FATOU–BIEBERBACH DOMAINS: A NEW CONSTRUCTION AND A THEME ON THE RUNGE PROPERTY

A Dissertation

by

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ABSTRACT

Fatou–Bieberbach domains are a phenomenon specific to several complex variables. Techniques for producing such domains are limited and fundamental questions about containment between two Fatou–Bieberbach are still being raised. We show that given a countable collection of Runge Fatou–Bieberbach domains with a ball in common and a common point omitted, there exists a Runge Fatou–Bieberbach domain that contains the union. Additionally, we provide a new construction for Fatou–Bieberbach domains modelled on the attracting basin, using right-side composition instead of the prototypical left-side composition. We use this construction to show that there exists a strictly decreasing family of Fatou–Bieberbach domains whose intersection contains a Fatou–Bieberbach domain. Additionally, we provide a generalized condition for constructing attracting basins from a sequence of automorphisms.

Even a blind squirrel finds a nut once in a while.

-Anonymous

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TABLE OF CONTENTS

1. INTRODUCTION

In the 1920's Fatou and Bieberbach proved the existence of proper domains in $\mathbb{C}^2$ that are biholomorphic to $\mathbb{C}^2$. Today these domains are called Fatou–Bieberbach domains. These domains are specific to Several Complex Variables: indeed it is easy to see using the Riemann mapping theorem that there is no proper domain in $\mathbb{C}$ that is biholomorphic to $\mathbb{C}$. Fatou and Bieberbach produced examples by using a basin of attraction, that is, the domain $\{z \in \mathbb{C}^n : \lim_{j\to\infty} F^j(z) = p\}$ where F is an automorphism that fixes point p and F^j denotes F composed j times. It was proved by Rudin and Rosay [2] in their seminal paper that if F is attracting at point p then the corresponding basin of attraction is biholomorphic to $\mathbb{C}^n$. Precisely, their theorem states:

Suppose that $F \in \text{Aut}(\mathbb{C}^n)$ fixes a point $p \in \mathbb{C}^n$ and that all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $F'(p)$ satisfy $|\lambda_i| < 1$. Let Ω be the set of all $z \in \mathbb{C}^n$ for which $\lim_{k\to\infty} F^k(z) = p$, where $F^k = F \circ F^{k-1}$, $F^1 = F$. Then there exists a biholomorphic map Φ from Ω onto $\mathbb{C}^n$.

(It should be noted that Reich [3] attempted to prove the above first and the ideas used seem to have motivated Rudin and Rosay in their method of proof.) Later, Wold [1] proved a related result for a sequence of automorphisms. Stated precisely, it says:

Let $0 < s < r < 1$ such that $r^2 < s$, let $\delta > 0$, and let $\{F_j\} \subseteq \text{Aut}_p(\mathbb{C}^n)$ such that $s||z-p|| \leq ||F_j(z) - p|| \leq r||z-p||$ for all $z \in B_\delta(p)$ and all $j \in \mathbb{N}$. Then there exists a biholomorphic map

$$\Phi : \Omega \to \mathbb{C}^n$$

where $\Omega = \{z \in \mathbb{C}^n : \lim_{j\to\infty} F_j \circ \cdots \circ F_1(z) = p\}$.

It is a long standing question whether or not "$r^2 < s$" can be removed from the hypothesis. I provide an interesting generalization to Wold's [1] Theorem 4.

In addition to this result, Wold [1] also proves that the union of nested increasing Runge Fatou–Bieberbach domains is biholomorphic to $\mathbb{C}^n$. It is a natural question whether or not the complementary idea is true. That is, is the interior of the intersection of decreasing nested Fatou–Bieberbach domains a Fatou–Bieberbach domain? Dixon and Esterle [4] have shown (Corollary 7.12) that the answer is no: it is possible for the intersection of decreasing nested Fatou–Bieberbach domains to have empty interior. However, I have shown that if each Fatou–Bieberbach domain contains a common ball and the Fatou–Bieberbach domains can exhibit a certain growth property, then the interior of the intersection of the decreasing nested Fatou–Bieberbach domains is a Fatou–Bieberbach domain!

The proof of this result uses a new construction of Fatou–Bieberbach domains. Fundamentally, the idea of the construction is to consider an attracting basin of a sequence of holomorphic mappings, but compose on the right hand side instead of the left hand side. We consider this new construction to be of real consequence because of the limited number of techniques available to construct Fatou–Bieberbach domains.

Constructing Fatou–Bieberbach domains that satisfy natural properties has been a notable part of the research of Fatou–Bieberbach domains. In particular, there is interest in being able to specify what a Fatou–Bieberbach domain can contain and simultaneously what its complement can contain. For example, Rosay and Rudin [2] have shown: If $K \subseteq \mathbb{C}^n$ is compact and strictly convex and $E \subseteq \mathbb{C}^n \setminus K$ is countable, then there is an injective holomorphic mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ so that $E \subseteq F(\mathbb{C}^n) \subseteq \mathbb{C}^n \setminus K$. We provide a result of this flavor, effectively showing that given a countable collection of Runge Fatou–Bieberbach domains that contain a common ball and omit a common point, there exists a Runge Fatou–Bieberbach domain that contains the union of the collection of Runge Fatou–Bieberbach domains.

The plan of this dissertation is as follows: In Chapter 2, we provide background information and context for the results herein as well as formal statements of these results. In Chapter 3, we provide a convergence result for composing a sequence of holomorphic mappings. This will then be used to provide a generalization for Theorem 4 in Wold [1]. Then we will show that given a countable collection of Fatou–Bieberbach domains under some conditions, we can find a Runge Fatou–Bieberbach domain that contains their union. In the second half of Chapter 3, we offer a new construction of Fatou–Bieberbach domains and provide an application for this construction. The Runge property will be used often throughout this dissertation. In fact, without this property it is the author's understanding that many of the related questions quickly become intractable. We state several conjectures throughout related to (removing) the Runge property. In Chapter 4, we provide concluding remarks.

2. BACKGROUND INFORMATION AND STATEMENT OF MAIN THEOREMS

2.1 Fatou–Bieberbach Domains

We start by supplying the basic definitions. Our first definition concerns the object of study throughout this work.

Definition 1. *We say that $\Omega \subseteq \mathbb{C}^n$ is a Fatou–Bieberbach domain if $\Omega \neq \mathbb{C}^n$ and Ω is biholomorphic to $\mathbb{C}^n$.*

Our second definition is for a property that will be used throughout this work.

Definition 2. *A domain in $\mathbb{C}^n$ is said to be Runge if for each holomorphic function defined on it and each compact set in it, there exists a sequence of polynomials that converges uniformly to the holomorphic function on the compact set.*

Next we define the basin of attraction. We will later see that under the right conditions it is biholomorphic to $\mathbb{C}^n$. In fact, throughout the literature it is the typical way a Fatou–Bieberbach domain is constructed.

Definition 3. *Let F be an automorphism of $\mathbb{C}^n$ with fixed point p (i.e. $F(p) = p$). Then we define the basin of attraction of F at p to be*

$$\{z \in \mathbb{C}^n : \lim_{j\to\infty} F^j(z) = p\}$$

where $F^j = F \circ F^{j-1}$, and $F^1 = F$.

Next we need a definition to describe when an automorphism will be attracting or repelling at point p.

Definition 4. *Let F be an automorphism of $\mathbb{C}^n$ that fixes point p. Then:*

1. *If each eigenvalue of matrix $F'(p)$ is less than 1 in modulus, we say that the fixed point p is*

attracting.

2. *If each eigenvalue of matrix $F'(p)$ is greater than 1 in modulus, we say that the fixed point p is repelling.*

This brings us to an important result by Rosay and Rudin:

Theorem 1. *Suppose that $F \in Aut(\mathbb{C}^n)$ has an attracting fixed point at p. Let Ω be the basin of attraction of F at p. Then there exists a biholomorphic map Φ from Ω onto $\mathbb{C}^n$.*

In their seminal paper, Rosay and Rudin used this theorem to produce many interesting examples of Fatou–Bieberbach domains.

We now provide a generalized definition of basin of attraction.

Definition 5. *Let $\{F_j\}$ be a sequence of automorphisms of $\mathbb{C}^n$ each with fixed point p. Then we define the basin of attraction of sequence F_j to be*

$$\{z \in \mathbb{C}^n : \lim_{j\to\infty} F_j \circ F_{j-1} \circ \cdots \circ F_1(z) = p\}.$$

Throughout this work, we will often abuse notation and write $\{F_j \circ \cdots \circ F_1 \to p\}$ instead of $\{z \in \mathbb{C}^n : \lim_{j\to\infty} F_j \circ F_{j-1} \circ \cdots \circ F_1(z) = p\}$.

Wold [1] Theorem 4 provides a semi-analogous result to the above Theorem by Rosay and Rudin, it states:

Theorem 2. *Wold [1] Theorem 4*

Let $\{F_j\} \subseteq Aut_p(\mathbb{C}^n)$. Suppose that there are $0 < s < r < 1$ and $\epsilon > 0$ such that

$$s||z-p|| \leq ||F_j(z) - p|| \leq r||z-p|| \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}.$$

Further suppose that $r^2 < s$. Then $\{F_j \circ \cdots \circ F_1 \to p\}$ is biholomorphic to $\mathbb{C}^n$.

Here $B_\epsilon(p)$ denotes the ball $\{z \in \mathbb{C}^n : ||z-p|| < \epsilon\}$.

One of our main results is a generalization to Wold's result:

Theorem 3. *Let $\{F_j\} \subseteq Aut_p(\mathbb{C}^n)$. Suppose that there are $0 < s_j \leq r_j \leq 1$ and $\epsilon > 0$ such that*

$$s_j||z-p|| \leq ||F_j(z)-p|| \leq r_j||z-p|| \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}.$$

Further suppose $\inf_i\{s_i\} > 0$ and $\sum_i \sqrt{(\frac{r_1^2}{s_1})\cdots(\frac{r_i^2}{s_i})} < \infty$.

Then $\{F_j \circ \cdots \circ F_1 \to p\}$ is biholomorphic to $\mathbb{C}^n$ and is Runge.

Here $\mathrm{Aut}_p(\mathbb{C}^n)$ is the set of automorphisms on $\mathbb{C}^n$ that fix the point $p \in \mathbb{C}^n$.

Notice that the hypothesis allows for the possibility that $\frac{r_j^2}{s_j} > 1$ for infinitely many j. This is the main distinction between this result and Wold's.

Important Aside

An important result used in the proof of the above theorem that is not used in Wold's proof follows.

Proposition 1. *Let $U \subseteq \mathbb{C}^n$ be a nonempty set. Let $U_i \subseteq \mathbb{C}^n$ ($i \in \mathbb{N}$) be connected open sets with $U \subseteq U_i$ for each $i \in \mathbb{N}$ and let $f_i : U_{i+1} \to U_i$ be holomorphic mappings. Suppose $\sum_i \sqrt{||f_i - id||_U} < \infty$. Then the sequence $f_1 \circ f_2 \circ \cdots \circ f_{j-1} \circ f_j$ converges uniformly on compacta on U.*

We believe this result to be interesting in its own right. Indeed, as far as the author is aware, there are few results of this flavor.

However, an analogous result that is easier to prove appears in Esterle and Dixon [4] as Lemma 8.3:

Proposition 2. *Let F_j be a sequence of holomorphic mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$. If $\sum_i ||\mathbb{I} - F_i||_{B_m(0)} < \infty$ for each $m \geq 1$, then the sequence $F_1 \circ \cdots \circ F_j$ converges uniformly* *on compact subsets of* $\mathbb{C}^n$ *to a mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$.

Here $|| \cdot ||_{B_m(0)}$ is the sup norm on the ball $B_m(0)$.

2.2 Containment of Fatou–Bieberbach Domains

Throughout the history of Fatou–Bieberbach domains there has been a desire to construction Fatou–Bieberbach domains with given properties. For example, Rudin and Rosay have shown precisely:

Theorem 4. *If* $K \subseteq \mathbb{C}^n$ *is compact and strictly convex and* $E \subseteq \mathbb{C}^n \setminus K$ *is countable, then there is a injective holomorphic mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$ *so that* $E \subseteq F(\mathbb{C}^n) \subseteq \mathbb{C}^n \setminus K$.

And as another example consider the following result by Globevnik [5]:

Theorem 5. *Let* $Q \subseteq \mathbb{C}$ *be a bounded open set with boundary of class* C^1 *whose complement is connected. Let* $0 < R < \infty$ *be such that* $\overline{Q} \subseteq R\Delta$. *There are a domain* $\Omega \subseteq \mathbb{C}^2$ *and a volume-preserving biholomorphic map from* Ω *onto* $\mathbb{C}^2$ *such that*

(i) $\Omega \subseteq \{(z, w) : |z| < \max\{R, |w|\}\}$

(ii) $\Omega \cap R(\Delta \times \Delta)$ *is a arbitrarily small* C^1*-perturbation of* $Q \times R\Delta$.

Here Δ denotes the unit disk in $\mathbb{C}$.

One of our main results shows that Fatou–Bieberbach domains are in some sense "big." Note: In what follows, we use the term "countable" to mean countably infinite or finite.

Theorem 6. *Let* $\{\Omega_j\}$ *be a nonempty countable set of Runge Fatou–Bieberbach domains in* $\mathbb{C}^n$ *such that* $\cup_j \Omega_j \neq \mathbb{C}^n$. *Let* K *be a compact set that is polynomially convex. Suppose there exist* $\epsilon > 0$ *and* $p \in \mathbb{C}^n$ *such that* $\overline{B_\epsilon(p)} \subseteq \cap_j \Omega_j$ *and* $\overline{B_\epsilon(p)} \subseteq K$. *Let* $\{a_j\}$ *be a countable set of points in* $(\cup_j \Omega_j)^{\mathsf{c}}$. *Let* $\{b_1, \ldots, b_l\}$ *be a nonempty finite set of points in* $(\cup_j \Omega_j)^{\mathsf{c}}$ *and suppose* $\{b_1, \ldots, b_l\} \cap (\cup_j \{a_j\} \cup K) = \varnothing$. *Then there exists a Runge Fatou–Bieberbach domain* Ω *such that* $\cup_j \Omega_j \subseteq \Omega$, $\cup_j \{a_j\} \subseteq \Omega$, $K \subseteq \Omega$, *and* $\{b_1, \ldots, b_l\} \subseteq \Omega^{\mathsf{c}}$.

2.3 A Construction: Reverse Semi-basin of Attraction

In this section, we provide the background information leading to a new type of construction for Fatou–Bieberbach domains. Throughout the literature on Fatou–Bieberbach domains, the typical construction uses the attracting basin construction. In fact, the author is aware of only a few constructions of Fatou–Bieberbach domains that do not rely on constructing an attracting basin. For instance see, Stensønes [6] and Weickert [7].

We now introduce the concepts necessary for a new construction of Fatou–Bieberbach domains.

Definition 6. *Let $A_j \subseteq \mathbb{C}^n$ be a sequence of sets. We define*

$$\textit{oliminf}_{j\to\infty} A_j = \{z \in \mathbb{C}^n | \text{ there exists } m > 0, \epsilon > 0 \text{ such that } B_\epsilon(z) \subseteq \cap_{i\geq m} A_i\}.$$

Definition 7. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. We define the reverse semi-basin of attraction for a ball $B_\epsilon(p)$ to be*

$$\begin{aligned}\Omega^{p,\epsilon}_{\{F_j\}} &= \textit{oliminf}_{j\to\infty} F_j^{-1}(B_\epsilon(p)) \cup \textit{oliminf}_{j\to\infty}(F_j \circ F_{j+1})^{-1}(B_\epsilon(p)) \cup \cdots \\ &= \cup_{i=0}^{\infty} \textit{oliminf}_{j\to\infty}(F(j+i,j))^{-1}(B_\epsilon(p))\end{aligned}$$

where $F(j+i,j) = F_j \circ F_{j+1} \circ \cdots \circ F_{j+i}$.

The reverse semi-basin of attraction should be viewed as semi-analogous to the attracting basin, but with taking composition on the right side instead of the left.

Theorem 7. *Let $\epsilon > 0$. Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Let $U_1 = \mathbb{C}^n$. Suppose that there are $0 < s_j \leq r_j < 1$, and $c_j > 0$ such that*

$$s_j||z-p|| \leq ||F_j(z) - p|| \leq r_j||z-p|| \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}$$

and

$$||A_j^{-1}(F_j(z)-p)-(z-p)|| \leq c_j||z-p||^2 \textit{ on } B_\epsilon(p) \textit{ for all } j \in \mathbb{N}.$$

Also assume $\sup_i\{r_i\} < 1$, $\inf_i\{s_i\} > 0$, *and* $\sup_k(c_k + \sum_{i=1}^{k-1} c_i(\frac{r_{i+1}^2}{s_{i+1}})\cdots(\frac{r_k^2}{s_k})) < \infty$. *Further, suppose there exists* $\delta > 0$ *such that for each* $k \in \mathbb{N}$, *there is a* $B_k < \infty$ *such that*

$$\sup_{\substack{z\in B_\delta(p)\\ i\in\mathbb{N}}} ||F_{i+k}^{-1}\circ\cdots\circ F_{i+1}^{-1}(z)|| < B_k.$$

Then there exists a domain FB*, biholomorphic to* $\mathbb{C}^n$*, such that*

$$\Omega_{\{F_j\}}^{p,\epsilon} \subseteq FB \subseteq \limsup_i U_i \setminus \cup_{k=1}^\infty \liminf_i (F_1\circ F_2\circ\cdots\circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(p)).$$

Additionally, if each U_i *is Runge, we may arrange that domain* FB *is Runge.*

3. RESULTS AND PROOFS

3.1 Generalization of Wold [1] Theorem 4

Our first lemma provides estimates for inverse holomorphic mappings given estimates on the holomorphic mappings.

Lemma 1. *Let $\{F_j\}$ be a set of holomorphic mappings that are injective on $B_\epsilon(p) \subseteq \mathbb{C}^n$, map into $\mathbb{C}^n$, and fix the point p. Suppose that there are $0 < s_j \leq r_j < \infty$ and $\epsilon > 0$ with $\inf_i\{s_i\} > 0$ and $\sup_i\{r_i\} < \infty$ such that*

$$s_j||z-p|| \leq ||F_j(z)-p|| \leq r_j||z-p|| \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}.$$

Then there exists an $\epsilon' > 0$ such that

$$\frac{1}{r_j}||z-p|| \leq ||F_j^{-1}(z)-p|| \leq \frac{1}{s_j}||z-p|| \text{ on } B_{\epsilon'}(p) \text{ for all } j \in \mathbb{N}.$$

Proof. Without loss of generality suppose $p = 0$. The assertion is clear once we can show there is an open ball about the origin in the set $\cap_i F_i(B_\epsilon(0))$.

Suppose, for sake of contradiction, that there is no open ball about the origin contained in the set $\cap_i F_i(B_\epsilon(0))$. Then we can find a sequence f_j in $\{F_i : i \in \mathbb{N}\}$ such that $B_{\frac{1}{j}}(0) \setminus f_j(B_\epsilon(0)) \neq \varnothing$. By Montel's Theorem, there is a subsequence f_{j_k} that converges on $B_\epsilon(0)$ to some holomorphic mapping f. Further, we have $\inf_i\{s_i\} \cdot ||z|| \leq ||f(z)||$ on $B_\epsilon(0)$. Therefore $Jf(0)$ is invertible. Thus there is some small ball $B_{\epsilon'}(0) \subseteq B_\epsilon(0)$ on which f is injective. Hence $f(B_{\epsilon'}(0))$ is open and so contains some small ball $B_\delta(0)$. But $f_{j_k}(z) \to f(z)$ uniformly on $B_{\epsilon'}(0)$, so for large k, $f_{j_k}(B_{\epsilon'}(0)) \supseteq B_{\frac{\delta}{2}}(0)$ contradicting $B_{\frac{1}{j_k}}(0) \setminus f_{j_k}(B_\epsilon(0)) \neq \varnothing$. $\square$

Lemma 2. *Let $\alpha, \beta > 0$ and let Γ be a family of holomorphic mappings that are injective on*

$B_\epsilon(0)$, fix the origin, and satisfy $\alpha||z|| \leq ||F(z)|| \leq \beta||z||$ for each $F \in \Gamma$ and $z \in B_\epsilon(0)$. Let $A_F = JF(0)$. Then there exist $C, C', \epsilon' > 0$ such that for each $F \in \Gamma$,

$$||A_F^{-1}F(z) - z|| \leq C||z||^2 \text{ for } z \in B_\epsilon(0)$$

$$||F^{-1}A_F(z) - z|| \leq C'||z||^2 \text{ for } z \in B_{\epsilon'}(0)$$

Proof. By Lemma 1, there exists $\epsilon'' > 0$ such that

$$\frac{1}{\beta}||z|| \leq ||F^{-1}(z)|| \leq \frac{1}{\alpha}||z||$$

on $B_{\epsilon''}(0)$. There exists $\epsilon' > 0$ such that $B_{\epsilon'}(0) \subseteq A_F^{-1}(B_{\epsilon''}(0))$ for each $F \in \Gamma$.

Using the hypothesis, we have for each $F \in \Gamma$,

$$||A_F^{-1}F(z) - z|| \leq ||A_F^{-1}F(z)|| + ||z|| \leq \frac{\beta}{\alpha}||z|| + ||z|| \leq (\frac{\beta}{\alpha} + 1)\epsilon$$

on $B_\epsilon(0)$. Similarly, we have for each $F \in \Gamma$,

$$||F^{-1}A_F(z) - z|| \leq ||F^{-1}A_F(z)|| + ||z|| \leq \frac{\beta}{\alpha}||z|| + ||z|| \leq (\frac{\beta}{\alpha} + 1)\epsilon$$

on $B_{\epsilon'}(0)$. The conclusion follows directly from the Schwarz lemma. □

The following is a technical result that is needed for the convergence proposition below.

Lemma 3. *Let $a_i > 0$ $(i \in \mathbb{N})$. Then the following are equivalent:*

(i.) $\sum_i \sqrt{a_i} < \infty$

(ii.) $\sum_i a_i < \infty$ *and there exists* $b_i > 0$ $(i \in \mathbb{N})$ *such that* $\sum_i b_i < \infty$ *and* $\sum_i \frac{a_i}{b_i} < \infty$

Proof. $(i.) \Rightarrow (ii.)$ Assume $\sum_i \sqrt{a_i} < \infty$. Clearly $\sum_i a_i < \infty$. Letting $b_i = \sqrt{a_i}$, the assertion follows.

$(ii.) \Rightarrow (i.)$ Assume $\sum_i a_i < \infty$ and that there exists $b_i > 0$ $(i \in \mathbb{N})$ such that $\sum_i b_i < \infty$ and $\sum_i \frac{a_i}{b_i} < \infty$. Using the inequality, $2\sqrt{xy} \leq x + y$ for $x, y \geq 0$, we see that

$$\sum_i 2\sqrt{a_i} \leq \sum_i \left(\frac{a_i}{b_i} + b_i\right) = \sum_i \frac{a_i}{b_i} + \sum_i b_i < \infty. \qquad \square$$

3.1.1 An Important Convergence Result

The following convergence result is necessary in the proof of Theorem 3 below. However, given its precise bound it seems to be of interest on its own.

Proposition 3. *Fix $r > 0$. Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $\{||z|| < r\} \subseteq U_i$ for each $i \in \mathbb{N}$ and let $f_i : U_{i+1} \to U_i$ be holomorphic mappings. Suppose*

$$\sum_i \sqrt{||f_i - id||_{\{||z||<r\}}} < \infty.$$

Then sequence $f_1 \circ f_2 \circ \cdots \circ f_{j-1} \circ f_j$ converges uniformly on compacta on $\{||z|| < r\}$.

Proof. Notice that if $||f_i - id||_{\{||z||<r\}} = 0$, then $f_i \equiv id$ by the identity theorem. So without loss of generality assume $||f_i - id||_{\{||z||<r\}} > 0$ for each $i \in \mathbb{N}$. Let $\epsilon > 0$ and $\epsilon < \frac{1}{2(r+2)}$. By Lemma 3, there exists a sequence $\epsilon_i > 0$ such that $\sum_i \epsilon_i < \infty$ and $\sum_i \frac{||f_i - id||_{\{||z||<r\}}}{\epsilon_i} < \infty$. Next, there exists an $N \in \mathbb{N}$ such that $\sum_{i \geq N} \epsilon_i < \epsilon$ and $\sum_{i \geq N} \frac{||f_i - id||_{\{||z||<r\}}}{\epsilon_i} < \epsilon$. (Note that this means $\epsilon_i > ||f_i - id||_{\{||z||<r\}}$ for $i \geq N$.) It suffices to show that the sequence $f_N \circ f_{N+1} \circ \cdots \circ f_j$ converges uniformly on $\{||z|| \leq r - \epsilon\}$.

We proceed with strong induction.

Base Case ($j = N$): Notice that for $(z, w) \in \{||z|| < r\}^2$,

$$||f_N(z) - f_N(w)|| \leq ||f_N(z)|| + ||f_N(w)|| \leq (\epsilon + ||z||) + (\epsilon + ||w||) \leq 2(\epsilon + r) \leq 2(2 + r).$$

Now by the Schwarz Lemma, we have for $(z, w) \in \{||z|| < r\}^2$,

$$||f_N(z) - f_N(w)|| \leq \frac{2(r+2)}{(r - ||w||)} ||z - w||.$$

For sake of notation, we write S_k in place of $\{||z|| < r - \sum_{N+1}^{k} \epsilon_p\}$.

Strong Induction Hypothesis: For $(z, w) \in S_j^2$,

$$||f_N \circ \cdots \circ f_j(z) - f_N \circ \cdots \circ f_j(w)|| \leq \frac{2(r+2)}{(r - \sum_{N+1}^{j} \epsilon_p - ||w||)} ||z - w||.$$

We prove this for $j + 1$. First observe that

$$\begin{aligned}
||f_N \circ f_{N+1} \circ \cdots \circ f_{j+1} - f_N||_{S_{j+1}} &\leq \sum_{i=N}^{j} ||f_N \circ \cdots \circ f_{i+1} - f_N \circ \cdots \circ f_i||_{S_{j+1}} \qquad (3.1) \\
&\leq \sum_{i=N}^{j} \frac{2(r+2)}{(r - \sum_{N+1}^{i} \epsilon_p - ||id||_{S_{j+1}})} ||f_{i+1} - id||_{S_{j+1}} \\
&\qquad \text{by the strong I.H.}(\dagger) \\
&= \sum_{i=N}^{j} \frac{2(r+2)}{\epsilon_{i+1} + \cdots + \epsilon_{j+1}} ||f_{i+1} - id||_{S_{j+1}} \\
&\leq \sum_{i=N}^{j} \frac{2(r+2)}{\epsilon_{i+1}} ||f_{i+1} - id||_{S_{j+1}} \\
&< 2(r+2)\epsilon \\
&< 1.
\end{aligned}$$

($\dagger$Note that since $\epsilon_{i+1} > ||f_{i+1} - id||_{\{||z||<r\}}$ we have that $(f_{i+1}(z), id(z)) \in S_i^2$ for $z \in S_{j+1} \subseteq$

S_{i+1}.)

Now since $||f_N \circ f_{N+1} \circ \cdots \circ f_{j+1} - f_N||_{S_{j+1}} < 1$, we have

$$
\begin{aligned}
||f_N \circ f_{N+1} \circ \cdots \circ f_{j+1}||_{S_{j+1}} &< 1 + ||f_N||_{S_{j+1}} \\
&< 1 + ||id||_{S_{j+1}} + \epsilon \\
&\leq r + 2.
\end{aligned}
$$

Thus, for $(z, w) \in S_{j+1}^2$,

$$
\begin{aligned}
||f_N \circ \cdots \circ f_{j+1}(z) - f_N \circ \cdots \circ f_{j+1}(w)|| &\leq ||f_N \circ \cdots \circ f_{j+1}(z)|| + ||f_N \circ \cdots \circ f_{j+1}(w)|| \\
&\leq ||f_N \circ \cdots \circ f_{j+1}||_{S_{j+1}} + ||f_N \circ \cdots \circ f_{j+1}||_{S_{j+1}} \\
&< 2(r+2)
\end{aligned}
$$

Now, with the aid of the Schwarz Lemma, we have that for $(z, w) \in {S_{j+1}}^2$,

$$
||f_N \circ \cdots \circ f_{j+1}(z) - f_N \circ \cdots \circ f_{j+1}(w)|| \leq \frac{2(r+2)}{(r - \sum_{N+1}^{j+1} \epsilon_p - ||w||)} ||z - w||.
$$

This completes the induction.

Notice that in the course of the induction we have shown from (3.1) that

$$
\sum_{i=N}^{\infty} ||f_N \circ \cdots \circ f_{i+1} - f_N \circ \cdots \circ f_i||_{\{||z|| \leq r - \epsilon\}} \leq 1
$$

since $\sum_{N+1}^{\infty} \epsilon_k < \epsilon$. Thus sequence $f_N \circ f_{N+1} \circ \cdots \circ f_j$ converges uniformly on $\{||z|| \leq r - \epsilon\}$.

$\square$

We are now ready to prove Proposition 1.

Proof. Let K be a compact set in U. Since K is compact there exist finitely many balls $B_{\epsilon_1}(p_1)$, $\ldots$, $B_{\epsilon_k}(p_k)$ such that $K \subseteq \cup_{i=1}^k B_{\epsilon_i}(p_i)$ and $\cup_{i=1}^k B_{2\epsilon_i}(p_i) \subseteq U$. Now by applying Proposition 3 it is straightforward to see that for given $i \in \{1, \ldots, k\}$, sequence $f_1 \circ \cdots \circ f_j$ converges uniformly on compacta on $B_{2\epsilon_i}(p_i)$. Thus, for given $i \in \{1, \ldots, k\}$, $f_1 \circ \cdots \circ f_j$ converges uniformly on $\overline{B_{\epsilon_i}(p_i)}$. Therefore $f_1 \circ \cdots \circ f_j$ converges uniformly on $K \subseteq \cup_{i=1}^k \overline{B_{\epsilon_i}(p_i)}$. □

Lemma 4. *Let U_i be Runge domains in $\mathbb{C}^n$. Let U be a domain in $\mathbb{C}^n$. Let $f_i : \mathbb{C}^n \to U_i \subseteq \mathbb{C}^n$ be a sequence of biholomorphic mappings that converges (uniformly on compacta) to a biholomorphic mapping $f : \mathbb{C}^n \to U \subseteq \mathbb{C}^n$. Then U is Runge.*

Proof. Let K be a compact in U and g be a holomorphic function defined on U. We show that g can be approximated by polynomials on K. Notice that $(g \circ f) \circ f_i^{-1} \to_{i\to\infty} g$ uniformly on K. Thus for some (large) $N \in \mathbb{N}$, $(g \circ f) \circ f_N^{-1}$ approximates g. Of course $g \circ f$ can be approximated by polynomials on $f^{-1}(K)$ and f_N^{-1} can be approximated by polynomials on K. □

3.1.2 The Proof

We are now ready to prove Theorem 3. Throughout, we will use the notation $F(j)$ to mean $F_j \circ \cdots \circ F_1$. Analogously, we will use the notation $A(j)$ and $A^{-1}(j)$.

Proof. First let us note that since $\sum_i \sqrt{(\frac{r_1^2}{s_1}) \cdots (\frac{r_i^2}{s_i})} < \infty$ we have $\prod_i \frac{r_i^2}{s_i} = 0$, and so $\prod_i r_i = 0$.

Without loss of generality it suffices to prove the assertion for $p = 0$. First we remark that the attracting basin $\{F_j \circ \cdots \circ F_1 \to 0\}$ is a connected open set. Notice that since $\prod_i r_i = 0$ and $\sup_i\{r_i\} \leq 1$, $\{F_j \circ \cdots \circ F_1 \to 0\} = \cup_i F(i)^{-1}(B_\epsilon(0))$ and $\cup_i F(i)^{-1}(B_\epsilon(0))$ is the union of connected open sets each containing the origin.

Define automorphisms $\Phi_j = A(j)^{-1}F(j)$ and $\Psi_j = F(j)^{-1}A(j)$. Clearly $\Phi_j \circ \Psi_j = \mathrm{id}_{\mathbb{C}^n}$ and $\Psi_j \circ \Phi_j = \mathrm{id}_{\mathbb{C}^n}$ for each $j \in \mathbb{N}$. We show that sequence Φ_j converges on $\{F_j \circ \cdots \circ F_1 \to 0\}$ and that sequence Ψ_j converges on $\mathbb{C}^n$.

Let C, C', ϵ' be as in Lemma 2 for $\Gamma = \{F_i : i \in \mathbb{N}\}, \alpha = \inf_i\{s_i\},$ and $\beta = \sup_i\{r_i\}$.

Let K be a compact in $\{F_j \circ \cdots \circ F_1 \to 0\}$. Then there exists an $l \in \mathbb{N}$ such that $F(l)(K) \subseteq B_\epsilon(0)$. Now notice that for $j > l$:

$$\begin{aligned}
||\Phi_{j+1}(z) - \Phi_j(z)|| &\leq ||A(j)^{-1}(A_{j+1}^{-1}F_{j+1}(F(j)(z)) - F(j)(z))|| \\
&\leq s_1^{-1} \cdots s_j^{-1} \cdot C||F(j)(z)||^2 \text{ by Lemma 2} \\
&\leq s_1^{-1} \cdots s_j^{-1} \cdot C \cdot r_j^2 r_{j-1}^2 \cdots r_{l+1}^2 \cdot \epsilon^2 \\
&\leq C\epsilon^2 s_1^{-1} \cdots s_l^{-1} \left(\frac{r_{l+1}^2}{s_{l+1}}\right) \cdots \left(\frac{r_j^2}{s_j}\right) \text{ for } z \in K.
\end{aligned}$$

Since $\sum_i (\frac{r_1^2}{s_1}) \cdots (\frac{r_i^2}{s_i}) < \infty$, Φ_j converges uniformly on K. Thus Φ_j converges uniformly on compacta to a holomorphic map $\Phi : \{F_j \circ \cdots \circ F_1 \to 0\} \to \mathbb{C}^n$.

Now we show sequence Ψ_j converges on $\mathbb{C}^n$. Let K' be a compact in $\mathbb{C}^n$. Since $\prod_i r_i = 0$, there exists an $l \in \mathbb{N}$ such that $A_l \cdots A_1(K') \subseteq B_{\epsilon'}(0)$. We show that sequence $F_l \cdots F_1 \Psi_j A_1^{-1} \cdots A_l^{-1}$ $= F_{l+1}^{-1} \cdots F_j^{-1} A_j \cdots A_{l+1}$ converges as $j \to \infty$. We see

$$\begin{aligned}
F_l \cdots F_1 \Psi_j A_1^{-1} \cdots A_l^{-1} &= F_{l+1}^{-1} \cdots F_j^{-1} A_j \cdots A_{l+1} \\
&= F_{l+1}^{-1} \cdots F_{j-1}^{-1} G_j A_{j-1} \cdots A_{l+1} \\
&= F_{l+1}^{-1} \cdots F_{j-1}^{-1} A_{j-1} \cdots A_{l+1} H_j \\
&= F_{l+1}^{-1} A_{l+1} H_{l+2} \cdots H_{j-1} H_j \\
&= H_{l+1} \cdots H_j
\end{aligned}$$

where we define automorphisms $G_i := F_i^{-1} A_i$ and $H_i := (A_{i-1} \cdots A_{l+1})^{-1} G_i A_{i-1} \cdots A_{l+1}$. Now notice $G_i(z) = z + g_i(z)$ where $||g_i(z)|| \leq C'||z||^2$ on $B_{\epsilon'}(0)$ by Lemma 2 and

$H_i(z) = z + h_i(z)$ where $h_i = (A_{i-1} \cdots A_{l+1})^{-1} g_i A_{i-1} \cdots A_{l+1}$ so

$$||h_i(z)|| \leq C' ||(A_{i-1} \cdots A_{l+1})^{-1}|| ||A_{i-1} \cdots A_{l+1}||^2 ||z||^2 \leq C' \left(\frac{r_{l+1}^2}{s_{l+1}}\right) \cdots \left(\frac{r_{i-1}^2}{s_{i-1}}\right) ||z||^2$$

on $B_{\epsilon'}(0)$. Since $||H_i(z) - z|| \leq C'(\frac{r_{l+1}^2}{s_{l+1}}) \cdots (\frac{r_{i-1}^2}{s_{i-1}})||z||^2$ on $B_{\epsilon'}(0)$ it follows by Proposition 1 that sequence $H_1 \cdots H_j$ converges uniformly on compacta on $B_{\epsilon'}(0)$. Thus sequence Ψ_j converges uniformly on K'. Thus $\Psi_j \to \Psi$ for some holomorphic mapping $\Psi : \mathbb{C}^n \to \mathbb{C}^n$.

Finally, we will show that $\Phi : \{F_j \circ \cdots \circ F_1 \to 0\} \to \mathbb{C}^n$ is bijective by using Theorem 5.2 in Dixon and Esterle [4]. To this end, we see that if $z \notin \{F_j \circ \cdots \circ F_1 \to 0\}$ then the sequence $\Phi_j(z)$ diverges: if $z \notin \{F_j \circ \cdots \circ F_1 \to 0\} = \cup_i F(i)^{-1}(B_\epsilon(0))$, then $|F(j)(z)| \geq \epsilon$ for each j so $||\Phi_j(z)|| = ||A(j)^{-1} F(j)(z)|| \geq (\frac{1}{r_1}) \cdots (\frac{1}{r_j})\epsilon \to_{j\to\infty} \infty$. Thus $\{F_j \circ \cdots \circ F_1 \to 0\}$ is the largest connected open set on which sequence Φ_j converges uniformly on compacta. Of course, $\mathbb{C}^n$ is the largest connected open set on which sequence Ψ_j converges uniformly on compacta. Also note that $\text{id} = J\Phi_j(0) \to J\Phi(0) = \text{id}$, thus Φ is nondegenerate. Therefore by Theorem 5.2 in Dixon and Esterle [4], Φ is injective. Similarly, Ψ is nondegenerate and thus by Theorem 5.2, image$(\Psi) \subseteq \{F_j \circ \cdots \circ F_1 \to 0\}$. Hence $\mathbb{C}^n = \text{image}(\Phi \circ \Psi) = \Phi(\text{image}(\Psi)) \subset \Phi(\{F_j \circ \cdots \circ F_1 \to 0\})$. Thus Φ is surjective.

Finally, applying the Lemma 4 to Ψ_j, we conclude $\Psi(\mathbb{C}^n) = \{F_j \circ \cdots \circ F_1 \to 0\}$ is Runge. $\square$

As remarked earlier, the main distinction between the above Theorem and Wold [1] Theorem 4, is that it is now possible that $\frac{r_i^2}{s_i} < 1$ for infinitely many i.

3.2 On Containment Between Fatou–Bieberbach Domains

The following is a minor generalization of Lemma 1.2 from Rosay and Rudin [2].

Lemma 5. *Suppose that $\epsilon > 0$ and that*

(i.) $a_1, \ldots, a_{m_1}$ *are points in a compact convex set* $K \subseteq \mathbb{C}^n$

(ii.) $b_1, \ldots, b_{m_2}$ *are points in* $\mathbb{C}^n \setminus K$

(iii.) p *and* q *are points in a hyperplane* $\Pi \subseteq \mathbb{C}^n$ *(of complex dimension* $n-1$*) which does not intersect* $K \cup \{b_1, \ldots, b_{m_2}\}$.

Then there is an automorphism (in particular, a shear) τ *which moves* p *to* q*, fixes every* a_i*, fixes every* b_i*, and moves no point of* K *by as much as* ϵ.

Proof. See the corresponding proof of Lemma 1.2 in Rosay and Rudin [2] and notice that polynomial g can additionally be chosen so that $g(\Lambda b_i) = 0$ $(1 \leq i \leq m_2)$. □

Analogous to Corollary 1.3 from Rosay and Rudin [2], we have:

Lemma 6. *If properties* (i) *and* (ii) *of Lemma 5 hold, and* p*,* q *are points in* $\mathbb{C}^n \setminus (K \cup \{b_1, \ldots, b_{m_2}\})$*, then some automorphism (in particular, some composition of two shears) moves* p *to* q*, fixes every* a_i*, fixes every* b_i*, and moves no point of* K *by as much as* ϵ.

Proof. There exist hyperplanes Π' and Π'', through p and q, respectively, which do not intersect $K \cup \{b_1, \ldots, b_{m_2}\}$ and which are not parallel. Choose $w \in \Pi' \cap \Pi''$ and apply Lemma 5 twice, moving p to w and then w to q. □

Lemma 7. *(Pushing-Points Lemma) Suppose that* $\epsilon > 0$ *and that*

(i.) $a_1, \ldots, a_{m_1}$ *are points in a compact convex set* $K \subseteq \mathbb{C}^n$

(ii.) $p_1, \ldots, p_{m_2}, q_1, \ldots, q_{m_2}$ *are distinct points in* $\mathbb{C}^n \setminus K$

Then there exists an automorphism (in particular, a composition of $2 \cdot m_2$ *shears) which moves* p_i *to* q_i *for each* $i \in \{1, \ldots, m_2\}$*, fixes every* a_i*, and moves no point of* K *by as much as* $\epsilon > 0$.

Proof. Use the previous lemma m_2 times: let ϕ_i denote a composition of two shears such that $\phi_i(q_k) = q_k$ for $k < i$, $\phi_i(p_i) = q_i$, and $\phi_i(p_k) = p_k$ for $k > i$. Take $\phi_{m_2} \circ \phi_{m_2 - 1} \circ \cdots \circ \phi_1$. □

A variation of the above lemma appears in Forstnerič's book [8, Corollary 4.12.7].

Lemma 8. *Let $U_1, U_2 \subseteq \mathbb{C}^n$ be nonempty connected open sets and let $f : U_1 \to U_2$ be a biholomorphic mapping. Suppose that U_2 is Runge and suppose that $V \subseteq U_1$ is a nonempty connected open set that is Runge. Then $f(V)$ is Runge.*

Proof. Let $g : f(V) \to \mathbb{C}^n$ be holomorphic. Write $g = (g \circ f) \circ f^{-1}$. Of course, $g \circ f$ can be approximated by polynomials on V and f^{-1} can be approximated by polynomials on U_2. $\square$

We are now ready to prove Theorem 6.

Proof. For simplicity we will assume sets $\{\Omega_j\}$ and $\{a_j\}$ are countably infinite, the other cases are similar. Without loss of generality assume $p = 0$. And let $\overline{B_\epsilon(0)} \subseteq K_j^1 \subseteq K_j^2 \subseteq \cdots$ be a compact exhaustion of Ω_j. To prove the assertion, we will construct an attracting basin and apply Theorem 3 (or Theorem 4 from [1]). By Dixon and Esterle [4] Corollary 5.3, the Fatou–Bieberbach domain that we obtain is Runge. As the sequence of automorphisms to be applied is rather complicated, we describe the automorphisms to be applied in stages where the set of functions in stage j is applied after stage $j-1$. We denote the composition of all automorphisms in stage j by S_j. S_j will have the following properties:

(a) $S_1(K) \subseteq B_\epsilon(0)$

(b) $S_j(S_{j-1} \circ \cdots \circ S_1(a_j)) \in B_\epsilon(0)$

(c) $S_j|_{\{b_1,\ldots,b_l\}} = \mathrm{id}|_{\{b_1,\ldots,b_l\}}$

(d) for $j > 1$, $S_j(S_{j-1} \circ \cdots \circ S_1(\cup_{i=1}^{j} K_i^j)) \subseteq B_\epsilon(0)$

Each automorphism in the composition of S_j will satisfy the hypothesis of Wold [1] Theorem 4. Thus, it is clear that Ω, the attracting basin that is constructed from the automorphisms is equal to $\{z \in \mathbb{C}^n : \lim_{i\to\infty} S_i \circ \cdots \circ S_1(z) = 0\}$. Noting this, together with the above properties ensure that the constructed Fatou–Bieberbach domain satisfies the properties of the conclusion of the assertion. In particular, since each S_j fixes $\{b_1, \ldots, b_l\}$, $\{b_1, \ldots, b_l\} \subseteq \Omega^{\mathsf{c}}$.

Stage 1: Let $r \in (0,1)$ be small enough such that $rK \subseteq B_\epsilon(0)$. Let $\delta \in (0,1)$ be small enough that $(r+\delta)^2 < r - \delta$ and $r + \delta < 1$. Now, applying Corollary 4.12.4 from Forstnerič [8] and then the above point-pushing lemma it is straightforward to see that, there exists $\phi_1 \in \mathrm{Aut}_0(\mathbb{C}^n)$ such that

$$(r-\delta)||z|| \leq ||\phi_1(z)|| \leq (r+\delta)||z||$$

on $B_\epsilon(0)$, $\phi_1(K) \subseteq B_\epsilon(0)$, and $\phi_1|_{\{b_1,\ldots,b_l\}} = \mathrm{id}|_{\{b_1,\ldots,b_l\}}$. [To see this, apply Corollary 4.12.4 from Forstnerič [8] to get automorphism τ_1 such that

$$\tau_1 \approx r\mathbb{I}$$

on K and

$$\tau_1 \approx \mathbb{I}$$

on $\{b_1, \ldots, b_l\}$. By a translation, we can assume without loss of generality $\tau_1(0) = 0$. By the Schwarz lemma,

$$(r-\varepsilon')||z|| \leq ||\tau_1(z)|| \leq (r+\varepsilon')||z||$$

on $B_\epsilon(0)$. Now by the point-pushing lemma there is an automorphism τ_2 such that $\tau_2 \approx \mathbb{I}$ on $B_\epsilon(0)$, $\tau_2(0) = 0$, and $\tau_2 \circ \tau_1|_{\{b_1,\ldots,b_l\}} = \mathrm{id}|_{\{b_1,\ldots,b_l\}}$. By the Schwarz lemma,

$$(1-\varepsilon'')||z|| \leq ||\tau_2(z)|| \leq (1+\varepsilon'')||z||$$

on $B_\epsilon(0)$. Thus

$$(1-\varepsilon'')(r-\varepsilon')||z|| \leq ||\tau_2 \circ \tau_1(z)|| \leq (1+\varepsilon'')(r+\varepsilon')||z||$$

on $B_\epsilon(0)$ and $\tau_2 \circ \tau_1(K) \subseteq B_\epsilon(0)$. Take $\phi_1 = \tau_2 \circ \tau_1$.]

Now we work on satisfying property (b). Using Forstnerič-Rosay [9] Theorem 2.3 and then the above point-pushing lemma it is straightforward to see that, there exists $\rho_1 \in \mathrm{Aut}_0(\mathbb{C}^n)$ such that

$$(r-\delta)||z|| \leq ||\rho_1(z)|| \leq (r+\delta)||z||$$

on $B_\epsilon(0)$, $\rho_1 \circ \phi_1(a_1) \in B_\epsilon(0)$, and $\rho_1 \circ \phi_1|_{\{b_1,\ldots,b_l\}} = \mathrm{id}|\{b_1,\ldots,b_l\}$. [To see this, apply Forstnerič-Rosay [9] Theorem 2.3 to get automorphism τ_1 such that

$$\tau_1 \approx r\mathbb{I}$$

on $B_\epsilon(0)$, $\tau_1(\phi_1(a_1)) \in B_\epsilon(0) \setminus \overline{B_{r\epsilon}(0)}$, and

$$\tau_1 \approx \mathbb{I}$$

on $\{b_1,\ldots,b_l\}$. Without loss of generality we can assume $\tau_1(0) = 0$. By the Schwarz lemma,

$$(r-\varepsilon')||z|| \leq ||\tau_1(z)|| \leq (r+\varepsilon')||z||$$

on $B_\epsilon(0)$. Now by the point-pushing lemma there is an automorphism τ_2 such that $\tau_2 \approx \mathbb{I}$ on $B_\epsilon(0)$, $\tau_2(0) = 0$, $\tau_2(\tau_1(\phi_1(a_1))) = \tau_1(\phi_1(a_1))$, and $\tau_2 \circ \tau_1|_{\{b_1,\ldots,b_l\}} = \mathrm{id}|_{\{b_1,\ldots,b_l\}}$. By the Schwarz lemma,

$$(1-\varepsilon'')||z|| \leq ||\tau_2(z)|| \leq (1+\varepsilon'')||z||$$

on $B_\epsilon(0)$. Thus

$$(1-\varepsilon'')(r-\varepsilon')||z|| \leq ||\tau_2 \circ \tau_1(z)|| \leq (1+\varepsilon'')(r+\varepsilon')||z||$$

on $B_\epsilon(0)$ and $\tau_2 \circ \tau_1(\phi_1(a_1)) \in B_\epsilon(0)$. Take $\rho_1 = \tau_2 \circ \tau_1$.]

Thus $S_1 := \rho_1 \circ \phi_1$ satisfies properties (a), (b), and (c) (and (d) vacuously.)

Stage j (for $j \geq 2$): Let $l_1 \in \mathbb{N}$ such that $r^{l_1}(S_{j-1} \circ \cdots \circ S_1)(K_1^j) \subseteq B_\epsilon(0)$. Since $\{b_1, \ldots, b_l\} \cap \Omega_1 = \varnothing$, we have $\{b_1, \ldots, b_l\} \cap S_{j-1} \circ \cdots \circ S_1(\Omega_1) = \varnothing$. Now since $\{b_1, \ldots, b_l\} \cap S_{j-1} \circ \cdots \circ S_1(\Omega_1) = \varnothing$ and $S_{j-1} \circ \cdots \circ S_1(\Omega_1)$ is Runge, by Wold [1] Lemma 4, there exists $\phi_1 \in \mathrm{Aut}(\mathbb{C}^n)$ such that $\phi_1 \approx \mathrm{id}$ on $S_{j-1} \circ \cdots \circ S_1(K_1^j)$ and $\phi_1(\{b_1, \ldots, b_j\}) \subseteq B_{(\frac{1}{r})^{l_1}\epsilon}(0)^{\mathsf{c}}$. Without loss of generality we can assume $\phi_1(0) = 0$. By Forstnerič-Rosay [9] Theorem 2.3 and then the above point-pushing lemma it is straightforward to see that, there exists $\psi_1 \in \mathrm{Aut}_0(\mathbb{C}^n)$ such that

$$(r - \delta)||z|| \leq ||\psi_1(z)|| \leq (r + \delta)||z||$$

on $B_\epsilon(0)$ and $\psi_1 \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_1-1} \circ (r\phi_1) = \mathrm{id}|_{\{b_1,\ldots,b_l\}}$.

Notice that $\psi_1 \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_1-1} \circ (r\phi_1)$ satisfies property (c) and $\psi_1 \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_1-1} \circ (r\phi_1) \circ S_{j-1} \circ \cdots \circ S_1(K_1^j) \subseteq B_\epsilon(0)$. We can do the same for $K_2^j, K_3^j, \ldots, K_j^j$. This gives

$$T := \psi_j \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_j-1} \circ (r\phi_j) \circ \psi_{j-1} \circ \cdots \circ \psi_1 \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_1-1} \circ (r\phi_1)$$

which satisfies properties (c) and (d).

Now we work on satisfying property (b). Using Forstnerič-Rosay [9] Theorem 2.3 and then the above point-pushing lemma it is straightforward to see that, there exists $\rho_j \in \mathrm{Aut}_0(\mathbb{C}^n)$ such that

$$(r - \delta)||z|| \leq ||\rho_j(z)|| \leq (r + \delta)||z||$$

on $B_\epsilon(0)$, $\rho_j \circ T(a_j) \in B_\epsilon(0)$, and $\rho_j \circ T|_{\{b_1,\dots,b_l\}} = \mathrm{id}|\{b_1,\dots,b_l\}$. Thus

$$S_j := \rho_j \circ \psi_j \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_j - 1} \circ (r\phi_j) \circ \psi_{j-1} \circ \cdots \circ \psi_1 \circ \underbrace{(r\mathbb{I}) \circ \cdots \circ (r\mathbb{I})}_{l_1 - 1} \circ (r\phi_1)$$

satisfies properties (a) (vacuously), (b) and (c) and (d). □

We conjecture that if the word "Runge" is removed from Theorem 6 that the statement remains true:

Conjecture 1. *Let $\{\Omega_j\}$ be a nonempty countable set of Fatou–Bieberbach domains in $\mathbb{C}^n$ such that $\cup_j \Omega_j \neq \mathbb{C}^n$. Let K be compact set that is polynomially convex. Suppose there exist $\epsilon > 0$ and $p \in \mathbb{C}^n$ such that $\overline{B_\epsilon(p)} \subseteq \cap_j \Omega_j$ and $\overline{B_\epsilon(p)} \subseteq K$. Let $\{a_j\}$ be a countable set in $(\cup_j \Omega_j)^{\mathsf{c}}$. Let $\{b_1, \dots, b_l\}$ be a nonempty finite set in $(\cup_j \Omega_j)^{\mathsf{c}}$ and suppose $\{b_1, \dots, b_l\} \cap (\cup_j \{a_j\} \cap K) = \varnothing$. Then there exists a Fatou–Bieberbach domain Ω such that $\cup_j \Omega_j \subseteq \Omega$, $\cup_j \{a_j\} \subseteq \Omega$, $K \subseteq \Omega$, and $\{b_1, \dots, b_l\} \subseteq \Omega^{\mathsf{c}}$.*

Of course, Theorem 6 above shows that given a Runge Fatou–Bieberbach domain we can find a strictly larger Runge Fatou–Bieberbach domain that contains it. We now record some properties for containment in the other direction.

Lemma 9. *Let $U_1, U_2 \subseteq \mathbb{C}^n$ be nonempty connected open sets and let $f : U_1 \to U_2$ be a biholomorphic mapping. Suppose that U_1 is Runge and suppose that $V \subseteq U_1$ is a nonempty connected open set that is non-Runge. Then $f(V)$ is non-Runge.*

Proof. If $f(V)$ is Runge, then by Lemma 8, $V = f^{-1}(f(V))$ is Runge, a contradiction. □

Corollary 1. *Every Runge Fatou–Bieberbach domain contains a proper Runge Fatou–Bieberbach domain.*

Proof. Let Ω be a Runge Fatou–Bieberbach domain. Let $F : \mathbb{C}^n \to \Omega$ be a biholomorphic mapping. Clearly $F(\Omega)$ is biholomorphic to $\mathbb{C}^n$, $F \circ F(\mathbb{C}^n) = F(\Omega) \subsetneq \Omega$, and by Lemma 8, $F(\Omega)$ is

Runge. $\square$

Corollary 2. *Every Runge Fatou–Bieberbach domain contains a (proper) non-Runge Fatou–Bieberbach domain.*

Proof. It was shown in Wold [10] that a non-Runge Fatou–Bieberbach exists in $\mathbb{C}^n$, say Ω'. Let Ω be an arbitrary Runge Fatou–Bieberbach domain in $\mathbb{C}^n$ and let $F : \mathbb{C}^n \to \Omega$ be a biholomorphic mapping. Then $F(\Omega') \subsetneq \Omega$ is a Fatou–Bieberbach domain and is non-Runge by the above lemma. $\square$

Corollary 3. *Every non-Runge Fatou–Bieberbach domain contains a proper non-Runge Fatou–Bieberbach domain.*

Proof. Let Ω' be a non-Runge Fatou–Bieberbach domain and let $F : \mathbb{C}^n \to \Omega'$ be a biholomorphic mapping. Consider the Fatou–Bieberbach domain $F(\Omega')$. Of course $F(\Omega') \subsetneq \Omega'$. If $F(\Omega')$ is non-Runge then we are done, so suppose $F(\Omega')$ is Runge. Then by Corollary 1, $F(\Omega')$ contains a proper non-Runge Fatou–Bieberbach domain. $\square$

Rather surprisingly, asking whether a non-Runge Fatou–Bieberbach domains contains a Runge Fatou–Bieberbach domain seems to be a hard question.

Conjecture 2. *Every non-Runge Fatou–Bieberbach domain contains a (proper) Runge Fatou–Bieberbach domain.*

3.3 A New Construction

The following is a technical lemma needed in the upcoming proof.

Lemma 10. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Suppose $0 < r_j < 1$ and $\epsilon > 0$ such that $\sup_i\{r_i\} < 1$ and $||F_j(z) - p|| \leq r_j||z - p||$ on $B_\epsilon(p)$. Then $\Omega^{p,\epsilon}_{\{F_j\}}$ is a nonempty connected open set and $\Omega^{p,\epsilon}_{\{F_j\}} = \Omega^{p,\epsilon'}_{\{F_j\}}$ for each $\epsilon' \in (0, \epsilon)$.*

Proof. The hypothesis implies that

$$\text{oliminf}_{j\to\infty}(F(j+i,j))^{-1}(B_\epsilon(p)) \subseteq \text{oliminf}_{j\to\infty}(F(j+i+1,j))^{-1}(B_\epsilon(p))$$

and so $\Omega^{p,\epsilon}_{\{F_j\}}$ is the union of nested connected open sets, and is therefore connected.

Clearly $\Omega^{p,\epsilon'}_{\{F_j\}} \subseteq \Omega^{p,\epsilon}_{\{F_j\}}$. For the other direction, note that there is an $l \in \mathbb{N}$ such that $\epsilon' > (\sup\{r_j\})^l\epsilon$. Thus

$$F_i \circ F_{i+1} \circ \cdots \circ F_{i+l-1}(B_\epsilon(p)) \subseteq B_{\epsilon'}(p)$$

for each $i \in \mathbb{N}$. Hence$(F_i \circ F_{i+1} \circ \cdots \circ F_{i+l-1})^{-1}(B_{\epsilon'}(p)) \supseteq B_\epsilon(p)$ for each $i \in \mathbb{N}$. It follows that $\Omega^{p,\epsilon'}_{\{F_j\}} \supseteq \Omega^{p,\epsilon}_{\{F_j\}}$ □

We now prove Theorem 7.

Proof. Without loss of generality assume $p = 0$.

Using the hypothesis and Lemma 1, we see

$$||F_j^{-1}A_j - z|| = ||(A_j^{-1}F_j - \mathbb{I})(F_j^{-1}A_j)(z)|| \leq c_j||F_j^{-1}A_jz||^2 \leq c_j\frac{1}{s_j^2}r_j^2||z||^2 \leq c_j\frac{[\sup_i\{r_i\}]^2}{[\inf_i\{s_i\}]^2}||z||^2$$

on $B_{\epsilon''}(0)$ for some small $\epsilon'' > 0$.

Thus by shrinking ϵ, allowed by the above lemma, we can without loss of generality assume that

$$||F_j^{-1}A_j - z|| \leq Dc_j||z||^2$$

on $B_\epsilon(0)$ where $D := \frac{[\sup_i\{r_i\}]^2}{[\inf_i\{s_i\}]^2} > 0$.

Define biholomorphic mappings $\Phi_j : U_{j+1} \to \mathbb{C}^n$ by $\Phi_j = A_j^{-1}\cdots A_1^{-1}F_1\cdots F_j$ and $\Psi_j : \mathbb{C}^n \to$

U_{j+1} by $\Psi_j = F_j^{-1} \cdots F_1^{-1} A_1 \cdots A_j$. (Note that Ψ_j is everywhere defined since $U_1 = \mathbb{C}^n$.) We show that there is a subsequence of Φ_j that converges on $\Omega^{p,\epsilon}_{\{F_j\}}$ and diverges on $\cup_{k=1}^{\infty} \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0))$. We also show that there is a subsequence of Ψ_j that converges on $\mathbb{C}^n$.

First we show Φ_j is uniformly bounded on $B_\epsilon(0)$. We write $A_i^{-1} F_i = z + f_i(z)$ where $||f_i(z)|| \leq c_i ||z||^2$ on $B_\epsilon(0)$. Then

$$\begin{aligned}
\Phi_j &= A_j^{-1} \cdots A_1^{-1} F_1 \cdots F_j \\
&= A_j^{-1} \cdots A_2^{-1} (\mathbb{I} + f_1) F_2 \cdots F_j \\
&= A_j^{-1} \cdots A_2^{-1} F_2 \cdots F_j + A_j^{-1} \cdots A_2^{-1} f_1 F_2 \cdots F_j \\
&\vdots \\
&= z + f_j + \sum_{i=1}^{j-1} A_j^{-1} \cdots A_{i+1}^{-1} f_i F_{i+1} \cdots F_j.
\end{aligned}$$

Thus for $z \in B_\epsilon(0)$,

$$\begin{aligned}
||\Phi_j(z)|| &\leq ||z|| + ||f_j(z)|| + \sum_{i=1}^{j-1} ||A_j^{-1} \cdots A_{i+1}^{-1} f_i F_{i+1} \cdots F_j|| \\
&\leq \epsilon + c_j ||z||^2 + \sum_{i=1}^{j-1} c_i (\frac{r_{i+1}^2}{s_{i+1}}) \cdots (\frac{r_j^2}{s_j}) ||z||^2 \\
&= \epsilon + (c_j + \sum_{i=1}^{j-1} c_i (\frac{r_{i+1}^2}{s_{i+1}}) \cdots (\frac{r_j^2}{s_j})) ||z||^2 \\
&< M \text{ for some } M > 0.
\end{aligned}$$

Next we show Φ_j is locally uniformly bounded on $\Omega^{0,\epsilon}_{\{F_j\}}$. Let $K \subseteq \Omega^{0,\epsilon}_{\{F_j\}}$ be a compact. By compactness there is an $l \in \mathbb{N}$ such that $K \subseteq \text{oliminf}_{i \to \infty} (F(i+l, i+1))^{-1}(B_\epsilon(0))$. Thus, for j

sufficiently large we have

$$
\begin{aligned}
||\Phi_{j+l}(z)|| &= ||A_{j+l}^{-1} \cdots A_{j+1}^{-1}(A_j^{-1} \cdots A_1^{-1} F_1 \cdots F_j) F_{j+1} \cdots F_{j+l}(z)|| \\
&\le \frac{1}{s_{j+l}} \cdots \frac{1}{s_{j+1}} \cdot \sup_{w \in B_\epsilon(0)} ||A_j^{-1} \cdots A_1^{-1} F_1 \cdots F_j(w)|| \\
&< \frac{1}{s_{j+l}} \cdots \frac{1}{s_{j+1}} M \\
&\le \left(\frac{1}{\inf_i \{s_i\}} \right)^l M.
\end{aligned}
$$

This shows that Φ_j is locally uniformly bounded on $\Omega^{0,\epsilon}_{\{F_j\}}$. Therefore by Montel's Theorem there is a subsequence $\Phi_{j(j')}$ that converges uniformly on compacta on $\Omega^{0,\epsilon}_{\{F_j\}}$.

Define $d_j = Dc_j$. Choose $\alpha > 0$ such that $\sqrt{\alpha} < \epsilon$ and $\sqrt{\alpha} < \delta$ and $[1 + \sup_k (d_k + \sum_{i=1}^{k-1} d_i (\frac{r_{i+1}^2}{s_{i+1}}) \cdots (\frac{r_k^2}{s_k}))]\alpha < \sqrt{\alpha}$. We show Ψ_j is uniformly bounded on $B_\alpha(0)$. We write $F_i^{-1} A_i = z + \tilde{f}_i(z)$ where $||\tilde{f}_i(z)|| \le d_i ||z||^2$ on $B_\epsilon(0)$. Then

$$
\begin{aligned}
\Psi_j &= F_j^{-1} \cdots F_1^{-1} A_1 \cdots A_j \\
&= F_j^{-1} \cdots F_2^{-1} (F_1^{-1} A_1) A_2 \cdots A_j \\
&= F_j^{-1} \cdots F_2^{-1} A_2 \cdots A_j H_1 \\
&= H_j \cdots H_2 H_1
\end{aligned}
$$

where $H_i = (A_{i+1} \cdots A_j)^{-1} (F_i^{-1} A_i)(A_{i+1} \cdots A_j)$. Now notice

$$
\begin{aligned}
||H_i(z)|| &= ||z + (A_{i+1} \cdots A_j)^{-1} \tilde{f}_i (A_{i+1} \cdots A_j)|| \\
&\le ||z|| + d_i (\frac{r_{i+1}^2}{s_{i+1}}) \cdots (\frac{r_j^2}{s_j}) ||z||^2 \text{ on } B_\epsilon(0).
\end{aligned}
$$

So for $z \in B_\alpha(0)$,

$$\begin{aligned}
||\Psi_j(z)|| &= ||H_j \cdots H_1(z)|| \\
&\leq \sup ||H_j \cdots H_2((\alpha + d_1(\frac{r_2^2}{s_2}) \cdots (\frac{r_j^2}{s_j})\alpha)B_1(0))|| \\
&\leq \sup ||H_j \cdots H_3((\alpha + d_1(\frac{r_2^2}{s_2}) \cdots (\frac{r_j^2}{s_j})\alpha + d_2(\frac{r_3^2}{s_3}) \cdots (\frac{r_j^2}{s_j})\alpha)B_1(0))|| \text{ by our choice of } \alpha \\
&\quad \vdots \\
&\leq [1 + d_j + \sum_{i=1}^{j-1} d_i(\frac{r_{i+1}^2}{s_{i+1}}) \cdots (\frac{r_j^2}{s_j})]\alpha \\
&\leq \sqrt{\alpha} < \delta.
\end{aligned}$$

Next we show Ψ_j is locally uniformly bounded on $\mathbb{C}^n$. Let $K' \subseteq \mathbb{C}^n$ be a compact. By compactness, there exists an $l' \in \mathbb{N}$ such that $(\sup_i\{r_i\})^{l'} K' \subseteq B_\alpha(0)$. Thus, we have for $z \in K'$,

$$\begin{aligned}
||\Psi_{j+l'}(z)|| &= ||F_{j+l'}^{-1} \cdots F_{j+1}^{-1}(F_j^{-1} \cdots F_1^{-1} A_1 \cdots A_j) A_{j+1} \cdots A_{j+l'}(z)|| \\
&\leq \sup ||F_{j+l'}^{-1} \cdots F_{j+1}^{-1}(F_j^{-1} \cdots F_1^{-1} A_1 \cdots A_j)(B_\alpha(0))|| \\
&\leq \sup ||F_{j+l'}^{-1} \cdots F_{j+1}^{-1}(B_\delta(0))|| < B_{l'}.
\end{aligned}$$

This shows Ψ_j is locally uniformly bounded on $\mathbb{C}^n$. Therefore by Montel's Theorem there is a subsequence $\Psi_{j(j'(j''))}$ that converges uniformly on compacta on $\mathbb{C}^n$.

Of course, $\Phi_{j(j'(j''))}$ converges uniformly on compacta on $\Omega^{0,\epsilon}_{\{F_j\}}$. Let FB denote the connected component containing 0 of the largest open set where $\Phi_{j(j'(j''))}$ converges uniformly on compacta. Obviously, $\mathbb{C}^n$ is the largest open set on which $\Psi_{j(j'(j''))}$ converges uniformly on compacta. By Dixon and Esterle [4] Theorem 5.2, it follows that FB is biholomorphic to $\mathbb{C}^n$ and that $FB \subseteq \limsup_i U_i$.

If each U_i is Runge, then applying Lemma 8 to $\Psi_{j(j'(j''))}$, we conclude that FB is Runge.

We have shown that $\Omega^{0,\epsilon}_{\{F_j\}} \subseteq FB$. It remains to show that $FB \subseteq \limsup U_i \setminus \cup_{k=1}^{\infty} \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0))$. It suffices to show that $\Phi_{j(j'(j''))}$ diverges on $\cup_{k=1}^{\infty} \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0))$. If $z \in \cup_{k=1}^{\infty} \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0))$, then $z \in \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_i)^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0))$ for some k; so for large j, $||\Phi_j(z)|| = ||A_j^{-1} \cdots A_1^{-1} F_1 \cdots F_j(z)|| \geq (\frac{1}{r_j}) \cdots (\frac{1}{r_1}) \frac{1}{k} \to_{j \to \infty} \infty$. $\square$

Because the reverse semi-basin of attraction is defined in terms of oliminf, it is possible that there are points that are in the "basin" infinitely often, but not eventually. In order to specify the semi-basin of attraction as much as possible, and include some points that are infinitely often in the "basin" we introduce the following notion:

Definition 8. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Let $\{n_j\}$ be a strictly increasing sequence of positive integers. We define the reverse semi-basin of attraction for a ball $B_\epsilon(p)$ with respect to $\{n_j\}$ to be*

$$\begin{aligned} {}_{\{n_j\}}\Omega^{p,\epsilon}_{\{F_j\}} &= \textit{oliminf}_{j\to\infty} F^{-1}_{n_j}(B_\epsilon(p)) \cup \textit{oliminf}_{j\to\infty}(F_{n_j-1} \circ F_{n_j})^{-1}(B_\epsilon(p)) \cup \cdots \\ &= \cup_{i=0}^{\infty} \textit{oliminf}_{j\to\infty}(F(n_j, n_j - i))^{-1}(B_\epsilon(p)) \end{aligned}$$

where $F(n_j, n_j - i) = F_{n_j - i} \circ F_{n_j - i + 1} \circ \cdots \circ F_{n_j}$.

Note we use the convention that F_j is the identity when $j \leq 0$.

The lemma below is a mild generalization of Lemma 10 with a near identical proof.

Lemma 11. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Let $\{n_j\}$ be a strictly increasing sequence of positive integers. Suppose $0 < r_j < 1$ and $\epsilon > 0$ such that $\sup_i\{r_i\} < 1$ and $||F_j(z) - p|| \leq r_j||z - p||$ on $B_\epsilon(p)$. Then ${}_{\{n_j\}}\Omega^{p,\epsilon}_{\{F_j\}}$ is a nonempty connected open set and ${}_{\{n_j\}}\Omega^{p,\epsilon}_{\{F_j\}} = {}_{\{n_j\}}\Omega^{p,\epsilon'}_{\{F_j\}}$ for each $\epsilon' \in (0, \epsilon)$.*

Theorem 8. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Let $U_1 = \mathbb{C}^n$. Let $\{n_j\}$ be a strictly increasing sequence of positive integers. Suppose that there are $0 < s_j \leq r_j < 1$, $c_j > 0$, and $\epsilon > 0$ such that*

$$s_j||z-p|| \leq ||F_j(z)-p|| \leq r_j||z-p|| \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}$$

and

$$||A_j^{-1}(F_j(z)-p)-(z-p)|| \leq c_j||z-p||^2 \text{ on } B_\epsilon(p) \text{ for all } j \in \mathbb{N}.$$

Also assume $\sup_i\{r_i\} < 1$, $\inf_i\{s_i\} > 0$, and $\sup_k(c_k + \sum_{i=1}^{k-1} c_i(\frac{r_{i+1}^2}{s_{i+1}})\cdots(\frac{r_k^2}{s_k})) < \infty$. Further, suppose there exists $\delta > 0$ such that for each $k \in \mathbb{N}$, there is a $B_k < \infty$ such that

$$\sup_{\substack{z\in B_\delta(p)\\ i\in\mathbb{N}}} ||F_{n_i}^{-1} \circ \cdots \circ F_{n_i-(k-1)}^{-1}(z)|| < B_k.$$

Then there exists a domain FB that is biholomorphic to $\mathbb{C}^n$ such that

$${}_{\{n_j\}}\Omega_{\{F_j\}}^{p,\epsilon} \subseteq FB \subseteq \limsup_i U_{n_i} \setminus \cup_{k=1}^\infty \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_{n_i})^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(p)).$$

Additionally, if each U_i is Runge, we may arrange that domain FB is Runge.

Proof. The proof is analogous to the one above, replacing Ψ_j by Ψ_{n_j} and Φ_j by Φ_{n_j}. □

Corollary 4. *Let $U_i \subseteq \mathbb{C}^n$ $(i \in \mathbb{N})$ be connected open sets with $B_\epsilon(p) \subseteq U_i$ for each $i \in \mathbb{N}$ and let $F_i : U_{i+1} \to U_i$ be biholomorphic mappings. Let $U_1 = \mathbb{C}^n$ and $\{n_j\}$ be a strictly increasing sequence of positive integers. Assume $0 < s \leq r < 1$ with $r^2 < s$, and $\epsilon > 0$ such that $s||z-p|| \leq ||F_j(z)-p|| \leq r||z-p||$ on $B_\epsilon(p)$ for all $j \in \mathbb{N}$. Also suppose there exists $\delta > 0$ such that for each $k \in \mathbb{N}$, there is a $B_k < \infty$ such that*

$$\sup_{\substack{z\in B_\delta(p)\\ i\in\mathbb{N}}} ||F_{n_i}^{-1} \circ \cdots \circ F_{n_i-(k-1)}^{-1}(z)|| < B_k. \tag{3.2}$$

Then there exists a domain FB that is biholomorphic to $\mathbb{C}^n$ such that

$$_{\{n_j\}}\Omega^{p,\epsilon}_{\{F_j\}} \subseteq FB \subseteq \limsup_i U_{n_i} \setminus \cup_{k=1}^{\infty} \liminf_i (F_1 \circ F_2 \circ \cdots \circ F_{n_i})^{-1}(\mathbb{C}^n \setminus B_{\frac{1}{k}}(p)).$$

Additionally, if each U_i is Runge, we may arrange that domain FB is Runge.

Proof. Apply Lemma 2 to Theorem 8. □

Theorem 9. *For each Runge Fatou–Bieberbach domain Ω_1, there exist Runge Fatou–Bieberbach domains $\Omega, \Omega_2, \Omega_3, \Omega_4, \ldots \subseteq \mathbb{C}^n$ such that $\Omega_1 \supsetneq \Omega_2 \supsetneq \cdots$ and $\Omega \subseteq \cap_i \Omega_i$.*

Proof. Let Ω_1 be a Runge Fatou–Bieberbach domain in $\mathbb{C}^n$. Let $p \in \Omega_1$. Without loss of generality assume $p = 0$. There exists $\epsilon > 0$ such that $B_\epsilon(0) \subseteq \Omega_1$. Let $\delta > 0$ be small enough so that $(\frac{1}{2} + \delta)^2 < (\frac{1}{2} - \delta)$. Now since Ω_1 is Runge, there exists a biholomorphic map $f : \Omega_1 \to \mathbb{C}^n$ such that

$$(\frac{1}{2} - \delta)||z|| < ||f(z)|| < (\frac{1}{2} + \delta)||z||$$

on $B_\epsilon(0)$. [To see this, notice that since Ω_1 is Runge, there exists a biholomorphic map $\phi : \Omega_1 \to \mathbb{C}^n$ such that ϕ is close to the identity map id on $B_\epsilon(0)$, and without loss of generality $\phi(0) = 0$. (See for instance the proof of Wold [1] Lemma 4.) Let $f := \frac{1}{2}\phi$.] For each $i \in \mathbb{N}$, let $\Omega_i := \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{i}(\mathbb{C}^n)$. Clearly, each Ω_i is Runge. Notice by construction that $\Omega_1 \supsetneq \Omega_2 \supsetneq \Omega_3 \supsetneq \cdots$. Now let $F_i = f|_{\Omega_i}$ and $n_i = i$ for each $i \in \mathbb{N}$. Note that (3.2) from Corollary 4 is satisfied since for fixed k, $F^{-1}_{j+k} \circ \cdots \circ F^{-1}_{j+1}(B_\epsilon(0))$ is the same for every j. Now apply Corollary 4. By construction, $\limsup_i \Omega_i \subseteq \cap_i \Omega_i$. □

Conjecture 3. *For each non-Runge Fatou–Bieberbach domain Ω_1, there exist non-Runge Fatou–Bieberbach domains $\Omega_2, \Omega_3, \ldots \subseteq \mathbb{C}^n$ and a Fatou–Bieberbach domain Ω such that $\Omega_1 \supsetneq \Omega_2 \supsetneq \cdots$ and $\Omega \subseteq \cap_i \Omega_i$.*

Conjecture 4. *Suppose* $\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \cdots$ *are (non-Runge) Fatou–Bieberbach domains such that* $\overline{B_\epsilon(p)} \subseteq \cap_i \Omega_i$ *for some* $p \in \mathbb{C}^n$ *and* $\epsilon > 0$*. Then int*$(\cap_i \Omega_i)$ *is a Fatou–Bieberbach domain.*

4. CONCLUSION

In conclusion, we have demonstrated the importance of the Runge property in a number of results. We have shown that given a Runge Fatou–Bieberbach domain there exists one strictly larger such that infinitely many points can be prescribed to be included in the domain. We have given a precise convergence result for composition of maps on the right-hand side. We have generalized Wold [1] Theorem 4. And we have given a new type of contruction for Fatou–Bieberbach domains and demonstrated its usefulness. And we hope that the conjectures provided herein will inspire future research.

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