GONČAROV POLYNOMIALS, PARTITION LATTICES AND PARKING SEQUENCES

A Dissertation

by

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ABSTRACT

Classical Gončarov polynomials arose in numerical analysis as a basis for the solutions of the Gončarov interpolation problem. These polynomials provide a natural algebraic tool in the enumerative theory of parking functions. By replacing the differentiation operator with a delta operator and using the theory of finite operator calculus, Lorentz, Tringali and Yan introduced the sequence of generalized Gončarov polynomials associated to a pair $(\Delta, \mathcal{Z})$ of a delta operator Δ and an interpolation grid $\mathcal{Z}$. Generalized Gončarov polynomials share many nice algebraic properties and have a connection with the theories of binomial enumeration and order statistics.

Parking functions are combinatorial objects which were introduced in 1966 by Konheim and Weiss. They have been well-studied in the literature due to their numerous connections and have several generalizations and extensions. Ehrenborg and Happ recently introduced a generalization of parking functions called parking sequences in which the n cars have different sizes, and each takes up a number of adjacent parking spaces after a trailer T parked on the first $z-1$ spots.

Consequently, this dissertation is divided into two major parts. In the first part, we give a complete combinatorial interpretation for any sequence of generalized Gončarov polynomials. First we show that they can be realized as weight enumerators in partition lattices. Then, we give a more concrete realization in exponential families and show that these polynomials enumerate various enriched structures of vector parking functions.

In the second part, we study increasing parking sequences and their representation via a special class of lattice paths. We also study two notions of invariance in parking sequences and prove some interesting results for a number of cases where the sequence of car sizes have special properties.

DEDICATION

To my wife, Uchechi.

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NOMENCLATURE

$[n]$	$\{1, 2, \ldots, n\}$
$[x, y]$	$\{x, x+1, ..., y\}$
$\mathbb{Z}_+$	The set $\{1, 2, 3, ...\}$
$\mathbb{N}_0$	The set $\{0, 1, 2, 3, ...\}$
$\mathbb{C}$	The set of complex numbers
$\mathbb{K}$	An algebraically closed field
$\mathscr{Z}$	An interpolation $\mathbb{K}$-grid
$S(n, k)$	The Stirling numbers of the second kind
$\mathfrak{S}_n$	The symmetric group on n elements
$(x_{(1)}, x_{(2)}, ..., x_{(n)})$	The order statistics of the sequence $(x_1, x_2, ..., x_n)$
$\Pi(E)$	The poset of all partitions π of E
Π_n	The poset of all partitions π of $[n]$
$\mathsf{LP}_{p,q}(b_1, b_2, ..., b_q)$	The set of lattice paths from $(0, 0)$ to (p, q) with strict right boundary $(b_1, b_2, ..., b_q)$
$\mathsf{PF}_n(\vec{u})$	The set of $\vec{u}$-parking functions of length n
PF_n	The set of classical parking functions of length n
$\mathsf{PS}(\vec{y}; z)$	The set of parking sequences for $(\vec{y}; z)$
$\mathsf{IPS}(\vec{y}; z)$	The set of increasing parking sequences for $(\vec{y}; z)$
$\mathsf{PS}_{inv}(\vec{y}; z)$	The set of invariant parking sequences for $(\vec{y}; z)$
$\mathsf{SPS}\{y_1, ..., y_n; z\}$	The set of strong parking sequences for the length set $\{y_1, ..., y_n; z\}$
$\mathsf{SPS}_k(n; z)$	The set of all k-strong parking sequences for n

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1. INTRODUCTION

This chapter gives an introduction to the problems that we consider in this dissertation. First, we give an introduction to our study of Gončarov polynomials in partition lattices and exponential families. Then we present an introduction to the study of parking sequences and the motivation for the associated problems we study in this regard.

1.1 Gončarov polynomials in partition lattices and exponential families

The classical Gončarov interpolation problem in numerical analysis was introduced by Gončarov [10, 11] and Whittaker[32]. It asks for a polynomial $f(x)$ of degree n such that the ith derivative of $f(x)$ at a given point a_i has value b_i for $i = 0, 1, 2, ..., n$. The solution is obtained by taking linear combinations of the (classical) Gončarov polynomials, or the Abel-Gončarov polynomials, which have been studied extensively by analysts; see e.g. [9, 10, 12, 21]. Gončarov polynomials also play a crucial role in Combinatorics due to their close relations to parking functions. The set of parking functions is central in algebraic and enumerative combinatorics, with many generalizations and connections to other research areas.

The connection between Gončarov polynomials and combinatorics was first found by Joseph Kung, who in a short note [18] of 1981 proved that classical Gončarov polynomials give the probability distribution of the order statistics of n independent uniform random variables, and its difference analog describes the order statistics of discrete, injective functions. These results were further developed in [19] to an explicit correspondence between classical Gončarov polynomials and vector parking functions. Inspired by the rich theory on delta operators and finite operator calculus, which is a unified theory on linear operators analogous to the differentiation operator D and special polynomials, Lorentz, Tringali, and Yan introduced the generalized Gončarov polynomials [22] as a basis for the solutions to the Gončarov interpolation problem with respect to a delta operator. Many algebraic and analytic properties of classical Gončarov polynomials have been extended to the generalized version.

A natural question that follows is to find the combinatorial interpretations for the generalized Gončarov polynomials. To answer this question we need to understand the combinatorial significance of delta operators. In the third paper of the seminal series *On the Foundations of Combinatorial Theory III*, Mullin and Rota [24] developed the basic theory of delta operators and their associated sequence of polynomials. Such sequences of polynomials are of binomial type and occur in many combinatorial problems when objects can be pieced together out of small, connected objects. Mullin and Rota's work provides a realization of binomial sequences in combinatorial problems. However, this realization is only valid for binomial sequences whose coefficients are non-negative integers, and so excludes many basic counting polynomials, for example, the falling factorial $x_{(n)} = x(x-1)\cdots(x-n+1)$. Mullin and Rota hint at a generalization of their theory to incorporate such cases. Using the language of partitions, partition types and partition categories, Ray [25] proved that every polynomial sequence of binomial type can be realized as a weighted enumerator in partition lattices.

At the end of their paper [22], Lorentz, Tringali and Yan remarked, among other things, that it would be interesting to investigate the role of generalized Gončarov polynomials in such weighted counting. This remark serves as the motivation for our study in order to answer the question posed earlier.

1.2 Parking Sequences

Parking sequences are objects in combinatorics whose study originates from the well-studied concept of parking functions. Parking functions were first introduced in 1966 by Konheim and Weiss [17] when they investigated the probability that a random hashing function would fill a hash table when linear probing was used to resolve collisions. One-by-one n drivers, each with a preferred parking spot attempt to park on a one-way street. If their preferred spot is unavailable, the drivers park in the first available space after their preferred one. If all drivers are able to park, the sequence of their preferences is called a parking function.

Parking functions have been well-studied in the literature. As stated earlier, they have numerous connections and have appeared in many places in algebraic and enumerative combinatorics:

chambers in Shi and braid arrangements, maximal chains in the lattice of non-crossing partitions, symmetric functions, polytope theory etc (see [29, 31, 36] and the references within). They have various specializations, applications and generalizations in other research areas such as storage problems in computer science, graph theory, interpolation theory, diagonal harmonics, representation theory, and cellular automaton. See the comprehensive survey [36] for more on the combinatorial theory of parking functions.

Some generalizations include vector parking functions, rational parking functions, parking distributions, parking on trees, subset parking functions and parking functions allowing backward movement among several others (see [4, 5, 6, 15, 16, 20, 28, 34]). The particular generalization we study in this dissertation arises from a situation where cars of various lengths attempt to park along a one-way street. These objects, called parking sequences, were introduced by Ehrenborg and Happ (see [7, 8]).

It is not difficult to see that any permutation of a parking function is also a parking function. This is however not true in the case of parking sequences. Thus, it is natural to ask how many parking sequences remain invariant under permutation. Another question is which sequence remains a parking sequence when the cars enter the street in different orders. In other words, we want to study which parking sequences remain valid when the length vector $\vec{y} = (y_1, ..., y_n)$ (y_i is the length of car C_i for each $i \in [n]$) is permuted to $(y_{\sigma(1)}, y_{\sigma(2)}, ..., y_{\sigma(n)})$ for some $\sigma \in \mathfrak{S}_n$. There are several interesting notions and variations associated with parking functions and their generalizations. Usually, these notions lead to a study of special classes of parking functions that have some interesting property. One such special class is the set of *increasing parking functions* which are counted by the ubiquitous Catalan numbers. It is only natural to ask for a generalization of this class in the set of parking sequences. This dissertation presents an initial foray into these subjects.

2. PRELIMINARIES

This chapter discusses the definitions and previous theorems appearing in the literature necessary for the following sections.

2.1 Delta Operators and Binomial Enumeration

We recall the basic theory of delta operators and their associated sequence of basic polynomials as developed by Rota, Kahaner, and Odlyzko [27]. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}[x]$ the vector space of all polynomials in the variable x over $\mathbb{K}$. For each $a \in \mathbb{K}$, let E_a denote the shift operator $\mathbb{K}[x] \to \mathbb{K}[x] : f(x) \mapsto f(x+a)$. A linear operator $\mathrm{s} : \mathbb{K}[x] \to \mathbb{K}[x]$ is called *shift-invariant* if $\mathrm{s}E_a = E_a\mathrm{s}$ for all $a \in \mathbb{K}$, where the multiplication is the composition of operators.

Definition 1. A delta operator Δ is a shift-invariant operator satisfying $\Delta(x) = a$ for some nonzero constant a.

Definition 2. A *polynomial sequence of binomial type* (or binomial sequence) is a sequence $\{p_n(x)\}_{n\geq 0}$ of polynomials such that $p_n(x)$ is of degree n and satisfies the equation

$$p_n(u+v) = \sum_{i\geq 0} \binom{n}{i} p_i(u)p_{n-i}(v), \tag{2.1}$$

for all $n \geq 0$.

Definition 3. Let Δ be a delta operator. A polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is called the sequence of basic polynomials, or the associated basic sequence of Δ if

(i) $p_0(x) = 1$;

(ii) Degree of $p_n(x)$ is n and $p_n(0) = 0$ for each $n \geq 1$;

(iii) $\Delta(p_n(x)) = np_{n-1}(x)$.

The following is a result due to Mullin and Rota [24].

Delta operators	Basic Polynomials
D	The standard power polynomials x^n
$E_1 - I$	lower factorial $x(x-1)(x-2)\cdots(x-n+1)$
$I - E_{-1}$	rising factorial $x(x+1)(x+2)\cdots(x+n-1)$
$E_a - E_b$	Gould polynomials $x\prod_{i=1}^{n-1}(x-ia-(n-i)b)$
E_aD	Abel Polynomials $x(x-na)^{n-1}$
$log(I+D)$	Bell Polynomials $b_n(x) = \sum_{k=1}^{n} S(n,k)x^k$
$D(D-I)^{-1}$	Laguerre Polynomials $L_n(x) = \sum_{k\geq 0}\frac{n!}{k!}\binom{n-1}{k-1}(-x)^k$

Table 2.1: Some delta operators and their associated basic polynomials

Theorem 1 ([24]). *Every delta operator has a unique sequence of basic polynomials, which is a sequence of binomial type. Conversely, every polynomial sequence of binomial type is the associated basic sequence of some delta operator.*

In Table 2.1, we show some examples of delta operators and their associated basic polynomials.

Let s be a shift-invariant operator, and Δ a delta operator. Then s can be expanded uniquely as a formal power series of Δ. If

$$\mathrm{s} = \sum_{k\geq 0}\frac{a_k}{k!}\Delta^k,$$

we say that $f(t) = \sum_{k\geq 0}\frac{a_k}{k!}t^k$ is the Δ-indicator of s. In fact, the correspondence

$$f(t) = \sum_{k\geq 0}\frac{a_k}{k!}t^k \longleftrightarrow \sum_{k\geq 0}\frac{a_k}{k!}\Delta^k$$

is an isomorphism from the ring $\mathbb{K}[[t]]$ of formal power series in t onto the ring of shift-invariant operators. Under this isomorphism, a shift-invariant operator is invertible if and only if its Δ-indicator $f(t)$ satisfies $f(0) \neq 0$, and it is a delta operator if and only if $f(0) = 0$ and $f'(0) \neq 0$, i.e., $f(t)$ has a compositional inverse $g(t)$ satisfying $f(g(t)) = g(f(t)) = t$.

Another important result is the generating function for the sequence of basic polynomials $\{p_n(x)\}_{n\geq 0}$ associated to a delta operator Δ. Let $f(t)$ be the D-indicator of Δ, where $D = d/dx$

is the differentiation operator. Let $g(t)$ be the compositional inverse of $f(t)$. Then,

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = \exp\left(xg(t)\right). \tag{2.2}$$

The operator $\Lambda = g(D)$ is called the *conjugate delta operator* of Δ, and $\{p_n(x)\}_{n\geq 0}$ is the *conjugate sequence* of Λ. It is easy to see that if $p_n(x) = \sum_{k\geq 1} p_{n,k}x^k$, then $g(t) = \sum_{k\geq 1} p_{k,1}\frac{t^k}{k!}$.

For example, when $p_n(x) = x^n$ (the power polynomials), the corresponding delta operator $\Delta = D$ has $f(t) = t$ as its D-indicator and $\Lambda = D$ as the conjugate delta operator. The exponential generating function given by (2.2) is

$$\sum_{n\geq 0} x^n\frac{t^n}{n!} = \exp\left(xt\right).$$

Another example is when $p_n(x) = x^{(n)}$ (the rising factorials). In this case, the corresponding delta operator is $\Delta = I - E_{-1}$ where I is the identity operator. The D-indicator is $f(t) = 1 - \exp(-t)$ and $\Lambda = \ln(I - D)^{-1}$ is the conjugate delta operator of Δ. The exponential generating function given by (2.2) is

$$\sum_{n\geq 0} x^{(n)}\frac{t^n}{n!} = (1-t)^{-x}.$$

Polynomial sequences of binomial type are closely related to the theory of binomial enumeration. Consider the following model. Assume $\mathcal{B}$ is a family of discrete structures. For a finite set E, let $\Pi(E)$ be the poset of all partitions π of E, ordered by refinement, and write $|\pi|$ for the number of blocks of π.

Definition 4. A *k-assembly of $\mathcal{B}$-structures on E* is a partition π of the set E into $|\pi| = k$ blocks such that each block of π is endowed with a $\mathcal{B}$-structure.

Let $B_k(E)$ denote the set of all such k-assemblies. For example, when $\mathcal{B}$ is a set of rooted trees, a k-assembly of $\mathcal{B}$-structures on E is a forest of k rooted trees with vertex set E. We can also take $\mathcal{B}$ to be other structures, such as permutations, complete graphs, posets, etc. Assume that the cardinality of $B_k(E)$ depends only on the cardinality of E, but not its content. In other words, there is a bijection between $B_k(E)$ and $B_k([n])$ where $[n] = \{1, 2, ..., n\}$ and $|E| = n$.

Definition 5. Let

$$b_{n,k} = \begin{cases} |B_k([n])|, & \text{if } k \leq n \\ 0, & \text{if } k > n, \end{cases}$$

where $b_{0,0} = 1$ and $b_{n,0} = 0$ for $n \geq 1$.

Theorem 2 ([24])**.** *Assume $b_{1,1} \neq 0$. If $b_n(x) = \sum_{k=1}^{n} b_{n,k} x^k$ is the enumerator for assemblies of $\mathcal{B}$-structures on $[n]$, then $(b_n(x))_{n \geq 0}$ is a sequence of polynomials of binomial type.*

Theorem 2 provides a realization of binomial sequences in combinatorial problems. If we think of x as a positive integer such that $|X| = x$ for some set X, then we can interpret $b_n(x)$ as the number of assemblies of $\mathcal{B}$-structures on $[n]$, where each block carries a label from X. From this viewpoint, it is easy to see that $(b_n(x))_{n \geq 0}$ is of binomial type.

This realization is only valid for binomial sequences whose coefficients are non-negative integers, and so excludes many polynomial sequences naturally appearing in combinatorics, for example, the falling factorials $x_{(n)}$. Mullin and Rota [24] expanded their construction slightly by considering the *monomorphic classes*, in which different blocks receive different labels from X, and hence the counting polynomial becomes $\tilde{b}_n(x) = \sum_{k=1}^{n} b_{n,k} x_{(k)}$. Ray [25] extended Mullin-Rota's theory and developed the concept of partition categories, and he proved that any binomial sequence can be realized as a weight enumerator in partition lattices. We will use Ray's model later on in Section 3.

2.2 Numerical Interpolation

We define the concept of interpolation in numerical analysis and specifically discuss the Gončarov interpolation theory and its generalization which is mentioned later on in this dissertation.

Numerical interpolation is a type of estimation. It is a method of constructing new numerical data points within the range of a discrete set of given data points. A well known interpolation method which yields Newton polynomials is called Newton interpolation. Hermite interpolation is another method of interpolating data points as a polynomial function. However,unlike Newton interpolation, Hermite interpolation matches an unknown function both in observed value, and the observed value of its first n derivatives.

2.2.1 Gončarov interpolation

The following interpolation problem is a special case of Hermite interpolation.

Gončarov interpolation problem: Given two sequences of real or complex numbers $a_0, a_1, ..., a_n$ and $d_0, d_1, ..., d_n$, find a polynomial $p(x)$ of degree n such that for each i, $0 \leq i \leq n$, the i-th derivative $p^{(i)}(x)$ evaluated at a_i equals d_i.

Gončarov polynomials arise as a natural basis for solving this interpolation problem. A special case of this is Abel interpolation, where the point a_i is the integer i. The Gončarov polynomials in this case are the Abel polynomials. One of the motivations for the paper [19] was the appearance of Abel polynomials in both the enumeration of parking functions and in Abel interpolation.

2.2.2 Generalized Gončarov Polynomials

Let $\mathcal{Z} = (z_i)_{i \geq 0}$ be a fixed sequence with values in $\mathbb{K}$, where $\mathbb{K}$ is a scalar field. For our purpose, it suffices to take $\mathbb{K}$ to be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. We call $\mathcal{Z}$ the interpolation grid and $z_i \in \mathcal{Z}$ the i-th interpolation node. Let $\mathcal{T} = (t_n(x; \Delta, \mathcal{Z}))_{n \geq 0}$ be the unique sequence of polynomials that satisfies

$$\varepsilon_{z_i} \Delta^i (t_n(x; \Delta, \mathcal{Z})) = n! \delta_{i,n}, \tag{2.3}$$

where ε_{z_i} is evaluation at z_i.

Definition 6. The polynomial sequence $\mathcal{T} = (t_n(x; \Delta, \mathcal{Z}))_{n \geq 0}$ determined by (2.3) is called the *sequence of generalized Gončarov polynomials* associated with the pair $(\Delta, \mathcal{Z})$ and $t_n(x; \Delta, \mathcal{Z})$ is the n-th generalized Gončarov polynomial relative to the same pair.

For example, in the classical case where $\Delta = D$, we have

$$t_0(x; D, \mathcal{Z}) = 1,$$

$$t_1(x; D, \mathcal{Z}) = x - a_0,$$

$$t_2(x; D, \mathcal{Z}) = x^2 - 2a_1 x + 2a_0 a_1 - a_0^2,$$

$$t_3(x; D, \mathcal{Z}) = x^3 - 3a_2 x^2 + (6a_1 a_2 - 3a_1^2)x - a_0^3 + 3a_0^2 a_2 - 6a_0 a_1 a_2 + 3a_0 a_1^2.$$

In general, this sequence $\mathcal{T}$ has a number of interesting algebraic properties. One of them is a recurrence formula described as follows: Let $t_n(x) = t_n(x; \Delta, \mathcal{Z})$ and $\{p_n(x)\}_{n\geq 0}$ be the basic sequence associated to Δ. Then

$$p_n(x) = \sum_{i=0}^{n} \binom{n}{i} p_{n-i}(z_i)\, t_i(x). \tag{2.4}$$

We remark that by definition, to compute the generalized Gončarov polynomials given the basic sequence, one would find the conjugate operator Λ via (2.2), compute Δ by solving for the compositional inverse of the D-indicator of Λ, and then find the n-th polynomial $t_n(x)$ of the sequence by using (2.3). The computation required in this process can be quite involved. However, (2.4) gives a recursive formula which can be used as an alternative definition for $t_n(x)$, which is much more convenient in combinatorial problems. For other algebraic properties of generalized Gončarov polynomials, see [22].

2.3 Parking Functions

Here we discuss both classical parking functions as well as vector parking functions, which are a well-studied generalization of the former.

2.3.1 Classical Parking Functions

Parking functions may be described as a sequence of drivers, each with a preferred parking space, searching for a place to park along a one-way street. Let $\mathrm{s} = (s_1, ..., s_n) \in [n]^n$ be a sequence of preferences and consider a street with n spots.

Parking Process: The drivers attempt to park according to the following process.

1) Beginning with $i = 1$, driver i begins at vertex s_i.

2) If the current spot is unoccupied, the driver parks there. If it is occupied and not the last spot on the street, the driver drives to the next spot ahead of the current one and repeats Step 2.

3) If driver i parks, the process continues with driver $i + 1$ attempting to park at vertex s_{i+1}. Otherwise, the process terminates.

If all n drivers parks, the sequence s is called a *parking function*. A more formal definition for parking functions can be stated as follows.

Definition 7. Let $\mathrm{s} = (s_1, s_2, ..., s_n)$ be a sequence of positive integers, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of s . Then the sequence s is a parking function if and only if $b_i \leq i$ for all indices i. Equivalently, s is a parking function if and only if, for all $i \in [n]$, we have

$$\#\{j : a_j \leq i\} \geq i. \tag{2.5}$$

For example, the preferences (1, 2, 3, 4), (2, 1, 3, 4) or (1, 2, 4, 1) all correspond to parking functions, while (2, 2, 4, 2) will have one car leave un-parked so is not a parking function. It is well-known that the number of classical parking functions is $(n + 1)^{n-1}$. An elegant proof by Pollak (see [26]) uses a circle with $(n + 1)$ spots where the parking functions are the preferences that could park all n cars without using the $(n + 1)$-th spot.

2.3.2 Vector parking functions

As earlier stated, parking functions have several generalizations. A well-known generalization are the vector parking functions (or $\vec{u}$-parking functions). Here the street is of length $x \geq n$, and we want to park n cars on this street with specified constraints. Let $\vec{u}$ be a non-decreasing sequence $(u_1, u_2, u_3, ...)$ of positive integers. We have the following definition.

Definition 8. A $\vec{u}$-parking function of length n is a sequence $(x_1, x_2, ..., x_n)$ of positive integers satisfying

$$\#\{j : x_j \leq u_i\} \geq i. \tag{2.6}$$

Classical parking functions correspond to the case that $u_i = i$. We denote the set of all $\vec{u}$-parking functions of length n by $\mathsf{PF}_n(\vec{u})$. In the special case where $u_i = a + b(i-1)$ for some $a, b \in \mathbb{Z}_+$, it is known that the number of such $\vec{u}$-parking functions is $a(a+bn)^{n-1}$; see e.g. [19]. We denote the set of all $\vec{u}$-parking functions of length n by $\mathsf{PF}_n(\vec{u})$.

2.4 Lattice Paths

A North-East (NE) lattice path is a lattice path in $\mathbb{Z}^2$ with steps $(0,1)$ and $(1,0)$. The $(0,1)$ steps are called North steps and denoted by N; the $(1,0)$ steps are called East steps and denoted by E. In this section, we discuss some lattice paths closely associated with parking functions called Dyck paths and lattice paths with strict right boundary.

2.4.1 Dyck Paths

A *Dyck path of semilength* n is a lattice path in $\mathbb{Z}^2$ with steps N $= (0,1)$ and E$= (1,0)$ starting from $(0,0)$ and ending at (n,n) such that the path never dips below the line $y = x$. We can also represent a Dyck path as a word of length $2n$ with letters $\{N, E\}$. Figure 2.1 gives an example of a Dyck path.

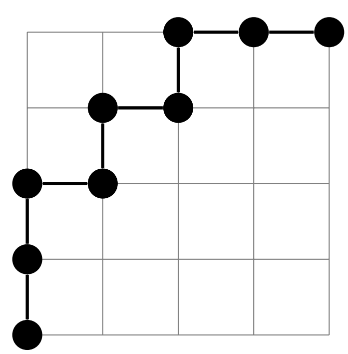

Figure 2.1: The Dyck path NNENENEE

The number of Dyck paths of semi-length n is well-known to be $C_n = \frac{1}{n+1}\binom{2n}{n}$. The numbers $\{C_n\}_{n \geq 0}$ are called the Catalan numbers and appear throughout combinatorics. One way to generalize the definition of Dyck path is to change the end point of a Dyck path. In this context, a generalized Dyck path is a lattice path from $(0, 0)$ to (kn, n) in $\mathbb{Z}^2$ which is below the diagonal line $x = ky$. The number of such paths is the Fuss-Catalan number $\frac{1}{kn+1}\binom{(k+1)n}{n}$.

We will also consider a special class of parking functions.

Definition 9 (Increasing Parking Functions)**.** A parking function $\mathbf{s} = (s_1, ..., s_n) \in [n]^n$ is *increasing* if $s_i \leq s_{i+1}$ for $1 \leq i \leq n - 1$.

There are C_n increasing parking functions of length n(see [30], problem 5.49).

Parking functions are closely connected to Dyck paths. As their count suggests, increasing parking functions are in bijection with Dyck paths. One bijection is, given a Dyck path, the number of drivers preferring i is the number of N-steps immediately before the i^{th} E-step. For example, the Dyck path in Figure 2.1 corresponds to the parking function $\mathbf{s} = (1, 1, 2, 3)$. Classical parking functions are in bijection with Dyck paths whose runs of N-steps are labeled by subsets of $[n]$. Given a parking function $\mathbf{s}'$, the corresponding Dyck path has a run of $\#\{i : s'_i = j\}$ N-steps labeled with the set $\{i : s'_i = j\}$ immediately before the j^{th} E-step. For example, Figure 2.2 corresponds to the parking function $\mathbf{s}' = (2, 1, 3, 1)$.

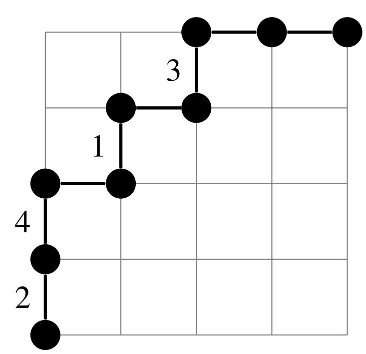

Figure 2.2: The labeled Dyck path $(2, 1, 3, 1)$

2.4.2 Lattice Paths with strict right boundary

A *lattice path with strict right boundary* is a more general lattice path. A lattice path with strict right boundary $(b_1, b_2, ..., b_q)$ from $(0, 0)$ to (p, q) can be represented by a sequence $(x_1, x_2, ..., x_q)$ of p east steps and q north steps on the integer lattice such that $0 \leq x_i < b_i$ for all $1 \leq i \leq q$. The north steps are at $(x_i, i-1) \rightarrow (x_i, i)$, for $i = 1, ..., q$. Figure 2.3 shows an example of a lattice path (2,3,3,7) from (0,0) to (8,4) with strict right boundary $\vec{b} = (3, 4, 5, 8)$ (the strict boundary is indicated by the blue dots).

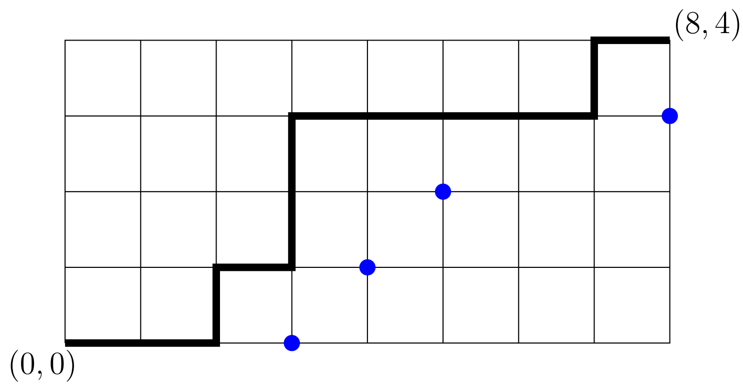

Figure 2.3: A lattice path $(2, 3, 3, 7)$ with strict right boundary at $(3, 4, 5, 8)$.

2.5 Parking Sequences

Ehrenborg and Happ [7, 8] recently introduced this generalization of parking functions. This time the car C_i has length $y_i \in \mathbb{Z}_+$ for each $i = 1, 2, ..., n$. Call $\vec{y} = (y_1, y_2, ..., y_n)$ the length vector. There is a trailer of length $z - 1$ parked at the beginning of the street after which the n cars park with each car taking up a number of adjacent parking spaces.

Definition 10 (Parking Process)**.** The drivers attempt to park according to the following process: Given a sequence $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{Z}_+^n$,

1) Beginning with $i = 1$, car C_i begins at vertex c_i, looking for the first empty spot $j \geq c_i$.

2) If the spaces j through $j + y_i - 1$ are empty, then the car parks in these spots. If any of the spots $j + 1$ through $j + y_i - 1$ is already occupied, then there will be a collision and the car cannot park and has to leave the street. In this case, we say the parking fails.

3) If car C_i parks, the process continues with car C_{i+1} attempting to park at vertex c_{i+1} repeating steps 1 and 2. Otherwise, the process terminates.

Thus, we have the following definition.

Definition 11. Assume there are $z - 1 + \sum_{i=1}^{n} y_i$ parking spots along a street, with the first $z - 1$ occupied by a trailer. The sequence $\mathbf{c} = (c_1, ..., c_n)$ is called a *parking sequence for* $(\vec{y}, z)$ where $\vec{y} = (y_1, ..., y_n)$ if all n cars can park without any collisions. We denote the set of all such parking sequences by $\mathsf{PS}(\vec{y}; z)$.

As given in [8], the number of parking sequences in $\mathsf{PS}(\vec{y}; z)$ is:

$$z \cdot (z + y_1 + n - 1) \cdot (z + y_1 + y_2 + n - 2) \cdots (z + y_1 + \cdots + y_{n-1} + 1). \tag{2.7}$$

In the special case where there is no trailer i.e. when $z = 1$, (2.7) yields the number

$$(y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_{n-1} + 2). \tag{2.8}$$

We will write $\mathsf{PS}(\vec{y}; 1) = \mathsf{PS}(\vec{y})$. It is clear that this is a generalization of parking functions, since when we set the size of all cars equal to 1 i.e. $y_i = 1$ for all $i = 1, 2, ..., n$, we obtain the number $(n + 1)^{n-1}$ which is the number of classical parking functions.

2.6 Partition Lattices

For any finite set S, let $\Pi(S)$ denote the set of all partitions of S, and write Π_n for $\Pi([n])$. Elements of $\Pi(S)$ are partially ordered by refinement: that is, define $\pi \leq \sigma$ if every block of π is contained in a block of σ. In particular, $\Pi(E)$ has a unique maximal element $\hat{1}$ that has only one block and a unique minimal element $\hat{0}$ for which every block is a singleton. Let $|\pi|$ be the number of blocks of π and $\Pi(\pi)$ be the partitions of the set that consists of blocks of π. When $\pi \leq \sigma$, the *induced partition* σ/π is the partition σ viewed as an element of $\Pi(\pi)$. Define the *class* of (π, σ) as the sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of non-negative integers such that λ_i is the number of blocks of size i in the partition σ/π, for $1 \leq i \leq |\pi|$. It follows that

$$\sum_{i\geq 1} i\lambda_i = |\pi| \qquad \text{and} \qquad \sum_{i\geq 1} \lambda_i = |\sigma|.$$

Example 1. Let $E = [8]$, $\pi = \{1\}, \{2\}, \{345\}, \{67\}, \{8\}$, $\sigma = \{1345\}, \{2\}, \{678\} \in \Pi_8$. Then, $\sigma/\pi = \{(1), (345)\}, \{(2)\}, \{(67), (8)\} \in \Pi(\pi)$. The class of (π, σ) is $\lambda = (1, 2, 0, 0, ...)$, where we have $\sum_{i\geq 1} i\lambda_i = |\pi| = 5$ and $\sum_{i\geq 1} \lambda_i = |\sigma| = 3$. □

2.7 Incidence algebra

We recall the basic notation in incidence algebra. Let P be a finite poset and A a commutative ring with unity. Denote by $Int(P)$ the set of all intervals of P, i.e., the set $\{(x, y) : x \leq y\}$. The *incidence algebra* $I(P, A)$ of P over A is the A-algebra of all functions

$$f : Int(P) \to A,$$

where multiplication is defined via the convolution

$$fg(x, y) = \sum_{x\leq z\leq y} f(x, z)g(z, y).$$

The algebra $I(P, A)$ is associative with identity δ, where

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

An element $f \in I(P, A)$ is invertible under the multiplication if and only if $f(x, x)$ is invertible in A for every $x \in P$.

In this dissertation, our focus will be the case $P = \Pi_n$, the partition lattice of $[n]$.

2.8 Formal Power Series

For a function $f : \mathbb{N}_0 \to \mathbb{C}$ we associate the formal power series $F(x) = \sum_{n\geq 0} f(n)x^n$, called the *ordinary generating function* of f. Rather than using the functional notation, we may write $f(n) = f_n = [x^n]F(x)$, depending on which is convenient. If we use generating functions to count an object, sometimes adjustment in the generating function proves useful. We say $\hat{F}(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ is the *exponential generating function* of f. While factors other than $1/n!$ appear, they are significantly less common.

2.9 Exponential Families

Exponential families are combinatorial models based on the partition lattices where the enumeration is captured by the exponential generating functions. The description of exponential families and their relation to the incidence algebra of Π_n can be found in standard textbooks, e.g., [30, Section 5.1]. Here we adopt Wilf's description of exponential families [33] in the context of 'playing cards' and 'hands'.

Suppose that there is given an abstract set P of 'pictures', which typically are the connected structures. A *card* $\mathcal{C}(S, p)$ is a pair consisting of a finite label set S of positive integers and a picture $p \in P$. The weight of $\mathcal{C}$ is $|S|$. If $S = [n]$, the card is called *standard*. A *hand* H is a set of cards whose label sets form a partition of $[n]$ for some n. The weight of a hand is the sum of the weights of the cards in the hand. The *n-th deck* $\mathcal{D}_n$ is the set of all standard cards of weights n. We require that $\mathcal{D}_n$ is always finite. An *exponential family* $\mathcal{F}$ is the collection of decks $\mathcal{D}_1, \mathcal{D}_2, \ldots$.

In an exponential family, let $d_i = |\mathcal{D}_i|$ and $h_{n,k}$ be the number of hands H of weight n that consist of k cards. Let $h_0(x) = 1$ and for $n \geq 1$,

$$h_n(x) = \sum_{k=1}^{n} h_{n,k}x^k. \tag{2.9}$$

Then the main counting theorem, *the exponential formula*, states that these polynomials satisfy the generating relation (alternatively, $\{h_n(x)\}_{n\geq 0}$ has the exponential generating function):

$$e^{xD(t)} = \sum_{n\geq 0} h_n(x)\frac{t^n}{n!}, \tag{2.10}$$

where $D(t) = \sum_{k\geq 1} d_k t^k/k!$. In other words, if $d_1 = h_{1,1} \neq 0$, $\{h_n(x)\}_{n\geq 0}$ is a sequence of binomial type that is conjugate to the delta operator $\sum_{k\geq 1} d_k D^k/k!$.

Example 2. *Set Partitions*: Here, a card is a label set $[n]$ with a 'picture' of n dots. Each deck $\mathcal{D}_n$ consists of the single card of weight n, and a hand is just a partition of the set $[n]$. Thus, $h_{n,k}$ is the number of partitions of the set $[n]$ into k classes, which is $S(n, k)$, the Stirling number of the second kind. □

Example 3. *Permutations and their Cycles*: Each card is a cyclic permutation on a label set S. The deck $\mathcal{D}_n$ consists of all distinct cyclic permutations on $[n]$ so $d_n = (n-1)!$ and a hand is a permutation of $[n]$ consisting of k cycles. Thus, $h_{n,k}$ is the number of permutations on $[n]$ that have k cycles, that is, the signless Stirling number of the first kind $c(n, k)$. □

Note that we can interpret x^k in $h_n(x)$ as the number of maps from the set of cards in a hand to the set $X = \{1, \ldots, x\}$ for some positive integer x. Hence $h_n(x)$ counts the number of hands of weight n in which each card is labeled by an element of X. This set-up gives a natural combinatorial interpretation for binomial polynomial sequences whose coefficients are positive integers.

In [24], Mullin and Rota introduced a structure called *reluctant functions*, which can be used to give a combinatorial interpretation for some generalized Gončarov polynomials; see [22].

Definition 12. Let S and X be finite disjoint sets. A *reluctant function from S to X* is a function $f : S \to S \cup X$, such that for every $s \in S$ there is a positive integer $k = k(s)$ with $f^k(s) \in X$.

The resulting partition that arises from this construction has a natural combinatorial structure of a rooted forest with its rooted trees belonging to some binomial class $\mathcal{B}$. The concept of exponential families generalizes reluctant functions since for each such binomial class $\mathcal{B}$, a card consists of all $s \in S$ such that $f^k(s) = x_i$ $(k > 0)$ for some fixed $x_i \in X$. The picture on a card is the underlying tree structure of $\mathcal{B}$. The deck D_n consists all trees on n nodes of the particular family $\mathcal{B}$, and a hand is a forest of rooted trees, where each such tree is in $\mathcal{B}$. Thus, $h(n, k)$ is the number of forests on n vertices consisting of k rooted trees, each of type $\mathcal{B}$.

3. GONČAROV POLYNOMIALS IN PARTITION LATTICES[1]

This chapter is an expansion of Section 3 of [3]. As earlier stated, we are concerned with the case $P = \Pi_n$, the partition lattice of $[n]$, and $A = \mathbb{K}[w_2, w_3, \ldots]$, where $w_2, w_3, \ldots$ are independent variables. In addition, we set $w_1 = 1$.

Definition 13. Assume $\pi \leq \sigma$ in Π_n and the class of (π, σ) is $\lambda = (\lambda_1, \lambda_2, \ldots)$. Define the *zeta-type function* $w(\pi, \sigma) \in I(\Pi_n, A)$ by letting

$$w(\pi, \sigma) = w_1^{\lambda_1} w_2^{\lambda_2} \ldots w_{|\pi|}^{\lambda_{|\pi|}}. \tag{3.1}$$

Note that $w(\pi, \pi) = 1$ for all π. Hence w is invertible. The inverse of w is called the *Möbius-type function* and denoted by μ^w. Explicitly, $\mu^w(\pi, \pi) = 1$ and for $\pi < \sigma$,

$$\mu^w(\pi, \sigma) = -\sum_{\pi \leq \tau < \sigma} \mu^w(\pi, \tau) w(\tau, \sigma).$$

When all $w_i = 1$, the zeta-type function and the Möbius-type function become the zeta function and the Möbius function of Π_n respectively.

Example 4. Consider the lattice Π_3. Then for all $\pi < \sigma$, $w(\pi, \sigma) = w_2$ except that $w(\hat{0}, \hat{1}) = w_3$. Consequently, $\mu^w(\pi, \sigma) = -w_2$ if $\pi < \sigma$ except that $\mu^w(\hat{0}, \hat{1}) = 3w_2^2 - w_3$. □

Define the *zeta-type enumerator* $\{a_n(x; w)\}_{n \geq 0}$ and *Möbius-type enumerator* $\{b_n(x; w)\}_{n \geq 0}$ as follows. Let $a_0(x; w) = b_0(x; w) = 1$ and for $n \geq 1$,

$$a_n(x; w) = \sum_{\pi \in \Pi_n} w(\hat{0}, \pi)\, x^{|\pi|}, \tag{3.2}$$

$$b_n(x; w) = \sum_{\pi \in \Pi_n} \mu^w(\hat{0}, \pi)\, x^{|\pi|}. \tag{3.3}$$

[1]Reprinted with permission from "Gončarov Polynomials in Partition Lattices and Exponential families" by A. Adeniran and C. H. Yan, 2020. *Advances in Applied Mathematics*, https://doi.org/10.1016/j.aam.2020.102045,

Theorem 3 ([25]). *1. The polynomial sequences $\{a_n(x;w)\}_{n\geq 0}$ and $\{b_n(x;w)\}_{n\geq 0}$ are of binomial type.*

2. Let Λ be the delta operator whose D-indicator is given by $g(t) = t + \sum_{i\geq 2} w_i t^i/i!$. Then $\{a_n(x;w)\}_{n\geq 0}$ is the conjugate sequence of Λ and $\{b_n(x;w)\}_{n\geq 0}$ is the basic sequence of Λ.

For $n = 0, 1, \ldots, 4$, the polynomials $a_n(w, x)$ and $b_n(w, x)$ are

$$\begin{aligned}
a_0(x;w) &= 1,\\
a_1(x;w) &= x,\\
a_2(x;w) &= x^2 + w_2 x,\\
a_3(x;w) &= x^3 + 3w_2x^2 + w_3x,\\
a_4(x;w) &= x^4 + 6w_2x^3 + (4w_3 + 3w_2^2)x^2 + w_4x,
\end{aligned}$$

and

$$\begin{aligned}
b_0(x;w) &= 1,\\
b_1(x;w) &= x,\\
b_2(x;w) &= x^2 - w_2 x,\\
b_3(x;w) &= x^3 - 3w_2x^2 + (3w_2^2 - w_3)x,\\
b_4(x;w) &= x^4 - 6w_2x^3 + (15w_2^2 - 4w_3)x^2 + (10w_2w_3 - w_4 - 15w_2^3)x.
\end{aligned}$$

The linear coefficient in $b_n(w;x)$ is $\mu_n^w = \mu^w(\hat{0}, \hat{1})$ in Π_n. Assume Δ is the conjugate delta operator of Λ. Then $\{a_n(x;w)\}_{n\geq 0}$ is the basic sequence of Δ and $\{b_n(x;w)\}_{n\geq 0}$ is the conjugate sequence of Δ. The operator Δ can be written as $\Delta = \sum_{n\geq 1} \mu_n^w D^n/n!$. Since $w_1 = 1$, each μ_n^w is a polynomial of $w_2, w_3, \ldots$. If we take w_1 to be a variable, μ_n^w would be a polynomial in $w_1^{-1}, w_2, w_3, \ldots$.

The condition $w_1 = 1$ is equivalent to the equation $a_1(x;w) = x$. Since the weight variables

$w_2, w_3, \ldots$ can take arbitrary values, Theorem 3 implies that any polynomial sequence $\{p_n(x)\}_{n\geq 0}$ of binomial type with $p_1(x) = x$ can be realized as the zeta-type weight enumerator or the Möbius-type weight enumerator over partition lattices. Note that for any scalar $k \neq 0$, if a sequence $\{p_n(x)\}_{n\geq 0}$ is the basic sequence of Δ and the conjugate sequence of Λ, then $\{p_n/k^n\}_{n\geq 0}$ is the basic sequence of $k\Delta$ and the conjugate sequence of $g(D/k)$ where $g(t)$ is the D-indicator of Λ. Hence, Theorem 3 covers all polynomial sequences of binomial type up to a scaling.

In the problem of counting assemblies of $\mathcal{B}$-structures outlined in Section 2.1, the enumerator $\sum_k b_{n,k}x^k$ in Theorem 2 is a specialization of the polynomial $a_n(x; w)$, where w_n is the number of $\mathcal{B}$-structures on a block of size n. For example, when $\mathcal{B}$ is the set of rooted trees, $w_n = n^{n-1}$ and hence $a_n(x; w) = x(x + n)^{n-1}$, the n-th Abel polynomial.

3.1 Gončarov polynomials in partition lattices

Our objective is to fit the generalized Gončarov polynomials into this model and present a combinatorial interpretation in terms of weight-enumeration in partition lattices. Following the notation of Theorem 3, let Δ be the conjugate delta operator of Λ. Given an interpolation grid $\mathcal{Z}$, we denote by $t_n(x; w, \mathcal{Z})$ the n-th generalized Gončarov polynomial relative to the pair $(\Delta, \mathcal{Z})$. We use this notation to emphasize the role of the zeta-type function $w(\pi, \sigma)$.

To get a formula for the polynomial $t_n(x; w, \mathcal{Z})$, we use the recurrence (2.4) in Section 2.2.2. Since $a_n(x; w)$ is the basic sequence of Δ, $\{t_n(x; w, \mathcal{Z})\}_{n\geq 0}$ is the unique sequence of polynomials that satisfies the recurrence

$$a_n(x; w) = \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(z_i; w)\, t_i(x; w, \mathcal{Z}). \tag{3.4}$$

In other words,

$$t_n(x; w, \mathcal{Z}) = a_n(x; w) - \sum_{i=0}^{n-1} \binom{n}{i} a_{n-i}(z_i; w)\, t_i(x; w, \mathcal{Z}). \tag{3.5}$$

In particular, $t_0(x; w, \mathcal{Z}) = 1$ and $t_1(x; w, \mathcal{Z}) = a_1(x; w) - a_1(z_0; w) = x - z_0$. Here we again assume $w_1 = 1$ and hence $a_1(x; w) = x$. Since if Δ is changed to $k\Delta$, the corresponding $t_n(x; w, \mathcal{Z})$ just changes to $t_n(x; w, \mathcal{Z})/k^n$, again we cover all the cases up to a scaling.

Assume x is a positive integer and $X = \{1, 2, \ldots, x\}$. Then $a_n(x; w)$ is the zeta-type weight enumerator of all the block-labeled partitions, where each block of the partition carries a label from X. In symbols,

$$a_n(x; w) = \sum_{\pi \in \Pi_n} w(\hat{0}, \pi) \cdot |\{f : \mathrm{Block}(\pi) \to X\}|,$$

where $\mathrm{Block}(\pi)$ is the set of blocks of π. For a partition π with a block-labeling f, we record the labeling by the list $f_\pi = (x_1, x_2, \ldots, x_n)$, where $x_i = f(B_j)$ whenever i is in the block B_j of π.

Let $\vec{z} = (z_0, z_1, \cdots, z_{n-1})$ be the initial segment of the grid $\mathcal{Z}$. Furthermore, assume that $z_0 \leq z_1 \cdots \leq z_{n-1}$ are positive integers with $z_{n-1} < x$.

Define the set $\mathcal{PF}_\pi(\mathcal{Z})$ as the set of all block-labelings of π that are also $\vec{z}$-parking functions, i.e.,

$$\mathcal{PF}_\pi(\mathcal{Z}) = \{f : \mathrm{Block}(\pi) \to X \mid f_\pi \text{ is a } \vec{z}\text{-parking function}\}. \tag{3.6}$$

More precisely, $\mathcal{PF}_\pi(\mathcal{Z})$ is the set of block-labelings of π such that the order statistics of $f_\pi = (x_1, x_2, \ldots, x_n)$ satisfies $x_{(i)} \leq z_{i-1}$ for $i = 1, \ldots, n$. Let $PF_\pi(\mathcal{Z})$ be the cardinality of $\mathcal{PF}_\pi(\mathcal{Z})$.

Our main result of this section is the following theorem.

Theorem 4. *Assume $t_n(x; w, \mathcal{Z})$ is the n-th generalized Gončarov polynomial defined by* (3.4) *with a positive increasing integer sequence $\mathcal{Z} = (z_0, z_1, ...)$. Let x be an integer larger than z_{n-1}. Then,*

$$t_n(0; \omega, -\mathcal{Z}) = t_n(x; \omega, x - \mathcal{Z}) = \sum_{\pi \in \Pi_n} w(\hat{0}, \pi) \cdot PF_\pi(\mathcal{Z}), \tag{3.7}$$

where $x - \mathcal{Z} = (x - z_0, x - z_1, x - z_2, \ldots)$ and $-\mathcal{Z} = (-z_0, -z_1, -z_2, \ldots)$.

The first equality follows from [22, Prop.3.5] that was proved by verifying the defining equation (2.3), and the second equality follows from the recurrence (3.4) and Lemma 1 proved next. Note that all three parts of (3.7) are polynomials of $z_0, z_1, \ldots, z_{n-1}$, hence (3.7) is a polynomial identity.

Lemma 1. *For every $n \geq 0$, it holds that*

$$a_n(x;w) = \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{\pi \in \Pi_i} w(\hat{0}, \pi) \cdot PF_\pi(\mathcal{Z}) \tag{3.8}$$

Proof. Again we assume that x and z_i are positive integers and $z_0 < z_1 < \cdots < z_{n-1} < x$. For a finite set E and P, let $\mathcal{S}(E, P)$ be the set of pairs (π, f) where π is a partition of the set E and f is a function from $\mathrm{Block}(\pi)$ to P. Then the left-hand side of (3.8) counts the set $\mathcal{S}([n], X)$ by the zeta-type weight function $w(\hat{0}, \pi)$. Note that if π has blocks $B_1, B_2, \ldots, B_k$, then

$$w(\hat{0}, \pi) = \prod_{j=1}^{k} w_{|B_j|}.$$

For a pair $(\pi, f) \in \mathcal{S}([n], X)$ with $f_\pi = (x_1, x_2, \ldots, x_n)$, let $\mathbf{inc}(f_\pi) = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ be the non-decreasing rearrangement of the terms of f_π. Set

$$i(f) = \max\{k : x_{(j)} \leq z_{j-1} \, \forall j \leq k\}.$$

Thus, the maximality of $i = i(f)$ means that

$$x_{(1)} \leq z_0, x_{(2)} \leq z_1, \ldots, x_{(i)} \leq z_{i-1}$$

and

$$z_i < x_{(i+1)} \leq x_{(i+2)} \leq \cdots \leq x_{(n)} \leq x.$$

In the case that $x_{(j)} > z_{j-1}$ for all j, we have $i(f) = 0$.

Assume $(x_{r_1}, \ldots, x_{r_i})$ is the subsequence of f_π from which the non-decreasing sequence $(x_{(1)}, x_{(2)}, \ldots, x_{(i)})$ is obtained. Let $R_1 = \{r_1, r_2, \ldots, r_i\} \subseteq [n]$. Then it is easy to see that R_1 must be a union of some blocks of π, while $R_2 = [n] \setminus R_1$ is the union of the remaining blocks of π. Let π_1 be the restriction of π on R_1 and π_2 the restriction of π on R_2. Thus π is a disjoint union of π_1 and π_2. Furthermore, let f_i be the restriction of f on R_i. Then f_1 is a map from the blocks of π_1 to $\{1, \ldots, z_i\}$ that is also a $\vec{z}$-parking function, and f_2 is a map from blocks of π_2 to the set $X \setminus [z_i] = \{z_i + 1, \ldots, x\}$.

Let $\mathcal{S}^P(E, X)$ be the subset of $\mathcal{S}(E, X)$ such that for each pair (π, f), the sequence f_π is a $\vec{z}$-parking function. Then the above argument defines a decomposition of $(\pi, f) \in \mathcal{S}([n], X)$ into pairs $(\pi_1, f_1) \in \mathcal{S}^P(R_1, X)$ and $(\pi_2, f_2) \in \mathcal{S}(R_2, X \setminus [z_i])$. Conversely, any pairs of (π_1, f_1) and (π_2, f_2) described above can be reassembled into a partition π of $[n]$ with labels in X. In other words, the set $\mathcal{S}([n], X)$ can be written as a disjoint union of Cartesian products as

$$\mathcal{S}([n], X) = \bigsqcup_{\substack{i; R_1 \subseteq [n] \\ |R_1| = i}} \mathcal{S}^P(R_1, X) \times \mathcal{S}(R_2, X \setminus [z_i]). \tag{3.9}$$

In addition, if π is the disjoint union of π_1 and π_2, then

$$w(\hat{0}, \pi) = w(\hat{0}, \pi_1) w(\hat{0}, \pi_2).$$

Putting the above results together, we have

$$\begin{aligned} a_n(x; w) &= \sum_{(\pi, f) \in \mathcal{S}([n], X)} w(\hat{0}, \pi) \\ &= \sum_{i=0}^{n} \sum_{R_1 : |R_1| = i} \left(\sum_{(\pi_1, f_1) \in \mathcal{S}^p(R_1, X)} w(\hat{0}, \pi_1) \cdot \sum_{(\pi_2, f_2) \in \mathcal{S}(R_2, X \setminus [z_i])} w(\hat{0}, \pi_2) \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{(\pi_1, f_1) \in \mathcal{S}^p(R_1, X)} w(\hat{0}, \pi_1) \\ &= \sum_{i=0}^{n} \binom{n}{i} a_{n-i}(x - z_i; w) \sum_{\pi \in \Pi_i} w(\hat{0}, \pi) PF_\pi(\mathcal{Z}). \end{aligned}$$

The last equation follows from the definition of $PF_\pi(\mathcal{Z})$. □

Example 5. From the recurrence (3.4) we get

$$t_2(x; w, \mathcal{Z}) = x^2 + (w_2 - 2z_1)x + (2z_0 z_1 - z_0^2 - w_2 z_0).$$

Hence $t_2(0; w, -\mathcal{Z}) = 2z_0z_1 - z_0^2 + w_2z_0$. On the other hand, there are two partitions in Π_2. For $\pi = \{12\}$, clearly $w(\hat{0}, \{12\}) = w_2$ and $PF_{\{12\}}(\mathcal{Z}) = z_0$. For $\pi = \{1\}\{2\}$, $w(\hat{0}, \pi) = 1$ and $PF_\pi(\mathcal{Z})$ is the number of pairs of positive integers (x, y) such that $\min(x, y) \leq z_0$ and $\max(x, y) \leq z_1$. It is easy to check that there are $2z_0z_1 - z_0^2$ such pairs. $\square$

Since $\{a_n(x; w)\}$ gives a generic form of the sequence of polynomials of binomial type, then $\{t_n(x; w, \mathcal{Z})\}$ is the generic form of the generalized Gončarov polynomials. In particular, from Theorem 4 we see that when $w_2 = w_3 = \cdots = 0$, $t_n(0; w, -\mathcal{Z})$ gives the number of $\vec{z}$-parking functions of length n.

4. GONČAROV POLYNOMIALS IN EXPONENTIAL FAMILIES[1]

This chapter is an expansion of Section 4 of [3]. As described earlier in Section 2.9, exponential families are combinatorial models based on the partition lattices where the enumeration is captured by the exponential generating functions.

We will show that by taking the type enumerator, an exponential family actually provides a combinatorial model for all generalized Gončarov polynomials.

4.1 Type Enumerator in Exponential Families

In a given exponential family $\mathcal{F}$, we have seen that

$$h_n(x) = \sum_{k=1}^{n} h_{n,k} x^k = \sum_{H} |\{f : \text{ cards in } H \ \to X\}| \tag{4.1}$$

where H ranges over all hands of weight n. For a hand H consisting of cards $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ of weights $t_1, t_2, \ldots, t_k$, define the type of H as

$$\text{type}(H) = y_{t_1} y_{t_2} \cdots y_{t_k},$$

where $y_1, y_2, \ldots,$ are free variables.

Let

$$h_n(x; \mathbf{y}) = \sum_{H:\text{ weight } n} \text{type}(H) \cdot |\{f : \text{ cards in } H \ \to X\}|. \tag{4.2}$$

Then we have the following form of the exponential formula.

Proposition 1. *The sequence of type enumerators $\{h_n(x; \mathbf{y})\}_{n \geq 0}$, viewed as a polynomial in x, is*

[1]Reprinted with permission from "Gončarov Polynomials in Partition Lattices and Exponential families" by A. Adeniran and C. H. Yan, 2020. *Advances in Applied Mathematics*, https://doi.org/10.1016/j.aam.2020.102045,

a sequence of polynomials of binomial type satisfying the equation

$$\sum_{n\geq 0} h_n(x;\mathbf{y})\frac{t^n}{n!} = \exp\left(x\sum_{k\geq 1} d_k y_k \frac{t^k}{k!}\right). \tag{4.3}$$

Proof. We compare the formula of $h_n(x;\mathbf{y})$ with that of $h_n(x)$. Note for $n \geq 1$, $h_n(x)$ can be computed by

$$h_n(x) = \sum_{k\geq 1}\sum_{H=\{\mathcal{C}_1,\ldots,\mathcal{C}_k\}} d_{t_1}d_{t_2}\cdots d_{t_k}x^k, \tag{4.4}$$

where $\{\mathcal{C}_1,\cdots,\mathcal{C}_k\}$ is a hand of weight n and t_i is the weight of card $\mathcal{C}_i$, while

$$h_n(x;\mathbf{y}) = \sum_{k\geq 1}\sum_{H=\{\mathcal{C}_1,\ldots,\mathcal{C}_k\}} d_{t_1}d_{t_2}\cdots d_{t_k}y_{t_1}y_{t_2}\cdots y_{t_k}x^k. \tag{4.5}$$

The exponential formula for $h_n(x)$ then implies Proposition 1. □

Remark 1. *Comparing to the generic form $a_n(x;w)$ in the previous section, we see that $h_n(x;\mathbf{y})$ corresponds to the case where the variables in the zeta-type function are determined by $w_n = d_n y_n$. As far as $d_1 \neq 0$, we can obtain arbitrary polynomial sequences of binomial type by taking suitable values for the y_i-variables.*

4.2 Sequence of Generalized Goncarov Polynomials

Let $Z = (z_i)_{i\geq 0}$ be an interpolation grid. For the binomial sequence $\{h_n(x;\mathbf{y})\}_{n\geq 0}$ defined in an exponential family $\mathcal{F}$, we can consider the associated generalized Gončarov polynomials given by (2.4) with $p_n(x)$ replaced by $h_n(x;\mathbf{y})$. Denote this Gončarov polynomial by $t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z})$ to emphasize that it has variables y_i and is defined in $\mathcal{F}$. Explicitly, $t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z})$ is obtained by the recurrence

$$t_n(x;\mathbf{y},\mathcal{F},\mathcal{Z}) = h_n(x;y) - \sum_{i=0}^{n-1}\binom{n}{i} h_{n-i}(z_i;y)t_i(x;\mathbf{y},\mathcal{F},\mathcal{Z}). \tag{4.6}$$

Suppose $X = \{1,2,...,x\}$ and assume that $z_0 \leq z_1 \leq \cdots \leq z_{n-1}$ are integers in X. Let $\vec{z} = (z_0,z_1,\ldots,z_{n-1})$. For a hand $H = \{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_k\}$ of weight n with a function f from $\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_k\}$ to X, denote by f_H the list $(x_1,x_2,\ldots,x_n)$, where $x_i = f(\mathcal{C}_j)$ if i is in the label set of $\mathcal{C}_j$. Let

$$\mathcal{PF}_H(\mathcal{Z}) = \{f : \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\} \to X \mid f_H \text{ is a } \vec{z}\text{-parking function}\},$$

and $PF_H(\mathcal{Z})$ the cardinality of $\mathcal{PF}_H(\mathcal{Z})$. Then we have the following analog of Theorem 4.

Theorem 5. *For* $n \geq 0$*,*

$$t_n(0; \mathbf{y}, \mathcal{F}, -\mathcal{Z}) = t_n(x; \mathbf{y}, \mathcal{F}, x - \mathcal{Z}) = \sum_{H:\text{ of weight } n} \operatorname{type}(H) \cdot PF_H(\mathcal{Z}). \tag{4.7}$$

Here by convention, the third term of (4.7) *equals* 1 *when* $n = 0$*.*

Theorem 5 follows from (4.6) and the following recurrence relation

$$h_n(x; \mathbf{y}) = \sum_{i=0}^{n} \binom{n}{i} h_{n-i}(x - z_i; \mathbf{y}) \sum_{H:\text{ of weight } i} \operatorname{type}(H) \cdot PF_H(\mathcal{Z}), \tag{4.8}$$

whose proof is similar to that of Lemma 1. In an exponential family $\mathcal{F}$, let $\mathcal{A}(S, X)$ be the set of pairs (H, f) such that H is a hand whose label sets form a partition of S and f is a function from the cards in H to X. Then the basic ingredients of the proof are that

1. $\operatorname{type}(H)$ is a multiplicative function only depending on the weights of cards in H, and

2. The set $\mathcal{A}([n], X)$ can be decomposed into a disjoint union of Cartesian products of the form

$$\mathcal{A}^P(R, X) \times \mathcal{A}([n] \setminus R, X \setminus [z_i]),$$

   where $\mathcal{A}^P(R, X) = \{(H, f) \in \mathcal{A}(R, X) : \ f_H \text{ is a } \vec{z}\text{-parking function}\}$, and the disjoint union is taken over all the subsets R of $[n]$.

We skip the details of the proof of Eq. (4.8).

We illustrate the above results and some connections to combinatorics in the exponential families given in Examples 2 and 3. There are many other exponential families in which the type enumerator and associated Gončarov polynomials have interesting combinatorial significance.

1. Let $\mathcal{F}_1$ be the exponential family of set partitions described in Example 2. In this family, $d_i = 1$ for all i and $h_n(x) = \sum_{k=0}^{n} S(n,k)x^k$. In the type enumerator, if we substitute $y_1 = 1$ and $y_i = w_i$ for $i \geq 2$, then $h_n(x; \mathbf{y})$ is exactly the same as the generic sequence $a_n(x; w)$ in (3.2), and consequently $t_n(x; \mathbf{y}, \mathcal{F}_1, \mathcal{Z})$ is the same as the generic Gončarov polynomial $t_n(x; w, \mathcal{Z})$ defined by (3.5). In particular, if all $y_i = 1$, $t_n(0; \mathbf{y}, \mathcal{F}_1, -\mathcal{Z})$ gives a formula for the number of $\vec{z}$-parking functions with the additional structure that cars arrive in disjoint groups, and drivers in the same group always prefer the same parking spot.

   When $y_i = 1$ and $z_i = 1 + i$ for all i, the first few terms of the Gončarov polynomials $t_n(x) = t_n(x; \mathbf{y}, \mathcal{F}_1, -\mathcal{Z})$ are

$$
\begin{aligned}
t_0(x) &= 1 \\
t_1(x) &= x + 1 \\
t_2(x) &= x^2 + 5x + 4 \\
t_3(x) &= x^3 + 12x^2 + 40x + 29 \\
t_4(x) &= x^4 + 22x^3 + 163x^2 + 453x + 311
\end{aligned}
$$

   In particular, for $x = 0$ we get the sequence $1, 1, 4, 29, 311, \ldots$. This is sequence A030019 in the On-Line Encyclopedia of Integer Sequences (OEIS) [1], where it is interpreted as the number of labeled spanning trees in the complete hypergraph on n vertices (all hyper-edges having cardinality 2 or greater). It would be interesting to find a direct bijection between the hyper-trees and the parking-function interpretation.

2. Let $\mathcal{F}_2$ be the exponential family of the permutations and their cycles, as described in Example 3. Here $d_n = (n-1)!$ and $h_n(x) = \sum_{k=0}^{n} c(n,k)x^k = x^{(n)}$, where the $c(n,k)$ are the signless Stirling numbers of the first kind and $x^{(n)}$ is the rising factorial $x(x+1)\cdots(x+n-1)$. When $y_1 = 1$, the Gončarov polynomial $t_n(x; \mathbf{y}, \mathcal{F}_2, \mathcal{Z})$ can be obtained from the generic

form $t_n(x; w, \mathcal{Z})$ by replacing w_n with $(n-1)!y_n$ for $n \geq 2$. When all $y_i = 1$, i.e. $y = \mathbf{1}$, $t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ gives a formula for the number of $\vec{z}$-parking functions with the additional requirement that cars are formed in disjoint cycles, and drivers in the same cycle prefer the same parking spot.

In addition, when $y = \mathbf{1}$, and $\mathcal{Z}$ is the arithmetic progression $z_i = a + bi$, the Goncarov polynomial is

$$t_n(x; \mathbf{1}, \mathcal{F}_2; -\mathcal{Z}) = (x+a)(x+a+nb+1)^{(n-1)}. \tag{4.9}$$

Another combinatorial interpretation of $t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ is given in [22, Section 6.7], where it shows that $t_n(0; \mathbf{1}, \mathcal{F}, -\mathcal{Z})$ is $n!$ times the number of lattice paths from $(0,0)$ to $(x-1, n)$ with strict right boundary $\mathcal{Z}$. For example, when $z_i = a + bi$ for some positive integers a and b, $\frac{1}{n!}t_n(0; \mathbf{1}, \mathcal{F}_2, -\mathcal{Z})$ is the number of lattice paths from $(0,0)$ to $(x-1, n)$ which stay strictly to the left of the points $(a + ib, i)$ for $i = 0, 1, \dots, n$. In particular for $a = 1$ and $b = k$, it counts the number of labeled lattice paths from the origin to (kn, n) that never pass below the line $x = yk$. In that case (4.9) gives $\frac{1}{1+kn}\binom{(k+1)n}{n}$, the n-th k-Fuss-Catalan number.

We can also consider the injective functions in the definition of $h_n(x)$ and $h_n(x; y)$ in (4.1) and (4.2), where the term x^k is replaced by the lower factorial $x_{(k)} = x(x-1)\cdots(x-k+1)$. In other words, cards of a hand are labeled by X with the additional property that different cards get different labels. Some examples are given in [22, Section 6] and called *monomorphic classes*. A result analogous to Theorem 5 still holds for the momomorphic classes of an exponential family.

As a final result we point out an explicit formula to compute the constant coefficient of the generalized Gončarov polynomial whenever we know the basic sequence $\{p_n(x)\}_{n\geq 0}$. It is proved in [22] and only depends on the recurrence (2.4) and the fact that $p_n(0) = 0$ for $n > 0$. The proof does not need an explicit formula for the delta operator Δ and hence the result is easier to use when we need to compute the value of $t_n(0; \mathbf{y}, \mathcal{F}, -\mathcal{Z})$ in a given exponential family.

Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of binomial type and $\mathcal{Z} = (z_0, z_1, \dots)$ be a given grid. Assume

$\{t_n(0; -\mathcal{Z})\}_{n\geq 0}$ is defined by the recurrence relation

$$t_n(0; -\mathcal{Z}) = -\sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(-z_i) t_i(0),$$

for $n \geq 1$ and $t_0(0; -\mathcal{Z}) = 1$. Then for $n \geq 1$, $t_n(0; -\mathcal{Z})$ can be expressed as a summation over ordered partitions.

Given a finite set S with n elements, an *ordered partition* of S is an ordered list $(B_1, ..., B_k)$ of disjoint nonempty subsets of S such that $B_1 \cup \cdots \cup B_k = S$. If $\rho = (B_1, ..., B_k)$ is an ordered partition of S, then we set $|\rho| = k$. For each $i = 1, 2, ..., k$, we let $b_i = b_i(\rho) = |B_i|$, and $s_i := s_i(\rho) := \sum_{j=1}^{i} b_j$. In particular, set $s_0(p) = 0$. Let $\mathcal{R}_n$ be the set of all ordered partitions of the set $[n]$.

Theorem 6 ([22]). *For $n \geq 1$,*

$$\begin{aligned} t_n(0; -\mathcal{Z}) &= \sum_{\rho \in \mathcal{R}_n} (-1)^{|\rho|} \prod_{i=0}^{k-1} p_{b_{i+1}}(-z_{s_i}) \\ &= \sum_{\rho \in \mathcal{R}_n} (-1)^{|\rho|} p_{b_1}(-z_0), \cdots p_{b_k}(-z_{s_{k-1}}). \end{aligned} \tag{4.10}$$

The following list gives the formulas for the first several Gončarov polynomials.

$$\begin{aligned} t_0(0; -\mathcal{Z}) &= 1 \\ t_1(0; -\mathcal{Z}) &= -p_1(-z_0) \\ t_2(0; -\mathcal{Z}) &= 2p_1(-z_0)p_1(-z_1) - p_2(-z_0) \\ t_3(0; -\mathcal{Z}) &= -p_3(-z_0) + 3p_2(-z_0)p_1(-z_2) + 3p_1(-z_0)p_2(-z_1) - 6p_1(-z_0)p_1(-z_1)p_1(-z_2). \end{aligned}$$

4.3 Degenerate Cases

In an exponential family, the polynomial $h_n(x)$ or $h_n(x; \mathbf{y})$ may not always have degree n, e.g., when $d_1 = h_{1,1} = 0$. We say that such polynomial sequences and the corresponding exponential families are *degenerate*. For a degenerate sequence of polynomials, there is no delta operator for which the sequence is the basic or the conjugate sequence. Nevertheless, the exponential formulas (2.10) and (4.3) are still true. Hence the sequences $\{h_n(x)\}_{n\geq 0}$ and $\{h_n(x;\mathbf{y})\}_{n\geq 0}$ still satisfy the binomial-type identity (2.1).

Without a delta operator, we cannot define the generalized Gončarov interpolation problems. However, we can still introduce the generalized Gončarov polynomials via the recurrence (2.4). Furthermore, we will prove in Theorem 7 that the shift invariance of Gončarov polynomials can also be derived from (2.4). Therefore, Theorems 5 and 6 still hold true for the degenerate exponential families since all the proofs follow from the binomial-type identity (2.1) and the recurrence (2.4).

Theorem 7. *Assume $\{p_n(x)\}_{n\geq 0}$ is a polynomial sequence of binomial type with $p_0(x) = 1$, but the degree of $p_n(x)$ is not necessarily n. Let $t_n(x;\mathcal{Z})$ be defined by the recurrence relation*

$$t_n(x;\mathcal{Z}) = p_n(x) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(x;\mathcal{Z}). \tag{4.11}$$

For any scalar η and the interpolation grid $\mathcal{Z} = \{z_0, z_1, z_2, \dots)$, let $\mathcal{Z}+\eta$ be the sequence $(z_0+\eta, z_1+\eta, z_2+\eta, \cdots)$. Then we have

$$t_n(x+\eta;\mathcal{Z}+\eta) = t_n(x;\mathcal{Z}) \tag{4.12}$$

for all $n \geq 0$.

Proof. We prove Theorem 7 by induction on n. The initial case $n = 0$ is trivial since $t_0(x;\mathcal{Z}) = 1$ for all x and any grid $\mathcal{Z}$. Assume Eq. (4.12) is true for all indices less than n. We compute $t_n(x+\eta;\mathcal{Z}+\eta)$. By definition

$$t_n(x+\eta;\mathcal{Z}+\eta) = p_n(x+\eta) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i+\eta) t_i(x+\eta;\mathcal{Z}+\eta). \tag{4.13}$$

By the inductive hypothesis $t_i(x+\eta;\mathcal{Z}+\eta) = t_i(x;\mathcal{Z})$ for $i < n$ and the binomial identity of

$p_n(x)$, the right-hand side of (4.13) can be written as

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(\eta)-\sum_{i=0}^{n-1}\binom{n}{i}\left(\sum_{j=0}^{n-i}\binom{n-i}{j}p_{n-i-j}(z_i)p_j(\eta)\right)t_i(x;\mathcal{Z})\\
=\;&\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(\eta)-\sum_{\substack{i+j\leq n\\ \text{except } (i,j)=(n,0)}}\binom{n}{i}\binom{n-i}{j}p_j(\eta)p_{n-i-j}(z_i)t_i(x;\mathcal{Z}) \qquad (4.14)
\end{aligned}$$

Since

$$\binom{n}{i}\binom{n-i}{j}=\frac{n!}{i!j!(n-i-j)!}=\binom{n}{j}\binom{n-j}{i},$$

then (4.14) can be expressed as

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(\eta)-\sum_{\substack{i+j\leq n\\ \text{except } (i,j)=(n,0)}}\binom{n}{j}\binom{n-j}{i}p_j(\eta)p_{n-i-j}(z_i)t_i(x;\mathcal{Z})\\
=\;&\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(\eta)-\sum_{j=1}^{n}\binom{n}{j}p_j(\eta)\sum_{i=0}^{n-j}\binom{n-j}{i}p_{n-i-j}(z_i)t_i(x;\mathcal{Z})\\
&-\sum_{i=0}^{n-1}\binom{n}{i}p_{n-i}(z_i)t_i(x;\mathcal{Z}). \qquad (4.15)
\end{aligned}$$

The last summation in (4.15) corresponds to the terms with $j=0$. Note that

$$\sum_{i=0}^{n-j}\binom{n-j}{i}p_{n-i-j}(z_i)t_i(x;\mathcal{Z})=p_{n-j}(x).$$

Hence For. (4.15) is equal to

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(\eta)-\sum_{j=1}^{n}\binom{n}{j}p_j(\eta)p_{n-j}(x)-\sum_{i=0}^{n-1}\binom{n}{i}p_{n-i}(z_i)t_i(x;\mathcal{Z})\\
=\;&p_n(x)-\sum_{i=0}^{n-1}\binom{n}{i}p_{n-i}(z_i)t_i(x;\mathcal{Z})\\
=\;&t_n(x;\mathcal{Z}).
\end{aligned}$$

This finishes the proof. □

The next example shows a degenerate exponential family.

Example 6. *2-Regular simple graphs.* In this exponential family a card is an undirected cycle on a label set $[m]$ (where $m \geq 3$). The deck $\mathcal{D}_n$ consists of all undirected circular arrangements of n letters so $d_n = \frac{1}{2}(n-1)!$ for $n \geq 3$ and $d_1 = d_2 = 0$. A hand is then a undirected simple graph on the vertex set $[n]$, which is 2-regular, that is, every vertex has degree 2. Thus, $h_{n,k}$ is the number of undirected 2-regular simple graphs on n vertices consisting of k cycles. Denote by $\mathcal{F}_3$ this exponential family.

For $\mathcal{F}_3$, the type enumerators are $h_0(x, \mathbf{y}) = 1$, $h_1(x; \mathbf{y}) = h_2(x; \mathbf{y}) = 0$, $h_3(x; \mathbf{y}) = y_3 x$, $h_4(x; \mathbf{y}) = 2y_4 x$, $h_5(x, \mathbf{y}) = 12y_5 x$, and $h_6(x; \mathbf{y}) = 60y_6 x + 10y_3^2 x^2$, etc. Although the degree of $h_n(x; \mathbf{y})$ is not n, the exponential formula still holds:

$$\sum_{k=0}^{n} h_n(x; \mathbf{y}) \frac{t^k}{k!} = \exp\left(x \sum_{k \geq 3} y_k \frac{t^k}{2k} \right).$$

We compute by the recurrence (4.6) that

$$\begin{aligned}
t_0(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= 1 \\
t_1(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= t_2(x; y, \mathcal{F}_3, \mathcal{Z}) = 0 \\
t_3(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= y_3(x - z_0), \\
t_4(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= 3y_3(x - z_0), \\
t_5(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= 12y_5(x - z_0), \\
t_6(x; \mathbf{y}, \mathcal{F}_3, \mathcal{Z}) &= 10y_3^2 x^2 + 60y_6 x - 20y_3^2 z_3 x - 60y_6 z_0 - 10y_3^2 z_0^2 + 20y_3^2 z_0 z_3.
\end{aligned}$$

The equation

$$t_n(0; \mathbf{y}, \mathcal{F}_3, -\mathcal{Z}) = \sum_{H \colon \text{of weight } n} \text{type}(H) \cdot PF_H(\mathcal{Z})$$

is still true. For example, for $n = 6$, $t_6(0; y, \mathcal{F}_3, -\mathcal{Z}) = 60y_6 z_0 + 20y_3^2 z_0 z_3 - 10y_3^2 z_0^2$. The term $60y_6 z_0$ comes from the $5!/2 = 60$ 6-cycles, and the terms $10y_3^2(2z_0 z_3 - z_0^2)$ comes from the 10 hands each with two 3-cycles. □

5. INCREASING AND INVARIANT PARKING SEQUENCES

This chapter covers work done on special classes of parking sequences. First, we study increasing parking sequences for any given length vector and then we study two different concepts of invariance in parking sequences. Parking sequences were first studied by Ehrenborg and Happ [7, 8] and serve as the entry point for our investigation into these classes of parking sequences.

In section 5.1, we discuss increasing parking sequences and their connection to lattice paths. Then, in section 5.2, we fix the length vector $\vec{y}$ and characterize all permutation-invariant parking sequences when $\vec{y}$ has some special characteristics. Lastly, in section 5.3, we characterize all parking sequences that remain invariant under all permutations of $\vec{y}$ for any given length set $\{y_1, ..., y_n\}$ as well as all possible invariant parking sequences on a given street with fixed length n.

5.1 Increasing Parking Sequences

In this section, we consider all non-decreasing parking sequences for any given pair $(\vec{y}; z)$. By convention, we write $[x] = \{1, 2, ..., x\}$ and the interval $[x, y] = \{x, x + 1, ..., y\}$, where $x, y \in \mathbb{Z}_+$ and $x < y$. Given any sequence $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{Z}_+^n$, let $\mathbf{b}_{inc} = (b_{(1)}, ..., b_{(n)})$ be the non-decreasing rearrangement of the entries of $\mathbf{b}$ and the i^{th} entry $b_{(i)}$ of $\mathbf{b}_{inc}$ is called the *i-th order statistic* of $\mathbf{b}$. Next, we define the final parking configuration for any given parking sequence.

Definition 14. The *final parking configuration* of a parking sequence $\mathbf{c}$ is the arrangement of cars $C_1, C_2, ..., C_n$ following the trailer T encoding their relative order on the street after they are done parking under the preference sequence $\mathbf{c}$.

For example, in Figure 2.3, the final parking configuration for $\mathbf{c} = (3, 7, 5, 3)$ is T, C_1, C_3, C_2, C_4.

First, we prove a necessary condition for parking sequences, which is analogous to inequality (2.5).

Lemma 2. *Suppose* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ *where* $\vec{y} = (y_1, ..., y_n)$*. Then,* $\#\{j \in [n] : c_j \leq$

$z\} \geq 1$ *and for each* $1 \leq t \leq n-1$,

$$\#\{j : c_j \leq z + \sum_{i=0}^{t-1} y_{(n-i)}\} \geq t+1. \tag{5.1}$$

Proof. Clearly, $\#\{j : c_j \leq z\} \geq 1$. Otherwise, there is no car whose preference is less or equal to z, thus no car parks on spot z and we obtain a contradiction. Suppose for some $t \in [2, n-1]$, $\#\{j : c_j \leq z + \sum_{i=0}^{t-1} y_{(n-i)}\} \leq t$. Then, in the final parking configuration on spots $[1, z + \sum_{i=0}^{t-1} y_{(n-i)}]$, there are at most 1 trailer and t cars occupying a total of at most $z-1+y_{(n)}+y_{(n-1)}+\cdots+y_{(n-t+1)}$ spots. Thus, not all spots are used in the final parking configuration and this contradicts the fact that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. □

The following result is immediate from Lemma 2.

Corollary 1. *Let* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ *where* $(\vec{y}; z) = (y_1, ..., y_n; z)$*. Let* $\vec{y}_{inc} = (y_{(1)}, ..., y_{(n)})$ *be the non-decreasing rearrangement of the length vector* $\vec{y}$*. Then,* $\mathbf{c}$ *can be rearranged to give an increasing sequence* $\mathbf{c}' = (c_{(1)}, ..., c_{(n)})$ *with* $c_{(1)} \leq z$ *and*

$$c_{(2)} \leq z + y_{(n)}, c_{(3)} \leq z + y_{(n)} + y_{(n-1)}, ..., c_{(n)} \leq z + y_{(n)} + y_{(n-1)} + \cdots + y_{(3)} + y_{(2)}. \tag{5.2}$$

It is important to note that the converse implication of Lemma 2 is not true. Using the same example as before, even though $\mathbf{c} = (1, 2)$ and $\mathbf{c}' = (2, 1)$ both satisfy (5.1) for $\vec{y} = (2, 2)$, $\mathbf{c}' \notin \mathsf{PS}(\vec{y})$.

Definition 15. A sequence $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ is an *increasing parking sequence for* $(\vec{y}; z)$ if $c_1 \leq c_2 \leq \cdots \leq c_n$. We denote the set of all increasing parking sequences for $(\vec{y}; z)$ by $\mathsf{IPS}(\vec{y}; z)$.

When $\vec{y} = (1, 1, ..., 1)$ and $z = 1$ (i.e. the trailer of length 0), we get the increasing classical parking functions, which are counted by the Catalan numbers.

We remark that for a parking sequence $\mathbf{c} \in PS(\vec{y}; z)$, its rearrangement $\mathbf{c}_{inc}$ is not necessarily a parking sequence. Consider the following example: $\mathbf{c} = (5, 6, 1) \in \mathsf{PS}(1, 1, 4)$ yields the increasing sequence $\mathbf{c}_{inc} = (1, 5, 6)$, which is not a parking sequence for $\vec{y} = (1, 1, 4)$.

We have the following characterization for increasing parking sequences.

Proposition 2. *Let* $(\vec{y}; z) = (y_1, ..., y_n; z)$*.Then,* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{IPS}(\vec{y}; z)$ *if and only if* $c_1 \leq c_2 \leq \cdots \leq c_n$ *and for all* $i \in [n]$*,*

$$c_i \leq z + \sum_{j=1}^{i-1} y_j. \tag{5.3}$$

Proof. First observe that if $\mathbf{c}$ is a preference sequence satisfying (5.3), then the cars will park in the final configuration $T, C_1, \ldots, C_n$. Hence such a sequence $\mathbf{c}$ is in $\mathsf{IPS}(\vec{y}; z)$.

Conversely, for a non-decreasing sequence $\mathbf{c}$ that allows all the cars to park, we need to prove that it satisfies (5.3). First by Corollary 1, $c_1 \leq z$. Thus, car C_1 parks right after the trailer leaving no gaps. By the rules of the parking process, if $c_i \leq c_{i+1}$, then car C_{i+1} will park after C_i if both are able to park. Hence for a non-decreasing $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$, the final parking configuration must be $T, C_1, C_2, \ldots, C_n$. It follows that the first spot occupied by car C_i is $z + y_1 + \cdots + y_{i-1}$, which must be larger than or equal to c_i.

□

Proposition 2 allows us to enumerate increasing parking sequences for any given length vector $\vec{y}$ and $z \in \mathbb{Z}_+$ using results in lattice path counting. Recall that a *lattice path* from $(0, 0)$ to (p, q) is a sequence of p east steps and q north steps. It can be represented by a sequence of non-decreasing integers $(x_1, x_2, \ldots, x_q)$ such that the north steps are at $(x_i, i-1) \to (x_i, i)$, for $i = 1, ..., q$. The lattice path is with strict right boundary $(b_1, b_2, ..., b_q)$ if $0 \leq x_i < b_i$ for all $1 \leq i \leq q$. Let $\mathsf{LP}_{p,q}(b_1, b_2, ..., b_q)$ denote the set of all lattice paths from $(0, 0)$ to (p, q) with strict right boundary $(b_1, b_2, \ldots, b_q)$.

We can represent increasing parking sequences in terms of lattice paths with strict right boundary as follows: Let $(\vec{y}; z) = (y_1, ..., y_n; z)$ and $M = z - 1 + y_1 + y_2 + \cdots + y_{n-1} + y_n$. Then by Proposition 2 there is a bijection from $\mathsf{IPS}(\vec{y}; z)$ to the set of lattice paths from $(0, 0)$ to (M, n) with strict right boundary $(z, z + y_1, z + y_1 + y_2, ..., z + y_1 + y_2 + \cdots + y_{n-1})$. (Here we have strict right boundary because in the lattice path, x_i can be 0 while in $\mathbf{c} \in \mathsf{IPS}(\vec{y}, z)$, $c_i \geq 1$.) There are well-known determinant formulas to count the number of lattice paths with general boundaries, see, for example, Theorem 1 of [23, p.32], which leads to the following determinant formula.

Corollary 2. *Suppose* $M = z - 1 + y_1 + y_2 + \cdots + y_{n-1} + y_n$. *Then,*

$$\begin{aligned}\#\mathsf{IPS}(\vec{y}; z) &= \#\mathsf{LP}_{M,n}(z, z + y_1, z + y_1 + y_2, ..., z + y_1 + y_2 + \cdots + y_{n-1}) \\ &= \det\left[\binom{b_i}{j - i + 1}\right]_{1 \le i,j \le n}\end{aligned}$$

where $b_1 = z$ *and* $b_i = z + y_1 + y_2 + \cdots + y_{i-1}$ *for* $i = 2, ..., n$.

For the special case that the length vector has constant entries, there are nicer closed formulae for the determinant. Specifically, when $\vec{y} = (k, k, \ldots, k)$ and $M = z + kn - 1$, $\mathsf{LP}_{M,n}(z, z + k, z + 2k, ..., z + (n-1)k)$ is the set of lattice paths from $(0, 0)$ to $(z + kn - 1, n)$ which never touch the line $x = z + ky$. Using the formula (1.11) (or Theorem 3) of [23, p.9], we have

Corollary 3. *Suppose* $(\vec{y}, z) = (k, k, ..., k; z)$ *and* $M = z + kn - 1$*, then*

$$\#\mathsf{IPS}(\vec{y}; z) = \#\mathsf{LP}_{M,n}(z, z + k, z + 2k, ..., z + (n-1)k) = \frac{z}{z + n(k+1)}\binom{z + n(k+1)}{n}.$$

This specializes to the Fuss-Catalan numbers when we set $z = 1$.

Corollary 4. *Suppose* $\vec{y} = (k, k, ..., k)$*, then*

$$\#\mathsf{IPS}(\vec{y}; 1) = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

When $\vec{y} = (1, 1, ..., 1)$ and $z = 1$, the increasing parking sequences are exactly the classical increasing parking functions, which are counted by the Catalan numbers. It is well-known that classical increasing parking functions of length n are in one-to-one correspondence with Dyck paths of semi-length n, which are lattice paths from $(0, 0)$ to (n, n) with strict right boundary $(1, 2, ..., n)$. Hence Corollaries 2–4 generalize the result in the classical case.

5.2 Invariant Parking Sequences

In this section, we study the first of two types of invariance in parking sequences. Recall that a permutation of a (classical) parking function is also a parking function but it is not true for parking sequences in general. This gives rise to the question of which parking sequences are invariant for a given length vector.

5.2.1 Invariant Parking Sequences for fixed length-vector

Fixing the length vector $\vec{y} \in \mathbb{Z}_+^n$ and a positive integer z, we investigate which parking sequence remains in the set $\mathsf{PS}(\vec{y}; z)$ after an arbitrary rearrangement of its terms.

Definition 16 (Permutation-invariant parking sequences)**.** Fix $\vec{y} = (y_1, ..., y_n)$ and $z \in \mathbb{Z}_+$. Let $\mathbf{c} = (c_1, ..., c_n)$ be a parking sequence for $(\vec{y}; z)$. Then, $\mathbf{c}$ is a *permutation-invariant parking sequence for* $(\vec{y}; z)$ if for any rearrangement $\mathbf{c}'$ of $\mathbf{c}$, we have $\mathbf{c}' \in PS_n(\vec{y}; z)$. We denote the set of all permutation-invariant parking sequences for $(\vec{y}; z)$ by $\mathsf{PS}_{inv}(\vec{y}; z)$.

For example, for $\vec{y} = (1, 2)$ and $z = 1$, $\mathsf{PS}(\vec{y}) = \{(1, 1), (1, 2), (2, 1), (3, 1)\}$ where $\mathsf{PS}_{inv}(\vec{y}; z) = \{(1, 1), (1, 2), (2, 1)\}$ since $(1, 3) \notin \mathsf{PS}(\vec{y})$. First, we prove a result that describes a minimal subset of the invariant parking sequences.

Proposition 3. *For any* $\mathbf{c} = (c_1, ..., c_n)$ *with* $c_i \in [z]$ *for all* $i = 1, ..., n$*, we have* $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$*.*

Proof. Clearly, for any such sequence $\mathbf{c}$, no matter what the value of c_i is for any $i \in [n]$, we obtain the final parking configuration $T, C_1, C_2, ..., C_n$, which means $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. Since this is true for any rearrangement of the $\mathbf{c}$, it follows that $\mathbf{c}$ is invariant. □

In general, $\mathsf{PS}_{inv}(\vec{y}; z)$ is larger than the set $[z]^n$, and the situation can be more complicated. The following two examples show that $\mathsf{PS}_{inv}(\vec{y}; z)$ depends not only on the relative order of the y_i's, but also on the exact value of y_i.

Example 7. Let $\vec{y} = (y_1, y_2)$ and $z = 1$. There are two possible cases. If $y_1 < y_2$, then $\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1)\}$. However, if $y_1 \geq y_2$, we have $\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1), (1, y_2{+}1), (y_2{+}1, 1)\}$.

Example 8. Suppose $\vec{y} = (4, 3, 2)$ and $\vec{y}' = (4, 3, 1)$. It is easy to see that

$$\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1)\}$$

and

$$\mathsf{PS}_{inv}(\vec{y}'; 1) = \{(1, 1, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1), (1, 1, 5), (1, 5, 1), (5, 1, 1)\}.$$

Note that the relative orders for the vectors $\vec{y}$ and $\vec{y}'$ are the same ($y_1 > y_2 > y_3$), but the invariant sets are not similar.

In this section we characterize the invariant set for some families of $\vec{y}$. First, we consider the case where the length vector is strictly increasing. Next, we look at the case where $\vec{y}$ is a constant sequence. Lastly, given $a, b \in \mathbb{Z}_+$, we consider two cases where the length vector is of the form (i) $\vec{y} = (a, ..., a, b, ..., b)$ where $a < b$ and (ii) $\vec{y} = (a, ..., a, b, ..., b)$ where $b = 1$ and $a > b$.

5.2.2 Strictly increasing length vector

When $\vec{y}$ is a strictly increasing sequence, we show that Proposition 3 gives all the invariant parking sequences.

Theorem 8. *Let* $(\vec{y}; z) = (y_1, y_2, ..., y_n; z)$ *where* $y_1 < y_2 < \cdots < y_n$. *Then,*

$$\mathsf{PS}_{inv}(\vec{y}; z) = [z]^n.$$

Proof. By Proposition 3, $[z]^n \subseteq \mathsf{PS}_{inv}(\vec{y}; z)$. Conversely, suppose $\mathbf{c} = (c_1, c_2, ..., c_n)$ is a parking sequence for $(\vec{y}; z)$ with some $c_i \notin [z]$. We claim that $\mathbf{c}$ is not invariant. To see this, let $x = \min\{c_i \in \mathbf{c} \mid c_i > z\}$. Then, we can consider the order statistics of $\mathbf{c}$ and the preference sequence associated with it i.e.

$$\mathbf{c}' = (c_{(1)}, c_{(2)}, ..., c_{(r)}, x, c_{(r+2)}, \ldots, c_{(n)})$$

where $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(r)} \leq z < x \leq c_{(r+2)} \leq \cdots \leq c_{(n)}$ and $r \geq 1$ by Corollary 1. Then, by (5.2), x satisfies the inequality: $z < x \leq z + \sum_{i=1}^{r} y_i$. Thus, we can choose the maximum s such that $x > z + \sum_{i=1}^{s} y_i$. ($s$ may be 0). Clearly, $s < r$. Consider the preference

$$\mathbf{c}'' = (c_{(1)}, c_{(2)}, ..., c_{(s)}, x, c_{(s+1)}, \ldots, c_{(r)}, c_{(r+2)}, \ldots, c_{(n)})$$

We try to park according to $\mathbf{c}''$. Clearly, the first s cars park in order after the trailer T without any gaps in between them. Then, the car C_{s+1} has preference x and parks after car C_s with h unoccupied spots in between C_s and C_{s+1}, where $h = x - (z + \sum_{i=1}^{s} y_i) \geq 1$, and $h \leq y_{s+1}$ by the maximality of s. Among the un-parked cars $C_{s+2}, \ldots, C_n$, the minimal length is y_{s+2}, where $y_{s+2} > y_{s+1} \geq h$. Hence no car can fill in these h unoccupied spots. It follows that $\mathbf{c}'' \notin \mathsf{PS}(\vec{y}; z)$, and thus $\mathbf{c} \notin \mathsf{PS}_{inv}(\vec{y}; z)$. □

Corollary 5. *Let $(\vec{y}; z) = (y_1, y_2, ..., y_n; z)$ where $y_1 < y_2 < \cdots < y_n$. Then,*

$$\#\mathsf{PS}_{inv}(\vec{y}; z) = z^n.$$

5.2.3 Constant length vector

In this subsection, we investigate the case where $\vec{y}$ is of the form $(k, k, ..., k)$. The following result gives a characterization for permutation-invariant parking sequences for such length vectors.

Theorem 9. *Suppose $(\vec{y}; z) = (k, k, ..., k; z)$ where $k \in \mathbb{Z}_+$ and $k > 1$. Then, $\mathsf{PS}_{inv}(\vec{y}; z)$ is the set of all sequences $(c_1, ..., c_n)$ whose order statistics satisfy* (5.2) *and such that for each $1 \leq i \leq n$,*

$$c_i \in \{1, 2, ..., z, z + k, z + 2k, ..., z + (n - 1)k\} \tag{5.4}$$

Proof. Let $\mathbf{c} = (c_1, ..., c_n)$ be a sequence that satisfies (5.2) and (5.4). That is, for each $1 \leq i \leq k$, $c_{(i)} \leq z + (i - 1)k$, and $c_i \leq z$ or $c_i = z + sk$ for some $s = 0, 1, ...n - 1$. We claim that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. Since conditions (5.2) and (5.4) are independent of the arrangement of the terms c_i's, the claim implies $\mathbf{c}$ is permutation-invariant.

We attempt to park using $\mathbf{c}$. First, C_1 either parks right after the trailer (if $c_1 \leq z$) or on spots $[c_1, c_1 + k - 1]$. We assume for our inductive hypothesis, that the first r cars are parked already, (where $2 \leq r \leq n-1$), and the following observations hold true at this stage in the parking process:

1. any car already parked on the street occupies spots of the form $[z + ks, z + k(s + 1) - 1]$ where $s \in \{0, 1, ..., n - 1\}$

2. any block of unoccupied spots is a multiple of k starting with $z + km$ (for some $m \in \{0, 1, ..., n - 1\}$.

Thus, for any car C_{r+1} with preference $c_{r+1} = z + kl$ coming in at a later stage, there are two possibilities:

- if spot $(z + kl)$ is empty, then C_{r+1} parks on spots $[z + kl, z + k(l + 1) - 1]$.

- if spot $(z + kl)$ is non-empty, then C_{r+1} drives forward to park in the first open block ahead. By the first observation we made earlier, the last car (say C_j) parked before the open spot parks on some interval $[k(s_0 - 1) + z, ks_0 + z - 1]$. Clearly, this cannot be the last spot on the street otherwise this contradicts (5.2). Thus, there is some open spot that is unoccupied in $[z + kl, z + nk - 1]$. Indeed, there are at least k consecutive unoccupied spots on this portion of the street by observation 2. Hence, C_{r+1} parks in this case.

This exhausts all possible cases for C_{r+1}. Thus, by induction, all cars can park and $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$.

Conversely, suppose for a contradiction that there is a parking sequence $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$ not satisfying (5.4). Then, there is some $j \in [n]$ such that $c_j = z + sk + t$ for some $s \in \{0, 1, ...n - 1\}$ and $1 \leq t < k$. Consider the following rearrangement of $\mathbf{c}$ given by: $\mathbf{c}' = (c_j, c_1, c_2, ..., c_{j-1}, c_{j+1}, ..., c_n)$. By our assumption, $\mathbf{c}' \in \mathsf{PS}(\vec{y}; z)$. We attempt to park using this preference. First, C_1 parks on $[z + sk + t, z + (s + 1)k + t - 1]$. However, between the trailer and C_1, there is now a continuous block of $(z + sk + t) - z = sk + t$ spots, which is clearly nonempty and not a multiple of k. Thus, no matter what preferences the remaining cars have, it is impossible to park all cars on this street. This yields a contradiction to our assumption and we are done. □

Recall that a $\vec{u}$-parking function of length n is a sequence $(x_1, x_2, ..., x_n)$ satisfying $1 \leq x_{(i)} \leq u_i$. We can use the results of vector parking functions to enumerate the number of sequences as described in Theorem 9.

Corollary 6. *Let* $(\vec{y}; z) = (k, k, ..., k; z)$. *Then,*

$$\#\mathsf{PS}_{inv}(\vec{y}; z) = z(n+z)^{n-1}.$$

Proof. By Theorem 9, any permutation-invariant parking sequence is of the form (5.4). Thus, for any $\mathbf{b} = (b_1, ..., b_n) \in \mathsf{PS}_{inv}(\vec{y}; z)$ we define the map $\mathsf{f} : \mathsf{PS}_{inv}(\vec{y}; z) \to \mathsf{PF}_n(\vec{u})$ by letting $\mathsf{f}(\mathbf{b}) = \mathbf{b}'$, where $\mathbf{b}'$ is the vector parking function associated to the vector $\vec{u} = (z, z+1, ..., z+n-1)$ with entries given by

$$b_i' = \begin{cases} b_i, \text{ if } 1 \leq b_i \leq z \\ \\ z + s, \text{ if } b_i = z + sk. \end{cases}$$

f is clearly a bijection since the map can be easily inverted. By [19, Corollary 5.5], the number of $\vec{u}$-parking functions is $z(z+n)^{n-1}$. □

REMARK. Note that Theorem 9 and Corollary 6 are also valid for $k = 1$, in which case $\mathsf{PS}_{inv}(1, \ldots, 1; z) = \mathsf{PS}(1, \ldots, 1; z)$ are exactly $\vec{u}$-parking functions associated to $\vec{u} = (z, z+1, \ldots, z+n-1)$.

5.2.4 Length vector $\vec{y} = (a, ..., a, b, ..., b)$ where $a < b$

Let $n \geq 2$ and z, a, b, r be positive integers with $a < b$ and $1 \leq r < n$. In this section fix $\vec{y} = (\underbrace{a, a, ..., a}_{r}, \underbrace{b, ..., b}_{n-r}) = (a^r, b^{n-r})$, i.e. the first r cars are of size a and the remaining $n-r$ cars are of size b. Next we prove a couple of Lemmas that characterize the set of permutation-invariant parking sequences for $(\vec{y}; z)$. In the following, we will refer to any car of size a (respectively, size b) as an A-car (respectively, B-car).

Lemma 3. *Assume* $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$. *Then in the final parking configuration of* $\mathbf{c}$*, all* A*-cars park in* $[z, z + ra - 1]$.

Proof. Suppose not. Then, there is some $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$ with at least one A-car not parked in $[z, z+ra-1]$ in its final parking configuration $\mathcal{F}$. In $\mathcal{F}$, between the trailer T and all A-cars there are blocks $L_1, \dots, L_m$ of consecutive spots occupied by B-cars. Assume the block L_i consists of $l_i b$ spots, where $l_1 + l_2 + \cdots + l_m = n - r$. Let C_j be the last A-car in the configuration $\mathcal{F}$. Then C_j occupies some spots in $[z+ra, z+ra+(n-r)b-1]$, and no other A-car has checked the spots C_j occupies in the parking process. In addition, let C_k be the first B-car in $\mathcal{F}$. Then $j \leq r < k$ and C_k parks before C_j in $\mathcal{F}$. Two possible cases arise:

1. Assume in $\mathcal{F}$ there are some other A-cars parked between C_k and C_j. Consider the rearrangement $\mathbf{c}' = (c_1, ..., c_{j-1}, c_k, c_{j+1}, ..., c_{k-1}, c_j, c_{k+1}, ..., c_n)$ obtained by exchanging the j-th and k-th terms in $\mathbf{c}$. Let the cars park according to the preference $\mathbf{c}'$. It is easy to see that all A-cars occupy the same spots as in $\mathcal{F}$ except that C_j now parks on a of the b spots originally occupied by car C_k in $\mathcal{F}$, leaving $(b-a)$ of these spots unused. Hence after all the A-cars are parked, the first block of consecutive open spots has size $l_1 b - a$, which is not a multiple of b. Hence it is impossible for the remaining B-cars to park. Thus $\mathbf{c}' \notin PS(\vec{y}; z)$.

2. There is no A-car parked between C_k and C_j in $\mathcal{F}$. Then $\mathcal{F}$ is of the form $A \cdots AB \cdots BAB \cdots B$, where there are $r-1$ A-cars before the first B-car C_k, and $c_j = z - 1 + (r-1)a + l_1 b$. Let $\mathbf{c}''$ be the following rearrangement of $\mathbf{c}$: the first r entries of $\mathbf{c}''$ are $c_1, \dots, c_{j-1}, c_k, c_{j+1}, \dots, c_r$, obtained from the first entries of $\mathbf{c}$ by replacing c_j with c_k; the preferences for B-cars are $c_j, c_{r+1}, \dots, c_{k-1}, c_{k+1}, \dots, c_n$. Let the cars park according to the preference $\mathbf{c}''$. Then the A-cars will occupy the spots $[z, z+ra-1]$, and the first B-car occupies spots $[c_j, c_j + b - 1]$. Now there are $c_j - (z - 1 + ra) = l_1 b - a$ spots between the last A-car and the first B-car; these spots cannot be filled by other B-cars. Hence $\mathbf{c}'' \notin \mathsf{PS}(\vec{y}; z)$.

In both cases we have a permutation of $\mathbf{c}$ that is not in $\mathsf{PS}(\vec{y}; z)$, contradicting the assumption that $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$. □

Lemma 4. *If* $(c_1, c_2, ..., c_n) \in \mathsf{PS}_{inv}(\vec{y}; z)$, *then* $c_i \leq z + (r-1)a$ *for each* $1 \leq i \leq n$.

Proof. Suppose not. Take any permutation of $\mathbf{c}$ starting with $\max\{c_i : i \in [n]\}$ and we contradict the conclusion of Lemma 3. $\square$

Lemma 5. *For $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$, let $c_{(1)} \le c_{(2)} \le \cdots \le c_{(n)}$ be the order statistics of* $\mathbf{c}$*. Then, $c_{(i)} \le z$ for each $1 \le i \le n-r+1$ and $c_{(n-r+j)} \in \{1, ...z, z+a, z+2a, ..., z+(j-1)a\}$ for each $2 \le j \le r$.*

Proof. By Lemmas 3 and 4, if $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$, then any r-term subsequence of $\mathbf{c}$, say $(c_{i_1}, c_{i_2}, ..., c_{i_r})$ parks all r A-cars in $[z, z+ra-1]$ and hence $c_i \le z+(r-1)a$ for all $i = 1, ..., n$. Furthermore, if we consider the last r terms of the order statistics of $\mathbf{c}$, this means $(c_{(n-r+1)}, c_{(n-r+2)}, ..., c_{(n)}) \in \mathsf{PS}_{inv}((a, a, ..., a); z)$. Thus, by Theorem 9, we obtain $c_{(n-r+j)} \in \{1, ...z, z+a, z+2a, ..., z+(j-1)a\}$ for each $2 \le j \le r$. Again, by the order statistics, $c_{(i)} \le c_{(n-r+1)} \le z$ for each $1 \le i \le n-r$. $\square$

Combining Lemmas 3, 4 and 5, we prove the following result.

Theorem 10. *Let $n \ge 2$ and z, a, b, r be positive integers with $a < b$ and $1 \le r < n$. Assume $\vec{y} = (a^r, b^{n-r})$. Let $\mathsf{PF}_n(\vec{u})$ be the set of $\vec{u}$-parking functions of length n where $\vec{u} = (\underbrace{z, z, ..., z}_{n-r+1}, z+1, z+2, ..., z+r-1)$. Then, there is a bijection between the sets $\mathsf{PS}_{inv}(\vec{y}; z)$ and $\mathsf{PF}_n(\vec{u})$.*

Proof. First, we claim that any $\mathbf{c}$ satisfying the inequalities in Lemma 5 is in $\mathsf{PS}_{inv}(\vec{y}; z)$. To see this, consider first the A-cars with preferences $(c_1, \dots, c_r)$. we have $c_i \in \{1, 2, ..., z, z+a, z+2a, ..., z+(r-1)a\}$ for all $1 \le i \le r$, and the order statistics of these r terms are no more than $(z, z+a, \cdots, z+(r-1)a)$ (coordinate-wise). By Theorem 9, $(c_1, \dots, c_r)$ is a parking sequence for $(a^r; z)$. Hence all A-cars must park on $[z, z+ra-1]$. Next, consider the B-cars. Since $c_i \le z+(r-1)a$ and all A-cars are parked without any unoccupied spots on $[z, z+ra-1]$, then all B-cars park in increasing order after the A-cars. In other words, the final parking configuration is $T, C'_1, ..., C'_r, C_{r+1}, ..., C_n$ where $C'_1, ..., C'_r$ is some rearrangement of the A-cars. This proves the claim.

Now, by the above claim and Lemma 5, we have shown that $\mathsf{PS}_{inv}(\vec{y}; z)$ is exactly the set of all sequences $\mathbf{c}$ whose order statistics satisfy $c_{(i)} \le z$ for each $1 \le i \le n-r+1$ and $c_{(n-r+j)} \in$

$\{1, ...z, z+a, z+2a, ..., z+(j-1)a\}$ for each $2 \le j \le r$. Let $\vec{u} = (u_1, u_2, \dots, u_n) = (z, z, ..., z, z+1, z+2, ..., z+r-1)$. Consider the map $\gamma_a : \mathsf{PS}_{inv}(\vec{y}; z) \to \mathsf{PF}_n(\vec{u})$ defined as follows.

$$\gamma_a : (c_1, ..., c_n) \mapsto (c'_1, ..., c'_n) = \mathbf{c}'$$

where for all $1 \le j \le n$

$$c'_j = \begin{cases} c_j, \text{ if } c_j \le z \\ z+s, \text{ if } c_j = z+sa. \end{cases}$$

The map γ_a is well-defined since the sequence $\mathbf{c}'$ has order statistics satisfying $1 \le c'_{(i)} \le u_i$ for each $i = 1, 2, ..., n$. Thus $\mathbf{c}' \in \mathsf{PF}_n(\vec{u})$. Clearly the map γ_a is invertible, hence γ_a is a bijection. □

Corollary 7. *Let $\vec{y}$ and $\vec{u}$ be as in Theorem 10. Then,*

$$\begin{aligned} \#\mathsf{PS}_{inv}(\vec{y}; z) &= \#\mathsf{PF}_n(\vec{u}) \\ &= \sum_{j=0}^{r-1} \binom{n}{j} (r-j)\, r^{j-1} z^{n-j}. \end{aligned}$$

In particular, when $r = 1$, $\#\mathsf{PS}_{inv}(a, b, b, ..., b; z) = z^n$.

Proof. The result follows from Theorem 10 and [[35], Theorem 3]. □

5.2.5 Length vector $\vec{y} = (a, 1, 1, \dots, 1)$ where $a > 1$

A natural question that follows the previous subsection is the case that $\vec{y} = (a^r, b^{n-r})$ with $a > b$. Unlike the preceding case, the number of sequences in $\mathsf{PS}_{inv}(\vec{y}; z)$ depends on the value of a and b. Table 5.1 shows the initial values for $\mathsf{PS}_{inv}(\vec{y}; z)$ where $z = b = 1$ and $a = 2, 3$.

A quick search in the On-Line Encyclopedia of Integer Sequences (OEIS) reveals that both these sequences do not seem to correspond to any known sequences. While we do not have a solution for the general case, in the following we present a small result for the special case where there is one A-car and $n-1$ cars each of size $b = 1$.

Some Initial Values					
$a = 2$:	Length $\vec{y}$	$(2,2)$	$(2,2,1)$	$(2,2,1,1)$	$(2,2,1,1,1)$
	$\#\mathsf{PS}_{inv}(\vec{y};1)$	3	7	31	81
$a = 3$:	Length $\vec{y}$	$(3,3)$	$(3,3,1)$	$(3,3,1,1)$	$(3,3,1,1,1)$
	$\#\mathsf{PS}_{inv}(\vec{y};1)$	3	7	13	51

Table 5.1: $\#\mathsf{PS}_{inv}((a,a,1,...,1);z)$ where $a > 1$.

Proposition 4. *Suppose $z, a \in \mathbb{Z}_+$ with $a > 1$. Let $\mathsf{PF}_n(\vec{u})$ be the set of $\vec{u}$-parking functions, where $\vec{u} = (z, z+1, ..., z+n-1)$. Then*

$$\mathsf{PS}_{inv}((a, 1^{n-1}); z) = \mathsf{PF}_n(\vec{u}).$$

Proof. Let $\mathbf{c} = (c_1, c_2, ..., c_n) \in \mathsf{PS}_{inv}(\vec{y}; z)$ and $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(n)}$ be its order statistics. If $c_{(i)} > z + i - 1$, consider the preference sequence $\mathbf{c}' = (c_{(n)}, c_{(n-1)}, \ldots,, c_{(1)})$. Under $\mathbf{c}'$ the first $n - i + 1$ cars all prefer spots in $[z + i, z + a + n - 2]$. There are only $a + n - i - 1$ spots in this interval yet the total length of the first $n - i + 1$ cars is $a + n - i$. It is impossible to park. Hence we must have $c_{(i)} \leq z + i - 1$ for all i and $\mathbf{c} \in \mathsf{PF}_n(\vec{u})$.

Conversely, given $\mathbf{x} \in \mathsf{PF}_n(\vec{u})$, we know $\mathsf{PF}_n(\vec{u})$ is permutation-invariant, thus we only needs to show that $\mathbf{x} \in \mathsf{PS}(\vec{y}; z)$ where $\vec{y} = (a, 1^{n-1})$. First, $x_1 \leq z + n - 1$ hence C_1 parks. We claim that all the remaining cars can park with the preference sequence $\mathbf{x}$. Assume not, then after all the cars have attempted parking, there are some cars that fail to park and there are empty spots left unoccupied. Let k be such an empty spot. Note that all the remaining cars are of length 1. A car C_i cannot park if and only if all the spots from x_i to the end are occupied when C_i enters. Since $x_i \leq z + n - 1$, it follows that $z \leq k \leq z + n - 1$. From $\mathbf{x} \in \mathsf{PF}_n(\vec{u})$ and condition (2.6), we have

$$\#\{j : x_j \leq k\} \geq k - (z - 1).$$

It means that there are at least $k - (z - 1)$ cars that attempted to park in the spots $[z, k]$, which has exactly $k-(z-1)$ spots. Therefore the spot k must be checked and cannot be left empty, a contradiction. □

Again using the counting formulas for $\vec{u}$-parking functions, we have

Corollary 8. $\#\mathsf{PS}_{inv}(y;z) = z(n+z)^{n-1}$.

5.3 Invariance with respect to the set of car sizes

5.3.1 Strong parking sequences

In this section, we study another type of invariance. Given a fixed set of cars of various lengths and a one-way street (whose length is equal to the sum of the car lengths and a trailer of length $z-1$), we consider all parking sequences for which all n cars can park on the street irrespective of the order in which they enter the street. We will refer to this as strong invariance. Denote by $\mathfrak{S}_n$ the set of all permutations on n letters. For a vector $\vec{y}$ and σ in $\mathfrak{S}_n$, let $\sigma(\vec{y}) = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

Definition 17. Let $\mathbf{c} = (c_1, ..., c_n)$ and $\vec{y} = (y_1, ..., y_n)$. Then, $\mathbf{c}$ is a *strong parking sequence for* $(\vec{y}; z)$ if and only if

$$\mathbf{c} \in \bigcap_{\sigma \in \mathfrak{S}_n} \mathsf{PS}(\sigma(\vec{y}); z).$$

We will denote the set of all strong parking sequences for $(y_1, ..., y_n; z)$ by $\mathsf{SPS}\{y_1, ..., y_n; z\}$ or $\mathsf{SPS}(\vec{y}; z)$. Note that $\mathsf{SPS}\{y_1, y_2, ..., y_n; z\}$ does not depend on the order of the y_i's.

Example 9. Consider the case $n = 2$. Let $\vec{y} = (a, b)$ with $a < b$. It is easy to see that

$$\mathsf{PS}(a,b;z) = \{(c_1,c_2) : 1 \le c_1 \le z, 1 \le c_2 \le z+a\} \cup \{(c_1,c_2) : c_1 = z+b, 1 \le c_2 \le z\},$$

$$\mathsf{PS}(b,a;z) = \{(c_1,c_2) : 1 \le c_1 \le z, 1 \le c_2 \le z+b\} \cup \{(c_1,c_2) : c_1 = z+a, 1 \le c_2 \le z\}.$$

This gives

$$\mathsf{SPS}\{a,b;z\} = \mathsf{PS}(a,b;z) \cap \mathsf{PS}(b,a;z) = \{(c_1,c_2) : 1 \le c_1 \le z, 1 \le c_2 \le z+a\}.$$

Note that $\mathsf{SPS}\{a, b; z\}$ is exactly the set of all preferences $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$ that yields the final parking configuration T, C_1, C_2.

By Ehrenborg and Happ's result (2.7), it is clear that if $\vec{y} = (s, s, ..., s)$, then

$$\#\mathsf{SPS}\{\vec{y}; z\} = \#\mathsf{PS}(\vec{y}; z) = z \cdot \prod_{i=1}^{n-1}(z + is + n - i).$$

In the following we consider the case that $\vec{y}$ does not have identical entries.

Definition 18. We say that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$ parks $\vec{y}$ in the *standard order* if the final parking configuration of $\mathbf{c}$ is given by $T, C_1, C_2, ..., C_n$.

For example, in the case where $(\vec{y}; z) = (2, 3, 1, 2, 1, 4; 3)$, the standard order is shown in Figure 5.1.

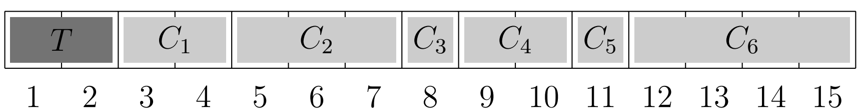

Figure 5.1: Standard order for $\vec{y} = (2, 3, 1, 2, 1, 4)$ and $z = 3$.

The following lemma is easily proved by induction.

Lemma 6. *Let* $\mathbf{c} = (c_1, c_2, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$*. Then,* $\mathbf{c}$ *parks* $\vec{y}$ *in the standard order if and only if*

$$c_k \leq z + y_1 + \cdots + y_{k-1} \text{ for all } k \in [n].$$

The following result characterizes strong invariance for any set of $n \geq 2$ cars with a given multi-set of lengths $\{y_1, y_2, ..., y_n\}$ and a trailer T of length $z - 1$.

Theorem 11. *Let* $n \geq 2$*. Assume that* $\vec{y} = (y_1, ..., y_n)$ *is not a constant sequence. Then* $\mathbf{c}$ *is a strong parking sequence for* $\{y_1, y_2, ..., y_n; z\}$ *if and only if* $\mathbf{c}$ *parks* $\vec{y}_{inc} = (y_{(1)}, y_{(2)}, ..., y_{(n)})$ *in standard order.*

Proof. Suppose $\mathbf{c}$ parks $\vec{y}_{inc}$ in the standard order. We need to check that $\mathbf{c}$ is a preference sequence for $(\sigma(\vec{y}_{inc}); z)$ for every $\sigma \in \mathfrak{S}_n$. This follows from Lemma 6 and the fact that $y_{(1)} + y_{(2)} + \cdots + y_{(i)} \leq y_{\sigma(1)} + y_{\sigma(2)} + \cdots + y_{\sigma(i)}$ for any $\sigma \in \mathfrak{S}_n$ and $i \in [n]$.

Conversely, let $\mathbf{c}$ be a parking sequence for $(\vec{y}_{inc}; z)$ that parks $\vec{y}_{inc}$ in a final configuration $\mathcal{F}$ which is not the standard order. We will construct a permutation σ such that for a sequence of cars with length vector $\sigma(\vec{y}_{inc})$, $\mathbf{c} \notin \mathsf{PS}(\sigma(\vec{y}_{inc}; z)$.

In the following, let C_i represent a car of length $y_{(i)}$, as shown in the table below. Hence $\mathcal{F}$ is the final parking configuration of $\mathbf{c}$ when we park the cars $C_1, \ldots, C_n$. In $\mathbf{c}$, let k_1 be the minimal index k such that $c_k > z + y_{(1)} + y_{(2)} + \cdots + y_{(k-1)}$. Then in $\mathcal{F}$ the trailer is followed by $C_1, \ldots, C_{k_1-1}$ with no gap, and there is a gap between C_{k_1-1} and C_{k_1}. Let C_t be the last car that parks right before car C_{k_1} in $\mathcal{F}$. Clearly $t > k_1$. There are two possibilities: either $y_{(k_1)} < y_{(t)}$ or $y_{(k_1)} = y_{(t)}$.

Car	C_1	C_2	$\cdots$	C_{k_1}	$\cdots$	C_t	$\cdots$	C_{n-1}	C_n
Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

1. Assume $y_{(k_1)} < y_{(t)}$. Let σ_1 be the transposition $(k_1 \longleftrightarrow t)$. Let D_i represent a car of length $y_{\sigma_1(i)}$, as shown in the table below.

σ_1	Car	D_1	D_2	$\cdots$	D_{k_1}	$\cdots$	D_t	$\cdots$	D_{n-1}	D_n
	Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

We park cars $D_1, ..., D_n$ using the preference sequence $\mathbf{c}$. If $\mathbf{c} \in \mathsf{PS}(\sigma_1(\vec{y}); z)$, then $D_1, ..., D_t$ can be parked and

(a) $D_1, D_2, ..., D_{k_1-1}$ have the same lengths and preferences as $C_1, C_2, ..., C_{k_1-1}$ and park in order right after the trailer.

(b) D_{k_1} is longer than C_{k_1} and occupies spots in $[c_{k_1}, c_{k_1} + y_{(t)} - 1]$

(c) Any car D_i for $i \in \{k_1 + 1, ..., t - 1\}$ has the same preference as C_i so it parks either before D_{k_1} and in the same spots as C_i in $\mathcal{F}$, or parks after D_{k_1}.

(d) D_t takes the first y_{k_1} spots of the ones occupied by C_t in $\mathcal{F}$.

After parking $D_1, ..., D_t$, there are $y_{(t)} - y_{(k_1)}$ unused spots between cars D_t and D_{k_1}. Any car trying to park after D_t has length $\geq y_{(t)} > y_{(t)} - y_{(k_1)}$. So the spots between D_t and D_{k_1} cannot be filled and hence $\mathbf{c} \notin \mathsf{PS}(\sigma_1(\vec{y}_{inc}), z)$.

2. Assume $y_{(k_1)} = y_{(t)}$. Then, since $\vec{y}_{inc}$ is not a constant sequence, either $y_{(1)} < y_{(k_1)}$ or $y_{(t)} < y_{(n)}$.

2a. Assume $y_{(t)} < y_{(n)}$. Let σ_2 be the transposition $(t \longleftrightarrow n)$ shown below and E_i be a car of length $\sigma_2(i)$.

σ_2	Car	E_1	E_2	$\cdots$	E_{k_1}	$\cdots$	E_t	$\cdots$	E_{n-1}	E_n
	Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(n)}$	$\cdots$	$y_{(n-1)}$	$y_{(t)}$

We park the cars $E_1, \ldots, E_n$ using the preference sequence $\mathbf{c}$. The cars $E_1, ..., E_{t-1}$ take the same spots as $C_1, \ldots, C_{t-1}$ in $\mathcal{F}$. Next, car E_t tries to park in the spots C_t occupies, at the interval $[c_{k_1} - y_{(t)}, c_{k_1} - 1]$. But E_t has length $y_{(n)}$ which is greater than $y_{(t)}$ and thus cannot fit. Hence, $\mathbf{c} \notin \mathsf{PS}(\sigma_2(\vec{y}); z)$.

2b. If $y_{(k_1)} = ... = y_{(t)} = ... = y_{(n)} = b$, then we must have $k_1 > 1$ and $y_{(1)} < y_{(k_1)}$. Let σ_3 be the transposition $(1 \longleftrightarrow k_1)$ and F_i be a car of length $\sigma_3(i)$.

σ_3	Car	F_1	F_2	$\cdots$	F_{k_1}	$\cdots$	F_t	$\cdots$	F_{n-1}	F_n
	Car Length	$y_{(k_1)}$	$y_{(2)}$	$\cdots$	$y_{(1)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

In the final configuration $\mathcal{F}$, at the time car C_{k_1} is parked, the lengths of all the blocks of consecutive empty spots left are multiples of b. Now for the cars $F_1, ..., F_n$, the cars $F_1, \ldots, F_{k_1-1}$ will take the spaces right after the trailer. The total length of $F_1, \ldots, F_{k_1-1}$ is no more than the total length of $C_1, \ldots, C_{k_1-1}$, and C_t, since $y_{(1)} + y_{(2)} + \cdots + y_{(k_1-1)} + y_{(t)} > y_{(2)} + \cdots + y_{(k_1-1)} + y_{(k_1)}$. So car F_{k_1} will park at the spots starting at c_k, just as C_{k_1}. But, as $y_{(1)} < y_{(k_1)}$, after F_{k_1} is parked, the available space after car F_{k_1} is nonempty and not a multiple of b, while all the remaining cars are of length b. Hence, it is not possible to park all of them.

□

Combining Lemma 6 and Theorem 11, we obtain the following counting formula.

Corollary 9. *Let $z \in \mathbb{Z}_+$ and $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}_+^n$. If $\vec{y} \neq (s^n)$ for any integer s, then*

$$\#\mathsf{SPS}\{y_1, y_2, ..., y_n; z\} = z \cdot \prod_{i=1}^{n-1}(z + y_{(1)} + \cdots + y_{(i)}).$$

where $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$ is the order statistics of $\vec{y}$.

5.3.2 Parking on a street with fixed length

Suppose instead of fixing the set of cars, we fix the total street length. Let $\mathfrak{C}_n = \{\vec{y} = (n_1, n_2, ..., n_k) \in \mathbb{Z}_+^k : n_1 + n_2 + \cdots + n_k = n\}$ i.e. $\mathfrak{C}_n$ is the set of all compositions of n into k parts. We consider all possible sequences that can park any set of k cars on the street of fixed length $z + n - 1$. More formally, we have the following definition.

Definition 19. Let $n, z \in \mathbb{Z}_+$ and $\vec{y} = (n_1, n_2, ..., n_k)$ be a composition of n into k parts. Then, $\mathbf{c} = (c_1, ..., c_k)$ is a *k-strong parking sequence for n* if and only if

$$\mathbf{c} \in \bigcap_{\vec{y} \in \mathfrak{C}_n} \mathsf{SPS}\{n_1, ..., n_k; z\}.$$

We will denote the set of all k-strong parking sequences for n by $\mathsf{SPS}_k(n; z)$ (or $\mathsf{SPS}_k(n)$ when $z = 1$). For example, when $n = 3$, we have the following sets:

$$\mathsf{SPS}_1(n) = \{(1)\}$$

$$\mathsf{SPS}_2(n) = \{(1,1), (1,2)\}$$

$$\mathsf{SPS}_3(n) = \{(1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2), (1,2,3), (1,3,1), (1,3,2), (2,1,1), (2,1,2), (2,1,3), (2,2,1), (2,3,1), (3,1,1), (3,1,2), (3,2,1)\}$$

We remark that in general, for any $n \in \mathbb{N}$, $\mathsf{SPS}_1(n) = \{(1)\}$ and $\mathsf{SPS}_n(n) = \mathsf{PF}_n$ where PF_n is the set of all parking functions of length n. The following proposition helps characterize $\mathsf{SPS}_k(n;z)$ for any $1 \le k \le n$ and $z \in \mathbb{Z}_+$.

Proposition 5. *Suppose $n, z \in \mathbb{Z}_+$ and let $\vec{y}_0 = (1^{k-1}, n-k+1)$ be the composition of n into k parts with $n_1 = n_2 = \cdots = n_{k-1} = 1$ and $n_k = n-k+1$. Then,*

$$\mathsf{SPS}_k(n;z) = \mathsf{SPS}\{\overbrace{1,1,...,1}^{k-1}, n-k+1; z\} = \bigcap_{\sigma \in \mathfrak{S}_n} \mathsf{PS}(\sigma(\vec{y}_0); z). \tag{5.5}$$

In other words, $\mathsf{SPS}_k(n;z)$ is the set of all sequences in $\mathsf{PS}(\vec{y}_0; z)$ that yield the standard order.

Proof. Follows from Lemma 6 and the fact that for any $\vec{y} = (n_1, ..., n_k)$ with $n_1 + \cdots + n_k = n$, we have for each $i \in [k-1]$,

$$\overbrace{1 + 1 + \cdots + 1}^{i} = i \le n_1 + \cdots + n_i.$$

□

Corollary 10.

$$\#\mathsf{SPS}_k(n;z) = \begin{cases} z^{(k)}, \ \textit{if } k \ne n \\ z(n+z)^{n-1}, \ \textit{if } k = n. \end{cases}$$

where $z^{(k)} = z(z+1)\cdots(z+n-1)$. In particular, when $z = 1$,

$$\#\mathsf{SPS}_k(n) = \begin{cases} k!, \ \textit{if } k \ne n \\ (n+1)^{n-1}, \ \textit{if } k = n. \end{cases}$$

Proof. Follows from Proposition 5 and Corollary 9. □

6. CONCLUSION AND FINAL REMARKS

This dissertation gives a new combinatorial interpretation to Gončarov polynomials. It also studies some special classes of parking sequences that opens many new avenues for research. In this section, we expand on several of these possibilities.

In chapters 3 and 4, we present the combinatorial interpretation of an arbitrary sequence of Gončarov polynomials associated with a polynomial sequence of binomial type. There are many other combinatorial problems that provide a formal framework of coalgebras, bialgebras, or Hopf algebras [14]. In those problems the counting sequences satisfy an identity that is analogous to the binomial-type identity (2.1), with the binomial coefficients $\binom{n}{i}$ replaced by some other section coefficients. For example, the theory of binomial enumeration proposed by Mullin and Rota [24] was generalized to an abstract context and applied to dissecting schemes by Henle [13]. It would be an interesting project to investigate the role of generalized Gončarov polynomials in these other dissecting schemes and discrete structures. As suggested by Henle, this research may lead to connections to rook polynomials, order invariants of posets, Tutte invariants of combinatorial geometries, cycle indices and symmetric functions, and many others.

We discussed increasing and invariant parking sequences in chapter 5. We looked at increasing parking sequences and their representations via lattice paths. We have also studied permutation-invariant parking sequences and length-invariant parking sequences. Specifically, we characterized all permutation-invariant parking sequences for some specific length vectors. While it may not be easy to find a general formula for all cases, a natural direction to go would be to study other special cases of car lengths. Furthermore, in the study of parking functions we encounter quite a number of other mathematical structures including trees, non-crossing partitions, hyperplane arrangements, polytopes etc. It will be interesting to find any connections between notions of invariance in parking sequences and other combinatorial structures. Recently in [2], parking sequences of [8] was extended to the case in which one or more trailers are placed anywhere on the street alongside n cars with length vector $\vec{y} = (1, 1, ..., 1)$. A natural generalization is to consider a similar scenario where $\vec{y}$ is any length vector with $y_i \geq 1$ for each $i \in [n]$.

One other class of parking functions that has been studied are the so-called prime parking functions. These are parking functions that satisfy only the strict part of the inequality (2.5). It is known that there are $(n-1)^{n-1}$ many prime parking functions. In section 5.1, we gave an analog for (2.5) in Lemma 2. Although it is necessary, the inequality (5.1) is not sufficient to characterize all parking functions. However, if we define *prime parking sequences* as all sequences that satisfy the strict inequality in (2.5), we believe it would be a worthwhile endeavor to study these sequences.

Finally, there are several statistics that have been considered in the set of parking functions e.g. the "lucky drivers" statistic (the number of drivers who park in their preferred spot), the sum statistic, the reversed sum or total displacement statistic, the number of distinct driver preferences and the number of spots the most unlucky driver had to check before parking. Studying any of these statistics on parking sequences may yield interesting results.

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