stopping time T.

DEDICATION

To my sister, my mother, and my father.

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First I would like to thank my advisor Prof. Eviatar Procaccia. He encouraged me to pursue a PhD degree in mathematics when I was an undergraduate student, and he continuously helped me in research and in life. This dissertation would not be possible without his guidance.

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Contributors

This article is based of three papers [1, 2, 3] that are joint work of Prof. Eviatar Procaccia, Prof. Yuan Zhang, and the author. These three papers are posted on arXiv (an open-access repository), but they are not published by the time of defense.

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NOMENCLATURE

DLA	Diffusion limited aggregation
FRI	Finitary random interlacements
RI	Random interlacements
SDLA	Stationary diffusion limited aggregation
$\mathcal{H}$	Stationary harmonic measure
$\mathtt{H}$	Harmonic measure
H	Continuous harmonic measure
$\mathbb{H}$	Upper half planar lattice
$\mathcal{I}^u$	Random interlacements at level $u > 0$
$\mathcal{FI}^{u,T}$	Finitary random interlacements with parameters $u, T > 0$

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1. INTRODUCTION

Random walk is one of the most basic and well-studied subjects in probability theory, and research about random walks is still active today. In this article, we will focus on simple random walks on the square lattice $\mathbb{Z}^d$. Let $x \in \mathbb{Z}^d$ be a vertex. We write $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ for the standard basis of $\mathbb{Z}^d$. We can consider a simple random walk on $\mathbb{Z}^d$ starting at x as a sum of i.i.d. random variables, i.e.

$$S_n = x + \sum_{i=1}^{n} X_i,$$

where $P(X_i = e_j) = P(X_i = -e_j) = 1/(2d)$, for all $1 \leq j \leq d$. For a more detailed description of simple random walks on $\mathbb{Z}^d$, readers are referred to [4, 5, 6].

The model of percolation was introduced by Broadbent and Hammersley [7] in 1957. Imagine water is flowing through a porous stone. We are interested in the question of macroscopic percolation of the water in the stone. In (bond) Bernoulli percolation on $\mathbb{Z}^d$, each edge is open with probability $p \in [0, 1]$ and is closed with probability $1 - p$, independent of all other edges. A site Bernoulli percolation is the same as the bond one except that each vertex is taken to be open or closed. Bernoulli percolation is particularly interesting because it is one of the most simple models that exhibit phase transitions:

Theorem 1.0.1 ([8, 9])**.** *For $d \geq 2$, there exists $p_c(\mathbb{Z}^d) \in (0, 1)$ such that:*

1. *(supercritical phase) for $p > p_c(\mathbb{Z}^d)$, there is a unique infinite component almost surely;*
2. *(subcritical phase) for $p < p_c(\mathbb{Z}^d)$, there is no infinite component almost surely.*

One of most significant results in percolation theory is Kesten's theorem [10] that shows $p_c(\mathbb{Z}^2) = 1/2$.

In recent years, probabilists are interested in "dependent" percolation models where states of edges/vertices are not independent. These "dependent" percolation models provide tools to study many phenomena. For example, Ising model introduced by Lenz [11] is a model to study ferromagnetism. We refer to [12] for more detailed description of Ising model. In particular, Peierls [13] developed a technique, which is now commonly known as "Peierls argument", to show the existence of a phase transition in the Ising model. We will use this technique in Chapter 3.

Another example of “dependent” percolation model is the random interlacement (RI) introduced in 2007 by Sznitman [14]. RI is defined as a Poisson point process on the space of doubly infinite random walk trajectories in the lattice $\mathbb{Z}^d$, with $d \geq 3$. A simple way to describe RI is the following: Fix $u > 0$ and a finite subset $A \subset \mathbb{Z}^d$. We sample a vertex x uniformly at random from all vertices of the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ and run a simple random walk from x up to time $\lfloor uN^d \rfloor$. This induces a measure on sites in A visited by the random walk. As N goes to infinity, this measure converges weakly to the measure on sites of the RI in A.

Finitary random interlacements (FRI) was recently introduced by Bowen [15] to solve a special case of the Gaboriau-Lyons problem. Informally speaking, FRI $\mathcal{FI}^{u,T}$ can be described as a Poisson cloud of geometrically killed random walks on $\mathbb{Z}^d$, $d \geq 3$, where $u > 0$ is the multiplicative parameter controlling the number of geometrically killed random walks and $T > 0$ is the expected length of each geometrically killed random walk. Bowen [15] showed that, for all $u > 0$, the measure of FRI $\mathcal{FI}^{u,T}$ converges to the one of RI $\mathcal{I}^u$ in the weak* topology as T goes to infinity.

In Chapter 3, we show a percolation phase transition in the connectivity of FRI $\mathcal{FI}^{u,T}$ with respect to the average stopping time T. For all $u > 0$, with probability one $\mathcal{FI}^{u,T}$ has no infinite connected component for all sufficiently small $T > 0$, and a unique infinite connected component for all sufficiently large $T < \infty$. This is different from RI. For all $u > 0$, the RI $\mathcal{I}^u$ is almost surely connected, so $\mathcal{I}^u$ has only one component and it is infinite.

Now we turn our focus to another probability model that at least might seem different from FRI. Diffusion limited aggregation (DLA) was introduced in 1983 by Witten and Sander [16] as a simple model to study aggregation systems governed by diffusive laws. DLA is defined recursively as a process on subsets $\{A_n\}$ of $\mathbb{Z}^2$. Let $A_0 = \{(0,0)\}$, and $A_{n+1} = A_n \cup a_{n+1}$, where a_{n+1} is a point sampled from the harmonic measure of $\partial^{out} A_n$, the external vertex boundary of A_n. Intuitively,

a_{n+1} is the point that a random walk starting from infinity first visits $\partial^{out} A_n$.

Although DLA is easy to define, little is known rigorously. One of the notable exception is Kesten's 1987 paper [17] which showed an upper bound on the growth rate of the DLA cluster. No non-trivial lower bound has been proved. The question whether the DLA cluster converges to a ball after suitable scaling is still open.

Inspired by Itai Benjamini, Eviatar Procaccia started studying stationary versions of different aggregation processes. In [18] Procaccia and Zhang defined a stationary version of the harmonic measure on subsets of $\mathbb{H}$, the upper half of the lattice $\mathbb{Z}^2$. In [18] they also showed an upper bound on the stationary harmonic measure and a dominating interacting particle system for the stationary DLA (SDLA) in the subsequent papers. In [19] Procaccia and Zhang showed that any subset in $\mathbb{H}$ with an appropriate sub-linear horizontal growth has non-zero stationary harmonic measure. On the other hand, any subset with super-linear horizontal growth has zero stationary harmonic measure everywhere.

In Chapter 2, we show that:

1. stationary harmonic measure can be written as a normalized harmonic measure from one point;

2. stationary harmonic measure of an infinite set can be represented as the proper scaling limit of the classical harmonic measure of truncations of the infinite set;

3. SDLA is well-defined up to a fixed time $t > 0$.

One can see that the geometry of FRI and SDLA are strongly related to properties of simple random walks on $\mathbb{Z}^d$.

Throughout this article, we will write $\mathbb{P}$ for probability and $\mathbb{E}$ for expectation. In addition, let $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | S_0 = x)$ be the probability law of a simple random walk on $\mathbb{Z}^d$ starting at x, and let $\mathbb{E}_x$ be the corresponding expectation. We denote positive constants by $c, C, c_1, c', \cdots$, and their values can be different from place to place. In Chapter 3, all positive constants will depend on dimension d by default.

2. STATIONARY DIFFUSION LIMITED AGGREGATION

The content of this chapter appears in [1, 2].

2.1 Notations and Definitions

Let $\mathbb{H} = \{(x, y) \in \mathbb{Z}^2 : y \geq 0\}$ be the upper half plane including the x-axis, and $(S_n)_{n\geq 0}$ be a 2-dimensional simple random walk. For any $x \in \mathbb{H}$, we write

$$x = (x^{(1)}, x^{(2)}),$$

where $x^{(i)}$ denoting the i-th coordinate of x. For each $n \geq 0$, define the subsets $L_n \subset \mathbb{H}$ as follows:

$$L_n = \{(x, n) : x \in \mathbb{Z}\},$$

i.e. L_n is the horizontal line of height n. For each subset $A \subset \mathbb{H}$, we define the stopping times

$$\tau_A = \min\{n \geq 1 : S_n \in A\},$$

and

$$\overline{\tau}_A = \min\{n \geq 0 : S_n \in A\}.$$

For any $R > 0$, let $B(0, R) = \{x \in \mathbb{Z}^2 : ||x||_2 < R\}$ be the discrete ball of radius R, and abbreviate

$$\tau_R = \tau_{B(0,R)},\ \bar{\tau}_R = \bar{\tau}_{B(0,R)}.$$

Let $|| \cdot ||_1$ be the l_1 norm. We define

$$\partial^{\text{out}} A := \{y \in \mathbb{H} \setminus A : \exists x \in A, ||x - y||_1 = 1\}$$

as the outer vertex boundary of A, and define

$$\partial^{\text{in}} A := \{y \in A : \exists x \in \mathbb{H} \setminus A, ||x - y||_1 = 1\}$$

as the inner vertex boundary of A. Let $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | S_0 = x)$. The stationary harmonic measure $\mathcal{H}_A$ on $\mathbb{H}$ is introduced in [18]. Let $A \subset \mathbb{H}$ be a connected set. For any edge $e = (x, y)$ with $x \in A$ and $y \in \mathbb{H} \setminus A$, define

$$\mathcal{H}_{A,N}(e) = \sum_{z \in L_N \setminus A} \mathbb{P}_z\big(S_{\bar{\tau}_{A \cup L_0}} = x, S_{\bar{\tau}_{A \cup L_0} - 1} = y\big).$$

Note that $\mathcal{H}_{A,N}(e) > 0$ if and only if $x \in \partial^{in} A$ and $||x - y||_1 = 1$. For all $x \in A$, define

$$\mathcal{H}_{A,N}(x) = \sum_{e \text{ starting from } x} \mathcal{H}_{A,N}(e),$$

and for all $y \in \mathbb{H} \setminus A$, define

$$\hat{\mathcal{H}}_{A,N}(y) = \sum_{e \text{ starting in } A \text{ ending at } y} \mathcal{H}_{A,N}(e).$$

Proposition 2.1.1 (Proposition 1 in [18])**.** *For any A and e above, there is a finite $\mathcal{H}_A(e)$ such that*

$$\lim_{N \to \infty} \mathcal{H}_{A,N}(e) = \mathcal{H}_A(e).$$

$\mathcal{H}_A(e)$ is called the stationary harmonic measure of e with respect to A. The limits

$$\mathcal{H}_A(x) := \lim_{N \to \infty} \mathcal{H}_{A,N}(x)$$

and

$$\hat{\mathcal{H}}_A(y) := \lim_{N \to \infty} \hat{\mathcal{H}}_{A,N}(y)$$

also exist, and $\mathcal{H}_A$ is called the stationary harmonic measure of x and y with respect to A.

Definition 2.1.2. We say that a set $L_0 \subset A \subset \mathbb{H}$ has a polynomial sub-linear growth if there exists a constant $\alpha \in (0,1)$ such that

$$|\{x = (x^{(1)}, x^{(2)}) \in A : x^{(2)} > |x^{(1)}|^{\alpha}\}| < \infty.$$

For any connected $A \subset \mathbb{H}$ such that $A \cap L_0 \neq \emptyset$, and any $x \in A$, $\mathcal{H}_A(x)$ was proved to have the following upper bounds that depends only on the height of x:

Theorem 2.1.3 (Theorem 1, [18])**.** *There is some constant $C < \infty$ such that for each connected $A \subset \mathbb{H}$ with $L_0 \subset A$ and each $x = (x_1, x_2) \in A \setminus L_0$, and any N sufficiently larger than x_2*

$$\mathcal{H}_{A,N}(x) \leq C x_2^{1/2}. \tag{2.1.1}$$

Remark 2.1.4. It is easy to note that for any $A \subset \mathbb{H}$ such that $A \cap L_0 \neq \emptyset$ and any $x = (x_1, x_2) \in A \setminus L_0$, $\mathcal{H}_A(x) = \mathcal{H}_{A \cup L_0}(x)$. Thus one may without loss of generality assume that $L_0 \subset A$.

Remark 2.1.5. Since the constant C above does not depend on subset A or point x, without loss of generality, one may (incorrectly) assume $C = 1$.

2.2 Stationary Harmonic Measure is Equivalent to Normalized Harmonic Measure

Lemma 2.2.1. *For all $x \in L_0$, $\mathcal{H}_{L_0}(x) = 1$.*

Proof. Like Proposition 1 in [18], the proof follows a coupling argument by translating one path starting from a fixed point of L_N horizontally. For each N, let $S_n^{(0,N)}$ be a simple random walk in the probability space $\mathbb{P}_{(0,N)}(\cdot)$ starting at $(0, N)$, and $S_n^{(k,N)} = S_n^{(0,N)} + (k, 0)$ for all $k \in \mathbb{Z}$. Note that $S_n^{(k,N)}$ is a simple random walk starting at (k, N). Let

$$\overline{\tau}_{L_0} = \inf\{n \geq 0 : S_n^{(0,N)} \in L_0\}$$

be a stopping time. Then we have

$$\overline{\tau}_{L_0} = \inf\{n \geq 0 : S_n^{(k,N)} \in L_0\}$$

for any $k \in \mathbb{Z}$, and

$$S_{\overline{\tau}_{L_0}}^{(k,N)} = S_{\overline{\tau}_{L_0}}^{(0,N)} + (k, 0).$$

Hence,

$$\mathcal{H}_{L_0,N}(x) = \sum_{k \in \mathbb{Z}} \mathbb{P}(S_{\overline{\tau}_{L_0}}^{(k,N)} = x) = 1.$$

By definition of the stationary harmonic measure,

$$\mathcal{H}_{L_0}(x) = \lim_{N \to \infty} \mathcal{H}_{L_0,N}(x) = 1.$$

□

We now define a new measure $\widetilde{\mathcal{H}}_A(\cdot)$ which can be shown equivalent to the stationary harmonic measure $\mathcal{H}_A(\cdot)$. For each $n > 0$, we first define

$$\widetilde{\mathcal{H}}_{A,n}(x) = \pi n \mathbb{P}_{(0,n)}(S_{\tau_{A \cup L_0}} = x).$$

Lemma 2.2.2. *For all* $x = (x^{(1)}, 0) \in L_0$*,*

$$\lim_{n \to \infty} \widetilde{\mathcal{H}}_{L_0,n}(x) = 1.$$

Proof. By Theorem 8.1.2 in Lawler and Limic [4],

$$\mathbb{P}_{(0,n)}(S_{\tau_{L_0}} = x) = \frac{n}{\pi(n^2 + (x^{(1)})^2)} \left(1 + O\left(\frac{n}{n^2 + (x^{(1)})^2}\right)\right) + O\left(\frac{1}{(n^2 + (x^{(1)})^2)^{3/2}}\right).$$

So,

$$\lim_{n \to \infty} \widetilde{\mathcal{H}}_{L_0,n}(x) = 1.$$

□

Similar to the construction of the stationary harmonic measure $\mathcal{H}_A(\cdot)$, we want to define a measure $\widetilde{\mathcal{H}}_A$ on $\mathbb{H}$ as following:

$$\widetilde{\mathcal{H}}_A(x) := \lim_{N\to\infty} \widetilde{\mathcal{H}}_{A,N}(x),$$

and denote it by the in-**harmonic measure**. We want to show that $\widetilde{\mathcal{H}}_A = \mathcal{H}_A$. We already proved that $\widetilde{\mathcal{H}}_{L_0} = \mathcal{H}_{L_0}$ in Lemma 2.2.1 and Lemma 2.2.2.

Proposition 2.2.3. *Let $A \subset \mathbb{H}$ be a connected finite subset. For any $x \in \mathbb{H}$,*

$$\widetilde{\mathcal{H}}_A(x) := \lim_{N\to\infty} \widetilde{\mathcal{H}}_{A,N}(x)$$

exists, and $\widetilde{\mathcal{H}}_A(x) = \mathcal{H}_A(x)$.

Proof. Without loss of generality, we assume $x \in \partial^{out} A$. Let

$$k = \max\{x^{(2)} : x = (x^{(1)}, x^{(2)}) \in A\},$$

and $n > m > k$ so that $L_m \cap A = \emptyset$. By the strong Markov property and translation invariance of simple random walk,

$$\begin{aligned}
&\widetilde{\mathcal{H}}_{A,n}(x) \\
&= \pi n \mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_0}} = x) \\
&= \pi n \sum_{y\in L_m} \mathbb{P}_{(0,n)}(S_{\tau_{L_m}} = y)\mathbb{P}_y(S_{\tau_{A\cup L_0}} = x) \\
&= \frac{n}{n-m} \sum_{y\in L_m} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x)\Big[\pi(n-m)\mathbb{P}_{(0,n)}(S_{\tau_{L_m}} = y)\Big] \\
&= \frac{n}{n-m} \sum_{y\in L_m} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x)\widetilde{\mathcal{H}}_{L_0,n-m}(y_0),
\end{aligned} \tag{2.2.1}$$

where $y_0 = (y^{(1)}, 0)$. Then by Dominated Convergence Theorem and Lemma 2.2.2,

$$\begin{aligned}
&\lim_{n\to\infty} \widetilde{\mathcal{H}}_{A,n}(x) \\
&= \lim_{n\to\infty} \sum_{y\in L_m} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x) \frac{n}{n-m} \widetilde{\mathcal{H}}_{L_0,n-m}(y_0) \\
&= \sum_{y\in L_m} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x) \left[\lim_{n\to\infty} \frac{n}{n-m} \widetilde{\mathcal{H}}_{L_0,n-m}(y_0) \right] \qquad (2.2.2) \\
&= \sum_{y\in L_m} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x) \\
&= \mathcal{H}_{A,m}(x).
\end{aligned}$$

We can apply Dominated Convergence Theorem in equation (2.2.2) because $\widetilde{\mathcal{H}}_{L_0,n-m}(y_0)$ is uniformly bounded from above for all n and $y_0 \in \mathbb{Z}$ by Theorem 8.1.2 of [4] and the fact that $\widetilde{\mathcal{H}}_{L_0,n-m}(0) \geq \widetilde{\mathcal{H}}_{L_0,n-m}(y_0)$ for all $y_0 \in \mathbb{Z}$. We claim that $\mathcal{H}_{A,m}(x) = \mathcal{H}_A(x)$. Let $m_1 > m$. By the strong Markov property and Lemma 2.2.1,

$$\begin{aligned}
&\mathcal{H}_{A,m_1}(x) \\
&= \sum_{y\in L_{m_1}} \mathbb{P}_y(S_{\tau_{A\cup L_0}} = x) \\
&= \sum_{y\in L_{m_1}} \sum_{z\in L_m} \mathbb{P}_y(S_{\tau_{L_m}} = z) \mathbb{P}_z(S_{\tau_{A\cup L_0}} = x) \\
&= \sum_{z\in L_m} \mathbb{P}_z(S_{\tau_{A\cup L_0}} = x) \left[\sum_{y\in L_{m_1}} \mathbb{P}_y(S_{\tau_{L_m}} = z) \right] \qquad (2.2.3) \\
&= \sum_{z\in L_m} \mathbb{P}_z(S_{\tau_{A\cup L_0}} = x) \mathcal{H}_{L_0,m_1-m}(z') \\
&= \sum_{z\in L_m} \mathbb{P}_z(S_{\tau_{A\cup L_0}} = x) \\
&= \mathcal{H}_{A,m}(x),
\end{aligned}$$

where $z' = z - (0, m)$. Hence,

$$\widetilde{\mathcal{H}}_A(x) = \mathcal{H}_{A,m}(x) = \lim_{N\to\infty} \mathcal{H}_{A,N}(x) = \mathcal{H}_A(x).$$

□

Our next goal is to show that the measures $\widetilde{\mathcal{H}}_A$ and $\mathcal{H}_A$ are equivalent for sets that satisfy polynomial sub-linear growth condition. We first prove the following combinatorial result: For any positive integer n, consider the following rectangle in $\mathbb{Z}^2$:

$$I_n = [-n, n] \times [0, n] \tag{2.2.4}$$

with height n and width $2n$. It is easy to see that $I_n \subset B(0, 2n)$. Moreover, we let $\partial^{in} I_n$ be the inner vertex boundary of A_n, and let

$$\partial_l^{in} I_n = \{-n\} \times [1, n], \;\; \partial_r^{in} I_n = \{n\} \times [1, n], \;\; \partial_u^{in} I_n = [-n, n] \times \{n\}, \;\; \partial_b^{in} I_n = [-n, n] \times \{0\}$$

be the four edges of $\partial^{in} I_n$.

Let $\{S_n, \; n \geq 0\}$ be a simple random walk starting from 0 and denote by $\mathbb{P}_0$ the probability distribution of S_n. Define the stopping time

$$T_n = \inf\{k > 0, \; S_k \in \partial^{in} I_n\}.$$

Using simple combinatorial arguments, we prove the following lemma:

Lemma 2.2.4. *For any integer* $n > 1$

$$\mathbb{P}_0 \left(S_{T_n} \in \partial_u^{in} I_n \right) \geq \mathbb{P}_0 \left(S_{T_n} \in \partial_l^{in} I_n \cup \partial_r^{in} I_n \right).$$

Proof. Let $\partial_{u,+}^{in} I_n = [1, n] \times \{n\}$ and $\partial_{u,-}^{in} I_n = [-n, -1] \times \{n\}$ be the left and right half of $\partial_u^{in} I_n$. By symmetry it suffices to prove that

$$\mathbb{P}_0 \left(S_{T_n} \in \partial_{u,+}^{in} I_n \right) \geq \mathbb{P}_0 \left(S_{T_n} \in \partial_r^{in} I_n \right). \tag{2.2.5}$$

By definition, we have

$$\mathbb{P}_0\left(S_{T_n}\in\partial^{in}_{u,+}I_n\right)=\sum_{k=1}^{\infty}\mathbb{P}_0\left(S_k\in\partial^{in}_{u,+}I_n,\ T_n=k\right)$$

and

$$\mathbb{P}_0\left(S_{T_n}\in\partial^{in}_{r}I_n\right)=\sum_{k=1}^{\infty}\mathbb{P}_0\left(S_k\in\partial^{in}_{r}I_n,\ T_n=k\right).$$

Moreover, for each k,

$$\mathbb{P}_0\left(S_k\in\partial^{in}_{u,+}I_n,\ T_n=k\right)=\frac{|\mathcal{U}^+_{n,k}|}{4^k},\ \ \mathbb{P}_0\left(S_k\in\partial^{in}_{r}I_n,\ T_n=k\right)=\frac{|\mathcal{R}_{n,k}|}{4^k}$$

where

$$\mathcal{U}^+_{n,k}=\{(a_0,a_1,\cdots,a_k),\text{ such that } a_0=0,\ \|a_{i+1}-a_i\|=1,\ \forall i=0,1,\cdots,k-1,$$
$$a_j\in A_n\setminus\partial^{in}A_n,\forall j=1,2,\cdots,k-1,\ a_k\in\partial^{in}_{u,+}I_n\}$$

and

$$\mathcal{R}_{n,k}=\{(a_0,a_1,\cdots,a_k),\text{ such that } a_0=0,\ \|a_{i+1}-a_i\|=1,\ \forall i=0,1,\cdots,k-1,$$
$$a_j\in A_n\setminus\partial^{in}A_n,\forall j=1,2,\cdots,k-1,\ a_k\in\partial^{in}_{r}I_n\}$$

give the subsets of the random walk trajectories in events $\{S_{T_n}\in\partial^{in}_{u,+}I_n\}$ and $\{S_{T_n}\in\partial^{in}_{r}I_n\}$.

Thus in order to show (2.2.5), we construct a one-to-one mapping φ between the trajectories in $\mathcal{R}_{n,k}$ and $\mathcal{U}^+_{n,k}$. For any trajectory $\vec{a}=(a_0,a_1,\cdots,a_k)\in\mathcal{R}_{n,k}$, define

$$m(\vec{a})=\sup\left\{i\geq 0,\ a_i^{(1)}=a_i^{(2)}\right\}$$

to be the last point in the trajectory lying on the diagonal. Here $a_i^{(1)}$ and $a_i^{(2)}$ are the two coordinates of a_i. In this paper, we use the convention that $\sup\{\emptyset\}=-\infty$. Then it is easy to see that $0\in\left\{i\geq 0,\ a_i^{(1)}=a_i^{(2)}\right\}$ and thus $m(\vec{a})\geq 0$ and that $m(\vec{a})<k$. The reason of the latter inequality

is that suppose $m(\vec{a}) = k$, then we must have $a_k = (n, n)$ which implies that $a_{k-1} = (n-1, n)$ or $(n, n-1)$, which contradicts with the definition of $\vec{a}$.

Now we can define

$$\varphi(\vec{a}) = \vec{a}' = (a'_0, a'_1, \cdots, a'_k)$$

such that

- $a'_i = a_i$ for all $i \le m(\vec{a})$.
- $a'_i = \left(a_i^{(2)}, a_i^{(1)}\right)$ for all $i > m(\vec{a})$.

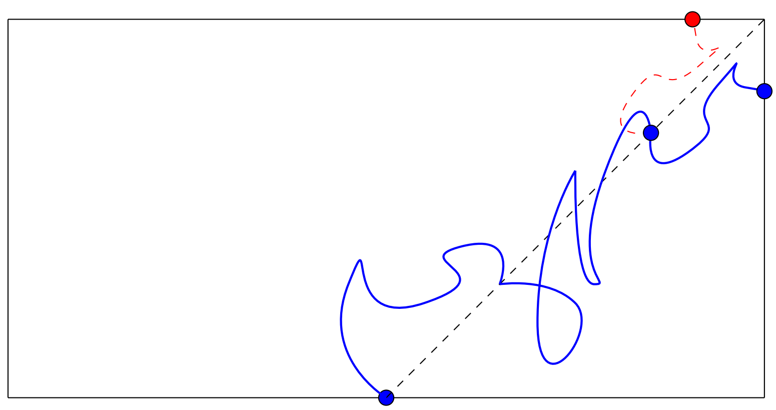

Figure 2.1: Mapping between trajectories in $\mathcal{R}_{n,k}$ and $\mathcal{U}^+_{n,k}$

I.e., we reflect the trajectory after the last time it visits the diagonal line $x = y$. See Figure 2.1 for illustration of the map φ. By definition

$$\left(a_{m(\vec{a})+1}, a_{m(\vec{a})+2}, \cdots, a_{k-1}\right)$$

stays within $\{(x, y) \in \mathbb{Z}^2,\ 0 < y < x < n\}$, while $a_k \in R_n$. Thus, under reflection we have

$$\left(a'_{m(\vec{a})+1}, a'_{m(\vec{a})+2}, \cdots, a'_{k-1}\right)$$

stays within $\{(x,y) \in \mathbb{Z}^2,\ 0 < x < y < n\}$, while $a'_k \in U^+_{n,k}$, which implies that $\vec{a}' \in \mathcal{U}^+_{n,k}$.

On the other hand, suppose we have two trajectories $\vec{a}$ and $\vec{b}$ both in $\mathcal{R}_{n,k}$, such that $\varphi(\vec{a}) = \varphi(\vec{b})$. Then one must have $m(\vec{a}) = m(\vec{b}) = m$ and that $a_i = b_i$ for all $i \le m$. Moreover, for all $i > m$, we have

$$\left(a_i^{(2)}, a_i^{(1)}\right) = a'_i = b'_i = \left(b_i^{(2)}, b_i^{(1)}\right)$$

which also implies that $a_i = b_i$. Thus we have shown that $\varphi(\vec{a}) = \varphi(\vec{b})$ if and only if $\vec{a} = \vec{b}$ and φ is a one-to-one mapping, which conclude the proof of this lemma. □

We define

$$F_m = F_{m,\alpha} = \{-\lfloor m^{1/\alpha} \rfloor, \lfloor m^{1/\alpha} \rfloor\} \times \mathbb{Z}_{\ge 0}$$

as two vertical lines on $\mathbb{H}$.

Lemma 2.2.5. *Fix $x \in \mathbb{H}$, then for all sufficiently large m,*

$$\mathbb{P}_x(\tau_{F_{m,\alpha}} < \tau_{L_0}) \le cm^{-1/\alpha}.$$

Proof. Let $m > 4|x_1|$, and $x' = (x^{(1)}, 0)$. There exists a constant $C > 0$ independent of m such that

$$C\mathbb{P}_x(\tau_{F_{m,\alpha}} < \tau_{L_0}) \le \mathbb{P}_{x'}(\tau_{F_{m,\alpha}} < \tau_{L_0}).$$

By translation invariance of simple random walk, we have

$$\mathbb{P}_{x'}(\tau_{F_{m,\alpha}} < \tau_{L_0}) \le \mathbb{P}_0(\tau_{I_{\lfloor m^{1/\alpha}/2 \rfloor}} < \tau_{L_0}).$$

By Lemma 2.2.4,

$$\mathbb{P}_0(\tau_{I_{\lfloor m^{1/\alpha}/2 \rfloor}} < \tau_{L_0}) \le 2\mathbb{P}_0(\tau_{L_{\lfloor m^{1/\alpha}/2 \rfloor}} < \tau_{L_0}) \le cm^{-1/\alpha}.$$

□

The next lemma claims that $\mathcal{H}_A$ is concentrated on the part arising from random walks starting from $y \in L_m$ such that $|y^{(1)}| \le \lfloor m^{1/\alpha} \rfloor$.

Lemma 2.2.6. *Let $A \subset \mathbb{H}$ be an infinite set that has polynomial sub-linear growth with parameter $\alpha \in (0,1)$. Let $1 > \alpha_1 = (\alpha+1)/2 > \alpha$, then for any $x \in \mathbb{H}$,*

$$\lim_{m\to\infty} \left| \sum_{y \in L_m, |y^{(1)}| \le \lfloor m^{1/\alpha_1} \rfloor} \mathbb{P}_y(S_{\tau_{A \cup L_0}} = x) - \mathcal{H}_{A,m}(x) \right| = 0.$$

Proof. Note that $\{y \in L_n, |y^{(1)}| \le \lfloor n^{1/\alpha_1} \rfloor\} \cap A = \emptyset$. Following the argument in [20, Lemma 2] on time reversibility and symmetry of simple random walk, we have

$$\begin{aligned} &\mathbb{P}_y(\tau_x = k, S_1, \cdots, S_{k-1} \notin \{x\} \cup L_0) \\ &= \mathbb{P}_x(\tau_y = k, S_1, \cdots, S_{k-1} \notin \{x\} \cup L_0) \\ &= \mathbb{P}_x(S_k = y, \tau_{\{x\} \cup L_0} > k). \end{aligned} \tag{2.2.6}$$

Then taking the summation over all k, we have

$$\begin{aligned} &\mathbb{P}_y(\tau_x \le \tau_{L_0}) \\ &= \sum_{k=1}^{\infty} \mathbb{P}_y(\tau_x = k, S_1, \cdots, S_{k-1} \notin \{x\} \cup L_0) \\ &= \sum_{k=1}^{\infty} \mathbb{P}_x(S_k = y, \tau_{\{x\} \cup L_0} > k) \\ &\le \mathbb{E}_x\left[\text{number of visits to } y \text{ in the time interval } [0, \tau_{\{x\} \cup L_0}) \right] \\ &\le \mathbb{E}_x\left[\text{number of visits to } y \text{ in the time interval } [0, \tau_{L_0}) \right] \end{aligned} \tag{2.2.7}$$

Then,

$$
\begin{aligned}
&\lim_{m\to\infty} \sum_{y\in L_m\setminus A, |y^{(1)}|\geq \lceil m^{1/\alpha_1}\rceil} \mathbb{P}_y(S_{\tau_A}=x) \\
&\leq \lim_{m\to\infty} \sum_{y\in L_m\setminus A, |y^{(1)}|\geq \lceil m^{1/\alpha_1}\rceil} \mathbb{P}_y(\tau_x \leq \tau_{L_0}) \\
&\leq \lim_{m\to\infty} \sum_{y\in L_m\setminus A, |y^{(1)}|\geq \lceil m^{1/\alpha_1}\rceil} \mathbb{E}_x\left[\text{number of visits to } y \text{ in the time interval } [0,\tau_{L_0})\right] \\
&\leq \lim_{m\to\infty} \mathbb{E}_x\left[\text{number of visits to } G_{m,\alpha_1} \text{ in the time interval } [0,\tau_{L_0})\right],
\end{aligned}
\tag{2.2.8}
$$

where $G_{m,\alpha_1} = \{y \in L_m : |y^{(1)}| \geq \lceil m^{1/\alpha_1}\rceil\}$. By Lemma 2.2.5, we have

$$
\begin{aligned}
&\lim_{m\to\infty} \left| \sum_{y\in L_m\setminus A, |y^{(1)}|\geq \lceil m^{1/\alpha_1}\rceil} \mathbb{P}_y(S_{\tau_A}=x)\right| \\
&\leq \lim_{m\to\infty} \mathbb{E}_x\left[\text{number of visits to } G_{m,\alpha_1} \text{ in the time interval } [0,\tau_{L_0})\right] \\
&\leq \lim_{m\to\infty} 4m\mathbb{P}_x(\tau_{G_{m,\alpha_1}} < \tau_{L_0}) \\
&\leq \lim_{m\to\infty} 4m\mathbb{P}_x(\tau_{F_{m,\alpha_1}} < \tau_{L_0}) \\
&= 0.
\end{aligned}
\tag{2.2.9}
$$

The proof is complete. □

Lemma 2.2.7. *Let $A \subset \mathbb{H}$ be an infinite set that has polynomial sub-linear growth with parameter $\alpha \in (0,1)$. Let $1 > \alpha_1 = (\alpha+1)/2 > \alpha$, then for all $x \in \mathbb{H}$ and for all $\epsilon > 0$ and for m and $n = n(m)$ large enough, we have*

$$
\left| \sum_{y\in L_m, |y^{(1)}|\leq \lfloor m^{1/\alpha_1}\rfloor} \mathbb{P}_y(S_{\tau_{A\cup L_0}}=x) - \widetilde{\mathcal{H}}_{A,n}(x)\right| < \epsilon.
$$

Proof. Fix $x \in \mathbb{H}$ and $\epsilon > 0$. Let $l = \max\{y^{(2)} : y \in A, y^{(2)} > |y^{(1)}|^{\alpha}\}$. Assume that n and m are large with $n > m > \max\{l, x^{(2)}\}$. Let $\alpha_1 = (\alpha+1)/2$ as defined in Lemma 2.2.6. By the strong

Markov property, we have

$$
\begin{aligned}
&\widetilde{\mathcal{H}}_{A,n}(x)\\
&= \pi n \mathbb{P}_{(0,n)}(S_{\tau_A} = x)\\
&= \sum_{y \in L_m \setminus A} \pi n \mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y)\mathbb{P}_y(S_{\tau_A} = x)\\
&\leq \sum_{y\in L_m, |y^{(1)}| \leq \lfloor m^{1/\alpha_1}\rfloor} \pi n \mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y)\mathbb{P}_y(S_{\tau_A} = x) + c \sum_{y \in L_m\setminus A, |y^{(1)}| \geq \lceil m^{1/\alpha_1}\rceil} \mathbb{P}_y(S_{\tau_A} = x),
\end{aligned}
\tag{2.2.10}
$$

where $c > 0$ is a constant. The last inequality of equation (2.2.10) is using Theorem 8.1.2 in [4] and the fact that

$$\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y) \leq \mathbb{P}_{(0,n)}(S_{\tau_{L_m}} = y).$$

By Lemma 2.2.6, we know

$$\lim_{m\to\infty} \sum_{y \in L_m\setminus A, |y^{(1)}| \geq \lceil m^{1/\alpha_1}\rceil} \mathbb{P}_y(S_{\tau_A} = x) = 0.$$

So there exists a $M_1 > \max\{l, x^{(2)}\}$ such that for all $m > M_1$ and all sufficiently large $n > m$,

$$\left| \widetilde{\mathcal{H}}_{A,n}(x) - \sum_{y\in L_m, |y^{(1)}| \leq \lfloor m^{1/\alpha_1}\rfloor} \pi n \mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y)\mathbb{P}_y(S_{\tau_A} = x) \right| < \frac{\epsilon}{2}.$$

Denote the set

$$\widetilde{A}_m = \{x \in \mathbb{H} : x^{(1)} > \lfloor m^{1/\alpha}\rfloor, m \leq x^{(2)} \leq |x^{(1)}|^{\alpha}\}.$$

Note that $\widetilde{A}_m$ contains the part of A that is above the horizontal line L_m. For $y \in L_m$ such that $|y^{(1)}| \leq m^{1/\alpha_1}$,we have

$$\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y) \leq \mathbb{P}_{(0,n)}(S_{\tau_{L_m}} = y) \tag{2.2.11}$$

while

$$\begin{aligned}\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}} = y) &\geq \mathbb{P}_{(0,n)}(S_{\tau_{\widetilde{A}_m\cup L_m}} = y) \\ &= \mathbb{P}_{(0,n)}(S_{\tau_{L_m}} = y) - \sum_{z\in\widetilde{A}_m} \mathbb{P}_{(0,n)}(S_{\tau_{\widetilde{A}_m\cup L_m}} = z)\mathbb{P}_z(S_{\tau_{L_m}} = y).\end{aligned} \tag{2.2.12}$$

Note that for $z \in \widetilde{A}_m$, $\mathbb{P}_{(0,n)}(S_{\tau_{\widetilde{A}_m\cup L_m}} = z) = 0$ unless z is in the upper inner boundary of $\widetilde{A}_m$, i.e. $z = (k, \lfloor k^\alpha \rfloor) \in \partial^{\text{in}}\widetilde{A}_m$ for some $k > \lfloor m^{1/\alpha} \rfloor$. Suppose $z = (k, \lfloor k^\alpha \rfloor) \in \partial^{\text{in}}\widetilde{A}_m$ with $k > \lfloor m^{1/\alpha} \rfloor$. Let $y \in L_m$ such that $|y^{(1)}| \leq m^{1/\alpha_1}$. By Theorem 8.1.2 in Lawler and Limic [4], we have

$$\begin{aligned}&\mathbb{P}_z(S_{\tau_{L_m}} = y) \\ &\leq \frac{c(\lfloor k^\alpha \rfloor - m)}{(\lfloor k^\alpha \rfloor - m)^2 + (k - \lfloor m^{1/\alpha_1} \rfloor)^2} \\ &\leq \frac{c(k^\alpha - m)}{(\lfloor k^\alpha \rfloor - m)^2 + (k - m^{1/\alpha_1})^2}.\end{aligned} \tag{2.2.13}$$

So,

$$\begin{aligned}&\sum_{z\in\widetilde{A}_m} \mathbb{P}_{(0,n)}(S_{\tau_{\widetilde{A}_m\cup L_m}} = z)\mathbb{P}_z(S_{\tau_{L_m}} = y) \\ &\leq \sum_{z\in\widetilde{A}_m} \mathbb{P}_z(S_{\tau_{L_m}} = y) \\ &\leq c \sum_{k=\lceil m^{1/\alpha} \rceil}^{\infty} \frac{k^\alpha - m}{(\lfloor k^\alpha \rfloor - m)^2 + (k - m^{1/\alpha_1})^2} \\ &\leq c \sum_{s=1}^{\infty} \frac{(s + m^{1/\alpha} + 1)^\alpha - m}{(\lfloor (s + \lfloor m^{1/\alpha} \rfloor)^\alpha \rfloor - m)^2 + (s + m^{1/\alpha} - m^{1/\alpha_1})^2}.\end{aligned} \tag{2.2.14}$$

It's easy to see that the sum above converges and goes to 0 if m goes to infinity. Moreover, let's consider the sum

$$S := cm^{3/(2\alpha)-1/2} \sum_{s=1}^{\infty} \frac{(s + m^{1/\alpha} + 1)^\alpha - m}{(\lfloor (s + \lfloor m^{1/\alpha} \rfloor)^\alpha \rfloor - m)^2 + (s + m^{1/\alpha} - m^{1/\alpha_1})^2}.$$

Note that

$$\begin{aligned}&cm^{3/(2\alpha)-1/2}\sum_{s=1}^{\infty}\frac{(s+m^{1/\alpha}+1)^{\alpha}-m}{(\lfloor(s+\lfloor m^{1/\alpha}\rfloor)^{\alpha}\rfloor-m)^2+(s+m^{1/\alpha}-m^{1/\alpha_1})^2}\\&\leq cm^{3/(2\alpha)-1/2}\sum_{s=1}^{\infty}\frac{(s+m^{1/\alpha}+1)^{\alpha}-m}{(s+m^{1/\alpha}-m^{1/\alpha_1})^2}.\end{aligned} \tag{2.2.15}$$

For all $0<\alpha<1$, there is a $M>0$ large enough such that for all $s>0$ and $m'>M$,

$$\frac{\partial}{\partial m}\left(cm^{3/(2\alpha)-1/2}\sum_{s=1}^{\infty}\frac{(s+m^{1/\alpha}+1)^{\alpha}-m}{(s+m^{1/\alpha}-m^{1/\alpha_1})^2}\right)\Bigg|_{m=m'}<0.$$

So the sum S goes to 0 if m goes to infinity. Hence, we can take $n=\lfloor m^{3/(2\alpha)-1/2}\rfloor$. Note that $3/(2\alpha)-1/2>1/\alpha$. Then for any $y\in L_m$ with $|y^{(1)}|\leq\lfloor m^{1/\alpha_1}\rfloor$, we have

$$\lim_{m\to\infty}n\sum_{z\in\widetilde{A}_m}\mathbb{P}_{(0,n)}(S_{\tau_{\widetilde{A}_m\cup L_m}}=z)\mathbb{P}_z(S_{\tau_{L_m}}=y)=0,$$

and

$$\lim_{m\to\infty}\pi n\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}}=y)=1.$$

Now fix $N>\max\{l,x_2\}$. From the proof of Theorem 1 in [18], we know that the sequence $H_{A,j}(x)$ is decreasing for $j\geq N$. There exists a $M_2>N$ such that for all $m>M_2$,

$$\left|\pi n\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}}=y)-1\right|<\frac{\epsilon}{2H_{A,N}(x)}.$$

Therefore,

$$\left|\sum_{y\in L_m,|y^{(1)}|\leq\lfloor m^{1/\alpha_1}\rfloor}\Big(\pi n\mathbb{P}_{(0,n)}(S_{\tau_{A\cup L_m}}=y)-1\Big)\mathbb{P}_y(S_{\tau_A}=x)\right|<\frac{\epsilon}{2}.$$

Now take $m>\max\{M_1,M_2\}$, and the proof is complete. □

The following theorem is a direct consequence of Lemma 2.2.6 and Lemma 2.2.7.

Theorem 2.2.8. *Let $A \subset \mathbb{H}$ be an infinite set that has polynomial sub-linear growth. For any $x \in \mathbb{H}$,*

$$\widetilde{\mathcal{H}}_A(x) := \lim_{N\to\infty} \widetilde{\mathcal{H}}_{A,N}(x)$$

exists, and $\widetilde{\mathcal{H}}_A(x) = \mathcal{H}_A(x)$.

Proof. Let $\epsilon > 0$. By Lemma 2.2.6 and Lemma 2.2.7, there is an $M > 0$ such that for all $m > M$,

$$|\mathcal{H}_{A,m}(x) - \widetilde{\mathcal{H}}_{A,m}(x)| < \epsilon.$$

We know

$$\lim_{m\to\infty} \mathcal{H}_{A,m}(x) = \mathcal{H}_A(x).$$

Hence,

$$\widetilde{\mathcal{H}}_A(x) := \lim_{m\to\infty} \widetilde{\mathcal{H}}_{A,m}(x)$$

exists and $\widetilde{\mathcal{H}}_A(x) = \mathcal{H}_A(x)$. □

2.3 Stationary Harmonic Measure is the Scaling Limit of Truncated Harmonic Measure

In this section, we show the asymptotic equivalence between the stationary harmonic measure of any given point with respect to subset A satisfying Definition 2.1.2 and the rescaled regular harmonic measure of the same point with respect to the truncations of A.

Theorem 2.3.1. *For any subset A satisfying Definition 2.1.2 and any positive integer n, let*

$$A_n = A \cap \Big\{[-n, n] \times \mathbb{Z}\Big\} \tag{2.3.1}$$

be the truncation of A with width $2n$. There is a constant $C \in (0, \infty)$, independent of the set A, such that any point $x \in A \setminus L_0$,

$$C \lim_{n\to\infty} n\mathtt{H}_{A_n}(x) = \mathcal{H}_A(x). \tag{2.3.2}$$

Moreover, $C = 2/\lim_{n\to\infty} n\mathtt{H}_{D_n}(0)$*, where* $D_n = \{[-n,n]\cap\mathbb{Z}\}\times\{0\}$.

Remark 2.3.2. For points in L_0, we can replace the regular harmonic measure $\mathtt{H}_{A_n}(x)$ in (2.3.2) by its edge version. I.e., we have for all $x \in L_0$,

$$C \lim_{n\to\infty} \lim_{\|y\|\to\infty} n\mathbb{P}_y\left(S_{\tau_{A_n}} = x, S^{(2)}_{\tau_{A_n}-1} > 0\right) = \mathcal{H}_A(x). \tag{2.3.3}$$

Later one can see the proof of (2.3.3) follows exactly the same argument as the one for (2.3.2).

In order to prove Theorem 2.3.1, we first show its special case when $A = L_0$. We denote the truncation of L_0 with width $2n$ by $D_n = \{[-n,n]\cap\mathbb{Z}\}\times\{0\}$.

Theorem 2.3.3. *There is a constant* $c \in (0,\infty)$ *such that*

$$\lim_{n\to\infty} n\mathtt{H}_{D_n}(0) = c. \tag{2.3.4}$$

The structure of this section is as follows: In subsections 2.3.1 and 2.3.2 we outline the proof of Theorem 2.3.3 and Theorem 2.3.1. Then in the following subsections, we give the detailed proof of the required propositions and lemmas.

2.3.1 Proof of Theorem 2.3.3

Theorem 2.3.3 can be proved according to the following outline: first, we show that $n\mathtt{H}_{D_n}(0)$ has finite and positive upper and lower limits:

Proposition 2.3.4. *There is a constant* $C \in (0,\infty)$ *such that*

$$\limsup_{n\to\infty} n\mathtt{H}_{D_n}(0) \le C. \tag{2.3.5}$$

Proposition 2.3.5. *There is a constant* $c \in (0,\infty)$ *such that*

$$\liminf_{n\to\infty} n\mathtt{H}_{D_n}(0) \ge c. \tag{2.3.6}$$

The two propositions above guarantee that the decaying rate of $\mathtt{H}_{D_n}(0)$ is of order $1/n$. To show $\limsup = \liminf$, we further show the following coupling result:

Proposition 2.3.6. *For any $\epsilon > 0$ there is a $\delta > 0$ such that for all sufficiently large n and any $x \in [-\delta n, \delta n] \times \{0\}$, we have*

$$\left|\mathtt{H}_{D_n}(0) - \mathtt{H}_{D_n}(x)\right| < \frac{\epsilon}{n} \tag{2.3.7}$$

Let $\bar{B}(0, R) = \{x \in \mathbb{R}^2 : ||x||_2 < R\}$ be the continuous ball of radius R in $\mathbb{R}^2$. For standard Brownian motion $B(t)$ and subset $A \subset \mathbb{R}^2$, define the stopping time

$$T_A = \inf\{t \geq 0,\ B(t) \in A\}.$$

For subset $A \subset \mathbb{R}^2$, H_A denotes the continuous harmonic measure with respect to A.

Lemma 2.3.7. *Fix $\delta \in (0, 1)$, then*

$$\lim_{n\to\infty} \mathtt{H}_{D_n}\left([-\delta n, \delta n] \times \{0\}\right) = \mathsf{H}_{[-1,1]\times\{0\}}([-\delta, \delta] \times \{0\}).$$

Once one has shown Proposition 2.3.4-2.3.7, the proof of Theorem 2.3.3 is mostly straightforward. Now suppose the limit in (2.3.4) does not exist. Then by Proposition 2.3.4 we must have

$$0 < \liminf_{n\to\infty} n\mathtt{H}_{D_n}(0) < \limsup_{n\to\infty} n\mathtt{H}_{D_n}(0) < \infty. \tag{2.3.8}$$

Let

$$\epsilon_0 = \frac{\limsup_{n\to\infty} n\mathtt{H}_{D_n}(0) - \liminf_{n\to\infty} n\mathtt{H}_{D_n}(0)}{5} > 0.$$

By Proposition 2.3.6, we have there are $\delta_0 > 0$ and $N_0 < \infty$ such that for all $n > N_0$ and any $x \in [-\delta_0 n, \delta_0 n] \times \{0\}$,

$$\left|\mathtt{H}_{D_n}(0) - \mathtt{H}_{D_n}(x)\right| < \frac{\epsilon_0}{n}.$$

Moreover, for any $N > N_0$, there are $n_1, n_2 > N$ such that

$$n_1 \mathtt{H}_{D_{n_1}}(0) < \liminf_{n\to\infty} n\mathtt{H}_{D_n}(0) + \epsilon_0$$

and that

$$n_2 \mathtt{H}_{D_{n_2}}(0) > \limsup_{n\to\infty} n\mathtt{H}_{D_n}(0) - \epsilon_0.$$

At the same time, we have for the $\delta_0 > 0$ defined above,

$$\begin{aligned} \mathtt{H}_{D_{n_1}}\left([-\delta_0 n_1, \delta_0 n_1] \times \{0\}\right) &= \sum_{x\in[-\delta_0 n_1, \delta_0 n_1]\times\{0\}} \mathtt{H}_{D_{n_1}}(x) \\ &\le \frac{\lfloor \delta_0 n_1 \rfloor + 1}{n_1} \left[\liminf_{n\to\infty} n\mathtt{H}_{D_n}(0) + 2\epsilon_0\right] \end{aligned} \tag{2.3.9}$$

and

$$\begin{aligned} \mathtt{H}_{D_{n_2}}\left([-\delta_0 n_2, \delta_0 n_2] \times \{0\}\right) &= \sum_{x\in[-\delta_0 n_2, \delta_0 n_2]\times\{0\}} \mathtt{H}_{D_{n_2}}(x) \\ &\ge \frac{\lfloor \delta_0 n_2 \rfloor + 1}{n_2} \left[\limsup_{n\to\infty} n\mathtt{H}_{D_n}(0) - 2\epsilon_0\right]. \end{aligned} \tag{2.3.10}$$

But by Lemma 2.3.7,

$$\lim_{n\to\infty} \mathtt{H}_{D_n}\left([-\delta_0 n, \delta_0 n] \times \{0\}\right) = \mathsf{H}_{[-1,1]\times\{0\}}([-\delta_0, \delta_0] \times \{0\}),$$

which contradicts with (2.3.9) and (2.3.10). □

2.3.2 Proof of Theorem 2.3.1

Define $\alpha_1 = (1+\alpha)/2 \in (0,1)$ and $Box(n) = [-n, n] \times [0, \lfloor n^{\alpha_1} \rfloor]$. Recalling the definition of regular harmonic measure, and the fact that $A_n \subset Box(n)$ for all sufficiently large n, we have for any $x \in A \setminus L_0$,

$$\mathtt{H}_{A_n}(x) = \sum_{y\in\partial^{in} Box(n)} \mathtt{H}_{Box(n)}(y) \mathbb{P}_y\left(S_{\bar\tau_{A_n}} = x\right).$$

Then define

$$\partial_u^{in} Box(n) = [-n,n] \times \{\lfloor n^{\alpha_1} \rfloor\}$$

$$\partial_d^{in} Box(n) = [-n,n] \times \{0\}$$

$$\partial_l^{in} Box(n) = \{-n\}, \times [1, \lfloor n^{\alpha_1} \rfloor - 1]$$

$$\partial_r^{in} Box(n) = \{n\}, \times [1, \lfloor n^{\alpha_1} \rfloor - 1]$$

to be the four edges of $\partial^{in} Box(n)$. Noting that $L_0 \subset A$, it is easy to see that for any $y \in \partial_d^{in} Box(n) = [-n,n] \times \{0\}$, $\mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) = 0$. Moreover, define $\alpha_2 = (7+\alpha)/8$, and

$$l_n = [-\lfloor n^{\alpha_2} \rfloor, \lfloor n^{\alpha_2} \rfloor] \times \{\lfloor n^{\alpha_1} \rfloor\}$$

to be the middle section of $\partial_u^{in} Box(n)$ and denote $l_n^c = \partial_l^{in} Box(n) \cup \partial_r^{in} Box(n) \cup \partial_u^{in} Box(n) \setminus l_n$. We further have the decomposition as follows:

$$\mathtt{H}_{A_n}(x) = \sum_{y \in l_n^c} \mathtt{H}_{Box(n)}(y) \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) + \sum_{y \in l_n} \mathtt{H}_{Box(n)}(y) \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right). \tag{2.3.11}$$

From (2.3.11), we first note that $\mathtt{H}_{Box(n)}(y)$ sums up to 1, which implies that

$$\sum_{y \in l_n^c} \mathtt{H}_{Box(n)}(y) \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) \le \max_{y \in l_n^c} \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right). \tag{2.3.12}$$

Thus our first step is to prove

Proposition 2.3.8. *For $Box(n)$, l_n and l_n^c defined as above, we have*

$$\lim_{n \to \infty} n \cdot \max_{y \in l_n^c} \mathbb{P}_y\left(S\bar{\tau}_{A_n} = x\right) = 0. \tag{2.3.13}$$

With Proposition 2.3.8, it sufficient for us to concentrate on the asymptotic of

$$\sum_{y \in l_n} \mathtt{H}_{Box(n)}(y) \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right).$$

We are to show that

Proposition 2.3.9. *For any $x \in A$ and the truncations A_n defined in* (2.3.1)

$$\lim_{n\to\infty} \sum_{y \in l_n} \mathbb{P}_y \left(S_{\bar{\tau}_{A_n}} = x \right) = \mathcal{H}_A(x). \tag{2.3.14}$$

and that

Proposition 2.3.10. *For any $\epsilon > 0$, there is a $N_0 < \infty$ such that for all $n \geq N_0$ and all $y \in l_n$,*

$$\left| 2\mathtt{H}_{Box(n)}(y) - \mathtt{H}_{D_n}(0) \right| < \epsilon/n. \tag{2.3.15}$$

Once we have proved the lemmas above, Theorem 2.3.1 follows immediately from the combination of Proposition 2.3.8- 2.3.10, together with Theorem 2.3.3. □

2.3.3 Existence of upper and lower limit

2.3.3.1 Bounds between harmonic measure and escaping probability

In this subsection we prove Proposition 2.3.4 and 2.3.5. First, recalling the notation

$$\mathtt{H}_D(y, x) = \mathbb{P}_y(\tau_D = \tau_x),$$

with standard time reversibility argument, see Lemma 2 of [20], we have for any n and $x \in D_n$

$$\begin{aligned} \mathtt{H}_{D_n}(x) &= \lim_{R\to\infty} \frac{1}{|\partial^{out} B(0,R)|} \sum_{y \in \partial^{out} B(0,R)} \mathtt{H}_{D_n}(y, x) \\ &= \lim_{R\to\infty} \frac{1}{|\partial^{out} B(0,R)|} \mathbb{E}_x \left[\text{number of visits to } \partial^{out} B(0,R) \text{ in } [0, \tau_{D_n}) \right]. \end{aligned}$$

Note that there is a finite constant C independent to R such that

$$\frac{1}{|\partial^{out} B(0,R)|} \leq \frac{C}{R}.$$

At the same time, define $C_n = [-\lfloor n/2 \rfloor, 0] \times \{0\} \subset D_n$ and apply Lemma 3-4 of [20] with $r = n$,

$$\begin{aligned}
&\mathbb{E}_x \left[\text{number of visits to } \partial^{out} B(0,R) \text{ in } [0, \tau_{D_n})\right] \\
&\leq \frac{\mathbb{P}_x(\tau_R < \tau_{D_n})}{\min_{w \in \partial^{out} B(0,R)} \mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)} \\
&\leq CR\log(R)\mathbb{P}_x(\tau_R < \tau_{D_n}) \\
&= CR\log(R)\left(\sum_{z \in \partial^{out} B(0,2n)} \mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right)\mathbb{P}_z(\tau_R < \tau_{D_n})\right) \\
&\leq CR\log(R)\left(\sum_{z \in \partial^{out} B(0,2n)} \mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right)\mathbb{P}_z(\tau_R < \tau_{C_n})\right) \\
&\leq CR\log(R)\mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}\right) \max_{z \in \partial^{out} B(0,2n)} \mathbb{P}_z(\tau_R < \tau_{C_n}) \\
&\leq CR\mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}\right).
\end{aligned}$$

Thus, there is a finite constant C independent to n such that

$$\mathtt{H}_{D_n}(x) \leq C\mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}\right). \tag{2.3.16}$$

On the other hand, by Lemma 3.2 of [18], there is a constant $C < \infty$ independent to the choice of n and $R \gg n$ such that for all $w \in \partial^{out} B(0,R)$

$$\mathbb{P}_w(\tau_{D_n} < \tau_R) \leq C[R\log(R)]^{-1}. \tag{2.3.17}$$

Thus

$$\begin{aligned}
&\mathbb{E}_x \left[\text{number of visits to } \partial^{out} B(0,R) \text{ in } [0, \tau_{D_n})\right] \\
&\geq \frac{\mathbb{P}_x(\tau_R < \tau_{D_n})}{\max_{w \in \partial^{out} B(0,R)} \mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)} \\
&\geq cR\log(R)\mathbb{P}_x(\tau_R < \tau_{D_n}).
\end{aligned}$$

At the same time, by Lemma 3.3 of [18], there are constants $2 < c_0 < \infty$ and $c > 0$ independent to the choice of n and $R \gg n$ such that for any $z \in \partial^{out} B(0, c_0 n)$

$$\mathbb{P}_z(\tau_R < \tau_{D_n}) \geq \frac{c}{\log(R)}. \tag{2.3.18}$$

Thus we have

$$\begin{aligned}\mathbb{P}_x(\tau_R < \tau_{D_n}) = &\sum_{z \in \partial^{out} B(0,c_0 n)} \mathbb{P}_x\left(\tau_{c_0 n} < \tau_{D_n}, S_{\tau_{c_0 n}} = z\right) \mathbb{P}_z(\tau_R < \tau_{D_n}) \\ &\geq cR\mathbb{P}_x\left(\tau_{c_0 n} < \tau_{D_n}\right).\end{aligned}$$

which implies that

$$\mathtt{H}_{D_n}(x) \geq c\mathbb{P}_x\left(\tau_{c_0 n} < \tau_{D_n}\right). \tag{2.3.19}$$

2.3.3.2 Proof of Proposition 2.3.4

With Lemma 2.2.4 and recalling the fact that $I_n \subset B(0, 2n)$, we have that

$$\begin{aligned}\mathbb{P}_0(\tau_{2n} < \tau_{D_n}) \leq \mathbb{P}_0(\tau_{I_n} < \tau_{D_n}) &\\ = \mathbb{P}_0\left(S_{T_n} \in L_n \cup \partial_r^{in} I_n \cup \partial_u^{in} I_n\right) &\\ \leq 2\mathbb{P}_0\left(S_{T_n} \in \partial_u^{in} I_n\right). &\end{aligned} \tag{2.3.20}$$

Moreover, note that

$$\mathbb{P}_0\left(S_{T_n} \in \partial_u^{in} I_n\right) \leq \mathbb{P}_0\left(\tau_{L_n} < \tau_{L_0}\right) = \frac{1}{4n}. \tag{2.3.21}$$

Thus by (2.3.16), (2.3.20) and (2.3.21), the proof of Proposition 2.3.4 is complete. □

2.3.3.3 Proof of Proposition 2.3.5

With (2.3.19), in order to Proposition 2.3.5, it is sufficient to show that

Lemma 2.3.11. *For any $k \geq 2$, there is a $c_k > 0$ such that*

$$\mathbb{P}_0\left(\tau_{kn} < \tau_{D_n}\right) \geq \frac{c_k}{n}.$$

Proof. Note that for a simple random walk starting from 0, it is easy to see that

$$\tau_{kn} \leq \tau_{L_{kn}},\ \tau_{L_0} \leq \tau_{D_n}.$$

Thus we have

$$\mathbb{P}_0\left(\tau_{kn} < \tau_{D_n}\right) \geq \mathbb{P}_0\left(\tau_{L_{kn}} < \tau_{L_0}\right) = \frac{1}{4kn}$$

and the proof of this lemma is complete. □

With Lemma 2.3.11, the proof of Proposition 2.3.5 is complete. □

2.3.4 Proof of Proposition 2.3.6

For the proof of Proposition 2.3.6, we without loss of generality assume that the first coordinate of x is an even number, see Remark 2.3.13 for details. With Proposition 2.3.4 and 2.3.5, by spatial translation it is easy to see there are constants $0 < c < C < \infty$ such that for all $x \in [-n/2, n/2]$

$$\frac{c}{n} < \mathtt{H}_{D_n}(x) < \frac{C}{n}. \tag{2.3.22}$$

Moreover, recall that

$$\begin{aligned}\mathtt{H}_{D_n}(x) &= \lim_{R\to\infty} \frac{1}{|\partial^{out}B(0,R)|} \sum_{y\in\partial^{out}B(0,R)} \mathtt{H}_{D_n}(y,x) \\ &= \lim_{R\to\infty} \frac{1}{|\partial^{out}B(0,R)|} \mathbb{E}_x\left[\text{number of visits to } \partial^{out}B(0,R) \text{ in } [0,\tau_{D_n})\right].\end{aligned}$$

Thus for any n and x, there has to be a R_0 such that for all $R \geq R_0$,

$$\left|\mathtt{H}_{D_n}(x) - \frac{1}{|\partial^{out}B(0,R)|}\mathbb{E}_x\left[\text{number of visits to } \partial^{out}B(0,R) \text{ in } [0,\tau_{D_n})\right]\right| < \frac{\epsilon}{4n}$$

and

$$\left|\mathtt{H}_{D_n}(0) - \frac{1}{|\partial^{out} B(0,R)|}\mathbb{E}_0\left[\text{number of visits to } \partial^{out}B(0,R) \text{ in } [0,\tau_{D_n})\right]\right| < \frac{\epsilon}{4n}.$$

At the same time

$$\begin{aligned}&\mathbb{E}_x\left[\text{number of visits to } \partial^{out}B(0,R) \text{ in } [0,\tau_{D_n})\right]\\ &= \sum_{z\in\partial^{out}B(0,2n)} \mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right) \sum_{w\in\partial^{out}B(0,R)} \frac{\mathbb{P}_z\left(\tau_R < \tau_{D_n}, S_{\tau_R} = w\right)}{\mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_0\left[\text{number of visits to } \partial^{out}B(0,R) \text{ in } [0,\tau_{D_n})\right]\\ &= \sum_{z\in\partial^{out}B(0,2n)} \mathbb{P}_0\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right) \sum_{w\in\partial^{out}B(0,R)} \frac{\mathbb{P}_z\left(\tau_R < \tau_{D_n}, S_{\tau_R} = w\right)}{\mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)}.\end{aligned}$$

Thus we have

$$\begin{aligned}&|\mathtt{H}_{D_n}(x) - \mathtt{H}_{D_n}(0)|\\ &\le \frac{1}{|\partial^{out}B(0,R)|} \sum_{z\in\partial^{out}B(0,2n)} \left|\mathbb{P}_0\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right) - \mathbb{P}_x\left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z\right)\right|\\ &\qquad \cdot \left(\sum_{w\in\partial^{out}B(0,R)} \frac{\mathbb{P}_z\left(\tau_R < \tau_{D_n}, S_{\tau_R} = w\right)}{\mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)}\right) + \frac{\epsilon}{2n}.\end{aligned} \tag{2.3.23}$$

Again by Lemma 3-4 of [20] with $r = n$, we have there is a constant $C < \infty$ such that for all n, $R \gg n$ and $z \in \partial^{out}B(0,2n)$

$$\begin{aligned}&\frac{1}{|\partial^{out}B(0,R)|}\left(\sum_{w\in\partial^{out}B(0,R)} \frac{\mathbb{P}_z\left(\tau_R < \tau_{D_n}, S_{\tau_R} = w\right)}{\mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)}\right)\\ &\le \frac{\mathbb{P}_z\left(\tau_R < \tau_{D_n}\right)}{|\partial^{out}B(0,R)| \min_{w\in\partial^{out}B(0,R)} \mathbb{P}_w\left(\tau_{D_n} < \tau_R\right)} \le C.\end{aligned} \tag{2.3.24}$$

Thus by (2.3.23) and (2.3.24), in order to prove Proposition 3.3.1, it suffices to show the following lemma:

Lemma 2.3.12. *For any $\epsilon > 0$ there is a $\delta > 0$ such that for all sufficiently large n and any*

$x \in [-\delta n, \delta n] \times \{0\}$, *we have*

$$\sum_{z \in \partial^{out} B(0,2n)} \left| \mathbb{P}_0 \left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z \right) - \mathbb{P}_x \left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z \right) \right| < \frac{\epsilon}{n}. \tag{2.3.25}$$

Proof. For any $\epsilon > 0$, define $\delta = e^{-\epsilon^{-1}} > 0$. In order to prove this lemma, we construct the following coupling between the simple random walk starting from 0 and $x \in [-\delta n, \delta n] \times \{0\}$:

(i) Define subset $A_n^\epsilon = [-\lfloor n/2 \rfloor, \lfloor n/2 \rfloor] \times [0, \lfloor \epsilon n \rfloor]$.

(ii) Let $\{\bar{S}_k\}_{k=0}^\infty$ be a simple random walk starting from 0, $\bar{T}_n^\epsilon = \inf\{k : \bar{S}_k \in \partial^{in} A_n^\epsilon\}$, and $x_n^\epsilon = \bar{S}_{\bar{T}_n^\epsilon}$.

(iii) For $k \leq \bar{T}_n^\epsilon$, let $S_{1,k} = \bar{S}_k$ and $S_{2,k} = \bar{S}_k + x$.

(iv) Let $\left\{\hat{S}_{1,k}\right\}_{k=0}^\infty$ and $\left\{\hat{S}_{2,k}\right\}_{k=0}^\infty$ be two simple random walks starting from x_n^ϵ and $x_n^\epsilon + x$ and coupled under the maximal coupling.

(v) For $k > \bar{T}_n^\epsilon$, let $S_{1,k} = \hat{S}_{1,k-T_n^\epsilon}$ and $S_{2,k} = \hat{S}_{2,k-T_n^\epsilon}$.

Remark 2.3.13. In Step (iv) we use the assumption that the first coordinate of x is an even number. Otherwise, one can construct $\hat{S}_{1,k}$ starting from x_n^ϵ and $\hat{S}_{2,k}$ starting uniformly from $B(x_n^\epsilon + x, 1)$ under maximal coupling.

By the strong Markov property, it is easy to see that $S_{1,k}$ and $S_{2,k}$ form two simple random walks starting from 0 and x. Let $\tau_\cdot^{(1)}$ and $\tau_\cdot^{(2)}$ be the stopping time with respect to $S_{1,k}$ and $S_{2,k}$ respectively. Thus

$$\begin{aligned} &\sum_{z \in \partial^{out} B(0,2n)} \left| \mathbb{P}_0 \left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z \right) - \mathbb{P}_x \left(\tau_{2n} < \tau_{D_n}, S_{\tau_{2n}} = z \right) \right| \\ = &\sum_{z \in \partial^{out} B(0,2n)} \left| \mathbb{P}_0 \left(\tau_{2n}^{(1)} < \tau_{D_n}^{(1)}, S_{1,\tau_{2n}^{(1)}} = z \right) - \mathbb{P}_x \left(\tau_{2n}^{(2)} < \tau_{D_n}^{(2)}, S_{2,\tau_{2n}^{(2)}} = z \right) \right|. \end{aligned}$$

Again we introduce

$$U_n^\epsilon = [-\lfloor n/2 \rfloor, \lfloor n/2 \rfloor] \times \lfloor \epsilon n \rfloor, \;\; B_n^\epsilon = [-\lfloor n/2 \rfloor, \lfloor n/2 \rfloor] \times 0$$

and

$$L_n^\epsilon = -\lfloor n/2 \rfloor \times [1, \lfloor \epsilon n \rfloor - 1] \;\; R_n^\epsilon = \lfloor n/2 \rfloor \times [1, \lfloor \epsilon n \rfloor - 1]$$

as the four edges of $\partial^{in} A_n^\epsilon$. Note that for all $\epsilon < 1/3$

$$\left\{ \tau_{2n}^{(1)} < \tau_{D_n}^{(1)} \right\} \cap \left\{ \bar{S}_{\bar{T}_n^\epsilon} \in B_n^\epsilon \right\} = \emptyset, \quad \left\{ \tau_{2n}^{(2)} < \tau_{D_n}^{(2)} \right\} \cap \left\{ \bar{S}_{\bar{T}_n^\epsilon} \in B_n^\epsilon \right\} = \emptyset.$$

Thus for any $z \in \partial^{out} B(0, 2n)$, we have

$$\begin{aligned} \mathbb{P}_0 \left(\tau_{2n}^{(1)} < \tau_{D_n}^{(1)}, S_{1,\tau_{2n}^{(1)}} = z \right) =& \mathbb{P}_0 \left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon, \; \tau_{2n}^{(1)} < \tau_{D_n}^{(1)}, \; S_{1,\tau_{2n}^{(1)}} = z \right) \\ &+ \mathbb{P}_0 \left(\bar{S}_{\bar{T}_n^\epsilon} \in L_n^\epsilon \cup R_n^\epsilon, \; \tau_{2n}^{(1)} < \tau_{D_n}^{(1)}, \; S_{1,\tau_{2n}^{(1)}} = z \right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_x \left(\tau_{2n}^{(2)} < \tau_{D_n}^{(2)}, S_{2,\tau_{2n}^{(2)}} = z \right) =& \mathbb{P}_x \left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon, \; \tau_{2n}^{(2)} < \tau_{D_n}^{(2)}, \; S_{2,\tau_{2n}^{(2)}} = z \right) \\ &+ \mathbb{P}_x \left(\bar{S}_{\bar{T}_n^\epsilon} \in L_n^\epsilon \cup R_n^\epsilon, \; \tau_{2n}^{(2)} < \tau_{D_n}^{(2)}, \; S_{2,\tau_{2n}^{(2)}} = z \right). \end{aligned}$$

Thus we have

$$
\begin{aligned}
&\sum_{z\in\partial^{out}B(0,2n)}\left|\mathbb{P}_0\left(\tau_{2n}^{(1)}<\tau_{D_n}^{(1)},S_{1,\tau_{2n}^{(1)}}=z\right)-\mathbb{P}_x\left(\tau_{2n}^{(2)}<\tau_{D_n}^{(2)},S_{2,\tau_{2n}^{(2)}}=z\right)\right|\\
\le&\sum_{z\in\partial^{out}B(0,2n)}\left|\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in U_n^\epsilon,\ \tau_{2n}^{(1)}<\tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}}=z\right)-\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in U_n^\epsilon,\ \tau_{2n}^{(2)}<\tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}}=z\right)\right|\\
+&\sum_{z\in\partial^{out}B(0,2n)}\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in L_n^\epsilon\cup R_n^\epsilon,\ \tau_{2n}^{(1)}<\tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}}=z\right)\\
+&\sum_{z\in\partial^{out}B(0,2n)}\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in L_n^\epsilon\cup R_n^\epsilon,\ \tau_{2n}^{(2)}<\tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}}=z\right)\\
\le&\sum_{z\in\partial^{out}B(0,2n)}\left|\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in U_n^\epsilon,\ \tau_{2n}^{(1)}<\tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}}=z\right)-\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in U_n^\epsilon,\ \tau_{2n}^{(2)}<\tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}}=z\right)\right|\\
&+2\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon}\in L_n^\epsilon\cup R_n^\epsilon\right).
\end{aligned}
\tag{2.3.26}
$$

In order to control the right hand side of (2.3.26), we first concentrate on controlling its second term. Note that by invariance principle it is easy to check that there is a constant $c>0$ such that for any integer $m>1$ and any integer j with $|j|\le m$, we have

$$\mathbb{P}_{(0,j)}\left(\tau_{\partial_l^{in}I_m\cup\partial_r^{in}I_m}<\tau_{\partial_u^{in}I_m\cup\partial_b^{in}I_m}\right)<1-c. \tag{2.3.27}$$

Moreover, by Lemma 2.2.4,

$$\mathbb{P}_{(0,0)}(\tau_{\partial_l^{in}I_m\cup\partial_r^{in}I_m}<\tau_{\partial_u^{in}I_m\cup\partial_b^{in}I_m})\le\mathbb{P}_{(0,0)}(\tau_{L_m}<\tau_{L_0})=\frac{1}{4\epsilon m}. \tag{2.3.28}$$

In the rest of the proof we call the event in (2.3.27) a side escaping event. The detailed proof of (2.3.27) follows exactly the same argument as the proof of Equation (11) in [19], which can also be illustrated in Figure 2.2.

Moreover, define $m(\epsilon,n)=\lfloor\epsilon n\rfloor$. Note that in the event $\{\bar{S}_{\bar{T}_n^\epsilon}\in L_n^\epsilon\cup R_n^\epsilon\}$, our simple random walk has to first escape $A_{m(\epsilon,n)}$ through $L_{m(\epsilon,n)}\cup R_{m(\epsilon,n)}$ and then has at least $K(\epsilon,n)=\lfloor\lfloor n/2\rfloor/m(\epsilon,n)\rfloor$ independent times of side escaping events. Thus by Lemma 2.2.4, (2.3.27),

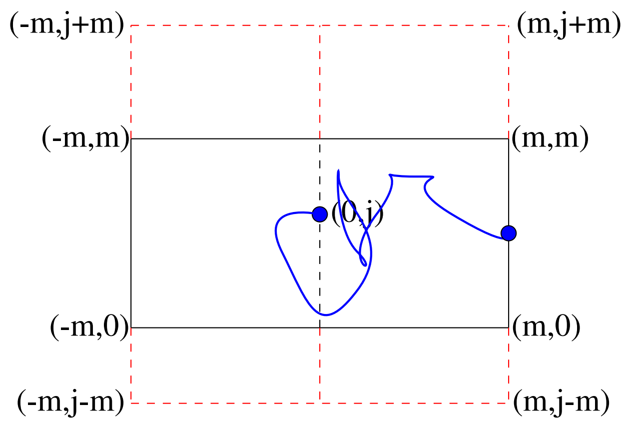

Figure 2.2: Invariance principle for (2.3.27)

(2.3.28), and the fact that for all sufficiently small $\epsilon > 0$,

$$K(\epsilon, n) = \lfloor \lfloor n/2 \rfloor / m(\epsilon, n) \rfloor \geq \frac{1}{3\epsilon}$$

we have

$$\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} \in L_n^\epsilon \cup R_n^\epsilon\right) \leq \frac{1}{4\epsilon n}(1-c)^{\frac{1}{3\epsilon}-1} \ll \frac{\epsilon}{n} \tag{2.3.29}$$

for all sufficiently small $\epsilon > 0$. Thus in order to prove Lemma 2.3.12, it suffices to show that

$$\sum_{z \in \partial^{out} B(0,2n)} \left| \mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(1)} < \tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}} = z\right) - \mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(2)} < \tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}} = z\right) \right|$$
$$\ll \frac{\epsilon}{n}. \tag{2.3.30}$$

Recall that in our construction, $\left\{\hat{S}_{1,k}\right\}_{k=0}^{\infty}$ and $\left\{\hat{S}_{2,k}\right\}_{k=0}^{\infty}$ are simple random walks coupled under the maximal coupling. Define events:

$$\mathcal{A}_1 = \left\{\hat{S}_{1,k} \notin D_n \cup \partial^{out} B(0, 2n),\ \forall k \leq \epsilon^4 n^2\right\},$$

$$\mathcal{A}_2 = \left\{\hat{S}_{2,k} \notin D_n \cup \partial^{out} B(0, 2n),\ \forall k \leq \epsilon^4 n^2\right\},$$

and

$$\mathcal{A}_3 = \left\{ \text{there exists a } k \le \epsilon^4 n^2 \text{ such that } \hat{S}_{1,j} = \hat{S}_{1,j}, \forall j \ge k \right\}.$$

By definition, one can easily see that

$$\begin{aligned} &\left\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(1)} < \tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}} = z \right\} \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \\ =& \left\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(2)} < \tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}} = z \right\} \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \end{aligned} \tag{2.3.31}$$

which implies that

$$\begin{aligned} &\sum_{z \in \partial^{out} B(0,2n)} \left| \mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(1)} < \tau_{D_n}^{(1)},\ S_{1,\tau_{2n}^{(1)}} = z \right) - \mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon,\ \tau_{2n}^{(2)} < \tau_{D_n}^{(2)},\ S_{2,\tau_{2n}^{(2)}} = z \right) \right| \\ &\le 2\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_1^c \right) + 2\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_2^c \right) + 2\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_3^c \right). \end{aligned} \tag{2.3.32}$$

Thus, it suffices to control the probabilities on the right hand side of (2.3.32). For its first term, we have by Proposition 2.1.2 of [4] there are constants $c, \beta \in (0, \infty)$, independent to n such that

$$\mathbb{P}\left(\mathcal{A}_1^c\right) \le c e^{-\beta/\epsilon^2},\ \ \mathbb{P}\left(\mathcal{A}_2^c\right) \le c e^{-\beta/\epsilon^2}.$$

By the strong Markov property, we have

$$\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_1^c \right) \le \frac{c e^{-\beta/\epsilon^2}}{\epsilon} n^{-1} \ll \frac{\epsilon}{n} \tag{2.3.33}$$

and

$$\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_2^c \right) \le \frac{c e^{-\beta/\epsilon^2}}{\epsilon} n^{-1} \ll \frac{\epsilon}{n} \tag{2.3.34}$$

for all sufficiently small $\epsilon > 0$. Finally, for the last term

$$\mathbb{P}\left(\{ \bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon \} \cap \mathcal{A}_3^c \right)$$

recall that the first coordinate of x is even and that $\left\{ \hat{S}_{1,k} \right\}_{k=0}^{\infty}$ and $\left\{ \hat{S}_{2,k} \right\}_{k=0}^{\infty}$ be two simple random walks starting from x_n^ϵ and $x_n^\epsilon + x$ and coupled under the maximal coupling. We have that

$$\mathbb{P}\left(\mathcal{A}_3^c\right) \leq d_{TV}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor}, \hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor}\right)$$

where $d_{TV}(\cdot,\cdot)$ stands for the total variation distance between the distributions of two random variables. On the other hand, note that

$$\begin{aligned}
d_{TV}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor}, \hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor}\right) &= \frac{1}{2}\sum_{z\in\mathbb{Z}^2}\left|\mathbb{P}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor} = z\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor} = z\right)\right| \\
&\leq \frac{1}{2}\left[\mathbb{P}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor} \in B^c(0,2n)\right) + \mathbb{P}\left(\hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor} \in B^c(0,2n)\right)\right. \\
&\left.+ \sum_{z\in B(0,2n)}\left|\mathbb{P}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor} = z\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor} = z\right)\right|\right].
\end{aligned}$$

And again by Proposition 2.1.2 of [4] there are constants $c, \beta \in (0,\infty)$, independent to n such that

$$\mathbb{P}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor} \in B^c(0,2n)\right) \leq ce^{-\beta/\epsilon^4},\ \mathbb{P}\left(\hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor} \in B^c(0,2n)\right) \leq ce^{-\beta/\epsilon^4}. \tag{2.3.35}$$

And for any $z \in B(0,2n)$, condition on $\bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon$, applying Proposition 4.1 of [21] with $x_0 = x_n^\epsilon$, $n_0 = \lfloor \epsilon^4 n^2 \rfloor$ and $R = \lfloor \epsilon^4 n \rfloor$, there are constant $h > 0$ and $C < \infty$ independent to n and the choice of x_n^ϵ,

$$\begin{aligned}
&\left|\mathbb{P}\left(\hat{S}_{1,\lfloor \epsilon^4 n^2 \rfloor} = z \middle| \bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor \epsilon^4 n^2 \rfloor} = z \middle| \bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right)\right| \\
&\leq C\left(\frac{e^{-\frac{1}{\epsilon}}}{\epsilon^4}\right)^h \sup_{(n,y)\in Q}\mathbb{P}_y\left(S_n = z\right),
\end{aligned}$$

where $Q = [n_0 - 2R^2, n_0] \times B(x_n^\epsilon, 2R)$. Moreover, by Local Central Limit Theorem, see Theorem 2.1.1 of [4] for example, there is a finite constant $C < \infty$ independent to n such that

$$\sup_{(n,y)\in Q}\mathbb{P}_y\left(S_n = z\right) \leq \frac{C}{\epsilon^4 n^2},$$

which implies that

$$\left(\frac{e^{-\frac{1}{\epsilon}}}{\epsilon^4}\right)^h \sup_{(n,y)\in Q} \mathbb{P}_y\left(S_n = z\right) \le Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}n^{-2}$$

and that

$$\begin{aligned}
&\left|\mathbb{P}\left(\hat{S}_{1,\lfloor\epsilon^4 n^2\rfloor} = z\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor\epsilon^4 n^2\rfloor} = z\right)\right| \\
&\le \sum_{x_n^\epsilon} \left|\mathbb{P}\left(\hat{S}_{1,\lfloor\epsilon^4 n^2\rfloor} = z \middle| \bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor\epsilon^4 n^2\rfloor} = z \middle| \bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right)\right| \mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right) \\
&\le Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}n^{-2}\sum_{x_n^\epsilon}\mathbb{P}\left(\bar{S}_{\bar{T}_n^\epsilon} = x_n^\epsilon\right) \\
&\le Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}n^{-2}.
\end{aligned} \tag{2.3.36}$$

Thus,

$$\begin{aligned}
&\sum_{z\in B(0,2n)} \left|\mathbb{P}\left(\hat{S}_{1,\lfloor\epsilon^4 n^2\rfloor} = z\right) - \mathbb{P}\left(\hat{S}_{2,\lfloor\epsilon^4 n^2\rfloor} = z\right)\right| \\
&\le \sum_{z\in B(0,2n)} Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}n^{-2} \\
&\le Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}.
\end{aligned} \tag{2.3.37}$$

Combining (2.3.35) and (2.3.37) we have

$$\mathbb{P}\left(\mathcal{A}_3^c\right) \le d_{TV}\left(\hat{S}_{1,\lfloor\epsilon^4 n^2\rfloor}, \hat{S}_{2,\lfloor\epsilon^4 n^2\rfloor}\right) \le \frac{1}{2}\big(2ce^{-\beta/\epsilon^4} + Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}\big). \tag{2.3.38}$$

By the strong Markov property,

$$\mathbb{P}\left(\{\bar{S}_{\bar{T}_n^\epsilon} \in U_n^\epsilon\} \cap \mathcal{A}_3^c\right) \le \frac{1}{8\epsilon n}\left(2ce^{-\beta/\epsilon^4} + Ce^{-\frac{h}{\epsilon}}\epsilon^{-4(1+h)}\right) \ll \frac{\epsilon}{n} \tag{2.3.39}$$

for all sufficiently large n and sufficiently small ϵ. Thus the proof of this lemma is complete. □

With Lemma 2.3.12, the proof of Proposition 2.3.6 is complete. □

2.3.5 Proof of Lemma 2.3.7

Let $M, M_0 \in \mathbb{Z}_+$ such that $M > M_0 > 1$. By the strong Markov property,

$$\begin{aligned}&\mathbb{P}_{(0,Mn)}(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \\ &= \sum_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_{(0,Mn)}(\tau_{\partial^{out}B(0,M_0n)} = y)\mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}).\end{aligned} \tag{2.3.40}$$

So by law of total probability,

$$\begin{aligned}&\min_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \\ &\leq \mathbb{P}_{(0,Mn)}(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \\ &\leq \max_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}).\end{aligned} \tag{2.3.41}$$

Notice that if we fix n,

$$\lim_{M\to\infty} \mathbb{P}_{(0,Mn)}(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) = \mathtt{H}_{D_n}([-\delta n, \delta n] \times \{0\}),$$

and thus

$$\begin{aligned}&\min_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \\ &\leq \mathtt{H}_{D_n}([-\delta n, \delta n] \times \{0\}) \\ &\leq \max_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}).\end{aligned} \tag{2.3.42}$$

Let $\{y_n : y_n \in \partial^{out}B(0, M_0n)\}$ be a sequence of points in $\mathbb{Z}^2$. Note that $||y_n||_2 \to \infty$ as $n \to \infty$. By invariance principle,

$$\limsup_{n\to\infty} \mathbb{P}_{y_n}(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \leq \sup_{z\in\partial\bar{B}(0,M_0)} \mathbb{P}_z^{BM}(\tau_{[-\delta,\delta]\times\{0\}} = \tau_{[-1,1]\times\{0\}}),$$

where $\mathbb{P}_z^{BM}$ is the law of a Brownian motion starting at the point $z \in \mathbb{R}^2$. Since the choice of $\{y_n\}$ is arbitrary,

$$\limsup_{n\to\infty} \max_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \le \sup_{z\in\partial\bar{B}(0,M_0)} \mathbb{P}_z^{BM}(\tau_{[-\delta,\delta]\times\{0\}} = \tau_{[-1,1]\times\{0\}}).$$

Similarly,

$$\liminf_{n\to\infty} \min_{y\in\partial^{out}B(0,M_0n)} \mathbb{P}_y(\tau_{[-\delta n,\delta n]\times\{0\}} = \tau_{D_n}) \ge \inf_{z\in\partial\bar{B}(0,M_0)} \mathbb{P}_z^{BM}(\tau_{[-\delta,\delta]\times\{0\}} = \tau_{[-1,1]\times\{0\}}).$$

Note that

$$\begin{aligned} &\lim_{M_0\to\infty} \sup_{z\in\partial\bar{B}(0,M_0)} \mathbb{P}_z^{BM}(\tau_{[-\delta,\delta]\times\{0\}} = \tau_{[-1,1]\times\{0\}}) \\ &= \lim_{M_0\to\infty} \inf_{z\in\partial\bar{B}(0,M_0)} \mathbb{P}_z^{BM}(\tau_{[-\delta,\delta]\times\{0\}} = \tau_{[-1,1]\times\{0\}}) \qquad (2.3.43) \\ &= \mathsf{H}_{[-1,1]\times\{0\}}([-\delta,\delta]\times\{0\}). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \mathtt{H}_{D_n}([-\delta n,\delta n]\times\{0\}) = \mathsf{H}_{[-1,1]\times\{0\}}([-\delta,\delta]\times\{0\}).$$

□

With Lemma 2.3.7, the proof of Theorem 2.3.3 is complete. □

2.3.6 Proof of Proposition 2.3.8

In order to prove

$$\lim_{n\to\infty} n \cdot \max_{y\in l_n^c} \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) = 0$$

we first recall that

$$l_n = [-\lfloor n^{\alpha_2}\rfloor, \lfloor n^{\alpha_2}\rfloor] \times \{\lfloor n^{\alpha_1}\rfloor\},$$

$\alpha_1 = (1+\alpha)/2$, $\alpha_2 = (7+\alpha)/8$, and that

$$l_n^c = \partial_l^{in} Box(n) \cup \partial_r^{in} Box(n) \cup \partial_u^{in} Box(n) \setminus l_n.$$

Thus for any point $y \in l_n^c$, define

$$T_y = \{\lfloor y^{(1)}/2 \rfloor\} \times [0, \infty)$$

to be the vertical line located in the exact midway between 0 and y. Noting that $\tau_{T_y} < \tau_x$, by the strong Markov property we have

$$\begin{aligned} \mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) &= \sum_{z \in T_y} \mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n},\ S_{\tau_{T_y}} = z\right) \mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right) \\ &= \sum_{z \in T_y,\ z^{(2)} \geq n^4} \mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n},\ S_{\tau_{T_y}} = z\right) \mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right) \\ &+ \sum_{z \in T_y,\ z^{(2)} < n^4} \mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n},\ S_{\tau_{T_y}} = z\right) \mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right) \\ &\leq \mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n},\ S^{(2)}_{T_y} \geq n^4\right) \\ &+ \max_{z \in T_y,\ z^{(2)} < n^4} \mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right) \mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n}\right). \end{aligned} \tag{2.3.44}$$

To control the right hand side of (2.3.44), we first define

$$\bar{D}_n = \left\{T_y \cup [\lfloor y/2 \rfloor, \infty) \times \{0\}\right\} \cap B(y, n^4)$$

and then note that

$$\mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n},\ S^{(2)}_{T_y} \geq n^4\right) \leq \mathbb{P}_y\left(\tau_{\partial^{out} B(y,n^4)} < \tau_{\bar{D}_n}\right).$$

Moreover, it is easy to see that

$$\text{rad}(\bar{D}_n) \geq n^4/2$$

for n sufficiently large, and that

$$d(\bar{D}_n, y) \leq \lfloor n^{\alpha_1} \rfloor.$$

We apply Theorem 1 in [22] with $\kappa = 1$ and $A = \bar{D}_n$ on the discrete ball $B(y, n^4)$, then there exists a constant $C > 0$ such that

$$\mathbb{P}_y\left(\tau_{\partial^{out} B(y,n^4)} < \tau_{\bar{D}_n}\right) \leq \mathbb{P}_y\left(\tau_{\partial^{out} B(y,n^4)} < \tau_{\bar{D}_{n_{[n^{\alpha_1}, n^4/2]}}}\right) \leq C\sqrt{\frac{n^{\alpha_1}}{n^4}} = o\left(\frac{1}{n}\right). \tag{2.3.45}$$

Note that this is a Beurling estimate for random walk. And for the second term in the right hand side of (2.3.44), note that for

$$\tilde{D}_n = L_0 \cap B\left(y, n^{\alpha_2}/2\right)$$

we have

$$\left\{\tau_{T_y} < \bar{\tau}_{A_n}\right\} \subset \left\{\tau_{\partial^{out} B(y, n^{\alpha_2}/2)} < \bar{\tau}_{\tilde{D}_n}\right\} \tag{2.3.46}$$

Using again the Theorem 1 of [22] to the right hand side of (2.3.46) we have

$$\mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n}\right) \leq \mathbb{P}_y\left(\tau_{\partial^{out} B(y, n^{\alpha_2}/2)} < \bar{\tau}_{\tilde{D}_n}\right) \leq C n^{-(\alpha_2 - \alpha_1)/2}. \tag{2.3.47}$$

At the same time, for any $z \in T_y$ such that $z^{(2)} < n^4$, again by the reversibility of simple random walk we have

$$\begin{aligned}
\mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right) &= \sum_{n=1}^{\infty} \mathbb{P}_z\left(S_1, S_2, \cdots, S_{n-1} \notin A_n,\ S_n = x\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}_x\left(S_1, S_2, \cdots, S_{n-1} \notin A_n,\ S_n = z\right) \\
&= \mathbb{E}_x\left[\#\text{ of visits to } z \text{ in } [0, \tau_{A_n})\right] \\
&= \mathbb{P}_x\left(\tau_z < \tau_{A_n}\right) \mathbb{E}_z\left[\#\text{ of visits to } z \text{ in } [0, \tau_{A_n})\right] \\
&= \frac{\mathbb{P}_x\left(\tau_z < \tau_{A_n}\right)}{\mathbb{P}_z\left(\tau_{A_n} < \tau_z\right)}.
\end{aligned} \tag{2.3.48}$$

To control the right hand side of (2.3.48), we first refer to the well known result:

Lemma 2.3.14. *(Lemma 1 of [20]) The series*

$$a(x) = \sum_{n=0}^{\infty} [P_0(S_n = 0) - P_0(S_n = x)] \tag{2.3.49}$$

converge for each $x \in \mathbb{Z}^2$, and the function $a(\cdot)$ has the following properties:

$$a(x) \geq 0, \ \forall x \in \mathbb{Z}^2, \ a(0) = 0, \tag{2.3.50}$$

$$a\big((\pm 1, 0)\big) = a\big((0, \pm 1)\big) = 1 \tag{2.3.51}$$

$$\mathbb{E}_x[a(S_1)] - a(x) = \delta(x, 0), \tag{2.3.52}$$

so $a(S_{n \wedge \tau_v} - v)$ is a nonnegative martingale, where $\tau_v = \tau_{\{v\}}$, for any $v \in \mathbb{Z}^2$. And there is some suitable c_0 such that

$$\left| a(x) - \frac{1}{2\pi} \log \|x\| - c_0 \right| = O(\|x\|^{-2}), \tag{2.3.53}$$

as $\|x\| \to \infty$.

Now we prove the following lower bound on the denominator:

Lemma 2.3.15. *There is a finite constant $C < \infty$ such that for any nonzero $x \in \mathbb{Z}^2$,*

$$\mathbb{P}_0(\tau_x < \tau_0) \geq \frac{C}{(\log \|x\|)^2}.$$

Proof. First, it suffices to show this lemma for all x sufficiently far away from 0. We consider stopping time

$$\Gamma = \tau_0 \wedge \tau_{\|x\|/2},$$

By Lemma 2.3.14, we have

$$1 = \mathbb{E}_0\left[a(S_\Gamma)\middle|\tau_{\|x\|/2} < \tau_0\right]\mathbb{P}_0\left(\tau_{\|x\|/2} < \tau_0\right).$$

Thus by (2.3.53),

$$\mathbb{P}_0\left(\tau_{\|x\|/2} < \tau_0\right) = \frac{1}{\mathbb{E}_0\left[a(S_\Gamma)\middle|\tau_{\|x\|/2} < \tau_0\right]} \geq \frac{\pi}{\log\|x\|} \tag{2.3.54}$$

for all x sufficiently far away from 0. By the strong Markov property,

$$\begin{aligned}\mathbb{P}_0(\tau_x < \tau_0) &= \sum_{y\in\partial^{out}B(0,\|x\|/2)} \mathbb{P}_0\left(\tau_{\|x\|/2} < \tau_0,\ S_{\tau_{\|x\|/2}} = y\right)\mathbb{P}_y\left(\tau_x < \tau_0\right) \\ &\geq \frac{\pi}{\log\|x\|}\min_{y\in\partial^{out}B(0,\|x\|/2)} \mathbb{P}_y\left(\tau_x < \tau_0\right).\end{aligned} \tag{2.3.55}$$

At the same time, for stopping time $\Gamma_1 = \tau_{\partial^{out}B(x,\|x\|/3)}$, and $\Gamma_2 = \tau_{\partial^{out}B(x,\|x\|/2)}$, we have

$$\mathbb{P}_y\left(\tau_x < \tau_0\right) \geq \sum_{z\in\partial^{out}B(x,\|x\|/3)} \mathbb{P}_y\left(\Gamma_1 < \tau_{\|x\|/3},\ S_{\Gamma_1} = z\right)\mathbb{P}_z\left(\tau_x < \Gamma_2\right). \tag{2.3.56}$$

For the right hand side of (2.3.56), we have by translation invariance of simple random walk,

$$\mathbb{P}_z\left(\tau_x < \Gamma_2\right) = \mathbb{P}_{z-x}\left(\tau_0 < \tau_{\|x\|/2}\right).$$

Moreover,

$$\left[1 - \mathbb{P}_{z-x}\left(\tau_0 < \tau_{\|x\|/2}\right)\right]\mathbb{E}_{z-x}\left[a(S_\Gamma)\middle|\tau_{\|x\|/2} < \tau_0\right] = a(z-x),$$

which implies that

$$\mathbb{P}_{z-x}\left(\tau_0 < \tau_{\|x\|/2}\right) = \frac{\mathbb{E}_{z-x}\left[a(S_\Gamma)\middle|\tau_{\|x\|/2} < \tau_0\right] - a(z-x)}{\mathbb{E}_{z-x}\left[a(S_\Gamma)\middle|\tau_{\|x\|/2} < \tau_0\right]}. \tag{2.3.57}$$

Again, by Lemma 2.3.14, we have that there are positive constants $c, C \in (0,\infty)$ such that uniformly for all n, x and z defined above,

$$\mathbb{E}_{z-x}\left[a(S_\Gamma)\middle|\tau_{\|x\|/2}<\tau_0\right]-a(z-x)\geq c,$$

while

$$\mathbb{E}_{z-x}\left[a(S_\Gamma)\middle|\tau_{\|x\|/2}<\tau_0\right]\leq C\log\|x\|.$$

Thus we have

$$\mathbb{P}_z\left(\tau_x<\Gamma_2\right)=\mathbb{P}_{z-x}\left(\tau_0<\tau_{\|x\|/2}\right)\geq\frac{c}{\log\|x\|}\tag{2.3.58}$$

uniformly for all n, x and z defined above.

On the other hand, by invariance principle, there is a constant $c>0$ such that for any $y\in\partial^{out}B(0,\|x\|/2)$,

$$\mathbb{P}_y\left(\Gamma_1<\tau_{\|x\|/3}\right)\geq c.$$

Thus,

$$\mathbb{P}_y\left(\tau_x<\tau_0\right)\geq\sum_{z\in\partial^{out}B(x,\|x\|/3)}\mathbb{P}_y\left(\Gamma_1<\tau_{\|x\|/3},\ X_{\Gamma_1}=z\right)\mathbb{P}_z\left(\tau_x<\Gamma_2\right)\geq\frac{c}{\log\|x\|}.\tag{2.3.59}$$

Now combining, (2.3.54), (2.3.55), and (2.3.59). The proof of this lemma is complete. □

With Lemma 2.3.15, we look back at the right hand side of (2.3.48). Noting that for any $z\in T_y$, $\tau_{T_y}\leq\tau_z$ and that $\tau_{A_n}\leq\tau_{D_n}$, we give the following upper bound estimate on its numerator:

Lemma 2.3.16. *Recall that $\alpha_2=(7+\alpha)/8$. Then for each $x\in A$,*

$$\mathbb{P}_x\left(\tau_{T_y}<\tau_{D_n}\right)\leq\frac{c}{n^{\alpha_2}}\tag{2.3.60}$$

for all sufficiently large n and all $y\in l_n^c$.

Proof. For any given $x\in A$, define $x_0=(x^{(1)},0)$ be the projection of x on L_0. Note that x_0 and x are connected by a path independent to n, which implies that there is a constant $c>0$ also independent to n such that

$$\mathbb{P}_{x_0}\left(\tau_{T_y} < \tau_{D_n}\right) \geq c\mathbb{P}_x\left(\tau_{T_y} < \tau_{D_n}\right).$$

Thus to prove Lemma 2.3.16 it suffices to replace x by x_0. Moreover, recall that $l_n^c = \partial_l^{in} Box(n) \cup \partial_r^{in} Box(n) \cup \partial_u^{in} Box(n) \setminus l_n$. For any $y \in l_n^c$, by the translation invariance of simple random walk, we have

$$\mathbb{P}_{x_0}\left(\tau_{T_y} < \tau_{D_n}\right) \leq \mathbb{P}_0\left(\tau_{I_{\lfloor n^{\alpha_2}/4\rfloor}} < \tau_{D_n}\right).$$

Here recall the definition of I_n in (2.2.4). Now by lemma 2.2.4,

$$\mathbb{P}_0\left(\tau_{I_{\lfloor n^{\alpha_2}/4\rfloor}} < \tau_{D_n}\right) \leq \frac{C}{\lfloor n^{\alpha_2}/4\rfloor}$$

and the proof of this lemma is complete. □

Now apply (2.3.47), (2.3.48), Lemma 2.3.15, and Lemma 2.3.16 together to the last term of (2.3.44), we have

$$\begin{aligned}\max_{z\in T_y,\, z^{(2)}<n^4} \mathbb{P}_z\left(S_{\bar{\tau}_{A_n}} = x\right)\mathbb{P}_y\left(\tau_{T_y} < \bar{\tau}_{A_n}\right) &\leq Cn^{-\alpha_2-(\alpha_2-\alpha_1)/2}(\log n)^2 \\ &\leq Cn^{-\frac{17}{16}+\frac{\alpha}{16}}(\log n)^2 \ll n^{-1}\end{aligned}$$

for all sufficiently large n. Thus, the proof of Proposition 2.3.8 is complete. □

2.3.7 Proof of Proposition 2.3.9

To show

$$\lim_{n\to\infty}\sum_{y\in l_n}\mathbb{P}_y\left(S_{\bar{\tau}_{A_n}} = x\right) = \mathcal{H}_A(x),$$

we first prove that

Lemma 2.3.17. *For any $x \in A$ and the truncations A_n defined in* (2.3.1)

$$\lim_{n\to\infty} \sum_{y\in l_n} \mathbb{P}_y \left(S_{\bar{\tau}_A} = x \right) = \mathcal{H}_A(x). \tag{2.3.61}$$

Proof. Recall that by definition that

$$\mathcal{H}_A(x) = \lim_{k\to\infty} \sum_{z\in L_k} \mathbb{P}_z \left(S_{\bar{\tau}_A} = x \right)$$

and that

$$l_n = [-\lfloor n^{\alpha_2} \rfloor, \lfloor n^{\alpha_2} \rfloor] \times \{\lfloor n^{\alpha_1} \rfloor\}.$$

Thus

$$\lim_{n\to\infty} \sum_{z\in L_{\lfloor n^{\alpha_1}\rfloor}} \mathbb{P}_z \left(S_{\bar{\tau}_A} = x \right) = \mathcal{H}_A(x),$$

while in order to prove Lemma 2.3.17, it suffices to show that

$$\lim_{n\to\infty} \sum_{z\in L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n} \mathbb{P}_z \left(S_{\bar{\tau}_A} = x \right) = 0. \tag{2.3.62}$$

Apply reversibility of simple random walk on each $z \in L_{\lfloor n^{\alpha_1}\rfloor} \setminus l_n$, we have

$$\begin{aligned} \sum_{z\in L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n} \mathbb{P}_z \left(S_{\bar{\tau}_A} = x \right) &= \mathbb{E}_x \left[\#\text{ of visits to } L_{\lfloor n^{\alpha_1}\rfloor} \setminus l_n \text{ in } [0, \bar{\tau}_A) \right] \\ &\le \frac{\mathbb{P}_x \left(\tau_{L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n} < \tau_{L_0} \right)}{\min_{z\in L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n} \mathbb{P}_z \left(\tau_{L_0} < \tau_{L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n} \right)}. \end{aligned} \tag{2.3.63}$$

First, for the denominator of (2.3.63), note that

$$\tau_{L_{\lfloor n^{\alpha_1}\rfloor}} \le \tau_{L_{\lfloor n^{\alpha_1}\rfloor}\setminus l_n}$$

We have for any $z \in L_{\lfloor n^{\alpha_1} \rfloor} \setminus l_n$

$$\mathbb{P}_z \left(\tau_{L_0} < \tau_{L_{\lfloor n^{\alpha_1} \rfloor} \setminus l_n} \right) \geq \mathbb{P}_z \left(\tau_{L_0} < \tau_{l_{\lfloor n^{\alpha_1} \rfloor}} \right) \geq \frac{c}{\lfloor n^{\alpha_1} \rfloor}. \tag{2.3.64}$$

On the other hand, using exactly the same argument as in the proof of Lemma 2.3.16

$$\mathbb{P}_x \left(\tau_{L_{\lfloor n^{\alpha_1} \rfloor} \setminus l_n} < \tau_{L_0} \right) \leq \frac{C}{\lfloor n^{\alpha_2} \rfloor}. \tag{2.3.65}$$

Thus, combining (2.3.63)-(2.3.65), the proof Lemma 2.3.17 is complete. □

Now with Lemma 2.3.17, it suffices to prove that

$$\lim_{n \to \infty} \sum_{y \in l_n} \left[\mathbb{P}_y \left(S_{\bar{\tau}_{A_n}} = x \right) - \mathbb{P}_y \left(S_{\bar{\tau}_A} = x \right) \right] = 0. \tag{2.3.66}$$

Again by reversibility,

$$\mathbb{P}_y \left(S_{\bar{\tau}_{A_n}} = x \right) = \mathbb{E}_x \left[\# \text{ of visits to } y \text{ in } [0, \tau_{A_n}) \right]$$

and

$$\mathbb{P}_y \left(S_{\bar{\tau}_A} = x \right) = \mathbb{E}_x \left[\# \text{ of visits to } y \text{ in } [0, \tau_A) \right],$$

which implies that for each y

$$\mathbb{P}_y \left(S_{\bar{\tau}_{A_n}} = x \right) - \mathbb{P}_y \left(S_{\bar{\tau}_A} = x \right) = \mathbb{E}_x \left[\# \text{ of visits to } y \text{ in } [\tau_A, \tau_{A_n}) \right]$$

and that

$$\begin{aligned} &\sum_{y \in l_n} \left[\mathbb{P}_y \left(S_{\bar{\tau}_{A_n}} = x \right) - \mathbb{P}_y \left(S_{\bar{\tau}_A} = x \right) \right] \\ &= \mathbb{E}_x \left[\# \text{ of visits to } l_n \text{ in } [\tau_A, \tau_{A_n}) \right]. \end{aligned} \tag{2.3.67}$$

Here we use the natural convention that the number of visits equals to 0 over an empty interval.

Moreover, define $\bar{T}_n = \{-n, n\} \times [0, \infty)$ and

$$\Gamma_4 = \inf\{n > \tau_A,\ S_n \in \bar{T}_n\}.$$

Noting that

$$\{\tau_A < \Gamma_4 < \tau_{A_n}\} \subset \{\tau_A < \tau_{A_n}\} \subset \{\tau_{\bar{T}_n} < \tau_{A_n}\},$$

thus by the strong Markov property, one can see that

$$\mathbb{E}_x\left[\#\text{ of visits to } l_n \text{ in } [\tau_A, \tau_{A_n})\right] \le \frac{\mathbb{P}_x\left(\tau_{\bar{T}_n} < \tau_{A_n}\right)}{\min_{z \in l_n} \mathbb{P}_z(\tau_{A_n} < \tau_{l_n})}. \tag{2.3.68}$$

First, for any $z = (z^{(1)}, z^{(2)}) \in l_n$, consider

$$(z^{(1)}, 0) + \Big\{[-\lfloor n^{\alpha_1} \rfloor, \lfloor n^{\alpha_1} \rfloor] \times [0, \lfloor n^{\alpha_1} \rfloor]\Big\}.$$

By Lemma 2.2.4 and translation/reflection invariance of simple random walk,

$$\begin{aligned}
\mathbb{P}_z(\tau_{A_n} < \tau_{l_n}) &\ge \mathbb{P}_0\left(\tau_{\partial_u^{in} I_{\lfloor n^{\alpha_1} \rfloor}} < \tau_{L_0}\right) \\
&\ge \mathbb{P}_0\left(\tau_{\partial_u^{in} I_{\lfloor n^{\alpha_1} \rfloor}} = \tau_{\partial^{in} I_{\lfloor n^{\alpha_1} \rfloor}}\right) \\
&\ge \frac{1}{2}\mathbb{P}_0\left(\tau_{\partial^{in} I_{\lfloor n^{\alpha_1} \rfloor}} < \tau_{L_0}\right) \\
&\ge \frac{1}{2}\mathbb{P}_0\left(\tau_{L_{\lfloor n^{\alpha_1} \rfloor}} < \tau_{L_0}\right) = \frac{1}{8\lfloor n^{\alpha_1} \rfloor}.
\end{aligned} \tag{2.3.69}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{P}_x\left(\tau_{\bar{T}_n} < \tau_{A_n}\right) &\le \mathbb{P}_x\left(\tau_{\bar{T}_n} < \tau_{L_0}\right) \\
&\le C\mathbb{P}_0\left(\tau_{\partial^{in} I_{\lfloor n/2 \rfloor}} < \tau_{L_0}\right) \\
&\le 2C\mathbb{P}_0\left(\tau_{\partial_u^{in} I_{\lfloor n^{\alpha_1} \rfloor}} = \tau_{\partial^{in} I_{\lfloor n/2 \rfloor}}\right) \quad \le 2C\mathbb{P}_0\left(\tau_{L_{\lfloor n/2 \rfloor}} < \tau_{L_0}\right) \le \frac{C}{n}.
\end{aligned} \tag{2.3.70}$$

Now combining (2.3.67)-(2.3.70), we have shown (2.3.66) and the proof of Proposition 2.3.9 is complete. $\square$

2.3.8 Proof of Proposition 2.3.10

At this point, in order to prove Theorem 2.3.1, we only need to show that for all sufficiently large n and any $y \in l_n$, $2\mathtt{H}_{Box(n)}(y)/\mathtt{H}_{D_n}(0)$ can be arbitrarily close to one. First, for any $y \in l_n$, define

$$M(y,n) = n + \left|y^{(1)}\right|, \;\; m(y,n) = n - \left|y^{(1)}\right|.$$

Recall that $Box(n) = [-n,n] \times [0, \lfloor n^{\alpha_1} \rfloor]$ and that $l_n = [-\lfloor n^{\alpha_2} \rfloor, \lfloor n^{\alpha_2} \rfloor] \times \{\lfloor n^{\alpha_1} \rfloor\}$. We have

$$n - \lfloor n^{\alpha_2} \rfloor \le m(y,n) \le n \le M(y,n) \le n + \lfloor n^{\alpha_2} \rfloor.$$

Moreover, noting that

$$Box(n) \subset \left[y^{(1)} - M(y,n), y^{(1)} + M(y,n)\right] \times [0, \lfloor n^{\alpha_1} \rfloor]$$

and that

$$\left[y^{(1)} - m(y,n), y^{(1)} + m(y,n)\right] \times [0, \lfloor n^{\alpha_1} \rfloor] \subset Box(n),$$

by definition we have

$$H_{\left[y^{(1)} - M(y,n), y^{(1)} + M(y,n)\right] \times [0, \lfloor n^{\alpha_1} \rfloor]}(y) \le \mathtt{H}_{Box(n)}(y)$$

and

$$H_{\left[y^{(1)} - m(y,n), y^{(1)} + m(y,n)\right] \times [0, \lfloor n^{\alpha_1} \rfloor]}(y) \ge \mathtt{H}_{Box(n)}(y).$$

Thus, combine translation invariance and Theorem 2.3.3, and note that for all $y \in l_n$, $M^{-1}(y,n) - n^{-1} = o(n^{-1})$, $m^{-1}(y,n) - n^{-1} = o(n^{-1})$. It is immediate to see that Proposition 2.3.10 is equivalent to the following statement:

Lemma 2.3.18. *For all integers $m, n > 0$, define*

$$\widehat{Box}(m,n) = [-n,n] \times [-m,0].$$

For any $\epsilon > 0$, we have

$$\mathtt{H}_{D_n}(0) - 2\mathtt{H}_{\widehat{Box}(m,n)}(0) \in \left[0,\ \frac{\epsilon}{n}\right) \tag{2.3.71}$$

for all sufficiently large n and all $0 < m \leq 2n^{\alpha_1}$.

Proof. First, for the lower bound estimate, note that

$$D_n \subset \widehat{Box}(m,n)$$

and that by the definition of harmonic measure, we have

$$\mathtt{H}_{D_n}(0) = \lim_{k\to\infty} \mathbb{P}_{(k,0)}\left(\tau_{D_n} = \tau_0\right)$$

and that

$$\mathtt{H}_{\widehat{Box}(m,n)}(0) = \lim_{k\to\infty} \mathbb{P}_{(k,0)}\left(\tau_{\widehat{Box}(m,n)} = \tau_0\right).$$

Moreover, by symmetry we have for all $k > n$,

$$\mathbb{P}_{(k,0)}\left(\tau_{D_n} = \tau_0\right) = 2\mathbb{P}_{(k,0)}\left(\tau_{D_n} = \tau_0,\ S_{\tau_0 - 1} = (0,1)\right).$$

At the same time on can see that in the event $\left\{\tau_{\widehat{Box}(m,n)} = \tau_0\right\}$, the random walk has to visit 0 through $(0,1)$, which implies that

$$\mathbb{P}_{(k,0)}\left(\tau_{D_n} = \tau_0,\ S_{\tau_0 - 1} = (0,1)\right) \geq \mathbb{P}_{(k,0)}\left(\tau_{\widehat{Box}(m,n)} = \tau_0\right).$$

Taking limit as $k \to \infty$, we have shown the lower bound estimate. For the upper bound estimate, again we note that for each sufficiently large k and a random walk starting from $(k, 0)$

$$\begin{aligned}&\{\tau_{D_n} = \tau_0,\ S_{\tau_0 - 1} = (0,1)\} \setminus \left\{\tau_{\widehat{Box}(m,n)} = \tau_0\right\}\\&= \{\tau_{D_n} = \tau_0,\ S_{\tau_0 - 1} = (0,1)\} \cap \left\{\tau_{\widehat{Box}(m,n)\setminus D_n} < \tau_{D_n}\right\},\end{aligned} \tag{2.3.72}$$

which, by the strong Markov property implies that

$$\begin{aligned}&\mathbb{P}_{(k,0)}\left(\tau_{D_n} = \tau_0,\ S_{\tau_0 - 1} = (0,1)\right) - \mathbb{P}_{(k,0)}\left(\tau_{\widehat{Box}(m,n)} = \tau_0\right)\\ \leq &\max_{y \in \widehat{Box}(m,n)\setminus D_n} \mathbb{P}_y\left(\tau_{(0,1)} < \tau_{D_n}\right).\end{aligned} \tag{2.3.73}$$

Now in order to find the upper bound of the right hand side of (2.3.73), we consider the following two cases based on the location of point $y = (y^{(1)}, y^{(2)}) \in \widehat{Box}(m,n) \setminus D_n$:

Case 1:

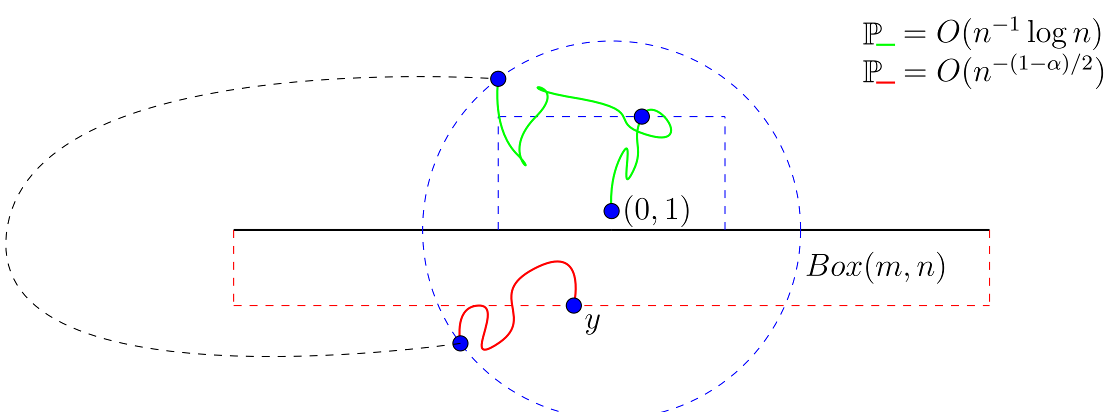

Figure 2.3: Illustration of proof for Case 1

If $\left|y^{(1)}\right| \leq n/3$, for all nearest neighbor paths starting at y which hit $(0,1)$ before D_n, they first have to hit $\partial^{out}B(0,n/2)$. Thus we have

$$\begin{aligned}\mathbb{P}_y\left(\tau_{(0,1)}<\tau_{D_n}\right)&=\sum_{z\in\partial^{out}B(0,n/2)}\mathbb{P}_y\left(\tau_{n/2}<\tau_{D_n},\ S_{\tau_{n/2}}=z\right)\mathbb{P}_z\left(\tau_{(0,1)}<\tau_{D_n}\right)\\&\le\mathbb{P}_y\left(\tau_{n/2}<\tau_{D_n}\right)\max_{z\in\partial^{out}B(0,n/2)}\mathbb{P}_z\left(\tau_{(0,1)}<\bar{\tau}_{D_n}\right).\end{aligned}\tag{2.3.74}$$

See Figure 2.3 for illustration of Case 1. For the first term of the right hand side of (2.3.74), recalling that $d(y,D_n)=\left|y^{(2)}\right|=m\le 2n^{\alpha_1}$ and that $\left|y^{(1)}\right|<n/3$, we have by the same Beurling estimate, there exists a constant $C<\infty$ independent to the choice of n,m and y satisfying Case 1, such that

$$\mathbb{P}_y\left(\tau_{n/2}<\tau_{D_n}\right)\le Cn^{-(1-\alpha_1)/2}.\tag{2.3.75}$$

At the same time, for any $z\in\partial^{out}B(0,n/2)$, to control the upper bound on $\mathbb{P}_z\left(\tau_{(0,1)}<\bar{\tau}_{D_n}\right)$, one can concentrate on the upper half plane, since each path from y to $(0,1)$ must pass through some point $z\in\partial^{out}B(0,n/2)\cap\{x\in\mathbb{H}:x^{(2)}>0\}$. Now for any such z, by reversibility, we have

$$\mathbb{P}_z\left(\tau_{(0,1)}<\bar{\tau}_{D_n}\right)=\mathbb{E}_{(0,1)}\left[\#\text{ of visits to }z\text{ in }[0,\tau_{D_n\cup\{(0,1)\}})\right]\le\frac{\mathbb{P}_{(0,1)}\left(\tau_z<\bar{\tau}_{D_n}\right)}{\mathbb{P}_z\left(\bar{\tau}_{D_n}<\tau_z\right)}.\tag{2.3.76}$$

For the numerator, note that for all sufficiently large n, $[-\lfloor n/3\rfloor,\lfloor n/3\rfloor]\times[0,\lfloor n/3\rfloor]\subset B(0,n/2)$. Applying the same argument as we repeatedly used in this paper, we have

$$\mathbb{P}_{(0,1)}\left(\tau_z<\bar{\tau}_{D_n}\right)\le\frac{C}{n}.$$

At the same time,

$$\begin{aligned}&\mathbb{P}_z\left(\bar{\tau}_{D_n}<\tau_z\right)\\&\quad\ge\sum_{w\in\partial^{out}B(z,\frac{z^{(2)}}{2})}\mathbb{P}_z\left(\bar{\tau}_{\partial^{out}B(z,\frac{z^{(2)}}{2})}<\tau_z,\bar{\tau}_{\partial^{out}B(z,\frac{z^{(2)}}{2})}=\tau_w\right)\mathbb{P}_w\left(\bar{\tau}_{D_n}<\tau_{\partial^{out}B(z,\frac{z^{(2)}}{3})}\right).\end{aligned}$$

And by invariance principle and the fact that $z^{(2)}\in(0,n]$, we have there is a constant $c>0$

independent to the choices of n, z and w, such that

$$\mathbb{P}_w\left(\bar{\tau}_{D_n} < \tau_{\partial^{out} B(z, \frac{z(2)}{3})}\right) \geq c.$$

Thus by Lemma 2.3.15,

$$\mathbb{P}_z\left(\bar{\tau}_{D_n} < \tau_z\right) \geq c\mathbb{P}_z\left(\bar{\tau}_{\partial^{out} B(z, \frac{z(2)}{2})} < \tau_z\right) \geq \frac{c}{\left(\log \frac{z(2)}{2}\right)^2} \geq \frac{c}{(\log n)^2},$$

which by (2.3.76) implies that

$$\mathbb{P}_z\left(\tau_{(0,1)} < \bar{\tau}_{D_n}\right) \leq \frac{C(\log n)^2}{n}. \tag{2.3.77}$$

Now combining (2.3.73), (2.3.74), (2.3.75), and (2.3.77),

$$\mathbb{P}_y\left(\tau_{(0,1)} < \tau_{D_n}\right) \leq Cn^{-(3-\alpha_1)/2}(\log n)^2 \ll n^{-1} \tag{2.3.78}$$

and thus our lemma hold when y in Case 1.

Case 2:

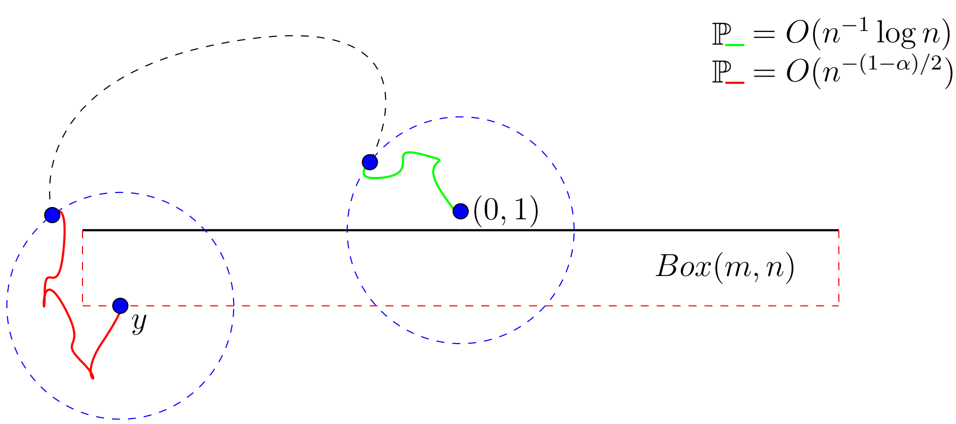

Figure 2.4: Illustration of proof for Case 2

Otherwise, if $\left|y^{(1)}\right| > n/3$, our proof follows the same techniques on slightly different stopping times. Consider two neighborhoods: $B(0, \frac{n}{7})$ and $B(y, \frac{n}{7})$. It is easy to see that

$$\partial^{out} B\left(0, \frac{n}{7}\right) \cap \partial^{out} B\left(y, \frac{n}{7}\right) = \emptyset.$$

Using the same argument as in Case 1,

$$\mathbb{P}_y\left(\tau_{(0,1)} < \tau_{D_n}\right) = \sum_{w \in \partial^{out} B(y, \frac{n}{7})} \mathbb{P}_y\left(\tau_{\partial^{out} B(y, \frac{n}{7})} < \tau_{D_n},\ S_{\tau_{\partial^{out} B(y, \frac{n}{7})}} = w\right) \mathbb{P}_w\left(\tau_{(0,1)} < \tau_{D_n}\right).$$

Moreover for any $w \in \partial^{out} B(y, \frac{n}{7})$ the random walk starting at w has to first visit $\partial^{out} B(0, \frac{n}{7})$ before ever reaches $(0, 1)$. This implies that

$$\begin{aligned}
\mathbb{P}_w\left(\tau_{(0,1)} < \tau_{D_n}\right) &= \sum_{z \in \partial^{out} B(0, \frac{n}{7})} \mathbb{P}_w\left(\tau_{n/7} < \tau_{D_n},\ S_{\tau_{n/7}} = z\right) \mathbb{P}_z\left(\tau_{(0,1)} < \tau_{D_n}\right) \\
&\leq \max_{z \in \partial^{out} B(0, \frac{n}{7})} \mathbb{P}_z\left(\tau_{(0,1)} < \tau_{D_n}\right).
\end{aligned}$$

See Figure 2.4 for illustration of Case 2. We have

$$\mathbb{P}_y\left(\tau_{(0,1)} < \tau_{D_n}\right) \leq \mathbb{P}_y\left(\tau_{\partial^{out} B(y, \frac{n}{7})} < \tau_{D_n}\right) \max_{z \in \partial^{out} B(0, \frac{n}{7})} \mathbb{P}_z\left(\tau_{(0,1)} < \tau_{D_n}\right). \tag{2.3.79}$$

Now since $y^{(2)} = -m \geq -2n^{\alpha_1}$, it is easy to see that

$$\text{rad}\left(B\left(y, \frac{n}{7}\right) \cap D_n\right) \geq \frac{n}{4}$$

for all sufficiently large n. Thus by (2.3.75) and (2.3.77), there exists a constant $C < \infty$ independent to the choice of n, m and y satisfying Case 2, such that

$$\mathbb{P}_y\left(\tau_{\partial^{out} B(y, \frac{n}{7})} < \tau_{D_n}\right) \leq C n^{-(1-\alpha_1)/2}. \tag{2.3.80}$$

and that

$$\max_{z\in\partial^{out}B(0,\frac{n}{7})}\mathbb{P}_z\left(\tau_{(0,1)}<\tau_{D_n}\right)\le\frac{C(\log n)^2}{n}. \tag{2.3.81}$$

Thus we also have

$$\mathbb{P}_y\left(\tau_{(0,1)}<\tau_{D_n}\right)\le Cn^{-(3-\alpha_1)/2}(\log n)^2\ll n^{-1} \tag{2.3.82}$$

and thus our lemma hold when y in Case 2 and the proof of Lemma 2.3.18 is complete. □

With Lemma 2.3.18, we have concluded the proof of Proposition 2.3.10. □

2.4 Stationary Diffusion Limited Aggregation is Well-defined

In this section, we define a (infinite) SDLA whose transition rate is given by the stationary harmonic measure, starting from the infinite initial configuration L_0.

Theorem 2.4.1. *Let $t>0$ and $A_0=L_0$, then there is a well defined SDLA process $\{A_s^\infty\}_{s\le t}$.*

Remark 2.4.2. The result remains true if one replace the initial state L_0 by any subset A_0 that can be seen as a connected forest of logarithmic horizontal growth rate. To be precise, A_0 can be written as $\cup_{n=-\infty}^{\infty}Tree_0^n$, where $Tree_0^n$ is connected for each n, with $Tree_0^n\cap L_0=(n,0)$ and moreover diam$(Tree_0^n)\ge\log n$ for only finite number of n's. We present the proof for $A_0=L_0$ for simplicity but without loss of (much) generality.

A major tool one obtains for the study of SDLA is ergodicity of the process.

Theorem 2.4.3. *For every $t>0$, A_t^∞ is ergodic with respect to shift in $\mathbb{Z}\times\{0\}$.*

2.4.1 Coupling construction

With the upper bounds of the harmonic measure on the upper half plane (see Theorem 2.1.3), a pure growth model called the **interface process** was introduced in [18] which can be used as a dominating process for both the DLA model in $\mathbb{H}$ and the stationary DLA model that will be introduced in this paper. Consider an interacting particle system $\bar{\xi}_t$ defined on $\{0,1\}^{\mathbb{H}}$, with 1 standing for an occupied site and 0 for a vacant site, with transition rates as follows:

(i) For each occupied site $x = (x_1, x_2) \in \mathbb{H}$, if $x_2 > 0$ it will try to give birth to each of its nearest neighbors at a Poisson rate of $\sqrt{x_2}$. If $x_2 = 0$, it will try to give birth to each of its nearest neighbors at a Poisson rate of 1.

(ii) If x attempts to give birth to a nearest neighbor y that is already occupied, the birth is suppressed.

We proved that an interacting particle system determined by the dynamic above is well-defined.

Proposition 2.4.4 (Proposition 3, [18]). *The interacting particle system $\bar{\xi}_t \in \{0,1\}^{\mathbb{H}}$ satisfying (i) and (ii) is well defined.*

Then when the initial aggregation V_0 is the origin or finite, we defined the DLA process in $\mathbb{H}$ starting from V_0 (Theorem 5, [18]), according to the graphic representation (see [23] for introduction) of the interface process $\bar{\xi}_t$ and a procedure of Poisson thinning, see Page 30-31 of [18] for details. Note that under this construction, the DLA model with finite initial aggregation is contained in the interface process.

Now in order to prove Theorem 2.4.1, we construct a sequence of processes $\{A_t^n\}_{n=1}^{\infty}$, each of which is the DLA in $\mathbb{H}$ with initial aggregation $V_0^n = [-n, n] \times 0$, coupled together with the same interface process. To be precise, recall the graphic representation in [18]:

- For each $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{H}$ such that $\|x - y\| = 1$, we associate the edge $\vec{e} = (x, y)$ with an independent Poisson process $N_t^{x \to y}, t \geq 0$ with intensity $\lambda_{x \to y} = \sqrt{x_2} \vee 1$.

- For each $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{H}$ such that $\|x - y\| = 1$ let $\{U_i^{x \to y}\}_{i=1}^{\infty}$ be i.i.d. sequences of $U(0,1)$ random variables independent of each other and of the Poisson processes.

At any time t when there is Poisson transition for edge $\vec{e} = (x, y)$, we draw the directed edge $(\vec{e}, t)$ in the phase space $\mathbb{H} \times [0, \infty)$. For any $x \in L_0$ and any fixed time t, recall that I_t^x is the set of all y's in $\mathbb{H}$ that are connected with x by a path going upwards vertically or following the directed edges. Then in [18] it has been proved that for all $V_0 \subset \mathbb{H}$,

$$\bar{\xi}_t^{V_0} = \bigcup_{x \in V_0} I_t^x$$

distributed as the interface process with initial state V_0. Moreover, it was proven that for each $t < \infty$ and all $x \in \mathbb{H}$, $|I_t^x| < \infty$ with probability one, and there can be only a finite number of different paths emanating from x by time t, which may only have finite transitions involved. Now for all finite V_0, in [18] we look at the finite set of all the transitions involved in the evolution of $\bar{\xi}_s^{V_0},\ s \in [0,t]$, and order them according to the time of occurrence. Then the following thinning was applied in order to define a process $A_t = (V_t, E_t)$ starting at $A_0 = (V_0, \emptyset)$: when a new transition arrives at time t_i, say it is the jth Poisson transition on the edge $\vec{e} = (x,y)$. Suppose one already knew $A_{t_i-} := \lim_{s \uparrow t_i} A_s$.

- If $x \notin V_{t_i-}$ or $y \in V_{t_i-}$, nothing happens.
- Otherwise:
  - If $U_j^{x \to y} \leq \mathcal{H}_{V_{t_i-}}(\vec{e})/\lambda_{\vec{e}}$, then $V_{t_i} = V_{t_i-} \cup \{y\}$, $E_t = E_{t-} \cup \{\vec{e}\}$.
  - Otherwise, nothing happens.

Thus we defined the process A_t up to all time t with V_t identically distributed as our DLA process starting from A_0. Now, for each n define A_t^n as the process with $A_0^n = ([-n,n] \times 0, \emptyset)$. Then we have coupled all A_t^n's using the same graphic representation and thinning factors. Now in order to prove Theorem 2.4.1, we first show the following theorem which states that for a finite space-time box, the discrepancy probabilities for our A^n's are summable.

Theorem 2.4.5. *For any compact subset $K \subset \mathbb{H}$ and any $T < \infty$, we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\exists t \leq T,\ s.t.\ A_t^n \cap K \neq A_t^{n+1} \cap K\right) < \infty. \tag{2.4.1}$$

Here for any $A = (V, E)$, we use the convention that $A \cap K = (V \cap K, \{\vec{e} = (x, y) \in E, \{x, y\} \cap K \neq \emptyset\})$.

Remark 2.4.6. Without loss of generality, we will assume that $T = 1$.

The proof of Theorem 2.4.5 is immediate once one proves that there exist constants $\alpha > 0$ and $C < \infty$ such that for all sufficiently large n

$$\mathbb{P}\left(\exists t \leq 1,\ s.t.\ A_t^n \cap K \neq A_t^{n+1} \cap K\right) \leq \frac{C}{n^{1+\alpha}}. \tag{2.4.2}$$

The same argument also implies

Corollary 2.4.7. *Let $A_t^{n,+}$ be the process with $A_0^{n,+} = ([-n, n+1] \times 0, \emptyset)$. Then for all sufficiently large n*

$$\mathbb{P}\left(\exists t \leq 1,\ s.t.\ A_t^n \cap K \neq A_t^{n,+} \cap K\right) \leq \frac{C}{n^{1+\alpha}}.$$

The same result holds for $A_t^{n,-}$ with $A_0^{n,-} = ([-n-1, n] \times 0, \emptyset)$.

Note that at $t = 0$, the initial aggregations A_0^n and A_0^{n+1} are different only by the two end points $(\pm(n+1), 0)$. Now we want to control the subset of the discrepancies so that they will not reach K by time 1. Intuitively, the idea we will follow in the detailed proof in the following sections can be summarized as the follows:

(I) With very high probability none of A_1^n and A_1^{n+1} can reach height $\log(n)$.

(II) For any $\alpha > 0$, with very high probability the two processes will have fewer than n^α discrepancies by time 1.

(III) For all these discrepancies ever created till time 1, with very high probability none of them will ever find its way to K.

2.4.2 Logarithmic growth of the interface process

In this section, we prove the logarithmic growth upper bound for A_t^n and A_t^{n+1} with $t \in [0, 1]$. Note that both are contained in the interface process $I_t^{[-n-1,n+1]\times 0}$. Thus it suffices to show that

Theorem 2.4.8. *For any $C < \infty$,*

$$\mathbb{P}\left(I_1^{[-n,n]\times 0} \not\subseteq [-n - \log n, n + \log n] \times [0, \log n]\right) < \frac{1}{n^C}$$

for all sufficiently large n.

Proof. First noting that

$$I_1^{[-n,n]\times 0} = \bigcup_{x\in[-n,n]\times 0} I_1^x.$$

By union bound, it suffices to show that for any $C < \infty$ and all sufficiently large k,

$$\mathbb{P}\left(\|I_1^0\|_2 \geq k\right) < \exp(-Ck), \tag{2.4.3}$$

where

$$\|A\|_2 = \max_{x\in A} \|x\|_2$$

for all finite $A \subset \mathbb{H}$. In order to get (2.4.3), one first proves

Lemma 2.4.9. *Let $\{T_i\}_{i=1}^k$ be independent exponential random variables with parameters $\lambda_i = 4\sqrt{i+1}$. Then, $\mathbb{P}(\|I_1^0\|_2 > k) \leq 4^k \mathbb{P}(\sum_{i=1}^k T_i < 1)$.*

Proof. Under the event $\{\|I_1^0\|_2 > k\}$, by definition and the fact that I_1^0 is a nearest neighbor growth model, there has to exist a nearest neighbor sequence of points $0 = x_0, x_1, \cdots, x_m$ with $\|x_m\| \geq k$ such that for stopping times

$$\eta_i = \inf\{s \geq 0 : \ x_i \in I_s^0\}$$

we have that

$$0 = \eta_0 < \eta_1 < \cdots < \eta_m < 1.$$

Noting that $x_0, x_1, \cdots, x_m$ is a nearest neighbor path with $\|x_m\| \geq k$, which implies $m \geq k$, we may without loss of generality assume $m = k$. More precisely, there exists a nearest neighbor sequence of points $0 = x_0, x_1, \cdots, x_k$ such that for stopping times

$$\eta_i = \inf\{s \geq 0 :\ x_i \in I_s^0\}$$

we have that

$$0 = \eta_0 < \eta_1 < \cdots < \eta_k < 1.$$

Note that there are no more than 4^k such different nearest neighbor sequences of points within $\mathbb{H}$ starting at 0. And for each given path $0 = x_0, x_1, \cdots, x_k$, and each $1 \leq i \leq k$, define

$$\Delta_i = \min_{y:\|y-x_i\|=1} \inf\left\{s > 0 :\ N_{\eta_{i-1}+s}^{y\to x_i} = N_{\eta_{i-1}}^{y\to x_i} + 1\right\}.$$

Then by definition and the strong Markov property, Δ_i is an exponential random variable with rate $\hat{\lambda}_i = \sum_{y:\|y-x_i\|=1} \lambda_{y\to x_i} \leq 4\sqrt{i+1}$, independent to $\mathcal{F}_{\eta_{i-1}}$. At the same time, note that by definition $\Delta_i \leq \eta_i - \eta_{i-1}$, which implies that $\Delta_i \in \mathcal{F}_{\eta_i}$, and that $\{\Delta_i\}_{i=1}^k$ is a sequence of independent random variables. Thus

$$\mathbb{P}(\eta_0 < \eta_1 < \cdots < \eta_k < 1) \leq \mathbb{P}\left(\sum_{i=1}^k \Delta_i < 1\right) \leq \mathbb{P}\left(\sum_{i=1}^k T_i < 1\right).$$

□

For some constants $c_1, c_2 > 0$ (to be chosen later) define the event

$$G = \left\{\left|\left\{1 \leq i \leq k : T_i \geq \frac{c_2}{\sqrt{i+1}}\right\}\right| > c_1 k\right\}.$$

Lemma 2.4.10. *For any $t > 0$ and $k \in \mathbb{N}$ large enough (depending on the choices of c_1 and c_2),*

$$\mathbb{P}\left(\sum_{i=1}^k T_i < 1\right) \leq \mathbb{P}(G^c).$$

Proof. Under the event G,

$$\sum_{i=1}^{k} T_i \geq \sum_{i:\ T_i \geq \frac{c_2}{\sqrt{i}}} T_i \geq c_1 k \frac{c_2}{\sqrt{k+1}} = \frac{1}{2} c_1 c_2 \sqrt{k} \geq 1, \tag{2.4.4}$$

where the last inequality holds for any sufficiently large k. $\square$

Lemma 2.4.11. *Let $t > 0$ any $\tilde{c} \in (0, \infty)$, then there exists $c_1, c_2 > 0$ such that for any sufficiently large k,*

$$\mathbb{P}(G^c) \leq \exp(-\tilde{c}k).$$

Proof. Define $X_i = \mathbb{1}_{\left\{T_i \geq \frac{c_2}{\sqrt{i+1}}\right\}}$, thus $\sum_{i=1}^{k} X_i$ is a binomial random variable with parameters k and $p = \mathbb{P}\left(T_i \geq \frac{c_2}{\sqrt{i+1}}\right) = e^{-4c_2}$, which converges to 1 when $c_2 \to 0$. By the large deviation principle for the binomial distribution

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i < c_1 k\right) \leq e^{-I(c_1, p)k}.$$

For p close enough to 1 we have $I(c_1, p) > \tilde{c}$ (see [24] for the exact rate function). $\square$

Proof of Theorem 2.4.8. For any $C \in (0, \infty)$, fix a $\tilde{c} = C + \log(4) + 1$. Then Theorem 2.4.8 follows from the combination of (2.4.3) and Lemma 2.4.9-2.4.11. $\square$

2.4.3 Truncated processes and number of discrepancies

In this section we complete Step (II) in the outline. But prior to that, we would like to use Theorem 2.4.8 to define a truncated version of coupled process (A_t^n, A_t^{n+1}). Define the stopping time

$$\Gamma = \inf\left\{t \geq 0 :\ V_t^n \cup V_t^{n+1} \not\subseteq [-n - \log n, n + \log n] \times [0, \log n]\right\}$$

to be the first time A_t^n or A_t^{n+1} grows outside the box $[-n - \log n, n + \log n] \times [0, \log n]$.

Remark 2.4.12. It is easy to see that V_t^n or V_t^{n+1} grows outside our box if and only if E_t^n or E_t^{n+1} does so.

Now we can define the **truncated processes**

$$(\hat{A}_t^n, \hat{A}_t^{n+1}) = \left(A_{t\wedge\Gamma}^n, A_{t\wedge\Gamma}^{n+1}\right).$$

I.e., we have the coupled processes stopped once either of them goes outside the box $[-n - \log n, n + \log n] \times [0, \log n]$. By definition, we have

$$(A_t^n, A_t^{n+1}) = (\hat{A}_t^n, \hat{A}_t^{n+1})$$

for all $t \in [0, \Gamma]$. At the same time, note that

$$V_t^n \cup V_t^{n+1} \subset \bigcup_{x\in[-n-1,n+1]\times 0} I_t^x$$

for all $t \geq 0$. Thus for all $C < \infty$ and all sufficiently large n,

$$\begin{aligned}
&\mathbb{P}\left(A_t^n \equiv \hat{A}_t^n, A_t^{n+1} \equiv \hat{A}_t^{n+1},\ \forall t \in [0,1]\right) \\
&\quad \leq \mathbb{P}\left(I_1^{[-n-1,n+1]\times\{0\}} \not\subseteq [-n-1-\log(n+1), n+1+\log(n+1)] \times [0, \log(n+1)]\right) \\
&\quad < \frac{1}{n^C}.
\end{aligned} \tag{2.4.5}$$

Thus in order to show Theorem 2.4.5, it suffices to prove that there exists constants $\alpha > 0$ and $C < \infty$ such that for all sufficiently large n

$$\mathbb{P}\left(\exists t \leq 1,\ s.t.\ \hat{A}_t^n \cap K \neq \hat{A}_t^{n+1} \cap K\right) \leq \frac{C}{n^{1+\alpha}}. \tag{2.4.6}$$

Now we formally define the set of discrepancies for the coupled process $(\hat{A}_t^n, \hat{A}_t^{n+1})$. For any $t < \infty$, define

$$V_t^{D,n} = \left\{x \in \mathbb{H},\ s.t.\ \exists s \leq t,\ x \in \hat{V}_s^n \triangle \hat{V}_s^{n+1}\right\}$$

as the set of **vertex discrepancies**, and

$$E_t^{D,n} = \left\{ \vec{e} = (x,y),\ x,y \in \mathbb{H},\ s.t.\ \exists s \le t,\ \vec{e} \in \hat{E}_s^n \triangle \hat{E}_s^{n+1} \right\}$$

as the set of **edge discrepancies**, where $\triangle$ stands for the symmetric difference of sets. From their definition, we list some basic properties of the sets of discrepancies as follows:

- $V_0^{D,n} = \{(\pm(n+1), 0)\}$, $E_0^{D,n} = \emptyset$.
- Both $V_t^{D,n}$ and $E_t^{D,n}$ are non-decreasing with respect to time.
- For any $x \in V_t^{D,n}$, then either $x = (\pm(n+1), 0)$ or there has to be an edge $\vec{e}_x \in E_t^{D,n}$ ending at x.
- For any $\vec{e} = (a, x) \in E_t^{D,n}$, x has to be in $x \in V_t^{D,n}$.
- Whenever a new vertex is added in $V_t^{D,n}$, there has to be a new edge added to $E_t^{D,n}$. However, when a new edge is added to $E_t^{D,n}$, there may or may not be a a new vertex added in $V_t^{D,n}$.

From the observations above, it is immediate to see that $V_t^{D,n}$ is the same as the collection of all ending points in $E_t^{D,n}$, which also implies that $|V_t^{D,n}| \le |E_t^{D,n}| + 2$.

Moreover, for the event of interest, we have

$$\left\{ \exists t \le 1,\ s.t.\ \hat{A}_t^n \cap K \ne \hat{A}_t^{n+1} \cap K \right\} = \left\{ V_1^{D,n} \cap K \ne \emptyset \right\}. \tag{2.4.7}$$

As we outlined in the previous section, in order to prove the event in (2.4.7) has a super-linearly decaying probability as $n \to \infty$, we first control the growth of $|E_t^{D,n}|$. I.e., by time 1 there cannot be too many discrepancies created in the coupled system. To be precise, we prove that

Lemma 2.4.13. *For any $\alpha > 0$, there is a $c > 0$ such that*

$$\mathbb{P}\left(|E_1^{D,n}| \ge n^\alpha \right) \le \exp(-n^c)$$

for all sufficiently large n*.*

Proof. Note that $|E_0^{D,n}| = 0$. For $i = 1, 2, \cdots$, define the stopping time $\Delta_i = \inf\{t \geq 0,\ |E_t^{D,n}| = i\}$, with the convention $\inf \emptyset = \infty$. Given the configuration of $(\hat{A}_t^n, \hat{A}_t^{n+1})$, we first discuss the rate at which a new discrepancy is created. If $t > \Gamma$, each such rate equals to zero by definition. Otherwise, each edge $\vec{e} = (x, y)$ in $\mathbb{H}$ can be classified according to the configuration as follows: define the indicator matrix

$$\mathbb{I}(\hat{A}_t^n, \hat{A}_t^{n+1})(\vec{e}) = \begin{pmatrix} \mathbb{1}_{x \in \hat{V}_t^n} & \mathbb{1}_{y \in \hat{V}_t^n} & \mathbb{1}_{\vec{e} \in \hat{E}_t^n} \\ \mathbb{1}_{x \in \hat{V}_t^{n+1}} & \mathbb{1}_{y \in \hat{V}_t^{n+1}} & \mathbb{1}_{\vec{e} \in \hat{E}_t^{n+1}} \end{pmatrix}.$$

Then by definition, the only edges that contribute to the increasing rate of $E_t^{D,n}$ are those with indicator matrices as one of the following:

$$\mathbb{I}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \mathbb{I}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\mathbb{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathbb{I}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\mathbb{I}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},\ \mathbb{I}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\mathbb{I}_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and we will denote the collections of such edges $E_1, E_2, \cdots, E_7$.

Now the rate that a new edge is added to $E_t^{D,n}$ can be written as the follows:

$$\begin{aligned}\lambda^D(\hat{A}_t^n, \hat{A}_t^{n+1}) = &\sum_{\vec{e}\in E_1} \left|\mathcal{H}_{\hat{V}_t^n}(\vec{e}) - \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e})\right| \\ &+ \sum_{\vec{e}\in E_2} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_3} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_4} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) \\ &+ \sum_{\vec{e}\in E_5} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_6} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_7} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}).\end{aligned} \tag{2.4.8}$$

For any $\vec{e} \in \cup_{i=2}^7 E_i$, note that at least one end point of $\vec{e}$ has to be within $\hat{V}_t^n \triangle \hat{V}_t^{n+1} \subset V_t^{D,n}$. Moreover, recall that for each point in $\mathbb{H}$, there can be no more than 4 directed edges emanating from it and 4 edges going towards it. Thus, $|\cup_{i=2}^7 E_i| \leq 8|V_t^{D,n}| \leq 8(|E_t^{D,n}| + 2)$. Now recalling $t < \Gamma$, $\hat{A}_t^n \cup \hat{A}_t^{n+1} \subset [-n - \log n, n + \log n] \times [0, \log n]$, which implies that for each $\vec{e} \in \cup_{i=2}^7 E_i$, the corresponding harmonic measure in (2.4.8) is bounded from above by $2\sqrt{\log n}$. Thus

$$\begin{aligned}&\sum_{\vec{e}\in E_2} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_3} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_4} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) \\ &+ \sum_{\vec{e}\in E_5} \mathcal{H}_{\hat{V}_t^n}(\vec{e}) + \sum_{\vec{e}\in E_6} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) + \sum_{\vec{e}\in E_7} \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e}) \leq 16(|E_t^{D,n}| + 2)\sqrt{\log n}.\end{aligned} \tag{2.4.9}$$

Now for each $\vec{e} = (x, y) \in E_1$, by definition x has to be in the inner boundary of $\hat{V}_t^n \cap \hat{V}_t^{n+1}$, while y is in the complement of $\hat{V}_t^n \cup \hat{V}_t^{n+1}$. Moreover, we have

$$\left|\mathcal{H}_{\hat{V}_t^n}(\vec{e}) - \mathcal{H}_{\hat{V}_t^{n+1}}(\vec{e})\right| \leq \mathcal{H}_{\hat{V}_t^n \cap \hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1}}(\vec{e}). \tag{2.4.10}$$

Using a similar method as in Section 5 of [18] and recalling the definition of stationary harmonic measure,

$$
\begin{aligned}
&\mathcal{H}_{\hat{V}_t^n \cap \hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1}}(\vec{e}) \\
&= \lim_{N\to\infty} \left(\mathcal{H}_{\hat{V}_t^n \cap \hat{V}_t^{n+1},N}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1},N}(\vec{e}) \right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \mathbb{P}_w \left(X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}-1} = y \right) \\
&- \lim_{N\to\infty} \sum_{w\in L_N} \mathbb{P}_w \left(X_{\tau_{(\hat{V}_t^n \cup \hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n \cup \hat{V}_t^{n+1})\cup L_0}-1} = y \right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \mathbb{P}_w \left(X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}-1} = y,\ X_{\tau_{(\hat{V}_t^n \cup \hat{V}_t^{n+1})\cup L_0}} \in \hat{V}_t^n \triangle \hat{V}_t^{n+1} \right) \\
&= \lim_{N\to\infty} \sum_{w\in L_N} \sum_{z\in \hat{V}_t^n \triangle \hat{V}_t^{n+1}} \mathbb{P}_w \left(X_{\tau_{(\hat{V}_t^n \cup \hat{V}_t^{n+1})\cup L_0}} = z \right) \mathbb{P}_z \left(X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}-1} = y \right).
\end{aligned}
$$

Taking the summation over all $\vec{e} \in E_1$, and note that for all $z \in \hat{V}_t^n \triangle \hat{V}_t^{n+1}$,

$$\sum_{\vec{e}=(x,y)\in E_1} \mathbb{P}_z \left(X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}} = x,\ X_{\tau_{(\hat{V}_t^n \cap \hat{V}_t^{n+1})\cup L_0}-1} = y \right) \le 1$$

since the summation above are over disjoint events. We have

$$\sum_{\vec{e}\in E_1} \mathcal{H}_{\hat{V}_t^n \cap \hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1}}(\vec{e}) \le \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1}}(\hat{V}_t^n \triangle \hat{V}_t^{n+1}).$$

Moreover, noting that by definition $\hat{V}_t^n \cup \hat{V}_t^{n+1}$ is connected in $\mathbb{H}$, and that

$$|\hat{V}_t^n \triangle \hat{V}_t^{n+1}| \le |V_t^{D,n}| \le |E_t^{D,n}| + 2,$$

one may, by Theorem 2.1.3 have,

$$\sum_{\vec{e}\in E_1} \mathcal{H}_{\hat{V}_t^n \cap \hat{V}_t^{n+1}}(\vec{e}) - \mathcal{H}_{\hat{V}_t^n \cup \hat{V}_t^{n+1}}(\vec{e}) \le (|E_t^{D,n}| + 2)\sqrt{\log n}. \tag{2.4.11}$$

Now combining (2.4.9)-(2.4.11) and plugging them back to (2.4.8) gives us

$$\lambda^D(\hat{A}_t^n, \hat{A}_t^{n+1}) \leq 17(|E_t^{D,n}| + 2)\sqrt{\log n} \tag{2.4.12}$$

Then recalling the definition of Δ_i, by Poisson thinning and the strong Markov property again we have

$$\mathbb{P}\left(|E_1^{D,n}| \geq n^\alpha\right) = \mathbb{P}\left(\sum_{i=0}^{n^\alpha-1} \Delta_i \leq 1\right) \leq \mathbb{P}\left(\sum_{i=0}^{n^\alpha-1} \sigma_i \leq 1\right)$$

where $\{\sigma_i\}_{i=0}^{n^\alpha-1}$ is an independent sequence of exponential random variables with $\tilde{\lambda}_i = 17(i + 2)\sqrt{\log n}$.

Thus, in order to prove Lemma 2.4.13, it suffices to prove the following result:

Lemma 2.4.14. *Let σ_i be defined as above. Then for all $\alpha < 1$, $\beta < \alpha$ and any $c_3 > 0$, for all n large enough*

$$\mathbb{P}\left(\sum_{i=0}^{n^\alpha-1} \sigma_i \leq 1\right) < e^{-c_3 n^\beta}$$

Proof. For $\beta < \alpha$ defined in the lemma and some constants $c_1, c_2 > 0$ (to be chosen later) define the events for $j \in [1, n^\alpha/n^\beta] \cap \mathbb{N}$,

$$G_j = \left\{\left|\left\{(j-1)n^\beta \leq i < jn^\beta : \sigma_i \geq \frac{c_2}{(i+2)\sqrt{\log n}}\right\}\right| > c_1 n^\beta\right\}.$$

Define $N_i = \mathbb{1}_{\left\{\sigma_i \geq \frac{c_2}{(i+2)\sqrt{\log n}}\right\}}$, thus $M_j = \sum_{i=(j-1)n^\beta}^{jn^\beta-1} N_i$ is a binomial random variable with parameters n^β and $p = \mathbb{P}\left(\sigma_i \geq \frac{c_2}{(i+2)\sqrt{\log n}}\right) = e^{-17c_2}$, which converges to 1 when $c_2 \to 0$. By the large deviation principle for binomial random variable

$$\mathbb{P}(G_j^c) = \mathbb{P}\left(M_j \leq c_1 n^\beta\right) \leq e^{-I(c_1,p)n^\beta} \leq e^{-c_3 n^\beta},$$

where the last inequality follows by taking p close enough to 1 such that $I(c_1, p) > c_3'$ (see [24] for the exact rate function). Since c_3' was arbitrary, for a slightly smaller c_3 we can obtain for large enough n

$$\mathbb{P}\left(\bigcup_{j\in[1,\dots,n^\alpha/n^\beta]\cap\mathbb{N}} G_j^c\right) \le n^{\alpha-\beta}e^{-c_3' n^\beta} \le e^{-c_3 n^\beta}.$$

But under the event $\left\{\bigcap_{j\in[1,\dots,n^\alpha/n^\beta]\cap\mathbb{N}} G_j\right\}$

$$\sum_{i=1}^{n^\alpha}\sigma_i = \sum_{j=1}^{n^{\alpha-\beta}}\sum_{(j-1)n^\beta}^{jn^\beta-1}\sigma_i \ge \frac{c_2}{\sqrt{\log n}}\left(\frac{c_1 n^\beta}{n^\beta+1}+\frac{c_1 n^\beta}{2n^\beta+1}+\cdots+\frac{c_1 n^\beta}{n^{\alpha-\beta}n^\beta+1}\right)$$
$$> \frac{1}{2}c_1c_2(\alpha-\beta)\sqrt{\log n} > 1,$$

where the last two inequalities require taking a large enough n. □

Thus the proof of Lemma 2.4.13 completes. □

2.4.4 Locations of discrepancies and proof of Theorem 2.4.5

In the previous section, we have shown that, for any $\alpha > 0$, by time 1 with stretch-exponentially high probability, there will be no more than n^α discrepancies. Now we show that it is highly unlikely that the first n^α possible discrepancies may ever reach our finite subset K.

To show this, note that now the truncated model $(\hat{A}_t^n, \hat{A}_t^{n+1})$ forms a finite state Markov process. In this section, it is more convenient to concentrate on the **embedded chain**

$$(\hat{A}_k^n, \hat{A}_k^{n+1}),\ k = 0, 1, 2, \cdots$$

where all configuration $(\hat{A}_k^n, \hat{A}_k^{n+1})$ with

$$\hat{V}_k^n \cup \hat{V}_k^{n+1} \not\subseteq [-n-\log n, n+\log n] \times [0, \log n]$$

are absorbing states.

Remark 2.4.15. Without causing further confusion, we will, in this section use the parallel notations such as $(\hat{A}_k^n, \hat{A}_k^{n+1})$, $V_k^{D,n}$ and $E_k^{D,n}$ etc., for the embedded chain without more specification.

Now we recall the stopping times for the creation of new discrepancies:

$$\Delta_i = \inf\{k \geq 0,\ |E_k^{D,n}| = i\},$$

with the convention $\inf \emptyset = \infty$. In order to show Step (III), we only need to prove the lemma as follows:

Lemma 2.4.16. *There exists an $\alpha > 0$ whose value will be specified later such that for any compact $K \subset \mathbb{H}$,*

$$\mathbb{P}\left(E_{\Delta_{n^\alpha}}^{D,n} \cap K \neq \emptyset\right) \leq n^{-1-\alpha}$$

for all sufficiently large n.

Proof. We define

$$\vec{e}_i = \begin{cases} E_{\Delta_i}^{D,n} \setminus E_{\Delta_{i-1}}^{D,n}, & \text{if } \Delta_i < \infty \\ \emptyset, & \text{otherwise} \end{cases}.$$

Note that $\vec{e}_i$ is either an empty set or a singleton with one edge. If it is a singleton, we do not distinguish between the singleton set and its unique element.

Now we are ready to introduce classifications on discrepancies as follows: Let $0 < \alpha < 1/5$.

- For any $i = 1$, we say $\vec{e}_1$ is **good** if either $\vec{e}_1 = \emptyset$ or

$$d(\vec{e}_1, (n+1, 0)) < n^{1-5\alpha}.$$

  Here $d(\cdot, \cdot)$ is defined as the minimum distance over all endpoints.

- For any $i \geq 1$, we say $\vec{e}_i$ is **good** if either $\vec{e}_i = \emptyset$ or

$$d(\vec{e}_i, E_{\Delta_{i-1}}^{D,n}) < n^{1-5\alpha}.$$

Otherwise, we will say $\vec{e}_i$ is **bad**.

- If an $\vec{e}_i$ is bad, we call it **devastating** if and only if $\vec{e}_i$ intersects with $[-n^{1-3\alpha}, n^{1-3\alpha}] \times [0, \log n]$.

Moreover, one can also define

$$\kappa = \inf\{i \geq 1,\ s.t.\ \vec{e}_i \text{ is bad}\}.$$

By definition, one may see that $E^{D,n}_{\Delta_{n^\alpha}} \cap K \neq \emptyset$ only if either of the following two events happens:

- Event A: $\kappa < n^\alpha$, and $\vec{e}_\kappa$ is devastating.
- Event B: $\kappa < n^\alpha$, $\vec{e}_\kappa$ is bad but not devastating, and there is at least one bad event within $\kappa + 1, \kappa + 2, \cdots, n^\alpha$.

To see the above assertion, one can from the definition of A and B see that $(A \cup B)^c$ can also be written as the union of $C \cup D$, where the events are defined as follows:

- Event C: $\vec{e}_i$ are good for all $i = 1, 2, \cdots, n^\alpha$.
- Event D: $\kappa < n^\alpha$, $\vec{e}_\kappa$ is bad but not devastating, and there are no bad events within $\kappa+1, \kappa+2, \cdots, n^\alpha$.

Moreover, for each i, we define

$$l_i^+ = \min\left\{x^{(1)} > 0:\ s.t.\ \exists x^{(2)} \text{ with } x = (x^{(1)}, x^{(2)}) \text{ a vertex for some edge within } E^{D,n}_{\Delta_i}\right\},$$

and

$$r_i^- = \max\left\{x^{(1)} < 0:\ s.t.\ \exists x^{(2)} \text{ with } x = (x^{(1)}, x^{(2)}) \text{ a vertex for some edge within } E^{D,n}_{\Delta_i}\right\}.$$

Thus under event C or D,

$$l_i^+ \geq n^{1-3\alpha} - n^{\alpha} \times n^{1-5\alpha} \geq n^{1-3\alpha}/2$$

and

$$r_i^- \leq -n^{1-3\alpha} + n^{\alpha} \times n^{1-5\alpha} \leq -n^{1-3\alpha}/2,$$

which implies no discrepancy may be within $[-n^{1-3\alpha}/2, n^{1-3\alpha}/2] \times [0, \log n] \supset K$ for all sufficiently large n.

Thus, now we only need to find the desired upper bound for the probability of events A and B. For any k, define event

$$G_k = \{\vec{e}_i \text{ is good for } i = 1, \cdots, k-1\}.$$

2.4.4.1 Upper bounds on $\mathbb{P}(A)$

For event A, by definition and the strong Markov property one has

$$\begin{aligned}\mathbb{P}(A) &= \sum_{k=1}^{n^\alpha} \mathbb{P}\left(G_k,\ \vec{e}_k \text{ is devastating}\right) \\ &= \sum_{k=1}^{n^\alpha} \sum_{j=0}^{\infty} \sum_{(\bar{A}_0, \tilde{A}_0)} \mathbb{P}\left(G_k,\ \Delta_{k-1} < \infty,\ \Delta_k - \Delta_{k-1} > j,\ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\right) \\ &\qquad\qquad \mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right),\end{aligned} \tag{2.4.13}$$

where $\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}$ stands for the distribution of the truncated embedded process $(\hat{A}^n_k, \hat{A}^{n+1}_k)$ starting from initial condition $(\bar{A}_0, \tilde{A}_0)$.

At the same time, with similar calculation we have for any $k = 1, 2, \cdots, n^\alpha$

$$\begin{aligned}&\mathbb{P}(G_k, \Delta_k < \infty) = \\ &\sum_{j=0}^{\infty} \sum_{(\bar{A}_0, \tilde{A}_0)} \mathbb{P}\left(G_k, \Delta_{k-1} < \infty, \Delta_k - \Delta_{k-1} > j, (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\right) \\ &\qquad \mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}\left(\Delta_1 = 1\right) \leq 1.\end{aligned} \tag{2.4.14}$$

Note that for any configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbb{P}\left(G_k,\ \Delta_{k-1} < \infty,\ \Delta_k - \Delta_{k-1} > j,\ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\right) \neq 0,$$

one must have $|\bar{E}_0 \triangle \tilde{E}_0| \leq k-1$. Now recalling the transition dynamic of the embedded chain, one has for all feasible $(\bar{A}_0, \tilde{A}_0)$ such that $\bar{V}_0 \cup \tilde{V}_0 \subset [-n-\log n, n+\log n] \times [0, \log n]$

$$\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1) = \frac{\lambda^D(\bar{A}_0, \tilde{A}_0)}{\lambda^T(\bar{A}_0, \tilde{A}_0)}$$

where $\lambda^D(\cdot, \cdot)$ was defined in (2.4.8) and

$$\lambda^T(\bar{A}_0, \tilde{A}_0) = \sum_{\vec{e}} \max\{\mathcal{H}_{\bar{V}_0}(\vec{e}), \mathcal{H}_{\tilde{V}_0}(\vec{e})\}.$$

Otherwise $\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1) = 0$. Now for

$$\mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(\Delta_1 = 1, \vec{e}_1 \text{ is devastating})$$

recall that in (2.4.8) we have

$$\begin{aligned}\lambda^D(\bar{A}_0, \tilde{A}_0) = &\sum_{\vec{e} \in E_1} \left|\mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e})\right| \\ &+ \sum_{\vec{e} \in E_2} \mathcal{H}_{\tilde{V}_0}(\vec{e}) + \sum_{\vec{e} \in E_3} \mathcal{H}_{\bar{V}_0}(\vec{e}) + \sum_{\vec{e} \in E_4} \mathcal{H}_{\bar{V}_0}(\vec{e}) \\ &+ \sum_{\vec{e} \in E_5} \mathcal{H}_{\bar{V}_0}(\vec{e}) + \sum_{\vec{e} \in E_6} \mathcal{H}_{\tilde{V}_0}(\vec{e}) + \sum_{\vec{e} \in E_7} \mathcal{H}_{\tilde{V}_0}(\vec{e}).\end{aligned}$$

For any $\vec{e} \in \cup_{i=2}^7 E_i$, recall that at least one of the endpoints of $\vec{e}$ has to be in $\bar{V}_0 \Delta \tilde{V}_0$. Thus it is easy to see

$$d(\vec{e}, E^{D,n}_{\Delta_{k-1}}) = 0.$$

Combining this with the fact that for all feasible $(\bar{A}_0, \tilde{A}_0)$, $\bar{E}_0 \triangle \tilde{E}_0 \subset (-\infty, -n+2n^{1-4\alpha}) \cup (n -$

$2n^{1-4\alpha}, \infty) \times [0, \log n]$, which is disjoint with $[-2n^{1-3\alpha}, 2n^{1-3\alpha}] \times [0, \log n]$, we have

$$\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right) \le \frac{\sum_{\vec{e}=(x,y)\in E_1, |x_1|\le 2n^{1-3\alpha}} \left|\mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e})\right|}{\lambda^T(\bar{A}_0, \tilde{A}_0)} \tag{2.4.15}$$

when $\bar{V}_0 \cup \tilde{V}_0 \subset [-n - \log n, n + \log n] \times [0, \log n]$ and equals to 0 otherwise. Thus for any configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbb{P}\left(G_k,\ \Delta_{k-1} < \infty,\ \Delta_k - \Delta_{k-1} > j,\ (\hat{A}^n_{\Delta_{k-1}+j}, \hat{A}^{n+1}_{\Delta_{k-1}+j}) = (\bar{A}_0, \tilde{A}_0)\right) \ne 0,$$

and that

$$\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right) \ne 0,$$

we have

$$\frac{\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1, \vec{e}_1 \text{ is devastating}\right)}{\mathbb{P}_{(\bar{A}_0,\tilde{A}_0)}\left(\Delta_1 = 1\right)} \le \frac{\sum_{\vec{e}=(x,y)\in E_1, |x_1|\le 2n^{1-3\alpha}} \left|\mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e})\right|}{\lambda^D(\bar{A}_0, \tilde{A}_0)}. \tag{2.4.16}$$

Now for the numerator of (2.4.16), again we have

$$\begin{aligned}
&\sum_{\vec{e}=(x,y)\in E_1, |x_1|\le 2n^{1-3\alpha}} \left|\mathcal{H}_{\bar{V}_0}(\vec{e}) - \mathcal{H}_{\tilde{V}_0}(\vec{e})\right| \\
&\quad \le \sum_{\vec{e}=(x,y)\in E_1, |x_1|\le 2n^{1-3\alpha}} \left[\mathcal{H}_{\bar{V}_0\cap\tilde{V}_0}(\vec{e}) - \mathcal{H}_{\bar{V}_0\cup\tilde{V}_0}(\vec{e})\right] \\
&\quad = \sum_{\vec{e}=(x,y)\in E_1, |x_1|\le 2n^{1-3\alpha}} \sum_{z\in\bar{V}_0\Delta\tilde{V}_0} \mathcal{H}_{\bar{V}_0\cup\tilde{V}_0}(z)\mathbb{P}_z\left(X_{\tau_{(\bar{V}_0\cap\tilde{V}_0)\cup L_0}-1} = y, X_{\tau_{(\bar{V}_0\cap\tilde{V}_0)\cup L_0}} = x\right) \\
&\quad \le \mathcal{H}_{\bar{V}_0\cup\tilde{V}_0}(\bar{V}_0\triangle\tilde{V}_0) \sup_{z\in\bar{V}_0\triangle\tilde{V}_0} \mathbb{P}_z\left(\tau_{Box} < \tau_{L_0}\right),
\end{aligned} \tag{2.4.17}$$

where

$$Box = [-2n^{1-3\alpha}, 2n^{1-3\alpha}] \times [0, \log n].$$

At the same time, note that for any feasible configuration $(\bar{A}_0, \tilde{A}_0)$,

$$\bar{V}_0 \triangle \tilde{V}_0 \subset Box_0 = [n - 2n^{1-4\alpha}, n + \log n] \cup [-n - \log n, -n + 2n^{1-4\alpha}] \times [0, \log n]$$

which implies that

$$\sup_{z \in \bar{V}_0 \triangle \tilde{V}_0} \mathbb{P}_z \left(\tau_{Box} < \tau_{L_0} \right) \leq \sup_{z \in Box_0} \mathbb{P}_z \left(\tau_{Box} < \tau_{L_0} \right). \tag{2.4.18}$$

Moreover, for each edge $\vec{e} = (z, w)$ such that $z \in \bar{V}_0 \triangle \tilde{V}_0$ and $w \notin \bar{V}_0 \cup \tilde{V}_0$, by definition it has to belong to $E_3 \cup E_6$ and thus by (2.4.8)

$$\lambda^D(\bar{A}_0, \tilde{A}_0) \geq \mathcal{H}_{\bar{V}_0 \cup \tilde{V}_0}(\bar{V}_0 \triangle \tilde{V}_0). \tag{2.4.19}$$

Now combining (2.4.13)-(2.4.19) we have

$$\mathbb{P}(A) \leq n^{\alpha} \sup_{x \in Box_0} \mathbb{P}_x \left(\tau_{Box} < \tau_{L_0} \right). \tag{2.4.20}$$

Now we prove the following lemma:

Lemma 2.4.17. *For all $\alpha < 1/5$ and all sufficiently large n*

$$\sup_{x \in Box_0} \mathbb{P}_x \left(\tau_{Box} < \tau_{L_0} \right) \leq n^{-1-2.5\alpha}.$$

Proof. The proof of Lemma 2.4.17 follows a similar argument as in [1]. Note that for any $x \in Box_0$,

$$\mathbb{P}_x \left(\tau_{Box} < \tau_{L_0} \right) \leq \sum_{y \in \partial^{in} Box} \mathbb{P}_x(\tau_y < \tau_{L_0}).$$

Then let $V_n = \{n/2\} \times [0, \infty)$, $V_n^1 = n/2 \times [0, n^4)$, and $V_n^2 = n/2 \times (n^4, \infty)$. By a similar argument as in [1] we have

$$\mathbb{P}_x \left(\tau_{V_n} < \tau_{L_0} \right) \leq n^{-1+\alpha/5} \tag{2.4.21}$$

while

$$\mathbb{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_{V_n^2}\right) \le \frac{1}{n^3}.$$

Thus by the strong Markov property,

$$\begin{aligned}\mathbb{P}_x(\tau_y < \tau_{L_0}) &= \sum_{z\in V_n} \mathbb{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_z\right)\mathbb{P}_z(\tau_y < \tau_{L_0}) \\ &\le \frac{1}{n^3} + \sum_{z\in V_n^1} \mathbb{P}_x\left(\tau_{V_n} < \tau_{L_0}, \tau_{V_n} = \tau_z\right)\mathbb{P}_z(\tau_y < \tau_{L_0}).\end{aligned} \tag{2.4.22}$$

Moreover, for each $z \in V_n^1$, by reversibility of random walk ([20]), we have

$$\mathbb{P}_z(\tau_y < \tau_{L_0}) \le \mathbb{P}_y(\tau_z < \tau_{L_0})\mathbb{E}_z[\#\text{ of visits to } z \text{ in } [0, \tau_{L_0})]. \tag{2.4.23}$$

For the first term in (2.4.23), the same argument for (2.4.21) implies that

$$\mathbb{P}_y(\tau_z < \tau_{L_0}) \le \mathbb{P}_y(\tau_{V_n} < \tau_{L_0}) \le n^{-1+\alpha/5}.$$

While for the second term in (2.4.23), by [1] we have there is a constant $C < \infty$ independent to n such that for all $z \in V_n^1$

$$\mathbb{E}_z[\#\text{ of visits to } z \text{ in } [0, \tau_{L_0})] \le C\log n.$$

Thus we have

$$\mathbb{P}_z(\tau_y < \tau_{L_0}) \le Cn^{-1+\alpha/5}\log n. \tag{2.4.24}$$

Combining (2.4.21)-(2.4.24), we have for any $x \in Box_0$, $y \in \partial^{in} Box$,

$$\mathbb{P}_x(\tau_y < \tau_{L_0}) \le Cn^{-2+2\alpha/5}\log n.$$

Finally, noting that $|\partial^{in} Box| \leq 5n^{1-3\alpha}$, we have

$$\sup_{x \in Box_0} \mathbb{P}_x\left(\tau_{Box} < \tau_{L_0}\right) \leq Cn^{-2+2\alpha/5} \log n \cdot n^{1-3\alpha} \leq n^{-1-2.5\alpha}$$

for all sufficiently large n. □

Combining (2.4.20) and Lemma 2.4.17, we have

$$\mathbb{P}(A) \leq n^{\alpha} \sup_{x \in Box_0} \mathbb{P}_x\left(\tau_{Box} < \tau_{L_0}\right) \leq n^{-1-1.5\alpha}. \tag{2.4.25}$$

2.4.4.2 Upper bounds on $\mathbb{P}(B)$

Now we find the upper bound for $\mathbb{P}(B)$. Recall that

- Event B: $\kappa < n^{\alpha}$, $\vec{e}_{\kappa}$ is bad but not devastating, and there is at least one bad event within $\kappa+1, \kappa+2, \cdots, n^{\alpha}$.

For any $k \geq 1$ define event

$$B_k = \left\{\vec{e}_1, \cdots, \vec{e}_{k-1} \text{ are good, } \vec{e}_k \text{ is bad}\right\}.$$

Then by Markov property, we have

$$\mathbb{P}(B) = \sum_{k=1}^{n^{\alpha}-1} \sum_{(\bar{A}_0, \tilde{A}_0)} \mathbb{P}\left(B_k, \vec{e}_k \text{ is not devastating, } (\hat{A}^n_{\Delta_k}, \hat{A}^{n-1}_{\Delta_k}) = (\bar{A}_0, \tilde{A}_0)\right) \left(\sum_{j=1}^{n^{\alpha}-k} \mathbb{P}_{(\bar{A}_0, \tilde{A}_0)}(B_j)\right). \tag{2.4.26}$$

Using the argument in Subsection 2.4.4.1 we have for all $k+j \leq n^{\alpha}$ and any feasible configuration $(\bar{A}_0, \tilde{A}_0)$ such that

$$\mathbb{P}\left(B_k, \vec{e}_k \text{ is not devastating, } (\hat{A}^n_{\Delta_k}, \hat{A}^{n-1}_{\Delta_k}) = (\bar{A}_0, \tilde{A}_0)\right) \neq 0$$

and such that $\mathbb{P}_{(\tilde{A}_0,\tilde{A}_0)}(B_i) > 0$ for some $i \leq n^\alpha - k$, we have

$$\mathbb{P}_{(\tilde{A}_0,\tilde{A}_0)}(B_j) \leq \mathbb{P}_{(\tilde{A}_0,\tilde{A}_0)}(G_j, \Delta_j < \infty)\mathbb{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \leq \mathbb{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right)$$

where $U_n = \{-n^{1-5\alpha}/2, n^{1-5\alpha}/2\} \times [0,\infty)$. Again from [1], we have

$$\mathbb{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \leq n^{-1+6\alpha}. \tag{2.4.27}$$

Thus by (2.4.26) and (2.4.27),

$$\mathbb{P}(B) \leq n^{-1+7\alpha}\left(\sum_{k=1}^{n^\alpha-1}\mathbb{P}(B_k)\right). \tag{2.4.28}$$

Again using the same argument, we have for any $k \leq n^\alpha - 1$,

$$\mathbb{P}(B_k) \leq \mathbb{P}(G_k, \Delta_k < \infty)\mathbb{P}_{(0,\log n)}\left(\tau_{U_n} < \tau_{L_0}\right) \leq n^{-1+6\alpha}$$

which implies that

$$\mathbb{P}(B) \leq n^{-2+14\alpha}. \tag{2.4.29}$$

Letting $\alpha = 1/16$, then Lemma 2.4.16 follows from Lemma 2.4.17 and (2.4.29). □

Proof of Theorem 2.4.5. At this point, Theorem 2.4.5 follows from the combination of Lemma 2.4.13 and Lemma 2.4.16. □

2.4.5 Proof of Theorem 2.4.1: Existence of the SDLA

Theorem 2.4.1 follows immediately once we show that the limiting process obtained by Theorem 2.4.5 has the desired property.

Lemma 2.4.18. *Fix a finite set K, $t > 0$ and some $\epsilon > 0$. $\exists N$ finite a.s., such that for all $n > N$, for all $0 \leq s \leq t$ and any $x \in K$,*

$$|\mathcal{H}_{L_0\cup A_s^n}(x) - \mathcal{H}_{L_0\cup A_s}(x)| < \epsilon. \tag{2.4.30}$$

Proof. By [1, Lemma 2.6] and the sub-linear growth of the interface model proved in Theorem 2.4.8 and the fact we constructed all A_s^n to be subsets of the interface model, there exists some $m > 0$ such that for every every $n \in \mathbb{N} \cup \{\infty\}$ and $x \in K$,

$$\left| \sum_{|y|<m^{1.1}} \mathbb{P}_{(y,m)} \left(S_{\tau_{L_0\cup A_s^n}} = x \right) - \mathcal{H}_{L_0\cup A_s^n}(x) \right| < \epsilon/2. \tag{2.4.31}$$

Let $K' \subset \mathbb{H}$ be a large finite subset such that

$$2m^{1.1} \max_{|y|<m^{1.1}} \mathbb{P}_{(y,m)}(\tau_{K'^c} < \tau_K) < \epsilon/2.$$

By Theorem 2.4.5 we know that there is some $N \in \mathbb{N}$ large enough such that for every $n > N$,

$$A_s^n \cap K' = A_s^N \cap K' = A_s \cap K'.$$

Thus

$$\left| \sum_{|y|<m^{1.1}} \mathbb{P}_{(y,m)} \left(S_{\tau_{L_0\cup A_s^n}} = x \right) - \sum_{|y|<m^{1.1}} \mathbb{P}_{(y,m)} \left(S_{\tau_{L_0\cup A_s}} = x \right) \right| < \epsilon/2.$$

Together with (2.4.31) we obtain (2.4.30).

□

It remains to prove that $\{A_s\}_{s\leq t}$ is Markov with the correct stationary harmonic measure as the infinitesimal generator.

Lemma 2.4.19. *For every finite subset $K \subset \mathbb{H}$ and any $t > 0$, for any $s \in [0,t]$ and $x \in K$,*

$$\lim_{\Delta s\to 0} \frac{\mathbb{P}\left(A_{s+\Delta s}(x) = 1 | A_s(x) = 0, \{A_\xi\}_{\xi\leq s}\right)}{\Delta s} = \mathcal{H}_{L_0\cup A_s}(x) \text{ a.s.}$$

Proof. Let $\epsilon > 0$ and G_n be the event that for all $s \leq t$ and for all $x \in K$, $A_s^n(x) = A_s(x)$ and in addition,

$$|\mathcal{H}_{L_0 \cup A_s^n}(x) - \mathcal{H}_{L_0 \cup A_s}(x)| < \epsilon.$$

By Lemma 2.4.18 and Theorem 2.4.5, $\lim_{n\to\infty} \mathbb{P}(G_n^c) = 0$. Now uniformly for all $s < t$ and Δs small enough, there is an $n \in \mathbb{N}$ such that

$$\begin{aligned}
&\mathbb{P}\left(A_{s+\Delta s}(x) = 1 | A_s(x) = 0, \{A_\xi\}_{\xi \le s}\right) \\
&\in \mathbb{P}\left(A_{s+\Delta s}(x) = 1 | A_s(x) = 0, \{A_\xi\}_{\xi \le s}, G_n\right) + (-\epsilon, \epsilon) \\
&= \mathbb{P}\left(A^n_{s+\Delta s}(x) = 1 | A^n_s(x) = 0, \{A_\xi\}_{\xi \le s}, G_n\right) + (-\epsilon, \epsilon) \\
&\in \mathbb{P}\left(A^n_{s+\Delta s}(x) = 1 | A^n_s(x) = 0, |\mathcal{H}_{L_0 \cup A_s^n}(x) - \mathcal{H}_{L_0 \cup A_s}(x)| < \epsilon, A_s\right) + (-2\epsilon, 2\epsilon) \\
&\in (1 - e^{-\Delta s(\mathcal{H}_{L_0 \cup A_s}(x) + \epsilon)}, 1 - e^{-\Delta s(\mathcal{H}_{L_0 \cup A_s}(x) - \epsilon)}) + (-2\epsilon, 2\epsilon),
\end{aligned}$$

where we use the dominated convergence theorem for the first and second approximations. Now taking $\epsilon \to 0$ and then $\Delta s \to 0$ we obtain the result. □

Proof of Theorem 2.4.1. By Lemma 2.4.19 we obtain that the almost sure limit

$$\{A_s\}_{s \le t} := \lim_{m\to\infty} \{A_s^m\}_{s \le t}$$

obtained in Theorem 2.4.5 is a SDLA. □

2.4.6 Proof of Theroem 2.4.3: Ergodocity of the SDLA

Proof. By Lemma 2.4.19 and the fact that the stationary harmonic measure is (well...) stationary, we obtain that A_t^∞ is stationary with respect to the translation $\lambda_n(A_t^\infty) = A_t^\infty + n$, for any $n \in \mathbb{Z}$. It is enough then to prove that A_t^∞ is strongly mixing. Let $t > 0$ and K_1, K_2 be two finite subsets of $\mathbb{H}$ of distance $\max\{|x_1 - x_2| : x_1 \in K_1, x_2 \in K_2\} > 4n$ (n will be chosen big enough). We now consider two copies of A_t^n constructed according to Poisson thinning of the same interface model, $A_t^n(1)$ is centered around an arbitrary point $x_1 \in K_1$ and $A_t^n(2)$ is centered around an arbitrary point $x_2 \in K_2$. For $i \in \{1,2\}$ and configurations $\xi_i \in \{0,1\}^{K_i}$. Define the events:

$$B_i = \{A_t^\infty \cap K_i = \xi_i\} \tag{2.4.32}$$

$$C_i = \{A_t^n(i) \cap K_i = \xi_i\} \tag{2.4.33}$$

$$D_i = \{\max_{x \in A_t^n(i)} |x - x_i| < 3n/2\} \tag{2.4.34}$$

Under the event $D_1 \cap D_2$ the events C_1 and C_2 are independent. This follows from the independence of Poisson processes on non intersecting domains. Moreover we know by Theorem 2.4.8 that

$$\lim_{n\to\infty} \mathbb{P}\left(D_1^c \cup D_2^c\right) = 0,$$

and by Theorem 2.4.5 that

$$\lim_{n\to\infty} \mathbb{P}\left(B_1 \setminus C_1 \cup B_2 \setminus C_2\right) = 0.$$

Thus

$$\lim_{n\to\infty} \mathbb{P}(B_1 \cap B_2) = \lim_{n\to\infty} \mathbb{P}(C_1 \cap C_2 | D_1 \cap D_2) = \lim_{n\to\infty} \mathbb{P}(C_1 | D_1 \cap D_2) \cdot \mathbb{P}(C_2 | D_1 \cap D_2) \tag{2.4.35}$$

$$= \lim_{n\to\infty} \mathbb{P}(B_1) \cdot \mathbb{P}(B_2) = \mathbb{P}(B_1) \cdot \mathbb{P}(B_2), \tag{2.4.36}$$

where in the last equality we used stationarity and abused notations to clarify that the limit is actually a constant sequence. □

We show that there exists a phase transition in FRI on $\mathbb{Z}^d$ with $d \geq 3$. This partially answered a question of Bowen (see Question 2, [25] for details). The content of this chapter appears in [3]. Consider $\mathcal{FI}^{u,T}$ as a random subgraph of $\mathbb{Z}^d$ (we will define $\mathcal{FI}^{u,T}$ in Section 3.1). For any two vertices $x, y \in \mathcal{FI}^{u,T}$, x and y are said to be connected if there exist vertices $x_0, x_1, \cdots, x_n \in \mathcal{FI}^{u,T}$ such that $x = x_0$, $y = x_n$, and (x_i, x_{i+1}) are edges in the graph $\mathcal{FI}^{u,T}$ for all $0 \leq i < n$.

Theorem 3.0.1 (Supercritical phase)**.** *For all $u > 0$, there is a $T_1(u, d) > 0$ such that for all $T > T_1$, $\mathcal{FI}^{u,T}$ has an unique infinite cluster almost surely.*

Theorem 3.0.2 (Subcritical phase)**.** *For all $u > 0$, there is a $T_0(u, d) > 0$ such that for all $0 < T < T_0$, $\mathcal{FI}^{u,T}$ has no infinite cluster almost surely.*

The proof of Theorem 3.0.1 relies on a renormalization/block construction argument along with coupling the FRI to RI. We define a good block event in Section 3.2, and we prove that this good event occurs with high probability in Section 3.3. In Section 3.4 we apply a standard renormalization/block construction argument to see the spread of our “good blocks" dominates a supercritical Bernoulli percolation. The proof of uniqueness is presented in Section 3.5. The proof of Theorem 3.0.2 is presented in Section 3.6.

3.1 Notations and Definitions

In this section, we collect some preliminary results on finitary random interlacements. Most of these results first appear in [25]. We begin with recalling the formal definition of FRI in [25]. Consider the lattice $\mathbb{Z}^d$, for $d \geq 3$. A finite walk on $\mathbb{Z}^d$ is a nearest-neighbor path $w : \{0, 1, \cdots, N\} \to \mathbb{Z}^d$, for some $N \in \mathbb{Z}_+ \cup \{0\}$. N is called the length of the finite walk w. Let $\mathrm{W}^{[0,\infty)}$ be the set of trajectories of all finite walks. And note that $\mathrm{W}^{[0,\infty)}$ is a countable set.

For $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, let $\mathbb{P}_x^n$ be the law of the simple random walk started at x and killed at time n. Define

$$\mathbb{P}_x^{(T)} = \left(\frac{1}{T+1}\right) \sum_{n=0}^{\infty} \left(\frac{T}{T+1}\right)^n \mathbb{P}_x^n.$$

I.e. $\mathbb{P}_x^{(T)}$ is the law of a geometrically killed simple random walk started at x with $1/(T+1)$ killing rate. The expected length is T. We sometimes call geometrically killed random walk a killed random walk.

For $0 < T < \infty$, let $v^{(T)}$ be the measure on $\mathrm{W}^{[0,\infty)}$ defined by

$$v^{(T)} = \sum_{x \in \mathbb{Z}^d} \frac{2d}{T+1} \mathbb{P}_x^{(T)}.$$

Note that $v^{(T)}$ is a σ-finite measure.

Definition 3.1.1. For $0 < u, T < \infty$, the finitary random interlacements (FRI) point process μ is a Poisson point process (PPP) on $\mathrm{W}^{[0,\infty)}$ with intensity measure $uv^{(T)}$.

Meanwhile, one may equivalently define $\mathcal{FI}^{u,T}$ constructively as follows:

Definition 3.1.2. For each vertex $x \in \mathbb{Z}^d$, define an independent Poisson random variable N_x with parameter $2du/(T+1)$. We start independent N_x geometrically killed random walks from x, and each of them has expected length T. The FRI can be defined as the point measure on $\mathrm{W}^{[0,\infty)}$ composed of all the geometrically killed random walk trajectories above from all vertices in $\mathbb{Z}^d$.

It is easy to see the two definitions above are equivalent:

Proposition 3.1.3. *The random point measure defined in Definition 3.1.2 is identically distributed as the Poisson point process defined in Definition 3.1.1.*

Proof. The equivalence follows directly from the standard construction of Poisson point process with a $\sigma-$finite intensity measure. See (4.2.1) of [26] for example. □

Remark 3.1.4. The construction in Definition 3.1.2 was informally described in Subsection 1.3.2, [25].

Remark 3.1.5. Without causing further confusion, we will use $\mathcal{FI}$ to denote both the Poisson point process on $\mathrm{W}^{[0,\infty)}$ and the random subgraph of $\mathbb{Z}^d$ it induces.

The rest of this section mainly concerns the distribution of paths within $\mathcal{FI}^{u,T}$ traversing a certain finite subset of $\mathbb{Z}^d$. Let $K \subset \mathbb{Z}^d$ be a finite subset. Let $W_K \subset \mathrm{W}^{[0,\infty)}$ be the set of all finite walks that visit K at least once. Define the stopping times

$$H_K(w) = \inf\{t \geq 0 : w(t) \in K\},$$

and

$$\tilde{H}_K(w) = \inf\{t \geq 1 : w(t) \in K\}.$$

For a finite path w, we say $H_K(w) = \infty$ if w vanishes before it hits the set K. Similar for $\tilde{H}_K(w) = \infty$. Define

$$W^{(2)} := \{(a, b) \in \mathrm{W}^{[0,\infty)} \times \mathrm{W}^{[0,\infty)} : a(0) = b(0)\}.$$

Let $K \subset L \subset \mathbb{Z}^d$ be finite subsets. For $x \in L \setminus K$, let $\xi_x^{(T)}$ be the measure on $W^{(2)}$ given by

$$\xi_x^{(T)}(\{(a, b)\}) = 2d \cdot 1_{\tilde{H}_L(a)=\infty} P_x^{(T)}(\{a\}) 1_{H_K(b)=\infty} P_x^{(T)}(\{b\}).$$

Define a measure $Q_{L,K}^{(T)}$ on $W^{(2)}$ by

$$Q_{L,K}^{(T)} = \sum_{x \in L \setminus K} \xi_x^{(T)}.$$

Define the concatenation map $\mathrm{Con} : W^{(2)} \to \mathrm{W}^{[0,\infty)}$ by

$$\mathrm{Con}(a, b) = \Big(a(len(a)), a(len(a) - 1), \cdots, a(0), b(1), \cdots, b(len(b))\Big).$$

Proposition 3.1.6 (Proposition 4.1 in [25]). *For any $0 < u, T < \infty$, let μ be FRI with parameters*

u, T and $K \subset L \subset \mathbb{Z}^d$ be finite subsets. Then $\mathbb{1}_{W_L \setminus W_K}\mu$ is a PPP with intensity measure $u \cdot Con_* Q_{L,K}^{(T)} = \mathbb{1}_{W_L \setminus W_K} uv^{(T)}$, where $Con_* Q_{L,K}^{(T)} = Q_{L,K}^{(T)} \circ Con^{-1}$ is the push-forward measure.

Corollary 3.1.7. *Let u, T, μ be as in Proposition 3.1.6 and $K \subset \mathbb{Z}^d$ be a finite subset. Then*

$$uv^{(T)}(W_K) = 2d \sum_{x \in K} \mathbb{P}_x^{(T)}(\tilde{H}_K = \infty).$$

Consequently,

$$\lim_{T \to \infty} \mathbb{P}\big(\mu(W_K) = 0\big) = e^{-2du \cdot cap(K)} = \mathbb{P}\big(\mathcal{I}^{2du} \cap K = \emptyset\big),$$

where $\mathcal{I}^u$ is the random interlacements at level u.

Proof. This follows from Proposition 3.1.6 and the fact that

$$\lim_{T \to \infty} \mathbb{P}_x^{(T)}(\tilde{H}_K = \infty) = \mathbb{P}_x(\tilde{H}_K = \infty).$$

□

Consider the space $\{0,1\}^{\mathbb{Z}^d}$ with the canonical product σ-algebra. For $u > 0$, let $\mathbb{P}^u$ be the unique probability measure on $\{0,1\}^{\mathbb{Z}^d}$ such that for all finite subset $K \subset \mathbb{Z}^d$,

$$\mathbb{P}^u(\{w \in \{0,1\}^{\mathbb{Z}^d} : w(x) = 0, \text{for all } x \in K\}) = e^{-u \cdot \text{cap}(K)},$$

i.e. $\mathbb{P}^u$ is the probability law for random interlacements at level u. For $0 < u, T < \infty$, let $\mathbb{P}^{u,T}$ be the probability measure on $\{0,1\}^{\mathbb{Z}^d}$ such that for all finite subset $K \subset \mathbb{Z}^d$,

$$\mathbb{P}^{u,T}(\{w \in \{0,1\}^{\mathbb{Z}^d} : w(x) = 0, \text{for all } x \in K\}) = e^{-2du \cdot \sum_{x \in K} P_x^{(T)}(\tilde{H}_K = \infty)},$$

i.e. $\mathbb{P}^{u,T}$ is the law for finitary random interlacements with parameters u, T.

Theorem 3.1.8 (Theorem A.2 in [25])**.** *For any $u > 0$, $\mathbb{P}^{u,T}$ converges to $\mathbb{P}^{2du}$ weakly as $T \to \infty$ in the space of probability measures on $\{0,1\}^{\mathbb{Z}^d}$.*

Let $K \subset \mathbb{Z}^d$ be a finite subset. Define the killed equilibrium measure by

$$e_K^{(T)}(x) := \mathbb{P}_x^{(T)}(\tilde{H}_K = \infty).$$

Define the killed capacity by

$$\text{cap}^{(T)}(K) := \sum_{x \in K} e_K^{(T)}(x).$$

Let

$$\tilde{e}_K^{(T)}(x) := \frac{e_K^{(T)}(x)}{\text{cap}^{(T)}(K)}$$

be the normalized equilibrium measure. Let $W_K^0 := \{w \in W_K : w(0) \in K\}$. Define a map

$$s_K : W_K \ni w \mapsto w^0 \in W_K^0,$$

where $w^0 = s_K(w)$ is the unique element of W_K^0 such that $w^0(i) = w(H_K(w)+i)$ for all $i \geq 0$ and $len(w^0) = len(w) - H_K(w)$. I.e. we keep the part of the trajectory of w after hitting K, and index the trajectory in a way such that the hitting of K occurs at time 0. If $m(\cdot)$ is a measure supported on K, then we define the measure

$$\mathbb{P}_m := \sum_{x \in K} m(x) \mathbb{P}_x^{(T)}$$

on W_K, for some $T > 0$.

Lemma 3.1.9. *For $0 < u, T < \infty$, let μ be FRI with parameters u, T and $K \subset \mathbb{Z}^d$ be a finite subset. Then $\mu_K = s_{K*}\mu$ is a PPP on W_K with intensity measure $2du \cdot cap^{(T)}(K)\mathbb{P}_{\tilde{e}_K^{(T)}}$.*

Proof. The proof follows from the Proposition 3.1.6 and properties of PPP (see Exercise 4.6(c) in [26]). □

As a consequence of Lemma 3.1.9, we have

$$K \cap \left(\bigcup_{w \in \text{Supp}(\mu_K)} \text{range}(w) \right) = K \cap \left(\bigcup_{w \in \text{Supp}(\mu)} \text{range}(w) \right),$$

where K, μ, μ_K are the same as in Lemma 3.1.9.

Lemma 3.1.10. *Let N_K be a Poisson random variable with parameter $2du \cdot cap^{(T)}(K)$, and $\{w_j\}_{j\geq 1}$ are i.i.d. killed random walks with distribution $\mathbb{P}_{\tilde{e}_K^{(T)}}$ and independent from N_K. Then the point measure*

$$\tilde{\mu}_K = \sum_{j=1}^{N_K} \delta_{w_j}$$

is a PPP on W_K with intensity measure $2du \cdot cap^{(T)}(K)\mathbb{P}_{\tilde{e}_K^{(T)}}$. In particular, $\tilde{\mu}_K$ has the same distribution as μ_K.

Proof. The proof follows from the construction of PPP (see section 4.2 in [26]) and the merging and thinning property of Poisson distribution. □

Remark 3.1.11. A similar result (Corollary 4.2) was proved in [25]. Here the previous two lemmas are stated in the form better suitable for the later use in this paper.

In this chapter, all positive constants $c, C, c_1, \cdots$ will depend on dimension d by default.

3.2 Definition of Good Boxes

In this section we define the "good" block event in which there is a locally generated large connected cluster in the corresponding “box". The viability of such event will be proved in the Section 3.3. Parts of the definition below are inspired by [27]. This also enables us to apply their estimates for regular interlacements in the next section.

Without loss of generality, we will always assume here the FRI's are constructed according to Definition 3.1.2. For any $u, T > 0$, the FRI $\mathcal{FI}^{u,T}$ is identically distributed as the union of two independent copies of FRI with intensity level $u/2$ and average stopping time T, i.e.

$$\mathcal{FI}^{u,T} = \mathcal{FI}_1^{u/2,T} \cup \mathcal{FI}_2^{u/2,T},$$

where $\mathcal{FI}_i^{u/2,T}$ is the i-th copy. Moreover, similar to [27], we may write

$$\mathcal{FI}_1^{u/2,T} = \bigcup_{j=1}^{d-2} \mathcal{FI}_{1,j}^{u/(2d-4),T}$$

where $\mathcal{FI}_{1,j}^{u/(2d-4),T}$ are i.i.d. copies of finitary interlacements with intensity level $u/(2d-4)$ and average stopping time T. For $x \in \mathbb{Z}^d$ and $R \in \mathbb{Z}_+$, let $B(x, R) := x + [-R, R]^d$ be a box of length R centered at x. Note that we define $B(x, R)$ differently in Chapter 2. We write $B(R) = B(0, R)$. Let $\hat{B}(R) := [-64R^2, 64R^2]^d$ be a box in the lattice $\mathbb{Z}^d$. We define some subboxes in $\hat{B}(R)$. For $0 \leq i \leq 8R$ and $1 \leq j \leq d$, let

$$x_{i,j} = (-32R^2 + 8Ri)\mathrm{e}_j,$$

where e_j is the j-th unit vector in $\mathbb{Z}^d$. Let

$$b_{i,j}(R) := x_{i,j} + [-R, R]^d \subset \hat{B}(R),$$

and

$$\hat{b}_{i,j}(R) := x_{i,j} + [-2R, 2R]^d \subset \hat{B}(R).$$

For any subset $A \subset \mathbb{Z}^d$, we define the internal vertex boundary of A by

$$\partial^{in} A := \{x \in A : \exists y \in \mathbb{Z}^d \setminus A \text{ such that } |x - y|_1 = 1\},$$

and define the external vertex boundary by

$$\partial^{out} A := \{x \in \mathbb{Z}^d \setminus A : \exists y \in A \text{ such that } |x - y|_1 = 1\}.$$

Recall the construction of FRI in Definition 3.1.2. Let $\mathcal{D}_i$ be the random subgraph in $\mathbb{Z}^d$ consisting of all trajectories of killed random walks starting in $B(0, 128R^2)$ in FRI $\mathcal{FI}_i^{u/2,T}$, for $i = 1, 2$, and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. For any subsets $A, B \subset \mathbb{Z}^d$ where A is connected, let $\mathcal{C}(A, B)$ be the connected component of $A \cup B$ containing A. Define the random set

$$\mathcal{C}_{i,j}(x) := \mathcal{C}\big(x, \hat{b}_{i,j}(R) \cap \mathcal{D}_1\big).$$

For $1 \le j \le d$, we define the "top" half of $\hat{B}(R)$ in the j-direction by

$$\hat{B}_j^+(R) = \big\{x \in \mathbb{R}^d : 0 < x_j \le 64R^2, \text{ and } -64R^2 \le x_i \le 64R^2, \text{ if } i \ne j\big\},$$

and define the "bottom" half of $\hat{B}(R)$ in the j-direction by

$$\hat{B}_j^-(R) = \big\{x \in \mathbb{R}^d : -64R^2 \le x_j < 0, \text{ and } -64R^2 \le x_i \le 64R^2, \text{ if } i \ne j\big\}.$$

Let

$$A_j^+(R) = \big\{x \in \mathbb{R}^d : 96R^2 \le x_j \le 128R^2, \text{ and } -128R^2 \le x_i \le 128R^2, \text{ if } i \ne j\big\},$$

and

$$A_j^-(R) = \big\{x \in \mathbb{R}^d : -128R^2 \le x_j \le -96R^2, \text{ and } -128R^2 \le x_i \le 128R^2, \text{ if } i \ne j\big\}.$$

Definition 3.2.1. We say $\hat{B}(R)$ is *good* if the following conditions hold:

1. For all $0 \le i \le 8R$ and $1 \le j \le d$, let

$$E_{i,j} := \Big\{x \in b_{i,j}(R) \cap \mathcal{D}_1 : \text{cap}\big(\mathcal{C}_{i,j}(x)\big) \ge R^{2(d-2)/3}\Big\}.$$

   We have $E_{i,j} \ne \emptyset$ for all i, j.

2. For all $0 \le i < 8R$ and $1 \le j \le d$, and for all $x \in E_{i,j}$, and $y \in E_{i+1,j}$,

$$\mathcal{C}_{i+1,j}(y) \cap \mathcal{C}\left(\mathcal{C}_{i,j}(x), \mathcal{D}_2\right) \ne \emptyset.$$

I.e., $\mathcal{C}_{i,j}(x)$ and $\mathcal{C}_{i+1,j}(y)$ are connected by $\mathcal{D}_2$.

3. For all $1 \leq j \leq d$, no geometrically killed random walks starting in $A_j^+(R)$ intersect with $\hat{B}_j^-(R)$, and no geometrically killed random walks starting in $A_j^-(R)$ intersects with $\hat{B}_j^+(R)$.

Remark 3.2.2. All conditions in Definition 3.2.1 are restrictions on the trajectories of the killed random walks starting in $B(0, 128R^2)$. This fact is crucial in the renormalization argument in Section 3.4.

Now we define the shift of the box $\hat{B}(R)$ in $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, let

$$\hat{B}_x(R) = 32R^2 x + \hat{B}(R).$$

We say that $\hat{B}_x(R)$ is *good* if $\hat{B}(R)$ is a good box in $\mathcal{FI}^{u,T} - 32R^2 x$.

Remark 3.2.3. Suppose x and y are two neighboring vertices in $\mathbb{Z}^d$, and both $\hat{B}_x(R)$ and $\hat{B}_y(R)$ are good, then by condition (3) in Definition 3.2.1 the connectivity event in $\hat{B}_x(R) \cap \hat{B}_y(R)$ can be generated only by the random walk paths starting in $B(x, 128R^2) \cap B(y, 128R^2)$, so we have a large connected component crossing $\hat{B}_x(R)$ and $\hat{B}_y(R)$.

Now we define a family $\{Y_x : x \in \mathbb{Z}^d\}$ of $\{0,1\}$-valued random variables given by

$$Y_x = \begin{cases} 1, & \text{if } \hat{B}_x(R) \text{ is good;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.1}$$

If there is an infinite open cluster in the lattice $\{Y_x\}_{x \in \mathbb{Z}^d}$, then by Remark 3.2.3 there is an infinite open cluster in the underlying original lattice. When $T = R^3$, we will show that $\hat{B}(R)$ is good with high probability for all sufficiently large R. Then we will use a renormalization argument to show that there is an infinite cluster in $\mathcal{FI}^{u,R^3}$ almost surely for large R.

Remark 3.2.4. For simplicity, we will assume $R \in \mathbb{Z}_+$ for the rest of this paper. For $R \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, one can replace R and R^2 by $\lfloor R \rfloor$ and $\lfloor R \rfloor^2$ respectively in the definition of good boxes, and all results will follow accordingly.

3.3 $\hat{B}(R)$ is Good with High Probability

In this section, we prove that $\hat{B}(R)$ is good with high probability. I.e.,

Theorem 3.3.1. *Consider the FRI $\mathcal{FI}^{u,R^3}$. For all $u > 0$, we have*

$$\lim_{R\to\infty} \mathbb{P}\big(Y_0 = 1\big) = 1.$$

To show Theorem 3.3.1, we will consider the following weaker version of conditions (1) and (2) in Definition 3.2.1:

(1*) For all $0 \le i \le 8R$ and $1 \le j \le d$, let

$$\tilde{\mathcal{C}}_{i,j}(x) := \mathcal{C}\big(x, \hat{b}_{i,j}(R) \cap \mathcal{FI}_1^{u,T}\big).$$

and

$$\tilde{E}_{i,j} := \Big\{x \in b_{i,j}(R) \cap \mathcal{FI}_1^{u,T} : \text{cap}\big(\tilde{\mathcal{C}}_{i,j}(x)\big) \ge R^{2(d-2)/3}\Big\}.$$

We have $\tilde{E}_{i,j} \neq \emptyset$ for all i, j.

(2*) For all $0 \le i < 8R$ and $1 \le j \le d$, and for all $x \in \tilde{E}_{i,j}$, and $y \in \tilde{E}_{i+1,j}$,

$$\tilde{\mathcal{C}}_{i+1,j}(y) \cap \mathcal{C}\left(\tilde{\mathcal{C}}_{i,j}(x), \mathcal{FI}_2^{u,T}\right) \neq \emptyset.$$

We first prove that condition (1^*) and (2^*) occur with high probability. Then we show that no killed random walk starting in $\mathbb{Z}^d \setminus B(128R^2)$ will reach $\hat{B}(R)$ with high probability. Combining these we know condition (1) and (2) in Definition 3.2.1 occur with high probability. We will show condition (3) occurs with high probability separately in Lemma 3.3.14.

We will often use the following large deviation bound for Poisson distributions.

Lemma 3.3.2 (equation 2.11 in [26])**.** *If X is a Poisson distribution with parameter λ, then*

$$\mathbb{P}\big(\lambda/2 \le X \le 2\lambda\big) \ge 1 - 2e^{-\lambda/10}.$$

3.3.1 Coupling of FRI and RI

In this subsection we introduce a coupling of FRI and RI that is crucial in the proof of Lemma 3.3.9. Let $K \subset \mathbb{Z}^d$ be a finite subset, and let $u, T > 0$. For any points $x \in K$, let $N_{x,u}$ be i.i.d. Poisson random variables with parameter $2du$. Let $\{Y_{x,T}^{(l,i)} + 1\}_{i=1}^{\infty}$ and $\{Y_{x,T}^{(r,i)} + 1\}_{i=1}^{\infty}$ be i.i.d. geometric random variables with parameter $1/(T+1)$. Moreover, for $i \in \mathbb{Z}_+$, let $\{S_{n,x}^{(l,i)}\}_{n=0}^{\infty}$ and $\{S_{n,x}^{(r,i)}\}_{n=0}^{\infty}$ be independent copies of simple random walks starting at x. Now we can construct a random point measure $\mathcal{I}^T(u, K)$ on $W^{[0,\infty)}$ as follows: for each $x \in K$ and $1 \le i \le N_{x,u}$, if

$$\{S_{n,x}^{(l,i)}\}_{n=0}^{Y_{x,T}^{(l,i)}} \cap K = \emptyset,$$

we add a delta measure on

$$\{S_{n,x}^{(r,i)}\}_{n=0}^{Y_{x,T}^{(r,i)}}$$

in $\mathcal{I}^T(u, K)$.

The following lemma is a consequence of Lemma 3.1.10. Let $\mu_K = \sum_{j=1}^{N_K} \delta_{w_j}$ be the restriction of FRI Poisson point measure on K with parameters u and T, where N_K is a Poisson random variable with parameter $2du \cdot \text{cap}^{(T)}(K)$, and $\{w_j\}_{j\ge 1}$ are i.i.d. killed random walks with distribution $\mathbb{P}_{\tilde{e}_K^{(T)}}$ and independent from N_K.

Lemma 3.3.3. *$\mathcal{I}^T(u, K)$ is identically distributed as μ_K.*

Proof. Notice that if we fix $x \in K$ and $1 \le i \le N_{x,u}$, then

$$\mathbb{P}\left(\{S_{n,x}^{(l,i)}\}_{n=0}^{Y_{x,T}^{(l,i)}} \cap K = \emptyset\right) = \mathbb{P}_x^{(T)}(\tilde{H}_K = \infty) = e_K^{(T)}(x).$$

By Lemma 3.1.10, μ_K is a PPP with intensity measure $2du \cdot \text{cap}^{(T)}(K)\mathbb{P}_{\tilde{e}_K^{(T)}}$, and by definition

$$e_K^{(T)}(x) = \text{cap}^{(T)}(K)\tilde{e}_K^{(T)}.$$

The result follows from the thinning property of Poisson distributions. □

Consider those trajectories in $\mathcal{I}^T(u, K)$ with length larger or equal to a fixed number $T_0 > 0$. We define the random point measure $\hat{\mathcal{I}}^{T,T_0}(u, K)$ as follows: for each $x \in K$ and $1 \leq i \leq N_{x,u}$, if

$$Y_{x,T}^{(r,i)} \geq T_0,$$

and

$$\{S_{n,x}^{(l,i)}\}_{n=0}^{Y_{x,T}^{(l,i)}} \cap K = \emptyset,$$

we add a delta measure on

$$\{S_{n,x}^{(r,i)}\}_{n=0}^{Y_{x,T}^{(r,i)}}$$

in $\hat{\mathcal{I}}^{T,T_0}(u, K)$. Note that by definition $\hat{\mathcal{I}}^{T,T_0}(u, K) \subset \mathcal{I}^T(u, K)$. Here we say $\mathcal{I}_1 \subset \mathcal{I}_2$ if all edges open in the support of $\mathcal{I}_1$ is also open in support of $\mathcal{I}_2$.

Now we construct a third random point measure $\bar{\mathcal{I}}^{T,T_0}(u, K)$ which is identically distributed as the collection of all trajectories within a RI traversing K, and we also define a $\tilde{\mathcal{I}}^{T,T_0}(u, K) \subset \bar{\mathcal{I}}^{T,T_0}(u, K)$ when all trajectories in $\bar{\mathcal{I}}^{T,T_0}(u, K)$ are truncated at a fixed time T_0. For each $x \in K$ and $1 \leq i \leq N_{x,u}$, if

$$Y_{x,T}^{(r,i)} \geq T_0,$$

and

$$\{S_{n,x}^{(l,i)}\}_{n=0}^{\infty} \cap K = \emptyset,$$

we add a delta measure on

$$\{S_{n,x}^{(r,i)}\}_{n=0}^{\infty}$$

in $\bar{\mathcal{I}}^{T,T_0}(u,K)$ and we add a delta measure on

$$\{S_{n,x}^{(r,i)}\}_{n=0}^{T_0}$$

in $\tilde{\mathcal{I}}^{T,T_0}(u,K)$. Note that by definition $\tilde{\mathcal{I}}^{T,T_0}(u,K) \subset \hat{\mathcal{I}}^{T,T_0}(u,K) \subset \bar{\mathcal{I}}^{T,T_0}(u,K)$ for any $T > 0$. Note that if $T_0 = 0$, $\bar{\mathcal{I}}^{T,0}(u,K)$ is identically distributed as set of all trajectories in $\mathcal{I}^{2du}$ traversing K but not including the backward parts before they enter K for the first time. We write $\bar{\mathcal{I}}^{T}(u,K) := \bar{\mathcal{I}}^{T,0}(u,K)$.

Lemma 3.3.4. *Let $Y+1$ be a geometric random variable with parameter $1/(T+1)$ independent from everything else, and $q = q(T,T_0) := P(Y \geq T_0)$. Let $\tilde{\mu}_K$ be restriction of RI at level $2duq$ on the set K. Then $\bar{\mathcal{I}}^{T,T_0}(u,K)$ is identically distributed to $\tilde{\mu}_K = \sum_{j=1}^{\tilde{N}_K} \delta_{\tilde{w}_j}$, where $\tilde{N}_K$ is Poisson random variable with parameter $2duq \cdot cap(K)$, and $\{\tilde{w}_j\}_{j\geq 1}$ are i.i.d. simple random walks with distribution $\mathbb{P}_{e_K}$ and independent from $\tilde{N}_K$.*

Proof. This is similar to the proof of Lemma 3.3.3. For $x \in \partial^{in} K$,

$$\mathbb{P}\bigg(\{S_{n,x}^{(l,i)}\}_{n=0}^{\infty} \cap K = \emptyset\bigg) = \mathbb{P}_x(\tilde{H}_K = \infty) = e_K(x).$$

Note that for all $x \in K \setminus \partial^{in} K$,

$$\mathbb{P}\bigg(\{S_{n,x}^{(l,i)}\}_{n=0}^{\infty} \cap K = \emptyset\bigg) = 0.$$

The result again follows from the thinning property of Poisson distributions. □

By Exercise 5.9 of [26], $\tilde{\mu}_K$ is the restriction of the PPP for RI at level $2duq$ on the set K.

3.3.2 Facts about capacity

We often use the following facts about capacity (or killed one) in our proof.

Lemma 3.3.5 (Proposition $6.5.2$ in [4])**.** *There are constants $c_1, c_2 > 0$ such that for all $R > 0$,*

$$c_1 R^{d-2} \leq \mathit{cap}\big(B(R)\big) \leq c_2 R^{d-2}.$$

Lemma 3.3.6 (Subadditivity of Capacity; Lemma 1.11 in [26])**.** *For any finite set $E_1, E_2 \subset \mathbb{Z}^d$,*

$$\mathit{cap}(E_1 \cup E_2) \leq \mathit{cap}(E_1) + \mathit{cap}(E_2).$$

Lemma 3.3.7 (Subadditivity of Killed Capacity)**.** *For any finite sets $E_1, E_2 \subset \mathbb{Z}^d$ and for all $T > 0$,*

$$\mathit{cap}^{(T)}(E_1 \cup E_2) \leq \mathit{cap}^{(T)}(E_1) + \mathit{cap}^{(T)}(E_2).$$

Proof. Follows the proof of Lemma 1.11 in [26] using the killed equilibrium measure. □

Lemma 3.3.8 (Monotonicity of Capacity; Exercise 1.15 in [26])**.** *For any finite sets $E_1 \subset E_2 \subset \mathbb{Z}^d$,*

$$\mathit{cap}(E_1) \leq \mathit{cap}(E_2).$$

3.3.3 Condition (1^*)

By translation invariance, one may without loss of generality prove the desired result for $i = 4R$ and $j = 1$. This this case, we have $x_{4R,1} = 0$, $b_{4R,1}(R) = B(R)$, and $\hat{b}_{4R,1}(R) = B(2R)$.

To begin with, let us consider the following random variable

$$N_{4R,1}^{(1)} = \left| \left\{ x \in B(R), \mathit{cap}\left(\mathcal{C}\left(x, \mathcal{FI}_{1,1}^{u/(2d-4),R^3} \cap B(R + R^{0.9}) \right) \right) > c_0 R^{0.7} \right\} \right|$$

and event $A_{4R,1}^{(1)} = \{N_{4R,1}^{(1)} \geq 1\}$, where $c_0 > 0$ the constant in Lemma 6, [27], which is independent to R. We first prove that

Lemma 3.3.9. *There is a constant $c = c(u) > 0$ such that for all sufficiently large R,*

$$\mathbb{P}(A_{4R,1}^{(1)}) \geq 1 - \exp(-cR).$$

Proof. Note that $N_{4R,1}^{(1)}$ is determined by trajectories within $\mathcal{FI}_{1,1}^{u/(2d-4),R^3}$ traversing $B(R)$, which can be sampled according to Subsection 3.3.1. Define

$$\hat{N}_{4R,1}^{(1)} := \Big|\Big\{(x,i) \in \partial^{in} B(R) \times \mathbb{Z}^+,\ s.t.\ i \le N_{x,u/(2d-4)},\ \{S_{n,x}^{(l,i)}\}_{n=1}^{\infty} \cap B(R) = \emptyset,$$
$$Y_{x,R^3}^{r,i} \ge R^{1.6},\ \{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}} \subset x + B(R^{0.9}),\ cap\left(\{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}}\right) > cR^{0.7}\Big\}\Big|.$$

By the definitions of $N_{4R,1}^{(1)}$, $\hat{N}_{4R,1}^{(1)}$, and Lemma 3.3.6, we have

$$\mathbb{P}\big(\hat{N}_{4R,1}^{(1)} \ge 1\big) \le \mathbb{P}\big(N_{4R,1}^{(1)} \ge 1\big) = \mathbb{P}\big(A_{4R,1}^{(1)}\big).$$

Note that for each (x,i), the events

$$\{i \le N_{x,u/(2d-4)}\},$$
$$\left\{\{S_{n,x}^{(l,i)}\}_{n=1}^{\infty} \cap B(R) = \emptyset\right\},$$
$$\left\{Y_{x,R^3}^{r,i} \ge R^{1.6}\right\},$$
$$\left\{\{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}} \subset x + B(R^{0.9}),\ cap\left(\{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}}\right) > cR^{0.7}\right\}$$

are independent to each other. At the same time

$$\mathbb{P}\left(\{S_{n,x}^{(l,i)}\}_{n=1}^{\infty} \cap B(R) = \emptyset\right) = e_{B(R)}(x)$$

while

$$\mathbb{P}\left(Y_{x,R^3}^{r,i} \ge R^{1.6},\ \{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}} \subset x + B(R^{0.9}),\ cap\left(\{S_{n,x}^{(r,i)}\}_{n=1}^{R^{1.6}}\right) > cR^{0.7}\right) = q_1(R) > 1/2$$

for all sufficiently large R. The last inequality is derived from

(1) The PMF estimate of geometric random variable $Y_{x,R^3}^{r,i}$.

(2) Hoeffding's inequality.

(3) Lemma 6, [27] with $T_1 = R^{1.6}$ and $\epsilon = 1/8$.

Thus we have

$$\hat{N}_{4R,1}^{(1)} \sim Poisson\Big(q_1(R)cap(B(R))u/(2d-4)\Big)$$

and the desired result follows from Lemma 3.3.2 and Lemma 3.3.5.

$\square$

Given event $A_{4R,1}^{(1)}$, one may sample a point uniformly at random from the random subset

$$S_{4R,1} = \Big\{x \in B(R), cap\left(\mathcal{C}\Big(x, \mathcal{FI}_{1,1}^{u/(2d-4),R^3} \cap B(R+R^{0.9})\Big)\right) > c_0 R^{0.7}\Big\}$$

and denote it by $x_{4R,1}^{(1)}$. Moreover, for the random subset

$$Com_{4R,1}^{(1)} = \mathcal{C}\Big(x_{4R,1}^{(1)}, \mathcal{FI}_{1,1}^{u/(2d-4),R^3} \cap B(R+R^{0.9})\Big)$$

by definition we have

$$cap\left(Com_{4R,1}^{(1)}\right) > cR^{0.7}.$$

Now for any $k = 2, 3, \cdots, d-2$ may define

$$Com_{4R,1}^{(k)} = \mathcal{C}\left(Com_{4R,1}^{(k-1)}, \mathcal{FI}_{1,k}^{u/(2d-4),R^3} \cap B(R+kR^{0.9})\right)$$

together with event

$$A_{4R,1}^{(k)} = \Big\{cap\big(Com_{4R,1}^{(k)}\big) > c_0^k R^{0.7k}\Big\}.$$

Note that for any $k = 2, 3, \cdots, d-2$, $Com_{4R,1}^{(k-1)}$ is measurable with respect to

$$\sigma_{k-1} = \sigma\left(\mathcal{FI}_{1,1}^{u/(2d-4),R^3}, \mathcal{FI}_{1,2}^{u/(2d-4),R^3}, \cdots, \mathcal{FI}_{1,k-1}^{u/(2d-4),R^3}\right)$$

which is independent to $\mathcal{FI}_{1,k}^{u/(2d-4),R^3}$. Thus for any connected component $\mathcal{C}_0^{(k-1)}$ within $B(R+$

$(k-1)R^{0.9}$) with

$$cap(\mathcal{C}_0^{(k-1)}) > c_0^k R^{0.7k}$$

given $Com_{4R,1}^{(k-1)} = \mathcal{C}_0^{(k-1)}$, the distribution of $Com_{4R,1}^{(k)}$ is determined by by the configuration of trajectories in $\mathcal{FI}_{1,k}^{u/(2d-4),R^3}$ traversing $\mathcal{C}_0^{(k-1)}$, which can again be sampled according to Subsection 5.1:

- For each $x \in \mathcal{C}_0^{(k-1)}$, let $N_{x,u/(2d-4)}^{(k)}$ be i.i.d. Poisson random variables independent to σ_{k-1} with intensity $u/(2d-4)$.
- For each $x \in \mathcal{C}_0^{(k-1)}$, and positive integer i, let $\{S_{n,x}^{(l,i,k)}\}_{n=1}^{\infty}$ and $\{S_{n,x}^{(r,i,k)}\}_{n=1}^{\infty}$ be independent simple random walks starting from x.
- For each $x \in \mathcal{C}_0^{(k-1)}$, and positive integer i, let $Y_{x,R^3}^{r,i,k}$ and $Y_{x,R^3}^{l,i,k}$ be independent geometric random variables with parameter $p = 1/(1+R^3)$.

Then recalling the construction in Subsection 5.1, one has

$$\begin{aligned}
&\mathbb{P}\left(A_{4R,1}^{(k)} \middle| Com_{4R,1}^{(k-1)} = \mathcal{C}_0^{(k-1)}\right) \\
\geq &\mathbb{P}\left(cap\left(\bigcup_{(x,i)\in I_{4R,1}^{(k-1)}} \{S_{n,x}^{(r,i,k)}\}_{n=1}^{R^{1.6}}\right) > c_0^k R^{0.7k},\ \{S_{n,x}^{(r,i,k)}\}_{n=1}^{R^{1.6}} \subset x + B(R^{0.9}),\ \forall (x,i) \in I_{4R,1}^{(k-1)}\right)
\end{aligned}$$

where

$$I_{4R,1}^{(k-1)} = \left\{(x,i) \in \partial^{in}\mathcal{C}_0^{(k-1)} \times \mathbb{Z}^+,\ s.t.\ i \leq N_{x,u/(2d-4)}^{(k)},\ \{S_{n,x}^{(l,i,k)}\}_{n=1}^{\infty} \cap \mathcal{C}_0^{(k-1)} = \emptyset,\ Y_{x,R^3}^{r,i,k} \geq R^{1.6}\right\}.$$

Then again by Lemma 6 and Lemma 8 of [27],

$$\mathbb{P}\left(A_{4R,1}^{(k)} \middle| Com_{4R,1}^{(k-1)} = \mathcal{C}_0^{(k-1)}\right) \geq 1 - \exp(-R^{1/17})$$

for all sufficiently large R. Thus we have proved that

$$\mathbb{P}(E_{4R,1} \neq \emptyset) \geq \mathbb{P}\left(\bigcap_{k=1}^{d-2} A_{4R,1}^{(k)}\right) \geq 1 - \exp(-R^{1/18}) \tag{3.3.1}$$

for all sufficiently large R.

3.3.4 Condition (2^*)

Again, Condition (2*) can be without loss of generality checked for $b_{4R,1}(R)$ and $b_{4R+1,1}(R)$. And one may follow a similar argument as Subsection 3.3.3 to check Condition (2*). To be precise, one can pick any two points x_0, x_1 from $E_{4R,1}$ and $E_{4R+1,1}$. Then we can look at the paths in $\mathcal{FI}_2^{u/2,R^3}$ (which is independent to $\mathcal{FI}_1^{u/2,R^3}$) traversing $\mathcal{C}_{4R,1}(x_0)$. We keep only those whose backward part never returning to $\mathcal{C}_{4R,1}(x_0)$ while the forward part is not truncated until the $R^{2.5}$th step. Then one can apply Lemma 11 and 12 in [27] for intensity $u/4$ to prove that with stretch exponentially high probability, at least one of the paths we kept in the procedure above has to intersect with $\mathcal{C}_{4R+1,1}(x_1)$ before they exit $B(4Re_1, CR)$, where C is the same constant as in Lemma 11 of [27].

However, since for the finitary interlacements, one can only guarantee that the first $R^{2.5}$ steps in the froward paths we keep are within $\mathcal{FI}_2^{u/2,R^3}$. So the only extra estimate needed is the following lower bound on the first exiting time of $B(CR)$.

Lemma 3.3.10. *There is a $c > 0$ independent to R such that*

$$\mathbb{P}_0(H_{\partial^{out} B(CR)} > R^{2.5}) < \exp(-cR^{0.5}).$$

Proof. By central limit theorem/invariance principle, there is a constant $c > 0$ such that

$$\sup_{x \in B(CR)} \mathbb{P}_x(H_{\partial^{out} B(CR)} > R^2) \leq \mathbb{P}_0(H_{\partial^{out} B(2CR)} > R^2) \leq 1 - c < 1. \tag{3.3.2}$$

Then for each $i = 1, 2, \cdots, \lfloor R^{0.5} \rfloor$, consider event

$$Es_i = \{H_{\partial^{out} B(CR)} > i \cdot R^2\}.$$

Then by (3.3.2) and Markov property we have

$$\mathbb{P}_0(Es_1) \leq 1 - c,$$

and

$$\mathbb{P}_0(Es_{i+1} | Es_i) \leq \sup_{x \in B(CR)} \mathbb{P}_x(H_{\partial^{out} B(CR)} > R^2) \leq 1 - c,$$

for all $i \geq 1$. Thus

$$\mathbb{P}_0(H_{\partial^{out} B(CR)} > R^{2.5}) \leq \mathbb{P}_0(Es_{\lfloor R^{0.5} \rfloor}) \leq (1-c)^{\lfloor R^{0.5} \rfloor} < \exp(-cR^{0.5}).$$

□

Remark 3.3.11. An alternative argument following (2.9) of [28] derives a slightly weaker result, but also suitable for the use here.

3.3.5 Condition (1) and (2)

We recall the construction of FRI in Definition 3.1.2. We first show that with high probability no killed random walks of $\mathcal{FI}^{u,R^3}$ starting in $\mathbb{Z}^d \setminus B(128R^2)$ intersect with $\hat{B}(R)$. Define the event

$$G(u,R) := \Big\{\text{No killed random walks of } \mathcal{FI}^{u,R^3} \text{ starting in } \mathbb{Z}^d \setminus B(128R^2) \text{ reach } \hat{B}(R)\Big\}.$$

Lemma 3.3.12. *For all $u > 0$, we have*

$$\lim_{R \to \infty} \mathbb{P}\big(G(u,R)\big) = 1.$$

Proof. We first fix $u > 0$ and $R > 0$. We define a sequence of subsets $\{A(m,R)\}_{m=1}^{\infty}$ of $\mathbb{Z}^d$. Let

$$A(1,R) := B\big((128+64)R^2\big) \setminus \hat{B}(R),$$

and for all $m > 1$,

$$A(m,R) := B\big((128+64m)R^2\big) \setminus B\big((128+64(m-1))R^2\big)$$

Note that $\{A(m,R)\}_{m=1}^{\infty}$ are pairwise disjoint, and

$$\mathbb{Z}^d = \left(\hat{B}(R) \cup \bigcup_{m=1}^{\infty} A(m,R)\right).$$

Let $x \in A(m,R) \cap \mathbb{Z}^d$ for some $m \geq 1$. Recall the construction of FRI in Definition 3.1.2. Let N_x be the number of killed random walks starting at x. So N_x is a Poisson distribution with parameter $2du/(R^3+1)$. By Markov inequality, for all sufficiently large R,

$$\mathbb{P}\left(N_x > \frac{2dumR^4}{R^3+1}\right) \leq \mathbb{E}[e^{N_x}]e^{-2dumR^4/(R^3+1)} \leq c_1 e^{-c_2 mR},$$

for some constants $c_1(u), c_2(u) > 0$. We also need to estimate the probability that a killed random walk escape from a big box. If Y is a geometric random variable with parameter R^3, then for all sufficiently large R,

$$\mathbb{P}(Y > R^{7/2}) \leq e^{-cR^{1/2}}, \tag{3.3.3}$$

for some $c > 0$ independent of R. By Azuma's inequality and the tail estimate of geometric distribution in (3.3.3), for all sufficiently large R,

$$\mathbb{P}_0^{(R^3)}\big(H_{B(64R^2)} < \infty\big) \leq e^{-c_3 R^{1/2}},$$

for some $c_3 > 0$. If $x \in A(m,R) \cap \mathbb{Z}^d$, then a geometrically killed random walk must escape from m boxes of size $64R^2$ before it reaches $\hat{B}(R)$. By the memoryless property of geometric distribution,

$$\mathbb{P}_x^{(R^3)}\big(H_{\hat{B}(R)} < \infty\big) \le e^{-c_3 m R^{1/2}}.$$

Note that the number of vertices in $A(m,R)$ is bounded above by $c_4 m^d R^{2d}$, for some $c_4 > 0$. So by union bound,

$$\mathbb{P}\big(G(u,R)^{\mathrm{c}}\big) \le \sum_{m=1}^{\infty}\left(c_4 m^d R^{2d} c_1 e^{-c_2 m R} + c_4 m^d R^{2d}\frac{2dumR^4}{R^3+1}e^{-c_3 m R^{1/2}}\right),$$

for all sufficiently large R. Let

$$S(R) := \sum_{m=1}^{\infty}\left(c_4 m^d R^{2d} c_1 e^{-c_2 m R} + c_4 m^d R^{2d}\frac{2dumR^4}{R^3+1}e^{-c_3 m R^{1/2}}\right).$$

Note that the sum $S(R)$ converges for all $R > 0$, and

$$S(R) \xrightarrow{R\to\infty} 0.$$

Therefore,

$$\mathbb{P}\big(G(u,R)^{\mathrm{c}}\big) \xrightarrow{R\to\infty} 0.$$

□

Lemma 3.3.13. *Let $u > 0$. Consider the FRI $\mathcal{FI}^{u,R^3}$. Then*

$$\lim_{R\to\infty} \mathbb{P}\Big(\textit{Conditions } (1) \textit{ and } (2) \textit{ are satisfied}\Big) = 1.$$

Proof. The result follows by the discussions in Subsections 3.3.3 and 3.3.4, and Lemma 3.3.12.

□

3.3.6 Condition (3)

By translation invariance and symmetry, it suffices to show the following lemma.

Lemma 3.3.14. *Let $u > 0$, then there are constants $c(u), C(u) > 0$ such that for all sufficiently large $R > 0$, we have*

$$\mathbb{P}\Big(\exists \text{ a killed random walk starting in } A_1^+(R) \text{ reach } \hat{B}_1^-(R)\Big) \leq cR^{2d+1}e^{-CR^{1/2}}.$$

Proof. One can easily adapt the calculations in the proof of Lemma 3.3.12. The result follows from Definition 3.1.2, and tail estimates of geometric and Poisson distributions, and Azuma's inequality. □

3.4 Renormalization and Proof of Existence

Recall the family $\{Y_x\}_{x\in\mathbb{Z}^d}$ of $\{0,1\}$-valued random variables defined in (3.2.1). In this section, we show that $\{Y_x\}$ stochastically dominates an i.i.d. supercritical site percolation when R is sufficiently large and thus it has an infinite open cluster almost surely.

Remark 3.4.1. Note that $\{Y_x\}_{x\in\mathbb{Z}^d}$ themselves form a finitely dependent percolation, and that the probability that each edge is open is high enough. An alternative "block construction" approach according to Durrett and Griffeath, [29] can also give us the desired result.

Lemma 3.4.2. *For any $u > 0$ and for all $R > 0$ that is sufficiently large (depending on u), the random field $\{Y_x\}_{x\in\mathbb{Z}^d}$ generated by $\mathcal{FI}^{u,R^3}$ stochastically dominates an i.i.d. site percolation $\{Z_x\}_{x\in\mathbb{Z}^d}$ such that $P(Z_0 = 1) > p_c(\mathbb{Z}^d)$, where $p_c(\mathbb{Z}^d)$ is the critical probability of site percolation on $\mathbb{Z}^d$.*

Proof. By the definition of good boxes in Section 3.2 and Remark 3.2.2, the random field $\{Y_x\}_{x\in\mathbb{Z}^d}$ is 9-dependent. The stochastic domination over an i.i.d supercritical site percolation follows from the domination by product measures result by Liggett, Schramm, and Stacey [30] and Theorem 3.3.1. □

Corollary 3.4.3. *For any $u > 0$ and for all $R > 0$ that is sufficiently large (depending on u), $\mathcal{FI}^{u,R^3}$ has an infinite cluster almost surely.*

Proof. We can choose the same R as in Lemma 3.4.2. By the definition of good boxes and Remark 3.2.3, $\mathcal{FI}^{u,R^3}$ has an infinite cluster if $\{Y_x\}_{x\in\mathbb{Z}^d}$ has one. $\square$

Now back to the proof of Theorem 3.0.1, for any $u > 0$ and sufficiently large T, one may let $R = \lfloor T^{1/3} \rfloor$ and the proof is complete. $\square$

3.5 Uniqueness of Infinite Cluster

We have shown that the FRI $\mathcal{FI}^{u,R^3}$ has an infinite cluster almost surely if $R > R_0(u)$, for some $R_0(u) > 0$. In this section, we show that the infinite cluster of $\mathcal{FI}^{u,R^3}$ is unique almost surely. Let $x \in \mathbb{Z}^d$, we define the canonical lattice shift

$$T_x : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$$

by $\big(T_x(\xi)\big)(y) = \xi(y+x)$, for any $\xi \in \{0,1\}^{\mathbb{Z}^d}$ and $y \in \mathbb{Z}^d$. We will first show that FRI is ergodic with respect to lattice shifts.

Lemma 3.5.1. *Let $\mu = \mu_{u,T}$ be the PPP measure for $\mathcal{FI}^{u,T}$. For any $x \in \mathbb{Z}^d$ and any $u, T > 0$, the map T_x preserves the measure μ.*

Proof. Fix $x \in \mathbb{Z}^d$. By Dynkin's π-λ Lemma, it suffices to show that for any finite subset $K \subset \mathbb{Z}^d$,

$$\mathbb{P}\big(\mathcal{FI}^{u,T} \cap (K - x) = \emptyset\big) = \mathbb{P}\big(\mathcal{FI}^{u,T} \cap K = \emptyset\big) = e^{-2du\cdot\text{cap}^{(T)}(K)}.$$

Note that

$$\mathbb{P}\big(\mathcal{FI}^{u,T} \cap (K - x) = \emptyset\big) = e^{-2du\cdot\text{cap}^{(T)}(K-x)} = e^{-2du\cdot\text{cap}^{(T)}(K)}.$$

The proof is complete. $\square$

Let $x \in \mathbb{Z}^d$, define the evaluation map

$$\Phi_x : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$$

by $\Phi_x(\xi) = \xi(x)$. We write $\sigma(\cdot)$ for the product σ-algebra generated by a set or the σ-algebra generated by a set of functions. The following lemma is a classical approximation result.

Lemma 3.5.2. *Let* $\big(\{0,1\}^{\mathbb{Z}^d}, \sigma(\{0,1\}^{\mathbb{Z}^d}), Q\big)$ *be a probability space, and let* $B \in \sigma(\{0,1\}^{\mathbb{Z}^d})$, *then for any* $\epsilon > 0$, *there is a finite subset* $K \subset \mathbb{Z}^d$ *and* $B_\epsilon \in \sigma(\Phi_x : x \in K)$ *such that*

$$Q\big(B \triangle B_\epsilon\big) \le \epsilon.$$

We need one more auxiliary lemma.

Lemma 3.5.3. *Let* $K \subset \mathbb{Z}^d$ *be a finite subset, and* $K_1 \subset K$, *and* $K_0 = K \setminus K_1$. *Then for all* $u, T > 0$,

$$\mathbb{P}\big(\mathcal{FI}^{u,T} \cap K = K_1\big) = \sum_{K' \subset K_1} (-1)^{|K'|} e^{-2du \cdot cap^{(T)}(K' \cup K_0)}.$$

Proof. This follows from inclusion-exclusion formula (see equaiton 2.1.3 of [26] for a similar result in RI). □

Proposition 3.5.4. *For any* $u, T > 0$ *and any* $0 \neq x \in \mathbb{Z}^d$, *the measure preserving map* T_x *is ergodic with respect to the FRI measure* $\mu = \mu_{u,T}$.

Proof. This is similar to the proof of ergodicity for RI (see Theorem 2.1 of [14]). Fix $0 \neq x \in \mathbb{Z}^d$ and $u, T > 0$. By Lemma 3.5.2, it suffices to show for any finite subset $K \subset \mathbb{Z}^d$ and $B_\epsilon \in \sigma(\Phi_x : x \in K)$, we have

$$\mu\big(B_\epsilon \cap T_x^n(B_\epsilon)\big) = \mu(B_\epsilon)^2. \tag{3.5.1}$$

From (3.5.1), one can deduce that for any invariant $A \in \sigma(\{0,1\}^{\mathbb{Z}^d})$,

$$\mu(A) = \mu(A)^2,$$

so $\mu(A) \in \{0,1\}$. In order to prove (3.5.1), we first claim for any finite subsets $K_1, K_2 \subset \mathbb{Z}^d$,

$$\lim_{|z| \to \infty} \text{cap}^{(T)}\big(K_1 \cup (K_2 + z)\big) = \text{cap}^{(T)}(K_1) + \text{cap}^{(T)}(K_2). \tag{3.5.2}$$

The proof of (3.5.2) is exactly the same as the RI case (see equation $2.2.5$ in [26]). If A is a cylinder event supported on a finite set $K \subset \mathbb{Z}^d$, i.e. A is of the from

$$A = \{\mathcal{FI}^{u,T} \cap K = K_1\},$$

where $K_1 \subset K$. Denote $K_0 := K \setminus K_1$. Take n large enough such that $K \cap (K + nx) = \emptyset$. By Lemma 3.5.3,

$$\begin{aligned} &\mu\big(A \cap T_x^n(A)\big) \\ &= \mu\Big(\mathcal{FI}^{u,T} \cap (K \cup (K+nx)) = K_1 \cup (K_1 + nx)\Big) \\ &= \sum_{K' \subset K_1} \sum_{K'' \subset K_1} (-1)^{|K'|+|K''|} \exp\Big(-2du \cdot \text{cap}^{(T)}\big((K' \cup K_0) \cup ((K'' \cup K_0) + nx)\big)\Big). \end{aligned} \tag{3.5.3}$$

By (3.5.2) and Lemma 3.5.3, we have

$$\begin{aligned} &\lim_{n\to\infty} \mu\big(A \cap T_x^n(A)\big) \\ &= \sum_{K' \subset K_1} \sum_{K'' \subset K_1} (-1)^{|K'|+|K''|} \exp\Big(-2du\big(\text{cap}^{(T)}(K' \cup K_0) + \text{cap}^{(T)}(K'' \cup K_0)\big)\Big) \\ &= \sum_{K' \subset K_1} (-1)^{|K'|} e^{-2du \cdot \text{cap}^{(T)}(K' \cup K_0)} \sum_{K'' \subset K_1} (-1)^{|K''|} e^{-2du \cdot \text{cap}^{(T)}(K'' \cup K_0)} \\ &= \mu(A)^2. \end{aligned} \tag{3.5.4}$$

Note that all events in $\sigma(\Phi_x : x \in K)$ can be extended by cylinder events in form of event A. The proof is complete. $\square$

Theorem 3.5.5. *For any $u > 0$ and for all sufficiently large $R > 0$ (depending on u), $\mathcal{FI}^{u,R^3}$ has a unique infinite open cluster almost surely.*

Proof. We adapt the proof of uniqueness in percolation model by Burton and Keane [9] (see Grimmett [8]). Fix $u > 0$. Let N be the number of infinite open clusters in $\mathcal{FI}^{u,R^3}$. Since N is translation-invariant, N is constant almost surely by Proposition 3.5.4. By Corollary 3.4.3, there is a $R_0(u) > 0$ such that for all $R > R_0$, $\mathcal{FI}^{u,R^3}$ has an infinite open cluster almost surely. We fix

$R > R_0$, so $P(N=0)=0$. Suppose $P(N=k)=1$ for $2 \le k < \infty$. Let $M_{B(n)}$ be the number of infinite open clusters in $\mathcal{FI}^{u,R^3}$ intersecting $B(n)$. Noting that

$$\mathbb{P}\big(M_{B(n)} \ge 2\big) \xrightarrow{n\to\infty} \mathbb{P}(N \ge 2) = 1,$$

there has to be a n such that

$$\mathbb{P}\big(M_{B(n)} \ge 2\big) > 0.$$

Recall Definition 3.1.2. Let $F_{1,0}$ be the subgraph in $\mathbb{Z}^d$ generated by paths starting from $B(n-1)$, $F_{1,1}$ be the subgraph in $\mathbb{Z}^d$ generated by paths starting from $\partial^{in} B(n)$, and $F_1 = F_{1,0} \cup F_{1,1}$. Moreover, let F_0 be the subgraph in $\mathbb{Z}^d$ generated by paths starting from $B^c(n)$.

Note that $F_{1,0}$ and $F_{1,1}$ may only have countable many configurations, there has to be a pair of (finite) configurations $\mathcal{F}_{1,0}$ and $\mathcal{F}_{1,1}$, and a $j \ge 2$ such that

$$\mathbb{P}\big(M_{B(n)} = j,\ F_{1,0} = \mathcal{F}_{1,0},\ F_{1,1} = \mathcal{F}_{1,1}\big) > 0,$$

which implies that

$$\mathbb{P}\big(F_0 \cup \mathcal{F}_{1,0} \cup \mathcal{F}_{1,1} \text{ has } k \text{ infinite components, among which } j \text{ components intersect } B(n)\big) > 0.$$

We denote the last event by A_0 and note that A_0 is measurable with respect to F_0 and thus independent to $F_{1,0}$ and $F_{1,1}$.

Now let $\hat{\mathcal{F}}_{1,1} = \mathcal{F}_{1,0} \cup \mathcal{F}_{1,1} \setminus B(n-1)$, and let

$$\hat{\mathcal{F}}_{1,0} = \{x \pm \mathrm{e}_j,\ x \in B(n-1),\ j = 1, 2, \cdots, d\}$$

be the collection of all edges starting from $B(n-1)$ (or all the edges within $B(n)$). One can immediately see that

$$\mathbb{P}\big(A_0, F_{1,0} = \hat{\mathcal{F}}_{1,0}, F_{1,1} = \hat{\mathcal{F}}_{1,1}\big) = \mathbb{P}\big(A_0\big)\mathbb{P}\big(F_{1,0} = \hat{\mathcal{F}}_{1,0}, F_{1,1} = \hat{\mathcal{F}}_{1,1}\big) > 0.$$

However, given the event above, note that

$$F_0 \cup F_1 = F_0 \cup \mathcal{F}_{1,0} \cup \mathcal{F}_{1,1} \cup \hat{\mathcal{F}}_{1,0}.$$

Since $\hat{\mathcal{F}}_{1,0}$ contains all the edges within $B(n)$, all the j components in $F_0 \cup \mathcal{F}_{1,0} \cup \mathcal{F}_{1,1}$ intersecting $B(n)$ merge to one, and the FRI with positive probability only has $k - j + 1$ infinite components. This contradicts with $\mathbb{P}(N = k) = 1$.

Now suppose $\mathbb{P}\big(N = \infty\big) = 1$. We say a point $x \in \mathbb{Z}^d$ is a trifurcation if:

1. x is in an infinite open cluster of $\mathcal{FI}^{u,R^3}$;
2. there exist exactly three open edges incident to x;
3. removing the three open edges incident to x will split this infinite open cluster of x into exactly three disjoint infinite open clusters.

Define the event $A_x := \{x \text{ is a trifurcation}\}$. By translation invariance, $\mathbb{P}(A_x)$ is constant for all $x \in \mathbb{Z}^d$. Therefore,

$$\frac{1}{|B(n)|}\mathbb{E}\left[\sum_{x \in B(n)} \mathbb{1}_{A_x}\right] = \mathbb{P}(A_0).$$

Recall that $M_{B(n)}$ is the number of infinite open clusters in $\mathcal{FI}^{u,R^3}$ intersecting $B(n)$. Note that

$$\mathbb{P}\big(M_{B(n)} \geq 3\big) \xrightarrow{n\to\infty} \mathbb{P}(N \geq 3) = 1.$$

Define the event

$$E_n := \Big\{\text{No killed random walks starting in } \mathbb{Z}^d \setminus B(2n) \text{ intersects } B(n)\Big\}.$$

By Lemma 3.3.12, the probability of event E_n^c decays stretch exponentially. We can choose n large enough such that

$$\mathbb{P}\big(M_{B(n)} \geq 3, E_n\big) > 1/2.$$

Similarly, let F_1 and F_2 be the random subgraphs in $\mathbb{Z}^d$ generated by the trace of all killed random walks starting in $B(n)$ and $B(2n) \setminus B(n)$, respectively. Note that F_1 and F_2 are independent. Since there are only countably many choices for F_1 and F_2, there exist two finite subgraphs $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathbb{Z}^d$ such that

$$\mathbb{P}\big(M_{B(n)} \geq 3, E_n, F_1 = \mathcal{F}_1, F_2 = \mathcal{F}_2\big) > 0.$$

If $\omega \in \{M_{B(n)} \geq 3, E_n, F_1 = \mathcal{F}_1, F_2 = \mathcal{F}_2\}$, then there exist $x(\omega), y(\omega), z(\omega) \in \partial^{in} B(n)$ lying in three distinct infinite open clusters in $\mathbb{Z}^d \setminus B(n)$. There are three paths connecting the origin and x, y, z, respectively, in the following way:

1. 0 is the unique common vertex in any two paths;
2. each path touches exactly one vertex in $\partial^{in} B(n)$.

Let $D_{x,y,z,n}$ be the event that:

1. there are exactly three killed random walks starting at the origin;
2. these three killed random walk paths end at x, y, z, respectively, and they satisfy the conditions above;
3. no killed random walks start at any vertices in $B(n) \setminus \{0\}$.

It is easy to see that $\mathbb{P}(D_{x,y,z,n}) > 0$ for all $n > 0$ and all distinct $x, y, z \in \partial^{in} B(n)$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are fixed and finite,

$$\mathbb{P}\Big(F_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \setminus B(n)\Big) > 0.$$

For $\omega \in \{M_{B(n)} \geq 3, E_n, F_1 = \mathcal{F}_1, F_2 = \mathcal{F}_2\}$, we can resample all N_x for $x \in B(2n)$, and then we resample all killed random walk paths starting in $B(2n)$ accordingly. Note that the resulting graph is still distributed as FRI $\mathcal{FI}^{u,R^3}$. If the events $D_{x,y,z,n}$ and $\{F_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \setminus B(n)\}$ occur after the resample, then 0 is a trifucation. Therefore,

$$\mathbb{P}\big(A_0\big) \geq \mathbb{P}\Big(D_{x,y,z,n}\Big)\mathbb{P}\Big(F_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \setminus B(n)\Big)\mathbb{P}\Big(M_{B(n)} \geq 3, E_n, F_1 = \mathcal{F}_1, F_2 = \mathcal{F}_2\Big) > 0.$$

Now we can apply the same finite energy argument in Burton and Keane [9]. For each trifurcation $t \in B(n)$, there is a one-to-one corresponding point $y_t \in \partial^{in} B(n)$. However, the number of trifurcation points grow in $B(n)$ as n^d, but $\partial^{in} B(n)$ grows as n^{d-1}. We have a contradiction. □

3.6 Subcritical Phase

In this section we present the proof of Theorem 3.0.2.

Proof of Theorem 3.0.2. We use the Peierls argument [13]. Fix $u > 0$. Let $\mathcal{C}$ be the connected component that contains the origin in the FRI, $\mathcal{FI}^{u,T}$. It suffices to show that there is a constant $T_0(u) > 0$ such that for all $0 < T < T_0$,

$$\mathbb{P}\big(|\mathcal{C}| = \infty\big) = 0.$$

We say a path is self-avoiding if it does not visit the same edge twice. Note that the number of self-avoiding paths in $\mathbb{Z}^d$ which have length n and start at the origin is bounded above by $(2d)^n$. Let $N(n)$ be the number of such paths which are open. If the origin belongs to an infinite open cluster, then there are open self-avoiding paths starting at the origin of all lengths. So for all $n > 0$,

$$\mathbb{P}\big(|\mathcal{C}| = \infty\big) \leq \mathbb{P}\big(N(n) \geq 1\big) \leq \mathbb{E}\big[N(n)\big].$$

Let γ be a self-avoiding path that has length n and starts at the origin. We want to estimate the probability that γ is open. Let N_γ be the number of killed random walks that traverse γ. Recall that N_γ is a Poisson random variable with parameter $2du \cdot \text{cap}^{(T)}(\gamma)$. Since the path γ has length

n, it has $n+1$ vertices. By the subadditivity of killed capacity,

$$\text{cap}^{(T)}(\gamma) \le n+1,$$

for all $T > 0$. By exponential Markov inequality,

$$\begin{aligned}
&\mathbb{P}\Big(N_\gamma > 2du \cdot e(n+1) + (n+1)\log(3d)\Big) \\
&\le \frac{\mathbb{E}\big[e^{N_\gamma}\big]}{\exp\big(2du \cdot e(n+1) + (n+1)\log(3d)\big)} \\
&= \frac{\exp\big(2du(e-1)\cdot \text{cap}^{(T)}(\gamma)\big)}{\exp\big(2du \cdot e(n+1) + (n+1)\log(3d)\big)} \\
&\le \exp\big(-(n+1)\log(3d)\big) \\
&= (3d)^{-n-1}.
\end{aligned} \tag{3.6.1}$$

If the path γ is open in $\mathcal{FI}^{u,T}$, then the N_γ killed random walks that traverse γ must travel more than n steps in total after they first enter γ. Assume $0 < T < 1$. Note that the survival rate for killed random walks at each step is $T/(T+1)$, which is smaller than T. Let $Y_1, Y_2, \cdots$ be i.i.d. geometric random variables with parameter $1-T$. Let

$$L := \lceil 2du \cdot e(n+1) + (n+1)\log(3d) \rceil.$$

Then,

$$\mathbb{P}\big(\gamma \text{ is open}\big| N_\gamma \le L\big) \le \mathbb{P}\left(\sum_{i=1}^{L} Y_i \ge L+n\right).$$

By Chernoff bound,

$$\mathbb{P}\left(\sum_{i=1}^{L} Y_i \ge L+n\right) \le e^{-t(L+n)}\left(\frac{(1-T)e^t}{1-Te^t}\right)^L = e^{-tn}\left(\frac{1-T}{1-Te^t}\right)^L,$$

for all $t > 0$ such that $Te^t < 1$. Take $t_0 = \log(6d)$. We choose $0 < T_0(u) < 1$ such that

$$T_0 e^{t_0} = 6dT_0 < 1,$$

and

$$\left(\frac{1 - T_0}{1 - T_0 e^{t_0}}\right)^{\lceil 2du \cdot e + \log(3d) \rceil} \leq 2.$$

Then for all $0 < T < T_0$,

$$\mathbb{P}\big(\gamma \text{ is open}\big|N_\gamma \leq L\big) \leq e^{-t_0 n}\left(\frac{1 - T}{1 - Te^{t_0}}\right)^L \leq (6d)^{-n} 2^{n+1} = 2(3d)^{-n}.$$

So,

$$\mathbb{P}\big(\gamma \text{ is open}\big) \leq \mathbb{P}\big(\gamma \text{ is open}\big|N_\gamma \leq L\big) + \mathbb{P}\big(N_\gamma > L\big) \leq 2(3d)^{-n} + (3d)^{-n-1}.$$

Since γ is arbitrary,

$$\mathbb{P}\big(|\mathcal{C}| = \infty\big) \leq \mathbb{E}\big[N(n)\big] \leq (2d)^n \Big(2(3d)^{-n} + (3d)^{-n-1}\Big) \xrightarrow{n \to \infty} 0.$$

The proof is complete. $\square$

4. FURTHER STUDY

Here we present two problems for future study.

1. **Finite branches of the SDLA**. Define

$$T_x(t) = \left\{ \text{connected component of } x \text{ in } A_t^\infty \setminus (L_0 \setminus \{x\}) \right\}$$

to be "branch" in A_t^∞ rooted at x. The following conjecture predicts that all branches finally fall under the shadow of other branches and stop growing:

Conjecture 1. *Define*

$$T_x = \bigcup_{t \geq 0} T_x(t)$$

Then with probability one, $|T_x| < \infty$ for all $x \in L_0$.

2. **Chemical distance in FRI**. Given Theorem 3.0.1, it is natural to ask about the chemical distance in the unique infinite cluster. In the case of random interlacements it was proved in [31, 32, 33] that the chemical distance in RI is proportional to the $\mathbb{Z}^d$ distance with high probability.

Conjecture 2. *The chemical distance in the unique infinite cluster of FRI is proportional to the $\mathbb{Z}^d$ distance with high probability. Moreover, We can denote by $d_{\mathcal{FI}^{u,T}}(\cdot,\cdot)$ and $d_{\mathcal{I}^u}(\cdot,\cdot)$ the chemical distances in FRI and RI respectively. Given Theorem 3.1.8, one may show that for every $u > 0$,*

$$\lim_{T\to\infty} \lim_{\|x\|_1\to\infty} \frac{d_{\mathcal{FI}^{u,T}}([0],[x])}{\|x\|_1} = \lim_{\|x\|_1\to\infty} \frac{d_{\mathcal{I}^{2du}}([0],[x])}{\|x\|_1},$$

where $[x]$ denotes the closest vertex in the appropriate infinite component of $\mathcal{FI}^{u,T}$ or $\mathcal{I}^{2du}$ to $x \in \mathbb{Z}^d$.

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