PERIODIC GROUPS VIA ORBITAL GRAPHS

A Dissertation

by

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ABSTRACT

In this work, we introduce a method of proving when an infinite group of homeomorphisms of a Cantor set is periodic using the geometry of its orbital graphs. In doing so, we expand a recent class of infinite finitely generated periodic groups introduced by Volodymyr Nekrashevych. In particular, we generalize his concept of fragmentation to arbitrary groups of homeomorphisms of a Cantor set, and give examples of finitely generated groups that can be fragmented to produce groups of Burnside type. Although some examples start with a group of isometries of the boundary of an infinite regular rooted tree, the fragmentations of such a group, in general, will not be a group of isometries.

It turns out that there is a strong relationship between fragmentations that produce a periodic group and certain subdirect products of a finite product of finite groups. We describe this relationship and give some results on when these types of subdirect products exists.

In order to study the orbital graphs of a group, we will realize the Cantor set as a space of infinite sequences, namely, as a space of infinite paths of a Bratelli diagram. Using partial actions of the group on finite paths, we can approximate certain connected infinite subgraphs of an orbital graph using finite graphs. There is a recursive procedure to building these approximating finite graphs described by the defining Bratteli diagram. We can then “paste" together some infinite subgraphs to form the orbital graph. In the best case scenario, a single such infinite subgraph will coincide with the entire orbital graph.

DEDICATION

To my wife Lisa Cantu for her love, patience, and support

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1. INTRODUCTION

1.1 History of the Burnside Problems

A *periodic* (or *torsion*) group G is a group in which every element has finite order. If there is a positive integer n such that $g^n = 1$ for all g in G, then we say G has *bounded exponent*. The least such n is the *exponent* of G. Clearly, any finite group G is periodic with exponent at most $|G|$. In 1902, William Burnside [4] posed the following question:

Problem 1.1.1 (General Burnside Problem)**.** Is every finitely generated periodic group necessarily finite?

In 1964, Golod and Shaferavich provided examples, for each prime p, of an infinite finitely generated group in which each element has order a power of p (see [7]), thus answering the General Burnside Problem in the negative. Their examples were attained as Galois groups of number field extensions and are now called Golod-Shafarevich groups. These groups, however, do not have bounded exponent.

Earlier work focused on Burnside's "easier" question of whether every finitely generated periodic group of exponent n is necessarily finite, known as the Bounded Burnside Problem, or just Burnside Problem. Let $S = \{s_1, s_2, \ldots, s_m\}$ and S^* denote the set of all finite words over S. The *free Burnside group* of rank m and exponent n, denoted $B(m, n)$, is the group with presentation $\langle S \mid g^n = 1, g \in S^* \rangle$. Any group with m generators and exponent n is a quotient of $B(m, n)$, thus the Burnside Problem is equivalent to asking whether every $B(m, n)$ is finite.

The case of rank 1 is trivial as $B(1, n)$ is just a cyclic group of order n. Burnside [4] proved $B(m, n)$ is finite for $n \leq 3$. In 1940, I. N. Sanov [20] showed that $B(m, 4)$ is finite. For exponent 5, it is still an open question whether $B(2, 5)$ is finite. Marshall Hall Jr. [12] proved that $B(m, 6)$ is finite in 1958.

In 1968, Adian and Novikov showed $B(m, n)$ is infinite for odd $n \geq 4381$ and $m > 1$, thus answering the Bounded Burnside Problem in the negative (see [18]). Several years later, Adian reduced the bound to odd $n \geq 665$. A corollary of this is that $B(m, kn)$ is infinite for any even k. Ivanov [14] and Lysenok [15] proved infiniteness of $B(m, n)$ for some even exponents n without odd divisor ≥ 665.

Some interesting examples of groups of Burnside type were discovered in the class of groups defined by their action on a rooted tree and groups generated by automata. The first such examples were constructed by S. Aleshin [1], V. Sushchanskii [21], and R. Grigorchuk [8].

A recent class of examples was constructed by V. Nekrashevych [17], where he developed a method (called *fragmentation*) to transform an arbitrary non-free minimal action of the infinite dihedral group D_∞ on a Cantor set into an orbit-equivalent action of an infinite finitely generated periodic group. This dissertation will generalize the definition of fragmentation to transform arbitrary groups of homeomorphisms of a Cantor set, and expand this class.

Infinite finitely generated periodic groups provided many firsts in group theory: first example of a group of intermediate growth [9], first examples of non-elementary amenable groups [10], first examples of non-amenable groups without free subgroups [19], and the first example of a simple group of intermediate growth [17]. Their existence was also used to show that the class of groups without free subgroups and the class of elementary amenable groups are distinct (see [5]). All previous examples of infinite finitely generated periodic groups, including these important examples, are part of four broad classes. The first class is the Golod-Shaferavich groups. The second class consists of solutions to the Bounded Burnside Problem. The third class contains groups defined by their action on rooted trees and groups generated by automata. The fourth class are those obtained as fragmentations of the infinite dihedral group.

1.2 Organization

This dissertation is organized as follows. Chapter 2 gives the necessary background, including definitions and notation, that will be used throughout the dissertation. This comprises of some graph theory, information on group actions, and an introduction to Bratteli diagrams.

Chapter 3 is dedicated to building the ingredients needed to state and prove the main theorem of periodicity. We begin by defining the domains of support of a finite group of homeomorphisms of a Cantor space. Under suitable conditions, the domains of support form a finite partition of the support of the group into invariant open sets. We then introduce germ-defining singular points. This special type of fixed point of a set of homeomorphisms gives us a handle on the structure of the graph of germs of the action at this point. Finally, we establish the notion of a thin graph, before stating and proving the periodicity theorem.

Chapter 4 discusses fragmentations of a homeomorphism and of a group of homeomorphisms. This is a generalization of the concepts of fragmentations of involutions and actions of the infinite dihedral group on a Cantor space introduced by Volodymyr Nekrashevych. We note a correspondence between certain fragmentations of finite groups and subdirect products. The chapter concludes with some results on subdirect products.

Chapter 5 is dedicated to examples to which we can apply the main periodicity theorem. Each example starts with a finitely generated group of homeomorphisms of a Cantor set (with properties analogous to the infinite dihedral group), and gives conditions on fragmentations of the group that result in periodicity. We mention that such conditions can be met, and thus give examples of infinite finitely generated periodic groups.

Chapter 6 summarizes the dissertation and discusses some natural questions related to the work in this paper.

2. PRELIMINARIES

We will assume that the reader is familiar with some group theory (at the level of chapter 1 of [13]) and topology (at the level of chapter 1 of [3]).

2.1 Graph theory

A *simple graph* Γ consists of a set $V = V(\Gamma)$ of *vertices* and a set $E = E(\Gamma)$ of 2-element subsets of V, called *edges*. If we instead let E consist of ordered pairs of vertices, called *directed edges* or *arrows*, then Γ is called a *directed graph*. Arrows of the form (v, v) for $v \in V$ are called *loops*. For $e = (u, v) \in E$, we say u is the *source* of e and v is the *target* of e, denoted $\mathsf{s}(e)$ and $\mathsf{t}(e)$, respectively. The vertices that make up an edge e (whether directed or undirected) are called the *endpoints* of e, and we say u, v are *adjacent* if there is an edge e with endpoints u and v. Allowing E to be a multiset gives the definition of a *multigraph*. We say Γ is *labeled (or edge-labeled)* if we are given a function $\ell : E \longrightarrow L$ for some finite set L of *labels*. A *rooted graph* is a pair (Γ, v), where Γ is a graph and v is a vertex of Γ.

For a vertex v of a simple graph, the *degree* of v is $\deg(v) = |\{e \in E : v \in e\}|$. If v is a vertex of a directed graph, then $\deg(v) = |\{e \in E : \mathsf{s}(e) = v\}| + |\{e \in E : \mathsf{t}(e) = v\}|$. All graphs in this paper are assumed to be *locally finite*, that is, $\deg(v)$ is finite for all $v \in V$. The *size* of a graph is the cardinality of its vertex set, denoted $|\Gamma|$ instead of $|V|$. Similarly, we occasionally use the notation $v \in \Gamma$ to mean v is a vertex of Γ.

A graph Δ is called a *subgraph* of Γ if $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. Furthermore, if Γ is labeled by ℓ, we require Δ be labeled by $\ell|_\Delta$. If Γ has root v and $v \in V(\Delta)$, we require Δ to have root v. Otherwise, Δ is not rooted. The *boundary vertices* of Δ, denoted $\partial_V(\Delta)$, are the vertices of Δ that are adjacent to a vertex outside of Δ.

Given a subset V' of V, we define the *vertex-induced* subgraph $\Gamma[V']$ of Γ with vertex set V' and all edges from E whose endpoints are in V'. Let $\Gamma \setminus V'$ denote the subgraph of Γ obtained by removing the vertices V' and any edges in E with an endpoint in V'. In other words, $\Gamma \setminus V' =$

$\Gamma[V \setminus V']$. Given a subset E' of E, we define the *edge-induced* subgraph $\Gamma[E']$ of Γ with edge set E' and all vertices from V which are an endpoint of an edge in E'. Let $\Gamma \setminus E'$ denote the subgraph of Γ with vertex set V and edge set $E \setminus E'$.

Let Γ_1, Γ_2 be two graphs of the same type (e.g., directed, labeled, rooted) with vertex sets V_1, V_2 and edge sets E_1, E_2. A *homomorphism* from Γ_1 to Γ_2, written $\phi : \Gamma_1 \longrightarrow \Gamma_2$, consists of a pair of bijections $\phi_V : V_1 \longrightarrow V_2$ and $\phi_E : E_1 \longrightarrow E_2$ (both written as just ϕ in the following) that respect the structure of the graphs. First, this means that if $e \in E_1$ has endpoints $u, v \in V_1$, then $\phi(e)$ has endpoints $\phi(u), \phi(v)$. If the graphs are directed and $e = (u, v)$, we require $\phi(e) = (\phi(u), \phi(v))$, that is, $\mathsf{s}(\phi(e)) = \phi(\mathsf{s}(e))$ and $\mathsf{t}(\phi(e)) = \phi(\mathsf{t}(e))$. If the graphs are labeled, say by ℓ_1 and ℓ_2, we require that ϕ preserves labels : $\ell_2(\phi(e)) = \ell_1(e)$. If Γ_1 has root v_1 and Γ_2 has root v_2, we require $\phi(v_1) = v_2$. If ϕ_V and ϕ_E are injective, we call ϕ an *embedding* and say Γ_1 *embeds* into Γ_2. If both ϕ_V and ϕ_E are bijections, we call ϕ an *isomorphism*. In particular, when $\Gamma_1 = \Gamma_2$, we call ϕ an *automorphism* of Γ_1.

A *walk* w in a graph Γ is a sequence $(v_0, e_1, v_1, \ldots, e_m, v_m)$ of alternating vertices v_i and edges e_i such that e_i has endpoints v_{i-1} and v_i. We say w *starts* at v_0 and *ends* at v_m. If Γ is directed and for each i the edge e_i has source v_{i-1} and target v_i, we call w *directed*. For $0 \leq j \leq k \leq m$, the walk $(v_j, e_{j+1}, \ldots, e_k, v_k)$ is called a *subwalk* of w. A walk consisting of only the vertex v_0 and no edges is called *trivial*. The *inverse* of w is the walk $w^{-1} = (v_m, e_m, \ldots, v_1, e_1, v_0)$. If $w = (v_0, e_1, v_1, \ldots, e_m, v_m)$ and $w' = (v'_0, e'_1, v'_1, \ldots, e'_n, v'_n)$ are walks such that $v_m = v'_0$, we define their *concatenation* $w_1 w_2 = (v_0, e_1, v_1, \ldots, e_m, v_m = v'_0, e'_1 \ \ldots, v'_n)$. The *length* of a walk is its number of edges and we say a walk is *closed* if $v_0 = v_m$. A walk is called a *path* if its vertices are distinct. A closed walk is a *cycle* if all of its edges are distinct and the vertices $v_0, v_1, \ldots, v_{m-1}$ are distinct.

Any walk $w = (v_0, e_1, v_1, \ldots, e_m, v_m)$ induces a subgraph $\Gamma(w)$ of Γ with vertices $v_0, v_1, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_m$. We also use the terminology walk, path, and cycle for these induced subgraphs.

If there is a path in Γ starting at vertex v and ending at vertex v', we say v is *connected to*

v'. This forms an equivalence relation on V. The subgraphs induced by the equivalence classes are called the *connected components* of Γ. A graph is called *connected* if it has one connected component. If the vertices v and v' are connected, the *distance* from v to v' is the length of a shortest path between them.

Let v be a vertex of Γ. For an integer $r \geq 0$, the *ball* of radius r centered at v, denoted $B_v(r)$, is the subgraph of Γ induced by the vertex set consisting of vertices at distance less than r from v. Let $\mathcal{G}_S$ be the set of all isomorphism classes of connected rooted directed graphs labeled by S. For $(\Gamma_1, v_1), (\Gamma_2, v_2) \in \mathcal{G}_S$, define the distance $d((\Gamma_1, v_1), (\Gamma_2, v_2))$ between them as 2^{-r}, where r is the maximal integer such that $B_{v_1}(r)$ and $B_{v_2}(r)$ are isomorphic (note that $B_v(0)$ is the null graph). This metric defines a topology on $\mathcal{G}_S$.

2.2 Group actions

Let G be a group and X be a nonempty set. An *action* of G on X is a homomorphism $\rho : G \longrightarrow \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the group of bijections of X under function composition. Every $g \in G$ defines a symmetry of X, $\rho_g = \rho(g) : X \longrightarrow X$. Since ρ is a homomorphism, ρ_{id} is the identity map on X and $\rho_{gh} = \rho_g \circ \rho_h$. If X has more structure (e.g., topological space), then we say G *acts by automorphisms* (e.g., homeomorphisms) if ρ_g is an automorphism of X for all $g \in G$. Let $\mathrm{Aut}(X)$ denote the set of automorphisms of X. Any subgroup G of $\mathrm{Aut}(X)$ defines a *natural* action of G on X under the natural embedding map of G.

Equivalently, we can define an action of G on X by a mapping $\phi : G \times X \longrightarrow X$ (with images denoted $g(x) = \phi(g, x)$) such that

1. $id(x) = x$ for all $x \in X$,

2. $(gh)(x) = g(h(x))$ for all $g, h \in G$ and $x \in X$.

For every $g \in G$, the map $\phi_g = \phi(g, \cdot) : X \longrightarrow X$ is a bijection of X. In particular, ϕ_{id} is the identity map on X. From condition (2), we have $\phi_{gh} = \phi_g \phi_h$. Thus the map $\rho : G \longrightarrow \mathrm{Sym}(X)$ defined by $\rho(g) = \phi_g$ is a homomorphism. Conversely, for any homomorphism ρ, the map $\phi(g, x) = [\rho(g)](x)$ satisfies both conditions above.

An action of G on X is called *faithful* if $g(x) = x$ for all $x \in X$ implies $g = id$. In other words, if the homomorphism $\rho : G \longrightarrow \text{Aut}(X)$ is injective and G can be realized as a subgroup of $\text{Aut}(X)$. An action is *free* if whenever $g(x) = x$ for some $x \in X$, we have $g = id$. Clearly all free actions are faithful.

Example 2.2.1. Let G be any group and $X = G$. Then G acts on X by left multiplication: $g(x) = gx$ for all $g, x \in G$. This action is free since if $g(x) = x$ for some $x \in X$, we get $gx = x$, and thus $g = id$ by multiplication on the right by x^{-1}. Notice that although X has group structure, each $g \in G \setminus \{id\}$ only defines a symmetry and not an automorphism of X. Indeed, $g(id) = g \neq id$, so g does not define a homomorphism.

The *orbit* of $x \in X$ is the set $G(x) = \{g(x) : g \in G\}$. If $y \in G(x)$, then $G(y) = G(x)$. An action is *transitive* if $G(x) = X$ for some (and thus all) $x \in X$. In other words, for every pair $x, y \in X$ there exists $g \in G$ such that $g(x) = y$. A subset A of X is called *G-invariant* if $g(A) = A$ for all $g \in G$. For example, any orbit is G-invariant. For any G-invariant subset A of X, we define $G|_A = \{g|_A : g \in G\}$, considered as a group of homeomorphisms of A (with subspace topology). The stabilizer of x, denoted G_x, is the subgroup of G consisting of all elements that fix x, that is, $G_x = \{g \in G : g(x) = x\}$. We say A is *g-invariant* if $g(A) = A$, and A is *fixed by g* if $g(a) = a$ for all $a \in A$.

Suppose G is generated by a finite set S. For $x \in X$, the *orbital graph* $\Gamma_x = \Gamma_x(G, X, S)$ is the graph with the set of vertices equal to the orbit $G(x)$ of x, in which for every $y \in G(x)$ and every $s \in S$ there is an arrow from y to $s(y)$ labeled by s. It follows that orbital graphs are labeled directed multigraphs. The edges of Γ_x are naturally identified with the set $S \times G(x)$ by $(x, s(x)) \leftrightarrow (s, x)$. Note that Γ_x has a natural root given by the vertex x. The action of G is completely determined by the action of its generators S, thus the orbital graphs encode all information about the action.

The graph Γ_x is naturally isomorphic to the *Schreier graph* of the group G modulo the stabilizer G_x. The Schreier graph of G modulo a subgroup H, denoted $\Gamma(G, H, S)$, is the graph with the set of vertices equal to the set of cosets $gH, g \in G$, in which for every coset gH and every generator $s \in S$ there is an arrow from gH to sgH labeled by s. In other words, the Schreier graph $\Gamma(G, H, S)$ is exactly the orbital graph $\Gamma_H(G, G/H, S)$ where G acts on G/H by left multiplication.

If s is an involution, undirected edges labeled by s are typically used in place of a pair of opposing arrows. Furthermore, loops corresponding to the identity element id of G are excluded in the case id is in S. The graph $\Gamma(G, S) = \Gamma(G, \{id\}, S)$ is called the *Cayley graph* of G (with respect to S). A Cayley graph Γ is *homogeneous* in the sense that (Γ, v_1) is isomorphic to (Γ, v_2) for any vertices v_1, v_2 of Γ, so that a Cayley graph looks the same everywhere.

When X is a topological space and G acts by homeomorphisms, we can define other types of actions, natural subgroups of G, and graphs associated to actions. We will focus on the case where X is a Cantor space. An action of G on X is called *minimal* if each orbit is dense in X. The terminology stems from the equivalence that an action is minimal if and only if X is the only nontrivial G-invariant closed subset of $\mathcal{X}$. Thus we can't (topologically) factor the action on X into actions on subspaces.

Denote by $G_{(x)}$ the subgroup of elements of G acting trivially on a neighborhood of x, called the *neighborhood stabilizer* of x. The *graph of germs* $\Gamma_{(x)}$ is the Schreier graph of G modulo $G_{(x)}$. Note that $G_{(x)}$ is a normal subgroup of G_x, hence the map $hG_{(x)} \mapsto hG_x$ induces a Galois covering of graphs $\Gamma_{(x)} \to \Gamma_x$ with the group of deck transformations $G_x/G_{(x)}$. We call $G_x/G_{(x)}$ the *group of germs* of the point x. The vertices of $\Gamma_{(x)}$ are identified with *germs* of elements of G at x. Here a *germ* is an equivalence class of a pair (g, x), where two pairs (g_1, x) and (g_2, x) are equivalent if there exists a neighborhood U of x such that $g_1|_U = g_2|_U$.

Let A be a subset of X that accumulates on x, that is, every neighborhood U of x intersects $A \setminus \{x\}$. Denote by $G_{(x,A)}$ the subgroup of elements of G acting trivially on a subset of A of the form $U \cap A$ for some neighborhood U of x, called the *neighborhood stabilizer of x relative to A*. It is easy to see that $G_{(x)} \leq G_{(x,A)} \leq G_x$. For example, $G_{(x,X)} = G_{(x)}$. The *graph of germs relative to A*, denoted $\Gamma_{(x,A)}$, is the Schreier graph of G modulo $G_{(x,A)}$. The vertices of $\Gamma_{(x,A)}$ can be identified with equivalence classes of triples (g, x, A), where (g_1, x, A) and (g_2, x, A) are equivalent if there exists a neighborhood U of x such that $g_1|_{U\cap A} = g_2|_{U\cap A}$.

Let h be a homeomorphism of X. A fixed point x of h is called *singular* if h fixes x but does not fix a neighborhood of x. For a set H of homeomorphisms, we say x is a singular point of H if x is a singular point of some $h \in H$. If x is not a singular point of H, then we say that it is H-*regular*, or just *regular* when H is understood. A subset of X is called regular (respectively, singular) if it consists entirely of regular (resp., singular) points.

For a homeomorphism h of X, we define the *support* of h to be $\mathrm{supp}(h) = \{x \in X : h(x) \neq x\}$. It is an open subset of X. For a set H of homeomorphisms, the *support* of H is the set $\mathrm{supp}(H) = \bigcup_{h\in H} \mathrm{supp}(h)$. The set of fixed points of h is a closed subset of $\mathcal{X}$ and is denoted by $\mathrm{Fix}(h)$. The set of singular points of a homeomorphism h is given by the common boundary of $\mathrm{supp}(h)$ and $\mathrm{Fix}(h)$. So we have $X = \mathrm{supp}(h) \bigsqcup \mathrm{Fix}(h)^\circ \bigsqcup \partial \mathrm{Fix}(h)$, where $\mathrm{Fix}(h)^\circ$ denotes the interior of $\mathrm{Fix}(h)$.

The point $x \in X$ is G-regular if and only if $G_{(x)} = G_x$. Since $gG_{(x)}g^{-1} = G_{(g(x))}$ and $gG_xg^{-1} = G_{g(x)}$ for all $x \in X$ and $g \in G$, the set of G-regular points is G-invariant. Consequently, any orbit $G(x)$ is either regular or singular.

2.2.1 Groupoids

A *groupoid* is a set $\mathcal{G}$ with partially defined multiplication and everywhere defined operation of taking inverses satisfying the following:

1. if g_1g_2 and g_2g_3 are defined, then $(g_1g_2)g_3$ and $g_1(g_2g_3)$ are both defined and are equal,

2. for every $g \in \mathcal{G}$ the products gg^{-1} and $g^{-1}g$ are defined,

3. if g_1g_2 is defined, then $(g_1^{-1}g_1)g_2$ and $g_1(g_2g_2^{-1})$ are both defined, and $(g_1^{-1}g_1)g_2 = g_2$ and $g_1(g_2g_2^{-1}) = g_1$.

Example 2.2.2. Let G be a group acting by homeomorphisms on a topological space X. Consider the equivalence relation on $G \times X$, where (g_1, x_1) and (g_2, x_2) are equivalent if $x_1 = x_2$ and there is a neighborhood U of x_1 such that $g_1|_U = g_2|_U$. The equivalence classes are called *germs*, and the germ (g, x) encodes the local action of g at x. The set of germs has a groupoid structure with multiplication $(g_1, g_2(x))(g_2, x) = (g_1 g_2, x)$ and inverses $(g, x)^{-1} = (g^{-1}, g(x))$.

2.3 Bratteli diagrams

A *Bratteli diagram* $\mathsf{D} = ((V_i)_{i\geq 0}, (E_i)_{i\geq 1}, \mathsf{s}, \mathsf{t})$ is defined by two sequences of finite sets $(V_0, V_1, \dots)$ and $(E_1, E_2, \dots)$, and maps $\mathsf{s} : \bigsqcup_{i\geq 1} E_i \longrightarrow \bigsqcup_{i\geq 0} V_i$, $\mathsf{t} : \bigsqcup_{i\geq 1} E_i \longrightarrow \bigsqcup_{i\geq 1} V_i$ such that $\mathsf{s}(E_i) = V_{i-1}$ and $\mathsf{t}(E_i) = V_i$ for $i \geq 1$. We interpret the sets $V = \bigsqcup_{i\geq 0} V_i$ and $E = \bigsqcup_{i\geq 1} E_i$ as *vertices* and *edges* of the diagram, partitioned into *levels*, and s and t as the *source* and *target* maps. An edge $e \in E_i$ connects $\mathsf{s}(e) \in V_{i-1}$ and $\mathsf{t}(e) \in V_i$. An example of a Bratteli diagram is shown in Figure 5.4. A *path of length* n, where n is a positive integer or infinity, in D is a sequence of edges $e_i \in E_i$, $1 \leq i \leq n$, such that $\mathsf{t}(e_i) = \mathsf{s}(e_{i+1})$ for all i. For n finite let $\Omega_n = \Omega_n(\mathsf{D})$ denote the set of paths in D of length n, and let $\Omega = \Omega(\mathsf{D})$ denote the set of infinite paths in D. Define $\Omega_* = \bigsqcup_{n\geq 1} \Omega_n$ to be the set of all finite paths in D, and extend the map s to $\Omega_* \cup \Omega$ by $\mathsf{s}(e_1, e_2, \dots) = \mathsf{s}(e_1)$, and the map t to Ω_* by $\mathsf{t}(e_1, e_2, \dots, e_n) = \mathsf{t}(e_n)$. We say $u \in \Omega_* \cup \Omega$ *starts* at $\mathsf{s}(u)$ and *ends* at $\mathsf{t}(u)$.

For $m \geq 1$, we can construct the Bratteli diagram $\mathsf{D}^{(m)}$ with vertex sequence $(V_m, V_{m+1}, \dots)$ and edge sequence $(E_{m+1}, E_{m+2}, \dots)$. We call the diagram $\mathsf{D}^{(m)}$ the *m-shift* of D, and consider it as a *subdiagram* of D. We then define $\Omega_n^{(m)} = \Omega_n(\mathsf{D}^{(m)})$ and $\Omega^{(m)} = \Omega(\mathsf{D}^{(m)})$, considered as length n paths and infinite paths in D starting in level m.

For $x \in V_n$, let $\Omega_x = \Omega_x(\mathsf{D})$ denote the set of paths in D ending at x. Similarly, for $m < n$, we define $\Omega_x^{(m)} = \Omega_x(\mathsf{D}^{(m)})$. Given $v = (e_{n+1}, e_{n+2}, \dots) \in \Omega^{(n)} \cup \Omega_x^{(n)}$ and $u = (e_{m+1}, \dots, e_n) \in \Omega_{\mathsf{s}(v)}^{(m)}$, their *concatenation* is the path $uv = (e_{m+1}, \dots, e_n, e_{n+1}, \dots) \in \Omega^{(m)} \cup \Omega_x^{(m)}$. We say $u \in \Omega_m$ is a *prefix* of $w \in \Omega \cup \Omega_x$ if there exists $v \in \Omega^{(m)} \cup \Omega_x^{(m)}$ such that $w = uv$. For $w \in \Omega$ and $n \geq 1$, let $w_n \in \Omega_n$ denote the length n prefix of w. The following is straightforward.

Proposition 2.3.1. *Let $x \in V_{n+m}$, then*

$$\Omega_x = \bigsqcup_{v\in\Omega_x^{(n)}} \Omega_{\mathsf{s}(v)} v,$$

where $\Omega_{\mathsf{s}(v)}v = \{uv \mid u \in \Omega_{\mathsf{s}(v)}\}$.

We can consider Ω to be a subspace of $\prod_{i\geq 1} E_i$ endowed with the prodiscrete topology. It is closed, compact, Hausdorff, and totally disconnected. For $w \in \Omega_n$, let $w\Omega$ denote the set of infinite paths beginning with w (note slight abuse of notation). The sets $w\Omega$, $w \in \Omega_*$, are clopen and form a basis of topology for Ω. We will assume Ω has no isolated points, so that it is homeomorphic to the Cantor set. This is if and only if $w\Omega$ is infinite for all $w \in \Omega_*$.

If $u, v \in \Omega_x$, then the map $\tau_{u,v} : uw \mapsto vw$ is a homeomorphism between $u\Omega$ and $v\Omega$. Given a homeomorphism h of Ω, let $\mathrm{supp}_x(h)$ be the subset of Ω_x consisting of elements u such that $h|_{u\Omega} = \tau_{u,v}$ for some $v \in \Omega_x$, $v \neq u$, and define $\mathrm{supp}_n(h) = \bigcup_{x\in V_n} \mathrm{supp}_x(h)$, called the *level* n *support* of h. Further, let $\mathrm{supp}_*(h) = \bigcup_{n\geq 1} \mathrm{supp}_n(h)$. Consider the poset $(\Omega_*, \leq)$, where $u \leq v$ if u is a prefix of v, and let $\mathcal{N}(h)$ be the set of minimal elements of $\mathrm{supp}_*(h)$. In other words, for $u \in \Omega_x$, $h|_{u\Omega} = \tau_{u,v}$ for some $v \in \Omega_x \setminus \{u\}$ if and only if there exists $u' \in \mathcal{N}(h)$ that is a prefix of u.

Definition 2.3.2. Let h be a homeomorphism of Ω and let $w \in \Omega$. We say w is a *critical point* of h if $h|_{w_n\Omega} \neq \tau_{w_n,v}$ for all $n \geq 1$ and $v \in \Omega_{\mathsf{t}(w_n)}$. We will use $\mathcal{C}_h$ to denote the set of critical points of h.

For a set S of homeomorphisms of Ω, define the set of critical points of S to be $\mathcal{C}_S = \bigcup_{s\in S} \mathcal{C}_s$. Note that a fixed critical point is singular (if w is a critical point of h, then $h|_{w_n\Omega} \neq \tau_{w_n,w_n} = id|_{w_n\Omega}$ for all $n \geq 1$).

Proposition 2.3.3. *Let* h *be a homeomorphism of* Ω*, then supp*$(h) = \bigsqcup_{u\in\mathcal{N}(h)} u\Omega \sqcup \{$*non-fixed* $\mathcal{C}_h\}$.

Proof. Let $w \in \mathrm{supp}(h)$. If w is not a critical point of h, then there exists a minimal $n \geq 1$ such that $h|_{w_n\mathcal{X}} = \tau_{w_n,v}$ for some $v \in \Omega_{\mathsf{t}(w_n)}$. We have $v \neq w_n$ since $w \in \mathrm{supp}(h)$, so that $w_n \in \mathcal{N}(h)$ (by minimality of n). It follows that $\mathrm{supp}(h) \subseteq \bigsqcup_{u\in\mathcal{N}(h)} u\Omega \sqcup \{\text{non-fixed } \mathcal{C}_h\}$.

Now let $uw \in u\Omega$ for some $u \in \mathcal{N}(h)$ of length n. Then $h|_{u\Omega}(uw) = \tau_{u,v}(uw) = vw$ for some $u \neq v \in \Omega_{\mathsf{t}(w_n)}$, so $uw \in \mathrm{supp}(h)$. □

In particular, if all critical points of h are fixed points, then $\mathrm{supp}(h) = \bigsqcup_{u \in \mathcal{N}(h)} u\Omega$. The set $\bigsqcup_{u \in \mathcal{N}(h)} u\Omega$ is called the *noncritical support* of h, and we say $\mathcal{N}(h)$ is its *generating set*.

2.3.1 Tiles

Let $G = \langle S \rangle$ be a finitely generated group acting faithfully on Ω by homeomorphisms. For $x \in V^{(1)}$, let T_x be the labeled directed graph with the set of vertices Ω_x and an edge labeled by $s \in S$ from u to v if $s|_{u\Omega} = \tau_{u,v}$. The graphs T_x, $x \in V_n$, are called *level n tiles*. For $u \in \Omega_n$, let $T_u = (T_{\mathsf{t}(u)}, u)$.

The *boundary* of T_x, denoted ∂T_x, consists of pairs $(s, u) \in S \times \Omega_x$ such that $s|_{u\Omega} \neq \tau_{u,v}$ for all $v \in \Omega_x$. We will also use the term *boundary edges* as we can interpret the boundary element (s, u) as an arrow in T_x labeled by s with source u and no target vertex.

Let $x \in V_{n+m}$. For every $v \in \Omega_x^{(n)}$, the map $\phi_v : u \mapsto uv$ gives an isomorphism of $T_{\mathsf{s}(v)}$ with a subgraph of T_x. We will denote this subgraph by $T_{\mathsf{s}(v)}v$. The graphs in $\{T_{\mathsf{s}(v)}v : v \in \Omega_x^{(n)}\}$ are disjoint and their union contains all vertices of T_x by Proposition 2.3.1. It follows that a level $n + m$ tile can be obtained by connecting tiles of the nth level along their boundary. In particular, we can obtain a level $n + 1$ tile from level n tiles.

Proposition 2.3.4 (Tile Inflation). *Fix $x \in V_{n+1}$ and consider the disjoint union of the graphs $T_{\mathsf{s}(e)}e$ for $e \in \Omega_x^{(n)} = \mathsf{t}^{-1}(x)$. Connect $ue \in T_{\mathsf{s}(e)}e$ to $u'e' \in T_{\mathsf{s}(e')}e'$ by an arrow labeled by s if $(s, u) \in \partial T_{\mathsf{s}(e)}$ and $s|_{ue\Omega} = \tau_{ue,u'e'}$. The obtained graph is equal to T_x. In particular, if $(s, ue) \in \partial T_x$, then $(s, u) \in \partial T_{\mathsf{s}(e)}$.*

Proof. Easily follows from the discussion in previous paragraph and the definition of tile graphs. □

For $x \in V_{n+1}$, we introduce the *model graph* $\mathcal{M}_x$ to describe how the boundaries of the level n tiles $T_{\mathsf{s}(e)}$, $e \in \mathsf{t}^{-1}(x)$, are connected to form T_x. It is the subgraph of T_x with vertex set $\{ue : e \in \mathsf{t}^{-1}(x), (s, u) \in \partial T_{\mathsf{s}(e)}\}$ with an edge from ue to $u'e'$ labeled by s if $(s, u) \in \partial T_{\mathsf{s}(e)}$ and $s|_{ue\Omega} = \tau_{ue,u'e'}$, or a dashed edge with source ue and no target vertex if $s|_{ue\Omega} \neq \tau_{ue,u'e'}$ for all $u'e'$. We can then imagine placing the graphs $T_{\mathsf{s}(e)}$, $e \in \Omega_x^{(n)}$, into the model graph so that if $(s,u) \in \partial T_{\mathsf{s}(e)}$, u aligns with ue, forming T_x as described in Proposition 2.3.4.

We say $w, z \in \Omega$ are *cofinal* if they differ in only finitely many edges. This defines an equivalence relation on Ω with equivalence classes denoted by $Cof(\cdot)$, called cofinality classes.

Let $z = (e_1, e_2, \dots) \in \Omega$. Consider the subgraph of Γ_z with vertex set $G(z)$ and an edge from w to w' labeled by $s \in S$ if there exists $n \geq 1$ such that $s|_{w_n\Omega} = \tau_{w_n, w_n'}$. The connected component of this graph containing z is the *orbital tile graph* T_z. The vertex set of T_z is $Cof(z) \cap G(z)$, and it is straightforward to see that the rooted graphs $(T_{\mathsf{t}(z_n)}, z_n)$ converge to (T_z, z). We also call the graphs T_z *infinite tiles*. We can then define the boundary of T_z, denoted by ∂T_z, as the set of pairs $(s, w) \in S \times (Cof(z) \cap G(z))$ such that $s|_{w_n\Omega} \neq \tau_{w_n, v}$ for all $v \in \Omega_{\mathsf{t}(w_n)}$ and $n \geq 1$ (i.e., such that w is a critical point of s). We can then construct orbital graphs from infinite tile graphs.

Proposition 2.3.5. *Fix $z \in \Omega$.*

1. *Then $G(z) \cap \mathcal{C}_S$ is empty if and only if $T_z = \Gamma_z$.*

2. *Suppose $G(z) \cap \mathcal{C}_S$ is nonempty. For every $w \in G(z) \cap \mathcal{C}_S$ and $(s, w) \in \partial T_w$ connect $w \in T_w$ to $s(w) \in T_{s(w)}$ by an arrow labeled by s. The obtained graph is equal to Γ_z.*

Proof. Suppose $G(z) \cap \mathcal{C}_S$ is empty. It is sufficient to show Γ_z is a subgraph of T_z. We first prove that $G(z) \subseteq Cof(z)$, so that T_z and Γ_z have the same vertex set. Let $w \in G(z)$, then there exists $g \in G$ such that $g(z) = w$. Write $g = s_m \cdots s_1$ as a product of elements of $S \cup S^{-1}$ and note that if S has no critical points in Gz, then $S \cup S^{-1}$ has no critical points in $G(z)$. Thus, z is not a critical point of s_1, $s_1(z)$ is not a critical point of $s_2, \dots$, and $s_{m-1} \cdots s_1(z)$ is not a critical point of s_m. It follows that g changes only finitely many letters of z, so that $w \in Cof(z)$. Now consider an arrow labeled by s from w to $s(w)$ in Γ_z. Since w is not a critical point of s, the same arrow exists in T_z. Thus Γ_z is a subgraph of T_z.

Conversely, if $\Gamma_z = T_z$, then ∂T_z is empty and $Cof(z) \cap G(z) = G(z)$. It follows that $G(z) \cap \mathcal{C}_S$ is empty.

Now suppose $G(z) \cap \mathcal{C}_S$ is nonempty. We first show that the vertices of $T_w \cup T_{s(w)}$ as w ranges over $G(z) \cap \mathcal{C}_S$ and $(s, w) \in \partial T_w$ cover the vertices of Γ_z, i.e., that

$$\bigcup_{w \in G(z) \cap \mathcal{C}_S} \bigcup_{(s,w) \in \partial T_w} \big(Cof(w) \cup Cof(s(w))\big) \cap G(z) = G(z).$$

The left containment is trivial. Let $w \in G(z)$ and suppose $Cof(w) \cap G(z) \cap \mathcal{C}_S$ is empty, then every vertex of T_w has $|S|$ outgoing arrows. There must be a vertex $w' \in G(z) \setminus Cof(w)$ and $s \in S$ such that $s(w') \in Cof(w)$. Indeed, if not, then $T_w = \Gamma_w$, a contradiction as $G(z) \cap \mathcal{C}_S = G(w) \cap \mathcal{C}_S$ is nonempty. Since w' and w differ in infinitely many letters, w' is a critical point of s. We have $w \in Cof(s(w'))$ as desired. Now suppose $Cof(w) \cap G(z) \cap \mathcal{C}_S$ is nonempty, so that there is an $s \in S$ and vertex w' of T_w such that w' is a critical point of s. We have that $w \in Cof(w')$ where $w' \in G(z) \cap \mathcal{C}_S$ and $(s, w') \in \partial T_{w'}$. □

Proposition 2.3.6. *If the finite tiles are eventually connected (i.e., there exists N such that any tile of level $n \geq N$ is connected), then $Cof(z) \subseteq G(z)$ for all $z \in \Omega$, so that the vertices of T_z can be identified with $Cof(z)$.*

Proof. There exists $N \in \mathbb{N}$ such that the tiles Ω_x are connected for all $x \in V_n$ and $n \geq N$. Suppose $z = (e_1, e_2, \ldots) \in \Omega$ and let $w = (f_1, f_2, \ldots) \in Cof(z)$ be different from z, then there exists $n \geq N$ such that $e_k = f_k$ for $k \geq n$. Since $T_{\mathsf{t}(z_n)}$ is connected, there exists $g \in G$ such that $g|_{z_n\Omega} = \tau_{z_n, w_n}$. Thus $g(z) = w$ and $w \in G(z)$. □

Let σ be the shift map on $\bigsqcup_{m \geq 0} \Omega^{(m)}$ sending $(e_{m+1}, e_{m+2}, e_{m+3}, \ldots)$ to $(e_{m+2}, e_{m+3}, e_{m+4}, \ldots)$. For $z \in \Omega$ and $n \geq 1$, we have $\sigma^n(z) \in \Omega^{(n)}$. The following is straightforward.

Proposition 2.3.7. *Let $z \in \Omega$ and $n \geq 1$, then*

$$Cof(z) = \bigsqcup_{w \in Cof(\sigma^n(z))} \Omega_{\mathsf{s}(w)} w.$$

Let $z \in \Omega$ and $n \geq 1$. Suppose the finite tiles are eventually connected, then the set of vertices of T_z is given by $Cof(z)$. For every $w \in Cof(\sigma^n(z))$, the map $\phi_w : v \mapsto vw$ gives an isomorphism of $T_{\mathsf{s}(w)}$ with a subgraph of T_z. Denote this subgraph by $T_{\mathsf{s}(w)}w$. The graphs $\{T_{\mathsf{s}(w)}w \mid w \in Cof(\sigma^n(z))\}$ are disjoint and their union contains all vertices of T_z by Proposition 2.3.7.

The study of orbital graphs descends to the study of orbital tile graphs, which are approximated by finite tiles. Thus, having a handle on finite tiles can lead to conclusions about orbital graphs. Dynamically, the action of G on Ω is approximated by a partial action of G on Ω that agrees with the complete action when defined. That is, $g \in G$ is defined on $w \in \Omega$ if there exists $n \geq 1$ and $v \in \Omega_{\mathsf{t}(w_n)}$ such that $g|_{w_n\Omega} = \tau_{w_n,v}$, and undefined if w is a critical point of g. The orbital graphs of this partial action are exactly the orbital tile graphs.

We also have partial actions of G on Ω_x, $x \in V^{(1)}$, whose "inductive limits" give the above partial action on Ω. That is, $g \in G$ is defined on $v \in \Omega_x$ if and only if there exists $v \in \Omega_{\mathsf{t}(w_n)}$ such that $g|_{w_n\Omega} = \tau_{w_n,v}$.

2.3.2 Homeomorphisms of bounded type

We want to use infinite tiles to get a handle on orbital graphs, and finite tiles to approximate infinite tiles. Thus, having a grasp on the size of tile boundaries will assist us in our study of orbital graphs.

Definition 2.3.8. Let h be a homeomorphism of Ω. For $x \in V^{(1)}$, let $\alpha_x(h)$ be the number of paths $u \in \Omega_x$ such that $h|_{u\Omega} \neq \tau_{u,v}$ for all $v \in \Omega_x$. We say h is of *bounded type* if $\alpha_x(h)$ is uniformly bounded and h has finitely many critical points.

The set of all homeomorphisms of bounded type form a group. If S is a finite generating set of homeomorphisms of Ω and $h \in S$, then $\alpha_x(h)$ is the number of elements $(s, u) \in \partial T_x$ with $s = h$.

Proposition 2.3.9. *Let h be a homeomorphism of $\Omega(\mathsf{D})$ and suppose $|V_i|$ is uniformly bounded. If $\alpha_x(h)$ is uniformly bounded, then h has finitely many critical points, so that h is of bounded type.*

Proof. There exists N, M such that $|V_i| \leq N$ for all i and $\alpha_x(h) \leq M$ for all x. Suppose $|\mathcal{C}_h| > MN$ and let $z^{(1)}, z^{(2)}, \ldots, z^{(MN+1)}$ be distinct critical points of h. There exists an m such that

$z_m^{(1)}, \ldots, z_m^{(MN+1)}$ are distinct paths of length m, and for $i = 1, 2, \ldots, MN+1$, we have $h|_{z_m^{(i)}\Omega} \neq \tau_{z_m^{(i)},v}$ for all $v \in \Omega_{\mathsf{t}(z_m^{(i)})}$. It follows that $\sum_{x \in V_m} \alpha_x(h) > MN$, a contradiction. Thus $|\mathcal{C}_h| \leq MN$, so that h has finitely many critical points. $\square$

Proposition 2.3.10. *Let h be a homeomorphism of Ω and suppose its set of critical points is finite.*

1. *If $\mathcal{N}(h)$ is finite, then h has no critical points and supp$(h) = \bigsqcup_{u \in \mathcal{N}(h)} u\Omega$. We say h is finitary of depth n, where n is the maximum of the lengths of elements of $\mathcal{N}(h)$.*

2. *If $\mathcal{N}(h)$ is infinite, then $\mathcal{C}_h$ is nonempty and equal to the boundary of $\bigsqcup_{u \in \mathcal{N}(h)} u\Omega$. We say h is almost finitary.*

Proof. (1) Suppose h has a critical point w and let n be the maximum of the lengths of elements of $\mathcal{N}(h)$. We have $h|_{w_n\mathcal{X}} \neq \tau_{w_n,v}$ for all $v \in \Omega_{\mathsf{t}(w_n)}$. It follows that every element of $w_n\Omega$ is a critical point, which is a contradiction. Indeed, suppose $w_n z$ is not a critical point, then there exists minimal $m \geq 1$ such that $h|_{w_n z_m \mathcal{X}} = \tau_{w_n z_m, v}$ for some $v \in \Omega_{\mathsf{t}(w_n z_m)}$. Then $w_n z_m \in \mathcal{N}(h)$ and has length $n+m$, a contradiction. Thus h has no critical points, and supp$(h) = \bigsqcup_{u \in \mathcal{N}(h)} u\Omega$ by Proposition 2.3.3.

(2) First note that the set $\bigsqcup_{u \in \mathcal{N}(h)} u\Omega$ is open and not closed, thus has nonempty boundary. If w is a boundary point of $\bigsqcup_{u \in \mathcal{N}(h)} u\Omega \subseteq$ supp(h), then w is clearly a critical point of h.

Now suppose w is a critical point of h and consider the basic clopen set $w_n\Omega$ for some n. There is a $z \in w_n\Omega$ that it not a critical point of h, so there exists minimal $m > n$ such that $h|_{z_m\Omega} = \tau_{z_m,v}$ for some $v \in \Omega_{\mathsf{t}(z_m)}$. It follows that $z_m \in \mathcal{N}(h)$ and $z_m\Omega \subset w_n\Omega$, so that w is a boundary point of $\bigsqcup_{u \in \mathcal{N}(h)} u\Omega$. $\square$

Note that h is finitary if and only if supp(h) is closed. If a homeomorphism $h : \Omega \longrightarrow \Omega$ is finitary of depth n, then for every $u \in \Omega_n$ there exists $v \in \Omega_{\mathsf{t}(u)}$ such that $h|_{u\Omega} = \tau_{u,v}$. In other words, h fixes the tail of each element of Ω starting at the nth edge (or sooner). The set of all finitary homeomorphisms of depth at most n form a group, and thus the set of all finitary homeomorphisms is also a group (as an increasing union).

Suppose S is a finite set of homeomorphisms of Ω of bounded type and consider the induced action of $G = \langle S \rangle$ on $\mathcal{X}$. It follows from Propositions 2.3.5 that for all $z \in \Omega$, either $T_z = \Gamma_z$ or we can construct Γ_z by connecting finitely many infinite tiles.

2.3.3 Groups acting on rooted trees

Consider the case when every V_i of the Bratteli Diagram D has exactly one vertex, say $V_i = \{v_i\}$. Then D is determined by the sequence $(E_1, E_2, E_3, \dots)$ of finite edge sets. We have $\Omega_n = \Omega_{v_n} = E_1 \times E_2 \cdots \times E_n$ and $\Omega = \prod_{i \geq 1} E_i$. The set $X^* = \Omega_* \cup \{v_0\}$ has a natural structure of a rooted tree with root v_0, level n vertices given by Ω_n, an edge between v_0 and every element of Ω_1, and an edge between $u \in \Omega_n$ and $v \in \Omega_{n+1}$ if $v = ue$ for some $e \in E_{n+1}$.

Let Aut(X^*) denote the automorphism group of the rooted tree X^* and let X^n denote the level n vertices Ω_n. The tree is level-transitive, that is, Aut(X^*) acts transitively on each of the levels X^n. Let $X^*_{(m)}$ denote the rooted tree corresponding to the diagram $\mathsf{D}^{(m)}$. It is naturally identified with any of the subtrees of X^* starting at a vertex of the mth level.

For every $g \in$ Aut(X^*) and $v \in X^n$, there exists $g|_v \in$ Aut$X^*_{(n)}$ such that

$$g(vw) = g(v)g|_v(w)$$

for all $w \in X^*_{(n)}$. The automorphism $g|_v$ is called the *section* of g at v. We have the following straightforward properties of sections:

$$(gh)|_v = g|_{h(v)}h|_v, \qquad g|_{vu} = (g|_v)|_v = g|_v|_u$$

for all $g, h \in$ Aut(X^*), $v \in X^*$, and $u \in X^*_{(n)}$, where $v \in X^n$. Any $g \in$ Aut(X^*) induces a homeomorphism of Ω by $g(e_1, e_2, e_3, \dots) = (g(e_1), g|_{e_1}(e_2), g|_{e_1 e_2}(e_3), \dots)$. The tile graph $T_n = T_{v_n}$ has vertex set X^n and an edge from u to v labeled by s if $s(u) = v$ and $s|_u = id$. The boundary ∂T_n consists of pairs $(s, u) \in S \times X^n$ such that $s|_u \neq id$.

If the edge sets E_i have the same cardinality, say $|E_i| = N$ for all i, then we will identify each

E_i with the set $X = \{0, 1, 2, \ldots, N-1\}$. We will generally write elements as finite and infinite words over X.

2.3.4 Automata

Definition 2.3.11. A (*transducer*) *automaton* is a 4-tuple $\mathcal{A} = (Q, X, \pi, \lambda)$, where

1. Q is a set (called the *set of states* of the automaton),
2. X is a finite set (called the *alphabet* of the automaton),
3. $\pi : Q \times X \longrightarrow Q$ is a map (called the *output function* of the automaton),
4. $\lambda : Q \times X \longrightarrow X$ is a map (called the *transition function* of the automaton).

We interpret $\mathcal{A}$ as a machine that when in state $q \in Q$ and reading $x \in X$ as input, outputs $\lambda(q, x)$ and then changes its state to $\pi(q, x)$. If we let X^* denote the set of finite words over X, including the empty word $\varnothing$, we can extend the maps π and λ to $Q \times X^*$ inductively by:

$$\pi(q, \varnothing) = q, \qquad \pi(q, x_1 x_2 \cdots x_n) = \pi(\pi(q, x_1), x_2 \cdots x_n),$$

$$\lambda(q, \varnothing) = \varnothing, \qquad \lambda(q, x_1 x_2 \cdots x_n) = \lambda(q, x_1)\lambda(\pi(q, x_1), x_2 \cdots x_n).$$

Then $\pi(q, x_1 x_2 \cdots x_n)$ is the state of the automaton after reading the word $x_1 x_2 \cdots x_n$ and starting in state q, and $\lambda(q, x_1 x_2 \cdots x_n)$ is the total output word (of length n).

If Q is finite, we say $\mathcal{A}$ is *finite-state*. An automaton is *invertible* if $\lambda(q, \cdot) : X \longrightarrow X$ is a permutation for all $q \in Q$. An automaton can be represented by a labeled directed graph, called a *Moore diagram*, which has vertex set Q and for every $q \in Q$ and $x \in X$ there is an arrow from q to $\pi(q, x)$ labeled by $x|\lambda(q, x)$. Figure 5.1 shows the Moore diagram for an invertible finite state automaton. We also use $\mathcal{A}$ to represent the Moore diagram of $\mathcal{A}$.

Let X^ω be the set of (right) infinite words over X. Every state $q \in Q$ inductively defines a transformation of $X^* \cup X^\omega$, also denoted q, by

$$q(x_1 x_2 x_2 \ldots) = \lambda(q, x_1)\lambda(\pi(q, x_1), x_2 x_3 \ldots).$$

In terms of the Moore diagram, given a word $x_1x_2x_3\dots$ in $X^*\cup X^\omega$, there is a unique directed path starting at q and successively labeled by $x_1|y_1, x_2|y_2, x_3|y_3, \dots$, and $q(x_1x_2x_3\dots) = y_1y_2y_3\dots$.

If $\mathcal{A}$ is invertible, then every transformation defined by a state of Q is invertible. We can then define the *automaton group* $G(\mathcal{A})$ generated by $\mathcal{A}$ to be the group of transformations generated by Q. The set X^* has a natural structure of a rooted tree, and any automaton group is a subgroup of $\mathrm{Aut}(X^*)$. For any $v \in X^*$ and $q \in Q$, we have $q|_v = \pi(q, v)$.

Definition 2.3.12. A finite invertible automaton $\mathcal{A}$ is called *bounded* if the number of left-infinite paths of $\mathcal{A} \setminus \{id\}$ is finite.

Here, a left-infinite path is a path in the Moore diagram of $\mathcal{A}$ where each edge is against orientation (or a loop). It is not difficult to show that $\mathcal{A}$ is bounded if and only if the number of right-infinite paths of $\mathcal{A}\setminus\{id\}$ is finite. There is a one-to-one correspondence between right infinite paths starting at state q and critical points of q. It follows that each $q \in Q$ is a homeomorphism of bounded type.

3. PERIODICITY

Let $\mathcal{X}$ be a Cantor space and h be a homeomorphism of $\mathcal{X}$. For $\zeta \in \mathcal{X}$, the *h-orbit* of ζ is the set $\mathcal{O}_h(\zeta) = \{h^n(\zeta) : n \in \mathbb{Z}\}$. For any integer $n \geq 2$, let $\operatorname{supp}(h, n) = \{\zeta \in \operatorname{supp}(h) : |\mathcal{O}_h(\zeta)| = n\}$. The order of h, denoted $\operatorname{ord}(h)$, is the smallest positive integer n such that h^n is the identity map on $\mathcal{X}$, or ∞ if no such n exists.

Reminder. The set of singular points of a homeomorphism h is given by the common boundary of $\operatorname{supp}(h)$ and $\operatorname{Fix}(h)$. We have $\mathcal{X} = \operatorname{supp}(h) \bigsqcup \operatorname{Fix}(h)^\circ \bigsqcup \partial\operatorname{Fix}(h)$, where $\operatorname{Fix}(h)^\circ$ denotes the interior of $\operatorname{Fix}(h)$.

3.1 Domains of support of a finite group of homeomorphisms

Let H be a nontrivial finite group of homeomorphisms of $\mathcal{X}$ and let $\mathcal{D}_H$ be the set of all nontrivial intersections of the form $\bigcap_{h \in H} U_h$, where $U_h \in \{\operatorname{supp}(h), \operatorname{Fix}(h)^\circ\}$ and $U_{h'} = \operatorname{supp}(h')$ for some $h' \in H$. For any U_{h_1} and $h_2 \in H$, we have $h_2(U_{h_1}) = U_{h_2 h_1 h_2^{-1}}$. The set $\mathcal{D}_H$ has the following properties:

1. $\mathcal{D}_H$ is finite,
2. the elements of $\mathcal{D}_H$ are open and pairwise disjoint,
3. for every $D \in \mathcal{D}_H$ and $h \in H$ either $D \subseteq \operatorname{supp}(h)$ or $D \subseteq \operatorname{Fix}(h)^\circ$,
4. $h(D) \in \mathcal{D}_H$ for every $h \in H$ and $D \in \mathcal{D}_H$,
5. $\bigsqcup_{D \in \mathcal{D}_H} D$ consists of the H-regular points of $\operatorname{supp}(H)$.

We call the elements of $\mathcal{D}_H$ the *domains of support* of H. The set of singular points of H is given by $\bigcup_{h \in H} \partial\operatorname{Fix}(h)$, and thus is nowhere dense. It follows that the domains of support of H cover $\operatorname{supp}(H)$ up to a meager set, i.e., up to a countable union of nowhere dense sets (such sets are considered "topologically small").

We will be interested in cases where the elements of $\mathcal{D}_H$ are H-invariant. For example, if H is abelian, then $h_2(U_{h_1}) = U_{h_2 h_1 h_2^{-1}} = U_{h_1}$ so that each $D \in \mathcal{D}_H$ is H-invariant. We say H has *invariant domains*. If $\text{supp}(h) = \text{supp}(H)$ for all $h \in H$ (i.e., H acts freely on $\text{supp}(H)$), then $\mathcal{D}_H = \{\text{supp}(H)\}$ and H trivially has invariant domains. If in addition, H has regular support, then $\mathcal{D}_H$ is a partition of $\text{supp}(H)$ into H-invariant open sets.

3.2 Germ-defining singular points

A nontrivial walk $w = (\zeta_0, e_1, \zeta_1, \ldots, e_n, \zeta_n)$ in an orbital graph of $G = \langle S \rangle \curvearrowright \mathcal{X}$ (with label function ℓ) corresponds to an element of G, namely $g_w = s_m^{\varepsilon_m} s_{m-1}^{\varepsilon_{m-1}} \cdots s_1^{\varepsilon_1}$, where $s_i = \ell(e_i)$ and $\varepsilon_i = 1$ if e_i has source v_{i-1} or $\varepsilon_i = -1$ if e_i has source v_i. The order of edges reverses as we are considering left actions. We have $g_{w_1 w_2} = g_{w_2} g_{w_1}$ and $g_{w^{-1}} = g_w^{-1}$. If w is trivial, then we define $g_w = id$.

Conversely, for a finite word $g = s_n s_{n-1} \cdots s_1$ over $S \cup S^{-1}$ and $\zeta_0 \in \mathcal{X}$, we define the walk $w_{g,\zeta_0} = (\zeta_0, e_1, \zeta_1, \ldots, e_n, \zeta_n)$ where $\zeta_i = s_i s_{i-1} \cdots s_1(\zeta)$ for $0 < i \leq n$, and e_i is the edge with endpoint(s) $\{\zeta_{i-1}, \zeta_i\}$ labeled by s_i following orientation if $s_i \in S$, and against orientation if $s_i \in S^{-1} \setminus S$. This is the walk starting at ζ and corresponding to g that is "as directed as possible". With a slight abuse of notation, we will denote this walk by $(\zeta_0, s_1, \zeta_1, \ldots, s_n, \zeta_n)$. Such walks are important in the following because the "journey" $\zeta_0, \zeta_1, \ldots, \zeta_n$ that ζ_0 takes when acted on by a product $s_n s_{n-1} \cdots s_1$ of elements of $S \cup S^{-1}$ will play a crucial role in analyzing the action of $\langle S \rangle$ on $\mathcal{X}$.

Definition 3.2.1. Let S be a finite set of homeomorphisms of $\mathcal{X}$ and consider the induced action of $G = \langle S \rangle$ on $\mathcal{X}$. A singular point ξ of S is called *germ-defining* if:

1. ξ is the only singular point of S in $G(\xi)$,

2. for any $\zeta \in G(\xi)$ and nonloop cycle c in Γ_ξ starting at ζ, we have $(g_c, \zeta) = (id, \zeta)$.

The second condition is really just a succint way to say that for any $\zeta_0 \in G(\xi)$ and product $g = s_m s_{m-1} \cdots s_1 \in G_{\zeta_0}$, $s_i \in S \cup S^{-1}$, such that $\zeta_i = s_i \cdots s_1(\zeta_0)$ are distinct for $i = 1, \ldots, m-1$, we have $(g, \zeta_0) = (id, \zeta_0)$.

Let S_ξ denote the subset of S consisting of elements that fix ξ and $S_{(\xi)}$ denote the subset of S consisting of elements that fix a neighborhood of ξ, then $S_\xi \setminus S_{(\xi)}$ consists of the elements of S that have ξ as a singular point. The *length* of $g \in G$ (with respect to S) is defined to be the minimal number of factors m needed for a decomposition $g = s_m \cdots s_1$, $s_i \in S \cup S^{-1}$, and is denoted $|g|_S$. In other words, it is the distance between id and g in the Cayley graph of G (with respect to S).

Proposition 3.2.2. *Let $G = \langle S \rangle$ and $H = \langle S_\xi \setminus S_{(\xi)} \rangle$. If ξ is a germ-defining singular point of S, then $G_\xi = G_{(\xi)} H$.*

Proof. We clearly have $G_{(\xi)} H \subseteq G_\xi$ since both $G_{(\xi)}$ and H are subgroups of G_ξ. We show $G_\xi \subseteq G_{(\xi)} H$ by strong induction on the length of elements of G_ξ. Let $g \in G_\xi$. The case of length 0 is trivial, as $g = id$. If g has length 1, then $g \in S_\xi \cup S_\xi^{-1}$. Either $g \in S_{(\xi)} \cup S_{(\xi)}^{-1}$, so that $g \in G_{(\xi)}$, or $g \notin S_{(\xi)} \cup S_{(\xi)}^{-1}$, so that $g \in H$. Therefore, $g \in G_{(\xi)} H$.

Suppose the claim is true for length less than some $n > 1$ and let $g = s_n \cdots s_1$ have length n. Consider the closed walk $w = w_{g,\xi} = (\xi_0, s_1, \xi_1, s_2, \ldots, s_n, \xi_n)$, where $\xi_0 = \xi_n = \xi$. If $\xi_k = \xi$ for some $0 < k < n$, then $g_1 = s_k \ldots s_1$ and $g_2 = s_n \ldots s_{k+1}$ are in G_ξ and have length less than n, so are in $G_{(\xi)} H$ by inductive hypothesis. But $G_{(\xi)} H$ is a group (since $G_{(\xi)}$ is normal in G_ξ), so $g = g_2 g_1 \in G_{(\xi)} H$.

Now suppose $\xi_k \neq \xi$ for $0 < k < n$. We will show the stronger conclusion $g \in G_{(\xi)}$. If w is a cycle, then we get $g = g_w \in G_{(\xi)}$ since ξ is germ-defining. If w is not a cycle, then it must contain a cycle subwalk c, say $c = (\xi_k, s_{k+1}, \ldots, s_{k+r}, \xi_{k+r})$, where $1 \leq r \leq n-2$ and $0 < k < n$. Indeed, if w is not a cycle, then by definition either its edges are not distinct or the vertices $\xi_0, \xi_1, \ldots, \xi_{n-1}$ are not distinct. Suppose the latter is true and let k be minimal such that ξ_k appears more than once in $\xi_0, \xi_1, \ldots, \xi_{n-1}$. By the assumption that $\xi_i \neq \xi$ for $0 < i < n$, we must have $k > 0$. Finally, there exists $1 \leq r \leq n-2$ such that $\xi_k = \xi_{k+r}$ and $\xi_k, \xi_{k+1}, \ldots, \xi_{k+r-1}$ are distinct. If the edges are not distinct, then $s_i = s_j^\varepsilon$ for some $1 \leq i < j \leq n$ and $\varepsilon \in \{-1, 1\}$. If $\varepsilon = 1$, then $\xi_{i-1} = \xi_{j-1}$. If $\varepsilon = -1$, then $\xi_i = \xi_{j-1}$. So this case reduces to the other.

We then have

$$\begin{aligned}(g,\xi) &= (s_n \cdots s_1, \xi)\\ &= (s_n \cdots s_{k+r+1}, \xi_{k+r})(s_{k+r} \cdots s_{k+1}, \xi_k)(s_k \cdots s_1, \xi)\\ &= (s_n \cdots s_{k+r+1}, \xi_{k+r})(id, \xi_k)(s_k \cdots s_1, \xi)\\ &= (s_n \cdots s_{k+r+1}, \xi_k)(s_k \cdots s_1, \xi)\\ &= (s_n \cdots s_{k+r+1} s_k \cdots s_1, \xi),\end{aligned}$$

where $g_1 = s_n \cdots s_{k+r+1} s_k \cdots s_1 \in G_\xi$ is a word over $S \cup S^{-1}$ of length $n - r$ and $2 \leq n - r < n$. If $w_1 = w_{g_1,\xi}$ is a nonloop cycle or $g_1 = id$ in G, then $(g, \xi) = (g_1, \xi) = (id, \xi)$. Otherwise, we can continue this until we obtain a word g_r of length at least 2 such that w_r is a nonloop cycle or $g_r = id$ in G, and $(g, \xi) = (g_r, \xi) = (id, \xi)$. □

Corollary 3.2.3. *If ξ is germ-defining, then $G_\xi/G_{(\xi)}$ is naturally isomorphic to $H/H_{(\xi)}$. In particular, for any $g \in G_\xi$ there exists $h \in H$ such that $(g, \xi) = (h, \xi)$.*

Proof. The Second Isomorphism Theorem gives $H/(H \cap G_{(\xi)})$ and $G_{(\xi)}H/G_{(\xi)}$ are isomorphic under the homomorphism $h(H \cap G_{(\xi)}) \mapsto hG_{(\xi)}$. From Proposition 3.2.2 we have $G_\xi = G_{(\xi)}H$. Since $H \cap G_{(\xi)} = H_{(\xi)}$, we get $H/H_{(\xi)}$ and $G_\xi/G_{(\xi)}$ are isomorphic under the homomorphism $\varphi : hH_{(\xi)} \mapsto hG_{(\xi)}$. The second statement follows from surjectivity of φ. □

Let Γ'_ξ be Γ_ξ without the loops at ξ labeled by $s \in S_\xi \setminus S_{(\xi)}$. Consider the graph Ξ with set of vertices $H/H_{(\xi)} \times G(\xi)$ obtained by taking $|H/H_{(\xi)}|$ copies of Γ'_ξ, and then connecting their roots ξ by the Schreier graph $\Gamma_{(\xi)}(H) = \Gamma(H, H_{(\xi)}, S_\xi \setminus S_{(\xi)})$, i.e., by the graph of germs of H at ξ.

Proposition 3.2.4. *If ξ is a germ-defining singular point of S, then $\Gamma_{(\xi)}$ is isomorphic to Ξ.*

Proof. Let T be a left transversal of G_ξ in G containing id, then any vertex ζ of Γ'_ξ can be written uniquely as $\zeta = g(\xi)$ for some $g \in T$. We show the map $\varphi : \Xi \longrightarrow \Gamma_{(\xi)}$ defined on vertices by $\varphi(hH_{(\xi)}, g(\xi)) = (gh, \xi)$ and on arrows by $\varphi((s, (hH_{(\xi)}, g(\xi))) = (s, (gh, \xi))$ is a well-defined isomorphism. First, if $(h_1H_{(\xi)}, g_1(\xi)) = (h_2H_{(\xi)}, g_2(\xi))$, then $h_1H_{(\xi)} = h_2H_{(\xi)}$ and $g_1 = g_2$. Thus $(g_1h_1, \xi) = (g_1, \xi)(h_1, \xi) = (g_2, \xi)(h_2, \xi) = (g_2h_2, \xi)$ and the map is well-defined. Now suppose $(g_1h_1, \xi) = (g_2h_2, \xi)$, then $g_1(\xi) = g_2(\xi)$ and thus $g_1 = g_2$. It follows that $(h_1, \xi) = (h_2, \xi)$, i.e., $h_1H_{(\xi)} = h_2H_{(\xi)}$, so φ is injective. Let $(g, \xi) \in G/G_{(\xi)}$. We have $g = g_1g_2$ for some $g_1 \in T$ and $g_2 \in G_\xi$. By Corollary 3.2.3, there exists $h \in H$ such that $(g_2, \xi) = (h, \xi)$, and so $(g, \xi) = (g_1g_2, \xi) = (g_1, \xi)(g_2, \xi) = (g_1, \xi)(h, \xi) = (g_1h, \xi)$, i.e., $\varphi(hH_{(\xi)}, g_1(\xi)) = (g, \xi)$, and thus φ is surjective.

Consider the s-labeled edge from $(hH_{(\xi)}, g(\xi))$ to $(hH_{(\xi)}, sg(\xi))$ in a copy of $\Gamma'_\xi \subseteq \Xi$. The vertices are sent to (gh, ξ) and $(sgh, \xi) = s(gh, \xi)$ by φ, which are connected by an edge labeled by s. Next consider the s-labeled edge, $s \in S_\xi \backslash S_{(\xi)}$, from $(hH_{(\xi)}, \xi)$ to $(shH_{(\xi)}, \xi)$ in $\Gamma_{(\xi)}(H) \subseteq \Xi$. The vertices are sent to (h, ξ) and $(sh, \xi) = s(h, \xi)$. Thus φ is an isomorphism. $\square$

Let S be a finite set of homeomorphisms of $\mathcal{X}$ with germ-defining singular point ξ. Let $G = \langle S \rangle$, $H = \langle S_\xi \setminus S_{(\xi)} \rangle$, and A be a subset of $\mathcal{X}$ accumulating on ξ. It is straightforward to show $G_{(\xi,A)} = G_{(\xi)}H_{(\xi,A)}$. Thus, we can go from the chain of subgroups $H_{(\xi)} < H_{(\xi,A)} < H$ to the chain $G_{(\xi)} < G_{(\xi,A)} < G_\xi$ by multiplication of $G_{(\xi)}$. The map $hH_{(\xi,A)} \mapsto hG_{(\xi,A)}$ is a bijection between $G_\xi/G_{(\xi,A)}$ and $H/H_{(\xi,A)}$, and the graph $\Gamma_{(\xi,A)}$ is isomorphic to the graph Ξ_A obtained by taking $|H/H_{(\xi,A)}|$ copies of Γ'_ξ and connecting their roots by the Schreier graph $\Gamma_{(\xi,A)}(H) = \Gamma(H, H_{(\xi,A)}, S_\xi \setminus S_{(\xi)})$. Let $\lambda_A : \Gamma_{(\xi)} \longrightarrow \Gamma_{(\xi,A)}$ denote the natural covering map. In terms of Ξ and Ξ_A, it is given by $\lambda_A(hH_{(\xi)}, v) = (hH_{(\xi,A)}, v)$. Note that if H is finite and $D \in \mathcal{D}_H$ is H-invariant and accumulates on ξ, then $H_{(\xi,D)} = H_D$, and thus $\Gamma_{(\xi,D)}(H) = \Gamma_D(H)$.

3.3 Thin graphs

A vertex of a connected graph is called a *cut vertex* if removing it disconnects the graph. A connected graph is called *separable* if it has a cut vertex, otherwise it is *biconnected*. A *block* of a graph is a maximal biconnected subgraph. The intersection of two different blocks is either empty, or consists of a single cut vertex, possibly with some loops. The *block-cut graph* $\mathcal{BC}(\Gamma)$ of a graph Γ is the simple graph with vertices given by the blocks and cut vertices of Γ, and an edge between a block B and cut vertex c if c belongs to B. If Γ is connected, then $\mathcal{BC}(\Gamma)$ is a tree, and is called a *block-cut tree*.

Definition 3.3.1. An infinite separable graph Γ is called *thin* if it has infinitely many blocks and there is a bound on the size of its blocks.

If Γ is thin, then $\mathcal{BC}(\Gamma)$ is an infinite tree. In other words, given a collection $\mathcal{F}$ of finite biconnected graphs of bounded size, we can create thin graphs whose blocks come from $\mathcal{F}$ by connecting elements of $\mathcal{F}$ in a "tree-like" way. In particular, infinite simple trees are examples of thin graphs. If Γ is also of bounded degree (i.e., there exists an integer d such that $\deg(v) \leq d$ for all vertices $v \in \Gamma$), then there are a finite number of "block types" that can occur in Γ.

3.3.1 Structure of thin graphs

Given an infinite rooted graph (Γ, ζ), a ray r in Γ is an infinite sequence of distinct vertices $v_0 = \zeta, v_1, v_2, \ldots$ in which each two consecutive vertices in the sequence are connected by an edge. Two rays r and r' are equivalent if for every finite subset $\Delta \subset V(\Gamma)$ infinitely many vertices of r and r' belong to the same connected component of $\Gamma \setminus \Delta$. An *end* of Γ is an equivalence class of rays. Let $V(r)$ denote the set of vertices of the ray r.

Definition 3.3.2. Let (Γ, ζ) be an infinite rooted thin graph with rays $\{r_i\}_{i \in I}$. The *spine* of Γ, denoted $|\Gamma$, is the subgraph induced by $\bigcup_{i \in I} V(r_i)$ minus loops.

Definition 3.3.3. For $v \in |\Gamma$, the *limb* at v is the connected component of $\Gamma \setminus E(|\Gamma)$ that contains v, denoted $\Lambda(v)$.

Every $\Lambda(v)$ is finite, and $|\Gamma$ along with $\Lambda(v)$, $v \in |\Gamma$, completely determine the graph (Γ, ζ). In other words, (Γ, ζ) consists of its infinite spine $|\Gamma$ with finite limbs $\Lambda(v)$, $v \in |\Gamma$, connected to it.

Definition 3.3.4. Let $v_1, v_2, \ldots, v_n$ be vertices of a connected graph Γ. The *hull* of $v_1, \ldots, v_n$, denoted $[v_1, \ldots, v_n]$, is the subgraph of Γ given by the union of all paths between these vertices.

It is not difficult to see that $[v_1, v_2, \ldots, v_n]$ is given by the union of all blocks (modulo loops) of Γ that contain an edge of some path between these vertices. Equivalently, if B_i is the block of

Γ containing v_i then $[v_1, v_2, \dots, v_n]$ is the union of all blocks (modulo loops) corresponding to a vertex of $[B_1, B_2, \dots, B_n]$. If Γ consists of a single block, then $\mathcal{BC}(\Gamma)$ is a single vertex and the hull of any set of vertices is the graph Γ minus loops.

For the rest of this section, assume ξ is a germ-defining singular point of S, Γ_ξ is a 1-ended thin graph, and $H = \langle S_\xi \setminus S_{(\xi)} \rangle$ is finite. Since $H_{(\xi)}$ is a proper normal subgroup of H, the graph of germs $\Gamma_{(\xi)}(H)$ is (isomorphic to) a finite Cayley graph with at least two vertices. It follows that this graph has no cut vertex. Therefore, $\Gamma_{(\xi)} \cong \Xi$ is also a thin graph, and $\Gamma_{(\xi)}(H)$ is a block in $\Gamma_{(\xi)}$ (modulo some loops coming from Γ'_ξ). In particular, $|\Gamma_{(\xi)}$ is obtained by connecting $|H/H_{(\xi)}|$ copies of $|\Gamma_\xi$ by $\Gamma_{(\xi)}(H)$. Similarly, the graph $\Gamma_{(\xi,D)}$ is thin for each $D \in \mathcal{D}_H$ that accumulates on ξ, except the graph $\Gamma_{(\xi,D)}(H) = \Gamma_D(H)$ may consist of several blocks. If D is H-invariant, then H_D is normal in H and $\Gamma_D(H)$ is (isomorphic to) a Cayley graph.

Let ζ_1, ζ_2 be vertices of Γ_ξ (or Γ'_ξ) that are either cut vertices or equal to ξ. If $\zeta_1 = \zeta_2 = \zeta$, then the hull $[\zeta, \zeta]$ consists of the vertex ζ without edges. If $\zeta_1 \neq \zeta_2$, by the structure of a thin graph $[\zeta_1, \zeta_2]$ is a finite union of blocks without loops, where the blocks are connected in a "path-like" way (meaning the corresponding subgraph of the block-cut tree is a path graph). The subgraph $\Sigma(\zeta_1, \zeta_2)$ given by the union of $[\zeta_1, \zeta_2]$ and $\Lambda(\zeta)$ for all $\zeta \in [\zeta_1, \zeta_2]$ is called the *segment* from ζ_1 to ζ_2. The vertices ζ_1, ζ_2 are its *endpoints*. Note that the endpoints of $\Sigma = \Sigma(\zeta_1, \zeta_2)$ are the boundary vertices $\partial_V(\Sigma)$ of Σ.

For $\zeta \in |\Gamma_\xi$ a cut vertex or equal to ξ, define the *core* subgraph $\Theta_{(\xi)}(\zeta)$ of $\Gamma_{(\xi)}$ to be the graph obtained by taking the union of $\Gamma_{(\xi)}(H)$ and the segment $\Sigma(\xi, \zeta)$ in each copy of Γ'_ξ. We can define the core subgraphs $\Theta_D(\zeta)$ of Γ_D analogously (we leave off the subscript of $\Theta(\zeta)$ when the parent graph is understood, or when talking about general core subgraphs). We say $\Theta(\zeta)$ has *segment length* m if the distance between ξ and ζ in Γ_ξ is equal to m. The vertices $\{(hH_{(\xi)}, \zeta)\}$ (or $\{(hH_D, \zeta)\}$) of $\Theta(\zeta)$ are its *endpoints*. As with segments, the endpoints of $\Theta = \Theta(\zeta)$ are the boundary vertices of Θ.

Proposition 3.3.5. *Suppose $D \in \mathcal{D}_H$ accumulates on ξ and $\zeta_n \in D$, $n \geq 1$, is a sequence of points converging to ξ, then Γ_{ζ_n} converges to $\Gamma_{(\xi,D)}$.*

Proof. Consider a ball $B_{(id,\xi,P)}(r)$ in $\Gamma_{(\xi,D)}$. It can be described by a set of equations and inequalities of the form $(g_1, \xi, D) = (g_2, \xi, D)$ or $(g_1, \xi, D) \neq (g_2, \xi, D)$ for $g_1, g_2 \in G$ of length at most r. If $(g_1, \xi, P) = (g_2, \xi, P)$, then there is a neighborhood U of ξ such that $g_1(\zeta) = g_2(\zeta)$ for all $\zeta \in U \cap P$.

Now suppose $(g_1, \xi, D) \neq (g_2, \xi, D)$, that is $(g_2 g_1^{-1}, \xi, D) \neq (id, \xi, D)$. If $g_1(\xi) \neq g_2(\xi)$, then $g_1(\zeta) \neq g_2(\zeta)$ for all $\zeta \in V = \text{supp}(g_2^{-1} g_1)$, which is an open set containing ξ. In particular, this is true for all $\zeta \in V \cap P$. If $g_1(\xi) = g_2(\xi)$, then $g_2^{-1} g_1 \in G_\xi$ so there exists $g \in G_{(\xi)}$ and $h \in H$ such that $g_2^{-1} g_1 = gh$. Then $(g_2^{-1} g_1, \xi, D) = (gh, \xi, D) = (h, \xi, D) \neq (id, \xi, D)$. It follows that $h|_D \neq id|_D$ and thus $D \subseteq \text{supp}(h)$. There exists a neighborhood W of ξ such that $g_2^{-1} g_1|_{W \cap D} = h|_{W \cap D}$, thus $g_2^{-1} g_1(\zeta) = h(\zeta) \neq \zeta$ for all $\zeta \in W \cap D$. That is, $g_1(\zeta) \neq g_2(\zeta)$ for all $\zeta \in W \cap D$.

It follows that there exists a neighborhood N of ξ such that for every $\zeta \in N \cap D$ the balls $B_\zeta(r)$ and $B_{(id,\xi,D)}(r)$ are isomorphic. $\square$

Since the set of G-regular points is dense in $\mathcal{X}$, we can find a sequence of regular points in D converging to ξ. Thus the balls in $\Gamma_{(\xi,D)}$ are isomorphic to balls in the orbital graphs of regular points. The following proposition shows that for minimal actions on $\mathcal{X}$, the balls in an orbital graph of a regular point are contained (as an isomorphic copy) in every orbital graph.

Proposition 3.3.6. *Suppose that the action of G on $\mathcal{X}$ is minimal and let $\zeta \in \mathcal{X}$ be a G-regular point. For every ball $B_\zeta(r)$ of Γ_ζ there exists $R(r) > 0$ such that for every $\eta \in \mathcal{X}$ there exists a vertex η' of Γ_η on distance at most $R(r)$ from η such that $B_\zeta(r)$ and $B_{\eta'}(r)$ are isomorphic.*

Proof. The ball $B_\zeta(r)$ is described by a finite set of equations and inequalities of the form $g_1(\zeta) = g_2(\zeta)$ and $g_1(\zeta) \neq g_2(\zeta)$ for $g_1, g_2 \in G$ of length at most r. Since ζ is G-regular, each such equation or inequality holds on a neighborhood of ζ. It follows that there exists a neighborhood U of ζ such that for every $\eta \in U$ the balls $B_\zeta(r)$ and $B_\eta(r)$ are isomorphic.

By minimality, for every $\eta \in \mathcal{X}$ there exists $g_\eta \in G$ such that $g_\eta(\eta) \in U$, i.e., $\eta \in g_\eta^{-1}(U)$. The sets $g_\eta^{-1}(U)$, $\eta \in \mathcal{X}$, cover $\mathcal{X}$, and by compactness there is a finite subcover $g_1^{-1}(U), \ldots, g_n^{-1}(U)$.

Let R be the maximum of the lengths of the elements g_i with respect to S. Then for every $\eta \in \mathcal{X}$ there exists g_i such that $\eta' = g_i(\eta) \in U$, and thus $B_\zeta(r)$ and $B_{\eta'}(r)$ are isomorphic. Clearly the distance between η and η' is at most R. There are finitely many isomorphism classes of balls of radius r in the orbital graphs of $G \curvearrowright \mathcal{X}$, so we can find an upper bound $R(r)$ independent of ζ. $\square$

We can apply the above proposition to every vertex of Γ_η so that for every vertex η' of Γ_η there exists η'' on distance at most $R(r)$ from η' such that $B_\zeta(r)$ is isomorphic to $B_{\eta''}(r)$. We say $B_\zeta(r)$ is *repetitive* in Γ_η.

Definition 3.3.7. Let Δ be a finite connected graph and Γ be an infinite connected graph. We say Δ is *strongly repetitive* in Γ if there exists infinitely many isomorphic copies $\{\Delta\}_{i\in I}$ of Δ in Γ such that $\Gamma \setminus E(\bigcup_{i\in I} \Delta_i)$ consists of finite connected components of bounded size. The set $\{\Delta_i\}_{i\in I}$ is called a Δ-*sieve*.

Without loss of generality, we can assume that each copy Δ_i of Δ has at least one full multiedge, that is, there exists distinct $u, v \in \Delta_i$ such that every edge of Γ with endpoints u, v is in Δ_i.

Proposition 3.3.8. *Suppose the size of the blocks of Γ_ξ is bounded by N and let Θ be a core with segment size greater than $N/2$. If Θ is strongly repetitive in Γ_ξ, then there is a copy that has two segments on the spine of Γ_ξ.*

Proof. Let $\{\Theta_i\}_{i\in I}$ be a Θ-sieve in Γ_ξ with embedding maps $\varphi_i : \Theta \longrightarrow \Theta_i$. Let $\{\Theta_j\}_{j\in J}$ be the subset of copies of Θ with an edge on the spine of Γ_ξ. Notice that J must be infinite, thus there exists $k \in J$ such that Θ_k does not contain ξ. We have $\partial_V(\Theta_k) \subseteq \varphi_k(\partial_V(\Theta))$ since the vertices of $\Theta \setminus \partial_V(\Theta)$ have degree $2|S|$, and every vertex of Γ_ξ has degree $2|S|$. There cannot be a path from distinct $u, v \in \partial_V(\Theta_k)$ with edges outside of Θ_k because this would imply Γ_ξ has a block of size greater than N, a contradiction. It follows that exactly one endpoint of Θ_k is connected to ξ by edges outside of Θ_k, and a second endpoint of Θ_k is on the unique infinite component of $\Gamma_\xi \setminus E(\Theta_k)$. In other words, two segments of Θ_k are on the spine of Γ_ξ. $\square$

3.4 Main theorem

Theorem 3.4.1. *Let S be a finite set of finite order homeomorphisms of the Cantor set $\mathcal{X}$. Suppose the following is true:*

1. *the induced action of $G = \langle S \rangle$ on $\mathcal{X}$ is minimal,*
2. *ξ is a germ-defining singular point of S,*
3. *Γ_ξ is thin and 1-ended,*
4. *$H = \langle S_\xi \setminus S_{(\xi)} \rangle$ is finite with invariant domains, and for every $h \in H$ there exists $D \in \mathcal{D}_H$ accumulating on ξ such that $h|_D = id|_D$,*
5. *for every $D \in \mathcal{D}_H$ that accumulates on ξ, any core subgraph of $\Gamma_{(\xi,D)}$ is strongly repetitive in Γ_ξ.*

Then G is periodic.

Proof. Let $g \in G \setminus \{id\}$ and suppose $|g|_S = m$. We will show g has finite order. Note that $\Gamma_{(\xi)}$ and each $\Gamma_{(\xi,D)}$, $D \in \mathcal{D}_H$, have the graph structure discussed in the previous section, with center given by the Cayley graph of $\Gamma_{(\xi)}(H)$ or $\Gamma_D(H)$. Let N be an upper bound for the block sizes of Γ_ξ.

Let Δ be a core subgraph of some $\Gamma_{(\xi,D)}$ with segment length greater than $\max\{m, N/2\}$. There is an embedding φ_ξ of Δ with two segments on the spine of Γ_ξ by Proposition 3.3.8. Denote its image by Δ_ξ. Let φ_{hH_D} and $\varphi_{hH_{(\xi)}}$ denote the natural embeddings of Δ_ξ into $\{hH_D\} \times \Gamma'_\xi$ and $\{hH_{(\xi)}\} \times \Gamma'_\xi$, respectively. Note that $\lambda_D \circ \varphi_{hH_{(\xi)}} = \varphi_{hH_D}$ and $g \circ \lambda_D = \lambda_D \circ g$ for all $D \in \mathcal{D}_H$ accumulating on ξ and $g \in G$.

Lemma 3.4.2. *For every vertex v of Δ_ξ on the segment closest to ξ, there exists a $\Gamma_{(\xi,D)}$ and integer $k \geq 1$ such that $g^k(\varphi_{H_D}(v)) \in \varphi_{H_D}(\Delta_\xi)$.*

Proof. By way of contradiction, suppose for all $\Gamma_{(\xi,D)}$ the sequence $g^k(\varphi_{H_D}(v))$ does not return to $\varphi_{H_D}(\Delta_\xi)$. In particular, $g(\varphi_{H_D}(v))$ jumps toward $\Gamma_D(H)$. Now fix a $\Gamma_{(\xi,D)}$, then the sequence

$g^k(\varphi_{H_D}(v))$ must converge to an end associated to $\{hH_D\} \times \Gamma'_\xi$ for some $h \notin H_D$. We have $g^k(\varphi_{H_D}(v)) = g^k(\lambda_D(\varphi_{H_{(\xi)}}(v))) = \lambda_D(g^k(\varphi_{H_{(\xi)}}(v)))$. This implies that in $\Gamma_{(\xi)}$ the sequence $g^k(\varphi_{H_{(\xi)}}(v))$ converges to an end $\{h'H_{(\xi)}\} \times \Gamma'_\xi$ where $h' \notin H_{(\xi)}$ and $h'H_D = hH_D$. By (4), there exists a $D' \in \mathcal{D}_H$ such that $\lambda_{D'}(\{h'H_{(\xi)}\} \times \Gamma'_\xi) = \{H_{D'}\} \times \Gamma'_\xi$, but then the sequence $\lambda_{D'}(g^k(\varphi_{H_{(\xi)}}(v)))$ will move from one connected component of $\Gamma_{(\xi,D')} \backslash \lambda_{D'}(\varphi_{H_{(\xi)}}(\Delta_\xi))$ to another, which is a contradiction as Δ has segment length m. $\square$

Corollary 3.4.3. *For every vertex v of Δ_ξ on the segment closest to ξ, there exists a $\Gamma_{(\xi,D)}$ and integer $k \geq 1$ such that $g^k(\varphi_{hH_D}(v)) \in \varphi_{hH_D}(\Delta_\xi)$ for all cosets hH_D.*

Proof. Follows from Lemma 3.4.2 and symmetry/homogeneity of $\Gamma_{(\xi,D)}$. $\square$

For simplicity, in the following we leave out any mention of embedding maps φ. Recall that two segments of Δ_ξ are on the spine of Γ_ξ. Let Σ_ξ denote the segment closest to ξ and Σ'_ξ denote the other segment. Let $v_1, v_2, \ldots, v_r$ be the vertices of Σ_ξ and $v'_1, v'_2, \ldots, v'_r$ be the vertices of Σ'_ξ. By Corollary 3.4.3, there exists a $\Gamma_{(\xi,D_1)}$ and $k_1 \geq 1$ such that $g^{k_1}(v_1) \in \Delta_\xi$ for each natural copy of Δ_ξ in $\Gamma_{(\xi,D_1)}$. Let Θ_1 be a core subgraph of $\Gamma_{(\xi,D_1)}$ containing each copy of Δ_ξ and the first-return orbit $\{g^k(v_1) : k = 1, \ldots, k_1\}$ for each copy of v_1.

By (5) and Proposition 3.3.8, there exists a copy of Θ_1 in Γ_ξ with two segments on the spine of Γ_ξ. It follows that two of the copies of $\Delta_\xi \subseteq \Theta_1$ are on the spine of Γ_ξ. Let $\Delta_{\xi,1}$ denote the copy closest to ξ, and consider $v'_1 \in \Sigma'_\xi \subseteq \Delta_{\xi,1}$. By Corollary 3.4.3, there exists a $\Gamma_{(\xi,D'_1)}$ and k'_1 such that $g^{k'_1}(v'_1) \in \Delta_{\xi,1}$ for each copy of $\Delta_{\xi,1}$ on the rays of $\Gamma_{(\xi,D'_1)}$. Let Θ'_1 be a core subgraph of $\Gamma_{(\xi,D'_1)}$ containing each ray's copy of Θ_1 and the first-return orbit $\{g^k(v'_1) : k = 1, \ldots, k'_1\}$ for each copy of v'_1.

Again, by (5) and Proposition 3.3.8, there exists a copy of Θ'_1 in Γ_ξ with two segments on the spine of Γ_ξ. It follows that two of the copies of $\Delta_{\xi,1} \subseteq \Theta'_1$ are on the spine of Γ_ξ. Let $\Delta'_{\xi,1}$ denote the copy on the segment of Θ'_1 closest to ξ, and consider $v_2 \in \Sigma_\xi \subseteq \Delta'_{\xi,1}$. By Corollary 3.4.3, there exists a $\Gamma_{(\xi,D_2)}$ and k_2 such that $g^{k_2}(v_2) \in \Delta'_{\xi,1}$ for each copy of $\Delta'_{\xi,1}$ on the rays of $\Gamma_{(\xi,D_2)}$. Let Θ_2 be a core subgraph of $\Gamma_{(\xi,D_2)}$ containing each ray's copy of Θ'_1 and the first-return orbit

$\{g^k(v_2) : k = 1, \ldots, k_2\}$ for each copy of v_2.

Continuing in this way, we get a core subgraph Θ'_r of $\Gamma_{(\xi, D'_r)}$ such that each vertex on the spine segments of each copy of $\Delta_{\xi,r}$ returns to $\Delta_{\xi,r}$ under some positive power of g, while also staying inside Θ'_r. Now let v be a vertex on a non spine segment and suppose the sequence $g^k(v)$, $k \geq 1$, does not return to $\Delta_{\xi,r}$, then the sequence is confined to the finite subgraph containing $g(v)$. It follows that there exists m_1, m_2 with $m_2 > m_1$ such that $g^{m_2}(v) = g^{m_1}(v)$, that is, $g^{m_2 - m_1}(v) = v$, a contradiction. Thus every vertex of each copy of $\Delta_{\xi,r}$ returns to $\Delta_{\xi,r}$ under some positive power of g, while also staying inside Θ'_r. Now by (5), $\Theta = \Theta'_r$ is strongly repetitive in Γ_ξ. Let $\{\Theta_i\}_{i \in I}$ be a Θ-sieve in Γ_ξ.

Lemma 3.4.4. *The collection of $|H/H_{P'_r}|$ copies of $\Delta_{\xi,r}$ in each Θ_i form a Δ-sieve in Γ_ξ.*

Proof. Let R_Θ be such that $\Gamma_\xi \setminus E(\bigcup_{i \in I} \Theta_i)$ consists of finite components of size less than or equal to R_Θ. If $\{\Delta_j\}_{j \in J}$ denotes the collection of copies of $\Delta_{\xi,r}$, then the components of $\Gamma_\xi \setminus E(\bigcup_{j \in J} \Delta_j)$ must be finite. Indeed, if there is an infinite component, it must contain an infinite piece of the spine of Γ_ξ. This would imply there are only finitely many copies of Δ on the spine of Γ_ξ, and thus only finitely many copies of Θ on the spine, a contradiction.

Let L be one of the finite components of $\Gamma_\xi \setminus E(\bigcup_{j \in J} \Delta_j)$, then either L is isomorphic to a subcore Θ' of Θ, or there is a unique component L' of $\Gamma_\xi \setminus E(\bigcup_{i \in I} \Theta_i)$ contained in L. In the first case, $|L| = |\Theta'|$. In the second case, the number of copies of Θ adjacent to L' in Γ_ξ is at most $|\partial_V(L')||S|$. Thus $|L| \leq |L'| + |\partial_V(L')||S||\Theta| \leq R_\Theta + R_\Theta|S||\Theta|$. It follows that the size of a component of $\Gamma_\xi \setminus E(\bigcup_{j \in J} \Delta_j)$ is bounded by $\max\{|\Theta'|, R_\Theta + R_\Theta|S||\Theta|\}$. $\square$

Let $\{\Delta_j\}_{j \in J}$ be the above Δ-sieve and let u be a vertex of Γ_ξ. (Case 1) If u is in some Δ_j, then the length of the orbit of u under g is not more than $M = |\Theta|$. (Case 2) Suppose u is in one of the finite components of $\Gamma_\xi \setminus E(\bigcup_{j \in J} \Delta_j)$ and each component has size less than or equal to R. Since $|g|_S = m$, the orbit of u under g cannot "jump over" a copy of Δ, so the orbit either stays in the finite component or eventually lands in a copy of Δ. (Case 2a) If the orbit stays in the finite component, then the length of the orbit of u under g is not more than R. (Case 2b) Suppose

$g^k(u) \in \Delta_j$ for some j and $k \geq 1$, then $g^{i+k}(u) = g^k(u)$ for $i \leq M$ by case 1 and thus $g^i(u) = u$.

It follows that the lengths of all g-orbits of vertices of Γ_ξ are uniformly bounded by $\max\{R, M\}$, hence there exists n such that g^n acts trivially on the vertices of Γ_ξ. But the set of vertices of Γ_ξ is dense in $\mathcal{X}$ by minimality, so $g^n = id$. $\square$

Remark. From the proof of the theorem, it is clear that it is sufficient to have the core subgraphs of each $\Gamma_{(\xi,D)}$ be *eventually* strongly repetitive in Γ_ξ, meaning there is an integer N such that all core subgraphs of segment length greater than N are strongly repetitive.

4. FRAGMENTATIONS

4.1 Introduction

We will come across examples of groups of homeomorphisms of $\mathcal{X}$ that satisfy all of the conditions of Theorem 3.4.1 except the condition that for every $h \in H$ there exists $D \in \mathcal{D}_H$ accumulating on ξ such that $h|_D = id|_D$. In this section we develop a method (called *fragmentation*) that can transform some such examples into groups satisfying all of the conditions of Theorem 3.4.1. Fragmentations (of involutions and the infinite dihedral group) were introduced by Nekrashevych in [17], where they were used to construct groups of Burnside type, including the first example of a simple group of intermediate growth.

Definition 4.1.1. Let h be a finite order homeomorphism of $\mathcal{X}$ and $\mathcal{P}_h$ be a finite partition of supp(h) into h-invariant open sets. A *fragmentation* of h (with respect to $\mathcal{P}_h$) is a group F_h of homeomorphisms of $\mathcal{X}$ such that:

1. for all $g \in F_h$ and $P \in \mathcal{P}_h$, there exists $k \in \mathbb{Z}$ such that $g|_P = h^k|_P$,
2. for all $g \in F_h$, $g|_{\text{Fix}(h)} = id|_{\text{Fix}(h)}$,
3. for all $P \in \mathcal{P}_h$ there exists $g \in F_h$ such that $g|_P = h|_P$.

The elements of $\mathcal{P}_h$ are called the *pieces* of the fragmentation.

Remark. We can be more general and allow h to have infinite order and $\mathcal{P}_h$ to be infinite in the definition of fragmentation, however, we are not interested in such fragmentations in this paper.

First note that such a partition $\mathcal{P}_h$ always exists since supp(h) itself is an h-invariant open set. The first two conditions tell us that for each $g \in F_h$ and $\zeta \in \mathcal{X}$ there exists $k \in \mathbb{Z}$ such that $g(\zeta) = h^k(\zeta)$, so that elements of F_h act as powers of h. If ζ is a regular point of $\langle h \rangle$, then this equation is true for a neighborhood of ζ. The third condition says that for every $\zeta \in \mathcal{X}$ there exists $g \in F_h$ such that $g(\zeta) = h(\zeta)$, so that F_h remembers h.

For example, we have the *trivial fragmentation* $F_h = \langle h \rangle$ (which doesn't depend on $\mathcal{P}_h$). On the other end of the spectrum, we have the *full fragmentation* $F_h = \langle h_P : P \in \mathcal{P}_h \rangle$, where

$$h_P(\zeta) = \begin{cases} h(\zeta) & \text{if } \zeta \in P, \\ \zeta & \text{otherwise.} \end{cases}$$

For any subset A of $\mathcal{X}$, we have $\langle h \rangle(A) = F_h(A)$. In particular, the elements of $\mathcal{P}_h$ are F_h-invariant and $(F_h)|_P = \langle h \rangle|_P$ for all $P \in \mathcal{P}_h$. Every $P \in \mathcal{P}_h$ defines an epimorphism $\pi_P : F_h \longrightarrow \langle h \rangle|_P$ by $\pi_P(g) = g|_P$, and the product map $(\pi_P)_{P \in \mathcal{P}} : F_h \longrightarrow \prod_{P \in \mathcal{P}_h} \langle h \rangle|_P$ is a subdirect embedding, i.e., is an embedding that is surjective on each factor. Conversely, any subdirect product of $\prod_{P \in \mathcal{P}_h} \langle h \rangle|_P$ naturally defines a fragmentation of h by the following faithful action on $\mathcal{X}$: for $\alpha \in \prod_{P \in \mathcal{P}_h} \langle h \rangle|_P$ define

$$\alpha(\zeta) = \begin{cases} h^k(\zeta) & \text{if } \zeta \in P \text{ and } \alpha_P = h^k|_P, \\ \zeta & \text{otherwise.} \end{cases}$$

This is a one-to-one correspondence. In particular, F_h is always a finite abelian group.

Definition 4.1.2. Let S be a set of finite order homeomorphisms of $\mathcal{X}$ and, for each $s \in S$, let $\mathcal{P}_s$ be a finite partition of $\mathrm{supp}(s)$ into s-invariant open sets. A *fragmentation* of the group $G = \langle S \rangle$ is a group $F_G = \langle \bigcup_{s \in S} F_s \rangle$, where F_s is a fragmentation of s (with respect to $\mathcal{P}_s$).

We will focus on the case where S is finite, so that S and $\bigcup_{s \in S} F_s$ are finite generating sets for G and F_G, respectively. First, we discuss some results about particular fragmentations of finite groups.

4.2 Fragmentations of finite groups

Suppose $H = \langle S \rangle$ is a finite group, then every element of S has finite order. Let $\mathcal{P}$ be a finite partition of $\mathrm{supp}(H)$ into H-invariant open sets and, for $s \in S$, let $\mathcal{P}|_s$ consist of the nonempty sets of the form $\mathrm{supp}(s) \cap P$ for $P \in \mathcal{P}$. It is straightforward to show that $\mathcal{P}|_s$ is a finite partition of supp(s) into s-invariant open sets.

Proposition 4.2.1. *Let $F_H = \langle \bigcup_{s\in S} F_s \rangle$ be a fragmentation of $H = \langle S \rangle$ such that $\mathcal{P}_s = \mathcal{P}|_s$ for all $s \in S$, then*

1. *every $P \in \mathcal{P}$ is F_H-invariant and $(F_H)|_P = H|_P$,*

2. *F_H subdirect embeds into $\prod_{P\in\mathcal{P}} (F_H)|_P = \prod_{P\in\mathcal{P}} H|_P$; consequently, F_H is finite.*

Proof. (1): Let $F_S = \bigcup_{s\in S} F_s$ denote the generating set of F_H and let $P \in \mathcal{P}$. Let $t \in F_S$ and suppose $t \in F_s$. We have $P = (\text{supp}(s) \cap P) \sqcup (\text{Fix}(s) \cap P)$, where $P' = \text{supp}(s) \cap P \in \mathcal{P}|_s$, and thus $t|_{P'} = s^k|_{P'}$ for some k. It follows that P is s-invariant and $t|_P = s^k|_P$ (since Fix$(s) \subseteq$ Fix(t)). Therefore, P is F_S-invariant, and thus F_H-invariant.

Let $g \in F_H$, say $g = t_m \cdots t_1$, $t_i \in F_S$, and suppose $t_i \in F_{s_i}$. Then

$$\begin{aligned} g|_P &= (t_m \cdots t_1)|_P \\ &= t_m|_P \cdots t_1|_P \\ &= s_m^{k_m}|_P \cdots s_1^{k_1}|_P \\ &= (s_m^{k_m} \cdots s_1^{k_1})|_P. \end{aligned}$$

for some k_i's. Thus $g|_P \in H|_P$. Now let $h \in H$, say $h = s_m \cdots s_1 \in H$, $s_i \in S$. For $i = 1, 2, \ldots, m$, there exists $t_i \in F_{s_i}$ such that $t_i|_P = s_i|_P$. Thus

$$\begin{aligned} h|_P &= (s_m \cdots s_1)|_P \\ &= s_m|_P \cdots s_1|_P \\ &= t_m|_P \cdots t_1|_P \\ &= (t_m \cdots t_1)|_P, \end{aligned}$$

so that $h|_P \in (F_H)|_P$.

(2): Clearly the homomorphism $(\pi_P)_{P\in\mathcal{P}} : F_H \longrightarrow \prod_{P\in\mathcal{P}}(F_H)|_P$ is surjective on each factor. Since $\text{supp}(F_s) = \text{supp}(s)$ for all $s \in S$, we have $\text{supp}(F_H) = \text{supp}(H) = \bigsqcup_{P\in\mathcal{P}} P$. It follows that $(\pi_P)_{P\in\mathcal{P}}$ is injective. $\square$

Conversely, suppose A is a subdirect product of $\prod_{P\in\mathcal{P}} H|_P$. For $s \in S$, the subgroup $A_s = A \cap \prod_{P\in\mathcal{P}}\langle s\rangle|_P$ is a subdirect product of $\prod_{P\in\mathcal{P}}\langle s\rangle|_P \cong \prod_{P\in\mathcal{P}|_s}\langle s\rangle|_P$. This corresponds to a fragmentation F_s of s with pieces $\mathcal{P}|_s$. We then have $A = \langle A_s\rangle \cong \langle F_s\rangle =: F_H$. In other words, there is a one-to-one correspondence between fragmentations $F_H = \langle\bigcup_{s\in S} F_s\rangle$ with $\mathcal{P}_s = \mathcal{P}|_s$ for all $s \in S$ and subdirect products of $\prod_{P\in\mathcal{P}} H|_P$.

Suppose H has regular support and invariant domains, then $\mathcal{D}_H$ is a finite partition of $\text{supp}(H)$ into H-invariant open sets. In this case, we have $\mathcal{X} = \text{supp}(H) \sqcup \text{Fix}(H)^\circ \sqcup \partial\text{Fix}(H)$ and $\partial\text{Fix}(H)$ consists exactly of the singular points of H. For example, if $H = \langle h\rangle$, it is straightforward to show $\mathcal{D}_{\langle h\rangle}$ consists of the sets $\text{supp}(h, n)$ for n dividing $\text{ord}(h)$.

A *refinement* $\mathcal{P}$ of $\mathcal{D}_H$ is a partition of $\text{supp}(H)$ into H-invariant open sets such that for every $P \in \mathcal{P}$ there exists $D \in \mathcal{D}_H$ with $P \subseteq D$. We also say P *refines* D. Since any subset of H also has regular support, the subgroup $\langle s\rangle$ has regular support for every $s \in S$. Let $\mathcal{P}$ be a refinement of $\mathcal{D}_H$, then $\mathcal{P}|_s$ is a refinement of $\mathcal{D}_{\langle s\rangle}$ for each $s \in S$. We say $\mathcal{P}|_s$ is the *refinement of $\mathcal{D}_{\langle s\rangle}$ induced by $\mathcal{P}$*.

Proposition 4.2.2. *Suppose $H = \langle S\rangle$ is finite with regular support and invariant domains. Let $\mathcal{P}$ be a refinement of $\mathcal{D}_H$. If $F_H = \langle\bigcup_{s\in S} F_s\rangle$ is a fragmentation of H such that $\mathcal{P}_s = \mathcal{P}|_s$ for all $s \in S$, then*

1. *$\prod_{P\in\mathcal{P}} H|_P \cong \prod_{D\in\mathcal{D}_H}(H|_D)^{r_D}$, where $r_D = |\{P \in \mathcal{P} : P \subseteq D\}|$,*

2. *$\mathcal{P}$ is a refinement of $\mathcal{D}_{F_H}$, which is a refinement of $\mathcal{D}_H$.*

Proof. (1): Suppose $P \in \mathcal{P}$ and $D \in \mathcal{D}_H$ such that $P \subseteq D$. We first show $H_P = H_D$. If $h \in H$ fixes P (pointwise), then it must fix D. Indeed, by definition of a domain of support, once h fixes a single point in D, it must fix all of D. The other containment is trivial.

The epimorphisms $h \mapsto h|_P$, $h \mapsto h|_D$ induce isomorphisms $H/H_P \cong H|_P$ and $H/H_D \cong H|_D$. It follows that $H|_P \cong H|_D$.

(2): Note that for any subset A of $\mathcal{X}$, we have $H(A) = F_H(A)$. In particular, a set is H-invariant if and only if it is F_H-invariant.

We first show $\mathcal{P}$ refines $\mathcal{D}_{F_H}$. Let $P \in \mathcal{P}$, then $P \subseteq D$ for some $D \in \mathcal{D}_H$, say $D = \bigcap_{h\in H} U_h$. For every $g \in F_H$ there exists $h_g \in H$ such that $g|_P = h_g|_P$, so that $P \cap \text{supp}(h_g) \subseteq \text{supp}(g)$ and $P \cap \text{Fix}(h_g)^\circ \subseteq \text{Fix}(h_g)^\circ$. For $g \in F_H$ let $U'_g = \text{supp}(g)$ if $U_{h_g} = \text{supp}(h_g)$, and $U'_g = \text{Fix}(h_g)^\circ$ otherwise, then $P \cap U_{h_g} \subseteq U'_g$. Define $D' = \bigcap_{g\in F_H} U'_g \in \mathcal{P}_{F_H}$. We have $P = P \cap D = \bigcap_{h\in H}(P \cap U_h) \subseteq \bigcap_{g\in F_H}(P \cap U_{h_g}) \subseteq \bigcap_{g\in F_H} U'_g = D'$. Recall $\text{supp}(F_H) = \text{supp}(H) = \bigsqcup_{P\in\mathcal{P}} P$, so that $\mathcal{P}$ is a partition of $\text{supp}(F_H)$ into open sets. Every $P \in \mathcal{P}$ is H-invariant, and thus F_H-invariant. It follows that the domains of F_H are F_H-invariant, and $\mathcal{P}$ refines $\mathcal{D}_{F_H}$.

We next show $\mathcal{D}_{F_H}$ refines $\mathcal{D}_H$. Let $D \in \mathcal{D}_{F_H}$, then $D = \bigsqcup_{i=1}^n P_i$, $P_i \in \mathcal{P}$. For each i there exists $D_i \in \mathcal{D}_H$ such that $P_i \subseteq D_i$. We claim $D_i = D_j$ for all i, j. By way of contradiction, suppose, without loss of generality, that $D_1 \neq D_2$. Then, without loss of generality, there exists $h \in H$ such that $D_1 \subseteq \text{supp}(h)$ and $D_2 \subseteq \text{Fix}(h)^\circ$. By Proposition 4.2.1 there is a $g \in F_H$ such that $g|_P = h|_P$, and thus $P \cap \text{supp}(h) \subseteq \text{supp}(g)$ and $D \cap \text{Fix}(h)^\circ \subseteq \text{Fix}(g)^\circ$. It follows that $P_1 \subseteq D \cap D_1 \subseteq D \cap \text{supp}(h) \subseteq \text{supp}(g)$ and $P_2 \subseteq D \cap D_2 \subseteq P \cap \text{Fix}(h)^\circ \subseteq \text{Fix}(g)^\circ$. But then only one of P_1, P_2 can be contained in D (depending on choice of U_g), a contradiction. Thus $D_i = D_j$ for all i, j. If we denote this common set by D', then $D \subseteq D'$. We have $\text{supp}(H) = \text{supp}(F_H) = \bigsqcup_{D\in\mathcal{D}_{F_H}} D$, so that $\mathcal{D}_{F_H}$ is a partition of $\text{supp}(H)$ into open sets. Every $D \in \mathcal{D}_{F_H}$ is F_H-invariant, and thus H-invariant. It follows that $\mathcal{D}_{F_H}$ refines $\mathcal{D}_H$. □

4.3 Orbital graphs of fragmentations

Let $F_G = \langle \bigcup_{s\in S} F_s \rangle$ be a fragmentation of $G = \langle S \rangle$ on $\mathcal{X}$. For every $s \in S$, $t \in F_s$, and $\zeta \in \mathcal{X}$ we have $t(\zeta) = s^k(\zeta)$ for some k. For every $s \in S$ and $\zeta \in \mathcal{X}$, there exists $t \in F_s$ such that $t(\zeta) = s(\zeta)$. It follows that $G(\zeta) = F_G(\zeta)$ so that $G \curvearrowright \mathcal{X}$ and $F_G \curvearrowright \mathcal{X}$ are orbit equivalent.

Let $\zeta \in \mathcal{X}$ and $s \in S$. Suppose first that $\zeta \in \text{Fix}(s)$, then s labels a loop at ζ in Γ_ζ and every $t \in F_s$ fixes ζ. In the corresponding orbital graph $\Gamma_\zeta(F_G)$ of $F_G \curvearrowright \mathcal{X}$, there are loops at ζ labeled by each $t \in F_s \setminus \{id\}$. It follows that fragmenting "multiplies loops".

Now suppose $\zeta \in \text{supp}(s)$ and let $\zeta_j = s^j(\zeta)$ for $j = 0, 1, \ldots, n-1$. Consider the directed cycle $(\zeta_0, s, \zeta_1, \ldots, \zeta_{n-1}, s, \zeta_0)$, each edge labeled by s, in the orbital graph Γ_ζ. This is precisely the orbital graph $\Gamma_\zeta(\langle s\rangle) = \Gamma_\zeta(\langle s\rangle, \mathcal{X}, \{s\})$. We have $\langle s\rangle(\zeta) = F_s(\zeta) = \{\zeta_0, \zeta_1, \ldots, \zeta_{n-1}\}$. Since $\zeta \in \text{supp}(s)$ and the elements of $\mathcal{P}_s$ are s-invariant, there exists $P \in \mathcal{P}_s$ that contains each ζ_j. Let $t \in F_s$, then there exists k such that $t|_P = s^k|_P$, and there will be an arrow labeled by t from ζ_i to ζ_{i+k} (subscripts taken modulo n) for each i. In other words, $\Gamma_\zeta(F_s) = \Gamma_\zeta(F_s, \mathcal{X}, F_s)$ is a "decorated complete" version of $\Gamma_\zeta(\langle s\rangle)$. It follows that fragmenting turns nontrivial directed cycles with all elements labeled by some s into "decorated complete graphs" with edges labeled by elements of $F_s \setminus \{id\}$. We call the graphs $\Gamma_\zeta(F_G)$ *fragmented orbital graphs*, and say $\Gamma_\zeta(F_G)$ is obtained by fragmenting Γ_ζ. Note that fragmenting preserves cut vertices, that is, v is a cut vertex of Γ_ζ if and only if v is a cut vertex of $\Gamma_\zeta(F_G)$.

Lemma 4.3.1. *Let $F_G = \langle\bigcup_{s\in S} F_s\rangle$ be a fragmentation of $G = \langle S\rangle$.*

1. *If $G \curvearrowright \mathcal{X}$ is minimal, then $F_G \curvearrowright \mathcal{X}$ is minimal.*

2. *If ξ is a germ-defining singular point of S, then ξ is a germ-defining singular point of $F_S = \bigcup_{s\in S} F_s$.*

3. *If Γ_ξ is thin and 1-ended, then $\Gamma_\xi(F_G)$ is thin and 1-ended.*

Proof. (1): Suppose the action of G on $\mathcal{X}$ is minimal. Recall that $G \curvearrowright \mathcal{X}$ and $F_G \curvearrowright \mathcal{X}$ are orbit equivalent. It immediately follows that the action of F_G on $\mathcal{X}$ is minimal.

(2): Suppose ξ is a germ-defining singular point of S. Let $\zeta \in F_G(\xi) \setminus \xi = G(\xi) \setminus \xi$, then ζ is not a singular point of any $s \in S$. It follows that ζ is not a singular point of any element of F_S.

We next show if c is a nonloop cycle in $\Gamma_\xi(F_G)$ starting at vertex $\zeta \in F_G(\xi)$, then $(g_c, \zeta) = (id, \zeta)$. Suppose $c = (\zeta, t_1, \zeta_1, t_2, \ldots, t_m, \zeta_m)$, where $\zeta_i = t_i t_{i-1} \cdots t_1(\zeta)$. It follows that $g_c = t_m t_{m-1} \cdots t_1$, $t_i \in F_S$. Suppose $t_i \in F_{s_i}$, then since t_i moves ζ_{i-1}, we have $(t_i, \zeta_{i-1}) = (s_i^{k_i}, \zeta_{i-1})$

for some $0 < k_i < ord(s_i)$. Then

$$\begin{aligned}
(g, \zeta) &= (t_m t_{m-1} \cdots t_1, \zeta) \\
&= (t_m, \zeta_{m-1})(t_{m-1}, \zeta_{m-2}) \cdots (t_1, \zeta) \\
&= (s_m^{k_m}, \zeta_{m-1})(s_{m-1}^{k_{m-1}}, \zeta_{m-2}) \cdots (s_1^{k_1}, \zeta) \\
&= (s_m^{k_m} s_{m-1}^{k_{m-1}} \cdots s_1^{k_1}, \zeta).
\end{aligned}$$

Let $\hat{g} = s_m^{k_m} s_{m-1}^{k_{m-1}} \cdots s_1^{k_1}$. Since c has no loops in $\Gamma_\xi(F_G)$, the directed closed walk $\hat{w}$ defined by $\hat{g}$ at $\zeta \in \Gamma_\xi$ has no loops. If $\hat{w}$ is a cycle at ζ in Γ_ξ, then $(\hat{g}, \zeta) = (id, \zeta)$. Otherwise, $\hat{w}$ has a nonloop cycle subwalk $\hat{c}$ and we can write $\hat{w} = u\hat{c}v$ for some walks u, v. We have $(\hat{g}, \zeta) = (g_v g_{\hat{c}} g_u, \zeta) = (g_v g_u, \zeta)$, and $\hat{w}_1 = uv$ is a closed walk at ζ without loops with length less than that of $\hat{w}$. After finitely many steps, we get a $\hat{w}_r$ that is a nonloop cycle at ζ and $(g, \zeta) = (g_{\hat{w}_r}, \zeta) = (id, \zeta)$.

(3): It is clear that if Γ_ξ is a 1-ended thin graph, then $\Gamma_\xi(F_G)$ is a 1-ended thin graph since fragmenting preserves cut vertices and block sizes. □

Theorem 4.3.2. *Let S be a finite set of homeomorphisms of $\mathcal{X}$ with singular point ξ. Suppose $H = \langle S_\xi \setminus S_{(\xi)} \rangle$ is finite with regular support and invariant pieces, and let $\mathcal{P}$ be a refinement of $\mathcal{D}_H$. Let $F_G = \langle \bigcup_{s \in S} F_s \rangle$ be a fragmentation of $G = \langle S \rangle$ such that $\mathcal{P}_s = \mathcal{P}|_s$ for all $s \in S_\xi \setminus S_{(\xi)}$. Suppose the following is true:*

1. *the induced action of G on $\mathcal{X}$ is minimal,*
2. *ξ is a germ-defining singular point of S,*
3. *Γ_ξ is thin and 1-ended,*
4. *for every $g \in F_H = \langle \bigcup_{s \in S_\xi \setminus S_{(\xi)}} F_s \rangle$ there exists $P \in \mathcal{P}$ accumulating on ξ such that $g|_P = id|_P$,*
5. *for every $P \in \mathcal{P}$ that accumulates on ξ, any core subgraph of $\Gamma_{(\xi,P)}(F_G)$ is strongly repetitive in $\Gamma_\xi(F_G)$.*

Then F_G is periodic.

Remark. Note that (1)-(3) are conditions associated to S and G, while (4)-(5) concern the fragmentation.

Proof. Let $F_S = \bigcup_{s\in S} F_s$ denote the generating set of F_G. We will show that the conditions of Theorem 3.4.1 are satisfied for $F_G = \langle F_S\rangle$ and ξ. Let (i) refer to the ith condition of Theorem 4.3.2 and (3.4.1.i) refer to the ith condition of Theorem 3.4.1.

The conditions (3.4.1.1) - (3.4.1.3) immediately follow from Lemma 4.3.1.

(3.4.1.4): We first show $F_H = \langle (F_S)_\xi \setminus (F_S)_{(\xi)}\rangle$. By definition, $F_H = \langle \bigcup_{s\in S_\xi\setminus S_{(\xi)}} F_s\rangle$. We have $(F_S)_\xi \setminus (F_S)_{(\xi)} = \bigcup_{s\in S}(F_s)_\xi \setminus \bigcup_{s\in S}(F_s)_{(\xi)} = \bigcup_{s\in S}[(F_s)_\xi \setminus (F_s)_{(\xi)}] = \bigcup_{s\in S_\xi\setminus S_{(\xi)}}[F_s \setminus (F_s)_{(\xi)}]$. Therefore, it is enough to show that $\langle F_s \setminus (F_s)_{(\xi)}\rangle = F_s$ for $s \in S_\xi \setminus S_{(\xi)}$. For this, it is enough to show that any $g \in (F_s)_{(\xi)}$ can be written as a product of elements in $F_s\setminus(F_s)_{(\xi)}$. Let $h \in F_s\setminus(F_s)_{(\xi)}$, then $hg, h^{-1} \in F_s \setminus (F_s)_{(\xi)}$ and $g = (h^{-1}h)g = h^{-1}(hg)$. Thus $\langle (F_S)_\xi \setminus (F_S)_{(\xi)}\rangle = F_H$.

By Proposition 4.2.1, we have F_H is finite. Let $g \in F_H$, then by (4) there exists $P \in \mathcal{P}$ accumulating on ξ such that $g|_P = id|_P$. By Proposition 4.2.2, F_H has invariant domains and $\mathcal{P}$ is a refinement of $\mathcal{D}_{F_H}$, thus there exists $D \in \mathcal{D}_{F_H}$ refined by P (therefore accumulating on ξ). Since g fixes the subset P of D, it must fix all of D, i.e., $g|_D = id|_D$.

(3.4.1.5): We start with a lemma.

Lemma 4.3.3. *If $D \in \mathcal{D}_{F_H}$ refines $D' \in \mathcal{D}_H$, then $\Gamma_{(\xi,D)}(F_G)$ is obtained by fragmenting $\Gamma_{(\xi,D')}$.*

Proof. Let $\zeta \in D$, then $(F_H)_\zeta = (F_H)_D$ and $H_\zeta = H_{D'}$ by definition of domains of support. Then $\Gamma_\zeta(F_H) = \Gamma_D(F_H)$ and $\Gamma_\zeta(H) = \Gamma_{D'}(H)$. It follows that $\Gamma_D(F_H)$ is obtained by fragmenting $\Gamma_{D'}(H)$ as described at the beginning of this section. Therefore $\Gamma_{(\xi,D)}(F_G)$ is obtained by fragmenting $\Gamma_{(\xi,D')}$. □

Now suppose $D \in \mathcal{D}_{F_H}$ accumulates on ξ, then there exists $P \in \mathcal{P}$ accumulating on ξ such that $P \subseteq D$. We have $(F_H)_D = (F_H)_P$, thus any core subgraph of $\Gamma_{(\xi,D)}(F_G) = \Gamma_{(\xi,P)}(F_G)$ is strongly repetitive in $\Gamma_\xi(F_G)$ by (5). □

4.4 Tiles of fragmentations

For the examples, it will be sufficient to describe infinite tiles $T_\xi(F_G)$ of a fragmentation F_G when $\partial T_\xi = (S_\xi \setminus S_{(\xi)}) \times \{\xi\}$.

Consider cycle $(\zeta, s, \zeta^1, \ldots, s, \zeta^m)$, $\zeta^i = s^i(\zeta)$, in T_ξ, so that ζ^i is not a critical point of s for all i. If $m = 1$, so that s labels a loop at ζ in T_ξ, then $s|_{\zeta_n\Omega} = \tau_{\zeta_n,\zeta_n} = id|_{\zeta_n\Omega}$ and $\zeta_n\Omega \subseteq \text{Fix}(s)^\circ$. For every $t \in F_s$, we have $\text{Fix}(s)^\circ \subseteq \text{Fix}(t)^\circ$ so that $t|_{\zeta_n\Omega} = id|_{\zeta_n\Omega} = \tau_{\zeta_n,\zeta_n}$. It follows that for every $t \in F_s \setminus \{id\}$ there is a loop at ζ labeled by t in $T_\xi(F_G)$.

Now suppose $m > 1$, then for each i there exists (minimal) $n_i \geq 1$ such that $s|_{\zeta^i_{n_i}\Omega} = \tau_{\zeta^i_{n_i},\zeta^{i+1}_{n_i}}$. Let $n = \max\{n_i\}$, then $s|_{\zeta^i_n\Omega} = \tau_{\zeta^i_n,\zeta^{i+1}_n}$ for all i. In particular, for all i and $k = 0, 1, \ldots, m-1$ we have $s^k|_{\zeta^i_n\Omega} = \tau_{\zeta^i_n,\zeta^{i+k}_n}$ and ζ^i is not a critical point of s^k. The same is true for any $N \geq n$. There exists $P \in \mathcal{P}_s$ containing each ζ^i. In the corresponding tile graph $T_\xi(F_G)$, there will be arrows from ζ^j to ζ^k labeled by all elements $t \in F_s \setminus \{id\}$ such that $t|_P = s^{k-j}|_P$. Indeed, suppose $t|_P = s^{k-j}|_P$ and consider the open set $\zeta^j_n\Omega \cap P$. Claim it is closed: the sets $\zeta^j_n\Omega \cap P$, $P \in \mathcal{P}_s$, finitely partition the clopen set $\zeta^j_n\Omega$ into open sets. It follows that each $\zeta^j_n\Omega \cap P$ is clopen. Suppose $\zeta^j_n\Omega \cap P = \bigsqcup_{i=1}^M u_i\Omega$. There exists i such that $\zeta^j \in u_i\Omega \subseteq \zeta^j_n\Omega \cap P$, then there exists $N \geq n$ such that $u_i = \zeta^j_N$. Therefore $t|_{\zeta^j_N\Omega} = s^{k-j}|_{\zeta^j_N\Omega} = \tau_{\zeta^j_N,\zeta^k_N}$. Thus the "fragmented cycle" is in $T_\xi(F_G)$.

For $s \in S_\xi \setminus S_{(\xi)}$, the loop at ξ labeled by s is missing from T_ξ. If $t \in (F_s)_{(\xi)} \setminus \{id\}$, then there will be a loop at ξ labeled by t. It is easy to see all of edges described above completely give $T_\xi(F_G)$. It follows that $\partial T_\xi(F_G) = ((F_S)_\xi \setminus (F_S)_{(\xi)}) \times \{\xi\}$.

Proposition 4.4.1. *Suppose ξ is a singular point of S and $\partial T_\xi = (S_\xi \setminus S_{(\xi)}) \times \{\xi\}$, then ξ is germ-defining. Furthermore, $T_\xi = \Gamma'_\xi$.*

Proof. Clearly ξ is the only singular point of S in $G(\xi)$. Let $c = (\zeta, e_1, \zeta^1, \ldots, e_m, \zeta^m)$ be a nonloop cycle in Γ_ξ and consider $g_c = \ell(e_m)^{\varepsilon_m} \cdots \ell(e_1)^{\varepsilon_1} = s_m \cdots s_1$, $s_i \in S \cup S^{-1}$. For each $0 \leq i < m$, there exists an n_i such that $s_{i+1}|_{\zeta^i_{n_i}\Omega} = \tau_{\zeta^i_{n_i},\zeta^{i+1}_{n_i}}$. It follows that there exists an N such that $g_c|_{\zeta_N\Omega} = id|_{\zeta_N\Omega}$.

By Proposition 2.3.5, we can obtain Γ_ξ from T_ξ by adding loops at ξ labeled by elements of

$S_\xi \setminus S_{(\xi)}$. In other words, $T_\xi = \Gamma'_\xi$. $\square$

4.5 Some results on subdirect products

Proposition 4.5.1. *For every prime p, there exists a subdirect product H of $C_p^{p+1} = \langle a \mid a^p = id\rangle^{p+1}$ such that every element of H has a coordinate equal to id.*

Proof. Consider the subgroup H generated by $x = (id, a, a^2, \ldots, a^{p-1}, a)$ and $y = (a, a, \ldots, a, id)$. Let $g \in H$, then there exists $c, d \in \{0, 1, \ldots, p-1\}$ such that

$$\begin{aligned} g &= x^c y^d, \\ &= (id, a^c, a^{2c}, \ldots, a^{(p-1)c}, a^c)(a^d, a^d, \ldots, a^d, id) \\ &= (a^d, a^{c+d}, a^{2c+d}, \ldots, a^{(p-1)c+d}, a^c) \end{aligned}$$

If c or d is 0, then the last or first coordinate of g is id. Suppose c, d are nonzero. We have $\{c, 2c, \ldots, (p-1)c\} = \{1, 2, \ldots, p-1\}$ since c is nonzero. In particular, there exists $1 \leq n \leq p-1$ such that $nc = -d \pmod p$. It follows that the $(n+1)$st coordinate of g is $a^{d+nc} = id$. $\square$

Corollary 4.5.2. *Let p, q be prime and r be coprime to p. There exists a subdirect product H of $(C_p \rtimes C_q)^{(p+1)(q+1)} = \langle a, b \mid a^p = id, b^q = id, bab^{-1} = a^r\rangle^{(p+1)(q+1)}$ such that every element of H has a coordinate equal to id.*

Proof. Partition the set of coordinates $\{1, 2, \ldots, (p+1)(q+1)\}$ into $q+1$ pieces $I_1, I_2, \ldots, I_{q+1}$ of size $p+1$, where $I_j = \{j(p+1) - p = (j-1)p + j, \ldots, j(p+1) = jp + j\}$. Define $x = (id, a, a^2, \ldots, a^{p-1}, a) \in C_p^{p+1}$, $y = (a, a, \ldots, a, id) \in C_p^{p+1}$, and $Id = (id, id, \ldots, id) \in C_p^{p+1}$. Let $G = C_p \rtimes C_q$. For $j = 1, 2, \ldots, q+1$ define $x_j = (Id, Id, \ldots, x, \ldots, Id) \in G^{(p+1)(q+1)}$ which is x in I_j and identity elsewhere. Similarly define y_j. Let $B = (b, b, \ldots, b) \in C_q^{p+1}$ and define $X = (Id, B, B^2, \ldots, B^{q-1}, B)$ and $Y = (B, B, \ldots, B, Id)$. Consider the subgroup $H = \langle x_1, \ldots, x_{q+1}, y_1, \ldots, y_{q+1}, X, Y\rangle$. It is straightforward to show that

$$H \cong \langle x_1, \ldots, x_{q+1}, y_1, \ldots, y_{q+1}\rangle \rtimes \langle X, Y\rangle \cong \Big(\prod_{j=1}^{q+1} \langle x_j, y_j\rangle\Big) \rtimes \langle X, Y\rangle.$$

We can write any element of H as h_1h_2 for $h_1 \in \langle x_1, \ldots, x_{q+1}, y_1, \ldots, y_{q+1}\rangle$ and $h_2 \in \langle X, Y\rangle$. The element h_2 will have some I_k coordinate equal to Id, and h_1 will have some coordinate k' inside of I_k equal to id, thus h_1h_2 will be id in coordinate k'. $\square$

When $q = 2$ and $r = -1$, we have $C_p \rtimes C_2 \cong D_p$, the dihedral group consisting of all symmetries of a regular p-gon. For example, $C_3 \rtimes C_2 \cong D_3 \cong \mathrm{Sym}(3)$ and the corollary implies there exists (by construction) a subdirect product of $\mathrm{Sym}(3)^{12}$ such that every element has a coordinate equal to id.

Proposition 4.5.3. *Suppose H has regular support and singular point $\xi \in Fix(H)$. For every $D \in \mathcal{D}_H$ and $n \geq 1$, there exists a refinement of D with n elements accumulating on ξ.*

Proof. Let U_k, $k \geq 0$, be a descending sequence of clopen neighborhoods of ξ such that $U_0 = \mathcal{X}$ and $\bigcap_{k\geq 0} U_k = \{\xi\}$. Then $V_k = \bigcap_{h\in H} h(U_k)$ is a descending sequence of clopen H-invariant neighborhoods of ξ such that $\bigcap_{k\geq 0} V_k = \{\xi\}$ and $\bigcup_{k\geq 0} V_k = \mathcal{X}$. Remove all repetitions, so that $V_k \neq V_{k+1}$ for every k. Also remove V_k if $V_k \setminus V_{k+1}$ does not intersect D. We're left with an infinite descending sequence W_k, $k \geq 0$, of clopen H-invariant neighborhoods of ξ such that $\bigcap_{k\geq 0} W_k = \{\xi\}$, $D \subseteq \bigcup_{k\geq 0} W_k$, $W_k \neq W_{k+1}$, and $W_k \setminus W_{k+1}$ intersects D nontrivially for all k.

Choose an arbitrary partition of the set of non-negative integers into $m \geq n$ disjoint subsets $I_1, I_2, \ldots, I_m$ with $I_1, \ldots, I_n$ infinite, and define $R_i = \bigcup_{k\in I_i} W_k \setminus W_{k+1}$. Then $P_i = D \cap R_i$ gives a refinement $\{P_i\}$ of D of size m with n elements accumulating on ξ. $\square$

5. EXAMPLES

In each the following examples, we begin with a finitely generated group $G = \langle S \rangle$ with singular point ξ that satisfies:

1. $G \curvearrowright \mathcal{X}$ is minimal,
2. ξ is a germ-defining singular point of S,
3. Γ_ξ is thin and 1-ended,
4. $H = \langle S_\xi \setminus S_{(\xi)} \rangle$ is finite with invariant domains,
5. for every $D \in \mathcal{D}_H$ that accumulates on ξ, any core subgraph of $\Gamma_{(\xi,D)}$ is strongly repetitive in Γ_ξ,

but does not satisfy the condition that for every $h \in H$ there exists $D \in \mathcal{D}_H$ such that $h|_D = id|_D$. Thus, it does not completely satisfy all of the conditions of Theorem 3.4.1. We know that any fragmentation of G will preserve 1-3 (Lemma 4.3.1). If we fragment the subgroup $H = \langle S_\xi \setminus S_{(\xi)} \rangle$ carefully, we can satisfy the conditions of Theorem 4.3.2, so that F_G is periodic. This will be most interesting when G is not periodic (or its periodicity is unknown), as *any* fragmentation of a periodic group is periodic. Thus, we present groups other than D_∞ that can be fragmented to produce groups of Burnside type.

5.1 Periodic fragmentations

Proposition 5.1.1. *Let S be a finite set of finite order homeomorphisms of $\mathcal{X}$. If $G = \langle S \rangle$ is periodic, then any fragmentation $F_G = \langle \bigcup_{s \in S} F_s \rangle$ of G is periodic.*

Proof. If G is periodic, then for any $g \in G$ there exists n such that $g^n = id$. Let F_G be a fragmentation of G and $g \in F_G$. Then $g = t_m t_{m-1} \cdots t_1$, $t_i \in \bigcup_{s \in S} F_s$, say $t_i \in F_{s_i}$. For any $\zeta \in \mathcal{X}$, there exist $k_1, k_2, \ldots, k_m$ (depending on ζ) such that $g(\zeta) = t_m t_{m-1} \cdots t_1(\zeta) =$

$s_m^{k_m} s_{m-1}^{k_{m-1}} \cdots s_1^{k_1}(\zeta)$. There are $\prod_{i=1}^{m} \operatorname{ord}(s_i)$ possible choices for the k_i's, and each choice gives an element of G of finite order. Multiplying all of these orders gives an N such that $g^N = id$. $\square$

Example 5.1.2. Let $X = \{0, 1\}$ and consider the set $\mathcal{X} = X^\omega$ of all right-infinite sequences over X. Define three automorphisms a, b, c of X^* by

$$\begin{aligned} a(0w) &= 1w, & b(0w) &= 0w, & c(0w) &= 0c(w), \\ a(1w) &= 0w, & b(1w) &= 1a(w), & c(1w) &= 1b(w). \end{aligned}$$

for $w \in X^*$. Recall that a, b, c induce actions on the boundary $\mathcal{X} = X^\omega$ of the tree. The group $G = \langle a, b, c \rangle$ is the iterated monodromy group of $h(z) = \left(\frac{2z}{q+1} - 1\right)^2$, where $q \approx -0.6478 + 1.7214i$ is a fixed point of $\left(\frac{p-1}{p+1}\right)^2$ (see [16]). It is easy to check that each generator has order two and cba has infinite order.

Recall $(s, u) \in \partial T_n$ if and only if $s|_u \neq id$. For $n \geq 2$, it can be shown (using the Moore diagram in Figure 5.1) that $\partial T_n = \{(c, \alpha_n), (c, \beta_n), (c, \gamma_n)\}$, where $\gamma = 0^{-\omega}, \beta = 0^{-\omega}1, \alpha = 0^{-\omega}11$. If n is understood, we will use α, β, γ, in place of $\alpha_n, \beta_n, \gamma_n$. Notice that the above set of left-infinite words is shift-closed. In particular, we have $\gamma = \gamma 0, \beta = \gamma 1, \alpha = \beta 1$.

The recursive procedure for constructing the level $n+1$ tile for $n \geq 2$ is shown in Figure 5.2 using model graphs. An easy induction shows that the tile graphs T_n are connected for all n. It will be convenient to consider the subgraphs $[\alpha, \beta, \gamma]_n$ of T_n given by the hull of the vertices $\alpha_n, \beta_n, \gamma_n$, i.e., by the union of all paths between these vertices. For $n \geq 3$, a simple induction using model graphs shows $[\alpha, \beta, \gamma]_n$ is a "star graph" with center 1^n and endpoints $\alpha_n, \beta_n, \gamma_n$ (see Figure 5.2).

Let $\mathbf{1} = 1^\omega$ and $\mathbf{1}_n = 1^n$. For $n \geq 3$, the graphs $[\alpha, \mathbf{1}]_n$, $[\beta, \mathbf{1}]_n$, and $[\gamma, \mathbf{1}]_n$ cover $[\alpha, \beta, \gamma]_n$ and pairwise intersect at 1^n. We have the following recursive descriptions of these subgraphs:

$$[\alpha, \mathbf{1}]_{n+1} \cong [\beta, \mathbf{1}]_n, \qquad [\beta, \mathbf{1}]_{n+1} \cong [\gamma, \mathbf{1}]_n, \qquad [\gamma, \mathbf{1}]_{n+1} \cong [\gamma, \mathbf{1}]_n [\mathbf{1}, \alpha]_n \overset{c}{\text{---}} [\alpha, \mathbf{1}]_n.$$

The only singular point of $S = \{a, b, c\}$ is $\xi = 0^\omega$, and $H = \langle S_\xi \setminus S_{(\xi)} \rangle = \langle c \rangle \cong \mathbb{Z}/2\mathbb{Z}$. It follows that $\mathcal{D}_H = \{\operatorname{supp}(c)\}$ and H is finite with regular support and invariant domains. We want

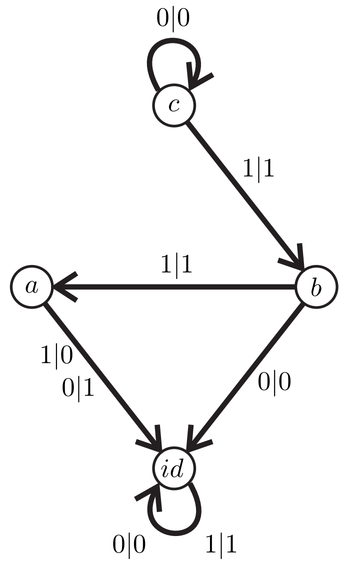
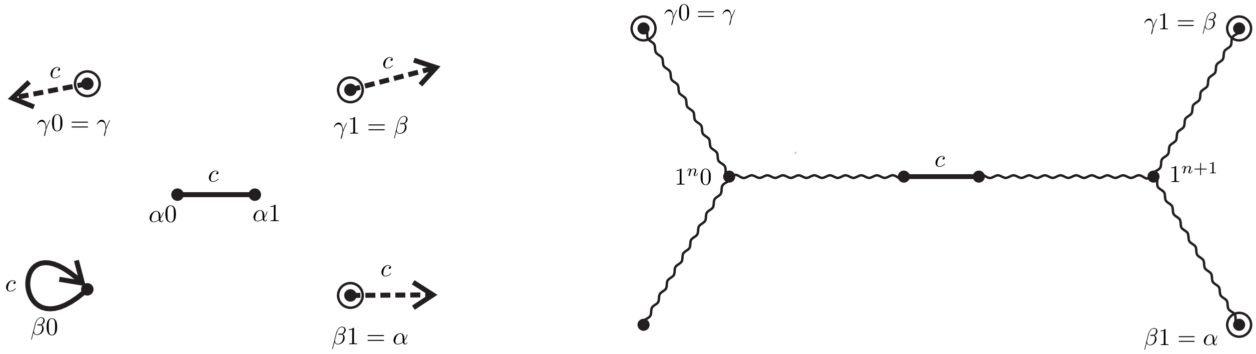


Figure 5.1: Moore diagram of IMG(h)



(a) $\mathcal{M}_{n+1}$ for $n \geq 2$ (b) $[\alpha, \beta, \gamma]_n$ for $n \geq 3$

Figure 5.2: Recursion for model graphs and hull graphs

to find fragmentations of G that satisfy the conditions of Theorem 4.3.2. The first three conditions are independent of the fragmentation.

All finite tiles T_n are connected. It follows from Proposition 2.3.6 that for each $\zeta \in \mathcal{X}$, we have $Cof(\zeta) \subseteq G(\zeta)$. Cofinality classes are dense in $\mathcal{X}$, thus G acts minimally on $\mathcal{X}$.

The generators a, b are finitary and do not have critical points, while c has the unique critical/singular point ξ. Thus ξ is germ-defining by Proposition 4.4.1.

Let $\mathcal{B}(\Gamma)$ denote the number of blocks of a finite graph Γ. From Figure 5.2, we see that

$\mathcal{B}(T_{n+1}) = 2\mathcal{B}(T_n) + 1$. It follows that T_ξ has infinitely many blocks. Furthermore, an easy induction shows each block of T_ξ consists of two vertices and a single edge (possibly with loops; or technically, a pair of opposing arrows with loops), so that T_ξ is thin. By Proposition 2.3.5, Γ_ξ is obtained from T_ξ by adding a loop at ξ labeled by c, so is also thin. By Propositions 3 and 4 of [2], the graph T_ξ is 1-ended, and thus Γ_ξ is 1-ended.

Conditions (4)-(5) of Theorem 4.3.2 depend on the fragmentation of $H = \langle c \rangle$, in other words, on the fragmentation of c. It will be useful to have a handle on the noncritical support structure of c. It is not difficult to show that

$$\begin{aligned} \mathcal{N}(c) &= \bigsqcup_{k \geq 0} \{0^k 110, 0^k 111\} \\ &= \bigsqcup_{k \geq 2} \{\alpha_k 0, \alpha_k 1\}, \end{aligned}$$

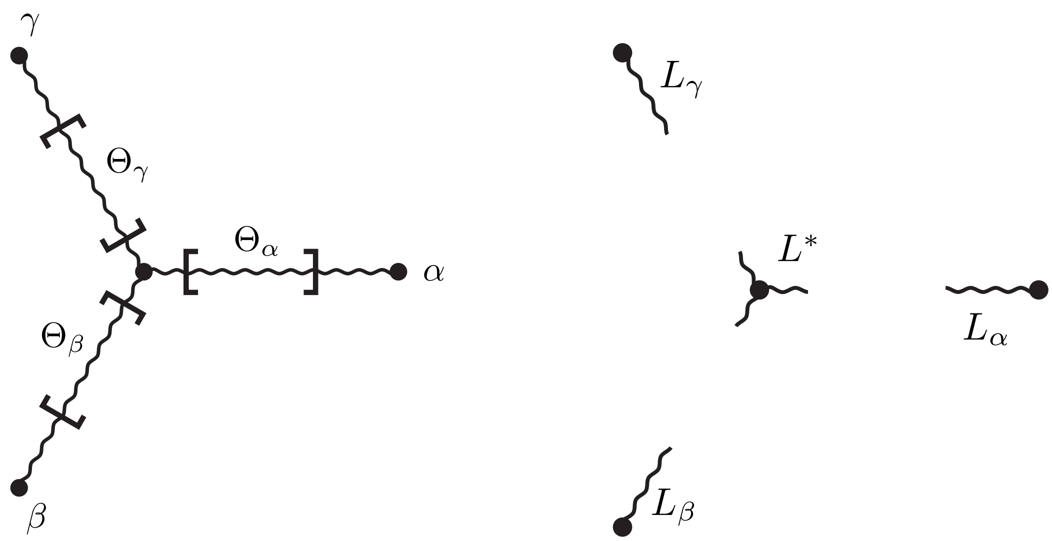
and thus $\text{supp}(c) = \bigsqcup_{k \geq 2} \alpha_k \mathcal{X}$. We first show that the core subgraphs of $\Gamma_{(\xi, \text{supp}(c))}$ for $G \curvearrowright \mathcal{X}$ are strongly repetitive in Γ_ξ, then give conditions for a fragmentation F_c that preserves the strong repetitivity for pieces in $\mathcal{P}_c$ that accumulate on ξ.

Proposition 5.1.3. *Any core subgraph of* $\Gamma_{(\xi, \textit{supp}(c))}$ *is strongly repetitive in* Γ_ξ.

Proof. Let Θ be a core subgraph of $\Gamma_{(\xi,D)}$, $D = \text{supp}(c) \in \mathcal{D}_H$, and let Σ be one of its segments, considered as a subgraph of Γ'_ξ. By Proposition 4.4.1, we have $\Gamma'_\xi = T_\xi$, so Σ is a subgraph of T_ξ. There exists $M \geq 2$ such that Σ is a subgraph of (T_n, ξ_n) for $n \geq M$. Let $N \geq M + 2$.

There is a copy of Σ in T_{N-2} with one endpoint $\xi_{N-2} = \gamma_{N-2}$ and the other endpoint also on $[\gamma, \mathbf{1}]_{N-2}$. From the model graphs (or the recursive definitions of the path hulls), we see after two inflations we will have a copy of Σ in T_N with one endpoint α_N and second endpoint also on $[\alpha, \mathbf{1}]_N$. In T_{N+1}, the copies of Σ at $\alpha_N 0$ and $\alpha_N 1$ are connected by the orbital graph $\Gamma_D(H)$ (i.e., the edge c), giving a copy of the core Θ on $[\gamma, \mathbf{1}]_{N+1}$. From the recursive description of path subgraphs, we see that there is a copy of $[\gamma, \mathbf{1}]_{N+1}$ in each path subgraph of level $N + 3$. Thus, there is a copy of Θ on each path subgraph of level $N + 3$.

Let $n = N + 3$ and denote the copies of Θ on each path subgraph by $\Theta_{\alpha_n}, \Theta_{\beta_n}, \Theta_{\gamma_n}$. Let $L_{\alpha_n}, L_{\beta_n}, L_{\gamma_n}$ be the components of $T_n \setminus E(\Theta_{\alpha_n} \cup \Theta_{\beta_n} \cup \Theta_{\gamma_n})$ containing $\alpha_n, \beta_n, \gamma_n$, respectively, and let L_n^* be the component containing 1^n (see Figure 5.3). We can cover the vertices of T_ξ by level n tiles and connect them along their boundary by edges (labeled by c) to obtain T_ξ. Let $\{\Theta_i\}_{i \in I}$ be the copies of Θ in this cover. A component of $T_\xi \setminus E(\bigcup_{i \in I} \Theta_i)$ will have size less than $\max\{|L_n^*|, m\}$, where $m = \max_{x,y \in \{\alpha,\beta,\gamma\}}\{|L_{x_n}| + |L_{y_n}|\}$, so that Θ is strongly repetitive in T_ξ, and thus Γ_ξ. $\square$



(a) Copies of Θ in $[\alpha, \beta, \gamma]_n$ (b) $[\alpha, \beta, \gamma]_n \setminus E(\Theta_\alpha \cup \Theta_\beta \cup \Theta_\gamma)$

Figure 5.3: Disconnecting $\alpha, \beta, \gamma, \mathbf{1}$ by copies of Θ

Corollary 5.1.4. *Suppose $F_G = \langle F_a \cup F_b \cup F_c \rangle$ is a fragmentation of G such that for every $P \in \mathcal{P}_c$ accumulating on ξ and $M \geq 2$, there exists $N \geq M$ such that $\alpha_N \mathcal{X} \subseteq P$. Then any core subgraph of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_c$, is strongly repetitive in $\Gamma_\xi(F_G)$.*

Proof. Let $\Theta(F_G)$ be a core of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_c$, and let $\Sigma(F_G)$ be one of its segments, considered as a subgraph of $\Gamma'_\xi(F_G)$. By Proposition 4.4.1, we have $\Gamma'_\xi(F_G) = T_\xi(F_G)$, so $\Sigma(F_G)$ is a subgraph of $T_\xi(F_G)$. There exists $M \geq 2$ such that $\Sigma(F_G)$ is a subgraph of $(T_n(F_G), \xi_n)$ for $n \geq M$. By assumption, there exists $N \geq M+2$ such that $\alpha_N \mathcal{X} \subseteq P$.

If we let Σ and Θ denote the “unfragmented" versions of Σ and Θ respectively, then as in the proof of Proposition 5.1.3, we get Θ is strongly repetitive in Γ_ξ. But the fragmented versions of each of these copies is exactly $\Theta(F_G)$, giving strong repetitivity in $\Gamma_\xi(F_G)$ (because vertices of c-edge are of the form $\alpha_N 0w$ and $\alpha_N 1w$). □

For example, any partition of $\{\alpha_k : k \geq 2\}$ induces a set of pieces $\mathcal{P}_c$ that satisfies the condition in the corollary. Recall that there is a one-to-one correspondence of fragmentations of c and subdirect products of $\prod_{P \in \mathcal{P}_c} H|_P \cong \prod_{D \in \mathcal{D}_H} (H|_D)^{r_D} = H^{|\mathcal{P}_c|} \cong (\mathbb{Z}/2\mathbb{Z})^{|\mathcal{P}_c|}$. Then if we use enough pieces that accumulate on ξ, to be specific, at least 3 (see Proposition 4.5.1), we can find a fragmentation of c where every element acts as identity on some piece accumulating on ξ, and thus obtain a periodic fragmentation F_G.

Example 5.1.5. Consider the Bratteli diagram D shown in Figure 5.4 where $V_n = \{(1,n),(2,n)\}$ for $n \geq 0$ and $E_n = E = \{a_1, c_1, a_2, b, c_2\}$ for all $n \geq 1$. Then $\Omega = \Omega(\mathsf{D})$ is a Markov shift with allowed transitions $\{a_1, c_1\} \cdot \{a_1, a_2\} \cup \{a_2, b, c_2\} \cdot \{b, c_1, c_2\}$. We will write elements of Ω as infinite words over E and finite tiles as $T_{1,n}, T_{2,n}$. Define transformations L, M, S of Ω by the following rules:

$$M(a_i w) = a_i w, \quad M(bw) = c_2 w, \quad M(c_1 w) = c_1 w, \quad M(c_2 w) = bw,$$

$$S(a_i w) = c_i w, \quad S(bw) = bM(w), \quad S(c_i w) = a_i w,$$

$$L(a_1 w) = bS(w), \quad L(a_2 w) = a_2 w, \quad L(bbw) = bS(bw), \quad L(bc_i w) = a_1 S(c_i w),$$

$$L(c_1 a_1 w) = c_2 L(a_1 w), \quad L(c_1 a_2 w) = c_1 a_2 w, \quad L(c_2 bbw) = c_2 L(bbw),$$

$$L(c_2 bc_i w) = c_1 L(bc_i w), \quad L(c_2 c_1 w) = c_2 L(c_1 w), \quad L(c_2 c_2 w) = c_2 L(c_2 w).$$

The transformations L, M, S have a partial nondeterministic automata structure as shown in Figure 5.5 (where $A_u = A|_{u\Omega}$ for $A \in \{L, M, S\}, u \in \Omega_*$). Using this, for $n \geq 2$, we get

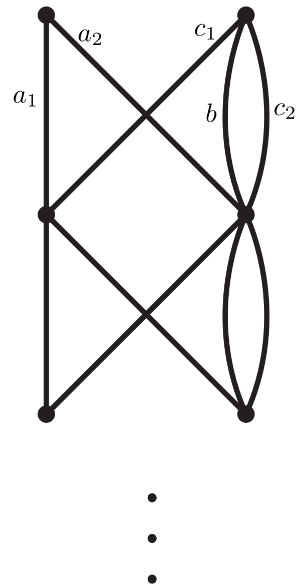


Figure 5.4: Bratteli diagram

$\partial T_{1,n} = \{(L, \lambda_{1,n}), (L, \sigma_{1,n})\}$ and $\partial T_{2,n} = \{(L, \lambda_{2,n}), (L, \sigma_{2,n}), (L, \mu_{2,n})\}$, where $\lambda_1 = c_2^{-\omega} c_1$, $\sigma_1 = c_2^{-\omega} c_1 a_1$, $\lambda_2 = c_2^{-\omega}$, $\sigma_2 = c_2^{-\omega} b$, and $\mu_2 = c_2^{-\omega} bb$. These left-infinite words are shift closed, and in particular, we have $\lambda_1 = \lambda_2 c_1$, $\sigma_1 = \lambda_1 a_1$, $\lambda_2 = \lambda_2 c_2$, $\sigma_2 = \lambda_2 b$, and $\mu_2 = \sigma_2 b$.

The recursive procedure for constructing the level $n+1$ tiles for $n \geq 2$ is shown in Figure 5.6. The tile graphs $T_{i,n}$ are connected for all n and $i = 1, 2$, so we can define the hull subgraphs $[\lambda, \mu]_{1,n}$ and $[\lambda, \mu, \sigma]_{2,n}$ as in the previous example. For $n \geq 3$, $[\lambda, \mu]_{1,n}$ is a path graph, and a simple induction using model graphs shows $[\lambda, \mu, \sigma]_{2,n}$ is a "star graph" with center b^n and endpoints $\lambda_{2,n}, \sigma_{2,n}$, and $\mu_{2,n}$ (see Figure 5.6).

Let $\mathbf{b} = b^\omega$ and $\mathbf{b}_n = b^n$. For $n \geq 3$, the graphs $[\lambda, \mathbf{b}]_{2,n}$, $[\mu, \mathbf{b}]_{2,n}$, and $[\sigma, \mathbf{b}]_{2,n}$ cover $[\lambda, \mu, \sigma]_{2,n}$ and pairwise intersect at b^n. We have the following recursive descriptions of the path subgraphs:

$$[\lambda, \sigma]_{1,n+1} \cong [\lambda, \mathbf{b}]_{2,n} [\mathbf{b}, \sigma]_{2,n} \overset{L}{\text{---}} [\sigma, \lambda]_{1,n}, \qquad [\mu, \mathbf{b}]_{2,n+1} \cong [\sigma, \mathbf{b}]_{2,n},$$

$$[\lambda, \mathbf{b}]_{2,n+1} \cong [\lambda, \mathbf{b}]_{2,n} [\mathbf{b}, \mu]_{2,n} \overset{L}{\text{---}} [\mu, \mathbf{b}]_{2,n}, \qquad [\sigma, \mathbf{b}]_{2,n+1} \cong [\lambda, \mathbf{b}]_{2,n+1}.$$

The only singular points of $\mathcal{S} = \{L, M, S\}$ is $\xi = c_2^\omega$, and $H = \langle \mathcal{S}_\xi \setminus \mathcal{S}_{(\xi)} \rangle = \langle L \rangle \cong \mathbb{Z}/2\mathbb{Z}$. It follows that $\mathcal{D}_H = \{\text{supp}(L)\}$ and H is finite with regular support and invariant domains. We

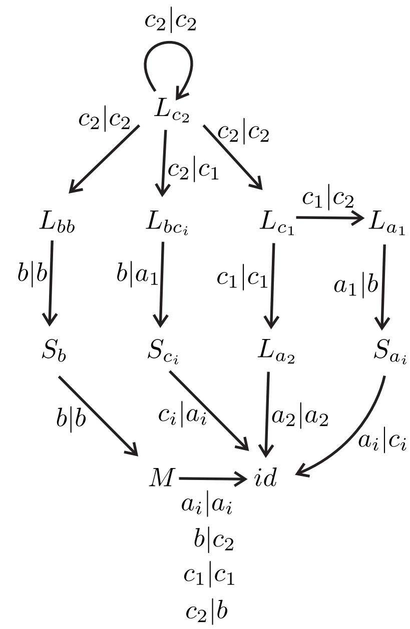


Figure 5.5: Partial nondeterministic automata describing the actions of L, M, S

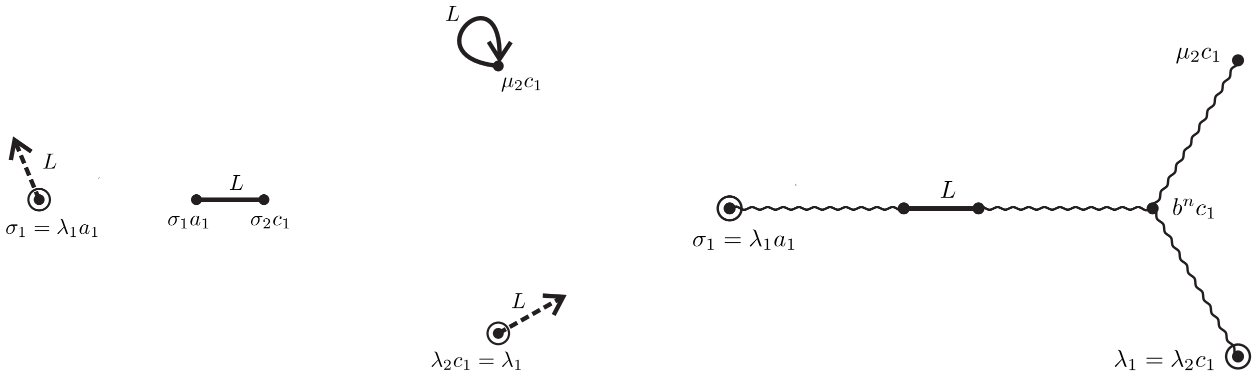
want to find fragmentations of $G = \langle S \rangle$ that satisfy the conditions of Theorem 4.3.2. The first three conditions are independent of the fragmentation.

All finite tiles $T_{i,n}$ are connected. It follows from Proposition 2.3.6 that for each $\zeta \in \Omega$, we have $Cof(\zeta) \subseteq G(\zeta)$. Cofinality classes are dense in $\mathcal{X}$, thus G acts minimally on Ω.

The generators M, S are finitary and do not have critical points, while L has the unique critical/singular point ξ. Thus ξ is rigid by Proposition 4.4.1.

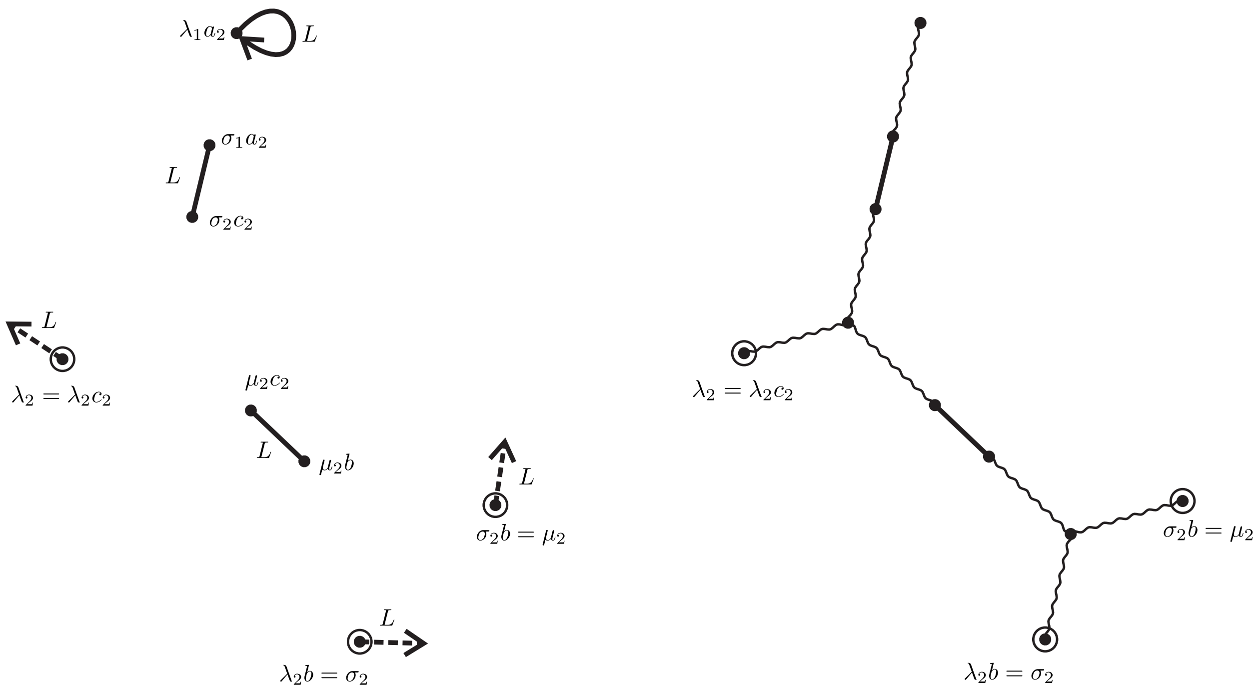
From Figure 5.6, we see that $\mathcal{B}(T_{2,n+1}) = 2\mathcal{B}(T_{2,n}) + \mathcal{B}(T_{1,n}) + 2$. It follows that T_ξ has infinitely many blocks. Furthermore, an easy induction shows each block of T_ξ consists of two vertices and a single edge (possibly with loops; or technically, a pair of opposing arrows with loops), so that T_ξ is thin. By Proposition 2.3.5, Γ_ξ is obtained from T_ξ by adding a loop at ξ labeled by L, so is also thin. By (the ideas of) Propositions 3 and 4 of [2], the graph T_ξ is 1-ended, and thus Γ_ξ is 1-ended.

Conditions (4)-(5) depend on the fragmentation of $H = \langle L \rangle$, in other words, on the fragmentation of L. It will be useful to have a handle on the noncritical support structure of L. Define $\mathcal{N}_n(L) = \mathcal{N}(L) \cap \Omega_n$, (then $\mathcal{N}_n(L)$ consists of the left labels of length n paths in the



(a) $\mathcal{M}_{1,n+1}$ for $n \geq 2$

(b) $[\lambda, \sigma]_{1,n+1}$ for $n \geq 3$



(c) $\mathcal{M}_{2,n+1}$ for $n \geq 2$

(d) $[\lambda, \mu, \sigma]_{2,n+1}$ for $n \geq 3$

Figure 5.6: Recursion for model graphs and hull graphs

automata starting at a vertex L_u and ending in id). It is not difficult to show that $\mathcal{N}_1(L) = \{a_2\}$, $\mathcal{N}_2(L) = \{c_1a_2, a_1a_1, a_1a_2, bc_1, bc_2\}$, and for $n \geq 2$

$$\begin{aligned}\mathcal{N}_{n+1}(L) &= c_2^{n-2}\{bbb, bbc_1, bbc_2, c_2bc_1, c_2bc_2, c_2c_1a_2, c_1a_1a_1, c_1a_1a_2\} \\ &= \{\mu_{2,n}b, \mu_{2,n}c_1, \mu_{2,n}c_2, \sigma_{2,n}c_1, \sigma_{2,n}c_2, \lambda_{1,n}a_2, \sigma_{1,n}a_1, \sigma_{1,n}a_2\}.\end{aligned}$$

We first show that the core subgraphs of $\Gamma_{(\xi,\mathrm{supp}(L))}$ for $G \curvearrowright \mathcal{X}$ are strongly repetitive in Γ_ξ, then give conditions on a fragmentation F_L that preserves the strong repetitivity for pieces in $\mathcal{P}_L$ that accumulate on ξ.

Proposition 5.1.6. *Any core subgraph of $\Gamma_{(\xi,supp(L))}$ is strongly repetitive in Γ_ξ.*

Proof. Let Θ be a core of $\Gamma_{(\xi,D)}$, $D = \mathrm{supp}(L) \in \mathcal{D}_H$, and let Σ be one of its segments, considered as a subgraph of Γ'_ξ. By Proposition 4.4.1, we have $\Gamma'_\xi = T_\xi$, so Σ is a subgraph of T_ξ. There exists $M \geq 2$ such that Σ is a subgraph of $(T_{2,n}, \xi_n)$ for $n \geq M$. Let $N \geq M + 2$.

There is a copy of Σ in T_{N-2} with one endpoint $\xi_{N-2} = \lambda_{2,N-2}$ and the other endpoint also on $[\lambda, \mathbf{b}]_{2,N-2}$. From the model graphs, we see after two inflations we will have a copy of Σ in $T_{2,N}$ with one endpoint $\mu_{2,N}$ and second endpoint also on $[\mu, \mathbf{b}]_{2,N}$. In $T_{2,N+1}$, the copies of Σ at $\mu_{2,N}b$ and $\mu_{2,N}c_2$ are connected by the orbital graph $\Gamma_D(H)$ (i.e., the edge L), giving a copy of the core Θ on $[\lambda, \mathbf{b}]_{2,N+1}$. From the recursive description of path subgraphs, we see that there is a copy of $[\lambda, \mathbf{b}]_{2,N+1}$ in each path subgraph of level $N+3$. Thus, there is a copy of Θ on each path subgraph of level $N+3$. Similar to the previous example, we get Θ is strongly repetitive in Γ_ξ. $\square$

Remark. Notice that from Figure 5.6, there were three choices for introducing an L-edge, i.e., a copy of $\Gamma_D(H)$. The edge labeled L connecting $\sigma_{1,N}a_2$ and $\sigma_{2,N}c_2$ is not in the hull $[\lambda, \mu, \sigma]_{2,N+1}$, so we cannot “distribute" the connection across all path subgraphs using the recursion to obtain strong repetivity. Similarly, although the edge labeled L connecting $\sigma_{1,N}a_1$ and $\sigma_{2,N}c_1$ is in the hull $[\lambda, \sigma]_{1,N+1}$, we cannot have it appear on a future hull of $\lambda_2, \mu_2, \sigma_2$.

Corollary 5.1.7. *Suppose $F_G = \langle F_L \cup F_M \cup F_S \rangle$ is a fragmentation of $G = \langle \mathcal{S} \rangle$ such that for every $P \in \mathcal{P}_L$ accumulating on ξ and $M \geq 2$, there exists $N \geq M$ such that $\mu_{2,N}b\Omega, \mu_{2,N}c_2\Omega \subseteq P$. Then any core subgraph of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_L$, is strongly repetitive in $\Gamma_\xi(F_G)$.*

Proof. Let $\Theta(F_G)$ be a core of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_L$, and let $\Sigma(F_G)$ be one of its segments, considered as a subgraph of $\Gamma'_\xi(F_G)$. By Proposition 4.4.1, we have $\Gamma'_\xi(F_G) = T_\xi(F_G)$, so $\Sigma(F_G)$ is a subgraph of $T_\xi(F_G)$. There exists $M \geq 2$ such that $\Sigma(F_G)$ is a subgraph of $(T_n(F_G), \xi_n)$ for $n \geq M$. By assumption, there exists $N \geq M + 2$ such that $\mu_{2,N}b\Omega \sqcup \mu_{2,N}c_2\Omega \subseteq P$.

If we let Σ and Θ denote the “unfragmented" versions of Σ and Θ respectively, then as in the proof of Proposition 5.1.6, we get Θ is strongly repetitive in Γ_ξ. But the fragmented versions of each of these copies is exactly $\Theta(F_G)$, giving strong repetitivity in $\Gamma_\xi(F_G)$ (because vertices of L-edge are of the form $\mu_{2,N}bw$ and $\mu_{2,n}c_2w$). $\square$

For example, any partition of $\{\{\mu_{2,k}b, \mu_{2,k}c_2\} : k \geq 2\}$ can be used to create a set of pieces $\mathcal{P}_L$ that satisfies the condition in the corollary. As in the last example, if we have at least 3 pieces, we can find a fragmentation of L where every element acts as identity on some piece accumulating on ξ, and thus produce periodic fragmentations F_G.

Example 5.1.8 (Gupta-Fabrykowski)**.** Let $X = \{0, 1, 2\}$ and consider the full shift $\mathcal{X} = X^\omega$ of all right-infinite sequences over X. Define two automorphisms a, b of X^* by

$$
\begin{aligned}
a(0w) &= 1w, & b(0w) &= 0a(w), \\
a(1w) &= 2w, & b(1w) &= 1w, \\
a(2w) &= 0w, & b(2w) &= 2b(w),
\end{aligned}
$$

for $w \in X^*$. The group $G = \langle a, b \rangle$ is the Gupta-Fabrykowski group [6]. Both generators have order three and ba has infinite order. See Figure 5.7 for the automata structure of G.

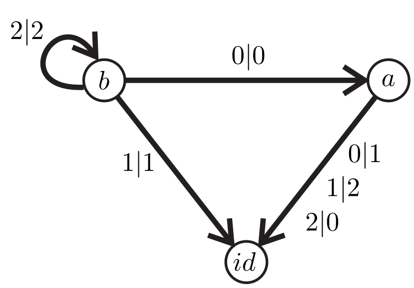
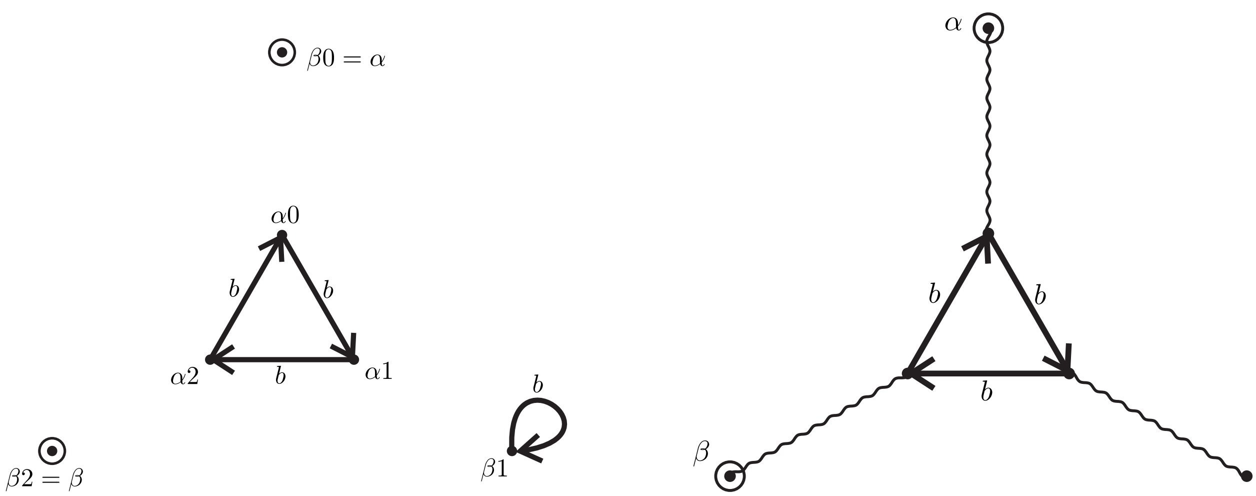


Figure 5.7: Moore diagram of Gupta-Fabrykowski group

For $n \geq 1$, it can be shown that $\partial T_n = \{(b, \alpha_n), (b, \beta_n)\}$, where $\alpha = 2^{-\omega}0$ and $\beta = 2^{-\omega}$. These left-infinite words are shift-closed: $\alpha = \beta 0$ and $\beta = \beta 2$.

The recursive procedure for constructing the level $n+1$ tile for $n \geq 1$ is shown in Figure 5.8. The tile graphs T_n are connected for all n, and the single hull subgraph $[\alpha, \beta]_n$ is a path subgraph for all n. In particular, $[\alpha, \beta]_n$ is a "path" of directed 3-cycles. In Figure 5.8, we have simplified this description with squiggly lines to denote the linear path structure.



(a) $\mathcal{M}_{n+1}$ for $n \geq 1$ (b) $[\alpha, \beta]_{n+1}$ for $n \geq 1$

Figure 5.8: Recursion for model graphs and hull graphs of Gupta-Fabrykowski group

The only singular point of $S = \{a, b\}$ is $\xi = 2^\omega$, and $H = \langle S_\xi \setminus S_{(\xi)} \rangle = \langle b \rangle \cong \mathbb{Z}/3\mathbb{Z}$. We have $\mathcal{D}_H = \{\text{supp}(b)\}$ and thus H is finite with regular support and invariant domains.

All finite tiles T_n are connected. It follows from Proposition 2.3.6 that for each $\zeta \in \mathcal{X}$, we have $Cof(\zeta) \subseteq G(\zeta)$. Cofinality classes are dense in $\mathcal{X}$, thus G acts minimally on $\mathcal{X}$.

The generator a is finitary and does not have critical points, while b has the unique critical/singular point ξ. Thus ξ is germ-defining by Proposition 4.4.1.

From Figure 5.8, we see that $\mathcal{B}(T_{n+1}) = 3\mathcal{B}(T_n) + 1$. It follows that T_ξ has infinitely many blocks. Furthermore, an easy induction shows each block of T_ξ consists of three vertices in a directed cycle with all labels either a or b, so that T_ξ is thin. By Proposition 2.3.5, Γ_ξ is obtained from T_ξ by adding a loop at ξ labeled by b, so is also thin. By Propositions 3 and 4 of [2] the graph T_ξ is 1-ended, and thus Γ_ξ is 1-ended.

Conditions (4)-(5) of Theorem 4.3.2 depend on the fragmentation of $H = \langle b \rangle$, in other words, on the fragmentation of b. It will be useful to have a handle on the noncritical support structure of b. It is not difficult to show that

$$\begin{aligned}\mathcal{N}(b) &= \bigsqcup_{k\geq 0}\{2^k00, 2^k01, 2^k02\} \\ &= \bigsqcup_{k\geq 1}\{\alpha_k 0, \alpha_k 1, \alpha_k 2\},\end{aligned}$$

and thus $\mathrm{supp}(b) = \bigsqcup_{k\geq 1} \alpha_k \mathcal{X}$. We first show that the core subgraphs of $\Gamma_{(\xi,\mathrm{supp}(b))}$ for $G \curvearrowright \mathcal{X}$ are strongly repetitive in Γ_ξ, then give conditions for a fragmentation F_b that preserves the strong repetitivity for pieces in $\mathcal{P}_b$ that accumulate on ξ.

Proposition 5.1.9. *Any core subgraph of* $\Gamma_{(\xi,supp(c))}$ *is strongly repetitive in* Γ_ξ.

Proof. Let Θ be a core of $\Gamma_{(\xi,D)}$, $D = \mathrm{supp}(b) \in \mathcal{D}_H$, and let Σ be one of its segments, considered as a subgraph of Γ'_ξ. By Proposition 4.4.1, we have $\Gamma'_\xi = T_\xi$, so Σ is a subgraph of T_ξ. There exists $M \geq 1$ such that Σ is a subgraph of (T_n, ξ_n) for $n \geq M$. Let $N \geq M + 1$.

There is a copy of Σ in T_{N-1} with one endpoint $\xi_{N-1} = \beta_{N-1}$ and the other endpoint also on $[\alpha, \beta]_{N-1}$. From the model graphs, we see after one inflation we will have a copy of Σ in T_N with one endpoint α_N and second endpoint also on $[\alpha, \beta]_N$. In T_{N+1}, the copies of Σ at $\alpha_N 0$, $\alpha_N 1$, and $\alpha_N 2$ are connected by the orbital graph $\Gamma_D(H)$ (i.e., the directed 3-cycle with b labels), giving a copy of the core Θ on $[\alpha, \beta]_{N+1}$.

Let $n = N+1$. Let $L_{\alpha_n}, L_{\beta_n}$, be the components of $T_n \setminus E(\Theta)$ containing α_n, β_n, respectively. We can cover the vertices of T_ξ by level n tiles and connect them along their boundary by 3-cycles (labeled by b) to obtain T_ξ. Let $\{\Theta_i\}_{i\in I}$ be the copies of Θ in this cover. A component of $T_\xi \setminus E(\bigcup_{i\in I} \Theta_i)$ will have size less than $3\max\{|L_{\alpha_n}|, |L_{\beta_n}|\}$, so that Θ is strongly repetitive in T_ξ, and thus Γ_ξ. $\square$

Corollary 5.1.10. *Suppose $F_G = \langle F_a \cup F_b \rangle$ is a fragmentation of G such that for every $P \in \mathcal{P}_b$ accumulating on ξ and $M \geq 1$, there exists $N \geq M$ such that $\alpha_N \mathcal{X} \subseteq P$. Then any core subgraph of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_b$, is strongly repetitive in $\Gamma_\xi(F_G)$.*

Proof. Let $\Theta(F_G)$ be a core of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}_b$, and let $\Sigma(F_G)$ be one of its segments, considered as a subgraph of $\Gamma'_\xi(F_G)$. By Proposition 4.4.1, we have $\Gamma'_\xi(F_G) = T_\xi(F_G)$, so $\Sigma(F_G)$ is a subgraph of $T_\xi(F_G)$. There exists $M \geq 1$ such that $\Sigma(F_G)$ is a subgraph of $(T_n(F_G), \xi_n)$ for $n \geq M$. By assumption, there exists $N \geq M + 1$ such that $\alpha_N \mathcal{X} \subseteq P$.

If we let Σ and Θ denote the “unfragmented" versions of Σ and Θ respectively, then as in the proof of Proposition 5.1.9, we get Θ is strongly repetitive in Γ_ξ. But the fragmented versions of each of these copies is exactly $\Theta(F_G)$, giving strong repetitivity in $\Gamma_\xi(F_G)$ (because vertices of b-labeled directed cycle are of the form $\alpha_N 0w$, $\alpha_N 1w$, and $\alpha_N 2w$). $\square$

For example, any partition of $\{\alpha_k : k \geq 1\}$ induces a set of pieces $\mathcal{P}_b$ that satisfies the condition in the corollary. There is a one-to-one correspondence of fragmentations of b and subdirect products of $\prod_{P \in \mathcal{P}_b} H|_P \cong \prod_{D \in \mathcal{D}_H} (H|_D)^{r_D} = H^{|\mathcal{P}_b|} \cong (\mathbb{Z}/3\mathbb{Z})^{|\mathcal{P}_b|}$. Then if we use enough pieces that accumulate on ξ, to be specific, at least 4 (see Proposition 4.5.1), we can find a fragmentation of b where every element acts as identity on some piece accumulating on ξ, and thus obtain a periodic fragmentation F_G.

Example 5.1.11. Let $X = \{0, 1, 2, 3, 4, 5\}$ and consider the set $\mathcal{X} = X^\omega$ of all right-infinite sequences over X. Define four automorphisms a_1, a_2, b_1, b_2 of X^* by

$$
\begin{aligned}
a_1(0w) &= 5w, & a_2(0w) &= 1w, & b_1(0w) &= 0a_1(w), & b_2(0w) &= 0a_2(w),\\
a_1(1w) &= 4w, & a_2(1w) &= 2w, & b_1(1w) &= 1b_1(w), & b_2(1w) &= 1b_2(w),\\
a_1(2w) &= 3w, & a_2(2w) &= 0w, & b_1(2w) &= 2w, & b_2(2w) &= 2w,\\
a_1(3w) &= 2w, & a_2(3w) &= 4w, & b_1(3w) &= 3w, & b_2(3w) &= 3w,\\
a_1(4w) &= 1w, & a_2(4w) &= 5w, & b_1(4w) &= 4w, & b_2(4w) &= 4w,\\
a_1(5w) &= 0w, & a_2(5w) &= 3w, & b_1(5w) &= 5w, & b_2(5w) &= 5w,
\end{aligned}
$$

for $w \in X^*$. Let $G = \langle a_1, a_2, b_1, b_2 \rangle$. The generators a_1, b_1 are involutions and a_2, b_2 have order three. The element $b_2 a_2$ has infinite order. See Figure 5.9 for the automata structure of G.

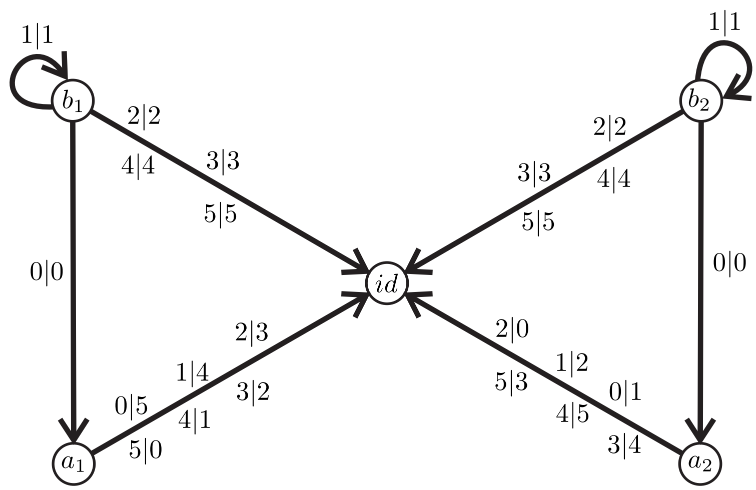


Figure 5.9: Automata describing the actions of a_1, a_2, b_1, b_2

For $n \geq 1$, it can be shown that $\partial T_n = \{(b_1, \alpha_n), (b_2, \alpha_n), (b_1, \beta_n), (b_2, \beta_n)\}$, where $\alpha = 1^{-\omega}0$ and $\beta = 1^{-\omega}$. These left-infinite words are shift-closed: $\alpha = \beta 0$ and $\beta = \beta 1$.

The recursive procedure for constructing the level $n+1$ tile for $n \geq 1$ is similar to the Gupta-Fabrykowski group, except the middle connections form the Cayley graph of $\mathrm{Sym}(3)$ with labels b_1, b_2. The tile graphs T_n are connected for all n, and the single hull subgraph $[\alpha, \beta]_n$ is a path subgraph for all n. In particular, $[\alpha, \beta]_n$ is a "path" of $\mathrm{Sym}(3)$ Cayley graphs.

The only singular point of $S = \{a_1, a_2, b_1, b_2\}$ is $\xi = 1^\omega$, and $H = \langle S_\xi \setminus S_{(\xi)} \rangle = \langle b_1, b_2 \rangle \cong \langle b_2 \rangle \rtimes \langle b_1 \rangle \cong C_3 \rtimes C_2 \cong \mathrm{Sym}(3)$. We have $\mathcal{D}_H = \{\mathrm{supp}(H)\}$ and thus H is finite with regular support and invariant domains.

All finite tiles T_n are connected. It follows from Proposition 2.3.6 that for each $\zeta \in \mathcal{X}$, we have $Cof(\zeta) \subseteq G(\zeta)$. Cofinality classes are dense in $\mathcal{X}$, thus G acts minimally on $\mathcal{X}$.

The generators a_1, a_2 are finitary and do not have critical points, while b_1 and b_2 have the unique critical/singular point ξ. Thus ξ is germ-defining by Proposition 4.4.1.

We have $\mathcal{B}(T_{n+1}) = 6\mathcal{B}(T_n) + 1$. It follows that T_ξ has infinitely many blocks. Furthermore, an easy induction shows each block of T_ξ consists of 6 vertices and is isomorphic to the Cayley graph of $\mathrm{Sym}(3)$ with either labels a_1, a_2 or labels b_1, b_2, so that T_ξ is thin. By Proposition 2.3.5, Γ_ξ is obtained from T_ξ by adding two loops at ξ labeled by b_1, b_2, so is also thin. By Propositions 3 and 4 of [2] the graph T_ξ is 1-ended, and thus Γ_ξ is 1-ended.

Conditions (4)-(5) of Theorem 4.3.2 depend on the fragmentation of $H = \langle b_1, b_2\rangle$ and the refinement $\mathcal{P}$ of $\mathcal{D}_H$ used to induce pieces $\mathcal{P}_{b_1} = \mathcal{P}|_{b_1}$, $\mathcal{P}_{b_2} = \mathcal{P}|_{b_2}$ for the fragmentations of b_1, b_2. It will be useful to have a handle on the noncritical support structure of each b_i. They are equal, and

$$\begin{aligned}\mathcal{N}(b_i) &= \bigsqcup_{k\geq 0}\{1^k00, 1^k01, 1^k02, 1^k03, 1^k04, 1^k05\}\\ &= \bigsqcup_{k\geq 1}\{\alpha_k 0, \alpha_k 1, \alpha_k 2, \alpha_k 3, \alpha_k 4, \alpha_k 5\},\end{aligned}$$

and thus $\mathrm{supp}(b_i) = \bigsqcup_{k\geq 1}\alpha_k\mathcal{X} = \mathrm{supp}(H)$. We first show that the core subgraphs of $\Gamma_{(\xi,\mathrm{supp}(H))}$ for $G \curvearrowright \mathcal{X}$ are strongly repetitive in Γ_ξ, then give conditions for a fragmentation F_H and refinement $\mathcal{P}$ of $\mathcal{D}_H$ that preserves the strong repetitivity for elements of $\mathcal{P}$ that accumulate on ξ.

Proposition 5.1.12. *Any core subgraph of $\Gamma_{(\xi,\mathit{supp}(H))}$ is strongly repetitive in Γ_ξ.*

Proof. Let Θ be a core of $\Gamma_{(\xi,D)}$, $D = \mathrm{supp}(H) \in \mathcal{D}_H$, and let Σ be one of its segments, considered as a subgraph of Γ'_ξ. By Proposition 4.4.1, we have $\Gamma'_\xi = T_\xi$, so Σ is a subgraph of T_ξ. There exists $M \geq 1$ such that Σ is a subgraph of (T_n, ξ_n) for $n \geq M$. Let $N \geq M + 1$.

There is a copy of Σ in T_{N-1} with one endpoint $\xi_{N-1} = \beta_{N-1}$ and the other endpoint also on $[\alpha, \beta]_{N-1}$. From the model graphs, we see after one inflation we will have a copy of Σ in T_N with one endpoint α_N and second endpoint also on $[\alpha, \beta]_N$. In T_{N+1}, the copies of Σ at $\alpha_N i$, $i = 0, 1, \ldots, 5$, are connected by the orbital graph $\Gamma_D(H)$ (i.e., the Cayley graph of $\mathrm{Sym}(3)$ labeled by b_1, b_2), giving a copy of the core Θ on $[\alpha, \beta]_{N+1}$.

Let $n = N + 1$. Let $L_{\alpha_n}, L_{\beta_n}$, be the components of $T_n \setminus E(\Theta)$ containing α_n, β_n, respectively.

We can cover the vertices of T_ξ by level n tiles and connect them along their boundary by Sym(3) Cayley graphs to obtain T_ξ. Let $\{\Theta_i\}_{i\in I}$ be the copies of Θ in this cover. A component of $T_\xi \setminus E(\bigcup_{i\in I}\Theta_i)$ will have size less than $6\max\{|L_{\alpha_n}|,|L_{\beta_n}|\}$, so that Θ is strongly repetitive in T_ξ, and thus Γ_ξ. □

Corollary 5.1.13. *Let $\mathcal{P}$ be a refinement of $\mathcal{D}_H$ such that for every $P \in \mathcal{P}$ accumulating on ξ and $M \geq 1$, there exists $N \geq M$ such that $\alpha_N \mathcal{X} \subseteq P$. Suppose $F_G = \langle F_{a_1} \cup F_{a_2} \cup F_{b_1} \cup F_{b_2}\rangle$ is a fragmentation of G such that $\mathcal{P}_{b_i} = \mathcal{P}|_{b_i} = \mathcal{P}$ for $i = 1, 2$. Then any core subgraph of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}$, is strongly repetitive in $\Gamma_\xi(F_G)$.*

Proof. Let $\Theta(F_G)$ be a core of $\Gamma_{(\xi,P)}(F_G)$, $P \in \mathcal{P}$, and let $\Sigma(F_G)$ be one of its segments, considered as a subgraph of $\Gamma'_\xi(F_G)$. By Proposition 4.4.1, we have $\Gamma'_\xi(F_G) = T_\xi(F_G)$, so $\Sigma(F_G)$ is a subgraph of $T_\xi(F_G)$. There exists $M \geq 1$ such that $\Sigma(F_G)$ is a subgraph of $(T_n(F_G), \xi_n)$ for $n \geq M$. By assumption, there exists $N \geq M + 1$ such that $\alpha_N \mathcal{X} \subseteq P$.

If we let Σ and Θ denote the “unfragmented" versions of Σ and Θ respectively, then as in the proof of Proposition 5.1.12, we get Θ is strongly repetitive in Γ_ξ. But the fragmented versions of each of these copies is exactly $\Theta(F_G)$, giving strong repetitivity in $\Gamma_\xi(F_G)$ (because vertices of b_1, b_1 labeled Sym(3) Cayley graph are of the form $\alpha_N iw$ for $i = 0, 1, 2, 3, 4, 5$). □

For example, any partition of $\{\alpha_k : k \geq 1\}$ induces a refinement $\mathcal{P}$ of $\mathcal{D}_H$ that satisfies the condition in the corollary. There is a one-to-one correspondence of fragmentations of H induced by $\mathcal{P}$ and subdirect products of $\prod_{P\in\mathcal{P}} H|_P \cong \prod_{D\in\mathcal{D}_H}(H|_D)^{r_D} = H^{|\mathcal{P}|} \cong$ Sym$(3)^{|\mathcal{P}|}$. Then if we use enough pieces that accumulate on ξ, to be specific, at least 12 (see Proposition 4.5.2), we can find a fragmentation of H where every element acts as identity on some piece accumulating on ξ, and thus obtain a periodic fragmentation F_G.

6. SUMMARY AND CONCLUSIONS

This dissertation had three main purposes. The first was to introduce a method of proving the periodicity of an infinite group of homeomorphisms of a Cantor set using the geometry of its orbital graphs. The second was to introduce the (generalized) concept of a fragmentation of a group of homeomorphisms of a Cantor set. The third was to expand the class of examples of groups of Burnside type containing fragmentations of the infinite dihedral group.

6.1 Questions and future work

Throughout this paper, there are several natural questions that arise. The first is the following.

Question 6.1.1. Are any conditions from Theorem 3.4.1 superfluous? In particular, is there a combination of four conditions that implies the fifth?

An affirmative answer to this question could greatly simplify examples, particularly, if strong repetivity of core subgraphs is the superfluous condition.

In the examples, it became apparent that constructing a periodic fragmentation relied on the existence of a subdirect product of a finite power of a finite group such that every element of the subdirect product had a coordinate equal to the identity. The mentioned finite group was the subgroup H of G generated by the subset of generators acting singularly on a point. It is not too difficult to construct examples where any finite group can appear as H. This motivates the following question.

Question 6.1.2. Given a finite group H, does there exist n and subdirect product A of H^n such that every element of A has a coordinate equal to id? If so, what is the minimal such n?

A majority of our examples started with a group generated by bounded automata. For some examples, we can fragment such a group and obtain a new bounded automata group. This is related to the following.

Question 6.1.3. Given a finite group H, does there exist n and subdirect product A of H^n such that A is closed under shifting coordinates: $(h_1, h_2, \ldots, h_n) \mapsto (h_2, h_3, \ldots, h_1)$?

In particular, is there an n and A such that A satisfies both conditions of Questions 6.1.2 and 6.1.3? For example, when $H = \mathbb{Z}/2\mathbb{Z}$, for any odd $n \geq 3$ we can take A to be the subgroup of H^n consisiting of all n-tuples with an even number of 1's.

Proving strong repetivity of core subgraphs heavily relied on the recursive procedure for building tile graphs. Is there a way to get around this dependence?

Question 6.1.4. Is there a notion of *strong minimality* for an action that implies balls of orbital graphs of regular points are eventually strongly repetitive in every orbital graph?

Finally, thin graphs are a natural generalization of infinite simple trees. In particular, simple thin graphs with block sizes bounded by 2 are precisely infinite simple trees.

Question 6.1.5. What other applications may thin graphs have?

The expanded fourth class of Burnside type groups containing generalized fragmentations are very similar to, and intersect, the third class of examples involving groups acting on rooted trees. However, there are examples in the third class, such as the Gupta-Sidki group [11], whose periodicity cannot be proved using the methods of this dissertation. It would be interesting to understand if there is a correct description of a big class encompassing all examples in these two classes.

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