REPRESENTATIONS OF THE NECKLACE BRAID GROUP

A Dissertation

by

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ABSTRACT

The necklace braid group $\mathcal{NB}_n$ is the motion group of the $n+1$ component necklace link $\mathcal{L}_n$ in Euclidean $\mathbb{R}^3$. The link $\mathcal{L}_n$ consists of n pairwise unlinked Euclidean circles each linked to an auxiliary circle. Partially motivated by physical considerations, we study representations of the necklace braid group $\mathcal{NB}_n$, especially those obtained as extensions of representations of the braid group $\mathcal{B}_n$. During this study, we show that any completely reducible $\mathcal{B}_n$ representation extends to $\mathcal{NB}_n$ in a standard way.

We also investigate non-standard extensions of several well-known $\mathcal{B}_n$-representations such as the Burau and Lawrence-Krammer-Bigelow representations. Moreover, we prove that any local representation of $\mathcal{B}_n$ (i.e. coming from a braided vector space) can be extended to $\mathcal{NB}_n$.

Motivated by the extensions of these local representations, we investigate local representations of $\mathcal{B}_n$ from the twisted tensor products of group algebras. We start by discussing the case of using the group $\mathbb{Z}_3 \times \mathbb{Z}_3$, and even give some explicit examples.

DEDICATION

This dissertation is dedicated to my family and their endless love and support throughout this endeavour.

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1. INTRODUCTION

Knot Theory owes much of its early development to Lord Kelvin and Tait's theory of the atom. Based off of the experiments of Helmholtz, they theorized that the atoms were knotted tubes of æther, and were distinguished by their knot type [15]. While this theory was dismissed, the tabulation of knots with few crossing by Tait arguably began modern knot theory.

Due to topological quantum computation [12, 17], the study of non-abelian statistics of anyons has become alluring. The exchanging of non-abelian anyons induces unitary representations of the braid group $\mathcal{B}_n$. This can yield braiding-only models for universal quantum computing. Thanks to the braid group acting faithfully on the fundamental group of the punctured plane, this is a mathematically rich theory. The well-studied theory of (2+1)-TQFTs can be used to systematically study such representations and their generalizations to mapping class groups of punctured surfaces of any genus.

The natural ambition is to extend these ideas to 3-dimensional topological materials. However, unlike (2+1)-TQFTs, (3+1)-TQFTs are not well-studied. Thus we can not obtain explicit descriptions as we could in the 2-dimensional case. When moving to 3-dimensions, we must be careful. This is due to the fact that the fundamental group of $\mathbb{R}^3$ with n points removed is trivial. However, excitations of loop or closed string particles naturally occur in condensed matter physics and string theory. The study of low dimensional local representations of the loop braid group, $\mathcal{LB}_n$, was done in [9, 3]. In Chapter 3 we instead will look at representations of the necklace braid group, $\mathcal{NB}_n$: the motion of n unlinked oriented circles that are linked to another auxiliary oriented circle (a visualization of $\mathcal{NB}_n$ is shown in Figure 1). This configuration is thought to be more physically feasible than the free loop setup.

In studying representations of $\mathcal{NB}_n$, it was discovered that local representations of the braid group will always extend to $\mathcal{NB}_n$. This is in contrast to the situation with $\mathcal{LB}_n$. This motivated looking at local representations of $\mathcal{B}_n$ (as they would also lead to local representations of $\mathcal{NB}_n$). One idea to pursue local representations comes from realizing the representations described in

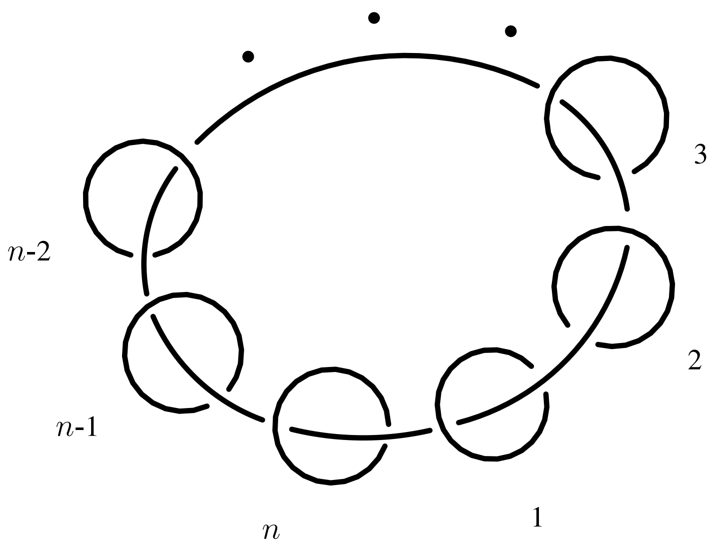

Figure 1.1: The necklace $\mathcal{L}_n$ as seen from a generic observation point

[13] and [6] as twisted tensors of group algebras. From this, we begin by trying to expand the representation in [6]. However, we show that its extension is not as compelling as one may hope. Therefore we consider a smaller case, the twisting of the group algebra $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]$ (denoted $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau n-1}$. We classify unitary representations of from $\mathcal{B}_n$ into $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau n-1}$, and then describe methods to then get a Braided Vector Space from these representations.

2. PRELIMINARIES

This chapter provides necessary definitions and properties in areas including representation and group theory that will be used throughout the dissertation.

For the remainder of the document, unless otherwise stated, V will denote a finite dimensional vector space. Let $\mathbf{e}_i$ be the ith standard basis vector for V (i.e. $\mathbf{e}_i$ has a 1 in the ith entry and a 0 elsewhere). We denote the group commutator with $[,]$, meaning, for elements g, h in a group G, $[g,h] = ghg^{-1}h^{-1}$.

2.1 Linear Representations

Let G be a group, and $GL(V)$ be the general linear group of V. A *linear representation of* G, is a pair (ρ, V), where $\rho : G \to GL(V)$ is a group homomorphism. This yields a natural action of G on V defined by $g.\mathbf{v} = \rho(g)\mathbf{v}$, for any $g \in G$, $\mathbf{v} \in V$. For ease of notation, we may omit the period in the G-action on V, simply writing $g\mathbf{v}$ instead of $g.\mathbf{v}$ (or $\rho(g)\mathbf{v}$). If the vector space V is clear, then often we denote (ρ, V) by just ρ. The *dimension of the representation* (ρ, V) is the dimension of V. If $W \subset V$ is a nontrivial proper subspace, and for all $g \in G$, $\rho(g)W \subset W$, then W is called *G-invariant*, and (ρ, W) is a *subrepresentation* of (ρ, V). The subrepresentation (ρ, W) is sometimes denoted $\rho|_W$. This notation is also used in a similar manner to describe a representation ρ of G that's had its domain restricted to a subgroup $H < G$, $\rho|_H$.

If the only G-invariant subspace of V are trivial (V and $\emptyset$), then (ρ, V) is an *irreducible representation*. Otherwise ρ is called *reducible*. A representation (ρ, V) is called *completely reducible* if it can be expressed as the direct sum of proper nonzero irreducible subrepresentations. Contrarily (ρ, V) is called *indecomposible* if and only if it can not be expressed as the direct sum of proper nonzero irreducible subrepresentations.

Example 2.1.1. *Consider the integers, $\mathbb{Z}$, as an additive group, and the representation $(\rho, \mathbb{C}^3)$,* *where ρ is defined by* $1 \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$*. Since subspace $W \subset \mathbb{C}^3$ spanned by the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ is $\mathbb{Z}$-invariant, ρ is reducible. However, notice that the orthogonal complement of W (which is the subspace spanned by $\mathbf{e}_3$) is not Z invariant, ρ is indecomposible.*

Example 2.1.2. *Similarly, the 2-dimensional subrepresentation (ρ, W) of (ρ, V), which is defined by* $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is also reducible and indecomposible. In fact, the only irreducible representations of $\mathbb{Z}$ are 1-dimensional.*

Example 2.1.3. *Let $\mathfrak{S}_3$ denote the symmetric group on 3 elements. The representation $(\rho, \mathbb{C}^2)$ defined by*

$$(1,2) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } (1,2,3) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

is an irreducible representation of $\mathfrak{S}_3$.

A representation (ρ, V) is a *unitary representation* if $\rho(g)$ is a unitary transformation for all $g \in G$. Note that the product of unitary transformations is also unitary, meaning that it is sufficient to check if $\rho(x)$ is unitary for all the generators x of G.

The following is a useful theorem:

Lemma 2.1.4 (Schur)**.** *If (ρ, V) and (ϕ, W) are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then*

1. *Either φ is an isomorphism, or $\varphi = 0$.*

2. *If $V = W$, then $\varphi = \lambda I_V$ for some scalar λ.*

2.2 The n-strand Braid Group

The n-strand braid group $\mathcal{B}_n$ is the finitely generated group, with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the following relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for all } i < n-1 \tag{2.2.1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1. \tag{2.2.2}$$

The relation (2.2.1) is often called the braid relation. We may also refer to the braid relations, and refer to the pair or relations (2.2.1) and (2.2.2). One notable element of $\mathcal{B}_n$, $\gamma = \sigma_1\sigma_2\cdots\sigma_{n-1}$ is called the *single twist*. It conjugates σ_i to σ_{i+1} (i.e. $\gamma\sigma_i\gamma^{-1} = \sigma_{i+1}$), therefore $\mathcal{B}_n$ can be generated by γ and σ_1, as $\gamma^k\sigma_1\gamma^{-k} = \sigma_{k+1}$ (for $1 \leq k < n-1$). It is also of interest as γ^n generates the center of $\mathcal{B}_n$.

2.2.1 A Few Famous Representations of $\mathcal{B}_n$

A few well studied representations of $\mathcal{B}_n$ that we will study include the *standard representation* β, the *reduced Burau representation* ϱ, and the *Lawrence-Krammer-Bigelow representation* (LKB). The first two are discussed in depth in [5] and [16], and the LKB is discussed in many papers, most notably in [2].

The standard representation is an n dimensional representation, and for $n \geq 3$, β is irreducible. β is defined as

$$\beta(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

where I_k is the kth dimensional identity, and $z \in \mathbb{C}\backslash\{0,1\}$. Notice that this is not irreducible in the $n = 2$ case, as $\sigma_1 = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$, and the subspace spanned by $\begin{pmatrix} \sqrt{z} \\ 1 \end{pmatrix}$ is fixed under the action of σ_1. One may also see that β is not irreducible for $n = 2$ by recalling that $\mathcal{B}_2 \cong \mathbb{Z}$, and the only irreducible representations of $\mathbb{Z}$ are one dimensional.

The reduced Burau representation is an $n-1$ dimensional representation ϱ, defined as:

$$\varrho(\sigma_1) = \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix} \oplus I_{n-3},\ \varrho(\sigma_{n-1}) = I_{n-3} \oplus \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix}$$

and

$$\varrho(\sigma_i) = I_{i-2} \oplus \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2} \text{ for } 1 < i < n-1$$

where t is a nonzero complex number. This representation is irreducible if $1+t+t^2+\cdots+t^{n-1} \neq 0$. The reduced Burau representation is the $n-1$ dimensional subrepresentation of the much studied n dimensional Burau representation. The Burau representation was thought to be a great candidate for showing that the braid group is linear (meaning it has a faithful representation). However it was shown to be faithful for $n = 3$, and unfaithful for $n > 4$. The Burau representation $\Phi : \mathcal{B}_n \to GL(V)$ is defined as $\Phi(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$, where $t \neq 0$. Notice that if $t = 1$, then $\Phi(\sigma_i)$ is just a permutation matrix. As stated, Φ is completely reducible. The fact that it is not irreducible can be quickly verified, as the $(n \times 1)$ vector of all 1's (in V) is fixed by $\Phi(\sigma_i)$ for all i. Sometimes Φ is referred to as the unreduced Burau, to distinguish it from the $n-1$ dimensional subrepresentation.

The next representation was shown to be faithful for $n \geq 3$ (and thus proving that $\mathcal{B}_n$ is a linear group). For the Lawrence-Krammer-Bigelow representation, let V be an $\binom{n}{2}$ dimensional vector space with basis $\mathbf{v}_{i,j}$ ($1 \leq i < j \leq n$). Assuming that the order of the indices do not matter (i.e. assuming that $\mathbf{v}_{i,j} = \mathbf{v}_{j,i}$), and letting t, q be two nonzero complex numbers, the LKB representation is defined as:

$$
\begin{aligned}
LKB(\sigma_i)\mathbf{v}_{i,i+1} &= tq^2\mathbf{v}_{i,i+1} && \\
LKB(\sigma_i)\mathbf{v}_{j,k} &= \mathbf{v}_{j,k} && \text{for } \{i,i+1\}\cap\{j,k\}=\emptyset \\
LKB(\sigma_i)\mathbf{v}_{i+1,j} &= \mathbf{v}_{i,j} && \text{for } j\neq i,i+1 \\
LKB(\sigma_i)\mathbf{v}_{i,j} &= tq(q-1)\mathbf{v}_{i,i+1}+(1-q)\mathbf{v}_{i,j}+q\mathbf{v}_{i+1,j} && \text{if } i+1<j \\
LKB(\sigma_i)\mathbf{v}_{j,i} &= (1-q)\mathbf{v}_{j,i}+q\mathbf{v}_{j,i+1}+q(q-1)\mathbf{v}_{i,i+1} && \text{if } j<i.
\end{aligned}
$$

As previously mentioned, LKB was used to show that $\mathcal{B}_n$ is a linear group [2]. It should also be noted that in general, this representation is irreducible. However, the following result explains when specializations of t and q make LKB reducible:

Theorem 2.2.1. *[11] Let $n \geq 3$ and assume that $q^k \neq 1$ for all integers $1 \leq k \leq n$. Then the LKB representation is reducible if*

$$t \in \left\{-1, \frac{1}{q}, \frac{1}{q^n}, \frac{1}{\sqrt{q}^n}, \frac{-1}{\sqrt{q}^n}\right\}.$$

2.2.2 Local Representations of $\mathcal{B}_n$

One source of representations of $\mathcal{B}_n$ is obtained from *braided vector spaces* (BVS). A BVS is a pair (R, V), where V is a vector space, and $R \in \mathrm{Aut}(V^{\otimes 2})$ satisfies the Yang-Baxer equation on $V^{\otimes 3}$:

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R).$$

A representation of $\mathcal{B}_n$ on $V^{\otimes n}$ is then yielded by assigning $\rho^R(\sigma_i) = I_V^{\otimes(i-1)} \otimes R \otimes I_V^{\otimes(n-i-1)}$. The reason this is called a *local representation* is due to the fact that each generator acts non-trivially only on two adjacent copies of V. Later when we extend this notion to $\mathcal{NB}_n$, we will consider the last and the first tensor copies of V in $V^{\otimes n}$ adjacent.

One notable property of local representations is the following:

Lemma 2.2.2. *[6, Lemma 2.1] If R has finite order, and $\rho^R(\mathcal{B}_n)$ is finite modulo its center Z^R, then $\rho^R(\mathcal{B}_n)$ itself is finite.*

An example of one such R is: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$. Next we introduce two local representations of $\mathcal{B}_n$ that come from first mapping the braid group into an algebra, and then obtaining a braided vector space from the algebra.

2.2.2.1 Gaussian Representations

The first local representation of $\mathcal{B}_n$ we introduce are Gaussian representations. They were first studied by Jones [8] and analyzed in [6]. As the matrices from these representations are rather cumbersome, we take a more algebraic approach.

Following [6], let $m \in \mathbb{N}$ and $q = \begin{cases} e^{2\pi i/m}, & \text{if } m \text{ odd} \\ e^{\pi i/m}, & \text{if } m \text{ even} \end{cases}$, and define $ES(m, n-1)$ as the algebra generated by $u_1, \ldots, u_{n-1}$ with the relations:

1. $u_i^m = 1$,
2. $[u_i, u_{i+1}] = q^2$,
3. $[u_i, u_j] = 1$ if $|i - j| > 1$.

Setting $\varphi_n(\sigma_i) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} q^{j^2} u_i^j$ defines a group homomorphism $\varphi_n : \mathcal{B}_n \to ES(m, n-1)^\times$. To get a braided vector space from $ES(m, n-1)$ it is enough to find a vector space V and $U \in \mathrm{Aut}(V^{\otimes 2})$ such that the map $u_i \mapsto I_V^{\otimes i-1} \otimes U \otimes I_V^{\otimes n-i-1}$ defines a representation of $ES(m, n-1)$ on $V^{\otimes n}$. Let $V \cong \mathbb{C}^m$, with standard basis $\{\mathbf{e}_i | 0 \le i \le m-1\}$. Define $\mathbf{e}_{i+m} = \mathbf{e}_i$, and $U \in \mathrm{End}(V^{\otimes 2})$ by $U(\mathbf{e}_i \otimes \mathbf{e}_j) = q^{j-i} \mathbf{e}_{i+1} \otimes \mathbf{e}_{j+1}$. In [6] it was shown that $u_i \mapsto U_i := I^{\otimes i-1} \otimes U \otimes I^{\otimes n-i-1}$ gives a $*$-algebra homomorphism (where $u_i^* = u_i^{-1}$) from $ES(m, n-1)$ to $\mathrm{End}(V^{\otimes n})$. It was also shown that $R := \dfrac{1}{\sqrt{m}} \displaystyle\sum_{j=0}^{m-1} q^{j^2} U^j$ is a unitary operator. Composing with φ_n we obtain a unitary representation $\phi_n : \mathcal{B}_n \to \mathrm{Aut}(V^{\otimes n})$, where $\phi_n(\sigma_i) = \dfrac{1}{\sqrt{m}} \sum_{j=0}^{m-1} q^{j^2} U_i^j$.

2.2.2.2 *Quaternionic Representation*

Similar to the subsection above, except this time following the presentation in [13], we define and algebra Q_n. For this representation, let $q = e^{2i\pi/6}$ (note that unlike before, q is fixed), and define Q_n to be the algebra generated by $u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}$ subject to the following relations:

1. $u_i^2 = v_i^2 = -1$ for all i,
2. $[u_i, v_j] = -1$ if $|i - j| < 2$,
3. $[u_i, v_j] = 1$ if $|i - j| \geq 2$,
4. $[u_i, u_j] = [v_i, v_j] = 1$.

The map $\upsilon : \mathcal{B}_n \to Q_n$ defined by $\sigma_i \mapsto \frac{-1}{2q}(1 + u_i + v_i + u_i v_i)$ gives a representation of $\mathcal{B}_n$ into Q_n. Unlike to the Gaussian case, Q_n does not have any obvious local representations. Instead, a 3-local representation (see [7, Theorem 5.28]).

2.3 The Necklace Braid Group

The necklace $\mathcal{L}_n$ (as shown in Figure 1), in $\mathbb{R}_3$, is the link of n oriented distinct circles $(L_1, \ldots, L_n)$ linked together by another auxiliary oriented circle (L_0). The *necklace braid group* is the fundamental group of the configuration space of $\mathcal{L}_n$. We denote this group as $\mathcal{NB}_n$. Similar to the case of $\mathcal{B}_n$, we let σ_i denote swapping objects in the ith and the $i + 1st$ spots. However in $\mathcal{NB}_n$, σ_i denotes passing the ith loop through the $i + 1$st. There are two elements of interest in $\mathcal{NB}_n$ that distinguish it from $\mathcal{B}_n$. The first is σ_n, which interchanges the nth and the 1st loops. Second is the generator denoted τ. The element τ^n can be thought of in two distinct ways. One way is by rotating L_0 (and thus the whole $\mathcal{L}_n$ system) in such a way that each of the loops $L_1, \ldots, L_n$ return to their original position. The other vision of τ^n can be seen as, first by assuming the loops $L_1, \ldots, L_n$ are sufficiently close, rotating the auxiliary loop L_0 around the other n. Figure 2.1,

Figure 2.1: The two realizations of τ^n
Reprinted from [1].

as seen in [1], helps visualize the two versions of τ^n. In the figure, the blue circles illustrate the rotations of L_0. Each realization is a full rotation about an axis, however we can continually swap between the two. For instance, we can alternate which axis τ is rotating around, and thus need to perform the operation τ $2n$ times to return to the original state.

The next theorem gives a presentation of $\mathcal{NB}_n$ by abstract generators and relations, cf. [1, Theorem 2.3].

Theorem 2.3.1 ([1]). *We have a presentation of a group isomorphic to $\mathcal{NB}_n$ by abstract generators $\sigma_1, \ldots, \sigma_n, \tau$ satisfying:*

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| \neq 1 \pmod{n}$*,*

(N1) $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ *for* $1 \leq i \leq n$

(N2) $\tau^{2n} = 1$

Where indices are taken modulo n, with $\sigma_{n+1} := \sigma_1$ and $\sigma_0 := \sigma_n$.

From this presentation, it is clear that the subgroup generated by the σ_i for $1 \leq i \leq n-1$ is a quotient of $\mathcal{B}_n$. As it turns out, both $(N1)$ and $(N2)$ do not induce any further relations among $\sigma_1, \ldots, \sigma_{n-1}$. Therefore, we in fact have $\mathcal{B}_n < \mathcal{NB}_n$.

The presentation also gives us that $\mathcal{NB}_n$ has a normal subgroup isomorphic to the *affine braid group* (of type A) on n strands $\mathcal{B}\tilde{A}_n \cong \langle \sigma_1, \ldots, \sigma_n \rangle \lhd \mathcal{NB}_n$. These facts, along with $[\mathcal{NB}_n :$

$\mathcal{B}\tilde{A}_n] = 2n$ and $\mathcal{NB}_n = \mathcal{B}\tilde{A}_n \rtimes \langle\tau\rangle$, were shown in [1]. From $\mathcal{NB}_n = \mathcal{B}\tilde{A}_n \rtimes \langle\tau\rangle$, we can write any element of $\mathcal{NB}_n$ as $\tau^k\beta$, where $\beta \in \mathcal{B}\tilde{A}_n$.

As it turns out, some of the relations (B1-N2) for $\mathcal{NB}_n$ are redundant. The below reduces the number of defining relations from $\frac{1}{2}n(n+1)+1$ to $2n-1$:

Lemma 2.3.2. *The relations (N1),(N2), (B1) for i=1 (i.e.* $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$*), and (B2) for* $i = 1$ *and* $3 \leq j \leq n-1$ *(i.e.* $\sigma_1\sigma_j = \sigma_j\sigma_1$ *for* $3 \leq j \leq n-1$*), imply all the relations of Theorem 2.3.1.*

Proof. To prove the lemma, we must show that all of (B1) and (B2) are satisfied for all i, j. First we verify (B1). Assuming (N1) gives us $\tau^{i-1}\sigma_1\tau^{-i+1} = \sigma_i$ for all i where indices are taken modulo n. Thus $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies that for any i:

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= (\tau^{i-1}\sigma_1\tau^{-i+1})(\tau^{i}\sigma_1\tau^{-i})(\tau^{i-1}\sigma_1\tau^{-i+1}) \\
&= \tau^{i-1}\sigma_1\tau^{1}\sigma_1\tau^{-1}\sigma_1\tau^{-i+1} \\
&= \tau^{i-1}\sigma_1\sigma_2\sigma_1\tau^{-i+1} \\
&= \tau^{i-1}\sigma_2\sigma_1\sigma_2\tau^{-i+1} \\
&= \tau^{i}\sigma_1\tau^{-1}\sigma_1\tau^{1}\sigma_1\tau^{-i} \\
&= (\tau^{i}\sigma_1\tau^{-i})(\tau^{i-1}\sigma_1\tau^{-i+1})(\tau^{i}\sigma_1\tau^{-i}) \\
&= \sigma_{i+1}\sigma_i\sigma_{i+1}.
\end{aligned}$$

To verify (B2), assume σ_1 commutes with σ_k with $3 \leq k \leq n-1$. We may assume $n \geq j > i > 1$ and $|j-i| \not\equiv 1 \pmod{n}$.

$$\begin{aligned}
\sigma_i\sigma_j &= \tau^{i-1}\sigma_1\tau^{-i+1}\tau^{i-1}\sigma_{j-i+1}\tau^{-i+1} \\
&= \tau^{i-1}\sigma_1\sigma_{j-i+1}\tau^{-i+1} \\
&= \tau^{i-1}\sigma_{j-i+1}\sigma_1\tau^{-i+1} \\
&= \tau^{i-1}\sigma_{j-i+1}\tau^{-i+1}\tau^{i-1}\sigma_1\tau^{-i+1} = \sigma_j\sigma_i
\end{aligned}$$

$\square$

While the reduction in the number of needed relations may not necessarily make it easier to understand $\mathcal{NB}_n$, it does make discovering representations of $\mathcal{NB}_n$ more manageable. Observe that for $n = 2$, $\mathcal{B}_2 \cong \mathbb{Z}$, and thus is less interesting than the $n \geq 3$ cases. However, for $\mathcal{NB}_n$ it is a different story. In fact $\mathcal{NB}_2 = \langle \sigma_1, \sigma_2, \tau \rangle$ with relations $\tau^4 = 1$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, $\tau\sigma_1\tau^{-1} = \sigma_2$, and $\tau\sigma_2\tau^{-1} = \sigma_1$, and therefore $\mathcal{NB}_2 \ncong \mathbb{Z}$. This means representations of $\mathcal{NB}_2$ may be of interest (in contrast to those of $\mathcal{B}_2$).

3. REPRESENTATIONS OF THE NECKLACE BRAID GROUP

In this chapter we will discuss representations of the necklace braid group. First will be a discussion of constructing representations from the ordinary braid group $\mathcal{B}_n$. Then we will discuss extending known representations of $\mathcal{B}_n$ to representations of $\mathcal{NB}_n$. Lastly we discuss low dimensional representations of $\mathcal{NB}_n$. Unless stated otherwise, V is a finite dimensional vector space, and I_V be the identity automorphism on V. Also, in showing that something is a representation of $\mathcal{NB}_n$, we will take advantage of the reduced set of relations as in Lemma 2.3.2. This chapter follows and includes many ideas from [4].

3.1 A Standard Extension

As stated in Section 2.3, the braid group on n strands, $\mathcal{B}_n$, is isomorphic to the subgroup of $\mathcal{NB}_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$. Leaving out the generators σ_n and τ means that the indices in (B1) and (B2) need not me considered modulo n. Also, since $\mathcal{B}_n < \mathcal{NB}_n$, any representation of $\mathcal{NB}_n$ will restrict to a representation of $\mathcal{B}_n$. Recall the element of $\mathcal{B}_n$, the single twist $\gamma = \sigma_1 \cdots \sigma_{n-1}$. As previously stated, $\gamma^k \sigma_1 \gamma^{-k} = \sigma_{k+1}$ for $1 \leq k \leq n-2$. If we were to define $\bar{\sigma}_n := \gamma^{n-1}\sigma_1\gamma^{1-n} = \gamma\sigma_{n-1}\gamma^{-1}$, then we have $\sigma_1, \ldots, \sigma_{n-1}, \bar{\sigma}_n$ will satisfy the relations (B1) and (B2) in 2.3.1. Also setting $\tau = \gamma$ will give us (N1), however, (N2) would not hold in general. However, this general set up will be advantageous to work with when producing representations of $\mathcal{NB}_n$.

Lemma 3.1.1. *Let $\rho : \mathcal{B}_n \to GL(V)$ be any finite-dimensional representation of $\mathcal{B}_n$ such that $\rho(\gamma^{2n}) = C_\rho I_V$ for some scalar C_ρ. Then ρ extends to an indecomposable representation of $\mathcal{NB}_n$ by $\rho(\sigma_n) = \rho(\gamma\sigma_{n-1}\gamma^{-1})$ and $\rho(\tau) = (C_\rho)^{-1/2n}\rho(\gamma)$.*

Proof. As shown in Lemma 2.3.2, we need only check (N1), (N2), $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and $\sigma_1\sigma_j = \sigma_j\sigma_1$ (for $3 \leq j \leq n-2$). The later two relations are satisfied, as ρ is defined to be a representation of $\mathcal{B}_n$. The fact that γ^n is central (in $\mathcal{B}_n$) provides

$$\begin{aligned}
\rho(\tau\sigma_n\tau^{-1}) &= \rho(\tau(\tau^{n-1}\sigma_1\tau^{1-n})\tau^{-1}) \\
&= \rho(\tau^n)\rho(\sigma_1)\rho(\tau^{-n}) = \left(C_p^{-1/2}\rho(\gamma^n)\right)\rho(\sigma_1)\left(C_p^{1/2}\rho(\gamma^{-n})\right) \\
&= \rho(\gamma^n\sigma_1\gamma^{-n}) = \rho(\sigma_1),
\end{aligned}$$

and therefore (N1) holds. For (N2), we note:

$$\rho(\tau^{2n}) = ((C_\rho)^{-1/2n}\rho(\gamma))^{2n} = \frac{1}{C_\rho}\rho(\gamma^{2n}) = I_V.$$

$\square$

Since the braid relations, (B1) and (B2), are homogeneous we can rescale the images $\rho(\sigma_i)$ by some nonzero a and get a new representation of $\mathcal{B}_n$ defined as $\bar{\rho}(\sigma_i) = a\rho(\sigma_i)$. By defining $\bar{\rho}(\tau) = \bar{\rho}(\gamma)$, the rescaling will not affect (B1), (B2), or (N1), however, a can be chosen to cancel the scalar with $\rho(\gamma^{2n})$.

Now consider ρ to be any finite dimensional representation of $\mathcal{B}_n$ and let ρ_i be indecomposable representations (also of $\mathcal{B}_n$) such that $\rho = \bigoplus_i \rho_i$. There exists a projection of ρ onto each of the ρ_i in $\mathrm{End}_{\mathcal{B}_n}(V)$. Summing each projection rescaled (as in Lemma 3.1.1) will give $\rho(\tau)$, and therefore a representation of $\mathcal{NB}_n$. In particular, we have the following:

Theorem 3.1.2. *Let $\rho : \mathcal{B}_n \to GL(V)$ be any completely reducible complex representation of $\mathcal{B}_n$. Then there exists a $D \in \mathrm{End}_{\mathcal{B}_n}(V)$ such that defining*

$$\rho(\tau) = D\rho(\gamma), \quad \rho(\sigma_n) = \rho(\gamma\sigma_{n-1}\gamma^{-1})$$

gives a representation of $\mathcal{NB}_n$.

Proof. By complete reducibility of ρ, there exist a collection W_{ii} nontrivial subspace of V such that $V \cong \bigoplus_i W_i$. Let ρ_{W_i} denote the projections of ρ onto W_i. Since γ^{2n} is a central element of $\mathcal{B}_n$, we have $\rho_{W_i}(\gamma^{2n}) = C_i I_{W_i}$ for some $C_i \in \mathbb{C}$. Define $D = \bigoplus_i (C_i)^{-1/2n} I_{W_i} \in \mathrm{End}_{\mathcal{B}_n}(V)$. This means that $(D\rho(\gamma))^{2n} = I_V$. Also note that D commutes with $\rho(\gamma)$ and $\rho(\sigma_i)$ for $1 \leq i \leq n-1$. Therefore defining $\rho(\tau) = D\rho(\gamma)$ and $\rho(\sigma_n) = \rho(\gamma\sigma_{n-1}\gamma^{-1})$ extends ρ to a representation of $\mathcal{NB}_n$ as desired. $\square$

Remark 3.1.3. Most notably, Theorem 3.1.2 applies to any irreducible or unitary representation of $\mathcal{B}_n$.

The notions in the above theorem resemble that of a standard extension found in [3]. Because of this we will adopt the same nomenclature, and call any representation ρ of $\mathcal{NB}_n$ with $\rho(\tau) = A\rho(\gamma)$ with $[A, \rho(\sigma_i)] = 1$ for all i, a *standard extension*. Note that A in this case is already in the image $\rho(\mathcal{NB}_n)$, as $A = \rho(\tau\gamma^{-1})$. This means that $\rho(\mathcal{NB}_n)$ can be generated by A and $\rho(\mathcal{B}_n)$. It should also be noted that standard extensions do not fully capitalize on the (3+1)-dimensional structure of $\mathcal{NB}_n$. Instead, the intriguing information contained in standard extensions are already present in $\mathcal{B}_n$, which pertains to (2+1)-dimensional topology.

Remark 3.1.4. *Schur's Lemma gives us that a standard extension of an irreducible representation of $\mathcal{B}_n$ has the form $\rho(\tau) = \lambda\rho(\gamma)$ for some scalar λ.*

Remark 3.1.5. *One observation is that if a representation is not completely reducible, it does not imply that it has no standard extension. For example: The $\mathcal{B}_n$ representation defined as $\rho(\sigma_i) = J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for all i is not completely reducible (as $\langle \mathrm{e}_1 \rangle$ is $\mathcal{B}_n$ invariant, but $\langle \mathrm{e}_2 \rangle$ is not). Observe that $\rho(\gamma) = J^{n-1}$ and J^{1-n} commutes with $\rho(\sigma_i)$ for all i. Hence $\rho(\tau) = I_2 = (J^{1-n})\rho(\gamma)$ is a standard extension of ρ.*

Another observation worth making is that, while Theorem 3.1.2 provides existence of standard extensions, it does not provide a way to construct these extensions. In this context, a computational distinction is made between each value of n, and construction that begins with a braid group representation for all n (such as the reduced Burau representation) that then determines a D for each n.

In the next few sections, we will present concrete examples of representations of $\mathcal{NB}_n$ acquired by extending well-studied representations of $\mathcal{B}_n$ (as defined in Section 2.2.1).

3.2 The Standard Representation

This section will require a bit of vigilance from the reader, as the nomenclature means we will now discuss standard (and non-standard) extensions of the standard representation. Recall the *standard representation* (β, V) is n dimensional, and defined as $\beta(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$.

Proposition 3.2.1. *For $n \geq 3$ any standard extension of the standard representation, β from $\mathcal{B}_n$ to a representation ρ of $\mathcal{NB}_n$ is of the form $\rho(\tau) = \lambda\beta(\gamma)$, where $\lambda \in \mathbb{C}$ such that $\lambda^{2n} = z^{-2(n-1)}$.*

Proof. From the fact that the standard representation is irreducible for $n \geq 3$ [16, Lemmas 5.3 and 5.4], we have that $\rho(\tau) = \lambda\beta(\gamma)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Next, observe that

$$I_n = \rho(\tau^{2n}) = (\rho(\tau))^{2n} = (\lambda\beta(\gamma))^{2n} = \lambda^{2n}(\beta(\gamma))^{2n},$$

and $(\beta(\gamma))^{2n} = z^{2(n-1)}I_n$. Hence $\lambda^{2n} = z^{-2(n-1)}$ as desired. $\square$

As mentioned in Chapter 2, when $n = 2$ also deserves investigation. Notice that in this case, $\gamma = \sigma_1$. Let $Z = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$. In this case, we want $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AZ = ZA$, $(AZ)^4 = I_2$, and $(AZ)^2Z(AZ)^{-2} = Z$. The last relation comes from the relation $\tau\sigma_2\tau^{-1} = \sigma_1$, and we are defining the image of σ_2 to be the image of $\tau\sigma_1\tau^{-1}$. The first relation ($AZ = ZA$) gives us that $a = d$ and $b = zc$. This, along with the other relations, we get the following possibilities for $\rho(\tau) = AZ$:

$$\left\{ \pm I_2, \pm \begin{pmatrix} 1 & 0 \\ 0 & i(z)^{-1} \end{pmatrix}, \xi_4 \begin{pmatrix} 0 & \sqrt{z} \\ (\sqrt{z})^{-1} & 0 \end{pmatrix}, \pm\frac{1}{2} \begin{pmatrix} 1 \pm i & (1 \mp i)\sqrt{z} \\ (1 \mp i)\sqrt{z}^{-1} & 1 \pm i \end{pmatrix} \right\},$$

where ξ_4 is a choice of 4th root of unity.

Next we examine non-standard extensions of β. In particular we have the following:

Theorem 3.2.2. *For $n \geq 3$, a representation, ϕ, of $\mathcal{NB}_n$ is an extension of the standard representation, β, of $\mathcal{B}_n$ if $\phi(\sigma_i) = \beta(\sigma_i)$ for $i = 1, \ldots, n-1$, $\phi(\tau) = \begin{pmatrix} 0 & t^{-n+1} \\ tI_{n-1} & 0 \end{pmatrix}$ (with $t \neq 0$), and $\phi(\sigma_n) = \phi(\tau)\beta(\sigma_{n-1})\phi(\tau^{-1})$.*

If $t^{2n} = z^{-2(n-1)}$, then the representation ϕ is a standard extension of β, i.e. the image of τ is a rescaling of the image of the single twist γ.

Proof. Let (ϕ, V) be a representation of $\mathcal{NB}_n$, along with $T \in \mathrm{End}(V)$ such that $\phi(\tau) = T$, and $T \neq \lambda\beta(\gamma)$ (for any choice of λ described in Proposition 3.2.1). This means that T must satisfy: $T^{2n} = I_V$, $T\beta(\sigma_i) = \beta(\sigma_{i+1})T$ for all $1 \leq i \leq n-2$, and that $T^2\beta(\sigma_{n-1}) = \beta(\sigma_1)T^2$. Let $T = (t_{i,j})_{i,j=1}^n$. The later two relations give us $t_{i,j} = 0$ if $j \not\equiv i - 1 \mod n$ and $t_{i,i-1} = t_{2,1}$ for $i = 2, \ldots, n-1$. This means that T has the following block form:

$$T = \begin{pmatrix} 0 & a \\ tI_{n-1} & 0 \end{pmatrix}.$$

From $T^{2n} = I_V$, we get that $a = t^{-n+1}$, meaning that $T = \begin{pmatrix} 0 & t^{-n+1} \\ tI_{n-1} & 0 \end{pmatrix}$. By Proposition 3.1.2 if $t^{2n} = z^{-2(n-1)}$, then we would have a standard extension of β. And therefore, a nonstandard extension of β would be when $t^{2n} \neq z^{-2(n-1)}$. $\square$

3.3 The Reduced Burau Representation

Recall that the reduced Burau representation, ϱ, is irreducible if $1+t+\cdots+t^{n-1} \neq 0$. Meaning that any standard extension ρ of ϱ, is one where $\rho(\tau) = \lambda\varrho(\gamma)$.

Proposition 3.3.1. *Any standard extension ρ of $\mathcal{NB}_n$ of the reduced Burau representation (with $1 + t + \cdots + t^{n-1} \neq 0$), ϱ has the form $\rho(\tau) = \lambda\varrho(\gamma)$ with $\lambda^{2n} = t^{-2n}$.*

Proof. From the reduced Burau representation being irreducible, any standard extension ρ of ϱ will send $\rho(\tau) = \lambda\varrho(\gamma)$ for some scalar λ. It can be computed that $\varrho(\gamma)^{2n} = t^{2n}I_V$. Therefore, from $I_V = \rho(\tau)^{2n} = (\lambda\varrho(\gamma))^{2n} = \lambda^{2n}\varrho(\gamma)^{2n}$, we get $\lambda^{2n} = t^{-2n}$ as desired. $\square$

3.4 The Lawrence-Krammer-Bigelow Representation

As in the definition of the Lawrence-Krammer-Bigelow, LKB, representation, let V be a $\binom{n}{2}$ dimensional vector space with basis $v_{i,j}$ $(1 \leq i, j \leq n)$ where the order of the indices (on the basis elements) does not matter. Also let t, q be the nonzero complex numbers in the definition of LKB. Recall in Theorem 2.2.1, there were specializations of t and q that would lead to LKB being reducible. The following proposition is stated for when t and q are left as generic scalars (and thus the LKB is irreducible).

Proposition 3.4.1. *Any $\binom{n}{2}$ dimensional representation, ρ, of $\mathcal{NB}_n$ is a standard extension of LKB if $\rho(\tau) = \lambda LKB(\gamma)$, where $\lambda = \omega_{2n}(t^{-1/n}q^{-2})$, and ω_{2n} is a choise of $2n$th root of unity.*

Proof. From our definition, one can compute that $LBK(\gamma)v_{i,j} = \begin{cases} tq^2 v_{i,i+1} & \text{if } j = n \\ q^2 v_{i+1,j+1} & \text{if } j < n \end{cases}$. Thus further computation gives us that $LBK(\gamma^{2n})v_{i,j} = t^2 q^{4n} v_{i,j}$, and thus $\lambda = \omega_{2n} t^{-1/n} q^{-2}$ as desired. $\square$

Remark 3.4.2. *If ω_{2n} is a nth root of unity, then τ^n is in the kernel of the standard extension, and therefore it would not be a faithful representation of $\mathcal{NB}_n$.*

3.4.1 Nonstandard Extensions of LKB

For $n = 2$, the LKB (and also any extension to $\mathcal{NB}_n$) is a 1 dimensional representation. Therefore, any extension to $\mathcal{NB}_n$ would be standard.

The case where $n = 3$ gives us the first nonstandard extension. Let α be a choice of cube root of $\pm t^{-1}$. The following gives a nonstandard extension of the LKB representation:

$$\tau \mapsto \alpha \begin{pmatrix} 0 & (q^2 - q + 1)q^{-2} & -(q-1)q^{-2} \\ 0 & -(q-1)q^{-1} & q^{-1} \\ tq^2 & (q-1)(tq^2 - q + 1)q^{-1} & (q-1)q^{-1} \end{pmatrix}.$$

If α is chosen to be the positive cube root of t^{-1}, then the extension is not faithful, as t^3 would be in the kernel.

For $n = 4$, again we assume that $q \neq 1$, and let $\alpha \in \{\pm\sqrt{\pm t}, (-t^2)^{\frac{1}{4}}\}$ and $p = q - 1$. With this, the following gives us an extension of LKB that is nonstandard:

$$\tau \mapsto \alpha \begin{pmatrix} 0 & 0 & (q^4t)^{-1}(q^3 - q + 1) & 0 & -(q^4t)^{-1}p & -(q^4t)^{-1}p \\ 0 & 0 & -(q^3t)^{-1}p & 0 & (q^3t)^{-1}(q^2 - q + 1) & -(q^3t)^{-1}p \\ 0 & 0 & -(q^2t)^{-1}p & 0 & -(q^2t)^{-1}p & (q^2t)^{-1} \\ q^2 & 0 & (q^3t)^{-1}p(q^3t - q + 1) & 0 & (q^3t)^{-1}(q^3 - 2q^2 + 2q - 1) & -(q^3t)^{-1}p^2 \\ 0 & q^2 & (q^2t)^{-1}p(q^3t - q + 1) & 0 & -(q^2t)^{-1}p^2 & (q^2t)^{-1}p \\ 0 & 0 & -(qt)^{-1}p^2 & q^2 & (qt)^{-1}(q^3t - q^2(t+1) + 2q - 1) & (qt)^{-1}p \end{pmatrix}.$$

Similar to the previous case, if $\alpha = \pm\sqrt{\pm t}$, then τ^4 is in the kernel of the extension.

3.5 Extending Irreducible Representation Gaps

There are gaps in the irreducible representation degrees of $\mathcal{B}_n$ (see [10] and references therein): for example $\mathcal{B}_n$ has no irreducible representations of dimension $2 \leq d \leq n - 3$, for $n \geq 5$. In this section, we show that these gaps extend to $\mathcal{NB}_n$. We first prove a useful lemma.

Lemma 3.5.1. *Let (ρ, V) to be an irreducible $\mathcal{NB}_n$ representation, $w \in V$, and $0 < \alpha$ be minimal such that $\tau^{-\alpha}w \in span\{\tau^{-\gamma}\mathbf{w} | 0 \leq \gamma < \alpha\}$. Then $\tau^{-(\alpha+1)}\mathbf{w} \in span\{\tau^{-\gamma}\mathbf{w} | 0 \leq \gamma < \alpha\}$.*

Proof. We have that $\tau^{-\alpha}\mathbf{w} = \sum_{i=0}^{\alpha-1} a_i\tau^{-i}\mathbf{w}$ for a_i scalars. Then

$$\begin{aligned} \tau^{-(\alpha+1)}\mathbf{w} = \tau^{-1}(\tau^{-\alpha}\mathbf{w}) &= \tau^{-1}\left(\sum_{i=0}^{\alpha-1} a_i\tau^{-i}\mathbf{w}\right) \\ &= \sum_{i=0}^{\alpha-1} a_i\tau^{-i-1}\mathbf{w} \\ &= \left(\sum_{i=1}^{\alpha-1} a_{i-1}\tau^{-i}\mathbf{w}\right) + \tau^{-\alpha}\mathbf{w} \end{aligned}$$

which is in the span of $\{\tau^{-\gamma}\mathbf{w} | 0 \leq \gamma < \alpha\}$. $\square$

Using this lemma, we can prove the following.

Theorem 3.5.2. *Let $n \geq 5$ and (ρ, V) be an irreducible $\mathcal{NB}_n$ representation with $\dim V = n-2$, then $\phi = \rho|_{\mathcal{B}_n}$ is an irreducible representation of $\mathcal{B}_n$.*

Proof. Assume that (ρ, V) is an irreducible $\mathcal{NB}_n$ representation and to the contrary that $\rho|_{\mathcal{B}_n}$ is not irreducible. So we have the existence a proper nonempty subspace W (of minimal dimension, $1 \leq d < n-2$) of V such that $(\phi|_W, W)$ is a $\mathcal{B}_n$ representation. Note that W being minimal dimension guarantees that $(\phi|_W, W)$ is irreducible. Since $n \geq 5$, we have that the only irreducible representations of $\mathcal{B}_n$ of dimension strictly less than $n-2$ are 1-dimensional. Let W be spanned by the vector $\mathbf{w} \in V$. Since W is a $\mathcal{B}_n$ invariant space, we have that $\sigma_i \mathbf{w} = \lambda_i \mathbf{w}$ for all $1 \leq i \leq n-1$ (i.e. $\mathbf{w}$ is an eigenvector for all $\rho(\sigma_i)$). As all the σ_i are conjugate to each other, we have that $\mathbf{w}$ is an eigenvector of the same eigenvalue. From $\tau^{2n} = 1$ and the assumption that V is irreducible, we also have that $\rho(\tau)^{\pm n} = \pm 1$. Note that, the subset of V $M = \{\mathbf{w}, \tau^{-1}\mathbf{w}, \cdots, \tau^{-n}\mathbf{w}\}$ is linearly dependent and that the span of M is τ invariant. We may extract a basis $\beta = \{\tau^{-\alpha_1}\mathbf{w}, \tau^{-\alpha_2}\mathbf{w}, \ldots, \tau^{-\alpha_k}\mathbf{w} | 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n\}$ for $Q = span(M)$. Since Q is τ invariant, we may instead use the basis $\beta' = \tau^{\alpha_1}\beta$. This gives us that $\mathbf{w} \in \beta'$. From the above lemma, we obtain $\beta' = \{\mathbf{w}, \tau^{-1}\mathbf{w}, \ldots, \tau^{-\alpha}\mathbf{w}\}$ (where $\alpha < n-2$) is a basis for Q. The restriction that $\alpha < n-2$ comes from the fact that Q is a subspace of V, and therefore $\dim Q \leq n-2$.

Let $0 \leq x \leq \alpha < n-2$. This means that $\sigma_{x+1} \in \mathcal{B}_n$. From the relation $\tau^x \sigma_1 \tau^{-x} = \sigma_{x+1}$, we get that $\sigma_1 \tau^{-x}\mathbf{w} = \tau^{-x}\sigma_{x+1}\mathbf{w} = \lambda(\tau^x \mathbf{w})$. Consequently Q is also σ_1 invariant. Hence Q is both τ and σ_1 invariant, and therefore $\mathcal{NB}_n$ invariant. Since V was irreducible, and Q is a $\mathcal{NB}_n$ invariant subspace of V, we have that $Q = V$. This means that β' is also a basis for V, and in this basis, $\rho(\sigma_1) = \lambda \cdot I_V$. Hence

$$\rho(\sigma_2) = \rho(\tau)\rho(\sigma_1)\rho(\tau^{-1}) = \lambda\rho(\tau)\rho(\tau^{-1}) = \lambda \cdot I_V = \rho(\sigma_1).$$

This gives us that $\rho(\tau)\rho(\sigma_1) = \rho(\sigma_1)\rho(\tau)$. Which gives us a contradiction that (ρ, V) is not an irreducible $\mathcal{NB}_n$ representation. $\square$

We can now show that $\mathcal{NB}_n$ also has gaps in its irreducible representation degrees:

Corollary 3.5.3. *For $n \geq 5$, the only irreducible representations of $\mathcal{NB}_n$ of dimension at most $n-3$ are 1-dimensional.*

Proof. Assume to the contrary that (ρ, V) is an irreducible representation of $\mathcal{NB}_n$ with $2 \leq \dim V < n-2$. Since $\mathcal{B}_n$ has no irreducible representations of dimension between 2 and $n-3$, $\rho|_{\mathcal{B}_n}$ can not be an irreducible $\mathcal{B}_n$ representation. Hence there exists a 1 dimensional subrepresentaion of $\rho|_{\mathcal{B}_n}$. Following the proof of Theorem 3.5.2, we would get that $\rho(\mathcal{NB}_n)$ is abelian, and therefore not irreducible. □

The previous theorem and corollary provide the question: Can we say something similar for other dimensions? The following theorem, and the remark thereafter, attempt to answer this.

Theorem 3.5.4. *Let $n \geq 5$ and (ρ, V) be an irreducible $\mathcal{NB}_n$ representation with $\dim V = n-1$. If $\rho|_{\mathcal{B}_n}$ is completely reducible, then $\rho|_{\mathcal{B}_n}$ is also an irreducible $\mathcal{B}_n$ representation.*

Proof. Assume to the contrary that $\rho|_{\mathcal{B}_n}$ is completely reducible and not irreducible. Then we have two possibilities, $V = \bigoplus_{i=1}^{n-1} W_i$ or $V = W \oplus U$ where W, W_i are all 1-dimensional subrepresentations, and U is an n-2 irreducible subrepresentation of V for $\mathcal{B}_n$. In either case, we have the existence of a 1-dimensional subrepresentation. From here, we follow the proof of Theorem 3.5.2, and note that the inequality that $\alpha < n-2$ becomes $\alpha \leq n-2$. This change in equality still ensures that Q is σ_1 invariant, because again, $\sigma_{n-1} = \tau^{n-2}\sigma_1\tau^{-n+2}$. So again, we would get the contradiction that (ρ, V) is not an irreducible representation of $\mathcal{NB}_n$. □

Remark 3.5.5. This idea, however, halts at $\dim V = n$. Recall the definition of the unreduced Burau representation Φ, and consider the mapping $\tau \mapsto \begin{pmatrix} 0 & a^{n-1} \\ \frac{1}{a}I_{n-1} & 0 \end{pmatrix}$. It can be checked that $\Phi(\tau^{-1}\sigma_i\tau) = \Phi(\sigma_{i+1})$ and $\Phi(\tau)^{2n} = I_n$. Thus we obtain a representation of $\mathcal{NB}_n$. Note that any invariant subspace of $\Phi(\mathcal{NB}_n)$ will also be invariant under $\Phi(\mathcal{B}_n)$ (but not necessarily vise versa). It is known that the Burau representation is reducible with invariant subspaces of dimension 1 and

$n-1$. The 1 dimensional subrepresentation is spanned by the vector of 1's. The other is the subspace of $\mathbb{C}^n$ of all vectors whose entries sum to 0. If $a \neq 1$, then we get that the vector of $1'$s is not fixed by τ. If $a^n \neq 1$, then we get that τ does not fix the $n-1$-dimensional subspace. This means that if $\mathcal{NB}_n$ has no invariant subspaces. Therefore if $a^n \neq 1$, then the extension of the Burau representation described above is an irreducible representation of $\mathcal{NB}_n$ whose restriction to $\mathcal{B}_n$ is reducible. As this shows, there exist irreducible representations of $\mathcal{NB}_n$ of dimension n whose restriction to $\mathcal{B}_n$ is no longer irreducible.

3.6 Irreducible Representations of dimension 2

From the fact that τ has order $2n$, we may assume that we have chosen a basis for V such that $\rho(\tau) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ where t_1, t_2 are $2n^{\text{th}}$ roots of unity. As stated before, for all $n \geq 5$, there are no irreducible 2 dimensional representations. This means we need only consider $n = 2, 3$, and 4. Since we are wanting irreducible reps, we have that $t_1 \neq t_2$. Similarly we have that $\rho(\sigma_1)$ is not upper or lower triangular, as otherwise $(1, 0)$ (or $(0, 1)$ repspectively) would generate a $\mathcal{NB}_n$ invariant subspace. Due to rescaling, we may assume that $\rho(\sigma_1) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$. Since we wish the image of σ_i to not be triangular, we see that that $c \neq 0$. Wanting our representations to be irreducible, we also have at least one of a or d are nonzero. Otherwise, if both are zero, then we could have a subspace

Proposition 3.6.1. *Any irreducible dimension 2 representation of $\mathcal{NB}_2, \mathcal{NB}_3$ or $\mathcal{NB}_4$ is isomorphic to one of those forms in Table A.1.*

In Table A.1, if no restriction on n is listed, then any $2 \leq n \leq 4$ works. Also t in $\rho(\tau)$ in the first row is a choice of $2n^{\text{th}}$ root of unity.

4. LOCAL REPRESENTATIONS

In this chapter we begin by showing that any local representation extends to $\mathcal{NB}_n$ in a (possibly) nonstandard way. Then we extend the Gaussian and Quaternionic representations from $\mathcal{B}_n$ to $\mathcal{NB}_n$ and examine whether the image of the extensions is still finite. Lastly, we dive into ways to obtain local representations of $\mathcal{B}_n$ from twisted tensors of group algebras. We continue with the notation stated previously, that ρ^R would be the representation induced by the braided vector space pair (R, V).

4.1 Extending Local Representations to $\mathcal{NB}_n$

As discussed in the preliminaries, Braided vector spaces (BVS) is one source of matrix representations of $\mathcal{B}_n$. By Theorem 3.1.2, if ρ^R is completely reducible, then it has a standard extension. Also discussed, the standard extension to $\mathcal{NB}_n$ does not bear any new information than ρ^R itself. However, there is another way to extend ρ^R by use of the symmetric group. For this, we define the flip operator $P(x \otimes y) = y \otimes x$ on $V \otimes V$. This gives us the following theorem:

Theorem 4.1.1. *Suppose that (R, V) is a BVS with corresponding $\mathcal{B}_n$ representation ρ^R. For $n \geq 3$, setting*

$$\rho^R(\tau) = (P \otimes I_V^{\otimes n-2})(I_V \otimes P \otimes I_V^{\otimes n-3}) \cdots (I_V^{\otimes n-2} \otimes P)$$

defines an extension of ρ^R to a representation to $\mathcal{NB}_n$.

Proof. From Lemma 2.3.2, we need only check (N1) and (N2). The relations (B1), for i=1, and (B2) for $i = 1$ and $3 \leq j \leq n-1$ are immediate because (R, V) is a BVS. Notice that the way we define $\rho^R(\tau)$, that it has order n (as it is just an n-cycle permutation on $V^{\otimes n}$). Hence (N2) is satisfied. This means that only (N1) is left. To prove this, it is sufficient to show that the following holds $\rho^R(\tau\sigma_1\tau^{-1}) = \rho^R(\sigma_2)$. Equivalently, we need to show:

$$(P \otimes I)(I \otimes P)(R \otimes I)(P \otimes I)(I \otimes P) = (I \otimes R).$$

To do this, we compare the two operations on a pure tensor of basis elements $v_1 \otimes v_2 \otimes v_3$ on $V^{\otimes 3}$. Doing this gives us the following:

$$\rho^R(\tau)(R \otimes I)(P \otimes I)(I \otimes P)(v_1 \otimes v_2 \otimes v_3) = (P \otimes I)(I \otimes P)[R(v_2 \otimes v_3) \otimes v_1] \tag{4.1.1}$$

$$(I \otimes R)(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes R(v_2 \otimes v_3) \tag{4.1.2}$$

for the left and right hand sides respectively. These two equations are easily seen as equal since $(P \otimes I)(I \otimes P)[R(v_2 \otimes v_3) \otimes v_1] = v_1 \otimes R(v_2 \otimes v_3)$. □

A few remarks are needed to clarify a few possibly subtle points.

Remark 4.1.2. While calling ρ^R for $\mathcal{B}_n$, a local representation made sense, the operator $\rho^R(\tau)$ defined in Theorem 4.1.1 is not local in the strict sense. While it does act non-trivially on all of the tensor factors, its action does not mix vectors within tensor factors. Instead it permutes the factors globally. despite of this, we will still call ρ^R a local representation of $\mathcal{NB}_n$ to keep the nomenclature consistent.

Remark 4.1.3. An interesting question: given R, how much bigger is the image $\rho^R(\mathcal{NB}_n)$ than $\rho^R(\mathcal{B}_n)$? While the subgroup $\mathcal{B}\tilde{A}_n$ has index $2n$ in $\mathcal{NB}_n$, $\mathcal{B}_n$ has infinite index. Therefore it is not immediate that if $|\rho^R(\mathcal{B}_n)| < \infty$, then $|\rho^R(\mathcal{NB}_n)| < \infty$.

Remark 4.1.4. The theorem is stated for $n \geq 3$. This is because in the $n = 2$ case, $\mathcal{B}_2 \cong \mathbb{Z}$, so *any* $R \in \mathrm{Aut}(V^{\otimes n})$ gives a representation of $\mathcal{B}_2$. Setting $\rho^R(\tau) = P$ defines an extension to $\mathcal{NB}_2$ if R is symmetric in the standard product basis of $V \otimes V$. We have $\rho^R(\sigma_1) = R$, $\rho^R(\tau) = P$, and $\rho^R(\sigma_2) = PRP$. The only relation requiring attention is $\rho^R(\sigma_1\sigma_2\sigma_1) = \rho^R(\sigma_2\sigma_1\sigma_2)$ (i.e. $RPRPR = PRPRPRP$). This is satisfied if R is symmetric (i.e. $PRP = R$). It should be noted that, in this $n = 2$ case, (B1) does not hold for every R satisfying the Yang-Baxter equation. Below is an example that clarifies this point.

Example 4.1.5. Consider $\dim V = 2$ (i.e. $V \cong \mathbb{C}^2$), and $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$. This gives us that

$$\rho^R(\sigma_1\sigma_2\sigma_1)(\mathbf{e}_2 \otimes \mathbf{e}_1) \neq \rho^R(\sigma_2\sigma_1\sigma_2)(\mathbf{e}_2 \otimes \mathbf{e}_1).$$

Hence mapping $\rho^R(\tau) = P$ would not give an extension of ρ^R from $\mathcal{B}_n$ to $\mathcal{NB}_n$.

For the R in this example, we have confirmed the following conjecture of $n \leq 6$.

Conjecture 4.1.6. *Given the above R, $n \geq 3$, and using the extension defined in Theorem 4.1.1, $|\rho^R(\mathcal{NB}_n)| = n|\rho^R(\mathcal{B}\tilde{A}_n)| = n2^n|\rho^R(\mathcal{B}_n)|$.*

This means that the image of $\mathcal{NB}_n$, while still being finite, can be significantly larger than that of $\mathcal{B}_n$.

4.2 Gaussian Braided Vector Spaces

In this section we take the idea outlined in Section 2.2.2.1 and we wish to extend it to $\mathcal{NB}_n$. To do this, we first define an algebra $NES(m,n)$, which will be an extension of $ES(m, n-1)$. Define $NES(m,n)$ to be the algebra generated by $u_1, \ldots, u_{n-1}, t$ with the following relations:

1. $u_i^m = 1 = t$
2. $[u_i, u_{i+1}] = q^2$ for all $1 \leq i \leq n-2$
3. $[u_i, u_j] = 1$ if $|i-j| \neq 1$
4. $tu_it^{-1} = u_{i+1}$ for all $1 \leq i \leq n-2$

Where q once again is either a $2m$th or mth root of unity, depending if m is odd or even (respectively).

This gives us that $NES(m,n)$ is almost a semidirect product of $ES(m, n-1)$ and $\mathbb{Z}_n$. To make the extension to $\mathcal{NB}_n$ more apparent, we introduce an ancillary generator (named u_n) to obtain a presentation more familiar to the modulo n relations in $\mathcal{NB}_n$.

Lemma 4.2.1. *Define $u_n := tu_{n-1}t^{-1}$, then u_n satisfies (1) above, relations (2) and (4) above hold for indices modulo n, and the condition $|i-j| \neq 1$ in (4) may be replaced with $|i-j| \not\equiv 1 \mod n$.*

Proof. From our definition, $u_n^m = (tu_{n-1}t^{-1})^m = t(u_{n-1}^m)t^{-1} = 1$. Notice that $u_{n-1} = t^{n-2}u_1t^{2-n}$ and $t^n = 1$ give us $tu_{n-1}t^{-1} = t^{n-1}u_1t^{1-n} = u_n$. Therefore we also get $tu_nt^{-1} = u_1$. Meaning (4) holds with indices modulo n. Next, we observe that

$$u_{n-1}u_n = tu_{n-2}u_{n-1}t^{-1} = t(q^2u_{n-1}u_{n-2})t^{-1} = q^2tu_{n-1}t^{-1}tu_{n-2}t^{-1} = q^2u_nu_{n-1}.$$

Which gives us that $[u_{n-1}, u_n] = q^2$. With this, $[u_1, u_n] = q^2$ is similarly verified. Lastly, to check (3), it is enough to verify that u_n commutes with u_{n-2} (for $n \geq 4$). This is also easily seen, as:

$$u_nu_{n-2} = tu_{n-1}u_{n-3}t^{-1} = tu_{n-3}u_{n-1}t^{-1} = u_{n-2}u_n.$$

□

Next, we show that $\mathcal{NB}_n$ admits a representation in $NES(m, n)$.

Theorem 4.2.2. *The map $\hat{\varphi}_n : \mathcal{NB}_n \to NES(m, n)^*$ defined by $\sigma_i \mapsto R_i(m) = \frac{1}{\sqrt{m}}\sum_{j=0}^{m-1} q^{j^2}u_i^j$ and $\tau \mapsto t$ is a group homomorphism.*

Proof. As shown in [6, Proposition 3.1], the relation $\hat{\varphi}_n(\sigma_1\sigma_2\sigma_1) = \hat{\varphi}_n(\sigma_2\sigma_1\sigma_2)$ and $\hat{\varphi}_n(\sigma_1\sigma_j) = \hat{\varphi}_n(\sigma_j\sigma_1)$ for $1 < j < n$ hold. From the definition of t, $\hat{\varphi}_n(\tau\sigma_i\tau^{-1}) = \hat{\varphi}_n(\sigma_{i+1})$ and $\hat{\varphi}_n(\tau)^{2n} = 1$. Therefore we have that $\hat{\varphi}_n$ is a group homomorphism, and thus a representation of $\mathcal{NB}_n$ into $NES(m, n)$ as desired. □

To obtain a BVS, we will again let $V \cong \mathbb{C}^m$, with standard basis $\{\mathbf{e}_0, \ldots, \mathbf{e}_{m-1}\}$, where $\mathbf{e}_{i+m} = \mathbf{e}_i$. Next we define $U, T \in \mathrm{End}(V^{\otimes 2})$ by $U(\mathbf{e}_i \otimes \mathbf{e}_j) = q^{j-i}\mathbf{e}_{i+1} \otimes \mathbf{e}_{j+1}$ and $T(\mathbf{e}_i \otimes \mathbf{e}_j) = \mathbf{e}_j \otimes \mathbf{e}_i$. We additionally define, for $n \geq 2$, elements $X, U_i \in \mathrm{Aut}(V^{\otimes n})$ by:

$$X := (T \otimes I_V^{\otimes n-2})(I_v \otimes T \otimes I_V^{\otimes n-3}) \cdots (I_V^{\otimes n-2} \otimes T),$$

$U_i := I_V^{\otimes i-1} \otimes U \otimes I_V^{n-i-1}$ for $1 \leq i \leq n-1$, and $U_n := XU_{n-1}X^{-1}$. To help alleviate any confusion in upcoming calculations, the following three equalities should be observed:

$$\begin{aligned} X(\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n}) &= \mathbf{e}_{i_n} \otimes \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \mathbf{e}_{i_{n-1}} \qquad (4.2.1) \\ X^{-1}(\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n}) &= \mathbf{e}_{i_2} \otimes \mathbf{e}_{i_3} \otimes \cdots \mathbf{e}_{i_n} \otimes \mathbf{e}_{i_1} \qquad (4.2.2) \end{aligned}$$

$$U^m(e_i \otimes e_j) = \begin{cases} q^{(m-j)(j-i)} q^{(j-i-m)(j-i)} q^{(j-i)i} e_i \otimes e_j = e_i \otimes e_j & i \leq j \\ q^{(m-i)(j-i)} q^{(i-j-m)(j-i)} q^{(j-i)j} e_i \otimes e_j = e_i \otimes e_j & i > j. \end{cases} \qquad (4.2.3)$$

To prove the following proposition, we need only verify the relations (1)-(4) for $NES(m, n)$.

Proposition 4.2.3. *The map* $\Psi : NES(m, n) \to \mathrm{Aut}(V^{\otimes n})$ *defined by* $u_i \mapsto U_i$ *and* $t \mapsto X$ *defines a representation of* $NES(m, n)$ *on* $V^{\otimes n}$.

Proof. From the definition of U_i, it is evident that U_i would commute with U_j if $|i-j| \ncong 1 \mod n$. This is because they would not "interact" with the same tensor factor in $V^{\otimes n}$. Thus proving (3). The relation (1) is satisfied, because of equation (4.2.3) and X has order n. For (4) observe the following calculation:

$$\begin{aligned} XU_1X^{-1}(\mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_n}) &= XU_1(\mathbf{e}_{j_2} \otimes \mathbf{e}_{j_3} \otimes \cdots \mathbf{e}_{j_n} \otimes \mathbf{e}_{j_1}) \\ &= X(q^{j_3-j_2} \mathbf{e}_{j_2+1} \otimes \mathbf{e}_{j_3+1} \otimes \cdots \otimes \mathbf{e}_{j_n} \otimes \mathbf{e}_{j_1}) \\ &= q^{j_3-j_2}(\mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2+1} \otimes \mathbf{e}_{j_3+1} \otimes \cdots \otimes \mathbf{e}_{j_n}) \\ &= U_2(\mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \otimes \mathbf{e}_{j_3} \otimes \cdots \otimes \mathbf{e}_{j_n}). \end{aligned}$$

While the above calculation was just for U_1 and U_2, the calculation for U_i and U_{i+1} is identical. Hence (4) holds. For the last relation, (2), it is sufficient to check for U_1, U_2 with $n = 3$. This has already be verified in [6], where they showed $U_1U_2(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = q^2 U_2 U_1(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$. $\square$

Notice that the composition $\Psi \circ \hat{\varphi}_n : \mathcal{NB}_n \to \mathrm{Aut}(V^{\otimes n})$ gives a local representation of $\mathcal{NB}_n$.

For the case of $m = 2$, a realization of R is given in Example 4.1. While the image $\rho^R(\mathcal{NB}_n)$ is conjectured to have finite image in that case, it was shown in [6], the image $\hat{\varphi}_n(\mathcal{B}_n)$ in $ES(m, n-1)$ is finite. Following a similar approach, we will obtain the following:

Theorem 4.2.4. *The image $\hat{\varphi}_n(\mathcal{NB}_n)$ in $NES(m, n)$ is a finite group.*

Proof. Notice that the monomials in $NES(m, n)$ have the following normal form: $t^\alpha u_1^{\alpha_1} \cdots u_n^{\alpha_n}$ where $0 \leq \alpha < n$ and $0 \leq \alpha_i < m$. In fact, we see that these $n(m)^n$ monomials form a basis for $NES(m, n)$ over $\mathbb{Q}(q)$. The structure of $NES(m, n)$ is more complicated than $ES(m, n-1)$, which is actually simple for n odd and has exactly m simple components for n even [14]. We let $\hat{\varphi}_n(\mathcal{NB}_n) \subset NES(m, n)$ act on the span of $\hat{U} = \{u_1^{\alpha_1} \cdots u_n^{\alpha_n}\}$ by conjugation. Since conjugation by t obviously permutes this spanning set, we first show that the conjugation action of $R_i(m)$ also permutes this set. The same approach as in [6] works here: (note we may omit the scalar $\frac{1}{\sqrt{m}}$ in $R_i(m)$ in these calculations):

$$\begin{aligned}
qu_i^{-1}u_{i+1}R_i(m) &= qu_i^{-1}u_{i+1}\sum_{j=0}^{m-1} q^{j^2}u_i^j = q^{-1}u_{i+1}u_i^{-1}\sum_{j=0}^{m-1} q^{j^2}u_i^j \\
&= q^{-1}\sum_{j=0}^{m-1} q^{j^2}u_{i+1}u_i^{j-1} = q^{-1}\sum_{j=0}^{m-1} q^{j^2}(q^{-2(j-1)})u_i^{j-1}u_{i+1} \\
&= \sum_{j=0}^{m-1} q^{-1}q^{j^2}q^{-2j+2}u_i^{j-1}u_{i+1} \\
&= \left(\sum_{j=0}^{m-1} q^{(j-1)^2}u_i^{j-1}\right)u_{i+1} = R_i(m)u_{i+1}
\end{aligned}$$

and

$$\begin{aligned}
q^{-1}u_{i-1}u_iR_i(m) &= qu_iu_{i-1}\sum_{j=0}^{m-1}q^{j^2}u_i^j \\
&= qu_i\sum_{j=0}^{m-1}q^{j^2}u_{i-1}u_i^j = qu_i\sum_{j=0}^{m-1}q^{j^2}q^{2j}u_i^ju_{i-1} \\
&= \sum_{j=0}^{m-1}qq^{j^2}q^{2j}u_i^{j+1}u_{i-1} \\
&= \left(\sum_{j=0}^{m-1}q^{(j+1)^2}u_i^{j+1}\right)u_{i-1} = R_i(m)u_{i-1}.
\end{aligned}$$

This shows that $R_i(m)u_{i+1}R_i(m)^{-1} = qu_i^{-1}u_{i+1}$ and $R_i(m)u_{i-1}R_i(m)^{-1} = q^{-1}u_{i-1}u_i$. Thus conjugation by $R_i(m)$ permutes the spanning set $\hat{U}$ up to scalars that are roots of unity (i.e. powers of q). This gives us that $\hat{\varphi}_n(\mathcal{NB}_n)$ is finite modulo the center. The subalgebra of $NES(m,n)$ generated by $\hat{\varphi}_n(\mathcal{NB}_n)$ is semisimple, so that the faithful representation of $\hat{\varphi}_n(\mathcal{NB}_n)$ on $NES(m,n)$ decomposes into full matrix algebras. Thus any element x of the center of $\hat{\varphi}_n(\mathcal{NB}_n)$ acts via a scalar matrix on each irreducible subrepresentation. But since the generators of $\hat{\varphi}_n(\mathcal{NB}_n)$ have determinant a root of unity (of degree m or n), the scalar x is also a root of unity of degree only depending on m and n (indeed the degree of each irreducible representation depends only on m,n). Thus the center of $\hat{\varphi}_n(\mathcal{NB}_n)$ is a finite group and has finite index, so $\hat{\varphi}_n(\mathcal{NB}_n)$ is a finite group. □

4.3 Quaternionic Representation

Similar to the Gaussian Braided Vector Space, the idea is to take a finite group (in this case the quaternion group K_8) and consider the group algebra with n copies of the group, where the generators will interact with 'close' neighbours in a specific way, but commute with 'far' neighbours. In this section we let $q = e^{2i\pi/6}$. Similar to the Gaussian case, we take the algebra Q_n defined in [13], and define another algebra to be (almost) a semi-direct product of Q_n with $\mathbb{Z}_n$. Define $\mathcal{Q}_n$ as the algebra generated by $t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}$ with the following relations:

1. $u_i^2 = v_i^2 = -1$ for all i,

2. $[u_i, v_j] = -1$ if $|i-j| < 2$,

3. $[u_i, v_j] = 1$ if $|i-j| \geq 2$,

4. $[u_i, u_j] = [v_i, v_j] = 1 = t^n$,

5. $tu_it^{-1} = u_{i+1}$, and $tv_it^{-1} = v_{i+1}$ for all i.

As was in the previous section, to make the connection to $\mathcal{NB}_n$ a bit clearer, we use the following lemma.

Lemma 4.3.1. *Defining, in $\mathcal{Q}_n$, $u_n := tu_{n-1}t^{-1}$ and $v_n := tv_{n-1}t^{-1}$, v_n, u_n satisfy relations $(1)-(5)$ with indices $\mod n$ (defining $v_0 = v_n$ and $v_{n+1} = v_1$).*

Proof. We must check that u_n and v_n also satisfy the relations $(1)-(5)$. For (1), note that $u_n^2 = (tu_{n-1}t^{-1})^2 = tu_{n-1}^2t^{-1} = -1 = tv_{n-1}^2t^{-1} = (tv_{n-1}t^{-1})^2 = v_n^2$. The relation (5) follows from the definition of u_n,v_n and $t^n = 1$; $tu_nt^{-1} = t(tu_{n-1}t^{-1})t^{-1} = t(t^{n-1}u_1t^{-n+1})t^{-1} = t^nu_1t^{-n} = u_1$ and similarly $tv_nt^{-1} = v_1$. For (2) and (3), we will first consider $[u_n, v_j]$. Doing so gives the following:

$$\begin{aligned} [u_n, v_j] = [tu_{n-1}t^{-1}, v_j] &= tu_{n-1}t^{-1}v_jtu_{n-1}^{-1}t^{-1}v_j^{-1} \\ &= tu_{n-1}v_{j+1}u_{n-1}^{-1}v_{j+1}^{-1}t^{-1} \\ &= t[u_{n-1}, v_{j+1}]t^{-1}. \end{aligned}$$

Similarly $[u_j, v_n] = t[u_{j+1}, v_{n-1}]t^{-1}$. Thus the relation (3) holds for $|i-j| \mod n \geq 2$. For (2) we need to check the above equation, $j = 1, n-1, n$.

$$[u_n, v_1] = [t^{-1}u_1t, v_1] = t^{-1}[u_1, v_2]t = -1$$
$$[u_n, v_{n-1}] = [tu_{n-1}t^{-1}, tv_{n-2}t^{-1}] = t[u_{n-1}, v_{n-2}]t^{-1} = -1$$
$$[u_n, v_n] = [tu_{n-1}t^{-1}, tv_{n-1}t^{-1}] = t[u_{n-1}, v_{n-1}]t^{-1} = -1.$$

Hence (2) holds for $|i-j| \mod n < 2$. Lastly (4) holds from $[u_n, u_j] = t[u_{n-1}, u_j]t^{-1}$ and $[v_n, v_j] = t[v_{n-1}, v_j]t^{-1}$. $\square$

Theorem 4.3.2. *The map $\xi_n : \mathcal{NB}_n \to \mathcal{Q}_n^\times$ given by $\xi_n(\sigma_i) = \dfrac{-1}{2q}(1+u_i+v_i+u_iv_i)$ and $\xi_n(\tau) = t$ defines a group homomorphism.*

Observe that this theorem is showing that ξ_n is an extension of υ from $\mathcal{B}_n$ to $\mathcal{NB}_n$.

Proof. It was shown in [13] that $\xi_n(\sigma_1\sigma_2\sigma_1) = \xi_n(\sigma_2\sigma_1\sigma_2)$ and $\xi_n(\sigma_1\sigma_j) = \xi_n(\sigma_j\sigma_1)$ for $1 < j < n$ hold. By Lemma 2.3.2 we just need to check $\xi_n(\tau\sigma_i\tau^{-1}) = \xi_n(\sigma_{i+1})$ and $[\xi_n(\tau)]^{2n} = 1$. However, these are both immediate from the (last two) relations in $\mathcal{Q}_n$. $\square$

As mentioned in Chapter 2, $\mathcal{Q}_n$ does not have an apparent local representation. It is however, easy to show the following:

Theorem 4.3.3. *The image $\xi_n(\mathcal{NB}_n)$ in $\mathcal{Q}_n$ is a finite group.*

Proof. First we show that,the conjugation action on the subalgebra $\hat{\mathcal{Q}}_n$ generated by $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ is finite as follows. Observe that $\hat{\mathcal{Q}}_n$ is spanned by monomials of the form

$$u_1^{\epsilon_1} \cdots u_n^{\epsilon_n} v_1^{\nu_1} \cdots v_n^{\nu_n}$$

where nonzero $\epsilon_i, \nu_i \in \{0, \pm 1\}$. The action of $\xi_n(\tau) = t$ obviously permutes this generating set. We can now compute, with $k = i \pm 1$:

$$\begin{aligned} u_i\xi_n(\sigma_i) &= \frac{-1}{2q}u_i(1+u_i+v_i+u_iv_i) = \frac{-1}{2q}(u_i - 1 + u_iv_i - v_i) \\ &= \frac{-1}{2q}(v_iu_iv_i + u_iv_iu_iv_i + u_iv_i + u_iu_iv_i) = \frac{-1}{2q}(v_i + u_iv_i + 1 + u_i)u_iv_i) \\ &= \xi_n(\sigma_i)u_iv_i \end{aligned}$$

and

$$\begin{aligned}
v_i\xi_n(\sigma_i) &= \frac{-1}{2q}v_i(1+u_i+v_i+u_iv_i) = \xi_n(\sigma_i)u_i \\
u_k\xi_n(\sigma_i) &= \frac{-1}{2q}u_k(1+u_i+v_i+u_iv_i) = \xi_n(\sigma_i)u_kv_i \\
v_k\xi_n(\sigma_i) &= \frac{-1}{2q}v_k(1+u_i+v_i+u_iv_i) = \xi_n(\sigma_i)(-u_iv_iv_k).
\end{aligned}$$

Thus the conjugation action of $\xi_n(\mathcal{NB}_n)$ permutes a spanning set up to roots of unity so that $\xi_n(\mathcal{NB}_n)$ is finite modulo its center.

Now again, as in the Gaussian case, we can see that the $\mathcal{Q}_n$ is a finite dimensional semisimple algebra and the restriction to the center of $\xi_n(\mathcal{NB}_n)$ on any irreducible subrepresentation of the faithful regular representation gives a scalar of finite order, hence $\xi_n(\mathcal{NB}_n)$ has finite center and is thus a finite group. $\square$

5. TWISTED TENSOR REPRESENTATIONS

In this chapter, we will discuss an idea inspired by analyzing the $\mathcal{B}_n$ representations in Sections 2.2.2.1 and 2.2.2.2. Motivation also comes from Chapter 4, as any local representation of $\mathcal{B}_n$ extend to a local representation of $\mathcal{NB}_n$ in a natural way.

Recall the algebra $ES(m, n-1)$ presented in Section 2.2.2.1. Upon further examination, $ES(m, n-1)$ can be realized as the twisted tensor product of $n-1$ copies of the group algebra $\mathbb{C}[\mathbb{Z}_m]$. Let u denote the generator for the group $\mathbb{Z}_m$. We define an algebra isomorphism $\mathfrak{A}$: $ES(m, n-1) \to \mathbb{C}[\mathbb{Z}_m]^{\bigotimes_\tau (n-1)}$ as

$$u_i \mapsto \underbrace{1 \otimes_\tau \cdots \otimes_\tau 1}_{i-1 \text{ copies}} \otimes_\tau u \otimes_\tau \underbrace{1 \otimes_\tau \cdots \otimes_\tau 1}_{n-i \text{ copies}}.$$

Because of this identification, we shall relax notation and omit the tensor products when talking about elements in $\mathbb{C}[\mathbb{Z}_m]^{\bigotimes_\tau (n-1)}$. It should be noted that the τ in $\bigotimes_\tau$ is not the generator of $\mathcal{NB}_n$, but instead just common notation for twisted tensor product. The twisting $\tau : \mathbb{C}[\mathbb{Z}_m] \otimes \mathbb{C}[Z_m] \to \mathbb{C}[\mathbb{Z}_m] \otimes \mathbb{C}[\mathbb{Z}_m]$ is described by the relations of $ES(m, n-1)$. In particular, $\tau(u_{i+1}, u_i) = q^{-2} u_i u_{i+1}$ and $\tau(u_i, u_j) = u_j u_i$ if $|i - j| > 1$. Looking at the algebra Q_n presented in Section 2.2.2.2, similarly, we observe Q_n is the twisted group algebra of the quaternion group. This leads to the idea of generating local representations of $\mathcal{B}_n$ from twisting $n-1$ copies of finite abelian group algebras.

5.1 Extending the Gaussian Representation to $\mathbb{Z}_m \times \mathbb{Z}_m$

A first step is to extend the idea from the Gaussian case $\left(\mathbb{C}[\mathbb{Z}_m]^{\bigotimes_\tau (n-1)}\right)$ to the twisted algebra $\mathbb{C}[\mathbb{Z}_m \times \mathbb{Z}_m]^{\bigotimes_\tau (n-1)}$. Similar to above, we will omit the tensor product in elements of $\mathbb{C}[\mathbb{Z}_m \times \mathbb{Z}_m]$ when apparent. Let u_i, v_i be the generators of the ith tensor copy of $\mathbb{C}[\mathbb{Z}_m \times \mathbb{Z}_m]$. We intend to define an algebra $\mathcal{A}_m^n$ generated by $u_1, v_1, \ldots, u_{n-1}, v_{n-1}$ which extends $ES(m, n-1)$. Because of this, we want $u_i^m = v_i^m = 1$, for all i, and $[u_i, u_{i+1}] = [v_i, v_{i+1}] = q^2$ for $1 \leq i \leq n-2$. This gives us two unknowns, $[u_i, v_{i+1}] = x$ and $[v_i, u_{i+1}] = y$.

Recall that in the gaussian representation, we had the image of σ_i in $ES(m, n-1)$ was $\frac{1}{\sqrt{m}}\sum_{j=0}^{m-1} q^{j^2} u_i^j$. Hence, in $\mathcal{A}_m^n$, we define $\rho_m^n : \mathcal{B}_n \to \mathcal{A}_m^n$ by

$$\rho_m^n(\sigma_i) = \left(\frac{1}{\sqrt{m}}\sum_{j=0}^{m-1} q^{j^2} u_i^j\right)\left(\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} q^{k^2} v_i^k\right) = \frac{1}{m}\sum_{j,k=0}^{m-1} q^{j^2+k^2} u_i^j v_i^k = \hat{R}_i.$$

From this, we find values for $[u_i, v_{i+1}] = x$ and $[v_i, u_{i+1}] = y$, which cause the R_i to satisfy the braid relation.

In [6], and stated in Section 2.2.2.1, a braided vector space (U, V) was used to get a linear representation of $\mathcal{B}_n$. For $\mathbb{Z}_m \times \mathbb{Z}_m$ however, we regard $\mathcal{V} = V \otimes V \cong \mathbb{C}^{m\times m}$. Further, as in [6], we define a basis of $\mathcal{V}$ to be $\{\mathbf{e}_i \otimes \mathbf{e}_j \mid 1 \leq i, j \leq m\}$, with $\mathbf{e}_{m+i} = \mathbf{e}_i$, and define $\mathcal{U}^u$,$\mathcal{U}^v \in \mathrm{Aut}(\mathcal{V}^{\otimes 2})$ as:

$$\mathcal{U}^u(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = q^{k-i}\mathbf{e}_{i+1} \otimes \mathbf{e}_j \otimes \mathbf{e}_{k+1} \otimes \mathbf{e}_l$$

$$\mathcal{U}^v(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = q^{l-j}\mathbf{e}_i \otimes \mathbf{e}_{j+1} \otimes \mathbf{e}_k \otimes \mathbf{e}_{l+1}.$$

The rationale that $\mathcal{U}^u$ and $\mathcal{U}^v$ are defined the way they are, comes from the realization that u_i interacts the first tensor product of V in $\mathcal{V}$, and v_i interacts with the second. Hence on $\mathcal{V}^{\otimes 2}$, $\mathcal{U}^u$ should interact with the first and third tensor copies of V (respectively $\mathcal{U}^v$ interacting with the second and forth copy). From their definition, the following on $\mathcal{V}^{\otimes 3}$ can readily be observed:

$$(\mathcal{U}^u \otimes I_{\mathcal{V}})(I_{\mathcal{V}} \otimes \mathcal{U}^v) = (I_{\mathcal{V}} \otimes \mathcal{U}^v)(\mathcal{U}^u \otimes I_{\mathcal{V}}) \tag{5.1.1}$$

$$(\mathcal{U}^v \otimes I_{\mathcal{V}})(I_{\mathcal{V}} \otimes \mathcal{U}^u) = (I_{\mathcal{V}} \otimes \mathcal{U}^u)(\mathcal{U}^v \otimes I_{\mathcal{V}}) \tag{5.1.2}$$

These equations, along with a desire for the map $\phi_m : \mathcal{A}_m^n \to \mathrm{End}(\mathcal{V}^{\otimes n})$ defined by

$$u_i \to \mathcal{U}_i^u := I_{\mathcal{V}}^{\otimes(i-1)} \otimes \mathcal{U}^u \otimes I_{\mathcal{V}}^{\otimes(n-i-1)} \text{ and } v_i \to \mathcal{U}_i^v := I_{\mathcal{V}^{\otimes(i-1)}} \otimes \mathcal{U}^v \otimes I_{\mathcal{V}}^{\otimes(n-i-1)}$$

to be an algebra homomorphism, imply that $[u_i, v_{i+1}] = [v_i, u_{i+1}] = 1$. To check that the above map will also satisfy $[u_i, u_{i+1}] = q^2 = [v_i, v_{i+1}]$, we show

$$\mathcal{U}_1^u\mathcal{U}_2^u(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y) = q^2\mathcal{U}_2^u\mathcal{U}_1^u(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y)$$

$$\mathcal{U}_1^v\mathcal{U}_2^v(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y) = q^2\mathcal{U}_2^v\mathcal{U}_1^v(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y).$$

To see that the above two are indeed true, we cite [6, Proposition 3.4], and observe:

$$\begin{aligned}
\mathcal{U}_1^u\mathcal{U}_2^u(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y) &= q^{x-k}q^{(k+1)-i}\mathbf{e}_{i+1} \otimes \mathbf{e}_j \otimes \mathbf{e}_{k+2} \otimes \mathbf{e}_l \otimes \mathbf{e}_{x+1} \otimes \mathbf{e}_y \\
&= q^2 q^{x-(k+1)}q^{k-i}\mathbf{e}_{i+1} \otimes \mathbf{e}_j \otimes \mathbf{e}_{k+2} \otimes \mathbf{e}_l \otimes \mathbf{e}_{x+1} \otimes \mathbf{e}_y \\
&= q^2\mathcal{U}_2^u\mathcal{U}_1^u(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_x \otimes \mathbf{e}_y).
\end{aligned}$$

The relation $\mathcal{U}_1^v\mathcal{U}_2^v = q^2\mathcal{U}_2^v\mathcal{U}_1^v$ is shown similarly. Thus the defining relations on $\mathcal{A}_m^n$ are:

1. $u_i^m = v_i^m = 1$ for all i
2. $[u_i, v_j] = 1$ for all i, j
3. $[u_i, u_{i+1}] = [v_i, v_{i+1}] = q^2$.

While this does give us a local representation of $\mathcal{B}_n$ $\left(\text{by } \varphi_m(\sigma_i) = \frac{1}{m}\sum_{j,k=0}^{m-1} q^{j^2+k^2}(\mathcal{U}_i^u)^j(\mathcal{U}_i^v)^k\right)$, it is just the tensor product of the Gaussian representation with itself. This in no way means that every algebra $\mathbb{C}[\mathbb{Z}_m \times \mathbb{Z}_m]^{\otimes_\tau n-1}$ with $[u_i, u_{i+1}] = q^2 = [v_i, v_{i+1}]$ must have the u and v generators commute. However, if we are wanting $R_i = \frac{1}{m}\sum_{j,k=0}^{m-1} q^{j^2+k^2}u_i^j v_i^k$ to be the image of σ_i, then the $u's$ and $v's$ will be forced to commute. In the pursuit of more alluring representations of $\mathcal{B}_n$, we begin analyzing $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$.

5.2 Local Representations from $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]$

For lucidity sake, we restate the setting. The desire is to find local representations of $\mathcal{B}_n$ by first mapping $\mathcal{B}_n$ into $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$, say by φ_m. After such a map has been established, we aim to ‘localize’ $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$. This is the process of finding $\mathcal{U}^u, \mathcal{U}^v \in \mathrm{Aut}(V)$, such that $u_i \mapsto \mathcal{U}_i^u$ and $v_i \mapsto \mathcal{U}_i^v$, provides a homomorphism and composing that with φ_m provides a local representation of $\mathcal{B}_n$. We call $\mathcal{U}^u, \mathcal{U}^v$ a *localization of* $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$, and φ_m composed with this localization a representation of *a representation of* $\mathcal{B}_n$ *through the algebra* $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$. The algebra $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$ is generated by $u_1, v_1, \ldots, u_{n-1}, v_{n-1}$ and subject to the following relations:

1. $u_i^3 = 1 = v_i^3$
2. $[u_i, v_j] = 1 = [v_i, u_j]$ for all $|i - j| \neq 1$.
3. $[u_i, u_{i+1}] = q_1$, $[u_i, v_{i+1}] = q_2$, $[v_i, u_{i+1}] = q_3$, and $[v_i, v_{i+1}] = q_4$,

where q_1, q_2, q_3, q_4 are choices of third root of unity. The goal now would be, for each tuple $\mathbf{q} = (q_1, q_2, q_3, q_4)$ to find $\alpha_{j,k}^{\mathbf{q}} \in \mathbb{C}$ such that

$$\sigma_i \to R_i := \sum_{j,k=0}^{m-1} \alpha_{j,k}^{\mathbf{q}} u_i^j v_i^k$$

defines a group homomorphism from $\mathcal{B}_n$ to $(\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)})^\times$. The solutions that are of most interest (due to their applications in TQC) are unitary solutions. Later we state a complete classification of unitary solutions.

Solving for the R_i’s involve solving a system with 9 unknowns for 81 distinct cases. There are many ways that we can narrow down these 81 cases into a less cumbersome amount. The first way would be to recognize that the algebra isomorphism on $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$ by $\omega \mapsto \omega^2$ would reduce the number of cases to 41. Another natural method to consider would be applying different compositions of the isomorphism that swap tensor copies of $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$ with each other (i.e. $u_i \leftrightarrow u_j$ and $v_i \leftrightarrow v_j$ for $i \neq j$) and the isomorphism swapping generators within a single tensor copy ($u_i \leftrightarrow v_i$). However, while applying these isomorphisms, one must be careful, as the coefficients of the different R_i’s would be permuted, meaning that the coefficient for $u_1 v_1^2$ in R_1 may be different from the coefficient of $u_2 v_2^2$ in R_2. Due to this, we pursue another method.

Instead of the above methods, we will use a notion of congruent matrices to reduce the number of tuples $\mathbf{q}$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}_3$ such that $\omega^{\alpha_i} = q_i$ (for $i = 1, 2, 3, 4$), $Q = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_3)$ correspond to the basis change of $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]$ in accordance with the basis change of $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$ of $u_i \mapsto u_i^a v_i^b$ and $v_i \mapsto u_i^c v_i^d$. Note that the collection of all possible Q is equivalent to $M_2(\mathbb{F}_3)$. A quick calculation shows that

$$AQA^T = \begin{pmatrix} a^2\alpha_1 + ab(\alpha_2 + \alpha_3) + b^2\alpha_4 & ac\alpha_1 + ad\alpha_2 + bc\alpha_3 + bd\alpha_4 \\ ac\alpha_1 + bc\alpha_2 + ad\alpha_3 + bd\alpha_4 & c^2\alpha_1 + cd(\alpha_2 + \alpha_3) + d^2\alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}.$$

The following calculates the new commutation relation for u_i, v_{i+1} under the basis change from A:

$$\begin{aligned} [u_i, v_{i+1}] &\to [u_i^a v_i^b, u_{i+1}^c v_{i+1}^d] \\ u_i v_{i+1} &\to u_i^a v_i^b u_{i+1}^c v_{i+1}^d \\ u_i^a v_i^b u_{i+1}^c v_{i+1}^d &= q_3^{bc} u_i^a u_{i+1}^c v_i^b v_{i+1}^d \\ &= q_3^{bc} q_1^{ac} q_4^{bd} q_2^{ad} u_{i+1}^c v_{i+1}^d u_i^a v_i^b \\ &= (\omega^{ac\alpha_1 + ad\alpha_2 + bc\alpha_3 + bd\alpha_4}) u_{i+1}^c v_{i+1}^d u_i^a v_i^b \\ &= \omega^{\beta_2} u_{i+1}^c v_{i+1}^d u_i^a v_i^b. \end{aligned}$$

Similarly, it can be shown that the β_i are the image of α_i under the basis change given by A. From this, if we consider the action of $GL_2(\mathbb{F}_3)$ on $M_2(\mathbb{F}_3)$ by $A.Q = AQA^T$ for $A \in GL_2(\mathbb{F}_3)$, $Q \in M_2(\mathbb{F}_3)$. The reason AQA^T is the proper choice, instead of some AQB, is that we wish for each tensor copy of $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]$ to have the same generators (i.e. we want u_1, v_1 to be sent to the same basis as u_j, v_j. In [18, Theorem 2] they showed the following general theorem.

Theorem 5.2.1 ([18])**.** *The number of congruence classes in $M_n(\mathbb{F}_p)$ is the coefficient of t^n in*

$$\prod_{k\geq 1}(1+t^k)^e(1-pt^2)^{-1}(1-t^k)^{-1}$$

where $e = 2^{p \mod 2}$.

A quick calculation shows that this means there are 7+p congruence classes in $M_2(\mathbb{F}_p)$. Thus we only have 10 cases. One considerable benefit to these congruence classes is that it corresponds to a basis change. Hence the coefficient in R_i for $u_i^x v_i^y$ is the same coefficient as in R_j for $u_j^x v_j^y$. Class representatives in $M_2(\mathbb{F}_3)$, are listed below. The congruence classes are listed in A.2 and A.3.

$$\mathcal{M}_1 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{M}_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathcal{M}_3 := \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \mathcal{M}_4 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathcal{M}_5 := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\mathcal{M}_6 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \mathcal{M}_7 := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \mathcal{M}_8 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{M}_9 := \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \mathcal{M}_{10} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that one can transform $\mathcal{M}_2$ to $\mathcal{M}_3$ (and also $\mathcal{M}_8$ to $\mathcal{M}_9$) by simply sending $\omega \to \omega^2$. As this is an algebra isomorphism, we further reduce the number of cases to 8, using the representatives $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_8, \mathcal{M}_{10}$. The only invertible solution for $\mathcal{M}_1$, is $R_i = 1$ (i.e. $\alpha_{j,k} = 0$ if $(j,k) \neq (0,0)$).

Proposition 5.2.2. *For $n \geq 3$, any unitary representation of $\mathcal{B}_n$ through $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau (n-1)}$, given by*

$$\sigma_i \mapsto R_i := \sum_{j,k=0}^{2} \alpha_{j,k} u_i^j v_i^k,$$

is isomorphic to a representation in Tables A.4 to A.10.

It should be noted, that Tables A.4 to A.10 only includes unitary representations. Some of which may need rescaling (i.e. to unitarize R, we just multiply it by some scalar λ). If such a scalar is needed, it is provided. Also the R_i coefficients are presented as a set, as writing out each solution as a polynomial with 9 terms is quite extensive.

The table does not contain all possible solutions for each congruence class $\mathcal{M}_i$. As the proposition states, they only contain the unitary solutions. To expand on this further, we consider the case $\mathcal{M}_5$. In this particular instance, the relations in $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$ are $[u_i, u_{i+1}] = \omega$, $[v_i, v_{i+1}] = \omega^2$, and $[u_i, v_\ell] = 1$ (for all i, ℓ). Before we solve for R_i that satisfy the braid relation, we set $\alpha_{0,0} = 1$, reducing our solution sets to 8 unknowns (this is the reason for possible rescaling needed to have R_i be unitary). Now we have

$$R_i = 1 + \alpha_{1,0}u_i + \alpha_{0,1}v_i + \alpha_{2,0}u_i^2 + \alpha_{0,2}v_i^2 + \alpha_{1,1}u_iv_i + \alpha_{2,1}u_i^2v_i + \alpha_{1,2}u_iv_i^2 + \alpha_{2,2}u_i^2v_i^2$$

and will find $\alpha_{j,k}$ such that $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$. There are many solutions to this that are not of much interest, for instance $\alpha_{j,k} = 0$ for all j, k (i.e. $R_i = 1$ for all i) is a solution, however it would only lead to trivial representations.

Another type of solution we wish to avoid are ones that would have a solution that is not invertible. In this section, we discuss eigenvalues of R_i, u_i, and v_i. By this we mean their eigenvalues after $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$ is mapped to $\mathrm{End}(V)$ for some finite dimensional vector spce V. To check the possible eigenvalues of R_i, one must only plug in the three possible values for u_i, v_i, which are $1, \omega, \omega^2$ (this stems from u_i, v_i both having order 3, so any eigenvalue must also have that order).

Example 5.2.3. *The solution $\alpha_{j,k} = \delta_{j,k}$ (where δ is the) has $R_i = 1 + u_iv_i + u_i^2v_i^2$. However, upon inspection, the only way that these R_i are invertible is if u_i, v_i are both mapped to the identity. This however would then give us that R_i would just be a multiple of the identity. And therefore a fairly frivolous representation.*

Another reason we wish to avoid solutions like in Example 5.2.3 is the fact that it can be seen as a subrepresentation, due to the fact that u_iv_i generates a subalgebra of $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$ isomorphic to $\mathbb{C}[\mathbb{Z}_3]$. In a similar vain the following two examples are unitary, however they are either isomorphic to a representation of $\mathbb{C}[\mathbb{Z}_3]^{\otimes_\tau (n-1)}$ or the product of two such representations.

Example 5.2.4. *The solution* $R_i = \dfrac{i}{\sqrt{3}}(1+\omega u_i + u_i^2)$ *is unitary, with eigenvalues* $1, \omega$.

Example 5.2.5. *The solution* $R_i = -\dfrac{1}{3}(1+\omega v_i + v_i^2)(1+\omega u_i + u_i^2)$ *is unitary with eigenvalues* $1, \omega, \omega^2$.

That being said, all unitary representations of the forms similar to Examples 5.2.4 and 5.2.5 have been expressed in Tables A.4 to A.10 as they may be of interest in some other manner.

Other solutions that are worth noting are solutions such as the one given in Example 5.2.6. This solution has all real eigenvalues, and is not unitary. While these do not have a direct application to TQC as their unitary counterparts, the solutions still warrant some attention.

Example 5.2.6. *The solution* $R_i = 1 + \left(\frac{-7}{2} + \frac{3\sqrt{5}}{2}\right)(u_i + v_i + u_i^2 + v_i^2 + u_i^2 v_i + u_i v_i^2 + u_i^2 v_i^2)$ *is not unitary, and has eigenvalues* $\frac{1}{2}(9 - 3\sqrt{5})$ *and* $-27 + 12\sqrt{5}$.

In fact, the R_i in Example 5.2.6 is a solution in not only the congruence class $\mathcal{M}_5$, but also for $\mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_8,$ and $\mathcal{M}_{10}$. This is not the only solution that is non-unitary with possible real eigenvalues. However those solutions are very messy and at the moment provide no further insight. These solutions have been omitted in the tables in this document for that reason, but may appear in future work.

5.2.1 Localizing $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$

Similar to Section 5.1, we wish to find $\mathcal{U}^u, \mathcal{U}^v \in \mathrm{Aut}(V^{\otimes 2})$, for some finite dimensional vector space V, where mapping $u_i \to \mathcal{U}_i^u$ and $v_i \to \mathcal{U}_i^v$ gives an algebra homomorphism. Unlike the previous section, we do not offer a complete classification of such $\mathcal{U}^u$ and $\mathcal{U}^v$, instead we provide a method for acquiring such automorphisms. First we let $V \cong \mathbb{C}^3$. Since we are looking for automorphisms on $V \otimes V$, we can look for $X, Y, J, K \in GL(V)$ such that $\mathcal{U}^u = X \otimes Y$ and $\mathcal{U}^v = J \otimes K$. From this, we would have $\mathcal{U}_1^u = X \otimes Y \otimes I_V, \mathcal{U}_2^u = I_V \otimes X \otimes Y, \mathcal{U}_1^v = J \otimes K \otimes I_V$, and $\mathcal{U}_2^v = I_V \otimes J \otimes K$. On account of wanting a $\mathcal{U}_1^u, \mathcal{U}_2^u, \mathcal{U}_1^v$, and $\mathcal{U}_2^v$ to satisfy the relations from the algebra $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$ (represented by some choice of $\mathcal{M}_i$). Recalling that we let $[a,b] = aba^{-1}b^{-1}$, it can be concluded that $[Y,X] = q_1$, $[Y,J] = q_2$, $[K,X] = q_3$, $[K,J] = q_4$, and

$[X, J] \otimes [Y, K] = 1$. Notice that the last equation is saying $[X, J] = [Y, K]^{-1}$. While we do not currently have a classification of all such 'localizations', we again highlight the $\mathcal{M}_5$ case. Define two elements of $\mathrm{Aut}(V \otimes V)$ by

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and } \mathcal{P} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence defining $\mathcal{U}^u := (A\mathcal{P})^{-2} \otimes (A\mathcal{P}^{-1})^2$ and $\mathcal{U}^v := (A^2(A\mathcal{P})^{-1}) \otimes (\mathcal{P}A)^{-1}$, will satisfy the relations for $\mathcal{M}_5$. Therefore, taking any of the $\alpha_{j,k}$ solutions in Table A.6, the map $\varphi : \mathcal{B}_n \to \mathrm{End}(V^{\otimes n})$, defined by $\sigma_i \mapsto R_i = \sum_{(j,k)\neq(0,0)}^{(2,2)} \alpha_{j,k}(\mathcal{U}_i^u)^j(\mathcal{U}_i^v)^k$, gives us a local representation of $\mathcal{B}_n$.

The reason for picking such an A and $\mathcal{P}$, is that they have eigenvalues $1, \omega$, and ω^2. From this, the current design is to find products of powers of A and $\mathcal{P}$ that satisfy the algebra relations associated with the various $\mathcal{M}_i$. While this will arrive at a local representation, it will not classify all algebra homomorphisms from $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\otimes_\tau (n-1)}$ to $\mathrm{End}(V)$. For example, it would not classify any $\mathcal{X} \in \mathrm{Aut}(V^{\otimes 2})$ such that $\mathcal{X}$ does not decompose into $X_1 \otimes X_2$. Also, instead of A or $\mathcal{P}$, one could for instance consider powers of ωA or even something like $\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$. Finding such localizations, or at least a locallization for each case of $\mathcal{M}_i$ is one goal in a current project with Paul Gustafson, Qing Zhang, and Dr Eric Rowell. Another goal is to extend this idea to $\mathbb{Z}_p \times \mathbb{Z}_p$, and to do that, one could consider similar A and $\mathcal{P}$, except being $p \times p$ matricies with ω now being a primitive pth root of unity.

Example 5.2.7. *Let $p \geq 3$ be a prime, A_p be the diagonal matrix with $(1, \xi, \xi^2, \ldots, \xi^{p-1})$ along the diagonal, $\mathcal{P}_p$ the permutation matrix associated with the permutation $(1, p, p-1, \ldots, 2)$, and $\xi = e^{2\pi i/p}$. Consider the algebra $\mathcal{T}_p^n := \mathbb{C}[\mathbb{Z}_p \times \mathbb{Z}_p]^{\otimes(n-1)}$ with generators u_i, v_i (for $1 \leq i \leq n-q$) subject to relations: $[u_i, u_{i+1}] = 1$, $[u_i, v_{i+1}] = \xi = [v_i, u_{i+1}]$, $[v_i, v_{i+1}] = \xi^{-2}$, $u_i^p = v_i^p = 1$, and if $|i-j| \neq 1$, the generators with index i, j will commute. Defining $X := A_p$, $Y := A_p^{-1}$,*

$J := (A_p\mathcal{P}_p)^{-1}$, $K := A_p\mathcal{P}_p^{-1}$, $\mathcal{U}^u = X \otimes Y$ and $\mathcal{U}^v = J \otimes K$, provides a localization of $\mathcal{T}_p^n$.

In the setting of $p = 3$, this corresponds to $\mathbf{q} = (1, \omega, \omega, \omega)$, which is in the congruence class of $\mathcal{M}_5$. This localization is isomorphic to the previously stated $\mathcal{U}^u = (A\mathcal{P})^{-2} \otimes (A\mathcal{P}^{-1})^2$, $\mathcal{U}^v = (A^2(A\mathcal{P})^{-1}) \otimes (\mathcal{P}A)^{-1}$, by the appropriate change of basis.

6. CONCLUSIONS AND FUTURE DIRECTIONS

To reiterate a few previously mentioned points, while there is a classification of the R_i solutions for all eight congruence classes $\mathcal{M}_j$, there currently is not a classification of localizations of the algebra $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes(n-1)}$ with defining relations from $\mathcal{M}_j$. It is believed that the idea described in 5.2.1 (defining $\mathcal{U}^u = X \otimes Y$ and $\mathcal{U}^v = J \otimes K$ for some $X, Y, J, K \in \mathrm{Aut}(\mathbb{C}^3)$) may not be fruitful for every congruence class $\mathcal{M}_j$. In fact it is conjectured that $\mathcal{M}_7$ has no such localization. This does not mean that there exists no localization for $\mathcal{M}_7$, alternatively, one could find $\mathcal{X}, \mathcal{Y} \in \mathrm{Aut}(V^{\bigotimes 2})$ such that (1) there does not exist $X_1, X_2, Y_1, Y_2 \in \mathrm{Aut}(V)$ with $\mathcal{X} = X_1 \otimes X_2$ and $\mathcal{Y} = Y_1 \otimes Y_2$, *and* (2) $u_i \mapsto \mathcal{U}_i^u = I_V^{\otimes(i-1)} \otimes \mathcal{X} \otimes I_V^{\otimes(n-i-1)}$, $v_i \mapsto \mathcal{U}_i^v = I_V^{\otimes(i-1)} \otimes \mathcal{Y} \otimes I_V^{\otimes(n-i-1)}$ defines an algebra homomorphism from $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau(n-1)}$ to $\mathrm{Aut}(V^{\otimes n})$.

Other questions about $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau(n-1)}$ that are of interest include: *What structure does $\mathbb{C}[\mathbb{Z}_3 \times \mathbb{Z}_3]^{\bigotimes_\tau(n-1)}$ admit under the different $\mathcal{M}_j$ congruence classes? Is the image of $\mathcal{B}_n$, under the natural map $\sigma_i \mapsto R_i$, finite for all solutions? What group is the image of $\mathcal{B}_n$ under this identification? Do these solutions extend to $\mathbb{C}[\mathbb{Z}_p \times \mathbb{Z}_p]^{\bigotimes(n-1)}$ in a standard or natural way?*

We briefly focus on the last idea and describe a method that is currently being pursued. The idea is, for each $\mathcal{M}_\ell$, find a complex valued function f_p^ℓ such that in $\mathbb{C}[\mathbb{Z}_p \times \mathbb{Z}_p]^{\bigotimes(n-1)}$, defining $\bar{R}_i = \sum_{j,k=0}^{m-1} f_p^\ell(j,k) u_i^j v_i^k$ will give rise to the natural homomorphism from $\mathcal{B}_n$. These $\bar{R}_i$, for $p = 3$, will correspond with one of the R_i in Proposition 5.2.2. This idea stems from not only wanting a solution for general p, but also the original construction of the Gaussian representation, described in 2.2.2.1, where they defined $R_i = \sum_{j=0}^{m-1} g_m(j) u_i^j$, with $g_m(j) = q^{j^2}$ (for the appropriate q).

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APPENDIX A

This appendix contains tables with a variety of classifications. They appear here, instead of the main text, do to their unwieldy nature.

$\rho(\sigma_1)$	$\rho(\tau)$	restrictions
$\begin{pmatrix} a & 1 \\ a^2-ad+d^2 & d \end{pmatrix}$	$\begin{pmatrix} -t_2 & 0 \\ 0 & t_2 \end{pmatrix}$	$a \neq d$
$\begin{pmatrix} a & 1 \\ -a^2+ad-d^2 & d \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \pm\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$	$a \neq d, n = 2$
$\begin{pmatrix} a & 1 \\ \frac{-1}{2}(a^2-ad+d^2) & d \end{pmatrix}$	$\pm\begin{pmatrix} e^{\pm i\pi/3} & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & e^{\pm i\pi/3} \end{pmatrix},$ $\pm\begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}, \pm\begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{2i\pi/3} \end{pmatrix}$	$a \neq d, n = 3$
$\begin{pmatrix} \omega d & 1 \\ c & d \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 \\ 0 & e^{\pm 2i\pi/3} \end{pmatrix}, \pm\begin{pmatrix} e^{\pm 2i\pi/3} & 0 \\ 0 & e^{\mp 2i\pi/3} \end{pmatrix}$	$d \neq 0, c \neq \omega d^2, n = 3$

Table A.1: Dimension 2 Irreducible Representations of $\mathcal{NB}_n$, for $2 \leq n \leq 4$.
In this table, ω is a primitive 3rd root of unity.

$\mathcal{M}_i$	Congruence Class
$\mathcal{M}_1 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathcal{M}_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
$\mathcal{M}_3 := \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
$\mathcal{M}_4 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix},$ $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$ $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},$ $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
$\mathcal{M}_5 := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix},$ $\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$

Table A.2: The congruence classes in $M_2(\mathbb{F}_3)$ for $\mathcal{M}_1$ to $\mathcal{M}_5$.

$\mathcal{M}_i$	Congruence Class
$\mathcal{M}_6 := \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\2 & 2\end{pmatrix}, \begin{pmatrix}1 & 1\\2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix},$ $\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\2 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\2 & 2\end{pmatrix}, \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 2\\1 & 2\end{pmatrix}$
$\mathcal{M}_7 := \begin{pmatrix}0 & 1\\2 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\2 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\1 & 0\end{pmatrix}$
$\mathcal{M}_8 := \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\2 & 1\end{pmatrix},$ $\begin{pmatrix}0 & 2\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\2 & 0\end{pmatrix}$
$\mathcal{M}_9 := \begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 2\\0 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\\2 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\1 & 2\end{pmatrix},$ $\begin{pmatrix}0 & 1\\2 & 2\end{pmatrix}, \begin{pmatrix}2 & 0\\2 & 2\end{pmatrix}, \begin{pmatrix}2 & 0\\1 & 2\end{pmatrix}, \begin{pmatrix}2 & 2\\1 & 0\end{pmatrix}$
$\mathcal{M}_{10} := \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}, \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}$

Table A.3: The congruence classes in $M_2(\mathbb{F}_3)$ for $\mathcal{M}_6$ to $\mathcal{M}_{10}$

Unitary R_i solutions for $\mathcal{M}_2$	scalar to unitarize
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega, \alpha_{2,0} = 0, \alpha_{0,2} = 1, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega, \alpha_{2,0} = 0, \alpha_{0,2} = \omega, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	-$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = 1, \alpha_{2,2} = 0 \}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = \omega , \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega , \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 1\}$	$\frac{i}{\sqrt{3}}$

Table A.4: Unitary solutions to R_i with relations given by $\mathcal{M}_2$.

Unitary R_i solutions for $\mathcal{M}_4$	scalar to unitarize
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega , \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 1\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega , \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = \omega \}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = 1, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = \omega , \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$

Table A.5: Unitary solutions to R_i with relations given by $\mathcal{M}_4$.

Unitary R_i solutions for $\mathcal{M}_5$	scalar to unitarize
$\{\alpha_{1,0} = \omega$, $\alpha_{0,1} = 0$, $\alpha_{2,0} = 1$, $\alpha_{0,2} = 0$, $\alpha_{1,1} = 0$, $\alpha_{2,1} = 0$, $\alpha_{1,2} = 0$, $\alpha_{2,2} = 0$ $\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = \omega$, $\alpha_{0,1} = 0$, $\alpha_{2,0} = \omega$, $\alpha_{0,2} = 0$, $\alpha_{1,1} = 0$, $\alpha_{2,1} = 0$, $\alpha_{1,2} = 0$, $\alpha_{2,2} = 0$ $\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = \omega$ $\alpha_{0,1} = \omega$, $\alpha_{2,0} = 1$, $\alpha_{0,2} = 1$, $\alpha_{1,1} = \omega^2$, $\alpha_{2,1} = \omega$, $\alpha_{1,2} = \omega$, $\alpha_{2,2} = 1$ $\}$ Factors into $(1 + \omega u_i + u_i^2)(1 + \omega v_i + v_i^2)$	-$\frac{1}{3}$
$\{\alpha_{1,0} = \omega^2$, $\alpha_{0,1} = \omega$, $\alpha_{2,0} = \omega^2$, $\alpha_{0,2} = \omega$, $\alpha_{1,1} = 1$, $\alpha_{2,1} = 1$, $\alpha_{1,2} = 1$, $\alpha_{2,2} = 1\}$	$\frac{1}{3}$
$\{\alpha_{1,0} = 1$, $\alpha_{0,1} = \omega$, $\alpha_{2,0} = \omega^2$, $\alpha_{0,2} = \omega$, $\alpha_{1,1} = \omega$, $\alpha_{2,1} = 1$, $\alpha_{1,2} = \omega$, $\alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = 1$, $\alpha_{0,1} = 1$, $\alpha_{2,0} = \omega^2$, $\alpha_{0,2} = \omega$, $\alpha_{1,1} = 1$, $\alpha_{2,1} = \omega^2$, $\alpha_{1,2} = \omega$, $\alpha_{2,2} = 1\}$	$\frac{1}{3}$

Table A.6: Unitary solutions to R_i with relations given by $\mathcal{M}_5$.

Unitary R_i solutions for $\mathcal{M}_6$	scalar to unitarize
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega, \alpha_{2,0} = 0, \alpha_{0,2} = 1, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega, \alpha_{2,0} = 0, \alpha_{0,2} = \omega, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega, \alpha_{1,2} = 1, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega, \alpha_{1,2} = \omega, \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 1\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = \omega\}$	$-\frac{i}{\sqrt{3}}$

Table A.7: Unitary solutions to R_i with relations given by $\mathcal{M}_6$.

Unitary R_i solutions for $\mathcal{M}_7$	scalar to unitarize
$\{\alpha_{1,0} = \omega\ , \alpha_{0,1} = \omega\ , \alpha_{2,0} = 1, \alpha_{0,2} = \omega\ , \alpha_{1,1} = \omega\ , \alpha_{2,1} = \omega\ ^2, \alpha_{1,2} = 1, \alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = 1, \alpha_{0,1} = \omega\ , \alpha_{2,0} = \omega\ , \alpha_{0,2} = \omega\ , \alpha_{1,1} = \omega\ ^2, \alpha_{2,1} = \omega\ , \alpha_{1,2} = 1, \alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = \omega, \alpha_{0,1} = \omega\ , \alpha_{2,0} = 1\ , \alpha_{0,2} = 1\ , \alpha_{1,1} = \omega\ ^2, \alpha_{2,1} = \omega\ , \alpha_{1,2} = \omega, \alpha_{2,2} = 1\ \}$ Factors into $(1 + \omega u_i + u_i^2)(1 + \omega v_i + v_i^2)$	$-\frac{1}{3}$
$\{\alpha_{1,0} = \omega\ ^2, \alpha_{0,1} = 1, \alpha_{2,0} = 1, \alpha_{0,2} = \omega\ , \alpha_{1,1} = \omega\ , \alpha_{2,1} = \omega\ , \alpha_{1,2} = \omega\ , \alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = \omega\ , \alpha_{0,1} = 1, \alpha_{2,0} = \omega\ , \alpha_{0,2} = \omega\ , \alpha_{1,1} = \omega\ ^2, \alpha_{2,1} = 1, \alpha_{1,2} = \omega\ , \alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = 1, \alpha_{0,1} = \omega\ , \alpha_{2,0} = \omega\ ^2, \alpha_{0,2} = \omega\ , \alpha_{1,1} = \omega\ , \alpha_{2,1} = 1, \alpha_{1,2} = \omega\ , \alpha_{2,2} = 1\}$	$-\frac{1}{3}$

Table A.8: Unitary solutions to R_i with relations given by $\mathcal{M}_7$

Unitary R_i solutions for $\mathcal{M}_8$	scalar to unitarize
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega\ , \alpha_{2,0} = 0, \alpha_{0,2} = 1, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = \omega\ , \alpha_{2,0} = 0, \alpha_{0,2} = \omega\ , \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega\ , \alpha_{1,2} = 1, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega\ , \alpha_{1,2} = \omega\ , \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$

Table A.9: Unitary solutions to R_i with relations given by $\mathcal{M}_8$

Unitary R_i solutions for $\mathcal{M}_{10}$	scalar to unitarize
$\{\alpha_{1,0} = \omega , \alpha_{0,1} = 0, \alpha_{2,0} = 1, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = \omega , \alpha_{0,1} = 0, \alpha_{2,0} = \omega , \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = 1, \alpha_{2,2} = 0\}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = 0, \alpha_{2,1} = \omega , \alpha_{1,2} = \omega , \alpha_{2,2} = 0\}$	$-\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega , \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = 1, \omega = \omega \}$	$\frac{i}{\sqrt{3}}$
$\{\alpha_{1,0} = \omega^2, \alpha_{0,1} = 1, \alpha_{2,0} = \omega , \alpha_{0,2} = 1, \alpha_{1,1} = \omega , \alpha_{2,1} = \omega^2, \alpha_{1,2} = 1, \alpha_{2,2} = 1\}$	$\frac{1}{3}$
$\{\alpha_{1,0} = \omega^2, \alpha_{0,1} = \omega , \alpha_{2,0} = \omega^2, \alpha_{0,2} = \omega , \alpha_{1,1} = 1, \alpha_{2,1} = 1, \alpha_{1,2} = 1, \alpha_{2,2} = 1\}$	$\frac{1}{3}$
$\{\alpha_{1,0} = \omega, \alpha_{0,1} = \omega , \alpha_{2,0} = 1, \alpha_{0,2} = 1 , \alpha_{1,1} = \omega^2, \alpha_{2,1} = \omega, \alpha_{1,2} = \omega, \alpha_{2,2} = 1\}$ Factors to $(1 + \omega u_i + u_i^2)(1 + \omega v_i + v_i^2)$	$-\frac{1}{3}$
$\{\alpha_{1,0} = 1, \alpha_{0,1} = \omega , \alpha_{2,0} = \omega^2, \alpha_{0,2} = \omega , \alpha_{1,1} = \omega , \alpha_{2,1} = 1, \alpha_{1,2} = \omega , \alpha_{2,2} = 1\}$	$-\frac{1}{3}$
$\{\alpha_{1,0} = 1, \alpha_{0,1} = 1, \alpha_{2,0} = \omega^2, \alpha_{0,2} = \omega , \alpha_{1,1} = 1, \alpha_{2,1} = \omega^2, \alpha_{1,2} = \omega , \alpha_{2,2} = 1\}$	$\frac{1}{3}$
$\{\alpha_{1,0} = 1, \alpha_{0,1} = \omega , \alpha_{2,0} = 1, \alpha_{0,2} = \omega^2, \alpha_{1,1} = \omega^2, \alpha_{2,1} = 1, \alpha_{1,2} = \omega , \alpha_{2,2} = 1\}$	$\frac{1}{3}$
$\{\alpha_{1,0} = 0, \alpha_{0,1} = 0, \alpha_{2,0} = 0, \alpha_{0,2} = 0, \alpha_{1,1} = \omega , \alpha_{2,1} = 0, \alpha_{1,2} = 0, \alpha_{2,2} = \omega\}$	$-\frac{i}{\sqrt{3}}$

Table A.10: Unitary solutions to R_i with relations given by $\mathcal{M}_{10}$