FINITE ELEMENT APPROXIMATION OF EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR

A Dissertation

by

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ABSTRACT

The surface finite element method is an important tool for discretizing and solving elliptic partial differential equations on surfaces. Recently the surface finite element method has been used for computing approximate eigenvalues and eigenfunctions of the Laplace-Beltrami operator, but no theoretical analysis exists to offer computational guidance. In this dissertation we develop approximations of the eigenvalues and eigenfunctions of the Laplace-Beltrami operator using the surface finite element method. We develop a priori estimates for the eigenvalues and eigenfunctions of the Laplace-Beltrami operator. We then use these a priori estimates to develop and analyze an optimal adaptive method for approximating eigenfunctions of the Laplace-Beltrami operator.

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NOMENCLATURE

FEM	Finite Element Method
SFEM	Surface Finite Element Method
AFEM	Adaptive Finite Element Method
DOF	Degrees of Freedom
PDE	Partial Differential Equation
γ	Exact surface
$\overline{\Gamma}$	Polyhedral approximation of γ
Γ	Piecewise polynomial approximation of γ
$\mathcal{T}$	Triangulation of the domain
$\mathbb{T}$	A family of triangulations

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1. INTRODUCTION

The need for accurate approximations of Laplace-Beltrami eigenpairs arises in a variety of applications. For example, there are approaches to shape classification based on the Laplace-Beltrami operator's spectral properties [2, 3, 4, 5, 6, 7, 8]. More specifically, the spectrum has been used as a "shape DNA" to create a fingerprint of a surface's shape. One possible application of shape classification is medical imaging. In this scenario the underlying surface γ is not known precisely, but is instead sampled via a medical scan. The spectrum that is studied is thus that of a reconstructed approximate surface rather than the underlying surface γ. Often this reconstruction is represented as a polyhedral approximation (triangulation). Bootstrap methods are another potential application of Laplace-Beltrami spectral calculations [9]. Finally, Laplace-Beltrami eigenvalues on subsurfaces of the sphere characterize singularities in solutions to elliptic PDE arising at vertices of polyhedral domains [10, 11, 12]. Many of these papers use the surface finite element method (SFEM) in order to calculate Laplace-Beltrami spectral properties. While these methods show empirical evidence of success, there has to date been no detailed analysis of the accuracy of the eigenpairs calculated using SFEM. Some of these papers employing SFEM also propose using higher-order finite element methods to improve accuracy, but do not suggest how to properly balance the discretization of γ with the degree of the finite element space. Guidance for understanding the interaction between geometric consistency and Galerkin errors in the context of spectral problems is needed.

In this work we study the approximation properties of SFEM when applied to computing eigenpairs of the Laplace-Beltrami operator. In the remaining sections of this chapter we provide an overview of both a priori and adaptive finite element results for the source problem and eigenvalue problem on flat (Euclidean) domains and surfaces that are relevant to the rest of this work. In Chapter 2 we present an a priori analysis of the SFEM approximations of eigenpairs of the Laplace-Beltrami operator based on our work in [1]. In Chapter 3 we develop and analyze an SFEM-based adaptive algorithm for approximating the eigenfunctions of the Laplace-Beltrami operator.

1.1 A Priori Estimates for the Source Problem on Flat Domains

We begin by examining a prototypical example problem for testing the finite element method (FEM). We solve the source problem with homogeneous Dirichlet boundary conditions defined on a convex domain Ω, i.e.

$$\begin{aligned} -\Delta u &= f \qquad \text{on } \Omega \\ u|_{\partial\Omega} &= 0 \end{aligned} \tag{1.1}$$

with $f \in L_2(\Omega)$. We can define a weak formulation of this problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \qquad \forall v \in H_0^1(\Omega). \tag{1.2}$$

We call the left hand side of (1.2) the bilinear form and denote it as

$$a(u, v) := \int_\Omega \nabla u \nabla v \, d\Omega.$$

We use the shorthand (f, v) for the L_2 inner product on the right hand side of (1.2).

We say a bilinear form $a(u, v)$ is continuous on a space V if there exists a constant $M > 0$ such that

$$|a(u, v)| \leq M \|u\|_V \|v\|_V \qquad \forall u, v \in V.$$

We say a bilinear form is V-elliptic if there exists a constant c such that

$$a(u, u) \geq c\|u\|_V^2 \qquad \forall u, v \in V.$$

The V-ellipticity and continuity properties of the bilinear form guarantee a unique solution to (1.2) thanks to the following result from Lax and Milgram.

Lemma 1.1 (Lax-Milgram)**.** *Let* $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ *be a continuous coercive bilinear form. Then* *for each $f \in V^*$ the variational equation*

$$a(u, v) = (f, v) \qquad \forall v \in V$$

has a unique solution $u \in V$, furthermore the a priori estimate

$$\|u\|_V \leq \frac{1}{c}\|f\|_{V^*}$$

is valid. Here, by V^* we mean the dual of V.

Essential to the finite element method is the creation of a mesh $\mathcal{T}$ which we will assume to be triangular unless specified otherwise. Given a mesh $\mathcal{T}$ of Ω, we can define a degree-r finite element space for the weak problem (1.2) as

$$\mathbb{V} := \{v_h \in H_0^1(\Omega) : v_h|_T \in \mathbb{P}^r(T),\ T \in \mathcal{T}\}.$$

We define the finite element problem as: find the solution $u_h \in \mathbb{V}$ such that

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in \mathbb{V}.$$

In this situation u_h can be thought of as a unique projection $\mathbf{G}_h : u \mapsto u_h$ onto the finite element space $\mathbb{V}$. Noting that since $\mathbb{V} \subset H_0^1(\Omega)$ we have for $v_h \in \mathbb{V}$

$$a(u, v_h) = (f, v_h) = a(u_h, v_h),$$

we then define the Galerkin projection operator $\mathbf{G}_h : H_0^1(\Omega) \to \mathbb{V}$ as the operator satisfying

$$a(u, v_h) = a(\mathbf{G}_h u, v_h) \qquad \forall v_h \in \mathbb{V}.$$

A natural question to ask is how close is the finite element solution u_h to u and in what sense?

Common norms used for measuring the finite element error are L_2, H^1, and L_∞. A priori error estimates seek to characterize error if we know the regularity of u.

We say that a family of meshes is shape-regular if there exists a constant C such that for any mesh $\mathcal{T}$ in the family and every triangle $T \in \mathcal{T}$ the ratio of the longest edge length h_T to that of the radius ρ_T of the largest inscribed ball is bounded by C, i.e.

$$\frac{h_T}{\rho_T} \leq C.$$

For a quasi-uniform mesh a priori estimates in the L_2 and H^1 norms for flat domains state that if $u \in H^{k+1}(\Omega)$, $f \in L_2(\Omega)$, and Ω is convex, then the error when using a degree-r finite element method with mesh size h satisfies the following bounds:

$$\|u - \mathbf{G}_h u\|_{H^1(\Omega)} \leq h^{\min(r,k)} |u|_{H^{k+1}(\Omega)} \tag{1.3}$$

$$\|u - \mathbf{G}_h u\|_{L_2(\Omega)} \leq h^{\min(r+1,k+1)} |u|_{H^{k+1}(\Omega)}. \tag{1.4}$$

1.2 A Priori Estimates for the Eigenvalue Problem on Flat Domains

The spectrum of the Laplacian is ubiquitous in the sciences and engineering. Consider the eigenvalue problem

$$-\Delta u = \lambda u$$

$$u|_{\partial\Omega} = 0$$

on a flat domain Ω. There is then a sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$ of eigenvalues with corresponding L_2-orthonormal eigenfunctions $\{u_i\}$. Given a finite element space $\mathbb{V} \subset H_0^1(\Omega)$, the natural finite element counterpart is to find $(u_h, \lambda_h) \in \mathbb{V} \times \mathbb{R}^+$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \lambda_h \int_\Omega u_h v_h \, d\Omega, \qquad \forall v_h \in \mathbb{V}.$$

Finite element methods (FEM) are a natural and widely used tool for approximating spectra of elliptic PDE. Analyzing the error behavior of such FEM is more challenging than for source problems because of the nonlinear nature of the problem. A priori error estimation for FEM approximations of the eigenvalues and eigenfunctions of the Laplacian and related operators in flat space is a classical topic in finite element theory; cf. [13, 14, 15, 16]. We highlight the review article [17] of Babuška and Osborn in this regard. These bounds are all asymptotic in the sense that they require an initial fineness condition on the mesh. More recently, sharp bounds for eigenvalues (but not eigenfunctions) appeared in [18]. These bounds are notable because they are truly a priori in the sense that they do not require a sufficiently fine mesh. Finally, over the past decade a number of papers have appeared analyzing convergence and optimality of adaptive finite element methods (AFEM) for eigenvalue problems [19, 20, 21, 22, 23, 24]. Because sharp a priori estimates are needed in order to analyze AFEM optimality properties, some of these papers also contain improved a priori estimates. We particularly highlight [21, 23] as our analysis of eigenfunction errors largely employs the framework of these papers.

Also, the constants in the first estimate are asymptotically independent of λ, while the constants in the second estimate depend in essence on the separation of λ from the remainder of the spectrum and the degree to which the discrete spectrum respects that separation. Corresponding "cluster-robust" estimates also hold for simultaneous approximation of clusters of eigenvalues.

Of particular interest to us will be what are known as eigenvalue clusters. Given an interval $[A, B]$, we define the eigenvalue cluster to be the set of all eigenvalues within the interval $[A, B]$ and their associated eigenfunctions. The assumptions on eigenvalue clusters are slightly weaker and only require that for any $\lambda_i \in [A, B]$ we have $\lambda_{h,i} \in [A, B]$.

When measuring the approximation errors for eigenfunctions one must be more careful and cannot just measure $u - u_h$ like in the context of the source problem. We illustrate this through a simple example. Suppose $-\Delta$ has an eigenvalue λ with geometric multiplicity two. There will then be two FEM eigenvalues $\lambda_{h,1}, \lambda_{h,2}$ converging to λ; $\lambda_{h,1}, \lambda_{h,2} \to \lambda$. The eigenvalue λ will have two eigenfunctions u_1 and u_2 and the FEM eigenvalues will have two eigenfunctions $u_{h,1}$

and $u_{h,2}$. The issue occurs when we try to ask if $u_{h,1}$ converge to u_1. It is just as valid to ask if $u_{h,1}$ converge to u_2 or any linear combination of u_1 and u_2, but $u_{h,1}$ can only converge to a single function.

The correct question to ask when studying eigenfunction approximation becomes how well does the subspace of FEM eigenfunctions associated with the cluster approximate the eigenfunctions of the cluster. We can express this through the language of projection operators. Define

$$\mathbb{W}_h := span\{u_{h,i}\}_{i \in J}.$$

Let $\mathbf{P}_h : L_2(\Omega) \to \mathbb{W}_h$ be the L_2 projection onto $\mathbb{W}_h$, i.e.

$$(u, w_h) = (\mathbf{P}_h u, w_h) \qquad \forall w_h \in \mathbb{W}_h.$$

In the case of eigenvalue clusters on flat domains, if $u \in H^{r+1}(\Omega)$ the a priori estimates when using degree-r finite elements are typically of the form

$$\begin{aligned} \|u - \mathbf{P}_h u\|_{H^1(\Omega)} &\lesssim \|u - \mathbf{G}_h u\|_{H^1(\Omega)} \lesssim h^r |u|_{H^{r+1}(\Omega)} \\ \|u - \mathbf{P}_h u\|_{L_2(\Omega)} &\lesssim h\|u - \mathbf{G}_h u\|_{H^1(\Omega)} \lesssim h^{r+1} |u|_{H^{r+1}(\Omega)} \\ |\lambda - \lambda_h| &\lesssim \|u - \mathbf{P}_h u\|^2_{H^1(\Omega)} \lesssim h^{2r} |u|^2_{H^{r+1}(\Omega)}. \end{aligned} \tag{1.5}$$

We see in (1.5) that the convergence rates for eigenfunctions match the rates in (1.3) and (1.4) that we've come to expect for FEM solutions to the source problem. However, the eigenvalues converge much faster. In fact, they converge at twice the rate of the eigenfunctions.

While $\mathbf{P}_h$ is commonly used in eigenfunction error estimates, it is sometimes convenient to characterize the eigenfunction error in terms of a different projection operator. We will regularly use the projection operator

$$\mathbf{Z}_h := \mathbf{P}_h \mathbf{G}_h : H^1_0(\Omega) \to \mathbb{W}_h.$$

We note that $\mathbf{Z}_h$ is the Galerkin projection onto $\mathbb{W}_h \subset \mathbb{V}$, i.e.

$$a(u, v_h) = a(\mathbf{Z}_h u, v_h), \qquad \forall v_h \in \mathbb{W}_h.$$

It can be shown that for $u \in H^{r+1}$, $\mathbf{Z}_h$ satisfies similar a priori estimates to those of $\mathbf{P}_h$:

$$\|u - \mathbf{Z}_h u\|_{H^1(\Omega)} \lesssim \|u - \mathbf{G}_h u\|_{H^1(\Omega)} \lesssim h^r |u|_{H^{r+1}(\Omega)}$$
$$\|u - \mathbf{Z}_h u\|_{L_2(\Omega)} \lesssim h\|u - \mathbf{G}_h u\|_{H^1(\Omega)} \lesssim h^{r+1} |u|_{H^{r+1}(\Omega)}.$$

1.3 A Posteriori Estimates and Adaptivity for the Source Problem on Flat Domains

While a priori estimates are a useful tool in the study of convergence, we sometimes do not know anything about the true solution u. A posteriori estimates seek to estimate the finite element error based solely on computable quantities. Early studies of a posteriori estimation were carried out in 1978 by Babuska and Rheinboldt [25]. We will focus solely on what is known as the residual estimator. For the source problem with a triangulation $\mathcal{T}$ and FEM solution U it takes the form

$$\eta_{\mathcal{T}}(U, T)^2 := h_T^2 \|f + \Delta U\|_{L_2(T)}^2 + \sum_{S \in \partial T} h_T \|\mathbf{n}^+ \cdot \nabla U^+ + \mathbf{n}^- \cdot \nabla U^-\|_{L_2(S)}^2, \tag{1.6}$$

$$\eta_{\mathcal{T}}(U)^2 := \sum_{T \in \mathcal{T}} \eta_{\mathcal{T}}(U, T)^2.$$

The estimator $\eta_{\mathcal{T}}(U, T)$ can be used to locally estimate the finite element error in the $H^1(\Omega)$ norm for the source problem. It can be shown that this estimator is equivalent to $\|u - u_h\|_{H^1(\Omega)}$ up to higher order terms. Let Π_n^2 denote the L_2 projection onto $\mathbb{P}^n(T)$. There exist constants C_{rel} and C_{eff} such that

$$\|u - U\|_{H^1(\Omega)}^2 \le C_{rel} \eta_{\mathcal{T}}^2, \tag{1.7}$$

$$C_{eff} \eta_{\mathcal{T}}^2 \le \|u - U\|_{H^1(\Omega)}^2 + Osc_{\mathcal{T}}(f, U)^2, \tag{1.8}$$

where

$$Osc_{\mathcal{T}}(f,U)^2 := \sum_{T\in\mathcal{T}} h_T^2 \|(\mathrm{id}-\Pi^2_{2r-2})(f+\Delta U)\|^2_{L_2(T)}$$
$$+ \sum_{S\in\partial T} h_T \|(\mathrm{id}-\Pi^2_{2r-1})(\mathbf{n}^+\cdot\nabla U^+ + \mathbf{n}^-\cdot\nabla U^-)\|^2_{L_2(S)}$$

is higher order than the finite element degree. The bounds in (1.7) and (1.8) are commonly referred to as a posteriori estimates. The upper bound in (1.7) is commonly referred to the reliability bound. It guarantees that the estimator $\eta_{\mathcal{T}}(U)$ does not underestimate the H^1 error. The lower bound in (1.8) is commonly referred to as the efficiency bound. It guarantees that up to the higher order oscillation term the estimator $\eta_{\mathcal{T}}(U)$ does not overestimate the error.

With the ability to estimate error contributions on individual mesh triangles comes the possibility of driving mesh refinement based on these estimates. Rather than creating a family of meshes that is quasi-uniform, it is possible to create a family of meshes $\mathbb{T}$ such that the mesh $\mathcal{T}_{\ell+1}$ is a refinement of $\mathcal{T}_\ell$ in regions where the error is estimated to be largest according to $\eta_{\mathcal{T}_\ell}(U_\ell, T)$. We call such systems of refinement adaptive algorithms. Typical adaptive mesh refinement algorithms consist of looping over the sequence of 4 steps:

$$SOLVE \to ESTIMATE \to MARK \to REFINE.$$

The 4 steps break down as follows:

1. SOLVE: Solve the finite element problem on the mesh $\mathcal{T}_\ell$ for $U_\ell \in \mathbb{V}_\ell$.

2. ESTIMATE: Estimate the error using $\eta_{\mathcal{T}_\ell}(U_\ell, T)$.

3. MARK: Mark a set of triangles $\mathcal{M}_\ell$ of minimum cardinality satisfying

$$\sum_{T\in\mathcal{M}_\ell} \eta_{\mathcal{T}_\ell}(U_\ell,T)^2 \ge \theta \sum_{T\in\mathcal{T}_\ell} \eta_{\mathcal{T}_\ell}(U_\ell,T)^2,$$

   where $0 < \theta \le 1$ is a user specified quantity called the bulk parameter which can roughly be interpreted as the fraction of the total error that the user wants marked for refinement. This style of marking is called Dörfler or bulk marking.

4. REFINE: Refine the elements of $\mathcal{M}_\ell$ $b \geq 1$ times and refine additional elements to ensure the new mesh $\mathcal{T}_{\ell+1}$ is conforming with $\mathbb{V}_\ell \subset \mathbb{V}_{\ell+1}$.

In 1984 Babuska and Vogelius gave an adaptive convergence analysis for 1D problems [26]. In 1996 Dörfler [27] provided foundational ideas for analysis of adaptive FEM. Binev, Dahmen, DeVore used ideas from nonlinear approximation theory in 2004 [28] to establish optimality of AFEM. Stevenson's work in 2007 [29] establishes optimality of a standard AFEM. In 2008 Cascon, Kreuzer, Nochetto, and Siebert [30] established a standard recipe for linear elliptic scalar problems. The ingredients are as follows:

1. A posteriori upper bounds as in (1.7).

2. Orthogonality

   Lemma 1.2 (Orthogonality)**.** *If for any pair $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T} \leq \mathcal{T}_*$ there holds $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}_*)$, then*

$$\|u - U_*\|^2_{H^1_0(\Omega)} = \|u - U\|^2_{H^1_0(\Omega)} + \|U - U_*\|^2_{H^1_0(\Omega)}$$

   for $U \in \mathbb{V}(\mathcal{T})$ and $U_* \in \mathbb{V}(\mathcal{T}_*)$.

3. Estimator Reduction

   Lemma 1.3 (Estimator Reduction)**.** *For $\mathcal{T} \in \mathbb{T}$ and $\mathcal{M} \subset \mathcal{T}$ let $\mathcal{T}_* \in \mathbb{T}$ be given by $\mathcal{T}_* := REFINE(\mathcal{T}, \mathcal{M})$. If $\lambda := 1 - 2^{\frac{b}{d}} > 0$ then there holds for any $V \in \mathbb{V}(\mathcal{T})$ and $V_* \in \mathbb{V}(\mathcal{T}_*)$ and any $\delta > 0$*

$$\eta_{\mathcal{T}_*}(\mathcal{T}_*; V_*)^2 \leq (1+\delta)(\eta_{\mathcal{T}}(\mathcal{T}; V)^2 - \lambda \eta_{\mathcal{T}}(\mathcal{M}; V)^2) + C_{red}(1+\delta^{-1})\|V_* - V\|^2_{H^1_0(\Omega)}.$$

4. Contraction Property

Theorem 1.4 (Contraction Property). *There exist constants $\beta > 0$ and $0 < \rho < 1$ depending on the shape regularity of $\mathcal{T}_0$, the refinement level b, and the bulk parameter θ such that*

$$\|u - U_{\ell+1}\|_{H_0^1(\Omega)} + \beta\eta_{\mathcal{T}_{\ell+1}}(\mathcal{T}_{\ell+1}; U_{\ell+1})^2 \leq \rho\left(\|u - U_\ell\|_{H_0^1(\Omega)} + \beta\eta_{\mathcal{T}_\ell}(\mathcal{T}_\ell; U_\ell)^2\right)$$

for any two consecutive mesh refinements $\mathcal{T}_0 \leq \mathcal{T}_\ell \leq \mathcal{T}_{\ell+1}$.

When the ingredients are combined the recipe yields a result that says if the solution data pair (u, f) belongs to a specific approximation class $\mathbb{A}_s$

$$\sigma(N; v, f) := \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{V\in\mathbb{V}(\mathcal{T})} \left(\|v - V\|_{H^1(\Omega)}^2 + Osc_{\mathcal{T}}(V, \mathcal{T})^2\right)^{\frac{1}{2}}$$

$$\mathbb{A}_s := \left\{(v, f) : |v, f|_{\mathbb{A}_s} := \sup_{N>0}(N^s\sigma(N; v, f)) < \infty\right\}$$

then the adaptive algorithm will recover the expected optimal order s convergence rate.

Theorem 1.5 (Theorem 5.11 of [30]). *If the bulk parameter $\theta << 1$ is sufficiently small, then the adaptive algorithm using the estimator in* (1.6) *generates dicrete solutions with optimal rate of convergence. Let u be the solution and let $\{\mathcal{T}_\ell, \mathbb{V}_\ell, U_\ell\}_{\ell\geq 0}$ be the sequence of meshes, finite element spaces, and discrete solutions produced by AFEM. Let $(u, f) \in \mathbb{A}_s$, then*

$$\left(\|u - U_\ell\|_{H_0^1(\Omega)}^2 + Osc_{\mathcal{T}_\ell}(U_\ell, f)^2\right)^{\frac{1}{2}} \leq C|u, f|_{\mathbb{A}_s}(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}.$$

1.4 Adaptivity for the Eigenvalue Problem on Flat Domains

Just as it is possible to create an a posteriori estimator to measure errors for the source problem, it is also possible to derive an a posteriori estimator to measure eigenfunction errors for the eigenvalue problem. For example, in [23, 24] the estimator used for eigenfunctions associated with an eigenvalue cluster indexed by the set J takes the form

$$\eta_{\mathcal{T}_\ell}(J,T)^2 := \sum_{j\in J} h_T^2 \|\Lambda_{\ell,j} U_{\ell,j} + \Delta U_{\ell,j}\|_{L_2(T)}^2 + \sum_{S\in\partial T} h_T \|\mathbf{n}^+ \cdot \nabla U_{\ell,j}^+ + \mathbf{n}^- \cdot \nabla U_{\ell,j}^-\|_{L_2(S)}^2. \quad (1.9)$$

Notice that this is just the residual estimator in (1.6) with $f = \Lambda_{\ell,j} U_{\ell,j}$ for individual eigenfunctions summed over the cluster.

Unfortunately, the form of this estimator does not lend itself well to deriving the necessary ingredients for proving optimality of an AFEM algorithm. In particular, it is difficult to compare FEM solutions on different meshes since the estimator depends explicitly on the discrete eigenfunctions. In [22] the idea of analyzing an AFEM algorithm for eigenfunctions using a theoretical estimator which avoids the pitfalls of working with the original estimator was introduced. The algorithm is analyzed as if the theoretical estimator is used for driving refinement and then the theoretical estimator is shown to be equivalent to the actual computable estimator. In the case of eigenvalue clusters this theoretical estimator which is equivalent to $\eta_{\mathcal{T}_\ell}(J,T)$ takes the form

$$\mu_{\mathcal{T}_\ell}(J,T)^2 := \sum_{j\in J} h_T^2 \|\lambda_j \mathbf{P}_\ell u_j + \Delta \mathbf{Z}_\ell u_j\|_{L_2(T)}^2 + \sum_{S\in\partial T} h_T \|\mathbf{n}^+ \cdot \nabla \mathbf{Z}_\ell u_j^+ + \mathbf{n}^- \cdot \nabla \mathbf{Z}_\ell u_j^-\|_{L_2(S)}^2.$$

The proof of optimality of adaptive eigenfunction approximation not only contains analogues to all of the steps presented in the work of [30] for the source problem, but it also contains additional assumptions on the maximum mesh size of $\mathcal{T}_0$ much like the a priori analysis for the eigenvalue problem. In fact, in order to handle the nonlinearities present in the eigenvalue problem in the theoretical analysis it is necessary to have a priori H^1 and L_2 estimates for eigenfunctions associated with the cluster. These estimates are used to measure the nonlinearity present in the problem and their presence in the analysis results in restrictions on the initial mesh resolution before the adaptive mesh refinement algorithm runs. The analysis culminates in the following theorem.

Theorem 1.6 ([23, 24]). *If $u_j \in \mathbb{A}_s$, $j \in J$, the bulk parameter $\theta << 1$ is sufficiently small, and the maximum mesh size H_0 is sufficiently small, then the adaptive algorithm using estimator*

(1.9) *generates a set* $\{\mathcal{T}_\ell, \mathbb{V}_\ell, \{(U_j, \lambda_{h,j})\}_{j\in J}\}_{\ell\geq 0}$ *of meshes, finite element spaces, and discrete eigenpairs satisfying*

$$\left(\sum_{j\in J} \|u_j - \mathbf{Z}_\ell u_j\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \left(\sum_{j\in J} |u_j|^2_{\mathbb{A}'_s}\right)^{\frac{1}{2}}.$$

1.5 A Priori Estimates for the Surface Finite Element Method

In this section we briefly explain the ideas behind the surface finite element method when applied to the source problem on surfaces. More details about the SFEM formalism specific to the topics in Chapters 2 and 3 can be found in their respective chapters.

Let $\gamma \subset \mathbb{R}^{d+1}$ be a smooth, closed, orientable d-dimensional surface, and let Δ_γ be the Laplace-Beltrami operator on γ. Given an f satisfying $\int_\gamma f\, d\sigma = 0$ one can define the source problem on γ as

$$-\Delta_\gamma u = f,$$
$$\int_\gamma u\, d\sigma = 0.$$

One can also write a weak form as: find $u \in H^1_0(\gamma)$

$$\int_\gamma \nabla_\gamma u \nabla_\gamma v\, d\sigma = \int_\gamma fv\, d\sigma \qquad \forall v \in H^1_0(\gamma), \tag{1.10}$$

where ∇_γ denotes the tangential gradient on γ and $H^1_0(\gamma)$ is to be interpreted as functions in $H^1(\gamma)$ satisfying $\int_\gamma u\, d\sigma = 0$. We call the left hand side of (1.10) the bilinear form and denote it as

$$a(u,v) := \int_\gamma \nabla_\gamma u \nabla_\gamma v\, d\sigma.$$

The right hand side of (1.10) is the L_2 inner product for functions defined on γ and will be denoted by the more compact notation $m(u,v) := \int_\gamma uv\, d\sigma$.

SFEM allows us to approximate solutions to (1.10). The SFEM corresponding to the cotangent method was introduced by Dziuk [31] in 1988. In [32] Demlow developed a natural higher order analogue to this method. Given a surface γ an approximate polyhedral surface $\overline{\Gamma}$ with shape-regular triangular faces of diameter h having vertices on γ is introduced. We assume the presence of a bijective mapping $\psi : \overline{\Gamma} \to \gamma$. This provides a piecewise linear approximation of γ. It is possible to create a higher order approximation if we take each of the triangular faces of the polyhedron $\overline{\Gamma}$ and create a degree-k Lagrange interpolant of ψ, $\psi_k := I_k\psi$. We then define the new piecewise polynomial surface $\Gamma = \psi_k(\overline{\Gamma})$. The mappings ψ and ψ_k give a way to relate functions defined on γ and Γ through compositions of mappings. These concepts are illustrated in Figure 1.1.

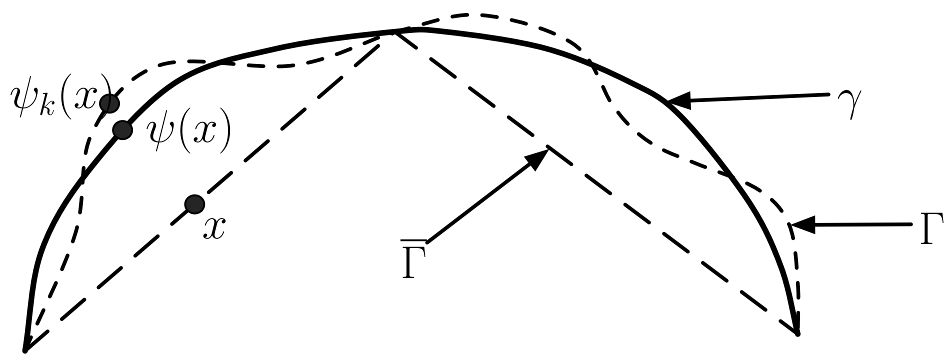

Figure 1.1: Typical surface approximation in SFEM. Surface γ with approximate surfaces $\overline{\Gamma}$, Γ, and mappings.

A weak formulation can then be introduced on the new surface Γ: Find $U \in H_0^1(\Gamma)$ such that

$$A(U,V) := \int_\Gamma \nabla_\Gamma U \nabla_\Gamma V \, d\Sigma = \int_\Gamma UF \, d\Sigma =: M(U,V), \qquad \forall V \in H_0^1(\Gamma).$$

Here F is a function related to f that satisfies the mean 0 condition on Γ. For instance, if the measures on γ and Γ satisfy a relation of the form $d\sigma = \frac{q}{Q} d\Sigma$, then $F = f\frac{q}{Q}$ is a valid choice.

Let $\overline{\mathcal{T}}$ denote the mesh on $\overline{\Gamma}$ and $\overline{V}$ denote functions originally defined on $\overline{\Gamma}$. We define a finite element space

$$\mathbb{V} := \{V \in H^1(\Gamma) \ : \ V = \overline{V} \circ \psi_k^{-1}, \text{ with } \overline{V}|_{\overline{T}} \in \mathbb{P}^r(\overline{T}) \quad \forall \overline{T} \in \overline{\mathcal{T}}\}.$$

Here $\mathbb{P}^r(\overline{T})$ denotes the space of polynomials of degree at most r on $\overline{T}$. Its subspace consisting of zero mean value functions is denoted $\mathbb{V}_{\#}$:

$$\mathbb{V}_{\#} := \mathbb{V}_{\#}(\Gamma) = \{V \in \mathbb{V} \ : \ \int_\Gamma V \, d\Sigma = 0\}.$$

The surface finite element problem is then: seek $U \in \mathbb{V}_{\#}$ such that

$$A(U,V) = M(F,V), \qquad \forall V \in \mathbb{V}_{\#}. \tag{1.11}$$

It is immediately clear from comparing (1.11) with (1.10) that the bilinear forms, L_2 inner products, and functions f and F differ. This leads to variational crimes. All of these crimes are due to the geometric approximations. We commonly refer to these errors as geometric consistency errors. Taking $\mathbf{G}$ to be the Galerkin projection onto $\mathbb{V}_{\#}$, the results of the a priori analysis of Demlow [32] are stated in Theorem 1.7. It can be seen that SFEM exhibits two error sources, a standard Galerkin error and a geometric consistency error due to the approximation of γ by Γ.

Theorem 1.7 (Corollary 4.2 of [32])**.** *Let $u \in H^{r+1}(\gamma)$ and γ be a C^∞ surface. If a degree-r finite element method is used and a degree-k surface interpolant is used, then*

$$\|u - \mathbf{G}u\|_{H^1(\gamma)} \le C(h^r \|u\|_{H^{r+1}(\gamma)} + h^{k+1}\|\nabla_\gamma u\|_{L_2(\gamma)}), \tag{1.12}$$

$$\left\| u - \mathbf{G}u - \left(\int_\gamma u - \mathbf{G}u \, d\sigma \right) \right\|_{L_2(\gamma)} \le C(h^{r+1} \|u\|_{H^{r+1}(\gamma)} + h^{k+1}\|\nabla_\gamma u\|_{L_2(\gamma)}). \tag{1.13}$$

1.6 An Adaptive Surface Finite Element Method

A posteriori estimators and adaptive algorithms based on SFEM have been developed for the source problem on surfaces [33, 34, 35, 36]. Our adaptive algorithm for eigenfunctions in Chapter

3 borrows from [35] and so here we summarize the ideas of [35] relevant to this work.

With the need to approximate the surface comes the need to modify the adaptive algorithm of Section 1.3. We envision a situation where the surface has low regularity and quasi-uniform refinement cannot recover the expected convergence rates. One might naively hope that the surface version of the estimator (1.6) in (1.14) could be used to drive mesh refinement, but that does not control the geometric consistency errors.

$$\eta_{\mathcal{T}}(U,T)^2 := h_T^2\|F + \Delta_\Gamma U\|_{L_2(T)}^2 + \sum_{S\in\partial T} h_T\|\mathbf{n}^+\cdot\nabla_\Gamma U^+ + \mathbf{n}^-\cdot\nabla_\Gamma U^-\|_{L_2(S)}^2 \tag{1.14}$$

The form of the a priori estimates in Theorem 1.7 offers some intuition as to what should be expected. In (1.12) the finite element approximation errors are bounded by the $h^r\|u\|_{H^{k+1}(\gamma)}$ term while the surface approximation errors are bounded by the $h^{k+1}\|\nabla_\gamma u\|_{L_2(\gamma)}$ term. There are two independent quantities bounding the total error. From this one should suspect that surface AFEM should consist of two estimators, one which locally estimates the surface approximation errors and another which estimates the FEM solution error. The geometric estimator used in [35] is equivalent to

$$\zeta(T) := \|\nabla(\psi - \psi_k)\|_{L_\infty(\overline{T})}. \tag{1.15}$$

This estimator measures the surface interpolation error in $W^{1,\infty}$.

When (1.15) is used in tandem with (1.14), these two estimators lead to a successful adaptive algorithm. We now summarize the AFEM algorithm of [35] and its two modules that drive mesh refinement, ADAPT_SURFACE which is driven by (1.15) and ADAPT_PDE which is driven by (1.14).

AFEM Algorithm: Given an initial triangulation $\mathcal{T}_0$ and parameters $\epsilon_0 > 0$, $0 < \rho < 1$, and $\omega > 0$, set $k = 0$.

1. $\mathcal{T}_k^+ =$ ADAPT_SURFACE$(\mathcal{T}_k, \omega\epsilon_k)$
2. $[U_{k+1}, \mathcal{T}_{k+1}] =$ ADAPT_PDE$(\mathcal{T}_k^+, \epsilon_k)$
3. $\epsilon_{k+1} = \rho\epsilon_k;\ k = k+1$

4. go to 1.

Module ADAPT_SURFACE: Given a tolerance $\epsilon > 0$ and an admissible subdivision $\mathcal{T}$, $\mathcal{T}_* =$ ADAPT_SURFACE$(\mathcal{T}, \omega\epsilon)$ refines the mesh until the new subdivision $\mathcal{T}_* \geq \mathcal{T}$ satisfies

$$\zeta_{\mathcal{T}}(\gamma) \leq \omega\epsilon,$$

i.e. until the geometric error as measured by the geometric estimator is sufficiently reduced. This module is based on a greedy algorithm and acts on a generic mesh $\mathcal{T}$:

$\mathcal{T}_* =$ ADAPT_SURFACE$(\mathcal{T}, \omega\epsilon)$

1. if $\mathcal{M} := \{T \in \mathcal{T} : \zeta_{\mathcal{T}}(\gamma, T) > \omega\epsilon\} = \emptyset$

   return$(\mathcal{T})$ and exit
2. $\mathcal{T} =$ REFINE$(\mathcal{T}, \mathcal{M})$
3. go to 1.

Module ADAPT_PDE: Given a tolerance $\epsilon > 0$ and an admissible subdivision $\mathcal{T}$, $[U_*, \mathcal{T}_*] =$ ADAPT_PDE$(\mathcal{T}, \epsilon)$ outputs a refinement $\mathcal{T}_* \geq \mathcal{T}$ and the associated FEM solution U_* such that

$$\eta_{\mathcal{T}_*}(U_*) \leq \epsilon.$$

This module is based on the sequence:

$[U_*, \mathcal{T}_*] =$ ADAPT_PDE$(\mathcal{T}, \epsilon)$

1. $U =$ SOLVE$(\mathcal{T})$
2. $\{\eta_{\mathcal{T}}(U, T)\}_{T \in \mathcal{T}} =$ ESTIMATE$(\mathcal{T}, U)$
3. if $\eta_{\mathcal{T}} < \epsilon$

   return$(\mathcal{T}, U_i)$ and exit
4. $\mathcal{M} =$ MARK$(\mathcal{T}, \{\eta_{\mathcal{T}}(U, T)\}_{T \in \mathcal{T}})$
5. $\mathcal{T} =$ REFINE$(\mathcal{T}, \mathcal{M})$
6. go to 1.

When ADAPT_SURFACE and ADAPT_PDE are combined in the AFEM algorithm the presence of the ω term in ADAPT_SURFACE guarantees that once ADAPT_SURFACE exits we have

$\zeta_{\mathcal{T}}(\gamma) < \eta_{\mathcal{T}}(U)$ for the duration of ADAPT_PDE. This means that during ADAPT_PDE the H^1 error satisfies the equivalent of flat domain a posteriori estimates with respect to $\eta_{\mathcal{T}}(U)$. It can then be shown that within ADAPT_PDE the rest of the ingredients necessary for the flat domain analysis of [30] are provably true. This eventually culminates in the following bound assuming (u, f, γ) belongs to the appropriate order s approximation class $\mathbb{A}_s$.

Theorem 1.8 (Theorem 8.3 of [35])**.** *Let the initial mesh $\mathcal{T}_0$ have an admissible labeling for refinement, and $\theta \in (0, \theta_*)$, $\omega \in (0, \omega_*)$ for θ_*, ω_* sufficiently small. If $(u, f, \gamma) \in \mathbb{A}_s$ then the sequence $\{\Gamma_\ell, \mathcal{T}_\ell, U_\ell\}$ generated by AFEM satisfies*

$$\|u - U_\ell\|_{H^1(\Omega)} + Osc_{\mathcal{T}}(U_\ell, F) + \omega^{-1}\zeta \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}.$$

In Chapter 3 we will combine the geometric estimator used in ADAPT_SURFACE with the surface equivalent of the eigenfunction cluster estimator (1.9) to create an adaptive surface finite element algorithm for eigenvalue clusters.

2. A PRIORI ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS TO EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR*

In this chapter we present a priori error estimates for the SFEM approximation of the eigenpairs of the Laplace-Beltrami operator based from our work in [1]. In particular, we develop a priori error estimates for the SFEM approximations to the solution of

$$-\Delta_\gamma u = \lambda u$$

on γ. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the Laplace-Beltrami eigenvalues with corresponding $L_2(\gamma)$-orthonormal eigenfunctions $\{u_i\}$. We show that the eigenvector error converges as the error for the source problem, up to a geometric term. Our first main result is:

$$\|u_i - \mathbf{Z}u_i\|_{H^1(\gamma)} \leq C\|u_i - \mathbf{G}u_i\|_{H^1(\gamma)} + C(\lambda_i)h^{k+1} \leq C(\lambda_i)(h^r + h^{k+1}). \tag{2.1}$$

We also prove L_2 error bounds and explicit upper bound for $C(\lambda_i)$ in terms of spectral properties. In addition to eigenfunction convergence rates, we prove the cluster robust estimate for the eigenvalue error:

$$|\lambda_i - \Lambda_i| \leq C(\lambda_i)(\|u_i - \mathbf{G}u_i\|^2_{H^1(\gamma)} + h^{k+1}) \leq C(\lambda_i)(h^{2r} + h^{k+1}), \tag{2.2}$$

where as above, explicit bounds for $C(\lambda_i)$ are given below.

Numerical results presented in Section 2.6 reveal that (2.2) is not sharp for $k > 1$. The deal.ii library [37] uses quadrilateral elements and Gauss-Lobatto points to interpolate the surface. The geometric consistency error for every shape we tested using deal.ii was found to be $O(h^{2k})$ rather than $O(h^{k+1})$ as in (2.2). This inspired our second main result which is stated in Theorem 2.20 in Section 2.5:

$$|\lambda_i - \Lambda_i| \lesssim h^{2r} + h^{2k} + h^{\ell}.$$

Here ℓ is the order of the quadrature rule associated with the interpolation points used to construct the surface. Thus with judicious choice of interpolation points, it is possible to obtain superconvergence for the geometric consistency error when $k > 1$. This phenomenon is novel as a geometric error of order h^{k+1} has been consistently observed in the literature for a variety of error notions. We also investigate this framework in the context of one-dimensional problems and triangular elements.

We finally comment on our proofs. Geometric consistency errors fit into the framework of variational crimes [38]. Banerjee and Osborn [39, 40] considered the effects of numerical integration on errors in finite element eigenvalue approximations, but did not provide a general variational crimes framework. Holst and Stern analyzed variational crimes analysis for surface FEM within the finite element exterior calculus framework and also briefly consider eigenvalue problems [41]. Their discussion of eigenvalue problems does not include convergence rates or a detailed description of the interaction of geometric and Galerkin errors. The recent paper [42] gives a variational crimes analysis for eigenvalue problems that applies to surface FEM. However, their analysis yields suboptimal convergence of the geometric errors in the eigenvalue analysis, considers a different error quantity than we do, and does not easily allow for determination of the dependence of constants in the estimates on spectral properties.

In Section 2.1 we give preliminaries. In Section 2.2, we prove a cluster-robust bound for the eigenvalue error which is sharp for the practically most important case $k = 1$. We also establish spectral convergence, which is foundational to all later results. In Section 2.3 we prove eigenfunction error estimates. In Section 2.4 we numerically confirm these convergence rates and investigate the sharpness of the constants in our bounds with respect to spectral properties. In Section 2.5 we prove superconvergence of eigenvalues and in Section 2.6 provide corresponding numerical results.

2.1 Surface Finite Element Method for Eigenclusters

2.1.1 Weak Formulation and Eigenclusters

We first define the set

$$H^1_{\#}(\gamma) := \left\{ v \in H^1(\gamma) : \int_\gamma v \, d\sigma = 0 \right\} \subset H^1(\gamma).$$

The problem of interest is to find (u, λ) satisfying $-\Delta_\gamma u = \lambda u$ with $\int_\gamma u = 0$. The corresponding weak formulation is: Find $(u, \lambda) \in H^1_{\#}(\gamma) \times \mathbb{R}^+$ such that

$$\int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v d\sigma = \lambda \int_\gamma uv \, d\sigma \qquad \forall v \in H^1_{\#}(\gamma). \tag{2.3}$$

In order to shorten the notation, we define the bilinear form on $H^1(\gamma)$ and the L_2 inner product on $L_2(\gamma)$ respectively as

$$\tilde{a}(u, v) := \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v \, d\sigma, \tag{2.4}$$

$$\tilde{m}(u, v) := \int_\gamma uv \, d\sigma. \tag{2.5}$$

We equip $H^1(\gamma)$ with the norm $\|.\|_{\tilde{a}} := \sqrt{\tilde{a}(.,.)}$.We also use the $\tilde{m}(.,.)$ bilinear form to define the L_2 norm on γ: $\|.\|_{\tilde{m}} := \sqrt{\tilde{m}(.,.)}$. We denote by $\{u_i\}_{i=1}^\infty$ a corresponding orthonormal basis (with respect to $\tilde{m}(\cdot,\cdot)$) of $H^1_{\#}(\gamma)$ consisting of eigenfunctions satisfying (2.3).

We wish to approximate an eigenvalue cluster. For $n \geq 1$ and $N \geq 0$, we assume

$$\lambda_{n-1} < \lambda_n \qquad \text{and} \qquad \lambda_{n+N} < \lambda_{n+N+1} \tag{2.6}$$

so that the targeted cluster of eigenvalues λ_i, $i \in J := \{n, ..., n+N\}$ is separated from the remainder of the spectrum.

2.1.2 Surface approximations

Distance Function. We assume that γ is a compact, orientable, C^∞, D-dimensional surface without boundary which is embedded in $\mathbb{R}^{D+1}$. Let d be the oriented distance function for γ taking negative values in the bounded component of $\mathbb{R}^{D+1}$ delimited by γ. The outward pointing unit normal of γ is then $\boldsymbol{\nu} := \nabla d$. We denote by $\mathcal{N} \subset \mathbb{R}^{D+1}$ a strip about γ of sufficiently small width so that any point $x \in \mathcal{N}$ can be uniquely decomposed as

$$x = \boldsymbol{\psi}(x) + d(x)\boldsymbol{\nu}(x). \tag{2.7}$$

$\boldsymbol{\psi}(x)$ is the unique orthogonal projection onto γ of $x \in \mathcal{N}$. We define the projection onto the tangent space of γ at $x \in \mathcal{N}$ as $P(x) := I - \boldsymbol{\nu}(x) \otimes \boldsymbol{\nu}(x)$ and the surface gradient satisfies $\nabla_\gamma = P\nabla$. From now, we assume that the diameter of the strip $\mathcal{N}$ about γ is small enough for the decomposition (2.7) to be well defined.

2.1.2.1 Approximations of γ

Multiple options for constructing polynomial approximations of γ have appeared. We prove our results under abstract assumptions in order to ensure broad applicability. Let $\overline{\Gamma}$ be a polyhedron or polytope (depending on $D = \dim(\gamma)$) whose faces are triangles or tetrahedra. This assumption is made for convenience but is not essential. The set of all triangular faces of $\overline{\Gamma}$ is denoted $\overline{\mathcal{T}}$.

The higher order approximation Γ of γ is constructed as follows. Letting $\overline{T} \in \overline{\mathcal{T}}$, we define the degree-$k$ approximation of $\boldsymbol{\psi}(\overline{T}) \subset \gamma$ via the Lagrange basis functions $\{\phi_1, ..., \phi_{n_k}\}$ with nodal points $\{x^1, ..., x^{n_k}\}$ on $\overline{T}$. For $x \in \overline{T}$, we have the discrete projection $\boldsymbol{L} : \overline{\Gamma} \to \Gamma$ defined by

$$\boldsymbol{L}(x) := \sum_{j=1}^{n_k} \boldsymbol{L}(x^j)\phi_j(x), \quad \text{where } |\boldsymbol{L}(x^j) - \boldsymbol{\psi}(x^j)| \leq Ch^{k+1}. \tag{2.8}$$

Since we have used the Lagrange basis we have a continuous piecewise polynomial approximation of γ which we define as

$$\Gamma := \{\boldsymbol{L}(x) \ : \ x \in \overline{\Gamma}\} \qquad \text{and} \qquad \mathcal{T} := \{\boldsymbol{L}(\overline{T}) \ : \ \overline{T} \in \overline{\mathcal{T}}\}. \tag{2.9}$$

The requirement $|\boldsymbol{L}(x^j) - \boldsymbol{\psi}(x^j)| \leq Ch^{k+1}$ ensures good approximation of γ by Γ while allowing for instances where Γ and γ do not intersect at interpolation nodes, or even possibly for $\gamma \cap \Gamma = \emptyset$. This could occur when Γ is constructed from imaging data or in free boundary problems. The assumption (2.8) also allows for maximum flexibility in constructing Γ, as we could for instance take $\boldsymbol{L}(x^j) = \boldsymbol{l}(x^j)$ with $\boldsymbol{l}$ a piecewise smooth bi-Lipschitz lift $\boldsymbol{l} : \overline{\Gamma} \to \gamma$ (cf. [43, 44, 35]).

2.1.2.2 Shape regularity and quasi-uniformity

Associated with a degree-k approximation Γ of γ, we follow [45] and let $\rho := \rho(\mathcal{T})$ be its shape regularity constant defined as the largest positive real number such that

$$\rho|\boldsymbol{\xi}| \leq |D\boldsymbol{F}_T(x)\boldsymbol{\xi}| \leq \rho^{-1}|\boldsymbol{\xi}|, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^D, \quad \forall T \in \mathcal{T} \text{ and } x \in T,$$

where

$$\boldsymbol{F}_T := \boldsymbol{L} \circ \overline{\boldsymbol{F}}_T \tag{2.10}$$

with $\overline{\boldsymbol{F}}_T$ the natural affine mapping from a Kuhn (reference) simplex $\widehat{T} \subset \mathbb{R}^D$ to $\overline{T}$. Further, the quasi-uniform constant $\eta := \eta(\mathcal{T})$ of $\mathcal{T}$ is the smallest constant such that

$$h := \max_{T \in \mathcal{T}} \text{diam}(T) \leq \eta \min_{T \in \mathcal{T}} \text{diam}(T).$$

We recall that $\boldsymbol{\nu} = \nabla d : \mathcal{N} \to \mathbb{R}^{D+1}$ is the normal vector on γ and let $\boldsymbol{N}$ be the normal vector on Γ. The assumption (2.8) yields

$$\|d\|_{L_\infty(\Gamma)} \leq Ch^{k+1}, \tag{2.11}$$

$$\|\boldsymbol{\nu} - \boldsymbol{N}\|_{L_\infty(\Gamma)} \leq Ch^k, \tag{2.12}$$

$$\|\boldsymbol{L} - \boldsymbol{\psi}\|_{W^{i,\infty}(\overline{T})} \leq Ch^{k+1-i}, \ \ \overline{T} \in \overline{\mathcal{T}}, \ \ 0 \leq i \leq k+1, \tag{2.13}$$

where C is a constant only depending on $\rho(\mathcal{T})$, $\eta(\mathcal{T})$ and γ.

2.1.2.3 Function Extensions

We assume Γ is contained in the strip $\mathcal{N}$. If $\tilde{u}$ is a function defined on γ, we extend it to $\mathcal{N}$ as $u = \tilde{u} \circ \boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is defined in (2.7). Note that $\boldsymbol{\psi}|_\Gamma : \Gamma \to \gamma$ is also a smooth bijection. We can leverage this to relate functions defined on the two surfaces. For a function u defined on Γ we define its lift to γ as $\tilde{u} = u \circ \boldsymbol{\psi}|_\Gamma^{-1}$. As a general rule, we use the tilde symbol to denote quantities defined on γ but when no confusion is possible, the tilde symbol is dropped.

2.1.2.4 Bilinear Forms on Γ

Given a degree-k approximation Γ of γ, let $H^1_\#(\Gamma) := \{v \in H^1(\Gamma) : \int_\Gamma v \; d\Sigma = 0\} \subset H^1(\Gamma)$ and define the forms on $H^1(\Gamma)$:

$$A(u,v) := \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \; d\Sigma, \qquad M(u,v) := \int_\Gamma uv \; d\Sigma. \tag{2.14}$$

The energy and L_2 norms on Γ are then $\|.\|_A := \sqrt{A(.,.)}$ and $\|.\|_M := \sqrt{M(.,.)}$.

We have already noted that $\boldsymbol{\psi}|_\Gamma$ provides a bijection from Γ to γ. Its smoothness (derived from the smoothness of γ) guarantees that $H^1(\gamma)$ and $H^1(\Gamma)$ are isomorphic. Moreover, the bilinear form $A(.,.)$ on $H^1(\Gamma)$ can be defined on $H^1(\gamma)$

$$\widetilde{A}(\tilde{u},\tilde{v}) := \int_\gamma A_\gamma \nabla_\gamma \tilde{u} \cdot \nabla_\gamma \tilde{v} \; d\sigma = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \; d\Sigma = A(u,v) \tag{2.15}$$

and similarly for the L^2 inner product

$$\widetilde{M}(\tilde{u},\tilde{v}) := \int_\gamma \tilde{u}\tilde{v}\frac{1}{Q}\,d\sigma = \int_\Gamma uv\,d\Sigma = M(u,v). \tag{2.16}$$

Here $Qd\Sigma = d\sigma$ and A_γ depends on the change of variable $\tilde{x} = \mathbf{\Psi}(x)$. We refer to [31, 32] for additional details. Again, we use the notations $\|.\|_{\widetilde{A}} := \sqrt{\widetilde{A}(.,.)}$ and $\|.\|_{\widetilde{M}} := \sqrt{\widetilde{M}(.,.)}$. For the majority of this paper we will work with these lifted forms.

2.1.3 Geometric approximation estimates

The results in this section are essential for estimating effects of approximation of γ by Γ. Recall that we assume that the diameter of the strip $\mathcal{N}$ about γ is small enough for the decomposition (2.7) to be well defined and that $\Gamma \subset \mathcal{N}$.

The following lemma provides a bound on the geometric quantities A_γ and Q appearing in (2.15) and (2.16); cf. [32] for proofs. As we make more precise in Section 2.1.4, we write $f \lesssim g$ when $f \leq Cg$ with C a nonessential constant.

Lemma 2.1 (Estimates on Q and A_γ)**.** *Let $P = I - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ be the projection onto the tangent plane of γ. Let A_γ and Q as in* (2.15) *and* (2.16) *respectively. Then*

$$\|1 - 1/Q\|_{L_\infty(\gamma)} + \|A_\gamma - P\|_{L_\infty(\gamma)} \lesssim h^{k+1}. \tag{2.17}$$

The above geometric estimates along with (2.15) and (2.16) immediately yield estimates for the approximations of $\tilde{m}(.,.)$ and $\tilde{a}(.,.)$ by $\widetilde{M}(.,.)$ and $\widetilde{A}(.,.)$ respectively.

Corollary 2.2 (Geometric estimates)**.** *The following relations hold:*

$$|(\tilde{m} - \widetilde{M})(v,w)| \lesssim h^{k+1}\|v\|_{\tilde{m}}\|w\|_{\tilde{m}}, \qquad \forall v,w \in L^2(\gamma) \tag{2.18}$$

$$|(\tilde{a} - \widetilde{A})(v,w)| \lesssim h^{k+1}\|v\|_{\tilde{a}}\|w\|_{\tilde{a}}, \qquad \forall v,w \in H^1(\gamma). \tag{2.19}$$

The following relations regarding the equivalence of norms are found e.g. in [32]:

$$\|.\|_{\widetilde{A}} \lesssim \|.\|_{\tilde{a}} \lesssim \|.\|_{\widetilde{A}} \qquad \text{and} \qquad \|.\|_{\widetilde{M}} \lesssim \|.\|_{\tilde{m}} \lesssim \|.\|_{\widetilde{M}}. \tag{2.20}$$

They are valid under the assumption that the diameter of the strip $\mathcal{N}$ around γ is small enough and that $\Gamma \subset \mathcal{N}$. We now provide a slight refinement of the above equivalence relations leading to sharper constants.

Corollary 2.3 (Equivalence of norms)**.** *Assume that the diameter of the strip $\mathcal{N}$ around γ is small enough. There exists a constant C only depending on γ and on the shape-regularity and quasi-uniformity constants $\rho(\mathcal{T})$, $\eta(\mathcal{T})$ such that*

$$\|.\|_{\widetilde{A}} \leq (1 + Ch^{k+1})\|.\|_{\tilde{a}}, \qquad \|.\|_{\tilde{a}} \leq (1 + Ch^{k+1})\|.\|_{\widetilde{A}}, \tag{2.21}$$

$$\|.\|_{\widetilde{M}} \leq (1 + Ch^{k+1})\|.\|_{\tilde{m}}, \qquad \|.\|_{\tilde{m}} \leq (1 + Ch^{k+1})\|.\|_{\widetilde{M}}. \tag{2.22}$$

Proof. For brevity, we only provide the proof of (2.21) as the arguments to guarantee (2.22) are similar and somewhat simpler. Let $v \in H^1(\gamma)$. We have

$$\|v\|^2_{\widetilde{A}} - \|v\|^2_{\tilde{a}} = \widetilde{A}(v, v) - \tilde{a}(v, v) = (\widetilde{A} - \tilde{a})(v, v) \tag{2.23}$$

so that in view of the geometric estimate (2.21), we arrive at

$$\|v\|^2_{\widetilde{A}} \leq \|v\|^2_{\tilde{a}} + |(\widetilde{A} - \tilde{a})(v, v)| \leq (1 + Ch^{k+1})\|v\|^2_{\tilde{a}}.$$

When $x \geq 0$, the slope of $\sqrt{1+x}$ is greatest at $x = 0$ with a value of $\frac{1}{2}$, so $\sqrt{1+x} \leq 1 + \frac{1}{2}x$. Thus $\sqrt{1 + Ch^{k+1}} \leq 1 + \frac{1}{2}Ch^{k+1}$, and the first estimate in (2.21) follows by taking a square root. The remaining estimates are derived similarly. $\square$

2.1.4 Surface Finite Element Methods

We construct approximate solutions to the eigenvalue problem (2.3) via surface FEM consisting of a finite element method on degree-k approximate surfaces. See [32, 31] for more details.

2.1.4.1 Surface Finite Elements

Recall that the degree-k approximate surface Γ and its associated subdivision $\mathcal{T}$ are obtained by lifting $\overline{\Gamma}$ and $\overline{\mathcal{T}}$ via (2.9). Similarly, finite element spaces on Γ consist of finite element spaces on the (flat) subdivision $\overline{\mathcal{T}}$ lifted to Γ using the interpolated lift $\boldsymbol{L}$ given by (2.8). More precisely, for $r \geq 1$ we set

$$\mathbb{V} := \mathbb{V}(\Gamma, \mathcal{T}) := \{V \in H^1(\Gamma) \ : \ V = \overline{V} \circ \boldsymbol{L}^{-1}, \text{ with } \overline{V}|_{\overline{T}} \in \mathbb{P}^r(\overline{T}) \quad \forall \overline{T} \in \overline{\mathcal{T}}\}. \tag{2.24}$$

Here $\mathbb{P}^r(\overline{T})$ denotes the space of polynomials of degree at most r on $\overline{T}$. Its subspace consisting of zero mean value functions is denoted $\mathbb{V}_\#$:

$$\mathbb{V}_\# := \mathbb{V}_\#(\Gamma) = \{V \in \mathbb{V} \ : \ \int_\Gamma V \, d\Sigma = 0\}.$$

2.1.4.2 Discrete Formulation

The proposed finite element formulation of the eigenvalue problem on Γ reads: Find $(U, \Lambda) \in \mathbb{V}_\# \times \mathbb{R}^+$ such that

$$A(U, V) = \Lambda \, M(U, V) \qquad \forall V \in \mathbb{V}_\#. \tag{2.25}$$

By the definitions (2.15), (2.16) of $\widetilde{A}(.,.)$ and $\widetilde{M}(.,.)$, relations (2.25) can be rewritten

$$\widetilde{A}(\tilde{U}, \tilde{V}) = \Lambda \, \widetilde{M}(\tilde{U}, \tilde{V}) \qquad \forall V \in \mathbb{V}_\#.$$

We denote by $0 < \Lambda_1 \leq ... \leq \Lambda_{\dim(\mathbb{V}_\#)}$ and $\{U_1, ..., U_{\dim(\mathbb{V}_\#)}\}$ the positive discrete eigenvalues and the corresponding M-orthonormal discrete eigenfunctions satisfying $M(U_i, 1) = 0$,

$i = 1, ..., \dim(\mathbb{V}_\#)$. From the definition (2.16) of $\widetilde{M}(.,.)$, $\{\tilde{U}_i\}_{i=1}^{\dim(\mathbb{V}_\#)}$ are pairwise $\widetilde{M}-$orthogonal and $\widetilde{M}(\tilde{U}_i, 1) = 0$, for $i = 1, ..., \dim(\mathbb{V}_\#)$.

2.1.4.3 Ritz projection

We define a Ritz projection for the discrete bilinear form

$$\boldsymbol{G} : H^1(\gamma) \to \mathbb{V}_\#$$

for any $\tilde{v} \in H^1(\gamma)$ as the unique finite element function $\boldsymbol{G}\tilde{v} := W \in \mathbb{V}_\#$ satisfying

$$\widetilde{A}(\tilde{W}, \tilde{V}) = \widetilde{A}(\tilde{v}, \tilde{V}), \qquad \forall V \in \mathbb{V}_\#. \tag{2.26}$$

2.1.4.4 Eigenvalue cluster approximation

We recall that we target the approximation of an eigencluster indexed by J satisfying the separation assumption (2.6). We denote the discrete eigencluster and orthonormal basis (with respect to $\widetilde{M}(\cdot, \cdot)$) by $\{\Lambda_n, ..., \Lambda_{n+N}\} \subset \mathbb{R}^+$ and $\{U_n, ..., U_{n+N}\} \subset \mathbb{V}_\#$. In addition, we use the notation

$$\mathbb{W}_\# := \text{span}\{U_i : i \in J\}$$

to denote the discrete invariant space. We also define the quantity

$$\mu(J) := \max_{\ell \in J} \max_{j \notin J} \left| \frac{\lambda_\ell}{\Lambda_j - \lambda_\ell} \right|, \tag{2.27}$$

which will play an important role in our eigenfunction estimates. It is finite provided h is sufficiently small, see Remark 2.7.

2.1.4.5 Projections onto $\mathbb{W}_\#$

We denote by $\boldsymbol{P} : H^1(\gamma) \to \mathbb{W}_\#$ the $\widetilde{M}(.,.)$ projection onto $\mathbb{W}_\#$, i.e., for $\tilde{v} \in H^1(\gamma)$, $\boldsymbol{P}v := W \in \mathbb{W}_\#$ satisfies

$$\widetilde{M}(\tilde{W}, \tilde{V}) = \widetilde{M}(\tilde{v}, \tilde{V}), \qquad \forall V \in \mathbb{W}_\#.$$

The other projection operator onto $\mathbb{W}_\#$ is defined by

$$\boldsymbol{Z} : H^1(\gamma) \to \mathbb{W}_\# \text{ s.t. } \widetilde{A}(\tilde{\boldsymbol{Z}}\tilde{v}, \tilde{V}) = \widetilde{A}(\tilde{v}, \tilde{V}), \qquad \forall V \in \mathbb{W}_\#.$$

Notice that $\boldsymbol{Z}$ can be thought of as the Galerkin projection onto $\mathbb{W}_\#$, since

$$\boldsymbol{Z}\tilde{v} = \boldsymbol{P}(\widetilde{\boldsymbol{G}}(\tilde{v})). \tag{2.28}$$

2.1.4.6 Alternate surface FEM

In our analysis of eigenvalue errors we employ a *conforming* parametric surface finite element method as an intermediate theoretical tool. For this, we introduce a finite element space on γ:

$$\widetilde{\mathbb{V}} := \{\tilde{V} \ : \ V \in \mathbb{V}\}.$$

The space of vanishing mean value functions (on γ) is denoted by $\widetilde{\mathbb{V}}_\#$:

$$\widetilde{\mathbb{V}}_\# := \{V \in \widetilde{\mathbb{V}} \ : \ \int_\gamma V \, d\sigma = 0\}.$$

For $i = 1, ..., \dim(\widetilde{\mathbb{V}}_\#)$, we let $(U^\gamma, \Lambda_i^\gamma) \in \widetilde{\mathbb{V}}_\# \times \mathbb{R}^+$ be finite element eigenpairs computed on the continuous surface γ, that is,

$$\tilde{a}(U_i^\gamma, V) = \Lambda_i^\gamma \tilde{m}(U_i^\gamma, V) \qquad \forall V \in \widetilde{\mathbb{V}}_\#. \tag{2.29}$$

2.1.4.7 Notation and constants

Generally we use small letters (γ, u, v,...) to denote quantities lying in infinite dimensional spaces in opposition to capital letters used to denote quantities defined by a finite number of parameters (Γ, U, V). We also recall that for every function $v : \Gamma \to \mathbb{R}$ defines uniquely (via the lift $\mathbf{\Psi}|_\Gamma$) a function $\tilde{v} : \gamma \to \mathbb{R}$ and conversely. We identify quantities defined on γ using a tilde but drop this convention when no confusion is possible, i.e. v could denote a function from Γ to $\mathbb{R}$ as well as its corresponding lift defined from γ to $\mathbb{R}$.

Whenever we write a constant C or c, we mean a generic constant that may depend on the regularity properties of γ and the Poincaré-Friedrichs constant C_F in the standard estimate $\|v\|_{L_2(\gamma)} \leq C_F \|v\|_a$, $v \in H^1_\#(\gamma)$ and on the shape-regularity $\rho(\mathcal{T})$ and quasi-uniformity $\eta(\mathcal{T})$ constants, but not otherwise on the spectrum of $-\Delta_\gamma$ and h. In addition, by $f \lesssim g$ we mean that $f \leq Cg$ for such a nonessential constant C. All other dependencies on spectral properties will be made explicit.

2.2 Clustered Eigenvalue Estimates

Theorem 3.3 of [46] gives a cluster-robust bound for cluster eigenvalue approximations in the conforming case. We utilize this result by employing the conforming surface FEM defined in (2.29) as an intermediate discrete problem. We first use the results of [46] to estimate $|\lambda_i - \Lambda_i^\gamma|$ in a cluster-robust fashion and then independently bound $|\Lambda_i^\gamma - \Lambda_i|$. Note that if λ_i is a multiple eigenvalue so that $\lambda_{i-\underline{k}} = ... = \lambda_i = ... = \lambda_{i+\bar{k}}$, then our bounds also immediately apply to $|\lambda_i - \Lambda_j|$, for $i - \underline{k} \leq j \leq i + \bar{k}$.

Because our setting is non-conforming, we introduce two different Rayleigh quotients defined for $v \in \widetilde{\mathbb{V}}$:

$$R_{\tilde{a}}(v) := \frac{\tilde{a}(v,v)}{\tilde{m}(v,v)} \qquad \text{and} \qquad R_{\widetilde{A}}(v) := \frac{\widetilde{A}(v,v)}{\widetilde{M}(v,v)},$$

where we exclude the case of division by zero. We invoke the min-max approach to characterize the approximate eigenvalues

$$\Lambda_j^\gamma = \min_{\substack{\mathbb{S}\subset\widetilde{\mathbb{V}} \\ \dim(\mathbb{S})=j+1}} \max_{V\in\mathbb{S}} R_{\tilde{a}}(V) \qquad \text{and} \qquad \Lambda_j = \min_{\substack{\mathbb{S}\subset\widetilde{\mathbb{V}} \\ \dim(\mathbb{S})=j+1}} \max_{V\in\mathbb{S}} R_{\widetilde{A}}(V). \tag{2.30}$$

Notice that we do not restrict the Rayleigh quotients to functions with vanishing mean values. Thus we consider subspaces of dimensions $\dim(S) = j+1$ rather than the usual $\dim(S) = j$. The extra dimension is the space of constant functions.

The bound for $|\Lambda_j^\gamma - \Lambda_j|$ given in the following lemma shows that this difference is only related to the geometric error scaled by the corresponding exact eigenvalue Λ_j^γ.

Lemma 2.4. *For $i = 1, ..., \dim(\mathbb{V}) - 1$, let Λ_i^γ and Λ_i be the discrete eigenvalues associated with the finite element method on γ and Γ respectively. Then, we have*

$$|\Lambda_i^\gamma - \Lambda_i| \lesssim \Lambda_i^\gamma h^{k+1}. \tag{2.31}$$

Proof. We use the characterization (2.30) and compare $R_a(.)$ and $R_{\widetilde{A}}(.)$. Using the finer norm equivalence properties (2.21) and (2.22), we have for $V \in \widetilde{\mathbb{V}}$

$$R_{\widetilde{A}}(V) \leq \frac{(1+Ch^{k+1})^2 \tilde{a}(V,V)}{\tilde{m}(V,V)/(1+Ch^{k+1})^2} = (1+Ch^{k+1})^4 R_{\tilde{a}}(V).$$

Thus

$$\begin{aligned} \Lambda_i &\leq \min_{\substack{\mathbb{S}\subset\mathbb{V} \\ \dim(\mathbb{S})=i+1}} \max_{V\in\mathbb{S}} (1+Ch^{k+1})^4 R_{\tilde{a}}(V) = (1+Ch^{k+1})^4 \Lambda_i^\gamma, \\ \Lambda_i - \Lambda_i^\gamma &\lesssim \Lambda_i^\gamma h^{k+1}. \end{aligned} \tag{2.32}$$

A similar argument gives $\Lambda_i^\gamma - \Lambda_i \lesssim \Lambda_i h^{k+1} \lesssim \Lambda_i^\gamma h^{k+1}$, where we used (2.32) in the last step. This implies (2.31), as claimed. □

We now translate Theorem 3.3 of [46] into our notation in order to bound $|\lambda_i - \Lambda_j^\gamma|$ in a cluster-robust manner. First, let $\boldsymbol{G}^\gamma$ be the Ritz projection calculated with respect to $\tilde{a}(\cdot,\cdot)$. That is, for

$v \in H^1(\gamma)$, $\boldsymbol{G}^\gamma v \in \widetilde{\mathbb{V}}_\#$ satisfies

$$\tilde{a}(\boldsymbol{G}^\gamma v, V) = \tilde{a}(v, V), \qquad \forall V \in \widetilde{\mathbb{V}}_\#.$$

Next, let $T : H^1_\#(\gamma) \to H^1_\#(\gamma)$ be the solution operator associated with the source problem (restricted to $H^1_\#(\gamma)$)

$$\tilde{a}(Tf, v) = \tilde{m}(f, v), \qquad \forall v \in H^1_\#(\gamma).$$

Finally, let $\boldsymbol{Z}^\gamma_n$ be the $\tilde{a}$-orthogonal projection onto the space spanned by $\{U^\gamma_i\}_{i=1,..,n-1}$, that is, onto the first $n-1$ discrete eigenfunctions calculated with respect to $\tilde{a}$ and $\tilde{m}$, see (2.29). Theorem 3.3 of [46] provides the following estimates.

Lemma 2.5 (Theorem 3.3 of [46])**.** *Let $j \in J$, and assume that*

$$\min_{i=1,\dots,n-1} |\Lambda^\gamma_i - \lambda_j| \neq 0. \tag{2.33}$$

Then,

$$0 \le \frac{\Lambda^\gamma_j - \lambda_j}{\lambda_j} \le \left(1 + \max_{i=1,..,n-1} \frac{(\Lambda^\gamma_i)^2 \lambda_j^2}{|\Lambda^\gamma_i - \lambda_j|^2} \sup_{\substack{v \in H^1_\#(\gamma) \\ \|v\|_{\tilde{a}}=1}} \|(I - \boldsymbol{G}^\gamma) T \boldsymbol{Z}^\gamma_n v\|^2_{\tilde{a}} \right)$$
$$\times \sup_{\substack{w \in span(u_k \,:\, k \in J) \\ \|w\|_{\tilde{a}}=1}} \|(I - \boldsymbol{G}^\gamma) w\|^2_{\tilde{a}}.$$

We now provide some interpretation of this result. Because $\boldsymbol{G}^\gamma$ is the Ritz projection defined with respect to $\tilde{a}(\cdot,\cdot)$, we have

$$\|(I - \boldsymbol{G}^\gamma) v\|_{\tilde{a}} = \inf_{V \in \widetilde{\mathbb{V}}_\#} \|v - V\|_{\tilde{a}}. \tag{2.34}$$

That is, the term $\sup_{w \in \text{span}(u_k \,:\, k \in J), \|w\|_{\tilde{a}}=1} \|(I - \boldsymbol{G}^\gamma) w\|^2_{\tilde{a}}$ measures approximability in the energy norm of the eigenfunctions in the targeted cluster span$(u_k \,:\, k \in J)$ by the finite element space.

Next, we unravel the term $\|(I - \boldsymbol{G}^\gamma) T \boldsymbol{Z}^\gamma_n v\|_{\tilde{a}}$. For $v \in H^1_\#(\gamma)$, we have $\boldsymbol{Z}^\gamma_n v \in \widetilde{\mathbb{V}}_\# \subset$

$H^1_\#(\gamma)$. Because γ is assumed to be smooth, a standard shift theorem guarantees that for $f := \boldsymbol{Z}_n^\gamma v \in H^1_\#(\gamma)$, $Tf \in H^3(\gamma) \cap H^1_\#(\gamma)$ and $\|Tf\|_{H^3(\gamma)} \lesssim \|f\|_{H^1(\gamma)}$. Thus, $T\boldsymbol{Z}_n^\gamma v \in H^3(\gamma)$, and $\|T\boldsymbol{Z}_n^\gamma v\|_{H^3(\gamma)} \lesssim \|v\|_{H^1(\gamma)}$. Therefore, $\|(I - \boldsymbol{G}^\gamma)T\boldsymbol{Z}_n^\gamma v\|_{\tilde{a}}$ measures the Ritz projection error of $v \in H^3(\gamma)$ in the energy norm, and so (cf. [32])

$$\sup_{v \in H^1_\#(\gamma),\ \|v\|_{\tilde{a}}=1} \|(I - \boldsymbol{G}^\gamma)T\boldsymbol{Z}_n^\gamma v\|_{\tilde{a}} \lesssim h^{\min\{2,r\}}. \tag{2.35}$$

Combining the previous two lemmas with these observations yields the following.

Theorem 2.6 (Cluster robust estimates)**.** *Let $j \in J$, and assume in addition that* $\min_{i=1,\dots,n-1} |\Lambda_i^\gamma - \lambda_j| \neq 0$. *Then*

$$\begin{aligned} |\lambda_j - \Lambda_j| \lesssim \Lambda_j^\gamma \left(1 + Ch^{\min\{2r,4\}} \max_{i=1,..,n-1} \frac{(\Lambda_i^\gamma)^2 \lambda_j^2}{|\Lambda_i^\gamma - \lambda_j|^2}\right) \\ \times \sup_{\substack{w \in span(u_k\,:\,k \in J) \\ \|w\|_{\tilde{a}}=1}} \inf_{V \in \widetilde{\mathbb{V}}_\#} \|w - V\|_{\tilde{a}}^2 + Ch^{k+1}\Lambda_j^\gamma. \end{aligned} \tag{2.36}$$

Remark 2.7 (Asymptotic nature of eigenvalue estimates)**.** *The constant* $\max_{i=1,\dots,n-1} \frac{\Lambda_i^\gamma \lambda_j}{|\Lambda_i^\gamma - \lambda_j|}$ *is not entirely a priori and could be undefined if by coincidence* $\Lambda_i^\gamma - \lambda_j = 0$ *for some* $i < n$. *Because this constant arises from a conforming finite element method, however, its properties are well understood; cf. [46, Section 3.2] for a detailed discussion. In short, convergence of the eigenvalues* $\Lambda_i^\gamma \to \lambda_i$ *is guaranteed as* $h \to 0$, *so* $\max_{i=1,\dots,n-1} \frac{\Lambda_i^\gamma \lambda_j}{|\Lambda_i^\gamma - \lambda_j|} \to \frac{\lambda_{n-1}\lambda_j}{|\lambda_{n-1} - \lambda_j|}$. *Because* $j \geq n$ *and we have assumed separation property* (2.6), *namely* $\lambda_n > \lambda_{n-1}$, *this quantity is well-defined.*

In the following section we prove eigenfunction error estimates under the assumption that the quantity $\mu(J) = \max_{\ell \in J} \max_{j \notin J} \left|\frac{\lambda_\ell}{\Lambda_{h,j} - \lambda_\ell}\right|$ *defined in* (2.27) *above is finite. The observation in the preceding paragraph and* (2.36) *guarantee the existence of* h_0 *such that* $\mu(J) < \infty$ *for all* $h \leq h_0$. *Thus there exists* h_0 *such that for all* $h \leq h_0$ *the discrete eigenvalue cluster respects the separation of the continuous cluster from the remainder of the spectrum in the sense that* $\Lambda_n > \lambda_{n-1}$ *and* $\Lambda_{n+N} < \lambda_{n+N+1}$.

Remark 2.8 (Constant in (2.36))**.** *The spectrally dependent constants in* (2.36) *are expressed with respect to the intermediate discrete eigenvalues Λ_j^γ instead of with respect to the computed discrete eigenvalues Λ_j. It is not difficult to essentially replace Λ_j^γ by Λ_j at least for h sufficiently small by noting that Lemma 2.4 may be rewritten as $|\Lambda_j - \Lambda_j^\gamma| \lesssim \Lambda_j h^{k+1}$. We do not pursue this change here.*

2.3 Eigenfunction Estimates

2.3.1 L_2 Estimate

We start by bounding the difference between the Galerkin projection $\boldsymbol{G}$ of an exact eigenfunction and its projection to the discrete invariant space. It is instrumental for deriving L^2 and energy bounds (Theorems 2.10 and 2.11).

Lemma 2.9. *Let $\{\lambda_j\}_{j\in J}$ be an exact eigenvalue cluster satisfying the separation assumption* (2.6)*. Let $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$ be the set of approximate FEM eigenvalues satisfying $\mu(J) < \infty$, where $\mu(J)$ is defined in* (2.27)*. Fix $i \in J$ and let $u_i \in H^1_\#(\gamma)$ be any eigenfunction associated with λ_i. Then for any $\alpha \in \mathbb{R}$, there holds*

$$\|\boldsymbol{G}u_i - \boldsymbol{Z}u_i\|_{\widetilde{M}} \lesssim (1 + \mu(J))\,(\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} + h^{k+1}\|u_i\|_{\widetilde{M}}). \tag{2.37}$$

Proof. Our proof essentially involves accounting for geometric variational crimes in an argument given for the conforming case in [21] (cf. [23]).

1 Recall that $\{U_j\}_{j=1}^{\dim(\mathbb{V}_\#)} \in \mathbb{V}_\#$ denotes the collection of discrete $\widetilde{M}$-orthonormal eigenfunctions associated with $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$. For $l \in \{1, ..., \dim(\mathbb{V}_\#)\} \setminus J$, $U_l \in \text{Ran}(I - \boldsymbol{P}) \subset \mathbb{V}_\#$ is $\widetilde{M}$-orthogonal to the approximate invariant space $\mathbb{W}_\# = \text{span}(U_j \ : \ j \in J\}$. According to relation (2.28), we then have $\widetilde{M}(\boldsymbol{Z}u_i, U_l) = \widetilde{M}(\boldsymbol{P}\boldsymbol{G}u_i, U_l) = 0$, which implies

$$\widetilde{M}(\boldsymbol{G}u_i - \boldsymbol{Z}u_i, U_l) = \widetilde{M}(\boldsymbol{G}u_i, U_l). \tag{2.38}$$

In addition, $W := \boldsymbol{G}u_i - \boldsymbol{Z}u_i = (I - \boldsymbol{P})\boldsymbol{G}u_i$ can be written as $W = \sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \beta_l U_l$ for some

$\beta_l \in \mathbb{R}$, so that, together with (2.38), we have

$$\|W\|^2_{\widetilde{M}} = \widetilde{M}(W,W) = \widetilde{M}\left(\boldsymbol{G}u_i, \sum_{\substack{l=1 \\ l \notin J}}^{\dim(\mathbb{V}_{\#})} \beta_l U_l\right). \tag{2.39}$$

$\boxed{2}$ We now proceed by deriving estimates for $\widetilde{M}\,(\boldsymbol{G}u_i, U_l)$, $l \notin J$. Since U_l is an eigenfunction of the approximate eigenvalue problem associated with Λ_l, we have

$$\Lambda_l \widetilde{M}(V, U_l) = \Lambda_l \widetilde{M}(U_l, V) = \widetilde{A}(U_l, V) = \widetilde{A}(V, U_l), \qquad \forall V \in \mathbb{V}_{\#}.$$

Choosing $V = \boldsymbol{G}u_i$ gives

$$\Lambda_l \widetilde{M}(\boldsymbol{G}u_i, U_l) = \widetilde{A}(\boldsymbol{G}u_i, U_l) = \widetilde{A}(u_i, U_l) = \tilde{a}(u_i, U_l) + (\widetilde{A} - \tilde{a})(u_i, U_l).$$

We now use the fact that u_i is an eigenfunction of the exact problem to get

$$\begin{aligned}\Lambda_l \widetilde{M}(\boldsymbol{G}u_i, U_l) &= \lambda_i \tilde{m}(u_i, U_l) + (\widetilde{A} - \tilde{a})(u_i, U_l) \\ &= \lambda_i \widetilde{M}(u_i, U_l) + \lambda_i(\tilde{m} - \widetilde{M})(u_i, U_l) + (\widetilde{A} - \tilde{a})(u_i, U_l).\end{aligned}$$

Subtracting $\lambda_i \widetilde{M}(\boldsymbol{G}u_i, U_l)$ from both sides yields

$$(\Lambda_l - \lambda_i)\widetilde{M}(\boldsymbol{G}u_i, U_l) = \lambda_i \widetilde{M}(u_i - \boldsymbol{G}u_i, U_l) + \lambda_i(\tilde{m} - \widetilde{M})(u_i, U_l) + (\widetilde{A} - \tilde{a})(u_i, U_l),$$

or

$$\widetilde{M}(\boldsymbol{G}u_i, U_l) = \frac{1}{\Lambda_l - \lambda_i}\left[\lambda_i \widetilde{M}(u_i - \boldsymbol{G}u_i, U_l) + \lambda_i(\tilde{m} - \widetilde{M})(u_i, U_l) + (\widetilde{A} - \tilde{a})(u_i, U_l)\right].$$

$\boxed{3}$ Returning to (2.39), we obtain

$$\|W\|_{\widetilde{M}}^2 = \widetilde{M}(W,W) = \widetilde{M}\left(u_i - \boldsymbol{G}u_i - \alpha, \sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \frac{\lambda_i}{\Lambda_l - \lambda_i}\beta_l U_l\right)$$
$$+ \left[(\tilde{m} - \widetilde{M}) + \frac{1}{\lambda_i}(\widetilde{A} - \tilde{a})\right]\left(u_i, \sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \frac{\lambda_i}{\Lambda_l - \lambda_i}\beta_l U_l\right)$$

where we used $\widetilde{M}(U_l, 1) = 0$ to incorporate $\alpha \in \mathbb{R}$ into the estimate. To continue further, we use the orthogonality property of the discrete eigenfunctions to obtain

$$\left\|\sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \frac{\lambda_i}{\Lambda_l - \lambda_i}\beta_l U_l\right\|_{\widetilde{M}}^2 = \sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \left(\frac{\lambda_i}{\Lambda_l - \lambda_i}\right)^2 \beta_l^2 \|U_l\|_{\widetilde{M}}^2 \le \mu(J)\|W\|_{\widetilde{M}}^2$$

and similarly $\left\|\sum_{\substack{l=1\\ l\notin J}}^{\dim(\mathbb{V}_\#)} \frac{\lambda_i}{\Lambda_l - \lambda_i}\beta_l U_l\right\|_{\widetilde{A}}^2 \le \mu(J)\|W\|_{\widetilde{A}}^2$ since $\tilde{A}(U_l, U_k) = \Lambda_l \widetilde{M}(U_l, U_k)$. Thus the geometric error estimates (Corollary 2.2) and a Young inequality imply

$$\begin{aligned}\|W\|_{\widetilde{M}}^2 &\le \mu(J)\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}}\|W\|_{\widetilde{M}} + Ch^{k+1}\mu(J)\|u_i\|_{\widetilde{M}}\|W\|_{\widetilde{M}} \\ &\quad + Ch^{2k+2}\frac{\mu(J)^2}{\lambda_i}\|u_i\|_{\widetilde{A}}^2 + \frac{1}{4\lambda_i}\|W\|_{\widetilde{A}}^2.\end{aligned} \tag{2.40}$$

$\boxed{4}$ To bound $\|W\|_{\tilde{A}}$, we recall that $\boldsymbol{P} \circ \boldsymbol{G}$ and $\boldsymbol{G}$ are the $\widetilde{A}(\cdot,\cdot)$ projections onto $\mathbb{W}_\#$ and $\mathbb{V}_\#$, respectively, and that $\boldsymbol{P}$ is the L_2 projection onto $\mathbb{W}_\#$. Thus

$$\begin{aligned}\|W\|_{\widetilde{A}}^2 &= \widetilde{A}(W,W) = \widetilde{A}((I - \boldsymbol{P})\boldsymbol{G}u_i, (I - \boldsymbol{P})\boldsymbol{G}u_i) = \widetilde{A}(\boldsymbol{G}u_i, (I - \boldsymbol{P})\boldsymbol{G}u_i) \\ &= \widetilde{A}(u_i, (I - \boldsymbol{P})\boldsymbol{G}u_i) = \widetilde{A}(u_i, W).\end{aligned}$$

To isolate the geometric error, we rewrite for any $\alpha \in \mathbb{R}$ the right hand side of the above equation as

$$
\begin{aligned}
&\tilde{a}(u_i, W) + (\widetilde{A} - \tilde{a})(u_i, W) = \lambda_i \tilde{m}(u_i, W) + (\widetilde{A} - \tilde{a})(u_i, W) \\
&= \lambda_i(\tilde{m} - \widetilde{M})(u_i, W) + \lambda_i \widetilde{M}(u_i - \boldsymbol{G}u_i, W) + \lambda_i \widetilde{M}(\boldsymbol{G}u_i - \boldsymbol{Z}u_i, W) + (\widetilde{A} - \tilde{a})(u_i, W) \\
&= \lambda_i(\tilde{m} - \widetilde{M})(u_i, W) + \lambda_i \widetilde{M}(u_i - \boldsymbol{G}u_i - \alpha, W) + \lambda_i \widetilde{M}(W, W) + (\widetilde{A} - \tilde{a})(u_i, W),
\end{aligned}
$$

upon invoking the orthogonality relations (2.38) and $\widetilde{M}(W, 1) = 0$. We take advantage again of the geometric error estimates (Corollary 2.2) to arrive at

$$
\begin{aligned}
\|W\|_{\widetilde{A}}^2 \le \lambda_i C h^{k+1} \|u_i\|_{\widetilde{M}} \|W\|_{\widetilde{M}} + \lambda_i \|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} \|W\|_{\widetilde{M}} + \lambda_i \|W\|_{\widetilde{M}}^2 \\
+ C h^{k+1} \|u_i\|_{\widetilde{A}} \|W\|_{\widetilde{A}}.
\end{aligned}
\tag{2.41}
$$

Now, noting that $\|u_i\|_{\widetilde{A}} \lesssim \|u_i\|_{\tilde{a}} = \sqrt{\lambda_i}\|u_i\|_{\tilde{m}}$ by (2.20) and using Young's inequality to absorb the last term by the left hand side gives

$$
\begin{aligned}
\|W\|_{\widetilde{A}}^2 \le C h^{k+1} \lambda_i \|u_i\|_{\widetilde{M}} \|W\|_{\widetilde{M}} + 2\lambda_i \|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} \|W\|_{\widetilde{M}} + 2\lambda_i \|W\|_{\widetilde{M}}^2 \\
+ C \lambda_i h^{2k+2} \|u_i\|_{\widetilde{M}}^2.
\end{aligned}
\tag{2.42}
$$

5 Using (2.42) in (2.40) gives

$$
\begin{aligned}
\|W\|_{\widetilde{M}}^2 \le \left(\frac{1}{2} + \mu(J)\right) \|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} \|W\|_{\widetilde{M}} + C h^{k+1} \left(1 + \mu(J)\right) \|u_i\|_{\widetilde{M}} \|W\|_{\widetilde{M}} \\
+ C h^{2k+2} \left(1 + \mu(J)^2\right) \|u_i\|_{\widetilde{M}}^2 + \frac{1}{2} \|W\|_{\widetilde{M}}^2.
\end{aligned}
$$

We apply Young's inequality again to arrive at

$$
\|W\|_{\widetilde{M}}^2 \lesssim (1 + \mu(J))^2 \left[\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}}^2 + h^{2k+2}\|u_i\|_{\widetilde{M}}^2 + h^{2k+2}\|u_i\|_{\widetilde{M}}^2\right],
$$

which yields the desired result upon taking a square root. □

Theorem 2.10 (L^2 error estimate)**.** *Let $\{\lambda_j\}_{j \in J}$ be an exact eigenvalue cluster satisfying the sep-*

aration assumption (2.6)*. Let* $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$ *be the set of approximate FEM eigenvalues satisfying* $\mu(J) < \infty$*. We fix* $i \in J$ *and denote by* $u_i \in H^1_\#(\gamma)$ *any eigenfunction associated with* λ_i*. Then for any* $\alpha \in \mathbb{R}$*, the following bound holds:*

$$\begin{aligned} \|u_i - \boldsymbol{P}u_i - \alpha\|_{\widetilde{M}} &\leq \|u_i - \boldsymbol{Z}u_i - \alpha\|_{\widetilde{M}} \\ &\lesssim (1+\mu(J))\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} + (1+\mu(J))\,\|u_i\|_{\widetilde{M}} h^{k+1}. \end{aligned} \tag{2.43}$$

Proof. Because $\boldsymbol{P}\alpha = \boldsymbol{Z}\alpha = 0$ and $\boldsymbol{P}$ is the $\widetilde{M}$-projection onto $\mathbb{W}_\#$, we have

$$\begin{aligned} \|(u_i-\alpha) - \boldsymbol{P}u_i\|_{\widetilde{M}} &= \|u_i - \alpha - \boldsymbol{P}(u_i-\alpha)\|_{\widetilde{M}} \leq \|(u_i-\alpha) - \boldsymbol{Z}(u_i-\alpha)\|_{\widetilde{M}} \\ &= \|u_i - \boldsymbol{Z}u_i - \alpha\|_{\widetilde{M}} \leq \|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} + \|\boldsymbol{G}u_i - \boldsymbol{Z}u_i\|_{\widetilde{M}}. \end{aligned}$$

The second leg is bounded using Lemma 2.9. □

2.3.2 Energy Estimate

We now focus on estimates for $\|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}}$.

Theorem 2.11 (Energy estimate)**.** *Let* $\{\lambda_j\}_{j\in J}$ *be an exact eigenvalue cluster satisfying the separation assumption* (2.6)*. Let* $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$ *be a set of approximate FEM eigenvalues satisfying* $\mu(J) < \infty$*. We fix* $i \in J$ *and denote by* $u_i \in H^1_\#(\gamma)$ *any eigenfunction associated with* λ_i*. Then for any* $\alpha \in \mathbb{R}$*, the following bound holds:*

$$\begin{aligned} \|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} \leq \|u_i - \boldsymbol{G}u_i\|_{\widetilde{A}} &+ C\sqrt{\lambda_i}(1+\mu(J))\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}} \\ &+ C\sqrt{\lambda_i}(1+\mu(J))h^{k+1}\|u_i\|_{\widetilde{M}}. \end{aligned} \tag{2.44}$$

Proof. Let $W := \boldsymbol{G}u_i - \boldsymbol{Z}u_i$. We restart from the estimate (2.42) for $\|W\|_{\widetilde{A}}$, apply Young's inequality, and take advantage of the L^2 error bound (2.37) to deduce

$$\begin{aligned} \|W\|^2_{\widetilde{A}} &\lesssim \lambda_i(h^{2k+2}\|u_i\|^2_{\widetilde{M}} + \|u_i - \boldsymbol{G}u_i - \alpha\|^2_{\widetilde{M}} + \|W\|^2_{\widetilde{M}}) \\ &\lesssim \lambda_i(1+\mu(J))^2(h^{2k+2}\|u_i\|^2_{\widetilde{M}} + \|u_i - \boldsymbol{G}u_i - \alpha\|^2_{\widetilde{M}}). \end{aligned}$$

The desired result follows from $\|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} \leq \|u_i - \boldsymbol{G}u_i\|_{\widetilde{A}} + \|W\|_{\widetilde{A}}$. $\square$

We end by commenting on (2.44). Because $\mathbf{G}$ is the Galerkin projection onto $\mathbb{V}_{\#}$ with respect to $\widetilde{A}(\cdot,\cdot)$, we have for the first term in (2.44) that

$$\|u_i - \mathbf{G}u_i\|_{\widetilde{A}} \leq \inf_{V \in \mathbb{V}_{\#}} \|u_i - V\|_{\widetilde{A}} = \inf_{V \in \mathbb{V}} \|u_i - V\|_{\widetilde{A}}. \tag{2.45}$$

Here we used that $\widetilde{A}(\tilde{v}, 1) = 0$, $v \in H^1(\gamma)$. The last term above may be bounded in a standard way (cf. [34] for definition of a suitable interpolation operator of Scott-Zhang type in any space dimension). Similar comments apply to (2.43).

Bounding $\|u_i - \mathbf{G}\|_{\widetilde{M}}$ is more complicated. Because Γ is not smooth, it is not possible to directly carry out a duality argument to obtain L_2 error estimates for $\mathbf{G}$ with no geometric error term. Abstract arguments of [32] however give error bounds for $u_i - \boldsymbol{G}u_i$ satisfying $\tilde{a}(u_i - \boldsymbol{G}u_i, V) = F(V)\ \forall V \in \mathbb{V}_{\#}$. Letting $F(V) = (\tilde{a} - \widetilde{A})(u_i - \boldsymbol{G}u_i, V)$, the fact that $\widetilde{A}(\tilde{v}, 1) = 0$ for any $v \in H^1(\gamma)$ yields

$$\tilde{a}(u_i - \boldsymbol{G}u_i, V) = F(V) \qquad \forall V \in \mathbb{V}.$$

Choosing $\alpha = \frac{1}{|\gamma|}\int_\gamma \boldsymbol{G}(u - u_i)$, [32, Theorem 3.1] along with (2.19) then yield

$$\|u_i - \boldsymbol{G}u_i - \alpha\|_{\tilde{m}} \lesssim h \min_{V \in \mathbb{V}} \|u_i - V\|_{\tilde{a}} + h^{k+1}\|u_i - \boldsymbol{G}u_i\|_{\tilde{a}} \lesssim h \min_{V \in \mathbb{V}} \|u_i - V\|_{\widetilde{A}}.$$

Thus the L_2 term above may also be bounded in a standard way.

2.3.3 Relationship between projection errors

Many classical papers on finite element eigenvalue approximations contain energy error bounds for the projection error $\|v - \boldsymbol{P}v\|_{\tilde{a}}$ [16, 17]. We briefly investigate the relationship between this error notion and our notion $\|v - \boldsymbol{Z}v\|_{\tilde{a}}$. Because $\boldsymbol{Z}$ is a Galerkin projection, we have $\|v - \boldsymbol{Z}v\|_{\widetilde{A}} \leq \|v - \boldsymbol{P}v\|_{\widetilde{A}}$. In Proposition 2.13 we show that the reverse inequality holds up to higher-order terms.

These two error notions are thus asymptotically equivalent.

Lemma 2.12. *Let $\{\lambda_j\}_{j\in J}$ be an exact eigenvalue cluster indexed by J satisfying the separation assumption* (2.6)*. Let $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$ be set of approximate FEM eigenvalues satisfying $\mu(J) < \infty$. We assume that for an absolute constant B, there holds $\max\{\Lambda_{n+N}\} \leq B$. Then for $v \in H^1(\gamma)$, we have*

$$\|\boldsymbol{P}v\|_{\widetilde{A}} \leq \sqrt{B}\|v\|_{\widetilde{M}}.$$

Proof. Since $\boldsymbol{P}v \in \mathbb{W}_\#$, there exists β_j, $j \in J$, such that $\boldsymbol{P}v = \sum_{j\in J}\beta_j U_j$. Thus

$$\begin{aligned}\|\boldsymbol{P}v\|_{\widetilde{A}}^2 = \widetilde{A}(\boldsymbol{P}v, \boldsymbol{P}v) &= \sum_{j\in J}\beta_j\widetilde{A}(U_j, \boldsymbol{P}v) = \sum_{j\in J}\beta_j\Lambda_j\widetilde{M}(U_j, \boldsymbol{P}v) \\ &= \sum_{j\in J}\beta_j\Lambda_j\widetilde{M}(U_j, \sum_{j\in J}\beta_j U_j) = \sum_{j\in J}\beta_j^2\Lambda_j\widetilde{M}(U_j, U_j) \leq B\|\boldsymbol{P}v\|_{\widetilde{M}}^2 \leq B\|v\|_{\widetilde{M}}^2,\end{aligned}$$

where we used that the discrete eigenfunctions $\{U_j\}$ are $\widetilde{M}$-orthogonal. □

Proposition 2.13. *Let $\{\lambda_j\}_{j\in J}$ be an exact eigenvalue cluster indexed by J satisfying the separation assumption* (2.6)*. Let $\{\Lambda_j\}_{j=1}^{\dim(\mathbb{V}_\#)}$ be set of approximate FEM eigenvalues satisfying $\mu(J) < \infty$. Furthermore, assume that for some absolute constant B, $\Lambda_{N+n} \leq B$. Let u_i be an eigenfunction with eigenvalues λ_i, for some $i \in J$. Then the following bound holds for any $\alpha \in \mathbb{R}$:*

$$\|u_i - \boldsymbol{P}u_i\|_{\widetilde{A}} \leq \|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} + \sqrt{B}\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}}.$$

Proof. By the triangle inequality we have:

$$\|u_i - \boldsymbol{P}u_i\|_{\widetilde{A}} \leq \|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} + \|\boldsymbol{Z}u_i - \boldsymbol{P}u_i\|_{\widetilde{A}} = \|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} + \|\boldsymbol{P}(u_i - \boldsymbol{G}u_i - \alpha)\|_{\widetilde{A}}.$$

Applying Lemma 2.12 for the last term gives

$$\|\boldsymbol{P}(u_i - \boldsymbol{G}u_i - \alpha)\|_{\widetilde{A}} \leq \sqrt{B}\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}},$$

and as a consequence

$$\|u_i - \boldsymbol{P}u_i\|_{\widetilde{A}} \leq \|u_i - \boldsymbol{Z}u_i\|_{\widetilde{A}} + \sqrt{B}\|u_i - \boldsymbol{G}u_i - \alpha\|_{\widetilde{M}}.$$

□

2.4 Numerical Results for Eigenfunctions

Let γ be the unit sphere in $\mathbb{R}^3$. The eigenfunctions of the Laplace-Beltrami operator are then the spherical harmonics. The eigenvalues are given by $\ell(\ell+1)$, $\ell = 1, 2, 3...$, with multiplicity $2\ell + 1$. Computations were performed on a sequence of uniformly refined quadrilateral meshes using deal.ii [37]; our proofs extend to this situation with modest modifications. When comparing norms of errors we took the first spherical harmonic for each eigenvalue $\ell(\ell+1)$ as the exact solution and then projected this function onto the corresponding discrete invariant space having dimension $2\ell + 1$.

2.4.1 Eigenfunction error rates

We calculated the eigenfunction error $\|u_1 - \boldsymbol{P}u_1\|_{\tilde{M}}$ and $\|u_1 - \boldsymbol{P}u_1\|_{\tilde{A}}$ for the lowest spherical harmonic corresponding to $\lambda_1 = 2$. From Theorem 2.10 and the results of [32], we expect

$$\|u_1 - \boldsymbol{P}u_1\|_{\tilde{M}} \lesssim C(\lambda)(h^{r+1} + h^{k+1}). \tag{2.46}$$

From Proposition 2.13 and Theorem 2.11, we expect

$$\|u_1 - \boldsymbol{P}u_1\|_{\tilde{A}} \lesssim C(\lambda)(h^{r} + h^{k+1}). \tag{2.47}$$

We postpone discussion of dependence of the constants on spectral properties to Section 2.4.2. When $r = 1$ and $k = 2$, the L_2 error is dominated by the PDE approximation (Figure 2.1), $h^{k+1} = h^3 \lesssim h^2 = h^{r+1}$. When $r = 3$ and $k = 1$ we see the L_2 error is dominated by the geometric approximation (Figure 2), $h^{r+1} = h^4 \lesssim h^2 = h^{k+1}$. This illustrate the sharpness of our theory with respect to the approximation degrees. The energy error behavior reported in Figure 2.1 similarly indicates that (2.47) is sharp.

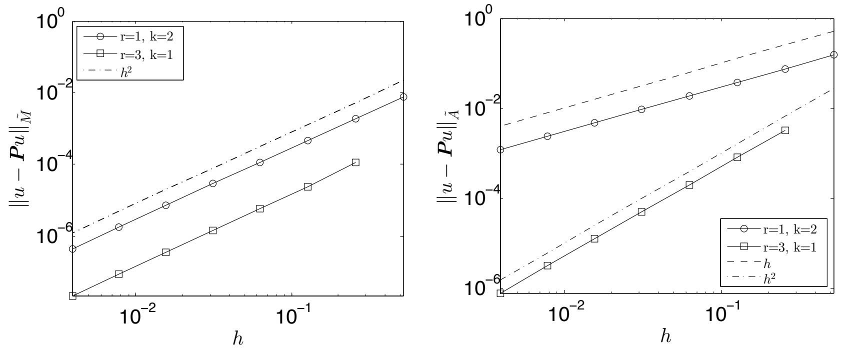

Figure 2.1: Convergence rates of the approximate invariant eigenspace corresponding to the first eigenvalue on the sphere: L_2 errors (left) and energy errors (right). Reprinted from [1].

2.4.2 Numerical evaluation of constants

In the left plot of Figure 2.2 we plot $\frac{\|u-\boldsymbol{P}u\|_{\tilde{A}}}{\sqrt{\lambda}(1+\mu(J))h^{k+1}}$ vs. h for $r = 3$ and $k = 1$ to evaluate the quality of our constant in Theorem 2.11. Here the Galerkin error is $O(h^4)$ and the geometric error $O(h^2)$, so the geometric error dominates. Consider the eigenvalues $\lambda = \ell(\ell+1)$, $\ell = 1, ..., 10$ and corresponding spherical harmonics. We chose two different exact spherical harmonics for $\ell = 10$ to determine whether the choice of harmonic would affect the computation. In the left plot of Figure 2.2, we see that the ratio $\frac{\|u-\boldsymbol{P}u\|_{\tilde{A}}}{\sqrt{\lambda}(1+\mu(J))h^{k+1}}$ decreases moderately as λ increases, indicating that the constant in Theorem 2.11 may not be sharp. We thus also plotted $\frac{\|u-\boldsymbol{P}u\|_{\tilde{A}}}{\sqrt{\lambda}(2+\sqrt{\mu(J)})h^{k+1}}$ and found this quantity to be more stable as λ increases (see the right plot of Figure 2.2). Thus it is possible that the dependence of the constant in front of the geometric error term in Theorem 2.11 is not sharp with respect to its dependence on $\mu(J)$. Our method of proof does not seem to provide a pathway to proving a sharper dependence, however, and our numerical experiments do confirm that the constant in front of the geometric error depends on spectral properties.

In Figure 2.3 we similarly test the sharpness of the geometric constant in the eigenvalue error estimate (2.36) by plotting $\frac{|\lambda-\Lambda|}{\lambda h^2}$. This quantity is very stable as λ increases, thus verifying the sharpness of the estimate as well as the correctness of the order, $O(h^{k+1})$ for $k = 1$. In Section 7 we observe that for $k \geq 2$ the geometric error is between h^{k+1} and h^{2k}. We delay giving numerical details until laying a theoretical foundation for explaining these superconvergence results.

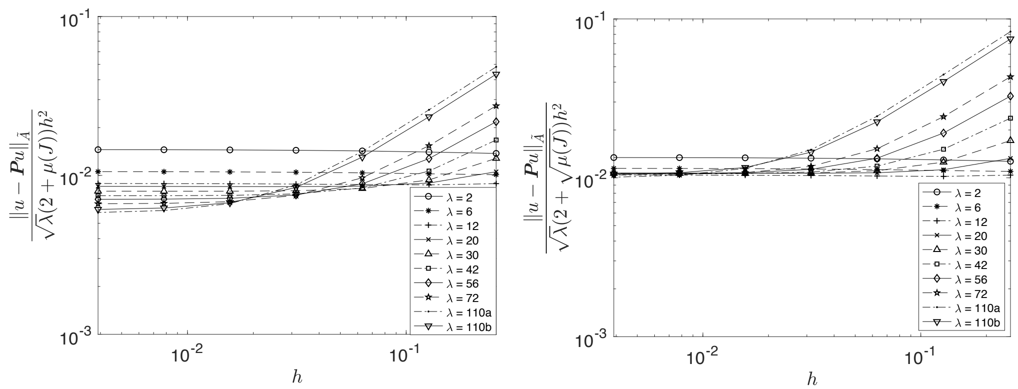

Figure 2.2: Dependence of geometric portion in energy errors on spectral constants: Theoretically established constant $\frac{\|u-\mathbf{P}u\|_{\tilde{A}}}{\sqrt{\lambda}(1+\mu(J))h^{k+1}}$ (left) and conjectured alternative constant $\frac{\|u-\mathbf{P}u\|_{\tilde{A}}}{\sqrt{\lambda}(2+\sqrt{\mu(J)})h^{k+1}}$ (right). Reprinted from [1].

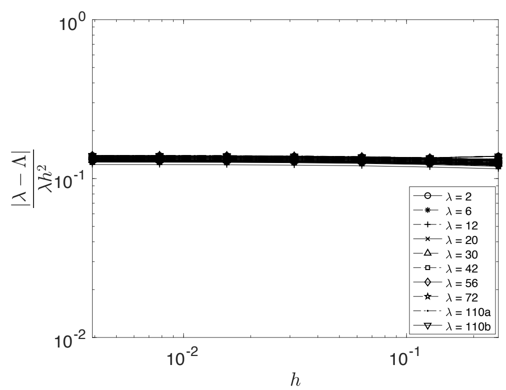

Figure 2.3: Dependence of geometric portion of eigenvalue errors on spectral constants, $k = 1$: Theoretically established constant $\frac{|\lambda-\Lambda|}{\lambda h^2}$ for eigenvalues $\ell(\ell+1)$, $\ell = 1, ..., 10$. Reprinted from [1].

2.5 Superconvergence of Eigenvalues

In this section we analyze the geometric error estimates (2.16) and (2.17) from the viewpoint of numerical integration. Our approach is not cluster robust, but allows us to analyze superconvergence effects and leads to a characterization of the relationship between the choice of interpolation points in the construction of Γ and the convergence rate for the eigenvalues. We show that we may obtain geometric errors of order $O(h^\ell)$ for $k+1 \le \ell \le 2k$ by choosing interpolation points in the construction of Γ that correspond to a quadrature scheme of order ℓ. Because these superconvergence effects require a more subtle analysis, we do not trace the dependence of constants on spectral properties in this section and are only interested in orders of convergence. We denote the untracked spectrally dependent constant by C_λ, which may change values throughout the calculations.

We first state a result similar to [39, Theorem 5.1], where effects of numerical quadrature on eigenvalue convergence were analyzed. Let λ_j be an eigenvalue of (2.3) with multiplicity N. Let $\mathbb{W}$ and $\mathbb{W}_\#$ be the spans of the eigenfunctions of λ_j and the N FEM eigenfunctions associated with the approximating eigenvalues of λ_j.

Lemma 2.14. *Eigenvalue Bound. Let $\boldsymbol{P}_{\lambda_j}$ be the projection onto $\mathbb{W}$ using the L_2 inner product $m(\cdot,\cdot)$. Let U_j be an eigenfunction in $\mathbb{W}_\#$ such that $\|U_j\|_m = 1$ and $A(U_j,U_j) = \Lambda_j M(U_j,U_j)$. Then*

$$
\begin{aligned}
|\lambda_j - \Lambda_j| = \left| \frac{a(\boldsymbol{P}_{\lambda_j}U_j, \boldsymbol{P}_{\lambda_j}U_j)}{m(\boldsymbol{P}_{\lambda_j}U_j, \boldsymbol{P}_{\lambda_j}U_j)} - \frac{\widetilde{A}(U_j,U_j)}{\widetilde{M}(U_j,U_j)} \right| \le \|\boldsymbol{P}_{\lambda_j}U_j - U_j\|_a^2 \\
+ \lambda_j \|\boldsymbol{P}_{\lambda_j}U_j - U_j\|_m^2 + \Lambda_j |m(U_j,U_j) - \widetilde{M}(U_j,U_j)| + |\widetilde{A}(U_j,U_j) - a(U_j,U_j)|.
\end{aligned}
\tag{2.48}
$$

Proof. Since $a(\boldsymbol{P}_{\lambda_j}U_j, U_j) = \lambda_j m(\boldsymbol{P}_{\lambda_j}U_j, U_j)$ and $\|\boldsymbol{P}_{\lambda_j}U_j\|_a^2 = \lambda_j \|\boldsymbol{P}_{\lambda_j}U_j\|_m^2$,

$$
\|\boldsymbol{P}_{\lambda_j}U_j - U_j\|_a^2 - \lambda_j \|\boldsymbol{P}_{\lambda_j}U_j - U_j\|_m^2 = \|\boldsymbol{P}_{\lambda_j}U_j\|_a^2 + \|U_j\|_a^2 - 2a(\boldsymbol{P}_{\lambda_j}U_j, U_j)
$$

$$
-\lambda_j \|\boldsymbol{P}_{\lambda_j}U_j\|_m^2 + 2\lambda_j m(\boldsymbol{P}_{\lambda_j}U_j, U_j) - \lambda_j \|U_j\|_m^2 = a(U_j,U_j) - \lambda_j \|U_j\|_m^2.
$$

Noting the assumption that $\|U_j\|_m = 1$, we get

$$-\lambda_j = \|\boldsymbol{P}_{\lambda_j} U_j - U_j\|_a^2 - \lambda_j \|\boldsymbol{P}_{\lambda_j} U_j - U_j\|_m^2 - a(U_j, U_j). \tag{2.49}$$

Because $\widetilde{A}(U_j, U_j) - \Lambda_j \widetilde{M}(U_j, U_j) = 0$ we get

$$-\lambda_j = \|\boldsymbol{P}_{\lambda_j} U_j - U_j\|_a^2 - \lambda_j \|\boldsymbol{P}_{\lambda_j} U_j - U_j\|_m^2 + [\widetilde{A}(U_j, U_j) - a(U_j, U_j)] - \Lambda_j \widetilde{M}(U_j, U_j).$$

Adding $\Lambda_j = \Lambda_j m(U_j, U_j)$ to both sides and taking absolute values gives the result. □

We now give a series of results bounding the terms on the right hand side of (2.48). Recall that $\boldsymbol{P}$ denotes the $\widetilde{M}$ projection onto $\mathbb{W}_{\#}$.

Lemma 2.15. *For h small enough, $\{\boldsymbol{P}u : u \in \mathbb{W}\}$ forms a basis for span$\{U : U \in \mathbb{W}_{\#}\}$. Moreover, for any $U \in \mathbb{W}_{\#}$ with $\|U\|_m = 1$,*

$$\sum_{i=1}^{N} |\alpha_i|^2 \leq C(N). \tag{2.50}$$

Proof. The proof follows the same steps given in the proof of [23, Lemma 5.1]. □

Lemma 2.16. *Let h be small enough that $\{\boldsymbol{P}u : u \in \mathbb{W}\}$ forms a basis for span$\{U : U \in \mathbb{W}_{\#}\}$. Let $\{u_i\}_{i=1}^N$ be an orthonormal basis for $\mathbb{W}$ with respect to $m(\cdot,\cdot)$. Then*

$$\|U - \boldsymbol{P}_{\lambda_j} U\|_a \leq C_\lambda \max_{i=1,\ldots,N} \|u_i - \boldsymbol{P}u_i\|_a \lesssim h^r + h^{k+1}, \tag{2.51}$$

$$\|U - \boldsymbol{P}_{\lambda_j} U\|_m \leq C_\lambda \max_{i=1,\ldots,N} \|u_i - \boldsymbol{P}u_i\|_m \lesssim h^{r+1} + h^{k+1} \tag{2.52}$$

for any $u \in \mathbb{W}$ and $U \in \mathbb{W}_{\#}$.

Proof. Recall that $N = \dim(\mathbb{W})$. Since $U \in \text{span}\{\boldsymbol{P}u : u \in \mathbb{W}\}$, there holds $U = \sum_{i=1}^N \alpha_i \boldsymbol{P}u_i$

with the coefficients satisfying (2.50). Thus

$$\boldsymbol{P}_{\lambda_j}U - U = \sum_{k=1}^{N} m(\sum_{i=1}^{N}\alpha_i \boldsymbol{P}u_i, u_k)u_k - \sum_{i=1}^{N}\alpha_i \boldsymbol{P}u_i.$$

Adding $-\sum_{i=1}^{N}\alpha_i m(u_i,u_i)u_i + \sum_{i=1}^{N}\alpha_i u_i = 0$ and using $m(u_i,u_k)=0$, $i\neq k$, yields

$$\boldsymbol{P}_{\lambda_j}U - U = \sum_{i=1}^{N}\alpha_i \left(\sum_{k=1}^{N} m(\boldsymbol{P}u_i - u_i, u_k)u_k + (u_i - \boldsymbol{P}u_i)\right). \tag{2.53}$$

Using $m(\boldsymbol{P}u_i - u_i, u_k) = \frac{1}{\lambda_j}a(\boldsymbol{P}u_i - u_i, u_k)$, noting (2.50) and applying $\|\cdot\|_a$ to both sides of (2.53) yields the first inequality in (2.51), while applying $\|\cdot\|_m$ to both sides of (2.53) yields similarly the first inequality in (2.52). The second inequality in (2.51) follows from Proposition 2.13 and (1.12).

To obtain the second inequality in (2.52), we first use (2.43) and $\|\cdot\|_m \simeq \|\cdot\|_{\widetilde{M}}$:

$$\begin{aligned}\|u_k - \boldsymbol{P}u_k\|_m &\lesssim \|u_k - \boldsymbol{G}u_k\|_m \\ &\leq \|u_k - \boldsymbol{G}u_k - m(u_k - \boldsymbol{G}u_k, 1)\|_m + \|m(u_k - \boldsymbol{G}u_k, 1)\|_m.\end{aligned} \tag{2.54}$$

Since $m(u_k,1) = \widetilde{M}(u_k,1) = 0$, we have from (2.18) that

$$\begin{aligned}\|m(u_k - \boldsymbol{G}u_k, 1)\|_m &= \|m(\boldsymbol{G}u_k, 1)\|_m \\ &= \sqrt{|\gamma|}|m(\boldsymbol{G}u_k,1) - \widetilde{M}(\boldsymbol{G}u_k,1)| \leq |\gamma|\|\boldsymbol{G}u_k\|_{\widetilde{M}}h^{k+1}.\end{aligned}$$

Also, $\|\boldsymbol{G}u_k\|_{\widetilde{M}} \lesssim \|\boldsymbol{G}u_k\|_{\widetilde{A}} \lesssim \|u_k\|_a \lesssim C_\lambda$. Bounding the first term on the right hand side of (2.54) using (1.13) completes the proof. □

Lemma 2.17. *Let $v \in H^1_{\#}(\gamma)$, let $d(x)$ be the signed distance function for γ, let $\psi(x)$ be the closest point projection onto γ, let ν be the normal vector of γ, let $\mathbf{N}$ be the normal vector of Γ,* *and $\{\mathbf{e}_i\}_{i=1}^n$ be the eigenvectors of the Hessian, $\mathbf{H}$, of γ, then*

$$\begin{aligned} |a(v,v)-\widetilde{A}(v,v)| \leq &\left|\int_\Gamma d(x)\mathcal{H}\left[\nabla_\Gamma v\right]^T \nabla_\Gamma v d\Sigma\right| \\ &+2\left|\int_\Gamma d(x)\left(\sum_{i=1}^n \kappa_i(\psi(x))\left[\nabla_\Gamma v\right]^T\left[\mathbf{e}_i\otimes\mathbf{e}_i\right]\nabla_\Gamma v\right)d\Sigma\right| + O(h^{2k}), \end{aligned} \tag{2.55}$$

$$\left|m(v,v)-\widetilde{M}(v,v)\right| \leq \left|\int_\Gamma v^2 d(x)\mathcal{H} d\Sigma\right| + O(h^{2k}). \tag{2.56}$$

Here $\mathcal{H}=\sum_{i=1}^n \kappa_i(\psi(x))$ is the scaled mean curvature of γ.

Proof. We shall need the two identities from [33]:

$$\nabla_\gamma v(x) = [(\mathbf{I}-d\mathbf{H})(x)]^{-1}\left[\mathbf{I}-\frac{\mathbf{N}\otimes\nu}{\mathbf{N}\cdot\nu}\right]\nabla_\Gamma v, \tag{2.57}$$

$$d\sigma = \nu\cdot\mathbf{N}\left[\prod_{i=1}^n\left(1-d(x)\frac{\kappa_i(\psi(x))}{1+d(x)\kappa_i(\psi(x))}\right)\right]d\Sigma := Qd\Sigma. \tag{2.58}$$

We note that since $|1-\nu\cdot\mathbf{N}| = \frac{1}{2}|\nu-\mathbf{N}|^2 \lesssim h^{2k}$ and $\|d\|_{L_\infty(\Gamma)} \lesssim h^{k+1}$,

$$Q = (1-d\mathcal{H}) + O(h^{2k}). \tag{2.59}$$

Using (2.57) and (2.59) we then have

$$\begin{aligned} |a(v,v)-\widetilde{A}(v,v)| &= \left|\int_\gamma [\nabla_\gamma v]^T\nabla_\gamma v d\sigma - \int_\Gamma [\nabla_\Gamma v]^T\nabla_\Gamma v d\Sigma\right| \\ &\leq \left|\int_\Gamma [\nabla_\Gamma v]^T\left[\mathbf{I}-\frac{\nu\otimes\mathbf{N}}{\mathbf{N}\cdot\nu}\right][[(\mathbf{I}-d\mathbf{H})(x)]^{-1}]^T[(\mathbf{I}-d\mathbf{H})(x)]^{-1}\right. \\ &\times\left.\left[\mathbf{I}-\frac{\mathbf{N}\otimes\nu}{\mathbf{N}\cdot\nu}\right]\nabla_\Gamma v\left[1-d(x)\mathcal{H}\right]-[\nabla_\Gamma v]^T\nabla_\Gamma v d\Sigma\right| + O(h^{2k}). \end{aligned} \tag{2.60}$$

Expanding the Hessian $\mathbf{H}$ as on page 425 of [33], we obtain:

$$[(\mathbf{I}-d\mathbf{H})(x)]^{-1} = \nu\otimes\nu + \sum_{i=1}^n [1+d(x)\kappa_i(\psi(x))]\mathbf{e}_i\otimes\mathbf{e}_i = \mathbf{I} + \sum_{i=1}^n d(x)\kappa_i(\psi(x))\mathbf{e}_i\otimes\mathbf{e}_i.$$

Using $\mathbf{e}_i \perp \nu$ and $\mathbf{e}_i \perp \mathbf{e}_j$, $1 \leq i, j \leq n$, yields

$$[[(\mathbf{I} - d\mathbf{H})(x)]^{-1}]^T [(\mathbf{I} - d\mathbf{H})(x)]^{-1} = \mathbf{I} + 2\sum_{i=1}^{n} d(x)\kappa_i(\psi(x))\mathbf{e}_i \otimes \mathbf{e}_i + O(h^{2k+2}).$$

Combining the above and carrying out a short calculation yields

$$\begin{aligned}
&\left[\mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N} \cdot \nu}\right] [[(\mathbf{I} - d\mathbf{H})(x)]^{-1}]^T [(\mathbf{I} - d\mathbf{H})(x)]^{-1} \left[\mathbf{I} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N} \cdot \nu}\right] \\
&= \left[\mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N} \cdot \nu}\right] [\mathbf{I} + 2\sum_{i=1}^{n} d(x)\kappa_i(\psi(x))\mathbf{e}_i \otimes \mathbf{e}_i] \left[\mathbf{I} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N} \cdot \nu}\right] + O(h^{2k}) \\
&= \mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N} \cdot \nu} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N} \cdot \nu} + \frac{\nu \otimes \nu}{(\mathbf{N} \cdot \nu)^2} \\
&+ 2\sum_{i=1}^{n} d(x)\kappa_i(\psi(x)) \left[\mathbf{e}_i \otimes \mathbf{e}_i - \frac{\mathbf{N} \cdot \mathbf{e}_i}{\mathbf{N} \cdot \nu} (\nu \otimes \mathbf{e}_i + \mathbf{e}_i \otimes \nu) + \left(\frac{\mathbf{N} \cdot \mathbf{e}_i}{\mathbf{N} \cdot \nu}\right)^2 \nu \otimes \nu\right] \\
&+ O(h^{2k}).
\end{aligned}$$

Let $P_\Gamma := \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$. Then

$$\begin{aligned}
\mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N} \cdot \nu} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N} \cdot \nu} + \frac{\nu \otimes \nu}{(\mathbf{N} \cdot \nu)^2} &= P_\Gamma + \left(\mathbf{N} - \frac{\nu}{\mathbf{N} \cdot \nu}\right) \otimes \left(\mathbf{N} - \frac{\nu}{\mathbf{N} \cdot \nu}\right) \\
&= P_\Gamma + O(h^{2k}).
\end{aligned}$$

We know $\|\mathbf{N} - \nu\|_\infty \lesssim h^k$, so $\mathbf{N} \cdot \mathbf{e}_i = O(h^k)$ which means all terms containing $d(x)\mathbf{N} \cdot \mathbf{e}_i$ are of order h^{2k+1}. Therefore we have

$$\begin{aligned}
&\left[\mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N} \cdot \nu}\right] [[(\mathbf{I} - d\mathbf{H})(x)]^{-1}]^T [(\mathbf{I} - d\mathbf{H})(x)]^{-1} \left[\mathbf{I} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N} \cdot \nu}\right] \\
&= P_\Gamma + 2\sum_{i=1}^{n} d(x)\kappa_i(\psi(x)) [\mathbf{e}_i \otimes \mathbf{e}_i] + O(h^{2k}).
\end{aligned} \tag{2.61}$$

Multiplying equations (2.61) and (2.59) gives

$$\left[\mathbf{I} - \frac{\nu \otimes \mathbf{N}}{\mathbf{N}\cdot\nu}\right] [[(\mathbf{I} - d\mathbf{H})(x)]^{-1}]^T [(\mathbf{I} - d\mathbf{H})(x)]^{-1} \left[\mathbf{I} - \frac{\mathbf{N} \otimes \nu}{\mathbf{N}\cdot\nu}\right] Q$$
$$= P_\Gamma(1 - d(x)\mathcal{H}) + 2\sum_{i=1}^{n} d(x)\kappa_i(\psi(x)) \left[\mathbf{e}_i \otimes \mathbf{e}_i\right] + O(h^{2k}).$$

Inserting the above into (2.60) and noting that $P_\Gamma \nabla_\Gamma v = \nabla_\Gamma v$ yields

$$|a(v,v) - \widetilde{A}(v,v)| \leq \left| \int_\Gamma d(x)\mathcal{H} \left|\nabla_\Gamma v\right|^2 d\Sigma \right|$$
$$+ 2\left| \int_\Gamma \left(\sum_{i=1}^{n} d(x)\kappa_i(\psi(x)) \left[\nabla_\Gamma v\right]^T \left[\mathbf{e}_i \otimes \mathbf{e}_i\right] \nabla_\Gamma v \right) d\Sigma \right| + O(h^{2k}).$$

This is (2.55). The proof of (2.56) follows directly from (2.59). □

We next define a quadrature rule on the reference element:

$$\int_{\hat{T}} \hat{\varphi}(\hat{x}) d\hat{\Sigma} \approx \sum_{i=1}^{L} \hat{w}_i \hat{\varphi}(\hat{q}_i),$$

where $\{\hat{w}_j\}_{j=1}^{L}$ are weights and $\{\hat{q}_j\}_{j=1}^{L}$ is a set of quadrature points. Recall the definition (2.10) of $\boldsymbol{F}_T : \hat{T} \to T$. The mapped rule on a physical element $T \subset \Gamma$ is

$$\int_T \varphi(x) d\Sigma \approx \sum_{i=1}^{L} w_i \varphi(q_i),$$

where $w_i = Q_{\boldsymbol{F}_T}(\hat{q}_i)\hat{w}_i$, $Q_{\boldsymbol{F}_T} = \sqrt{\det(J^T J)}$ with J the Jacobian matrix of $\boldsymbol{F}_T$, and $q_i = \boldsymbol{F}_T(\hat{q}_i)$. The quadrature errors on the unit and physical elements are

$$E_{\hat{T}}(\varphi) := \int_{\hat{T}} \hat{\varphi}(\hat{x}) d\hat{\Sigma} - \sum_{i=1}^{L} \hat{w}_i \hat{\varphi}(\hat{q}_i), \qquad E_T(\varphi) := \int_T \varphi(x) d\Sigma - \sum_{i=1}^{L} w_i \varphi(q_i). \tag{2.62}$$

We say that a mapping $\boldsymbol{F}_T$ is *regular* if $|\boldsymbol{F}_T|_{W^{i,\infty}(\hat{T})} \leq h^i$, $0 \leq i \leq k$. This is implied by assumption (2.13). Note also that $|\boldsymbol{F}_T|_{W^{i,\infty}(\hat{T})} = 0$, $i > k$.

Lemma 2.18. *Suppose $E_{\hat{T}}(\hat{\chi}) = 0\ \forall \hat{\chi} \in \mathbb{P}^{\ell-1}(\hat{T})$, $d \in W^{\ell,\infty}(T)$, and $\boldsymbol{F}_T$ is a regular mapping. Then there is a constant C, independent of T, such that*

$$|E_T(d\varphi\psi)| \le C\|d\|_{W^{\ell,\infty}(T)} h^{\ell} |\varphi|_{H^{\min\{r,\ell\}}(T)} |\psi|_{H^{\min\{r,\ell\}}(T)}, \quad \forall \hat{\varphi}, \hat{\psi} \in \mathbb{P}^r(\hat{T}). \tag{2.63}$$

Proof. We use standard steps from basic finite element theory [47]. For each T,

$$E_T(d\varphi\psi) = E_{\hat{T}}\left(d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T} \hat{\varphi}\hat{\psi}\right). \tag{2.64}$$

Since $E_{\hat{T}}(\hat{\chi}) = 0, \forall \hat{\chi} \in \mathbb{P}^{\ell-1}(\hat{T})$, it follows from the Bramble-Hilbert Lemma and (2.62) that

$$|E_{\hat{T}}(\hat{g})| = \inf_{\chi \in \mathbb{P}^{\ell-1}} |E_{\hat{T}}(\hat{g} - \chi)| \le \inf_{\chi \in \mathbb{P}^{\ell-1}} \|\hat{g} - \chi\|_{L_\infty(\hat{T})} \le \hat{C} |\hat{g}|_{W^{\ell,\infty}(\hat{T})}.$$

Substituting $\hat{g} = d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T} \hat{\varphi}\hat{\psi}$, we thus have

$$\left| E_{\hat{T}}\left(d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T} \hat{\varphi}\hat{\psi}\right)\right| \le \hat{C} \left| d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T} \hat{\varphi}\hat{\psi}\right|_{W^{\ell,\infty}(\hat{T})}.$$

We now apply equivalence of norms over finite dimensional spaces and scaling arguments noting that $D^\alpha \hat{\varphi} = D^\alpha \hat{\psi} = 0$ for $|\alpha| > r$ to get

$$\begin{aligned} \left| d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T} \hat{\varphi}\hat{\psi}\right|_{W^{\ell,\infty}(\hat{T})} &\le \sum_{\substack{i,j=0 \\ \ell-i-j\ge 0}}^{\min\{r,\ell\}} |d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T}|_{W^{\ell-i-j,\infty}(\hat{T})} |\hat{\varphi}|_{W^{i,\infty}(\hat{T})} |\hat{\psi}|_{W^{j,\infty}(\hat{T})} \\ &\lesssim \sum_{\substack{i,j=0 \\ \ell-i-j\ge 0}}^{\min\{r,\ell\}} |d(\boldsymbol{F}_T) Q_{\boldsymbol{F}_T}|_{W^{\ell-i-j,\infty}(\hat{T})} |\hat{\varphi}|_{H^{i}(\hat{T})} |\hat{\psi}|_{H^{j}(\hat{T})}. \end{aligned}$$

Through standard arguments we have

$$|\hat{\varphi}|_{H^i(\hat{T})} |\hat{\psi}|_{H^j(\hat{T})} \lesssim h^{i+j} \|Q_{\boldsymbol{F}_T^{-1}}\|_{L_\infty(T)} |\hat{\varphi}|_{H^i(T)} |\hat{\psi}|_{H^j(T)}.$$

Noting that $|Q_{\boldsymbol{F}_T}|_{W^{k,\infty}(\hat{T})} \lesssim h^{n+j}$ and $\|Q_{\boldsymbol{F}_T^{-1}}\|_{L_\infty(T)} \lesssim h^{-n}$ along with

$$|d(\boldsymbol{F}_T)Q_{\boldsymbol{F}_T}|_{W^{\ell-i-j,\infty}(\hat{T})} \lesssim \sum_{k=0}^{\ell-i-j} |Q_{\boldsymbol{F}_T}|_{W^{k,\infty}(\hat{T})} |d(\boldsymbol{F}_T)|_{W^{\ell-i-j-k,\infty}(\hat{T})}$$

and

$$|d(\boldsymbol{F}_T)|_{W^{\ell-i-j-k,\infty}(\hat{T})} \lesssim h^{\ell-i-j-k} \, \|d\|_{W^{\ell-i-j-k,\infty}(T)}$$

gives

$$\left| d(\boldsymbol{F}_T)Q_{\boldsymbol{F}_T}\hat{\varphi}\hat{\psi} \right|_{W^{\ell,\infty}(\hat{T})} \lesssim h^\ell \|d\|_{W^{\ell,\infty}(\Omega)} \|\varphi\|_{H^{\min\{r,\ell\}}(T)} \|\psi\|_{H^{\min\{r,\ell\}}(T)},$$

which is the desired result. □

We now consider the effects of constructing Γ by interpolating $\boldsymbol{\psi}$.

Lemma 2.19 (Superconvergent Geometric Consistency)**.** *Let QUAD$_{\hat{T}}$ be a degree $\ell - 1$, R point quadrature rule on the unit element with quadrature points $\{\hat{q}_i\}_{i=1}^R$, $V \in \mathbb{V}$ be degree-r function, and assume that $d(x)\mathcal{H} \in W^{\ell,\infty}(\mathcal{N})$. If the points $\{\boldsymbol{L}(x^j)\}_{j=1}^{n_k}$ in* (2.8) *and $\{q_i\}_{i=1}^L$ coincide and in addition $\boldsymbol{L}(x^j) = \boldsymbol{\psi}(x^j)$, then*

$$|a(V,V) - \widetilde{A}(V,V)| \le h^\ell \, \|d(x)\mathcal{H}\|_{W^{\ell,\infty}_{\mathcal{T}}(\Gamma)} |V|^2_{H^{\min\{r,\ell\}}_{\mathcal{T}}(\Gamma)} + O(h^{2k}), \tag{2.65}$$

$$|m(V,V) - \widetilde{M}(V,V)| \lesssim h^\ell \, \|d(x)\mathcal{H}\|_{W^{\ell,\infty}_{\mathcal{T}}(\Gamma)} |V|^2_{H^{\min\{r,\ell\}}_{\mathcal{T}}(\Gamma)} + O(h^{2k}). \tag{2.66}$$

Here a subscript $\mathcal{T}$ denotes a broken (elementwise) version of the given norm.

Proof. We prove (2.66). (2.65) follows from similar arguments. Recalling (2.56) and partition the first integral based on the underlying mesh.

$$\left| \int_\Gamma V^2 d(x)\mathcal{H} d\Sigma \right| \le \sum_{j=1}^{\#elements} \left| \int_{T_j} V^2 d(x)\mathcal{H} d\Sigma \right|.$$

Let q be a quadrature point on T_j. By assumption $\boldsymbol{L}(q) = \boldsymbol{\psi}(q)$, so $d(q) = 0$ and

$$\left|\int_{T_j} V^2 d(x)\mathcal{H} d\Sigma\right| = \left|\int_{T_j} V^2 d(x)\mathcal{H} d\Sigma - \text{QUAD}_{T_j}\left(V^2 d(x)\mathcal{H}\right)\right|$$

$$= E_{T_j}(d(x)\mathcal{H}V^2) \lesssim h^{\ell} \|d(x)\mathcal{H}\|_{W_{\mathcal{T}}^{\ell,\infty}(\Gamma)} |V|^2_{H_{\mathcal{T}}^{\min\{r,\ell\}}(T_j)}$$

by Lemma 2.18. Summing over all of the elements yields (2.66). □

Theorem 2.20 (Order of eigenvalue error)**.** *If* Γ *be constructed using interpolation points that correspond to a degree* $\ell - 1$ *quadrature rule as in Lemma 2.19, then*

$$|\lambda_j - \Lambda_j| \lesssim h^{2r} + h^{2k} + h^{\ell}. \tag{2.67}$$

Proof. Standard arguments (adding and subtracting an interpolant and applying inverse inequalities) yield $\|U\|_{H^k} \lesssim \|\mathbf{P}_{\lambda_j} U\|_{H^{k+1}}$. Combining Lemma 2.19 and Lemma 2.16 into Lemma 2.14 completes the proof. □

Remark 2.21. *Our proofs carry over to the setting of quadrilateral elements with appropriate modification of the definition of regularity of the mapping* $\boldsymbol{F}_T$*. If Gauss-Lobatto points are used on the faces of* $\overline{\Gamma}$ *as the Lagrange interpolation points to define the surface* Γ*, then the* $O(h^{\ell})$ *term in* (2.67) *is the error due to tensor-product* $k+1$*-point Gauss-Lobatto quadrature, which is exact for polynomials of order* $2k-1$*. Thus* $\ell = 2k$ *and* $|\lambda_j - \Lambda_j| \lesssim h^{2r} + h^{2k}$*. We demonstrate this numerically below.*

Remark 2.22. *It follows from* (2.66) *that computation of* $\text{area}(\gamma)$ *using quadrature may also be superconvergent. This has been observed numerically when using deal.ii [37, Step 10 Tutorial].*

2.6 Numerical results for eigenvalue superconvergence

In this section we numerically investigate the convergence rate of the geometric term in the eigenvalue estimate of Theorem 2.20. Using the upper bound we derived as a guide, we set the order r of the PDE approximation so that h^{2r} is higher order in the experiments.

We first approximated the unit circle using a sequence of polygons with uniform faces. For higher order approximations we interpolated the circle using equally spaced points and points based on Gauss-Lobatto quadrature. The left plot in Figure 2.4 shows convergence rates for λ_1 for various choices of k for both spacings. The error when using Gauss-Lobatto points follows a trend of h^{2k} as predicted by our analysis in Section 2.5. The errors when using equally spaced Lagrange points are $O(h^{k+1})$ for odd values of k and $O(h^{k+2})$ for even values of k. These quadrature errors arise from the Newton-Cotes rule corresponding to standard Lagrange points, yielding for example Simpson's rule with error $O(h^4) = O(h^{k+2})$ when $k = 2$.

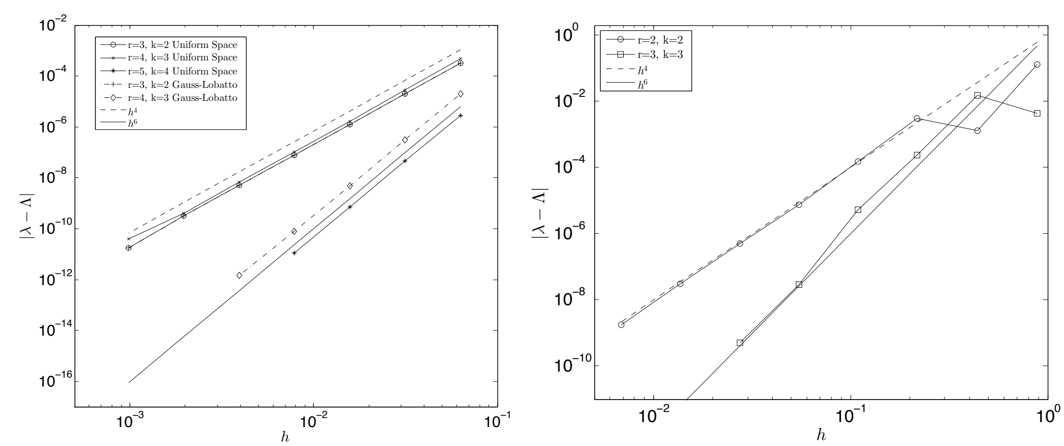

Figure 2.4: *Left:* Convergence rates of the first eigenvalue for the circle using typical equally spaced Lagrange basis points and Gauss-Lobatto Lagrange basis points. *Right:* Convergence rates of the first eigenvalue for $(x - z^2)^2 + y^2 + z^2 + \frac{1}{2}(x - 0.1)(y + 0.1)(z + 0.2) - 1 = 0$ surface using a quadrilateral mesh with Gauss-Lobatto Lagrange basis points. Reprinted from [1].

In our next experiment we used a quadrilateral mesh to approximate the surface $(x - z^2)^2 + y^2 + z^2 + \frac{1}{2}(x - 0.1)(y + 0.1)(z + 0.2) - 1 = 0$. We used Gauss-Lobatto quadrature points on each face to construct the interpolated surface. Convergence rates for the first eigenvalue using $k = 2, 3$ are seen in the right plot in Figure 2.4. The trend of order h^{2k} convergence predicted by our analysis holds for surfaces in 2D when using Gauss-Lobatto interpolation points. Experiments yielding similar convergence rates were also performed on the sphere and torus.

We next investigated convergence on triangular meshes. We first created a triangulated approximation of the level set $(x - z^2)^2 + y^2 + z^2 - 1 = 0$ using standard Lagrange basis points. These points do not correspond to a known higher order quadrature rule. In the left plot in Figure 2.5, we see convergence rates of order h^{k+1} for odd values of k and h^{k+2} for even values of k. Unlike in one space dimension, these results cannot be directly proved using our framework above. More subtle superconvergence phenomenon may provide an explanation. For example, it is easy to show that the Newton-Cotes rule for $k = 2$ corresponding to standard Lagrange interpolation points exactly integrates cubic polynomials on any two triangles forming a parallelogram. It has previously been observed that meshes in which most triangle pairs form approximate parallelograms may lead to superconvergence effects, and it has been argued that many practical meshes fit within this framework; cf. [48].

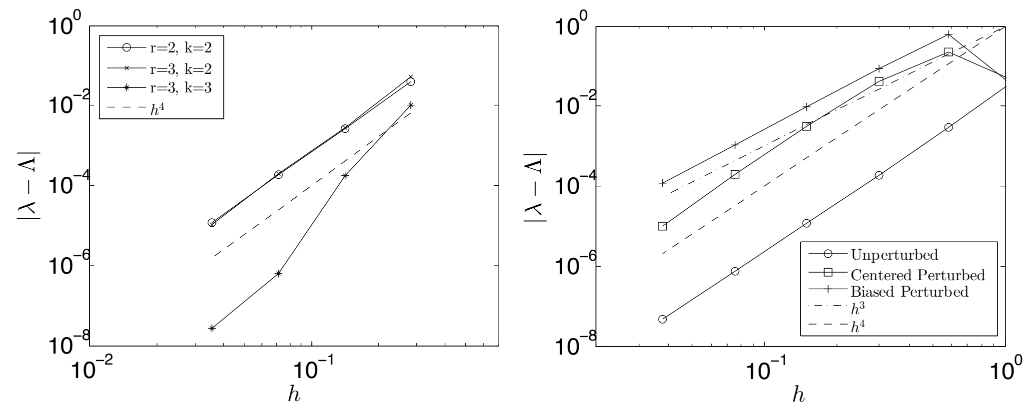

Figure 2.5: *Left:*. Convergence rates of an eigenvalue for $(x - z^2)^2 + y^2 + z^2 - 1 = 0$ surface using triangular mesh and typical Lagrange basis points. *Right:* Convergence rates of the first eigenvalue for spherical surface using triangular mesh and unperturbed interpolation points, randomly perturbed interpolation points from a uniform distribution centered at 0 displacement, and randomly perturbed interpolation points from a uniform distribution centered at $0.5h^{k+1}$ displacement. Reprinted from [1].

Finally, we attempted to break this even-odd superconvergence behavior by perturbing the points used to interpolate the sphere. First we perturbed points by $O(h^{k+1})$ using a uniform distribution on $h^{k+1}(-1, 1)$. In expectation we then have a radial perturbation of 0. The superconvergence of $O(h^{k+2})$ for even k values persisted for this situation. We then biased the previous distribution to be $h^{k+1}(-0.5, 1.5)$ so that perturbations tended to be outward of the surface of the sphere. This led to convergence of $O(h^{k+1})$ for both even and odd values of k. Numerical results for the error of the first eigenvalue of the sphere when $r = 3$ and $k = 2$ for an unperturbed sphere as well as these two perturbations are seen in the right plot in Figure 2.5.

Remark 2.23. *The perturbations of interpolation points on the sphere described above satisfy the abstract assumptions* (2.11) *through* (2.13) *and so fit within the basic eigenvalue convergence theory of Section 2.2. That theory is thus sharp without additional assumptions, but clearly does not satisfactorily explain many cases of interest.*

Remark 2.24. *The superconvergence effects we have observed appear to be relatively robust. They may still occur even in applications where the continuous surface is not interpolated exactly as long as surface approximation errors at the interpolation points are uniformly distributed inside and outside of* γ *with zero mean.*

3. OPTIMALITY OF ADAPTIVE FINITE ELEMENT METHODS FOR EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR

In this chapter we develop a quasi-optimal adaptive finite element method (AFEM) based on SFEM to approximate eigenfunctions associated with a cluster J of eigenvalues associated with the eigenvalue problem

$$-\Delta_\gamma u = \lambda u \quad \text{on} \quad \gamma. \tag{3.1}$$

We take $\gamma \subset \mathbb{R}^{d+1}$ to be a closed hypersurface, which is globally Lipschitz and piecewise in a Besov class that embeds into $C^{1,\alpha}$ with $\alpha \in (0,1]$. Here, Δ_γ is the Laplace-Beltrami operator on γ.

We combine tools from SFEM with those from the analysis of AFEM and eigenvalue problems. In addition to the usual Galerkin approximation errors, the SFEM framework introduces geometric consistency errors. In the adaptive context this means that not only must the solution be resolved by the mesh, but the surface must also be sufficiently resolved by the mesh. One approach for adaptively approximating solutions to the source problem via SFEM has been analyzed in [35]. In the framework of [35] two estimators are used to drive refinements, one for the Galerkin error and another for the geometric error with each estimator having a separate dedicated refinement procedure.

Unlike the source problem, the eigenvalue problem is nonlinear which makes for a more challenging problem to analyze. Over the past decade a number of papers have appeared analyzing convergence and optimality of AFEM for eigenvalue problems on flat domains. These papers require a priori L_2 estimates which are used to control the nonlinearity of the eigenvalue problem and lead to minimal mesh resolution requirements to guarantee the nonlinearity has been resolved. We summarize these requirements in Table 3.1. For the purposes of proving a priori L_2 estimates we require and prove a new regularity result for the source problem on globally $W^{1,\infty}$ piecewise $C^{1,\alpha}$ surfaces.

(H1)	$D(J) := \max_{\ell \in J} \max_{j \notin J} \left\| \frac{\lambda_\ell}{\Lambda_j - \lambda_\ell} \right\| < \infty$	Lemma 3.14
(H2)	$\max_{j \in J} \|u_j - \mathbf{Z} u_j\|_{L_2(\gamma)} \leq \sqrt{1 + (2\|J\|)^{-1}} - 1$	Lemma 3.22
(H3)	$\lambda_{\max}^2 K_0 H_0^{2s} \leq \frac{1}{2}$	Lemma 3.24
(H4)	$\left(\frac{K_1}{2B_3} \lambda_{\max}^2 H_0^2 + (1 + \lambda_{\max}^2) K_2 + \lambda_{\max}^2 \right) C H_0^{2s} < \min \left\{ \frac{1}{8B_3(2C_1 + C_2)} \frac{\xi^2 \theta^4}{4 - 2\xi\theta^2}, \frac{\xi\theta^2}{8} \right\}$	Theorem 3.31
(H5)	$(B_3 K_0 + K_1 H_0^2) \lambda_{\max}^2 C H_0^{2s} \leq 0.2$	Lemma 3.41
(H6)	$\left(K_2(1 + \lambda_{\max}^2) + \lambda_{\max}^2 + \frac{3}{2} K_0 \lambda_{\max}^2 \right) C H_0^{2s} \leq 0.2$	Lemma 3.41
(H7)	$C(1 + 2B_3)\lambda_{\max}^2 H_0^{2s} + C(1 + 2B_3) K_2 (1 + \lambda_{\max}^2) H_0^{2s} + C K_1 \lambda_{\max}^2 H_0^{2+2s} \leq \frac{1}{2}$	Lemma 3.43
(H8)	$C K_1 \lambda_{\max}^2 H_0^2 \leq \frac{1}{2}$	Lemma 3.43

Table 3.1: The restrictions on the maximum mesh size H_0 and where they are first used.

Our approach borrows heavily from [35]. Our adaptive algorithm (AFEM) consists of loops made up of an ADAPT_SURFACE step followed by an ADAPT_EIGENFUNCTION step.
The ADAPT_SURFACE step is the same as that of [35] and employs the same geometric estimator ζ. It should be noted that in [1] the a priori analysis for approximating eigenfunctions with SFEM found that the geometric error for a piecewise degree-k approximation of the surface was $O(h^{k+1})$ when the surface is of regularity C^{k+1}, $k \geq 1$. Unfortunately, the geometric estimator we use from [35] is heuristically only $O(h^k)$ for C^{k+1} surfaces. A new geometric estimator for C^{k+1}, $k \geq 1$ surfaces that is heuristically $O(h^{k+1})$ has recently been introduced in [36]. We choose

to use the geometric estimator used in [35] since it can handle surfaces of lower regularity and the adaptive algorithm for the source problem has been shown to be optimal unlike the algorithm using the heuristically $O(h^{k+1})$ estimator in [36]. Even in the context of the source problem, the discrepancy between the order h^{k+1} geometric errors observed for C^2 surfaces and the order h^k geometric errors that have been proved for less regular surfaces is not well understood. Our ADAPT_EIGENFUNCTION step uses a residual type estimator, η, to control the eigenfunction error. This estimator will be analyzed via a theoretical estimator, μ, in a similar fashion to what was done for flat domains in [23]. We show that μ and η are equivalent up to geometric error terms which can be controlled by ζ. We will show in Section 3 that this implies any results proven for the estimator μ are also true for η within ADAPT_EIGENFUNCTION. We will also develop an L_2 a priori estimate for eigenfunction approximations on piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces. In order to extend the a priori estimates of [1] from C^2 surfaces to these lower regularity surfaces we must prove a regularity estimate in Section 2 for eigenfunctions on on piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces which is our first main result:

Let γ be a piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ d-dimensional closed surface in $\mathbb{R}^{d+1}$. Let $f \in L_2(\gamma)$. Let $u \in H^1(\gamma)$ solve $-\Delta_\gamma u = f$ in a weak sense subject to $\int_\gamma u \, d\sigma = 0$. Then

$$\|\nabla_\gamma u\|_{H^s(\gamma)} \lesssim \|f\|_{L_2(\gamma)}.$$

It should be noted that this result not only extends the a priori estimates for eigenfunctions on surfaces of [1], but it can also be used to extend the a priori estimates of [32] for the source problem from C^2 surfaces to piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces.

The rest of the paper is laid out as follows. In Section 2 we introduce the surface approximation scheme used as well as the geometric estimator ζ, the Galerkin formulation, surface finite element method, and our regularity result stated above. In Section 3 we prove a posteriori estimates for the theoretical estimator. In Section 4 we introduce our modified eigenfunction version of the AFEM algorithm used in [35] for the source problem. The major change is the replacement of the residual based PDE estimator of [35] with our residual based eigenfunction estimator η. In Section 5 we prove our second main result, a conditional contraction property:

If the parameter $\omega > 0$ is small enough and the largest mesh size H_0 is small enough, then there exist constants $\beta > 0$ and an $0 < \alpha < 1$ such that for all iterate $0 \le j < R$ of ADAPT_EIGENFUNCTION we have

$$\sum_{i\in J} \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + \beta(\mu^{j+1}(J))^2 \le \alpha^2\Bigg(\sum_{i\in J} \|\nabla_\gamma e_i^{j}\|^2_{L_2(\gamma)} + \beta(\mu^{j}(J))^2\Bigg).$$

Moreover, the number of inner iterates of ADAPT_EIGENFUNCTION is uniformly bounded.

In Section 6 we give a partial characterization of our approximation classes $\mathbb{A}'_s$ in terms of Besov regularity. In Section 7 we prove convergence rates:

If $\{(u_j, \lambda_j, \gamma)\}_{j\in J} \in \mathbb{A}'_s(J)$ for some $0 < s \le n/d$, $\omega > 0$ is small enough, and the largest mesh size H_0 is small enough, then there exists a constant C, depending on the Lipschitz constant L of γ, $\lambda_{\max}$, the refinement depth b, the initial triangulation $\mathcal{T}_0$, and AFEM parameters θ, ω, ρ such that

$$\sum_{j\in J} e(\mathbf{Z}_k u_j) + Osc_{\mathcal{T}_k}(\lambda_j \mathbf{P}_k u_j, \mathbf{Z}_k u_j, \gamma) + |J|\zeta_{\mathcal{T}_k}(\gamma) \le C|J,\gamma|_{\mathbb{A}'_s(J)}(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

. In Section 8 we verify the convergence rates with numerical experiments.

3.1 Preliminaries

In this section we present the necessary background information from the differential geometry of surfaces as well as a priori estimates for the approximation of eigenfunctions. The latter estimates will be necessary for handling higher order terms which appear in the a posteriori analysis. Our presentation in this section closely follows that of [35], and we refer to that work for more details.

3.1.1 Parametric Surfaces

We assume that the surface γ is the deformation of a

d-dimensional polyhedral surface $\overline{\Gamma}_0$, with vertices on γ, and described by a globally Lipschitz homeomorphism $P_0 : \overline{\Gamma}_0 \to \gamma \subset \mathbb{R}^{d+1}$. The overline notation for $\overline{\Gamma}_0$ is to emphasize the piecewise affine nature of the surface. If $\overline{\Gamma}_0 = \bigcup_{i=1}^{F} \overline{\Gamma}_0^i$ is made up of F polyhedral faces $\overline{\Gamma}_0^i$, $i = 1, \ldots, F$, we denote by $P_0^i : \overline{\Gamma}_0^i \to \mathbb{R}^{d+1}$ the restriction of P_0 to $\overline{\Gamma}_0^i$. The macro-elements $\overline{\Gamma}_0^i$ induce a partition $\{\gamma^i\}_{i=1}^F$ of γ upon setting

$$\gamma^i := P_0^i(\overline{\Gamma}_0^i).$$

This non-overlapping decomposition allows for piecewise smooth surfaces γ with possible kinks aligned with the decomposition $\{\gamma^i\}_{i=1}^F$. We assume that the macro-elements are simplices, i.e. there exists a closed reference simplex $\Omega \subset \mathbb{R}^d$ and a mapping $X_0^i : \mathbb{R}^d \to \mathbb{R}^{d+1}$ such that $\overline{\Gamma}_0^i = X_0^i(\Omega)$.

We define $\chi^i := P_0^i \circ \overline{X}_0^i : \Omega \to \gamma^i$ to be the mapping from the reference simplex to a member γ^i of the partition of γ which is also bi-Lipschitz, i.e. there exists a constant $L \geq 1$ such that for $1 \leq i \leq F$

$$L^{-1}|\hat{\mathbf{x}} - \hat{\mathbf{y}}| \leq |\chi^i(\hat{\mathbf{x}}) - \chi^i(\hat{\mathbf{y}})| \leq L|\hat{\mathbf{x}} - \hat{\mathbf{y}}|, \qquad \forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Omega,$$

which implies

$$L^{-1}|\mathbf{w}| \leq |\widehat{\nabla}\chi^i(\hat{\mathbf{x}})\mathbf{w}| \leq L|\mathbf{w}|, \qquad \forall \mathbf{w} \in \mathbb{R}^d.$$

We also note that a function $v^i : \gamma \to \mathbb{R}$ uniquely defines two other functions $\widehat{v}^i : \Omega \to \mathbb{R}$ and $\overline{v}^i : \overline{\Gamma}_0^i \to \mathbb{R}$

$$\widehat{v}^i(\widehat{\mathbf{x}}) = v^i(\chi^i(\widehat{\mathbf{x}})) \quad \forall \widehat{\mathbf{x}} \in \Omega \quad \text{and} \quad \overline{v}^i(\overline{\mathbf{x}}) = v^i(P_0(\overline{\mathbf{x}})) \quad \forall \overline{\mathbf{x}} \in \overline{\Gamma}_0^i.$$

In fact, given any of the functions v^i, $\widehat{v}^i$, or $\overline{v}^i$ the other two are automatically and uniquely defined. Throughout this paper we will drop the superscript and denote all three functions by v. The meaning should be clear from context.

3.1.2 Finite Element Spaces and Surface Approximations

The partition of $\overline{\Gamma}_0$ into macro-elements creates a conforming triangulation of $\overline{\Gamma}_0$ which we will denote by $\overline{\Gamma}_0^i$. We denote the class of conforming meshes generated by successive bisections of $\overline{\mathcal{T}}_0$ as $\overline{\mathbb{T}}$. A triangulation $\overline{\mathcal{T}} \in \overline{\mathbb{T}}$ produces triangulations of F copies of Ω and a piecewise polynomial approximation Γ of γ. Any number of conforming graded bisections of each macro-element $\overline{\Gamma}_0^i$ generate via $(X_0^i)^{-1}$ a partition of the local parametric domain $\Omega \subset \mathbb{R}^d$ denoted $\widehat{\mathcal{T}}^i(\Omega)$ or $\widehat{\mathcal{T}}^i$ for short.

Let $n \geq 1$ and $\widehat{\mathbb{V}}^i := \mathbb{V}(\widehat{\mathcal{T}}^i)$ denote the finite element space of globally continuous piecewise polynomials of degree $\leq n$ subordinate to the partition $\widehat{\mathcal{T}}^i$. Let $I_{\mathcal{T}^i} : C^0(\overline{\Omega}) \to \widehat{\mathbb{V}}^i$ be the Lagrange interpolation operator. We shall also denote the componentwise Lagrange interpolant operator as $I_{\mathcal{T}^i} : (C^0(\overline{\Omega}))^d \to (\widehat{\mathbb{V}}^i)^d$. We define

$$X_{\mathcal{T}^i}^i := I_{\mathcal{T}^i}\chi^i \qquad \Gamma^i := X_{\mathcal{T}^i}^i(\Omega) \qquad \mathcal{T}^i := \{T := X_{\mathcal{T}^i}^i(\widehat{T}) : \widehat{T} \in \widehat{\mathcal{T}}^i\}$$

to be the piecewise polynomial interpolations of χ^i, γ^i, and the associated mesh. The global parametric space denoted Ω^F consists of F copies of Ω. Its subdivision is denoted $\widehat{\mathcal{T}}$ and defined as

$$\widehat{\mathcal{T}} := \bigcup_{i=1}^{F} \widehat{\mathcal{T}}^i.$$

Each triangulation $\overline{\mathcal{T}} \in \overline{\mathbb{T}}$ uniquely determines $\widehat{\mathbb{T}}$. This allows us to define the forest

$$\widehat{\mathbb{T}} := \mathbb{T}(\widehat{\mathcal{T}}_0) := \{\widehat{\mathcal{T}} : \overline{\mathcal{T}} \in \overline{\mathbb{T}}\}.$$

It should be noted that $\widehat{\mathbb{T}}$ does not necessarily correspond to F copies of the same forest, but rather F different compatible forests. The bisection rule is governed by the topology of $\overline{\mathcal{T}}_0$ and dictates which initial bisection of each separate Ω is performed. Refinement of a macro-element in $\overline{\mathcal{T}}_0$ induces a partition of its boundary which must be compatible with refinements of adjacent macro-elements. The global subdivision $\mathcal{T}$ is given by

$$\mathcal{T} := \bigcup_{i=1}^{F} \mathcal{T}^i$$

and the corresponding forest is

$$\mathbb{T} := \mathbb{T}(\mathcal{T}_0) := \{\mathcal{T} : \overline{\mathcal{T}} \in \overline{\mathbb{T}}\}.$$

It should be noted that $\mathcal{T}_0 = \overline{\mathcal{T}}_0$ only for polynomials of degree $n = 1$. The global polynomial surface and parameterization $\mathbf{X}_{\mathcal{T}}$ of Γ are given by

$$\Gamma := \bigcup_{\overline{T} \in \overline{\mathcal{T}}} T, \qquad \mathbf{X}_{\mathcal{T}} = \{X^i_{\mathcal{T}^i}\}_{i=1}^{F}.$$

We say that $(\mathcal{T}, \Gamma)$ is a mesh-surface approximation pair when $\mathcal{T} \in \mathbb{T}$ and $\Gamma = \Gamma_{\mathcal{T}}$. For a subdivision $\mathcal{T} \in \mathbb{T}$ we denote by $S_{\mathcal{T}}$ the set of interior faces.

We define the two finite element spaces on Γ:

$$\mathbb{V}(\mathcal{T}) := \{V \in C^0(\Gamma) : V|_{\Gamma^i} \text{ is the lift of some } \widehat{V}^i \in \widehat{\mathbb{V}}^i \text{ via } X^i_{\mathcal{T}^i}\}$$

and

$$\mathbb{V}_{\#}(\mathcal{T}) := \{V \in \mathbb{V}(\mathcal{T}) : V = 0 \text{ on } \partial\Gamma \text{ or } \int_\Gamma V = 0 \text{ if } \partial\Gamma = \emptyset\}.$$

The refinement procedure consists of bisecting elements in $\overline{\mathcal{T}}_0$ and propagating its effects on $\widehat{\mathcal{T}}$ and $\mathcal{T}$ via the mappings $\mathbf{X}_0^{-1}$ and $\mathbf{X}_{\mathcal{T}} \circ \mathbf{X}_0^{-1}$, respectively. For $\overline{\mathcal{T}}, \overline{\mathcal{T}}_* \in \overline{\mathbb{T}}$, we use the notation $\overline{\mathcal{T}}_* \geq \overline{\mathcal{T}}$ to mean that $\overline{\mathcal{T}}_*$ is a conforming refinement of $\overline{\mathcal{T}}$. Given two subdivisions $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$, we write $\mathcal{T}_* \geq \mathcal{T}$to indicate that $\overline{\mathcal{T}}_* \geq \overline{\mathcal{T}}$. It should be noted that given $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T}_* \geq \mathcal{T}$, the finite element space $\mathbb{V}(\mathcal{T})$ is not a subspace of $\mathbb{V}(\mathcal{T}_*)$ since the associated surface approximations Γ and Γ_* do not match. Fortunately, we still have the nested property $\mathbb{V}(\widehat{\mathcal{T}}^i) \subset \mathbb{V}(\widehat{\mathcal{T}}^i_*)$ for $1 \leq i \leq F$.

The three different subdivisions $\overline{\mathcal{T}}$, $\widehat{\mathcal{T}}$, and $\mathcal{T}$ have various uses. $\overline{\mathcal{T}}$ is the triangulation on the flat surface and drives the refinement procedure; $\widehat{\mathcal{T}}$ is the triangulation on Ω and used to evaluate such quantities as oscillation and the geometric estimator; $\mathcal{T}$ is made of the curved faces that define $\Gamma = \Gamma_{\mathcal{T}}$ where the approximate eigenvalue problem is solved.

3.1.3 Shape Regularity and Geometric Estimators

Definition 3.1 (Shape regularity)**.** *We say that the class of conforming meshes $\mathbb{T}$ is shape regular if there is a constant C_0 only depending on $\overline{\mathcal{T}}_0$, such that $\widehat{\mathcal{T}} \in \widehat{\mathbb{T}}$, and all $i = 1, \ldots, F$,*

$$C_0^{-1}|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}| \leq |X^i_{\mathcal{T}^i}(\widehat{\mathbf{x}}) - X^i_{\mathcal{T}^i}(\widehat{\mathbf{y}})| \leq C_0|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}| \qquad \forall \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in \widehat{T}, \qquad \forall \widehat{T} \in \widehat{\mathcal{T}}^i. \tag{3.2}$$

It should be noted that (3.2) also implies

$$C_0^{-1}|\mathbf{w}| \leq |\widehat{\nabla} X^i_{\mathcal{T}^i}(\widehat{\mathbf{x}})\mathbf{w}| \leq C_0|\mathbf{w}| \qquad \forall \mathbf{w} \in \mathbb{R}^d.$$

We define the elementwise geometric error estimator to be

$$\zeta_{\mathcal{T}^i}(\gamma, T) := \|\widehat{\nabla}(\chi^i - X^i_{\mathcal{T}^i})\|_{L_\infty(\widehat{T})} = \|\widehat{\nabla}(\chi^i - I_{\mathcal{T}^i}\chi^i)\|_{L_\infty(\widehat{T})}$$

and the corresponding global geometric error estimator to be

$$\zeta_{\mathcal{T}}(\gamma) := \max_{i=1,\ldots,F} \max_{T \in \mathcal{T}^i} \zeta_{\mathcal{T}^i}(\gamma, T). \tag{3.3}$$

We will need the following two Lemmas about $\zeta_{\mathcal{T}}(\gamma)$ from [35].

Lemma 3.2 (Quasi-monotonicity of the geometric estimator (Lemma 2.3 of [35]))**.** *There exists a constant $B_0 > 1$ solely dependent on $\overline{\mathcal{T}}_0$, the polynomial degree k used to construct Γ, and the dimension d such that*

$$\zeta_{\mathcal{T}_*}(\gamma) \leq B_0 \zeta_{\mathcal{T}}(\gamma) \tag{3.4}$$

for any $\mathcal{T}_*, \mathcal{T} \in \mathbb{T}$ with $\mathcal{T}_* \geq \mathcal{T}$. This bound holds elementwise as well.

Lemma 3.3 (Shape regularity (Lemma 2.4 of [35])). *The forest $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ is shape-regular with constant $C_0 = 2L$ provided*

$$\zeta_{\mathcal{T}_0}(\gamma) \leq \frac{1}{2B_0 L}$$

with $L \geq 1$ the non-degeneracy constant.

3.1.4 The Laplace-Beltrami Operator

We now introduce the basic notions from differential geometry that will be necessary for the remainder of the paper. Let $\mathbf{T} \in \mathbb{R}^{(n+1)\times n}$ be the matrix

$$\mathbf{T} := [\widehat{\partial}_1 \chi, \cdots, \widehat{\partial}_n \chi].$$

The first fundamental form of γ is the symmetric positive-definite matrix $\mathbf{G} \in \mathbb{R}^{n\times n}$ defined by

$$\mathbf{G} := (\widehat{\partial}_i \chi^T \widehat{\partial}_j \chi)_{1\leq i,j\leq d} = \mathbf{T}^T \mathbf{T}.$$

We define $\mathbf{D}$ to be the matrix satisfying

$$\nabla_\gamma v = \widehat{\nabla} v \mathbf{D}.$$

The inverse of $\mathbf{G}$ can be expressed as $\mathbf{G}^{-1} = \mathbf{D}\mathbf{D}^T$. We define the measure on γ to be

$$q := \sqrt{\det \mathbf{G}}.$$

We denote the measure on subfaces of $\chi(\widehat{T}) \in \chi(\widehat{\mathcal{T}})$ by q^s. We denote the unit normal vector of γ as $\mathbf{n}$. We denote the analogues of $\mathbf{G}$, $\mathbf{D}$, q, q^s, and $\mathbf{n}$ on Γ by appending a Γ subscript. When

$\chi \in C^2(\Omega)$ and $v \in H^2(\gamma)$, we have

$$\Delta_\gamma v := \frac{1}{q}\widehat{\text{div}}(q\widehat{\nabla} v \mathbf{G}^{-1}).$$

In this case we also have the integration by parts formula

$$\int_\gamma \nabla_\gamma w \nabla_\gamma v \, d\sigma = \int_\gamma -\Delta_\gamma w v \, d\sigma + \int_{\partial\gamma} (\nabla_\gamma w \cdot \mathbf{n}) v \, d(\partial\gamma).$$

3.1.5 Variational Formulation and Galerkin Method

We define the space $H^1(\gamma)$ to be

$$H^1(\gamma) := \Big\{ v \in L_2(\gamma) : \nabla_\gamma v^i \in [L_2(\gamma^i)]^{d+1}, v^i = v^j \text{ on } \gamma^i \cap \gamma^j, 1 \leq i, j \leq F \Big\},$$

and the space $H^1_\#(\gamma)$ to be

$$H^1_\#(\gamma) := \Big\{ v \in H^1(\gamma) : \int_\gamma v \, d\sigma = 0 \Big\}.$$

We define the bilinear form and L_2 inner product on γ to be:

$$a(u, v) := \int_\gamma \nabla_\gamma u \nabla_\gamma v \, d\sigma \quad \text{and} \quad m(u, v) := \int_\gamma uv \, d\sigma$$

respectively. The weak formulation of (3.1) is then: Find an eigenpair $(u, \lambda) \in H^1_\#(\gamma) \times \mathbb{R}^+$ such that

$$a(u, v) = \lambda m(u, v) \quad \forall v \in H^1_\#(\gamma), \tag{3.5}$$

where the eigenvalues satisfy $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$. We define the bilinear form and L_2 inner product on Γ to be:

$$A(u, v) := \int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \, d\Sigma \quad \text{and} \quad M(u, v) := \int_\Gamma uv \, d\Sigma$$

respectively. The corresponding finite element problem associated with (3.5) is: Find an eigenpair $(U, \Lambda) \in \mathbb{V}_{\#}(\Gamma) \times \mathbb{R}^{+}$ such that

$$A(U, V) = \Lambda M(U, V) \quad \forall V \in \mathbb{V}_{\#}(\Gamma). \tag{3.6}$$

3.1.6 Eigenvalue Clusters

We now define what it is to be an eigenvalue cluster as well as various projection operators essential to our analysis. Let $p \geq 1$, $N \geq 0$ and

$$\lambda_{p-1} < \lambda_p \qquad \text{and} \qquad \lambda_{p+N} < \lambda_{p+N+1}. \tag{3.7}$$

We define the set of eigenvalues λ_i associated with the index set

$$J := \{p, ..., p+N\}$$

to be an eigenvalue cluster. We denote by $\{u_i\}_{i=1}^{\infty}$ a corresponding orthonormal basis (with respect to $m(\cdot,\cdot)$) of $H^1_{\#}(\gamma)$. We denote the span of the FEM eigenfunctions $\{U_j\}_{j\in J}$ associated with the eigenvalue cluster with index set J as

$$\mathbb{W}(J) := \text{span}\{U_j\}_{j\in J}. \tag{3.8}$$

We define the Galerkin projection operator $\mathbf{R} : H^1(\Gamma) \to \mathbb{V}_{\#}(\Gamma)$ with respect to the discrete bilinear form as the operator satisfying:

$$A(u, V) = A(\mathbf{R}u, V) \quad \forall V \in \mathbb{V}_{\#}(\Gamma).$$

We define the L_2-projection $\mathbf{P} : L_2(\Gamma) \to \mathbb{W}(J)$ as the operator satisfying:

$$M(u, V) = M(\mathbf{P}u, V) \quad \forall V \in \mathbb{W}(J). \tag{3.9}$$

We define the $\mathbf{Z}$ projection operator as:

$$\mathbf{Z} := \mathbf{P}\mathbf{R}. \tag{3.10}$$

It should be noted that $\mathbf{Z}$ satisfies:

$$A(u, V) = A(\mathbf{Z}u, V) \quad \forall V \in \mathbb{W}(J) \tag{3.11}$$

as seen in [1].

We next present Lemma 3.4 which is the surface finite element analogue of Lemma 2.2 of [23]. Lemma 2.2 of [23] gives an algebraic identity relating $\mathbf{Z}u$ to $\lambda\mathbf{P}u$ in terms of the bilinear form and L_2 inner product on flat domains. Our version is similar, but the effect of using an approximate surface manifests itself in the form of extra geometric consistency terms.

Lemma 3.4 (Algebraic Identity). *Let* $\{(u_j, \lambda_j)\}_{j\in J}$ *be the set of eigenpairs associated with a cluster that satisfies* (3.5). *Let* $\mathbb{W}(J)$, $\mathbf{P}$, *and* $\mathbf{Z}$ *be defined as in* (3.8), (3.9), *and* (3.10). *Then for any exact eigenpair* $(u, \lambda) \in H^1_{\#}(\gamma) \times \mathbb{R}^+$ *associated with the cluster, we have*

$$A(\mathbf{Z}u, V) = \lambda M(\mathbf{P}u, V) + [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] + \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)] \quad \forall V \in \mathbb{V}(\Gamma). \tag{3.12}$$

Proof. Noting that $\mathbf{Z}u = \sum_{j\in J} M(\mathbf{R}u, U_j)U_j = \sum_{j\in J} \frac{A(\mathbf{R}u, U_j)}{\Lambda_j} U_j = \sum_{j\in J} \frac{A(u, U_j)}{\Lambda_j} U_j$, we get the following:

$$\begin{aligned}
A(\mathbf{Z}u,V) &= A(\sum_{j\in J}\frac{A(u,U_j)}{\Lambda_j}U_j,V) = \sum_{j\in J}\frac{1}{\Lambda_j}A(u,U_j)A(U_j,V)\\
&= \sum_{j\in J}(\lambda M(u,U_j) + [A(u,U_j) - \lambda M(u,U_j)])M(U_j,V)\\
&= \lambda M(\mathbf{P}u,V) + \sum_{j\in J}[A(u,U_j) - a(u,U_j) + \lambda m(u,U_j) - \lambda M(u,U_j)]M(U_j,V)\\
&= \lambda M(\mathbf{P}u,V) + [A(u,\mathbf{P}V) - a(u,\mathbf{P}V)] + \lambda[m(u,\mathbf{P}V) - M(u,\mathbf{P}V)].
\end{aligned}$$

□

3.1.7 Geometric Error and Estimator

We now elaborate on the relationship between the geometric estimator defined in (3.3) and the geometric consistency errors. Following [35], we express the difference between the bilinear forms as

$$\int_\Gamma \nabla_\Gamma v \nabla_\Gamma w \, d\Sigma - \int_\gamma \nabla_\gamma v \nabla_\gamma w \, d\sigma = \int_\gamma \nabla_\gamma v \mathbf{E}_\Gamma \nabla_\gamma w \, d\sigma \quad \forall v,w \in H^1(\gamma),$$

where $\mathbf{E}_\Gamma \in \mathbb{R}^{(n+1)\times(n+1)}$ stands for the error matrix

$$\mathbf{E}_\Gamma := \frac{1}{q}\mathbf{T}(q_\Gamma \mathbf{G}_\Gamma^{-1} - q\mathbf{G}^{-1})\mathbf{T}^T.$$

The following lemma shows that if the initial mesh $\mathcal{T}_0$ is sufficiently refined as measured by ζ, then the difference of the bilinear forms is bounded by the geometric estimator ζ.

Lemma 3.5 (Corollary 5.1 of [44]). *If $\zeta_{\mathcal{T}_0}(\gamma)$ satisfies*

$$\zeta_{\mathcal{T}_0}(\gamma) \lesssim \frac{1}{6B_0L^3}, \tag{3.13}$$

then we have for $\mathcal{T} \in \mathbb{T}$,

$$\|\mathbf{E}_\Gamma\|_{L_\infty(\widehat{T})} \lesssim \zeta_{\mathcal{T}}(\gamma,T) \quad \forall T \in \mathcal{T}. \tag{3.14}$$

The next lemma from [35] states that all other forms of geometric consistency errors we will encounter are bounded by the geometric estimator ζ.

Lemma 3.6 (Geometric Consistency Error Bounds (Lemma 4.1 of [35]))**.** *If $\zeta_{\mathcal{T}_0}(\gamma)$ satisfies* (3.13)*, then the matrices $\mathbf{G}$ and $\mathbf{G}_\Gamma$ have eigenvalues in the interval $[L^{-2}, L^2]$ and $[\frac{1}{2}L^{-2}, \frac{3}{2}L^2]$, respectively. Moreover, the forest $\mathbb{T}$ is shape regular, $L^{-n} \lesssim q, q_\Gamma \lesssim L^n$, and for $\mathcal{T} \in \mathbb{T}$*

$$\begin{aligned} \|q - q_\Gamma\|_{L_\infty(\widehat{T})} + \|q^s - q_\Gamma^s\|_{L_\infty(\partial\widehat{T})} + \|\nu - \nu_\Gamma\|_{L_\infty(\widehat{T})} \\ + \|\mathbf{G} - \mathbf{G}_\Gamma\|_{L_\infty(\widehat{T})} + \|\mathbf{D} - \mathbf{D}_\Gamma\|_{L_\infty(\widehat{T})} \lesssim \zeta_{\mathcal{T}}(\gamma, T) \quad \forall T \in \mathcal{T} \end{aligned} \tag{3.15}$$

where we recall that $\Gamma = \Gamma_{\mathcal{T}}$.

It follows from (3.15) that the difference of the L_2 inner products satisfies

$$\int_\Gamma vw \, d\Sigma - \int_\gamma vw \, d\sigma \lesssim \zeta_{\mathcal{T}}(\gamma)\|v\|_{L_2(\gamma)}\|w\|_{L_2(\gamma)}.$$

Lemma 3.7 (Equivalence of norms (Lemma 4.2 of [35]))**.** *If $\zeta_{\mathcal{T}_0}(\gamma)$ satisfies* (3.13)*, then the following equivalence of norms holds for all $\mathcal{T} \in \mathbb{T}$ with constants depending on $\mathcal{T}_0$ and L:*

$$\|v\|_{L_2(\widetilde{T})} \approx \|v\|_{L_2(T)} \approx \|v\|_{L_2(\widehat{T})}, \qquad |v|_{H^1(\widetilde{T})} \approx |v|_{H^1(T)} \approx |v|_{H^1(\widehat{T})} \qquad \forall T \in \mathcal{T}, \tag{3.16}$$

where $\widehat{T} = \chi_{\mathcal{T}}^{-1}(T)$ and $\widetilde{T} = \chi(\widehat{T})$. It follows that the equivalence of norms holds globally.

Lemma 3.8. *Let $\mathbf{Z}$ and $\mathbf{P}$ be the projection operators onto $\mathbb{W}(J)$ defined in* (3.9) *and* (3.10)*. Let $v \in H^1(\gamma)$ and $V \in \mathbb{V}(\Gamma)$, then*

$$\|\nabla_\gamma \mathbf{Z} v\|_{L_2(\gamma)} \lesssim \|\nabla_\gamma v\|_{L_2(\gamma)} \tag{3.17}$$

$$\|\nabla_\gamma \mathbf{P} V\|_{L_2(\gamma)} \lesssim \|\nabla_\gamma V\|_{L_2(\gamma)} \tag{3.18}$$

$$\|\mathbf{P} v\|_{L_2(\gamma)} \lesssim \|v\|_{L_2(\gamma)}. \tag{3.19}$$

Proof. Inequality (3.17) follows from (3.16) and (3.11):

$$\begin{aligned}\|\nabla_\gamma \mathbf{Z}v\|^2_{L_2(\gamma)} &\lesssim \|\nabla_\Gamma \mathbf{Z}v\|^2_{L_2(\Gamma)} = A(\mathbf{Z}v, \mathbf{Z}v) = A(v, \mathbf{Z}v) \\ &\leq \|\nabla_\Gamma v\|_{L_2(\Gamma)} \|\nabla_\Gamma \mathbf{Z}v\|_{L_2(\Gamma)} \lesssim \|\nabla_\gamma v\|_{L_2(\gamma)} \|\nabla_\gamma \mathbf{Z}v\|_{L_2(\gamma)}.\end{aligned}$$

Inequality (3.18) follows from (3.16), the definition of $\mathbf{P}$, and eigenfunction properties:

$$\begin{aligned}\|\nabla_\gamma \mathbf{P}V\|^2_{L_2(\gamma)} \lesssim \|\nabla_\Gamma \mathbf{P}V\|^2_{L_2(\Gamma)} &= A(\mathbf{P}V, \mathbf{P}V) = A(\sum_{j\in J} M(V, U_j)U_j, \mathbf{P}V) \\ &= \sum_{j\in J} M(V, U_j)A(U_j, \mathbf{P}V) = \sum_{j\in J} M(V, U_j)\Lambda_j M(U_j, \mathbf{P}V) \\ &= \sum_{j\in J} M(V, U_j)\Lambda_j M(U_j, V) = \sum_{j\in J} M(V, U_j)A(U_j, V) \\ &= A(\sum_{j\in J} M(V, U_j)U_j, V) = A(\mathbf{P}V, V) \lesssim \|\nabla_\Gamma \mathbf{P}V\|_{L_2(\Gamma)} \|\nabla_\Gamma V\|_{L_2(\Gamma)} \\ &\lesssim \|\nabla_\gamma \mathbf{P}V\|_{L_2(\gamma)} \|\nabla_\gamma V\|_{L_2(\gamma)}.\end{aligned}$$

Inequality (3.19) follows from the fact that $\mathbf{P}$ is an L_2-projection with respect to $L_2(\Gamma)$:

$$\|\mathbf{P}v\|_{L_2(\gamma)} \lesssim \|\mathbf{P}v\|_{L_2(\Gamma)} \leq \|v\|_{L_2(\Gamma)} \lesssim \|v\|_{L_2(\gamma)}.$$

□

3.1.8 Regularity of Solutions on $C^{1,\alpha}$ Surfaces

We now prove that weak solutions to $-\Delta_\gamma u = f$ on piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces have more than H^1 regularity. The main result of this subsection is the following theorem.

Theorem 3.9 (Regularity of u)**.** *Let γ be a piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ d-dimensional closed surface in $\mathbb{R}^{d+1}$. Let $f \in L_2(\gamma)$. Let $u \in H^1(\gamma)$ solve $-\Delta_\gamma u = f$ in a weak sense subject to*

$\int_\gamma u \, d\sigma = 0$. *Then there exists some* $\overline{s} \leq \alpha$ *such that for all* $s \in [0, \overline{s}]$ *we have*

$$\|\nabla_\gamma u\|_{H^s(\gamma)} \lesssim \|f\|_{L_2(\gamma)}.$$

Remark 3.10. *We note that in contrast to our regularity result in Theorem 3.9, it was recently shown in Lemma 4 of [49] that if* γ *satisfies the stronger assumption of being globally* W^2_p *(which embeds into* $C^{1,\alpha}$ *for* $0 < \alpha = 1 - \frac{d}{p}$*) then the solution to* $-\Delta_\gamma u = f$ *is* H^2 *for* $f \in L_2(\gamma)$*.*

In order to prove Theorem 3.9 we require some results from [50]. Let $\Omega_F \subset \mathbb{R}^d$ be a flat domain. Define $A : H^1_0(\Omega_F) \to H^{-1}(\Omega_F)$ by

$$\langle Au, v\rangle_{H^{-1}(\Omega_F), H^1_0(\Omega_F)} := \int_{\Omega_F} E \nabla u \nabla v \, d\Omega_F \quad \forall v \in H^1_0(\Omega_F)$$

where $E \in L_\infty(\Omega_F, \mathbb{C}^{d \times d})$ denotes a matrix-valued function which is assumed to be uniformly positive definite with multiplier property

$$Ef \in H^{s_0}(\Omega_F) \quad \text{for all } f \in H^{s_0}(\Omega_F) \text{ with some } s_0 \in (0, 1/2).$$

The results of [50] address the regularity of the weak solution to the problem with flat domain Ω_F and data $f \in H^{-1}(\Omega_F)$: Find $u \in H^1_0(\Omega_F)$ such that

$$\langle Au, v\rangle_{H^{-1}(\Omega_F), H^1_0(\Omega_F)} = \langle f, v\rangle_{H^{-1}(\Omega_F), H^1_0(\Omega_F)} \quad \forall v \in H^1_0(\Omega_F). \tag{3.20}$$

The following theorem is a reformulation of Theorem 3 of [50] for the weak problem in (3.20). It is also the flat domain version of Theorem 3.9.

Theorem 3.11 (Theorem 3 of [50]). *Let* Ω_F *be a flat domain of* $\mathbb{R}^d$*. There exists some* $\overline{s} \in (0, s_0)$*, depending on* $\Omega_F \subset \mathbb{R}^d$*, the boundary* $\partial\Omega_F$*, and* E*, such that for all* $s \in [0, \overline{s}]$ *and* $u \in H^1_0(\Omega_F)$

with $Au \in H^{s-1}(\Omega_F)$ one has $\nabla u \in H^s(\Omega_F)$ and

$$\|\nabla u\|_{H^s(\Omega_F)} \leq C_0\Big(\|\nabla u\|_{L_2(\Omega_F)} + \|Au\|_{H^{s-1}(\Omega_F)}\Big),$$

with some constant $C_0 \in (0, \infty)$ independent of s and u.

In addition to Theorem 3.11 we will need the following two lemmas to prove Theorem 3.9.

Lemma 3.12 (Lemma 1 of [50]). *Let $\Omega_F \subset \mathbb{R}^d$ be a Lipschitz domain and let $s \in (0, \frac{1}{2}]$. Then there exists a constant $C_s \in (0, \infty)$ such that*

$$\left|\int_{\Omega_F} w \nabla g \; d\Omega_F\right| \leq C_s \|w\|_{H^s(\Omega_F)} \|g\|_{H^{1-s}(\Omega_F)}$$

for all vector fields $w \in H^s(\Omega, \mathbb{C}^d)$ and functions $g \in H^1(\Omega)$.

Lemma 3.13 (Lemma 2 of [50]). *Let $\Omega_F \subset \mathbb{R}^d$ be a Lipschitz domain and let $s \in (0, \frac{1}{2}]$. Assume further that the function $h : \mathbb{R}^d \to \mathbb{C}$ has the form*

$$h = \sum_{k=1}^{n} \chi_k h_k,$$

where the bounded functions $h_k \in C^\alpha(\mathbb{R}^d)$ are Holder continuous for $\alpha > s$, and the χ_k are characteristic functions for Lipschitz domains in $\mathbb{R}^d$. Then

$$gh \in H^s(\Omega_F) \text{ for all } g \in H^s(\Omega_F)$$

Proof of Theorem 3.9. Let $\{U^n\}_{n=1}^N$ be a finite covering of γ with charts $\{\varphi_n\}_{n=1}^N$ and let $\{\psi_n\}_{n=1}^N$ form a partition of unity subordinate to $\{U^n\}_{n=1}^N$. For $u, v \in H_0^1(\gamma)$ we have

$$\int_\gamma \nabla_\gamma u \nabla_\gamma v \; d\sigma = \sum_{n=1}^N \int_\gamma \nabla_\gamma(\psi_n u) \nabla_\gamma v \; d\sigma = \sum_{n=1}^N \int_{U^n} \nabla_\gamma(\psi_n u) \nabla_\gamma v \; d\sigma.$$

Noting that the presence of ψ_n in the integrands makes integrals over a single patch equal to integrals over all of γ, we then have

$$\begin{aligned}\int_{U^n} \nabla_\gamma(\psi_n u)\nabla_\gamma v \, d\sigma &= \int_\gamma \nabla_\gamma u \nabla_\gamma(\psi_n v)\, d\sigma - \int_\gamma (\nabla_\gamma u \nabla_\gamma \psi_n) v \, d\sigma + \int_\gamma (\nabla_\gamma v \nabla_\gamma \psi_n) u \, d\sigma \\ &= \int_\gamma f(\psi_n v)\, d\sigma - \int_\gamma (\nabla_\gamma u \nabla_\gamma \psi_n) v \, d\sigma + \int_\gamma (\nabla_\gamma v \nabla_\gamma \psi_n) u \, d\sigma.\end{aligned} \tag{3.21}$$

We now map the integral (3.21) from U^n to a flat Lipschitz domain $\Omega_F^n := \varphi_n^{-1}(U^n) \subset \mathbb{R}^d$. Let $V \in H_0^1(\Omega_F^n)$ and $\widetilde{V} := V \circ \varphi_n^{-1} \in H_0^1(U^n)$. Then

$$\int_{U^n} \nabla_\gamma(\psi_n u)\nabla_\gamma \widetilde{V} d\sigma = \int_{\Omega_F^n} \nabla \Big[(\psi_n \circ \varphi_n)(u \circ \varphi_n)\Big] \mathbf{G}^{-1} \nabla V \sqrt{|\det(\mathbf{G})|}\, d\Omega_F^n.$$

We define $A^n : H_0^1(\Omega_F^n) \to H^{-1}(\Omega_F^n)$ to be the operator satisfying:

$$\langle A^n U, V \rangle_{H^{-1}(\Omega_F^n), H_0^1(\Omega_F^n)} = \int_{\Omega_F^n} \nabla U \mathbf{G}^{-1} \nabla V \sqrt{|\det(\mathbf{G})|}\, d\Omega_F^n \quad \forall V \in H_0^1(\Omega_F^n).$$

By the regularity of u, ψ_n, and φ_n we have $[(\psi_n \circ \varphi_n)(u \circ \varphi_n)] \in H_0^1(\Omega_F^n)$. We now prove

$A^n[(\psi_n\circ\varphi_n)(u\circ\varphi_n)] \in H^{s-1}(\Omega_F^n)$ in order to apply Theorem 3.11. From equation (3.21) we have

$$\begin{aligned}
\langle A^n[(\psi_n\circ\varphi_n)(u\circ\varphi_n)], V\rangle_{H^{-1}(\Omega_F^n),H_0^1(\Omega_F^n)} &= \langle f,\psi_n\widetilde{V}\rangle_{H^{-1}(\gamma),H_0^1(\gamma)} - \int_\gamma (\nabla_\gamma u\nabla_\gamma\psi_n)\widetilde{V}\,d\sigma \\
&+ \int_\gamma (\nabla_\gamma\widetilde{V}\nabla_\gamma\psi_n)u\,d\sigma \\
&= \langle f,\psi_n\widetilde{V}\rangle_{H^{-1}(\gamma),H_0^1(\gamma)} - \int_{U^n} (\nabla_\gamma u\nabla_\gamma\psi_n)\widetilde{V}\,d\sigma \\
&+ \int_{U^n} (\nabla_\gamma\widetilde{V}\nabla_\gamma\psi_n)u\,d\sigma \\
&\le |\langle f,\psi_n\widetilde{V}\rangle_{H^{-1}(U^n),H_0^1(U^n)}| + \left|\int_{U^n} (\nabla_\gamma u\nabla_\gamma\psi_n)\widetilde{V}\,d\sigma\right| \\
&+ \left|\int_{\Omega_F^n} \Big[\nabla V\mathbf{G}^{-1}\nabla(\psi_n\circ\varphi_n)\Big](u\circ\varphi_n)\sqrt{|\det(\mathbf{G})|}\,d\Omega_F^n\right| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Noting that $0\le\psi_n\le 1$, we have

$$\begin{aligned}
I_1 &:= |\langle f,\psi_n\widetilde{V}\rangle_{H^{-1}(U^n),H_0^1(U^n)}| = \left|\int_{U^n} f\psi_n\widetilde{V}\,d\sigma\right| \le \|\psi_n f\|_{L_2(\gamma)}\|\widetilde{V}\|_{L_2(\gamma)} \le \|f\|_{L_2(\gamma)}\|\widetilde{V}\|_{L_2(U^n)} \\
&= \|f\|_{L_2(\gamma)}\sqrt{\int_{\Omega_F^n} V^2\sqrt{|\det(\mathbf{G})|}\,dx} \le \|f\|_{L_2(\gamma)}\sqrt{\|\sqrt{|\det(\mathbf{G})|}\|_{L_\infty(\gamma)}}\|V\|_{L_2(\Omega_F^n)} \\
&\lesssim \|f\|_{L_2(\gamma)}\|V\|_{H^a(\Omega_F^n)}.
\end{aligned}$$

with $0\le a\le 1$ and the boundedness of $\sqrt{|\det(\mathbf{G})|}$ owing to $\varphi_n\in W^{1,\infty}(\Omega^n)$. Using similar arguments to those used to bound I_1, we get the following for I_2:

$$\begin{aligned}
I_2 &:= \left|\int_{U^n} (\nabla_\gamma u\nabla_\gamma\psi_n)\widetilde{V}\,d\sigma\right| \le \|\nabla_\gamma\psi_n\|_{L_\infty(\gamma)}\|\nabla_\gamma u\|_{L_2(\gamma)}\|\widetilde{V}\|_{L_2(U^n)} \\
&\le \sqrt{\left\|\sqrt{|\det(\mathbf{G})|}\right\|_{L_\infty(\gamma)}}\|\nabla_\gamma\psi_n\|_{L_\infty(\gamma)}\|\nabla_\gamma u\|_{L_2(\gamma)}\|V\|_{H^a(\Omega_F^n)} \\
&\lesssim \|f\|_{L_2(\gamma)}\|V\|_{H^a(\Omega_F^n)}.
\end{aligned}$$

We now bound I_3 by applying Lemma 3.12 with $g=V$ and

$w = \mathbf{G}^{-1}\nabla(\psi_n \circ \varphi_n)(u \circ \varphi_n)\sqrt{|\det(\mathbf{G})|}$ and noting that in w the C^α terms are Sobolev multipliers of $H^s(\Omega_F^n)$ to get

$$
\begin{aligned}
I_3 &\leq C_s \left\| \mathbf{G}^{-1}\nabla(\psi_n \circ \varphi_n)(u \circ \varphi_n)\sqrt{|\det(\mathbf{G})|} \right\|_{H^s(\Omega_F^n)} \|V\|_{H^{1-s}(\Omega_F^n)} \\
&\lesssim \|u\|_{H^1(\gamma)}\|V\|_{H^{1-s}(\Omega_F^n)} \\
&\lesssim \|f\|_{L_2(\gamma)}\|V\|_{H^{1-s}(\Omega_F^n)}.
\end{aligned}
$$

Here we have used that $w \in H^s(\Omega_F^n)$, which we justify as follows. Since $\mathbf{G}$ is constructed from the derivatives of piecewise $C^{1,\alpha}$ maps and $\mathbf{G}\mathbf{G}^{-1} = \mathbf{1}$, we have that both $\mathbf{G}$ and $\mathbf{G}^{-1}$ are piecewise C^α. Since $\mathbf{G}$ is a Riemannian metric, it is positive definite. Thus $\sqrt{|\det(\mathbf{G})|}$ is piecewise C^α as well. By the definition of ψ_n, we have that $\psi_n \circ \varphi_n \in C^\infty$. Finally, since $\varphi_n \in W^{1,\infty}(\Omega_F^n)$ and $u \in H_0^1(\gamma)$, we have that $u \circ \varphi_n \in H^1(\Omega_F^n)$. Noting that $\sqrt{|\det(\mathbf{G})|}\mathbf{G}^{-1}$ is of the same form as h in Lemma 3.13, we can apply Lemma 3.13 to w to get $w = \sqrt{|\det(\mathbf{G})|}\mathbf{G}^{-1}\nabla(\psi_n \circ \varphi_n)(u \circ \varphi_n) \in H^s(\Omega_F^n)$ for $s \in (0, \min(\frac{1}{2}, \alpha)]$.

Combining the bounds on I_1, I_2, and I_3 and noting that the regularity of V is limited by that of I_3 then gives

$$
\langle A^n[(\psi_n \circ \varphi_n)(u \circ \varphi_n)], V\rangle_{H^{-1}(\Omega_F^n), H_0^1(\Omega_F^n)} \lesssim \|f\|_{L_2(\gamma)}\|V\|_{H^{1-s}(\Omega_F^n)}, \quad s \in (0, \min(1/2, \alpha).
$$

By Theorem 3.11 we then have

$$
\nabla[(\psi_n \circ \varphi_n)(u \circ \varphi_n)] \in H^{s_n}(\Omega_F^n)
$$

for each Ω_F^n. We now map everything back to γ. It is well known from differential geometry that

$$
(\nabla_\gamma v)|_{U^n} = \nabla(v \circ \varphi_n)\mathbf{G}^{-1}(\nabla\varphi_n)^T,
$$

where on Ω_F^n we have $\mathbf{G} = (\nabla\varphi_n)^T(\nabla\varphi_n)$. Taking $v = \psi_n u$, we know $\nabla(v \circ \varphi_n) \in H^s(\Omega_F^n)$.

Noting that $\mathbf{G}$ and $\nabla\varphi_n$ are piecewise C^α and applying Lemma 3.13 then gives

$$\nabla_{U^n}(\psi_n u) \in H^{s_n}(U^n).$$

Since $(\psi_n u)|_{\partial U^n} = 0$, we can extend by 0 to all of γ to get $\nabla_{U^n}(\psi_n u) \in H^{s_n}(\gamma)$. Let $s_0 = \min_n\{s_n\}_{n=1}^N$. Exploiting the linearity of ∇_γ and stitching the patches together then gives

$$\sum_{n=1}^N \nabla_\gamma(\psi_n u) = \nabla_\gamma(\sum_{n=1}^N \psi_n u) = \nabla_\gamma u \in H^{s_0}(\gamma) \qquad \text{and} \qquad \|\nabla_\gamma u\|_{H^s(\gamma)} \lesssim \|f\|_{L_2(\gamma)}.$$

□

3.1.9 A Priori Estimates

We will need the following lemma which is a modification of Theorem 4.2 from [1]. We assume lower surface regularity than that of [1] and thus use a different geometric estimator. We bound the geometric consistency errors of Theorem 4.2 of [1] with the geometric estimator (3.3) rather than a power of h since geometric consistency errors in the AFEM algorithm are controlled through the geometric estimator. The steps of the proof are essentially the same and we do not repeat them here.

Lemma 3.14 (L_2 Eigenfunction Bound)**.** *Let γ be a piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surface. Given a mesh of maximum mesh size H_0 with H_0 sufficiently small so that*

$$D(J) := \max_{\ell\in J}\max_{j\notin J}\left|\frac{\lambda_\ell}{\Lambda_j - \lambda_\ell}\right| < \infty \tag{H1}$$

and an eigencluster J with eigenpairs $\{(u_i, \lambda_i)\}_{i\in J}$, fix $i \in J$. Then the following bound holds:

$$\begin{aligned}\|u_i - \mathbf{P}u_i - \overline{u_i - \mathbf{P}u_i}\|_{L_2(\gamma)} &\lesssim \|u_i - \mathbf{Z}u_i - \overline{u_i - \mathbf{Z}u_i}\|_{L_2(\gamma)} \\ &\lesssim (1 + D(J))\|u_i - \mathbf{R}u_i - \overline{u_i - \mathbf{R}u_i}\|_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma).\end{aligned}$$

We now give an a priori estimate for $\|u_i - \mathbf{R}u_i - \overline{u_i - \mathbf{R}u_i}\|_{L_2(\gamma)}$ in terms of the maximum mesh size H_0. The following lemma can be viewed as a generalization to $C^{1,\alpha}$ surfaces of the L_2 estimate in Theorem 3.1 of [32].

Lemma 3.15 (L_2 A Priori Estimate)**.** *Let γ be a piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surface. Given a mesh of maximum mesh size H_0 satisfying* (H1) *and an eigencluster J with eigenpairs $\{(u_i, \lambda_i)\}_{i\in J}$, fix $i \in J$. Then the following bound holds:*

$$\|u_i - \mathbf{R}u_i - \overline{u_i - \mathbf{R}u_i}\|_{L_2(\gamma)} \lesssim H_0^s \|\nabla_\gamma (u_i - \mathbf{R}u_i)\|_{L_2(\gamma)}$$

Proof. Let $e := u_i - \mathbf{R}u_i$. Let z be the solution to the dual problem: Find $z \in H^1_\#(\gamma)$ such that

$$a(v, z) = m(e - \overline{e}, v) \qquad \forall v \in H^1_\#(\gamma).$$

Since $\|\nabla_\gamma z\|^2_{L_2(\gamma)} = a(z, z) = m(e - \overline{e}, z) \le \|e - \overline{e}\|_{L_2(\gamma)} \|z\|_{L_2(\gamma)}$, we also have $\|\nabla_\gamma z\|_{L_2(\gamma)} \lesssim \|e - \overline{e}\|_{L_2(\gamma)}$. Using this bound on $\|\nabla_\gamma z\|_{L_2(\gamma)}$, Galerkin orthogonality, equivalence of norms, and the results of Theorem 3.9, we have

$$\begin{aligned}
\|e - \overline{e}\|^2_{L_2(\gamma)} &= a(e, z) = A(e, z) + (a - A)(e, z) = A(e, z - I_h z) + (a - A)(e, z) \\
&\lesssim \|\nabla_\Gamma e\|_{L_2(\Gamma)} \|z - I_h z\|_{L_2(\Gamma)} + \zeta_{\mathcal{T}}(\gamma) \|\nabla_\gamma e\|_{L_2(\gamma)} \|\nabla_\gamma z\|_{L_2(\gamma)} \\
&\lesssim \|\nabla_\gamma e\|_{L_2(\gamma)} \|z - I_h z\|_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma) \|\nabla_\gamma e\|_{L_2(\gamma)} \|\nabla_\gamma z\|_{L_2(\gamma)} \qquad (3.22) \\
&\lesssim \|\nabla_\gamma e\|_{L_2(\gamma)} h^s \|\nabla_\gamma z\|_{H^s(\gamma)} + \zeta_{\mathcal{T}}(\gamma) \|\nabla_\gamma e\|_{L_2(\gamma)} \|\nabla_\gamma z\|_{L_2(\gamma)} \\
&\lesssim \|\nabla_\gamma e\|_{L_2(\gamma)} h^s \|e - \overline{e}\|_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma) \|\nabla_\gamma e\|_{L_2(\gamma)} \|e - \overline{e}\|_{L_2(\gamma)}
\end{aligned}$$

which gives

$$\|u_i - \mathbf{R}u_i - \overline{u_i - \mathbf{R}u_i}\|_{L_2(\gamma)} \lesssim (H_0^s + \zeta_{\mathcal{T}}(\gamma)) \|\nabla_\gamma (u_i - \mathbf{R}u_i)\|_{L_2(\gamma)}.$$

Recalling that $\zeta_{\mathcal{T}}(\gamma) = \max_{i=1,\dots,F} \max_{T \in \mathcal{T}^i} \|\widehat{\nabla}(\chi^i - I_{\mathcal{T}^i}\chi^i)\|_{L_\infty(\widehat{T})}$, and applying standard arguments gives $\zeta_{\mathcal{T}}(\gamma) \lesssim H_0^\alpha |\chi|_{W^{1+\alpha,\infty}(\Omega)}$. By Theorem 3.9 we have that $\alpha \leq s$, so

$$\|u_i - \mathbf{R}u_i - \overline{u_i - \mathbf{R}u_i}\|_{L_2(\gamma)} \lesssim H_0^s \|\nabla_\gamma(u_i - \mathbf{R}u_i)\|_{L_2(\gamma)}.$$

□

Combining Lemma 3.14 with 3.15 yields the following corollary.

Corollary 3.16 (A Priori Estimates)**.** *Given a mesh of maximum mesh size H_0 satisfying* (H1) *and an eigencluster J with eigenpairs $\{(u_i, \lambda_i)\}_{i\in J}$, fix $i \in J$. Then for any $\alpha \in \mathbb{R}$, the following bound holds:*

$$\begin{aligned} \|u_i - \mathbf{P}u_i - \overline{u_i - \mathbf{P}u_i}\|_{L_2(\gamma)} &\lesssim \|u_i - \mathbf{Z}u_i - \overline{u_i - \mathbf{Z}u_i}\|_{L_2(\gamma)} \\ &\lesssim (1 + D(J))H_0^s \|\nabla_\gamma(u_i - \mathbf{Z}u_i)\|_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma) \end{aligned} \tag{3.23}$$

Lemma 3.17 (H^1 Eigenfunction Bound)**.** *Let γ be a piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surface. Given a mesh of maximum mesh size H_0 satisfying* (H1) *and an eigencluster J with eigenpairs $\{(u_i, \lambda_i)\}_{i\in J}$, fix $i \in J$. Then the following bound holds:*

$$\|\nabla_\gamma(u_i - \mathbf{Z}u_i)\|_{L_2(\gamma)} \lesssim (1 + D(J))\|\nabla_\gamma(u_i - \mathbf{R}u_i)\|_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma).$$

3.2 A Posteriori Error Analysis

3.2.1 Upper and Lower Bounds for Energy Error

In this section we introduce the computable and theoretical estimator for the eigenfunctions. The FEM eigenpair $(U_j, \Lambda_j)_{\mathcal{T}}$, generated on the mesh $\mathcal{T}$ may not approximate the same eigenpair as $(U_j, \Lambda_j)_{\mathcal{T}_*}$ for $\mathcal{T}_* > \mathcal{T}$. The computable estimator's dependence on (U_j, Λ_j) causes difficulty in comparing estimators $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{T}_*}$. The theoretical estimator approach, first introduced in [22] for flat domains, overcomes this comparison difficulty through the use of projection operators onto $\mathbb{W}(J)$. Our analysis throughout the paper will be carried out using the theoretical estimator which allows for comparison between finite element solutions on different meshes. We show that the theoretical estimator is reliable and efficient. We also show that the two estimators are equivalent up to geometric consistency errors and our results for the theoretical eigenfunction estimator are indeed enough to prove optimality of the AFEM algorithm.

Given an FEM eigenpair (U, Λ) and a triangle $T \in \mathcal{T}$, we define the computable local error indicator $\eta_{\mathcal{T}}(\Lambda, U, T)$ for the eigenpair as

$$\eta_{\mathcal{T}}(\Lambda, U, T)^2 := h_T^2 \|\Lambda U + \Delta_\Gamma U\|_T^2 + h_T \|[\![\nabla_\Gamma U]\!]\|_{\partial T}^2 \qquad \forall T \in \mathcal{T}.$$

We define the computable estimator for the eigenpairs $\{(U_j, \Lambda_j)\}_{j \in J}$ of a cluster as

$$\eta_{\mathcal{T}}(J, T)^2 := \sum_{j \in J} \eta_{\mathcal{T}}(\Lambda_j, U_j, T)^2 \qquad \forall T \in \mathcal{T}.$$

Let (u, λ) be an eigenpair of (3.5) associated with the cluster of eigenpairs $\{(u_j, \lambda_j)\}_{j \in J}$. Let $\mathbf{Z}$ and $\mathbf{P}$ be the projection operators associated with the cluster. We now derive a theoretical residual estimator for $\|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\gamma)}$ which will depend entirely on projections $\mathbf{P}$ and $\mathbf{Z}$ onto $\mathbb{W}(J)$ rather than explicitly on $\{(U_j, \Lambda_j)\}_{j \in J}$. We introduce the theoretical interior and jump residual

$$\mathcal{R}_T := \lambda \mathbf{P}u|_T + \Delta_\Gamma \mathbf{Z}u|_T, \qquad \mathcal{J}_{\partial T} := \{\mathcal{J}_S\}_{S \subset \partial T} \qquad \forall T \in \mathcal{T},$$

$$\mathcal{J}_S := \nabla_\Gamma \mathbf{Z}u^+|_S \mathbf{n}_S^+ + \nabla_\Gamma \mathbf{Z}u^-|_S \mathbf{n}_S^- \qquad \forall S \in \mathcal{S}_{\mathcal{T}}.$$

Let (u, λ) be an eigenpair associated with the cluster $\{(u_j, \lambda_j)\}_{j \in J}$ and let $v \in H^1(\gamma)$. Our strategy will be to express $a(u - \mathbf{Z}u, v)$ in terms of the computable bilinear form and L_2 inner product on Γ. This will result in the expected residual terms plus geometric consistency errors and a higher order L_2 projection error term which can be reabsorbed later on. Through standard manipulaitons we have

$$
\begin{aligned}
a(u-\mathbf{Z}u,v) &= \lambda m(u,v) - a(\mathbf{Z}u,v) = \lambda m(u,v) - A(\mathbf{Z}u,v) + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&= \lambda m(u-\mathbf{P}u,v) + \lambda m(\mathbf{P}u,v) - A(\mathbf{Z}u,v) + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&= \lambda m(u-\mathbf{P}u,v) + \lambda M(\mathbf{P}u,v) - A(\mathbf{Z}u,v) \\
&+ \lambda[m(\mathbf{P}u,v) - M(\mathbf{P}u,v)] + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)]\,.
\end{aligned}
$$

Adding and subtracting $V \in \mathbb{V}(\Gamma)$ and applying Lemma 3.4 to $A(\mathbf{Z}u,V)$ then gives

$$
\begin{aligned}
a(u-\mathbf{Z}u,v) &= \lambda m(u-\mathbf{P}u,v) + \lambda M(\mathbf{P}u,v) - A(\mathbf{Z}u,v-V) - A(\mathbf{Z}u,V) \\
&+ \lambda[m(\mathbf{P}u,v) - M(\mathbf{P}u,v)] + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&= \lambda m(u-\mathbf{P}u,v) + \lambda M(\mathbf{P}u,v) - A(\mathbf{Z}u,v-V) \\
&- \lambda M(\mathbf{P}u,V) - [A(u,\mathbf{P}V) - a(u,\mathbf{P}V)] - \lambda[m(u,\mathbf{P}V) - M(u,\mathbf{P}V)] \\
&+ \lambda[m(\mathbf{P}u,v) - M(\mathbf{P}u,v)] + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&= \lambda m(u-\mathbf{P}u,v) + [\lambda M(\mathbf{P}u,v-V) - A(\mathbf{Z}u,v-V)] \\
&+ [a(u,\mathbf{P}V) - A(u,\mathbf{P}V)] + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&+ \lambda[M(u,\mathbf{P}V) - m(u,\mathbf{P}V)] + \lambda[m(\mathbf{P}u,v) - M(\mathbf{P}u,v)].
\end{aligned}
$$

After integration by parts, we then have

$$
\begin{aligned}
a(u-\mathbf{Z}u,v) &= \lambda m(u-\mathbf{P}u,v) + \sum_{T\in\mathcal{T}} \int_T \mathcal{R}_T(v-V)\,d\Sigma - \sum_{S\in\mathcal{S}_\mathcal{T}} \int_S \mathcal{J}_S(v-V)\,d\Sigma \\
&+ [a(u,\mathbf{P}V) - A(u,\mathbf{P}V)] + [A(\mathbf{Z}u,v) - a(\mathbf{Z}u,v)] \\
&+ \lambda[M(u,\mathbf{P}V) - m(u,\mathbf{P}V)] + \lambda[m(\mathbf{P}u,v) - M(\mathbf{P}u,v)] \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}
\tag{3.24}
$$

Motivated by the above calculations, the higher order nature of I_1, and the fact that I_4 through I_6

are geometric consistency errors, we define the theoretical eigenfunction error indicator as

$$\mu_{\mathcal{T}}(\lambda \mathbf{P}u, \mathbf{Z}u, T)^2 := h_T^2 \|\mathcal{R}_T\|_{L_2(T)}^2 + h_T \|\mathcal{J}_{\partial T}\|_{\partial T}^2 \qquad \forall T \in \mathcal{T},$$

and the eigenfunction error indicator for the cluster $\{(u_j, \lambda_j)\}_{j\in J}$ as

$$\mu_{\mathcal{T}}(J, T)^2 := \sum_{j\in J} \mu_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, T)^2 \qquad \forall T \in \mathcal{T}.$$

We now show that the theoretical estimator is reliable. Similar to the reliability estimate in [23] for flat domains, a higher order term of the form $H_0^s \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\gamma)}$ is present. This term may be reabsorbed for H_0 sufficiently small.

Theorem 3.18 (Reliability of Theoretical Estimator)**.** *Let H_0 satisfy* (H1)*. Let $\{(u_j, \lambda_j)\}_{j\in J}$ be an exact eigenvalue cluster indexed by J and satisfying the separation assumption* (3.7)*. Let $(u, \lambda) \in \{(u_j, \lambda_j)\}_{j\in J}$ be any eigenpair associated with the cluster. Let $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy* (3.13)*and $(\mathcal{T}, \Gamma)$ be a pair of mesh-surface approximations. Then there exist constants C_1, B_1, and K_0 depending only on $\mathcal{T}_0$, the Lipschitz constant of γ, and λ, such that*

$$\|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\gamma)}^2 \le C_1 \mu_{\mathcal{T}}(\lambda, u)^2 + B_1 \zeta_{\mathcal{T}}(\gamma)^2 + K_0 H_0^{2s} \lambda^2 \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\gamma)}^2.$$

Proof. We begin by bounding the terms in (3.24) with $v \in H^1_\#(\gamma)$. Throughout the proof we take V to be the Scott-Zhang interpolant built on $\overline{\Gamma}$ and lifted to Γ. Noting that for $v \in H^1_\#(\gamma)$ we have that $m(\alpha, v) = 0$ for $\alpha \in \mathbb{R}$, we compute

$$\begin{aligned} I_1 &:= \int_\gamma \lambda(u - \mathbf{P}u - \alpha) v \, d\sigma \le \lambda \|u - \mathbf{P}u - \alpha\|_{L_2(\gamma)} \|v\|_{L_2(\gamma)} \\ &\le \lambda \|u - \mathbf{P}u - \alpha\|_{L_2(\gamma)} C_F \|\nabla_\gamma v\|_{L_2(\gamma)} \\ &\lesssim \lambda H_0^s \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\gamma)} C_F \|\nabla_\gamma v\|_{L_2(\gamma)} + \lambda \zeta_{\mathcal{T}}(\gamma) C_F \|\nabla_\gamma v\|_{L_2(\gamma)}. \end{aligned}$$

We now turn to bounding I_2 and I_3:

$$I_2 := \sum_{T\in\mathcal{T}}\int_T \mathcal{R}_T(v-V)\,d\Sigma = \sum_{T\in\mathcal{T}}\|\mathcal{R}_T\|_{L_2(T)}\|v-V\|_{L_2(T)} \lesssim \sqrt{\sum_{T\in\mathcal{T}} h_T^2\|\mathcal{R}_T\|_{L_2(T)}^2}\|\nabla_\gamma v\|_{L_2(\gamma)},$$

$$I_3 := -\sum_{S\in\mathcal{S}_\mathcal{T}}\int_S \mathcal{J}_S(v-V)\,d\Sigma \lesssim \sqrt{\sum_{S\in\mathcal{S}_\mathcal{T}} h_T\|\mathcal{J}_S\|_{L_2(T)}^2}\|\nabla_\gamma v\|_{L_2(\gamma)}$$

Combining I_2 and I_3 and enforcing shape regularity gives

$$I_2 + I_3 \lesssim \left(\sqrt{\sum_{T\in\mathcal{T}} h_T^2\|\mathcal{R}_T\|_{L_2(T)}^2} + \sqrt{\sum_{S\in\mathcal{S}_\mathcal{T}} h_T\|\mathcal{J}_S\|_{L_2(T)}^2}\right)\|\nabla_\gamma v\|_{L_2(\gamma)}$$

The bounds for I_4 through I_7 follow from (3.14), (3.15), and Lemma 3.8

$$\begin{aligned} I_4 &:= \left[\int_\Gamma \nabla_\Gamma \mathbf{Z}u\nabla_\Gamma^T v\,d\Sigma - \int_\gamma \nabla_\gamma \mathbf{Z}u\nabla_\gamma^T v\,d\sigma\right] = \int_\gamma \nabla_\gamma \mathbf{Z}u\mathbf{E}_\Gamma\nabla_\gamma^T v\,d\sigma \\ &\lesssim \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma \mathbf{Z}u\|_{L_2(\gamma)}\|\nabla_\gamma v\|_{L_2(\gamma)} \lesssim \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma u\|_{L_2(\gamma)}\|\nabla_\gamma v\|_{L_2(\gamma)} \\ &= \zeta_\mathcal{T}(\gamma)\sqrt{\lambda}\|\nabla_\gamma v\|_{L_2(\gamma)} \end{aligned}$$

$$\begin{aligned} I_5 &:= \left[\int_\gamma \nabla_\gamma u\nabla_\gamma \mathbf{P}V\,d\sigma - \int_\Gamma \nabla_\Gamma u\nabla_\Gamma \mathbf{P}V\,d\Sigma\right] = \int_\gamma \nabla_\gamma u\mathbf{E}_\Gamma\nabla_\gamma^T \mathbf{P}V\,d\sigma \\ &\lesssim \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma u\|_{L_2(\gamma)}\|\nabla_\gamma \mathbf{P}V\|_{L_2(\gamma)} \lesssim \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma u\|_{L_2(\gamma)}\|\nabla_\gamma V\|_{L_2(\gamma)} \\ &\lesssim \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma u\|_{L_2(\gamma)}\|\nabla_\gamma v\|_{L_2(\gamma)} = \zeta_\mathcal{T}(\gamma)\sqrt{\lambda}\|\nabla_\gamma v\|_{L_2(\gamma)} \end{aligned}$$

$$\begin{aligned} I_6 &:= \left[\int_\Gamma \lambda u\mathbf{P}V\,d\Sigma - \int_\gamma \lambda u\mathbf{P}V\,d\sigma\right] = \sum_{\widehat{T}\in\widehat{\mathcal{T}}}\int_{\widehat{T}} \lambda u\mathbf{P}V(q_\Gamma - q)\,d\widehat{\Sigma} \\ &\lesssim \lambda\zeta_\mathcal{T}(\gamma)\|u\|_{L_2(\gamma)}\|\mathbf{P}V\|_{L_2(\gamma)} \lesssim \lambda\zeta_\mathcal{T}(\gamma)\|u\|_{L_2(\gamma)}\|V\|_{L_2(\gamma)} \\ &\lesssim \lambda\zeta_\mathcal{T}(\gamma)\|u\|_{L_2(\gamma)}\|v\|_{L_2(\gamma)} \lesssim \lambda\zeta_\mathcal{T}(\gamma)C_F\|\nabla_\gamma v\|_{L_2(\gamma)} \end{aligned}$$

$$
\begin{aligned}
I_7 &:= \left[\int_\gamma \lambda \mathbf{P}uv\, d\sigma - \int_\Gamma \lambda \mathbf{P}uv\, d\Sigma\right] = \sum_{\widehat{T}\in\widehat{\mathcal{T}}} \int_{\widehat{T}} \lambda \mathbf{P}uv(q_\Gamma - q)\, d\widehat{\Sigma} \\
&\lesssim \lambda\zeta_{\mathcal{T}}(\gamma)\|\mathbf{P}u\|_{L_2(\gamma)}\|v\|_{L_2(\gamma)} \lesssim \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)}\|v\|_{L_2(\gamma)} \\
&\leq \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)} C_F \|\nabla_\gamma v\|_{L_2(\gamma)} = \lambda\zeta_{\mathcal{T}}(\gamma) C_F \|\nabla_\gamma v\|_{L_2(\gamma)}.
\end{aligned}
$$

The rest follows from taking $\alpha = \overline{u - \mathbf{P}u} := m(u - \mathbf{P}u, 1)$ and applying Young's inequality. □

Before proving the efficiency bound, we introduce the oscillation for a single eigenpair (u, λ) of the cluster $\{(u_j, \lambda_j)\}_{j\in J}$ on a triangle $T \in \mathcal{T}$:

$$
\begin{aligned}
Osc_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, T)^2 := h_T^2 &\left\|(\mathrm{id} - \Pi^2_{2n-2})\left(\lambda\mathbf{P}uq_\Gamma + \widehat{\mathrm{div}}(q_\Gamma \widehat{\nabla}\mathbf{Z}u\mathbf{G}_\Gamma^{-1})\right)\right\|^2_{L_2(\widehat{T})} \\
&+ h_T\|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\|^2_{L_2(\partial\widehat{T})},
\end{aligned}
\tag{3.25}
$$

where $\mathbf{P}$ and $\mathbf{Z}$ are the projection operators associated with the cluster.

Remark 3.19. *It is possible to remove the q_Γ term on $\lambda\mathbf{P}u$, but this comes at the cost of changing the other terms in the definition of the oscillation. We choose this form for ease of comparison with the oscillations involved in the definition of the approximation classes used in [35].*

We now show that the theoretical estimator is efficient. Once again, like the reliability estimate, a higher order term appears.

Theorem 3.20 (Efficiency)**.** *Let H_0 satisfy* (H1)*. Let $\{(u_j, \lambda_j)\}_{j\in J}$ be an exact eigenvalue cluster indexed by J and satisfying the separation assumption* (3.7)*. Let $(u, \lambda) \in \{(u_j, \lambda_j)\}_{j\in J}$ be any eigenpair associated with the cluster. Let $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy* (3.13)*and $(\mathcal{T}, \Gamma)$ be a pair of mesh-surface approximations. Then there exist constants C_2, B_1, and K_0 depending only on $\mathcal{T}_0$, the Lipschitz constant of γ, and λ, such that*

$$
\begin{aligned}
C_2\mu_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u)^2 \leq \|\nabla_\gamma(u - \mathbf{Z}u)\|^2_{L_2(\gamma)} &+ Osc_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, \gamma)^2 + B_1\zeta_{\mathcal{T}}(\gamma)^2 \\
&+ K_0 H_0^{2s+1}\lambda^2\|\nabla_\gamma(u - \mathbf{Z}u)\|^2_{L_2(\gamma)}
\end{aligned}
\tag{3.26}
$$

Proof. We begin by pulling the residual on a triangle $T \in \Gamma$ back to the reference element $\widehat{T}$:

$$
\begin{aligned}
\|\mathcal{R}_T\|_{L_2(T)} &= \left\| \left(\lambda \mathbf{P} u + q_\Gamma^{-1} \widehat{\operatorname{div}} (q_\Gamma (\widehat{\nabla} \mathbf{Z} u) \mathbf{G}_\Gamma^{-1}) \right) \sqrt{q_\Gamma} \right\|_{L_2(\widehat{T})} \\
&\leq \left\| \frac{1}{q_\Gamma} \right\|_{L_\infty(\widehat{T})}^{1/2} \left\| \left(\lambda \mathbf{P} u q_\Gamma + \widehat{\operatorname{div}} (q_\Gamma (\widehat{\nabla} \mathbf{Z} u) \mathbf{G}_\Gamma^{-1}) \right) \right\|_{L_2(\widehat{T})} \\
&\leq \left\| \frac{1}{q_\Gamma} \right\|_{L_\infty(\widehat{T})}^{1/2} \left\| (\mathrm{id} - \Pi_{2n-2}^2) \left(\lambda \mathbf{P} u q_\Gamma + \widehat{\operatorname{div}} (q_\Gamma (\widehat{\nabla} \mathbf{Z} u) \mathbf{G}_\Gamma^{-1}) \right) \right\|_{L_2(\widehat{T})} \\
&+ \left\| \frac{1}{q_\Gamma} \right\|_{L_\infty(\widehat{T})}^{1/2} \left\| \Pi_{2n-2}^2 \left(\lambda \mathbf{P} u q_\Gamma + \widehat{\operatorname{div}} (q_\Gamma (\widehat{\nabla} \mathbf{Z} u) \mathbf{G}_\Gamma^{-1}) \right) \right\|_{L_2(\widehat{T})} \\
&\lesssim \left\| \mathcal{R}_T - \overline{\mathcal{R}}_T \right\|_{L_2(\widehat{T})} + \left\| \overline{\mathcal{R}}_T \right\|_{L_2(\widehat{T})}
\end{aligned}
\tag{3.27}
$$

where Π_m^p denotes the best L_p approximation operator onto the space $\mathbb{P}_m$ of polynomials of degree $\leq m$ and

$$
\overline{\mathcal{R}}_T := \Pi_{2n-2}^2 \left(\lambda \mathbf{P} u q_\Gamma + q_\Gamma^{-1} \widehat{\operatorname{div}} (q_\Gamma (\widehat{\nabla} \mathbf{Z} u) \mathbf{G}_\Gamma^{-1}) \right)
$$

on $\widehat{T}$ and zero elsewhere. We now set out to bound $\|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})}$. Let ϕ_T be the standard bubble function on $\widehat{T}$ given by the product of the barycentric coordinates on $\widehat{T}$. Using the equivalence of norms on different triangles (3.16) gives

$$
\begin{aligned}
\|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})}^2 \lesssim \|\sqrt{\phi_T} \overline{\mathcal{R}}_T\|_{L_2(\widehat{T})}^2 \lesssim \|\sqrt{\phi_T} \overline{\mathcal{R}}_T\|_{L_2(T)}^2 &= \int_T \phi_T (\overline{\mathcal{R}}_T - \mathcal{R}_T + \mathcal{R}_T) \overline{\mathcal{R}}_T \, d\Sigma \\
&\lesssim \|\sqrt{\phi_T} (\mathcal{R}_T - \overline{\mathcal{R}}_T)\|_{L_2(T)} \|\overline{\mathcal{R}}_T\|_{L_2(T)} + \int_T \mathcal{R}_T (\overline{\mathcal{R}}_T \phi_T) \, d\Sigma \\
&\lesssim \|\sqrt{\phi_T} (\mathcal{R}_T - \overline{\mathcal{R}}_T)\|_{L_2(\widehat{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + \int_T \mathcal{R}_T (\overline{\mathcal{R}}_T \phi_T) \, d\Sigma.
\end{aligned}
\tag{3.28}
$$

Using (3.24) with $V = 0$ and $v = \overline{\mathcal{R}}_T \phi_T \in H^1(\gamma)$, and noting that $\phi_T = 0$ on ∂T we have

$$
\begin{aligned}
\int_\gamma \nabla_\gamma (u - \mathbf{Z} u) \nabla_\gamma (\overline{\mathcal{R}}_T \phi_T) \, d\sigma = \lambda \int_\gamma (u - \mathbf{P} u) (\overline{\mathcal{R}}_T \phi_T) \, d\sigma &+ \int_\Gamma \mathcal{R}_T (\overline{\mathcal{R}}_T \phi_T) \, d\Sigma \\
&+ \left[\int_\Gamma \nabla_\Gamma \mathbf{Z} u \nabla_\Gamma (\overline{\mathcal{R}}_T \phi_T) \, d\Sigma - \int_\gamma \nabla_\gamma \mathbf{Z} u \nabla_\gamma (\overline{\mathcal{R}}_T \phi_T) \, d\sigma \right] \\
&+ \lambda \left[\int_\gamma \mathbf{P} u (\overline{\mathcal{R}}_T \phi_T) \, d\sigma - \int_\Gamma \mathbf{P} u (\overline{\mathcal{R}}_T \phi_T) \, d\Sigma \right],
\end{aligned}
$$

which reduces to

$$\int_{\widetilde{T}} \nabla_\gamma (u - \mathbf{Z}u) \nabla_\gamma (\overline{\mathcal{R}}_T \phi_T)\, d\sigma = \lambda \int_{\widetilde{T}} (u - \mathbf{P}u)(\overline{\mathcal{R}}_T \phi_T)\, d\sigma + \int_T \mathcal{R}_T (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma$$
$$+ \left[\int_T \nabla_\Gamma \mathbf{Z}u \nabla_\Gamma (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma - \int_{\widetilde{T}} \nabla_\gamma \mathbf{Z}u \nabla_\gamma (\overline{\mathcal{R}}_T \phi_T)\, d\sigma \right]$$
$$+ \lambda \left[\int_{\widetilde{T}} \mathbf{P}u (\overline{\mathcal{R}}_T \phi_T)\, d\sigma - \int_T \mathbf{P}u (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma \right].$$

This then implies

$$\begin{aligned}\int_T \mathcal{R}_T (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma &\le \lambda \|u - \mathbf{P}u\|_{L_2(\widetilde{T})} \|(\overline{\mathcal{R}}_T \phi_T)\|_{L_2(\widetilde{T})} \\ &\quad + \|\nabla_\gamma (u - \mathbf{Z}u)\|_{L_2(\widetilde{T})} \|\nabla_\gamma (\overline{\mathcal{R}}_T \phi_T)\|_{L_2(\widetilde{T})} \\ &\quad + \left| \int_T \nabla_\Gamma \mathbf{Z}u \nabla_\Gamma (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma - \int_{\widetilde{T}} \nabla_\gamma \mathbf{Z}u \nabla_\gamma (\overline{\mathcal{R}}_T \phi_T)\, d\sigma \right| \\ &\quad + \lambda \left| \int_{\widetilde{T}} \mathbf{P}u (\overline{\mathcal{R}}_T \phi_T)\, d\sigma - \int_T \mathbf{P}u (\overline{\mathcal{R}}_T \phi_T)\, d\Sigma \right|. \end{aligned} \tag{3.29}$$

Noting that

$$\begin{aligned}\|(u - \mathbf{P}u)\|_{L_2(\widetilde{T})} &\le \|u - \mathbf{P}u - \overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} + \|\overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} \\ &= \|u - \mathbf{P}u - \overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} + |\overline{u - \mathbf{P}u}||\widetilde{T}|^{1/2} \\ &\lesssim \|u - \mathbf{P}u - \overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} + |\overline{u - \mathbf{P}u}| h_T^{d/2}, \end{aligned}$$

we now bound $|\overline{u - \mathbf{P}u}|$. Since $u \in H^1_\#(\gamma)$, $m(u,1) = 0$. We also know that since $\mathbf{P}u \in \mathbb{V}_\#(\Gamma)$, $M(\mathbf{P}u, 1) = 0$. These mean 0 conditions then yield

$$\begin{aligned}\overline{u - \mathbf{P}u} &= \frac{1}{|\gamma|} m(u - \mathbf{P}u, 1) = \frac{1}{|\gamma|} m(u,1) - \frac{1}{|\gamma|} m(\mathbf{P}u, 1) = -\frac{1}{|\gamma|} m(\mathbf{P}u, 1) \\ &= \frac{1}{|\gamma|} (M - m)(\mathbf{P}u, 1) - \frac{1}{|\gamma|} M(\mathbf{P}u, 1) = \frac{1}{|\gamma|} (M - m)(\mathbf{P}u, 1) \lesssim \zeta_{\mathcal{T}}(\gamma) \|u\|_{L_2(\gamma)}. \end{aligned}$$

Combining these results yields

$$\|(u - \mathbf{P}u)\|_{L_2(\widetilde{T})} \lesssim \|u - \mathbf{P}u - \overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} + \zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)} h_T^{d/2}. \tag{3.30}$$

Combining the inequalities (3.30), (3.29), and (3.28) then yields

$$\begin{aligned}
\|\overline{\mathcal{R}}_T\|^2_{L_2(\widehat{T})} &\lesssim \|\sqrt{\phi_T}(\mathcal{R}_T - \overline{\mathcal{R}}_T)\|_{L_2(\widehat{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} \\
&+ \lambda\|u - \mathbf{P}u - \overline{u - \mathbf{P}u}\|_{L_2(\widetilde{T})} \|(\overline{\mathcal{R}}_T \phi_T)\|_{L_2(\widetilde{T})} \\
&+ \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)} h_T^{d/2} \|(\overline{\mathcal{R}}_T \phi_T)\|_{L_2(\widetilde{T})} + \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\widetilde{T})} \|\nabla_\gamma(\overline{\mathcal{R}}_T \phi_T)\|_{L_2(\widetilde{T})} \\
&+ \left| \int_T \nabla_\Gamma \mathbf{Z}u \nabla_\Gamma(\overline{\mathcal{R}}_T \phi_T)\, d\Sigma - \int_{\widetilde{T}} \nabla_\gamma \mathbf{Z}u \nabla_\gamma(\overline{\mathcal{R}}_T \phi_T)\, d\sigma \right| \\
&+ \lambda \left| \int_{\widetilde{T}} \mathbf{P}u(\overline{\mathcal{R}}_T \phi_T)\, d\sigma - \int_T \mathbf{P}u(\overline{\mathcal{R}}_T \phi_T)\, d\Sigma \right|.
\end{aligned}$$

Applying an inverse inequality yields

$$\|\nabla_\Gamma(\overline{\mathcal{R}}_T \phi_T)\|_{L_2(T)} \lesssim h_T^{-1}\|\overline{\mathcal{R}}_T \phi_T\|_{L_2(T)} \lesssim h_T^{-1}\|\overline{\mathcal{R}}_T\|_{L_2(T)}. \tag{3.31}$$

Applying (3.31), equivalence of norms, and $\|\sqrt{\phi_T}(\mathcal{R}_T - \overline{\mathcal{R}}_T)\|_{L_2(\widehat{T})} \lesssim \|(\mathcal{R}_T - \overline{\mathcal{R}}_T)\|_{L_2(\widehat{T})}$ we get

$$\begin{aligned}
\|\overline{\mathcal{R}}_T\|^2_{L_2(\widehat{T})} &\lesssim \|\mathcal{R}_T - \overline{\mathcal{R}}_T\|_{L_2(T)} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + \lambda\|(u - \mathbf{P}u - \overline{u - \mathbf{P}u})\|_{L_2(\widetilde{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} \\
&+ \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)} h_T^{d/2} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + h_T^{-1}\|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\widetilde{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} \\
&+ \left| \int_T \nabla_\Gamma \mathbf{Z}u \nabla_\Gamma(\overline{\mathcal{R}}_T \phi_T)\, d\Sigma - \int_{\widetilde{T}} \nabla_\gamma \mathbf{Z}u \nabla_\gamma(\overline{\mathcal{R}}_T \phi_T)\, d\sigma \right| \\
&+ \lambda \left| \int_{\widetilde{T}} \mathbf{P}u(\overline{\mathcal{R}}_T \phi_T)\, d\sigma - \int_T \mathbf{P}u(\overline{\mathcal{R}}_T \phi_T)\, d\Sigma \right| \\
&\lesssim \|\mathcal{R}_T - \overline{\mathcal{R}}_T\|_{L_2(T)} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + \lambda\|(u - \mathbf{P}u - \overline{u - \mathbf{P}u})\|_{L_2(\widetilde{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} \\
&+ \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)} h_T^{d/2} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + h_T^{-1}\|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\widetilde{T})} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} \\
&+ \zeta_{\mathcal{T}}(\gamma)\|\nabla_\gamma \mathbf{Z}u\|_{L_2(T)} h_T^{-1}\|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})} + \lambda\zeta_{\mathcal{T}}(\gamma)\|\mathbf{P}u\|_{L_2(T)} \|\overline{\mathcal{R}}_T\|_{L_2(\widehat{T})}.
\end{aligned} \tag{3.32}$$

Combining (3.27) with (3.32) and equivalence of norms on triangles then gives

$$
\begin{aligned}
h_T\|\mathcal{R}_T\|_{L_2(\widehat{T})} \lesssim{}& h_T\|\mathcal{R}_T - \overline{\mathcal{R}}_T\|_{L_2(T)} + h_T\lambda\|(u - \mathbf{P}u - \overline{u - \mathbf{P}u})\|_{L_2(\widetilde{T})} \\
&+ \lambda\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)}h_T^{d/2+1} + \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\widetilde{T})} \\
&+ \zeta_{\mathcal{T}}(\gamma)\|\nabla_\gamma\mathbf{Z}u\|_{L_2(\widetilde{T})} + h_T\lambda\zeta_{\mathcal{T}}(\gamma)\|\mathbf{P}u\|_{L_2(\widetilde{T})}.
\end{aligned}
\tag{3.33}
$$

Squaring (3.33), summing over the triangulation, and using the bound (3.23) gives

$$
\begin{aligned}
\sum_{T\in\mathcal{T}} h_T^2\|\mathcal{R}_T\|^2_{L_2(\widehat{T})} \lesssim{}& \sum_{T\in\mathcal{T}} h_T^2\|\mathcal{R}_T - \overline{\mathcal{R}}_T\|^2_{L_2(T)} + \left(1 + H_0^{2+2s}\lambda^2\right)\|\nabla_\gamma(u - \mathbf{Z}u)\|^2_{L_2(\gamma)} \\
&+ \left(h_T^2\lambda^2|\gamma|\|u\|^2_{L_2(\gamma)} + \|\nabla_\gamma\mathbf{Z}u\|^2_{L_2(\gamma)} + h_T^2\lambda^2\|\mathbf{P}u\|^2_{L_2(\gamma)}\right)\zeta_{\mathcal{T}}(\gamma)^2.
\end{aligned}
\tag{3.34}
$$

The jump terms follow in a similar manner

$$
\begin{aligned}
\|\mathcal{J}_S\|_{L_2(S)} &= \left\|\nabla_\Gamma\mathbf{Z}u^+|_S\mathbf{n}_S^+ + \nabla_\Gamma\mathbf{Z}u^-|_S\mathbf{n}_S^-\right\|_{L_2(S)} \\
&= \left\|\left(\frac{q_\Gamma^+}{r_\Gamma}\widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\widehat{\mathbf{n}}^+ + \frac{q_\Gamma^-}{r_\Gamma}\widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1}\widehat{\mathbf{n}}^-\right)\sqrt{r_\Gamma}\right\|_{L_2(\widehat{S})} \\
&= \left\|\frac{1}{\sqrt{r_\Gamma}}\right\|_{L_\infty(\widehat{S})}\left\|q_\Gamma^+\widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\widehat{\mathbf{n}}^+ + q_\Gamma^-\widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1}\widehat{\mathbf{n}}^-\right\|_{L_2(\widehat{S})} \\
&\lesssim \left\|(\mathrm{id} - \Pi^2_{2n-1})\left(q_\Gamma^+\widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\widehat{\mathbf{n}}^+ + q_\Gamma^-\widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1}\widehat{\mathbf{n}}^-\right)\right\|_{L_2(\widehat{S})} \\
&+ \left\|\Pi^2_{2n-1}\left(q_\Gamma^+\widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\widehat{\mathbf{n}}^+ + q_\Gamma^-\widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1}\widehat{\mathbf{n}}^-\right)\right\|_{L_2(\widehat{S})} \\
&\lesssim \|\mathcal{J}_S - \overline{\mathcal{J}_S}\|_{L_2(\widehat{S})} + \|\overline{\mathcal{J}_S}\|_{L_2(\widehat{S})},
\end{aligned}
$$

where we define

$$
\overline{\mathcal{J}}_S := \Pi^2_{2n-1}\left(q_\Gamma^+\widehat{\nabla}\mathbf{Z}u^+(\mathbf{G}_\Gamma^+)^{-1}\widehat{\mathbf{n}}^+ + q_\Gamma^-\widehat{\nabla}\mathbf{Z}u^-(\mathbf{G}_\Gamma^-)^{-1}\widehat{\mathbf{n}}^-\right)
$$

to be the L_2 projection onto $\mathbb{P}_{2n-1}(\widehat{S})$ and extending it to the edge patch ω_S by extending constantly along the normals to $\widehat{S}$. Let ϕ_S be the standard bubble function on the edge patch ω_S of S. We then have

$$\|\overline{\mathcal{J}}_S\|^2_{L_2(\widehat{S})} \leq \|\sqrt{\phi_S}\overline{\mathcal{J}}_S\|^2_{L_2(\widehat{S})} \lesssim \|\sqrt{\phi_S}\overline{\mathcal{J}}_S\|^2_{L_2(S)} = \int_S \phi_S(\overline{\mathcal{J}}_S - \mathcal{J}_S + \mathcal{J}_S)\overline{\mathcal{J}}_S \, dS$$
$$\lesssim \|\sqrt{\phi_S}(\overline{\mathcal{J}}_S - \mathcal{J}_S)\|_{L_2(\widehat{S})}\|\overline{\mathcal{J}}_S\|_{L_2(\widehat{S})} + \int_S \mathcal{J}_S\overline{\mathcal{J}}_S\phi_S \, dS \tag{3.35}$$

Using (3.24) with $V = 0$ and $v = \overline{\mathcal{J}}_S\phi_S$ gives

$$\begin{aligned}\int_\gamma \nabla_\gamma(u - \mathbf{Z}u)\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S) \, d\sigma = \lambda \int_\gamma (u - \mathbf{P}u)(\overline{\mathcal{J}}_S\phi_S) \, d\sigma + \int_\Gamma \mathcal{R}_T(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma \\ - \int_S \mathcal{J}_S(\overline{\mathcal{J}}_S\phi_S) \, dS \\ + \left[\int_\Gamma \nabla_\Gamma \mathbf{Z}u\nabla_\Gamma(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma - \int_\gamma \nabla_\gamma \mathbf{Z}u\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S) \, d\sigma\right] \\ + \lambda\left[\int_\gamma \mathbf{P}u(\overline{\mathcal{J}}_S\phi_S) \, d\sigma - \int_\Gamma \mathbf{P}u(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma\right].\end{aligned}$$

Rearranging terms and restricting to supports then gives

$$\begin{aligned}\int_S \mathcal{J}_S(\overline{\mathcal{J}}_S\phi_S) \, dS = -\int_{\widetilde{\omega}_S} \nabla_\gamma(u - \mathbf{Z}u)\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S) \, d\sigma + \lambda \int_{\widetilde{\omega}_S} (u - \mathbf{P}u)(\overline{\mathcal{J}}_S\phi_S) \, d\sigma \\ + \int_{\omega_S} \mathcal{R}_T(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma + \left[\int_{\omega_S} \nabla_\Gamma \mathbf{Z}u\nabla_\Gamma(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma - \int_{\widetilde{\omega}_S} \nabla_\gamma \mathbf{Z}u\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S) \, d\sigma\right] \\ + \lambda\left[\int_{\widetilde{\omega}_S} \mathbf{P}u(\overline{\mathcal{J}}_S\phi_S) \, d\sigma - \int_{\omega_S} \mathbf{P}u(\overline{\mathcal{J}}_S\phi_S) \, d\Sigma\right].\end{aligned}$$

Applying Cauchy-Schwarz then gives:

$$\begin{aligned}\int_S \mathcal{J}_S(\overline{\mathcal{J}}_S\phi_S) \, dS \lesssim \|\nabla_\gamma(u - \mathbf{Z}u)\|_{L_2(\widetilde{\omega}_S)}\|\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S)\|_{L_2(\widetilde{\omega}_S)} + \lambda\|u - \mathbf{P}u\|_{L_2(\widetilde{\omega}_S)}\|\overline{\mathcal{J}}_S\phi_S\|_{L_2(\widetilde{\omega}_S)} \\ + \|\mathcal{R}_T\|_{L_2(\omega_S)}\|\overline{\mathcal{J}}_S\phi_S\|_{L_2(\omega_S)} + \zeta_\mathcal{T}(\gamma)\|\nabla_\gamma \mathbf{Z}u\|_{L_2(\widetilde{\omega}_S)}\|\nabla_\gamma(\overline{\mathcal{J}}_S\phi_S)\|_{L_2(\widetilde{\omega}_S)} \\ + \lambda\zeta_\mathcal{T}(\gamma)\|\mathbf{P}u\|_{L_2(\widetilde{\omega}_S)}\|\overline{\mathcal{J}}_S\phi_S\|_{L_2(\widetilde{\omega}_S)}.\end{aligned}$$

Applying (3.30) on the patch ω_S and standard arguments then give

$$\begin{aligned}\int_S \mathcal{J}_S(\overline{\mathcal{J}}_S\phi_S)\,dS &\lesssim \|\nabla_\gamma(u-\mathbf{Z}u)\|_{L_2(\widetilde{\omega}_S)}h_T^{-1/2}\|(\overline{\mathcal{J}}_S)\|_{L_2(S)}\\ &+\lambda\|u-\mathbf{P}u-\overline{u-\mathbf{P}u}\|_{L_2(\widetilde{\omega}_S)}\|\overline{\mathcal{J}}_S\|_{L_2(S)}\\ &+\|\mathcal{R}_T\|_{L_2(\omega_S)}\|\overline{\mathcal{J}}_S\|_{L_2(S)}+\zeta_{\mathcal{T}}(\gamma)\|\nabla_\gamma\mathbf{Z}u\|_{L_2(\widetilde{\omega}_S)}h_T^{-1/2}\|\overline{\mathcal{J}}_S\|_{L_2(S)}\\ &+\lambda\zeta_{\mathcal{T}}(\gamma)\|\mathbf{P}u\|_{L_2(\widetilde{\omega}_S)}\|\overline{\mathcal{J}}_S\|_{L_2(S)}+\zeta_{\mathcal{T}}(\gamma)\|u\|_{L_2(\gamma)}h_T^{d/2}\|\overline{\mathcal{J}}_S\|_{L_2(S)}.\end{aligned} \tag{3.36}$$

Using (3.36) in (3.35) just as we did when using (3.29) to handle (3.28) yields the result final result. □

We now show that the theoretical estimator satisfies a discrete reliability condition.

Theorem 3.21 (Discrete Reliability)**.** *Let* H_0 *satisfy* (H1). *Let* $\{(u_j,\lambda_j)\}_{j\in J}$ *be an exact eigenvalue cluster indexed by* J *and satisfying the separation assumption* (3.7). *Let* $(u,\lambda)\in\{(u_j,\lambda_j)\}_{j\in J}$ *be any eigenpair associated with the cluster. Let* $\zeta_{\mathcal{T}_0}(\gamma)$ *satisfy* (3.13). *Given any pair* $(\mathcal{T}_*,\Gamma_*)$, $(\mathcal{T},\Gamma)$ *of mesh-surface approximations with* $\mathcal{T}\le\mathcal{T}_*$*, let* $\mathcal{R}:=\mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}\subset\mathcal{T}$ *be the set of elements refined in* $\mathcal{T}$ *to create* $\mathcal{T}_*$*. Let* $\mathbf{Z}_*$ *and* $\mathbf{Z}$ *be the cluster projections associated with* Γ_* *and* Γ*, respectively. Then the following bound holds*

$$\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|^2_{L_2(\gamma)}\le C_1\mu_{\mathcal{T}}(\lambda\mathbf{P}u,\mathbf{Z}u,\mathcal{R})^2+B_1\zeta_{\mathcal{T}}(\gamma)^2+\lambda^2K_0\|\mathbf{P}_*u-\mathbf{P}u\|^2_{L_2(\gamma)}, \tag{3.37}$$

with constants C_1*,* B_1 *and* K_0 *defined as in Theorem 3.18.*

Proof. We use the shorthand $E_*=\mathbf{Z}_*u-\mathbf{Z}u$. We also use the $*$ subscript throughout to denote quantities defined on Γ_*. We follow similar arguments to those used in the derivation of the reliability estimate in Theorem 3.18, but with Γ_* used in place of γ. For $V\in\mathbb{V}(\mathcal{T})$ we have

$$A_*(\mathbf{Z}_*u-\mathbf{Z}u,E_*)=-A(\mathbf{Z}u,V)-A(\mathbf{Z}u,E_*-V)+A_*(\mathbf{Z}_*u,E_*)+[A(\mathbf{Z}u,E_*)-A_*(\mathbf{Z}u,E_*)] \tag{3.38}$$

We now apply (3.12) to $A(\mathbf{Z}u, V)$ to get

$$A(\mathbf{Z}u, V) = \lambda M(\mathbf{P}u, V) + [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] + \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)]. \tag{3.39}$$

Combining (3.38) and (3.39) then yields:

$$\begin{aligned} A_*(\mathbf{Z}_*u - \mathbf{Z}u, E_*) &= -\lambda M(\mathbf{P}u, V)) - A(\mathbf{Z}u, E_* - V) \\ &+ A_*(\mathbf{Z}_*u, E_*) - [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] - \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)] \\ &+ [A(\mathbf{Z}u, E_*) - A_*(\mathbf{Z}u, E_*)] \\ &= \lambda M(\mathbf{P}u, E_* - V)) - A(\mathbf{Z}u, E_* - V) \\ &- [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] - \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)] \\ &+ [A(\mathbf{Z}u, E_*) - A_*(\mathbf{Z}u, E_*)] + A_*(\mathbf{Z}_*u, E_*) - \lambda M(\mathbf{P}u, E_*)). \end{aligned} \tag{3.40}$$

We now apply (3.12) to $A_*(\mathbf{Z}_*u, E_*)$ to get

$$A_*(\mathbf{Z}_*u, E_*) = \lambda M_*(\mathbf{P}_*u, E_*) + [A_*(u, \mathbf{P}_*E_*) - a(u, \mathbf{P}_*E_*)] + \lambda[m(u, \mathbf{P}_*E_*) - M_*(u, \mathbf{P}_*E_*)]. \tag{3.41}$$

Using (3.41) in (3.40) we then have

$$\begin{aligned}
A_*(\mathbf{Z}_* u - \mathbf{Z}u, E_*) &= \lambda M(\mathbf{P}u, E_* - V)) - A(\mathbf{Z}u, E_* - V) \\
&\quad - [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] - \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)] \\
&\quad + [A(\mathbf{Z}u, E_*) - A_*(\mathbf{Z}u, E_*)] + \lambda[M_*(\mathbf{P}_* u, E_*) - M(\mathbf{P}u, E_*))] \\
&\quad + [A_*(u, \mathbf{P}_* E_*) - a(u, \mathbf{P}_* E_*)] + \lambda[m(u, \mathbf{P}_* E_*) - M_*(u, \mathbf{P}_* E_*)] \\
&= \lambda M(\mathbf{P}_* u - \mathbf{P}u, E_*)) + [\lambda M(\mathbf{P}u, E_* - V)) - A(\mathbf{Z}u, E_* - V)] \\
&\quad - [A(u, \mathbf{P}V) - a(u, \mathbf{P}V)] - \lambda[m(u, \mathbf{P}V) - M(u, \mathbf{P}V)] \\
&\quad + [A(\mathbf{Z}u, E_*) - A_*(\mathbf{Z}u, E_*)] + \lambda[M_*(\mathbf{P}_* u, E_*) - M(\mathbf{P}_* u, E_*))] \\
&\quad + [A_*(u, \mathbf{P}_* E_*) - a(u, \mathbf{P}_* E_*)] + \lambda[m(u, \mathbf{P}_* E_*) - M_*(u, \mathbf{P}_* E_*)].
\end{aligned} \tag{3.42}$$

We now follow the arguments given in the proof of Lemma 4.6 of [35] to bound $\lambda M(\mathbf{P}u, E_* - V)) - A(\mathbf{Z}u, E_* - V)$ in (3.42). We first construct an approximation $V \in \mathbb{V}(\mathcal{T})$ of $E_* \in \mathbb{V}(\mathcal{T}_*)$. Let ω be the union of elements of $\mathcal{R} = \mathcal{T} \setminus \mathcal{T}_*$. Let $\overline{\omega}$ be the corresponding union in $\overline{\mathcal{T}}$. Let ω_j (resp. $\overline{\omega}_j$), $1 \leq j \leq N$ be the connected components of the interior of ω (resp. $\overline{\omega}_j$). Let $\overline{\mathcal{T}}_j$ be the subset of elements of $\overline{\mathcal{T}}$ contained in $\overline{\omega}_j$. Let $\mathbb{V}(\overline{\mathcal{T}}_j)$ be the restriction of $\mathbb{V}(\overline{\mathcal{T}})$ to $\overline{\omega}_j$. We construct the Scott-Zhang operator on each $\overline{\omega}_j$ and use the map $X_{\mathcal{T}} \circ X_0^{-1}$ to lift it to Γ. We denote the lifted interpolant as $\pi_j : H^1(\overline{\omega}_j) \to \mathbb{V}(\mathcal{T}_j)$, with

$$\mathcal{T}_j := \{T = X_{\mathcal{T}} \circ X_0^{-1}(\overline{T}) : \overline{T} \in \overline{\mathcal{T}}_j\} \subset \mathcal{T}.$$

Let $V \in \mathbb{V}(\mathcal{T})$ be the following approximation of $E_* \in \mathbb{V}(\mathcal{T}_*)$:

$$V := \pi_j E_* \in \omega_j, \qquad V := E_* \text{ elsewhere.}$$

Since $V = E_*$ on $\Gamma \setminus \omega$, we get via the same arguments in the proof of Theorem 3.18 that

$$\lambda M(\mathbf{P}u, E_* - V)) - A(\mathbf{Z}u, E_* - V) \lesssim \mu_{\mathcal{T}}(\lambda \mathbf{P}u, \mathbf{Z}u, \mathcal{R}) \|\nabla_\Gamma E_*\|_{L_2(\Gamma).}$$

Applying the same arguments as in the proof of Theorem 3.18 and equivalence of norms on surfaces to the rest of the terms in (3.42) then yields the final result. □

3.2.2 Properties of the Eigenfunction Estimator and Oscillation

We begin with a lemma that gives the relationship between the theoretical and computable eigenfunction estimators.

Lemma 3.22 (Relation between Eigenfunction Estimators). *Let $\mu_{\mathcal{T}}(J,T)$ and $\eta_{\mathcal{T}}(J,T)$ be the theoretical and computable estimators, respectively for the cluster indexed by J on a subset $\mathcal{S} \subseteq \mathcal{T}$. Let $\zeta_{\mathcal{T}}(\gamma)$ be the geometric estimator for the entire domain γ. Let the largest mesh size H_0 be small enough so that*

$$\max_{j\in J} \|u_j - \mathbf{Z}u_j\|_{L_2(\Gamma)} \leq \sqrt{1+(2|J|)^{-1}} - 1. \tag{H2}$$

Then

$$\mu_{\mathcal{T}}(J,T)^2 \leq 3\eta_{\mathcal{T}}(J,T)^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2 \tag{3.43}$$

$$\eta_{\mathcal{T}}(J,T)^2 \leq 4\mu_{\mathcal{T}}(J,T)^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2, \tag{3.44}$$

where $|J|$ is the cardinality of J.

Proof. Let (u_i,λ_i) be in the cluster $\{(u_j,\lambda_j)\}_{j\in J}$. Let $\{(U_j,\Lambda_j)\}_{j\in J}$ be the set of eigenpairs associated with the FEM cluster. Then by the definition of $\mathbf{P}$ and the properties of eigenfunctions

$$\begin{aligned}
\lambda_i\mathbf{P}u_i &= \lambda_i\sum_{j\in J} M(u_i,U_j)U_j = \lambda_i\sum_{j\in J} m(u_i,U_j)U_j + \lambda_i\sum_{j\in J}[M(u_i,U_j)-m(u_i,U_j)]U_j \\
&= \sum_{j\in J} a(u_i,U_j)U_j + \lambda_i\sum_{j\in J}[M(u_i,U_j)-m(u_i,U_j)]U_j \\
&= \sum_{j\in J} \Lambda_j M(\mathbf{Z}u_i,U_j)U_j + \sum_{j\in J}[a(u_i,U_j)-A(u_i,U_j)]U_j \\
&+ \lambda_i\sum_{j\in J}[M(u_i,U_j)-m(u_i,U_j)]U_j \\
&= \sum_{j\in J} \Lambda_j M(\mathbf{Z}u_i,U_j)U_j + \sum_{j\in J}[\lambda_i M(u_i,U_j)-A(u_i,U_j)]U_j.
\end{aligned} \tag{3.45}$$

By the definition (3.10) of $\mathbf{Z}$ we have

$$\mathbf{Z}u_i = \sum_{j\in J} M(\mathbf{Z}u_i, U_j)U_j. \tag{3.46}$$

Taking the Laplacian of (3.46) and adding it to (3.45) yields

$$\begin{aligned}[\lambda_i \mathbf{P}u_i + \Delta_\Gamma \mathbf{Z}u_i]_T &= \sum_{j\in J} M(\mathbf{Z}u_i, U_j)[\Lambda_j U_j + \Delta_\Gamma U_j]_T \\ &\quad + \sum_{j\in J} [\lambda_i M(u_i, U_j) - A(u_i, U_j)]U_j|_T\end{aligned}$$

when restricted to a single triangle. We can write this in matrix format as $\vec{v} = \mathbf{F}\vec{V} + \mathbf{M}\vec{U}$, where

$$\vec{v}_i = [\lambda_i \mathbf{P}u_i + \Delta_\Gamma \mathbf{Z}u_i]_T, \qquad \vec{V}_i = [\Lambda_i U_i + \Delta_\Gamma U_i]_T, \qquad \vec{U}_i = U_i|_T,$$

$$\mathbf{F}_{i,j} = M(\mathbf{Z}u_i, U_j), \qquad \mathbf{M}_{i,j} = [\lambda_i M(u_i, U_j) - A(u_i, U_j)].$$

Squaring $\vec{v} = \mathbf{F}\vec{V} + \mathbf{M}\vec{U}$, then gives

$$\begin{aligned}|\vec{v}|^2 = \vec{v}^T\vec{v} &= (\mathbf{F}\vec{V} + \mathbf{M}\vec{U})^T(\mathbf{F}\vec{V} + \mathbf{M}\vec{U}) \\ &= (\vec{V}^T\mathbf{F}^T + \vec{U}^T\mathbf{M}^T)(\mathbf{F}\vec{V} + \mathbf{M}\vec{U}) \\ &= \vec{V}^T\mathbf{F}^T\mathbf{F}\vec{V} + \vec{U}^T\mathbf{M}^T\mathbf{M}\vec{U} + \vec{U}^T\mathbf{M}^T\mathbf{F}\vec{V} + \vec{V}^T\mathbf{F}^T\mathbf{M}\vec{U}.\end{aligned}$$

Applying Young's inequality then gives $|\vec{v}|^2 \leq 2\vec{V}^T\mathbf{F}^T\mathbf{F}\vec{V} + 2\vec{U}^T\mathbf{M}^T\mathbf{M}\vec{U}$. Since (H2) is satisfied, we can employ the arguments made in Lemma 3.1 of [24] to get $\|\mathbf{F}^T\mathbf{F}\|_2 \leq \frac{3}{2}$. We now have $|\vec{v}|^2 \leq 3|\vec{V}|^2 + 2\|\mathbf{M}^T\mathbf{M}\|_2|\vec{U}|^2$. Taking a closer look at the elements of $\mathbf{M}$ we see that

$$\mathbf{M}_{i,j} = \lambda_i M(u_i, U_j) - A(u_i, U_j) = \lambda_i[M(u_i, U_j) - m(u_i, U_j)] + [a(u_i, U_j) - A(u_i, U_j)] \lesssim \zeta_{\mathcal{T}}(\gamma),$$

which implies

$$|(\mathbf{M}^T\mathbf{M})_{i,j}| \lesssim |J|\zeta_{\mathcal{T}}(\gamma)^2.$$

The largest eigenvalue, $\sigma_{\max}$, of $\mathbf{M}^T\mathbf{M}$ equals $\|\mathbf{M}^T\mathbf{M}\|_2$. By Gershgorin's theorem, $\sigma_{\max} \lesssim |J|^2\zeta_{\mathcal{T}}(\gamma)^2$ We now have $|\vec{v}|^2 \leq 3|\vec{V}|^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2$ which implies

$$\mu_{\mathcal{T}}(J,T)^2 \leq 3\eta_{\mathcal{T}}(J,T)^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2$$

Going the other way, we have $\vec{V} = \mathbf{F}^{-1}\vec{v} - \mathbf{F}^{-1}\mathbf{M}\vec{U}$. After squaring we get

$$|\vec{V}|^2 \leq 2\vec{v}^T(\mathbf{F}^{-1})^T\mathbf{F}^{-1}\vec{v} + 2\vec{U}\mathbf{M}^T(\mathbf{F}^{-1})^T\mathbf{F}^{-1}\mathbf{M}\vec{U}.$$

Using arguments from Lemma 3.1 of [24] to handle $(\mathbf{F}^{-1})^T\mathbf{F}^{-1}$ and the above arguments again we have $|\vec{V}|^2 \leq 4|\vec{v}|^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2$ which implies

$$\eta_{\mathcal{T}}(J,T)^2 \leq 4\mu_{\mathcal{T}}(J,T)^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2.$$

The jump computations follow directly from (3.46). □

The next lemma shows that for a sufficiently small ADAPT_SURFACE parameter ω, if the computable estimator $\eta_{\mathcal{T}}(J)$ bounds the geometric estimator within ADAPT_EIGENFUNCTION, then so does the theoretical estimator $\mu_{\mathcal{T}}(J)$. This will be important for proving the equivalence of the error and estimator within the ADAPT_EIGENFUNCTION loop.

Lemma 3.23. *Let H_0 satisfy* (H2). *Define*

$$\omega_1 := \frac{1}{\sqrt{2B_0^2B_1|J|^2}}. \tag{W1}$$

If $\omega < \omega_1$ and $\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\eta_{\mathcal{T}}(J)^2$, then

$$\zeta_{\mathcal{T}}(\gamma)^2 \leq 8B_0^2\omega^2\mu_{\mathcal{T}}(J)^2. \tag{3.47}$$

Proof. If $\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\eta_{\mathcal{T}}(J)^2$, then (3.44) gives

$$\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\eta_{\mathcal{T}}(J)^2 \leq 4B_0^2\omega^2\mu_{\mathcal{T}}(J)^2 + B_0^2B_1|J|^2\omega^2\zeta_{\mathcal{T}}(\gamma)^2,$$

which implies $\zeta_{\mathcal{T}}(\gamma)^2 \leq \frac{4B_0^2\omega^2}{1-B_0^2B_1|J|^2\omega^2}\mu_{\mathcal{T}}(J)^2$. To simplify the algebra later on we note that $bx > \frac{x}{1-ax}$ in the interval $[0, \frac{b-1}{ab})$ for $b > 1$ and $a > 0$. Taking $a = B_0^2B_1|J|^2$, $b = 2$, and $x = \omega^2$, we then have for $\omega < \frac{1}{\sqrt{2B_0^2B_1|J|^2}}$ that

$$\zeta_{\mathcal{T}}(\gamma)^2 \leq \frac{4B_0^2\omega^2}{1 - B_0^2B_1|J|^2\omega^2}\mu_{\mathcal{T}}(J)^2 \leq 8B_0^2\omega^2\mu_{\mathcal{T}}(J)^2.$$

□

As is customary for flat domains, the definition (3.25) of oscillation guarantees that $Osc_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, \gamma, T)$ is dominated by $\mu_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, T)$, namely

$$Osc_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, \gamma, T)^2 \leq C_3\mu_{\mathcal{T}}(\lambda\mathbf{P}u, \mathbf{Z}u, T)^2 \qquad T \in \mathcal{T}, \tag{3.48}$$

where the constant C_3 depends on the surface γ. We define the eigenfunction error for the cluster as

$$\mathcal{E}_{\mathcal{T}}(J)^2 := \sum_{j\in J}\left(\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2\right). \tag{3.49}$$

We now show that if the initial mesh is sufficiently refined and the parameter $\omega > 0$ is sufficiently small, then the eigenfunction error for the cluster is equivalent to the theoretical eigenfunction estimator for the cluster within ADAPT_EIGENFUNCTION.

Lemma 3.24 (Equivalence of error and estimator)**.** *Let C_1, C_2, B_1 be given in Theorems 3.18 and 3.26 and C_3 be as in 3.48. Let $\lambda_{\max}$ be the maximum eigenvalue in the cluster $\{\lambda_j\}_{j\in J}$. Let H_0 satisfy* (H1)*,* (H2)*, and*

$$\lambda_{\max}^2 K_0 H_0^{2s} \leq \frac{1}{2}. \tag{H3}$$

Define

$$\omega_2 := \sqrt{\frac{C_2}{16B_0^2 B_1|J|}}. \tag{W2}$$

If $\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\eta_{\mathcal{T}}(J)^2$ *and* $\omega < \min\{\omega_1, \omega_2\}$ *as in Lemma 3.23, then there exist constants* $C_4 \geq C_5 > 0$*, depending on* C_1*,* C_2*, and* C_3*, such that*

$$C_5\mu_{\mathcal{T}}(J) \leq \mathcal{E}_{\mathcal{T}}(J) \leq C_4\mu_{\mathcal{T}}(J). \tag{3.50}$$

Proof. Combining Theorem 3.18 summed over J and (3.47) we get

$$\frac{1}{2}\sum_{j\in J}\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} \leq \left(C_1 + \frac{C_2}{2}\right)\mu_{\mathcal{T}}(J)^2. \tag{3.51}$$

Combining this with (3.48) gives

$$\begin{aligned}\frac{1}{2}\mathcal{E}_{\mathcal{T}}(J)^2 &= \frac{1}{2}\left(\sum_{j\in J}\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2\right)\\ &\leq \left(C_1 + \frac{C_2}{2} + \frac{C_3}{2}\right)\mu_{\mathcal{T}}(J)^2.\end{aligned}$$

For the lower bound we combine (3.26) and (3.47) to get

$$\left(C_2 - \frac{C_2}{2}\right)\mu_{\mathcal{T}}(J)^2 \leq \frac{3}{2}\left(\sum_{j\in J}\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2\right) = \frac{3}{2}\mathcal{E}_{\mathcal{T}}(J)^2$$

which then implies $\frac{C_2}{3}\mu_{\mathcal{T}}(J)^2 \leq \mathcal{E}_{\mathcal{T}}(J)^2$. □

Lemma 3.25 (Residual Estimator Reduction)**.** *Let* $(u, \lambda) \in \{(u_j, \lambda_j)\}_{j\in J}$ *be an eigenpair associated with the cluster of eigenpairs. Given a mesh-surface pair* $(\mathcal{T}, \Gamma)$*, let* $\mathcal{M} \subset \mathcal{T}$ *be the subset of elements bisected at least* $b \geq 1$ *times in refining* $\mathcal{T}$ *to obtain* $\mathcal{T}_* \geq \mathcal{T}$*. If* $\xi := (1 - 2^{b/d})$*, then there exists a constant* δ *to be defined in Theorem 3.31 as well as constants* B_2 *and* B_3*, solely depending* *on the shape regularity of* $\mathbb{T}$, *and the Lipschitz constant* L *of* γ *such that a*

$$\begin{aligned}\mu_{\mathcal{T}_*}(\lambda\mathbf{P}_*u,\mathbf{Z}_*u)^2 &\leq (1+\delta)\Big(\mu_{\mathcal{T}}(\lambda\mathbf{P}u,\mathbf{Z}u)^2 - \xi\mu_{\mathcal{T}}(\lambda\mathbf{P}u,\mathbf{Z}u,\mathcal{M})^2\Big)\\ &+ (1+\delta^{-1})\left(B_3\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|^2_{L_2(\gamma)} + B_2\zeta_{\mathcal{T}}(\gamma)^2 + K_1H_0^2\lambda^2\|\mathbf{P}_*u-\mathbf{P}u\|^2_{L_2(\gamma)}\right)\end{aligned} \tag{3.52}$$

Proof. We begin by bounding the residual on triangles. Let $T_* \in \mathcal{T}_*$ and $T \in \mathcal{T}$ satisfy $\widehat{T}_* \subset \widehat{T}$. We also define $T' := X_{\mathcal{T}} \circ X^{-1}_{\mathcal{T}_*}(T_*) \subset T$. The residual term in the error estimator on the triangle T_* then satisfies

$$\begin{aligned}\|\mathcal{R}_{T_*}(\lambda\mathbf{P}_*u,\mathbf{Z}_*u)\|_{L_2(T_*)} &= \|q^{\frac{1}{2}}_{\Gamma_*}\mathcal{R}_{T_*}(\lambda\mathbf{P}_*,\mathbf{Z}_*u)\|_{L_2(\widehat{T}_*)}\\ &\leq \|q^{\frac{1}{2}}_{\Gamma_*}(\mathcal{R}_{T_*}(\lambda\mathbf{P}_*,\mathbf{Z}_*u)-\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u))\|_{L_2(\widehat{T}_*)}\\ &+ \|q^{\frac{1}{2}}_{\Gamma_*}\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(\widehat{T}_*)}\\ &\leq \|q^{\frac{1}{2}}_{\Gamma_*}(\mathcal{R}_{T_*}(\lambda\mathbf{P}_*,\mathbf{Z}_*u)-\mathcal{R}_T(\lambda\mathbf{P},\mathbf{Z}u))\|_{L_2(\widehat{T}_*)}\\ &+ \|(q^{\frac{1}{2}}_{\Gamma_*}-q^{\frac{1}{2}}_{\Gamma})\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(\widehat{T}_*)} + \|q^{\frac{1}{2}}_{\Gamma}\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(\widehat{T}_*)}.\end{aligned} \tag{3.53}$$

Noting the boundedness of $\|q_{\Gamma_*}\|_{L_\infty(\Gamma_*)}$, the first term on the right hand side of (3.53) can be bounded as

$$\begin{aligned}\|q^{\frac{1}{2}}_{\Gamma_*}(\mathcal{R}_{T_*}(\lambda\mathbf{P}_*u,\mathbf{Z}_*u)-\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u))\|_{L_2(\widehat{T}_*)} &\lesssim \lambda\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(\widehat{T}_*)}\\ &+ \|\Delta_{\Gamma_*}(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(\widehat{T}_*)}\\ &+ \|(\Delta_{\Gamma_*}-\Delta_\Gamma)\mathbf{Z}u\|_{L_2(\widehat{T}_*)}\\ &\lesssim \lambda\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(T')}\\ &+ \frac{1}{h_{T_*}}\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(T')}\\ &+ \frac{1}{h_{T_*}}\zeta_{\mathcal{T}}(\gamma,T')\|\nabla_\Gamma(\mathbf{Z}u)\|_{L_2(T')},\end{aligned} \tag{3.54}$$

where the last line follows from inverse inequalities, equivalence of norms, and

$$\begin{aligned}\|(\Delta_{\Gamma_*}-\Delta_\Gamma)\mathbf{Z}u\|_{L_2(\widehat{T}_*)} &\le \|(q_{\Gamma_*}^{-1}-q_\Gamma^{-1})\widehat{\mathrm{div}}(q_{\Gamma_*}\widehat{\nabla}\mathbf{Z}u\mathbf{G}_{\Gamma_*}^{-1})\|_{L_2(\widehat{T}_*)}\\ &+\|q_\Gamma^{-1}\widehat{\mathrm{div}}((q_{\Gamma_*}-q_\Gamma)\widehat{\nabla}\mathbf{Z}u\mathbf{G}_{\Gamma_*}^{-1})\|_{L_2(\widehat{T}_*)}\\ &+\|q_\Gamma^{-1}\widehat{\mathrm{div}}(q_\Gamma\widehat{\nabla}\mathbf{Z}u(\mathbf{G}_{\Gamma_*}^{-1}-\mathbf{G}_\Gamma^{-1}))\|_{L_2(\widehat{T}_*)}\\ &\lesssim \frac{1}{h_{T_*}}\zeta_{\mathcal{T}}(\gamma,T')\|\nabla\mathbf{Z}u\|_{L_2(T')}.\end{aligned}$$

Noting the boundedness of $\|(q_{\Gamma_*}^{\frac{1}{2}}+q_\Gamma^{\frac{1}{2}})^{-1}\|_{L_\infty(\Gamma_*)}$ and using (3.15) to bound $\|(q_{\Gamma_*}-q_\Gamma)^{-1}\|_{L_\infty(\Gamma_*)}$; the second term on the right hand side of (3.53) can be bounded as

$$\begin{aligned}\|(q_{\Gamma_*}^{\frac{1}{2}}-q_\Gamma^{\frac{1}{2}})\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(\widehat{T}_*)} &= \|(q_{\Gamma_*}-q_\Gamma)(q_{\Gamma_*}^{\frac{1}{2}}+q_\Gamma^{\frac{1}{2}})^{-1}\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(\widehat{T}_*)}\\ &\lesssim \zeta_{\mathcal{T}}(\gamma,T')\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}\\ &\lesssim \zeta_{\mathcal{T}}(\gamma,T')\left(\frac{1}{h_{T_*}}\|\nabla_\Gamma\mathbf{Z}u\|_{L_2(T')}+\|\lambda\mathbf{P}u\|_{L_2(T')}\right).\end{aligned} \tag{3.55}$$

Using (3.54) and (3.55) to bound the terms on the right hand side of (3.53) and multiplying by h_{T_*} yields

$$\begin{aligned}h_{T_*}\|\mathcal{R}_{T_*}(\lambda\mathbf{P}_*u,\mathbf{Z}u)\|_{L_2(T_*)} &\le h_{T_*}\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}+h_{T_*}\lambda\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(T')}\\ &+C\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(T')}+C\zeta_{\mathcal{T}}(\gamma,T')\left(2\|\nabla_\Gamma\mathbf{Z}u\|_{L_2(T')}+h_{T_*}\|\lambda\mathbf{P}u\|_{L_2(T')}\right)\\ &=\left(h_{T_*}\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}\right)\\ &+\Bigg(h_{T_*}\lambda\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(T')}+C\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(T')}\\ &+C\zeta_{\mathcal{T}}(\gamma,T')\left(2\|\nabla_\Gamma\mathbf{Z}u\|_{L_2(T')}+h_{T^*}\|\lambda\mathbf{P}u\|_{L_2(T')}\right)\Bigg),\end{aligned}$$

where C represents the generic constant in the (3.54) and (3.55) bounds.

Squaring both sides, applying Young's inequality (with constant δ), and summing over triangles then yields

$$\sum_{T^*\in\mathcal{T}_*} h_{T_*}^2\|\mathcal{R}_{T_*}(\lambda\mathbf{P}_*,\mathbf{Z}_*u)\|_{L_2(T')}^2 \leq (1+\delta)\bigg(\sum_{T^*\in\mathcal{T}_*} h_{T_*}^2\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}^2\bigg) + (1+\delta^{-1})\bigg(K_1H_0^2\lambda^2\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(\gamma)}^2 + C\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(\gamma)}^2 + C\zeta_{\mathcal{T}}(\gamma)^2\bigg).$$

Noting that

$$\sum_{T^*\in\mathcal{T}_*} h_{T_*}^2\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}^2 \leq \sum_{T^*\in\mathcal{T}_*} h_T^2\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}^2 - \sum_{T^*\in\mathcal{M}} \xi h_T^2\|\mathcal{R}_T(\lambda\mathbf{P}u,\mathbf{Z}u)\|_{L_2(T')}^2$$

completes the proof. The same steps apply to the jumps and can be found in [35]. □

Lemma 3.26. *Let $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy* (3.13). *Let $(\mathcal{T},\Gamma)$, $(\mathcal{T}_*,\Gamma_*)$ be mesh-surface pairs with $\mathcal{T}\leq\mathcal{T}_*$. Then, there exist constants C_6, B_2, and B_3 depending only on $\mathcal{T}_0$, the Lipschitz constant L of γ, and λ, such that*

$$Osc_{\mathcal{T}_*}(\lambda\mathbf{P}_*u,\mathbf{Z}_*u,\gamma)^2 \leq C_6 Osc_{\mathcal{T}}(\lambda\mathbf{P}u,\mathbf{Z}u,\gamma)^2 + B_3\|\nabla_\gamma(\mathbf{Z}_*u-\mathbf{Z}u)\|_{L_2(\gamma)}^2 + B_2\zeta_{\mathcal{T}}(\gamma)^2 + K_1H_0^2\lambda^2\|\mathbf{P}_*u-\mathbf{P}u\|_{L_2(\gamma)}^2 \tag{3.56}$$

Proof. Noting that $\mathrm{id}-\Pi_{2n-2}^2$ is a projection, so $\|(\mathrm{id}-\Pi_{2n-2}^2)v\|_{L_2(\widehat{T}^*)} \leq \|v\|_{L_2(\widehat{T}^*)}$ and $h_{T_*}\leq h_T$, we have

$$
\begin{aligned}
Osc_{\mathcal{T}_*}(\lambda \mathbf{P}_* u, \mathbf{Z}_* u, \gamma) &= h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P}_* u q_{\Gamma_*} + \widehat{\mathrm{div}}(q_{\Gamma_*} \widehat{\nabla} \mathbf{Z}_* u \mathbf{G}^{-1}_{\Gamma_*}) \right) \right\|_{L_2(\widehat{T}_*)} \\
&\le h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P} u q_{\Gamma} + \widehat{\mathrm{div}}(q_{\Gamma} \widehat{\nabla} \mathbf{Z} u \mathbf{G}^{-1}_{\Gamma}) \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P}_* u q_{\Gamma_*} - \lambda \mathbf{P} u q_{\Gamma} \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(q_{\Gamma} \Delta_{\Gamma} \mathbf{Z} u - q_{\Gamma_*} \Delta_{\Gamma_*} \mathbf{Z}_* u \right) \right\|_{L_2(\widehat{T}_*)} \\
&\le h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P} u q_{\Gamma} + \widehat{\mathrm{div}}(q_{\Gamma} \widehat{\nabla} \mathbf{Z} u \mathbf{G}^{-1}_{\Gamma}) \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P}_* u q_{\Gamma_*} - \lambda \mathbf{P} u q_{\Gamma} \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left((q_{\Gamma} \Delta_{\Gamma} - q_{\Gamma_*} \Delta_{\Gamma_*}) \mathbf{Z} u \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| (\mathrm{id} - \Pi^2_{2n-2}) q_{\Gamma_*} \Delta_{\Gamma_*} \left(\mathbf{Z} u - \mathbf{Z}_* u \right) \right\|_{L_2(\widehat{T}_*)} \\
&\le h_{T} \left\| (\mathrm{id} - \Pi^2_{2n-2}) \left(\lambda \mathbf{P} u q_{\Gamma} + \widehat{\mathrm{div}}(q_{\Gamma} \widehat{\nabla} \mathbf{Z} u \mathbf{G}^{-1}_{\Gamma}) \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| \left(\lambda \mathbf{P}_* u q_{\Gamma_*} - \lambda \mathbf{P} u q_{\Gamma} \right) \right\|_{L_2(\widehat{T}_*)} \\
&+ h_{T_*} \left\| \left((q_{\Gamma} \Delta_{\Gamma} - q_{\Gamma_*} \Delta_{\Gamma_*}) \mathbf{Z} u \right) \right\|_{L_2(\widehat{T}_*)} + h_{T_*} \left\| q_{\Gamma_*} \Delta_{\Gamma_*} \left(\mathbf{Z} u - \mathbf{Z}_* u \right) \right\|_{L_2(\widehat{T}_*)}
\end{aligned}
$$

for the residual portion of the oscillation. The rest of the proof follows from the steps taken in the proof of Lemma 3.25. □

Remark 3.27 (Local perturbation of data oscillation)**.** *The previous Lemma is also valid locally, that is for any subset $\tau \subset \mathcal{T}_*$. If $\tau = \mathcal{T} \cap \mathcal{T}_*$, the same proof gives* (3.56) *with $C_6 = 2$,*

$$
\begin{aligned}
Osc_{\mathcal{T}_*}(\lambda \mathbf{P}_* u, \mathbf{Z}_* u, \tau)^2 &\le 2 Osc_{\mathcal{T}}(\lambda \mathbf{P} u, \mathbf{Z} u, \tau)^2 + B_3 \|\nabla_\gamma (\mathbf{Z}_* u - \mathbf{Z} u)\|^2_{L_2(\gamma)} + B_2 \zeta_{\mathcal{T}}(\gamma)^2 \\
&+ K_1 H_0^2 \lambda^2 \|\mathbf{P}_* u - \mathbf{P} u\|^2_{L_2(\gamma)}
\end{aligned}
\tag{3.57}
$$

3.3 AFEM: Design and Properties

In this section we discuss the AFEM algorithm in detail.

AFEM Algorithm: Given $\mathcal{T}_0$ and parameters $\epsilon_0 > 0$, $0 < \rho < 1$, and $\omega > 0$, set $k = 0$.

1. $\mathcal{T}_k^+ =$ ADAPT_SURFACE$(\mathcal{T}_k, \omega\epsilon_k)$

2. $[\{(U_{k+1,i}, \Lambda_{k+1,i})\}_{i=1}^N, \mathcal{T}_{k+1}] =$ ADAPT_EIGENFUNCTION$(\mathcal{T}_k^+, \epsilon_k)$

3. $\epsilon_{k+1} = \rho\epsilon_k; k = k+1$

4. go to 1.

3.3.1 Module ADAPT_SURFACE

Given a tolerance $\epsilon > 0$ and an admissible subdivision $\mathcal{T}$, $\mathcal{T}_* =$ ADAPT_SURFACE$(\mathcal{T}, \epsilon)$ improves the surface resolution until the new subdivision $\mathcal{T}_* \geq \mathcal{T}$ satisfies

$$\zeta_{\mathcal{T}}(\gamma) \leq \epsilon, \tag{3.58}$$

where $\zeta_{\mathcal{T}}$ is the geometric estimator. This module is based on a greedy algorithm and acts on a generic mesh $\mathcal{T} = \cup_{i=1}^F \mathcal{T}^i \in \mathbb{T}$:

$\mathcal{T}_* =$ ADAPT_SURFACE$(\mathcal{T}, \epsilon)$

1. if $\mathcal{M} := \{T \in \mathcal{T} : \zeta_{\mathcal{T}}(\gamma, T) > \epsilon\} = \emptyset$
   return$(\mathcal{T})$ and exit

2. $\mathcal{T} =$ REFINE$(\mathcal{T}, \mathcal{M})$

3. go to 1.

We require ADAPT_SURFACE to be t-optimal, i.e. there exists a constant $C(\gamma)$ such that the set $\mathcal{M}$ of all the elements marked for refinement in a call to ADAPT_SURFACE$(\mathcal{T}, \epsilon)$ satisfies

$$\#\mathcal{M} \leq C(\gamma)\epsilon^{-1/t}$$

It is shown in [35] and Lemma 3.32 that this assumption is satisfied by a greedy algorithm as long as $\chi^i \in B_q^{1+td}(L_q(\Omega))$ with $tq > 1$, $0 < q \leq \infty$ and $td \leq k$ for all $1 \leq i \leq F$.

3.3.2 Module ADAPT_EIGENFUNCTION

Given a tolerance $\epsilon > 0$ and an admissible subdivision $\mathcal{T} \in \mathbb{T}$, $[\{(U_{*,i}, \Lambda_{*,i})\}_{i\in J}, \mathcal{T}_*] =$ ADAPT_EIGENFUNCTION$(\mathcal{T}, \epsilon)$ outputs a refinement $\mathcal{T}_* \geq \mathcal{T}$ and an associated set of FEM eigenpairs $\{(U_{*,i}, \Lambda_{*,i})\}_{i\in J}$ such that

$$\eta_{\mathcal{T}_*}(J) \leq \epsilon. \tag{3.59}$$

This module is based on the sequence:

$[\{(U_{*,i}, \Lambda_{*,i})\}_{i\in J}, \mathcal{T}_*] =$ ADAPT_EIGENFUNCTION$(\mathcal{T}, \epsilon)$

1. $\{(U_i, \Lambda_i)\}_{i\in J} =$ SOLVE$(\mathcal{T})$
2. $\{\eta_{\mathcal{T}}(J, T)\}_{T\in\mathcal{T}} =$ ESTIMATE$(\mathcal{T}, \{(U_i, \Lambda_i)\}_{i\in J})$
3. if $\eta_{\mathcal{T}} < \epsilon$

   return$(\mathcal{T}, \{(U_i, \Lambda_i)\}_{i\in J})$
4. $\mathcal{M} =$ MARK$(\mathcal{T}, \{\eta_{\mathcal{T}}(T)\}_{T\in\mathcal{T}})$
5. $\mathcal{T} =$ REFINE$(\mathcal{T}, \mathcal{M})$
6. go to 1.

Procedure ESTIMATE. Given the FEM eigenpairs $\{(U_i, \Lambda_i)\}_{i\in J} \in \mathbb{V}_{\#}(\mathcal{T}) \times \mathbb{R}^+$ associated with the cluster, we want ω to satisfy

$$\omega < \min\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}, \tag{3.60}$$

where we have summarized the restrictions on ω in Table 3.2.

(W1)	$\omega_1 := \frac{1}{\sqrt{2B_0^2 B_1 \lvert J\rvert^2}}$	Lemma 3.23
(W2)	$\omega_2 := \sqrt{\frac{C_2}{16B_0^2 B_1 \lvert J\rvert}}$	Lemma 3.24
(W3)	$\omega_3 := \frac{\xi\theta^2}{\sqrt{512\left(B_2\left(1+\frac{1}{2B_3}\right)+\frac{CK_1}{2B_3}\lambda_{\max}^2+(1+\lambda_{\max}^2)K_2C+C\lambda_{\max}^2\right)B_3B_0}}$	Theorem 3.31
(W4)	$\omega_4 := \frac{C_5}{\sqrt{\left(\frac{9}{2}B_1+2B_3B_2+5B_2+3CK_2+\left(\frac{9}{2}CK_0+3CK_2+3C+2CB_3K_0+2CK_1H_0^2\right)\lambda_{\max}^2\right)8B_0^2\lvert J\rvert}}$	Lemma 3.41
(W5)	$\omega_5 := \frac{\theta}{\sqrt{6B_1 \lvert J\rvert^2}}$	Lemma 3.42

Table 3.2: The set of upper bounds on ω and where they first appear.

Given a tolerance $\epsilon > 0$ to be reached by ADAPT_EIGENFUNCTION and starting from the input subdivision $\mathcal{T}$ satisfying $\zeta_{\mathcal{T}}(\gamma) \leq \omega\epsilon$, we observe that (3.4) guarantees that $\mathcal{T}$ as well as all subdivisions $\mathcal{T}_* \geq \mathcal{T}$ constructed within the inner iterates of ADAPT_EIGENFUNCTION satisfy

$$\zeta_{\mathcal{T}_*}(\gamma)^2 \leq B_0^2\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\epsilon^2.$$

Within the while loop of ADAPT_EIGENFUNCTION we have $\eta_{\mathcal{T}}(J) > \epsilon$, so we deduce

$$\zeta_{\mathcal{T}}(\gamma)^2 \leq B_0^2\omega^2\eta_{\mathcal{T}}(J)^2 \tag{3.61}$$

and so by Lemma 3.23 we have $\zeta_{\mathcal{T}}(\gamma)^2 \leq \frac{C_2}{2\lvert J\rvert B_1}\mu_{\mathcal{T}}(J)^2$. Thus (3.50) is valid within the ADAPT_EIGENFUNCTION loop.

Procedure MARK. We rely on an optimal Dörfler marking strategy for the selection of elements. Given the set of computable indicators $\{\eta_{\mathcal{T}}(J,T)\}_{T\in\mathcal{T}}$ and a marking parameter $\theta \in (0,1]$, MARK outputs a subset of marked elements $\mathcal{M} \subset \mathcal{T}$ such that

$$\eta_{\mathcal{T}}(J,\mathcal{M}) \geq \theta \eta_{\mathcal{T}}(J). \tag{3.62}$$

Procedure REFINE. Given a subdivision $\mathcal{T}$ and a set $\mathcal{M} \subset \mathcal{T}$ of marked elements, the call REFINE($\mathcal{T}, \mathcal{M}$) bisects all elements in $\mathcal{M}$ at least $b \geq 1$ times and performs additional refinements necessary to maintain conformity. The resulting subdivision is denoted by $\mathcal{T}_*$. Recall that the bisection procedure is first executed on faces of the corresponding flat subdivision $\overline{\mathcal{T}}$ and its effext is transferred to the actual subdivision via interpolation of maps $X^i_{\mathcal{T}^i} \circ (X^i_0)^{-1}$ for $i = 1, \dots, F$.

Since the refinement procedure is performed on $\overline{\mathcal{T}}$ or on $\widehat{\mathcal{T}}$, the complexity results of the overall refinement algorithm proved by Binev, Dahmen, and DeVore for $d = 2$ [28] and Stevenson [29] for $d > 2$ hold in our setting.

Remark 3.28. *For $d = 1$, any subdivision is said to have admissible labeling. For $d = 2$ we say $\mathcal{T}_0$ has an admissible labeling is each edge of $\mathcal{T}_0$ has an admissible labeling if each edge of $\mathcal{T}_0$ has either a 0 or 1 such that each element of $\mathcal{T}_0$ has two edges labeled as 1 and one edge labeled as 0 [28]; refining an element consists of connecting the middle of the edge labeled zero with the opposite angle. For $d > 2$ the condition is much more technical and we omit it. In short the admissible initial labeling guarantees that the bisection procedure terminates in a finite number of steps with a conforming mesh and any uniform refinement is conforming.*

Lemma 3.29 (Complexity of REFINE)**.** *Assume that the initial triangulation $\mathcal{T}_0$ has an admissible labeling. Let $\{\mathcal{T}_k\}_{k\geq 0}$ be a sequence of triangulations produced by successive calls to $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$, where $\mathcal{M}_k$ is any subset of $\mathcal{T}_k$, $k \geq 0$. Then, there exists a constant C_7 solely ddependent on $\mathcal{T}_0$, its labeling, and the refinement depth b, such that*

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq C_7 \sum_{j=0}^{k-1} \#\mathcal{M}_j, \qquad \forall k \geq 1.$$

3.4 Conditional Contraction Property

We show that the procedure

ADAPT_EIGENFUNCTION yields a contraction property. Given a single eigenpair $(u_i, \lambda_i) \in \{(u_j, \lambda_j)\}_{j\in J}$, the following shorthand will be used:

$$e_i^j := u_i - \mathbf{Z}_j u_i \quad E_i^j := \mathbf{Z}_{j+1} u_i - \mathbf{Z}_j u_i$$

$$\mu_i^j := \mu_{\mathcal{T}_j}(\lambda_i, u_i), \quad \mu_i^{j+1} := \mu_{\mathcal{T}_{j+1}}(\lambda_i, u_i), \quad \zeta^j := \zeta_{\mathcal{T}_j}(\gamma)$$

$$(\mu^j(J))^2 := \sum_{i\in J} (\mu_i^j)^2, \quad (\mu^{j+1}(J))^2 := \sum_{i\in J} (\mu_i^{j+1})^2.$$

We begin by proving a quasi-orthogonality property which will be important for the proof of the conditional contraction property.

Lemma 3.30 (Quasi-orthogonality). *Let (u_i, λ_i) be an eigenpair of the cluster with $\|u_i\|_{L_2(\gamma)} = 1$ and $\mathcal{T}_{j+1} \geq \mathcal{T}_j$. There exists a constant $B_2 > 0$ depending only on λ_i and the Lipschitz constant L of γ such that for $j \geq 0$, we have*

$$\begin{aligned} \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} \leq \|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} - \frac{1}{2}\|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)} + B_2(\zeta^j)^2 \\ + \lambda_i^2 \|u_i - \mathbf{P}_{j+1} u_i\|^2_{L_2(\gamma)} + K_2(1+\lambda_i^2)\|E_i^j\|^2_{L_2(\gamma)} \end{aligned} \tag{3.63}$$

$$\begin{aligned} \|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} - \frac{3}{2}\|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)} - B_2(\zeta^j)^2 \leq \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} \\ + \lambda^2 \|u_i - \mathbf{P}_{j+1} u_i\|^2_{L_2(\gamma)} + K_2(1+\lambda^2)\|E_i^j\|^2_{L_2(\gamma)}. \end{aligned} \tag{3.64}$$

Proof. We begin with the identity

$$\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} = \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + \|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)} + 2a(u_i - \mathbf{Z}_{j+1} u_i, E_i^j). \tag{3.65}$$

Rearranging terms in (3.65) then yields

$$\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} - \|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)} \leq \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + 2|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \tag{3.66}$$

and

$$\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} \leq \|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} - \|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)} + 2|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)|. \tag{3.67}$$

It is apparent from (3.66) and (3.67) that we must bound the term $|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)|$. Expanding $a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)$ gives

$$\begin{aligned} a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j) &= \lambda_i m(u_i, E_i^j) - a(\mathbf{Z}_{j+1}u_i, E_i^j) \\ &= \lambda_i m(u_i - \mathbf{P}_{j+1}u_i, E_i^j) + \lambda_i m(\mathbf{P}_{j+1}u_i, E_i^j) - a(\mathbf{Z}_{j+1}u_i, E_i^j) \\ &= \lambda_i m(u_i - \mathbf{P}_{j+1}u_i, E_i^j) + \lambda_i[m(\mathbf{P}_{j+1}u_i, E_i^j) - M_{j+1}(\mathbf{P}_{j+1}u_i, E_i^j)] \\ &+ \lambda_i M_{j+1}(\mathbf{P}_{j+1}u_i, E_i^j) - a(\mathbf{Z}_{j+1}u_i, E_i^j). \end{aligned} \tag{3.68}$$

Using the algebraic identity (3.12) on $\lambda_i M_{j+1}(\mathbf{P}_{j+1}u_i, E_i^j)$ gives

$$\begin{aligned} \lambda M_{j+1}(\mathbf{P}_{j+1}u_i, E_i^j) &= A_{j+1}(\mathbf{Z}_{j+1}u_i, E_i^j) + [a(u_i, \mathbf{P}_{j+1}E_i^j) - A_{j+1}(u_i, \mathbf{P}_{j+1}E_i^j)] \\ &+ \lambda_i[M_{j+1}(u_i, \mathbf{P}_{j+1}E_i^j) - m(u_i, \mathbf{P}_{j+1}E_i^j)]. \end{aligned} \tag{3.69}$$

Combining (3.69) with (3.68) then gives

$$\begin{aligned} a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j) &= \lambda_i m(u_i - \mathbf{P}_{j+1}u_i, E_i^j) + \lambda_i[m(\mathbf{P}_{j+1}u_i, E_i^j) - M_{j+1}(\mathbf{P}_{j+1}u_i, E_i^j)] \\ &+ [A_{j+1}(\mathbf{Z}_{j+1}u_i, E_i^j) - a(\mathbf{Z}_{j+1}u_i, E_i^j)] + [a(u_i, \mathbf{P}_{j+1}E_i^j) - A_{j+1}(u_i, \mathbf{P}_{j+1}E_i^j)] \\ &+ \lambda_i[M_{j+1}(u_i, \mathbf{P}_{j+1}E_i^j) - m(u_i, \mathbf{P}_{j+1}E_i^j)]. \end{aligned}$$

Applying Cauchy-Schwarz in combination with (3.15) gives

$$\begin{aligned}|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \leq{}& \lambda_i \|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \\ &+ C\lambda_i \|\mathbf{P}_{j+1}u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \zeta^{j+1} + C\lambda \|u_i\|_{L_2(\gamma)} \|\mathbf{P}_{j+1}E_i^j\|_{L_2(\gamma)} \zeta^{j+1} \\ &+ C\|\nabla_\gamma \mathbf{Z}_{j+1}u_i\|_{L_2(\gamma)} \|\nabla_\gamma E_i^j\|_{L_2(\gamma)} \zeta^{j+1} \\ &+ C\|\nabla_\gamma u_i\|_{L_2(\gamma)} \|\nabla_\gamma \mathbf{P}_{j+1}E_i^j\|_{L_2(\gamma)} \zeta^{j+1}.\end{aligned}$$

We now apply (3.19) to $\|\mathbf{P}_{j+1}u_i\|_{L_2(\gamma)}$ and $\|\mathbf{P}_{j+1}E_i^j\|_{L_2(\gamma)}$, (3.17) to $\|\nabla_\gamma \mathbf{Z}_{j+1}u_i\|_{L_2(\gamma)}$, and (3.18) to $\|\nabla_\gamma \mathbf{P}_{j+1}E_i^j\|_{L_2(\gamma)}$ to get

$$\begin{aligned}|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \leq{}& \lambda_i \|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \\ &+ C\lambda_i \|u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \zeta^{j+1} + C\lambda_i \|u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \zeta^{j+1} \\ &+ C\|\nabla_\gamma u_i\|_{L_2(\gamma)} \|\nabla_\gamma E_i^j\|_{L_2(\gamma)} \zeta^{j+1} + C\|\nabla_\gamma u_i\|_{L_2(\gamma)} \|\nabla_\gamma E_i^j\|_{L_2(\gamma)} \zeta^{j+1}.\end{aligned}$$

Noting that our assumption $\|u_i\|_{L_2(\gamma)} = 1$ implies $\|\nabla_\gamma u_i\|^2_{L_2(\gamma)} = \lambda_i \|u_i\|^2_{L_2(\gamma)} = \lambda_i$, we then have

$$\begin{aligned}|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \leq{}& \lambda_i \|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)} \|E_i^j\|_{L_2(\gamma)} \\ &+ C\lambda_i \|E_i^j\|_{L_2(\gamma)} \zeta^{j+1} + C\sqrt{\lambda_i} \|\nabla_\gamma E_i^j\|_{L_2(\gamma)} \zeta^{j+1}.\end{aligned}$$

Applying Young's inequality and being careful to make the coefficient of $\|\nabla_\gamma E_i^j\|_{L_2(\gamma)}$ equal to $\frac{1}{4}$ gives

$$\begin{aligned}|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \leq{}& \frac{\lambda_i^2}{2} \|u_i - \mathbf{P}_{j+1}u_i\|^2_{L_2(\gamma)} + \frac{1}{2}(1 + C\lambda_i^2) \|E_i^j\|^2_{L_2(\gamma)} \\ &+ \frac{1}{2}(1 + 2C^2\lambda_i)(\zeta^{j+1})^2 + \frac{1}{4}\|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)}.\end{aligned}$$

Using (3.4) on ζ^{j+1} we then have

$$\begin{aligned}|a(u_i - \mathbf{Z}_{j+1}u_i, E_i^j)| \leq{}& \frac{\lambda_i^2}{2} \|u_i - \mathbf{P}_{j+1}u_i\|^2_{L_2(\gamma)} + \frac{1}{2}(1 + C\lambda_i^2) \|E_i^j\|^2_{L_2(\gamma)} \\ &+ \frac{B_0}{2}(1 + 2C^2\lambda_i)(\zeta^j)^2 + \frac{1}{4}\|\nabla_\gamma E_i^j\|^2_{L_2(\gamma)}.\end{aligned} \tag{3.70}$$

Plugging (3.70) into (3.66) and (3.67) gives the final result. □

With the quasi-orthogonality result just presented, we now prove a conditional contraction property which holds true within the ADAPT_EIGENFUNCTION module of the AFEM algorithm described in Section 4.

Theorem 3.31 (Contraction Property)**.** *Let $\theta \in (0,1]$ be the marking parameter of MARK and let $\{\mathcal{T}^j, \mathbf{Z}_j, \mathbf{P}_j\}_{j=1}^R$ be a sequence of meshes and projection operators associated with the cluster produced by a call to ADAPT_EIGENFUNCTION($\mathcal{T}^0, \epsilon$) inside AFEM, i.e., $\zeta_{\mathcal{T}_0} \leq \omega\epsilon$. Assume H_0 satisfies* (H1)*,* (H2)*, and*

$$\left(\frac{K_1}{2B_3}\lambda_{\max}^2 H_0^2 + (1+\lambda_{\max}^2)K_2 + \lambda_{\max}^2\right) C H_0^{2s} < \min\left\{\frac{1}{8B_3\,(2C_1 + C_2)}\frac{\xi^2\theta^4}{4-2\xi\theta^2}, \frac{\xi\theta^2}{8}\right\}. \tag{H4}$$

Define

$$\omega_3 := \frac{\xi\theta^2}{\sqrt{512\left(B_2\left(1+\frac{1}{2B_3}\right) + \frac{CK_1}{2B_3}\lambda_{\max}^2 + (1+\lambda_{\max}^2)K_2 C + C\lambda_{\max}^2\right) B_3 B_0}}. \tag{W3}$$

Assume that the AFEM algortihm parameter ω used in ADAPT_SURFACE satisfies

$$\omega < \min\{\omega_1, \omega_2, \omega_3\},$$

where ω_1 and ω_2 are given in (W1) *and* (W2)*, respectively. Then there exist constants $\beta > 0$ and an $0 < \alpha < 1$ such that*

$$\sum_{i\in J} \|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2 + \beta(\mu^{j+1}(J))^2 \leq \alpha^2\left(\sum_{i\in J}\|\nabla_\gamma e_i^j\|_{L_2(\gamma)}^2 + \beta(\mu^j(J))^2\right), \qquad \forall 0 \leq j < R.$$

Moreover, the number of inner iterates of ADAPT_EIGENFUNCTION is uniformly bounded.

Proof. We apply the quasi-orthogonality upper bound (3.63) in combination with the estimator reduction property (3.52) for a single eigenpair (u_i, λ_i) of the cluster J to get

$$\begin{aligned}
\|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2 + \beta(\mu_i^{j+1})^2 \le \|\nabla_\gamma e_i^j\|_{L_2(\gamma)}^2 - \frac{1}{2}\|\nabla_\gamma E_i^j\|_{L_2(\gamma)}^2 + B_2(\zeta^j)^2 \\
+ \lambda_i^2\|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)}^2 + K_2(1+\lambda_i^2)\|E_i^j\|_{L_2(\gamma)}^2 \\
+ \beta(1+\delta)\left((\mu_i^j)^2 - \xi\mu_i^j(\mathcal{M}^j)^2\right) \\
+ \beta(1+\delta^{-1})\left(B_3\|\nabla_\gamma E_i^j\|_{L_2(\gamma)}^2 + B_2(\zeta^j)^2 + K_1\lambda_i^2H_0^2\|\mathbf{P}_{j+1}u_i - \mathbf{P}_j u_i\|_{L_2(\gamma)}^2\right).
\end{aligned}$$

Regrouping terms then gives

$$\begin{aligned}
\|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2 + \beta(\mu_i^{j+1})^2 \le \|\nabla_\gamma e_i^j\|_{L_2(\gamma)}^2 + \left(-\frac{1}{2} + \beta(1+\delta^{-1})B_3\right)\|\nabla_\gamma E_i^j\|_{L_2(\gamma)}^2 \\
+ \beta(1+\delta)\left((\mu_i^j)^2 - \xi(\mu_i^j(\mathcal{M}^j))^2\right) + B_2(1+\beta(1+\delta^{-1}))(\zeta^j)^2 \\
+ \beta(1+\delta^{-1})K_1\lambda_i^2H_0^2\|\mathbf{P}_{j+1}u_i - \mathbf{P}_j u_i\|_{L_2(\gamma)}^2 \\
+ \lambda_i^2\|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)}^2 + K_2(1+\lambda_i^2)\|E_i^j\|_{L_2(\gamma)}^2.
\end{aligned} \tag{3.71}$$

We make the coefficient of $\|\nabla_\gamma E_i^j\|_{L_2(\gamma)}^2$ on the right hand side of (3.71) zero by choosing β such that

$$\beta(1+\delta^{-1})B_3 = \frac{1}{2} \quad \Longrightarrow \quad \beta(1+\delta) = \frac{\delta}{2B_3}. \tag{3.72}$$

We now have

$$\begin{aligned}
\|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2 + \beta(\mu_i^{j+1})^2 \le \|\nabla_\gamma e_i^j\|_{L_2(\gamma)}^2 + \beta(1+\delta)\left((\mu_i^j)^2 - \xi\mu_i^j(\mathcal{M}^j)^2\right) \\
+ B_2\left(1 + \frac{1}{2B_3}\right)(\zeta^j)^2 + \frac{1}{2B_3}K_1\lambda_i^2H_0^2\|\mathbf{P}_{j+1}u_i - \mathbf{P}_j u_i\|_{L_2(\gamma)}^2 \\
+ \lambda_i^2\|u_i - \mathbf{P}_{j+1}u_i\|_{L_2(\gamma)}^2 + K_2(1+\lambda_i^2)\|E_i^j\|_{L_2(\gamma)}^2.
\end{aligned} \tag{3.73}$$

Next, we combine (3.30) and (3.23) with the monotonicity condition $\zeta^{j+1} \lesssim \zeta^j$, to get

$$\begin{aligned}
&\|\mathbf{P}_{j+1}u_i - \mathbf{P}_j u_i\|^2_{L_2(\gamma)} \leq CH_0^{2s}(\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} + \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)}) + C(\zeta^j)^2 \\
&\|E_i^j\|^2_{L_2(\gamma)} = \|\mathbf{Z}_{j+1}u_i - \mathbf{Z}_j u_i\|^2_{L_2(\gamma)} \leq CH_0^{2s}(\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} + \|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)}) + C(\zeta^j)^2 \\
&\|u_i - \mathbf{P}_{j+1}u_i\|^2_{L_2(\gamma)} \leq CH_0^{2s}\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + C(\zeta^j)^2,
\end{aligned} \tag{3.74}$$

where C is a generic constant derived from the a priori estimates in Section 3. Combining (3.74) with (3.73) we then have

$$\begin{aligned}
\left(1 - \frac{CK_1}{2B_3}\lambda_i^2 H_0^{2s+2} - (1+\lambda_i^2)K_2CH_0^{2s} - \lambda_i^2 CH_0^{2s}\right)\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + \beta(\mu_i^{j+1})^2 \\
\leq \left(1 + \frac{CK_1}{2B_3}\lambda_i^2 H_0^{2s+2} + (1+\lambda_i^2)K_2CH_0^{2s}\right)\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} \\
+ \left(B_2\left(1 + \frac{1}{2B_3}\right) + \frac{CK_1}{2B_3}\lambda_i^2 H_0^2 + CK_2(1+\lambda_i^2) + C\lambda_i^2\right)(\zeta^j)^2 \\
+ \beta(1+\delta)\left((\mu_i^j)^2 - \xi\mu_i^j(\mathcal{M}^j)^2\right).
\end{aligned} \tag{3.75}$$

Since $\omega < \omega_1$, we have that $(\zeta^j)^2 \leq 8B_0^2\omega^2(\mu^j(J))^2$ by Lemma 3.23. We now sum (3.75) over the entire cluster in order to invoke the Dörfler marking property, (3.62), for the theoretical estimator. Dörfler marking yields

$$(\mu^j(J))^2 - \xi\mu^j(J,\mathcal{M}^j)^2 \leq (1 - \xi\theta^2)(\mu^j(J))^2.$$

We also bound all λ_i in the sum by $\lambda_{\max}$ to get

$$\begin{aligned}
&\left(1 - \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} - (1+\lambda_{\max}^2)K_2CH_0^{2s} - \lambda_{\max}^2 CH_0^{2s}\right)\sum_{i\in J}\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + \beta(\mu^{j+1}(J))^2 \\
&\leq \left(1 + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} + (1+\lambda_{\max}^2)K_2CH_0^{2s}\right)\sum_{i\in J}\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} \\
&+ \left(B_2\left(1 + \frac{1}{2B_3}\right) + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^2 + (1+\lambda_{\max}^2)K_2C + C\lambda_{\max}^2\right)8B_0^2\omega^2(\mu^j(J))^2 \\
&+ \beta(1+\delta)\left(1 - \xi\theta^2\right)(\mu^j(J))^2.
\end{aligned}$$

We now split the θ^2 term and regroup to get

$$\begin{aligned}
&\left(1-\frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2}-(1+\lambda_{\max}^2)K_2CH_0^{2s}-\lambda_{\max}^2CH_0^{2s}\right)\sum_{i\in J}\|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2\\
&+\beta(\mu^{j+1}(J))^2\\
&\le\left(1+\frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2}+(1+\lambda_{\max}^2)K_2CH_0^{2s}\right)\sum_{i\in J}\|\nabla_\gamma e_i^{j}\|_{L_2(\gamma)}^2\\
&-\left(\beta(1+\delta)\frac{\xi\theta^2}{2}\right)(\mu^j(J))^2\\
&+\left(B_2\left(1+\frac{1}{2B_3}\right)+\frac{CK_1}{2B_3}\lambda_{\max}^2H_0^2+(1+\lambda_{\max}^2)K_2C+C\lambda_{\max}^2\right)8B_0^2\omega^2(\mu^j(J))^2\\
&+\beta(1+\delta)\left(1-\frac{\xi\theta^2}{2}\right)(\mu^j(J))^2.
\end{aligned} \tag{3.76}$$

Since $\omega<\min\{\omega_1,\omega_2\}$, (3.51) holds:

$$\sum_{i\in J}\|\nabla_\gamma e_i^j\|_{L_2(\gamma)}^2\le(2C_1+C_2)\,(\mu^j(J))^2. \tag{3.77}$$

Bounding the $(\mu^j(J))^2$ in the $-\left(\beta(1+\delta)\frac{\xi\theta^2}{2}\right)(\mu^j(J))^2$ term by (3.77) and replacing β using (3.72) then gives

$$\begin{aligned}
&\epsilon(H_0)\left(\sum_{i\in J}\|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2+\beta(\mu^{j+1}(J))^2\right)\\
&\le\left(1+\frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2}+(1+\lambda_{\max}^2)K_2CH_0^{2s}-\frac{\delta}{2B_3\,(2C_1+C_2)}\frac{\xi\theta^2}{2}\right)\sum_{i\in J}\|\nabla_\gamma e_i^{j}\|_{L_2(\gamma)}^2\\
&+\frac{1}{\beta}\left(B_2\left(1+\frac{1}{2B_3}\right)+\frac{CK_1}{2B_3}\lambda_{\max}^2H_0^2+(1+\lambda_{\max}^2)K_2C+C\lambda_{\max}^2\right)8B_0^2\omega^2\beta(\mu^j(J))^2\\
&+(1+\delta)\left(1-\frac{\xi\theta^2}{2}\right)\beta(\mu^j(J))^2,
\end{aligned} \tag{3.78}$$

where $\epsilon(H_0):=1-\frac{CK_1}{2B_3}\lambda_{\max}^2H_0^{2s+2}-(1+\lambda_{\max}^2)K_2CH_0^{2s}-\lambda_{\max}^2CH_0^{2s}$ is a monotonic function of H_0 and $\epsilon(H_0) < 1$. Inequality (3.78) then implies

$$\epsilon(H_0)\left(\sum_{i\in J}\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} + \beta(\mu^{j+1}(J))^2\right) \le \alpha_1(\delta)^2\sum_{i\in J}\|\nabla_\gamma e_i^j\|^2_{L_2(\gamma)} + \alpha_2(\delta)^2\beta(\mu^j(J))^2,$$

where

$$\alpha_1(\delta)^2 := 1 + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} + (1+\lambda_{\max}^2)K_2CH_0^{2s} - \frac{\delta}{2B_3\left(2C_1+C_2\right)}\frac{\xi\theta^2}{2}$$

$$\alpha_2(\delta)^2 := \left(B_2\left(1+\frac{1}{2B_3}\right) + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^2 + (1+\lambda_{\max}^2)K_2C + C\lambda_{\max}^2\right)\frac{8B_0\omega^2}{\beta} + (1+\delta)\left(1-\frac{\xi\theta^2}{2}\right).$$

We now choose δ such that

$$(1+\delta)\left(1-\frac{\xi\theta^2}{2}\right) = 1-\frac{\xi\theta^2}{4} \quad\implies\quad \delta = \frac{\xi\theta^2}{4-2\xi\theta^2}. \tag{3.79}$$

From (3.72) we then have $\beta = \frac{\xi\theta^2}{2B_3(4-\xi\theta^2)} \ge \frac{\xi\theta^2}{8B_3}$ which implies

$$\alpha_2(\delta)^2 \le \left(B_2\left(1+\frac{1}{2B_3}\right) + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^2 + (1+\lambda_{\max}^2)K_2C + C\lambda_{\max}^2\right)\frac{64B_3B_0\omega^2}{\xi\theta^2} + 1 - \frac{\xi\theta^2}{4}.$$

The assumption that $\omega \le \omega_3$ then guarantees that $\alpha_2^2 \le 1 - \frac{\xi\theta^2}{8} < 1$. With the choice of δ in (3.79) we have

$$\alpha_1^2 \le 1 + \frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} + (1+\lambda_{\max}^2)K_2CH_0^{2s} - \frac{1}{4B_3\left(2C_1+C_2\right)}\frac{\xi^2\theta^4}{4-2\xi\theta^2}.$$

We choose H_0 small enough so that

$$\frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} + (1+\lambda_{\max}^2)K_2CH_0^{2s} < \frac{1}{8B_3\left(2C_1+C_2\right)}\frac{\xi^2\theta^4}{4-2\xi\theta^2}. \tag{3.80}$$

This then implies that

$$\alpha_1^2 < 1 - \frac{1}{8B_3\left(2C_1 + C_2\right)} \frac{\xi^2\theta^4}{4 - 2\xi\theta^2} < 1.$$

We then have

$$\begin{aligned}
\alpha_1^2, \alpha_2^2 &\le \max\left\{1 - \frac{1}{8B_3\left(2C_1 + C_2\right)} \frac{\xi^2\theta^4}{4 - 2\xi\theta^2}, 1 - \frac{\xi\theta^2}{8}\right\} \\
&= 1 - \min\left\{\frac{1}{8B_3\left(2C_1 + C_2\right)} \frac{\xi^2\theta^4}{4 - 2\xi\theta^2}, \frac{\xi\theta^2}{8}\right\}.
\end{aligned}$$

For a contraction property we then need H_0 small enough so that

$$\frac{1 - \min\left\{\frac{1}{8B_3(2C_1+C_2)} \frac{\xi^2\theta^4}{4-2\xi\theta^2}, \frac{\xi\theta^2}{8}\right\}}{\epsilon(H_0)} < 1,$$

or

$$\frac{CK_1}{2B_3}\lambda_{\max}^2 H_0^{2s+2} + (1 + \lambda_{\max}^2)K_2 C H_0^{2s} + \lambda_{\max}^2 C H_0^{2s} < \min\left\{\frac{1}{8B_3\left(2C_1 + C_2\right)} \frac{\xi^2\theta^4}{4 - 2\xi\theta^2}, \frac{\xi\theta^2}{8}\right\}, \tag{3.81}$$

which also implies (3.80). We then have that there exists an α satisfying

$$\sum_{i\in J} \|\nabla_\gamma e_i^{j+1}\|_{L_2(\gamma)}^2 + \beta(\mu^{j+1}(J))^2 \le \alpha\left(\sum_{i\in J} \|\nabla_\gamma e_i^{j}\|_{L_2(\gamma)}^2 + \beta(\mu^{j}(J))^2\right)$$

which completes the contraction proof.

The contraction property guarantees that ADAPT_EIGENFUNCTION stops in a finite number of iterations I. To show that I is independent of the outer iteration counter k, take $k \ge 1$ and note that before the call to ADAPT_EIGENFUNCTION($\mathcal{T}_k^+, \epsilon_k$) in AFEM, we have

$$\eta_k := \eta_{\mathcal{T}_k}(J) \le \epsilon_{k-1} = \frac{\epsilon_k}{\rho}, \qquad \zeta_k := \zeta_{\mathcal{T}_k}(\gamma) \le B_0 \zeta_{\mathcal{T}_{k-1}^+}(\gamma) \le \frac{B_0\omega}{\rho}\epsilon_k.$$

From (3.52) with $\delta = 1$ and (3.63) we have

$$\mu_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j)^2 \leq 2\mu_{\mathcal{T}}(\lambda_j \mathbf{P} u_j, \mathbf{Z} u_j)^2 + 2B_3 \|\nabla_\gamma(\mathbf{Z}_* u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)} + 2B_2 \zeta_{\mathcal{T}}(\gamma)^2 + 2K_1 H_0^2 \lambda_j^2 \|\mathbf{P}_* u_j - \mathbf{P} u_j\|^2_{L_2(\gamma)}, \tag{3.82}$$

$$\|\nabla_\gamma(\mathbf{Z}_* u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)} \leq 2\|\nabla_\gamma(u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)} + 2B_2 \zeta_{\mathcal{T}}(\gamma)^2 + 2\lambda_j^2 \|u_j - \mathbf{P}_* u_j\|^2_{L_2(\gamma)} + 2K_2(1+\lambda_j^2)\|\mathbf{Z}_* u_j - \mathbf{Z} u_j\|^2_{L_2(\gamma)}. \tag{3.83}$$

Combining (3.82) with (3.83) yields

$$\begin{aligned}\mu_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j)^2 \leq{}& 2\mu_{\mathcal{T}}(\lambda_j \mathbf{P} u_j, \mathbf{Z} u_j)^2 + 4B_3 \|\nabla_\gamma(u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)} \\ &+ (4B_3 B_2 + 2B_2)\zeta_{\mathcal{T}}(\gamma)^2 + 4B_3 \lambda_j^2 \|u_j - \mathbf{P}_* u_j\|^2_{L_2(\gamma)} \\ &+ 4B_3 K_2 (1+\lambda_j^2)\|\mathbf{Z}_* u_j - \mathbf{Z} u_j\|^2_{L_2(\gamma)} + 2K_1 H_0^2 \lambda_j^2 \|\mathbf{P}_* u_j - \mathbf{P} u_j\|^2_{L_2(\gamma)}.\end{aligned} \tag{3.84}$$

Applying (3.74) to (3.84) yields

$$\begin{aligned}\mu_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j)^2 \leq{}& 2\mu_{\mathcal{T}}(\lambda_j \mathbf{P} u_j, \mathbf{Z} u_j)^2 \\ &+ \left(4B_3 + 4B_3 K_2(1+\lambda_j^2) C H_0^{2s} + 2K_1 H_0^2 \lambda_j^2 C H_0^{2s}\right)\|\nabla_\gamma(u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)} \\ &+ \left(4B_3 \lambda_j^2 C + 4B_3 K_2 (1+\lambda_j^2) C + 2K_1 H_0^2 \lambda_j^2 C\right) H_0^{2s} \|\nabla_\gamma(u_j - \mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} \\ &+ \left(4B_3 B_2 + 2B_2 + 4B_3 \lambda_j^2 C + 4B_3 K_2(1+\lambda_j^2) C + 2K_1 H_0^2 \lambda_j^2 C\right)\zeta_{\mathcal{T}}(\gamma)^2.\end{aligned} \tag{3.85}$$

Applying Theorem 3.18 to $\|\nabla_\gamma(u_j - \mathbf{Z} u_j)\|^2_{L_2(\gamma)}$ and $\|\nabla_\gamma(u_j - \mathbf{Z}_* u_j)\|^2_{L_2(\gamma)}$ in (3.85) with

$K_0 H_0^{2s} \lambda_{\max}^2 \leq \frac{1}{2}$ yields

$$\begin{aligned}
&\mu_{\mathcal{T}_*}(\lambda \mathbf{P}_* u_j, \mathbf{Z}_* u_j)^2 \leq 2\mu_{\mathcal{T}}(\lambda \mathbf{P} u_j, \mathbf{Z} u_j)^2 \\
&+ \left(4B_3 + 4B_3 K_2(1+\lambda_j^2) C H_0^{2s} + 2K_1 H_0^2 \lambda_j^2 C H_0^{2s}\right)\left(2C_1 \mu_{\mathcal{T}}(\lambda \mathbf{P} u_j, \mathbf{Z} u_j)^2 + 2B_1 \zeta_{\mathcal{T}}(\gamma)^2\right) \\
&+ \left(4B_3 \lambda_j^2 C + 4B_3 K_2 (1+\lambda_j^2) C + 2K_1 H_0^2 \lambda_j^2 C\right) H_0^{2s} \\
&\times \left(2C_1 \mu_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j)^2 + 2B_0 B_1 \zeta_{\mathcal{T}}(\gamma)^2\right) \\
&+ \left(4B_3 B_2 + 2B_2 + 4B_3 \lambda_j^2 C + 4B_3 K_2 (1+\lambda_j^2) C + 2K_1 H_0^2 \lambda_j^2 C\right) \zeta_{\mathcal{T}}(\gamma)^2.
\end{aligned} \tag{3.86}$$

Rearranging terms and summing over the cluster then yields

$$\mu_0^2 := \mu_{\mathcal{T}_k^+}(J)^2 \lesssim \mu_k^2 + \zeta_k^2 \lesssim \epsilon_k.$$

The contraction property and the equivalence between μ and η within ADAPT_EIENFUNCTION then implies that the number of iterates I will be bounded independent of k. □

3.5 Approximation Classes

In this section we define our notions of total error and introduce an associated approximation class $\mathbb{A}_s'$ for our eigenfunctions. We then show that our approximation classes are equivalent to the $\mathbb{A}_s$ ones used in [35]. With this equivalence in approximation classes in hand, the partial characterizations of $\mathbb{A}_s$ in terms of Besov regularity from [35] carry over to our approximation classes $\mathbb{A}_s'$.

3.5.1 Total Error

We define the total error for an eigenpair (u_j, λ_j) for $j \in J$ as

$$E_{\mathcal{T}}(u_j, \lambda_j, \gamma)^2 := \|\nabla_\gamma (u_j - \mathbf{Z} u_j)\|_{L_2(\gamma)}^2 + Osc_{\mathcal{T}}(\lambda_j \mathbf{P} u_j, \mathbf{Z} u_j, \gamma)^2 + \zeta_{\mathcal{T}}(\gamma)^2. \tag{3.87}$$

We then have that the total error for the cluster is

$$E_{\mathcal{T}}(J,\gamma)^2 := \sum_{j\in J}\left(\|\nabla_\gamma(u_j-\mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j,\mathbf{Z}u_j,\gamma)^2\right) + |J|\zeta_{\mathcal{T}}(\gamma)^2.$$

or $E_{\mathcal{T}}(J,\gamma)^2 = \mathcal{E}_{\mathcal{T}}(\mathbf{Z},J)^2 + |J|\zeta_{\mathcal{T}}(\gamma)^2$. Mapping back to Ω we obtain the following equivalent notions of total error provided $\zeta_{\mathcal{T}_0}(\gamma)$ satisfies (3.13):

$$\widehat{E}_{\mathcal{T}}(u_j,\lambda_j,\gamma)^2 := \sum_{T\in\mathcal{T}}\|\widehat{\nabla}(u_j-\mathbf{Z}u_j)\|^2_{L_2(\widehat{T})} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j,\mathbf{Z}u_j,\gamma)^2 + \zeta_{\mathcal{T}}(\gamma)^2,$$
$$\widehat{E}_{\mathcal{T}}(J,\gamma)^2 := \sum_{j\in J}\left(\sum_{T\in\mathcal{T}}\|\widehat{\nabla}(u_j-\mathbf{Z}u_j)\|^2_{L_2(\widehat{T})} + Osc_{\mathcal{T}}(\lambda_j\mathbf{P}u_j,\mathbf{Z}u_j,\gamma)^2\right) + |J|\zeta_{\mathcal{T}}(\gamma)^2.$$

For a single eigenpair $(u_j,\lambda_j) \in \{(u_i,\lambda_i)\}_{i\in J}$ we define the quality of the best approximation based on the total error for meshes in $\mathbb{T}_N$ to be

$$\sigma'(N;u_j,\lambda_j,\gamma) := \inf_{\mathcal{T}\in\mathbb{T}_N}\widehat{E}_{\mathcal{T}}(u_j,\lambda_j,\gamma),$$

where $\mathbb{T}_N := \{\mathcal{T}\in\mathbb{T} : \#\mathcal{T} - \#\mathcal{T}_0 \leq N\}$. We define our approximation class to be

$$\mathbb{A}'_s := \{(u_j,\lambda_j,\gamma) : j\in J,\ |u_j,\lambda_j,\gamma|_{\mathbb{A}'_s} := \sup_{N\geq 1}(N^s\sigma'(N;u_j,\lambda_j,\gamma)) < \infty\}.$$

We will use the shorthand

$$|J,\gamma|^2_{\mathbb{A}'_s} := \sum_{j\in J}|u_j,\lambda_j,\gamma|^2_{\mathbb{A}'_s}. \tag{3.88}$$

3.5.2 Constructive Approximation of u and γ

Lemma 3.32 (Constructive Approximation of γ, Corollary 7.4 of [35])**.** *Let γ be globally of class W^1_∞ and be parameterized by $\chi \in [B^{1+td}_q(L_q(\Omega))]^F$ with $tq > 1$, $0 < q \leq \infty$, $td \leq n$. Let $\mathcal{T}_0$ have an admissible labeling. Then $\mathcal{T}_*$=ADAPT_SURFACE($\mathcal{T},\delta$) is t-optimal, i.e.*

$$\zeta_{\mathcal{T}_*} \leq \delta, \qquad \#\mathcal{M} \lesssim C_1(\gamma)^{1/t}\delta^{-1/t}$$

where $\mathcal{M}$ denotes the number of elements marked during the execution of the procedure ADAPT_SURFACE($\mathcal{T},\delta$) and $C_1(\gamma) \le |\chi|_{B_q^{1+td}(L_q(\Omega))}$.

Lemma 3.33 (Constructive Approximation of u, Corollary 7.5 of [35])**.** *Let $u \in H^1_{\#}(\gamma)$ be piecewise of class*

$B_p^{1+sd}(L_p(\Omega))$, namely $u \in [B_p^{1+sd}(L_p(\Omega))]^F$, with $s-1/p+1/2>0$, $0<p\le\infty$ and $0<sd\le n$. Let $\mathcal{T}_0$ have an admissible labeling. Then, given $\delta>0$ there exists a triangulation $\mathcal{T}\in\mathbb{T}$ such that

$$\inf_{v\in\mathbb{V}(\mathcal{T})} \|\widehat{\nabla}(u-V)\|_{L_2(\Omega)} \lesssim \delta, \qquad \#\mathcal{M} \lesssim C(u)^{1/s}\delta^{-1/s},$$

where $\mathcal{M}$ is the set of marked elements to create $\mathcal{T}$ and $C(u)=|u|_{B_p^{1+sd}(L_p(\Omega))}$.

We now define a notion of oscillation for any function $V\in\mathbb{V}(\mathcal{T})$. Let $T\in\mathcal{T}$, then we define

$$\begin{aligned} Osc_{\mathcal{T}}(V,T)^2 := h_T^2 &\left\|(\mathrm{id}-\Pi^2_{2n-2})\widehat{\mathrm{div}}(q_\Gamma\widehat{\nabla}V\mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\ &+ h_T\|(\mathrm{id}-\Pi^2_{2n-1})(q_\Gamma^+\widehat{\nabla}V^+(\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^-\widehat{\nabla}V^-(\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\|^2_{L_2(\partial\widehat{T})}, \end{aligned} \tag{3.89}$$

and

$$\begin{aligned} Osc_{\mathcal{T}}(V)^2 := \sum_{T\in\mathcal{T}} h_T^2 &\left\|(\mathrm{id}-\Pi^2_{2n-2})\widehat{\mathrm{div}}(q_\Gamma\widehat{\nabla}V\mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\ &+ h_T\|(\mathrm{id}-\Pi^2_{2n-1})(q_\Gamma^+\widehat{\nabla}V^+(\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^-\widehat{\nabla}V^-(\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\|^2_{L_2(\partial\widehat{T})}. \end{aligned}$$

Lemma 3.34 (Uniform Decay of $Osc_{\mathcal{T}}(V)$, Proposition 7.12 of [35])**.** *Let γ be globally of class W^1_∞ and be parameterized by $\chi\in[B_q^{1+td}(L_q(\Omega))]^F$ with $tq>1$, $0<q\le\infty$, $td\le n$. Let $\mathcal{T}_0$ have an admissible labeling and let $\mathcal{T}\ge\mathcal{T}_0$ be a refinement of $\mathcal{T}_0$. Then, for any tolerance $\delta>0$ there exists a subdivision $\mathcal{T}_\delta\in\mathbb{T}$ such that $\mathcal{T}_\delta\ge\mathcal{T}$ and*

$$\max_{V\in\mathbb{V}(\mathcal{T}_\delta)} \frac{Osc_{\mathcal{T}_\delta}(V)}{\|\widehat{\nabla}V\|_{L_2(\Omega)}} \lesssim \delta,\ \zeta_{\mathcal{T}_\delta}(\gamma)\lesssim\delta, \qquad \#\mathcal{M}\lesssim C_2(\gamma)^{1/t}\delta^{-1/t},$$

where $\mathcal{M}$ is the set of elements marked to create $\mathcal{T}_\delta$ from $\mathcal{T}$ and the constant $C_2(\gamma)$ depends on γ.

Lemma 3.35 (Proposition 7.1 of [35]). *Let* $\mathcal{T} := \bigcup_{i=1}^{F} \mathcal{T}^i$ *be created by successive bisection of* $\mathcal{T}_0$*, which has an admissible labeling. Let* $0 < p \le \infty$ *and let* $\mathbf{g} := \{g^i\}_{i=1}^{F} : \Omega \to \mathbb{R}^F$ *be a vector-valued function and* $\{\xi_{\mathcal{T}}(\mathbf{g}, T)\}_{T\in\mathcal{T}}$ *be corresponding local error estimators that satisfy*

$$\xi_{\mathcal{T}}(\mathbf{g}, T) \lesssim h_T^r |g^i|_T, \quad r > 0, \quad T \in \mathcal{T}^i, \quad 1 \le i \le F, \tag{3.90}$$

where $h_T := |\widehat{T}|^{1/d}$ *and* $\left(\sum_{i=1}^{F} \sum_{T\in\mathcal{T}^i} |g^i|_T^p\right)^{1/p} \le |\mathbf{g}|_\Omega$ *is a given semi-(quasi) norm.*

If $|\mathbf{g}|_\Omega < \infty$*, then the module* $GREEDY(\mathbf{g}, \mathcal{T}, \delta)$ *terminates in a finite number of steps and the number of elements marked* $\mathcal{M}$ *within* $GREEDY$ *satisfies*

$$\#\mathcal{M} \lesssim |\mathbf{g}|_\Omega^{\frac{dp}{d+rp}} \delta^{-\frac{dp}{d+rp}}.$$

Lemma 3.36 (Corollary 7.2 of [35]). *Let* $\xi_{\mathcal{T}}(\mathbf{g})$ *satisfy* (3.90) *with* $r = d(s - 1/p + 1/q) > 0$*. Let the initial subdivision* $\mathcal{T}_0$ *have an admissible labeling. Given* $\delta > 0$ *there exists a conforming mesh refinement* $\mathcal{T} \in \mathbb{T}$ *such that*

$$\|\xi_{\mathcal{T}}(\mathbf{g})\|_{\ell_q} \lesssim \delta, \qquad \#\mathcal{T} - \#\mathcal{T}_0 \lesssim \#\mathcal{M} \lesssim |\mathbf{g}|_\Omega^{1/s} \delta^{-1/s}.$$

Lemma 3.37. *Let* γ *be globally of class* W^1_∞ *and be parameterized by* $\chi \in [B_q^{1+td}(L_q(\Omega))]^F$ *with* $tq > 1$, $0 < q \le \infty$, $td \le n$*. Let* $\mathcal{T}_0$ *have an admissible labeling. Then, for any tolerance* $\delta > 0$ *there exists a triangulation* $\mathcal{T} \in \mathbb{T}$ *such that*

$$\sqrt{\sum_{T\in\mathcal{T}} h_T^2 \left\|(id - \Pi_{2n-2}^2)(\mathbf{P}u_j q)\right\|_{L_2(\gamma)}^2} \lesssim \delta, \qquad \#\mathcal{T} - \#\mathcal{T}_0 \lesssim C_3(\gamma)^{1/s} \delta^{-1/s},$$

where

$$C_3(\gamma) := \|\chi\|_{B_q^{1+td}(L_q(\Omega))} + \|\chi\|_{B_q^{1+td}(L_q(\Omega))}^k.$$

Proof. We first note the following bound from Lemma 3.2 of [30],

$$\|(\mathrm{id}-\Pi_n^2)(vV)\|_{L_2(\omega)} \le \|(\mathrm{id}-\Pi_{n-m}^\infty)v\|_{L_\infty(\omega)}\|V\|_{L_2(\omega)}, \tag{3.91}$$

for $0 \le m \le n$, $V \in \mathbb{P}_m(\omega)$, and $v \in L_\infty(\omega)$ for any domain ω of $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$. Using (3.91) we have

$$\begin{aligned} h_T \left\|(\mathrm{id}-\Pi_{2n-2}^2)(\mathbf{P}u_j q)\right\|_{L_2(\widehat{T})} &\lesssim h_T \left\|(\mathrm{id}-\Pi_{n-2}^\infty)q\right\|_{L_\infty(\widehat{T})} \|\mathbf{P}u_j\|_{L_2(\widehat{T})} \\ &\lesssim h_T \left\|(\mathrm{id}-\Pi_{n-2}^\infty)q\right\|_{L_\infty(\widehat{T})} \|\mathbf{P}u_j\|_{L_2(\widehat{T})} \\ &\lesssim h_T^r \, |q|_{B_q^{td}(L_q(\widehat{T}))} \|\mathbf{P}u_j\|_{L_2(\widehat{T})}, \end{aligned}$$

with $r = td - d/q$ and $0 < r \le n$. The rest follows from Lemmas 3.35 and 3.36. □

3.5.3 Membership in $\mathbb{A}'_s$

We now state a bound for the oscillation in terms of quantities we have shown to be t-optimal in Section 6.2.

Lemma 3.38. *Let $(u_j, \lambda_j) \in \{(u_i, \lambda_i)\}_{i \in J}$, $V \in \mathbb{V}(\mathcal{T})$, and the assumptions of Lemma 3.17 be satisfied. Then*

$$\begin{aligned} &Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2 \\ &\qquad \lesssim \sum_{T \in \mathcal{T}} \lambda_j^2 h_T^2 \left\|(id-\Pi_{2n-2}^2)(\mathbf{P}u_j q_\Gamma)\right\|_{L_2(\widehat{T})}^2 + Osc_{\mathcal{T}}(V)^2 + \|\nabla_\gamma(u_j - V)\|_{L_2(\gamma)}^2 + \zeta_{\mathcal{T}}(\gamma). \end{aligned} \tag{3.92}$$

Proof. By Lemma 3.17 we have the following H^1 a priori bound

$$\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 \lesssim \|\nabla_\gamma(u_j - \mathbf{R}u_j)\|_{L_2(\gamma)}^2 + \zeta_{\mathcal{T}}(\gamma)^2.$$

By equivalence of norms and the definition of $\mathbf{R}$ we have for $V \in \mathbb{V}(\mathcal{T})$

$$\|\nabla_\gamma(u_j - \mathbf{R}u_j)\|^2_{L_2(\gamma)} \lesssim \|\nabla_\gamma(u_j - V)\|^2_{L_2(\gamma)},$$

and so

$$\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} \lesssim \|\nabla_\gamma(u_j - V)\|^2_{L_2(\gamma)} + \zeta_{\mathcal{T}}(\gamma)^2. \tag{3.93}$$

By the definition of $Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)$ and standard arguments we have

$$\begin{aligned}
Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2 &= \sum_{T\in\mathcal{T}} h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\left(\lambda_j \mathbf{P}u_j q_\Gamma + \widehat{\mathrm{div}}(q_\Gamma \widehat{\nabla}\mathbf{Z}u_j \mathbf{G}_\Gamma^{-1})\right)\right\|^2_{L_2(\widehat{T})} \\
&\quad + h_T \left\|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla}\mathbf{Z}u_j^+ (\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla}\mathbf{Z}u_j^- (\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\right\|^2_{L_2(\partial\widehat{T})} \\
&\lesssim \sum_{T\in\mathcal{T}} h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\lambda_j \mathbf{P}u_j q_\Gamma\right\|^2_{L_2(\widehat{T})} + h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\widehat{\mathrm{div}}(q_\Gamma \widehat{\nabla}\mathbf{Z}u_j \mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\
&\quad + h_T \left\|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla}\mathbf{Z}u_j^+ (\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla}\mathbf{Z}u_j^- (\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\right\|^2_{L_2(\partial\widehat{T})} \\
&\lesssim \sum_{T\in\mathcal{T}} h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\lambda_j \mathbf{P}u_j q_\Gamma\right\|^2_{L_2(\widehat{T})} + h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-1})\widehat{\mathrm{div}}(q_\Gamma \widehat{\nabla}V \mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\
&\quad + h_T \left\|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla}V^+ (\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla}V^- (\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\right\|^2_{L_2(\partial\widehat{T})} \\
&\quad + h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\widehat{\mathrm{div}}(q_\Gamma \widehat{\nabla}(\mathbf{Z}u_j - V)\mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\
&\quad + h_T \left\|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla}(\mathbf{Z}u_j - V)^+ (\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla}(\mathbf{Z}u_j - V)^- (\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\right\|^2_{L_2(\partial\widehat{T})}.
\end{aligned}$$

Noting that since $(\mathrm{id} - \Pi^2_{2n-2})$ is a projection we have $\|(\mathrm{id} - \Pi^2_{2n-2})\|_{L_2(\widehat{T}) \to L_2(\widehat{T})} \leq 1$ and applying inverse inequalities yields

$$
\begin{aligned}
Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2 \lesssim \sum_{T\in\mathcal{T}} & h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})\lambda_j \mathbf{P}u_j q_\Gamma\right\|^2_{L_2(\widehat{T})} \\
& + h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-1})\widehat{\mathbf{div}}(q_\Gamma \widehat{\nabla} V \mathbf{G}_\Gamma^{-1})\right\|^2_{L_2(\widehat{T})} \\
& + h_T \|(\mathrm{id} - \Pi^2_{2n-1})(q_\Gamma^+ \widehat{\nabla} V^+ (\mathbf{G}_\Gamma^+)^{-1}\mathbf{n}^+ + q_\Gamma^- \widehat{\nabla} V^- (\mathbf{G}_\Gamma^-)^{-1})\mathbf{n}^-\|^2_{L_2(\partial\widehat{T})} \\
& + \|\nabla_\gamma(\mathbf{Z}u_j - V)\|^2_{L_2(\gamma)} \\
& \lesssim \sum_{T\in\mathcal{T}} \lambda_j^2 h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P}u_j q_\Gamma)\right\|^2_{L_2(\widehat{T})} + Osc_{\mathcal{T}}(V)^2 + \|\nabla_\gamma(\mathbf{Z}u_j - V)\|^2_{L_2(\gamma)} \\
& \lesssim \sum_{T\in\mathcal{T}} \lambda_j^2 h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P}u_j q)\right\|^2_{L_2(\widehat{T})} + \lambda_j^2 h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P}u_j (q_\Gamma - q))\right\|^2_{L_2(\widehat{T})} \\
& + Osc_{\mathcal{T}}(V)^2 + \|\nabla_\gamma(\mathbf{Z}u_j - V)\|^2_{L_2(\gamma)} \\
& \lesssim \sum_{T\in\mathcal{T}} \lambda_j^2 h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P}u_j q)\right\|^2_{L_2(\widehat{T})} + \lambda_j^2 H_0^2 \|q_\Gamma - q\|^2_{L_\infty(\gamma)} + Osc_{\mathcal{T}}(V)^2 \\
& + \|\nabla_\gamma(\mathbf{Z}u_j - V)\|^2_{L_2(\gamma)} \\
& \lesssim \sum_{T\in\mathcal{T}} \lambda_j^2 h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P}u_j q_\Gamma)\right\|^2_{L_2(\widehat{T})} + Osc_{\mathcal{T}}(V)^2 + \|\nabla_\gamma(u_j - V)\|^2_{L_2(\gamma)} \\
& + \zeta_{\mathcal{T}}(\gamma)^2,
\end{aligned}
$$

where in the last line we used (3.93) and (3.15). □

From (3.92) and (3.93) we immediately get that the total error (3.87) is bounded by the optimal terms in Lemmas 3.32, 3.34, and 3.37.

Corollary 3.39. *Let $(u_j, \lambda_j) \in \{(u_j, \lambda_j)\}_j \in J$, $V \in \mathbb{V}(\mathcal{T})$, and the assumptions of Lemma 3.17 be satisfied. Then*

$$
\begin{aligned}
E_{\mathcal{T}}(u_j, \lambda_j, \gamma)^2 &= \|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2 + \zeta_{\mathcal{T}}(\gamma)^2 \\
&\lesssim \sum_{T\in\mathcal{T}} \|\widehat{\nabla}(u_j - V)\|^2_{L_2(\widehat{T})} + h_T^2 \lambda_j^2 \left\|(id - \Pi^2_{2n-2})(\mathbf{P}u_j q_\Gamma)\right\|^2_{L_2(\widehat{T})} \qquad (3.94) \\
&+ Osc_{\mathcal{T}}(V)^2 + \zeta_{\mathcal{T}}(\gamma)^2.
\end{aligned}
$$

Theorem 3.40 (Membership in $\mathbb{A}'_s$)**.** *Let H_0 satisfy* (H1). *Let γ be globally of class W^1_∞ and be parameterized by $\chi \in [B_q^{1+td}(L_q(\Omega))]^F$ with $tq > 1$, $0 < q \leq \infty$, $td \leq n$, and let $k := \lfloor td \rfloor + 1$. Let $u \in H^1_\#(\gamma)$ be piecewise of class $B_p^{1+sd}(L_p(\Omega))$, namely $u \in [B_p^{1+sd}(L_p(\Omega))]^F$, with $s - 1/p + 1/2 > 0$, $0 < p \leq \infty$, and $0 < sd \leq n$. Let $\mathcal{T}_0$ have an admissible labeling and $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy* (3.13)*. Then,*

$$(u_j, \lambda_j, \gamma) \in \mathbb{A}'_s,$$

i.e. given a $\delta > 0$ there exists a conforming refinement $\mathcal{T}$ such that

$$E_{\mathcal{T}}(u_j, \lambda_j, \gamma) \lesssim \delta, \qquad \#\mathcal{T} - \#\mathcal{T}_0 \lesssim |u_j, \lambda_j, \gamma|_{\mathbb{A}'_{s\wedge t}}^{\frac{1}{s\wedge t}} \delta^{-\frac{1}{s\wedge t}}.$$

Proof of Theorem 3.40. By Lemmas 3.33 and 3.37, there exist triangulations $\mathcal{T}_u, \mathcal{T}_q \in \mathbb{T}$ such that

$$\inf_{V \in \mathbb{V}(\mathcal{T}_u)} \|\widehat{\nabla}(u - V)\|_{L_2(\Omega)} \lesssim \delta, \qquad \#\mathcal{M}_u \lesssim \delta^{-1/s},$$

$$\sqrt{\sum_{T \in \mathcal{T}} h_T^2 \left\|(\mathrm{id} - \Pi^2_{2n-2})(\mathbf{P} u_j q)\right\|^2_{L_2(\gamma)}} \lesssim \delta, \qquad \#\mathcal{T}_q - \#\mathcal{T}_0 \lesssim C_3(\gamma)^{1/s} \delta^{-1/s},$$

By Lemma 3.34 there exists a mesh $\mathcal{T}_\gamma \in \mathbb{T}$ with $\mathcal{T}_\gamma \geq \mathcal{T}_u$ such that

$$\max_{V \in \mathbb{V}(\mathcal{T}_\gamma)} \frac{Osc_{\mathcal{T}_\gamma}(V)}{\|\widehat{\nabla} V\|_{L_2(\Omega)}} \lesssim \delta, \; \zeta_{\mathcal{T}_\gamma}(\gamma) \lesssim \delta, \qquad \#\mathcal{M}_\gamma \lesssim C_2(\gamma)^{1/t} \delta^{-1/t}.$$

By Lemma 3.29 we then have that

$$\begin{aligned} \#\mathcal{T}_\gamma - \#\mathcal{T}_0 &\lesssim \#\mathcal{M}_\gamma + \#\mathcal{M}_u \lesssim C(\gamma)^{1/t} \delta^{-1/t} + C(u)^{1/s} \delta^{-1/s} \\ &\lesssim (C(\gamma)^{1/(s\wedge t)} + C(u_j, \lambda_j)^{1/(s\wedge t)}) \delta^{-1/(s\wedge t)}. \end{aligned}$$

Let $\mathcal{T} = \mathcal{T}_q \oplus \mathcal{T}_\gamma$ be the overlay of the meshes $\mathcal{T}_q$ and $\mathcal{T}_\gamma$. By Lemma 3.7 of [30] the cardinality

$\#\mathcal{T} - \#\mathcal{T}_0$ is bounded by $\#\mathcal{T}_q + \#\mathcal{T}_\gamma - 2\#\mathcal{T}_0$, whence

$$\#\mathcal{T} - \#\mathcal{T}_0 \lesssim |u_j, \lambda_j, \gamma|_{\mathbb{A}'_{s\wedge t}}^{\frac{1}{s\wedge t}} \delta^{-\frac{1}{s\wedge t}}$$

By (3.94) $E_{\mathcal{T}_\gamma}(u_j, \lambda_j, \gamma) \lesssim \delta$ which implies $(u_j, \lambda_j, \gamma) \in \mathbb{A}'_s$. □

3.6 Convergence Rates

Define

$$\omega_4 := \frac{C_5}{\sqrt{\left(\frac{9B_1}{2} + 2B_3B_2 + 5B_2 + 3CK_2 + \left(\frac{9K_0}{2} + 3K_2 + 3 + 2B_3K_0 + 2K_1H_0^2\right)\lambda_{\max}^2\right)8CB_0^2|J|}} \tag{W4}$$

and

$$\theta_* := \frac{C_5}{\sqrt{2\left(\frac{9}{4}C_1 + C_3 + C_1B_3\right)}}. \tag{3.95}$$

Lemma 3.41 (Dörfler Marking)**.** *Let H_0 satisfy* (H1)*,* (H2)*,* (H3)*,*

$$(B_3K_0 + K_1H_0^2)\lambda_{\max}^2 CH_0^{2s} \leq 0.2, \tag{H5}$$

and

$$\left(K_2(1 + \lambda_{\max}^2) + \lambda_{\max}^2 + \frac{3}{2}K_0\lambda_{\max}^2\right) CH_0^{2s} \leq 0.2. \tag{H6}$$

Let $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy (3.13)*, and the parameter θ and ω satisfy*

$$0 < \theta < \theta_*, \qquad 0 < \omega \leq \min\{\omega_1, \omega_2, \omega_4\},$$

where θ_*, ω_4 are defined in (3.95) *and* (W4)*. Let $\kappa := \frac{1}{2}\sqrt{1 - \frac{\theta^2}{\theta_*^2}}$ and $(\Gamma, \mathcal{T}, \mathbf{Z})$ be the approximate surface, mesh, and discrete projection operator produced by an inner iterate of*

ADAPT_EIGENFUNCTION. If $(\Gamma_*, \mathcal{T}_*, \mathbf{Z}_*)$ *is a surface-mesh-resolution triple with* $\mathcal{T}_* \geq \mathcal{T}$*, such that the eigenfunction error satisfies*

$$\mathcal{E}_{\mathcal{T}_*}(\mathbf{Z}_*, J) \leq \kappa \mathcal{E}_{\mathcal{T}}(\mathbf{Z}, J), \tag{3.96}$$

then the refined set $\mathcal{R} := \mathcal{T} \setminus \mathcal{T}_*$ *satisfies Dörfler property with parameter* θ*, namely*

$$\mu_{\mathcal{T}}(\mathbf{Z}, \mathcal{R}, J) \geq \theta \mu_{\mathcal{T}}(J). \tag{3.97}$$

Proof. Since $\omega \leq \min\{\omega_1, \omega_2\}$ and $\lambda_{\max}^2 K_0 H_0^{2s} \leq \frac{1}{2}$, we combine the lower bound of (3.50) with (3.96) to get

$$\begin{aligned}(1-2\kappa^2)C_5\mu_{\mathcal{T}}^2 &\leq (1-2\kappa^2)\sum_{j\in J}(\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2 + Osc_{\mathcal{T}}(\mathbf{Z}u_j,\gamma)^2) \\ &\leq \sum_{j\in J}\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2 - 2\|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 + Osc_{\mathcal{T}}(\mathbf{Z}u_j,\gamma)^2 - 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2\end{aligned} \tag{3.98}$$

We begin by bounding the oscillation terms. From (3.48) we know that

$$Osc_{\mathcal{T}}(\mathbf{Z}u_j, \mathcal{R})^2 \leq C_3\mu_{\mathcal{T}}(\mathbf{Z}u_j, \mathcal{R})^2.$$

For $\mathcal{T}_* \cap \mathcal{T}$, (3.57) with the roles of $\mathcal{T}_*$ and $\mathcal{T}$ reversed, yields

$$\begin{aligned}Osc_{\mathcal{T}}(\mathbf{Z}u_j, \mathcal{T}\cap\mathcal{T}_*)^2 - 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2 &\leq B_3\|\nabla_\gamma(\mathbf{Z}_*u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 + B_2\zeta_{\mathcal{T}}(\gamma)^2 \\ &\quad + K_1H_0^2\lambda_j^2\|\mathbf{P}_*u_j - \mathbf{P}u_j\|_{L_2(\gamma)}^2.\end{aligned}$$

Combining these estimates with (3.37) and noting that $\mathcal{T} = (\mathcal{T} \cap \mathcal{T}_*) \cup \mathcal{R}$ yields

$$\begin{aligned} Osc_{\mathcal{T}}(\mathbf{Z}u_j,\gamma)^2 - 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2 &= Osc_{\mathcal{T}}(\mathbf{Z}u_j,\mathcal{R})^2 + Osc_{\mathcal{T}}(\mathbf{Z}u_j,\mathcal{T}\cap\mathcal{T}_*)^2 \\ &- 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2 \\ &\le (C_3 + B_3C_1)\mu_{\mathcal{T}}(\mathbf{Z}u_j,\mathcal{R})^2 + (B_3B_2 + B_2)\zeta_{\mathcal{T}}(\gamma)^2 \\ &+ (B_3K_0 + K_1H_0^2)\lambda_j^2\|(\mathbf{P}_*u_j - \mathbf{P}u_j\|_{L_2(\gamma)}^2. \end{aligned}$$

Applying (3.74) to the $\|(\mathbf{P}_*u_j - \mathbf{P}u_j\|_{L_2(\gamma)}^2$ term then gives

$$\begin{aligned} Osc_{\mathcal{T}}(\mathbf{Z}u_j,\gamma)^2 - 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2 &\le (C_3 + B_3C_1)\mu_{\mathcal{T}}(\mathbf{Z}u_j,\mathcal{R})^2 \\ &+ (B_3B_2 + B_2 + C[B_3K_0\lambda_j^2 + K_1\lambda_j^2H_0^2])\zeta_{\mathcal{T}}(\gamma)^2 \\ &+ (B_3K_0 + K_1H_0^2)\lambda_j^2CH_0^{2s}\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 \\ &+ (B_3K_0 + K_1H_0^2)\lambda_j^2CH_0^{2s}\|\nabla_\gamma(u_j - \mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2. \end{aligned} \tag{3.99}$$

Using (3.99) in (3.98) then yields

$$\begin{aligned} (1-2\kappa^2)C_5\mu_{\mathcal{T}}^2 &\le \sum_{j\in J}\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2 - 2\|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \\ &+ Osc_{\mathcal{T}}(\mathbf{Z}u_j,\gamma)^2 - 2Osc_{\mathcal{T}_*}(\mathbf{Z}_*u_j,\gamma)^2 \\ &\le \sum_{j\in J}(C_3 + C_1B_3)\mu_{\mathcal{T}}(\mathbf{Z}u_j,\mathcal{R})^2 \\ &+ \sum_{j\in J}\left(1 + (B_3K_0 + K_1H_0^2)\lambda_j^2CH_0^{2s}\right)\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 \\ &- \sum_{j\in J}\left(2 - (B_3K_0 + K_1H_0^2)\lambda_j^2CH_0^{2s}\right)\|\nabla_\gamma(u_j - \mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \\ &+ |J|\left(B_3B_2 + B_2 + CB_3K_0\lambda_j^2 + CK_1\lambda_j^2H_0^2\right)\zeta_{\mathcal{T}}(\gamma)^2 \end{aligned}$$

We enforce that $(B_3K_0 + K_1H_0^2)\lambda_{\max}^2 CH_0^{2s} \leq 0.2$ to get

$$
\begin{aligned}
(1-2\kappa^2)C_5\mu_{\mathcal{T}}^2 \leq & \sum_{j\in J}(C_3 + C_1B_3)\mu_{\mathcal{T}}(\mathbf{Z}u_j, \mathcal{R})^2 \\
& + \sum_{j\in J} 1.2\|\nabla_\gamma(u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 - 1.8\|\nabla_\gamma(u_j - \mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \\
& + |J|\left(B_3B_2 + B_2 + CB_3K_0\lambda_j^2 + CK_1\lambda_j^2H_0^2\right)\zeta_{\mathcal{T}}(\gamma)^2.
\end{aligned} \tag{3.100}
$$

From (3.64) we have

$$
\begin{aligned}
\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2 - \|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \leq & \frac{3}{2}\|\nabla_\gamma(\mathbf{Z}_*u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 + B_2\zeta_{\mathcal{T}}(\gamma)^2 \\
& + \lambda_j^2\|u_j - \mathbf{P}_*u_j\|_{L_2(\gamma)}^2 + K_2(1+\lambda_j^2)\|\mathbf{Z}_*u_j - \mathbf{Z}u_j\|_{L_2(\gamma)}^2.
\end{aligned}
$$

Reducing using (3.74) then gives

$$
\begin{aligned}
\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2 - \|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \leq & \frac{3}{2}\|\nabla_\gamma(\mathbf{Z}_*u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 \\
& + (B_2 + CK_2(1+\lambda_j^2) + C\lambda_j^2)\zeta_{\mathcal{T}}(\gamma)^2 \\
& + (K_2(1+\lambda_j^2) + \lambda_j^2)CH_0^{2s}\|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \\
& + (K_2(1+\lambda_j^2) + \lambda_j^2)CH_0^{2s}\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2
\end{aligned} \tag{3.101}
$$

From (3.37) and (3.74) we have

$$
\begin{aligned}
\|\nabla_\gamma(\mathbf{Z}_*u_j - \mathbf{Z}u_j)\|_{L_2(\gamma)}^2 & \leq C_1\mu_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 + B_1\zeta_{\mathcal{T}}(\gamma)^2 + \lambda_j^2K_0\|\mathbf{P}_*u_j - \mathbf{P}u_j\|_{L_2(\gamma)}^2 \\
& \leq C_1\mu_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 + (B_1 + CK_0\lambda_j^2)\zeta_{\mathcal{T}}(\gamma)^2 \\
& + CK_0\lambda_j^2H_0^{2s}\|\nabla_\gamma e(\mathbf{Z}_*u_j)\|_{L_2(\gamma)}^2 \\
& + CK_0\lambda_j^2H_0^{2s}\|\nabla_\gamma e(\mathbf{Z}u_j)\|_{L_2(\gamma)}^2.
\end{aligned} \tag{3.102}
$$

Combining (3.101) and (3.102) then gives

$$\begin{aligned}\|\nabla_\gamma e(\mathbf{Z}u_j)\|^2_{L_2(\gamma)} - \|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} \le \frac{3}{2}C_1\mu_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 \\ + \left(\frac{3}{2}B_1 + \frac{3}{2}CK_0\lambda_j^2 + B_2 + CK_2(1+\lambda_j^2) + C\lambda_j^2\right)\zeta_{\mathcal{T}}(\gamma)^2 \\ + \left(K_2(1+\lambda_j^2) + \lambda_j^2 + \frac{3}{2}K_0\lambda_j^2\right) CH_0^{2s}\|\nabla_\gamma e_i^{j+1}\|^2_{L_2(\gamma)} \\ + \left(K_2(1+\lambda_j^2) + \lambda_j^2 + \frac{3}{2}K_0\lambda_j^2\right) CH_0^{2s}\|\nabla_\gamma e_i^{j}\|^2_{L_2(\gamma)}.\end{aligned}$$

Using the H_0 assumption $\left(K_2(1+\lambda_{\max}^2) + \lambda_{\max}^2 + \frac{3}{2}K_0\lambda_{\max}^2\right) CH_0^{2s} \le 0.2$ then gives

$$\begin{aligned}0.8\|\nabla_\gamma e(\mathbf{Z}u_j)\|^2_{L_2(\gamma)} - 1.2\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} \le \frac{3}{2}C_1\mu_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 \\ + \left(\frac{3}{2}B_1 + \frac{3}{2}CK_0\lambda_j^2 + B_2 + CK_2(1+\lambda_j^2) + C\lambda_j^2\right)\zeta_{\mathcal{T}}(\gamma)^2,\end{aligned}$$

or

$$\begin{aligned}1.2\|\nabla_\gamma e(\mathbf{Z}u_j)\|^2_{L_2(\gamma)} - 1.8\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} \le \frac{9}{4}C_1\mu_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 \\ + \frac{3}{2}\left(\frac{3}{2}B_1 + \frac{3}{2}CK_0\lambda_j^2 + B_2 + CK_2(1+\lambda_j^2) + C\lambda_j^2\right)\zeta_{\mathcal{T}}(\gamma)^2.\end{aligned} \tag{3.103}$$

Combining (3.103) with (3.100) then gives

$$\begin{aligned}(1-2\kappa^2)C_5\mu_{\mathcal{T}}^2 \le \sum_{j\in J}\left(\frac{9}{4}C_1 + C_3 + C_1B_3\right)\mu_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 \\ + |J|\Bigg(\frac{9}{4}B_1 + \frac{9}{4}CK_0\lambda_{\max}^2 + \frac{3}{2}B_2 + \frac{3}{2}CK_2(1+\lambda_{\max}^2) \\ + \frac{3}{2}C\lambda_{\max}^2 + B_3B_2 + B_2 + CB_3K_0\lambda_{\max}^2 + CK_1\lambda_{\max}^2 H_0^2\Bigg)\zeta_{\mathcal{T}}(\gamma)^2.\end{aligned} \tag{3.104}$$

By the assumption $\omega \le \min\{\omega_1, \omega_2\}$, we have within ADAPT_EIGENFUNCTION that (3.47) holds which implies $\zeta_{\mathcal{T}}(\gamma) \leq \sqrt{8}B_0\omega\mu_{\mathcal{T}}(J)$. Using this inequality with (3.104) then gives

$$(1-2\kappa^2)C_5^2\mu_{\mathcal{T}}(J)^2 \leq \left(\frac{9}{4}C_1 + C_3 + C_1B_3\right)\sum_{j\in J}\mu_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2$$
$$+ \left(\frac{9}{4}B_1 + B_3B_2 + \frac{5}{2}B_2 + \frac{3}{2}CK_2\right.$$
$$\left. + \left(\frac{9}{4}CK_0 + \frac{3}{2}CK_2 + \frac{3}{2}C + CB_3K_0 + CK_1H_0^2\right)\lambda_{\max}^2\right)8B_0^2|J|\omega^2\mu_{\mathcal{T}}(J)^2.$$

By the assumption that $\omega \leq \omega_4$ and the definition of θ_* we then have

$$(1-2\kappa^2)C_5^2\mu_{\mathcal{T}}(J)^2 \leq \frac{C_5^2}{2\theta_*^2}\sum_{j\in J}\mu_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2 + \frac{C_5^2}{2}\mu_{\mathcal{T}}(J)^2$$

which implies

$$(1-4\kappa^2)\theta_*^2\mu_{\mathcal{T}}(J)^2 \leq \sum_{j\in J}\mu_{\mathcal{T}}(\lambda_j\mathbf{P}u_j, \mathbf{Z}u_j, \mathcal{R})^2.$$

The choice of κ implies the final result. □

Lemma 3.42. *Let H_0 satisfy* (H2). *Let $\mathcal{M} \subset \mathcal{T}$ and define*

$$\omega_5 := \frac{\theta}{\sqrt{6B_1|J|^2}}. \tag{W5}$$

Let $\omega \leq \omega_5$. Within the ADAPT_EIGENFUNCTION loop we have that if $\theta\mu_{\mathcal{T}}(J) \leq \mu_{\mathcal{T}}(\mathcal{M}, J)$, then

$$\frac{\theta}{\sqrt{72}}\eta_{\mathcal{T}}(J) \leq \eta_{\mathcal{T}}(\mathcal{M}, J).$$

Proof. From (3.43) we have

$$\theta^2\mu_{\mathcal{T}}(J)^2 \leq \mu_{\mathcal{T}}(\mathcal{M}, J)^2 \leq 3\eta_{\mathcal{T}}(\mathcal{M}, J)^2 + B_1|J|^2\zeta_{\mathcal{T}}(\gamma)^2. \tag{3.105}$$

Since $\omega \le \omega_5$ within ADAPT_EIGENFUNCTION we have $\omega^2 \le \frac{\theta^2}{6B_1|J|^2}$, which implies

$$\zeta_{\mathcal{T}}(\gamma)^2 \le \frac{\theta^2}{6B_1|J|^2}\eta_{\mathcal{T}}(J)^2. \tag{3.106}$$

Combining (3.106) with (3.44) then gives

$$\frac{1}{4}\left(1-\frac{\theta^2}{6}\right)\theta^2\eta_{\mathcal{T}}(J)^2 = \frac{1}{4}\left(\theta^2-\frac{\theta^4}{6}\right)\theta^2\eta_{\mathcal{T}}(J)^2 \le \theta^2\mu_{\mathcal{T}}(J)^2. \tag{3.107}$$

Using (3.106) and (3.107) in (3.105) and noting that $\theta^4 \le \theta^2$ for $0 < \theta < 1$ then gives

$$\frac{1}{3}\left(\frac{\theta^2}{4}-\frac{\theta^4}{24}-\frac{\theta^2}{6}\right)\eta_{\mathcal{T}}(J)^2 \le \frac{1}{3}\left(\frac{\theta^2}{4}-\frac{\theta^2}{24}-\frac{\theta^2}{6}\right)\eta_{\mathcal{T}}(J)^2 = \frac{\theta^2}{72}\eta_{\mathcal{T}}(J)^2 \le \eta_{\mathcal{T}}(\mathcal{M},J)^2,$$

which completes the proof. □

Lemma 3.43 (Cardinality of $\mathcal{M}$)**.** *Let H_0 satisfy* (H1)*,* (H2)*,* (H3)*,* (H5)*,* (H6)*,*

$$C(1+2B_3)\lambda_{\max}^2 H_0^{2s} + C(1+2B_3)K_2(1+\lambda_{\max}^2)H_0^{2s} + CK_1\lambda_{\max}^2 H_0^{2+2s} \le \frac{1}{2}, \tag{H7}$$

and

$$CK_1\lambda_{\max}^2 H_0^2 \le \frac{1}{2}. \tag{H8}$$

Let $\zeta_{\mathcal{T}_0}(\gamma)$ satisfy (3.13) *and the procedure MARK select a set $\mathcal{M}$ with minimum cardinality and bulk parameter $\frac{\theta}{\sqrt{72}}$. Let the parameters θ and ω satisfy*

$$0 < \theta < \theta_*, \qquad 0 < \omega \le \min\{\omega_1, \omega_2, \omega_4, \omega_5\}$$

with θ_*, ω_1, ω_2, ω_4, and ω_5 given in (3.95)*,* (W1)*,* (W2)*,* (W4)*, and* (W5) *respectively. Let $\{(u_j,\lambda_j)\}_{j\in J}$ be solutions of* (3.5)*, and let $(\Gamma, \mathcal{T}, \mathbf{Z})$ be produced within*

ADAPT_EIGENFUNCTION. If $\{(u_j, \lambda_j, \gamma)\}_{j\in J} \in \mathbb{A}'_s$, *then*

$$\#\mathcal{M} \lesssim |J,\gamma|_{\mathbb{A}'_s}^{\frac{1}{s}} \mathcal{E}_{\mathcal{T}}(\mathbf{Z}, J)^{-\frac{1}{s}}.$$

Proof. We set

$$\delta^2 = \widehat{\kappa}^2 \mathcal{E}_{\mathcal{T}}(\mathbf{Z}, J)^2 = \widehat{\kappa}^2 \left(\sum_{j\in J} e(\mathbf{Z}u_j)^2 + Osc_{\mathcal{T}}(\mathbf{Z}u_j)^2 \right)$$

for $0 < \widehat{\kappa} < \kappa = \frac{1}{2}\sqrt{1 - \frac{\theta^2}{\theta_*^2}} < 1$ sufficiently small to be determined later. Throughout we shall use the shorthand notation $e(\mathbf{Z}u) = u - \mathbf{Z}u$. Since $(u_j, \lambda_j, \gamma) \in \mathbb{A}'_s$, there exists a subdivision $\mathcal{T}_\delta \in \mathbb{T}$ with projection $\mathbf{Z}_\delta$ satisfying

$$\#\mathcal{T}_\delta - \#\mathcal{T}_0 \lesssim |J,\gamma|_{\mathbb{A}'_s}^{\frac{1}{s}} \delta^{-\frac{1}{s}},$$

$$\widehat{E}_{\mathcal{T}_\delta}(J,\gamma)^2 = \sum_{j\in J} \|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|_{L_2(\gamma)}^2 + Osc_{\mathcal{T}_\delta}(\mathbf{Z}_\delta u_j)^2 + |J|\zeta_{\mathcal{T}_\delta}(\gamma)^2 \le \delta^2.$$

Let $\mathcal{T}_* = \mathcal{T} \oplus \mathcal{T}_\delta$ be the overlay of $\mathcal{T}$ and $\mathcal{T}_\delta$, i.e. the smallest common refinement which satisfies

$$\#\mathcal{T}_* \le \#\mathcal{T} + \#\mathcal{T}_\delta - \#\mathcal{T}_0 \tag{3.108}$$

Let $\mathbf{Z}_*$ be the projection operator associated with $\mathcal{T}_*$. We observe that $\mathcal{T}_* \ge \mathcal{T}_\delta, \mathcal{T}$, and invoke the upper bound (3.63) along with (3.56) to get

$$\begin{aligned} \|\nabla_\gamma e(\mathbf{Z}_* u_j)\|_{L_2(\gamma)}^2 + Osc_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j, \gamma)^2 \le{}& \|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|_{L_2(\gamma)}^2 + B_2 \zeta_{\mathcal{T}_\delta}(\gamma)^2 \\ &+ \lambda_j^2 \|u_j - \mathbf{P}_* u_j\|_{L_2(\gamma)}^2 + K_2(1+\lambda_j^2)\|\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j\|_{L_2(\gamma)}^2 \\ &+ C_6 Osc_{\mathcal{T}_\delta}(\lambda_j \mathbf{P}_\delta u_j, \mathbf{Z}_\delta u_j, \gamma)^2 + B_3 \|\nabla_\gamma(\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j)\|_{L_2(\gamma)}^2 \\ &+ B_2 \zeta_{\mathcal{T}_\delta}(\gamma)^2 + K_1 H_0^2 \lambda_j^2 \|\mathbf{P}_* u_j - \mathbf{P}_\delta u_j\|_{L_2(\gamma)}^2. \end{aligned} \tag{3.109}$$

Using (3.63) once again to bound $\|\nabla_\gamma(\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j)\|_{L_2(\gamma)}$ gives

$$\begin{aligned}\|\nabla_\gamma(\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j)\|^2_{L_2(\gamma)} &\le 2\|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|^2_{L_2(\gamma)} + 2B_2\zeta_{\mathcal{T}_\delta}(\gamma)^2 \\ &\quad + 2\lambda_j^2\|u_j - \mathbf{P}_* u_j\|^2_{L_2(\gamma)} + 2K_2(1+\lambda_j^2)\|\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j\|^2_{L_2(\gamma)}.\end{aligned} \tag{3.110}$$

Using (3.110) in (3.109) then gives

$$\begin{aligned}\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j, \gamma)^2 &\le (1+2B_3)\|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|^2_{L_2(\gamma)} \\ &\quad + (2+2B_3)B_2\zeta_{\mathcal{T}_\delta}(\gamma)^2 + (1+2B_3)\lambda_j^2\|u_j - \mathbf{P}_* u_j\|^2_{L_2(\gamma)} \\ &\quad + (1+2B_3)K_2(1+\lambda_j^2)\|\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j\|^2_{L_2(\gamma)} \\ &\quad + C_6 Osc_{\mathcal{T}_\delta}(\lambda_j \mathbf{P}_\delta u_j, \mathbf{Z}_\delta u_j, \gamma)^2 + K_1 H_0^2 \lambda_j^2 \|\mathbf{P}_* u_j - \mathbf{P}_\delta u_j\|^2_{L_2(\gamma)}.\end{aligned} \tag{3.111}$$

We now use (3.74) to bound $\|u_j - \mathbf{P}_* u_j\|_{L_2(\gamma)}$, $\|\mathbf{P}_* u_j - \mathbf{P}_\delta u_j\|_{L_2(\gamma)}$, and $\|\mathbf{Z}_* u_j - \mathbf{Z}_\delta u_j\|_{L_2(\gamma)}$ in (3.111) to get

$$\begin{aligned}&\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j, \gamma)^2 \\ &\le \left(1 + 2B_3 + C(1+2B_3)K_2(1+\lambda_j^2)H_0^{2s} + CK_1 H_0^{2+2s}\lambda_j^2\right)\|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|^2_{L_2(\gamma)} \\ &+ \left(C(1+2B_3)\lambda_j^2 H_0^{2s} + C(1+2B_3)K_2(1+\lambda_j^2)H_0^{2s} + CK_1 H_0^{2+2s}\lambda_j^2\right)\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} \\ &+ \left((2+2B_3)B_2 + (1+2B_3)\lambda_j^2 C + (1+2B_3)K_2(1+\lambda_j^2)C + K_1 H_0^2\lambda_j^2 C\right)\zeta_{\mathcal{T}_\delta}(\gamma)^2 \\ &+ C_6 Osc_{\mathcal{T}_\delta}(\lambda_j \mathbf{P}_\delta u_j, \mathbf{Z}_\delta u_j, \gamma)^2.\end{aligned} \tag{3.112}$$

Enforcing

$$C(1+2B_3)\lambda_{\max}^2 H_0^{2s} + C(1+2B_3)K_2(1+\lambda_{\max}^2)H_0^{2s} + CK_1\lambda_{\max}^2 H_0^{2+2s} \le \frac{1}{2}$$

and

$$CK_1\lambda_{\max}^2 H_0^2 \le \frac{1}{2}$$

in (3.112) and rearranging terms yields

$$\begin{aligned}\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j, \gamma)^2 \leq 2\left(\frac{3}{2} + 2B_3\right)\|\nabla_\gamma e(\mathbf{Z}_\delta u_j)\|^2_{L_2(\gamma)} \\ + 2\left((2+2B_3)B_2 + (1+2B_3)\lambda_j^2 C + (1+2B_3)K_2(1+\lambda_j^2)C + \frac{1}{2}\right)\zeta_{\mathcal{T}_\delta}(\gamma)^2 \\ + 2C_6 Osc_{\mathcal{T}_\delta}(\lambda_j \mathbf{P}_\delta u_j, \mathbf{Z}_\delta u_j, \gamma)^2.\end{aligned} \tag{3.113}$$

We now choose $\widehat{\kappa} = \frac{\kappa}{\sqrt{\max\left\{2C_6, 3+4B_3, 2\left((2+2B_3)B_2+(1+2B_3)\lambda_j^2 C+(1+2B_3)K_2(1+\lambda_j^2)C+\frac{1}{2}\right)\right\}}}$ to end up with

$$\begin{aligned}&\sum_{j\in J}\|\nabla_\gamma e(\mathbf{Z}_* u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}_*}(\lambda_j \mathbf{P}_* u_j, \mathbf{Z}_* u_j, \gamma)^2 \\ &\leq \max\left\{2C_6, 3+4B_3, 2\big((2+2B_3)B_2 + (1+2B_3)\lambda_j^2 C + (1+2B_3)K_2(1+\lambda_j^2)C + \frac{1}{2}\big)\right\}\delta^2 \\ &= \kappa^2\left(\sum_{j\in J}|\nabla_\gamma e(\mathbf{Z}u_j)\|^2_{L_2(\gamma)} + Osc_{\mathcal{T}}(\lambda_j \mathbf{P}u_j, \mathbf{Z}u_j, \gamma)^2\right).\end{aligned}$$

Thus by Lemma 3.41 the set $\mathcal{R}_{\mathcal{T}\to\mathcal{T}_*} \subset \mathcal{T}$ satisfies the Dörfler property (3.97)

$$\theta\mu_{\mathcal{T}}(J) \leq \mu_{\mathcal{T}}(\mathbf{Z}, \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}, J)$$

which implies

$$\frac{\theta}{\sqrt{72}}\eta_{\mathcal{T}}(J) \leq \eta_{\mathcal{T}}(\mathbf{Z}, \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}, J).$$

Since $\mathcal{M}$ is the smallest cardinality set satisfying

$$\frac{\theta}{\sqrt{72}}\eta_{\mathcal{T}}(J) \leq \eta_{\mathcal{T}}(\mathcal{M}, J)$$

and thus we have

$$\#\mathcal{M} \leq \#\mathcal{R} \leq \#\mathcal{T}_* - \#\mathcal{T} \leq \#\mathcal{T}_\delta - \#\mathcal{T}_0 \lesssim |J,\gamma|^{\frac{1}{s}}_{\mathbb{A}'_s}\delta^{-\frac{1}{s}}.$$

□

Theorem 3.44 (Convergence Rate of AFEM)**.** *Let H_0 satisfy conditions* (H1) *through* (H8)*. Let $\epsilon_0 \le \frac{1}{\omega 6 K_0 L^3}$ be the initial tolerance, and the parameters θ, ω, ρ of AFEM satisfy*

$$0 < \theta \le \theta_*, \qquad 0 < \omega \le \omega_* := \min\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}, \qquad 0 < \rho < 1,$$

where θ_*, ω_1, ω_2, ω_3, ω_4, and ω_5 are given in (3.95)*,* (W1)*,* (W2)*,* (W3)*,* (W4)*, and* (W5)*, respectively. Let $\mathcal{T}_0$ have an admissible labeling, and let the procedure MARK select sets with minimal cardinality. Let $\{(u_j, \lambda_j)\}_{j\in J}$ be the solutions of* (3.5) *and $\{\Gamma_k, \mathcal{T}_k, \mathbf{Z}_k u\}_{k\ge 0}$ be the sequence of approximate surfaces, meshes, and solutions generated by AFEM.*

If $\{(u_j, \lambda_j, \gamma)\}_{j\in J} \in \mathbb{A}'_s$ for some $0 < s \le n/d$, then there exists a constant C, depending on the Lipschitz constant L of γ, $\lambda_{\max}$, the refinement depth b, the initial triangulation $\mathcal{T}_0$, and AFEM parameters θ, ω, ρ such that

$$\sum_{j\in J} e(\mathbf{Z}_k u_j) + Osc_{\mathcal{T}_k}(\lambda_j \mathbf{P}_k u_j, \mathbf{Z}_k u_j, \gamma) + |J|\zeta_{\mathcal{T}_k}(\gamma) \le C|J,\gamma|_{\mathbb{A}'_s}(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}, \qquad (3.114)$$

where $|J,\gamma|_{\mathbb{A}'_s}$ is defined in (3.88)*.*

Proof. We start by noting that since $\omega\epsilon_0 \le \frac{1}{6K_0L^3}$, the output of the procedure ADAPT_SURFACE fulfills $\zeta_{\mathcal{T}_0^+}(\gamma) \le \frac{1}{6K_0L^3}$ which is (3.13) and implies that $\mathbb{T}(\mathcal{T}_0^+)$ is shape regular.

There are two instances where elements are added, inside ADAPT_SURFACE and inside ADAPT_EIGENFUNCTION. We observe that ADAPT_SURFACE is s-optimal with $C(\gamma) \lesssim |J,\gamma|_{\mathbb{A}'_s}^{1/s}$, whence the set of all the elements marked for refinement in the k-th call to ADAPT_SURFACE satisfies

$$\#\mathcal{M}_k \lesssim C(\gamma)\omega^{-\frac{1}{s}}\epsilon_k^{-\frac{1}{s}} \lesssim |J,\gamma|_{\mathbb{A}'_s}^{1/s}\epsilon_k^{-\frac{1}{s}}$$

Within the ADAPT_EIGENFUNCTION loop we apply Lemma 3.43 for the ith loop to get

$$\#\mathcal{M}_k^i \lesssim |J,\gamma|_{\mathbb{A}_s'}^{1/s} \left(\sum_{j\in J} e(\mathbf{Z}_k^i u_j) + Osc_{\mathcal{T}_k^i}(\lambda \mathbf{P}_k^i u_j, \mathbf{Z}_k^i u_j, \gamma)\right)^{-\frac{1}{s}} \qquad 0 \le i \le I.$$

We also have (3.50) within ADAPT_EIGENFUNCTION which implies

$$\sum_{j\in J} e(\mathbf{Z}_k^i u_j) + Osc_{\mathcal{T}_k^i}(\mathbf{Z}_k u_j) \approx \sum_{j\in J} e(\mathbf{Z}_k^i u_j) + \mu_{\mathcal{T}_k^i}(J). \tag{3.115}$$

Finally, the contraction property (Theorem 3.31) holds within ADAPT_EIGENFUNCTION and gives

$$\sum_{j\in J} e(\mathbf{Z}_k^{I-1} u_j) + \mu_{\mathcal{T}_k^{I-1}}(J) \lesssim \alpha^{I-1-i} \left(\sum_{j\in J} e(\mathbf{Z}_k^i u_j) + Osc_{\mathcal{T}_k^i}(\mathbf{Z}_k^i u_j)\right). \tag{3.116}$$

Combining (3.115) with (3.116) then gives

$$\begin{aligned}\left(\sum_{j\in J} e(\mathbf{Z}_k^i u_j) + Osc_{\mathcal{T}_k^i}(\mathbf{Z}_k^i u_j)\right)^{-\frac{1}{s}} &\lesssim \alpha^{\frac{I-1-i}{s}} \left(\sum_{j\in J} e(\mathbf{Z}_k^{I-1} u_j) + \mu_{\mathcal{T}_k^{I-1}}(J)\right)^{-\frac{1}{s}} \\ &\lesssim \alpha^{\frac{I-1-i}{s}} \epsilon_k^{-\frac{1}{s}}.\end{aligned}$$

Summing over the inner iterates then gives

$$\sum_{i=0}^{I-1} \#\mathcal{M}_k^i \lesssim |J,\gamma|_{\mathbb{A}_s'}^{1/s} \epsilon_k^{-\frac{1}{s}} \sum_{i=0}^{I-1} \alpha^{\frac{I-i-1}{s}} \lesssim |J,\gamma|_{\mathbb{A}_s'}^{1/s} \epsilon_k^{-\frac{1}{s}}.$$

Counting the marked elements from ADAPT_SURFACE, we then have by Lemma 3.29

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \le C_7 \sum_{j=0}^{k-1} \left(\#\mathcal{M}_j + \sum_{i=0}^{I-1} \#\mathcal{M}_j^i\right) \lesssim |J,\gamma|_{\mathbb{A}_s'}^{1/s} \sum_{j=0}^{k-1} \epsilon_j^{-\frac{1}{s}}$$

Noting that $\epsilon_{k+1} = \rho\epsilon_k$ as part of the AFEM algorithm together with $\rho < 1$ we obtain

$$\sum_{j=0}^{k-1} \epsilon_j^{-\frac{1}{s}} = \epsilon_{k-1}^{-\frac{1}{s}} \sum_{j=0}^{k-1} \rho^{\frac{j}{s}} \lesssim \epsilon_k^{-\frac{1}{s}}.$$

Hence

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim |J,\gamma|_{\mathbb{A}_s'}^{1/s} \epsilon_k^{-\frac{1}{s}}. \tag{3.117}$$

Noting the stopping criteria (3.58) and (3.59) are satisfied gives

$$\sum_{j\in J} e(\mathbf{Z}_k u_j) + Osc_{\mathcal{T}_k}(\lambda_j \mathbf{P}_k u_j, \mathbf{Z}_k u_j, \gamma) + \omega^{-1}\zeta_{\mathcal{T}_k}(\gamma) \lesssim \epsilon_k.$$

The bound (W1) implies $|J| \lesssim \omega^{-1}$ which implies

$$\sum_{j\in J} e(\mathbf{Z}_k u_j) + Osc_{\mathcal{T}_k}(\lambda_j \mathbf{P}_k u_j, \mathbf{Z}_k u_j, \gamma) + |J|\zeta_{\mathcal{T}_k}(\gamma) \lesssim \epsilon_k. \tag{3.118}$$

Combining (3.117) with (3.118) yields the final result. □

3.7 Numerical Experiments

In this section we numerically investigate the rates of convergence for our adaptive algorithm. We choose the cluster associated with the interval $[1, 50]$. We use the $C^{1,\alpha}$ surface defined as the graph of $z(x, y) = \left(\frac{3}{4} - x^2 - y^2\right)_+^{1+\alpha}$ on the unit square $\Omega = (0, 1)^2$ and assume homogeneous Dirichlet boundary conditions. We use piecewise linear finite elements to approximate the solution and surface. It is shown in Section 9 of [44] that for $\alpha = \frac{3}{5}$ γ is a member of an order $\frac{1}{2}$ approximation class when measured by $\zeta_{\mathcal{T}}(\gamma)$, i.e. in terms of degrees of freedom $\zeta_{\mathcal{T}}(\gamma)$ converges as $DOF^{-1/2}$. Our analysis of the regularity of the eigenfunctions showed that the eigenfunction had regularity H^{1+s} for some $s \leq \alpha$. In Figure 3.1 we have plotted the values of the geometric estimator $\zeta_{\mathcal{T}}(\gamma)$ during both the ADAPT_SURFACE and ADAPT_EIGENFUNCTION loops. We have plotted the values of $\eta_{\mathcal{T}}(J)$ only during ADAPT_EIGENFUNCTION. We see the eigenfunction estimator is converging at the best rate we could expect, $DOF^{-1/2}$. We also see strong evidence that $\zeta_{\mathcal{T}}(\gamma)$ is indeed order $\frac{1}{2}$. For γ corresponding to $\alpha = \frac{2}{5}$ the results of [44] say that the

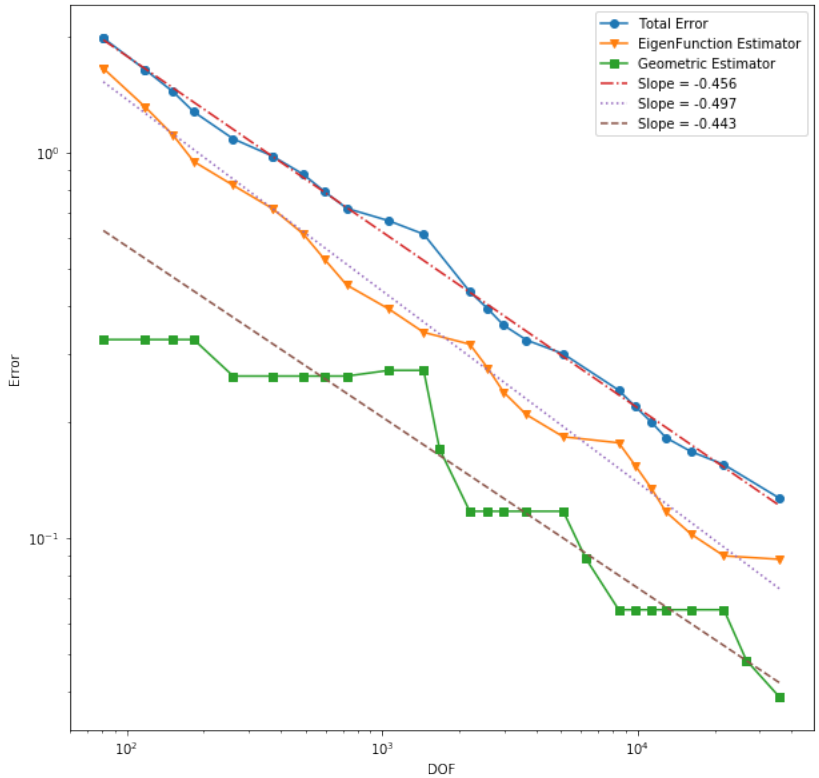

Figure 3.1: Convergence rates of total error, geometric error estimator, and eigenfunction cluster estimator in AFEM when $\alpha = \frac{3}{5}$.

expected rate of $\zeta_{\mathcal{T}}(\gamma)$ is $DOF^{-0.4}$. We see in Figure 3.2 that $\zeta_{\mathcal{T}}(\gamma)$ is indeed order 0.4 and so is the eigenfunction estimator.

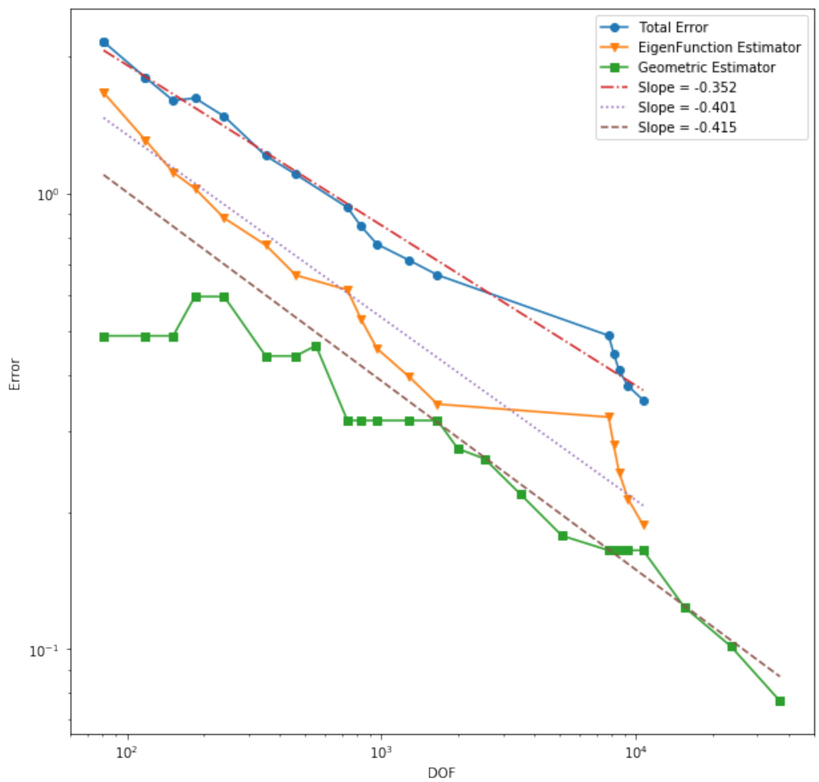

Figure 3.2: Convergence rates of total error, geometric error estimator, and eigenfunction cluster estimator in AFEM when $\alpha = \frac{2}{5}$.

4. SUMMARY AND CONCLUSIONS

In this dissertation we have developed the theory of approximation of eigenvalues and eigenfunctions of the Laplace-Beltrami operator using the surface finite element method. In Chapter 2 we presented a priori estimates for eigenpairs of the Laplace-Beltrami operator on C^∞ surfaces based on joint work with Andrea Bonito and Alan Demlow. Our analysis showed that the SFEM eigenfunctions converge with the same rates as the SFEM solutions to the source problem. There is the usual FEM approximation error we see when solving problems on flat domains plus an $O(h^{k+1})$ term which accounts for the geometric consistency errors introduced by the SFEM framework. We were able to verify that our estimates for the convergence rates were sharp through numerical tests which matched the theoretical rates. We developed a priori estimates for the convergence of the SFEM eigenvalues and showed that for clusters of eigenvalues we could not guarantee better than the usual error for flat domains plus an $O(h^{k+1})$ term which accounts for the geometric consistency errors introduced by the SFEM framework. However, for single eigenvalues we developed new theoretical tools based on the theory of numerical quadrature for analyzing geometric consistency errors. Using this new theoretical framework we were able to show that it is possible to attain superconvergent geometric consistency errors on quadrilateral meshes. The framework also showed a clear way of improving geometric consistency errors by using surface interpolation points in the construction of Γ which coincide with a quadrature rule. This culminated in a best possible convergence rate of $O(h^{2k})$ when using Gauss-Lobatto quadrature points for interpolation rather than the typical $O(h^{k+1})$ rate that results from using equally spaced Lagrange interpolation points. We also proved that these theoretical results were sharp via numerical experiments.

In Chapter 3 we developed and analyzed an adaptive algorithm for approximating eigenfunctions with SFEM. We used a modified version of the adaptive algorithm for the source problem on surfaces presented in Section 1.6 as our template. We employed the theoretical tools discussed in Section 1.4 for an adaptive eigenfunction algorithm on flat domains with proper modifications for SFEM to analyze our algorithm and prove it is optimal. In the process we introduced a new regularity result for solutions to the source problem on piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces. We then used this result to extend the a priori estimates for eigenfunctions results of Chapter 2 to piecewise $C^{1,\alpha}$ globally $W^{1,\infty}$ surfaces. We then used these new a priori estimates to show that if our eigenfunctions and surface belong to an approximation class $\mathbb{A}'_s$, then for a sufficiently fine initial mesh $\mathcal{T}_0$ our eigenfunction estimator does indeed lead to optimal order s convergence rates. We also provided a partial characterization of our approximation classes in terms of Besov spaces.

There is still room for future improvement and extension of our results. The geometric estimator we used is heuristically $O(h^k)$ for C^∞ surfaces. Based on a priori analysis we would hope to find an estimator that is heuristically $O(h^{k+1})$ for C^∞ surfaces. Recently a new heuristically correct geometric estimator for surfaces of regularity C^2 or better was introduced in [36]. This estimator should bound all of the geometric consistency errors encountered in our analysis from Chapter 3. Algorithmic performance has not yet been theoretically analyzed, but we would expect the performance of this new estimator when used in our adaptive algorithm to offer an advantage when working with C^2 surfaces. In the future we hope to analyze this new estimator and improve our algorithmic performance on higher regularity surfaces.

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