CONVERGENCE OF THE NEUMANN LAPLACIAN ON OPEN BOOK STRUCTURES

A Dissertation

by

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ABSTRACT

We consider a compact C^∞-stratified $2D$ variety M in $\mathbb{R}^3$ and its ϵ–neighborhood M_ϵ, which we call a "fattened open book structure." Assuming absence of zero-dimensional strata, i.e. "corners," we show that the (discrete) spectrum of the Neumann Laplacian in M_ϵ converges when $\epsilon \to 0$ to the spectrum of a differential operator on M.

Similar results have been obtained before for the case of fattened graphs, i.e. M being one-dimensional. In the case of a $2D$ smooth submanifold M, the problem has been studied well. However, having singularities along strata of lower dimensions significantly complicates considerations. As in the quantum graph case, such considerations are triggered by various applications such as micro-electronics, photonic devices, and dynamical systems with two "slow" and one "fast" degrees of freedom.

The results are obtained under two restrictions: 1) there are no zero dimensional strata (corners); 2) the pages are transverse at the bindings (no cusps).

We begin with the "uniformly fattened case:" width of the fattened domain shrinks with the same speed around "pages" and "bindings." Next we consider more general fattened open book structures with a finite number of parameters which control the size of the fattened neighborhood around each point. In particular we consider ϵ^β-sized neighborhoods around the bindings and ϵ-sized neighborhood around the pages. By properly tuning these parameters, we demonstrate three classes of limit operators on M. We show that there is a relative length scale (controlled by β) between the "fattened pages" and "fattened binding" which causes the system to undergo phase transitions. Two such phases have novel boundary currents along the bindings.

DEDICATION

To my mother, for her great personal sacrifices for my education, and to Sarah, for inspiring me to push through.

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First and foremost, none of this would have been possible without the limitless support and encouragement of my adviser Dist. Prof. Peter Kuchment. Through the trials of this program, he has only provided guidance and inspiration. If I were to forget the "why's" of all this, I was re-inspired with his earnest love of mathematics and the dignity of a life of learning he presented. My completion of this dissertation is only a small testament to his care and concern in helping me. I cannot begin to express my gratitude for his patience and mentorship which has helped me move on academically and as a person.

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Contributors

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All work for this dissertation was completed by the student, partially in collaboration with Professor Peter Kuchment of the Department of Mathematics.

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1. INTRODUCTION[1]

Consider a compact C^∞-stratified[2] $2D$ variety M in $\mathbb{R}^3$ without zero-dimensional strata, i.e. M locally (in a neighborhood of any point) looks like either a smooth submanifold or like an "open book" with smooth two-dimensional "pages" meeting transversely along a common smooth one-dimensional "binding," see Fig. 1.1.

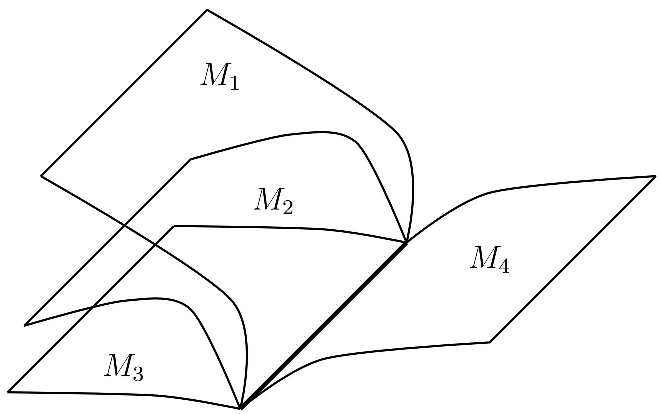

Figure 1.1: An open book structure with "pages" M_k meeting at a "binding."

Clearly, any compact smooth submanifold of $\mathbb{R}^3$ (with or without a boundary) qualifies as an open book structure with a single page. Another example of such structure is shown in Fig. 1.2.

The $2D$-strata need be neither contractible nor orientable.

We then consider a "fattened" version M_ϵ of M, which is an (appropriately defined) ϵ–neighborhood of M, which we call a "fattened open book structure."

Consider now the Laplace operator $-\Delta$ on the domain M_ϵ with Neumann boundary conditions

[1]Portions of this chapter have been adapted from: "Spectra of 'fattened' open book type structures," by J. E. Corbin and P. Kuchment, to appear in *The Mathematical Legacy of Victor Lomonosov*, De Gruyter.

[2]We do not provide here the general definition of what is called Whitney stratification, see e.g. [1, 19, 26, 43, 44], resorting to a simple description through local models.

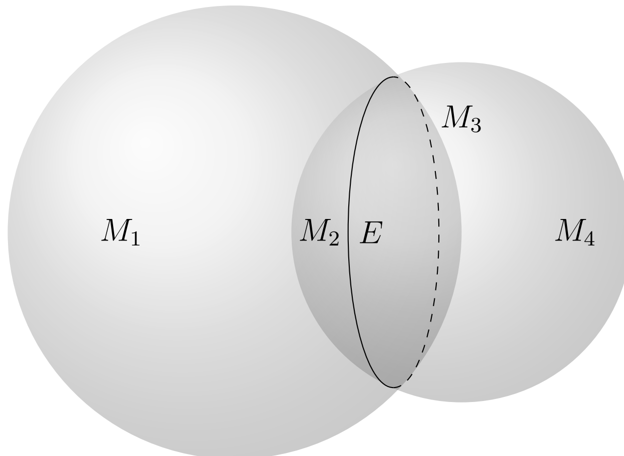

Figure 1.2: A transverse intersection of two spheres yields an open book structure with four pages and a circular binding. The requirement of absence of zero-dimensional strata prohibits adding a third sphere with a generic triple intersection. Tangential contacts of spheres are also disallowed.

(the "**Neumann Laplacian**"). We denote this operator[3] A_ϵ. As a (non-negative) elliptic operator on a compact manifold, it has discrete finite multiplicity spectrum $\lambda_n^\epsilon := \lambda_n(A^\epsilon)$ with the only accumulation point at infinity. The primary result of this dissertation, Theorem 2.2.6, states that when $\epsilon \to 0$, each eigenvalue λ_n^ϵ converges to the corresponding eigenvalue λ_n of an operator A on M, which acts as $-\Delta_M$ (2D Laplace-Beltrami) on each 2D stratum (**page**) of M, with appropriate junction conditions along 1D strata (**bindings**). This result is also announced in a forthcoming publication: "Spectra of 'fattened' open book type structures," by the author and P. Kuchment [4].

Similar results have been obtained previously for the case of fattened graphs (see [14,15,24,36], as well as books [2, 16, 30] and references therein), i.e. M being one-dimensional. They have been triggered by various applications [7, 12, 13, 22, 36–39]. In the case of a *smooth* submanifold $M \subset \mathbb{R}^3$, the problem is not that hard and has been studied well under a variety of "hard" and "soft" constraints set near M (see, e.g. [17, 20, 22]). However, having singularities along strata of lower dimensions significantly complicates considerations, even in the quantum graph case [5–7, 10, 20, 22–25, 36, 40].

[3]Throughout this work ϵ-dependent spaces, functions, operators, and coefficients will carry an ϵ subscript or superscript.

In Chapter 2 the results are obtained under two restrictions: that the width of the fattened domain shrinks “with the same speed” around all strata and the binding of the book does not have a cusp. The more complex case of slower shrinkage of the neighborhoods of lower dimensional strata, which leads to phase transition phenomena (see [25,30] for the quantum graph case), is handled in Chapter 3. There we consider more general families of fattened domains $\{M_\epsilon\}$ ($\epsilon \in (0, \epsilon_0]$, $\epsilon_0 > 0$). For instance, the width of the fattened pages vary by some positive differentiable function or the fattened binding shrinks at a different rate than the fattened page. Given some restrictions on the family of fattened domains, we ask whether we can identify a corresponding operator A on M such that A_ϵ converges to A in spectrum where A_ϵ is again the Neumann Laplacian. In fattened graph literature [11, 15, 25, 30] the analogous problem has already been considered, and it was shown that the limit operator A falls into distinct classes according to a heuristic based on the relative volume of the fattened strata (in that case “fattened edges” and “fattened vertices”). E.g. if the region around a vertex were much larger than the regions around the adjacent edges in the $\epsilon \to 0$ limit, the limit operator A is densely defined on a larger space of functions than the limit operator in the uniformly fattened graph case.

Our results here corroborate that heuristic. The main results of Chapter 3, Theorem 3.2.4, demonstrate spectral convergence of the Neumann Laplacian on a parameterized family of fattened domains to three classes of limit operators (see Propositions 3.1.12, 3.1.17, and 3.1.19). As in Chapter 2, the results are obtained under the restriction that the binding of the book does not have a cusp or a corner.

The dissertation is structured as follows: the first section of each main chapter, Section 2.1 and Section 3.1, contains the descriptions of the main objects: open book structures and their fattened versions, the limit operator A, etc. Following those are the formulation of the main result in each chapter in Sections 2.2 and 3.2 The proofs are provided in Sections 2.3 and 3.3 with proofs to the more technical propositions appearing in Appendix B.1. In Chapter 3, we reserve a section for a different construction of a fattened domain, called the thin-junction domain, to its own Section 3.4. We conclude with final remarks concerning generalizations and future work in Chapter 4.

2. SPECTRAL CONVERGENCE OF THE NEUMANN LAPLACIAN ON A UNIFORMLY FATTENED DOMAIN[1]

2.1 The Main Notions

This chapter is dedicated to what we call the "uniformly fattened" case. Here we introduce the main geometric objects to be studied. With the exception of our definition of the open book structure, the definitions as written here are specified strictly for the uniformly fattened case; however, all other definition developed here are rephrased in the Chapter 3 with appropriate modifications.

2.1.1 Open Book Structures

We start introducing the notion we will be using throughout the text:

Definition 2.1.1. *Let M denote a connected compact C^∞ stratified two-dimensional variety in $\mathbb{R}^3$ with the following properties:*

- *Zero dimensional strata are absent.*
- *M is composed of finitely many smooth $2D$ strata $\{M_k\}$ ($k \leq n_M$) (open smooth surfaces) called **pages** and smooth $1D$ strata $\{E_m\}$ ($m \leq n_E$) (closed smooth curves (edges)) called **bindings**.*
- *The pages are transverse at the bindings.*

*For the purpose of this dissertation, we call M an **open book structure**.*[2]

Simply put, to say M is a stratified surface means that it consists of finitely many connected, compact smooth submanifolds (with or without boundary) of $\mathbb{R}^3$, called **strata**, of dimensions two, one, and zero (i.e., points in the latter case), such that they may only intersect along their boundaries

[1]Portions of this chapter have been adapted from: "Spectra of 'fattened' open book type structures," by J. E. Corbin and P. Kuchment, to appear in *The Mathematical Legacy of Victor Lomonosov*, De Gruyter.

[2]One can find open book structures in a somewhat more general setting being discussed in algebraic topology literature, e.g. in [31, 45].

and each stratum's boundary is the union of some lower dimensional strata [19]. We assume additionally that the strata intersect at their boundaries transversely and that there are no zero-dimensional strata. In other words, locally M looks either as a smooth surface, or an "open book" with pages meeting at a non-zero angle at a "binding." Up to a diffeomorphism, a neighborhood of the binding looks like the picture in Fig. 2.1.

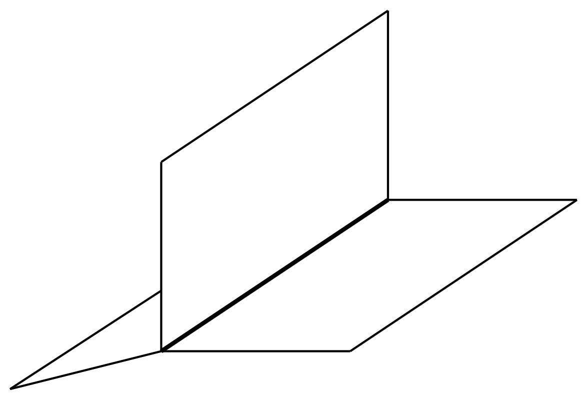

Figure 2.1: A local model of a binding neighborhood.

2.1.2 The Fattened Open Book Structure

If M were a smooth surface in $\mathbb{R}^3$, we could "fatten" it by considering its ϵ-neighborhood M_ϵ. The **fattened domain** M_ϵ for some $\epsilon > 0$ consists of all points at the distance of order ϵ from M, plus possibly "fatter" neighborhoods $E_{m,\epsilon}$ of the bindings E_m. Let us make this more precise.

The following statement is rather obvious:

Lemma 2.1.2. *There exists $\epsilon > 0$ so small that for any two points $x_1, x_2 \in M$ outside of an ϵ_0-neighborhood of the bindings, the closed intervals of radius ϵ_0 normal to M at these points do not intersect.*

This ensures that for $\epsilon < \epsilon_0$, the ϵ-fattened neighborhoods do not form a connecting bridge between two points that are otherwise far away from each other along M.

We can now define the **uniformly fattened open book structure** M_ϵ for M, our subject for this chapter. We denote the ball of radius r about x in $\mathbb{R}^3$ as $B(x,r)$ and similarly denote the ball of radius r about x in $\mathbb{R}^2$ as $D(x,r)$. In this dissertation we typically refer to a $2D$ ball as a disk.

Definition 2.1.3. *Let M denote an open book structure in $\mathbb{R}^3$, as defined above and $\epsilon_0 > 0$ from Lemma 2.1.2. We define for any $\epsilon < \epsilon_0$ the corresponding uniformly fattened domain M_ϵ as follows:*

$$M_\epsilon := \bigcup_{x\in M} B(x,\epsilon). \tag{2.1}$$

The similar notation R_ϵ will be used for the fattened version of any subset $R \in \mathbb{R}^3$.

Remark 2.1.4. *Later on we will assume that ϵ tends to zero. This will explain the meaning of the notations like $O(\epsilon)$ or $o(1)$. In particular, the assumption $\epsilon < \epsilon_0$ will be satisfied automatically and thus not mentioned.*

In this text "c" denotes a positive constant uniform with respect to ϵ, particularly as a bound in an inequality. E.g. if an expression $f(\epsilon)$ is $O(\epsilon)$, this means $|f(\epsilon)| \le c\epsilon$ as $\epsilon \to 0$ for some $c > 0$. Furthermore, if c appears as a bound in an inequality on some space of functions, c is understood to be a uniform bound on that space. E.g. an implication of the Sobolev embedding theorem ([27], see Theorem A.1.1 in Appendix A) is if $u \in H^1((0,1))$ there is a $c > 0$ such that the following holds true for all u,

$$||u||_{L_\infty((0,1))} \le c||u||_{H^1((0,1))}. \tag{2.2}$$

We often use subscripts (e.g. c_M) either for labeling or to denote implicit dependence on some parameter.

2.1.3 The Local Structure of the Uniformly Fattened Open Book

For any binding E_m, the parts of the adjacent pages M_k that are $O(\epsilon)$–close to E_m are called **sleeves** and denoted $S_{k,m,\epsilon}$. More precisely,

Definition 2.1.5. *Let M be an open book structure. Let $\{a_m\}_{m\le n_E}$ denote a finite set of positive* *numbers independent of ϵ. The sleeve $S_{k,m,\epsilon}$ on page M_k at E_m is defined as*

$$S_{k,m,\epsilon} := \{x \in M_k : \mathsf{dist}_{M_k}(x, E_m) < a_m \epsilon\}, \tag{2.3}$$

where $\mathsf{dist}_{M_k}(x, E_m)$ denotes the geodesic distance from E_m to x on M_k (see Fig. 2.2).

We use the following shorthand notation for the page without its sleeves:

$$M_{k,S} := M_k \backslash \bigcup_m S_{k,m,\epsilon}. \tag{2.4}$$

The next statement is easy to establish due to the non-tangential nature of pages' intersections:

Lemma 2.1.6. *Under appropriate choice (which we will fix) of $\{a_m\}$, the ϵ-neighborhoods of $M_{k,S}$ do not intersect each other for different values of k and any binding E_m.*

Definition 2.1.7. *Assuming a choice of orientation of M_k, we denote the positive unit normal vector to M_k at a point $x \in M_k$ as $\mathcal{N}_k(x)$. If M_k is non-orientable, a local choice of normal orientation will be sufficient for our purposes.*

We denote by $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$ the interval of the normal to M_k at x consisting of points at distance less than ϵ from x. The fattened page $M_{k,S,\epsilon}$ is thus foliated into normal fibers $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$.

$$M_{k,S,\epsilon} := \bigcup_{x \in M_{k,S}} \mathcal{I}_{\mathcal{N}_k(x),\epsilon}. \tag{2.5}$$

The latter foliation will be used to define the local averaging operator on $M_{k,S,\epsilon}$ in Subsection 2.3.7.

Definition 2.1.8. *Let M be an open book structure as in Definition 2.1.1. The fattened binding $E_{m,\epsilon}$ about E_m is the union of the ϵ-neighborhood of E_m and the 2ϵ width normal fibers over the sleeves $S_{k,m,\epsilon}$:*

$$E_{m,\epsilon} := \bigcup_{x \in E_m} B(x,\epsilon) \bigcup \Big(\bigcup_{k; x \in S_{k,m,\epsilon}} \mathcal{I}_{\mathcal{N}_k(x),\epsilon} \Big). \tag{2.6}$$

Definition 2.1.9. *We can also define a* ***cross-section*** $\omega_{m,\epsilon}(x)$*. For a point* x *in* E_m*,* N_x *is the normal plane of* E_m *at* x*, an affine subspace of* $\mathbb{R}^3$*. The cross-section* $\omega_{m,\epsilon}(x)$ *is the connected component of the intersection of* N_x *with* $M_\epsilon \setminus \bigcup_k M_{k,S,\epsilon}$ *containing* x*.*

The fattened binding can also be defined as the union of these cross-sections.

$$E_{m,\epsilon} := \bigcup_{x \in E_m} \omega_{m,\epsilon}(x). \tag{2.7}$$

Definition 2.1.10. *The* ***interface*** $\Gamma_{k,m,\epsilon}$ *between* $M_{k,S,\epsilon}$ *and* $E_{m,\epsilon}$ *is the strip-like domain shared between* $\partial M_{k,S,\epsilon}$ *and* $\partial E_{m,\epsilon}$ *(see Fig. 2.2).*

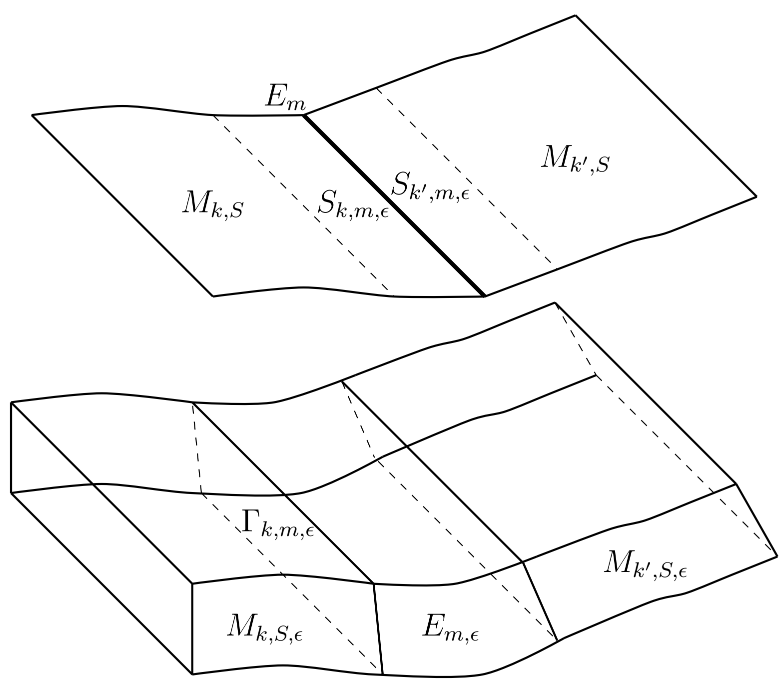

Figure 2.2: A neighborhood of a binding and the corresponding uniformly fattened neighborhood.

2.1.4 Quadratic Forms and Operators

We adopt the standard notation for Sobolev spaces (see, e.g. [27]). Thus, $H^1(\Omega)$ denotes the space of square integrable with respect to the Lebesgue measure functions on a domain $\Omega \subset \mathbb{R}^n$

with square integrable first order weak derivatives, and $L_p^k(\Omega)$ denotes the space of functions whose k-th order derivatives are in $L_p(\Omega)$.

Definition 2.1.11. *Let Q_ϵ denote the closed non-negative quadratic form with domain $H^1(M_\epsilon)$, given by*

$$Q_\epsilon(u) = \int_{M_\epsilon} |\nabla u|^2 \, dM_\epsilon. \tag{2.8}$$

*We also refer to $Q_\epsilon(u)$ as the **energy** of u.*

This form is associated with a unique self-adjoint operator A_ϵ in $L_2(M_\epsilon)$. The following statement is standard (see, e.g. [9, 27]):

Proposition 2.1.12. *The form Q_ϵ generates the **Neumann Laplacian** on M_ϵ. I.e. $A_\epsilon = -\Delta$ with its domain consisting of functions in $H^2(M_\epsilon)$ whose normal derivatives at the boundary ∂M_ϵ vanish.*

Its spectrum $\sigma(A_\epsilon)$ is discrete and non-negative.

We equip M with the surface measure dM (or dM_k when referring to a particular page) induced from $\mathbb{R}^3$.

Definition 2.1.13. *Let Q be the closed, non-negative quadratic form (**energy**) on $L_2(M)$ given by*

$$Q(u) = \sum_k \int_{M_k} |\nabla_{M_k} u|^2 \, dM \tag{2.9}$$

*with **domain** $\mathcal{G}^1$ consisting of functions u for which $Q(u)$ is finite and that are continuous across the bindings between pages M_k and $M_{k'}$:*

$$u|_{\partial M_k \cap E_m} = u|_{\partial M_{k'} \cap E_m}. \tag{2.10}$$

Here ∇_{M_k} is the gradient along M_k and restrictions in (2.10) to the binding E_m coincide as elements of $H^{1/2}(E_m)$.

Unlike the fattened graph case, by the Sobolev embedding theorem ([9], Theorem A.1.1) the restriction to the binding is not continuous as an operator from $\mathcal{G}^1$ to $C(E_m)$; it only maps to

$H^{1/2}(E_m)$. This distinction significantly complicates the analysis of fattened stratified surfaces in comparison with fattened graphs.

Proposition 2.1.14. *The operator A associated with the quadratic form Q acts on each M_k as*

$$Au := -\Delta_{M_k} u, \tag{2.11}$$

with the domain $\mathcal{G}^2$ consisting of functions on M such that the following conditions are satisfied:

$$||u||^2_{L_2(M)} + ||Au||^2_{L_2(M)} < \infty, \tag{2.12}$$

continuity across common bindings E_m of pairs of pages $M_k, M_{k'}$:

$$u|_{\partial M_k \cap E_m} = u|_{\partial M_{k'} \cap E_m}, \tag{2.13}$$

*and **Kirchhoff condition** at the bindings:*

$$\sum_{k:\partial M_k \supset E_m} D_{\nu_k} u(E_m) = 0. \tag{2.14}$$

Here $-\Delta_{M_k}$ is the Laplace-Beltrami operator on M_k and D_{ν_k} denotes the normal derivative to ∂M_k along M_k.

The spectrum of A is discrete and non-negative.

The proof is simple, standard, and similar to the graph case. We thus omit it.

Definition 2.1.15. *For a real number Λ not in the spectrum of A_ϵ, we denote by $\mathcal{P}^\epsilon_\Lambda$ the **spectral projector** of A_ϵ in $L_2(M_\epsilon)$ onto the spectral subspace corresponding to the half-line $\{\lambda \in \mathbb{R} \mid \lambda < \Lambda\}$.*

Similarly, $\mathcal{P}_\Lambda$ denotes the analogous spectral projector for A. We then denote the corresponding (finite dimensional) spectral subspaces as $\mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$ and $\mathcal{P}_\Lambda L_2(M)$ for M_ϵ and M respectively.

Proposition 2.1.16. *Functions from these (finite-dimensional) spectral subspaces satisfy the "reverse" embedding inequality. Namely, if* $u \in \mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$ *and* $\Lambda \notin \sigma(A_\epsilon)$ *then* $u \in H^1(M_\epsilon)$ *with*

$$||u||^2_{H^1(M_\epsilon)} \leq (1+\Lambda)||u||^2_{L_2(M_\epsilon)} \tag{2.15}$$

and similarly $u \in \mathcal{P}_{\Lambda} L_2(M)$ *and* $\Lambda \notin \sigma(A)$

$$||u||^2_{\mathcal{G}^1} \leq (1+\Lambda)||u||^2_{L_2(M)}. \tag{2.16}$$

It also follows for $u \in \mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$

$$||u||^2_{H^2(M_\epsilon)} \leq (1+\Lambda^2)||u||^2_{L_2(M_\epsilon)}. \tag{2.17}$$

Proof: Since $\Lambda \notin \sigma(A)$, the projector $\mathcal{P}_\Lambda$ is continuous. As established, the spectrum of A is of finite multiplicity with only one accumulation point at infinity, so $\mathcal{P}_\Lambda L_2(M)$ is finite dimensional. Therefore AP_Λ is a finite rank operator that is diagonalized in a spectral basis, and the diagonal entries are non-negative and bounded above by Λ. This gives the following:

$$||A\mathcal{P}_\Lambda u||^2_{L_2(M)} \leq \Lambda^2 ||u||^2_{L_2(M)}. \tag{2.18}$$

We then express the form Q on the $\mathcal{P}_\Lambda L_2(M)$ as $Q(u) = (u, Au)$ (and consequentially the norm of $\mathcal{G}^1$ by $Q(u) + (u,u)$) giving us the desired inequality. The statement for the other projector $\mathcal{P}^{\epsilon}_{\Lambda}$ follows from identical arguments. □

2.2 Formulation of Spectral Convergence of the Neumann Laplacian on a Uniformly Fattened Domain

We denote the non-decreasingly ordered eigenvalues of A as $\{\lambda_n\}_{n\in\mathbb{N}}$, and those of A_ϵ as $\{\lambda^{\epsilon}_n\}_{n\in\mathbb{N}}$.

Definition 2.2.1. *We say the operators A_ϵ **converge in spectra to** A as ϵ tends to zero if for each n*

$$|\lambda_n - \lambda_n^\epsilon| = o(1),$$

where $o(1)$ is not necessarily uniform with respect to n.

We now introduce two families of operators needed for the formulation and proof of the main result.

Definition 2.2.2. *A family of linear operators J_ϵ from $H^1(M_\epsilon)$ to $\mathcal{G}^1$ is called **averaging operators** if for any $\Lambda \notin \sigma(A_\epsilon)$ there is an ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$ the following conditions are satisfied:*

- *For $u \in \mathcal{P}_\Lambda^\epsilon L_2(M_\epsilon)$, J_ϵ is "nearly an isometry" from $L_2(M_\epsilon)$ to $L_2(M)$ with an $o(1)$ error, i.e.*

$$\left| ||u||_{L_2(M_\epsilon)}^2 - ||J_\epsilon u||_{L_2(M)}^2 \right| \leq o(1)||u||_{H^1(M_\epsilon)}^2 \tag{2.19}$$

  where $o(1)$ is uniform with respect to u.

- *For $u \in \mathcal{P}_\Lambda^\epsilon L_2(M_\epsilon)$, J_ϵ asymptotically "does not increase the energy," i.e.*

$$Q(J_\epsilon u) - Q_\epsilon(u) \leq o(1) Q_\epsilon(u) \tag{2.20}$$

  where $o(1)$ is uniform with respect to u.

Definition 2.2.3. *A family of linear operators K_ϵ from $\mathcal{G}^1$ to $H^1(M_\epsilon)$ is called **extension operators** if for any $\Lambda \notin \sigma(A)$ there is an ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$ the following conditions are satisfied:*

- *For $u \in \mathcal{P}_\Lambda L_2(M)$, K_ϵ is "nearly an isometry" from $L_2(M)$ to $L_2(M_\epsilon)$ with $o(1)$ error, i.e.*

$$\left| ||u||_{L_2(M)}^2 - ||K_\epsilon u||_{L_2(M_\epsilon)}^2 \right| \leq o(1)||u||_{\mathcal{G}^1}^2 \tag{2.21}$$

  where $o(1)$ is uniform with respect to u.

- *For $u \in \mathcal{P}_\Lambda L_2(M)$, K_ϵ asymptotically "does not increase" the energy, i.e.*

$$Q_\epsilon(K_\epsilon u) - Q(u) \leq o(1)Q(u) \tag{2.22}$$

where $o(1)$ is uniform with respect to u.

Existence of such averaging and extension operators is known to be sufficient for spectral convergence of A_ϵ to A (see [30]). For the sake of completeness, we formulate and prove this in our situation.

Theorem 2.2.4. *Let M be an open book structure as in Definition 2.1.1 and $\{M_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be its fattened partner as in Definition 2.1.3. Let A_ϵ and A be operators on M and M_ϵ as in Propositions 2.1.12 and 2.1.14.*

Suppose there exist averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ and extension operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ as stated in Definitions 2.2.2 and 2.2.3. Then, for any n

$$\lambda_n(A_\epsilon) \underset{\epsilon\to 0}{\to} \lambda_n(A). \tag{2.23}$$

We start with the following standard (see, e.g. [32]) min-max characterization of the spectrum.

Proposition 2.2.5. *Let B be a self-adjoint non-negative operator with discrete spectrum of finite multiplicity and $\lambda_n(B)$ be its eigenvalues listed in non-decreasing order. Let also q be its quadratic form with the domain D. Then*

$$\lambda_n(B) = \underset{W\subset D}{min} \underset{x\in W\setminus\{0\}}{max} \frac{q(x,x)}{(x,x)}, \tag{2.24}$$

where the minimum is taken over all n-dimensional subspaces W in the quadratic form domain D.

Proof of Theorem 2.2.4 Let us now employ Proposition 2.2.5 and the averaging and extension operators J, K to "transplant" the test spaces W in (2.28) between the domains of the quadratic forms Q and Q_ϵ.

Let us first notice that due to the definition of these operators (the near-isometry property), for any fixed finite-dimensional space W in the corresponding quadratic form domain, for sufficiently small ϵ the operators are injective on W and thus preserve its dimension. Since we are only interested in the limit $\epsilon \to 0$, we will assume below that ϵ is sufficiently small for these operators to preserve the dimension of W. Thus, taking also into account the inequalities (2.19)-(2.22), one concludes that on any fixed finite dimensional subspace W one has the following estimates of Rayleigh ratios:

$$\frac{Q(J_\epsilon u)}{||J_\epsilon u||^2_{L_2(M)}} \leq \big(1+o(1)\big)\frac{Q_\epsilon(u)}{||u||^2_{L_2(M_\epsilon)}}, \tag{2.25}$$

$$\frac{Q_\epsilon(K_\epsilon u)}{||K_\epsilon u||^2_{L_2(M_\epsilon)}} \leq \big(1+o(1)\big)\frac{Q(u)}{||u||^2_{L_2(M)}}. \tag{2.26}$$

Let now $W_n \subset \mathcal{G}^1$ and $W_n^\epsilon \subset H^1(M_\epsilon)$ be n, such that

$$\lambda_n = \max_{u\in W_n\setminus\{0\}} \frac{Q(u,u)}{(u,u)}, \tag{2.27}$$

and

$$\lambda_n^\epsilon = \max_{u\in W_n^\epsilon\setminus\{0\}} \frac{Q_\epsilon(u,u)}{(u,u)}. \tag{2.28}$$

Due to the min-max description and inequalities (2.25) and (2.26), one gets

$$\lambda_n \leq \sup_{u\in J_\epsilon(W_n^\epsilon)} \frac{Q(J_\epsilon u)}{||J_\epsilon u||^2_{L_2(M)}} \leq \big(1+o(1)\big)\lambda_n^\epsilon, \tag{2.29}$$

and

$$\lambda_n^\epsilon \leq \sup_{u\in K_\epsilon(W_n)} \frac{Q_\epsilon(K_\epsilon u)}{||K_\epsilon u||^2_{L_2(M_\epsilon)}} \leq \big(1+o(1)\big)\lambda_n. \tag{2.30}$$

Thus, $\lambda_n - \lambda_n^\epsilon = o(1)$, which proves the theorem. □

We will construct the required averaging and extension operators, which then will lead to the main result of this chapter:

Theorem 2.2.6. [3] *Let M be an open book structure as in Definition 2.1.1 and $\{M_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be its fattened partner as in Definition 2.1.3. Let A_ϵ and A be operators on M and M_ϵ as in Propositions 2.1.12 and 2.1.14. There exist averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ and extension operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ as stated in Definitions 2.2.2 and 2.2.3. Thus, for any n*

$$\lambda_n(A_\epsilon) \underset{\epsilon\to 0}{\to} \lambda_n(A). \tag{2.31}$$

2.3 The Proof of the Main Result (Theorem 2.2.6)

In order to define these averaging and extension operators, we must first consider the different local geometries of M. We define a local averaging operator on each of the fattened strata and a local extension operator from each of the pages into M_ϵ. Then we find a way to reconcile these local operators defined on different geometries. This is somewhat similar to the analysis on the fattened graph; however, different embedding theorems in dimensions higher than 1 require a more careful analysis than in the graph case.

2.3.1 Fattened Binding Geometry

In this subsection we describe the geometry of the fattened binding and, in particular, specify the lengths a_m. We describe carefully the geometry in order for the domain to admit a suitable partition of unity. This partition of unity is chosen as to allow good estimates with regards to ϵ dependence on the norms of trace and extension operators.

Definition 2.3.1. *Let M be an open book structure. Let $\theta_{m,k,k'}(x)$ be the (smaller) angle between two tangent vectors normal to two intersecting page boundaries ∂M_k and $\partial M_{k'}$ at $x \in E_m$. The sleeve width a_m ($m \le n_E$) (see Fig. 2.3) is*

$$a_m = \begin{cases} \max_{x\in E_m}(1+\cot(\min_{k,k'}\theta_{m,k,k'}(x)/2)) & \min_{x,k,k'}\theta_{m,k,k'}(x) < \pi/2 \\ 2 & \min_{x,k,k'}\theta_{m,k,k'}(x) \ge \pi/2 \end{cases}. \tag{2.32}$$

[3] This result is to appear in [4].

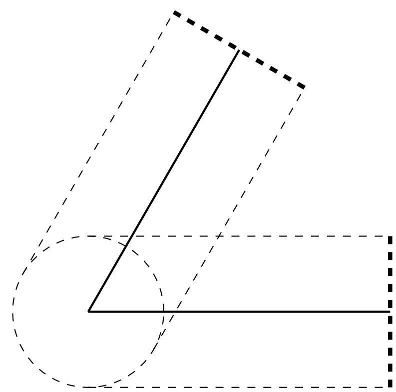

Figure 2.3: A cross-section of a uniformly fattened binding neighborhood. Dashed lines denote the boundary of a fattened stratum. Thickest dashed lines denote the cross-section of the boundary $\Gamma_{k,m,\epsilon}$ between the fattened binding and fattened page.

Consequentially, the closure of the normal fibers $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$ and $\mathcal{I}_{\mathcal{N}_{k'}(x'),\epsilon}$ do not touch for two distinct fattened pages $M_{k,S,\epsilon}$ and $M_{k',S,\epsilon}$.

Definition 2.3.2. *Let l_{E_m} denote the length of the E_m. We define $\gamma_m(y) : U = [0, l_{E_m}]/\{0, l_{E_m}\} \mapsto E_m$ to be a smooth parameterization of E_m. We suppose around each point x on E_m (with $x = \gamma_m(y)$) there is a neighborhood $V \subset U$ of y such that there exists two smooth orthogonal unit length vectors $v_{m,1}$ and $v_{m,2}$ on $\gamma_m(V)$ that span $N_{\gamma_m(y)}$.*

We equip the normal plane N_x ($x \in E_m$) with the following coordinate chart $\phi_x : N_x \mapsto \mathbb{R}^2$ where $\phi_x(x) = 0$, ϕ_x is an isometry, and $\phi_x(v_{m,i}(x))$ is the standard basis vector $\boldsymbol{e}_{y_i}$. The image of $\omega_{m,\epsilon}(x)$ through this chart ϕ_x is denoted $\varpi_{m,\epsilon}(x)$, an open region in $\mathbb{R}^2$. We call $\varpi_{m,\epsilon}(x)$ a cross-section as well.

Remark 2.3.3. *The cross-section $\omega_{m,\epsilon}(x)$ is a slice of $E_{m,\epsilon}$ cut by a plane in $\mathbb{R}^3$. It is clear that $E_{m,\epsilon}$ is the union of all these slices. This cross-section can be identified with a region in the plane which we denote $\varpi_{m,\epsilon}(x)$. Next subsection we define a fibration over E_m given by the collection of cross-sections $\varpi_{m,\epsilon}(x)$. We then define an averaging operator on these cross-sectional fibers in Subsection 2.3.7 that satisfies a Poincaré-type inequality with a Poincaré constant of order ϵ.*

Definition 2.3.4. *A domain* $\Omega \subset \mathbb{R}^n$ *is called a special Lipschitz domain if there is an orthogonal transformation* T *of Cartesian coordinates such that*

$$T\Omega = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n > \varphi(x')\} \tag{2.33}$$

where φ *is a uniformly Lipschitz function on* $\mathbb{R}^{n-1}$*. We call* φ *the boundary graph function to* Ω*.*

This following proposition follows from our definition of the fattened binding. The statements in the proposition establish the requirements needed for some embedding and extension theorems.

Proposition 2.3.5. *Let* $\{E_{m,\epsilon}\}$ $(0 < \epsilon \leq \epsilon_0)$ *be a family of fattened bindings as previously described. The following properties hold uniformly for each cross-section* $\varpi_{m,\epsilon}(x)$ $(x \in E_m)$*:*

1. *The inner and outer diameters over each cross-section are bounded of order* ϵ*:*

$$D(0, c_1\epsilon) \subset \varpi_{m,\epsilon}(x) \subset D(0, c_2\epsilon) \tag{2.34}$$

   where c_1 *and* c_2 *are constants.*

2. *There is a positive number* c_r *such that each cross-section* $\varpi_{m,\epsilon}(x)$ *is star-shaped with respect to the disk* $D(0, c_r\epsilon)$ *(see Fig. 2.4).*

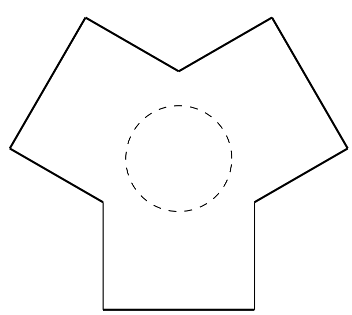

Figure 2.4: A view of $\varpi_{m,\epsilon}(x)$ and the disk it is star-shaped with respect to.

3. *There exists numbers c_M, c_N, c_U, and c_3 such for each $\epsilon \in (0, \epsilon_0]$ there is a finite collection of open sets $\{\tilde{U}_{i,\epsilon}\}$ ($i \leq c_U$) in $\mathbb{R}^2$ where*

   (a) *if $y \in \partial\varpi_{m,\epsilon}(x)$ then $D(y, c_3\epsilon) \subset \tilde{U}_{i,\epsilon}$ for some i,*

   (b) *each $y \in \partial\varpi_{m,\epsilon}(x)$ is contained in at most c_N sets $\tilde{U}_{i,\epsilon}$,*

   (c) *and for any i there is a special Lipschitz domain $\tilde{\Omega}_{i,\epsilon}$ with boundary graph function $\tilde{\phi}_{i,\epsilon}$ such that $\tilde{U}_{i,\epsilon} \cap \varpi_{m,\epsilon}(x) = \tilde{U}_{i,\epsilon} \cap \tilde{\Omega}_{i,\epsilon}$ and*

$$|\tilde{\phi}_{i,\epsilon}(z) - \tilde{\phi}_{i,\epsilon}(z')| \leq c_M|z - z'|, \qquad z,\, z' \in \mathbb{R}. \tag{2.35}$$

We extend (3) to a statement about the existence of a partition of unity on $E_{m,\epsilon}$ that has the properties that we will need later.

Corollary 2.3.6. *Let $\{E_{m,\epsilon}\}$ be a family of fattened binding neighborhoods as previously described. For each $\epsilon \in (0, \epsilon_0]$ there exists a partition of unity $\{\varphi_{i,\epsilon}\}$ (i is a counting number up to $N_{U,\epsilon}$ which depends on ϵ) subordinate to the finite open cover $\{U_{i,\epsilon}\}$ of $E_{m,\epsilon}$ with the following properties:*

1. *$\bigcup_i U_{i,\epsilon}$ is contained in $\bigcup_{x\in E_m} B(x, c_0\epsilon)$.*

2. *Each point contained in the covering is in at most c_N sets. In this sense we say the finite intersection property of these coverings holds in the $\epsilon \to 0$ limit.*

3. *Each open set $U_{i,\epsilon}$ contains a ball of radius $c_1\epsilon$ and is contained in a ball of radius $c_2\epsilon$.*

4. *If $x \in \partial E_{m,\epsilon}$, then $B(x, c_3\epsilon) \subset U_{i,\epsilon}$ for some i and $U_{i,\epsilon} \cap \partial E_{m,\epsilon}$ is a connected subset of some special Lipschitz domain $\Omega_{i,\epsilon}$ whose boundary graph function $\phi_{i,\epsilon}$ has a (Lipschitz) norm bounded above by a constant c_M.*

5. *There is a positive constant c_φ such that for each ϵ the gradient of each $\varphi_{i,\epsilon}$ has a uniform bound $c_\varphi\epsilon^{-1}$:*

$$|\nabla\varphi_{i,\epsilon}| \leq c_\varphi\epsilon^{-1}. \tag{2.36}$$

We will return to this partition of unity later in this chapter. The purpose of this partition of unity is to set up a generalization of the following well-known theorem attributed to Calderón and later improved on by Stein [3, 42] regarding boundedness of extension operators.

Theorem 2.3.7. *Let Ω be an open set in $\mathbb{R}^n$ and let there be positive numbers r, m, N (an integer) and a sequence $\{U_i\}_{i\geq 1}$ of open sets satisfying the conditions:*

1. *if $x \in \partial\Omega$, then $B(x, r) \subset U_i$ for some i,*
2. *every point $x \in \mathbb{R}^n$ is contained in at most N sets U_i,*
3. *for any $i \geq 1$ there is a special Lipschitz domain Ω_i with boundary graph function φ_i such that $U_i \cap \Omega = U_i \cap \Omega_i$ and*

$$|\varphi_i(x') - \varphi_i(y')| \leq m|x' - y'|,\ x',\ y' \in \mathbb{R}^{n-1}. \tag{2.37}$$

Then there exists a linear operator E mapping functions defined on Ω into functions defined on $\mathbb{R}^n$ and having the following properties:

1. *$Eu\big|_{\Omega} = u$.*
2. *E is a continuous operator: $\bigcap_{0\leq k\leq l} L_p^k(\Omega) \to \bigcap_{0\leq k\leq l} L_p^k(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ and a positive integer l .*
3. *The norm $||E||_{V_p^l(\Omega)\to V_p^l(\mathbb{R}^n)}$ ($V_p^l(\Omega) := \bigcap_{0\leq k\leq l} L_p^k(\Omega)$) is bounded by a constant depending only on n, p, l, r, m, N.*

This theorem has been extended to more general domains [21, 33]. We are dealing with a family of domains $\{E_{m,\epsilon}\}$ ($\epsilon \in (0, \epsilon_0]$) that have zero volume in the $\epsilon \to 0$ limit, and in particular this family does not admit an constant r such that the conditions in Theorem 2.3.7 hold. We approach constructing a family of extension or trace operators by carefully rescaling each subset of the covering in Corollary 2.3.6.

2.3.2 The Fattened Binding Foliation

Given our foliations of the fattened pages $M_{k,S,\epsilon}$ and $M_{k',S,\epsilon}$ (in terms of the normal lines $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$), we wish to extend these foliations into $E_{m,\epsilon}$. We accomplish this by introducing regions of the fattened binding called sectors. Breaking up the fattened binding into sectors, we can describe a vector field whose image "connects" the foliation of one fattened page to another foliation (see Fig. 2.6).

Definition 2.3.8. *Let E_m be a binding and $\{M_k\}$ ($k \leq n_m$) is the collection of at least two pages that meet at E_m all of which are orientable. We call the connected components of $E_{m,\epsilon} \backslash (E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))$ **sectors**, and we denote them as $\{\Sigma_{m,i,\epsilon}\}$ for $i \leq n_m$. A sector's boundary contains two sleeves of which we say that pair is associated with that sector (see Fig. 2.5).*

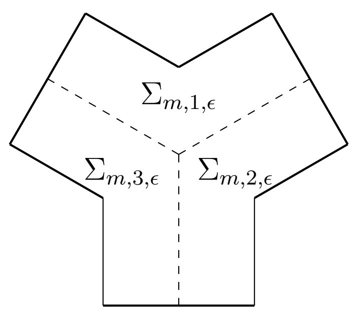

Figure 2.5: Sectors.

If E_m is a binding connected to non-orientable pages, then taking a partition into local neighborhoods is sufficient for our discussion. The case of only one page meeting at a binding is handled separately.

Definition 2.3.9. *Let E_m be a binding and $\{M_k\}$ ($k \leq n_m$) is the collection pages that meet at E_m all of which are orientable and there are at least two such pages. We say that the image of family of vector fields $\{tv_{m,i,\epsilon}\}$ ($t \in (0,1)$)*

$$v_{m,i,\epsilon}(x) : E_m \cup S_{k,m,\epsilon} \cup S_{k',m,\epsilon} \mapsto \mathbb{R}^3 \qquad S_{k,m,\epsilon},\ S_{k',m,\epsilon} \subset \partial\Sigma_{m,i,\epsilon} \tag{2.38}$$

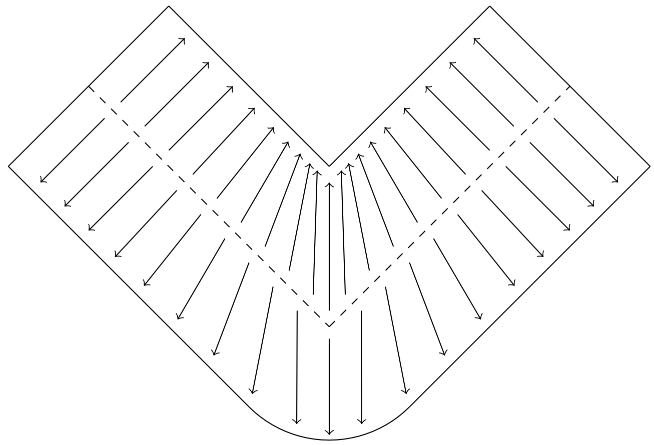

Figure 2.6: Cross sectional view of a pair of vector fields on each of the sleeves yielding a foliation of uniformly fattened binding.

is a ***foliation of the sector matching the foliation of fattened pages*** *(see Fig. 2.6) if:*

1. $v_{m,i,\epsilon}$ *is Lipschitz.*

2. $x \mapsto x + v_{m,i,\epsilon}(x)$ *is a homeomorphism between the domain of* $v_{m,i,\epsilon}$ *and the outward boundary of the sector:* $\partial\Sigma_{m,i,\epsilon} \cap \partial\left(E_{m,\epsilon} \setminus \bigcup_k \partial M_{k,S,\epsilon}\right)$

3. *The limit of* $v_{m,i,\epsilon}(x)$ *as* $x \to x' \in \partial S_{k,m,\epsilon} \cap M_k$ *is* $\pm\epsilon\mathcal{N}_k(x')$*.*

If E_m *is attached to only one page* M_k*, we say a family of vector fields* $\{v_{m,i,\epsilon}\}$ *(*$i = 1, 2$*)*

$$v_{m,i,\epsilon} : S_{k,m,\epsilon} \mapsto \mathbb{R}^3 \tag{2.39}$$

extends the foliation of the fattened page (see Figure 2.7) if:

1. $v_{m,i,\epsilon}$ *is Lipschitz.*

2. $x \mapsto x + v_{m,i,\epsilon}(x)$ *is a homeomorphism between the domain of* $v_{m,i,\epsilon}$ *and a subset boundary of the the fattened binding:* $\partial E_{m,\epsilon} \setminus \partial M_{k,S,\epsilon}$*.*

3. *The limit of* $v_{m,i,\epsilon}(x)$ *as* $x \to x' \in \partial S_{k,m,\epsilon} \cap M_k$ *is* $\pm\epsilon\mathcal{N}_k(x')$*.*

4. *The limits of* $v_{m,1,\epsilon}(x)$ *and* $v_{m,2,\epsilon}(x)$ *match at* E_m*.*

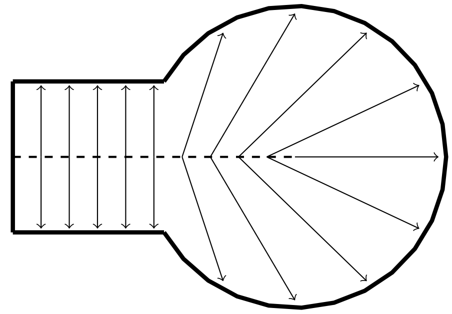

Figure 2.7: Cross sectional view of a pair of vector-valued functions on the sleeves that yield a foliation of fattened binding (non-uniformly fattened as pictured).

We expand on (2) and describe the construction of functions $\{v_{m,i,\epsilon}\}$ for all small, positive ϵ that have uniformly bounded gradients (where they exists).

Proposition 2.3.10. *There is a family of vector-valued functions* $\{v_{m,i,\epsilon}\}$ *(*$\epsilon \in (0, \epsilon_0]$*) that extends the foliation of the fattened pages that has length of* $O(\epsilon)$ *and uniformly bounded gradient (where it exists). I.e. there exists a* c_1 *and* c_2 *such that*

$$\max_{x \in D(v_{m,i,\epsilon})} |v_{m,i,\epsilon}(x)| \leq c_1 \epsilon, \tag{2.40}$$

and

$$\max_{x \in D(v_{m,i,\epsilon})} |\nabla v_{m,i,\epsilon}(x)| \leq c_2. \tag{2.41}$$

Proof: In Appendix B.1.1.

Corollary 2.3.11. *Each sector* $\Sigma_{m,i,\epsilon}$ *can be parameterized using* $v_{m,i,\epsilon}$*. Namely, a point* $x \in \Sigma_{m,i,\epsilon}$ *can be written as* $x = y + z v_{m,i,\epsilon}(y)$ $(y \in E_m \bigcup \left(\bigcup_k S_{k,m,\epsilon}\right), \, z \in (0,1))$*.*

2.3.3 Approximating the Geometry of Fattened Strata

Here we approximate each fattened page by the product of the corresponding page with an interval. Although this is not crucial for the proof, we assume that the page M_k is simply connected, otherwise one can partition it further. Because M_k is partitioned into simply connected patches, the normal $\mathcal{N}_k(x)$ is well-defined locally. A similar analysis is applied to E_m and its fattened partner

$E_{m,\epsilon}$.

Definition 2.3.12. *Suppose U is an open region of $\mathbb{R}^2$ with coordinates $y = (y_1, y_2)$. We define $X_{k,S}$ to be a smooth parameterization of $M_{k,S}$ on U:*

$$X_{k,S} : (y_1, y_2) \in U \subset \mathbb{R}^2 \mapsto M_{k,S} \subset \mathbb{R}^3. \tag{2.42}$$

In this subsection, we denote the coefficient functions of the first fundamental form of an immersed surface, in this case $M_{k,S}$, (see [41]) as E, F, and G which are functions on U. The derivatives of the parameterization $X_{k,S}$ with respect to y_1 and y_2 are functions to $\mathbb{R}^3$, and so we have

$$\begin{aligned} E &= D_{y_1} X_{k,S} \cdot D_{y_1} X_{k,S}, \\ F &= D_{y_1} X_{k,S} \cdot D_{y_2} X_{k,S}, \\ G &= D_{y_2} X_{k,S} \cdot D_{y_2} X_{k,S}. \end{aligned} \tag{2.43}$$

where the symbol "$\cdot$" denotes the inner product on $\mathbb{R}^3$.

Proposition 2.3.13. *Let $\boldsymbol{e}_{y_1}$ and $\boldsymbol{e}_{y_2}$ denote standard basis vectors of the tangent space $T_yU = \mathbb{R}^2$. The parameterization $X_{k,S}$ induces a metric $g_{M_{k,S}}$ on U. I.e. $g_{M_{k,S}}$ is the following positive definite bilinear form on T_yU:*

$$g_{M_{k,S}}(\boldsymbol{a}, \boldsymbol{b}) := \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{2.44}$$

where a_i, b_j are the respective coefficients of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the $(\boldsymbol{e}_{y_1}, \boldsymbol{e}_{y_2})$ basis of T_yU. We also use $g_{M_{k,S}}$ to denote the matrix in (2.44).

Proof: This is standard (see [41]).

Definition 2.3.14. *For sufficiently small ϵ, $M_{k,S,\epsilon}$ admits a parameterization $X_{k,S,\epsilon}$ on $U \times (-\epsilon, \epsilon)$ ($y \in U$, $z \in (-\epsilon, \epsilon)$) where*

$$X_{k,S,\epsilon}(y, z) := X_{k,S}(y) + z\mathcal{N}_k(X_{k,S}(y)). \tag{2.45}$$

We denote the coefficient functions of the second fundamental form of an orientable immersed surface, in this case $M_{k,S}$, (see [41]) as e, f, and g:

$$\begin{aligned} e &= -D_{y_1}X_{k,S} \cdot D_{y_1}\mathcal{N}_k(X_{k,S}(y)),\\ f &= -D_{y_1}X_{k,S} \cdot D_{y_2}\mathcal{N}_k(X_{k,S}(y)),\\ g &= -D_{y_2}X_{k,S} \cdot D_{y_2}\mathcal{N}_k(X_{k,S}(y)). \end{aligned} \tag{2.46}$$

Proposition 2.3.15. *The parameterization $X_{k,S,\epsilon}$ induces a metric $g_{M_{k,S,\epsilon}}$ on $U \times (-\epsilon, \epsilon)$.*

$$g_{M_{k,S,\epsilon}} := \begin{bmatrix} E - ze & F - zf & 0 \\ F - zf & G - zg & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.47}$$

Definition 2.3.16. *We define $\tilde{M}_{k,S,\epsilon}$ to be the product space of $M_{k,S}$ and $(-\epsilon, \epsilon)$.*

$$\tilde{M}_{k,S,\epsilon} := M_{k,S} \times (-\epsilon, \epsilon). \tag{2.48}$$

Proof: This follows from an explicit calculation of $D_i X_{k,S,\epsilon} \cdot D_j X_{k,S,\epsilon}$ from (2.45) and simplifying using (2.43) and (2.46). □

Definition 2.3.17. *The product space $\tilde{M}_{k,S,\epsilon}$ admits a parameterization $\tilde{X}_{k,S,\epsilon}$ on an open region $U \times (-\epsilon, \epsilon)$ in $\mathbb{R}^3$ of the form*

$$\tilde{X}_{k,S,\epsilon} = (X_{k,S}, z). \tag{2.49}$$

Proposition 2.3.18. *The parameterization $\tilde{X}_{k,S,\epsilon}$ (2.49) induces a metric on $U \times (-\epsilon, \epsilon)$:*

$$g_{\tilde{M}_{k,S,\epsilon}} := \begin{bmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.50}$$

Definition 2.3.19. *For sufficiently small ϵ, there exists a diffeomorphism $\phi_{M_{k,S,\epsilon}}$ from $M_{k,S,\epsilon}$ to $\tilde{M}_{k,S,\epsilon}$ of the form*

$$\phi_{M_{k,S,\epsilon}}(x) = \tilde{X}_{M_{k,S,\epsilon}}(X^{-1}_{M_{k,S,\epsilon}}(x)). \tag{2.51}$$

Proposition 2.3.20. *The linear operator $\Phi_{M_{k,S,\epsilon}}$ from $H^1(M_{k,S,\epsilon})$ to $H^1(\tilde{M}_{k,S,\epsilon})$ induced by the diffeomorphism $\phi_{M_{k,S,\epsilon}}$ (i.e. $\Phi_{M_{k,S,\epsilon}} u = u(\phi_{M_{k,S,\epsilon}})$) preserves H^1-norm of a function up to an $O(\epsilon^{1/2})$ error.*

$$\big|\, ||u||^2_{H^1(M_{k,S,\epsilon})} - ||\Phi_{M_{k,S,\epsilon}} u||^2_{H^1(\tilde{M}_{k,S,\epsilon})} \,\big| \leq c\epsilon ||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{2.52}$$

This inequality (2.52) also holds true for other Sobolev spaces H^n and in particular L_2.

Proof: First, we show that the metrics $g_{\tilde{M}_{k,S,\epsilon}}$ and $g_{M_{k,S,\epsilon}}$ are close.

Lemma 2.3.21. *On the domain $U \times (-\epsilon, \epsilon)$, the two metrics $g_{M_{k,S,\epsilon}}$ and $g_{\tilde{M}_{k,S,\epsilon}}$ are close:*

$$g_{M_{k,S,\epsilon}} - g_{\tilde{M}_{k,S,\epsilon}} = B g_{M_{k,S,\epsilon}} \tag{2.53}$$

where matrix B is $O(\epsilon)$ in the Frobenius norm.

Proof: This can be explicitly calculated:

$$g_{M_{k,S,\epsilon}} - g_{\tilde{M}_{k,S,\epsilon}} = \frac{z}{EG - F^2} \begin{bmatrix} e & f & 0 \\ f & g & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & -F & 0 \\ -F & E & 0 \\ 0 & 0 & 0 \end{bmatrix} g_{\tilde{M}_{k,S,\epsilon}}. \tag{2.54}$$

Because $|z| \leq \epsilon$, it is clear the right hand side is small. □

Having demonstrated the metrics are close, we then calculate the perturbation of two matrix valued functions about $g_{\tilde{M}_{k,S,\epsilon}}$:

Corollary 2.3.22. *The square root of the determinant and inverses of the two metrics $g_{M_{k,S,\epsilon}}$ and $g_{\tilde{M}_{k,S,\epsilon}}$ are also close:*

$$\sqrt{\mathsf{det} g_{M_{k,S,\epsilon}}} = \sqrt{\mathsf{det} g_{\tilde{M}_{k,S,\epsilon}}} \left(1 + \frac{1}{2}\mathsf{det}(B) + O(\epsilon^2)\right), \tag{2.55}$$

$$g^{-1}_{M_{k,S,\epsilon}} = g^{-1}_{\tilde{M}_{k,S,\epsilon}}(1 - B + O(\epsilon^2)). \tag{2.56}$$

Fixing the target space $U \times (-\epsilon, \epsilon)$ of our coordinate charts on $M_{k,S,\epsilon}$ and $\tilde{M}_{k,S,\epsilon}$, we can now compare functions on $M_{k,S,\epsilon}$ and $\tilde{M}_{k,S,\epsilon}$.

Corollary 2.3.23. *Let $u \in L_2(U \times (-\epsilon, \epsilon))$, then*

$$\begin{aligned} \Big| \int_{U\times(-\epsilon,\epsilon)} |u(y,z)|^2 \sqrt{\mathsf{det} g_{M_{k,S,\epsilon}}}\, dydz - \int_{U\times(-\epsilon,\epsilon)} |u(y,z)|^2 \sqrt{\mathsf{det} g_{\tilde{M}_{k,S,\epsilon}}}\, dydz \Big| \\ \leq c\epsilon \int_{U\times(-\epsilon,\epsilon)} |u(y,z)|^2 \sqrt{\mathsf{det} g_{M_{k,S,\epsilon}}}\, dydz. \end{aligned} \tag{2.57}$$

Corollary 2.3.24. *Let u be a function in $H^1(U \times (-\epsilon, \epsilon))$, then*

$$\begin{aligned} &\Big| \int_{U\times(-\epsilon,\epsilon)} (\nabla u)^* g^{-1}_{M_{k,S,\epsilon}} \nabla u \sqrt{\mathsf{det} g_{M_{k,S,\epsilon}}}\, dydz \\ &- \int_{U\times(-\epsilon,\epsilon)} (\nabla u)^* g^{-1}_{\tilde{M}_{k,S,\epsilon}} \nabla u \sqrt{\mathsf{det} g_{\tilde{M}_{k,S,\epsilon}}}\, dydz \Big| \\ &\leq c\epsilon \int_{U\times(-\epsilon,\epsilon)} (\nabla u)^* g^{-1}_{M_{k,S,\epsilon}} \nabla u \sqrt{\mathsf{det} g_{M_{k,S,\epsilon}}}\, dydz \end{aligned} \tag{2.58}$$

where $\nabla u = (D_{y_1}u, D_{y_2}u, D_z u)$.

These last two statements prove Proposition 2.3.20. □

The cross-sections $\omega_{m,\epsilon}(x)$ vary with $x \in E_m$ due to the curvature of the pages. Consequentially, more work is needed in defining the parameterization of $E_{m,\epsilon}$.

Definition 2.3.25. *Let $\gamma_m(y)$ be a smooth parameterization E_m on $U = (0, l_{E_m})$. We invoke the notation from Definition 2.3.2: $v_{m,1}$ and $v_{m,2}$ are the pair of orthonormal functions that span the normal planes of E_m and $\boldsymbol{e}_{z_1}$ and $\boldsymbol{e}_{z_2}$ denote standard basis vectors in the normal planes of E_m ($(z_1, z_2) = z \in \varpi_{m,\epsilon}(x)$).*

We define a fibration $\tilde{U}$ over U as follows:

$$\tilde{U} := \coprod_{y\in U} \varpi_{m,\epsilon}(\gamma_m(y)). \tag{2.59}$$

Let $\Omega_{m,\epsilon}(y,z) := z_1 v_{m,1}(y) + z_2 v_{m,2}(y)$. We can parameterize the $E_{m,\epsilon}$ with γ_m and $\Omega_{m,\epsilon}$:

$$Y_{m,\epsilon}(y,z) = \gamma_m(y) + \Omega_{m,\epsilon}(y,z). \tag{2.60}$$

Proposition 2.3.26. *Parameterization $Y_{m,\epsilon}$ in Definition 2.3.25 has the following conditions:*

1. *The image of $\gamma_m(y) + \Omega_{m,\epsilon}(y,\cdot)$ is $\omega_{m,\epsilon}(\gamma_m(y))$.*
2. *$D_{z_1}\Omega_{m,\epsilon}(y,\cdot)$ and $D_{z_2}\Omega_{m,\epsilon}(y,\cdot)$ lie in the normal plane of E_m at $\gamma_m(y)$.*
3. *$c_1 \le |D_{z_i}\Omega_{m,\epsilon}| \le c_2$.*
4. *$|D_y\Omega_{m,\epsilon}| \le c_3\epsilon$.*

The parameterization $Y_{m,\epsilon}$ induces a metric $g_{E_{m,\epsilon}}$ on $\tilde{U}$:

$$g_{E_{m,\epsilon}} = \begin{bmatrix} 1 + D_y\gamma_m \cdot D_y\Omega_{m,\epsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.61}$$

Proof: Since $v_{m,1}$ and $v_{m,2}$ are orthogonal to $D_y\gamma_m$, $D_y\gamma_m \cdot D_z\Omega_{m,\epsilon} = 0$. Because of vectors $v_{m,1}$ and $v_{m,2}$ are orthogonal, we have: $D_{z_i}\Omega_{m,\epsilon} \cdot D_{z_i}\Omega_{m,\epsilon} = 1$, and $D_{z_i}\Omega_{m,\epsilon} \cdot D_{z_j}\Omega_{m,\epsilon} = 0$ for $i \neq j$. □

Definition 2.3.27. *We denote by $\tilde{E}_{m,\epsilon}$ the fibration of E_m with fibers $\varpi_{m,\epsilon}(x)$:*

$$\tilde{E}_{m,\epsilon} := \coprod_{x \in E_m} \varpi_{m,\epsilon}(x). \tag{2.62}$$

Proposition 2.3.28. *The fibration $\tilde{E}_{m,\epsilon}$ space admits a parameterization $\tilde{Y}_{m,\epsilon}$ on $\tilde{U}$:*

$$\tilde{Y}_{m,\epsilon} = (y, z_1, z_2) \tag{2.63}$$

with an induced metric

$$g_{\tilde{E}_{m,\epsilon}} = \mathsf{Id}_{\mathbb{R}^3} \tag{2.64}$$

where $\mathsf{Id}_{\mathbb{R}^3}$ *is the identity matrix.*

Proposition 2.3.29. *For sufficiently small* ϵ*, there exists a diffeomorphism* $\phi_{E_{m,\epsilon}}$ *from* $E_{m,\epsilon}$ *to* $\tilde{E}_{m,\epsilon}$ *such that the induced linear operator* $\Phi_{E_{m,\epsilon}}$ *on* $H^1(E_{m,\epsilon})$ *(i.e.* $\Phi_{E_{m,\epsilon}}u = u(\phi_{E_{m,\epsilon}})$*) preserves* H^1*-norm up to an* $O(\epsilon^{1/2})$ *error:*

$$\Big|\, ||u||^2_{H^1(E_{m,\epsilon})} - ||\Phi_{E_{m,\epsilon}}u||^2_{H^1(\tilde{E}_{m,\epsilon})}\,\Big| \leq c\epsilon ||u||^2_{H^1(E_{m,\epsilon})}. \tag{2.65}$$

This inequality also holds true for other Sobolev spaces H^n *and in particular* L_2*.*

2.3.4 Bounds on the Sleeves

This subsection introduces two needed inequalities. The proof of the first inequality uses the calculations on induced metrics to show that stretching $M_{k,S}$ back to M_k induces only a small change of a function's norm. The second inequality involves bounding the L_2-norm of a function on a sleeve its H^1-norm on the page.

Proposition 2.3.30. *There exists a diffeomorphism* ψ_{M_k} *from* M_k *to* $M_{k,S}$ *such that*

- *each column vector of the Jacobian of* ψ_{M_k} *has length* $1 + O(\epsilon)$*,*
- *for any unit speed differentiable curve* γ *on* $\bar{M}_{k,S}$ *that is normal to* $\partial M_{k,S}$*, its image* $\psi_{M_k}(\gamma)$ *has unit speed and is normal to the boundary* ∂M_k*,*
- *the induced operator* Ψ_{M_k} *(i.e.* $\Psi_{M_k}u = u(\psi_{M_k})$*) preserves* H^1*-norm up to an* $O(\epsilon^{1/2})$ *error:*

$$\Big|\, ||u||^2_{H^1(M_k)} - ||\Psi_{M_k}u||^2_{H^1(M_{k,S})}\,\Big| \leq c\epsilon ||u||^2_{H^1(M_k)}. \tag{2.66}$$

This inequality also holds true for other Sobolev spaces H^n *and in particular* L_2*.*

Proof: A sufficiently small neighborhood V of ∂M_k admits a normal coordinate system, i.e. there is a parameterization X_k on $U \subset \mathbb{R}^2$ of V:

$$\begin{aligned} X_k : (y_1, y_2) \in U = (0, l_{E_m}) \times (0, a) \mapsto M_k, \\ \text{such that} \quad \mathsf{dist}_{M_k}(E_m, X_k(y_1, y_2)) = y_2. \end{aligned} \tag{2.67}$$

For sufficiently small ϵ, the set $\partial M_{k,S}$ is contained in V. By Definition 2.1.5, $\partial M_{k,S} \cap E_m$ is the image of $X_k(\cdot, a_m\epsilon)$. We define a smooth "shortening" function

$$\begin{aligned} \varphi_\epsilon : (0, a) \mapsto (a_m\epsilon, a) \quad \text{such that} \quad D\varphi_\epsilon \geq 0, \\ D\varphi_\epsilon(0) = D\varphi_\epsilon(a) = 1, \qquad |D\varphi_\epsilon - 1| \leq c\epsilon \end{aligned} \tag{2.68}$$

for some $c > 0$. We can now construct ψ_{M_k}:

$$\psi_{M_k}(x) := X_k((y_1, \varphi_\epsilon(y_1))), \quad \text{where} \quad (y_1, y_2) = X_k^{-1}(x). \tag{2.69}$$

The remainder of the proof follows from the calculating the induced metric from ψ_{M_k} as done in Corollaries 2.3.23 and 2.3.24. □

Proposition 2.3.31. *Let M_k be a smooth page with boundary $\bigcup_m E_m$. The L_2-norm of a function on $S_{k,m,\epsilon}$ is $O(\epsilon^{1/2})$-bounded by the function's H^1-norm on M_k:*

$$\int_{S_{k,m,\epsilon}} |u|^2 \, dM_k \leq c\epsilon \int_{M_k} |u|^2 + |\nabla_{M_k} u|^2 \, dM_k. \tag{2.70}$$

Proof: Appears in Appendix B.1.2.

2.3.5 Local Extensions of Functions on a Stratum to the Fattened Domain

We can extend a function form $M_{k,S}$ into $M_{k,S,\epsilon}$ by first extending along the fibers and then applying the diffeomorphism operator in Proposition 2.3.20. The extension from the binding and the sleeves is handled by extending along the foliation derived in Definition 2.3.9 by means of its associated coordinate system (Corollary 2.3.11).

Definition 2.3.32. *Let $u \in L_2(M_{k,S})$. We denote a point in the fibration $\tilde{M}_{k,S,\epsilon}$ as (y,z) for $y \in M_{k,S}$ and $z \in (-\epsilon, \epsilon)$. We define $\tilde{\mathcal{E}}_{k,z,\epsilon}$ to be the extension operator from $M_{k,S}$ to $\tilde{M}_{k,S,\epsilon}$, a bounded linear operator from $L_2(M_{k,S})$ to $L_2(\tilde{M}_{k,S,\epsilon})$ given by:*

$$\mathcal{E}_{k,z,\epsilon} u(y,z) = u(y). \tag{2.71}$$

Definition 2.3.33. *Let $u \in L_2(M_{k,S})$. We define $\mathcal{E}_{k,z,\epsilon}$ to be the extension operator from $M_{k,S}$ to $M_{k,S,\epsilon}$ given by*

$$\mathcal{E}_{k,z,\epsilon} := \Phi^{-1}_{M_{k,S,\epsilon}} \tilde{\mathcal{E}}_{k,z,\epsilon}. \tag{2.72}$$

Proposition 2.3.34. *For $u \in H^1(M_{k,S})$, one has:*

$$||u||^2_{L_2(M_{k,S})} = ||(2\epsilon)^{-1/2} \tilde{\mathcal{E}}_{k,z,\epsilon} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} \tag{2.73}$$

and

$$||\nabla_{M_k} u||^2_{L_2(M_{k,S})} = ||\nabla (2\epsilon)^{-1/2} \tilde{\mathcal{E}}_{k,z,\epsilon} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}. \tag{2.74}$$

Proof: Because

$$\int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} |\tilde{\mathcal{E}}_{k,z,\epsilon} u(y,z)|^2 \, dz = |u(y)|^2, \tag{2.75}$$

it follows that

$$\begin{aligned} \Big| \, ||u||^2_{L_2(M_{k,S})} &- ||(2\epsilon)^{-1/2} \tilde{\mathcal{E}}_{k,z,\epsilon} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} \Big| \\ &= \left| \int_{M_{k,S}} |u|^2 \, dM_k - \int_{M_{k,S}} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} |\tilde{\mathcal{E}}_{k,z,\epsilon} u(y,z,\epsilon)|^2 \, dz \, dM_k \right| \\ &= 0. \end{aligned} \tag{2.76}$$

Turning to the norm of the gradient, we have

$$\begin{aligned}&\big|\, ||\nabla_{M_k} u||^2_{L_2(M_{k,S})} - ||\nabla (2\epsilon)^{-1/2}\tilde{\mathcal{E}}_{k,z,\epsilon}u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}\,\big| \\ &= \left| \int_{M_{k,S}} |\nabla_{M_k} u|^2 \, dM_k - \int_{M_{k,S}} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} |\nabla \tilde{\mathcal{E}}_{k,z,\epsilon} u|^2 \, dz \, d\tilde{M}_k \right|.\end{aligned} \tag{2.77}$$

Clearly $D_z \tilde{\mathcal{E}}_{k,z,\epsilon} u = 0$, so we can rewrite (2.77) to get:

$$\left| \int_{M_{k,S}} |\nabla_{M_k} u|^2 \, dM_k - \frac{1}{2\epsilon} \int_{M_{k,S}} \int_{-\epsilon}^{\epsilon} |\nabla_{M_k} \tilde{\mathcal{E}}_{k,z,\epsilon} u|^2 + |D_z \tilde{\mathcal{E}}_{k,z,\epsilon} u|^2 \, dz \, d\tilde{M}_k \right| = 0. \qquad \square \tag{2.78}$$

Proposition 2.3.35. *For $u \in H^1(M_{k,S})$, one has:*

$$\big|\, ||u||^2_{L_2(M_{k,S})} - ||(2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})}\,\big| \le c\epsilon ||u||^2_{L_2(M_{k,S})} \tag{2.79}$$

and

$$\big|\, ||\nabla_{M_k} u||^2_{L_2(M_{k,S})} - ||\nabla (2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})}\,\big| \le c\epsilon ||u||^2_{H^1(M_{k,S})}. \tag{2.80}$$

Proof: An application of Proposition 2.3.20 to the results in Proposition 2.3.34.

Definition 2.3.36. *For the fattened binding $E_{m,\epsilon}$, we suppose its sectors $\Sigma_{m,i,\epsilon}$ are equipped with coordinate system described in Corollary 2.3.11 generated by $v_{m,i,\epsilon}$, the vector-valued function as described in Definition 2.3.9 and Proposition 2.3.10. We define $\mathcal{E}_{m,S,z,\epsilon}$ to be the extension operator on $L_2(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))$ to $L_2(E_{m,\epsilon})$ given by sector as*

$$\mathcal{E}_{m,S,z,\epsilon} u(y,z) = u(y) \qquad y \in E_m \bigcup S_{k,m,\epsilon} \bigcup S_{k',m,\epsilon} \mapsto \Sigma_{m,i,\epsilon} \ni (y,z). \tag{2.81}$$

Proposition 2.3.37. *The extension operators $(2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}$ from $H^1(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))$ to $H^1(E_{m,\epsilon})$ satisfy the following bound:*

$$||(2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{H^1(E_{m,\epsilon})} \le c||u||^2_{H^1(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))}. \tag{2.82}$$

Proof: Corollary 2.3.11 prescribes a coordinate system $x = y + zv_{m,i,\epsilon}(y)$ ($y \in E_m \bigcup (\bigcup_k S_{k,m,\epsilon})$, $z \in (0,1)$) on each sector $\Sigma_{m,i,\epsilon}$. It is a straightforward matter to calculate the norm of a function on a sector. We break a sector into three pieces for this calculation. The first of which, the set $\{y + zv_{m,i,\epsilon} : y \in E_m,\ z \in (0,1)\}$, is a zero measure set with respect to the Lebesgue measure on $\mathbb{R}^3$. The two other sets are of the form $\{y + zv_{m,i,\epsilon}(y) : y \in S_{k,m,\epsilon},\ z \in (0,1)\}$ for some page index k. We calculate the induced metric on this region to demonstrate the determinant of the metric is the correct order of ϵ such that the L_2 part of (2.82) holds. To accomplish that, we use the parameterization of $S_{k,m,\epsilon}$ in (2.67) renaming the parameterized variable as t ($X_k(t) = y$), and we denote the induced metric on the domain of t for $S_{k,m,\epsilon}$ as g_{M_k}. The induced metric $g_{\Sigma_{m,i,\epsilon},k}$ on (t,z) is

$$g_{\Sigma_{m,i,\epsilon},k} = \begin{bmatrix} g_{M_k} + D_tX_k \cdot zD_tv_{m,i,\epsilon}(X_k) & D_tX_k \cdot v_{m,i,\epsilon} \\ D_tX_k \cdot v_{m,i,\epsilon} & |v_{m,i,\epsilon}|^2 \end{bmatrix}. \tag{2.83}$$

It follows,

$$\mathsf{det}(g_{\Sigma_{m,i,\epsilon},k}) \leq c\epsilon\mathsf{det}(g_{M_k}). \tag{2.84}$$

Thus it follows:

$$||(2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c||u||^2_{L_2(E_m \bigcup(\bigcup_k S_{k,m,\epsilon}))}. \tag{2.85}$$

To calculate the gradient at the point, we calculate the divided difference between the $\mathcal{E}_{m,S,z,\epsilon}u(x)$ and $\mathcal{E}_{m,S,z,\epsilon}u(x+\delta)$. Writing $x + \delta = y + \delta_y + (z + \delta_z)v_{m,i,\epsilon}(y + \delta_y)$, we have

$$\begin{aligned} |\nabla_{\hat{\delta}}\mathcal{E}_{m,S,z,\epsilon}| &= \big|\limsup_{\delta\to 0} \frac{\mathcal{E}_{m,S,z,\epsilon}u(y+\delta_y, z+\delta_z) - \mathcal{E}_{m,S,z,\epsilon}u(y,z)}{|\delta|}\big| \\ &= \big|\limsup_{\delta\to 0} \frac{u(y+\delta_y) - u(y)}{|\delta_y|}\frac{|\delta_y|}{|\delta|}\big| \leq c_{\nabla v_{m,i,\epsilon}}|\nabla_{M_k}u|. \end{aligned} \tag{2.86}$$

This lets us conclude:

$$|\nabla\mathcal{E}_{m,S,z,\epsilon}u(y,z)| \leq c_{\nabla v_{m,i,\epsilon}}|\nabla_{M_k}u(y)|. \tag{2.87}$$

Hence we arrive at a bound on the derivative giving us (2.82) with (2.85):

$$||\nabla(2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq \sum_k c||\nabla_{M_k}u||^2_{L_2(S_{k,m,\epsilon})}. \qquad \square \tag{2.88}$$

2.3.6 Extension Operator K_ϵ

Now we can define the extension operators in the sense of Definition 2.2.3.

Proposition 2.3.38. *Let M be an open book structure. Let $\Lambda \leq c\epsilon^{-1+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A)$. For some $\epsilon_0 > 0$, the family of linear operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the conditions in Definition 2.2.3 is ($u \in \mathcal{P}_\Lambda L_2(M)$)*

$$K_\epsilon u := \begin{cases} (2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u & M_{k,S} \\ (2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u & E_m \bigcup \left(\bigcup_k S_{k,m,\epsilon}\right). \end{cases} \tag{2.89}$$

Proof: Beginning with $E_m \bigcup (\bigcup_k S_{k,m,\epsilon})$, we apply Proposition 2.3.37 to get

$$||(2\epsilon)^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{H^1(E_{m,\epsilon})} \leq c||u||^2_{H^1(E_m \bigcup(\bigcup_k S_{k,m,\epsilon}))}. \tag{2.90}$$

Applying the spectral embedding Proposition 2.1.16, the previously expression is bounded by

$$c(1+\Lambda)||u||^2_{L_2(E_m \bigcup(\bigcup_k S_{k,m,\epsilon}))} \tag{2.91}$$

which in turn is bounded by the energy on M (Proposition 2.3.31). This yields an upper bound of

$$c(1+\Lambda)\epsilon||u||^2_{\mathcal{G}^1} = o(1)||u||^2_{\mathcal{G}^1}. \tag{2.92}$$

Therefore (2.90) is negligible both in L_2 and H^1. For the $M_{k,S}$ pieces, we show that they are not only close to their extension $(2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u$ in L_2 but also in H^1. Starting with the following norm difference

$$\Big|\sum_k ||(2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u||_{H^1(M_{k,S,\epsilon})} - ||u||_{\mathcal{G}^1}\Big|, \tag{2.93}$$

we break $||u||_{\mathcal{G}^1}$ into page terms and sleeve terms and use the triangle inequality. We get an upper bound of (2.93) of

$$\sum_k \Big|||(2\epsilon)^{-1/2}\mathcal{E}_{k,z,\epsilon}u||_{H^1(M_{k,S,\epsilon})} - ||u||_{H^1(M_{k,S})}\Big| + ||u||_{H^1(E_m\bigcup(\bigcup_k S_{k,m,\epsilon}))}. \tag{2.94}$$

The first term of (2.94) is bounded by Proposition 2.3.35. After a norm bound on the sleeve (Propositions 2.3.31 and 2.1.16), we conclude (2.93) is bounded by $(1+\Lambda)^{1/2}O(\epsilon^{1/2})||u||_{\mathcal{G}^1}$. We conclude K_ϵ is a near isometry in both L_2 and H^1, provided the function u is restricted to the spectral subspace $\mathcal{P}_{c\epsilon^{-1+\delta}}L_2(M)$. □

2.3.7 Local Averaging Operators

This subsection concerns an averaging operator on the fattened page and an averaging operation on the fattened binding constructed by means of an integral representation. These averaging operators satisfy some Poincaré-type inequalities. I.e. the norm of the difference between a function and a constant (in the simplest formulation this constant is the average) is bounded by the norm of the function's derivative. We first define these operators in the fibrations $\tilde{M}_{k,S,\epsilon}$ and $\tilde{E}_{m,\epsilon}$ (Definitions 2.3.17 and 2.3.27) then apply the operators $\Phi_{M_{k,S,\epsilon}}$ and $\Phi_{E_{m,\epsilon}}$.

Definition 2.3.39. *Let $\tilde{N}_{k,\epsilon}$ denote the following bounded linear operator on $L_2(\tilde{M}_{k,S,\epsilon})$:*

$$\tilde{N}_{k,\epsilon}u(y,z) := \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} u(y,\zeta)\,d\zeta \qquad y\in M_{k,S},\ z\in(-\epsilon,\epsilon). \tag{2.95}$$

We also let $\tilde{N}_{k,\epsilon}$ denote the bounded linear operator from $L_2(\tilde{M}_{k,S,\epsilon})$ to $L_2(M_{k,S})$ by restricting $\tilde{N}_{k,\epsilon}u$ to $M_{k,S}$ ($\tilde{N}_{k,\epsilon}u(y,z=0)$).

Proposition 2.3.40. *The family of averaging operators $\{\tilde{N}_{k,\epsilon}\}$ on $L_2(\tilde{M}_{k,S,\epsilon})$ has a uniform bound c.*

Proof: Boundedness is clear from the Cauchy-Schwartz Inequality.

Definition 2.3.41. *The averaging operator $N_{k,\epsilon}$ on $M_{k,S,\epsilon}$ is given by composition with the corresponding diffeomorphism:*

$$N_{k,\epsilon} := \Phi^{-1}_{M_{k,S,\epsilon}} \tilde{N}_{k,\epsilon} \Phi_{M_{k,S,\epsilon}}. \tag{2.96}$$

We also let $N_{k,\epsilon}$ denote a bounded linear operator from $L_2(M_{k,S,\epsilon})$ to $L_2(M_{k,S})$ by restricting $N_{k,\epsilon}u$ to $M_{k,S}$ ($N_{k,\epsilon}u|_{M_{k,S}} = \tilde{N}_{k,\epsilon}\Phi_{M_{k,S,\epsilon}}u(y, z = 0)$).

Proposition 2.3.42. *For $u \in H^1(\tilde{M}_{k,S,\epsilon})$, $\tilde{N}_{k,\epsilon}$ satisfies a Poincaré-type inequality:*

$$\int_{\tilde{M}_{k,S,\epsilon}} |u - \tilde{N}_{k,\epsilon}u|^2 \, d\tilde{M}_{k,S,\epsilon} \leq c\epsilon^2 \int_{\tilde{M}_{k,S,\epsilon}} |\nabla u|^2 \, d\tilde{M}_{k,S,\epsilon}. \tag{2.97}$$

Proof: Because the lowest non-constant Neumann eigenfunction for the interval $(-1, 1)$ is $\sin(\pi x/2)$, the Poincaré inequality for an ϵ-interval yields

$$\int_{-\epsilon}^{\epsilon} |u - \tilde{N}_{k,\epsilon}u|^2 \, dz \leq \frac{4\epsilon^2}{\pi^2} \int_{-\epsilon}^{\epsilon} |D_z u(y, z)|^2 \, dz. \tag{2.98}$$

We then integrate (2.98) over $M_{k,S}$. Because $\tilde{M}_{k,S,\epsilon}$ is a product of $M_{k,S}$ and $(-\epsilon, \epsilon)$, the result follows from Fubini's theorem. □

Corollary 2.3.43. *For $u \in H^1(M_{k,S,\epsilon})$, the averaging operator $N_{k,\epsilon}$ admits a Poincaré-type inequality:*

$$||u - N_{k,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})} \leq c\epsilon^2 ||\nabla u||^2_{L_2(M_{k,S,\epsilon})}. \tag{2.99}$$

Proof: The inequality (2.99) is straightforward application of Proposition 2.3.20:

$$\begin{aligned} ||u - \Phi^{-1}_{M_{k,S,\epsilon}} \tilde{N}_{k,\epsilon} \Phi_{M_{k,S,\epsilon}} u||^2_{L_2(M_{k,S,\epsilon})} &\leq (1 + O(\epsilon)) ||\Phi_{M_{k,S,\epsilon}} u - \tilde{N}_{k,\epsilon} \Phi_{M_{k,S,\epsilon}} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} \\ &\leq (1 + O(\epsilon)) c\epsilon^2 ||\nabla \Phi_{M_{k,S,\epsilon}} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} \leq (1 + O(\epsilon)) c\epsilon^2 ||\nabla u||^2_{L_2(M_{k,S,\epsilon})}. \quad \square \end{aligned} \tag{2.100}$$

Proposition 2.3.44. *For $u \in H^1(\tilde{M}_{k,S,\epsilon})$, one has:*

$$\left| ||u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} - ||(2\epsilon)^{1/2}\tilde{N}_{k,\epsilon}u||^2_{L_2(M_{k,S})} \right| \leq c\epsilon ||u||^2_{H^1(\tilde{M}_{k,S,\epsilon})}. \tag{2.101}$$

Proof: Bounding the difference squared, we get:

$$\begin{aligned}
&\int_{\tilde{M}_{k,S,\epsilon}} |u|^2 \, d\tilde{M}_{k,S,\epsilon} - \int_{M_{k,S}} |\tilde{N}_{k,\epsilon}u|^2 \, 2\epsilon \, dM_k \\
&\leq \int_{\tilde{M}_{k,S,\epsilon}} |u|^2 \, d\tilde{M}_{k,S,\epsilon} - \int_{M_{k,S}} \Big(\int_{-\epsilon}^{\epsilon} |\tilde{N}_{k,\epsilon}u|^2 \, dz \Big) \, dM_k \\
&\leq (1 + O(\epsilon)) ||u - \tilde{N}_{k,\epsilon}u||_{L_2(\tilde{M}_{k,S,\epsilon})} ||u + \tilde{N}_{k,\epsilon}u||_{L_2(\tilde{M}_{k,S,\epsilon})} \\
&\leq 2\epsilon(1 + O(\epsilon)) ||u||^2_{H^1(\tilde{M}_{k,S,\epsilon})}. \qquad \square
\end{aligned} \tag{2.102}$$

Corollary 2.3.45. *For $u \in H^1(M_{k,S,\epsilon})$, one has:*

$$\left| ||u||^2_{L_2(M_{k,S,\epsilon})} - ||(2\epsilon)^{1/2}N_{k,\epsilon}u||^2_{L_2(M_{k,S})} \right| \leq c\epsilon ||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{2.103}$$

Proof: This corollary is an application of Proposition 2.3.20 on (2.101).

Proposition 2.3.46. *The linear operator $\tilde{N}_{k,\epsilon}$ is bounded on $H^1(\tilde{M}_{k,S,\epsilon})$,*

$$\int_{M_{k,S}} |\nabla_{M_k}(2\epsilon)^{1/2}\tilde{N}_{k,\epsilon}u|^2 \, dM_k \leq \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k}u|^2 \, d\tilde{M}_{k,S,\epsilon}. \tag{2.104}$$

Proof: We begin with rewriting the integral over $M_{k,S}$ in (2.104):

$$\int_{M_{k,S}} |\nabla_{M_k}(2\epsilon)^{1/2}\tilde{N}_{k,\epsilon}u|^2 \, dM_k = \int_{M_{k,S}} \Big(\int_{-\epsilon}^{\epsilon} |\nabla_{M_k}\tilde{N}_{k,\epsilon}u|^2 dz \Big) \, dM_k. \tag{2.105}$$

Using the reverse Fatou Lemma (see Lemma A.1.2 in Appendix A), we demonstrate the derivative of an average is bounded above by absolute value of the average of the derivative:

$$\begin{aligned}
&\int_{\tilde{M}_{k,S,\epsilon}} \left| D_{y_i} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} u \, d\zeta \right|^2 d\tilde{M}_{k,S,\epsilon} \\
&\leq \int_{\tilde{M}_{k,S,\epsilon}} \left| \limsup_{\delta \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \frac{u(y_i+\delta, y_j, \zeta) - u(y_i, y_j, \zeta)}{\delta} \, d\zeta \right|^2 d\tilde{M}_{k,S,\epsilon} \\
&\leq \int_{\tilde{M}_{k,S,\epsilon}} \left| \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} D_{y_i} u \, d\zeta \right|^2 d\tilde{M}_{k,S,\epsilon}.
\end{aligned} \tag{2.106}$$

We then use the embedding of L_1 in L_2 on a compact interval and the Cauchy-Schwartz Inequality:

$$\begin{aligned}
&\int_{\tilde{M}_{k,S,\epsilon}} \left| \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \nabla_{M_k} u \, d\zeta \right|^2 d\tilde{M}_{k,S,\epsilon} \\
&\leq \int_{\tilde{M}_{k,S,\epsilon}} \left(\frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |\nabla_{M_k} u|^2 \, d\zeta \right) d\tilde{M}_{k,S,\epsilon} \\
&\leq \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, d\tilde{M}_{k,S,\epsilon}. \qquad \square
\end{aligned} \tag{2.107}$$

Corollary 2.3.47. *The linear operator $N_{k,\epsilon}$ is bounded on $H^1(M_{k,S,\epsilon})$,*

$$\int_{M_{k,S}} |\nabla_{M_k} (2\epsilon)^{1/2} N_{k,\epsilon} u|^2 \, dM_k \leq (1 + O(\epsilon)) \int_{M_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, dM_\epsilon. \tag{2.108}$$

Proof: This is an application of Proposition 2.3.20 on (2.104).

Lemma 2.3.48. *Let Ω be a bounded domain in $\mathbb{R}^n$ with diameter D. Suppose $l > 0$ and*

$$Ru(z) := \int_\Omega \frac{u(\zeta)}{|z-\zeta|^{n-l}} d\zeta. \tag{2.109}$$

Then R is a continuous linear operator on $L_p(\Omega)$, $1 \leq p \leq \infty$, and

$$||R|| \leq n \, |B(0,1)| D^l / l. \tag{2.110}$$

Proof 2.3.48: Let χ be the characteristic function of $B(0, D)$. Letting our test function be zero outside of Ω and $K = |z|^{l-n}\chi(z)$, we observe $Ru(z) = (K * u)|_\Omega$. Therefore the inequality (2.110) follows from the Young inequality. □

The kernel in (2.109) appears in the remainder term in the following integral representation (see [28]):

Theorem 2.3.49. *Let Ω be a bounded domain star-shaped with respect to a ball $B(0,\delta) \subset \Omega$ in $\mathbb{R}^n$ and let $u \in L_p^l(\Omega)$. Then for almost all $x \in \Omega$*

$$u(z) = \delta^{-n} \sum_{|\alpha|<l} \left(\frac{z}{\delta}\right)^{\alpha} \int_{B(0,\delta)} \phi_\alpha \left(\frac{\zeta}{\delta}\right) u(\zeta) d\zeta + \sum_{|\alpha|=l} \int_{\Omega} \frac{f_\alpha(z,r,\theta)}{r^{n-l}} D^\alpha u(\zeta) d\zeta, \tag{2.111}$$

where $r = |z-\zeta|$, $\theta = (\zeta - z)/r$, $\phi_\alpha \in C_0^\infty(B(0,1))$, and f_α are infinitely differentiable functions such that

$$|f_\alpha| \leq c(D/\delta)^{n-1}. \tag{2.112}$$

c is a constant independent of Ω and D is the diameter of Ω.

Remark 2.3.50. *Let $\varphi \in C_0^\infty(B(0,1))$ such that $\int_{B(0,1)} \varphi = 1$. The function f_α in the integral representation (2.111) has an explicit expression in terms of φ; in particular (2.111) can be written as:*

$$\begin{aligned} u(z) = \delta^{-n} \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_{B(0,\delta)} \varphi\left(\frac{\zeta}{\delta}\right) (z-\zeta)^\alpha D^\alpha u(\zeta) d\zeta \\ + \sum_{|\alpha|=l} \frac{(-1)^l l}{\alpha!} \int_{\Omega} \Big(\int_r^\infty \varphi\left(\frac{z+\rho\theta}{\delta}\right) \rho^{n-1}\, d\rho \Big) \frac{\theta^\alpha}{r^{n-l}} D^\alpha u(\zeta) d\zeta. \end{aligned} \tag{2.113}$$

Proof: These are standard results in the theory of differentiable functions [27, 28]. We present the full proof in Appendix B.1.3.

We note this representation (2.111) in particular holds on almost every slice of a fibration like $\tilde{E}_{m,\epsilon}$.

Definition 2.3.51. *We define $\tilde{P}_{m,\epsilon}$ to denote the following bounded linear operator on $L_2(\tilde{E}_{m,\epsilon})$.*

$$\tilde{P}_{m,\epsilon} u(y,z) := \frac{1}{|D(0,c_r\epsilon)|} \int_{D(0,c_r\epsilon)} \varphi\left(\frac{\zeta}{c_r\epsilon}\right) u(y,\zeta)\, d\varpi_{m,\epsilon}(y)\ , \tag{2.114}$$

where $\varphi \in C_0^\infty(D(0,1))$ *such that* $\int_{D(0,1)} \varphi = 1.$

Let $\tilde{P}_{m,\epsilon}$ *also denote a bounded linear operator from* $L_2(\tilde{E}_{m,\epsilon})$ *to* $L_2(E_m)$ *by restricting* $\tilde{P}_{m,\epsilon}u$ *to* E_m *(*$\tilde{P}_{m,\epsilon}u(y, z = 0)$*).*

Proposition 2.3.52. *The norms of the family of averaging operators* $\{\tilde{P}_{m,\epsilon}\}$ *on* $L_2(\tilde{E}_{m,\epsilon})$ *has a uniform upper bound* c*.*

As with the operator $\tilde{N}_{k,\epsilon}$, boundedness of $\tilde{P}_{m,\epsilon}$ is clear from the Cauchy-Schwartz Inequality.

Definition 2.3.53. *The averaging operator* $P_{m,\epsilon}$ *on* $E_{m,\epsilon}$ *is given by composition with the corresponding diffeomorphism:*

$$P_{m,\epsilon} := \Phi_{E_{m,\epsilon}}^{-1} \tilde{P}_{m,\epsilon} \Phi_{E_{m,\epsilon}}. \tag{2.115}$$

We also let $P_{m,\epsilon}$ *denote a bounded linear operator from* $L_2(E_{m,\epsilon})$ *to* $L_2(E_m)$ *by restricting* $P_{m,\epsilon}u$ *to* E_m *(*$P_{m,\epsilon}u|_{E_m} = \tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u(y, z = 0)$*).*

Proposition 2.3.54. *The linear operator* $\tilde{P}_{m,\epsilon}$ *is bounded on* $H^1(\tilde{E}_{m,\epsilon})$*:*

$$\int_{\tilde{E}_{m,\epsilon}} |\nabla \tilde{P}_{m,\epsilon}u|^2 \, d\tilde{E}_{m,\epsilon} \leq \int_{D(0,1)} |\varphi|^2 \int_{\tilde{E}_{m,\epsilon}} |\nabla u|^2 \, d\tilde{E}_{m,\epsilon}. \tag{2.116}$$

Proof: Using the reverse Fatou Lemma (Lemma A.1.2), we get:

$$\begin{aligned}
&\int_{\tilde{E}_{m,\epsilon}} \Big|\nabla \frac{1}{|D(0,c_r\epsilon)|} \int_{D(0,c_r\epsilon)} \varphi\left(\frac{\zeta}{c_r\epsilon}\right) u \, d\varpi_{m,\epsilon}(y)\Big|^2 d\tilde{E}_{m,\epsilon} \\
&\leq \int_{\tilde{E}_{m,\epsilon}} \Big|\limsup_{\delta\to 0} \frac{1}{|D(0,c_r\epsilon)|} \int_{D(0,c_r\epsilon)} \frac{u(y+\delta,\zeta) - u(y,\zeta)}{\delta} \varphi\left(\frac{\zeta}{c_r\epsilon}\right) \, d\varpi_{m,\epsilon}(y)\Big|^2 d\tilde{E}_{m,\epsilon} \\
&\leq \int_{\tilde{E}_{m,\epsilon}} \Big|\frac{1}{|D(0,c_r\epsilon)|} \int_{D(0,c_r\epsilon)} \varphi\left(\frac{\zeta}{c_r\epsilon}\right) D_y u \, d\varpi_{m,\epsilon}(y)\Big|^2 d\tilde{E}_{m,\epsilon}.
\end{aligned} \tag{2.117}$$

We use the embedding of L_1 in L_2 on a compact interval and Cauchy-Schwartz Inequality:

$$\begin{aligned}
&\int_{\tilde{E}_{m,\epsilon}} \Big|\frac{1}{|D(0,c_r\epsilon)|}\int_{D(0,c_r\epsilon)} \varphi\left(\frac{\zeta}{c_r\epsilon}\right) D_y u\, d\varpi_{m,\epsilon}(y)\Big|^2 d\tilde{E}_{m,\epsilon} \\
&\leq \int_{\tilde{E}_{m,\epsilon}} \Big(\frac{1}{|D(0,c_r\epsilon)|}\int_{D(0,c_r\epsilon)} |\varphi\left(\frac{\zeta}{c_r\epsilon}\right) D_y u|^2\, d\varpi_{m,\epsilon}(y)\Big) d\tilde{E}_{m,\epsilon} \\
&\leq \frac{||\varphi(\zeta/c_r\epsilon)||^2_{L_2(D(0,c_r\epsilon))}}{|D(0,c_r\epsilon)|}\int_{\tilde{E}_{m,\epsilon}} |D_y u|^2\, d\tilde{E}_{m,\epsilon}. \qquad \square
\end{aligned} \tag{2.118}$$

Proposition 2.3.55. *For $u \in H^1(\tilde{E}_{m,\epsilon})$, the averaging operator $\tilde{P}_{m,\epsilon}$ satisfies a Poincaré-type inequality:*

$$\int_{\tilde{E}_{m,\epsilon}} |u - \tilde{P}_{m,\epsilon}u|^2\, d\tilde{E}_{m,\epsilon} \leq c\epsilon^2 \int_{\tilde{E}_{m,\epsilon}} |\nabla u|^2\, d\tilde{E}_{m,\epsilon}. \tag{2.119}$$

Proof: Calculating the difference squared on each cross-section:

$$\begin{aligned}
|u - \tilde{P}_{m,\epsilon}u|^2 &= \Big|\int_{\varpi_{m,\epsilon}(y)} \frac{f_{y,\zeta}(z,r,\theta)}{r} D_\zeta u(y,\zeta)\, d\varpi_{m,\epsilon}(y)\Big|^2 \\
&\leq c\Big|\int_{\varpi_{m,\epsilon}(y)} \frac{D_\zeta u(y,\zeta)}{r}\, d\varpi_{m,\epsilon}(y)\Big|^2 \leq c' R_y D_\zeta u(y,\zeta)
\end{aligned} \tag{2.120}$$

where R_y is the operator of the form of Lemma 2.3.48 on $L_2(\varpi_{m,\epsilon}(y))$ (in this case, it is the convolution with $1/r$). From (2.110) the norm of R_y is bounded by $c\epsilon$. $\square$

Corollary 2.3.56. *For $u \in H^1(E_{m,\epsilon})$, the averaging operator $P_{m,\epsilon}$ satisfies a Poincaré-type inequality:*

$$||u - P_{m,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^2 ||\nabla u||^2_{L_2(E_{m,\epsilon})}. \tag{2.121}$$

Proof: This is a simple of applying Proposition 2.3.29 on (2.119):

$$\begin{aligned}
||u - \Phi^{-1}_{E_{m,\epsilon}} \tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u||^2_{L_2(E_{m,\epsilon})} &\leq (1+O(\epsilon))||\Phi_{E_{m,\epsilon}}u - \tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u||^2_{L_2(\tilde{E}_{m,\epsilon})} \\
\leq (1+O(\epsilon))c'\epsilon^2||\nabla\Phi_{E_{m,\epsilon}}u||^2_{L_2(\tilde{E}_{m,\epsilon})} &\leq (1+O(\epsilon))c'\epsilon^2||\nabla u||^2_{L_2(E_{m,\epsilon})}. \qquad \square
\end{aligned} \tag{2.122}$$

2.3.8 The Case of Fattened Smooth Manifold (No Binding)

We state here the spectral convergence result for a fattened closed surface in $\mathbb{R}^3$.

Theorem 2.3.57. *Let M be a smooth closed surface in $\mathbb{R}^3$ and let M_ϵ ($M_\epsilon = \bigcup_{x\in M} B(x,\epsilon)$) be the corresponding ϵ-fattened domain to M.*

Let A_ϵ denote the Neumann Laplacian on M_ϵ and A denote the Laplace-Beltrami operator $-\Delta_M$ on M. The (non-decreasingly ordered) eigenvalues $\lambda_n^\epsilon(A_\epsilon)$ converge to $\lambda_n(A)$ as ϵ tends to zero.

Proof: We define the product space $\tilde{M}_\epsilon := M \times (-\epsilon, \epsilon)$, and adapt Proposition 2.3.20 to a page without a boundary. I.e. there is a bounded linear operator $\Phi_\epsilon : H^n(M_\epsilon) \mapsto H^n(\tilde{M}_\epsilon)$ that is nearly an isometry. We define local extension and averaging operators in the vein of Definitions 2.3.33 and 2.3.39:

$$\tilde{\mathcal{E}}_\epsilon u(y,z) = u(y) \qquad y \in M \tag{2.123}$$

and

$$\tilde{N}_\epsilon u(y) = \int_{-\epsilon}^{\epsilon} u(y,z)\,dz \qquad (y,\zeta) \in \tilde{M}_\epsilon. \tag{2.124}$$

The inequalities in Propositions 2.3.34, 2.3.44, and 2.3.46 are applicable to a single page. Our global averaging and extension operators are

$$J_\epsilon : H^1(M_\epsilon) \mapsto H^1(M), \qquad J_\epsilon := \sqrt{2\epsilon}\tilde{N}_\epsilon\Phi_\epsilon \tag{2.125}$$

and

$$K_\epsilon : H^1(M) \mapsto H^1(M_\epsilon), \qquad K_\epsilon := \frac{1}{\sqrt{2\epsilon}}\Phi_\epsilon^{-1}\tilde{\mathcal{E}}_\epsilon. \tag{2.126}$$

These operators satisfy Definitions 2.2.2 and 2.2.3 leading to the same conclusion as in Theorem 2.2.4 applied to a single page with no binding. □

2.3.9 Bounding the Norm on the Uniformly Fattened Binding

Having established the required estimations for an averaging operator on each stratum, we now need to combine these different averaging operators into a global one. To do so, here we establish several propositions regarding the trace on the interface $\Gamma_{k,m,\epsilon}$ between $M_{k,S,\epsilon}$ and $E_{m,\epsilon}$.

Definition 2.3.58. *The **trace** or **restriction operator** from $M_{k,S,\epsilon}$ to $\Gamma_{k,m,\epsilon}$ is denoted $T_{k,m,\epsilon}$.*

The trace operator from $E_{m,\epsilon}$ to $\Gamma_{k,m,\epsilon}$ is denoted $T_{m,k,\epsilon}$.

The standard embedding theorem claims that the trace space $T_{k,m,\epsilon}H^1(M_{k,S,\epsilon})$ is isomorphic to $H^{1/2}(\Gamma_{k,m,\epsilon})$ [4]. However, it these ϵ-dependent spaces are in general not uniformly equivalent as metric spaces as was shown in [28]. Let us expand on this, consider a family of homothetically scaled bounded domains $\{\Omega_\epsilon\}$ ($\Omega_\epsilon := \{\epsilon x : x \in \Omega \subset \mathbb{R}^n\}$ for some Lipschitz domain Ω) where $\epsilon \in (0, \epsilon_0]$. We say $TH^1(\Omega_\epsilon)$ and $H^{1/2}(\Omega_\epsilon)$ are isomorphic but not ϵ-uniformly equivalent as metric spaces if $u \in TH^1(\Omega_\epsilon)$ if and only if $u \in H^{1/2}(\Omega_\epsilon)$ and for any positive functions f and g such that

$$f(\epsilon)||u||^2_{H^{1/2}(\Gamma_{k,m,\epsilon})} \leq ||u||^2_{T_{k,m,\epsilon}H^1(M_{k,S,\epsilon})} \leq g(\epsilon)||u||^2_{H^{1/2}(\Gamma_{k,m,\epsilon})} \tag{2.128}$$

for all u then either $f(\epsilon)$ tends to zero or $g(\epsilon)$ tends to infinity as ϵ tends to zero. As shown in [28], the correct asymptotic metric of the trace space of a thin cylinder is nontrivial.

Definition 2.3.59. *Let Γ be an n-dimensional domain. Then $[f]_\Gamma$ denotes the following seminorm*

$$[f]^2_\Gamma = \int_{\Gamma\times\Gamma} \frac{|f(x) - f(y)|^2}{|x-y|^{1+n}}\, dx\, dy. \tag{2.129}$$

The $H^{1/2}$ norm is given by $||u||^2_{H^{1/2}(\Gamma)} = ||u||^2_{L_2(\Gamma)} + [u]^2_\Gamma$.

Let us estimate the trace on the fattened bindings. First, we state a result that connects Corollary 2.3.6 to a trace estimation.

Lemma 2.3.60. *Let Ω be a special Lipschitz domain and let φ be the associated graph function with bounded Lipschitz norm c_Ω. Let T_φ denote the operator from $L_2(\Omega)$ to $L_2(\mathbb{R}^n_+)$ (the half-*

[4] The trace space of Ω restricted to Γ is given by the norm:

$$||v||_{TH^1(\Omega)} := \inf_{u\in H^1(\Omega):\, u|_\Gamma = v} ||u||_{H^1(\Omega)} \tag{2.127}$$

space) given by

$$T_\varphi u = u(x', x_n + \varphi(x')) \qquad x = (x', x_n) \in \mathbb{R}^n_+. \tag{2.130}$$

Then T_φ is also a bounded linear operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^n_+)$ whose norm depends only on the φ and in particular c_Ω.

Proof: We begin with calculating the derivative (for $i < n$):

$$D_{x_i} T_\varphi u = u_{x_i}(x', x_n + \varphi(x')) + u_{x_n}(x', x_n + \varphi(x')) D_{x_i}\varphi(x') \tag{2.131}$$

and (for $i = n$)

$$D_{x_n} T_\varphi u = u_{x_n}(x', x_n + \varphi(x')). \tag{2.132}$$

The Jacobian of the transformation $(x', x_n) \mapsto (x', x_n + \varphi(x'))$ also only depends on φ and its derivatives. Consequentially, the norm $T_\varphi : H^1(\Omega) \mapsto H^1(\mathbb{R}^n_+)$ has an upperbound that depends only on the maximum of $|\varphi|$ and c_Ω. $\square$

Definition 2.3.61. *Following notation in Corollary 2.3.6, we have a partition of unity $\{\varphi_{i,\epsilon}\}$ subordinate to the finite open cover $\{U_{i,\epsilon}\}$ on $E_{m,\epsilon}$. Since each $U_{i,\epsilon} \cap \partial E_{m,\epsilon}$ (for a set $U_{i,\epsilon}$ near the boundary) is a connected subset of some special Lipschitz domain $\Omega_{i,\epsilon}$ with boundary graph function $\phi_{i,\epsilon}$, we define $T_{\phi_{i,\epsilon}} : L_2(U_{i,\epsilon}) \mapsto L_2(\mathbb{R}^3)$ to be an operator in the sense of Lemma 2.3.60 for the subset $U_{i,\epsilon}$.*

We denote the coordinate transformation from $U_{i,\epsilon}$ to $\mathbb{R}^3_+$ as $\chi_{i,\epsilon}$.

Lemma 2.3.62. *Let $\{E_{m,\epsilon}\}$ be a family of fattened bindings ($\epsilon \in (0, \epsilon_0]$). Let $u \in H^1(E_{m,\epsilon})$, then one has*

$$\epsilon^{-1}||T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})} + [T_{m,k,\epsilon}(u - P_{m,\epsilon}u)]^2_{\Gamma_{k,m,\epsilon}} \leq c_m||u||^2_{H^1(E_{m,\epsilon})}. \tag{2.133}$$

Proof: This is laid out in full in Appendix B.1.4.

With a norm estimate on the trace space of $E_{m,\epsilon}$, we can now construct an extension operator from $\Gamma_{k,m,\epsilon}$ to $M_{k,S,\epsilon}$.

Proposition 2.3.63. *For $u \in H^1(E_{m,\epsilon})$, the complement of the cross-sectional average $u - P_{m,\epsilon}u$ has an extension into M_ϵ denoted $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ such that*

$$||\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)||^2_{H^1(M_\epsilon)} \leq c_m ||u||^2_{H^1(E_{m,\epsilon})}. \tag{2.134}$$

Furthermore, $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ is supported within an $O(\epsilon)$ neighborhood of E_m.

Proof: This proof follows the ideas laid out in the Calderón-Stein Theorem (Theorem 2.3.7) along with using a homothetic scaling. This proof appears in Appendix B.1.5.

Corollary 2.3.64. *For $u \in H^1(E_{m,\epsilon})$, one has:*

$$||P_{m,\epsilon}u - T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^2 ||u||^2_{H^1(E_{m,\epsilon})}. \tag{2.135}$$

Proof: While $T_{k,m,\epsilon}N_{k,\epsilon}u$ is a function on the interface $\Gamma_{k,m,\epsilon}$, we can express it as a function on E_m by noting it is constant valued on $\partial\omega_{m,\epsilon}(x)$. With an abuse of notation, we can set $N_{k,\epsilon}u(x \in E_m) := N_{k,\epsilon}u|_{\partial\omega_{m,\epsilon}(x)}$. Beginning with an application of Proposition 2.3.29, we have

$$\begin{aligned}
&||\Phi^{-1}_{E_{m,\epsilon}}\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \\
&\leq (1 + O(\epsilon))||\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})} \\
&= (1 + O(\epsilon)) \int_{E_m}\int_{\varpi_{m,\epsilon}(y)} |\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u|^2 \, d\varpi_{m,\epsilon}(y)dE_m.
\end{aligned} \tag{2.136}$$

Noting $\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u$ can be extended to the boundary, (2.136) is bounded by

$$\frac{\max_{y\in E_m}|\varpi_{m,\epsilon}(y)|}{2\epsilon}\int_{\tilde{\Gamma}_{k,m,\epsilon}} |\tilde{N}_{k,\epsilon}[\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u]|^2 \, d\tilde{\Gamma}_{k,m,\epsilon}. \tag{2.137}$$

Because the norm of $\tilde{N}_{k,\epsilon}$ is bounded independently of ϵ, the above (2.137) is bounded by

$$c\epsilon \int_{\tilde{\Gamma}_{k,m,\epsilon}} |\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u|^2 \, d\tilde{\Gamma}_{k,m,\epsilon}. \tag{2.138}$$

After applying the operator $\Phi^{-1}_{E_{m,\epsilon}}$, we have (2.138) is equal to

$$(1 + O(\epsilon))||P_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - T_{k,m,\epsilon}u||^2_{L_2(\Gamma_{k,m,\epsilon})} \tag{2.139}$$

This is the L_2 term in (2.133), so we use Lemma 2.3.62. Consequentially, the desired bound for (2.135) is achieved. □

Lemma 2.3.65. *For $u \in H^1(M_{k,S,\epsilon})$, one has:*

$$||T_{k,m,\epsilon}u||^2_{L_2(\Gamma_{k,m,\epsilon})} \leq c_k||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{2.140}$$

Proof: Placed in Appendix B.1.6.

Corollary 2.3.66. *For $u \in H^1(M_\epsilon)$, one has:*

$$||T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{2.141}$$

Proof: It is analogous to the proof of Corollary 2.3.64 using Lemma 2.3.65.

Theorem 2.3.67. *For $u \in H^1(M_\epsilon)$, the L_2 norm of u on $E_{m,\epsilon}$ is small:*

$$||u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon||u||^2_{H^1(M_\epsilon)}. \tag{2.142}$$

Proof: We use the triangle inequality:

$$\begin{aligned} ||u||_{L_2(E_{m,\epsilon})} \leq\ & ||u - P_{m,\epsilon}u||_{L_2(E_{m,\epsilon})} \\ & + ||P_{m,\epsilon}u - T_{k,m,\epsilon}N_{k,\epsilon}u||_{L_2(E_{m,\epsilon})} + ||T_{k,m,\epsilon}N_{k,\epsilon}u||_{L_2(E_{m,\epsilon})}. \end{aligned} \tag{2.143}$$

With Corollaries 2.3.56, 2.3.64, and 2.3.66, the theorem is proven. □

Corollary 2.3.68. *Assuming $u \in \mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$ for $\Lambda \leq c\epsilon^{-1+\delta}$, $\delta > 0$, and $\Lambda \notin \sigma(A_\epsilon)$, then the H^1-norm of u on $E_{m,\epsilon}$ is $o(1)$ with respect to H^1-norm on M_ϵ.*

Proof: Due to the embedding of $\mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$ into $L_2(M_\epsilon)$, we can write

$$||\nabla u||^2_{L_2(E_{m,\epsilon})} \leq \Lambda ||u||^2_{L_2(E_{m,\epsilon})} \leq c\Lambda\epsilon ||u||^2_{H^1(M_\epsilon)} = c\epsilon^{\delta} ||u||^2_{H^1(M_\epsilon)} \qquad \square \tag{2.144}$$

2.3.10 Averaging Operator J_ϵ

At last we can then define the averaging and extension operators in the sense of Definition 2.2.2.

Lemma 2.3.69. *For any complex numbers a and b and for $d \in (0,1)$, one has:*

$$(1-d)|a|^2 + (1-d^{-1})|b|^2 \leq |a+b|^2 \leq (1+d)|a|^2 + (1+d^{-1})|b|^2. \tag{2.145}$$

Proof: Let us first assume both a and b are real. Because $(d^{1/2}a \pm d^{-1/2}b)^2$ is non-negative,

$$-da^2 - d^{-1}b^2 \leq 2ab \leq da^2 + d^{-1}b^2. \tag{2.146}$$

This completes the argument for the real case. For two complex numbers a and b, we first observe we can without loss of generality suppose the argument of a is zero. For the sum $a+b$ we may factor $\exp(\imath \mathsf{Arg}(a))$, which has a modulus of one, out of $|a+b|$. We have $|a+b|^2 - |a|^2 - |b|^2 = a(b+\bar{b})$. The term $a(b+\bar{b})$ is real and bounded between $\pm 2|a||b|$. So, we can apply the results from the real case and arrive at (2.145). □

Proposition 2.3.70. *Let M be an open book domain (Definition 2.1.1) and M_ϵ be the corresponding uniformly fattened domain (Definition 2.1.3). Assume $\Lambda \leq c\epsilon^{-1+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. For some $\epsilon_0 > 0$, the family of linear operators $\{J_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ that satisfies the conditions in Definition*

2.2.2 for the open book structure M is ($u \in \mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$)

$$J_\epsilon u := \begin{cases} \sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big] & M_{k,S,\epsilon} \mapsto M_k \\ \sqrt{2\epsilon} P_{m,\epsilon} u & E_{m,\epsilon} \mapsto E_m \end{cases} \tag{2.147}$$

Proof: First, we check whether $J_\epsilon u$ satisfies the boundary conditions on $\mathcal{G}^1$.

$$\begin{aligned} \lim_{x' \to x \in \partial M_{k,S,\epsilon} \cap \partial E_{m,\epsilon}} & N_{k,\epsilon}\big[u(x') + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)(x')\big] \\ &= N_{k,\epsilon} P_{m,\epsilon} u(x) = P_{m,\epsilon} u(x). \end{aligned} \tag{2.148}$$

Thus J_ϵ is in $\mathcal{G}^1$. Because each $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ is supported in a small $O(\epsilon)$ neighborhood around E_m, these extensions have disjoint supports. Using Lemma 2.3.69, we break up the terms on $M_{k,S,\epsilon}$,

$$\begin{aligned} (1-d)\big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon} u\big|^2 \; &+ (1-d^{-1}) \sum_m \big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big|^2 \\ &\le \Big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big]\Big|^2 \\ &\le (1+d)\big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon} u\big|^2 \\ &+ (1+d^{-1}) \sum_m \big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big|^2. \end{aligned} \tag{2.149}$$

To demonstrate the L_2 near isometry property, we first assume that $||J_\epsilon u||^2_{L_2(M_k)} \ge ||u||^2_{L_2(M_{k,S,\epsilon})}$. The other case $||J_\epsilon u||^2_{L_2(M_k)} \le ||u||^2_{L_2(M_{k,S,\epsilon})}$ can be handled by appropriately modifying the subsequent inequality (2.150) (i.e. flipping signs and switching upper and lower bounds). This results in a largely redundant calculation, so it is omitted. We calculate the upper and lower bound on the norm difference:

$$\begin{aligned}\sum_k (1-d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{L_2(M_k)} + (1-d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{L_2(M_k)}\\
-\sum_k ||u||^2_{L_2(M_{k,S,\epsilon})} - ||u||^2_{L_2(E_{m,\epsilon})}\\
\leq ||J_\epsilon u||^2_{L_2(M)} - ||u||^2_{L_2(M_\epsilon)}\\
\leq \sum_k (1+d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})}\\
+(1+d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{L_2(M_{k,S,\epsilon})}\\
-\sum_k ||u||^2_{L_2(M_{k,S,\epsilon})} - ||u||^2_{L_2(E_{m,\epsilon})}.\end{aligned} \tag{2.150}$$

Since we only require demonstrating that $||J_\epsilon u||_{H^1(M)}$ is bounded above (2.20), we begin with assuming $||J_\epsilon u||_{H^1(M)} \geq ||u||_{H^1(M_\epsilon)}$ and write:

$$\begin{aligned}||J_\epsilon u||^2_{H^1(M)} - ||u||^2_{H^1(M_\epsilon)}\\
\leq \sum_k (1+d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{H^1(M_{k,S,\epsilon})}\\
+(1+d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{H^1(M_{k,S,\epsilon})}\\
-\sum_k ||u||^2_{H^1(M_{k,S,\epsilon})} - ||u||^2_{H^1(E_{m,\epsilon})}.\end{aligned} \tag{2.151}$$

Having established these two inequalities (2.150) and (2.151), we collect terms in these inequalities and apply various propositions established in this chapter to demonstrate which terms are negligible (are $o(1)$ in an appropriate norm) and which terms are nearly an isometry (are $1+o(1)$ in an appropriate norm).

By Proposition 2.3.30, we have

$$
\begin{aligned}
&\Big|\int_{M_k} |\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u|^2\,dM_k - \int_{M_{k,S,\epsilon}} |u|^2\,dM_\epsilon\Big| \\
&\qquad\qquad \le \Big|(1+O(\epsilon))\int_{M_{k,S}} |\sqrt{2\epsilon}N_{k,\epsilon}u|^2\,dM_k - \int_{M_{k,S,\epsilon}} |u|^2\,dM_\epsilon\Big| \qquad (2.152)\\
&\qquad\qquad \le c\epsilon||u||^2_{H^1(M_\epsilon)}
\end{aligned}
$$

where the last inequality results from Corollary 2.3.45. We note the energy bound only needs to be demonstrated from above, so we see

$$
\int_{M_k} |\nabla_{M_k}\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u|^2\,dM_k - \int_{M_{k,S,\epsilon}} |\nabla u|^2\,dM_\epsilon \le c\epsilon||u||^2_{H^1(M_\epsilon)} \qquad (2.153)
$$

which follows from Proposition 2.3.30 and Corollary 2.3.47.

This leaves the extensions from the fattened bindings into the page ($\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$) and the norm of the binding unaccounted for in (2.150) and (2.151). We estimate the H^1-norm of the extensions. Using Proposition 2.3.30, Corollaries 2.3.45 and 2.3.47, and the disjoint supports of $E_{m,\epsilon}(u - P_{m,\epsilon}u)$:

$$
\begin{aligned}
&\sum_m ||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)||^2_{H^1(M_{k,S})} + \sum_m ||u||^2_{H^1(E_{m,\epsilon})} \\
&\le (1+O(\epsilon))\sum_m ||\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)||^2_{H^1(M_{k,S,\epsilon})} + \sum_m ||u||^2_{H^1(E_{m,\epsilon})}.
\end{aligned} \qquad (2.154)
$$

By Proposition 2.3.63, this is bounded by

$$
(1+O(\epsilon))c\sum_m ||u||^2_{H^1(E_{m,\epsilon})}. \qquad (2.155)
$$

Because $u \in \mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$ and Corollary 2.3.68, we arrive to the following upper bound on the norm of (3.157):

$$
c\epsilon^\delta||u||^2_{H^1(M_\epsilon)}. \qquad (2.156)
$$

Hence by setting $d = \epsilon^{\delta/2}$, we conclude that $J_\epsilon u|_{M_k}$ is close in L_2 to u and $J_\epsilon u|_{M_k}$ does not exceed the energy on M_ϵ by more than an $o(1)$ factor.

Thus J_ϵ is an averaging operator as required in Theorem 2.2.6 completing the proof of Proposition 2.3.70 and consequentially Theorem 2.2.6. □

3. FATTENED DOMAINS OF VARIABLE WIDTH

3.1 The Main Notions

Here we reintroduce the main geometric objects and differential operators to be studied. Most of these terms will be familiar from Chapter 2. Some are modified more than others; if a proposition requires only small modification, its proof will be omitted.

The definition of the open book structure Definition 2.1.1 needs no change in this second main chapter, so all references to an open book structure M refer to that definition.

3.1.1 The Non-Uniformly Fattened Structure

The **fattened domain** M_ϵ for some $\epsilon > 0$ consists of all points at the distance of order $o(1)$ from M.

Definition 3.1.1. *Let M denote an open book in $\mathbb{R}^3$ as in Definition 2.1.1. Let $\epsilon_0 > 0$ and let $\beta \in (0, 1]$. Suppose $\{r_m\}$ ($m \leq n_E$) is a collection of positive functions in $C^2(E_m)$ (with no dependence on ϵ) and $\{r_k\}$ ($k \leq n_M$) is a collection of positive functions in $C^2(M_k) \cap C(\bar{M}_k)$ (also independent of ϵ) where $\epsilon r_k|_{E_m} \leq \epsilon^\beta r_m$ for $\epsilon \in (0, \epsilon_0]$.*

*We say $\{M_\epsilon\}$ ($0 < \epsilon \leq \epsilon_0$) is a **model family of fattened domains (of type I, II, or III)** if*

$$M_\epsilon := \Big(\bigcup_{m; x \in E_m} B(x, r_m \epsilon^\beta) \Big) \bigcup \Big(\bigcup_{k; x \in M_k} B(x, r_k \epsilon) \Big). \tag{3.1}$$

In particular, if

- $\beta > 1/2$, *$\{M_\epsilon\}$ is a type I family,*
- $\beta < 1/2$, *$\{M_\epsilon\}$ is a type II family,*
- $\beta = 1/2$, *$\{M_\epsilon\}$ is a type III family.*

Our results hold for more relaxed conditions. For instance, we may consider some ϵ dependent family $\{r_{m,\epsilon}\}$ in place of $\{r_m\}$ where $r_{m,\epsilon} \to r_m$ as $\epsilon \to 0$. However this does not add more substance to the results, and so we work with the simplified paradigm. It is important to consider even thinner ($\beta > 1$) neighborhoods around the binding, but this requires more setup. These thin junctions are reserved for Section 3.4.

Remark 3.1.2. *An observation about each type of fattened domain can be made:*

- *If $\beta > 1/2$, the ratio $|\bigcup_{x\in E_m} B(x, r_m\epsilon^\beta)|/|\bigcup_{x\in M_k} B(x, r_k\epsilon)|$ tends to zero as ϵ tends to zero.*
- *If $\beta < 1/2$, the ratio $|\bigcup_{x\in E_m} B(x, r_m\epsilon^\beta)|/|\bigcup_{x\in M_k} B(x, r_k\epsilon)|$ tends to infinity as ϵ tends to zero.*
- *If $\beta = 1/2$, the ratio $|\bigcup_{x\in E_m} B(x, r_m\epsilon^\beta)|/|\bigcup_{x\in M_k} B(x, r_k\epsilon)|$ has a finite, positive limit as ϵ tends to zero.*

*Hence we say type I domains correspond to the **small binding case**, type II domains correspond to the **large binding case**, and type III domains correspond to the **critical case**.*

3.1.2 The Local Structure of Non-Uniformly Fattened Domains

We need the notion of a small neighborhood of the binding E_m in a page M_k which we call a **sleeve** and denoted it by $S_{k,m,\epsilon}$. In this chapter these sleeves are the parts of the page M_k that are $O(\epsilon^\beta)$-close to E_m.

Definition 3.1.3. *Let M have the open book structure as pictured in Fig. 2.1. Let $\{a_m\}_{m\le n_E}$ denote a finite set of positive functions in $C^2(E_m)$ (independent of ϵ). The sleeve $S_{k,m,\epsilon}$ on page M_k at E_m is defined as*

$$S_{k,m,\epsilon} := \{x \in M_k : \mathsf{dist}_{M_k}(x, y \in E_m) < a_{m,\epsilon}(y)\epsilon^\beta\} \tag{3.2}$$

where $a_{m,\epsilon} = a_m(1 + o(1))$ and dist_{M_k} denotes the geodesic distance along M_k (see Fig. 2.2). We use the following shorthand notation for the page without its sleeves:

$$M_{k,S} := M_k \backslash \bigcup_m S_{k,m,\epsilon}. \tag{3.3}$$

The next statement is easy to establish due to the non-tangential nature of pages' intersections:

Lemma 3.1.4. *Under appropriate choice (which we will fix) of* $\{a_m\}$*, the* ϵ*-neighborhoods of* $M_{k,S}$ *and* $M_{k',S}$ *do not intersect each other for distinct indices* k *and* k'*.*

Definition 3.1.5. *Assuming a choice of orientation of* M_k*, we denote the unit normal vector to* M_k *at a point* $x \in M_k$ *as* $\mathcal{N}_k(x)$*. If* M_k *is non-orientable, a local choice of normal orientation will be sufficient for our purposes.*

We denote by $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$ *the largest open interval of the normal to* M_k *containing* x *contained in* M_ϵ*. Upon picking some local orientation of* M_k*, we let* $I_{k,\epsilon}(x) \subset \mathbb{R}$ *denote the image of the fiber* $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$*:*

$$I_{k,\epsilon}(x) := (-r_k(x)\epsilon, r_k(x)\epsilon). \tag{3.4}$$

The ***fattened page*** $M_{k,S,\epsilon}$ *is thus foliated into normal fibers* $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$*:*

$$M_{k,S,\epsilon} := \bigcup_{x \in M_{k,S}} \mathcal{I}_{\mathcal{N}_k(x),\epsilon}. \tag{3.5}$$

Remark 3.1.6. *The latter foliation in terms of normal intervals is used to define the local averaging operator on* $M_{k,S,\epsilon}$ *in Subsection 3.3.6.*

Definition 3.1.7. *Let* M *be an open book structure as in Definition 2.1.1. We define a* ***cross-section*** $\omega_{m,\epsilon}(x)$ *of the fattened binding. For a point* x *in* E_m*,* N_x *is the normal plane of* E_m *at* x*, an affine subspace of* $\mathbb{R}^3$*. The cross-section* $\omega_{m,\epsilon}(x)$ *is the connected component of the intersection of* N_x *with* $M_\epsilon \backslash \bigcup_k M_{k,S,\epsilon}$ *containing* x*.*

The ***fattened binding*** *is defined to be the union of these cross-sections.*

$$E_{m,\epsilon} := \bigcup_{x \in E_m} \omega_{m,\epsilon}(x). \tag{3.6}$$

Definition 3.1.8. *The* ***interface*** $\Gamma_{k,m,\epsilon}$ *between* $M_{k,S,\epsilon}$ *and* $E_{m,\epsilon}$ *is the strip-like domain shared between* $\partial M_{k,S,\epsilon}$ *and* $\partial E_{m,\epsilon}$ *(see Fig. 2.2).*

Remark 3.1.9. *The foliation of $E_{m,\epsilon}$ in terms of the cross-sections is used to construct a local averaging operator on the fattened binding.*

We leave the explicit description of the fattened binding in Subsection 3.3.1.

3.1.3 Quadratic Forms and Operators of Types I, II, and III

The quadratic form Q_ϵ (Definition 2.1.11) and operator A_ϵ (Proposition 2.1.12) are again the energy and Neumann Laplacian on M_ϵ, respectively.

We equip M with the surface measure dM (for a particular page we use dM_k) induced from $\mathbb{R}^3$. Similarly dE_m denotes the induced measure on the $1D$-submanifold E_m from $\mathbb{R}^3$. For a domain $\Omega \subset \mathbb{R}^3$, we denote the square-integrable space weighted by function w as $L_2(\Omega, w\, d\Omega)$.

Definition 3.1.10. *Let $\mathcal{G}_1$ denote Hilbert space on M where each page's surface measure is weighted by $2r_k$. I.e.*

$$\mathcal{G}_1 := \{u : u|_{M_k} \in L_2(M_k, 2r_k dM_k)\}. \tag{3.7}$$

Definition 3.1.11. *Let Q_1 be the closed, non-negative quadratic form on $\mathcal{G}_1$ given by*

$$Q_1(u) = \sum_k \int_{M_k} |\nabla_{M_k} u|^2 \, 2r_k \, dM \tag{3.8}$$

*with **domain** $\mathcal{G}_1^1$ consisting of functions u for which $Q_1(u)$ is finite and that are continuous across the bindings between pages M_k and $M_{k'}$:*

$$u|_{\partial M_k \cap E_m} = u|_{\partial M_{k'} \cap E_m}. \tag{3.9}$$

Here ∇_{M_k} is the gradient on M_k and restrictions to the binding E_m coincide as elements of $H^{1/2}(E_m)$.

The previous chapter covered uniformly fattened domains, and that class of problem corresponds directly to $\beta = 1$ and $r_m = r_k = 1$. The induced operator from Q_1 is the weighted Laplace-Beltrami operator which is consistent with previous results (i.e. the weight is a constant if

M_ϵ is uniformly fattened).

Proposition 3.1.12. *The operator A_1 associated with the quadratic form Q_1 is*

$$A_1 u := -\frac{1}{2r_k}\nabla_{M_k} \cdot 2r_k \nabla_{M_k} u \quad \text{on } M_k \tag{3.10}$$

with domain $\mathcal{G}_1^2$ consisting of $\mathcal{G}_1$ functions such that

$$\sum_k \int_{M_k} |A_1 u|^2 \, 2r_k dM_k < \infty \tag{3.11}$$

with continuity at the binding:

$$u|_{\partial M_k \cap E_m} = u|_{\partial M_{k'} \cap E_m}, \tag{3.12}$$

*and **Kirchhoff conditions** at the binding*

$$\sum_{k:\partial M_k \supset E_m} r_k D_{\nu_k} u(E_m) = 0 \tag{3.13}$$

where D_{ν_k} denotes the normal derivative at ∂M_k.

The spectrum of A_1 is discrete and non-negative.

Remark 3.1.13. *By the trace theorem [27, 29], a function in $\mathcal{G}_1^2$ is bounded and continuous.*

Type II and type III scenarios involve both surface and line measures.

Definition 3.1.14. *For an open book structured M, we say $\tilde{M}$ is the **decomposition of** M if $\tilde{M}$ is the topological space given by the disjoint union of each stratum (each embedded in its own copy of $\mathbb{R}^3$). I.e.,*

$$\tilde{M} = (\oplus_k M_k) \oplus (\oplus_m E_m). \tag{3.14}$$

In this topology, ∂M_k does not intersect E_m for any m.

Definition 3.1.15. *Let $\mathcal{G}_2$ denote the Hilbert space on the decomposed domain $\tilde{M}$ where each page has the surface measure dM_k in $\mathbb{R}^3$ weighted by $2r_k$ and each binding has the line measure dE_m*

weighted by πr_m^2.

$$\mathcal{G}_2 := \{(u,v) : u|_{M_k} \in L_2(M_k, 2r_k dM_k) \\ v = (v_1, ..., v_{N_E}) \quad v_m \in L_2(E_m, \pi r_m^2 dE_m)\}. \tag{3.15}$$

We abbreviate the pair (u,v) *as* w.

The next quadratic form is for type II fattened domains ($\beta < 1/2$) wherein the fattened bindings are so large as to have a non-negligible contribution to the total energy in the $\epsilon \to 0$ limit.

Definition 3.1.16. *Let* Q_2 *be the closed, non-negative quadratic form on* $\mathcal{G}_2$ *given by*

$$Q_2(w) = \sum_k \int_{M_k} |\nabla_{M_k} u|^2 \, 2r_k dM_k + \sum_m \int_{E_m} |\nabla v|^2 \, \pi r_m^2 dE_m \tag{3.16}$$

*with **domain*** $\mathcal{G}_2^1$ *consisting of functions* $w = (u,v)$ *for which* $Q_2(w)$ *is finite and* u *vanishes on the boundary of* M_k*:*

$$u(\partial M_k) = 0. \tag{3.17}$$

Equation 3.17 is exactly the reason for providing an alternative topology of M. The class of smooth test functions on which we define the weak differentiability of a function in $\mathcal{G}_2$ is $C_c^\infty(\tilde{M})$ not $C^\infty(M)$.

Because u and v are independent of one another, deriving the induced operator is trivial. Thus, we have a weighted Dirichlet Laplace-Beltrami operator on each page and a weighted Laplacian with periodic boundary conditions on the binding.

Proposition 3.1.17. *The operator* A_2 *associated with the quadratic form* Q_2 *is*

$$A_2 w := \begin{cases} -\dfrac{1}{2r_k} \nabla_{M_k} \cdot 2r_k \nabla_{M_k} u & M_k \\ -\dfrac{1}{\pi r_m^2} \nabla \cdot \pi r_m^2 \nabla v & E_m. \end{cases} \tag{3.18}$$

The domain $\mathcal{G}_2^2$ consists of $\mathcal{G}_2$ functions such that

$$\sum_k \int_{M_k} |A_2 u|^2 \, 2r_k dM_k + \sum_m \int_{E_m} |A_2 v|^2 \, \pi r_m^2 dE_m < \infty \tag{3.19}$$

with Dirichlet conditions on the boundary of M_k:

$$u|_{\partial M_k \cap E_m} = 0. \tag{3.20}$$

The spectrum of A_2 is discrete and non-negative.

Type III domains also possess the energy term due to the binding; however, the limit operator demands continuity between pages and bindings.

Definition 3.1.18. *Let Q_3 be the closed, non-negative quadratic form on $\mathcal{G}_2$ given by*

$$Q_3(w) = \sum_k \int_{M_k} |\nabla_{M_k} u|^2 \, 2r_k dM_k + \sum_m \int_{E_m} |\nabla v|^2 \, \pi r_m^2 dE_m \tag{3.21}$$

with ***domain*** *$\mathcal{G}_3^1$ consisting of functions $w = (u, v)$ for which $Q_3(w)$ is finite and u and v_m agree on E_m:*

$$u|_{\partial M_k \cap E_m} = v_m. \tag{3.22}$$

Proposition 3.1.19. *The operator A_3 associated with the quadratic form Q_3 is*

$$A_3 w := \begin{cases} -\dfrac{1}{2r_k} \nabla_{M_k} \cdot 2r_k \nabla_{M_k} u & M_k \\ -\dfrac{1}{\pi r_m^2} \nabla \cdot \pi r_m^2 \nabla v & E_m. \end{cases} \tag{3.23}$$

The domain $\mathcal{G}_3^2$ consists of pairs (u, v) in $\mathcal{G}_2$ such that

$$\sum_k \int_{M_k} |A_3 u|^2 \, 2r_k dM_k + \sum_m \int_{E_m} |A_3 v|^2 \, \pi r_m^2 dE_m < \infty, \tag{3.24}$$

u and v agree on the boundary of ∂M_k

$$u|_{\partial M_k \cap E_m} = v_m, \tag{3.25}$$

and u also satisfies Kirchhoff conditions:

$$\sum_{k:\partial M_k \supset E_m} r_k D_{\nu_k} u(E_m) = 0. \tag{3.26}$$

The spectrum of A_3 is discrete and non-negative.

Throughout the remainder of this text $\mathcal{G}$, $\mathcal{G}^1$, and A stand for one of the type I, II, or III spaces or operators. The spectral projector $\mathcal{P}_\Lambda$ (see Definition 2.1.15) is understood to project to the spectral subspace of the respective operator A_1, A_2, or A_3.

3.2 Formulation of Spectral Convergence for Types I, II, and III

We denote the non-decreasingly ordered eigenvalues of A as $\{\lambda_n\}_{n\in\mathbb{N}}$, and those of A_ϵ as $\{\lambda_n^\epsilon\}_{n\in\mathbb{N}}$. As in the preceding chapter the proof is reduced to finding operators satisfying some norm conditions. The definition of the averaging and extension operators are not different from those in Chapter 2, but are replicated here for completeness.

Definition 3.2.1. *A family of linear operators J_ϵ from $H^1(M_\epsilon)$ to $\mathcal{G}^1$ is called **averaging operators** if for any $\Lambda \notin \sigma(A_\epsilon)$ there is an ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$ the following conditions are satisfied:*

- *For $u \in \mathcal{P}_\Lambda^\epsilon L_2(M_\epsilon)$, J_ϵ is "nearly an isometry" from $L_2(M_\epsilon)$ to $\mathcal{G}$ with an $o(1)$ error, i.e.*

$$\left| ||u||^2_{L_2(M_\epsilon)} - ||J_\epsilon u||^2_{\mathcal{G}} \right| \le o(1)||u||^2_{H^1(M_\epsilon)} \tag{3.27}$$

  where $o(1)$ is uniform with respect to u.

- *For $u \in \mathcal{P}_\Lambda^\epsilon L_2(M_\epsilon)$, J_ϵ asymptotically "does not increase the energy," i.e.*

$$Q(J_\epsilon u) - Q_\epsilon(u) \le o(1)Q_\epsilon(u) \tag{3.28}$$

where $o(1)$ is uniform with respect to u.

Definition 3.2.2. *A family of linear operators K_ϵ from $\mathcal{G}^1$ to $H^1(M_\epsilon)$ is called* ***extension operators*** *if for any $\Lambda \notin \sigma(A)$ there is an ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$ the following conditions are satisfied:*

- *For $u \in \mathcal{P}_\Lambda \mathcal{G}$, K_ϵ is "nearly an isometry" from $\mathcal{G}$ to $L_2(M_\epsilon)$ with $o(1)$ error, i.e.*

$$\left| \big|\big|u\big|\big|_{\mathcal{G}}^2 - \big|\big|K_\epsilon u\big|\big|_{L_2(M_\epsilon)}^2 \right| \leq o(1)\big|\big|u\big|\big|_{\mathcal{G}^1}^2 \tag{3.29}$$

  where $o(1)$ is uniform with respect to u.

- *For $u \in \mathcal{P}_\Lambda \mathcal{G}$, K_ϵ asymptotically "does not increase" the energy, i.e.*

$$Q_\epsilon(K_\epsilon u) - Q(u) \leq o(1)Q(u) \tag{3.30}$$

  where $o(1)$ is uniform with respect to u.

Theorem 3.2.3. *Let M be an open book structure as in Definition 2.1.1 and $\{M_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be its fattened partner as in Definition 3.1.1 for either type I, II, or III. Let A_ϵ be the operator as in Proposition 2.1.12. Let A and be the corresponding operator according to the type M_ϵ as in Propositions 3.1.12, 3.1.17, or 3.1.19 for type I, II, or III respectively.*

Suppose there exist averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ and extension operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ as stated in Definitions 3.2.1 and 3.2.2. Then, for any n

$$\lambda_n(A_\epsilon) \underset{\epsilon\to 0}{\to} \lambda_n(A). \tag{3.31}$$

We will construct the required averaging and extension operators, which will lead to the main result of this chapter:

Theorem 3.2.4. *Let M be an open book structure as in Definition 2.1.1 and $\{M_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be its fattened partner as in Definition 3.1.1. Let A_ϵ be the operator on M_ϵ as given in Proposition 2.1.12. Let A be an operator on M accordingly defined in Propositions 3.1.12, 3.1.17, or 3.1.19 for* *type I, II, or III respectively. There exist averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ and extension operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ as stated in Definitions 3.2.1 and 3.2.2. Thus, for any n*

$$\lambda_n(A_\epsilon) \underset{\epsilon\to 0}{\to} \lambda_n(A). \tag{3.32}$$

3.3 Proof of Spectral Convergence for Non-Uniformly Fattened Domains (Theorem 3.2.4)

As in the uniformly fattened case, we need to define a local averaging operator on each of the fattened strata and a local extension operator from each of the strata on M into M_ϵ. The presence of more parameters introduces several new technicalities. As well as requiring some new propositions, some proofs and definitions have to be considerably modified. Because of that we specify for what β the proposition is applicable to if there are restrictions.

3.3.1 Fattened Binding Geometry

In this subsection we describe the geometry of the fattened binding and, in particular, specify the length a_m. We also define a partition of unity for $E_{m,\epsilon}$ that is used in the estimation of a trace operator.

Definition 3.3.1. *Let M be an open book structure. Let $\theta_{m,k,k'}(x)$ be the (smaller) angle between two tangent vectors normal to two intersecting page boundaries ∂M_k and $\partial M_{k'}$ at $x \in E_m$. In the $\beta = 1$ case, $a_{m,\epsilon}$ is a_m. The sleeve width a_m ($m \le n_E$) for $\beta = 1$ domains is (see Fig. 3.1):*

$$a_m = \begin{cases} \max_{x\in E_m}(r_m(x) + r_k(x)\cot(\min_{k,k'}\theta_{m,k,k'}(x)/2)) & \min_{x,k,k'}\theta_{m,k,k'}(x)/2 < \pi/2 \\ \max_{x\in E_m}(r_m(x) + r_k(x)) & \min_{x,k,k'}\theta_{m,k,k'}(x)/2 \ge \pi/2. \end{cases} \tag{3.33}$$

For $\beta < 1$ the limit ($\epsilon \to 0$) sleeve width a_m ($m \le n_E$) is

$$a_m = r_m, \tag{3.34}$$

and the sleeve width $a_{m,\epsilon}$ is the distance from the binding at which the fattened page first emerges

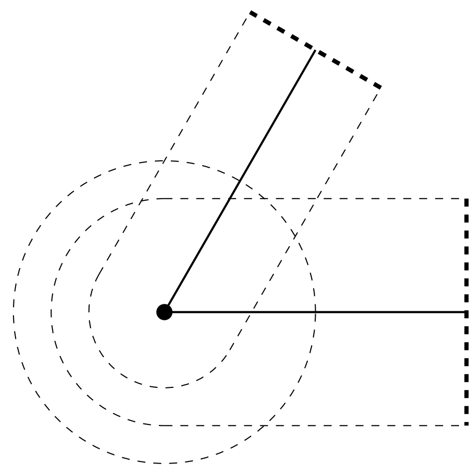

Figure 3.1: A cross-section of a $\beta = 1$ fattened binding neighborhood with distinct values of r_m and r_k. Dashed lines denote the boundary of a fattened stratum. The thickest dashed lines each denote a cross-section of $\Gamma_{k,m,\epsilon}$.

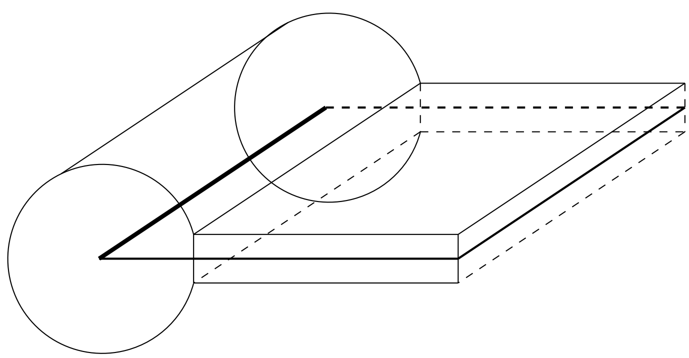

Figure 3.2: When $\beta < 1$, the fattened binding is a cylindrical-like domain cut by an $O(\epsilon)$-width strip.

out of the fattening of the binding (see Fig. 3.2):

$$a_{m,\epsilon}(y \in E_m) = \min_{x \in U} \mathsf{dist}_{M_k}(y, x) \tag{3.35}$$

where

$$U := \{x \in M_k : x + \epsilon r_k(x)\mathcal{N}_k(x) \in \partial\big(\bigcup_{y \in E_m} B(y, r_m(y)\epsilon^\beta)\big)\}. \tag{3.36}$$

Thus the closure of the normal fibers $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$ and $\mathcal{I}_{\mathcal{N}_{k'}(x'),\epsilon}$ do not touch for two distinct fattened *pages $M_{k,S,\epsilon}$ and $M_{k',S,\epsilon}$.*

Lemma 3.3.2. *Let $\beta < 1$ then the area of $\omega_{m,\epsilon}(x)$ is $\pi r_m^2 \epsilon^{2\beta}(1 + O(\epsilon^{2-2\beta}))$.*

Proof: Given a disk with radius $r_m\epsilon^\beta$, we cut it with a chord of length $2r_k\epsilon(1 + o(1))$. After removing the smaller region given by the cut, the residual area is $\pi r_m^2 \epsilon^{2\beta} - O(\epsilon^2)$. □

Definition 3.3.3. *Let l_{E_m} denote the length of the E_m. We define $\gamma_m(y) : U = [0, l_{E_m}]/\{0, l_{E_m}\} \mapsto E_m$ to be a smooth parameterization of E_m. We suppose around each point x on E_m (with $x = \gamma_m(y)$) there is a neighborhood $V \subset U$ of y such that there exists two smooth orthogonal unit length vectors $v_{m,1}$ and $v_{m,2}$ on $\gamma_m(V)$ that span $N_{\gamma_m(y)}$.*

We equip the normal plane N_x ($x \in E_m$) with the following coordinate chart $\phi_x : N_x \mapsto \mathbb{R}^2$ where $\phi_x(x) = 0$, ϕ_x is an isometry, and $\phi_x(v_{m,i}(x))$ is the standard basis vector $\mathbf{e}_{y_i}$. The image of $\omega_{m,\epsilon}(x)$ through this chart ϕ_x is denoted $\varpi_{m,\epsilon}(x)$, an open region in $\mathbb{R}^2$. We call $\varpi_{m,\epsilon}(x)$ a cross-section as well.

This following lemma follows from our definition of the fattened binding. The reader should note that cutting the fattened binding when $\beta < 1$ to have no protruding region (see Fig. 3.2 and Definition 3.3.1) is necessary for the following to hold:

Proposition 3.3.4. *Let $\{E_{m,\epsilon}\}$ ($0 < \epsilon \leq \epsilon_0$) denote a family of fattened binding as previously described. The following properties hold uniformly for each cross-section $\varpi_{m,\epsilon}(x)$ ($x \in E_m$):*

1. *The inner and outer diameters over each cross-section are bounded of order ϵ^β:*

$$D(0, c_1\epsilon^\beta) \subset \varpi_{m,\epsilon}(x) \subset D(0, c_2\epsilon^\beta). \tag{3.37}$$

2. *There is a positive number c_r such that each cross-section $\varpi_{m,\epsilon}(x)$ is star-shaped with respect to the disk $D(0, c_r\epsilon^\beta)$ (see Fig. 3.3).*

3. *There exists numbers c_M, c_N, c_U, and c_3 such for each $\epsilon \in (0, \epsilon_0]$ there is a finite collection of open sets $\{\tilde{U}_{i,\epsilon}\}$ ($i \leq c_U$) in $\mathbb{R}^2$ where*

Figure 3.3: Left: a view of $\varpi_{m,\epsilon}(x)$ when $\beta = 1$. Given $r_m \geq r_k$, there is an c_r such that this cross-section is star-shaped with respect to $D(0, c_r\epsilon)$. Right: a view of $\varpi_{m,\epsilon}(x)$ when $\beta < 1$.

(a) if $y \in \partial\varpi_{m,\epsilon}(x)$ then $D(y, c_3\epsilon^\beta) \subset \tilde{U}_{i,\epsilon}$ for some i,

(b) each $y \in \partial\varpi_{m,\epsilon}(x)$ is contained in at most c_N sets $\tilde{U}_{i,\epsilon}$,

(c) and for any i there is a special Lipschitz domain $\tilde{\Omega}_{i,\epsilon}$ with boundary graph function $\tilde{\phi}_{i,\epsilon}$ such that $\tilde{U}_{i,\epsilon} \cap \varpi_{m,\epsilon}(x) = \tilde{U}_{i,\epsilon} \cap \tilde{\Omega}_{i,\epsilon}$ and

$$|\tilde{\phi}_{i,\epsilon}(z) - \tilde{\phi}_{i,\epsilon}(z')| \leq c_M|z - z'|, \qquad z,\, z' \in \mathbb{R}. \tag{3.38}$$

We extend (3) to a statement about the existence of a partition of unity on $E_{m,\epsilon}$ that we will need later.

Corollary 3.3.5. *Let $\{E_{m,\epsilon}\}$ denote a family of fattened binding neighborhoods as previously described. For each $\epsilon \in (0, \epsilon_0]$ there exists a partition of unity $\{\varphi_{i,\epsilon}\}$ (i is a counting number up to $N_{U,\epsilon}$ which depends on ϵ) subordinate to the finite open cover $\{U_{i,\epsilon}\}$ of $E_{m,\epsilon}$ with the following properties:*

1. *$\bigcup_i U_{i,\epsilon}$ is contained in $\bigcup_{x\in E_m} B(x, c_0\epsilon^\beta)$.*
2. *Each point contained in the covering is in at most c_N sets.*
3. *Each open set $U_{i,\epsilon}$ contains a ball of radius $c_1\epsilon^\beta$ and is contained in a ball of radius $c_2\epsilon^\beta$.*
4. *If $x \in \partial E_{m,\epsilon}$, then $B(x, c_3\epsilon^\beta) \subset U_{i,\epsilon}$ for some i and $U_{i,\epsilon} \cap \partial E_{m,\epsilon}$ is a connected subset of some special Lipschitz domain $\Omega_{i,\epsilon}$ whose boundary graph function $\phi_{i,\epsilon}$ has a (Lipschitz) norm bounded above by a constant c_M.*

5. *There is a positive constant c_φ such that for each ϵ the gradient of each $\varphi_{i,\epsilon}$ has a uniform bound $c_\varphi \epsilon^{-\beta}$:*

$$|\nabla \varphi_{i,\epsilon}| \leq c_\varphi \epsilon^{-\beta}. \tag{3.39}$$

3.3.2 Fattened Binding Foliation

Given our foliations of the fattened pages $M_{k,S,\epsilon}$ and $M_{k',S,\epsilon}$ (in terms of the normal lines $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$), we wish to extend those foliations into $E_{m,\epsilon}$. As in the uniformly fattened case, we accomplish this by introducing regions of the fattened binding called sectors. Breaking up the fattened binding into sectors, we can describe a vector field whose image "connects" the foliation of one fattened page to another foliation (see Fig. 3.4). These results are used for type I domains only. Type II and III domains are handled differently.

Definition 3.3.6. *Let E_m be a binding and $\{M_k\}$ ($k \leq n_m$) is the collection of at least two pages that meet at E_m all of which are orientable. We call the connected components of $E_{m,\epsilon} \backslash (E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))$ **sectors** denoted $\{\Sigma_{m,i,\epsilon}\}$ for $i \leq n_m$. A sector's boundary contains two sleeves of which we say that pair is associated with that sector (see Fig. 2.5).*

If E_m is a binding connected to non-orientable pages, then taking a partition into trivializable neighborhoods is sufficient for our discussion. The case of only one page meeting at a binding is handled separately.

Definition 3.3.7. *Let E_m be a binding and $\{M_k\}$ ($k \leq n_m$) is the collection pages that meet at E_m all of which are orientable and there are at least two such pages. We say that the image of family of vector fields $\{tv_{m,i,\epsilon}\}$ ($t \in (0,1)$)*

$$v_{m,i,\epsilon}(x) : E_m \cup S_{k,m,\epsilon} \cup S_{k',m,\epsilon} \mapsto \mathbb{R}^3 \qquad S_{k,m,\epsilon},\, S_{k',m,\epsilon} \subset \partial \Sigma_{m,i,\epsilon} \tag{3.40}$$

*is a **foliation of the sector matching the foliation of fattened pages** (see Fig. 3.4) if*

1. *$v_{m,i,\epsilon}$ is Lipschitz.*

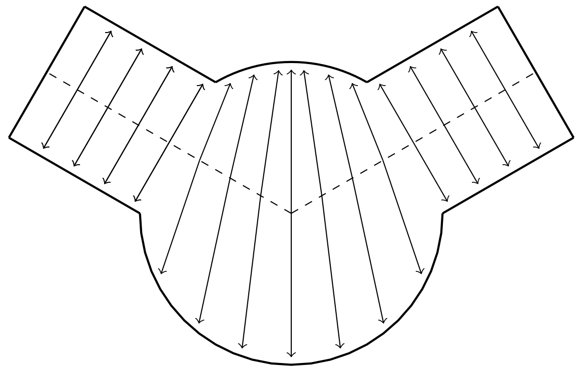

Figure 3.4: Cross sectional view of a pair of vector fields on each of the sleeves yielding a foliation of the two sectors.

2. *$x \mapsto x + v_{m,i,\epsilon}(x)$ is a homeomorphism between the domain of $v_{m,i,\epsilon}$ and the outward boundary of the sector: $\partial\Sigma_{m,i,\epsilon} \cap (\partial E_{m,\epsilon} \setminus \cup_k \partial M_{k,S,\epsilon})$.*

3. *The limit of $v_{m,i,\epsilon}(x)$ as $x \to x' \in \partial S_{k,m,\epsilon} \cap M_k$ is $\pm\epsilon r_k(x')\mathcal{N}_k(x')$.*

If E_m meets only one sleeve M_k, we say a family of vector fields $\{v_{m,i,\epsilon}\}$ ($i = 1, 2$)

$$v_{m,i,\epsilon} : S_{k,m,\epsilon} \mapsto \mathbb{R}^3 \tag{3.41}$$

extends the foliation of the fattened page (see Fig. 2.7) if:

1. *$v_{m,i,\epsilon}$ is Lipschitz.*

2. *$x \mapsto x + v_{m,i,\epsilon}(x)$ is a homeomorphism between the domain of $v_{m,i,\epsilon}$ and a subset boundary of the the fattened binding: $\partial E_{m,\epsilon} \backslash \partial M_{k,S,\epsilon}$.*

3. *The limit of $v_{m,i,\epsilon}(x)$ as $x \to x' \in \partial S_{k,m,\epsilon} \cap M_k$ is $\pm\epsilon r_k(x')\mathcal{N}_k(x')$.*

4. *The limits of $v_{m,1,\epsilon}(x)$ and $v_{m,2,\epsilon}(x)$ match at E_m.*

We expand on (2) and describe the construction of $\{v_{m,i,\epsilon}\}$ for all small, positive ϵ that has a uniformly bounded gradient (where it exists).

Proposition 3.3.8. *There is a family of vector-valued functions* $\{v_{m,i,\epsilon}\}$ *(*$\epsilon \in (0, \epsilon_0]$*) that extends the foliation of the fattened pages that have length of* $O(\epsilon^\beta)$ *and uniformly bounded gradient (where it exists). I.e. there exists a* c_1 *and* c_2 *such that*

$$\max_{x \in D(v_{m,i,\epsilon})} ||v_{m,i,\epsilon}(x)|| \leq c_1 \epsilon^\beta, \tag{3.42}$$

$$\max_{x \in D(v_{m,i,\epsilon})} ||\nabla v_{m,i,\epsilon}(x)|| \leq c_2. \tag{3.43}$$

Proof: For $\beta < 1$, this proof is simpler than the $\beta = 1$ case because of the convexity of the cross-sections. When $\beta = 1$ and $r_m \neq 1 \neq r_k$, the proof requires small modification. The proof is contained along side the analogous proposition for the uniformly fattened case. See Appendix B.1.1.

Corollary 3.3.9. *Each sector* $\Sigma_{m,i,\epsilon}$ *can be parameterized using* $v_{m,i,\epsilon}$*. Namely, a point* $x \in \Sigma_{m,i,\epsilon}$ *can be written as* $x = y + z v_{m,i,\epsilon}(y)$ *(*$y \in E_m \bigcup \left(\bigcup_k S_{k,m,\epsilon}\right)$*,* $z \in (0,1)$*).*

3.3.3 Approximating the Geometry of Non-Uniformly Fattened Strata

The propositions and their proofs in this subsection are not qualitatively different than the uniformly fattened case in Chapter 2 except $\tilde{M}_{k,S,\epsilon}$ is not a fiber bundle (locally trivializable as a product) but a fibration (locally trivializable as a disjoint union of fibers).

Definition 3.3.10. *We define the fibration of* $M_{k,S}$ *with fibers* $I_{k,\epsilon}(x)$ *as* $\tilde{M}_{k,S,\epsilon}$*:*

$$\tilde{M}_{k,S,\epsilon} := \coprod_{x \in M_{k,S}} I_{k,\epsilon}(x). \tag{3.44}$$

Proposition 3.3.11. *For sufficiently small* ϵ*, there exists a diffeomorphism* $\phi_{M_{k,S,\epsilon}}$ *from* $M_{k,S,\epsilon}$ *to* $\tilde{M}_{k,S,\epsilon}$ *such that the induced linear operator* $\Phi_{M_{k,S,\epsilon}}$ *on* $H^1(M_{k,S,\epsilon})$ *(i.e.* $\Phi_{M_{k,S,\epsilon}} u = u(\phi_{M_{k,S,\epsilon}})$*) preserves* H^1*-norm of a function up to an* $O(\epsilon^{1/2})$ *error.*

$$\left| \, ||u||^2_{H^1(M_{k,S,\epsilon})} - ||\Phi_{M_{k,S,\epsilon}} u||^2_{H^1(\tilde{M}_{k,S,\epsilon})} \, \right| \leq c\epsilon ||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{3.45}$$

The inequality (3.45) also holds true for other Sobolev spaces H^n and in particular L_2.

Definition 3.3.12. *We define the fibration over E_m with fibers $\varpi_{m,\epsilon}(x)$ as $\tilde{E}_{m,\epsilon}$:*

$$\tilde{E}_{m,\epsilon} := \coprod_{x \in E_m} \varpi_{m,\epsilon}(x). \tag{3.46}$$

Proposition 3.3.13. *For sufficiently small ϵ, there exists a diffeomorphism $\phi_{E_{m,\epsilon}}$ from $E_{m,\epsilon}$ to $\tilde{E}_{m,\epsilon}$ such that the induced linear operator $\Phi_{E_{m,\epsilon}}$ on $H^1(E_{m,\epsilon})$ (i.e. $\Phi_{E_{m,\epsilon}}u = u(\phi_{E_{m,\epsilon}})$) preserves H^1-norm up to an $o(1)$ error:*

$$\big| \, ||u||^2_{H^1(E_{m,\epsilon})} - ||\Phi_{E_{m,\epsilon}}u||^2_{H^1(\tilde{E}_{m,\epsilon})} \big| \leq c\epsilon^{\beta}||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.47}$$

This inequality (3.47) also holds true for other Sobolev spaces H^n and in particular L_2.

3.3.4 Bounds on the Sleeves for Non-Uniform Case

In this subsection we demonstrate there is a diffeomorphism from M_k to $M_{k,S}$ satisfying certain properties that lets us subsequently bound the L_2-norm of a function on a sleeve with respect to its H^1-norm on the page. Unlike the uniformly fattened case (see Subsection 2.3.4), when $\beta < 1$ the sleeves have varying width $a_{m,\epsilon}\epsilon^{\beta}$. The analogous proof in Chapter 2 requires a change in the smooth contracting function.

Proposition 3.3.14. *There exists a diffeomorphism ψ_{M_k} from M_k to $M_{k,S}$ where*

- *each column vector of the Jacobian of ψ_{M_k} has length $1 + O(\epsilon^{\beta})$,*
- *for any unit speed differentiable curve γ on $\bar{M}_{k,S}$ that is normal to $\partial M_{k,S}$, $\psi_{M_k}(\gamma)$ is unit speed and normal to the boundary ∂M_k,*
- *and the induced operator Ψ_{M_k} (i.e. $\Psi_{M_k}u = u(\psi_{M_k})$) preserves H^1-norm up to an $O(\epsilon^{\beta/2})$ error:*

$$\big| \, ||u||^2_{H^1(M_k)} - ||\Psi_{M_k}u||^2_{H^1(M_{k,S})} \big| \leq c\epsilon^{\beta}||u||^2_{H^1(M_k)}. \tag{3.48}$$

This inequality also holds true for other Sobolev spaces H^n and in particular L_2.

Proof: A sufficiently small neighborhood V of ∂M_k admits a normal coordinate system. Meaning there is a parameterization X_k on $U \subset \mathbb{R}^2$ of V:

$$\begin{aligned} X_k : (y_1, y_2) \in U = (0, l_{E_m}) \times (0, a) \mapsto M_k, \\ \text{such that} \quad \mathsf{dist}_{M_k}(E_m, X_k(y_1, y_2)) = y_2. \end{aligned} \tag{3.49}$$

For sufficiently small ϵ, $\partial M_{k,S}$ is contained in V. Clearly $\partial M_{k,S}$ is the image of $X_k(\cdot, a_{m,\epsilon}\epsilon^\beta)$. We define a smooth shortening function

$$\varphi_\epsilon : (0, l_{E_m}) \times (0, a) \mapsto (0, l_{E_m}) \times (a_{m,\epsilon}\epsilon^\beta, a) \tag{3.50}$$

whose Jacobian J_{φ_ϵ} satisfies

$$J_{\varphi_\epsilon} = \begin{bmatrix} 1 + O(\epsilon^\beta) & O(\epsilon^\beta) \\ O(\epsilon^\beta) & 1 + O(\epsilon^\beta) \end{bmatrix} \tag{3.51}$$

and

$$J_{\varphi_\epsilon}|_{(0,l_{E_m})\times\{0\}} = J_{\varphi_\epsilon}|_{(0,l_{E_m})\times\{a\}} = \mathsf{Id}_{\mathbb{R}^2}. \tag{3.52}$$

This is sufficient to construct ψ_{M_k}:

$$\psi_{M_k}(x) = X_k(\varphi_\epsilon(y_1, y_2)) \qquad (y_1, y_2) = X_k^{-1}(x). \tag{3.53}$$

The remainder of the proof follows from the calculating the induced metric from ψ_{M_k} as done in Corollaries 2.3.23 and 2.3.24. □

Proposition 3.3.15. *Let M_k be a smooth page with boundary $\bigcup_m E_m$. The L_2 norm of a function* *on $S_{k,m,\epsilon}$ is $O(\epsilon^{\beta/2})$-bounded by the function's H^1 norm on M_k:*

$$\int_{S_{k,m,\epsilon}} |u|^2 \, dM_k \leq c\epsilon^{\beta} \int_{M_k} |u|^2 + |\nabla_{M_k} u|^2 \, dM_k. \tag{3.54}$$

The proof of this proposition only requires small modification from the corresponding proposition covered in the uniformly fattened case, and so it is omitted.

3.3.5 Extensions from Each Stratum Non-Uniform Case

We define extensions from the strata on M into the "trivialized" spaces $\tilde{M}_{k,S,\epsilon}$ and $\tilde{E}_{m,\epsilon}$ and relate them to the original space by the diffeomorphism operators in Propositions 3.3.11 and 3.3.13. There is a new extension operator introduced for type II and type III domains which does not have a counterpart in the uniformly fattened case. This operator $\mathcal{E}_{m,z,\epsilon}$ fills the roll of $\mathcal{E}_{m,S,z,\epsilon}$ for the cases where the energy on the binding is non-negligible ($\beta \leq 1/2$).

Definition 3.3.16. *Let $u \in L_2(M_{k,S})$ and $\tilde{M}_{k,S,\epsilon}$ be as defined in Definition 3.3.10. We denote a point in the fibration $\tilde{M}_{k,S,\epsilon}$ as (y,z) for $y \in M_{k,S}$ and $z \in I_{k,\epsilon}(y)$. We define $\tilde{\mathcal{E}}_{k,z,\epsilon}$ to be the extension operator from $M_{k,S}$ to $\tilde{M}_{k,S,\epsilon}$, a bounded linear operator from $L_2(M_{k,S})$ to $L_2(\tilde{M}_{k,S,\epsilon})$, given by:*

$$\tilde{\mathcal{E}}_{k,z,\epsilon} u(y,z) := u(y). \tag{3.55}$$

Definition 3.3.17. *Let $u \in L_2(M_{k,S})$. We define $\mathcal{E}_{k,z,\epsilon}$ to be the extension operator from $M_{k,S}$ to $M_{k,S,\epsilon}$ given by*

$$\mathcal{E}_{k,z,\epsilon} := \Phi^{-1}_{M_{k,S,\epsilon}} \tilde{\mathcal{E}}_{k,z,\epsilon}. \tag{3.56}$$

Lemma 3.3.18. *For $u \in H^1(M_{k,S}, 2r_k dM_k)$, one has:*

$$||u||^2_{L_2(M_{k,S},2r_k dM_k)} = ||\epsilon^{-1/2} \tilde{\mathcal{E}}_{k,z,\epsilon} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}. \tag{3.57}$$

Proof: Because

$$\frac{1}{\epsilon} \int_{I_{k,\epsilon}(y)} \tilde{\mathcal{E}}_{k,z,\epsilon} u(y,z) \, dI_{k,\epsilon}(y) = 2r_k(y) \, |u|^2, \tag{3.58}$$

we have

$$\begin{aligned}
&\big|\, ||u||^2_{L_2(M_{k,S},2r_k dM_k)} - ||\epsilon^{-1/2}\tilde{\mathcal{E}}_{k,z,\epsilon}u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}\,\big| \\
&= \left| \int_{M_{k,S}} |u|^2\, 2r_k dM_k - \int_{M_{k,S}} \int_{I_{k,\epsilon}(y)} \epsilon^{-1} |\tilde{\mathcal{E}}_{k,z,\epsilon}u(y,z)|^2\, dI_{k,\epsilon}(y)\, dM_k \right|
\end{aligned} \tag{3.59}$$

which is zero. □

Lemma 3.3.19. *For* $u \in H^1(M_{k,S}, 2r_k dM_k)$*, one has:*

$$\big|\, ||u||^2_{H^1(M_{k,S},2r_k dM_k)} = ||\epsilon^{-1/2}\tilde{\mathcal{E}}_{k,z,\epsilon}u||^2_{H^1(\tilde{M}_{k,S,\epsilon})}. \tag{3.60}$$

Proof: Beginning with the difference of the norms of the derivative,

$$\big|\, ||\nabla_{M_k} u||^2_{L_2(M_{k,S},2r_k dM_k)} - ||\nabla \epsilon^{-1/2}\tilde{\mathcal{E}}_{k,z,\epsilon}u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}\,\big|, \tag{3.61}$$

we note $D_z\tilde{\mathcal{E}}_{k,z,\epsilon}u = 0$. We can then write:

$$\begin{aligned}
&\left| \int_{M_{k,S}} |\nabla_{M_k} u|^2\, 2r_k dM_k - \int_{M_{k,S}} \int_{I_{k,\epsilon}(y)} \epsilon^{-1} |\nabla \tilde{\mathcal{E}}_{k,z,\epsilon}u|^2\, dI_{k,\epsilon}(y)\, d\tilde{M}_k \right| \\
&= \left| \int_{M_{k,S}} |\nabla_{M_k} u|^2\, 2r_k dM_k \right. \\
&\left. - \epsilon^{-1} \int_{M_{k,S}} \int_{I_{k,\epsilon}(y)} |\nabla_{M_k} \tilde{\mathcal{E}}_{k,z,\epsilon}u|^2 + |D_z \tilde{\mathcal{E}}_{k,z,\epsilon}u|^2\, dI_{k,\epsilon}(y)\, d\tilde{M}_{k,S,\epsilon} \right| \\
&= 0. \qquad \square
\end{aligned} \tag{3.62}$$

Corollary 3.3.20. *For* $u \in H^1(M_{k,S}, 2r_k dM_k)$*, one has:*

$$\big|\, ||u||^2_{H^1(M_{k,S},2r_k dM_k)} - ||\epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}u||^2_{H^1(M_{k,S,\epsilon})}\,\big| \le c\epsilon ||u||^2_{H^1(M_{k,S},2r_k dM_k)}. \tag{3.63}$$

Proof: This is a straightforward application of Proposition 3.3.11 to (3.60).

Definition 3.3.21. *For the fattened binding* $E_{m,\epsilon}$*, we suppose its sectors* $\Sigma_{m,i,\epsilon}$ *are equipped with coordinate system described in Corollary 3.3.9 generated by* $v_{m,i,\epsilon}$*, the vector-valued function as* *described in Definition 3.3.7 and Proposition 3.3.8. We define $\mathcal{E}_{m,S,z,\epsilon}$ to be the extension operator on $L_2(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}))$ to $L_2(E_{m,\epsilon})$ given by sector as*

$$\mathcal{E}_{m,S,z,\epsilon}u(y,z) = u(y) \qquad y \in S_{k,m,\epsilon} \bigcup S_{k',m,\epsilon} \mapsto \Sigma_{m,i,\epsilon} \ni (y,z). \tag{3.64}$$

Proposition 3.3.22. *The extension operators $\epsilon^{-1/2}\mathcal{E}_{m,S,z,\epsilon}$ from $H^1(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}), 2r_k dM_k)$ to $H^1(E_{m,\epsilon})$ satisfy the following bound:*

$$||\epsilon^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{H^1(E_{m,\epsilon},2r_k dM_k)} \leq c\epsilon^{\beta-1}||u||^2_{H^1(E_m \bigcup(\bigcup_k S_{k,m,\epsilon}),2r_k dM_k)}. \tag{3.65}$$

Proof: Corollary 3.3.9 prescribes a coordinate system $x = y+zv_{m,i,\epsilon}(y)$ ($y \in E_m \bigcup (\bigcup_k S_{k,m,\epsilon})$, $z \in (0,1)$) on each sector $\Sigma_{m,i,\epsilon}$. With only small modification we conclude that the induced metric on $\Sigma_{m,i,\epsilon}$ in to the proof of Proposition 2.3.37 is ϵ^β bounded:

$$\mathsf{det}(g_{\Sigma_{m,i,\epsilon},k}) \leq c\epsilon^\beta \mathsf{det}(g_{M_k}). \tag{3.66}$$

The remainder of the proof is not qualitatively different than the uniformly fattened case leading to (3.65). □

Definition 3.3.23. *Let $u \in L_2(E_m)$. We denote a point in the fibration $\tilde{E}_{m,\epsilon}$ as (y,z) for $y \in E_m$ and $z \in \varpi_{m,\epsilon}(y)$. We define an extension operator $\tilde{\mathcal{E}}_{m,z,\epsilon}$ from E_m to $\tilde{E}_{m,\epsilon}$ given by:*

$$\tilde{\mathcal{E}}_{m,z,\epsilon}u(y,z) = u(y). \tag{3.67}$$

Definition 3.3.24. *Let $u \in L_2(E_m)$. We define $\mathcal{E}_{m,z,\epsilon}$ to be the bounded linear operator from E_m to $E_{m,\epsilon}$ given by:*

$$\mathcal{E}_{m,z,\epsilon} := \Phi^{-1}_{E_{m,\epsilon}} \tilde{\mathcal{E}}_{m,z,\epsilon}. \tag{3.68}$$

Lemma 3.3.25. *For $u \in H^1(E_m, \pi r_m^2 dE_m)$, one has:*

$$\big|\, ||u||^2_{L_2(E_m,\pi r_m^2 dE_m)} - ||\epsilon^{-\beta}\tilde{\mathcal{E}}_{m,z,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})}\,\big| \le c\epsilon^{2-2\beta}||u||^2_{H^1(E_m,\pi r_m^2 dE_m)}. \tag{3.69}$$

Proof: We begin with the left hand side of (3.69):

$$\begin{aligned}
&\big|\, ||u||^2_{L_2(E_m,\pi r_m^2 dE_m)} - ||\epsilon^{-\beta}\tilde{\mathcal{E}}_{m,z,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})}\,\big| \\
&= \left| \int_{E_m} |u|^2\, \pi r_m^2 dE_m - \int_{E_m}\int_{\varpi_{m,\epsilon}(y)} \epsilon^{-2\beta}|\tilde{\mathcal{E}}_{m,z,\epsilon}u|^2\, d\varpi_{m,\epsilon}(y)\, dE_m \right| \\
&\le \max_{y\in E_m} \frac{|\pi r_m(y)^2 - |\varpi_{m,\epsilon}(y)|\epsilon^{-2\beta}|}{\pi r_m^2}||u||^2_{L_2(E_m,\pi r_m(y)^2 dE_m)}.
\end{aligned} \tag{3.70}$$

By Lemma 3.3.2, we get the $O(\epsilon^{2-2\beta})$ bound. □

Lemma 3.3.26. *For $u \in H^1(E_m, \pi r_m^2 dE_m)$, one has:*

$$\big|\, ||u||^2_{H^1(E_m,\pi r_m^2 dE_m)} - ||\epsilon^{-\beta}\tilde{\mathcal{E}}_{m,z,\epsilon}u||^2_{H^1(\tilde{E}_{m,\epsilon})}\,\big| \le c\epsilon^{2-2\beta}||u||^2_{H^1(E_m,\pi r_m(y)^2 dE_m)}. \tag{3.71}$$

Proof: The transverse derivatives D_{z_i} $(i = 1, 2)$ of $\tilde{\mathcal{E}}_{m,z,\epsilon}u$ vanish, so

$$\begin{aligned}
&\big|\, ||D_y u||^2_{L_2(E_m,\pi r_m^2 dE_m)} - ||\nabla\epsilon^{-\beta}\tilde{\mathcal{E}}_{m,z,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})}\,\big| \\
&= \left| \int_{E_m} |D_y u|^2\, \pi r_m^2 dE_m - \int_{E_m}\int_{\varpi_{m,\epsilon}(y)} \epsilon^{-2\beta}|\nabla\tilde{\mathcal{E}}_{m,z,\epsilon}u|^2\, d\tilde{E}_{m,\epsilon} \right| \\
&\le \max_{y\in E_m} \frac{|\pi r_m(y)^2 - |\varpi_{m,\epsilon}(y)|\epsilon^{-2\beta}|}{\pi r_m(y)^2}||D_y u||^2_{L_2(E_m,\pi r_m^2 dE_m)}
\end{aligned} \tag{3.72}$$

which is again $O(\epsilon^{2-2\beta})$ bounded by Lemma 3.3.2. □

Corollary 3.3.27. *For $u \in H^1(E_m, \pi r_m^2 dE_m)$, one has:*

$$\big|\, ||u||^2_{H^1(E_m,\pi r_m^2 dE_m)} - ||\epsilon^{-\beta}\mathcal{E}_{m,z,\epsilon}u||^2_{H^1(E_{m,\epsilon})}\,\big| \le c\epsilon^{\min(2-2\beta,\beta)}||u||^2_{H^1(E_m,\pi r_m(y)^2 dE_m)}. \tag{3.73}$$

Proof: This is a straightforward application of Proposition 3.3.13 to (3.71).

3.3.6 Local Averaging Operators

This subsection concerns an averaging operator on the fattened binding and an averaging operation on the fattened binding by means of an integral representation. These averaging operators satisfy some Poincaré-type inequalities. We begin with defining averaging operators in fibrations $\tilde{M}_{k,S,\epsilon}$ and $\tilde{E}_{m,\epsilon}$ (see Definitions 3.3.10 and 3.3.12) and apply the transformation operators $\Phi_{M_{k,S,\epsilon}}$ and $\Phi_{E_{m,\epsilon}}$ to get our desired local averaging operator. For type II and III domains, we need a mollification operator for the fattened binding which regularizes the averaged function enough so we can apply a Poincaré-type inequality on the higher order derivatives.

In the case of the fattened pages, each fiber contains (and is obviously star-shaped with respect to) an interval of some fixed length. This time we use the integral representation (Theorem 2.3.49) for the local averaging operator on non-uniformly fattened domains

Definition 3.3.28. *Let* $c_l = \inf_{k;y\in M_k} r_k(y)$.

Let $\varphi \in C_0^\infty((-1,1))$ *such that* $\int_{(-1,1)} \varphi = 1$. $\tilde{N}_{k,\epsilon}$ *denotes the following bounded linear operator on* $L_2(\tilde{M}_{k,S,\epsilon})$*:*

$$\tilde{N}_{k,\epsilon} u(y,z) = \frac{1}{|2c_l\epsilon|} \int_{I_{k,\epsilon}(y)} \varphi\left(\frac{\zeta}{c_l\epsilon}\right) u(y,\zeta)\, dI_{k,\epsilon}(y). \tag{3.74}$$

We also let $\tilde{N}_{k,\epsilon}$ *denote the bounded linear operator from* $L_2(\tilde{M}_{k,S,\epsilon})$ *to* $L_2(M_{k,S}, 2r_k dM_k)$ *by means of restricting* $\tilde{N}_{k,\epsilon}u$ *to* $M_{k,S}$ *(*$\tilde{N}_{k,\epsilon}u(y,z=0)$*).*

Proposition 3.3.29. *The family of averaging operators* $\{N_{k,\epsilon}\}$ *on* $L_2(\tilde{M}_{k,S,\epsilon})$ *has a uniformly bound* c *on their norms.*

Boundedness is clear from the Cauchy-Schwartz Inequality.

Definition 3.3.30. *The averaging operator* $N_{k,\epsilon}$ *on* $M_{k,S,\epsilon}$ *is given by composition with the corresponding diffeomorphism:*

$$N_{k,\epsilon} := \Phi^{-1}_{M_{k,S,\epsilon}} \tilde{N}_{k,\epsilon} \Phi_{M_{k,S,\epsilon}}. \tag{3.75}$$

We also let $N_{k,\epsilon}$ to denote a bounded linear operator from $L_2(M_{k,S,\epsilon})$ to $L_2(M_{k,S})$ by restricting $N_{k,\epsilon}u$ to $M_{k,S}$ ($N_{k,\epsilon}u|_{M_{k,S}} = \tilde{N}_{k,\epsilon}\Phi_{M_{k,S,\epsilon}}u(y, z = 0)$).

Proposition 3.3.31. *For $u \in H^1(\tilde{M}_{k,S,\epsilon})$, $\tilde{N}_{k,\epsilon}$ satisfies a Poincaré-type inequality:*

$$\int_{\tilde{M}_{k,S,\epsilon}} |u - \tilde{N}_{k,\epsilon}u|^2 \, d\tilde{M}_{k,S,\epsilon} \leq c_k\epsilon^2 \int_{\tilde{M}_{k,S,\epsilon}} |\nabla u|^2 \, d\tilde{M}_{k,S,\epsilon}. \tag{3.76}$$

Proof: We begin with our integral representation:

$$\int_{\tilde{M}_{k,S,\epsilon}} |u(y,z) - \tilde{N}_{k,\epsilon}u(y,z)|^2 \, d\tilde{M}_{k,S,\epsilon} = \int_{\tilde{M}_{k,S,\epsilon}} \Big| \int_{I_{k,\epsilon}(y)} f_{y,\zeta}(z,r,\theta) D_\zeta u(y,\zeta) \, d\zeta \Big|^2 \, d\tilde{M}_{k,S,\epsilon}. \tag{3.77}$$

After applying the bounds of $f_{y,\zeta}$ according to Theorem 2.3.49, we then use the embedding of L_1 into L_2 over a compact space:

$$\begin{aligned} c \int_{\tilde{M}_{k,S,\epsilon}} \Big| \int_{I_{k,\epsilon}(y)} & D_\zeta u(y,\zeta) \, d\zeta \Big|^2 \, d\tilde{M}_{k,S,\epsilon} \\ & \leq c' \int_{\tilde{M}_{k,S,\epsilon}} |I_{k,\epsilon}(y)| \int_{I_{k,\epsilon}(y)} |D_\zeta u(y,\zeta)|^2 \, d\zeta d\tilde{M}_{k,S,\epsilon} \\ & \leq c \max_{y \in M_{k,S}} |I_{k,\epsilon}(y)|^2 \, ||\nabla u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}. \end{aligned} \tag{3.78}$$

Corollary 3.3.32. *For $u \in H^1(M_{k,S,\epsilon})$, the averaging operator $N_{k,\epsilon}$ satisfies a Poincaré-type inequality:*

$$||u - N_{k,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})} \leq c\epsilon^2 ||\nabla u||^2_{L_2(M_{k,S,\epsilon})}. \tag{3.79}$$

Proof: This is a straightforward application of Proposition 3.3.11 to (3.76).

Proposition 3.3.33. *For $u \in H^1(\tilde{M}_{k,S,\epsilon})$, one has:*

$$\left| ||u||^2_{L_2(\tilde{M}_{k,S,\epsilon})} - ||\epsilon^{1/2}\tilde{N}_{k,\epsilon}u||^2_{L_2(M_{k,S}, 2r_k dM_k)} \right| \leq c\epsilon ||u||^2_{H^1(\tilde{M}_{k,S,\epsilon})}. \tag{3.80}$$

Proof: Bounding the difference squared,

$$\Big|\int_{\tilde{M}_{k,S,\epsilon}} |u|^2\, d\tilde{M}_{k,S,\epsilon} - \int_{M_{k,S}} |\tilde{N}_{k,\epsilon} u|^2\, 2r_k \epsilon\, dM_k\Big|, \tag{3.81}$$

we have

$$\begin{aligned}\Big|\int_{\tilde{M}_{k,S,\epsilon}} |u|^2\, d\tilde{M}_{k,S,\epsilon} - \int_{M_{k,S}} \Big(\int_{I_{k,\epsilon}(y)} |\tilde{N}_{k,\epsilon} u|^2\, dI_{k,\epsilon}(y)\Big)\, dM_k\Big| \\ \leq (1+O(\epsilon))||u - \tilde{N}_{k,\epsilon} u||_{L_2(\tilde{M}_{k,S,\epsilon})} ||u + \tilde{N}_{k,\epsilon} u||_{L_2(\tilde{M}_{k,S,\epsilon})} \\ \leq 2\epsilon(1+O(\epsilon))||u||^2_{H^1(\tilde{M}_{k,S,\epsilon})}. \qquad \square\end{aligned} \tag{3.82}$$

Proposition 3.3.34. *The linear operator $\tilde{N}_{k,\epsilon}$ is bounded on $H^1(\tilde{M}_{k,S,\epsilon})$:*

$$\int_{M_{k,S}} |\nabla_{M_k} \epsilon^{1/2} \tilde{N}_{k,\epsilon} u|^2\, 2r_k dM_k \leq \int_{(-1,1)} |\varphi|^2 \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2\, d\tilde{M}_{k,S,\epsilon}. \tag{3.83}$$

Furthermore, $\epsilon^{1/2}\tilde{N}_{k,\epsilon} u$ satisfies the following energy bound for $u \in H^2(\tilde{M}_{k,S,\epsilon})$:

$$\int_{M_{k,S}} |\nabla_{M_k} \epsilon^{1/2} \tilde{N}_{k,\epsilon} u|^2\, 2r_k dM_k - \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2\, d\tilde{M}_{k,S,\epsilon} \leq c\epsilon ||u||^2_{H^2(\tilde{M}_{k,S,\epsilon})}. \tag{3.84}$$

Proof: We relate the integral on the weighted page with the fattened domain:

$$\int_{M_{k,S}} |\nabla_{M_k} \epsilon^{1/2} \tilde{N}_{k,\epsilon} u|^2\, 2r_k dM_k = \int_{M_{k,S}} \Big(\int_{I_{k,\epsilon}(y)} |\nabla_{M_k} \tilde{N}_{k,\epsilon} u|^2 dI_{k,\epsilon}(y)\Big)\, dM_k. \tag{3.85}$$

Using the reverse Fatou Lemma (Lemma A.1.2), we have:

$$\begin{aligned}\int_{\tilde{M}_{k,S,\epsilon}} \Big|\nabla_{M_k} \frac{1}{2c_l\epsilon} \int_{\mathcal{I}(y)} \varphi\left(\frac{\zeta}{c_l\epsilon}\right) u\, dI_{k,\epsilon}(y)\Big|^2\, d\tilde{M}_{k,S,\epsilon} \\ \leq \int_{\tilde{M}_{k,S,\epsilon}} \Big|\limsup_{\delta\to 0} \frac{1}{2c_l\epsilon} \int_{I_{k,\epsilon}(y)} \frac{u(y+\delta,\zeta) - u(y,\zeta)}{\delta} \varphi\left(\frac{\zeta}{c_l\epsilon}\right)\, dI_{k,\epsilon}(y)\Big|^2\, d\tilde{M}_{k,S,\epsilon} \\ \leq \int_{\tilde{M}_{k,S,\epsilon}} \Big|\frac{1}{2c_l\epsilon} \int_{I_{k,\epsilon}(y)} \varphi\left(\frac{\zeta}{c_l\epsilon}\right) \nabla_{M_k} u\, dI_{k,\epsilon}(y)\Big|^2\, d\tilde{M}_{k,S,\epsilon}.\end{aligned} \tag{3.86}$$

We then use the embedding of L_1 in L_2 on a compact interval and the Cauchy-Schwartz Inequality:

$$\begin{aligned}\int_{\tilde{M}_{k,S,\epsilon}} \Big|\frac{1}{2c_l\epsilon}\int_{I_{k,\epsilon}(y)} \varphi\left(\frac{\zeta}{c_l\epsilon}\right)\nabla_{M_k} u \, dI_{k,\epsilon}(y)\Big|^2 d\tilde{M}_{k,S,\epsilon} \\ \leq \int_{\tilde{M}_{k,S,\epsilon}} \left(\frac{1}{2c_l\epsilon}\int_{I_{k,\epsilon}(y)} |\varphi\left(\frac{\zeta}{c_l\epsilon}\right)\nabla_{M_k} u|^2 \, dI_{k,\epsilon}(y)\right) d\tilde{M}_{k,S,\epsilon} \\ \leq \frac{||\varphi(\zeta/c_l\epsilon)||^2_{L_2(I_{k,\epsilon})}}{2c_l\epsilon}\int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, d\tilde{M}_{k,S,\epsilon}.\end{aligned} \tag{3.87}$$

To demonstrate the other energy bound, we note

$$\int_{\tilde{M}_{k,S,\epsilon}} \Big|\frac{1}{2c_l\epsilon}\int_{I_{k,\epsilon}(y)} \varphi\left(\frac{\zeta}{c_l\epsilon}\right)\nabla_{M_k} u \, dI_{k,\epsilon}(y)\Big|^2 d\tilde{M}_{k,S,\epsilon} = ||\tilde{N}_{k,\epsilon}\nabla_{M_k} u||^2_{L_2(\tilde{M}_{k,S,\epsilon})}. \tag{3.88}$$

Taking the difference of the energy of the averaged function with the function's energy, we have

$$\begin{aligned}\int_{M_{k,S}} |\nabla_{M_k}\epsilon^{1/2}\tilde{N}_{k,\epsilon}u|^2 \, 2r_k dM_k - \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, d\tilde{M}_{k,S,\epsilon} \\ \leq \int_{\tilde{M}_{k,S,\epsilon}} |\tilde{N}_{k,\epsilon}\nabla_{M_k} u|^2 \, d\tilde{M}_{k,S,\epsilon} - \int_{\tilde{M}_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, d\tilde{M}_{k,S,\epsilon} \\ \leq ||\tilde{N}_{k,\epsilon}\nabla_{M_k} u - \nabla_{M_k} u||_{L_2(\tilde{M}_{k,S,\epsilon})} ||\tilde{N}_{k,\epsilon}\nabla_{M_k} u + \nabla_{M_k} u||_{L_2(\tilde{M}_{k,S,\epsilon})}.\end{aligned} \tag{3.89}$$

Here we apply Theorem 2.3.49. Assuming $u \in H^2(\tilde{M}_{k,S,\epsilon})$, it follows $D_{y_1}u,\ D_{y_2}u \in H^1(\tilde{M}_{k,S,\epsilon})$. We suppose $\nabla_{M_k} u = (w, v)$ with $w, v \in H^1(\tilde{M}_{k,S,\epsilon})$. Letting R^α_y denote the remainder operator in the integral representation (2.111), we have

$$\nabla_{M_k} u = (w + R^{y_1}_y D_{y_1} w + R^{y_2}_y D_{y_2} w, v + R^{y_1}_y D_{y_1} v + R^{y_2}_y D_{y_2} v). \tag{3.90}$$

Hence $\tilde{N}_{k,\epsilon}\nabla_{M_k} u - \nabla_{M_k} u$ is $(R^\alpha_y D^\alpha w, R^\alpha_y D^\alpha v)$. Therefore by the Cauchy inequality, this expression is proportional to $R^\alpha_y \Delta u$ and bounded by $c\epsilon||u||^2_{H^2(\tilde{M}_{k,S,\epsilon})}$. □

Proposition 3.3.35. *For $u \in H^1(M_{k,S,\epsilon})$, one has:*

$$\left| ||u||^2_{L_2(M_{k,S,\epsilon})} - ||\epsilon^{1/2}N_{k,\epsilon}u||^2_{L_2(M_{k,S},2r_k dM_k)} \right| \leq c\epsilon||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{3.91}$$

Proposition 3.3.36. *The linear operator $N_{k,\epsilon}$ is bounded on $H^1(M_{k,S,\epsilon})$,*

$$\int_{M_{k,S}} |\nabla_{M_k} \epsilon^{1/2} N_{k,\epsilon} u|^2 \, 2r_k dM_k \leq \int_{(-1,1)} |\varphi|^2 \int_{M_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, dM_\epsilon. \tag{3.92}$$

Furthermore, $\epsilon^{1/2} N_{k,\epsilon} u$ satisfies the following energy bound for $u \in H^2(M_{k,S,\epsilon})$:

$$\int_{M_{k,S}} |\nabla_{M_k} \epsilon^{1/2} N_{k,\epsilon} u|^2 \, 2r_k dM_k - \int_{M_{k,S,\epsilon}} |\nabla_{M_k} u|^2 \, dM_\epsilon \leq c\epsilon ||u||^2_{H^2(M_{k,S,\epsilon})}. \tag{3.93}$$

Proof: This an application of Proposition 3.3.11 on Proposition 3.3.34.

Definition 3.3.37. *We denote $\tilde{P}_{m,\epsilon}$ to be the following bounded linear operator on $L_2(\tilde{E}_{m,\epsilon})$:*

$$\tilde{P}_{m,\epsilon} u(y,z) = \frac{1}{|D(0, c_r \epsilon^\beta)|} \int_{D(0,c_r\epsilon^\beta)} \varphi\left(\frac{\zeta}{c_r \epsilon^\beta}\right) u(y,\zeta) \, d\varpi_{m,\epsilon}(y) \ , \tag{3.94}$$

where $\varphi \in C_0^\infty(D(0,1))$ such that $\int_{D(0,1)} \varphi = 1$.

Let $\tilde{P}_{m,\epsilon}$ also denote a bounded linear operator from $L_2(\tilde{E}_{m,\epsilon})$ to $L_2(E_m, \pi r_m^2 dE_m)$ by means of restricting $\tilde{P}_{m,\epsilon}$ to E_m ($\tilde{P}_{m,\epsilon} u(y, z = 0)$).

Proposition 3.3.38. *The family of averaging operators $\{\tilde{P}_{m,\epsilon}\}$ on $L_2(\tilde{E}_{m,\epsilon})$ has a uniform bound c.*

As with the operator $\tilde{N}_{k,\epsilon}$, boundedness of $\tilde{P}_{m,\epsilon}$ is clear from the Cauchy-Schwartz Inequality.

Definition 3.3.39. *The averaging operator $P_{m,\epsilon}$ on $E_{m,\epsilon}$ is given by composition with the corresponding diffeomorphism:*

$$P_{m,\epsilon} := \Phi^{-1}_{E_{m,\epsilon}} \tilde{P}_{m,\epsilon} \Phi_{E_{m,\epsilon}}. \tag{3.95}$$

We also let $P_{m,\epsilon}$ to denote a bounded linear operator from $L_2(E_{m,\epsilon})$ to $L_2(E_m)$ by restriction onto E_m ($P_{m,\epsilon} u|_{E_m} = \tilde{P}_{m,\epsilon} \Phi_{E_{m,\epsilon}} u(y, z = 0)$).

Proposition 3.3.40. *The linear operator* $\tilde{P}_{m,\epsilon}$ *is bounded on* $H^1(\tilde{E}_{m,\epsilon})$*,*

$$\int_{\tilde{E}_{m,\epsilon}} |\nabla \tilde{P}_{m,\epsilon} u|^2 \, d\tilde{E}_{m,\epsilon} \leq \int_{D(0,1)} |\varphi|^2 \int_{\tilde{E}_{m,\epsilon}} |\nabla u|^2 \, d\tilde{E}_{m,\epsilon}. \tag{3.96}$$

Proof: Using the reverse Fatou Lemma (Lemma A.1.2):

$$\begin{aligned}
&\int_{\tilde{E}_{m,\epsilon}} \Big|\nabla \frac{1}{|D(0,c_r\epsilon^\beta)|} \int_{D(0,c_r\epsilon^\beta)} \varphi\left(\frac{\zeta}{c_r\epsilon^\beta}\right) u \, d\varpi_{m,\epsilon}(y)\Big|^2 \, d\tilde{E}_{m,\epsilon} \\
&\quad \leq \int_{\tilde{E}_{m,\epsilon}} \Big| \limsup_{\delta \to 0} \frac{1}{|D(0,c_r\epsilon^\beta)|} \\
&\quad \int_{D(0,c_r\epsilon^\beta)} \frac{u(y+\delta,\zeta) - u(y,\zeta)}{\delta} \varphi\left(\frac{\zeta}{c_r\epsilon^\beta}\right) \, d\varpi_{m,\epsilon}(y)\Big|^2 \, d\tilde{E}_{m,\epsilon} \\
&\quad \leq \int_{\tilde{E}_{m,\epsilon}} \Big|\frac{1}{|D(0,c_r\epsilon^\beta)|} \int_{D(0,c_r\epsilon^\beta)} \varphi\left(\frac{\zeta}{c_r\epsilon^\beta}\right) D_y u \, d\varpi_{m,\epsilon}(y)\Big|^2 \, d\tilde{E}_{m,\epsilon}.
\end{aligned} \tag{3.97}$$

We use the embedding of L_1 in L_2 on a compact interval and Cauchy-Schwartz:

$$\begin{aligned}
&\int_{\tilde{E}_{m,\epsilon}} \Big|\frac{1}{|D(0,c_r\epsilon^\beta)|} \int_{D(0,c_r\epsilon^\beta)} \varphi\left(\frac{\zeta}{c_r\epsilon^\beta}\right) D_y u \, d\varpi_{m,\epsilon}(y)\Big|^2 \, d\tilde{E}_{m,\epsilon} \\
&\quad \leq \int_{\tilde{E}_{m,\epsilon}} \Big(\frac{1}{|D(0,c_r\epsilon^\beta)|} \int_{D(0,c_r\epsilon^\beta)} |\varphi\left(\frac{\zeta}{c_r\epsilon^\beta}\right) D_y u|^2 \, d\varpi_{m,\epsilon}(y)\Big) \, d\tilde{E}_{m,\epsilon} \\
&\quad \leq \frac{||\varphi(\zeta/c_r\epsilon^\beta)||^2_{L_2(D(0,c_r\epsilon^\beta))}}{|D(0,c_r\epsilon^\beta)|} \int_{\tilde{E}_{m,\epsilon}} |D_y u|^2 \, d\tilde{E}_{m,\epsilon}. \qquad \square
\end{aligned} \tag{3.98}$$

Proposition 3.3.41. *For* $u \in H^1(\tilde{E}_{m,\epsilon})$*, the averaging operator* $\tilde{P}_{m,\epsilon}$ *satisfies a Poincaré-type inequality:*

$$\int_{\tilde{E}_{m,\epsilon}} |u - \tilde{P}_{m,\epsilon} u|^2 \, d\tilde{E}_{m,\epsilon} \leq c\epsilon^{2\beta} \int_{\tilde{E}_{m,\epsilon}} |\nabla u|^2 \, d\tilde{E}_{m,\epsilon}. \tag{3.99}$$

Proof: We apply our integral representation:

$$\begin{aligned}
|u - \tilde{P}_{m,\epsilon} u|^2 &= \Big| \int_{\varpi_{m,\epsilon}(y)} \frac{f_{y,\zeta}(z,r,\theta)}{r} D_\zeta u(y,\zeta) \, d\varpi_{m,\epsilon}(y)\Big|^2 \\
&\leq c\Big| \int_{\varpi_{m,\epsilon}(y)} \frac{D_\zeta u(y,\zeta)}{r} \, d\varpi_{m,\epsilon}(y)\Big|^2 \leq c' ||R_y D_\zeta u(y,\zeta)||^2_{L_2(\varpi_{m,\epsilon}(y))}
\end{aligned} \tag{3.100}$$

where R_y is the operator from Lemma 2.3.48 on $L_2(\varpi_{m,\epsilon}(y))$ (in this case it is the convolution with $1/r$). Using the upper bound on the norm of R_y ($c\epsilon^\beta$), we get the desired result. □

Corollary 3.3.42. *For $u \in H^1(E_{m,\epsilon})$, the averaging operator $P_{m,\epsilon}$ satisfies a Poincaré-type inequality:*

$$||u - P_{m,\epsilon}\, u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^{2\beta}||\nabla u||^2_{L_2(E_{m,\epsilon})}. \tag{3.101}$$

Proof: This is another application of Proposition 3.3.13 to (3.100).

We cannot in general commute an averaging operator on the fibers with the derivative with respect to the transverse variables. For the binding, we need a tighter approximation of a function, and so we introduce a mollifier that lets us commute derivatives with the averages over the fibers. The following lemma appears in "Differential Functions on Bad Domains" [28] where it is used to develop a generalization of the Poincaré inequality for a function on an ϵ-radius cylinder.

Lemma 3.3.43. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $K \in C_0^\infty(\mathbb{R}^m)$. For a function v defined on the cylinder $D = \mathbb{R}^m \times \Omega \subset \mathbb{R}^{m+n}$,*

$$\mathcal{T}v(x) = \int_{\mathbb{R}^m} K(t)v(y + |z|t, z)\, dt \qquad x = (y, z) \in D. \tag{3.102}$$

Let $l \in \mathbb{Z}^+$. Suppose that

$$\int_{\mathbb{R}^m} K(t)t^\nu dt = 0 \tag{3.103}$$

for all multi-indices $\nu \in \mathbb{Z}^m_+$, $|\nu| \leq l - 1$. Then if $\mathcal{T} : L^l_p(D) \to L^l_p(D)$ for $1 \leq p \leq \infty$ and the following estimate holds:

$$||\nabla_l \mathcal{T} v||_{L_p(D)} \leq c||\nabla_l v||_{L_p(D)}. \tag{3.104}$$

Furthermore, if $\int K(t)dt = 1$, then the following estimate holds

$$||\nabla_k(\mathcal{T}v - v)||_{L_p(D)} \leq cr^{l-k}||\nabla_l v||_{L_p(D)} \tag{3.105}$$

where $0 \leq k \leq l$, $r = \sup\{|z| : z \in \Omega\}$ and v an arbitrary function in $L^l_p(D)$.

Proof: The proof is included in Appendix B.1.7.

Remark 3.3.44. *Clearly, one can rewrite $t' = y + |z|t$ to get explicit convolution of v with K. Thus it follows the mollifier commutes with the longitudinal derivative: $D_y \mathcal{T} v = \mathcal{T} D_y v$. We note that the lemma holds if the longitudinal dimensions $\mathbb{R}^m$ are instead compact without a boundary (i.e. holds for $\mathbb{T}^m$, see proof of Lemma 3.3.43).*

While $\mathcal{T}$ as it is written is defined for a cylindrical domain, we can define an action of an operator of the form in Lemma 3.3.43 on $P_{m,\epsilon}u$ since its values are uniquely determined on $E_m \times D(0, c_r\epsilon^{\beta})$. Let us expand:

Definition 3.3.45. *Let $\mathcal{T}_{m,\epsilon}$ be a bounded linear operator on $H^1(E_m \times D(0, r\epsilon^{\beta}))$ in the sense of Lemma 3.3.43. I.e. for $u \in H^1(E_m \times D(0, r\epsilon^{\beta}))$*

$$\mathcal{T}_{m,\epsilon}u = \int_{E_m} K(t)u(y + |z|t, z)dt \qquad y \in E_m \; z \in D(0, r\epsilon^{\beta}) \tag{3.106}$$

where $K(t) \in C_0^{\infty}(E_m)$ and $K(t)$ satisfies

$$\int_{E_m} K(t)\, dE_m = \int_{E_m} K(t)t\, dE_m = 0. \tag{3.107}$$

The function $\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u$ is constant valued on each cross-section $(y, D(0, r\epsilon^{\beta}))$, and so this operator product can be extended to $H^1(\tilde{E}_{m,\epsilon})$ by means of extending the value on $(y, D(0, r\epsilon^{\beta}))$ to $\varpi_{m,\epsilon}(y)$. Hence we define an bounded linear operator $\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}$ on $H^1(\tilde{E}_{m,\epsilon})$.

Corollary 3.3.46. *For $u \in H^2(\tilde{E}_{m,\epsilon})$, one has the following:*

$$||\Delta(\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u - \tilde{P}_{m,\epsilon}u)||_{L_2(\tilde{E}_{m,\epsilon})} \leq c||\Delta u||_{L_2(\tilde{E}_{m,\epsilon})}, \tag{3.108}$$

$$||\nabla\big(\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u - \tilde{P}_{m,\epsilon}u\big)||_{L_2(\tilde{E}_{m,\epsilon})} \leq c\epsilon^{\beta}||\Delta u||_{L_2(\tilde{E}_{m,\epsilon})}, \tag{3.109}$$

and

$$||\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u - \tilde{P}_{m,\epsilon}u||_{L_2(\tilde{E}_{m,\epsilon})} \leq c\epsilon^{2\beta}||\Delta u||_{L_2(\tilde{E}_{m,\epsilon})}. \tag{3.110}$$

Furthermore, $\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}$ commutes with D_y on $\tilde{E}_{m,\epsilon}$.

Introducing $\mathcal{T}_{m,\epsilon}$ leads to the following close approximation of function in H^1:

Proposition 3.3.47. *Let $u \in H^2(\tilde{E}_{m,\epsilon})$. The function $\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u$ is close with respect to the H^1-norm to u:*

$$||u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u||^2_{H^1(\tilde{E}_{m,\epsilon})} \leq c\epsilon^{2\beta}||u||^2_{H^2(\tilde{E}_{m,\epsilon})}. \tag{3.111}$$

Proof: Bounding the L_2-norm, we have:

$$u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u = \big(u - \tilde{P}_{m,\epsilon}u\big) + \big(\tilde{P}_{m,\epsilon}u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u\big) \tag{3.112}$$

which is bounded by Proposition 3.3.41 and (3.110). To bound the derivative we write

$$\begin{aligned} \nabla(u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u) &= \nabla u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}\nabla u \\ &= \big(\nabla u - \tilde{P}_{m,\epsilon}\nabla u\big) + (\tilde{P}_{m,\epsilon}\nabla u - \mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}\nabla u). \end{aligned} \tag{3.113}$$

The first term can be reformulated in terms of the Poincaré inequality on ∇u (see Proposition 3.3.41 and proof of Proposition 3.3.34), and the second term is handled by (3.109). □

Proposition 3.3.48. *Let $\tilde{E}_{m,\epsilon}$ be a fattened binding of type II or III ($\beta \leq 1/2$). For $u \in H^1(\tilde{E}_{m,\epsilon})$, one has:*

$$\Big|\, ||\epsilon^{\beta}\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u||^2_{L_2(E_m,\pi r_m^2 dE_m)} - ||u||^2_{L_2(\tilde{E}_{m,\epsilon})} \Big| \leq c\epsilon^{\beta}||u||^2_{H^1(\tilde{E}_{m,\epsilon})}. \tag{3.114}$$

Proof: Rewriting the first term as an integral over $\tilde{E}_{m,\epsilon}$, we get:

$$\begin{aligned}\Big|\int_{E_m}\epsilon^{2\beta}|\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2\pi r_m^2\,dE_m-\int_{\tilde{E}_{m,\epsilon}}|u|^2\,d\tilde{E}_{m,\epsilon}\Big|\\ =\Big|\int_{E_m}\frac{\pi r_m(y)^2\epsilon^{2\beta}}{|\varpi_{m,\epsilon}(y)|}\Big(\int_{\varpi_{m,\epsilon}(y)}d\varpi_{m,\epsilon}(y)\Big)|\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2\,dy\\ -\int_{E_m}\int_{\varpi_{m,\epsilon}(y)}|u|^2\,d\varpi_{m,\epsilon}(y)\,dy\Big|.\end{aligned} \tag{3.115}$$

Since $\beta < 1$, we use the estimate in Lemma 3.3.2. This gives us an upper bound on (3.115) of:

$$\begin{aligned}&\max_y\frac{\pi r_m(y)^2\epsilon^{2\beta}-|\varpi_{m,\epsilon}(y)|}{|\varpi_{m,\epsilon}(y)|}||\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u||^2_{L_2(E_m,\pi r_m^2 dE_m)}\\ &+\int_{\tilde{E}_{m,\epsilon}}|\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2-|u|^2\,d\tilde{E}_{m,\epsilon}\le c\epsilon^{\min(\beta,2-2\beta)}||u||^2_{H^1(\tilde{E}_{m,\epsilon})}.\end{aligned} \tag{3.116}$$

Since $\beta \le 1/2$, we get $O(\epsilon^{\beta})$ bounds on (3.114). □

Proposition 3.3.49. *Let $\tilde{E}_{m,\epsilon}$ be a fattened binding of type I or II ($\beta \le 1/2$). If $u \in H^2(\tilde{E}_{m,\epsilon})$, then $\epsilon^{\beta}\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u$ satisfies the following energy bound:*

$$\int_{E_m}|D_y\epsilon^{\beta}\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2\,\pi r_m^2 dE_m-\int_{\tilde{E}_{m,\epsilon}}|\nabla u|^2\,d\tilde{E}_{m,\epsilon}\le c\epsilon^{\beta}||u||^2_{H^2(\tilde{E}_{m,\epsilon})}. \tag{3.117}$$

Proof: Starting with

$$\int_{E_m}|D_y\epsilon^{\beta}\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2\,\pi r_m^2 dE_m-\int_{\tilde{E}_{m,\epsilon}}|\nabla u|^2\,d\tilde{E}_{m,\epsilon}, \tag{3.118}$$

we use Lemma 3.3.2 on the first term to get:

$$\begin{aligned}&\int_{\tilde{E}_{m,\epsilon}}(1+O(\epsilon^{2-2\beta}))|D_y\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}u|^2\,d\tilde{E}_{m,\epsilon}-\int_{\tilde{E}_{m,\epsilon}}|\nabla u|^2\,d\tilde{E}_{m,\epsilon}\\ &\le(1+O(\epsilon^{2-2\beta}))\int_{\tilde{E}_{m,\epsilon}}|\mathcal{T}_{m,\epsilon}\tilde{P}_{m,\epsilon}D_z u|^2-|D_z u|^2\,d\tilde{E}_{m,\epsilon}\\ &\le(1+O(\epsilon^{2-2\beta}))||\mathcal{T}_{m,\epsilon}D_z u-D_z u||_{L_2(\tilde{E}_{m,\epsilon})}||\mathcal{T}_{m,\epsilon}D_z u+D_z u||_{L_2(\tilde{E}_{m,\epsilon})}.\end{aligned} \tag{3.119}$$

Lastly using Corollary 3.3.46, (3.119) is bounded by

$$(1 + O(\epsilon^{2-2\beta}))\epsilon^{\beta}||u||^2_{H^2(\tilde{E}_{m,\epsilon})}. \qquad \square \tag{3.120}$$

Definition 3.3.50. *We define $P_{T,m,\epsilon}$ to be an operator on $H^1(E_{m,\epsilon})$:*

$$P_{T,m,\epsilon} = \Phi^{-1}_{E_{m,\epsilon}} \mathcal{T}_{m,\epsilon} \tilde{P}_{m,\epsilon} \Phi_{E_{m,\epsilon}}. \tag{3.121}$$

We also let $P_{T,m,\epsilon}$ be an operator from $H^1(E_{m,\epsilon})$ to $H^1(E_m)$ by restricting $P_{T,m,\epsilon}u(x,y)$ to E_m ($P_{T,m,\epsilon}u(x, y = 0)$).

Proposition 3.3.51. *Let $E_{m,\epsilon}$ be a fattened binding of type II or III ($\beta \leq 1/2$). For $u \in H^1(E_{m,\epsilon})$, one has:*

$$\big|\, ||\epsilon^{\beta} P_{T,m,\epsilon}u||^2_{L_2(E_m, \pi r_m^2 dE_m)} - ||u||^2_{L_2(E_{m,\epsilon})} \,\big| \leq c\epsilon^{\beta}||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.122}$$

Proof: This is an application of Proposition 3.3.11 on Proposition 3.3.48.

Proposition 3.3.52. *Let $E_{m,\epsilon}$ be a fattened binding of type I or II ($\beta \leq 1/2$). If $u \in H^2(E_{m,\epsilon})$, then $\epsilon^{\beta} P_{T,m,\epsilon}u$ satisfies the following energy bound:*

$$\int_{E_m} |D_y \epsilon^{\beta} P_{T,m,\epsilon}u|^2 \, \pi r_m^2 dE_m - \int_{E_{m,\epsilon}} |\nabla u|^2 \, dM_{\epsilon} \leq c\epsilon^{\beta}||u||^2_{H^2(E_{m,\epsilon})}. \tag{3.123}$$

Proof: This is an application of Proposition 3.3.11 on Proposition 3.3.49.

3.3.7 Bounding the Norm on the Type I Fattened Binding and Extending the Average on the Type II Fattened Binding

Having established the required estimations for local averaging operators on each stratum, we now need to combine these different local averaging operators. In this subsection, we establish several lemmas regarding the trace on the interface $\Gamma_{k,m,\epsilon}$ between $M_{k,S,\epsilon}$ and $E_{m,\epsilon}$. We also extend the averaged component $P_{m,\epsilon}u$ from the fattened binding to the fattened pages. This leads to an important observation concerning type II domains.

Definition 3.3.53. *The* ***trace*** *or* ***restriction operator*** *from $M_{k,S,\epsilon}$ to $\Gamma_{k,m,\epsilon}$ is denoted $T_{k,m,\epsilon}$.*

The trace operator from $E_{m,\epsilon}$ to $\Gamma_{k,m,\epsilon}$ is denoted $T_{m,k,\epsilon}$.

Lemma 3.3.54. *Let $\{E_{m,\epsilon}\}$ be a family of fattened bindings ($\epsilon \in (0, \epsilon_0]$). Let $u \in H^1(E_{m,\epsilon})$, then one has:*

$$\epsilon^{-\beta}||T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})} + [T_{m,k,\epsilon}(u - P_{m,\epsilon}u)]^2_{\Gamma_{k,m,\epsilon}} \leq c_m||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.124}$$

The same inequality holds for $P_{T,m,\epsilon}$ in place of $P_{m,\epsilon}$.

Proof 3.3.54: This proof only requires small modification from the uniformly fattened case. We refer to the proof of Lemma 2.3.62 and note the following differences: the homothetic scaling map θ is changed to $\theta : x \mapsto x/\epsilon^\beta$ and the partition of unity used in the previous proof has already been adjusted for fattened bindings of $\beta < 1$ in Corollary 3.3.5. □

With a norm estimate on the trace space of $E_{m,\epsilon}$, we may now construct an extension operator from $\Gamma_{k,m,\epsilon}$ to $M_{k,S,\epsilon}$.

Proposition 3.3.55. *For $u \in H^1(E_{m,\epsilon})$, the complement of the cross-sectional average $u - P_{m,\epsilon}u$ has an extension into M_ϵ denoted $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ such that*

$$||\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)||^2_{H^1(M_\epsilon)} \leq c_m||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.125}$$

Furthermore, $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ is supported within an $O(\epsilon^\beta)$ neighborhood of E_m. The same inequality holds for $P_{T,m,\epsilon}$ in place of $P_{m,\epsilon}$.

Proof: This proof does not significantly differ from the proof of Proposition 2.3.63 (see Appendix B.1.5).

Corollary 3.3.56. *For $u \in H^1(E_{m,\epsilon})$, one has:*

$$||P_{m,\epsilon}u - T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^{3\beta-1}||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.126}$$

Proof: While $T_{k,m,\epsilon}N_{k,\epsilon}u$ is a function on the interface $\Gamma_{k,m,\epsilon}$, we can express it as a function on E_m by noting it is constant valued on $\partial\omega_{m,\epsilon}(x)$. With an abuse of notation, we can set $N_{k,\epsilon}u(x \in E_m) := N_{k,\epsilon}u|_{\partial\omega_{m,\epsilon}(x)}$. Beginning with an application of Proposition 3.3.13, we have

$$\begin{aligned}
&||\Phi^{-1}_{E_{m,\epsilon}}\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \\
&\leq (1+O(\epsilon^\beta))||\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})} \\
&= (1+O(\epsilon^\beta))\int_{E_m}\int_{\varpi_{m,\epsilon}(y)}|\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u|^2\,d\varpi_{m,\epsilon}(y)dE_m.
\end{aligned} \tag{3.127}$$

Noting $\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u$ can be extended to the boundary, (3.127) is bounded by

$$\frac{\max_{y\in E_m}|\varpi_{m,\epsilon}(y)|}{\min_{y\in E_m}|I_{k,\epsilon}(y,a_{m,\epsilon}(y))|}\int_{\tilde{\Gamma}_{k,m,\epsilon}}|\tilde{N}_{k,\epsilon}[\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u]|^2\,d\tilde{\Gamma}_{k,m,\epsilon}. \tag{3.128}$$

Because the norm of $\tilde{N}_{k,\epsilon}$ is bounded independently of ϵ, the above is bounded by

$$c\epsilon^{2\beta-1}\int_{\tilde{\Gamma}_{k,m,\epsilon}}|\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - \Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u|^2\,d\tilde{\Gamma}_{k,m,\epsilon}. \tag{3.129}$$

Observe this is the same L_2 term from Lemma 3.3.54 up $\Phi^{-1}_{E_{m,\epsilon}}$ and a scaling. Thus the desired bound is achieved for (3.126). □

Lemma 3.3.57. *For $u \in H^1(M_{k,S,\epsilon})$, one has:*

$$||T_{k,m,\epsilon}u||^2_{L_2(\Gamma_{k,m,\epsilon})} \leq c||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{3.130}$$

Proof: This proof appears in Appendix B.1.6.

Corollary 3.3.58. *For $u \in H^1(M_\epsilon)$, one has:*

$$||T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^{2\beta-1}||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{3.131}$$

Proof: The proof is analogous to Corollary 3.3.56. While $T_{k,m,\epsilon}N_{k,\epsilon}u$ is a function on the interface $\Gamma_{k,m,\epsilon}$, we can express it as a function on E_m by noting it is constant valued on $\partial\omega_{m,\epsilon}(x)$. With an abuse of notation, we can set $N_{k,\epsilon}u(x \in E_m) := N_{k,\epsilon}u|_{\partial\omega_{m,\epsilon}(x)}$.

$$\begin{aligned}
&||T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq (1+O(\epsilon^\beta))||\Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})} \\
&= (1+O(\epsilon^\beta))\int_{E_m}\int_{\varpi_{m,\epsilon}(y)} |\Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}N_{k,\epsilon}u|^2 \, d\varpi_{m,\epsilon}(y)dE_m \\
&= O(\epsilon^{2\beta-1})\int_{\tilde{\Gamma}_{k,m,\epsilon}} |\tilde{N}_{k,\epsilon}\Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u|^2 \, d\tilde{\Gamma}_{k,m,\epsilon} \\
&\leq O(\epsilon^{2\beta-1})\int_{\tilde{\Gamma}_{k,m,\epsilon}} |\Phi_{E_{m,\epsilon}}T_{k,m,\epsilon}u|^2 \, d\tilde{\Gamma}_{k,m,\epsilon}.
\end{aligned} \tag{3.132}$$

Observe this is the same L_2 term from Corollary 3.3.58 up $\Phi^{-1}_{E_{m,\epsilon}}$. Consequentially the desired bound is achieved. □

Theorem 3.3.59. *Let M_ϵ be a type I domain ($1/2 < \beta \leq 1$). For $u \in H^1(M_\epsilon)$, the L_2-norm of u on $E_{m,\epsilon}$ is small:*

$$||u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^{2\beta-1}||u||^2_{H^1(M_\epsilon)}. \tag{3.133}$$

Proof: We use the triangle inequality:

$$\begin{aligned}
||u||_{L_2(E_{m,\epsilon})} &\leq ||u - P_{m,\epsilon}u||_{L_2(E_{m,\epsilon})} \\
&\quad + ||P_{m,\epsilon}u - T_{k,m,\epsilon}N_{k,\epsilon}u||_{L_2(E_{m,\epsilon})} + ||T_{k,m,\epsilon}N_{k,\epsilon}u||_{L_2(E_{m,\epsilon})}.
\end{aligned} \tag{3.134}$$

With Corollaries 3.3.42, 3.3.56, and 3.3.58, the theorem is proven. □

Corollary 3.3.60. *Let M_ϵ be a type I domain ($1/2 < \beta \leq 1$). Assume $u \in \mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$ for $\Lambda \leq c\epsilon^{-(2\beta-1)+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. The H^1-norm of u on $E_{m,\epsilon}$ is $o(1)$ with respect to the H^1-norm of u on M_ϵ.*

Proof: By embedding $\mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$ into $L_2(M_\epsilon)$, we can write:

$$||\nabla u||^2_{L_2(E_{m,\epsilon})} \leq \Lambda||u||^2_{L_2(E_{m,\epsilon})} \leq c\Lambda\epsilon^{2\beta-1}||u||^2_{H^1(M_\epsilon)} \leq c\epsilon^\delta||u||^2_{H^1(M_\epsilon)}. \qquad \square \tag{3.135}$$

Proposition 3.3.61. *Let $E_{m,\epsilon}$ be type II ($\beta < 1/2$) and $u \in H^1(E_{m,\epsilon})$. There is an extension $\mathcal{E}_{m,\epsilon}P_{T,m,\epsilon}u$ of $P_{T,m,\epsilon}u$ to $H^1(M_\epsilon)$ such that*

$$||\mathcal{E}_{m,\epsilon}P_{T,m,\epsilon}u||^2_{H^1(M_\epsilon)} \leq (1 + c\epsilon^{1-2\beta})||P_{T,m,\epsilon}u||^2_{H^1(E_{m,\epsilon})}. \tag{3.136}$$

Proof: The proof is placed in Appendix B.1.8.

Remark 3.3.62. *The bound $1 + c\epsilon^{1-2\beta}$ in the previous proposition can be reformulated in terms of capacity. The **capacity** of a set $F \subset \Omega$ be define as*

$$\mathsf{cap}(F, H^1(\Omega)) := \inf\{||u||^2_{H^1(\Omega)} : u \in H^1(\Omega) \qquad u|_F \geq 1\}. \tag{3.137}$$

Then it follows for $\beta < 1/2$ that

$$\left|\frac{\mathsf{cap}(E_{m,\epsilon}, M_\epsilon)}{|E_{m,\epsilon}|} - 1\right| = O(\epsilon^{1-2\beta}). \tag{3.138}$$

Corollary 3.3.63. *Let $E_{m,\epsilon}$ be type II or III ($\beta \leq 1/2$). There is a family of operators $\mathcal{E}_{m,\epsilon} : H^1(E_{m,\epsilon}) \to H^1(M_\epsilon)$ whose norms have a uniform bound independent of ϵ. For $u \in H^1(E_{m,\epsilon})$, we have:*

$$||\mathcal{E}_{m,\epsilon}u||^2_{H^1(M_\epsilon)} \leq c||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.139}$$

Proof: While the ranges of $P_{T,m,\epsilon}$ and $(1 - P_{T,m,\epsilon})$ are not orthogonal ($P_{T,m,\epsilon}$ is not an orthogonal projector), a function u can still be uniquely written as $u = P_{T,m,\epsilon}u + (u - P_{T,m,\epsilon}u)$. The averaged component is extended by Proposition 3.3.61 and the zero-average component is extended by Proposition 3.3.55 and each of these functions is norm bounded by some positive constant c', so their sum is norm bounded. □

Corollary 3.3.64. *Let $E_{m,\epsilon}$ be type II or III ($\beta \leq 1/2$). For $u \in H^2(E_{m,\epsilon})$ there is an extension $\mathcal{E}_{m,\epsilon}(u - P_{T,m,\epsilon}u)$ of $u - P_{T,m,\epsilon}u$ into M_ϵ that is negligible:*

$$||\mathcal{E}_{m,\epsilon}(u - P_{T,m,\epsilon}u)||^2_{H^1(M_\epsilon)} \leq c\epsilon^{2\beta}||u||^2_{H^2(E_{m,\epsilon})}. \tag{3.140}$$

Proof: Since $\mathcal{E}_{m,\epsilon}$ is norm bounded, we apply Proposition 3.3.47.

Corollary 3.3.65. *Let $E_{m,\epsilon}$ be type II ($\beta < 1/2$) and let $\Lambda \leq c\epsilon^{-2\beta+\delta}$ where $\delta > 0$. For $u \in \mathcal{P}_\Lambda^\epsilon \mathcal{G}_2$ there is an extension operator $\mathcal{E}_{m,\epsilon} : H^1(E_{m,\epsilon}) \mapsto H^1(M_\epsilon)$ such that*

$$||\mathcal{E}_{m,\epsilon} u||^2_{H^1(M_\epsilon \setminus E_{m,\epsilon})} \leq c\epsilon^{\min(\delta,1-2\beta)} ||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.141}$$

Proof: We can embed H^1 into H^2 by Proposition 2.1.16. The result then follows from Corollary 3.3.64 and Proposition 3.3.61. □

3.3.8 Extension Operator K_ϵ

Now we can define the extension operators in the sense of Definition 3.2.2. The first extension operator is analogous to the operator constructed in the uniformly fattened case (Proposition 2.3.38).

Proposition 3.3.66. *Let M be an open book structure and let M_ϵ be a corresponding type I model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $1/2 < \beta \leq 1$. Let $\Lambda \leq c\epsilon^{-\beta+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A)$. For some $\epsilon_0 > 0$, the family of linear operators $\{K_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.2 is ($u \in \mathcal{P}_\Lambda \mathcal{G}_1$):*

$$K_\epsilon u := \begin{cases} \epsilon^{-1/2} \mathcal{E}_{k,z,\epsilon} u & M_{k,S} \\ \epsilon^{-1/2} \mathcal{E}_{m,S,z,\epsilon} u & E_m \bigcup \left(\bigcup_k S_{k,m,\epsilon}\right). \end{cases} \tag{3.142}$$

Proof: Beginning with $E_m \bigcup (\bigcup_k S_{k,m,\epsilon})$, we apply Proposition 3.3.22 to get

$$||\epsilon^{-1/2} \mathcal{E}_{m,S,z,\epsilon} u||^2_{H^1(E_{m,\epsilon})} \leq c||u||^2_{H^1(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}), 2r_k dM_k)}. \tag{3.143}$$

Applying the spectral embedding Proposition 2.1.16, the previously expression is bounded by

$$c(1+\Lambda)||u||^2_{L_2(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}), 2r_k dM_k)} \tag{3.144}$$

which in turn is bounded by the energy on M (Proposition 3.3.15). This yields an upper bound of

$$c(1+\Lambda)\epsilon^{\beta}||u||^2_{\mathcal{G}_1^1} = o(1)||u||^2_{\mathcal{G}_1^1}. \tag{3.145}$$

Therefore (3.143) is negligible both in L_2 and H^1. For the $M_{k,S}$ pieces, we show that they are not only close to their extension $\epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}u$ in L_2 but also in H^1. Starting with the following norm difference

$$\left|\sum_k ||\epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}u||_{H^1(M_{k,S,\epsilon})} - ||u||_{\mathcal{G}_1^1}\right|, \tag{3.146}$$

we break $||u||_{\mathcal{G}_1^1}$ into page terms and sleeve terms and use the triangle inequality. We get an upper bound of (3.146) of

$$\sum_k \left|||\epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}u||_{H^1(M_{k,S,\epsilon})} - ||u||_{H^1(M_{k,S})}\right| + ||u||_{H^1(E_m \bigcup (\bigcup_k S_{k,m,\epsilon}), 2r_k dM_k)}. \tag{3.147}$$

The first term of (3.147) is $o(1)$-bounded by Corollary 3.3.20. After a norm bound on the sleeve (Propositions 3.3.15 and 2.1.16), we conclude (2.93) is bounded by $(1+\Lambda)^{1/2}O(\epsilon^{1/2})||u||_{\mathcal{G}_1^1}$. We conclude K_ϵ is a near isometry in both L_2 and H^1 for u in $\mathcal{P}_{c\epsilon^{-\beta+\delta}}\mathcal{G}_1$. □

The extension operator for type II scenario works as follows: the pages are shortened then the function is extended along the normal fibers. The function along the binding is also extended along the cross-sections. To ensure the function is in H^1, we extend function on the binding into the page.

Proposition 3.3.67. *Let M be an open book structure and let M_ϵ be a type II model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $\beta < 1/2$. Let $\Lambda \leq c\epsilon^{-\beta+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A)$. For some $\epsilon_0 > 0$, the family of linear operators $\{K_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.2 for the open book structure M is ($u \in \mathcal{P}_\Lambda \mathcal{G}_2$):*

$$K_\epsilon w := \epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}\Psi_{M_k}u + \mathcal{E}_{m,\epsilon}\epsilon^{-\beta}\mathcal{E}_{m,z,\epsilon}v. \tag{3.148}$$

Proof: On $E_{m,\epsilon}$, we have using Corollary 3.3.27,

$$\big|\, ||\epsilon^{-\beta}\mathcal{E}_{m,z,\epsilon}v||_{H^1(E_m,\pi r_m^2 dE_m)} - ||v||^2_{H^1(E_m,\pi r_m^2 dE_m)} \,\big| \leq c\epsilon^{2-2\beta}||v||^2_{H^1(E_m,\pi r_m^2 dE_m)}. \tag{3.149}$$

On $M_{k,S,\epsilon}$, we use Proposition 3.3.61 to dispense with the $\mathcal{E}_{m,\epsilon}\epsilon^{-\beta}\mathcal{E}_{m,z,\epsilon}v$ term. Next we see $\epsilon^{-1/2}$ $\mathcal{E}_{k,z,\epsilon}\Psi_{M_k}u$ is close in L_2 and in energy by Proposition 3.3.14 and reasoning following (3.146). □

For type III domains, we do not need an extension operator since the continuity condition between the pages and the binding: $\lim_{x\to y\in E_m} u(x) = v(y)$.

Proposition 3.3.68. *Let M be an open book structure and M_ϵ be a corresponding type III model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $\beta < 1/2$. Let $\Lambda \leq c\epsilon^{-\beta+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A)$. For some $\epsilon_0 > 0$, the family of linear operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.2 for the open book structure M is ($u \in \mathcal{P}_\Lambda\mathcal{G}_2$):*

$$K_\epsilon w := \epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}\Psi_{M_k}u + \epsilon^{-1/2}\mathcal{E}_{m,z,\epsilon}v. \tag{3.150}$$

Proof: Only requires a small modification on proof of Proposition 3.3.67, so it is omitted.

3.3.9 Averaging Operator J_ϵ

We define the averaging operators in the sense of Definition 3.2.1 thereby completing the main spectral convergence theorems.

The type I case is analogous to the uniformly fattened case.

Proposition 3.3.69. *Let M be an open book structure and M_ϵ be a corresponding type I model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $1/2 < \beta \leq 1$. Let $\Lambda \leq c\epsilon^{-2\beta+1+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. For some $\epsilon_0 > 0$, the family of averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the* *conditions in Definition 3.2.1 for the open book structure M is ($u \in \mathcal{P}^{\epsilon}_{\Lambda} L_2(M_\epsilon)$):*

$$J_\epsilon u := \begin{cases} \epsilon^{1/2}\Psi^{-1}_{M_k} N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big] & M_{k,S,\epsilon} \mapsto M_k \\ \epsilon^{1/2} P_{m,\epsilon} u & E_{m,\epsilon} \mapsto E_m. \end{cases} \tag{3.151}$$

Proof: As seen in the uniformly fattened case (see Proposition 2.3.70), $J_\epsilon u$ satisfies the boundary conditions on $\mathcal{G}^1_1$. Because each $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ is supported in a small $O(\epsilon)$ neighborhood around E_m, these extensions have disjoint supports. Using Lemma 2.3.69, we break up the terms on $M_{k,S,\epsilon}$,

$$\begin{aligned} (1-d)\big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon} u\big|^2 \; &+ (1-d^{-1})\sum_m \big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big|^2 \\ &\le \Big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big]\Big|^2 \\ &\le (1+d)\big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon} u\big|^2 \\ &+ (1+d^{-1})\sum_m \big|\sqrt{2\epsilon}\Psi^{-1}_{M_k} N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big|^2. \end{aligned} \tag{3.152}$$

To demonstrate the L_2 near isometry property, we first assume that $||J_\epsilon u||^2_{L_2(M_k)} \ge ||u||^2_{L_2(M_{k,S,\epsilon})}$. The other case $||J_\epsilon u||^2_{L_2(M_k)} \le ||u||^2_{L_2(M_{k,S,\epsilon})}$ can be handled by appropriately modifying the subsequent inequality (3.153) (i.e. flipping signs and switching upper and lower bounds). This results in a largely redundant calculation, so it is omitted. We calculate the upper and lower bound on the norm difference:

$$\begin{aligned}\sum_k (1-d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{L_2(M_k)} + (1-d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{L_2(M_k)}\\ -\sum_k ||u||^2_{L_2(M_{k,S,\epsilon})} - ||u||^2_{L_2(E_{m,\epsilon})}\\ \leq ||J_\epsilon u||^2_{L_2(M)} - ||u||^2_{L_2(M_\epsilon)}\\ \leq \sum_k (1+d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{L_2(M_{k,S,\epsilon})}\\ +(1+d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{L_2(M_{k,S,\epsilon})}\\ -\sum_k ||u||^2_{L_2(M_{k,S,\epsilon})} - ||u||^2_{L_2(E_{m,\epsilon})}.\end{aligned} \tag{3.153}$$

Since we only require demonstrating that $||J_\epsilon u||_{H^1(M)}$ is bounded above (3.28), we begin with assuming $||J_\epsilon u||_{H^1(M)} \geq ||u||_{H^1(M_\epsilon)}$ and write:

$$\begin{aligned}||J_\epsilon u||^2_{H^1(M)} - ||u||^2_{H^1(M_\epsilon)}\\ \leq \sum_k (1+d)||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}u||^2_{H^1(M_{k,S,\epsilon})}\\ +(1+d^{-1})\sum_{k,m}||\sqrt{2\epsilon}\Psi_{M_k}^{-1}N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u-u)||^2_{H^1(M_{k,S,\epsilon})}\\ -\sum_k ||u||^2_{H^1(M_{k,S,\epsilon})} - ||u||^2_{H^1(E_{m,\epsilon})}.\end{aligned} \tag{3.154}$$

Having established these two inequalities (3.153) and (3.154), we collect terms in these inequalities and apply various propositions established in this chapter to demonstrate which terms are negligible (are $o(1)$ in an appropriate norm) and which terms are nearly an isometry (are $1+o(1)$ in an appropriate norm).

By Proposition 3.3.14, we have

$$\begin{aligned}
&\Big|\int_{M_k} |\sqrt{2\epsilon}\Psi_{M_k}^{-1} N_{k,\epsilon} u|^2 \, dM_k - \int_{M_{k,S,\epsilon}} |u|^2 \, dM_\epsilon\Big| \\
&\qquad \le \Big|(1+O(\epsilon))\int_{M_{k,S}} |\sqrt{2\epsilon} N_{k,\epsilon} u|^2 \, dM_k - \int_{M_{k,S,\epsilon}} |u|^2 \, dM_\epsilon\Big| \\
&\qquad \le c\epsilon ||u||^2_{H^1(M_\epsilon)}
\end{aligned} \tag{3.155}$$

where the last inequality results from Proposition 3.3.35. We note the energy bound only needs to be demonstrated from above, so we see

$$\int_{M_k} |\nabla_{M_k}\sqrt{2\epsilon}\Psi_{M_k}^{-1} N_{k,\epsilon} u|^2 \, dM_k - \int_{M_{k,S,\epsilon}} |\nabla u|^2 \, dM_\epsilon \le c\epsilon ||u||^2_{H^1(M_\epsilon)} \tag{3.156}$$

which follows from Propositions 3.3.14 and 3.3.36.

This leaves the extensions from the fattened bindings into the page ($\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$) and the norm of the binding unaccounted for in (3.153) and (3.154). We estimate the H^1-norm of the extensions. Using Propositions 3.3.14, 3.3.35, and 3.3.36, and the disjoint supports of $E_{m,\epsilon}(u - P_{m,\epsilon}u)$:

$$\begin{aligned}
&\sum_m ||\sqrt{2\epsilon}\Psi_{M_k}^{-1} N_{k,\epsilon}\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)||^2_{H^1(M_{k,S})} + \sum_m ||u||^2_{H^1(E_{m,\epsilon})} \\
&\le (1+O(\epsilon))\sum_m ||\mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)||^2_{H^1(M_{k,S,\epsilon})} + \sum_m ||u||^2_{H^1(E_{m,\epsilon})}.
\end{aligned} \tag{3.157}$$

By Proposition 3.3.55, this is bounded by

$$(1+O(\epsilon))c\sum_m ||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.158}$$

Because $u \in \mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$ and Corollary 3.3.60, we arrive to the following upper bound on the norm of (3.157):

$$c\epsilon^\delta ||u||^2_{H^1(M_\epsilon)}. \tag{3.159}$$

Hence by setting $d = \epsilon^{\delta/2}$, we conclude that $J_\epsilon u|_{M_k}$ is close in L_2 to u and $J_\epsilon u|_{M_k}$ does not exceed the energy on M_ϵ by more than an $o(1)$ factor.

Thus J_ϵ is an averaging operator in the sense of Definition 3.2.1 as required in Theorem 3.2.4. This completes the proof of Proposition 3.3.69 and consequentially Theorem 3.2.4 for type I fattened domains. □

Proposition 3.3.70. *Let M be an open book structure and M_ϵ be a corresponding type II model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $\beta < 1/2$. Let $\Lambda \leq c\epsilon^{-\beta+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. For some $\epsilon_0 > 0$, the family of averaging operators $\{J_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.1 for the open book structure M is ($u \in \mathcal{P}_\Lambda^\epsilon L_2(M_\epsilon)$):*

$$J_\epsilon u := \begin{cases} \epsilon^{1/2}\Psi_{M_k}^{-1} N_{k,\epsilon}\big[u - \sum_m \mathcal{E}_{m,\epsilon} u\big] & M_\epsilon \mapsto M_k \\ \epsilon^\beta P_{T,m,\epsilon} u & E_{m,\epsilon} \mapsto E_m. \end{cases} \tag{3.160}$$

Proof: First we note $J_\epsilon u$ is zero at the boundary of M_k. Beginning with the calculation on M_k, we estimate the extension term:

$$\begin{aligned} &||\, \epsilon^{1/2}\Psi_{M_k}^{-1} N_{k,\epsilon} \sum_m \mathcal{E}_{m,\epsilon} u \,||_{H^1(M_k, 2r_k dM_k)} \\ &\leq \sum_m (1 + O(\epsilon^\beta)) ||\, \epsilon^{1/2} N_{k,\epsilon} \mathcal{E}_{m,\epsilon} u \,||_{H^1(M_{k,S}, 2r_k dM_k)} \\ &\leq \sum_m (1 + O(\epsilon^\beta)) ||\, \mathcal{E}_{m,\epsilon} u \,||_{H^1(M_{k,S,\epsilon})}. \end{aligned} \tag{3.161}$$

Here we use Corollary 3.3.65 to get an $o(1)||u||_{H^1(M_\epsilon)}$ bound of (3.161). Next looking at the averaging operator on the binding, we evaluate:

$$\left| ||\epsilon^\beta \, P_{T,m,\epsilon} u||^2_{H^1(E_m, \pi r_m^2 dE_m)} - ||u||^2_{H^1(E_{m,\epsilon})} \right| \leq c\epsilon^\beta ||u||^2_{H^2(E_{m,\epsilon})}. \tag{3.162}$$

The above (3.162) follows from Propositions 3.3.52 and 3.3.51. We then use the spectral subspace bounds ($\Lambda \leq c\epsilon^{-\beta+\delta}$) and Proposition 2.1.16 to bound (3.162). Consequentially (3.162) is bounded by $c\epsilon^{\delta}||u||^2_{H^1(E_{m,\epsilon})}$. Since we have achieved a near isometry in L_2 on both the pages and the binding as well as the required energy bound (3.28) for the energy on both the pages and bindings (see 3.16), we conclude (3.160) is the averaging operator as required from Definition 3.2.1. □

Proposition 3.3.71. *Let M be an open book structure and M_ϵ be a corresponding type III model fattened domain with parameters $\{r_m\}$, $\{r_k\}$, and $\beta = 1/2$. Let $\Lambda \le c\epsilon^{-\beta+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. For some $\epsilon_0 > 0$, the family of averaging operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.1 for the open book structure M is ($u \in \mathcal{P}^\epsilon_\Lambda L_2(M_\epsilon)$):*

$$J_\epsilon u := \begin{cases} \epsilon^{1/2}\Psi^{-1}_{M_k} N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}\left(P_{T,m,\epsilon}u - u\right)\big] & M_\epsilon \mapsto M_k \\ \epsilon^{1/2} P_{T,m,\epsilon}u & E_{m,\epsilon} \mapsto E_m. \end{cases} \tag{3.163}$$

Proof: This proof is similar to Proposition 3.3.70.

With these averaging operators being constructed, the main theorem of this chapter, Theorem 3.2.4, is proven.

3.4 Thin Junctions

Thin junction domains (see Fig. 3.5) are domains where the fattened binding is thinner than the fattened page. As before the size of the binding is controlled by a parameter β which is greater than 1 for these domains. We present partial results on this problem: if $\beta < 2$, the resulting operator is type I. We conjecture that the $\beta > 2$ case should yield a new type IV operator A_4 with Neumman conditions at the binding, but these results are incomplete. We begin with the description of domains with thin junctions.

3.4.1 Statement of Thin Junction Type Convergence

Definition 3.4.1. *Let M be an open book structure as in Definition 2.1.1. Let $\beta > 1$ and $\{r_k\}$ denote a set of positive functions where $r_k \in C^2(M_k) \cap C(\bar{M}_k)$ and $r_k > 0$. We denote the sleeves*

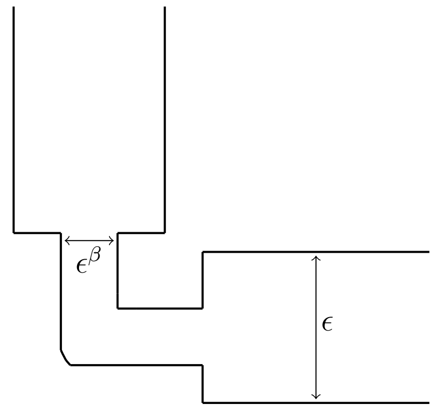

Figure 3.5: The cross-section of a thin junction.

as $S_{k,m,\epsilon}$ as in Definition 2.1.5 with sleeve width a_m ($a_{m,\epsilon} = a_m$) is given by

$$a_m = a_{m,\epsilon} = \begin{cases} \max_{x\in E_m}(1 + r_k(x)\cot(\min_{k,k'}\theta_{m,k,k'}(x)/2)) & \min_{x,k,k'}\theta_{m,k,k'}(x)/2 < \pi/2 \\ \max_{x\in E_m}(1 + r_k(x)) & \min_{x,k,k'}\theta_{m,k,k'}(x)/2 \geq \pi/2 \end{cases} \tag{3.164}$$

where $\theta_{m,k,k'}(x)$ is (smaller) angle between two touching pages M_k and $M_{k'}$ at x (see Fig. 3.1).

Definition 3.4.2. *The normal fibers $\mathcal{I}_{\mathcal{N}_k(x),\epsilon}$ are the same as in Definition 3.1.5; i.e. $\mathcal{I}_{\mathcal{N}_k(x)}$ is the normal fiber of length $2r_k(x)\epsilon$ centered at $x \in M_{k,S}$. The* ***fattened page for a thin junction domain*** *is:*

$$M_{k,S,\epsilon} := \bigcup_{x\in M_{k,S}} \mathcal{I}_{\mathcal{N}_k(x),\epsilon}. \tag{3.165}$$

Definition 3.4.3. *The* ***fattened binding*** $E_{m,\epsilon}$ ***of a thin junction domain*** *is: a $2\epsilon^\beta$ tube about the sleeves. I.e.*

$$E_{m,\epsilon} := \bigcup_{k;x\in S_{k,m,\epsilon}} B(x,\epsilon^\beta)\backslash \bigcup_k M_{k,S,\epsilon}. \tag{3.166}$$

Definition 3.4.4. *We say the family of fattened domains $\{M_\epsilon\}$ is a* ***model thin junction domain*** *if M_ϵ is the union of fattened pages $M_{k,S,\epsilon}$ as defined in Definition 3.4.2, which are fattened by parameters ϵ and $\{r_k\}$, and fattened bindings $E_{m,\epsilon}$ as defined in Definition 3.4.3, which are tubes of width $2\epsilon^\beta$ for $\beta > 1$.*

Any two fattened pages consequentially do not touch.

Theorem 3.4.5. *Let M be an open book structure as in Definition 2.1.1 and M_ϵ be a corresponding thin junction domain with $1 < \beta < 2$ (Definition 3.4.4). Let A be the type I operator in Proposition 3.1.12 and A_ϵ be the Neumann Laplacian (Proposition 2.1.12). There exist averaging operators $\{J_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ and extension operators $\{K_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ be as stated in Definitions 3.2.1 and 3.2.2 for these domains, and thus $\lambda_n(A_\epsilon) \to_s \lambda_n(A)$ for all n as ϵ tends to zero.*

3.4.2 Averaging and Extension Operators for Thin Junctions

To define the averaging and extension operators, J_ϵ and K_ϵ, we adapt several ancillary operators used earlier in this chapter to these thin junction domains. As before $P_{m,\epsilon}$ is a local averaging operator on the binding (see Appendix B.1.9: Lemma B.1.3).

Proposition 3.4.6. *Let M_ϵ be a thin junction domain with $\beta < 2$. For $u \in H^1(M_\epsilon)$, one has:*

$$||u||^2_{L_2(E_{m,\epsilon})} \le c\epsilon^{4-2\beta}||u||^2_{L_2(M_\epsilon)}. \tag{3.167}$$

Proof: This is technical and requires a few secondary lemmas which we reserve for Appendix B.1.9.

We also have need of an analogue to $\mathcal{E}_{m,\epsilon}$, an extension operator in the sense of Proposition 3.3.55. Following our previous calculations, we need an estimate of the trace of a zero mean function. That estimate appears in the Appendix B.1.9 under Lemma B.1.4. This lets us conclude the following:

Proposition 3.4.7. *For $u \in H^1(E_{m,\epsilon})$, the complement of the cross-sectional average $u - P_{m,\epsilon}u$ has an extension into M_ϵ denoted $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ such that*

$$||\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)||^2_{H^1(M_\epsilon)} \le c_m||u||^2_{H^1(E_{m,\epsilon})}. \tag{3.168}$$

Furthermore, $\mathcal{E}_{m,\epsilon}(u - P_{m,\epsilon}u)$ is supported within an $O(\epsilon^\beta)$ distance neighborhood of E_m.

Proposition 3.4.8. *The fattened binding $E_{m,\epsilon}$ of a thin junction domain admits a decomposition into sectors $\Sigma_{m,i,\epsilon}$ (Definition 2.3.8) and admits a vector field $v_{m,i,\epsilon}$ in the sense of Definition 2.3.9 except point (3) reads "the limit of $v_{m,i,\epsilon}(x)$ as $x \to x' \in \partial S_{k,m,\epsilon} \cap M_k$ is $\pm\epsilon^{\beta}\mathcal{N}_k(x')$." We can define an extensions operator $\mathcal{E}_{m,S,z,\epsilon}$ on $L_2(E_m \bigcup(\bigcup_m S_{k,m,\epsilon}))$ in the sense of Definition 2.3.36. This operator satisfies the conclusion of Proposition 2.3.37 with ϵ replaced with ϵ^{β}:*

$$||(2\epsilon^{\beta})^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u||^2_{H^1(E_{m,\epsilon})} \le c||u||^2_{H^1(E_m \bigcup(\bigcup_k S_{k,m,\epsilon}))}. \tag{3.169}$$

Besides the normal averaging operator $N_{k,\epsilon}$ which is the same for this domain as it appears in Definition 3.3.30, these are all the operators needed to recover the results of type I domains for thin junction domains of $\beta < 2$.

Proposition 3.4.9. *Let M be an open book structure and M_ϵ be a corresponding thin junction domain ($1 < \beta < 2$) with parameters $\{r_k\}$ as in Definition 3.4.4. Let $\Lambda \le c\epsilon^{2\beta-4+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A_\epsilon)$. For some $\epsilon_0 > 0$, the family of linear operators $\{J_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the conditions in Definition 3.2.1 for the open book structure M is ($u \in \mathcal{P}^{\epsilon}_{\Lambda}L_2(M_\epsilon)$):*

$$J_\epsilon u := \begin{cases} \epsilon^{1/2}\Psi^{-1}_{M_k}N_{k,\epsilon}\big[u + \sum_m \mathcal{E}_{m,\epsilon}(P_{m,\epsilon}u - u)\big] & M_{k,S,\epsilon} \mapsto M_k \\ \epsilon^{1/2}P_{m,\epsilon}u & E_{m,\epsilon} \mapsto E_m. \end{cases} \tag{3.170}$$

Proof: The proof does not differ from the proof of Proposition 3.3.69 with the exception of the specific order of ϵ in the bounding term Proposition 3.4.6 causing our choice of spectral bound to be $O(\epsilon^{2\beta-4+\delta})$.

Proposition 3.4.10. *Let M be an open book structure and M_ϵ be a corresponding thin junction domain ($1 < \beta < 2$) with parameters $\{r_k\}$ as in Definition 3.4.4. Let $\Lambda \le c\epsilon^{\beta-2+\delta}$ where $\delta > 0$ and $\Lambda \notin \sigma(A)$. For some $\epsilon_0 > 0$, the family of linear operators $\{K_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ that satisfies the* *conditions in Definition 3.2.2 for the open book structure M is ($u \in \mathcal{P}_{\Lambda}^{\epsilon} L_2(M_\epsilon)$):*

$$K_\epsilon u := \begin{cases} \epsilon^{-1/2}\mathcal{E}_{k,z,\epsilon}u & M_{k,S} \\ \epsilon^{-1/2}\mathcal{E}_{m,S,z,\epsilon}u & E_m \bigcup \left(\bigcup_k S_{k,m,\epsilon}\right). \end{cases} \tag{3.171}$$

Proof: It follows Corollary 3.3.20 holds for thin junction domains since the construction of the fattened pages has not been modified from the model fattened domains seen in Definition 3.1.1. The remained of the proof follows from Proposition 3.4.8 and Proposition 3.3.15. □

Consequentially Theorem 3.4.5 is proven, so the operator $\beta < 2$ thin junction domains fall under the type I operator class.

4. CONCLUSION AND REMARKS

In this dissertation we explored the problem of the spectral convergence of Neumann Laplacians on a fattened open book structure. In Chapter 2 we demonstrated spectral convergence of the Neumann Laplacian on a uniformly fattened open book structure to an operator on the open book structure. These results were extended by considering a parameterized family of fattened open book structures in Chapter 3.

The results here build off the results in the fattened graph literature toward answering a more general problem: the resolvent convergence of elliptic operators on $(m+n)$-dimensional domains that retract to an m-dimensional stratified domain. Resolvent convergence of elliptic operators for any arbitrary retraction to a geometry not in general position is seemingly not a tractable problem. Let us illustrate the complexities of this family of problems by discussing the development of the fattened graph problem.

The starting point for the problem was considering the Neumann Laplacian on uniformly fattened domains whose underlying graph is compact and has no cusps [15,24,35]. We remark that the formulation of the problem holds true for periodic graphs with only small modifications [25, 30]. There are four primary fronts of increased complexity to this problem: first, while adding bounded potentials (in the Schödinger sense) does not present a serious problem, changing the boundary conditions on the fattened domain to Dirichlet or Robin requires delicate analysis in order to consistently project onto the lowest modes [8]. We claim that modifying our analysis of type I fattened domains over an open book structure to allow Schrödinger operators with bounded potentials with Neumann boundary conditions should be a straightforward exercise. In this instance the Schrödinger operator result can be extrapolated from the results in this dissertation and the results in fattened graph literature. Observe $H^1(\mathbb{R}^2)$ allows for singular potentials, so there is room for interesting physics particularly for singular potentials define on type II and III fattened domains over an open book structure. An interesting operator to research would be a Schrödinger operator with a logarithmic potential in a large fattened binding which would describe a Coulomb potential due to the binding being a charged wire.

Second, along with considering the problem of resolvent convergence on compact graphs, there is a similar problem of considering resonances on unbounded graphs [5–7, 30]. This is important for the physical application of scattering dynamics [18]. Third, there is the problem we considered in part in Chapter 3 – general retractions of fattened domains [11, 25, 30]. Fourth front is considering graphs where a pair of edges meet tangentially [5–7, 10, 20, 22–25, 36, 40]. The third and the fourth front delve into issue of know what it means for a domain to be "good." It is well-known in the study PDEs that there are few known necessary and sufficient conditions for determining whether a domain admits classical results of PDEs – e.g. solvability of the Dirichlet problem or density (in some Sobolev space) of smooth functions defined on the closure of the domain [9, 27]. As seen with the difficulties in reconciling the local averaging operators, care must be taken to ensure a family of fairly regular shrinking domains satisfy appropriate certain "classical" estimates in the limit of the domains shrinking to zero measure.

When considering stratified spaces of dimension higher than $1D$, it becomes clear that the "problem of dimensionality" exacerbates the issues that are already present in $1D$. Namely, as seen here, the embedding of H^1 into continuous functions falters for dimension 2 and higher. We have resolved that issue in this dissertation, but further difficulties lie ahead in extending results to more general stratified spaces.

We have not considered here the case of the presence of zero-dimensional stata (corners). Partial results suggest that phase transitions should also be seen in a non-uniformly fattened polyhedra where the phase boundaries are determined by capacity heuristics. The presences of corners in an embedded $2D$ stratified space complicates local topologies and leads to several classes of singularities such as those modeled by the tangential contact of two spheres, which may be of interest in applications.

The remaining parameter space ($\beta \geq 2$) for the "thin junction" domains have not been presented here. The analogue from the open graph case suggests that in this parameter range, there should be a "disconnected" limit operator (i.e., each page has Neumann conditions imposed at its boundary).

The result of this dissertation opens up exploration of the scattering problem on thin microelectronic or photonic devices modeled by the fattened open book structures.

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APPENDIX A

ESTABLISHED THEOREMS AND PROPOSITIONS

A.1 Sobolev Embedding Theorem

Here we include the statements mentioned in passing in the main body.

As per [27], $V_p^l(\Omega)$ is defined as the space $\bigcap_{k\leq l} L_p^k(\Omega)$ (i.e. all derivatives up to order l are in $L_p(\Omega)$).

Theorem A.1.1. *[9, 27] Let Ω be a domain in $\mathbb{R}^n$ with compact closure and let it be the union of a finite number of domains of the class EV_p^l (i.e. extensionable domains in the sense of Stein (Theorem 2.3.7); this assumption holds if Ω has the cone property).*

Futhermore, let μ be a measure on Ω satisfying:

$$\sup_{x\in\mathbb{R}^n,\rho>0} \rho^{-s}\mu(\Omega\cap B(x,\rho)) < \infty, \tag{A.1}$$

where $s > 0$ (e.g. if s is an integer, then μ can be the s-dimensional Lebesgue measure on $\Omega\cap\mathbb{R}^s$).

Then for any $u \in C^\infty(\Omega)\cap V_p^l(\Omega)$,

$$\sum_{j=0}^{k} ||\nabla_j u||_{L_p(\Omega,\mu)} \leq c||u||_{V_p^l(\Omega)}, \tag{A.2}$$

where c is a constant independent of u, and the parameters q, s, p, l, and k satisfy the inequalities:

1. *$p > 1$, $0 < n - p(l-k) < s \leq n$, $q \leq sp(n-p(l-k))^{-1}$;*

2. *$p = 1$, $0 < n - l + k \leq s \leq n$, $q \leq s(n-l+k)^{-1}$;*

3. *$p > 1$, $n = p(l-k)$, $s \leq n$, q is any positive number.*

If either of the follow conditions hold:

4. *$p > 1$, $n < p(l-k)$;*

5. $p = 1$, $n \leq l - k$;

then

$$\sum_{j=0}^{k} \sup_{\Omega} |\nabla_j u| \leq c||u||_{V_p^l(\Omega)}. \tag{A.3}$$

If Ω *belongs to the class* EV_p^l *(for example,* Ω *is in* $C^{0,1}$*), then in the case of* (4) *the Theorem can be refined as follows:*

- *If* $p \geq 1$, $(l-k-1)p < n < (l-k)p$ *and* $\lambda = l - k - n/p$*, then for all* $u \in V_p^l(\Omega) \cap C^\infty(\Omega)$

$$\sup_{x, x+h \in \Omega, h \neq 0} \frac{|\nabla_k u(x+h) - \nabla_k u(x)|}{|h|^\lambda} \leq c||u||_{V_p^l(\Omega)}. \tag{A.4}$$

- *If* $(l-k-1)p = n$*, then the inequality (A.4) holds for all* $0 < \lambda < 1$ *and* $u \in V_p^l(\Omega) \cap C^\infty(\Omega)$.

Lemma A.1.2. *[34] Let* $\Omega \subset \mathbb{R}^n$ *be a Lebesgue measurable set. Let* g *be a non-negative integrable function over* Ω *and suppose* $\{f_n\}$ *is a sequence of measurable functions on* Ω *such that for each* n, $|f_n| \leq g$ *almost everywhere on* Ω*. It then follows:*

$$\int_\Omega \liminf f_n \leq \liminf \int_\Omega f_n \leq \limsup \int_\Omega f_n \leq \int_\Omega \limsup f_n. \tag{A.5}$$

APPENDIX B

COLLECTED PROOFS

B.1 Proofs of Several Propositions Appearing in the Text

Here we place some of the longer proofs of the text. We often combine the uniform and non-uniform versions of a proposition within the same proof.

B.1.1 Proof of Proposition 2.3.10 and 3.3.8

First, we direct the reader to the relevant Figures 2.6, 2.7, and 3.4. The goal of the proof will be to demonstrate the construction of the desired vector functions for the cases of a uniformly fattened domain (Fig. 2.6), $\beta = 1$ and $r_m \neq r_k$ fattened domain domain, and $\beta < 1$ fattened domain (Fig. 3.4). The proof of the single page case (Fig. 2.7) can be adapted from the following arguments.

Consider the region of the sector contained in a cross-section $\sigma_{m,i,\epsilon}(x) := \Sigma_{m,i,\epsilon} \cap \omega_{m,\epsilon}(x)$. We further define to continuous curves: $\Gamma_{1,m,i,\epsilon}(x) = \sigma_{m,i,\epsilon}(x) \cap D(v_{m,i,\epsilon})$ (the domain of $v_{m,i,\epsilon}$) and $\Gamma_{2,m,i,\epsilon}(x) = \sigma_{m,i,\epsilon}(x) \cap (\partial E_{m,\epsilon} \setminus \bigcup_k \partial M_{k,S,\epsilon})$. Summarily $\Gamma_{1,m,i,\epsilon}(x)$ is the part of the pair of sleeves in a cross-section, and $\Gamma_{2,m,i,\epsilon}(x)$ is the outward boundary of that sector in a cross-section.

We construct a map ϕ between $\Gamma_{1,m,i,\epsilon}(x)$ and $\Gamma_{2,m,i,\epsilon}(x)$. Per Definition 2.3.9:(2) the displacement vector between y and $\phi(y)$ gives us $v_{m,i,\epsilon}$. I.e. $v_{m,i,\epsilon}(y) = \phi(y) - y$ where $y \in E_m \bigcup (\bigcup_k S_{k,m,\epsilon}) \subset \mathbb{R}^3$.

The boundary segments $\Gamma_{j,m,i,x}(x)$ are of length $O(\epsilon)$. In particular there exists constants c_3 and c_4 such that

$$|\Gamma_{1,m,i,\epsilon}(x)| = 2a_m\epsilon \qquad c_3\epsilon \leq l_2 = |\Gamma_{2,m,i,\epsilon}(x)| \leq c_4\epsilon \tag{B.1}$$

for the *uniform case*, and

$$|\Gamma_{1,m,i,\epsilon}(x)| = 2a_{m,\epsilon}\epsilon^\beta \qquad c_3\epsilon^\beta \leq l_2 = |\Gamma_{2,m,i,\epsilon}(x)| \leq c_4\epsilon^\beta \tag{B.2}$$

for the *non-uniform case*.

We parameterize $\Gamma_{1,m,i,\epsilon}(x)$ and $\Gamma_{2,m,i,\epsilon}(x)$ with unit speed parameterizations $\gamma_{j,x}$:

$$\gamma_{1,x} : (0, 2a_m\epsilon) \mapsto \Gamma_{1,m,i,\epsilon}(x) \qquad \gamma_{2,x} : (0, l_2) \mapsto \Gamma_{2,m,i,\epsilon}(x) \tag{B.3}$$

for the *non-uniform case* and similarly for the non-uniformly fattened case.

Clearly, there is only one such way to match the end points of $\Gamma_{1,m,i,\epsilon}(x)$ and $\Gamma_{2,m,i,\epsilon}(x)$ in order that Definition 2.3.9:(3) holds, so we assume the parameterizations are oriented correctly. Because $\varpi_{m,\omega}(x)$ is not convex in the uniformly fattened case (or $\beta = 1$), we must take to ensure the segment connecting the two curves is in the sector.

The mapping for uniformly fattened domains (and $\beta = 1$):

$$\phi(x) = \begin{cases} \gamma_{2,x}(\gamma_{1,x}^{-1}(y)) & \gamma_{1,x}^{-1}(y) < (a_m - r_m)\epsilon \\ \gamma_{2,x}\left(\dfrac{l_2 - 2(a_m - r_m)\epsilon}{2\epsilon r_m}\gamma_{1,x}^{-1}(y)\right) & (a_m - r_m)\epsilon \leq \gamma_{1,x}^{-1}(y) \leq (a_m + r_m)\epsilon \\ \gamma_{2,x}(\gamma_{1,x}^{-1}(y)) & \gamma_{1,x}^{-1}(y) > (a_m + r_m)\epsilon \end{cases} \tag{B.4}$$

where $r_m = 1$ in the uniformly fattened case.

For $\beta < 1$ the expression is simpler because the cross-section is convex:

$$\phi(x) = \gamma_{2,x}\left(\frac{l_2}{2a_{m,\epsilon}\epsilon^\beta}\gamma_{1,x}^{-1}(y)\right). \tag{B.5}$$

Inequality 3.42 follows since the diameter of the cross-section is bounded by $O(\epsilon^\beta)$. Because the pages intersect transversely at E_m, $\gamma_{1,x}$ is Lipschitz. $\gamma_{2,x}$ is also Lipschitz because $E_{m,\epsilon}$ is a Lipschitz graph domain (both Lipschitz norms are independent of ϵ). Consequentially, $v_{m,i,\epsilon}$ is Lipschitz (with Lipschitz norm independent of ϵ). We conclude that where the in-plane derivatives of $v_{i,m,\epsilon}$ exist in $\sigma_{m,i,\epsilon}(x)$, the derivative is bounded by a constant uniform with respect to ϵ.

For the derivatives of $v_{i,m,\epsilon}$ with respect to the direction out-of-plane of $\omega_{m,\epsilon}(x)$, we note this depends on the angle between the pages $\theta_{m,k,k'}$ and the curvature of the pages. These functions do not depend on ϵ, so there is an ϵ-independent bound on the out-of-plane derivative of $v_{m,i,\epsilon}$. $\square$

B.1.2 Proof of Proposition 2.3.31

Using the triangle inequality, we write

$$\begin{aligned}
\int_{S_{k,m,\epsilon}} |u|^2 \, dM_k &\leq \Big| \int_{M_k} |u|^2 \, dM_k - \int_{M_{k,S}} |\Psi_{M_k} u|^2 \, dM_k \Big| \\
&\quad + \left| \int_{M_{k,S}} |u|^2 \, dM_k - \int_{M_{k,S}} |\Psi_{M_k} u|^2 \, dM_k \right| \\
&\leq O(\epsilon) ||u||^2_{L_2(M_k)} + \left| \int_{M_{k,S}} |u|^2 - |\Psi_{M_k} u|^2 \, dM_k \right| \\
&\leq O(\epsilon) ||u||^2_{L_2(M_k)} + \big|\big|u - \Psi_{M_k} u\big|\big|_{L_2(M_{k,S})} \big|\big|u + \Psi_{M_k} u\big|\big|_{L_2(M_{k,S})}
\end{aligned} \tag{B.6}$$

To bound $||u - \Psi_{M_k} u||_{L_2(M_{k,S})}$, we use the coordinate system provided in the proof of Proposition 2.3.30. Let X_k be the coordinate patch on $U = (0, l_{E_m}) \times (0, a)$ (2.67), and φ_ϵ be the smooth shortening function from (2.68). We define a family of curves that go from y to $\psi_{M_k}(y)$:

$$\gamma_{\varphi_\epsilon, y} : t \in [0,1] \mapsto U \qquad \gamma_{\varphi_\epsilon, y}(0) = (y_1, y_2) \quad \gamma_{\varphi_\epsilon, y}(1) = (y_1, \varphi_\epsilon(y_2)). \tag{B.7}$$

In particular we can choose $\gamma_{\varphi_\epsilon, y}$ to be constant speed. Outside of $X_k(U) \subset M_k$, $u = \Psi_{M_k} u$, and so we need to concern ourselves only with the function on $X_k(U)$. Let $U' = (0, l_{E_m}) \times (a_m \epsilon, a)$ and let $v(y_1, y_2) = u(X_k(y_1, y_2))$. Then we have

$$\begin{aligned}
&\big|\big|u - \Psi_{M_k} u\big|\big|^2_{L_2(M_{k,S})} \\
&\quad = \int_{U'} \left| v(y) - \left(v(y) + \int_0^1 \nabla v\Big(y + \gamma_{\varphi_\epsilon, y}\Big) \cdot \gamma'_{\varphi_\epsilon, y} dt \right) \right|^2 \sqrt{\mathsf{det} g_{M_k}}(y) \, dy \\
&\quad = \int_{U'} \left| \int_0^1 D_{y_2} v\big(y_1, y_2 + t(\varphi_\epsilon(y_2) - y_2)\big) \, |\varphi_\epsilon(y_2) - y_2| \, dt \right|^2 \sqrt{\mathsf{det} g_{M_k}}(y) \, dy.
\end{aligned} \tag{B.8}$$

Let $\xi = y_2 + t(\varphi_\epsilon(y_2) - y_2)$, and so $d\xi = dy_2(1 - t + tD\varphi_\epsilon(y_2))$. Because $D\varphi_\epsilon(y_2) = 1 + O(\epsilon)$, we can then write $d\xi = dy_2(1 - tO(\epsilon))$. Thus, the Jacobian J from $(y_1, y_2) \mapsto (y_1, \xi)$ is of the form $1 + O(\epsilon)$. Applying $|\varphi_\epsilon(y_2) - y_2| = O(\epsilon)$, we have

$$\begin{aligned} &\big|\big|u - \Psi_{M_k} u\big|\big|^2_{L_2(M_{k,S})} \\ &\quad\le \int_{U'} \int_0^1 \Big| D_\xi v(y_1,\xi)\Big|^2 \frac{O(\epsilon^2)}{1 - tO(\epsilon)}\, dt \sqrt{\mathsf{det} g_{M_k}}(y_1,\xi)\, dy_1 d\xi \\ &\quad\le \big|\big|D_\xi v\big|\big|^2_{L_2(U', \mathsf{det} g^{1/2}_{M_k})} \int_0^1 \frac{O(\epsilon^2)}{1 - tO(\epsilon)}\, dt \\ &\quad\le O(\epsilon^2) ||\nabla_{M_k} u||^2_{L_2(M_k)}. \end{aligned} \tag{B.9}$$

Applying that to (B.6) we get the $O(\epsilon)$ bounds. □

B.1.3 Proof of Theorem 2.3.49

It is sufficient to only consider $\delta = 1$ and later recover the full results by homothetically scaling the coordinates. We also note another result of Sobolev theory [27, 28]: that if Ω is a bounded Lipschitz graph class domain in $\mathbb{R}^n$, $C^\infty(\bar{\Omega})$ is dense in $L^l_p(\Omega)$ for $p < \infty$

Let $\varphi \in C^\infty_0(B(0,1))$ and we begin with assuming $u \in C^\infty(\Omega)$. Let $x \in \Omega$ and $z \in B(0,1)$. By star-shapedness the segment $z + (x - z)t$ ($t \in [0,1]$) is contained in Ω. Thus, using Taylor's theorem we have

$$u(x) = \sum_{|\alpha|<l} \frac{D^\alpha u(z)}{\alpha!}(x-z)^\alpha + l \int_0^1 (1-t)^{l-1} \sum_{|\alpha|=l} \frac{1}{\alpha!} D^\alpha u(z + t(x-z))(x-z)^\alpha\, dt. \tag{B.10}$$

Multiplying this equality by $\varphi(z)$ and integrating, we get

$$\begin{aligned} u(x) = &\sum_{|\alpha|<l} \int_{B(0,1)} \frac{D^\alpha u(z)}{\alpha!}(x-z)^\beta \varphi(z)\, dz \\ &+ l \sum_{|\alpha|=l} \int_0^1 \int_{B(0,1)} (1-t)^{l-1} \frac{1}{\alpha!} D^\alpha u(z + t(x-z))(x-z)^\alpha \varphi(z)\, dt. \end{aligned} \tag{B.11}$$

A simple integration by parts gives ($|\alpha| < l$)

$$\int_{B(0,1)} D^\alpha u(z)(x-z)^\alpha \varphi(z)\, dz = (-1)^{|\alpha|} \int_{B(0,1)} u(z) D^\alpha_z((x-z)^\alpha \varphi(z))\, dz. \tag{B.12}$$

Returning the to the remainder term in the Taylor expansion, we use the following change of variables:

$$x - z = (1-t)^{-1}(x-y), \qquad dz = (1-t)^{-n}dy. \tag{B.13}$$

This results in

$$\begin{aligned} l\sum_{|\alpha|=l}\int_0^1\int_{B(0,1)}(1-t)^{l-1}\frac{1}{\alpha!}D^\alpha u(z+t(x-z))(x-z)^\alpha\varphi(z)\,dt \\ = \frac{1}{\alpha!}\int_{\mathbb{R}^n}\int_0^1 D^\alpha u(y)(x-y)^\alpha\varphi\left(\frac{y-tx}{1-t}\right)\frac{1}{(1-t)^{n+1}}\,dt\,dy. \end{aligned} \tag{B.14}$$

To get the last result in Remark 2.3.50, we identify the kernel in the last expression and perform one last variable transformation:

$$\int_0^1 \varphi\left(\frac{y-tx}{1-t}\right)\frac{1}{(1-t)^{n+1}} = r^{-n}\int_r^\infty \varphi(x+\rho\theta)\rho^{n-1}\,d\rho. \tag{B.15}$$

The coordinates can be scaled to recover δ. Because $C^\infty(\bar{\Omega})$ is dense in $L_p^l(\Omega)$ and each of these integral operators is continuous in $L_p^l(\Omega)$, we can pass a converging sequence $u_i \to u$ in $C^\infty(\bar{\Omega})$ to have an integral representation of u. $\square$

B.1.4 Proof of Lemma 2.3.62

We apply the partition of unity $\{\varphi_{i,\epsilon}\}$ as laid out in Corollary 2.3.6 and use Lemma 2.3.60 in the scaled domain. We denote the homothetic scaling on $\mathbb{R}^3$: $\theta : x \to x/\epsilon$ and Θ the induced operator on functions ($\Theta u = u(\theta)$). Beginning with the left hand side of (2.133), we have:

$$\begin{aligned} &\epsilon^{-1}||T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})} + [T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\Gamma_{k,m,\epsilon}} \\ &= \epsilon||\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\theta(\Gamma_{k,m,\epsilon}))} + \epsilon[\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\theta(\Gamma_{k,m,\epsilon})} \\ &\le \epsilon\sum_i ||\varphi_{i,\epsilon}(\theta)\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\theta(\Gamma_{k,m,\epsilon}))} + [\varphi_{i,\epsilon}(\theta)\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\theta(\Gamma_{k,m,\epsilon})}. \end{aligned} \tag{B.16}$$

Recalling Corollary 2.3.6, we first note that the gradient of all the partition functions $\varphi_{i,\epsilon}(\theta)$ is uniformly bounded above by a constant $c_{\nabla\varphi}$ uniform with respect to ϵ. We also note the bounding balls about $\varphi_{i,\epsilon}(\theta)$ have an upper bound on their diameter also independent of ϵ. Local finiteness of the partition holds as well (number of intersections is bounded about by c_U). The support of $\varphi_{i,\epsilon}$ ($U_{i,\epsilon}$) can be identified with a local neighborhood of a special Lipschitz domain $\Omega_{i,\epsilon}$. All of which have Lipschitz graph norms bounded about by an ϵ-independent constant c_M.

Using Lemma 2.3.60, each support set of $\varphi_{i,\epsilon}$ can be mapped to an half-space $\mathbb{R}^3_+$ by $\chi_{i,\epsilon}$ (Definition 2.3.61) with norm bounded independently of ϵ. On each copy of $\mathbb{R}^3_+$ we invoke the Sobolev embedding theorem:

$$\begin{aligned} &||\Theta T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}(\theta))\cap\theta(\Gamma_{k,m,\epsilon}))} + [\Theta T_{m,k,\epsilon}(u - P_{m,\epsilon}u)]^2_{\mathsf{supp}(\varphi_{i,\epsilon}(\theta)\cap\theta(\Gamma_{k,m,\epsilon})} \\ &\quad \leq c\big(||\Theta T_{\phi_{i,\epsilon}} T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}(\theta(\chi_{i,\epsilon}))\cap\theta(\chi_{i,\epsilon}(\Gamma_{k,m,\epsilon}))} \\ &\quad + [\Theta T_{\phi_{i,\epsilon}} T_{m,k,\epsilon}(u - P_{m,\epsilon}u)]^2_{\mathsf{supp}(\varphi_{i,\epsilon}(\theta(\chi_{i,\epsilon})))\cap\theta(\chi_{i,\epsilon}(\Gamma_{k,m,\epsilon}))}\big). \end{aligned} \tag{B.17}$$

Denoting the upper bound of the norm of the embedding as c_{em} (depending only on c_M, the upper bound on the Lipschitz norms of the boundary graphs), the (B.17) is bounded by:

$$c_{em}||\Theta T_{\phi_{i,\epsilon}}(u - P_{m,\epsilon}u)||^2_{H^1(\mathsf{supp}(\varphi_{i,\epsilon}(\theta(\chi_{i,\epsilon}))}. \tag{B.18}$$

Thus (B.16) is bounded by

$$\epsilon c' \sum_i ||\varphi_{i,\epsilon}(\theta)\Theta(u - P_{m,\epsilon}u)||^2_{H^1(\mathsf{supp}(\varphi_{i,\epsilon}(\theta)))}. \tag{B.19}$$

After imputing all the constants associated with our partition of unity, the (B.19) is bounded by

$$\epsilon c' c_U(1 + c_{\nabla\varphi})||\Theta(u - P_{m,\epsilon}u)||^2_{H^1(\theta(E_{m,\epsilon}))}. \tag{B.20}$$

Lastly, we scale the domain back to ϵ size to get the bound $c||u||^2_{H^1(E_{m,\epsilon})}$. $\square$

B.1.5 Proof of Proposition 2.3.63

Let R_0 denote the continuous (lowest order) reflection operator on $\mathbb{R}^n_+$. Namely, for $u(x', x_n) \in C(\mathbb{R}^n_+)$ where $x_n \geq 0$ we define

$$R_0 u(x', x_n) = u(x', |x_n|) \qquad (x', x_n) \in \mathbb{R}^n. \tag{B.21}$$

Because continuous functions are dense in $H^1(\mathbb{R}^n_+)$, R_0 can be extended to a linear operator from $H^1(\mathbb{R}^n_+)$ to $H^1(\mathbb{R}^n)$ of norm 2.

Using the machinery laid out in Corollary 2.3.6 and Lemma 2.3.62, we begin with the covering $\{U_{i,\epsilon}\}$ and homothetic scaling map θ. By application of $T_{\phi_{i,\epsilon}}$ of Lemma 2.3.60, we can rectify each $\theta(U_{i,\epsilon})$ along the interface $\theta(\Gamma_{k,m,\epsilon})$ into being a subset of the half space $\mathbb{R}^3_+$.

In particular the transformation laid out in Lemma 2.3.60 takes $\theta(\Gamma_{k,m,\epsilon} \cap U_{i,\epsilon})$ into a subset of the hyperplane $\partial\mathbb{R}^n_+$. We then use R_0 to reflect a function across $\theta(\Gamma_{k,m,\epsilon})$ into $\theta(M_{k,S,\epsilon})$:

$$R_0 : \Theta T_{\phi_{i,\epsilon}} H^1(U_{i,\epsilon}) = H^1(V \subset \mathbb{R}^3_+) \mapsto H^1(\mathbb{R}^3). \tag{B.22}$$

Subsequently, we may take a smooth cutoff function ψ with respect to the normal distance from the planar set $\theta(\chi_{i,\epsilon}(\Gamma_{k,m,\epsilon} \cap U_{i,\epsilon}))$:

$$\psi \in C^\infty(\mathbb{R}) \qquad \psi(x \geq 0) = 1 \qquad \psi(x \leq -c) = 0 \tag{B.23}$$

where $c > 0$. We then define

$$\begin{aligned} \mathcal{E}_{m,i,1} :& H^1(\theta(\chi_{i,\epsilon}((U_{i,\epsilon}))) \mapsto H^1(\theta(M_{k,S,\epsilon})) \\ \mathcal{E}_{m,i,1} =& T^{-1}_{\phi_{i,\epsilon}} \psi R_0. \end{aligned} \tag{B.24}$$

This operator has norm bounded uniformly bounded above by a constant c_m independent of i and ϵ. The last matter is to observe that the collection the supports of these extensions have a finite intersection property in the limit of ϵ tends to zero. This property follows because of the limit finite intersection property of $U_{i,\epsilon}$ and the finite support length parameter c. We then define

$$\mathcal{E}_{m,\epsilon} = \sum_i \Theta^{-1}\mathcal{E}_{m,i,1}\Theta T_{\phi_{i,\epsilon}}\varphi_{i,\epsilon}. \tag{B.25}$$

Because

$$||\Theta(u - P_{m,\epsilon}u)||^2_{H^1(\theta(E_{m,\epsilon}))} \leq c\epsilon^{-1}||u - P_{m,\epsilon}u||^2_{H^1(E_{m,\epsilon})} \tag{B.26}$$

and

$$||\Theta^{-1}v||^2_{H^1(E_{m,\epsilon})} \leq c\epsilon||v||^2_{H^1(\theta(E_{m,\epsilon}))}, \tag{B.27}$$

the extension operator $\mathcal{E}_{m,\epsilon}$ satisfies the inequality in (2.134). □

B.1.6 Proof of Lemmas 2.3.65 and 3.3.57

In the uniformly fattened case the domain $M_{k,S,\epsilon}$ is a “slab” of width 2ϵ, and in the non-uniformly fattened case the domain $M_{k,S,\epsilon}$ is a “slab” of variable width of $2\epsilon r_k$.

We cover a neighborhood in $M_{k,S,\epsilon}$ of the interface $\Gamma_{k,m,\epsilon}$ with a partition of unity satisfying similar requirements to Corollary 2.3.6 with some differences. Let $\{U_{i,\epsilon}\}$ be a locally finite open cover of $\Gamma_{k,m,\epsilon}$ such that the maximum number of intersections is bounded above by n_U for all $\epsilon > 0$. We also suppose the intersection $U_{i,\epsilon} \cap U_{i',\epsilon}$ contains a set of diameter larger than $c_1\epsilon$. The inner and outer diameters of each $U_{i,\epsilon}$ have lower and upper bounds of $c_2\epsilon$ and $c_3\epsilon$ respectively.

We consider cylindrical domains over each $U_{i,\epsilon}$ called $V_{i,\epsilon}$ in $M_{k,S,\epsilon}$. For some point $y \in U_{i,\epsilon}$, we denote the normal vector to $\Gamma_{k,m,\epsilon}$ at y pointing into $M_{k,S,\epsilon}$ as $\nu_{k,m,\epsilon}(y)$. For some constant c_4 (depending only on the geometry of $M_{k,S}$), the collection of sets $\{V_{i,\epsilon}\}$ where

$$V_{i,\epsilon} := \{x \in M_{k,S,\epsilon} : x = y + z\nu_{k,m,\epsilon}(y) \qquad y \in U_{i,\epsilon},\ z \in (0, c_4)\} \tag{B.28}$$

has the finite intersection property as $\epsilon \to 0$. I.e. there is an n_V such that at most n_V sets $V_{i,\epsilon}$ (for a collection of i) have non-trivial intersection.

The requires some more clarification. The constant c_4 must be appropriately chosen to avoid caustics, so c_4 must be less than half of the inner diameter (as defined geodesically) of $M_{k,S}$. As an elementary example, we consider the disk and a covering of the boundary of the disk with $O(\epsilon)$ intervals $\{U_{i,\epsilon}\}$ with some finite intersection property and associated constant n_U. Taking strips of length of $1/4$ each with their base one of these $O(\epsilon)$ intervals along the boundary, these strips still have a finite intersection property. Furthermore, the maximum number of intersections of these strips will be $\dfrac{4\pi n_U}{3\pi}$. The maximum number of intersections of $V_{i,\epsilon}$ is a function of n_U, c_4, and the curvature of $\Gamma_{k,m,\epsilon}$.

We equip $M_{k,S,\epsilon}$ with a local normal coordinate system (y_1, y_2, z), where y_2 denotes the distance from the boundary $\Gamma_{k,m,\epsilon}$. Considering the scaling $\theta : (y_1, y_2, z) \to (y_1/\epsilon, y_2, z/\epsilon)$ (and induced operator Θ). Let $\{\varphi_{i,\epsilon}\}$ be a smooth partition of unity subordinate to $\{V_{i,\epsilon}\}$. Applying the scaling θ and the partition of unity, we have

$$||T_{k,m,\epsilon}u||^2_{L_2(\Gamma_{k,m,\epsilon})} \leq \sum_i \epsilon^2 ||T_{k,m,\epsilon}\Theta\varphi_{i,\epsilon}(\theta)u||^2_{L_2(\theta(\Gamma_{k,m,\epsilon}))}. \tag{B.29}$$

Under the scaling θ, the support sets $\mathsf{supp}(\varphi_{i,\epsilon}(\theta))$ are contained in a ball of radius c_1 uniform with respect to i and ϵ and contain a ball of radius c_2 also uniform with respect to i and ϵ. As before, the each of these domains is equivalent to a subset of a special Lipschitz domain whose graph function has Lipschitz norm bounded above by M (also a uniform constant). The right hand side of (B.29) is bounded by:

$$c_{em}\epsilon^2 ||\Theta\varphi_{i,\epsilon}(\theta)u||^2_{H^1(\theta(M_{k,S,\epsilon}))} \leq c||u||^2_{H^1(M_{k,S,\epsilon})}. \qquad \square \tag{B.30}$$

B.1.7 Proof of Lemma 3.3.43

This proof will be broken up further into several lemmas:

Lemma B.1.1. *Let $l \geq 1$ and suppose*

$$\int_{\mathbb{R}^m} K(t)t^\nu \, dt = 0 \tag{B.31}$$

for all multi-indices $\nu \in \mathbb{Z}_+^m$ and $|\nu| \leq l-1$. If $u \in C^\infty(D)$, then the following estimate

$$||D_y^\gamma(\mathcal{T}u)(\cdot, z)||_{L_p(\mathbb{R}^m)} \leq c|z|^{l-|\gamma|}||\nabla_l u(\cdot, z)||_{L_2(\mathbb{R}^m)} \tag{B.32}$$

holds for $z \in \Omega \backslash \{0\}$, $|\gamma| \leq l$, $p \geq 1$.

Proof B.1.1: Applying the Taylor expansion to $u(\cdot, z)$, we have:

$$\mathcal{T}u(y,z) = l|z|^l \int_{\mathbb{R}^m} \int_0^1 \sum_{|\alpha|=l} \frac{t^\alpha}{\alpha!}(D_y^\alpha u)(y+\tau|z|t, z)(t-\tau)^{l-1}\, d\tau. \tag{B.33}$$

The case $\gamma = 0$ follows immediately from the Minkowski inequality. The following is a proof by induction on l. Setting $l = 1$, we get

$$\begin{aligned} D_{z_i}\mathcal{T}u = \frac{z_i}{|z|}\sum_{j=1}^m \int_{\mathbb{R}^m} t_j K(t) D_{y_j} u(y+|z|t, z)\, dt \\ + \int_{\mathbb{R}^m} D_{z_i} u(y+|z|t, z)\, dt = \mathcal{T}D_{z_i}u + \frac{z_i}{|z|}\sum_{j=1}^m \mathcal{T}_j D_{y_j} u \end{aligned} \tag{B.34}$$

where $\mathcal{T}_j$ is an operator of the form of $\mathcal{T}$ with a kernel $K_j = t_j K$. Using the Minkowski inequality again, we arrive at:

$$||D_{z_i}\mathcal{T}u||_{L_p(\mathbb{R}^n)} \leq c||\nabla u||_{L_p(\mathbb{R}^m)}. \tag{B.35}$$

For the induction step, let $l \geq 2$ and assume the Lemma holds for all orders up to $l-1$. Let $|\alpha| = |\gamma| - 1 \leq l-1$ and $D_z^\gamma = D_z^\alpha D_{z_i}$ for some i. From before we have

$$D_z^\gamma = D_z^\alpha \mathcal{T} D_{z_i} u + \sum_{j=1}^m D_z^\alpha(z_i|z|^{-1}\mathcal{T}_j D_{y_j} u). \tag{B.36}$$

We say for multi-index δ that $\delta \leq \alpha$ if $\delta_i \leq \alpha_i$ index-wise. The last term in (B.36) is bounded by

$$||D_z^\alpha(z_i|z|^{-1}(\mathcal{T}_j D_{y_j} u)||_{L_p(\mathbb{R}^m)} \leq c\sum_{\delta \leq \alpha} |z|^{|\delta|-|\alpha|}||D_z^\delta \mathcal{T}_j D_{y_j} u||_{L_p(\mathbb{R}^m)}. \tag{B.37}$$

The kernel K_j of $\mathcal{T}_j$ also satisfies the integrability condition (B.31). Thus, it follows

$$||D_z^\delta \mathcal{T}_j D_{y_j} u||_{L_p(\mathbb{R}^m)} \leq c|z|^{l-1-|\delta|}||\nabla_{l-1} D_{y_j} u||_{L_p(\mathbb{R}^m)}. \tag{B.38}$$

Lastly, combining these inequalities we have

$$||D_z^\alpha \mathcal{T} D_{z_i} u||_{L_p(\mathbb{R}^m)} \leq c|z|^{l-1-|\alpha|}||\nabla_{l-1} D_{z_i} u||_{L_p(\mathbb{R}^m)}. \qquad \square \tag{B.39}$$

Lemma B.1.2. *Let (B.31) be satisfied for all ν with $1 \leq |\nu| \leq l-1$. Then $\mathcal{T}$ is a bounded operator and the following estimates hold*

$$||\nabla_l \mathcal{T} u||_{L_p(D)} \leq c||\nabla_l u||_{L_p(D)}. \tag{B.40}$$

Proof B.1.2: This is proven by an induction on l. We suppose $u \in C^\infty(D) \cap L_p^l(D)$ and bootstrap our results to $L_p^l(D)$ by the density argument. The case $l = 0$ is trivial, and $l = 1$ follows since $\mathcal{T} D_{y_j} = D_{y_j} \mathcal{T}$. Let $l \geq 2$ and assume

$$||\nabla_k \mathcal{T} u||_{L_p(D)} \leq c||\nabla_k u||_{L_p(D)} \tag{B.41}$$

holds for all $k \leq l-1$ and smooth functions $u \in L_p^k(D)$. Let γ and β be multi-indices. Because $\mathcal{T}$ commutes with the longitudinal derivative, we have

$$D_z^\gamma D_y^\beta \mathcal{T} u = D_z^\gamma(\mathcal{T} D_y^\beta u). \tag{B.42}$$

Let $|\beta| + |\gamma| = l$. If $\gamma = 0$ then simply commuting the derivative with $\mathcal{T}$ gives us the required result. If $0 < |\gamma| < l$, by the induction hypothesis we have

$$||D_z^\gamma D_y^\beta \mathcal{T} u||_{L_p(D)} \leq c||\nabla_{|\gamma|} D^\beta u||_{L_p(D)} \leq c||\nabla_l u||_{L_p(D)}. \tag{B.43}$$

Let $|\beta| = 0$, $|\gamma| = l$ and $D_z^\gamma = D_z^\alpha D_{z_i}$. As established in the previous lemma, we have

$$||D_z^\gamma \mathcal{T} u||_{L_p(\mathbb{R}^m)} \leq ||D_z^\alpha \mathcal{T} D_{z_i} u||_{L_p(\mathbb{R}^m)} + c \sum_{j=1}^{m} \sum_{\delta \leq \alpha} |z|^{|\delta| - |\alpha|} ||D_z^\delta \mathcal{T}_j D_{y_j} u||_{L_p(\mathbb{R}^m)}. \tag{B.44}$$

Note again that the kernel K_j of $\mathcal{T}_j$ satisfies (B.31). By integrating with respect to $z \in \Omega$, we conclude that this is bounded by the right hand side of (B.32). □

Returning back to the original statement Lemma 3.3.43, the proof is again an induction on l using a density argument. Letting $u \in L_p^l(D) \cap C^\infty(D)$, we have

$$\begin{aligned} |\mathcal{T} u - u| &= \Big| \int_{\mathbb{R}^m} K(t)(u(y + |z|t, z) - u(y, z)\, dt \Big| \\ &\leq c|z|^l \int_{\mathbb{R}^m} \int_0^1 |K(t)| \sum_{|\alpha| = l} |D_y^\alpha (y + |z|t\tau, z)|\, d\tau dt. \end{aligned} \tag{B.45}$$

For $l = 1$, $k = 0$ the inequality has already been verified and $k = 1$ follows from the Lemma B.1.2. Let $l \geq 2$ and suppose the lemma holds true for all orders up to $l - 1$. β and γ again denote multi-indices such that $|\beta| + |\gamma| = k$ and $|\beta| > 0$. By the commutation property of $\mathcal{T}$, we get:

$$D_z^\gamma D_y^\beta (\mathcal{T} u - u) = D_z^\gamma (\mathcal{T} D_y^\beta u - D_y^\beta u). \tag{B.46}$$

The induction hypothesis gives

$$||D^\beta D^\gamma (\mathcal{T} u - u)||_{L_p(D)} \leq c r^{l - |\beta| - |\gamma|} ||\nabla_{l - |\beta|} D^\beta u||_{L_p(D)}. \tag{B.47}$$

Suppose first $|\beta| = 0$. $D_z^\gamma = D_z^\alpha D_{z_i}$ for some i. From before we have

$$||D_z^\gamma||_{L_p(D)} \leq ||D_z^\alpha \mathcal{T} D_{z_i} u||_{L_p(D)} + \sum_{j=1}^{m} ||D_z^\alpha (z_i |z|^{-1} \mathcal{T}_j D_{y_j} u)||_{L_p(D)}. \tag{B.48}$$

The last term is dominated by

$$cr^{l-1-|\alpha|}||\nabla_{l-1}D_{y_j}u||_{L_p(D)}, \tag{B.49}$$

while the other term is bounded by the induction hypothesis

$$||D^\alpha(\mathcal{T}D_{z_i}u - D_{z_i}u)||_{L_p(D)} \leq cr^{l-1-|\alpha|}||\nabla_{l-1}D_{z_i}u||_{L_p(D)}. \tag{B.50}$$

Consequentially, the result (B.40) is proven. □

B.1.8 Proof of Proposition 3.3.61

From Proposition 3.3.55, we constructed an extension operator on $u - P_{m,\epsilon}$. We now extend the averaged component of u. Observe that $P_{m,\epsilon}u$ (and $P_{T,m,\epsilon}u$) is constant along each cross-section. Thus, extending $P_{T,m,\epsilon}u$ as a constant with a smooth cutoff function along the cross-sections into a larger, bounded cylindrical domain is a bounded operator. Let us formulate this.

Let $\{U_i\}$ be a finite covering of $U = [0, l_{E_m}]/\{0, l_{E_m}\}$, and so $\{\gamma_m(U_i)\}$ is a finite covering of E_m. For a sufficiently small distance $b_i > 0$, there is a neighborhood of $\gamma_m(U_i)$ admits a coordinate system $(t, z) \in U_i \times D(0, b_i)$ such that $\mathsf{dist}_{\mathbb{R}^3}(\gamma_m(U_i), (t, z)) = |z|$. Let $b = \min b_i$. We suppose ϵ_0 is sufficiently small such that $E_{m,\epsilon}$ is contained in this neighborhood of distance b. Let ϕ denote a compactly supported function on $\mathbb{R}$ and ϕ is 1 in $(-1, 1)$ and 0 outside $(-2, 2)$. In this coordinate system, we can define the extension of $P_{T,m,\epsilon}u$ locally. We first extend to $\mathbb{R}^3$ and take the restriction to M_ϵ:

$$\mathcal{E}_{m,\epsilon}P_{T,m,\epsilon}(t, z) := \phi\left(\frac{|z|}{2b}\right) P_{T,m,\epsilon}u(\gamma_m(t)). \tag{B.51}$$

Clearly, when ϵ is sufficiently small, this extension is supported in $U_i \times D(0, b)$. Secondly, this extension is well-defined in the overlap $U_i \cap U_j$. We then calculate the H^1-norm of this extension.

Note that the derivative of ϕ is only non-zero outside the fattened binding. We get

$$\begin{aligned}
&||\mathcal{E}_{m,\epsilon} P_{T,m,\epsilon} u||^2_{H^1(M_\epsilon)} - ||P_{T,m,\epsilon} u||^2_{H^1(E_{m,\epsilon})} \\
&\quad = \int_{M_\epsilon \setminus E_{m,\epsilon}} (|\phi(|z|/2b)|^2 + |D_z \phi(|z|/2b)|^2)|P_{T,m,\epsilon} u|^2 \\
&\quad + 2D_z \phi(|z|/2b) \cdot D_z P_{T,m,\epsilon} u + |\phi(|z|/2b)|^2 |\nabla P_{T,m,\epsilon} u|^2 \, dM_\epsilon.
\end{aligned} \tag{B.52}$$

Now make a volume comparison: $M_{k,S,\epsilon}$ is a slab of volume $O(\epsilon)$ while $E_{m,\epsilon}$ is a tube of volume $O(\epsilon^{2\beta})$. Using the volume analysis, we conclude (B.52) is bounded by

$$\frac{2 \max_y |I_{k,\epsilon}(y)|}{\min_y |\omega_{m,\epsilon}(y)|} \Big((2b)^{-2} \max_z |\phi'(|z|/2b)|^2 + 1 \Big) ||u||^2_{H^1(E_{m,\epsilon})}. \tag{B.53}$$

I.e., for any ϕ, the difference in H^1-norms is bounded by the ratio of the volume of the fattened pages to the volume of the fattened binding in the $\epsilon \to 0$ limit. $\square$

B.1.9 Proof of Proposition 3.4.6

This proof is broken up into several statements.

In the thin junction cases several things can be observed: $E_{m,\epsilon}$ is still given by cross-sections. These cross-sections have diameter $O(\epsilon)$ and are star-shaped with respect to a ball of radius $O(\epsilon^\beta)$.

Corollary B.1.3. *Let $P_{m,\epsilon}$ (see Definitions 3.3.37 and 3.3.39) be analogous averaging operator for the thin junction domain . For $u \in H^1(E_{m,\epsilon})$, the averaging operator $P_{m,\epsilon}$ satisfies a Poincaré-type inequality:*

$$||u - P_{m,\epsilon} u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon^{4-2\beta} ||\nabla u||^2_{L_2(E_{m,\epsilon})}. \tag{B.54}$$

From Lemma 2.3.48 and Theorem 2.3.49, the Poincaré constant on a cross-section is proportional to D^2/δ where D is the diameter of the cross-section and δ is the diameter of the ball which it is star-shaped with respect to. In this case $D = O(\epsilon)$ and $\delta = O(\epsilon^\beta)$. $\square$

Lemma B.1.4. *Let M_ϵ be a thin junction type domain with $1 < \beta < 2$. For $u \in H^1(E_{m,\epsilon})$, one* *has:*

$$\epsilon^{-\beta}||T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})} + [T_{m,k,\epsilon}(u - P_{m,\epsilon}u)]^2_{\Gamma_{k,m,\epsilon}} \leq c\epsilon^{3-3\beta}||u||^2_{H^1(E_{m,\epsilon})}. \quad \text{(B.55)}$$

Proof: Begin with a partition of unity $\{\varphi_{i,\epsilon}\}$ subordinate to an open cover $\{U_{i,\epsilon}\}$ as described in Corollary 3.3.5. Observe that the thin junction fattened binding can be covered with balls of radius $O(\epsilon^{\beta})$ and the fattened binding has the properties described in Corollary 3.3.5. Let $\theta : x \to x/\epsilon^{\beta}$ be a homothetic transformation and Θ be the induced operator on functions. This proof begins with following the set up to the proof of Lemma 2.3.65. Let $\{U_{i,\epsilon}\}$ be a locally finite open cover of $\Gamma_{k,m,\epsilon}$ consisting of sets of diameter $O(\epsilon^{\beta})$. We also suppose if $U_{i,\epsilon} \cap U_{j,\epsilon}$ overlap in such a way that there exists a smooth partition of unity subordinate to $\{U_{i,\epsilon}\}$ such that each function has derivative uniformly bounded by $c_{\varphi}\epsilon^{-\beta}$. Noting the geometry of a cross-section looks like a rectangle of with $2\epsilon^{\beta}$ with height $O(\epsilon)$, we consider cylindrical sets $V_{i,\epsilon}$ with base $U_{i,\epsilon}$ of height $O(\epsilon)$ in $E_{m,\epsilon}$ (similar to the construction in the proof of Lemma 2.3.65). We suppose $\{V_{i,\epsilon}\}$ maintains a finite intersection property in the $\epsilon \to 0$ limit, i.e. at most c_U sets ever intersect. Let $\{\varphi_{i,\epsilon}\}$ be a partition of unity subordinate to $\{V_{i,\epsilon}\}$. Suppose each $V_{i,\epsilon}$ admits a geodesic coordinate system $x = (y, z_1, z_2)$ where the z_2 the distance from $\Gamma_{k,m,\epsilon}$. Let θ denote a non-uniform scale transformation $\theta : (y, z_1, z_2) \mapsto (y/\epsilon^{\beta}, z_1/\epsilon^{\beta}, z_2/\epsilon)$ and Θ denotes the induced operator on functions. Following previous calculations, we scale the trace norm and this time estimate the change in the derivative in each direction:

$$
\begin{aligned}
&\epsilon^{-\beta}||T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})}+[T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\Gamma_{k,m,\epsilon}}\\
&=\epsilon^{\beta}||\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\theta(\Gamma_{k,m,\epsilon}))}+\epsilon^{\beta}[\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\theta(\Gamma_{k,m,\epsilon})}\\
&\leq \epsilon^{\beta}\sum_i\Big(||\varphi_{i,\epsilon}(\theta)\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\theta(\Gamma_{k,m,\epsilon}))}+[\varphi_{i,\epsilon}(\theta)\Theta T_{m,k,\epsilon}(u-P_{m,\epsilon}u)]^2_{\theta(\Gamma_{k,m,\epsilon})}\Big)\\
&\leq \epsilon^{\beta}c_{em}\sum_i\big(||\varphi_{i,\epsilon}(\theta)\Theta(u-P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}(\theta)))}+||\nabla\varphi_{i,\epsilon}(\theta)\Theta(u-P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}(\theta)))}\big)\\
&\leq c_{em}\sum_i\big(\epsilon^{-\beta-1}||\varphi_{i,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}))}+\epsilon^{\beta-1}||\nabla_{(y,z_1)}\varphi_{i,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}))}\\
&+\epsilon^{1-\beta}||D_{z_2}\varphi_{i,\epsilon}(u-P_{m,\epsilon}u)||^2_{L_2(\mathsf{supp}(\varphi_{i,\epsilon}))}\big)\\
&\leq c_{em}\sum_i\epsilon^{3-3\beta}||\varphi_{i,\epsilon}(u-P_{m,\epsilon}u)||^2_{H^1(\mathsf{supp}(\varphi_{i,\epsilon}))}\\
&\leq \epsilon^{3-3\beta}c_{em}c_U(1+c_{\nabla\varphi})||(u-P_{m,\epsilon}u)||^2_{H^1(E_{m,\epsilon})}. \qquad \square
\end{aligned}
\tag{B.56}
$$

Definition B.1.5. *Let $1<\beta<2$. We define $N_{\Gamma,\epsilon}$ to be the averaging operator averaging over $\mathcal{I}_{\mathcal{N}_k,\epsilon}(y)$ intersecting $\Gamma_{k,m,\epsilon}$. I.e.*

$$N_{\Gamma,\epsilon}u(y)=\frac{1}{|\mathcal{I}_{\mathcal{N}_k(y),\epsilon}\cap\Gamma_{k,m,\epsilon}|}\int_{\mathcal{I}_{\mathcal{N}_k(y),\epsilon}\cap\Gamma_{k,m,\epsilon}}u(y,\zeta)d\zeta. \tag{B.57}$$

Lemma B.1.6. *For $u\in H^1(E_{m,\epsilon})$, one has:*

$$||P_{m,\epsilon}u-T_{k,m,\epsilon}N_{\Gamma,\epsilon}u||^2_{L_2(E_{m,\epsilon})}\leq c\epsilon^{4-2\beta}||u||^2_{H^1(E_{m,\epsilon})}. \tag{B.58}$$

Proof: Following similar calculations, we identify a normal fiber of $\Gamma_{k,m,\epsilon}$ (normal with respect to M_k) with a point on E_m by the boundary of the cross-section of E_m. We then get

$$
\begin{aligned}
||P_{m,\epsilon}u - N_{\Gamma,\epsilon}T_{m,k,\epsilon}u||^2_{L_2(\tilde{E}_{m,\epsilon})} \\
&= \int_{\varpi_{m,\epsilon}} d\omega_{m,\epsilon} \int_{E_m} |\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - N_{\Gamma,\epsilon}T_{m,k,\epsilon}u|^2 \, dE_m \\
&\leq \frac{\max_{y\in E_m} |\varpi_{m,\epsilon}(y)|}{\min |\mathcal{I}_{\mathcal{N}_k(y),\epsilon} \cap \Gamma_{k,m,\epsilon}|} \int_{E_m} |N_{\Gamma,\epsilon}T_{m,k,\epsilon}\left(\tilde{P}_{m,\epsilon}\Phi_{E_{m,\epsilon}}u - u\right)|^2 \, dE_m \\
&\leq \epsilon^{1+\beta}\big(\epsilon^{-\beta}||T_{m,k,\epsilon}(u - P_{m,\epsilon}u)||^2_{L_2(\Gamma_{k,m,\epsilon})}\big) \\
&\leq c\epsilon^{4-2\beta}||u||^2_{H^1(E_{m,\epsilon})}. \qquad \square
\end{aligned}
\tag{B.59}
$$

Lemma B.1.7. *For $u \in H^1(M_\epsilon)$, one has:*

$$
||N_{\Gamma,\epsilon}T_{m,k,\epsilon}u||^2_{L_2(E_{m,\epsilon})} \leq c\epsilon||u||^2_{H^1(M_{k,S,\epsilon})}. \tag{B.60}
$$

Proof: This requires only small modification of the proof of Lemma 2.3.65. Because $\epsilon^{-1}||u||^2_{L_2(\Gamma_{k,m,\epsilon})} \leq c||u||^2_{H^1(M_{k,S,\epsilon})}$, we only need to show

$$
||N_{\Gamma,\epsilon}T_{m,k,\epsilon}u||^2_{L_2(\Gamma_{k,m,\epsilon})} \leq ||u||^2_{L_2(\partial M_{k,S,\epsilon})}. \tag{B.61}
$$

This is a simple application of the Cauchy-Schwartz Inequality. $\square$

Returning to proving Proposition 3.4.6, we use the triangle inequality:

$$
\begin{aligned}
||u||_{L_2(E_{m,\epsilon})} &\leq ||P_{m,\epsilon}u - u||_{L_2(E_{m,\epsilon})} \\
&\quad + ||P_{m,\epsilon}u - N_{\Gamma,\epsilon}T_{k,m,\epsilon}u||^2_{L_2(E_{m,\epsilon})} + ||N_{\Gamma,\epsilon}T_{m,k,\epsilon}u||^2_{L_2(E_{m,\epsilon})}.
\end{aligned}
\tag{B.62}
$$

The upper bound estimate follows from Corollary B.1.3 and Lemmas B.1.6 and B.1.7. $\square$