THE DISTRIBUTION OF FOURIER COEFFICIENTS OF WEAK MAASS FORMS

A Dissertation

by

WEI-LUN TSAI

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Chair of Committee,	Riad Masri
Committee Members,	Matthew A. Papanikolas
	Matthew P. Young
	Lan Zhou
Head of Department,	Sarah Witherspoon

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ABSTRACT

The theory of weak Maass forms has been studied extensively in recent years, resulting in many striking results. Examples include the fundamental work of Bruinier and Ono which relates the Fourier coefficients of half-integral weight harmonic (weak) Maass forms to periods and central values/central derivatives of modular L–functions.

In this work, we investigate the distribution of Fourier coefficients of a generic family of weak Maass forms. For integral weight forms, we prove a quantitative Sato-Tate distribution for normalized Fourier coefficients of these forms of integral weight k and prime level p as $p \to \infty$. As a direct application, we prove similar results for harmonic Maass forms of integral weight $k \leq 0$ and prime level p along with the results for weakly holomorphic modular forms of integral weight $k \geq 2$ and prime level p. The proofs involve geometrical method related to bounding the analytic conductor of a suitable ℓ-adic Fourier sheaf and approximating the normalized Fourier coefficients of the weak Maass forms by normalized Kloosterman sums.

For half-integral weight forms, we prove that these coefficients are quantitatively equidistributed with respect to the pushforward of the Haar measure on the unitary group $U(1)$. Similarly, we prove quantitative equidistribution for normalized Fourier coefficients of these forms of half-integral weight $k \leq 1/2$ and level $4p$ as $p \to \infty$ along with the results for weakly holomorphic modular forms of half-integral weight $k \geq 3/2$ and level $4p$. The proofs involve analytic methods and approximating the normalized Fourier coefficients of the weak Maass forms by normalized Salié sums. As a crucial part of our analysis, we prove quantitative vertical equidistribution of Salié sums, and a uniform bound for sums of half-integral weight (opposite sign) Kloosterman sums with θ-multiplier.

DEDICATION

To my parents and wife, whose love, sacrifice and support made everything of me possible.

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All work for the dissertation was completed by the student, under the advisement of Professor Riad Masri of the Department of Mathematics.

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1. INTRODUCTION AND MOTIVATION

In this work, we focus on studying the distribution of Fourier coefficients of an arithmetically rich class of automorphic forms called *weak Maass forms*. The theory of weak Maass forms has been developed extensively over the past 15 years leading to many striking results. In particular, their Fourier coefficients have been shown to be related to a wide range of arithmetic objects such as integer partitions, central values and derivatives of modular L–functions, and singular moduli (see e.g. [5, 6, 28]). To motivate the distribution problems, first recall the classical Sato-Tate conjecture.

An elliptic curve E over $\mathbb{Q}$ is given by an equation of the form

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}$$

such that the discriminant $d_E := -16(4a^3 + 27b^2) \neq 0$. Given a prime p which does not divide d_E, one can reduce the coefficients of E modulo p and obtain a new elliptic curve $E_p : y^2 = x^3 + \bar{a}x + \bar{b}$ over the finite field $\mathbb{F}_p$. The set of all points $(x, y) \in \mathbb{F}_p^2$ satisfying this equation form a finite group $E_p(\mathbb{F}_p)$. The trace of Frobenius of E_p is then defined by

$$t_p := p + 1 - |E_p(\mathbb{F}_p)|.$$

A classical bound of Hasse asserts that $|t_p| \leq 2\sqrt{p}$. Hence the normalized trace $\lambda_p := t_p/2\sqrt{p}$ lies in the interval $[-1, 1]$.

There is a natural probability measure on $[-1, 1]$ called the *Sato-Tate measure* which is defined by

$$\mu_{\text{ST}}(t) := \frac{2}{\pi}\sqrt{1 - t^2}dt.$$

If the endomorphism ring of E satisfies $\text{End}(E) = \mathbb{Z}$ (that is, E is "non-CM"), then a celebrated conjecture of Sato and Tate asserts that the sets $\{\lambda_p : \ p \nmid d_E, \ p \leq X\}$ become equidistributed on $[-1, 1]$ with respect to μ_{ST} as $X \to \infty$; namely, for any subinterval $[\alpha, \beta]$ of $[-1, 1]$, we have

$$\lim_{X \to \infty} \frac{|\{p \leq X : \alpha \leq \lambda_p \leq \beta\}|}{|\{p \leq X\}|} = \int_{\alpha}^{\beta} d\mu_{\mathrm{ST}}(t).$$

The Sato-Tate conjecture was proved in a landmark series of papers by Clozel, Harris, Shepherd-Barron and Taylor ([8, 19, 37]).

Remark 1.0.1. *By the Eichler-Shimura theory, the Sato-Tate conjecture implies that the normalized p-th Fourier coefficients of a weight 2 non-CM newform f are equidistributed with respect to μ_{ST}.*

In view of Remark 1.0.1, we can phase the question in term of the distribution of normalized p-th Fourier coefficients of a weight 2 non-CM newform in the interval $[-1, 1]$. It is natural to ask how the normalized coefficients of weak Maass forms distribute in the interval $[-1, 1]$. That is our main goal for this work. In the following two chapters, we will show quantitative equidistribution for certain weak Maass forms of integral and half-integral weights, respectively.

2. SATO-TATE FOR WEAK MAASS FORMS OF INTEGRAL WEIGHT

In this chapter, we prove a Sato-Tate law for normalized Fourier coefficients of weak Maass forms of integral weight and prime level. Roughly speaking, for each fixed $m \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$, we consider a family of weak Maass forms $P_{m,k,p}(z,s)$ of weight k and prime level p. We define a finite set $X_p \subset [-1,1]$ of normalized Fourier coefficients of $P_{m,k,p}(z,s)$, and prove that the sets $\{X_p\}$ become quantitatively equidistributed on $[-1,1]$ with respect to the Sato-Tate measure

$$\mu_{\mathrm{ST}}(t) = \frac{2}{\pi}\sqrt{1-t^2}dt$$

as $p \to \infty$. When $s = 1-k/2$ and $k \leq 0$, the family $\{P_{m,k,p}(z, 1-k/2)\}_{m\in\mathbb{Z}^+}$ spans the vector space of harmonic Maass forms of weight k and level p. See Theorem 2.1.1 and Corollary 2.1.2 for precise statements of our results.

2.1 Introduction and statement of results

2.1.1 Background on weak Maass forms of integral weight

In this section we review some facts concerning the weak Maass forms whose coefficients we will study. More details can be found in the fundamental works of Bruinier [2, 3], Bruinier/Funke [4] and Bruinier/Ono [5, 6].

A *weak Maass form* of weight $k \in \mathbb{Z}$ and level N is a smooth function $f : \mathbb{H} \to \mathbb{C}$ on the complex upper half-plane $\mathbb{H} := \{z = x+iy : y > 0\}$ satisfying the following conditions:

(1) $f_{|_k}\gamma = f$ for all $\gamma \in \Gamma_0(N)$.

(2) f is an eigenfunction of the weight k hyperbolic Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(3) There is a non-constant polynomial $P_{f,\infty}(z) := \sum_{n\leq 0} a_f(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) -$

$P_{f,\infty}(z) = O(e^{-\epsilon y})$ as $y \to \infty$ for some $\epsilon > 0$, where $q := e^{2\pi i z}$. A similar growth condition holds for all other cusps.

One can construct an ample supply of weak Maass forms as follows. For $m \in \mathbb{Z}^+$, define the *Maass-Poincaré series* (see e.g. [3, 6])

$$P_{m,k,N}(z,s) := \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} [\mathbf{M}_{s,k}(4\pi m y) e(-mx)] \,|_k \gamma, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1, \quad (2.1.1)$$

where $e(z) := e^{2\pi i z}$ and

$$\mathbf{M}_{s,k}(t) := t^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(t)$$

with $M_{\kappa,\mu}$ equal to the usual M-Whittaker function (see [29, Section 13.14]).

The Maass-Poincaré series $P_{m,k,N}(z,s)$ is a weak Maass form of weight k and level N with Fourier expansion at ∞ given by (see Proposition 2.2.1)

$$\begin{aligned} P_{m,k,N}(z,s) = {} & \frac{\mathbf{M}_{s,k}(4\pi m y)}{\Gamma(2s)} e(-mx) + a_{m,k,N}(0,s) y^{1-s-k/2} \\ & + \sum_{n \in \mathbb{Z}^+} a_{m,k,N}(\pm n, s) \frac{\mathbf{W}_{s,k}(\pm 4\pi n y)}{\Gamma(s+k/2)} e(\pm n x) \end{aligned} \quad (2.1.2)$$

where

$$\mathbf{W}_{s,k}(t) := |t|^{-k/2} W_{\mathrm{sign}(t)\frac{k}{2}, s-\frac{1}{2}}(|t|)$$

with $W_{\kappa,\mu}$ equal to the usual Whittaker function (see [29, Section 13.14]).

Note that $P_{m,k,N}(z,s)$ is an eigenfunction for Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$.

Importantly, the family $P_{m,k,N}(z,s)$ can be used to generate certain vector spaces of harmonic Maass forms.

A *harmonic Maass form* (resp. *weakly holomorphic modular form*) of weight $k \in \mathbb{Z}$ and level N with a pole only at the cusp $\mathfrak{a} = \infty$ is a smooth (resp. holomorphic) function $f : \mathbb{H} \to \mathbb{C}$ on the complex upper half-plane $\mathbb{H}$ satisfying $(1)-(3)$, and the following additional conditions:

(4) $\Delta_k f = 0$.

(5) $f_{|_k}\gamma_{\mathfrak{a}}(z)$ is bounded as $y \to \infty$ for any matrix $\gamma_{\mathfrak{a}} \in SL_2(\mathbb{Z})$ such that $\gamma_{\mathfrak{a}}(\infty) = \mathfrak{a} \not\approx \infty$.

Let $H_k^{\#}(N)$ (resp. $M_k^{\#}(N)$) denote the vector space of harmonic Maass forms (resp. the subspace of weakly holomorphic modular forms) of this type.

We define

$$P_{m,k,N}(z) := P_{m,k,N}(z, 1-k/2), \quad k \leq 0,$$
$$Q_{m,k,N}(z) := \Gamma(k) P_{m,k,N}(z, k/2), \quad k \geq 2,$$

where by $P_{m,k,N}(z,1)$, we mean the value of the analytic continuation of $P_{m,k,N}(z,s)$ at $s = 1$ (see Section 2.2). Then $P_{m,k,N} \in H_k^{\#}(N)$ is a harmonic Maass form with Fourier expansion at ∞ given by (see Proposition 2.2.2)

$$\begin{aligned} P_{m,k,N}(z) = q^{-m} &+ \sum_{n=0}^{\infty} a_{m,k,N}(n, 1-k/2) q^n \\ &+ \sum_{n=1}^{\infty} (a_{m,k,N}(-n, 1-k/2) - \delta_m(n)) \frac{\Gamma(1-k, 4\pi n y)}{\Gamma(1-k)} q^{-n}, \end{aligned} \tag{2.1.3}$$

where $\Gamma(s,x)$ is the incomplete Gamma function and $\delta_m(n)$ is the Kronecker delta function.

Similarly, $Q_{m,k,N} \in M_k^{\#}(N)$ is a weakly holomorphic modular form with Fourier expansion at ∞ given by (see Proposition 2.2.3)

$$Q_{m,k,N}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{m,k,N}(n, k/2) q^n. \tag{2.1.4}$$

In Appendix B.1, we explain how these Fourier expansions give the decompositions

$$H_k^{\#}(N) = \text{Span}(\delta_0(k), \{P_{m,k,N}(z)\}_{m \in \mathbb{Z}^+}), \qquad k \leq 0 \tag{2.1.5}$$

and

$$M_k^{\#}(N) = \text{Span}(\{Q_{m,k,N}(z)\}_{m\in\mathbb{Z}^+}) \sqcup M_k(N), \quad k \geq 2 \tag{2.1.6}$$

where $M_k(N)$ denotes the vector space of holomorphic modular forms of weight k for $\Gamma_0(N)$.

2.1.2 Main results

Let $s > 3/4$,

$$\alpha \geq \max\left\{5, 8+\log\left(\frac{(4s-1)^8}{65536\pi^8 m^4}\right), 8+\log\left(\frac{6561}{65536\pi^8 m^4}\right)\right\},$$

and p be a prime number satisfying

$$p \geq \max\left\{3, \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha-2)^{\frac{8}{\alpha-2}}\right\}.$$

In Section 2.6, we show there is a natural choice of normalizing factor $N(m,k,s,p,\alpha,n)$ such that if $n \geq p^\alpha$, then the normalized Fourier coefficients

$$\lambda_{m,k,s,p}(n) := a_{m,k,p}(n,s)/N(m,k,s,p,\alpha,n),$$

lie in the interval $[-1,1]$.

Now, given an interval $I_p \subset \mathbb{F}_p^\times$, choose a complete set of residue classes

$$I_p = \{[n_{p,1}], \ldots, [n_{p,|I_p|}]\}$$

such that the class representatives $n_{p,i}$ satisfy the bound $n_{p,i} \geq p^\alpha$ for $i = 1, \ldots, |I_p|$. This choice determines a set

$$S_{\alpha,I_p} := \{n_{p,1}, \ldots, n_{p,|I_p|}\}.$$

Let $BV([-1,1])$ be the space of functions $f : [-1,1] \to \mathbb{C}$ whose total variation $\mathrm{Var}(f)$ is bounded.

The following is our main result.

Theorem 2.1.1. *Let $I_p \subset \mathbb{F}_p^\times$ be an interval of length $|I_p| > \sqrt{p}$. Then if $f \in BV([-1,1])$, we have*

$$\frac{1}{|I_p|} \sum_{n \in S_{\alpha, I_p}} f(\lambda_{m,k,s,p}(n)) = \int_{-1}^{1} f(t) d\mu_{\mathrm{ST}}(t) + R(f,m,k,s,p)$$

where

$$|R(f,m,k,s,p)| \leq (7208 + (4864)C(m,k,s))\mathrm{Var}(f) \log\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1/3}$$

for the explicit constant $C(m,k,s) > 0$ defined by (2.3.10).

Theorem 2.1.1 implies that for "short" intervals $I_p \subset \mathbb{F}_p^\times$ satisfying the growth condition $|I_p|/\sqrt{p} \to \infty$, the Fourier coefficients $\lambda_{m,k,s,p}(n)$ for $n \in S_{\alpha,I_p}$ become quantitatively equidistributed on $[-1,1]$ with respect to the Sato-Tate measure μ_{ST} as $p \to \infty$.

More precisely, let X be a locally compact Hausdorff space and μ be a Borel probability measure on X. A sequence $\{X_\ell\}$ of finite subsets $X_\ell \subset X$ is said to become *equidistributed* on X with respect to μ if for any continuous function $f : X \to \mathbb{C}$ we have

$$\frac{1}{|X_\ell|} \sum_{x \in X_\ell} f(x) \longrightarrow \int_X f(x) d\mu(x)$$

as $\ell \to \infty$.

Define the sequence of finite subsets

$$X_{\alpha,I_p} := \{\lambda_{m,k,s,p}(n) : \ n \in S_{\alpha,I_p}\} \subset [-1,1].$$

The following result is an immediate consequence of Theorem 2.1.1.

Corollary 2.1.2. *Let $I_p \subset \mathbb{F}_p^\times$ be as in Theorem 2.1.1, and assume further that $|I_p|/\sqrt{p} \to \infty$ as $p \to \infty$. Then the sets $\{X_{\alpha,I_p}\}$ become equidistributed on $[-1,1]$ with respect to the Sato-Tate measure μ_{ST} as $p \to \infty$.*

2.1.3 Numerical examples

We used `SageMath` [36] to give examples which illustrate the equidistribution of the sets $\{X_{\alpha,\mathbb{F}_p^\times}\}$ with respect to μ_{ST} as $p \to \infty$. Our calculations suggest that the rate of equidistribution is $O_\varepsilon(p^{-1/2-\varepsilon})$, which is plainly much faster than the rate $O_\varepsilon(p^{-1/6+\varepsilon})$ implied by Theorem 2.1.1.

Let $m = 1$, $k = 0$, $s = 1$, $\alpha = 5.3$, and $p \geq 3$. Divide the interval $[-1,1]$ into $N_p := 2[(p-1)^{1/2}]$ subintervals $T_{p,i}$ of length $|T_{p,i}| = (p-1)^{-1/2}$. Let $\chi_{T_{p,i}}$ be the characteristic function of $T_{p,i}$. An approximation argument shows that if the sets $\{X_{5.3,\mathbb{F}_p^\times}\}$ become equidistributed with respect to μ_{ST} as $p \to \infty$, then

$$\frac{1}{p-1} \sum_{i=1}^{N_p} \sum_{n \in S_{5.3,\mathbb{F}_p^\times}} \chi_{T_{p,i}}(\lambda_{1,0,1,p}(n)) \longrightarrow \int_{-1}^{1} d\mu_{\mathrm{ST}}(t) \tag{2.1.7}$$

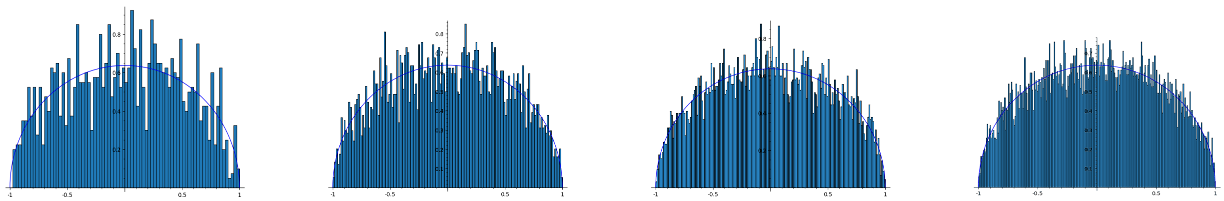
as $p \to \infty$. Now, define the rectangles $R_{p,i} := T_{p,i} \times [0, H_{p,i}]$ where the height of $R_{p,i}$ is defined to be

$$H_{p,i} := (p-1)^{-1/2} |\{n \in S_{5.3,\mathbb{F}_p^\times} : \lambda_{1,0,1,p}(n) \in T_{p,i}\}|.$$

Then (2.1.7) is equivalent to

$$A_p := \sum_{i=1}^{N_p} \mathrm{Area}(R_{p,i}) \longrightarrow \int_{-1}^{1} d\mu_{\mathrm{ST}}(t)$$

as $p \to \infty$. The histograms in Figure 2.1 display how A_p approximates the area bounded by the function $(2/\pi)\sqrt{1-t^2}$ on the interval $[-1,1]$ for successively larger values of p.



(a) $p = 1601$ (b) $p = 5477$ (c) $p = 8101$ (d) $p = 15377$

Figure 2.1: Histograms for $X_{5.3,\mathbb{F}_p^\times}$

2.2 Coefficients of Maass-Poincaré series

In this section, we give formulas for the coefficients appearing in the Fourier expansions (2.1.2), (2.1.3) and (2.1.4).

Recall that the Kloosterman sum of modulus c is defined by

$$S(u, v; c) := \sum_{\substack{d \pmod{c} \\ (d,c)=1}} e\left(\frac{u\bar{d} + vd}{c}\right).$$

Let I_v and J_v denote the I and J-Bessel functions of order v, respectively (see [29, Section 10.2 and 10.25]).

Proposition 2.2.1. We have

$$\begin{aligned} P_{m,k,N}(z, s) = &\frac{\mathbf{M}_{s,k}(4\pi my)}{\Gamma(2s)} e(-mx) + a_{m,k,N}(0, s) y^{1-s-k/2} \\ &+ \sum_{n \in \mathbb{Z}^+} a_{m,k,N}(\pm n, s) \frac{\mathbf{W}_{s,k}(\pm 4\pi ny)}{\Gamma(s + k/2)} e(\pm nx) \end{aligned}$$

where the coefficients $a_{m,k,N}(\pm n, s)$ are given by

$$a_{m,k,N}(0, s) = \frac{2^{2-k}\pi^{1+s-\frac{k}{2}} i^{-k} m^{s-\frac{k}{2}}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{S(-m, 0; c)}{c^{2s}},$$

$$a_{m,k,N}(n,s) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{S(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$
$$a_{m,k,N}(-n,s) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{S(-m,-n;c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. We may express the Maass-Poincaré series defined by (2.1.1) as

$$P_{m,k,N}(z,s) = \frac{1}{\Gamma(2s)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \left[\psi_{s,k}(y)e(-mz)\right]|_k\gamma$$

where

$$\psi_{s,k}(y) := (4\pi my)^{-\frac{k}{2}} M_{-\frac{k}{2},s-\frac{1}{2}}(4\pi my)e^{-2\pi my}.$$

We have the double coset decomposition

$$\Gamma_0(N) = \Gamma_\infty \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_\infty \bigsqcup \bigsqcup_{\substack{c>0 \\ c\equiv 0 \pmod N \\ d\in(\mathbb{Z}/c\mathbb{Z})^\times}} \Gamma_\infty w_{d/c}\Gamma_\infty$$

where

$$w_{d/c} := \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then using this decomposition we get

$$\begin{aligned} P_{m,k,N}(z,s) &= \frac{1}{\Gamma(2s)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \left[\psi_{s,k}(y)e(-mz)\right]|_k\gamma \\ &= \frac{1}{\Gamma(2s)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \psi_{s,k}(\mathrm{Im}(\gamma z))e(-m(\gamma z))(cz+d)^{-k} \end{aligned}$$

$$= \frac{\mathbf{M}_{s,k}(4\pi my)}{\Gamma(2s)} e(-mz) + \frac{1}{\Gamma(2s)} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \sum_{\substack{d \pmod c \\ (c,d)=1}} \sum_{n\in\mathbb{Z}} \phi_{n,m,k,}(z),$$

where

$$\phi_{n,s,k}(z) := \psi_{s,k}(\mathrm{Im}(w_{d/c}(z+n)))e(-m(w_{d/c}(z+n)))(cz+cn+d)^{-k}.$$

Now, by Poisson summation we have

$$\sum_{n\in\mathbb{Z}} \phi_{n,s,k}(z) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{s,k}(\mathrm{Im}(w_{d/c}(z+t)))e(-m(w_{d/c}(z+t)))(cz+ct+d)^{-k}e(-nt)dt. \tag{2.2.1}$$

Note that

$$w_{d/c}(z+t) = \frac{a}{c} - \frac{1}{c^2(t+x+d/c+iy)}.$$

Then using the change of variables $t \mapsto t - x - d/c$, the integral in (2.2.1) becomes

$$c^{-k} e\left(nx + \frac{nd}{c}\right) \times \int_{\mathbb{R}} \psi_{s,k}\left(\mathrm{Im}\left(\frac{a}{c} - \frac{1}{c^2(t+iy)}\right)\right) e\left(-m\left(\frac{a}{c} - \frac{1}{c^2(t+iy)}\right)\right)(t+iy)^{-k}e(-nt)dt.$$

Hence

$$P_{m,k,N}(z,s) = \frac{\mathbf{M}_{s,k}(4\pi my)}{\Gamma(2s)} e(-mz) + \frac{1}{\Gamma(2s)} \sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{S(-m,n;c)}{c^k} \alpha_{n,s,k}(c,y) \tag{2.2.2}$$

where

$$\alpha_{n,s,k}(c,y) := \int_{\mathbb{R}} \psi_{s,k}\left(\frac{y}{c^2(t^2+y^2)}\right) e\left(\frac{m}{c^2(t+iy)} - nt\right)(t+iy)^{-k} dt. \tag{2.2.3}$$

Finally, by Lemma A.1.1 we have

$$\alpha_{n,s,k}(c,y) = \begin{cases} 2\pi i^{-k} \frac{\Gamma(2s)}{\Gamma(s+k/2)} c^{k-1} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \mathbf{W}_{s,k}(4\pi n y), & n \geq 1, \\ \frac{2^{2-k}\pi^{1+s-\frac{k}{2}} i^{-k} m^{s-\frac{k}{2}} y^{1-s-\frac{k}{2}} c^{k-2s}\Gamma(2s)}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}, & n = 0, \\ 2\pi i^{-k} \frac{\Gamma(2s)}{\Gamma(s-k/2)} c^{k-1} \left(\frac{|n|}{m}\right)^{\frac{k-1}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{m|n|}}{c}\right) \mathbf{W}_{s,k}(4\pi n y), & n \leq -1. \end{cases}$$

Then substituting these identities in (2.2.2) gives the desired result. □

Using the Weil bound

$$|S(m,n;c)| \leq \tau(c)(m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}},$$

one can show that the Fourier expansion (2.1.2) is absolutely and uniformly convergent for $\mathrm{Re}(s) > 3/4$, and hence gives an analytic continuation of $P_{m,k,N}(z,s)$ to this region. In particular, we can analytically continue $P_{m,k,N}(z,s)$ to $\mathrm{Re}(s) > 3/4$.

Now, recall that $P_{m,k,N}(z) = P_{m,k,N}(z, 1-k/2)$ for $k \leq 0$.

Proposition 2.2.2. We have

$$\begin{aligned} P_{m,k,N}(z) = q^{-m} &+ \sum_{n=0}^{\infty} a_{m,k,N}(n, 1-k/2) q^n \\ &+ \sum_{n=1}^{\infty} (a_{m,k,N}(-n, 1-k/2) - \delta_m(n)) \frac{\Gamma(1-k, 4\pi n y)}{\Gamma(1-k)} q^{-n} \end{aligned}$$

where

$$a_{m,k,N}(0,1-k/2) = (2\pi i)^{2-k}\frac{m^{1-k}}{\Gamma(2-k)}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,0;c)}{c^{2-k}},$$

$$a_{m,k,N}(n,1-k/2) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,n;c)}{c}I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

$$a_{m,k,N}(-n,1-k/2) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,-n;c)}{c}J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. Let $s = 1-k/2$ with $k \le 0$ in (2.1.2). Using [29, Eq. 13.18.4] we get

$$M_{-\frac{k}{2},\frac{1-k}{2}}(4\pi my) = (1-k)e^{2\pi my}(4\pi my)^{\frac{k}{2}}[\Gamma(1-k) - \Gamma(1-k,4\pi my)].$$

Then the functional equation $\Gamma(2-k) = (1-k)\Gamma(1-k)$ yields

$$\frac{\mathbf{M}_{1-\frac{k}{2},k}(4\pi my)}{\Gamma(2-k)} = e^{2\pi my} - e^{2\pi my}\frac{\Gamma(1-k,4\pi my)}{\Gamma(1-k)}$$

so that

$$\frac{\mathbf{M}_{1-\frac{k}{2},k}(4\pi my)}{\Gamma(2-k)}e(-mx) = q^{-m} - \frac{\Gamma(1-k,4\pi my)}{\Gamma(1-k)}q^{-m}.$$

Next, by applying [29, Eq. 13.18.2] we have

$$\mathbf{W}_{1-\frac{k}{2},k}(4\pi ny) = (4\pi ny)^{-\frac{k}{2}}W_{\frac{k}{2},\frac{1-k}{2}}(4\pi ny) = e^{-2\pi ny},$$

and by applying [29, Eq. 13.18.5] we have

$$\mathbf{W}_{1-\frac{k}{2},k}(-4\pi ny) = (4\pi ny)^{-\frac{k}{2}}W_{-\frac{k}{2},\frac{1-k}{2}}(4\pi ny) = \Gamma(1-k,-4\pi ny)e^{-2\pi ny}.$$

Hence after combining the preceding identities we get

$$a_{m,k,N}(0,1-k/2) = (2\pi i)^{2-k}\frac{m^{1-k}}{\Gamma(2-k)}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,0;c)}{c^{2-k}},$$
$$a_{m,k,N}(n,1-k/2) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,n;c)}{c}I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$
$$a_{m,k,N}(-n,1-k/2) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,-n;c)}{c}J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

□

Similarly, recall that $Q_{m,k,N}(z) := P_{m,k,N}(z,k/2)$ for $k \geq 2$.

Proposition 2.2.3. We have

$$Q_{m,k,N}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{m,k,N}(n,k/2)q^n$$

where

$$a_{m,k,N}(n,k/2) = 2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{S(-m,n;c)}{c}I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. Let $s = k/2$ with $k \geq 2$ in (2.1.2). Using [29, Eq. 13.18.3] we get

$$M_{-\frac{k}{2},\frac{k-1}{2}}(4\pi my) = e^{2\pi my}(4\pi my)^{\frac{k}{2}}.$$

It follows that

$$\mathbf{M}_{\frac{k}{2},k}(4\pi my) = e^{2\pi my}$$

and thus

$$\mathbf{M}_{\frac{k}{2},k}(4\pi my)e(-mx) = q^{-m}.$$

Next, by applying [29, Eq. 13.18.2] we have

$$\mathbf{W}_{\frac{k}{2},k}(4\pi ny) = (4\pi ny)^{-\frac{k}{2}} W_{\frac{k}{2},\frac{k-1}{2}}(4\pi ny) = e^{-2\pi ny}.$$

Finally, since $\Gamma(s)$ has a simple pole at $s = 0$, the constant and non-holomorphic terms in the Fourier expansion vanish. Hence by combining the preceding facts we get

$$\begin{aligned} &a_{m,k,N}(0, k/2) = a_{m,k,N}(-n, k/2) = 0, \\ &a_{m,k,N}(n, k/2) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{S(-m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned}$$

□

2.3 Approximation by Kloosterman sums

Let p be a prime number. By Proposition 2.2.1, the Fourier coefficient $a_{m,k,p}(n,s)$ of $P_{m,k,p}(z,s)$ is given by

$$a_{m,k,p}(n,s) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod p}} \frac{S(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.3.1}$$

We define the modified Fourier coefficients

$$\widetilde{a}_{m,k,p}(n,s) := \frac{a_{m,k,p}(n,s)}{\mathcal{N}(m,k,p,n)},$$

where

$$\mathcal{N}(m,k,p,n) := 2^{1/2} i^{-k} m^{\frac{1-2k}{4}} n^{\frac{2k-3}{4}} \exp\left(\frac{4\pi\sqrt{mn}}{p}\right).$$

Then from (2.3.1) we get the decomposition

$$\widetilde{a}_{m,k,p}(n,s) = \frac{S(-m,n;p)}{2\sqrt{p}} A(m,k,s,p,n) + B(m,k,s,p,n), \tag{2.3.2}$$

where

$$A(m,k,s,p,n) := \pi 2^{3/2} (mn)^{1/4} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right) p^{-1/2} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right),$$
$$B(m,k,s,p,n) := \pi 2^{1/2} (mn)^{1/4} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) \sum_{\substack{c>p \\ c\equiv 0 \pmod p}} \frac{S(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

We will need the following estimates.

Proposition 2.3.1. Let $\mathrm{Re}(s) > 3/4$ and

$$\alpha \geq \max\left\{8 + \log\left(\frac{[1+2\mathrm{Re}(2s-1)]^8}{65536\pi^8 m^4}\right), 8 + \log\left(\frac{6561}{65536\pi^8 m^4}\right)\right\}.$$

(i) There is a function $C(m,k,s,p,n)$ such that if $n \geq p^\alpha$, then

$$A(m,k,s,p,n) = 1 + C(m,k,s,p,n)$$

where

$$|C(m,k,s,p,n)| \leq C_{m,k,s,p} := C_{m,k,s,p,1} + \exp\left(-8\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\{1 + C_{m,k,s,p,1}\},$$
$$C_{m,k,s,p,1} := \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{4\pi\sqrt{m}} p^{\frac{2-\alpha}{2}} \exp\left(\frac{|(2s-1)^2-\frac{1}{4}|}{4\pi\sqrt{m}} p^{\frac{2-\alpha}{2}}\right).$$

(ii) If $n \geq p^{\alpha}$, then

$$|B(m,k,s,p,n)| \leq B_{m,k,s,1} \exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right) + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}\exp\left(-\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)$$

where

$$B_{m,k,s,1} := 2^{\frac{1}{2}+\mathrm{Re}(2s-1)}\pi^{1+\mathrm{Re}(2s-1)}m^{\frac{3}{4}+\frac{\mathrm{Re}(2s-1)}{2}}\zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right)\sum_{\ell=0}^{\infty}\frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!},$$

$$B_{m,k,s,2} := 2^{\frac{7}{2}}3^{-\frac{1}{2}}\pi^2 m^{\frac{5}{4}}.$$

Proof. (i) Consider the following asymptotic formula for the I-Bessel function (see [27, p. 269] or [29, Eq. 10.40.12])

$$I_v(z) = \frac{\exp(z)}{\sqrt{2\pi z}}[1+E_v(z)] - i\frac{\exp(-z-v\pi i)}{\sqrt{2\pi z}}[1+E_v(z)], \tag{2.3.3}$$

where

$$|E_v(z)| \leq \frac{|(4v^2-1)(4v^2-9)|}{|z|}\exp\left(\left|\frac{v^2-\frac{1}{4}}{z}\right|\right). \tag{2.3.4}$$

Then we can write

$$A(m,k,s,p,n) = I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right)2^{\frac{3}{2}}\pi(mn)^{\frac{1}{4}}p^{-\frac{1}{2}}\exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) = 1 + C(m,k,s,p,n),$$

where

$$C(m,k,s,p,n) := E_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right)$$

$$- i \exp\left(-\frac{8\pi\sqrt{mn}}{p} - (2s-1)\pi i\right)\left[1 + E_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right)\right].$$

By (2.3.4), if $n \geq p^{\alpha}$ then

$$\begin{aligned} |C(m,k,s,p,n)| &\leq \left|E_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right)\right| + \exp\left(-\frac{8\pi\sqrt{mn}}{p}\right)\left[1 + \left|E_{2s-1}\left(\frac{4\pi\sqrt{mn}}{p}\right)\right|\right] \\ &\leq C_{m,k,s,p} \end{aligned} \tag{2.3.5}$$

where

$$\begin{aligned} C_{m,k,s,p} &= C_{m,k,s,p,1} + \exp\left(-8\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\left\{1 + C_{m,k,s,p,1}\right\}, \\ C_{m,k,s,p,1} &= \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{4\pi\sqrt{m}} p^{\frac{2-\alpha}{2}} \exp\left(\frac{|(2s-1)^2 - \frac{1}{4}|}{4\pi\sqrt{m}} p^{\frac{2-\alpha}{2}}\right). \end{aligned}$$

(ii) Observe that (see [29, Eq. 10.25.2])

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{c=0}^{\infty} \frac{1}{\Gamma(v+1+c)c!}\left(\frac{z^2}{4}\right)^c.$$

If $0 < z < 1$, then we have

$$|I_v(z)| \leq \left|\left(\frac{z}{2}\right)\right|^{\mathrm{Re}(v)} \sum_{c=0}^{\infty} \frac{1}{4^c |\Gamma(v+1+c)| c!}. \tag{2.3.6}$$

Now, consider the decomposition

$$\begin{aligned} B(m,k,s,p,n) &= 2^{\frac{1}{2}}\pi(mn)^{\frac{1}{4}} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) \sum_{\substack{c>p \\ c\equiv 0 \pmod p}} \frac{S(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &= 2^{\frac{1}{2}}\pi(mn)^{\frac{1}{4}} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) \sum_{c=2}^{\infty} \frac{S(-m,n;cp)}{cp} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{cp}\right) \\ &= 2^{\frac{1}{2}}\pi(mn)^{\frac{1}{4}} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right)\left\{R_1 + R_2\right\}, \end{aligned}$$

where

$$R_1 := \sum_{c > \frac{4\pi\sqrt{mn}}{p}} \frac{S(-m,n;cp)}{cp} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{cp}\right),$$

$$R_2 := \sum_{c=2}^{\lfloor \frac{4\pi\sqrt{mn}}{p} \rfloor} \frac{S(-m,n;cp)}{cp} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{cp}\right).$$

By the Weil bound

$$|S(-m,n;cp)| \leq \tau(cp)(-m,n,cp)^{\frac{1}{2}}(cp)^{\frac{1}{2}} \tag{2.3.7}$$

and (2.3.6), we have

$$\begin{aligned} |R_1| &\leq \sum_{c > \frac{4\pi\sqrt{mn}}{p}} \frac{|S(-m,n;cp)|}{cp} \left(\frac{2\pi\sqrt{mn}}{cp}\right)^{\mathrm{Re}(2s-1)} \sum_{\ell=0}^{\infty} \frac{1}{4^\ell |\Gamma(2s+\ell)|\ell!} \\ &\leq m^{\frac{1}{2}}(2\pi\sqrt{mn})^{\mathrm{Re}(2s-1)} \sum_{\ell=0}^{\infty} \frac{1}{4^\ell |\Gamma(2s+\ell)|\ell!} \sum_{c > \frac{4\pi\sqrt{mn}}{p}} \frac{\tau(cp)}{(cp)^{\frac{1}{2}+\mathrm{Re}(2s-1)}}. \end{aligned}$$

Moreover, we have

$$\sum_{c > \frac{4\pi\sqrt{mn}}{p}} \frac{\tau(cp)}{(cp)^{\frac{1}{2}+\mathrm{Re}(2s-1)}} \leq \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{\frac{1}{2}+\mathrm{Re}(2s-1)}} = \zeta^2\left(\frac{1}{2} + \mathrm{Re}(2s-1)\right).$$

Hence

$$|R_1| \leq m^{\frac{1}{2}}(2\pi\sqrt{mn})^{\mathrm{Re}(2s-1)} \zeta^2\left(\frac{1}{2} + \mathrm{Re}(2s-1)\right) \sum_{\ell=0}^{\infty} \frac{1}{4^\ell |\Gamma(2s+\ell)|\ell!}. \tag{2.3.8}$$

Similarly, by (2.3.3), (2.3.5), (2.3.7) and the bound

$$\tau(c) \leq \sqrt{3}c^{1/2},$$

we get

$$\begin{aligned}
|R_2| &\leq \sum_{c=2}^{\lfloor \frac{4\pi\sqrt{mn}}{p} \rfloor} \frac{|S(-m,n;cp)|}{cp} \frac{\sqrt{cp}}{\sqrt{4\pi}(mn)^{\frac{1}{4}}} \exp\left(\frac{4\pi\sqrt{mn}}{cp}\right)(1+C_{m,k,s,p}) \\
&\leq \frac{m^{\frac{1}{2}}(1+C_{m,k,s,p})}{\sqrt{4\pi}(mn)^{\frac{1}{4}}} \exp\left(\frac{2\pi\sqrt{mn}}{p}\right) \sum_{c=2}^{\lfloor \frac{4\pi\sqrt{mn}}{p} \rfloor} \tau(cp) \\
&\leq \frac{m^{\frac{1}{2}}(1+C_{m,k,s,p})}{\sqrt{4\pi}(mn)^{\frac{1}{4}}} \exp\left(\frac{2\pi\sqrt{mn}}{p}\right) \frac{2}{3}\sqrt{3p}\left(\frac{4\pi\sqrt{mn}}{p}\right)^{\frac{3}{2}}. \qquad (2.3.9)
\end{aligned}$$

Combining (2.3.8) and (2.3.9) now gives

$$\begin{aligned}
|B(m,k,s,p,n)| \leq{}& 2^{\frac{1}{2}+\mathrm{Re}(2s-1)}\pi^{1+\mathrm{Re}(2s-1)} m^{\frac{3}{4}+\frac{\mathrm{Re}(2s-1)}{2}} \zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right) \sum_{\ell=0}^{\infty} \frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!} \\
&\times n^{\frac{1}{4}+\frac{\mathrm{Re}(2s-1)}{2}} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) \\
&+ 2^{\frac{7}{2}} 3^{-\frac{1}{2}} \pi^2 m^{\frac{5}{4}} (1+C_{m,k,s,p}) p^{-1} n^{\frac{3}{4}} \exp\left(-\frac{2\pi\sqrt{mn}}{p}\right).
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
& \alpha \geq \max\left\{8+\log\left(\frac{[1+2\mathrm{Re}(2s-1)]^8}{65536\pi^8 m^4}\right), 8+\log\left(\frac{6561}{65536\pi^8 m^4}\right)\right\} \\
\implies \quad & n \geq p^{\alpha} \geq \max\left\{\frac{6561 p^8}{65536\pi^8 m^4}, \frac{(1+2\mathrm{Re}(2s-1))^8 p^8}{65536\pi^8 m^4}\right\},
\end{aligned}$$

it follows that

$$\begin{aligned}
n^{\frac{1}{4}+\frac{\mathrm{Re}(2s-1)}{2}} \exp\left(-\frac{4\pi\sqrt{mn}}{p}\right) &\leq \exp\left(-\frac{2\pi\sqrt{mn}}{p}\right), \\
n^{\frac{3}{4}} \exp\left(-\frac{2\pi\sqrt{mn}}{p}\right) &\leq \exp\left(-\frac{\pi\sqrt{mn}}{p}\right), \\
-\frac{\sqrt{n}}{p} &\leq -p^{\frac{\alpha-2}{2}}.
\end{aligned}$$

Hence

$$|B(m,k,s,p,n)| \leq B_{m,k,s,1} \exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right) \\ + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}\exp\left(-\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)$$

where

$$B_{m,k,s,1} = 2^{\frac{1}{2}+\mathrm{Re}(2s-1)}\pi^{1+\mathrm{Re}(2s-1)}m^{\frac{3}{4}+\frac{\mathrm{Re}(2s-1)}{2}}\zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right)\sum_{\ell=0}^{\infty}\frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!},$$
$$B_{m,k,s,2} = 2^{\frac{7}{2}}3^{-\frac{1}{2}}\pi^2 m^{\frac{5}{4}}.$$

□

Now, by the Weil bound

$$|S(-m,n;p)| \leq 2\sqrt{p},$$

there exists a unique real number $\theta_{m,p}(n) \in [0,\pi]$ called the *Kloosterman angle* such that

$$\frac{S(-m,n;p)}{2\sqrt{p}} = \cos(\theta_{m,p}(n)).$$

We will use Proposition 2.3.1 to deduce the following effective asymptotic formula.

Theorem 2.3.2. *Let* $\mathrm{Re}(s) > 3/4$*,*

$$\alpha \geq \max\left\{3, 8+\log\left(\frac{[1+2\mathrm{Re}(2s-1)]^8}{65536\pi^8 m^4}\right), 8+\log\left(\frac{6561}{65536\pi^8 m^4}\right)\right\},$$

and p *be a prime number satisfying*

$$p \geq \max\left\{3, \left|(2s-1)^2-\frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha-2)^{\frac{8}{\alpha-2}}\right\}.$$

Then if $n \geq p^{\alpha}$*, we have*

$$\widetilde{a}_{m,k,p}(n,s) = \cos(\theta_{m,p}(n)) + E(m,k,s,p,n),$$

where

$$|E(m,k,s,p,n)| \leq C(m,k,s)p^{\frac{2-\alpha}{2}}$$

with

$$C(m,k,s) := B_{m,k,s,1} + B_{m,k,s,2} + (1 + B_{m,k,s,2})\left(1 + \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}\right). \tag{2.3.10}$$

Proof. By (2.3.2) and Proposition 2.3.1 we have

$$\widetilde{a}_{m,k,p}(n,s) = \cos(\theta_{m,p}(n)) + E(m,k,s,p,n),$$

where the error term

$$E(m,k,s,p,n) := \cos(\theta_{m,p}(n))C(m,k,s,p,n) + B(m,k,s,p,n)$$

satisfies the bound

$$\begin{aligned}|E(m,k,s,p,n)| &\leq |C(m,k,s,p,n)| + |B(m,k,s,p,n)| \\ &\leq C_{m,k,s,p} + B_{m,k,s,1}\exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right) \\ &\quad + B_{m,k,s,2}(1 + C_{m,k,s,p})p^{-1}\exp\left(-\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right).\end{aligned}$$

We have

$$p \geq \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}} \quad \Longrightarrow \quad p \geq \left(\frac{|(2s-1)^2 - \frac{1}{4}|}{4\pi\sqrt{m}\log 2}\right)^{\frac{2}{\alpha-2}}$$
$$\Longrightarrow \quad \exp\left(\frac{|(2s-1)^2 - \frac{1}{4}|}{4\pi\sqrt{m}} p^{\frac{2-\alpha}{2}}\right) \leq 2.$$

Hence

$$C_{m,k,s,p} \leq \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{2\pi\sqrt{m}} p^{\frac{2-\alpha}{2}} \left[1 + \exp\left(-8\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\right]$$
$$+ \exp\left(-8\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right).$$

Similarly, we have

$$p \geq \max\left\{3, (\alpha-2)^{\frac{8}{\alpha-2}}\right\} \quad \Longrightarrow \quad p \geq \max\left\{3, \left(\frac{\alpha-2}{2\pi\sqrt{m}}\right)^{\frac{8}{\alpha-2}}\right\}$$
$$\Longrightarrow \quad \exp\left(-\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right) \leq p^{\frac{2-\alpha}{2}}.$$

It follows that

$$C_{m,k,s,p} \leq \left(1 + \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}\right) p^{\frac{2-\alpha}{2}}.$$

Hence

$$C_{m,k,s,p} + B_{m,k,s,1}\exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right) + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}\exp\left(-\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)$$
$$\leq C_{m,k,s,p} + B_{m,k,s,1}p^{\frac{2-\alpha}{2}} + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}p^{\frac{2-\alpha}{2}}$$
$$\leq B_{m,k,s,1}p^{\frac{2-\alpha}{2}} + B_{m,k,s,2}p^{\frac{2-\alpha}{2}}$$
$$+ (1+B_{m,k,s,2})\left(1 + \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}\right) p^{\frac{2-\alpha}{2}}.$$

□

2.4 Discrepancy bounds

In this section we give some bounds concerning the star discrepancy of a set which we will need.

Given any finite set $X \subset [-1, 1]$, we define the averaged Dirac measures on $[-1, 1]$ by

$$\mu_X(t) := \frac{1}{|X|} \sum_{x \in X} \chi_{(x,1]}(t),$$

where $\chi_{(x,1]}(t)$ denotes the characteristic function of the interval $(x, 1] \subset [-1, 1]$. It follows that for any $y \in [-1, 1]$ we have the corresponding distribution function

$$\mu_X(y) = \int_{-1}^{y} d\mu_X(t).$$

On the other hand, recall the Sato-Tate measure

$$\mu_{\mathrm{ST}}(t) = \frac{2}{\pi}\sqrt{1 - t^2}dt.$$

Then we define the distribution function of $\mu_{\mathrm{ST}}(t)$ on $[-1, 1]$ by

$$G(y) := \int_{-1}^{y} d\mu_{\mathrm{ST}}(t).$$

The *star discrepancy* of X with respect to the measures μ_X and μ_{ST} is defined by

$$D_X^{\star} := \sup_{y \in [-1,1]} |\mu_X(y) - G(y)|.$$

Recall that $BV([-1, 1])$ denotes the space of functions $f : [-1, 1] \to \mathbb{C}$ whose total variation $\mathrm{Var}(f)$ is bounded.

We have the following Koksma-Hlawka type inequality for the Sato-Tate measure.

Lemma 2.4.1. *Given any finite set $X \subset [-1,1]$ and function $f \in BV([-1,1])$, we have*

$$\left| \frac{1}{|X|} \sum_{x \in X} f(x) - \int_{-1}^{1} f(t) d\mu_{\mathrm{ST}}(t) \right| \leq D_X^{\star} \mathrm{Var}(f).$$

Proof. Define the function

$$R_X(y) := \mu_X(y) - G(y).$$

Then using integration by parts for the Riemann-Stieltjes integral, we get

$$\begin{aligned} \int_{-1}^{1} R_X(y) df(y) &= \int_{-1}^{1} \mu_X(y) df(y) - \int_{-1}^{1} G(y) df(y) \\ &= \int_{-1}^{1} \mu_X(y) df(y) - G(1)f(1) + G(-1)f(-1) + \int_{-1}^{1} f(y) dG(y) \\ &= \int_{-1}^{1} \mu_X(y) df(y) - f(1) + \int_{-1}^{1} f(t) d\mu_{\mathrm{ST}}(t). \end{aligned}$$

On the other hand, we have

$$\int_{-1}^{1} \mu_X(y) df(y) = \frac{1}{|X|} \sum_{x \in X} \int_{x}^{1} df(y) = \frac{1}{|X|} \sum_{x \in X} (f(1) - f(x)) = f(1) - \frac{1}{|X|} \sum_{x \in X} f(x).$$

Combining these identities gives

$$\int_{-1}^{1} R_X(y) df(y) = \int_{-1}^{1} f(t) d\mu_{\mathrm{ST}}(t) - \frac{1}{|X|} \sum_{x \in X} f(x).$$

Finally, it follows that

$$\left| \frac{1}{|X|} \sum_{x \in X} f(x) - \int_{-1}^{1} f(t) d\mu_{\mathrm{ST}}(t) \right| = \left| \int_{-1}^{1} R_X(y) df(y) \right| \leq D_X^{\star} \int_{-1}^{1} |df(y)| \leq D_X^{\star} \mathrm{Var}(f).$$

□

Now, recall that for $r \in \mathbb{Z}^+$ the Chebyshev polynomials of the second kind are defined by

$$U_r(t) := \frac{\sin((r+1)\cos^{-1}(t))}{\sqrt{1-t^2}}, \quad t \in [-1,1].$$

The polynomials $\{U_r(t)\}_{r=0}^{\infty}$ form an orthonormal system with respect to the measure μ_{ST} on $[-1,1]$.

We will need the following Erdös-Turán type inequality due to Niederreiter [26, Lemma 3].

Proposition 2.4.2. Given any finite set $X \subset [-1,1]$ and positive odd integer d, we have

$$D_X^\star \leq \frac{8}{(0.362)\pi d + 4} + \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \left| \frac{1}{|X|} \sum_{x \in X} U_r(x) \right|.$$

2.5 Analytic conductor of the Kloosterman sheaf

We will need an explicit formula for the analytic conductor of the r-th symmetric power of the rank 2 Kloosterman sheaf.

Let Kl_2 be the rank 2 Kloosterman sheaf on the affine line $\mathbf{A}^1_{\mathbb{F}_p}$ and consider the Fourier sheaf defined by the r-th symmetric power

$$\mathrm{Kl}_2^{(r)} := \mathrm{Sym}^r(\mathrm{Kl}_2).$$

The *analytic conductor* of $\mathrm{Kl}_2^{(r)}$ is defined by (see e.g. [14, Section 2.2])

$$\mathbf{c}(\mathrm{Kl}_2^{(r)}) := \mathrm{rank}(\mathrm{Kl}_2^{(r)}) + |S(\mathrm{Kl}_2^{(r)})| + \sum_{x \in S(\mathrm{Kl}_2^{(r)})} \mathrm{Swan}_x(\mathrm{Kl}_2^{(r)})$$

where $\mathrm{rank}(\mathrm{Kl}_2^{(r)})$ is the rank of $\mathrm{Kl}_2^{(r)}$,

$$S(\mathrm{Kl}_2^{(r)}) \subset \mathbf{P}^1_{\overline{\mathbb{F}}_p}$$

is the set of singularities of $\mathrm{Kl}_2^{(r)}$, and $\mathrm{Swan}_x(\mathrm{Kl}_2^{(r)})$ is the Swan conductor at x.

We will deduce the following result from Fu and Wan [15, Theorem 3.1].

Proposition 2.5.1. For $p > 2$ we have

$$\mathbf{c}(\mathrm{Kl}_2^{(r)}) = r + 3 + \frac{1}{2}\left(r + 1 - d_r(2, p)\right)$$

where $d_r(2, p)$ is the number of 2-tuples of non-negative integers (a_1, a_2) satisfying $a_1 + a_2 = r$ and $a_1 \equiv a_2 \pmod{p}$.

Proof. Since Kl_2 has rank 2, the rank of $\mathrm{Kl}_2^{(r)}$ equals the number of 2-tuples of non-negative integers (a_1, a_2) satisfying $a_1 + a_2 = r$, which is $r + 1$ (see e.g. [20, Exercise 5.16]). Hence

$$\mathrm{rank}(\mathrm{Kl}_2^{(r)}) = r + 1.$$

Now, by Deligne [9] (see e.g. [24, Theorem 4.1.1]) the Kloosterman sheaf Kl_2 is lisse except at the two ramified points 0 (tame) and ∞ (wild). Hence $\mathrm{Kl}_2^{(r)}$ is also lisse except at the two ramified points 0 (tame) and ∞ (wild). In particular,

$$S(\mathrm{Kl}_2^{(r)}) = \{0, \infty\}.$$

Since $\mathrm{Kl}_2^{(r)}$ is tamely ramified at 0, by [24, Proposition 1.9] we have

$$\mathrm{Swan}_0(\mathrm{Kl}_2^{(r)}) = 0.$$

Let

$$I_\infty < \mathrm{Gal}(\mathbb{F}_p(X)^{\mathrm{sep}}/\mathbb{F}_p(X))$$

be the inertia group at ∞ and $I_\infty(2) < I_\infty$ be the unique index 2 open subgroup. By [24, 1.13.1] we have

$$\mathrm{Swan}_\infty(\mathrm{Kl}_2^{(r)}) = \frac{1}{2} \times \mathrm{Swan}_\infty([2]^*(\mathrm{Kl}_2^{(r)}))$$

where $[2]^*(\mathrm{Kl}_2^{(r)})$ is the restriction of $\mathrm{Kl}_2^{(r)}$ to $I_\infty(2)$. Then by (the proof of) [15, Theorem 3.1] we have

$$\mathrm{Swan}_\infty([2]^*(\mathrm{Kl}_2^{(r)})) = r + 1 - d_r(2, p)$$

where $d_r(2, p)$ is the number of 2-tuples of non-negative integers (a_1, a_2) satisfying $a_1 + a_2 = r$ and $a_1 \equiv a_2 \pmod p$.

The result now follows by combining the preceding facts. □

2.6 Proof of Theorem 2.1.1

Let $s > 3/4$,

$$\alpha \geq \max\left\{5, 8 + \log\left(\frac{(4s-1)^8}{65536\pi^8 m^4}\right), 8 + \log\left(\frac{6561}{65536\pi^8 m^4}\right)\right\},$$

and p be a prime number satisfying

$$p \geq \max\left\{3, \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha-2)^{\frac{8}{\alpha-2}}\right\}.$$

For $n \geq p^\alpha$, we define the normalized Fourier coefficients

$$\lambda_{m,k,s,p}(n) := \frac{a_{m,k,p}(n,s)}{N(m,k,s,p,\alpha,n)}$$

where

$$N(m,k,s,p,\alpha,n) := (1 + C(m,k,s)p^{\frac{2-\alpha}{2}})\mathcal{N}(m,k,p,n).$$

Then by Theorem 2.3.2 we have

$$|\lambda_{m,k,s,p}(n)| \leq 1,$$

so that $\lambda_{m,k,s,p}(n) \in [-1,1]$.

Given an interval $I_p \subset \mathbb{F}_p^\times$, choose a complete set of residue classes

$$I_p = \{[n_{p,1}], \ldots, [n_{p,|I_p|}]\}$$

such that the class representatives $n_{p,i}$ satisfy the bound $n_{p,i} \geq p^\alpha$ for $i = 1, \ldots, |I_p|$. This choice determines a set

$$S_{\alpha,I_p} := \{n_{p,1}, \ldots, n_{p,|I_p|}\}.$$

Now, define the sequence of finite sets

$$X_{\alpha,I_p} := \{\lambda_{m,k,s,p}(n) : \ n \in S_{\alpha,I_p}\} \subset [-1,1].$$

We will deduce Theorem 2.1.1 from the following effective bound for the star discrepancy of X_{α,I_p}.

Proposition 2.6.1. Let $I_p \subset \mathbb{F}_p^\times$ be an interval of length $|I_p| > \sqrt{p}$. Then we have

$$D^\star_{X_{\alpha,I_p}} \leq (7208 + (4864)C(m,k,s)) \log\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1/3}.$$

Proof. By Proposition 2.4.2, for any positive odd integer d we have

$$D^\star_{X_{\alpha,I_p}} \leq \frac{8}{(0.362)\pi d + 4} + \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \left| \frac{1}{|I_p|} \sum_{n \in S_{\alpha,I_p}} U_r(\lambda_{m,k,s,p}(n)) \right|.$$

Now, by the triangle inequality,

$$D^{\star}_{X_{\alpha,I_p}} \le S_1 + S_2$$

where

$$S_1 := \frac{8}{(0.362)\pi d + 4} + \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \left| \frac{1}{|I_p|} \sum_{n \in S_{\alpha,I_p}} U_r(\cos(\theta_{m,p}(n))) \right|,$$
$$S_2 := \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \frac{1}{|I_p|} \sum_{n \in S_{\alpha,I_p}} \left| U_r(\lambda_{m,k,s,p}(n)) - U_r(\cos(\theta_{m,p}(n))) \right|.$$

In order to estimate S_1 we will use of the following bound of Fouvry, Kowalski, Michel, Raju, Rivat and Soundararajan [14] for sums of complex-valued functions on short intervals in cyclic groups which goes beyond the Pólya-Vinogradov range.

Theorem 2.6.2 ([14], Theorem 1.1.). *Let $\varphi : \mathbb{Z} \to \mathbb{C}$ be an m-periodic function. Then for any interval $I \subset \mathbb{Z}$ of length $\sqrt{m} < |I| \le m$ we have*

$$\left| \sum_{n \in I} \varphi(n) \right| \le c(\varphi) \log\left(\frac{4e^8 |I|}{\sqrt{m}} \right) \sqrt{m},$$

where $c(\varphi) := \max\{||\varphi||_\infty, ||\widehat{\varphi}||_\infty\}$ and $\widehat{\varphi}$ is the normalized finite Fourier transform of φ,

$$\widehat{\varphi}(h) := \frac{1}{\sqrt{m}} \sum_{n \pmod m} \varphi(n) e\left(\frac{hn}{m}\right), \quad h \in \mathbb{Z}.$$

Since $S(m, n+p; p) = S(m, n; p)$, the function $\varphi : \mathbb{Z} \to \mathbb{R}$ defined by

$$\varphi(n) := U_r(\cos(\theta_{m,p}(n)))$$

is a p-periodic function. Hence by Theorem 2.6.2 we have

$$\left|\sum_{n\in S_{\alpha,I_p}} U_r(\cos(\theta_{m,p}(n)))\right| \leq c(\varphi)\log\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\sqrt{p}. \tag{2.6.1}$$

Now, it is known that φ is the Frobenius trace function of the Fourier sheaf $\mathrm{Kl}_2^{(r)}$ (see e.g. [24, Chapter 13]). Then by [14, (2.6)], which utilizes [13, Proposition 8.2], we have the crucial bound

$$c(\varphi) \leq 10\mathbf{c}(\mathrm{Kl}_2^{(r)})^2.$$

It follows from Proposition 2.5.1 that

$$c(\varphi) \leq 10\left(\frac{3r+7}{2}\right)^2 \leq 5(3r+7)^2. \tag{2.6.2}$$

For notational convenience, we define

$$\beta(p) := \frac{|I_p|}{\sqrt{p}}.$$

Then by (2.6.1) and (2.6.2) we get

$$\left|\frac{1}{|I_p|}\sum_{n\in S_{\alpha,I_p}} U_r(\cos(\theta_{m,p}(n)))\right| \leq 5(3r+7)^2\frac{\log\left(4e^8\beta(p)\right)}{\beta(p)}. \tag{2.6.3}$$

Applying the bound (2.6.3) gives

$$\begin{aligned}
S_1 &\leq \frac{8}{(0.362)\pi d+4} + \frac{4d-3}{(0.362)\pi d+2\pi}\sum_{r=1}^{2d-1}\frac{5(r+1)(3r+7)^2}{r(r+2)}\frac{\log(4e^8\beta(p))}{\beta(p)} \\
&\leq \frac{8}{(0.362)\pi d+4} + \frac{4d-3}{(0.362)\pi d+2\pi}\sum_{r=1}^{2d-1}(100)r\frac{\log(4e^8\beta(p))}{\beta(p)} \\
&\leq \frac{8}{(0.362)\pi d+4} + \frac{100(4d-3)(2d-1)d}{(0.362)\pi d+2\pi}\frac{\log(4e^8\beta(p))}{\beta(p)}
\end{aligned}$$

$$\leq 8d^{-1} + (800)d^2 \frac{\log(4e^8\beta(p))}{\beta(p)}.$$

We choose d such that

$$\beta(p)^{1/3} \leq d < \beta(p)^{1/3} + 2.$$

Then we have

$$S_1 \leq 8\beta(p)^{-1/3} \left[1 + 900 \log(4e^8\beta(p))\right].$$

To estimate S_2 we will require the following lemma.

Lemma 2.6.3. *We have*

$$||U_r'||_\infty \leq r(r+1)^2.$$

Proof. For $\theta \in [0, \pi]$, we have

$$\begin{aligned}
\left|\frac{d}{d\theta} U_r(\cos(\theta))\right| &= \left|\frac{d}{d\theta} \frac{\sin((r+1)\theta)}{\sin(\theta)}\right| \\
&= \left|\frac{d}{d\theta} \sum_{n=0}^{r} \cos(n\theta)(\cos(\theta))^{r-n}\right| \\
&= \left|-\sum_{n=0}^{r} \left[n\sin(n\theta)(\cos(\theta))^{r-n} + (r-n)\cos(n\theta)\sin(\theta)(\cos(\theta))^{r-n-1}\right]\right|.
\end{aligned} \tag{2.6.4}$$

On the other hand, we have

$$\left|\frac{d}{d\theta} U_r(\cos(\theta))\right| = |U_r'(\cos(\theta))| \times |-\sin(\theta)|. \tag{2.6.5}$$

Hence, by combining (2.6.4) and (2.6.5), we get

$$
\begin{aligned}
|U_r'(\cos(\theta))| &= \left|\sum_{n=0}^{r} \frac{n\sin(n\theta)(\cos(\theta))^{r-n} + (r-n)\cos(n\theta)\sin(\theta)(\cos(\theta))^{r-n-1}}{\sin(\theta)}\right| \\
&\leq \sum_{n=0}^{r} \left|\frac{n\sin(n\theta)(\cos(\theta))^{r-n}}{\sin(\theta)}\right| + \sum_{n=0}^{r} \left|(r-n)\cos(n\theta)(\cos(\theta))^{r-n-1}\right| \\
&\leq \sum_{n=0}^{r} \left|\frac{n\sin(n\theta)}{\sin(\theta)}\right| + r(r+1) \\
&= \sum_{n=0}^{r} n \left|\sum_{j=0}^{n-1} \cos(n\theta)(\cos(\theta))^{n-1-j}\right| + r(r+1) \\
&\leq \sum_{n=0}^{r} n^2 + r(r+1) \\
&\leq r^2(r+1) + r(r+1) \\
&= r(r+1)^2.
\end{aligned}
$$

□

By the mean-value theorem, Lemma 2.6.3, and Theorem 2.3.2, we get

$$
\begin{aligned}
S_2 &= \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \frac{1}{|I_p|} \sum_{n \in S_{\alpha, I_p}} |U_r(\lambda_{m,k,s,p}(n)) - U_r(\cos(\theta_{m,p}(n)))| \\
&\leq \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \frac{1}{|I_p|} \sum_{n \in S_{\alpha, I_p}} ||U_r'||_\infty \, |\lambda_{m,k,s,p}(n) - \cos(\theta_{m,p}(n))| \\
&\leq \frac{4d-3}{(0.362)\pi d + 2\pi} \sum_{r=1}^{2d-1} \frac{r+1}{r(r+2)} \frac{r(r+1)^2}{|I_p|} \sum_{n \in S_{\alpha, I_p}} |\lambda_{m,k,s,p}(n) - \cos(\theta_{m,p}(n))| \\
&\leq \frac{(4d)^2(2d-1)(4d-3)}{(0.362)\pi d + 2\pi} \frac{1}{|I_p|} \sum_{n \in S_{\alpha, I_p}} \left| \frac{\cos(\theta_{m,p}(n)) + E(m,k,s,p,n)}{1 + C(m,k,s)p^{\frac{2-\alpha}{2}}} - \cos(\theta_{m,p}(n)) \right| \\
&\leq \frac{16d^2(2d-1)(4d-3)}{(0.362)\pi d + 2\pi} \times 2C(m,k,s)p^{\frac{2-\alpha}{2}} \\
&\leq 256(\beta(p)^{1/3} + 2)^3 C(m,k,s)p^{\frac{2-\alpha}{2}}.
\end{aligned}
$$

Finally, by combining our bounds for S_1 and S_2, we have

$$D^{\star}_{X_{\alpha,I_p}} \leq 8\beta(p)^{-1/3}\left[1+900\log(4e^8\beta(p))\right]+256(\beta(p)^{1/3}+2)^3C(m,k,s)p^{\frac{2-\alpha}{2}}.$$

Moreover, by our assumption on α we have $\alpha \geq 5$, hence

$$\begin{aligned} p^{\frac{2-\alpha}{2}} &< \beta(p)^{\frac{4-\alpha}{2}} < \beta(p)^{-1/3}, \\ p^{\frac{2-\alpha}{2}} &< \beta(p)^{\frac{10-3\alpha}{6}} < \beta(p)^{-1/3}, \\ p^{\frac{2-\alpha}{2}} &< \beta(p)^{\frac{8-3\alpha}{6}} < \beta(p)^{-1/3}. \end{aligned}$$

It follows that

$$\begin{aligned} D^{\star}_{X_{\alpha,I_p}} &\leq 8\beta(p)^{-1/3}+7200\log(4e^8\beta(p))\beta(p)^{-1/3}+256C(m,k,s)\beta(p)^{\frac{4-\alpha}{2}} \\ &\quad + 1536C(m,k,s)\beta(p)^{\frac{10-3\alpha}{6}}+3072C(m,k,s)\beta(p)^{\frac{8-3\alpha}{6}} \\ &\leq (7208+(4864)C(m,k,s))\beta(p)^{-1/3}\log(4e^8\beta(p)). \end{aligned}$$

□

Proof of Theorem 2.1.1. By Lemma 2.4.1 we have

$$\left|\frac{1}{|I_p|}\sum_{n\in S_{\alpha,I_p}} f(\lambda_{m,k,s,p}(n)) - \int_{-1}^{1} f(t)d\mu_{\mathrm{ST}}(t)\right| \leq D^{\star}_{X_{\alpha,I_p}}\mathrm{Var}(f).$$

The result now follows immediately from Proposition 2.6.1. □

3. EQUIDISTRIBUTION FOR WEAK MAASS FORMS OF HALF-INTEGRAL WEIGHT

In this chapter, we prove quantitative equidistribution for normalized Fourier coefficients of weak Maass forms of half-integral weight and level $4p$. Briefly, for each fixed $m \in \mathbb{Z}^+$ and $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, we consider a family of weak Maass forms $P_{m,k,p}(z,s)$ of weight k and level $4p$. We define a finite set $X_p \subset [-1,1]$ of normalized Fourier coefficients of $P_{m,k,p}(z,s)$, and prove that the sets $\{X_p\}$ become quantitatively equidistributed on $[-1,1]$ with respect to the probability measure

$$\mu_{\mathrm{Arc}}(t) := \frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}dt$$

as $p \to \infty$. This measure is the pushforward of the Haar measure on the unitary group $U(1)$.

When $s = 1 - k/2$ and $k \leq 1/2$, the family $\{P_{m,k,p}(z, 1-k/2)\}_{m \in \mathbb{Z}^+}$ spans the vector space of harmonic Maass forms of weight k and level $4p$. For the precise statements of these results, see Theorem 3.1.1 and Corollary 3.1.2.

3.1 Introduction and statement of results

3.1.1 Background on weak Maass forms of half-integral weight

In this section we review some facts concerning the weak Maass forms whose coefficients we will study. More details can be found, for example, in the fundamental works of Bruinier [2, 3], Bruinier/Funke [4] and Bruinier/Ono [5, 6].

Let $f : \mathbb{H} \to \mathbb{C}$ be a function on the complex upper half-plane $\mathbb{H} := \{z = x + iy : y > 0\}$. Recall that the slash operator for half-integral weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ is defined by

$$f_{|_k}\gamma(z) := \left[\theta_k(\gamma)(cz+d)^k\right]^{-1} f\left(\frac{az+b}{cz+d}\right)$$

for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N),$$

where the θ-multiplier of weight k is defined by

$$\theta_k(\gamma) := \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k}$$

with

$$\varepsilon_d := \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv 3 \pmod 4. \end{cases}$$

Here $\left(\frac{c}{d}\right)$ is the extended Kronecker symbol in the sense of Shimura [32].

Now, we introduce weak Maass forms of half-integral weight.

A *weak Maass form* of weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for $\Gamma_0(4N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(1) $f_{|_k}\gamma = f$ for all $\gamma \in \Gamma_0(4N)$.

(2) f is an eigenfunction of the weight k hyperbolic Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(3) There is a non-constant polynomial $P_{f,\infty}(z) := \sum_{n \le 0} a_f(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_{f,\infty}(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$, where $q := e^{2\pi i z}$. A similar growth condition holds for all other cusps.

Similarly, there is a generic family of weak Maass forms of half-integral weight called *Maass*

Poincaré series. For $m \in \mathbb{Z}^+$, define the Maass-Poincaré series of weight k for $\Gamma_0(4N)$ by

$$P_{m,k,N}(z,s) := \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} [\mathbf{M}_{s,k}(4\pi m y) e(-mx)] \, |_k \gamma, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1, \quad (3.1.1)$$

where $e(z) := e^{2\pi i z}$ and

$$\mathbf{M}_{s,k}(y) := y^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(y)$$

with $M_{\kappa,\mu}$ equal to the usual M-Whittaker function (see [29, Section 13.14]).

$P_{m,k,N}(z,s)$ is an eigenfunction for Δ_k with eigenvalue $s(1-s)+(k^2-2k)/4$. Also, the Maass-Poincaré series $P_{m,k,N}(z,s)$ has the Fourier expansion at ∞ given by (see Proposition 3.4.1)

$$\begin{aligned} P_{m,k,N}(z,s) = & \frac{\mathbf{M}_{s,k}(4\pi m y)}{\Gamma(2s)} e(-mx) + a_{m,k,N}(0,s) y^{1-s-k/2} \\ & + \sum_{n \in \mathbb{Z}^+} a_{m,k,N}(\pm n, s) \frac{\mathbf{W}_{s,k}(\pm 4\pi n y)}{\Gamma(s+k/2)} e(\pm n x) \end{aligned} \quad (3.1.2)$$

where

$$\mathbf{W}_{s,k}(t) := |t|^{-k/2} W_{\mathrm{sign}(t)\frac{k}{2}, s-\frac{1}{2}}(|t|)$$

with $W_{\kappa,\mu}$ equal to the usual Whittaker function (see [29, Section 13.14]).

It is important that the family $P_{m,k,N}(z,s)$ can be used to generate certain vector spaces of harmonic Maass forms.

A *harmonic Maass form* (resp. *weakly holomorphic modular form*) of weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for $\Gamma_0(4N)$ with a pole only at the cusp $\mathfrak{a} = \infty$ is a smooth (resp. holomorphic) function $f : \mathbb{H} \to \mathbb{C}$ on the complex upper half-plane $\mathbb{H}$ satisfying $(1)-(3)$, and the following additional conditions:

(4) $\Delta_k f = 0$.

(5) $f_{|_k} \gamma_\mathfrak{a}(z)$ is bounded as $y \to \infty$ for any matrix $\gamma_\mathfrak{a} \in SL_2(\mathbb{Z})$ such that $\gamma_\mathfrak{a}(\infty) = \mathfrak{a} \not\approx \infty$.

Let $H_k^{\#}(4N)$ (resp. $M_k^{\#}(4N)$) denote the vector space of harmonic Maass forms (resp. the subspace of weakly holomorphic modular forms) of this type.

We define

$$P_{m,k,N}(z) := P_{m,k,N}(z, 1-k/2), \quad k \leq 1/2$$
$$Q_{m,k,N}(z) := \Gamma(k) P_{m,k,N}(z, k/2), \quad k \geq 3/2$$

where by $P_{m,k,N}(z, 3/4)$ we mean the value of the analytic continuation of $P_{m,k,N}(z, s)$ at $s = 3/4$ (see Section 3.4). Then $P_{m,k,N} \in H_k^{\#}(4N)$ is a harmonic Maass form with Fourier expansion at ∞ given by

$$\begin{aligned} P_{m,k,N}(z) = q^{-m} &+ \sum_{n=0}^{\infty} a_{m,k,N}(n, 1-k/2) q^n \\ &+ \sum_{n=1}^{\infty} (a_{m,k,N}(-n, 1-k/2) - \delta_m(n)) \frac{\Gamma(1-k, 4\pi n y)}{\Gamma(1-k)} q^{-n}, \end{aligned} \tag{3.1.3}$$

where $\Gamma(s, x)$ is the incomplete Gamma function and $\delta_m(n)$ is the Kronecker delta function.

Similarly, $Q_{m,k,N} \in M_k^{\#}(4N)$ is a weakly holomorphic modular form with Fourier expansion at ∞ given by

$$Q_{m,k,N}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{m,k,N}(n, k/2) q^n. \tag{3.1.4}$$

In Appendix B.2, we explain how these Fourier expansions give the decompositions

$$H_k^{\#}(4N) = \text{Span}(\{P_{m,k,N}(z)\}_{m \in \mathbb{Z}^+}), \qquad k \leq -1/2, \tag{3.1.5}$$

$$H_{1/2}^{\#}(4N) = \text{Span}(\{P_{m,1/2,N}(z)\}_{m \in \mathbb{Z}^+}) \sqcup \Theta(4N), \quad k = 1/2, \tag{3.1.6}$$

and

$$M_k^{\#}(4N) = \mathrm{Span}(\{Q_{m,k,N}(z)\}_{m\in\mathbb{Z}^+}) \sqcup M_k(4N), \quad k \geq 3/2, \tag{3.1.7}$$

where $\Theta(4N)$ consists of those twisted theta functions which generate $M_{1/2}(4N)$, where $M_k(4N)$ denotes the vector space of holomorphic modular forms of weight k for $\Gamma_0(4N)$.

Convention. If $s = 3/4$, then we only deal with $P_{m,k,N}(z,s)$ for the most interesting cases $k = 1/2$ and $k = 3/2$.

3.1.2 Equidistribution of weak Maass form coefficients for half-integral weight

Let $s = 3/4$ ($k = 1/2$ or $k = 3/2$) or $s > 3/4$. Fix a positive real number

$$\alpha \geq \max\left\{3, 8+\log\left(\frac{(4s-1)^8}{256\pi^8 m^4}\right), 8+\log\left(\frac{6561}{\pi^8 m^4}\right)\right\},$$

and p be a prime number satisfying

$$p \geq \max\left\{8, \left|(2s-1)^2-\frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha-2)^{\frac{2}{\alpha-2}}\right\}.$$

In Section 3.8, we show there is a natural choice of normalizing factor $N(m,k,s,p,\alpha,n)$ such that if $n \geq p^\alpha$, then the normalized Fourier coefficients

$$\lambda_{m,k,s,p}(n) := a_{m,k,p}(n,s)/N(m,k,s,p,\alpha,n),$$

lie in the interval $[-1,1]$.

Now, given an interval $I_p \subset \mathbb{F}_p^\times$, we choose a complete set of residue classes

$$I_p = \{[n_{p,1}], \ldots, [n_{p,|I_p|}]\}$$

such that the class representatives $n_{p,i}$ satisfy the bound $n_{p,i} \geq p^\alpha$ for $i = 1, \ldots, |I_p|$. This choice determines a set

$$S_{\alpha,I_p} := \{n_{p,1}, \dots, n_{p,|I_p|}\}.$$

Let $BV([-1,1])$ be the space of functions $f : [-1,1] \to \mathbb{C}$ whose total variation $\mathrm{Var}(f)$ is bounded.

Finally, recall the measure

$$\mu_{\mathrm{Arc}}(t) = \frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}dt.$$

We will prove the following quantitative equidistribution theorem.

Theorem 3.1.1. *Let $I_p \subset \mathbb{F}_p^\times$ be an interval of length $|I_p| > \sqrt{p}$ such that each $n \in I_p$ satisfies $(\frac{-mn}{p}) = 1$. Then if $f \in BV([-1,1])$, we have*

$$\frac{1}{|I_p|}\sum_{n\in S_{\alpha,I_p}} f(\lambda_{m,k,s,p}(n)) = \int_{-1}^{1} f(t)d\mu_{\mathrm{Arc}}(t) + R(f,m,k,s,p)$$

where

$$|R(f,m,k,s,p)| \le (14\pi^5 + 42\pi^5 C(m,k,s))\mathrm{Var}(f)\log^2\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1}$$

for the explicit constant $C(m,k,s) > 0$ defined by (3.7.11).

Theorem 3.1.1 implies that for "short" subintervals $I_p \subset \mathbb{F}_p^\times$ satisfying the growth condition $|I_p|/\sqrt{p} \to \infty$, the Fourier coefficients $\lambda_{m,k,s,p}(n)$ for $n \in S_{\alpha,I_p}$ become quantitatively equidistributed on $[-1,1]$ with respect to the measure μ_{Arc} as $p \to \infty$.

Define the sequence of finite subsets

$$X_{\alpha,I_p} := \{\lambda_{m,k,s,p}(n) :\ n \in S_{\alpha,I_p}\} \subset [-1,1].$$

The following result is an immediate consequence of Theorem 3.1.1.

Corollary 3.1.2. *Let $I_p \subset \mathbb{F}_p^\times$ be as in Theorem 3.1.1, and assume further that $|I_p|/\sqrt{p} \to \infty$ as $p \to \infty$. Then the sets $\{X_{\alpha,I_p}\}$ become equidistributed on $[-1,1]$ with respect to the measure μ_{Arc} as $p \to \infty$.*

3.1.3 Vertical equidistribution of Salié sums

As an important step in the proof of Theorem 3.1.1, we will require the "vertical" equidistribution of Salié sums.

Recall that the *Salié sum* of prime modulus p is defined by

$$T(m,n;p) := \sum_{x \pmod p} \left(\frac{x}{p}\right) e\left(\frac{m\overline{x}+nx}{p}\right), \quad m,n \in \mathbb{Z}.$$

Assume that p satisfies $(p,2n)=1$ and $(\frac{mn}{p})=1$. A fundamental identity of Sarnak [34] asserts that

$$T(m,n;p) = \left(\frac{n}{p}\right) \varepsilon_p \sqrt{p} \sum_{v^2 \equiv 4mn \pmod p} e\left(\frac{v}{p}\right). \tag{3.1.8}$$

In particular, let $v \pmod p$ be a solution of $v^2 \equiv 4mn \pmod p$, then one can write

$$T(m,n;p) =: 2\left(\frac{n}{p}\right) \varepsilon_p \sqrt{p} \cos(\theta_{m,p}(n))$$

for a real number

$$\theta_{m,p}(n) = \frac{2\pi v}{p} \in [0,\pi] \tag{3.1.9}$$

called the *Salié angle*. Accordingly, the *normalized Salié sum* is defined by

$$\frac{T(m,n;p)}{2(\frac{n}{p})\varepsilon_p\sqrt{p}} \in [-1,1].$$

A remarkable theorem of Duke, Friedlander, and Iwaniec [11] shows that the Salié sums become "horizontally" equidistributed with respect to μ_{Arc} as $p \to \infty$; more precisely, given any interval $[\alpha, \beta] \subset [-1, 1]$, we have

$$\lim_{X\to\infty} \frac{|\{p \le X : \ \cos(\theta_{m,p}(n)) \in [\alpha, \beta]\}|}{|\{p \le X\}|} = \int_{\alpha}^{\beta} d\mu_{\text{Arc}}(t).$$

Here we will prove the quantitative "vertical" equidistribution of Salié sums in short intervals.

Theorem 3.1.3. *Let $I_p \subset \mathbb{F}_p^\times$ be a interval of length $|I_p| > \sqrt{p}$ such that each $n \in I_p$ satisfies $(\frac{mn}{p}) = 1$. Then if $f \in BV([-1, 1])$, we have*

$$\frac{1}{|I_p|} \sum_{n\in I_p} f(\cos(\theta_{m,p}(n))) = \int_{-1}^{1} f(t) d\mu_{\text{Arc}}(t) + R(f, m, p)$$

where

$$|R(f, m, p)| \le 14\pi^5 \text{Var}(f) \log^2 \left(\frac{4e^8|I_p|}{\sqrt{p}}\right) \left(\frac{|I_p|}{\sqrt{p}}\right)^{-1}.$$

Define the sequence of finite subsets consisting of the normalized Salié sums

$$X_{I_p} := \{\cos(\theta_{m,p}(n)) : \ n \in I_p\} \subset [-1, 1].$$

The following result is an immediate consequence of Theorem 3.1.3.

Corollary 3.1.4. *Let $I_p \subset \mathbb{F}_p^\times$ be as in Theorem 3.1.3, and assume further that $|I_p|/\sqrt{p} \to \infty$ as $p \to \infty$. Then the sets $\{X_{I_p}\}$ become equidistributed on $[-1, 1]$ with respect to the measure μ_{Arc} as $p \to \infty$.*

Remark 3.1.5. *A fundamental theorem of Katz [24] shows that the Kloosterman sums become vertically equidistributed with respect to the Sato-Tate measure on $[-1, 1]$ as $p \to \infty$. Corollary 3.1.4 is an analog of this result for Salié sums.*

3.1.4 Bounds for sums of half-integral weight Kloosterman sums

For the proof of Theorem 3.1.1, we will also require a power-saving bound for sums of (opposite sign) half-integral weight Kloosterman sums with θ-multiplier which is uniform in all parameters.

Recall that for $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, the Kloosterman sum with θ-multiplier is defined by

$$S_k(u, v; c) := \sum_{\substack{d \pmod{c} \\ (d,c)=1}} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} e\left(\frac{u\bar{d} + vd}{c}\right).$$

We will prove the following bound.

Theorem 3.1.6. *Assume that $m, n, N \in \mathbb{Z}^+$ and $k = 1/2$ or $k = 3/2$. Then for any $\varepsilon > 0$, we have*

$$\sum_{\substack{0<c\leq X \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m, n; c)}{c} \ll_\varepsilon (mn)^{\frac{1}{4}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{\frac{1}{6}+\varepsilon}.$$

Several years ago, Sarnak and Tsimerman [35] proved the analogous bound for sums of classical Kloosterman sums over $SL_2(\mathbb{Z})$. Our approach is modeled on theirs. The crucial tool is a Kuznetsov-Proskurin formula for half-integral weight opposite sign Kloosterman sums with θ-multiplier due to Blomer [7]. We note that by modifying our analysis we can give a same sign version of the bound in Theorem 3.1.6. We focus here on the opposite sign case since this is what is needed for our applications.

3.1.5 A numerical example

Here we use `SageMath` [36] to give an example which illustrates that the sets $\{X_{(\mathbb{F}_p^\times)^2}\}$ become equidistributed with respect to the measure μ_{Arc} as $p \to \infty$. Our calculations indicate that the rate of equidistribution is $O_\varepsilon(p^{-1/2-\varepsilon})$, which is faster than the rate $O_\varepsilon(p^{-1/2+\varepsilon})$ given by Theorem 3.1.3.

Let $m = 1$ and $p \geq 3$. Divide the interval $[-1, 1]$ into $N_p := 2[(p-1)^{1/2}]$ subintervals

$T_{p,i}$ of length $|T_{p,i}| = (p-1)^{-1/2}$. Let $\chi_{T_{p,i}}$ be the characteristic function of $T_{p,i}$. A standard approximation argument shows that the equidistribution of the sets $\{X_{(\mathbb{F}_p^\times)^2}\}$ implies that

$$\frac{2}{p-1} \sum_{i=1}^{N_p} \sum_{n \in (\mathbb{F}_p^\times)^2} \chi_{T_{p,i}}(\cos(\theta_{1,p}(n))) \longrightarrow \int_{-1}^{1} d\mu_{\mathrm{Arc}}(t) \tag{3.1.10}$$

as $p \to \infty$. Now, define the rectangles

$$R_{p,i} := T_{p,i} \times [0, H_{p,i}]$$

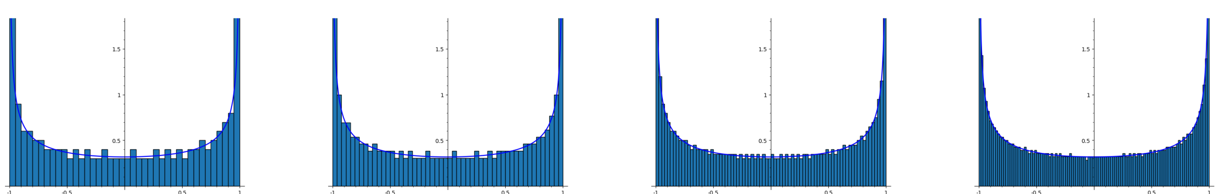
where the height is given by

$$H_{p,i} := 2(p-1)^{-1/2} |\{n \in (\mathbb{F}_p^\times)^2 : \ \cos(\theta_{1,p}(n)) \in T_{p,i}\}|.$$

Then (3.1.10) is equivalent to

$$A_p := \sum_{i=1}^{N_p} \mathrm{Area}(R_{p,i}) \longrightarrow \int_{-1}^{1} d\mu_{\mathrm{Arc}}(t)$$

as $p \to \infty$. The histograms in Figure 3.1 display how A_p approximates the area bounded by the function $1/(\pi\sqrt{1-t^2})$ on the interval $[-1, 1]$ for successively larger values of p.



(a) $p = 401$ (b) $p = 677$ (c) $p = 1061$ (d) $p = 3137$

Figure 3.1: Histograms for $X_{(\mathbb{F}_p^\times)^2}$

3.2 Erdös-Turán for μ_{Arc}

Given any finite subset $X \subset [-1, 1]$, we define the averaged Dirac measures on $[-1, 1]$ by

$$\mu_X(t) := \frac{1}{|X|} \sum_{x \in X} \chi_{(x,1]}(t)$$

where $\chi_{(x,1]}(t)$ denotes the characteristic function of the interval $(x, 1] \subset [-1, 1]$. It follows that for any $y \in [-1, 1]$, we have the corresponding distribution function

$$\mu_X(y) = \int_{-1}^{y} d\mu_N(t).$$

Additionally, recall the measure

$$\mu_{\text{Arc}}(t) = \frac{1}{\pi} \frac{1}{\sqrt{1 - t^2}} dt.$$

Then we define the distribution function of $\mu_{\text{Arc}}(t)$ on $[-1, 1]$ by

$$G(y) := \int_{-1}^{y} d\mu_{\text{Arc}}(t).$$

Finally, we define the star discrepancy of X by

$$D_X^{\star} = \sup_{y \in [-1,1]} \left| \mu_X(y) - G(y) \right|.$$

Now, recall that the Chebyshev polynomials of the first kind are defined by

$$T_r(t) := \cos(r \cos^{-1}(t)), \quad t \in [-1, 1].$$

The polynomials $\{T_r(t)\}_{r=0}^{\infty}$ form an orthonormal system with respect to the measure μ_{Arc} on $[-1, 1]$.

We use the framework of Chebyshev, Markov and Stieltjes [1, Chapter 3] and Feldheim and

Sodin [16] to prove the following effective Erdös-Turán type inequality for the measure μ_{Arc}.

Proposition 3.2.1. Let X be a finite subset of $[-1,1]$. Then for any $d \in \mathbb{Z}^+$, we have

$$D_X^\star \leq \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \left| \frac{1}{|X|} \sum_{x \in X} T_r(x) \right|.$$

Proof. We apply the argument in [16, Section 3 and (3)] to the measures μ_X and μ_{Arc} to get the upper bound

$$D_X^\star \leq S_0 + \sum_{r=1}^{2n_0-2} \left| S_r \int_{-1}^{1} T_r(t) d\mu_X(t) \right|,$$

valid for any $n_0 \in \mathbb{Z}^+$, where

$$\begin{aligned} S_0 &:= \int_{-1}^{1} \frac{T_{n_0}(t)^2}{T'_{n_0}(y_0)^2 (t-y_0)^2} d\mu_{\mathrm{Arc}}(t), \\ S_r &:= \int_{-1}^{1} \frac{T_{n_0}(t)^2 |T_r(t)|}{T'_{n_0}(y_0)^2 (t-y_0)^2} d\mu_{\mathrm{Arc}}(t) + \int_{y_0}^{1} T_r(t) d\mu_{\mathrm{Arc}}(t), \end{aligned}$$

and y_0 is any one of non-negative roots of $T_{n_0}(t)$.

We first estimate S_0. Recall the following identities for Chebyshev polynomials of the first kind,

$$\begin{aligned} T_r(\cos\theta) &= \cos(r\theta), \\ T'_r(\cos\theta) &= \frac{r \sin(r\theta)}{\sin\theta}. \end{aligned} \tag{3.2.1}$$

By setting $t = \cos\theta$ and (3.2.1), we have

$$\begin{aligned} S_0 = \int_{-1}^{1} \frac{T_{n_0}(t)^2}{T'_{n_0}(y_0)^2 (t-y_0)^2} d\mu_{\mathrm{Arc}}(t) &= \frac{\sin^2\theta_0}{4\pi n_0^2} \int_0^{\pi} \frac{\cos^2(n_0\theta)}{\sin^2\left(\frac{\theta+\theta_0}{2}\right) \sin^2\left(\frac{\theta-\theta_0}{2}\right)} d\theta \\ &\leq \frac{\theta_0^2}{4\pi n_0^2} \int_0^{\pi} \frac{\cos^2(n_0\theta)}{\sin^2\left(\frac{\theta+\theta_0}{2}\right) \sin^2\left(\frac{\theta-\theta_0}{2}\right)} d\theta. \end{aligned} \tag{3.2.2}$$

Now, by applying

$$\frac{2}{\pi}\theta \leq \sin\theta \leq \theta, \tag{3.2.3}$$

we get

$$\begin{aligned}
\int_0^{\theta_0-\frac{\pi}{3n_0}} \frac{\cos^2(n_0\theta)}{\sin^2\left(\frac{\theta+\theta_0}{2}\right)\sin^2\left(\frac{\theta-\theta_0}{2}\right)}d\theta &\leq \pi^4\int_0^{\theta_0-\frac{\pi}{3n_0}} \frac{1}{(\theta+\theta_0)^2(\theta-\theta_0)^2}d\theta \\
&\leq \frac{\pi^4}{\theta_0^2}\int_0^{\theta_0-\frac{\pi}{3n_0}} \frac{1}{(\theta-\theta_0)^2}d\theta \\
&= \frac{\pi^4}{\theta_0^2}\int_{-\theta_0}^{-\frac{\pi}{3n_0}} \frac{1}{\theta^2}d\theta \\
&\leq \frac{3\pi^3 n_0}{\theta_0^2}. \qquad (3.2.4)
\end{aligned}$$

Similarly, by (3.2.3) and the same argument, we have

$$\int_{\theta_0+\frac{\pi}{3n_0}}^{\pi} \frac{\cos^2(n_0\theta)}{\sin^2\left(\frac{\theta+\theta_0}{2}\right)\sin^2\left(\frac{\theta-\theta_0}{2}\right)}d\theta \leq \frac{3\pi^3 n_0}{\theta_0^2}. \tag{3.2.5}$$

On the other hand, if $\theta_0 - \frac{\pi}{3n_0} \leq \theta \leq \theta_0 + \frac{\pi}{3n_0}$, then by $(3.2.1)$ we have

$$|T'_{n_0}(\cos\theta)| \leq \left(\frac{\theta_0+\frac{\pi}{3n_0}}{\sin(\theta_0+\frac{\pi}{3n_0})}\right)^2 |T'_{n_0}(\cos\theta_0)| \leq \left(\frac{\frac{\pi}{2}+\frac{\pi}{3}}{\sin(\frac{\pi}{6})}\right)^2 |T'_{n_0}(\cos\theta_0)| = \frac{25\pi^2}{9}|T'_{n_0}(\cos\theta_0)|. \tag{3.2.6}$$

Moreover, by mean value theorem and $(3.2.6)$, we get

$$\begin{aligned}
\int_{\theta_0-\frac{\pi}{3n_0}}^{\theta_0+\frac{\pi}{3n_0}} \frac{\cos^2(n_0\theta)}{\sin^2\left(\frac{\theta+\theta_0}{2}\right)\sin^2\left(\frac{\theta-\theta_0}{2}\right)}d\theta &= \int_{\theta_0-\frac{\pi}{3n_0}}^{\theta_0+\frac{\pi}{3n_0}} \frac{T_{n_0}(\cos\theta)^2}{T'_{n_0}(\cos\theta_0)^2(\cos\theta-\cos\theta_0)^2}d\theta \\
&\leq \int_{\theta_0-\frac{\pi}{3n_0}}^{\theta_0+\frac{\pi}{3n_0}} \frac{T'_{n_0}(\cos\theta)^2}{T'_{n_0}(\cos\theta_0)^2}d\theta \\
&\leq \frac{1300\pi^5}{243 n_0}. \qquad (3.2.7)
\end{aligned}$$

Hence, by combining (3.2.2), (3.2.4), (3.2.5) and (3.2.7), we have

$$S_0 \le \frac{3\pi^2}{2n_0} + \frac{325\pi^5}{486n_0^3} \le \frac{3\pi^5}{n_0}. \tag{3.2.8}$$

Next, we estimate S_r. By definition of S_r, we have

$$|S_r| \le S_0 + \int_{y_0}^{1} |T_r(t)| d\mu_{\mathrm{Arc}}(t) \le S_0 + \frac{1}{\pi r}.$$

Then by (3.2.8) it follows that

$$|S_r| \le \frac{3\pi^5}{n_0} + \frac{1}{\pi r} \le \frac{6\pi^5}{r} + \frac{1}{\pi r} \le \frac{7\pi^5}{r}. \tag{3.2.9}$$

Hence, by (3.2.8) and (3.2.9), and letting $n_0 = \lceil \frac{d}{2} \rceil + 1$, we get

$$\begin{aligned} D_X^\star \le \frac{3\pi^5}{n_0} + 7\pi^5 \sum_{r=1}^{2n_0-2} \frac{1}{r} \left| \int_{-1}^{1} T_r(t) d\mu_X(t) \right| &\le \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \left| \int_{-1}^{1} T_r(t) d\mu_X(t) \right| \\ &= \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \left| \frac{1}{|X|} \sum_{x \in X} T_r(x) \right|. \end{aligned}$$

□

3.3 Quantitative vertical equidistribution of Salié sums

We briefly recall the setup from the introduction. Let $m, n \in \mathbb{Z}$ and p be a prime which satisfies $(p, 2n) = 1$ and $(\frac{mn}{p}) = 1$. The Salié sum of modulus p is defined by

$$T(m, n; p) := \sum_{x \pmod p} \left(\frac{x}{p} \right) e \left(\frac{m\overline{x} + nx}{p} \right),$$

and the normalized Salié sum is defined by

$$\cos(\theta_{m,p}(n)) := \frac{T(m, n; p)}{2(\frac{n}{p})\varepsilon_p \sqrt{p}} \in [-1, 1].$$

Let $I_p \subset \mathbb{F}_p^\times$ be an interval such that each $n \in I_p$ satisfies $(\frac{mn}{p}) = 1$ and define the sequence of finite subsets

$$X_{I_p} := \{\cos(\theta_{m,p}(n)) :\ n \in I_p\} \subset [-1, 1].$$

We will deduce Theorem 3.1.3 from the following effective upper bound for the star discrepancy $D^\star_{X_{I_p}}$.

Proposition 3.3.1. Assume that $|I_p| > \sqrt{p}$. Then we have

$$D^\star_{X_{I_p}} \leq 14\pi^5 \log^2\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1}.$$

Proof. By Proposition 3.2.1, we have

$$D^\star_{X_{I_p}} \leq \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r}\left|\frac{1}{|I_p|}\sum_{n\in I_p} T_r(\cos(\theta_{m,p}(n)))\right|.$$

Again, we will use the following result of Fouvry, Kowalski, Michel, Raju, Rivat and Soundararajan [14] which gives a bound for sums of complex-valued functions on short intervals in cyclic groups.

Theorem 3.3.2 ([14], Theorem 1.1.)**.** *Let $\varphi : \mathbb{Z} \to \mathbb{C}$ be an m-periodic function. Then for any interval $I \subset \mathbb{Z}$ of length $\sqrt{m} < |I| \leq m$ we have*

$$\left|\sum_{n\in I} \varphi(n)\right| \leq c(\varphi) \log\left(\frac{4e^8|I|}{\sqrt{m}}\right)\sqrt{m},$$

where $c(\varphi) := \max\{||\varphi||_\infty, ||\widehat{\varphi}||_\infty\}$ and $\widehat{\varphi}$ is the normalized finite Fourier transform of φ,

$$\widehat{\varphi}(h) := \frac{1}{\sqrt{m}} \sum_{n \pmod m} \varphi(n) e\left(\frac{hn}{m}\right), \quad h \in \mathbb{Z}.$$

By $T(m, n+p; p) = T(m, n; p)$ the function $\varphi : \mathbb{Z} \to \mathbb{R}$ defined by

$$\varphi(n) := T_r(\cos(\theta_{m,p}(n)))$$

is a p-periodic function. Hence by Theorem 3.3.2 we have

$$\left| \sum_{n \in I_p} T_r(\cos(\theta_{m,p}(n))) \right| \leq c(\varphi) \log \left(\frac{4e^8 |I_p|}{\sqrt{p}} \right) \sqrt{p}, \tag{3.3.1}$$

where

$$c(\varphi) := \max \left\{ ||\varphi||_\infty, ||\widehat{\varphi}||_\infty \right\}.$$

For convenience, we define

$$\beta(p) := \frac{|I_p|}{\sqrt{p}}. \tag{3.3.2}$$

Then applying (3.3.1) gives

$$D^\star_{X_{I_p}} \leq \frac{6\pi^2}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \frac{c(\varphi) \log(4e^8 \beta(p))}{\beta(p)}.$$

Lemma 3.3.3. *We have*

$$c(\varphi) \leq 1.$$

Proof. By Sarnak's identity (3.1.8) and the identity (3.1.9), we have

$$\varphi(n) = T_r(\cos(\theta_{m,p}(n))) = \cos(r\theta_{m,p}(n)) = \frac{1}{2} \sum_{v^2 \equiv 4mn \pmod{p}} e\left(\frac{rv}{p} \right). \tag{3.3.3}$$

Hence $|\varphi(n)| \leq 1$ so that

$$||\varphi||_\infty \leq 1.$$

Now, using (3.3.3) we compute the Fourier transform of φ as

$$\widehat{\varphi}(h) = \frac{1}{\sqrt{p}} \sum_{n \pmod p} \varphi(n) e\left(\frac{hn}{p}\right) = \frac{1}{2\sqrt{p}} \sum_{n \pmod p} \sum_{v^2 \equiv 4mn \pmod p} e\left(\frac{rv + hn}{p}\right).$$

If $h \equiv 0 \pmod p$, then by the one-to-one correspondence

$$\mathbb{F}_p \longleftrightarrow \left\{v \in \mathbb{F}_p : v^2 \equiv 4mn \pmod p, \text{ for some } n \in \mathbb{F}_p\right\}$$

and the orthogonal relations for Dirichlet characters, we have

$$\widehat{\varphi}(0) = \frac{1}{2\sqrt{p}} \sum_{n \pmod p} \sum_{v^2 \equiv 4mn \pmod p} e\left(\frac{rv}{p}\right) = \frac{1}{2\sqrt{p}} \sum_{x \pmod p} e\left(\frac{x}{p}\right) = 0.$$

On the other hand, if $h \not\equiv 0 \pmod p$, then by the change of variables

$$n \equiv \overline{4m} v^2 \pmod p$$

we have

$$\widehat{\varphi}(h) = \frac{1}{2\sqrt{p}} \sum_{n \pmod p} \sum_{v^2 \equiv 4mn \pmod p} e\left(\frac{rv + hn}{p}\right) = \frac{1}{2\sqrt{p}} \sum_{v \pmod p} e\left(\frac{\overline{4m} hv^2 + rv}{p}\right).$$

Now, it is known that (see e.g. [22, Lemma 4.8])

$$\sum_{v \pmod p} e\left(\frac{av^2 + bv}{p}\right) = \varepsilon_p \sqrt{p} \left(\frac{a}{p}\right) e\left(\frac{-b^2 \overline{4a}}{p}\right). \tag{3.3.4}$$

Hence, by (3.3.4) we have

$$|\widehat{\varphi}(h)| = \left| \frac{1}{2\sqrt{p}} \varepsilon_p \sqrt{p} \left(\frac{\overline{4m}h}{p} \right) e\left(\frac{-r^2\overline{4}4m\overline{h}}{p} \right) \right| = \left| \frac{\varepsilon_p}{2} \left(\frac{mh}{p} \right) e\left(\frac{-r^2 m\overline{h}}{p} \right) \right| = \frac{1}{2}.$$

It follows that

$$||\widehat{\varphi}||_\infty \leq \frac{1}{2}.$$

□

Continuing, by Lemma 3.3.3 we have

$$D^\star_{X_{I_p}} \leq \frac{6\pi^2}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \frac{\log(4e^8\beta(p))}{\beta(p)} \leq \frac{6\pi^2}{d} + 7\pi^5(1+\log d)\frac{\log(4e^8\beta(p))}{\beta(p)}.$$

Finally, we choose d such that $\beta(p) \leq d < \beta(p)+1$. Then we get

$$\begin{aligned} D^\star_{X_{I_p}} &\leq \frac{6\pi^2}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \frac{\log(4e^8\beta(p))}{\beta(p)} \\ &\leq \frac{6\pi^2}{\beta(p)} + 7\pi^5(1+\log(\beta(p)+1))\frac{\log(4e^8\beta(p))}{\beta(p)} \\ &\leq 14\pi^5 \frac{\log^2(4e^8\beta(p))}{\beta(p)}. \end{aligned}$$

This completes the proof of Proposition 3.3.1. □

Proof of Theorem 3.1.3. If $f \in BV([-1,1])$, then by the same argument as in Lemma 2.4.1 with the Sato-Tate measure replace with μ_{Arc}, we have the Koksma-Hlwaka type inequality

$$\left| \frac{1}{|I_p|} \sum_{n\in I_p} f(\cos(\theta_{m,p}(n))) - \int_{-1}^{1} f(t)d\mu_{\text{Arc}}(t) \right| \leq \text{Var}(f) D^\star_{X_{I_p}}.$$

Then by combining this bound with Proposition 3.3.1, we complete the proof. □

3.4 Fourier coefficients of Maass-Poincaré series

In this section, we give formulas for the coefficients appearing in the Fourier expansions (3.1.2), (3.1.3) and (3.1.4).

Recall that the Kloosterman sum with θ-multiplier of weight $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ is defined by

$$S_k(u,v;c) := \sum_{\substack{d \pmod c \\ (d,c)=1}} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} e\left(\frac{u\bar{d}+vd}{c}\right).$$

Also, we recall the special function

$$\mathbf{W}_{s,k}(y) = |y|^{-k/2} W_{\mathrm{sign}(y)\frac{k}{2}, s-\frac{1}{2}}(|y|)$$

where $W_{\kappa,\mu}$ is the usual Whittaker function, and let I_v and J_v denote the I and J-Bessel functions of order v, respectively.

Proposition 3.4.1. We have

$$\begin{aligned} P_{m,k,N}(z,s) = {} & \frac{\mathbf{M}_{s,k}(4\pi m y)}{\Gamma(2s)} e(-mx) + a_{m,k,N}(0,s) y^{1-s-k/2} \\ & + \sum_{n\in\mathbb{Z}^+} a_{m,k,N}(\pm n, s) \frac{\mathbf{W}_{s,k}(\pm 4\pi n y)}{\Gamma(s+k/2)} e(\pm nx) \end{aligned} \tag{3.4.1}$$

where the coefficients $a_{m,k,N}(\pm n, s)$ are given by

$$a_{m,k,N}(0,s) = \frac{2^{2-k}\pi^{1+s-\frac{k}{2}} i^{-k} m^{s-\frac{k}{2}}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,0;c)}{c^{2s}},$$

$$a_{m,k,N}(n,s) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

$$a_{m,k,N}(-n,s) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,-n;c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. We may express the Maass-Poincaré series defined in (3.1.1) as

$$P_{m,k,N}(z,s) = \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} [\psi_{s,k}(y)e(-mz)]\,|_k \gamma$$

where

$$\psi_{s,k}(y) := (4\pi m y)^{-\frac{k}{2}} M_{-\frac{k}{2}, s-\frac{1}{2}}(4\pi m y) e^{-2\pi m y}.$$

We have the double coset decomposition

$$\Gamma_0(4N) = \Gamma_\infty \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_\infty \bigsqcup \bigsqcup_{\substack{c>0 \\ c \equiv 0 \pmod{4N} \\ d \in (\mathbb{Z}/c\mathbb{Z})^\times}} \Gamma_\infty w_{d/c} \Gamma_\infty$$

where

$$w_{d/c} := \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(4N).$$

Then using this decomposition we get

$$\begin{aligned} P_{m,k,N}(z,s) &= \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} [\psi_{s,k}(y)e(-mz)]\,|_k \gamma \\ &= \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} \psi_{s,k}(\mathrm{Im}(\gamma z)) e(-m(\gamma z)) \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} (cz+d)^{-k} \\ &= \frac{\mathbf{M}_{s,k}(4\pi m y)}{\Gamma(2s)} e(-mz) + \frac{1}{\Gamma(2s)} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{4N}}} \sum_{\substack{d \pmod{c} \\ (c,d)=1}} \sum_{n \in \mathbb{Z}} \phi_{n,m,k,}(z), \end{aligned}$$

where

$$\phi_{n,s,k}(z) := \psi_{s,k}(\mathrm{Im}(w_{d/c}(z+n))) e(-m(w_{d/c}(z+n))) \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} (cz+cn+d)^{-k}.$$

By Poisson summation, we have

$$\sum_{n\in\mathbb{Z}} \phi_{n,s,k}(z)$$
$$= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{s,k}(\mathrm{Im}(w_{d/c}(z+t)))e(-m(w_{d/c}(z+t))) \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k}(cz+ct+d)^{-k}e(-nt)dt. \tag{3.4.2}$$

Note that

$$w_{d/c}(z+t) = \frac{a}{c} - \frac{1}{c^2(t+x+d/c+iy)}.$$

Then using the change of variables $t \mapsto t - x - d/c$, the integral in (3.4.2) becomes

$$c^{-k}\left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} e\left(nx + \frac{nd}{c}\right)$$
$$\times \int_{\mathbb{R}} \psi_{s,k}\left(\mathrm{Im}\left(\frac{a}{c} - \frac{1}{c^2(t+iy)}\right)\right) e\left(-m\left(\frac{a}{c} - \frac{1}{c^2(t+iy)}\right)\right)(t+iy)^{-k}e(-nt)dt.$$

Hence

$$P_{m,k,N}(z,s) = \frac{\mathbf{M}_{s,k}(4\pi my)}{\Gamma(2s)}e(-mz) + \frac{1}{\Gamma(2s)}\sum_{n\in\mathbb{Z}} e(nx) \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c^k}\alpha_{n,s,k}(c,y) \tag{3.4.3}$$

where

$$\alpha_{n,s,k}(c,y) := \int_{\mathbb{R}} \psi_{s,k}\left(\frac{y}{c^2(t^2+y^2)}\right) e\left(\frac{m}{c^2(t+iy)} - nt\right)(t+iy)^{-k}dt.$$

Now, by Lemma A.1.1 we have

$$\alpha_{n,s,k}(c,y) = \begin{cases} 2\pi i^{-k}\frac{\Gamma(2s)}{\Gamma(s+k/2)}c^{k-1}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\mathbf{W}_{s,k}(4\pi ny), & n \geq 1, \\ \frac{2^{2-k}\pi^{1+s-\frac{k}{2}} i^{-k} m^{s-\frac{k}{2}} y^{1-s-\frac{k}{2}} c^{k-2s}\Gamma(2s)}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}, & n = 0, \\ 2\pi i^{-k}\frac{\Gamma(2s)}{\Gamma(s-k/2)}c^{k-1}\left(\frac{|n|}{m}\right)^{\frac{k-1}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{m|n|}}{c}\right)\mathbf{W}_{s,k}(4\pi ny), & n \leq -1. \end{cases}$$

Then substituting these identities in (3.4.3) gives the desired result. □

Using the Weil-type bound (see e.g. [21, (1.6)])

$$|S_k(m,n;c)| \leq \tau(c)(m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}, \tag{3.4.4}$$

one can show that the Fourier expansion (3.1.2) is absolutely and uniformly convergent for $\mathrm{Re}(s) > 3/4$, and hence gives an analytic continuation of $P_{m,k,N}(z,s)$ to this region. Moreover, using the bound for sums of Kloosterman sums in Theorem 3.1.6, one can show that the Fourier expansion (3.1.2) is conditionally convergent at $s = 3/4$.

Now, recall that $P_{m,k,N}(z) = P_{m,k,N}(z, 1-k/2)$ for $k \leq 1/2$.

Proposition 3.4.2. We have

$$\begin{aligned} P_{m,k,N}(z) = q^{-m} &+ \sum_{n=0}^{\infty} a_{m,k,N}(n, 1-k/2)q^n \\ &+ \sum_{n=1}^{\infty}(a_{m,k,N}(-n,1-k/2) - \delta_m(n))\frac{\Gamma(1-k,4\pi ny)}{\Gamma(1-k)}q^{-n} \end{aligned}$$

where

$$a_{m,k,N}(0,1-k/2) = (2\pi i)^{2-k}\frac{m^{1-k}}{\Gamma(2-k)}\sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}}\frac{S_k(-m,0;c)}{c^{2-k}},$$

$$a_{m,k,N}(n, 1-k/2) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

$$a_{m,k,N}(-n, 1-k/2) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,-n;c)}{c} J_{1-k}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. Let $s = 1 - k/2$ with $k \leq 1/2$ in (3.1.2). Using the identity [29, Eq. 13.18.4] we get

$$M_{-\frac{k}{2},\frac{1-k}{2}}(4\pi my) = (1-k)e^{2\pi my}(4\pi my)^{\frac{k}{2}}[\Gamma(1-k) - \Gamma(1-k, 4\pi my)].$$

Then the functional equation $\Gamma(2-k) = (1-k)\Gamma(1-k)$ yields

$$\frac{\mathbf{M}_{1-\frac{k}{2},k}(4\pi my)}{\Gamma(2-k)} = e^{2\pi my} - e^{2\pi my}\frac{\Gamma(1-k, 4\pi my)}{\Gamma(1-k)}$$

so that

$$\frac{\mathbf{M}_{1-\frac{k}{2},k}(4\pi my)}{\Gamma(2-k)} e(-mx) = q^{-m} - \frac{\Gamma(1-k, 4\pi my)}{\Gamma(1-k)} q^{-m}.$$

Also, by applying [29, Eq. 13.18.2] we have

$$\mathbf{W}_{1-\frac{k}{2},k}(4\pi ny) = (4\pi ny)^{-\frac{k}{2}} W_{\frac{k}{2},\frac{1-k}{2}}(4\pi ny) = e^{-2\pi ny},$$

and by applying [29, Eq. 13.18.5] we obtain

$$\mathbf{W}_{1-\frac{k}{2},k}(-4\pi ny) = (4\pi ny)^{-\frac{k}{2}} W_{-\frac{k}{2},\frac{1-k}{2}}(4\pi ny) = \Gamma(1-k, -4\pi ny)e^{-2\pi ny}.$$

Hence, after combining the preceding identities we get the desired result.

□

Similarly, recall that $Q_{m,k,N}(z) := P_{m,k,N}(z, k/2)$ for $k \geq 3/2$.

Proposition 3.4.3. We have

$$Q_{m,k,N}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{m,k,N}(n, k/2) q^n$$

where

$$a_{m,k,N}(n,k/2) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. Let $s = k/2$ with $k \geq 3/2$ in (3.1.2). Then by the identity [29, Eq. 13.18.3], we have

$$M_{-\frac{k}{2},\frac{k-1}{2}}(4\pi my) = e^{2\pi my}(4\pi my)^{\frac{k}{2}}.$$

It follows that

$$\mathbf{M}_{\frac{k}{2},k}(4\pi my) = e^{2\pi my}$$

and thus

$$\mathbf{M}_{\frac{k}{2},k}(4\pi my)e(-mx) = q^{-m}.$$

Also, by the identity [29, Eq. 13.18.2], we get

$$\mathbf{W}_{\frac{k}{2},k}(4\pi ny) = (4\pi ny)^{-\frac{k}{2}} W_{\frac{k}{2},\frac{k-1}{2}}(4\pi ny) = e^{-2\pi ny}.$$

Finally, since the Gamma function has a simple pole at $s = 0$, the constant term vanishes and the non-holomorphic terms vanish. Hence, by combining the preceding facts we complete the proof. □

3.5 Bounds for integral transforms

In preparation for bounding sums of half-integral weight Kloosterman sums, we record some bounds for integral transforms which will be needed.

Suppose that $X > 1$ and $1 < Z < X/2$. Following the setup in [17, 35], we let $\phi : \mathbb{R} \to [0, 1]$ be a test function satisfying the following conditions:

(1) $\phi(t) = 1$ for $\frac{a}{2X} \le t \le \frac{a}{X}$, where $a = 4\pi\sqrt{mn}$.

(2) $\phi(t) = 0$ for $t \le \frac{a}{2X+2Z}$ and $t \ge \frac{a}{X-Z}$.

(3) $\phi'(t) \ll \left(\frac{a}{X-Z} - \frac{a}{X}\right)^{-1}$.

(4) $\phi(t)$ and $\phi'(t)$ are piecewise monotone on a fixed number of intervals in $\left[\frac{a}{2X+2Z}, \frac{a}{2X}\right]$ and $\left[\frac{a}{X}, \frac{a}{X-Z}\right]$.

We use Figure 3.2 to illustrate the function $\phi(t)$ satisfying $(1) - (4)$.

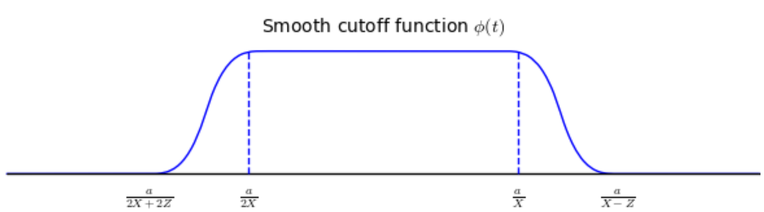


Figure 3.2: Smooth cutoff function $\phi(t)$

Define the integral transform

$$\check{\phi}(t) := 2e\left(\frac{k}{2}\right)\cosh(\pi t)\int_0^\infty K_{2it}(y)\phi(y)\frac{dy}{y}.$$

Lemma 3.5.1. *We have the following bounds.*

(1) $\check{\phi}(t) \ll 1, \quad \text{if } t \in (0, \frac{i}{4})$.

(2) $\check{\phi}(t) \ll (1+|t|)^{-\frac{1}{2}}, \quad \text{if } |t| \leq 1$.

(3) $\check{\phi}(t) \ll |t|^{-\frac{3}{2}}, \quad \text{if } |t| \geq 1$.

(4) $\check{\phi}(t) \ll \frac{X}{Z}|t|^{-\frac{5}{2}}, \quad \text{if } |t| \text{ sufficiently large}$.

Proof. (1) Let $t = ir \in (0, \frac{i}{4})$. We first assume that $X \ll a$. Recall the integral representation of $K_{2it}(y)$ (see [29, Eq. 10.32.9])

$$K_{2it}(y) = \int_0^\infty \exp(-y\cosh(u))\cos(2tu)du. \tag{3.5.1}$$

Then following the approach in [10, Lemma 7.1], we integrate by parts to get

$$\begin{aligned}
\int_0^\infty \exp(-y\cosh(u))\frac{\phi(y)}{y}dy &= \frac{1}{\cosh(u)}\int_0^\infty \left(\frac{\phi(y)}{y}\right)' \exp(-y\cosh(u))dy \\
&\ll \frac{1}{\cosh(u)}\int_{\frac{a}{2X+2Z}}^{\frac{a}{X-Z}} \frac{1}{y^2}\exp(-y\cosh(u))dy \\
&\ll \exp(-u)\int_{\frac{a}{2X+2Z}}^{\frac{a}{X-Z}} \frac{1}{y^2}\exp(-y\cosh(u))dy \\
&\ll \min\left\{\exp(-u)\left(\frac{a}{2X+2Z}\right)^{-1}, 1\right\}.
\end{aligned} \tag{3.5.2}$$

Hence by combining (3.5.1), (3.5.2) and $\cos(2iru) \ll \exp(2ru)$, we obtain

$$\begin{aligned}
\check{\phi}(t) = \check{\phi}(ir) &= 2e\left(\frac{k}{2}\right)\cosh(\pi ir)\int_0^\infty K_{-2r}(y)\phi(y)\frac{dy}{y} \\
&= 2e\left(\frac{k}{2}\right)\cosh(\pi ir)\int_0^\infty\int_0^\infty \exp(-y\cosh(u))\cos(2iru)\frac{\phi(y)}{y}dydu \\
&\ll \int_0^\infty \exp(2ru)\times\min\left\{\exp(-u)\left(\frac{a}{2X+2Z}\right)^{-1}, 1\right\}du \\
&\ll \frac{1+\left(\frac{a}{2X+2Z}\right)^{-2r}}{1+\frac{a}{2X+2Z}} \\
&\ll 1.
\end{aligned}$$

We next assume that $X \gg a$ (y sufficiently small). Then by [29, Eq. 10.27.4] and the mean value theorem, we get

$$\begin{aligned}
\check{\phi}(ir) &= 2e\left(\frac{k}{2}\right)\cosh(\pi ir)\int_0^\infty K_{-2r}(y)\phi(y)\frac{dy}{y} \\
&= \pi e\left(\frac{k}{2}\right)\cosh(\pi ir)\int_0^\infty \frac{I_{2r}(y)-I_{-2r}(y)}{\sin(-2r\pi)}\phi(y)\frac{dy}{y} \\
&\ll \int_{\frac{a}{2X+2Z}}^{\frac{a}{X-Z}} \frac{dy}{y} \\
&\ll \log\left(\frac{X+Z}{X-Z}\right) \\
&\ll 1.
\end{aligned}$$

(2) We apply an argument similar to that in [25, Lemma 7.1]. Recall the following identities for the I and K-Bessel functions (see [29, Eq. 10.25.2 and 10.27.4])

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{c=0}^\infty \frac{1}{\Gamma(v+1+c)c!}\left(\frac{z^2}{4}\right)^c,$$

and

$$K_v(z) = \frac{\pi}{2}\frac{1}{\sin(\pi v)}(I_{-v}(z) - I_v(z)).$$

This yields the identity

$$\check{\phi}(t) = \Phi(t) - \Phi(-t),$$

where

$$\Phi(t) := \pi e\left(\frac{k}{2}\right)\frac{\cosh(\pi t)}{\sin(2\pi it)}\sum_{c=0}^\infty \frac{\tilde{\phi}(-2it+2c)}{2^{-2it+2c}\Gamma(-2it+1+c)c!}$$

and

$$\tilde{\phi}(s) := \int_0^\infty \phi(u) u^{s-1} du$$

is the Mellin transform of ϕ. Integrating by parts yields

$$\tilde{\phi}(s) \ll \left(\frac{a}{2X+2Z}\right)^{\mathrm{Re}(s)} (1+|s|)^{-1}. \tag{3.5.3}$$

Since $\check{\phi}(t) = \check{\phi}(-t)$, we only deal with the case $0 \le t \le 1$. By Stirling's formula, we know that if $c+1 \ge 2t$ then $\Gamma(c+1-2it) \gg c^{\frac{1}{2}}(c+2)^c \exp(-\frac{\pi|t|}{2} - c)$. By applying this bound and using (3.5.3) and $\sin(2\pi i t) \asymp \exp(2\pi t)$, we have

$$\Phi(t) \ll (1+|2t|)^{-1} \ll (1+|t|)^{-\frac{1}{2}}.$$

Similarly, the same upper bound holds for $\Phi(-t)$, and hence holds for $\check{\phi}(t)$. We remark that (2) can be viewed as a special case of Lemma 6 in [7].

(3) We let $t \ge 1$ and follow the same argument in [35, p. 629-630]. Recall the asymptotic expansion of K-Bessel function of purely imaginary order ([12, 7.13.2 (19)])

$$K_{it}(y) = \frac{\sqrt{2\pi}}{(t^2-y^2)^{\frac{1}{4}}} \exp\left(-\frac{\pi}{2}t\right)\left[\sin\left(\frac{\pi}{4} - \sqrt{t^2-y^2} + t\cosh^{-1}\left(\frac{t}{y}\right)\right) + O\left(\frac{1}{y}\right)\right].$$

Moreover, by replacing $\sin(x)$ with $(\exp(ix) - \exp(-ix))/2i$, it follows that

$$K_{it}(y) = \frac{\sqrt{2\pi}}{(t^2-y^2)^{\frac{1}{4}}} \exp\left(i\xi\left(\frac{y}{t}\right)t - \frac{\pi}{2}t + \frac{\pi i}{4}\right) + \text{"lower order terms"},$$

where

$$\xi(s) := \cosh^{-1}\left(\frac{1}{s}\right) - \sqrt{1-s^2}.$$

Moreover, by making the substitution $y = ts$, we are reduced to bounding

$$\begin{aligned}\check{\phi}(t) &\ll t^{-\frac{1}{2}} \left| \int_0^\infty \frac{\exp(i\xi(s)t)}{(1-s^2)^{\frac{1}{4}}} \phi(ts) \frac{ds}{s} \right| \\ &= t^{-\frac{3}{2}} \left| \int_0^\infty \left[\exp(i\xi(s)t)t\xi'(s)\right] \frac{\phi(ts)}{\xi'(s)s(1-s^2)^{\frac{1}{4}}} ds \right|. \end{aligned} \tag{3.5.4}$$

Since $\xi'(s) = -\sqrt{1-s^2}/s$, we know that $\xi'(s)$ is uniformly bounded away from zero and $\xi'(s) \sim -1/s$, as $s \to 0$. Also, ϕ is monotonic implies that $\phi(ts)/(\xi'(s)s(1-s^2)^{\frac{1}{4}})$ is monotonic. Now, by the mean value theorem for integrals (see [35, (46)]), we have

$$\int_0^\infty \left[\exp(i\xi(s)t)t\xi'(s)\right] \frac{\phi(ts)}{\xi'(s)s(1-s^2)^{\frac{1}{4}}} ds \ll \left| \int_{\frac{a}{t(2X+2Z)}}^{\frac{a}{t(X-Z)}} \left[\exp(i\xi(s)t)\right]' ds \right| \ll 1.$$

Then by (3.5.4), we obtain

$$\check{\phi}(t) \ll t^{-\frac{3}{2}}.$$

We can apply the same argument for the case $t \leq -1$ to get the same upper bound.

(4) For t sufficiently large, we have $\phi(ts) = 0$. Also, by construction of ϕ, we know $\phi(0) = 0$. Hence, we can get a better bound by integration by parts in (3.5.4). It follows that

$$\begin{aligned}\check{\phi}(t) &\ll t^{-\frac{3}{2}} \left| \int_0^\infty \left[\exp(i\xi(s)t)\right]' \frac{\phi(ts)}{\xi'(s)s(1-s^2)^{\frac{1}{4}}} ds \right| \\ &= t^{-\frac{3}{2}} \left| \int_0^\infty \exp(i\xi(s)t) \left(\frac{\phi(ts)}{(1-s^2)^{\frac{3}{4}}} \right)' ds \right| \\ &\ll t^{-\frac{3}{2}} \left| \int_0^\infty \exp(i\xi(s)t) \frac{t\phi'(ts)}{(1-s^2)^{\frac{3}{4}}} ds \right| + t^{-\frac{3}{2}} \left| \int_0^\infty \exp(i\xi(s)t) \frac{s\phi(ts)}{(1-s^2)^{\frac{7}{4}}} ds \right| \\ &\ll t^{-\frac{3}{2}} \left| \int_{\frac{a}{t(2X+2Z)}}^{\frac{a}{t(X-Z)}} \left[\exp(i\xi(s)t)\right]' \frac{s\phi'(ts)}{(1-s^2)^{\frac{5}{4}}} ds \right| + t^{-\frac{5}{2}} \left| \int_{\frac{a}{t(2X+2Z)}}^{\frac{a}{t(X-Z)}} \left[\exp(i\xi(s)t)\right]' \frac{s^2\phi(ts)}{(1-s^2)^{\frac{9}{4}}} ds \right|. \end{aligned} \tag{3.5.5}$$

Moreover, by the condition (3) of ϕ, we have

$$\phi'(ts) \ll \left(\frac{a}{X-Z} - \frac{a}{X}\right)^{-1} \ll \left(\frac{aZ}{X^2}\right)^{-1}.$$

It follows that

$$\frac{s\phi'(ts)}{(1-s^2)^{\frac{5}{4}}} \ll \frac{s}{(1-s^2)^{\frac{5}{4}}}\left(\frac{aZ}{X^2}\right)^{-1} \ll \frac{1}{(1-s^2)^{\frac{5}{4}}}\left(\frac{a}{tX}\right)\left(\frac{aZ}{X^2}\right)^{-1} = \frac{1}{(1-s^2)^{\frac{5}{4}}}\frac{X}{Zt}. \tag{3.5.6}$$

Observe that $1/(1-s^2)^{\frac{5}{4}}$ and $s^2\phi(ts)/(1-s^2)^{\frac{9}{4}}$ are both monotonic. Hence by the mean value theorem for integrals and combining (3.5.5) and (3.5.6), we get

$$\check{\phi}(t) \ll \frac{X}{Z}t^{-\frac{5}{2}} + t^{-\frac{5}{2}} \ll \frac{X}{Z}t^{-\frac{5}{2}}.$$

□

3.6 Bounds for sums of half-integral weight Kloosterman sums

3.6.1 Background

We assume throughout this section that $k = 1/2$ or $k = 3/2$ and $N \in \mathbb{Z}^+$.

A cusp $\mathfrak{a}$ is *singular* for $\Gamma_0(4N)$ and weight k if

$$\left(\frac{c_\mathfrak{a}}{d_\mathfrak{a}}\right)\varepsilon_{d_\mathfrak{a}}^{-2k} = 1$$

where

$$\gamma_\mathfrak{a} := \begin{pmatrix} * & * \\ c_\mathfrak{a} & d_\mathfrak{a} \end{pmatrix} \in \Gamma_0(4N)$$

is a generator of the stabilizer $\Gamma_\mathfrak{a}$ of the cusp $\mathfrak{a}$. For each singular cusp $\mathfrak{a}$, we define the Eisenstein series attached to $\mathfrak{a}$ by

$$E_{\mathfrak{a}}(z,s) := \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma_0(4N)} j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-2k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s, \quad \mathrm{Re}(s)>1$$

where $\sigma_{\mathfrak{a}}$ is a scaling matrix for $\mathfrak{a}$ and the automorphy factor is defined by

$$j(\gamma,z) := \left(\frac{c}{d}\right)\varepsilon_d^{-1}\left(\frac{|cz+d|}{cz+d}\right)^{-1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N).$$

The Fourier expansion of $E_{\mathfrak{a}}(z;s)$ is given by (see e.g. [30, p. 3876])

$$E_{\mathfrak{a}}(z;s) = \delta_{\mathfrak{a}=\infty}y^s + \phi_{\mathfrak{a}}(0,s)y^{1-s} + \pi^s e\left(-\frac{k}{4}\right)\sum_{n\neq 0}|n|^{s-1}\phi_{\mathfrak{a}}(n,s)\frac{W_{\mathrm{sgn}(n)\frac{k}{2},\frac{1}{2}-s}(4\pi|n|y)}{\Gamma(s+\mathrm{sgn}(n)\frac{k}{2})}e(nx),$$

where

$$\phi_{\mathfrak{a}}(n,s) := \sum_{\substack{0\le d<c \\ \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}\in\Gamma_0(4N)}} \left(\frac{c}{d}\right)\varepsilon_d^{2k} e\left(\frac{nd}{c}\right)c^{-2s}, \qquad n\neq 0.$$

Now, let $\mathbf{H}_k(4N)$ be the Hilbert space of L^2-integrable functions f such that

$$f(\gamma z) = j(r,z)^{2k}f(z)$$

for all $\gamma\in\Gamma_0(4N)$. Let $\{u_j\}\subset\mathbf{H}_k(4N)$ be an orthonormal basis of eigenfunctions of Δ_k with eigenvalues λ_j and spectral parameters t_j such that $\lambda_j = 1/4 + t_j^2$.

If $t_j\notin\mathbb{R}$ (so that $\lambda_j < 1/4$), then λ_j is a so-called exceptional eigenvalue and it is known that $\lambda_j \ge 3/16$ (see [31]). If $t_j = i/4$ (so that $\lambda_j = 3/16$) then we have

$$y^{-\frac{k}{2}}u_j \in M_k(4N). \tag{3.6.1}$$

In all other cases, by the bound of Kim and Sarnak [23] we have

$$|\mathrm{Im}(t_j)| \leq \frac{7}{128}. \tag{3.6.2}$$

Recall that u_j has the Fourier expansion

$$u_j(z) = \rho_j(0, y) + \sum_{n \neq 0} \rho_j(n) W_{\mathrm{sgn}(n)\frac{k}{2}, it_j}(4\pi|n|y)e(nx).$$

We will need the following version of the Kuznetsov-Proskurin formula for opposite signs given by Blomer [7, Proposition 2].

Proposition 3.6.1. Let $m, n \in \mathbb{Z}^+$. Further, let $\phi : [0, \infty) \to \mathbb{R}$ be a smooth function satisfying

$$\phi(0) = \phi'(0) = \phi''(0) = 0 \quad \text{and} \quad \phi^{(j)}(x) \ll x^{-2-\varepsilon} \tag{3.6.3}$$

for $0 \leq j \leq 3$ and $x \to \infty$. Then we have

$$\begin{aligned} &\sum_{\substack{c>0 \\ c \equiv 0 \pmod{4N}}} \frac{S_k(-m, n; c)}{c} \phi\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &= 4\sqrt{mn} \sum_{j \geq 0} \frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)} \check{\phi}(t_j) \\ &\quad + \sum_{\mathfrak{a}\text{: singular cusp}} \int_{\mathbb{R}} \left(\frac{n}{m}\right)^{it} \frac{\overline{\phi_{\mathfrak{a}}(-m, \frac{1}{2} + it)}\phi_{\mathfrak{a}}(n, \frac{1}{2} + it)}{\cosh(\pi t)\Gamma(\frac{k+1}{2} - it)\Gamma(\frac{1-k}{2} + it)} \check{\phi}(t) dt. \end{aligned} \tag{3.6.4}$$

We will also need the following crucial estimate from [7, Lemma 5].

Lemma 3.6.2. *Let $T \geq 1$. Then we have*

$$\begin{aligned} &\sum_{|t_j| \leq T} \frac{n|\rho_j(\pm n)|^2(1 + |t_j|)^{\pm k - \frac{1}{2}}}{\cosh(\pi t_j)} \\ &+ \sum_{\mathfrak{a}\text{: singular cusp}} \int_{-T}^{T} \frac{|\phi_{\mathfrak{a}}(\pm n, \frac{1}{2} + it)|^2(1 + |t|)^{\pm k - \frac{1}{2}}}{\cosh(\pi t)|\Gamma(\pm\frac{k}{2} + \frac{1}{2} + it)|^2} dt \ll_\varepsilon T^{\frac{3}{2}} + (nT)^{\frac{1}{2} + \varepsilon}. \end{aligned} \tag{3.6.5}$$

3.6.2 Bounding the weighted sum

Let ϕ be a test function satisfying conditions $(1)-(4)$ of Section 3.5 and the condition (3.6.3).

Proposition 3.6.3. For all $\varepsilon > 0$, we have

$$\sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{mn}}{c}\right) \ll_\varepsilon (XZ^{-1})^{\frac{1}{2}} + (mn)^{\frac{1}{4}+\varepsilon}.$$

Proof. We begin by estimating the contribution from the discrete spectrum in the Kuznetsov-Proskurin formula (3.6.4).

Assume first that $t_j \notin \mathbb{R}$ (the exceptional eigenvalues). By (3.6.1), there is no contribution from the discrete spectrum when $t_j = i/4$. Hence by Lemma 3.5.1 (1) we have

$$\check{\phi}(t_j) \ll 1. \tag{3.6.6}$$

It follows from (3.6.2), (3.6.6), the Cauchy-Schwarz inequality, and (3.6.5) that

$$\begin{aligned} 4\sqrt{mn}\sum_{t_j\notin\mathbb{R}} \frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)}\check{\phi}(t_j) &\ll 4\sqrt{mn}\sum_{|t_j|\leq 7/128} \frac{|\overline{\rho_j(-m)}\rho_j(n)|}{\cosh(\pi t_j)} \\ &\ll \sqrt{\sum_{|t_j|\leq 7/128}\frac{m|\rho_j(-m)|^2}{\cosh(\pi t_j)}}\sqrt{\sum_{|t_j|\leq 7/128}\frac{n|\rho_j(n)|^2}{\cosh(\pi t_j)}} \\ &\ll_\varepsilon (mn)^{\frac{1}{4}+\varepsilon} \end{aligned} \tag{3.6.7}$$

Next assume that $t_j \in \mathbb{R}$ with $|t_j| \leq 1$. By Lemma 3.5.1 (2), we have

$$\check{\phi}(t_j) \ll (1+|t_j|)^{-\frac{1}{2}}. \tag{3.6.8}$$

Then by (3.6.8), the Cauchy-Schwarz inequality, and (3.6.5) we get

$$4\sqrt{mn}\sum_{|t_j|\leq 1} \frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)}\check{\phi}(t_j) \ll \sqrt{mn}\sum_{|t_j|\leq 1} \frac{|\overline{\rho_j(-m)}\rho_j(n)|}{\cosh(\pi t_j)}(1+|t_j|)^{-\frac{1}{2}}$$

$$\ll \sqrt{\sum_{|t_j|\le 1} \frac{m|\rho_j(-m)|^2(1+|t_j|)^{-1}}{\cosh(\pi t_j)}} \sqrt{\sum_{|t_j|\le 1} \frac{n|\rho_j(n)|^2}{\cosh(\pi t_j)}}$$
$$\ll_\varepsilon (mn)^{\frac{1}{4}+\varepsilon}. \tag{3.6.9}$$

Last, we assume that $t_j \in \mathbb{R}$ with $|t_j| \ge 1$. By Lemma 3.5.1 (3) and (4) we have

$$\check{\phi}(t_j) \ll \min\left\{ |t_j|^{-\frac{3}{2}}, \frac{X}{Z}|t_j|^{-\frac{5}{2}} \right\}. \tag{3.6.10}$$

Let $A \ge 1$. Then using (3.6.10) and the Cauchy-Schwarz inequality we get

$$4\sqrt{mn} \sum_{A\le |t_j|\le 2A} \frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)} \check{\phi}(t_j)$$
$$\ll \min\left\{ A^{-\frac{3}{2}}, \frac{X}{Z}A^{-\frac{5}{2}} \right\} \sqrt{\sum_{A\le |t_j|\le 2A} \frac{m|\rho_j(-m)|^2(1+|t_j|)^{-1}}{\cosh(\pi t_j)}} \sqrt{\sum_{A\le |t_j|\le 2A} \frac{n|\rho_j(n)|^2}{\cosh(\pi t_j)}}.$$

Now, by (3.6.5) we have

$$\sum_{A\le |t_j|\le 2A} \frac{n|\rho_j(\pm n)|^2}{\cosh(\pi t_j)} + \sum_{\mathfrak{a}:\text{ singular cusp}} \int_{A\le |t|\le 2A} \frac{|\phi_{\mathfrak{a}}(\pm n, \frac{1}{2}+it)|^2}{\cosh(\pi t)|\Gamma(\pm\frac{k}{2}+\frac{1}{2}+it)|^2} dt$$
$$\ll_\varepsilon A^{\mp k+\frac{1}{2}} \left(A^{\frac{3}{2}} + (nA)^{\frac{1}{2}+\varepsilon} \right).$$

Then it follows that

$$4\sqrt{mn} \sum_{A\le |t_j|\le 2A} \frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)} \check{\phi}(t_j)$$
$$\ll_\varepsilon \min\left\{ A^{-\frac{3}{2}}, \frac{X}{Z}A^{-\frac{5}{2}} \right\} \left[A \left(A^{\frac{3}{2}} + (mA)^{\frac{1}{2}+\varepsilon} \right) \left(A^{\frac{3}{2}} + (nA)^{\frac{1}{2}+\varepsilon} \right) \right]^{\frac{1}{2}}$$
$$\ll_\varepsilon \min\left\{ A^{-\frac{3}{2}}, \frac{X}{Z}A^{-\frac{5}{2}} \right\} A^2 \left(1 + n^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + m^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + (mn)^{\frac{1}{4}+\varepsilon}A^{-1+\varepsilon} \right)$$
$$\ll_\varepsilon \left(A^{-\frac{3}{2}} \times \frac{X}{Z}A^{-\frac{5}{2}} \right)^{\frac{1}{2}} A^2 \left(1 + n^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + m^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + (mn)^{\frac{1}{4}+\varepsilon}A^{-1+\varepsilon} \right)$$
$$\ll_\varepsilon (XZ^{-1})^{\frac{1}{2}} \left(1 + n^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + m^{\frac{1}{4}+\varepsilon}A^{-\frac{1}{2}+\varepsilon} + (mn)^{\frac{1}{4}+\varepsilon}A^{-1+\varepsilon} \right).$$

Summing over the dyadic intervals gives

$$4\sqrt{mn}\sum_{|t_j|\geq 1}\frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)}\check{\phi}(t_j)\ll_\varepsilon (XZ^{-1})^{\frac{1}{2}}+(mn)^{\frac{1}{4}+\varepsilon}. \tag{3.6.11}$$

Finally, by combining (3.6.7), (3.6.9), and (3.6.11) we get

$$4\sqrt{mn}\sum_{j\geq 0}\frac{\overline{\rho_j(-m)}\rho_j(n)}{\cosh(\pi t_j)}\check{\phi}(t_j)\ll_\varepsilon (XZ^{-1})^{\frac{1}{2}}+(mn)^{\frac{1}{4}+\varepsilon}.$$

To estimate the contribution from the continuous spectrum in (3.6.4), we proceed in exactly the same way by considering the cases $|t|\leq 1$ and $|t|\geq 1$ separately to get

$$\sum_{\mathfrak{a}:\text{ singular cusp}}\int_{\mathbb{R}}\left(\frac{n}{m}\right)^{it}\frac{\overline{\phi_{\mathfrak{a}}(-m,\frac{1}{2}+it)}\phi_{\mathfrak{a}}(n,\frac{1}{2}+it)}{\cosh(\pi t)\Gamma(\frac{k+1}{2}-it)\Gamma(\frac{1-k}{2}+it)}\check{\phi}(t)dt\ll_\varepsilon (XZ^{-1})^{\frac{1}{2}}+(mn)^{\frac{1}{4}+\varepsilon}.$$

This completes the proof. □

3.6.3 Proof of Theorem 3.1.6

We will require the following unsmoothed approximation.

Lemma 3.6.4. *Given any $\varepsilon>0$, we have*

$$\sum_{\substack{c>0\\ c\equiv 0\ (\mathrm{mod}\ 4N)}}\frac{S_k(-m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{mn}}{c}\right)-\sum_{\substack{X\leq c\leq 2X\\ c\equiv 0\ (\mathrm{mod}\ 4N)}}\frac{S_k(-m,n;c)}{c}\ll_\varepsilon (mn)^\varepsilon N^{\varepsilon-1}X^{\varepsilon-\frac{1}{2}}Z.$$

Proof. By construction of ϕ (condition (4) and $|\phi|\leq 1$), we have

$$\begin{aligned}&\left|\sum_{\substack{c>0\\ c\equiv 0\ (\mathrm{mod}\ 4N)}}\frac{S_k(-m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{mn}}{c}\right)-\sum_{\substack{X\leq c\leq 2X\\ c\equiv 0\ (\mathrm{mod}\ 4N)}}\frac{S_k(-m,n;c)}{c}\right|\\ &\leq\sum_{\substack{X-Z\leq c\leq X\\ 2X\leq c\leq 2X+2Z\\ c\equiv 0\ (\mathrm{mod}\ 4N)}}\left|\frac{S_k(-m,n;c)}{c}\right|.\end{aligned}$$

Then by applying the Weil-type bound

$$|S_k(-m,n;c)| \leq \tau(c)(m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}$$

gives

$$\begin{aligned}
&\left|\sum_{\substack{c>0\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{mn}}{c}\right) - \sum_{\substack{X\leq c\leq 2X\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c}\right| \\
&\leq \sum_{\substack{X-Z\leq c\leq X\\ 2X\leq c\leq 2X+2Z\\ c\equiv 0 \pmod{4N}}} \frac{\tau(c)(m,n,c)^{\frac{1}{2}}}{c^{\frac{1}{2}}} \\
&\leq \frac{\tau(4N)}{(4N)^{\frac{1}{2}}} \sum_{\substack{\frac{X-Z}{4N}\leq d\leq \frac{X}{4N}\\ \frac{2X}{4N}\leq d\leq \frac{2X+2Z}{4N}}} \frac{\tau(d)(m,n,d)^{\frac{1}{2}}}{d^{\frac{1}{2}}} \\
&\leq \frac{\tau(4N)}{(4N)^{\frac{1}{2}}} \sum_{d_1|(m,n)} \tau(d_1) \sum_{\substack{\frac{X-Z}{4N}\leq d_2\leq \frac{X}{4N}\\ \frac{2X}{4N}\leq d_2\leq \frac{2X+2Z}{4N}}} \frac{\tau(d_2)}{d_2^{\frac{1}{2}}} \\
&\leq \frac{\tau(4N)}{(X-Z)^{\frac{1}{2}}} \sum_{d_1|(m,n)} \tau(d_1) \sum_{\substack{\frac{X-Z}{4N}\leq d_2\leq \frac{X}{4N}\\ \frac{2X}{4N}\leq d_2\leq \frac{2X+2Z}{4N}}} \tau(d_2) \\
&\ll \frac{\tau(4N)}{X^{\frac{1}{2}}}\frac{Z\log X}{4N} \sum_{d_1|(m,n)} \tau(d_1) \\
&\ll_\varepsilon (mn)^\varepsilon N^{\varepsilon-1}X^{\varepsilon-\frac{1}{2}}Z.
\end{aligned}$$

□

Now, by Lemma 3.6.4 and Proposition 3.6.3 we have

$$\sum_{\substack{X\leq c\leq 2X\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} \ll_\varepsilon (XZ^{-1})^{\frac{1}{2}} + (mn)^{\frac{1}{4}+\varepsilon} + (mn)^\varepsilon N^{\varepsilon-1}X^{\varepsilon-\frac{1}{2}}Z.$$

Then choosing $Z = (NX)^{\frac{2}{3}}$ to balance the exponent of X gives

$$\sum_{\substack{X \le c \le 2X \\ c \equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} \ll_\varepsilon (mn)^{\frac{1}{4}+\varepsilon} + (mn)^\varepsilon N^{-\frac{1}{3}+\varepsilon} X^{\frac{1}{6}+\varepsilon} \ll_\varepsilon (mn)^{\frac{1}{4}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{\frac{1}{6}+\varepsilon}. \tag{3.6.12}$$

Since the total number of dyadic segments is at most $\ll \log X$, Theorem 3.1.6 follows from (3.6.12). □

3.6.4 An auxiliary bound

We will require the following bound in our proof of Theorem 3.1.1.

Proposition 3.6.5. For all $\varepsilon > 0$, we have

$$\sum_{\substack{8N \le c \le X \\ c \equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ \ll_\varepsilon m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right) + (mn)^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{-\frac{1}{3}+\varepsilon} + m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon}.$$

In particular, letting $0 < \varepsilon < 1/3$ and $X \to \infty$ gives

$$\sum_{\substack{c > 4N \\ c \equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right) \ll_\varepsilon m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right) + m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon}. \tag{3.6.13}$$

Proof. We may assume that $k = 1/2$ since the same argument holds for $k = 3/2$ by the duality

$$S_k(-m,n;c) = \overline{S_{2-k}(m,-n;c)}.$$

Recall that

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh(z) \le \sqrt{\frac{2}{\pi z}} \frac{\exp(z)}{2}, \quad z > 0. \tag{3.6.14}$$

Then by (3.4.4) and (3.6.14) we get

$$\sum_{\substack{8N\le c\le 4\pi\sqrt{mn}\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right) \ll_\varepsilon m^{\frac{1}{4}} n^{-\frac{1}{4}} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right) \sum_{\substack{8N\le c\le 4\pi\sqrt{mn}\\ c\equiv 0 \pmod{4N}}} \tau(c)$$
$$\ll_\varepsilon m^{\frac{1}{4}} n^{-\frac{1}{4}} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right) \int_0^{4\pi\sqrt{mn}} t^\varepsilon dt$$
$$\ll_\varepsilon m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right).$$

On the other hand, we have (see [29, Eq. 10.25.2])

$$I_{\frac{1}{2}}(z) \le \sqrt{\frac{z}{2}} \sum_{c=0}^{\infty} \frac{1}{4^c\Gamma(3/2+c)c!} \le I_{\frac{1}{2}}(1)\sqrt{z}, \quad 0<z<1 \tag{3.6.15}$$

and

$$I'_{\frac{1}{2}}(z) \le \frac{I_{\frac{1}{2}}(1)}{2\sqrt{z}}, \quad 0<z<1. \tag{3.6.16}$$

Then by Abel summation, Theorem 3.1.6, (3.4.4), (3.6.14), (3.6.15) and (3.6.16) we get

$$\sum_{\substack{4\pi\sqrt{mn}<c\le X\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$
$$= I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{X}\right) \sum_{\substack{c\le X\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} - I_{\frac{1}{2}}(1) \sum_{\substack{c\le 4\pi\sqrt{mn}\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c}$$
$$- \int_{4\pi\sqrt{mn}}^{X} \sum_{\substack{c\le t\\ c\equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} \left[I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{t}\right)\right]' dt$$
$$\ll_\varepsilon (mn)^{\frac{1}{4}} X^{-\frac{1}{2}} \times (mn)^{\frac{1}{4}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{\frac{1}{6}+\varepsilon} + \int_0^{4\pi\sqrt{mm}} m^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon} dt$$
$$+ (mn)^{\frac{1}{4}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} \int_{4\pi\sqrt{mn}}^{X} t^{\frac{1}{6}+\varepsilon} \left[4\pi\sqrt{mn} I'_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{t}\right) t^{-2}\right] dt$$
$$\ll_\varepsilon (mn)^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{-\frac{1}{3}+\varepsilon} + m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon} + (mn)^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} \int_{4\pi\sqrt{mn}}^{X} t^{-\frac{4}{3}+\varepsilon} dt$$

$$\ll_\varepsilon (mn)^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{-\frac{1}{3}+\varepsilon} + m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon}.$$

It follows from these two bounds that

$$\sum_{\substack{8N \le c \le X \\ c \equiv 0 \pmod{4N}}} \frac{S_k(-m,n;c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$
$$\ll_\varepsilon m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon} \exp\left(\frac{\pi\sqrt{mn}}{2N}\right) + (mn)^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{3}+\varepsilon} X^{-\frac{1}{3}+\varepsilon} + m^{\frac{3}{4}+\varepsilon} n^{\frac{1}{4}+\varepsilon}.$$

□

3.7 Approximation by the Salié sum

Let p be a prime number. Then by Propositions 3.4.1 the Fourier coefficient $a_{m,k,p}(n,s)$ of $P_{m,k,p}(z,s)$ is given by

$$a_{m,k,p}(n,s) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{4p}}} \frac{S_k(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

On the other hand, if $p \ge 3$, then by [21, Lemma 2] and a short calculation, we have

$$S_k(-m,n;4p) = K_k(m,n,p) T(-m,n;p), \tag{3.7.1}$$

where

$$K_k(m,n,p) := \begin{cases} \exp\left(\frac{(n-m)}{2}\pi i\right) + i^{2k}\exp\left(\frac{(m-n)}{2}\pi i\right), & \text{if } p \equiv 1 \pmod 4, \\ \exp\left(\frac{(m-n)}{2}\pi i\right) + i^{2k-2}\exp\left(\frac{(n-m)}{2}\pi i\right), & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

In particular, we have

$$K_k(m,n,p) = \pm 1 \pm i. \tag{3.7.2}$$

Now, we define the modified Fourier coefficients

$$\widetilde{a}_{m,k,p}(n,s) := \frac{a_{m,k,p}(n,s)}{\mathcal{N}(m,k,p,n)}$$

where

$$\mathcal{N}(m,k,p,n) := 2^{-1/2} i^{-k} m^{\frac{1-2k}{4}} n^{\frac{2k-3}{4}} \varepsilon_p \left(\frac{n}{p}\right) K_k(m,n,p) \exp\left(\frac{\pi\sqrt{mn}}{p}\right).$$

Then by (3.7.1) we get the decomposition

$$\widetilde{a}_{m,k,p}(n,s) = \frac{T(-m,n;p)}{2\left(\frac{n}{p}\right)\varepsilon_p\sqrt{p}} A(m,k,s,p,n) + B(m,k,s,p,n), \tag{3.7.3}$$

where

$$\begin{aligned} A(m,k,s,p,n) &:= \pi 2^{1/2}(mn)^{1/4} I_{2s-1}\left(\frac{\pi\sqrt{mn}}{p}\right) p^{-1/2} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right), \\ B(m,k,s,p,n) &:= \pi 2^{3/2}(mn)^{1/4} \varepsilon_p^{-1} \left(\frac{n}{p}\right) K_k(m,n,p)^{-1} \\ &\quad \times \ \exp\left(-\frac{\pi\sqrt{mn}}{p}\right) \sum_{\substack{c>4p \\ c\equiv 0 \pmod{4p}}} \frac{S_k(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned}$$

We will need the following estimates.

Proposition 3.7.1. Let $s = 3/4$ or $\mathrm{Re}(s) > 3/4$ and

$$\alpha \geq \max\left\{8 + \log\left(\frac{[1+2\mathrm{Re}(2s-1)]^8}{256\pi^8 m^4}\right), 8 + \log\left(\frac{6561}{\pi^8 m^4}\right)\right\}.$$

(1) There is a function $C(m,k,s,p,n)$ such that if $n \geq p^\alpha$, then

$$A(m,k,s,p,n) = 1 + C(m,k,s,p,n)$$

where

$$|C(m,k,s,p,n)| \leq C_{m,k,s,p} := C_{m,k,s,p,1} + \exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\{1 + C_{m,k,s,p,1}\},$$

$$C_{m,k,s,p,1} := \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}p^{\frac{2-\alpha}{2}}\exp\left(\frac{|(2s-1)^2-\frac{1}{4}|}{\pi\sqrt{m}}p^{\frac{2-\alpha}{2}}\right).$$

(2) If $n \geq p^\alpha$ and $\mathrm{Re}(s) > 3/4$, then

$$\begin{aligned}|B(m,k,s,p,n)| \leq B_{m,k,s,1}\exp\left(-\frac{\pi\sqrt{m}}{2}p^{\frac{\alpha-2}{2}}\right) \\ + B_{m,k,s,2}(1 + C_{m,k,s,p})p^{-1}\exp\left(-\frac{\pi\sqrt{m}}{4}p^{\frac{\alpha-2}{2}}\right)\end{aligned}$$

where

$$\begin{aligned}B_{m,k,s,1} &:= 2^{1+\mathrm{Re}(2s-1)}\pi^{1+\mathrm{Re}(2s-1)}m^{\frac{3}{4}+\frac{\mathrm{Re}(2s-1)}{2}}\zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right)\sum_{\ell=0}^{\infty}\frac{1}{4^\ell|\Gamma(2s+\ell)|\ell!},\\ B_{m,k,s,2} &:= 2^{\frac{3}{2}}3^{-\frac{1}{2}}m^{\frac{5}{4}}\pi^2.\end{aligned}$$

(3) If $n \geq p^\alpha$, $s = 3/4$ ($k = 1/2$ or $k = 3/2$) then

$$B(m,k,s,p,n) \ll m^{\frac{5}{4}}\exp\left(-\frac{\pi\sqrt{m}}{4}p^{\frac{\alpha-2}{2}}\right)$$

where the implied constant is independent of m, p and n.

Proof. (1) Consider the following asymptotic formula for the I-Bessel function valid for $z > 0$ and $v \geq 1/2$ (see [27, p. 269] or [29, Eq. 10.40.12])

$$I_v(z) = \frac{\exp(z)}{\sqrt{2\pi z}}\left[1 + E_v(z)\right] - i\frac{\exp(-z - v\pi i)}{\sqrt{2\pi z}}[1 + E_v(z)], \tag{3.7.4}$$

where

$$|E_v(z)| \le \frac{|(4v^2-1)(4v^2-9)|}{|z|} \exp\left(\left|\frac{v^2-\frac{1}{4}}{z}\right|\right). \tag{3.7.5}$$

Then we can write

$$A(m,k,s,p,n) = \pi 2^{\frac{1}{2}} (mn)^{\frac{1}{4}} I_{|k-1|}\left(\frac{\pi\sqrt{mn}}{p}\right) p^{-\frac{1}{2}} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right) = 1 + C(m,k,s,p,n),$$

where

$$\begin{aligned} C(m,k,s,p,n) := E_{2s-1}&\left(\frac{\pi\sqrt{mn}}{p}\right) \\ &- i\exp\left(-\frac{2\pi\sqrt{mn}}{p} - (2s-1)\pi i\right)\left[1 + E_{2s-1}\left(\frac{\pi\sqrt{mn}}{p}\right)\right]. \end{aligned}$$

By (3.7.5), if $n \ge p^\alpha$ then

$$\begin{aligned} |C(m,k,s,p,n)| &\le \left|E_{2s-1}\left(\frac{\pi\sqrt{mn}}{p}\right)\right| + \exp\left(-\frac{2\pi\sqrt{mn}}{p}\right)\left[1 + \left|E_{2s-1}\left(\frac{\pi\sqrt{mn}}{p}\right)\right|\right] \\ &\le C_{m,k,s,p} \end{aligned} \tag{3.7.6}$$

where

$$\begin{aligned} C_{m,k,s,p} &= C_{m,k,s,p,1} + \exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\{1 + C_{m,k,s,p,1}\}, \\ C_{m,k,s,p,1} &= \frac{|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}} p^{\frac{2-\alpha}{2}} \exp\left(\frac{|(2s-1)^2-\frac{1}{4}|}{\pi\sqrt{m}} p^{\frac{2-\alpha}{2}}\right). \end{aligned}$$

(2) Observe that (see [29, Eq. 10.25.2])

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{c=0}^{\infty} \frac{1}{\Gamma(v+1+c)c!}\left(\frac{z^2}{4}\right)^c.$$

If $0 < z < 1$, then we have

$$|I_v(z)| \leq \left|\left(\frac{z}{2}\right)\right|^{\mathrm{Re}(v)} \sum_{c=0}^{\infty} \frac{1}{4^c |\Gamma(v+1+c)| c!}. \tag{3.7.7}$$

Now, consider the decomposition

$$\begin{aligned}
&B(m,k,s,p,n) \\
&= 2^{\frac{3}{2}} \pi (mn)^{\frac{1}{4}} \varepsilon_p^{-1} \left(\frac{n}{p}\right) K_k(m,n,p)^{-1} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right) \\
&\quad \times \sum_{\substack{c>4p \\ c\equiv 0 \pmod{4p}}} \frac{S_k(-m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\
&= 2^{\frac{3}{2}} \pi (mn)^{\frac{1}{4}} \varepsilon_p^{-1} \left(\frac{n}{p}\right) K_k(m,n,p)^{-1} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right) \sum_{c=2}^{\infty} \frac{S_k(-m,n;4cp)}{4cp} I_{2s-1}\left(\frac{\pi\sqrt{mn}}{cp}\right) \\
&= 2^{-\frac{1}{2}} \pi (mn)^{\frac{1}{4}} \varepsilon_p^{-1} \left(\frac{n}{p}\right) K_k(m,n,p)^{-1} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right) \{R_1 + R_2\},
\end{aligned}$$

where

$$\begin{aligned}
R_1 &:= \sum_{c > \frac{\pi\sqrt{mn}}{p}} \frac{S_k(-m,n;4cp)}{cp} I_{2s-1}\left(\frac{\pi\sqrt{mn}}{cp}\right), \\
R_2 &:= \sum_{c=2}^{\lfloor \frac{\pi\sqrt{mn}}{p} \rfloor} \frac{S_k(-m,n;4cp)}{cp} I_{2s-1}\left(\frac{\pi\sqrt{mn}}{cp}\right).
\end{aligned}$$

By the Weil-type bound (3.4.4)

$$|S_k(-m,n;4cp)| \leq \tau(4cp)(-m,n,4cp)^{\frac{1}{2}}(4cp)^{\frac{1}{2}} \tag{3.7.8}$$

and (3.7.7), we have

$$\begin{aligned}|R_1| &\leq \sum_{c>\frac{\pi\sqrt{mn}}{p}} \frac{|S_k(-m,n;4cp)|}{cp}\left(\frac{\pi\sqrt{mn}}{2cp}\right)^{\mathrm{Re}(2s-1)}\sum_{\ell=0}^{\infty}\frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!}\\ &\leq 4m^{\frac{1}{2}}\left(2\pi\sqrt{mn}\right)^{\mathrm{Re}(2s-1)}\sum_{\ell=0}^{\infty}\frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!}\sum_{c>\frac{\pi\sqrt{mn}}{p}}\frac{\tau(4cp)}{(4cp)^{\frac{1}{2}+\mathrm{Re}(2s-1)}}.\end{aligned}$$

Moreover, we have

$$\sum_{c>\frac{\pi\sqrt{mn}}{p}}\frac{\tau(4cp)}{(4cp)^{\frac{1}{2}+\mathrm{Re}(2s-1)}} \leq \sum_{c=1}^{\infty}\frac{\tau(c)}{c^{\frac{1}{2}+\mathrm{Re}(2s-1)}} = \zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right).$$

Hence

$$|R_1| \leq 4m^{\frac{1}{2}}(2\pi\sqrt{mn})^{\mathrm{Re}(2s-1)}\zeta^2\left(\frac{1}{2}+\mathrm{Re}(2s-1)\right)\sum_{\ell=0}^{\infty}\frac{1}{4^{\ell}|\Gamma(2s+\ell)|\ell!}. \tag{3.7.9}$$

Similarly, by (3.7.2), (3.7.4), (3.7.6), (3.7.8) and the bound

$$\tau(\ell) \leq \sqrt{3}\ell^{\frac{1}{2}},$$

we get

$$\begin{aligned}|R_2| &\leq \sum_{c=2}^{\lfloor\frac{\pi\sqrt{mn}}{p}\rfloor}\frac{|S_k(-m,n;4cp)|}{cp}\frac{\sqrt{cp}}{\sqrt{2\pi}(mn)^{\frac{1}{4}}}\exp\left(\frac{\pi\sqrt{mn}}{cp}\right)(1+C_{m,k,s,p})\\ &\leq \frac{2m^{\frac{1}{2}}(1+C_{m,k,s,p})}{\sqrt{2\pi}(mn)^{\frac{1}{4}}}\exp\left(\frac{\pi\sqrt{mn}}{2p}\right)\sum_{c=2}^{\lfloor\frac{\pi\sqrt{mn}}{p}\rfloor}\tau(4cp)\\ &\leq \frac{2m^{\frac{1}{2}}(1+C_{m,k,s,p})}{\sqrt{2\pi}(mn)^{\frac{1}{4}}}\exp\left(\frac{\pi\sqrt{mn}}{2p}\right)\frac{4}{3}\sqrt{3p}\left(\frac{\pi\sqrt{mn}}{p}\right)^{\frac{3}{2}}.\end{aligned} \tag{3.7.10}$$

Combining (3.7.9) and (3.7.10) now gives

$$\begin{aligned}|B(m,k,s,p,n)| \leq 2^{1+\text{Re}(2s-1)}\pi^{1+\text{Re}(2s-1)}m^{\frac{3}{4}+\frac{\text{Re}(2s-1)}{2}}\zeta^2\left(\frac{1}{2}+\text{Re}(2s-1)\right)\\ \times n^{\frac{1}{4}+\frac{\text{Res}(2s-1)}{2}}\exp\left(-\frac{\pi\sqrt{mn}}{p}\right)\\ + 2^{\frac{3}{2}}3^{-\frac{1}{2}}\pi^2 m^{\frac{5}{4}}(1+C_{m,k,s,p})p^{-1}n^{\frac{3}{4}}\exp\left(-\frac{\pi\sqrt{mn}}{2p}\right).\end{aligned}$$

Furthermore, since

$$\begin{aligned}&\alpha \geq \max\left\{8+\log\left(\frac{[1+2\text{Re}(2s-1)]^8}{256\pi^8 m^4}\right), 8+\log\left(\frac{6561}{\pi^8 m^4}\right)\right\}\\ \implies\quad & n \geq p^\alpha \geq \max\left\{\frac{6561p^8}{\pi^8 m^4}, \frac{(1+2\text{Re}(2s-1))^8 p^8}{256\pi^8 m^4}\right\},\end{aligned}$$

it follows that

$$\begin{aligned}n^{\frac{1}{4}+\frac{\text{Res}(2s-1)}{2}}\exp\left(-\frac{\pi\sqrt{mn}}{p}\right) &\leq \exp\left(-\frac{\pi\sqrt{mn}}{2p}\right),\\ n^{\frac{3}{4}}\exp\left(-\frac{\pi\sqrt{mn}}{2p}\right) &\leq \exp\left(-\frac{\pi\sqrt{mn}}{4p}\right),\\ -\frac{\sqrt{n}}{p} &\leq -p^{\frac{\alpha-2}{2}}.\end{aligned}$$

Hence

$$|B(m,k,s,p,n)| \leq B_{m,k,s,1}\exp\left(-\frac{\pi\sqrt{m}}{2}p^{\frac{\alpha-2}{2}}\right) + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}\exp\left(-\frac{\pi\sqrt{m}}{4}p^{\frac{\alpha-2}{2}}\right)$$

where

$$\begin{aligned}B_{m,k,s,1} &= 2^{1+\text{Re}(2s-1)}\pi^{1+\text{Re}(2s-1)}m^{\frac{3}{4}+\frac{\text{Re}(2s-1)}{2}}\zeta^2\left(\frac{1}{2}+\text{Re}(2s-1)\right)\sum_{\ell=0}^{\infty}\frac{1}{4^\ell|\Gamma(2s+\ell)|\ell!},\\ B_{m,k,s,2} &= 2^{\frac{3}{2}}3^{-\frac{1}{2}}m^{\frac{5}{4}}\pi^2.\end{aligned}$$

(3) By the same argument and taking $\varepsilon = 1/4$ in (3.6.13), we have

$$B(m,k,3/4,p,n) \ll m^{\frac{5}{4}} n^{\frac{3}{4}} \exp\left(-\frac{\pi\sqrt{mn}}{2p}\right) + m^{\frac{5}{4}} n^{\frac{3}{4}} \exp\left(-\frac{\pi\sqrt{mn}}{p}\right)$$
$$\ll m^{\frac{5}{4}} n^{\frac{3}{4}} \exp\left(-\frac{\pi\sqrt{mn}}{2p}\right).$$

Moreover, since

$$\alpha \geq 8 + \log\left(\frac{6561}{\pi^8 m^4}\right) \quad \Longrightarrow \quad n \geq p^\alpha \geq \frac{6561 p^8}{\pi^8 m^4},$$

it follows that

$$n^{\frac{3}{4}} \exp\left(-\frac{\pi\sqrt{mn}}{2p}\right) \leq \exp\left(-\frac{\pi\sqrt{mn}}{4p}\right),$$
$$-\frac{\sqrt{n}}{p} \leq -p^{\frac{\alpha-2}{2}}.$$

Hence

$$B(m,k,3/4,p,n) \ll m^{\frac{5}{4}} \exp\left(-\frac{\pi\sqrt{m}}{4} p^{\frac{\alpha-2}{2}}\right).$$

□

Now, recall that there is a unique real number $\theta_{-m,p}(n) \in [0,\pi]$ called the Salié angle such that the normalized Salié sum is given by

$$\cos(\theta_{-m,p}(n)) = \frac{T(-m,n;p)}{2(\frac{n}{p})\varepsilon_p\sqrt{p}}.$$

We will use Proposition 3.7.1 to deduce the following effective asymptotic formula.

Theorem 3.7.2. *Let* $s = 3/4$ *or* $\mathrm{Re}(s) > 3/4$ *and*

$$\alpha \geq \max\left\{3, 8 + \log\left(\frac{[1+2\mathrm{Re}(2s-1)]^8}{256\pi^8 m^4}\right), 8 + \log\left(\frac{6561}{\pi^8 m^4}\right)\right\},$$

and p be a prime number satisfying

$$p \geq \max\left\{3, \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha-2)^{\frac{2}{\alpha-2}}\right\}.$$

Then if $n \geq p^\alpha$, we have

$$\widetilde{a}_{m,k,p}(n,s) = \cos(\theta_{-m,p}(n)) + E(m,k,s,p,n),$$

where

$$|E(m,k,s,p,n)| \leq C(m,k,s)p^{\frac{2-\alpha}{2}}$$

with

$$C(m,k,s) := \begin{cases} C_{1/2,3/2}m^{\frac{5}{4}}, & \text{if } s = 3/4, \\ B_{m,k,s,1} + B_{m,k,s,2} + (1 + B_{m,k,s,2})\left(1 + \frac{4|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}\right), & \text{otherwise,} \end{cases} \tag{3.7.11}$$

and the constant $C_{1/2,3/2}$ is independent of m, p and n.

Proof. We first deal with the case that $\mathrm{Re}(s) > 3/4$. By (3.7.3) and Proposition 3.7.1 we have

$$\widetilde{a}_{m,k,p}(n,s) = \cos(\theta_{-m,p}(n)) + E(m,k,s,p,n),$$

where the error term

$$E(m,k,s,p,n) := \cos(\theta_{-m,p}(n))C(m,k,s,p,n) + B(m,k,s,p,n)$$

satisfies the bound

$$\begin{aligned}|E(m,k,s,p,n)| &\le |C(m,k,s,p,n)| + |B(m,k,s,p,n)| \\ &\le C_{m,k,s,p} + B_{m,k,s,1}\exp\left(-\frac{\pi\sqrt{m}}{2}p^{\frac{\alpha-2}{2}}\right) \\ &\quad + B_{m,k,s,2}(1+C_{m,k,s,p})p^{-1}\exp\left(-\frac{\pi\sqrt{m}}{4}p^{\frac{\alpha-2}{2}}\right).\end{aligned}$$

We have

$$\begin{aligned}p \ge \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}} \quad &\implies \quad p \ge \left(\frac{|(2s-1)^2-\frac{1}{4}|}{\pi\sqrt{m}\log 2}\right)^{\frac{2}{\alpha-2}} \\ &\implies \quad \exp\left(\frac{|(2s-1)^2-\frac{1}{4}|}{\pi\sqrt{m}}p^{\frac{2-\alpha}{2}}\right) \le 2.\end{aligned}$$

Hence

$$\begin{aligned}C_{m,k,s,p} \le\ & \frac{2|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}p^{\frac{2-\alpha}{2}}\left[1+\exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right)\right] \\ & + \exp\left(-2\pi\sqrt{m}p^{\frac{\alpha-2}{2}}\right). \qquad (3.7.12)\end{aligned}$$

Similarly, we have

$$\begin{aligned}p \ge \max\left\{3, (\alpha-2)^{\frac{2}{\alpha-2}}\right\} \quad &\implies \quad p \ge \max\left\{3, \left(\frac{2\alpha-4}{\pi\sqrt{m}}\right)^{\frac{8}{\alpha-2}}\right\} \\ &\implies \quad \exp\left(-\frac{\pi\sqrt{m}}{4}p^{\frac{\alpha-2}{2}}\right) \le p^{\frac{2-\alpha}{2}}. \qquad (3.7.13)\end{aligned}$$

It follows that

$$C_{m,k,s,p} \le \left(1 + \frac{4|(4(2s-1)^2-1)(4(2s-1)^2-9)|}{\pi\sqrt{m}}\right)p^{\frac{2-\alpha}{2}}.$$

Hence

$$
\begin{aligned}
&C_{m,k,s,p} + B_{m,k,s,1} \exp\left(-\frac{\pi\sqrt{m}}{2} p^{\frac{\alpha-2}{2}}\right) + B_{m,k,s,2}(1 + C_{m,k,s,p}) p^{-1} \exp\left(-\frac{\pi\sqrt{m}}{4} p^{\frac{\alpha-2}{2}}\right) \\
&\leq C_{m,k,s,p} + B_{m,k,1} p^{\frac{2-\alpha}{2}} + B_{m,k,s,2}(1 + C_{m,k,s,p}) p^{-1} p^{\frac{2-\alpha}{2}} \\
&\leq B_{m,k,s,1} p^{\frac{2-\alpha}{2}} + B_{m,k,s,2} p^{\frac{2-\alpha}{2}} \\
&\quad + (1 + B_{m,k,s,2}) \left(1 + \frac{4|(4(2s-1)^2 - 1)(4(2s-1)^2 - 9)|}{\pi\sqrt{m}}\right) p^{\frac{2-\alpha}{2}}.
\end{aligned}
$$

Next, if $s = 3/4$ and $k = 1/2$ or $3/2$, then by combining Proposition 3.7.1, (3.7.12) and (3.7.13), we directly get

$$
E(m, k, s, p, n) \ll m^{\frac{5}{4}} p^{\frac{2-\alpha}{2}} \leq C(m, k, s) p^{\frac{2-\alpha}{2}}.
$$

□

3.8 Proof of Theorem 3.1.1

Let $s \geq 3/4$ and

$$
\alpha \geq \max\left\{3, 8 + \log\left(\frac{(4s-1)^8}{256\pi^8 m^4}\right), 8 + \log\left(\frac{6561}{\pi^8 m^4}\right)\right\},
$$

and p be a prime number satisfying

$$
p \geq \max\left\{8, \left|(2s-1)^2 - \frac{1}{4}\right|^{\frac{2}{\alpha-2}}, (\alpha - 2)^{\frac{2}{\alpha-2}}\right\}.
$$

For $n \geq p^\alpha$, we define the normalized Fourier coefficients

$$
\lambda_{m,k,s,p}(n) := \frac{a_{m,k,p}(n, s)}{N(m, k, s, p, \alpha, n)}
$$

where

$$N(m,k,s,p,\alpha,n) := (1 + C(m,k,s)p^{\frac{2-\alpha}{2}})\mathcal{N}(m,k,p,n).$$

Then by Theorem 3.7.2 we have

$$|\lambda_{m,k,s,p}(n)| \leq 1,$$

so that $\lambda_{m,k,s,p}(n) \in [-1,1]$.

Let $I_p \subset \mathbb{F}_p^\times$ be an interval such that each $n \in I_p$ satisfies $(\frac{-mn}{p}) = 1$, and choose a complete set of residue classes

$$I_p = \{[n_{p,1}], \ldots, [n_{p,|I_p|}]\}$$

such that the class representatives $n_{p,i}$ satisfy the bound $n_{p,i} \geq p^\alpha$ for $i = 1, \ldots, |I_p|$. This choice determines a set

$$S_{\alpha,I_p} := \{n_{p,1}, \ldots, n_{p,|I_p|}\}.$$

Define the set

$$X_{\alpha,I_p} := \{\lambda_{m,k,s,p}(n) : \ n \in S_{\alpha,I_p}\}.$$

We will deduce Theorem 3.1.1 from the following effective bound for the star discrepancy $D^\star_{X_{\alpha,I_p}}$.

Proposition 3.8.1. Assume that $|I_p| > \sqrt{p}$. Then we have

$$D^\star_{X_{\alpha,I_p}} \leq (14\pi^5 + 42\pi^5 C(m,k,s)) \log^2\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1}.$$

Proof. By Proposition 3.2.1, for any $d \in \mathbb{Z}^+$ we have the Erdös-Turán type inequality

$$D^\star_{X_{\alpha,I_p}} \leq \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \left| \frac{1}{|I_p|} \sum_{n \in S_{\alpha,I_p}} T_r(\lambda_{m,k,s,p}(n)) \right|.$$

Now, by the triangle inequality we have

$$D^\star_{X_{\alpha,I_p}} \leq S_1 + S_2,$$

where

$$S_1 := \frac{6\pi^5}{d} + 7\pi^5 \sum_{r=1}^{d} \frac{1}{r} \left| \frac{1}{|I_p|} \sum_{n \in S_{\alpha,I_p}} T_r(\cos(\theta_{-m,p}(n))) \right|,$$
$$S_2 := 7\pi^5 \sum_{r=1}^{d} \frac{1}{r|I_p|} \sum_{n \in S_{\alpha,I_p}} \left| T_r(\lambda_{m,k,s,p}(n)) - T_r(\cos(\theta_{-m,p}(n))) \right|.$$

We first estimate S_1. By Proposition 3.3.1 and (3.3.2), we have

$$S_1 \leq 14\pi^5 \log^2\left(\frac{4e^8|I_p|}{\sqrt{p}}\right)\left(\frac{|I_p|}{\sqrt{p}}\right)^{-1} = 14\pi^5 \log^2\left(4e^8\beta(p)\right)\beta(p)^{-1}.$$

We next estimate S_2.

Lemma 3.8.2. *We have*

$$||T_r'||_\infty \leq r^2.$$

Proof. For $\theta \in [0, \pi]$, we have

$$\left| \frac{d}{d\theta} T_r(\cos(\theta)) \right| = \left| \frac{d}{d\theta} \cos(r\theta) \right| = \left| -r \sin(r\theta) \right|. \tag{3.8.1}$$

On the other hand, we have

$$\left|\frac{d}{d\theta}T_r(\cos(\theta))\right| = |T_r'(\cos(\theta))| \times |-\sin(\theta)|. \tag{3.8.2}$$

Hence, by combining (3.8.1) and (3.8.2), it follows that

$$|T_r'(\cos(\theta))| = \left|\frac{r\sin(r\theta)}{\sin(\theta)}\right| = \left|r\sum_{n=0}^{r-1}\cos(n\theta)(\cos(\theta))^{r-1-n}\right| \leq r^2.$$

□

By the mean value theorem, Lemma 3.8.2, Theorem 3.7.2 and (3.3.2) we get

$$\begin{aligned}
S_2 &= 7\pi^5\sum_{r=1}^{d}\frac{1}{r|I_p|}\sum_{n\in S_{\alpha,I_p}}|T_r(\lambda_{m,k,s,p}(n)) - T_r(\cos(\theta_{-m,p}(n)))| \\
&\leq 7\pi^5\sum_{r=1}^{d}\frac{1}{r|I_p|}\sum_{n\in S_{\alpha,I_p}}||T_r'||_\infty\,|\lambda_{m,k,s,p}(n) - \cos(\theta_{-m,p}(n))| \\
&\leq 7\pi^5\sum_{r=1}^{d}\frac{r}{|I_p|}\sum_{n\in S_{\alpha,I_p}}|\lambda_{m,k,s,p}(n) - \cos(\theta_{-m,p}(n))| \\
&\leq \frac{7}{2}\pi^5(d(d+1))\frac{1}{|I_p|}\sum_{n\in S_{\alpha,I_p}}\left|\frac{\cos(\theta_{-m,p}(n)) + E(m,k,s,p,n)}{1 + C(m,k,s)p^{\frac{2-\alpha}{2}}} - \cos(\theta_{-m,p}(n))\right| \\
&\leq \frac{7}{2}\pi^5(1+\beta(p))(2+\beta(p)) \times 2C(m,k,s)p^{\frac{2-\alpha}{2}} \\
&\leq 42\pi^5C(m,k,s)p^{\frac{2-\alpha}{2}}\beta(p)^2.
\end{aligned}$$

By combining our bounds for S_1 and S_2, we have

$$D^\star_{X_{\alpha,I_p}} \leq 14\pi^5\log^2\left(4e^8\beta(p)\right)\beta(p)^{-1} + 42\pi^5C(m,k,s)p^{\frac{2-\alpha}{2}}\beta(p)^2.$$

Moreover, by our assumption on α we have $\alpha \geq 8$, hence

$$p^{\frac{2-\alpha}{2}} \leq \beta(p)^{\frac{2-\alpha}{2}} \leq \beta(p)^{-3}.$$

It follows that

$$\begin{aligned} D^{\star}_{X_{\alpha,I_p}} &\le 14\pi^5 \log^2\left(4e^8\beta(p)\right)\beta(p)^{-1} + 42\pi^5 C(m,k,s)\beta(p)^{-1} \\ &\le (14\pi^5 + 42\pi^5 C(m,k,s))\beta(p)^{-1}\log^2(4e^8\beta(p)). \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1.1. Again, if $f \in BV([-1,1])$, then by the same argument as in Lemma 2.4.1 with the Sato-Tate measure replaced with μ_{Arc}, we have the Koksma-Hlwaka type inequality

$$\left|\frac{1}{|I_p|}\sum_{n\in S_{\alpha,I_p}} f(\lambda_{m,k,s,p}(n)) - \int_{-1}^{1} f(t)d\mu_{\text{Arc}}(t)\right| \le \text{Var}(f) D^{\star}_{X_{\alpha,I_p}}.$$

Then by combining this bound with Proposition 3.8.1, we complete the proof. □

4. SUMMARY

The main theme of this work is to prove that the normalized Fourier coefficients of weak Maass forms of both integral and half-integral weights are quantitatively equidistributed with respect to the Sato-Tate measure and the pushforward of the Haar measure on the unitary group of degree one, respectively. Therefore, this work can be viewed as an analogue of the equidistribution results for holomorphic cusp forms. In fact, we prove that these equidistribution results also hold in short intervals.

The crucial insight of our approach is that if n is sufficiently large compared to the level p, then we can approximate the normalized Fourier coefficients $\lambda_{m,k,s,p}(n)$ of the weak Maass forms by normalized Kloosterman sums of modulus p. By doing so, we transfer (in essence) to a study of the distribution of normalized Kloosterman sums. A fundamental theorem of Katz [24] asserts that these Kloosterman sums become equidistributed with respect to the Sato-Tate measure as $p \to \infty$; this is the so-called "vertical" equidistribution of Kloosterman sums. Rather than appealing directly to Katz's work, we instead use an elegant bound of Fouvry, Kowalski, Michel, Raju, Rivat and Soundararajan [14] for periodic functions on short intervals in $\mathbb{Z}$ which goes beyond the Polya-Vinogradov range. This bound is quite versatile for proving equidistribution in short intervals (see e.g. [14, Section 1.3]). For example, if the test function arising in the Weyl criterion for the equidistribution problem is p-periodic and can be realized as the Frobenius trace function of a suitable ℓ-adic Fourier sheaf, then equidistribution boils down to giving a bound which is uniform in p for the sup-norm of the Fourier transform of the trace function. Further, if this bound can be made effective (which can be difficult to do in practice and is one feature of our work), then combined with a suitable Erdös-Turán type inequality, one can prove quantitative equidistribution as in Theorem 2.1.1.

Finally, the proof of the half-integral weight case involves two fundamental parts which are of independent interest (see Chapter 3). First, we need the quantitative "vertical" equidistribution of Salié sums which is analogous to Katz's vertical equidistribution for Kloosterman sums [24].

Second, we need a power-saving bound for sums of half-integral weight Kloosterman sums which is uniform in all parameters. As a crucial step in the proof of Theorem 3.1.3, we must approximate the normalized Fourier coefficients $\lambda_{m,k,p,s}(n)$ by normalized Salié sums of modulus p. These coefficients can be expressed as infinite sums of half-integral weight (opposite sign) Kloosterman sums with θ-multiplier, which can then be related to infinite sums of Salié sums by a calculation of Iwaniec [21]. We require a power saving bound for these sums of Kloosterman sums which is uniform in all parameters. If the weight $k = 1/2$ or $k = 3/2$, these sums are only *conditionally* convergent. Hence, getting the necessary cancellation and uniformity in this case is a difficult problem which requires advanced methods from the spectral theory of automorphic forms.

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APPENDIX A

INTEGRAL TRANSFORMS

A.1 Calculating integral transforms

In this Appendix we calculate the integral transforms defined by (2.2.3).

Lemma A.1.1. *Let* $\alpha_{n,s,k}(c,y)$ *be defined by* (2.2.3)*. Then we have*

$$\alpha_{n,s,k}(c,y)=\begin{cases} 2\pi i^{-k}\frac{\Gamma(2s)}{\Gamma(s+k/2)}c^{k-1}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\mathbf{W}_{s,k}(4\pi ny), & n\geq 1,\\ \frac{2^{2-k}\pi^{1+s-\frac{k}{2}}i^{-k}m^{s-\frac{k}{2}}y^{1-s-\frac{k}{2}}c^{k-2s}\Gamma(2s)}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}, & n=0,\\ 2\pi i^{-k}\frac{\Gamma(2s)}{\Gamma(s-k/2)}c^{k-1}\left(\frac{|n|}{m}\right)^{\frac{k-1}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{m|n|}}{c}\right)\mathbf{W}_{s,k}(4\pi ny), & n\leq -1.\end{cases}$$

Proof. By a direct calculation we have

$$\alpha_{n,s,k}(c,y)=c^k(4\pi my)^{-\frac{k}{2}}\int_{\mathbb{R}} M_{-\frac{k}{2},s-\frac{1}{2}}\left(\frac{4\pi my}{c^2(t^2+y^2)}\right)(t^2+y^2)^{\frac{k}{2}}e\left(\frac{mt}{c^2(t^2+y^2)}-nt\right)(t+iy)^{-k}dt.$$

Using [29, Eq. 13.14.2] we get

$$M_{-\frac{k}{2},s-\frac{1}{2}}\left(\frac{4\pi my}{c^2(t^2+y^2)}\right)=\exp\left(\frac{-2\pi my}{c^2(t^2+y^2)}\right)\left(\frac{4\pi my}{c^2(t^2+y^2)}\right)^s M\left(s+\frac{k}{2},2s,\frac{4\pi my}{c^2(t^2+y^2)}\right)$$

where $M(a,b,z)$ is the Kummer Confluent Hypergeometric function (see [29, Section 13.2]). Then the change of variables $t=yu, A=1/c^2y, B=-ny$ gives

$$\frac{\alpha_{n,s,k}(c,y)}{\Gamma(2s)}=i^{-k}c^k(-4\pi m)^{-\frac{k}{2}}y^{-\frac{k}{2}+1}L_n \tag{A.1.1}$$

where

$$L_n := \int_{\mathbb{R}} G\left(\frac{4\pi m A}{1+u^2}\right) e^{2\pi i\left(\frac{muA}{1+u^2}+Bu\right)} \left(\frac{1-iu}{1+iu}\right)^{-\frac{k}{2}} du$$

and

$$G(z) := \frac{z^s e^{-z/2} M(s+\frac{k}{2}, 2s, z)}{\Gamma(2s)}.$$

Using [18, Lemma 5.5, p. 357] we get

$$L_n = \begin{cases} \frac{2\pi(4\pi ny)^s}{\Gamma(s+k/2)} e^{-2\pi ny} U(s-k/2, 2s, 4\pi ny) \sqrt{\frac{m}{nc^2y^2}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), & n \geq 1, \\ \frac{2\pi^{s+1}}{(s-1/2)\Gamma(s+k/2)\Gamma(s-k/2)} \left(\frac{m}{c^2y}\right)^s, & n = 0, \\ \frac{2\pi(-4\pi ny)^s}{\Gamma(s-k/2)} e^{2\pi ny} U(s+k/2, 2s, -4\pi ny) \sqrt{\frac{m}{-nc^2y^2}} J_{2s-1}\left(\frac{4\pi\sqrt{-mn}}{c}\right), & n \leq -1, \end{cases}$$

where $U(a,b,z)$ is the Tricomi Confluent Hypergeometric function (see [29, Section 13.2]). Moreover, by [29, Eq. 13.14.5] we have

$$U(s-k/2, 2s, 4\pi ny) = e^{2\pi ny}(4\pi ny)^{-s} W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi ny),$$
$$U(s+k/2, 2s, -4\pi ny) = e^{-2\pi ny}(-4\pi ny)^{-s} W_{-\frac{k}{2}, s-\frac{1}{2}}(-4\pi ny).$$

Finally, we substitute these identities into (A.1.1) and simplify to complete the evaluation. □

APPENDIX B

CONSTRUCTING HARMONIC MAASS FORMS

B.1 Constructing harmonic Maass Forms of integral weight

In this Appendix we briefly explain how to verify the decompositions (2.1.5) and (2.1.6) asserted in the introduction.

We first assume that $k \le 0$. Each $f \in H_k^{\#}(N)$ has a Fourier expansion at ∞ of the form

$$f(z) = \sum_{m=1}^{M_\infty} a_f^+(-m)q^{-m} + \sum_{n=0}^{\infty} a_f^+(n)q^n + \sum_{n=1}^{\infty} a_f^-(-n)\Gamma(1-k, 4\pi ny)q^{-n}.$$

Define the function

$$P(z) := \sum_{m=1}^{M_\infty} a_f^+(-m)P_{m,k,N}(z).$$

Then by (2.1.3) we have

$$P(z) = \sum_{m=1}^{M_\infty} a_f^+(-m)q^{-m} + \sum_{n=0}^{\infty} a_P^+(n)q^n + \sum_{n=1}^{\infty} a_P^-(-n)\frac{\Gamma(1-k, 4\pi ny)}{\Gamma(1-k)}q^{-n}$$

where

$$a_P^+(n) := \sum_{m=1}^{M_\infty} a_f^+(-m)a_{m,k,N}(n, 1-k/2),$$
$$a_P^-(-n) := \sum_{m=1}^{M_\infty} a_f^+(-m)(a_{m,k,N}(-n, 1-k/2) - \delta_m(n)).$$

The function $P(z)$ has the same principal part as $f(z)$ at ∞. Moreover, since $k \le 0$ the growth condition (5) implies that the function $f(\gamma_{\mathfrak{a}}(z))$ is bounded as $y \to \infty$. It follows that $f(z) - P(z)$ is a bounded harmonic function on $\mathbb{H}$ and thus constant. In particular, we have

$f(z) - P(z) = a_f^+(0) - a_P^+(0)$, or equivalently

$$f(z) = P(z) + 1 \cdot (a_f^+(0) - a_P^+(0)). \tag{B.1.1}$$

Thus

$$f \in \text{Span}(\delta_0(k), \{P_{m,k,N}\}_{m \in \mathbb{Z}_+})$$

which verifies (2.1.5). Note that if $k \leq -1$, then by applying the slash operator $|_k$ to both sides (B.1.1) we find that

$$a_f^+(0) - a_P^+(0) = 0,$$

in which case $f \in \text{Span}(\{P_{m,k,N}\}_{m \in \mathbb{Z}_+})$. Also, (B.1.1) implies that $a_P^-(-n) = 0$ for all $n \geq 1$.

If $k \geq 2$, a similar argument can be used to verify (2.1.6). The difference is that although the analogous function $Q(z)$ will have the same principal part as $f \in M_k^{\#}(N)$ at ∞, when $k \geq 2$ the growth condition (5) no longer implies that the harmonic function $f(z) - Q(z)$ is bounded on $\mathbb{H}$, and hence $f(z) - Q(z)$ not necessarily constant. In any case, $f(z) - Q(z)$ is a holomorphic modular form of weight k, which gives (2.1.6).

B.2 Constructing harmonic Maass Forms of half-integral weight

Next, we briefly explain how to verify the decompositions (3.1.5), (3.1.6), and (3.1.7) asserted in the introduction.

First assume that $k \leq -1/2$. Each $f \in H_k^{\#}(4N)$ has a Fourier expansion at ∞ given by

$$f(z) = \sum_{m=1}^{M_\infty} a_f^+(-m)q^{-m} + \sum_{m=0}^{\infty} a_f^+(m)q^m + \sum_{n=1}^{\infty} a_f^-(-n)\Gamma(1-k, 4\pi ny)q^{-n}.$$

Define the function

$$P(z) := \sum_{m=1}^{M_\infty} a_f^+(-m) P_{m,k,N}(z, 1-k/2).$$

Then by (3.1.3) we have

$$P(z) = \sum_{m=1}^{M_\infty} a_f^+(-m) q^{-m} + \sum_{n=0}^{\infty} a_P^+(n) q^n + \sum_{n=1}^{\infty} a_P^-(-n) \frac{\Gamma(1-k, 4\pi n y)}{\Gamma(1-k)} q^{-n},$$

where

$$a_P^+(n) := \sum_{m=1}^{M_\infty} a_f^+(-m) a_{m,k,N}(n, 1-k/2),$$
$$a_P^-(-n) := \sum_{m=1}^{M_\infty} a_f^+(-m)(a_{m,k,N}(-n, 1-k/2) - \delta_m(n)).$$

The function $P(z)$ has the same principal part as $f(z)$ at ∞. Moreover, since $k \leq -1/2$ the growth condition (5) implies that the function $f(\gamma_{\mathfrak{a}}(z))$ is bounded as $y \to \infty$. It follows that $f(z) - P(z)$ is a bounded harmonic function on $\mathbb{H}$ and is thus constant. In particular, we have $f(z) - P(z) = a_f^+(0) - a_P^+(0)$, or equivalently,

$$f(z) = P(z) + 1 \cdot (a_f^+(0) - a_P^+(0)). \tag{B.2.1}$$

Moreover, since $k \leq -1/2$, by applying the slash operator $|_k$ to both sides (B.2.1) we find that

$$a_f^+(0) - a_P^+(0) = 0,$$

and thus

$$f \in \text{Span}(\{P_{m,k,N}(z)\}_{m \in \mathbb{Z}^+})$$

which verifies (3.1.5).

Now, if $k = 1/2$ the condition (5) no longer implies that $f(z) - P(z)$ is bounded on $\mathbb{H}$. However, since $f(z) - P(z) \in M_{1/2}(4N)$, then by the Serre-Stark basis theorem [33] we have

$$f(z) - P(z) \in \Theta(4N).$$

Hence

$$f \in \text{Span}(\{P_{m,1/2,N}(z)\}_{m\in\mathbb{Z}^+}) \sqcup \Theta(4N)$$

which verifies (3.1.6).

Finally, assume that $k \geq 3/2$. As before, given $f \in M_k^{\#}(4N)$ we define the function

$$Q(z) := \sum_{m=1}^{M_\infty} a_f^+(-m) Q_{m,k,N}(z).$$

Then by (3.1.4) we have

$$Q(z) = \sum_{m=1}^{M_\infty} a_f^+(-m) q^{-m} + \sum_{n=0}^{\infty} b_Q^+(n) q^n$$

where

$$b_Q^+(n) := \sum_{m=1}^{M_\infty} a_f^+(-m) b_{m,k,N}^+(n).$$

The function $Q(z)$ has the same principal part as f at ∞. Again, since $k \geq 3/2$ the growth condition (5) no longer implies that $f(z) - Q(z)$ is bounded on $\mathbb{H}$. However, since $f(z) - Q(z) \in M_k(4N)$ we have

$$f \in \text{Span}(\{Q_{m,k,N}(z)\}_{m\in\mathbb{Z}^+}) \sqcup M_k(4N)$$

which verifies (3.1.7).