STRONG RELATIVE NOVIKOV CONJECTURE

A Dissertation

by

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ABSTRACT

This dissertation can be said to consider Relative Strong Novikov Conjecture for a pair of countable discrete groups.

The first part of the dissertation is about formulation of the relative Baum-Connes assembly map for a pair of countable discrete groups. Our goal is to extend the theory to relative case so that it becomes applicable to relative Novikov conjecture for manifold with boundary. Different from the classical case, we have to consider maximal group C^*-algebras since it is functorial in nature.

In the second part of the dissertation, we study when the strong relative Novikov conjecture is true. Yu and Skandalis-Tu-Yu proved that if a group (viewed as metric spaces with respect to a word metric) admits a coarse embedding into a Hilbert space, then the strong Novikov conjecture is true. Suppose $h : G \to \Gamma$ is a group homomorphism. In the relative case, we will prove that if G is an a-T-menable group, Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture is true. Secondly, we will prove that if $\ker(h)$ is trival and Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture is true.

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TABLE OF CONTENTS

Page

1. INTRODUCTION

A central problem in mathematics is the Novikov conjecture [31]. Roughly speaking, the Novikov conjecture claims that compact smooth manifolds are rigid at an infinitesimal level. More precisely, the Novikov conjecture states that the higher signatures of compact oriented smooth manifolds are invariant under orientation preserving homotopy equivalences. Recall that a compact manifold is called aspherical if its universal cover is contractible. In the case of aspherical manifolds, the Novikov conjecture is an infinitesimal version of the Borel conjecture [2], which states that all compact aspherical manifolds are topologically rigid, i.e. if another compact manifold N is homotopy equivalent to the given compact aspherical manifold M, then N is homeomorphic to M. A deep theorem of Novikov says that the rational Pontryagin classes are invariant under orientation preserving homeomorphisms [32]. The Novikov conjecture for compact aspherical manifolds follows from the Borel conjecture and Novikov's theorem since for aspherical manfolds, the information about higher signatures is equivalent to that of rational Pontryagin classes.

The Novikov conjecture has inspired a lot of beautiful theories. It motivated the development of Kasparov's KK-theory [26, 27], Connes' cyclic cohomology theory [8], Gromov-Connes-Moscovici theory of almost flat bundles [12], Connes-Higson's E-theory [10], and quantitative operator K-theory [34]. The Novikov conjecture has been proven for a large number of cases [27, 11, 12, 44, 28, 45, 39]. The general philosophy is that the conjecture should be true if the fundamental group of the manifold arises from nature.

The following concept is given by Gromov which makes precise of the idea of drawing a good picture of a metric space in a Hilbert space.

Definition 1.0.1. Let X be a metric space and H be a Hilbert space. A map $f : X \to H$ is said to be a coarse embedding if there exist non-decreasing functions ρ_1 and ρ_2 on $[0, \infty)$

such that

(1) $\rho_1(d(x,y)) \leqslant d_H(f(x), f(y)) \leqslant \rho_2(d(x,y))$ for all $x, y \in X$;

(2) $\lim_{r\to\infty} \rho_1(r) = +\infty$.

Coarse embeddability of a countable group is independent of the choice of proper length metrics. Examples of groups coarsely embeddable into Hilbert space include groups acting properly and isometrically on a Hilbert space (in particular amenable groups [5]), groups with Property A [45], countable subgroups of connected Lie groups [18], hyperbolic groups [15, 38], groups with finite asymptotic dimension [16, 44, 37], Coxeter groups [14], mapping class groups [30, 19], and semi-direct products of groups of the above types. One can refer to [33] for more details. Note that there are groups which do not admit coarse embedding into Hilbert space, see [17] for details.

The following theorem proved by G. Yu [45] and Skandalis-Tu-Yu [39] shows that the strong Novikov conjecture holds for groups coarsely embeddable into Hilbert spaces which implies the Novikov conjecture.

Theorem 1.0.2. Let Γ be a countable group and A be any Γ-C^*-algebra. Suppose Γ admits coarse embedding into a Hilbert space H, then the strong Novikov conjecture with coefficients in A holds for Γ, i.e. the Baum-Connes assembly map

$$\mu : K_*^\Gamma(\mathcal{E}\Gamma, A) \to K_*(C^*_{red}(\Gamma, A))$$

is injective.

Kasparov-Yu [29] generalize the definition of Gromov and discuss the connection of the strong Novikov conjecture with geometry of Banach spaces.

Definition 1.0.3. A real Banach space X is said to have Property (H) if there exists an increasing sequence of finite dimensional subspaces $\{V_n\}$ of X and an increasing sequence of finite dimensional subspaces $\{W_n\}$ of a real Hilbert space such that

(1) $V = \cup_n V_n$ is dense in X;

(2) if W denotes $\cup_n W_n,$ and $S(V), S(W)$ denote respectively the unit spheres of $V, W,$ then there exists a uniformly continuous map $\psi : S(V) \to S(W)$ such that the restriction of ψ to $S(V_n)$ is a homeomorphism (or more generally a degree one map) onto $S(W_n)$ for each $n.$

As an example, let X be the Banach space $\ell^p(\mathbb{N})$ for some $p \geqslant 1$. Let V_n and W_n be respectively the subspaces of $\ell^p(\mathbb{N})$ and $\ell^2(\mathbb{N})$ consisting of all sequences whose coordinates are zero after the n-th terms. We define a map ψ from $S(V)$ to $S(W)$ by $\psi(c_1, \cdots, c_k, \cdots) = (c_1|c_1|^{p/2-1}, \cdots, c_k|c_k|^{p/2-1}, \cdots)$. ψ is called the Mazur map. It is not difficult to verify that ψ satisfies the conditions in the definition of Property (H). For each $p \geqslant 1$, one can similarly prove that C_p, the Banach space of all Schatten p-class operators on a Hilbert space, has Property (H).

Kasparov-Yu proved the following theorem which can be seen as a generalization of theorem 1.0.2.

Theorem 1.0.4. Let Γ be a countable group and A be any Γ-C^*-algebra. Suppose Γ admits coarse embedding into a Banach space with property (H), then the strong Novikov conjecture with coefficients in A holds for Γ, i.e. the Baum-Connes assembly map

$$\mu : K_*^{\Gamma}(\mathcal{E}\Gamma, A) \to K_*(C^*_{red}(\Gamma, A))$$

is injective.

In the first part of this dissertation, we develop the general framework for the relative Baum-Connes assembly map.

In the second part of this dissertation, we study in what condition the strong relative Novikov conjecture holds.

Let A be a C^*-algebra and let a countable discrete group Γ act on A by $*$-automorphisms. One may then form the reduced crossed product C^*-algebra $C^*_{red}(\Gamma, A)$. The usual Baum-Connes conjecture with coefficients posits that a certain homomorphism

$$\mu : K_*^{\Gamma}(\mathcal{E}\Gamma; A) \to K_*(C^*_{red}(\Gamma, A))$$

is an isomorphism [3, 4], where the left-hand side is the equivariant K-homology with coefficients in A of the classifying space $\mathcal{E}\Gamma$ for proper Γ-actions, and the right-hand side is the K-theory of the reduced crossed product C^*-algebra. We will consider a particular model for $\mathcal{E}\Gamma$, namely $\bigcup_{s\geq 0} P_s(\Gamma)$ equipped with the ℓ_1 metric (cf. [4, Section 2]), where $P_s(\Gamma)$ is the Rips complex of Γ at scale s, i.e., it is the simplicial complex with vertex set Γ, and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g, h) \leq s$ for all $g, h \in E$. Here we assume that Γ is equipped with a proper length function and d is the associated metric. One may then reformulate the Baum-Connes map as

$$\lim_{s\to\infty} K_*(C^*_L(\Gamma, P_s(\Gamma), A)) \xrightarrow{e_*} \lim_{s\to\infty} K_*(C^*_{red}(\Gamma, P_s(\Gamma), A)) \cong K_*(C^*_{red}(\Gamma, A)),$$

where $C^*_L(\Gamma, P_s(\Gamma), A)$ is Yu's localization algebra with coefficients in A, $C^*_{red}(\Gamma, P_s(\Gamma), A)$ is the reduced equivariant Roe C^*-algebra with coefficients in A, and e is (induced by) the evaluation-at-zero map. The fact that K-homology can be identified with the K-theory of the localization algebra was shown for finite-dimensional simplicial complexes in [43], and in full generality in [35]. The fact that the equivariant Roe algebra with coefficients is stably isomorphic to the reduced crossed product forms the basis for the coarse-geometric approach to the Baum-Connes conjecture with coefficients (see [36] for the case without coefficients).

In the relative framework, we have to use the maximal group C^*-algebra. Let A be a C^*-algebra and let $h : G \to \Gamma$ be a group homomorphism. Let G and Γ act on A simultaneously, and $g \cdot a = h(g) \cdot a$ for any $g \in G$, $a \in A$. We call A a (G, Γ)-C^*-algebra.

The relative Baum-Connes assembly map can be formulated as follows. If $h : G \to \Gamma$ is a group homomorphism and A is a (G, Γ)-C^*-algebra. There exist left-invariant metrics d_G and d_Γ on G and Γ such that $d_\Gamma(h(g_1), h(g_2)) \leq 2d_G(g_1, g_2)$ for any g_1, g_2 in G. Hence there exists a homomorphism (also denoted by h) from $C^*_L(G, P_sG, A)$ to $C^*_L(\Gamma, P_{2s}\Gamma, A)$. We can formulate the mapping cone of h and denote it by $C^*_L(G, P_sG, \Gamma, P_{2s}\Gamma, A)$. Similarly, we have $C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)$. Now the relative Baum-Connes assembly map can be defined as

$$\begin{array}{ccc} \lim\limits_{s\to\infty} K_*(C^*_L(G, P_sG, \Gamma, P_{2s}\Gamma, A)) & \xrightarrow{e_*} \lim\limits_{s\to\infty} & K_*(C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)) \\ & & \downarrow\cong \\ & & K_*(C^*_{max}(G, \Gamma, A)). \end{array}$$

We denote the left hand side by $K^{G,\Gamma}_*(\mathcal{E}G, \mathcal{E}\Gamma, A)$, the relative Baum-Connes assembly map can be written as

$$\mu_{max} : K^{G,\Gamma}_*(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A)).$$

When h is injective, we can formulate the reduced version of relative Baum-Connes assembly map

$$\mu_{red} : K^{G,\Gamma}_*(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A)).$$

Strong Relative Novikov Conjecture with coefficients. Let G and Γ be countable discrete groups, $h : G \to \Gamma$ be a group homomorphism and A any (G, Γ)-C^*-algebra. Then the *maximal relative Baum-Connes assembly map*

$$\mu_{max} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A))$$

is injective. If h is injective, then the *reduced relative Baum-Connes assembly map*

$$\mu_{red} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A)).$$

is injective.

Our main result may then be stated as follows.

Theorem 1.0.5. Let $h : G \to \Gamma$ be a group homomorphism and A any (G, Γ)-C^*-algebra. If G is an a-T-menable group and Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture with coefficients in A holds for (G, Γ, h), i.e. the maximal relative Baum-Connes assembly map

$$\mu_{max} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A))$$

is injective.

Theorem 1.0.6. Let $h : G \to \Gamma$ be an injective group homomorphism and A any (G, Γ)-C^*-algebra. Suppose Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture with coefficients in A holds for (G, Γ, h), i.e. the reduced relative Baum-Connes assembly map

$$\mu_{red} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A))$$

is injective.

2. PRELIMINARIES

2.1 C^*-Algebras

In this section, we record some basic facts about C^*-algebras that can be found, for instance, in [13] or [24]. Throughout this dissertation, we will only work with C^*-algebras.

Definition 2.1.1. A Banach algebra A is an algebra equipped with a submultiplicative norm, i.e., $||ab|| \leq ||a|| ||b||$ for all $a, b \in A$, and such that $(A, || \cdot ||)$ is a Banach space.

A is said to be unital if there exists $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. If there is no such element, then A is said to be non-unital.

Definition 2.1.2. Let A be an algebra. A map $* : A \to A, a \mapsto a^*$, is called an involution if it satisfies

1. $(a + b)^* = a^* + b^*$ for all $a, b \in A$,
2. $(\lambda a)^* = \bar{\lambda} a^*$ for all $\lambda \in \mathbb{C}$, $a \in A$,
3. $(ab)^* = b^* a^*$ for all $a, b \in A$,
4. $(a^*)^* = a$ for all $a \in A$.

If A is a Banach algebra equipped with an isometric involution, i.e., $||a^*|| = ||a||$ for all $a \in A$, then A is called a Banach *-algebra.

If the involution also satisfies $||a^* a|| = ||a||^2$ for all $a \in A$, then A is called a C^*-algebra.

Example 2.1.3.

1. Let X be a compact Hausdorff space. Then $C(X)$, the set of continuous functions on X, is a unital C^*-algebra when equipped with pointwise multiplication and the norm $||f|| := \sup_{x \in X} |f(x)|$.

2. If X is locally compact but not compact, then $C_0(X)$, the set of continuous functions on X vanishing at infinity, is a non-unital C^*-algebra with the above multiplication and norm.

3. Let H be a complex Hilbert space. Then $B(H)$, the set of bounded linear operators on H, is a unital C^*-algebra with composition as multiplication and the operator norm.

4. Let H be a complex Hilbert space. Then $K(H)$, the set of compact operators on H, is a non-unital C^*-algebra with composition as multiplication and the operator norm.

Definition 2.1.4. Suppose A is an algebra. Define $A^+ = A \times \mathbb{C}$ equipped with the operation $(a, z)(b, w) = (ab + zb + wa, zw)$ for $a, b \in A$ and $z, w \in \mathbb{C}$. Then A^+ is a unital algebra with unit $(0, 1)$. We identify A as a subalgebra in A^+ via the map $a \mapsto (a, 0)$. We call A^+ the unitization of A.

Note that this construction makes sense even when A is already unital, but the original unit in A is not the unit in A^+.

The unitization A^+ of A can be equipped with a submultiplicative norm extending the norm on A such that $(0, 1) \in A^+$ has norm 1. One such norm is given by

$$||(a, z)||_1 = ||a|| + |z|$$

for all $(a, z) \in A^+$. If there is an isometric algebra homomorphism $\phi : A \to B$, where B is a unital normed algebra with $||1_B|| = 1$ and $1_B \notin \phi(A)$, then the homomorphism $\phi^+ : A^+ \to B$ is injective, and we can also define a submultiplicative norm $|| \cdot ||'$ on A^+ by

$$||(a, z)||' = ||\phi^+(a, z)||.$$

If A is a Banach algebra, then any submultiplicative norm on A^+ extending the norm on A such that $(0, 1) \in A^+$ has norm 1 is in fact equivalent to the norm $|| \cdot ||_1$ defined above. This is a consequence of the open mapping theorem once one observes that any such norm is dominated by $|| \cdot ||_1$.

The invertible elements in a unital Banach algebra play an important role in the theory of Banach algebras, and also in K-theory.

Lemma 2.1.5. Suppose A is a unital Banach algebra, and $a \in A$ is invertible. Suppose that $b \in A$ satisfies $||b - a|| < \frac{1}{||a^{-1}||}$. Then b is also invertible, and

$$||b^{-1} - a^{-1}|| \leq \frac{||a^{-1}||^2||b - a||}{1 - ||a^{-1}||||b - a||}.$$

Proof. Define $y = 1 - a^{-1}b$. Then $||y|| = ||a^{-1}(a - b)|| \leq ||a^{-1}||||a - b|| < 1$. Since $||y^n|| \leq ||y||^n$ for all $n \geq 1$, the series $\sum_{n=0}^{\infty} y^n$ converges to an element $z \in A$, and we have

$$||1 - z|| \leq \sum_{n=1}^{\infty} ||y||^n \leq \sum_{n=1}^{\infty} (||a^{-1}||||b - a||)^n = \frac{||a^{-1}||||b - a||}{1 - ||a^{-1}||||b - a||}.$$

By the definition of z, we have $z(1 - y) = (1 - y)z = 1$, so $a^{-1}b = 1 - y$ is invertible, and $(a^{-1}b)^{-1} = z$. Since a is invertible, it follows that b is invertible with inverse $b^{-1} = za^{-1}$. We then have

$$||b^{-1} - a^{-1}|| = ||(1 - z)a^{-1}|| \leq ||1 - z||||a^{-1}|| \leq \frac{||a^{-1}||^2||b - a||}{1 - ||a^{-1}||||b - a||}.$$

$\square$

Corollary 2.1.6. If A is a unital Banach algebra, then the set of invertible elements in A, denoted by $GL(A)$, is open in A, and inversion is continuous.

Definition 2.1.7. Let A be a unital Banach algebra. For $a \in A$, the set

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}$$

is called the spectrum of a (relative to A).

Theorem 2.1.8. Let A be a unital Banach algebra. For any $a \in A$, the set $\sigma_A(a)$ is compact and non-empty.

Sketch of proof. Given $a \in A$, consider the map $\psi : \mathbb{C} \to A$ given by $\psi(\lambda) = \lambda 1 - a$. Then $\mathbb{C} \setminus \sigma_A(a) = \psi^{-1}(GL(A))$. The fact that $\mathbb{C} \setminus \sigma_A(a)$ is open is a consequence of the continuity of ψ and the fact that $GL(A)$ is open.

If $|\lambda| > ||a||$, then the element $x = \lambda 1 - a$ satisfies $||x - \lambda 1|| = ||a|| < |\lambda| = ||(\lambda 1)^{-1}||^{-1}$ so x is invertible, which means that $\lambda \notin \sigma_A(a)$. Hence $\sigma_A(a)$ is bounded as $\sigma_A(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq ||a||\}$.

If $\sigma_A(a) = \emptyset$, define $F : \mathbb{C} \to A$ by $F(\zeta) = (\zeta 1 - a)^{-1}$. Then F is continuous, and one shows that $\lim_{\zeta \to \infty} ||F(\zeta)|| = 0$. For each continuous linear functional $\theta \in A^*$, one shows that the map $F_\theta = \theta \circ F : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\lim_{\zeta \to \infty} F_\theta(\zeta) = 0$. By Liouville's theorem, it follows that $F_\theta \equiv 0$. Fix $\zeta \in \mathbb{C}$. The fact that $\theta(F(\zeta)) = F_\theta(\zeta) = 0$ for all $\theta \in A^*$, combined with the Hahn-Banach theorem, forces $F(\zeta) = 0$, which is impossible. $\square$

Example 2.1.9.

1. For $T \in M_n(\mathbb{C})$, we have $\sigma_{M_n(\mathbb{C})}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue for } T\}$.

2. Let X be a compact Hausdorff space. For $f \in C(X)$, we have $\sigma_{C(X)}(f) = f(X)$.

The notion of functional calculus is also an important one in the theory of Banach algebras. It allows one to make sense of expressions like $f(a)$, where a is an element of a Banach algebra and $f : \mathbb{C} \to \mathbb{C}$ is an appropriate function.

Fix some element a in a unital Banach algebra A. Suppose that $p : \mathbb{C} \to \mathbb{C}$ is a polynomial, i.e., $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ with $c_0, \ldots, c_n \in \mathbb{C}$. We can then define $p(a) = c_0 1 + c_1 a + \cdots + c_n a^n$. Now let U be an open subset of $\mathbb{C}$ containing $\sigma_A(a)$, and denote by $R(U)$ the set of all rational functions on U, i.e., $f \in R(U)$ if and only if $f = (\frac{p}{q})|_U$, where p and q are polynomials with $q(z) \neq 0$ for all $z \in U$. Since $\sigma_A(q(a)) = q(\sigma_A(a))$, we have that $0 \notin \sigma_A(q(a))$ so we may define $f(a) \in A$ by $f(a) = p(a)q(a)^{-1}$. If we let $R(a) = \bigcup\{R(U) : U \text{ open}, U \supseteq \sigma_A(a)\}$, then $R(a)$ is an algebra, and $f(a) \in A$ is well-defined for every $f \in R(a)$. In fact, the mapping $f \mapsto f(a)$ is a homomorphism from $R(a)$ into A, and satisfies $\sigma_A(f(a)) = f(\sigma_A(a))$.

More generally, we can also make sense of $f(a)$ for a function f that is holomorphic on a neighborhood of $\sigma_A(a)$. Given an open subset of $\mathbb{C}$, let $H(U)$ denote the algebra of all holomorphic functions on U. For $a \in A$, let $H(a)$ be the set of all functions that are holomorphic in some neighborhood of $\sigma_A(a)$. Then $H(a)$ is an algebra under pointwise operations.

Proposition 2.1.10. [24, Proposition 3.15] Let A be a unital Banach algebra, let $a \in A$, and let U be an open neighborhood of $\sigma_A(a)$. Suppose that $\gamma_1, \ldots, \gamma_n$ are closed, piecewise smooth curves in $U \setminus \sigma_A(a)$ such that for any holomorphic function f on U and $z \in \sigma_A(a)$,

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z} dw.$$

Then for any rational function f on U,

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(z 1_A - a)^{-1} dz.$$

We would like to define, for $f \in H(a)$, an element $f(a) \in A$ by setting

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(z1_A - a)^{-1} dz,$$

where $\gamma_1, \ldots, \gamma_n$ are as above. Indeed, this definition does not depend on the choice of U and of the curves $\gamma_1, \ldots, \gamma_n$ [24, Lemma 3.16]. The set of mappings $H(a) \to A, f \mapsto f(a)$, is referred to as the holomorphic functional calculus.

Theorem 2.1.11. (cf. [24, Theorem 3.18]) Let A be a unital Banach algebra, and let $a \in A$.

1. The mapping $f \mapsto f(a)$ is a homomorphism from $H(a)$ into A.
2. Suppose that f and f_n ($n \in \mathbb{N}$) are holomorphic functions on some open set U containing $\sigma_A(a)$ and that f_n converges uniformly to f on every compact subset of U. Then $||f_n(a) - f(a)|| \to 0$.

2.2 Group C^*-Algebras

In this section, we review some basic facts about group C^*-algebras that can be found, for instance, in [21]. Throughout this dissertation, we will only work with countable discrete groups.

Definition 2.2.1. (cf. [21, Definition 2.17]) Let G be a discrete group and let A be a G-C^*-algebra. A covariant representation of A in a C^*-algebra B is a pair (φ, π) consisting of a $*$-homomorphism φ from A into a C^*-algebra B and a group homomorphism π from G into the unitary group of the multiplier algebra of B which are related by the formulas

$$\pi(g)\varphi(a)\pi(g^{-1}) = \varphi(ga), \; for \; all \; a \in A, g \in G.$$

Definition 2.2.2. (cf. [21, Definition 2.18]) Let G be a discrete group and let A be a G-C^*-algebra. The linear space $C_c(G, A)$ of finitely supported, A-valued functions on G is an involutive algebra with respect to the convolution multiplication and involution defined by

$$f_1 \star f_2(g) = \sum_{\gamma \in G} f_1(\gamma)(\gamma \cdot (f_2(\gamma^{-1}g)))$$

and

$$f^*(g) = g \cdot (f(g^{-1})^*).$$

Observe that a covariant representation of A in a C^*-algebra B determines a $*$-homomorphism $\varphi \times \pi$ from $C_c(G, A)$ into B by the formula

$$\varphi \times \pi(f) = \sum_{g \in G} \varphi(f(g))\pi(g), \ for \ all \ f \in C_c(G, A).$$

Definition 2.2.3. (cf. [21, Definition 2.19]) The full crossed product C^*-algebra $C^*_{max}(G, A)$ is the completion of the $*$-algebra $C_c(G, A)$ in the smallest C^*-algebra norm which makes all the $*$-homomorphisms $\varphi \times \pi$ continuous.

Example 2.2.4. Let $A = \mathbb{C}$, we obtain the full group C^*-algebras $C^*_{max}(G)$.

Next we review the reduced group C^*-algebra.

Definition 2.2.5. (cf. [21, Definition 2.20]) Let G be a discrete group and let A be a G-C^*-algebra. Denote by $\ell^2(G, A)$ the Hilbert A-module comprised of functions $\xi : G \to A$ for which the series $\sum_g \xi(g)^*\xi(g)$ is norm-convergent in A. The regular representation of A is the covariant representation $\varphi \times \pi$ into the bounded, adjoinable operators on $\ell^2(G, A)$ given by the formulas

$$(\varphi(a)\xi)(g) = (g^{-1} \cdot a)\xi(g), \ \xi \in \ell^2(G, A)$$

and

$$(\pi(g)\xi)(\gamma) = \xi(g^{-1}\gamma), \ \xi \in \ell^2(G, A).$$

The regular representation determines a $*$-homomorphism from the full crossed product algebra $C^*_{max}(G, A)$ into the C^*-algebra of all bounded, adjoinable operators on $\ell^2(G, A)$.

Definition 2.2.6. (cf. [21, Definition 2.21]) Let G be a discrete group and let A be a G-C^*-algebra. The reduced crossed product algebra $C^*_{red}(G, A)$ is the image of $C^*_{max}(G, A)$ under the regular representation.

Example 2.2.7. Let $A = \mathbb{C}$, we obtain the reduced group C^*-algebras $C^*_{red}(G)$.

Remark 2.2.1. Notice that given a group homomorphism $h : G \to \Gamma$ and a (G, Γ)-C^*-algebra A, there does not exist a natural homomorphism (except for injective homomorphism) from $C^*_{red}(G, A)$ to $C^*_{red}(\Gamma, A)$ which is induced from h. Hence we have to consider the homomorphism from $C^*_{max}(G, A)$ to $C^*_{max}(\Gamma, A)$ in general.

2.3 K-Theory for C^*-Algebras

In this section, we record some basic facts about the K-theory of Banach algebras, details of which can be found in [6] (or [41] when restricted to C^*-algebras).

In order to define the K_0 group of a Banach algebra A, we consider idempotents not only in A, but in $M_\infty(A) := \bigcup_{n\in\mathbb{N}} M_n(A)$, where we regard $M_n(A)$ as embedded in $M_{n+1}(A)$ via $a \mapsto \mathrm{diag}(a, 0)$.

Definition 2.3.1. An idempotent in a Banach algebra A is an element e satisfying $e^2 = e$. Two idempotents e and f are orthogonal if $ef = fe = 0$.

Definition 2.3.2. Let e and f be idempotents in a Banach algebra A.

1. We say that e and f are similar, and write $e \sim_s f$, if there is an invertible element $z \in A^+$ such that $zez^{-1} = f$.

2. We say that e and f are homotopic, and write $e \sim_h f$, if there is a norm-continuous path of idempotents in A from e to f.

Proposition 2.3.3. [6, Proposition 4.3.2] Let e and f be idempotents in a Banach algebra A. If $||e-f|| < \frac{1}{||2e-1||}$, then $e \sim_s f$. In fact, there exists $z \in A^+$ with $||z-1|| < \frac{||2e-1||}{||e-f||}$ and $z^{-1}ez = f$. Also, $e \sim_h f$.

Proposition 2.3.4. [6, Proposition 4.3.3] If $e \sim_h f$ via the path e_t, then there is a path z_t of invertibles with $z_0 = 1$ and $z_t^{-1}ez_t = e_t$ for all t. Thus $e \sim_s f$.

In general, it is not true that $e \sim_s f$ implies $e \sim_h f$.

Proposition 2.3.5. [6, Proposition 4.4.1] If $e \sim_s f$, then

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}.$$

Since we will consider simultaneously all matrix algebras over A, the two equivalence relations become interchangeable (up to doubling matrix sizes).

Definition 2.3.6. Let A be a Banach algebra. Define $V(A)$ to be the set of all homotopy classes of idempotents in $M_\infty(A)$. On $V(A)$, define addition by $[e]+[f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right]$.

It is straightforward to check that this addition operation is well-defined and makes $V(A)$ into an abelian semigroup with identity $[0]$.

Example 2.3.7.

1. $V(\mathbb{C}) = V(M_n(\mathbb{C})) = V(K(H)) = \mathbb{N} \cup \{0\}$, where $K(H)$ denotes the algebra of compact operators on a separable Hilbert space H.

2. $V(B(H)) = \mathbb{N} \cup \{0, \infty\}$, where $B(H)$ denotes the algebra of bounded linear operators on a separable Hilbert space H.

If $\phi : A \to B$ is a homomorphism between Banach algebras, then ϕ extends to a homomorphism from $M_\infty(A)$ to $M_\infty(B)$, which induces a semigroup homomorphism

$\phi_* : V(A) \to V(B)$ given by $\phi_*([e]) = [\phi(e)]$.

Definition 2.3.8. Let A and B be Banach algebras. Two bounded homomorphisms $\phi, \psi : A \to B$ are said to be homotopic if there is a path of bounded homomorphisms $\omega_t : A \to B$ for $0 \leq t \leq 1$, continuous in t in the topology of pointwise norm-convergence, with $\omega_0 = \phi$ and $\omega_1 = \psi$.

Equivalently, ϕ and ψ are homotopic if there exists a bounded homomorphism $\omega : A \to C([0,1], B)$ with $\pi_0 \circ \omega = \phi$ and $\pi_1 \circ \omega = \psi$, where $\pi_t : C([0,1], B) \to B$ is evaluation at t.

From the definitions, one sees that if $\phi, \psi : A \to B$ are homotopic, then $\phi(e) \sim_h \psi(e)$ for any idempotent $e \in M_\infty(A)$, so $\phi_* = \psi_* : V(A) \to V(B)$. This property is known as homotopic invariance.

One can also verify that

- if $A = A_1 \oplus A_2$, then $V(A) \cong V(A_1) \oplus V(A_2)$;
- if $A = \varinjlim A_i$, then $V(A) \cong \varinjlim V(A_i)$.

Definition 2.3.9. For a unital Banach algebra A, define $K_0(A)$ to be the Grothendieck group of $V(A)$.

For a non-unital Banach algebra A, define $K_0(A)$ to be $\ker(\pi_* : K_0(A^+) \to K_0(\mathbb{C}))$, where $\pi : A^+ \to \mathbb{C}$ is the homomorphism given by $\pi(a, z) = z$.

Example 2.3.10.

1. $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = K_0(K(H)) = \mathbb{Z}$;
2. $K_0(B(H)) = 0$.

Let A be a Banach algebra. Let $GL_n(A) = \{x \in GL_n(A^+) : x \equiv I_n \mod M_n(A)\}$. We embed $GL_n(A)$ into $GL_{n+1}(A)$ via the map $u \mapsto \operatorname{diag}(u, 1)$, and let $GL_\infty(A) =$

$\varinjlim GL_n(A)$, which can be thought of as the group of invertible infinite matrices that have diagonal elements in $1_{A^+} + A$, off-diagonal elements in A, and only finitely many entries different from 0 or 1.

Definition 2.3.11. Let u and v be invertible elements in a unital Banach algebra A. We say that u and v are homotopic if there is a norm-continuous path of invertible elements in A from u to v.

Definition 2.3.12. Let A be a Banach algebra. Define $K_1(A)$ to be the set of homotopy classes of invertible elements in $GL_\infty(A)$.

$K_1(A)$ has an abelian group structure under the operation $[u] + [v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]$.

Example 2.3.13. $K_1(\mathbb{C}) = 0$ since every invertible matrix with entries in $\mathbb{C}$ can be connected to the identity matrix.

The properties that we stated for K_0 also hold for K_1, i.e.,

- If $\phi : A \to B$ is a homomorphism between Banach algebras, then it extends to a unital homomorphism $A^+ \to B^+$, thereby inducing a homomorphism $\phi_* : K_1(A) \to K_1(B)$.
- If $\phi, \psi : A \to B$ are homotopic, then $\phi_* = \psi_*$.
- $K_1(A_1 \oplus A_2) \cong K_1(A_1) \oplus K_1(A_2)$.
- $K_1(\varinjlim A_i) \cong \varinjlim K_1(A_i)$.

Definition 2.3.14. Let A be a Banach algebra. The suspension of A, denoted by SA, is $C_0(\mathbb{R}, A)$ equipped with pointwise operations and the sup norm.

Using suspensions, one can view K_1 groups as K_0 groups. More precisely, we have

Theorem 2.3.15. [6, Theorem 8.2.2] There is an isomorphism $\theta_A : K_1(A) \to K_0(SA)$ such that whenever $\phi : A \to B$ is a homomorphism, we have the following commutative diagram:

$$\begin{tikzcd} K_1(A) \arrow[r, "\phi_*"] \arrow[d, "\theta_A"] & K_1(B) \arrow[d, "\theta_B"] \\ K_0(SA) \arrow[r, "(S\phi)_*"] & K_0(SB) \end{tikzcd}$$

Definition 2.3.16. A sequence $N \xrightarrow{f} G \xrightarrow{g} \Gamma$ of groups and group homomorphisms is said to be exact if $\mathrm{im} f = \ker g$.

Theorem 2.3.17. If J is a closed two-sided ideal in A, then we have the following exact sequence:

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{q_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/J),$$

where $i : J \to A$ is the inclusion, $q : A \to A/J$ is the quotient homomorphism, and $\partial : K_1(A/J) \to K_0(J)$ is defined as follows: Let $u \in GL_n(A/J)$, and let $w \in GL_{2n}(A)$ be a lift of diag(u, u^{-1}). Then $\partial([u]) = [wp_nw^{-1}] - [p_n] \in K_0(J)$, where p_n is the matrix with n 1's along the diagonal and 0 everywhere else.

In fact, one can connect $K_0(A/J)$ to $K_1(J)$ to make the sequence a cyclic six-term exact sequence. This is a consequence of Bott periodicity, which we will now briefly describe.

If e is an idempotent in $M_n(A^+)$, write $f_e(z) = ze + (1 - e) \in C(S^1, GL_n(A^+))$. Such loops represent elements in $K_1(SA)$. Consider the homomorphism $\beta_A : K_0(A) \to K_1(SA)$ given by $\beta_A([e] - [p_n]) = [f_e f_{p_n}^{-1}]$, called the Bott map. If $\phi : A \to B$ is a homomorphism, then we have the following commutative diagram:

$$\begin{tikzcd} K_0(A) \arrow[r, "\phi_*"] \arrow[d, "\beta_A"] & K_0(B) \arrow[d, "\beta_B"] \\ K_1(SA) \arrow[r, "(S\phi)_*"] & K_1(SB) \end{tikzcd}$$

Theorem 2.3.18. [6, Theorem 9.2.1](Bott Periodicity) β_A is an isomorphism.

Define $\partial : K_0(A/J) \to K_1(J)$ to be the composition

$$K_0(A/J) \stackrel{\beta_A}{\to} K_1(S(A/J)) \to K_0(SJ) \stackrel{\theta_J^{-1}}{\to} K_1(J).$$

Theorem 2.3.19. [6, Theorem 9.3.1] If J is a closed two-sided ideal in A, then we have the following six-term exact sequence:

$$\begin{tikzcd} K_1(J) \arrow[r, "i_*"] & K_1(A) \arrow[r, "q_*"] & K_1(A/J) \arrow[d, "\partial"] \\ K_0(A/J) \arrow[u, "\partial"] & K_0(A) \arrow[l, "q_*"] & K_0(J) \arrow[l, "i_*"] \end{tikzcd}$$

This six-term exact sequence is one of the standard computational tools in K-theory. Another useful computational tool is the Mayer-Vietoris sequence.

Definition 2.3.20. A pushout diagram of C^*-algebras is a diagram of the form

$$\begin{tikzcd} I \cap J \arrow[r, "\iota^I"] \arrow[d, "\iota^J"] & I \arrow[d, "\kappa^I"] \\ J \arrow[r, "\kappa^J"] & A \end{tikzcd}$$

where I and J are ideals in A, the arrows are the obvious inclusions, and where the sum

$I + J$ is dense in A.

Theorem 2.3.21. (cf. [42, Proposition 2.7.15]) Let A, I, J be as in definition 2.3.20 above. Then we have the following six-term exact sequence:

$$\begin{tikzcd}
K_1(I \cap J) \arrow[r] & K_1(I) \oplus K_1(J) \arrow[r] & K_1(A) \arrow[d] \\
K_0(A) \arrow[u] & K_0(I) \oplus K_0(J) \arrow[l] & K_0(I \cap J) \arrow[l]
\end{tikzcd}$$

which is natural for commutative diagrams of pushout diagrams. The morphisms

$$K_*(I \cap J) \to K_*(I) \oplus K_*(J) \text{ and } K_*(I) \oplus K_*(J) \to K_*(A)$$

in the above are given by

$$x \mapsto \iota_*^I(x) \oplus \iota_*^J(x) \text{ and } y \oplus z \mapsto \kappa_*^I(y) - \kappa_*^J(z)$$

respectively.

3. RELATIVE BAUM-CONNES ASSEMBLY MAP

3.1 Relative Equivariant K-Homology

3.1.1 Review of analytic K-homology

In this section, we will review the development of analytic K-homology [23]. In literature, analytic K-homology draws together ideas from algebraic topology, functional analysis and geometry. It is a tool of conveying information among these three subjects and it has been used with spectacular success to prove and indeed discover remarkable theorems across a wide span of mathematics. These include results in operator theory which make no mention of topology or geometry at all, and results in topology and geometry which are apparently far removed from functional analysis. The subject of analytic K-homology had two separate beginnings, one in the index theory of Atiyah and Singer and one in operator theory.

In one direction, the index theory of Atiyah and Singer presents a view of the Fredholm index pairing between K-theory and K-homology. In 1969, M. Atiyah [1] began to realized the K-homology in terms of abstract elliptic operators. Suppose that X is a compact manifold and that D is an linear elliptic operator on X. Then D has a Fredholm index. But in addition if V is a vector bundle on X then a standard construction in index theory (essentially a tensor product) produces a new linear elliptic operator D_V "with coefficients in V", and the assignment $V \mapsto \text{Index}(D_V)$ determines a homomorphism $\text{Index}_D : K^0(X) \to \mathbb{Z}$ In order to extend this discussion to spaces other than manifolds, Atiyah identified the key functional analytic properties of an elliptic operator on a manifold and so developed an abstract notion of elliptic operator, now called a Fredholm module. However he could not give the appropriate relation on them. Kasparov [25] developed Atiyah's idea and showed that the abelian group generated by a homotopy classes of Fredholm modules is an analytic model for K-homology, this time for the degree-zero K-homology group of X.

In another direction, operator theory has long considered the problem of classifying Hilbert space operators "modulo compact operators". Weyl and von Neumann showed that two self-adjoint operators are unitarily equivalent modulo compact operators if and only if they have the same spectrum apart from isolated eigenvalues of finite multiplicity. In the 1960's, Brown, Douglas and Fillmore began an investigation of essentially normal operators, meaning those for which $T^*T = TT^*$ modulo compact operators, by asking themselves the following question: is the unilateral shift operator on the Hilbert space $\ell^2(\mathbb{N})$ unitarily equivalent modulo compact operators to the bilateral shift operator on $\ell^2(\mathbb{Z})$? The essential spectrum, meaning the part of the spectrum which is stable under compact perturbations, is for both operators the unit circle S^1 in the complex plane. According to the Weyl-von Neumann Theorem the essential spectrum is a complete classification invariant for self-adjoint operators. But in the present case a new invariant emerges, namely the Fredholm index. Indeed the index of the unilateral shift is -1, whereas the index of the bilateral shift is 0, while the stability properties of the index show that it is an invariant for unitary equivalence modulo compact operators. Using simple operator theory techniques it is not hard to show that two essentially normal operators with essential spectrum S^1 are unitarily equivalent modulo compact operators if and only if they have the same Fredholm index. But the situation for other essential spectra $X \subseteq \mathbb{C}$ (for example, the closed unit disk) is considerably more complicated. Brown, Douglas and Fillmore [7] introduced the classifying structure Ext(X) to help attack the problem, and then they proved two very unexpected things: first, Ext(X) is actually an abelian group, and second, Ext(X) is the degree-one K-homology group of X. This is the so-called Brown-Douglas-Fillmore Theorem. The determination of Ext(X), which is to say the classification of essentially normal operators, was thereby carried out by reducing the classification problem to a computational problem in algebraic topology.

In 1980's, Kasparov [26] unified the ideas of Fredholm modules by Atiyah and the extension theory of C^*-algebras by Brown-Douglas-Fillmore to create an extremely powerful and flexible tool (called Kasparov K-homology) in index theory. Roughly speaking, Kasparov K-homology is defined in terms of Fredholm modules for degree-zero and degree-one simutaneously. Kasparov K-homology in degree-one is equivalent to the homology of Brown, Douglas and Fillmore.

Kasparov's K-homology has proved to be an extremely powerful and flexible tool in application. For example the proof of the Atiyah-Singer Index Theorem itself can be presented very simply and conceptually using the product structure on K-homology. Moreover Kasparov's work has allowed a considerable strengthening of the index theory of Atiyah and Singer. Kasparov developed his theory as a tool in differential topology, and indeed some of the most powerful theorems in the topological theory of manifolds (pertaining particularly to the Novikov conjecture) rely very heavily on Kasparov's machinery. In several cases no proofs of these theorems are known which do not employ functional analysis to a very considerable extent.

In 1994's, Yu [43] constructed a new analytic model of K-homology by the language of localization algebras. It turns out that it is very useful for proving coarse Baum-Connes conjecture and the strong Novikov conjecture in a more general sense. Since we are going to use this K-homology, we give more terminology of it.

Definition 3.1.1. [42, Definiton 4.1.1] Let X be locally compact, second countable, metric space. An (geometric) module over X is a separable Hilbert space H_X equipped with a non-degenerate $*$-representation $\rho : C_0(X) \to B(H_X)$.

A geometric module H_X is ample if no non-zero element of $C_0(X)$ acts as a compact operator and H_X is infinite dimensional.

We will often say something like 'let H_X be a geometric module', leaving X implicit in the notation.

Definition 3.1.2. [42, Definiton 4.1.6, Definiton 4.1.6] Let H_X be a geometric module and let ϕ be a $*$-homomorphism from $C_0(X)$ to $B(H_X)$, the C^*-algebra of all bounded operators on H_X. Let T be an operator in $B(H_X)$.

(1) The support of T is defined to be the complement (in $X \times X$) of the set of all points $(x, y) \in X \times X$ for which there exists $f \in C_0(X)$ and $g \in C_0(X)$ satisfying $\phi(f)T\phi(g) = 0$ and $f(x) \neq 0$ and $g(y) \neq 0$;

(2) The propagation of T is defined to be

$$prop(T) := \sup\{d(x, y) : (x, y) \in Supp(T)\};$$

(3) T is said to be locally compact if $\phi(f)T$ and $T\phi(f)$ are in $K(H_X)$ for all $f \in C_0(X)$, where $K(H_X)$ is defined to be the operator norm closure of all finite rank operators on the Hilbert space H_X.

Definition 3.1.3. [42, Definition 5.1.4] Let H_X be a geometric module. The Roe $*$-algebra of H_X, denoted $\mathbb{C}[H_X]$, is the $*$-algebra of all finite propagation, locally compact operators on H_X.

The Roe C^*-algebra, or just Roe algebra, of H_X, denoted $C^*(H_X)$, is the norm closure of $\mathbb{C}[H_X]$ in $B(H_X)$.

Definition 3.1.4. [42, Definition 6.2.4] Let H_X be a geometric module. The algebraic localization algebra $\mathbb{C}_L[H_X]$ is defined to be the algebra of all bounded and uniformly continuous functions $f : [0, \infty) \to \mathbb{C}[H_X]$ such that the propagation of $f(t)$ goes to 0 as $t \to \infty$.

The localization algebra $C_L^*(H_X)$ is the norm closure of $\mathbb{C}_L[H_X]$ with respect to the norm

$$||f|| := \sup_{t\in[0,\infty)} ||f(t)||.$$

Definition 3.1.5. [42, Definition 6.3.1] The K-homology groups of X are defined by the formula

$$K_n(X) := K_{-n}(C_L^*(H_X)), \quad K_*(X) := K_0(X) \oplus K_1(X).$$

Notice that the K-homology groups do not depend on the choices of geometric module over X. Hence we always write $C_L^*(X)$ instead of $C_L^*(H_X)$.

Example 3.1.6. If X is a single point space, then

$$K_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \; mod \; 2 \\ 0, & n = 1 \; mod \; 2 \end{cases}$$

Theorem 3.1.7. [35, Theorem 3.4] Let H_X be a direct sum of infinitely many copies of some ample geometric module over X. Then

$$KK_*(C_0(X), \mathbb{C}) \to K_*(C_L^*(X))$$

is an isomorphism, where $KK_*(C_0(X), \mathbb{C})$ is the Kasparov K-homology of X.

Since we are going to consider relative Baum-Connes assembly map for groups, we want to give more details about the equivariant K-homology in the end of this section.

Let Γ be a countable discrete group, and X be a locally compact, second countable metric space on which Γ acts properly by isometries. We denote the induced action of Γ on $C_0(X)$ by α, where

$$\alpha_\gamma(f)(x) := f(\gamma^{-1}x)$$

for all $\gamma \in \Gamma$, $f \in C_0(X)$, and $x \in X$.

Definition 3.1.8. [42, Definition 4.5.1] A (geometric) Γ-module over X, is an module H_X equipped with a unitary $U : \Gamma \to \mathcal{U}(H_X)$ that spatially implements the action of Γ of $C_0(X)$.

Definition 3.1.9. [42, Definition 4.5.2] An Γ-module H_X is locally free if for any finite subgroup F of Γ and any F-invariant Borel subset E of X, there is a Hilbert space H_E (possibly zero) equipped with the trivial representation of F such that $\chi_E H_X$ and $\ell^2(F) \otimes H_E$ are isomorphic as F representations.

The Γ-module H_X is ample if it is locally free, and ample as a module over X in the sense of definition 3.1.1.

Proposition 3.1.10. [42, Lemma 4.5.5] Ample Γ-modules over X always exist.

Definition 3.1.11. [42, Definition 5.2.1] Let H_X be a geometric Γ-module, and let $\mathbb{C}[H_X]$ be the associated Roe $*$-algebra. The equivariant Roe $*$-algebra of H_X is defined to be the algebra of fixed points $\mathbb{C}[H_X]^\Gamma$, under the conjugation Γ action on $\mathbb{C}[H_X]$ defined by

$$T \mapsto U_\gamma T U_\gamma^*.$$

The *reduced equivariant Roe C^*-algebra* of H_X, denoted $C^*_{red}(H_X)^\Gamma$, is the closure of $\mathbb{C}[H_X]^\Gamma$ in the operator norm in $B(H_X)$.

The *maximal equivariant Roe C^*-algebra* of H_X, denoted $C^*_{max}(H_X)^\Gamma$, is the closure of $\mathbb{C}[H_X]^\Gamma$ under the under the maximal norm:

$$||a||_{max} = \sup_\phi \{||\phi(a)|| \mid \phi : \mathbb{C}[H_X]^\Gamma \to B(H) \text{ is a } *\text{-representation}\}.$$

Definition 3.1.12. [42, Definition 6.5.1] Let H_X be a geometric Γ-module. The equivariant localization $*$-algebra $\mathbb{C}_L[H_X]^\Gamma$ is defined to be the algebra of all bounded and uniformly continuous functions $f : [0, \infty) \to \mathbb{C}[H_X]^\Gamma$ such that the propagation of $f(t)$ goes to 0 as

$t \to \infty$.

The *reduced equivariant localization C^*-algebra* $C^*_{L,red}(H_X)^\Gamma$ is the norm closure of $\mathbb{C}_L[H_X]^\Gamma$ with respect to the norm

$$||f|| := \sup_{t\in[0,\infty)} ||f(t)||.$$

The *maximal equivariant localization C^*-algebra* $C^*_{L,max}(H_X)^\Gamma$ is the norm closure of $\mathbb{C}_L[H_X]^\Gamma$ with respect to the maximal norm:

$$||f||_{max} = \sup_{\phi}\{||\phi(f)|| \mid \phi : \mathbb{C}_L[H_X]^\Gamma \to B(H) \text{ is a } *\text{-representation}\}.$$

Remark 3.1.1. Note that we assume Γ acts on X properly by isometries, it it not hard to show that $C^*_{L,max}(H_X)^\Gamma \to C^*_{L,red}(H_X)^\Gamma$ is an isomorphism. Hence we use notation $C^*_L(H_X)^\Gamma$ for both of them.

Definition 3.1.13. [42, Definition 6.5.8] The *equivariant K-homology* groups of X are defined by the formula

$$K_n^\Gamma(X) := K_{-n}(C^*_L(H_X)^\Gamma), \quad K_*^\Gamma(X) := K_0^\Gamma(X) \oplus K_1^\Gamma(X).$$

Note that the equivariant K-homology groups do not depend on the choices of geometric Γ-module over X. Hence we always write $C^*_L(X)^\Gamma$ instead of $C^*_L(H_X)^\Gamma$.

We can also define the versions of equivariant localization C^*-algebras and equivariant Roe C^*-algebras with cofficients in a Γ-C^*-algebra A.

Let A be a Γ-C^*-algebra. Let H be a (countably generated) Γ-Hilbert module over A. Let $\rho : C_0(X) \to B(H)$ be a $*$-homomorphism which is covariant in the sense that

$$\rho(\gamma f)h = (\gamma\rho(f)\gamma^{-1})h,$$

for all $\gamma \in \Gamma$, $f \in C_0(X)$ and $h \in H$. Such a triple $(C_0(X), \Gamma, \rho)$ is called a covariant system.

Definition 3.1.14. [29, Definition 3.2] The covariant system $(C_0(X), \Gamma, \rho)$ is called *admissible* if

(1) the Γ-action on X is proper and cocompact;

(2) there exist a Γ-Hilbert space H_X and a separable and infinite dimensional Γ-Hilbert space E such that

(a) H is isomorphic to $H_X \otimes E \otimes A$ as Γ-Hilbert modules over A;

(b) $\rho = \rho_0 \otimes I$ for some Γ-equivariant $*$-homomorphism ρ_0 from $C_0(X)$ to $B(H_X)$ such that $\rho_0(f)$ is not in $K(H_X)$ for any nonzero function $f \in C_0(X)$ and ρ_0 is nondegenerate;

(c) for each $x \in X$, E is isomorphic to $\ell^2(\Gamma_x) \otimes H_x$ as Γ_x-Hilbert spaces for some Hilbert space H_x with a trivial Γ_x action, where Γ_x is the subgroup of Γ stablising x.

Let $(C_0(X), \Gamma, \rho)$ be an admissible covariant system. We can define equivariant localization C^*-algebras and equivariant Roe C^*-algebras with cofficients in a Γ-C^*-algebra A as Definition 3.1.11 and Definition 3.1.12. We denote them by $C^*_L(\Gamma, X, A)$, $C^*_{red}(\Gamma, X, A)$ and $C^*_{max}(\Gamma, X, A)$.

Definition 3.1.15. [29, Theorem 3.6] The *equivariant K-homology* groups of X with coefficients in A are defined by the formula

$$K_n^{\Gamma}(X, A) := K_{-n}(C^*_L(\Gamma, X, A)), \quad K_*^{\Gamma}(X, A) := K_0^{\Gamma}(X, A) \oplus K_1^{\Gamma}(X, A).$$

Theorem 3.1.16. [42, Theorem 6.5.15] If Γ acts freely on X, then

$$K_*^{\Gamma}(X) \cong K_*(X/\Gamma).$$

3.1.2 Formulation of relative equivariant K-homology

We fix a pair of countable discrete groups G, Γ, and a group homomorphism between them, $h : G \to \Gamma$. In this section, we introduce the relative equivariant K-homology (with coefficients in A) of the pair of classifying spaces $\mathcal{E}G$ and $\mathcal{E}\Gamma$ with proper G and Γ-actions respectively.

Proposition 3.1.17. Let $h : G \to \Gamma$ be a group homomorphism. There exist left-invariant metrics d_G and d_Γ on G and Γ such that $d_\Gamma(h(g_1), h(g_2)) \leq 2d_G(g_1, g_2)$ for any g_1, g_2 in G.

Proof. Denote Γ_0 to be the image of h. Then it is a subgroup of Γ.

Firstly, we show that there exists a list of elements in G and Γ_0, $G = \{e, g_1^{\pm 1}, g_2^{\pm 1}, ...\}$, $\Gamma_0 = \{e, \gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \cdots\}$ such that $h(g_k^{\pm 1}) = \gamma_n^{\pm 1}$ for any $n \leq k$ and any $g_k \in G$.

Since Γ_0 is countable, we can write Γ_0 as $\{e, \gamma_1^{\pm 1}, \gamma_2^{\pm 1}, ...\}$. Then $G = \bigsqcup_{k=0}^{\infty} g_k^{\pm 1} N$, where $g_k^{\pm 1}$ are chosen such that $h(g_k^{\pm 1}) = \gamma_k^{\pm 1}$ for $k > 0$, $g_0 = e$ and N is $\ker h$.

For $k > 0$, $\{g_k N\}^{-1} = g_k^{-1} N$ and $\{g_k^{-1} N\}^{-1} = g_k N$, so we can write $g_k N \cup g_k^{-1} N$ as $\{g_{k,1}^{\pm 1}, g_{k,2}^{\pm 1}, ...\}$. For $k = 0$, we write $g_0 N$ as $\{e, g_{0,1}^{\pm 1}, g_{0,2}^{\pm 1}, ...\}$. In summary, G can be written as $\{e, g_{0,1}^{\pm 1}, g_{0,2}^{\pm 1}, g_{1,1}^{\pm 1}, g_{0,3}^{\pm 1}, g_{1,2}^{\pm 1}, g_{2,1}^{\pm 1} ...\}$. Such lists of G and Γ_0 satisfy the properties we need.

Secondly, we define length functions on G and Γ_0 as follows,

$$l_G(g) = \min\{\sum_{i=1}^{k} a_i n_i | g = g_{n_1}^{\pm a_1} \cdots g_{n_k}^{\pm a_k}, \text{ where } a_i \in \mathbb{N}\}, \text{ for } g \in G,$$

and

$$l_{\Gamma_0}(\gamma) = \min\{\sum_{i=1}^{k} b_i m_i | \gamma = \gamma_{m_1}^{\pm b_1} \cdots \gamma_{m_k}^{\pm b_k} \text{ where } b_i \in \mathbb{N}\}, \text{ for } \gamma \in \Gamma_0.$$

For $g = g_{n_1}^{\pm a_1} \cdots g_{n_k}^{\pm a_k}$, we have

$$h(g) = h(g_{n_1})^{\pm a_1} \cdots h(g_{n_k})^{\pm a_k} = \gamma_{m_1}^{\pm a_1} \cdots \gamma_{m_k}^{\pm a_k},$$

where $m_i \leq n_i$ for any i. Hence $l_G(g) \geq l_{\Gamma_0}(h(g))$. Hence $d_{\Gamma_0}(h(g_1), h(g_2)) \leq d_G(g_1, g_2)$ for all $g_1, g_2 \in G$.

Finally, we write $\Gamma - \Gamma_0$ as $\{\widetilde{\gamma}_1^{\pm 1}, \widetilde{\gamma}_2^{\pm 1}, \cdots\}$ and write Γ as $\{e, \alpha_1^{\pm 1}, \alpha_2^{\pm 1}, \cdots\}$, where $\alpha_{2k-1} = \gamma_k$ and $\alpha_{2k} = \widetilde{\gamma}_k$.

Define the lenge function on Γ as follows,

$$l_\Gamma(\gamma) = \min\{\sum_{i=1}^{k} b_i m_i | \gamma = \alpha_{m_1}^{\pm b_1} \cdots \alpha_{m_k}^{\pm b_k} \text{ where } b_i \in \mathbb{N}\}, \text{ for } \gamma \in \Gamma.$$

For any $\gamma = \gamma_{n_1}^{\pm a_1} \cdots \gamma_{n_k}^{\pm a_k} \in \Gamma_0$, we have

$$\gamma = \gamma_{n_1}^{\pm a_1} \cdots \gamma_{n_k}^{\pm a_k} = \alpha_{2n_1 - 1}^{\pm a_1} \cdots \alpha_{2n_k - 1}^{\pm a_k}.$$

Since

$$(2n_1 - 1)a_1 + \cdots + (2n_k - 1)a_k \leq 2(a_1 n_1 + \cdots + a_k n_k),$$

we have $l_\Gamma(\gamma) \leq 2 l_{\Gamma_0}(\gamma)$.

Hence $d_\Gamma(\gamma_1, \gamma_2) \leq 2 d_{\Gamma_0}(\gamma_1, \gamma_2)$ for any $\gamma_1, \gamma_2 \in \Gamma_0$ which implies that $d_\Gamma(h(g_1), h(g_2)) \leq 2 d_G(g_1, g_2)$ for any g_1, g_2 in G. □

Remark 3.1.18. If G and Γ are finitely generated group, then the metrics d_G and d_Γ on G and Γ can be chosen such that $d_\Gamma(h(g_1), h(g_2)) \leq d_G(g_1, g_2)$ for any g_1, g_2 in G. However the constant is not important. Note that different proper left-invariant metrics on a finitely generated group are quasi-isometry, hence from above proposition, we know for any proper left-invariant metrics d_G and d_Γ on G and Γ, there exist constants $L \geqslant 1$ and

$C \geqslant 0$, such that $d_\Gamma(h(g_1), h(g_2)) \leq L \cdot d_G(g_1, g_2) + C$ for any g_1, g_2 in G. It turns out that the relative equivariant K-homology does not depend on the metrics we choose. Hence we consider relative equivariant K-homology in term of the metrics in above proposition without specifying finitely generated groups.

Let us recall the definition of Rips complexes.

Definition 3.1.19. Let Γ be a countable discrete group with a proper length function $l : \Gamma \to \mathbb{R}^+$. Let $s \geqslant 0$. The ***Rips complex*** of Γ at scale s, denoted $P_s(\Gamma)$, is the simplicial complex with vertex set Γ, and where a subset $\{\gamma_0, \cdots, \gamma_n\}$ of Γ spans a simplex if and only if $d(\gamma_i, \gamma_j) \leqslant s$ for all i, j.

Write any point x of $P_s(\Gamma)$ as formal linear combinations $x = \sum_{\gamma\in\Gamma} t_\gamma \gamma$, where each coefficient t_γ is in $[0, 1]$, $\sum_{\gamma\in\Gamma} t_\gamma = 1$, and only finitely many coefficients are non-zero. In this way, $P_s(\Gamma)$ identifies with a subset of $\ell^1(\Gamma)$, and we equip it with the topology defined by the induced metric. Concretely, if $x = \sum t_\gamma \gamma$ and $y = \sum t'_\gamma \gamma$, then

$$d(x, y) = \sum_{\gamma\in\Gamma} |t_\gamma - t'_\gamma|.$$

Proposition 3.1.20. [42, Lemma 7.3.2] Let Γ be a countable, discrete group equipped with a proper length function $l : \Gamma \to \mathbb{R}^+$.

(i) The Rips complex $P_s(\Gamma)$ is a locally compact, and second countable metric space.

(ii) For each $s \geqslant r$, the canonical inclusion $i_{sr} : P_r(\Gamma) \to P_s(\Gamma)$ is an isometry onto its image, and an equivariant coarse equivalence.

(iii) The action of Γ on $P_s(\Gamma)$ defined by

$$g \cdot (\sum t_\gamma \gamma) := \sum t_\gamma (g\gamma)$$

is isometric.

Corollary 3.1.21. Under the assumption in proposition 3.1.17, h induces continuous maps as follows,

$$\begin{aligned} h: P_s(G) &\rightarrow P_{2s}(\Gamma) \\ \sum t_g g &\mapsto \sum t_g h(g) \end{aligned}, \qquad \begin{aligned} h_q: P_s(G)/G &\rightarrow P_{2s}(\Gamma)/\Gamma \\ \left[\sum t_g g\right] &\mapsto \left[\sum t_g h(g)\right] \end{aligned},$$

where the metrics on $P_s(G)/G$ and $P_{2s}(\Gamma)/\Gamma$ are induced from $P_s(G)$ and $P_{2s}(\Gamma)$, and the second map is proper.

Proof. It is not hard to show that they are continuous. It suffices to prove the properness of the second map.

Given any $\{g_1, \cdots, g_k\} \subseteq G$ such that $\{h(e), h(g_1), \cdots, h(g_k)\}$ spans a simplex in $P_{2s}(\Gamma)$. Let us denote $[h(e), h(g_1), \cdots, h(g_k)]$ for the simplex spaned by $\{h(e), h(g_1), \cdots,$ $h(g_k)\}$ in $P_{2s}(\Gamma)$ and $\big[[h(e), h(g_1), \cdots, h(g_k)]\big]$ for the image of $[h(e), h(g_1), \cdots, h(g_k)]$ under the quotient map. We will show $h_q^{-1}(\big[[h(e), h(g_1), \cdots, h(g_k)]\big])$ is compact in $P_s(G)/G$.

One can check that

$$h_q^{-1}(\big[[h(e), h(g_1), \cdots, h(g_k)]\big]) = \pi(h^{-1}([h(e), h(g_1), \cdots, h(g_k)])),$$

where π is the quotient map.

Since

$$h^{-1}([h(e), h(g_1), \cdots, h(g_k)]) = \bigcup_{g \in G} g \cdot \left(\bigcup_{n_j \in \ker(h), l(n_j) \leqslant C, \forall j} [e, n_1 g_1, \cdots, n_k g_k] \right),$$

where C is some constant, we have

$$\begin{aligned}\pi(h^{-1}([h(e),\cdots,h(g_k)])) &= \pi(\bigcup_{g\in G} g\cdot\left(\bigcup_{n_j\in\ker(h),l(n_j)\leqslant C,\forall j}[e,n_1g_1,\cdots,n_kg_k]\right)) \\ &= \pi(\bigcup_{n_j\in\ker(h),l(n_j)\leqslant C,\forall j}[e,n_1g_1,\cdots,n_kg_k]) \\ &= \bigcup_{n_j\in\ker(h),l(n_j)\leqslant C,\forall j}\big[[e,n_1g_1,\cdots,n_kg_k]\big].\end{aligned}$$

The last union of the above equality is a finite union of compact sets, and hence it is compact which means $h_q^{-1}(\big[[h(e),h(g_1),\cdots,h(g_k)]\big])$ is compact in $P_s(G)/G$.

For a general compact set E in $P_{2s}(\Gamma)/\Gamma$, since E intersects with a finite union of sets as $\big[[h(e),h(g_1),\cdots,h(g_k)]\big]$, we have $h_q^{-1}(E)$ sits inside of a finite union of compact sets in $P_s(G)/G$. Moreover $P_{2s}(\Gamma)/\Gamma$ is Hausdorff space and h_q is continuous, so we have $h_q^{-1}(E)$ is compact. $\square$

Lemma 3.1.22. For $P_s(G)$, there exists a countable G-invariant dense subset $Z\subseteq P_s(G)$ such that the group action of G on Z is free. Moreover, Z/G is a countable dense subset of $P_s(G)/G$.

Proof. We denote the i-th skeleton by X_i for $i\geqslant 0$. Let $\widetilde{X}_k=X_k-X_{k-1}$ for $k\geqslant 1$, and let $\widetilde{X}_0=X_0=G$. Define

$$W_k^{ij}:=\{\alpha_0g_0+\cdots+\alpha_ig_i+\cdots+\alpha_jg_j+\cdots+\alpha_kg_k\mid \alpha_i=\alpha_j,\sum_{l=0}^{k}\alpha_l=1,\alpha_l\geq 0\}$$

and

$$W_k:=\bigcup_{\substack{[g_0,g_1,\cdots,g_k]\\ \text{simplex of }\widetilde{X}_k}}\ \bigcup_{\substack{i,j=0,1,\cdots,k,\\ i\neq j}} W_k^{ij}.$$

Let $Y_k=\widetilde{X}_k-W_k$.

CLAIM: (1) W_k is a nowhere dense subset of $\widetilde{X}_k$. (2) For any $x\in Y_k$, $G_x=\{e\}$. (3) W_k and Y_k are G-invariant subsets of $P_s(G)$.

(1). Since W_k^{ij} is a $(k-1)$-dimensional super subspace of $\widetilde{X}_k$ and W_k is a locally finite union of W_k^{ij}, W_k is a nowhere dense subset of $\widetilde{X}_k$.

(2). Fix any $x \in \widetilde{X}_k$, assume $gx = x$, $g \neq e$, $x = \alpha_0 g_0 + \cdots + \alpha_m g_m$ where $\alpha_i > 0$ for any i. Then

$$\alpha_0 g_0 + \cdots + \alpha_m g_m = x = gx = \alpha_0 g g_0 + \cdots + \alpha_m g g_m$$

which implies that $\{g_0, \cdots, g_m\} = \{gg_0, \cdots, gg_m\}$. Since $g \neq e$, there must exist $k_0, k_1, \cdots, k_j$, $1 \leq j \leq m$ such that $gg_{k_0} = g_{k_1}, gg_{k_1} = g_{k_2}, \cdots, gg_{k_j} = g_{k_0}$. So

$$\alpha_{k_0} = \alpha_{k_1} = \cdots = \alpha_{k_j}.$$

Hence $x \in W_k$. So for any $x \in Y_k$, $G_x = \{e\}$.

(3). It is not hard to prove.

Now take any countable dense subset Z_k of Y_k, then $Z = \bigcup_{k \geq 0} G \cdot Z_k$ has the property.

□

Lemma 3.1.23. Given any countable discrete group G, a G-C^*-algebra A, and its Rips complex $P_s(G)$. Let $Z \subseteq P_s(G)$ be given as lemma 3.1.22 and H be a countably infinite complex Hilbert space with trivial G-action. Then

$$\rho : C_0(P_s(G)) \to B(\ell^2(Z) \otimes H \otimes A)$$
$$f \mapsto M_f \otimes I \otimes I$$

is a covariant $*$-homomorphism, where M_f is the multiplication operator and $\ell^2(Z) \otimes H \otimes A$ is equipped with the diagonal action of G. Moreover $(C_0(P_s(G)), G, \rho)$ is an admissible covariant system.

Proof. It is not hard to prove the $*$-homomorphism ρ is covariant; the G-action on $P_s(G)$ is proper and cocompact.

Since the group action of G on Z is free, there exists a domain $\triangle \subseteq Z$ such that $G \cdot \triangle = Z$ and $g \cdot \triangle \cap g' \cdot \triangle = \emptyset$ when $g \neq g'$.

Let $E = \ell^2(G) \otimes H$ with the left regular multiplication on $\ell^2(G)$ and trivial G-action on H. By Fell's trick,

$$\ell^2(Z)\otimes H\otimes A = \ell^2(G)\otimes\ell^2(\triangle)\otimes H\otimes A \cong \ell^2(G)\otimes\ell^2(\triangle)\otimes\ell^2(G)\otimes H\otimes A = \ell^2(Z)\otimes E\otimes A$$

as G-Hilbert modules over A.

$\rho = \rho_0 \otimes I$ for some G-equivariant $*$-homomorphism ρ_0 from $C_0(P_s(G))$ to $B(\ell^2(Z))$, where $\rho_0(f) = M_f$, such that $\rho_0(f)$ is not in $K(\ell^2(Z))$ for any nonzero function $f \in C_0(P_s(G))$ and ρ_0 is nondegenerate.

By Fell's trick, for each $x \in X$, E is isomorphic to $\ell^2(G_x) \otimes H_x$ as G_x-Hilbert spaces for some Hilbert space H_x with a trivial G_x action. $\square$

Next we will formulate a natural map from $C^*_{max}(G, P_s(G), A)$ to $C^*_{max}(\Gamma, P_{2s}(\Gamma), A)$ which preserves propagations. Hence it will induce a natural map from $C^*_L(G, P_s(G), A)$ to $C^*_L(\Gamma, P_{2s}(\Gamma), A)$. First, we need a definition called (G, Γ)-C^*-algebra.

Definition 3.1.24. A C^*-algebra A is called (G, Γ)-C^*-algebra if A is a G-C^*-algebra and Γ-C^*-algebra simultaneously, and $g \cdot a = h(g) \cdot a$ for any $g \in G$, $a \in A$.

Remark 3.1.2. Note that if the kernel of h has infinite many elements, one can not find a (G, Γ)-C^*-algebra A, such that G and Γ act on A properly. This is why we can not imitate the classical method to the case of strong relative Novikov conjecture in general.

Lemma 3.1.25. Given a surjective homomorphism $h : G \to \Gamma_0$. Then $G/Ker(h)$ ($\cong \Gamma_0$) acts on $P_s(G)/Ker(h)$ properly and $q : P_s(G)/Ker(h) \to P_s(\Gamma_0)$ is an equivariant proper continuous map.

Proof. The first part is trivial. We only need to prove the properness of second statement. Fix any simplex $[\gamma_1, \gamma_2, \cdots, \gamma_m]$ in $P_s(\Gamma_0)$. Since

$$h^{-1}([\gamma_1, \gamma_2, \cdots, \gamma_m]) = \bigcup_{diag\{g_1,\cdots,g_k\}\le n, \{h(g_j)\}_{j=1}^k \subseteq \{\gamma_j\}_{j=1}^m, k\in\mathbb{N}} [g_1, g_2, \cdots, g_k],$$

we have

$$q^{-1}([\gamma_1, \gamma_2, \cdots, \gamma_m]) = \bigcup_{diag\{g_1,\cdots,g_k\}\le n, \{h(g_j)\}_{j=1}^k \subseteq \{\gamma_j\}_{j=1}^m, k\in\mathbb{N}} \pi([g_1, g_2, \cdots, g_k]),$$

where $\pi : P_s(G) \to P_s(G)/Ker(h)$ is the quotient map.

We will show that it is acturally a finite union in the last formula.

Fix any $\widetilde{g}_i \in h^{-1}(\gamma_i)$ for $i = 1, \cdots, m$. For any $\{g_i\}_{i=1,\cdots,m}$ which satisfied $diag\{g_1, \cdots, g_k\} \le s$ and $\{h(g_j)\}_{j=1}^k \subseteq \{\gamma_j\}_{j=1}^m$, there must exist $g_0 \in ker(h)$ such that $g_0 g_i = \widetilde{g}_j$ for some $1 \le i \le k, 1 \le j \le m$. So $g_0[g_1, g_2, \cdots, g_k] = [g_0 g_1, \cdots, \widetilde{g}_j, \cdots, g_0 g_k]$. Hence

$$\pi([g_1, g_2, \cdots, g_k]) = \pi(g_0[g_1, g_2, \cdots, g_k]) = \pi([g_0 g_1, \cdots, \widetilde{g}_j, \cdots, g_0 g_k]).$$

It follows that

$$\bigcup_{\substack{diag\{g_1,\cdots,g_k\}\le n,\\ \{h(g_j)\}_{j=1}^k \subseteq \{\gamma_j\}_{j=1}^m, k\in\mathbb{N}}} \pi([g_1, g_2, \cdots, g_k]) = \bigcup_{j=1}^m \bigcup_{\substack{diag\{\widetilde{g}_j, g_1,\cdots,g_k\}\le n,\\ \{h(g_i)\}_{i=1}^k \subseteq \{\gamma_i\}_{i=1}^m, k\in\mathbb{N}}} \pi([\widetilde{g}_j, g_1, \cdots, g_k]).$$

Since d_G is a proper metric, we have that it is a finite union in the last formula. Hence $q^{-1}([\gamma_1, \gamma_2, \cdots, \gamma_m])$ is a finite union of compact sets which will be compact. $\square$

Proposition 3.1.26. Given a surjective homomorphism $h : G \to \Gamma_0$ and a (G, Γ_0)-C^*-algebra A. For any $s > 0$, there exists a $*$-homomorphism (also denoted by h)

$$h : C^*_{max}(G, P_s(G), A) \to C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A)$$

such that if $k \in C^*_{max}(G, P_s(G), A)$ has finite propagation and is represented as a kernel on Z with values in $K(H) \otimes A$, then $h(k)$ has finite propagation and $Prop(h(k)) \leq Prop(k)$. Hence h induces a $*$-homomorphism (also denoted by h)

$$h : C^*_L(G, P_s(G), A) \to C^*_L(\Gamma_0, P_s(G)/Ker(h), A).$$

Proof. We fix admissible covariant systems $(C_0(P_s(G)), G, \rho)$ and $(C_0(P_s(G)/Ker(h)),$ $\Gamma_0,\sigma)$, where

$$\rho : C_0(P_s(G)) \to B(\ell^2(Z) \otimes H \otimes A)$$
$$f \mapsto M_f \otimes I \otimes I$$

and

$$\sigma : C_0(P_s(G)/Ker(h)) \to B(\ell^2(Z/Ker(h)) \otimes H \otimes A)$$
$$f \mapsto M_f \otimes I \otimes I,$$

Z is taken as lemma 3.1.22.

Given any operator $k \in C^*_{max}(G, P_s(G), A)$ with finite propagation and is represented as a kernel on Z with values in $K(H) \otimes A$, we define $h(k) \in B(\ell^2(Z/Ker(h)) \otimes H \otimes A)$

as follows,

$$h(k)([x],[y]) = \sum_{r \in Ker(h)} k(rx, y)$$

for $[x],[y] \in Z/Ker(h)$.

Firstly, it is well-defined. Fix any $r_1, r_2 \in Ker(h)$, then

$$\begin{aligned}
\sum_{r \in Ker(h)} k(rr_1x, r_2y) &= \sum_{r \in Ker(h)} k(r_2r_2^{-1}rr_1x, r_2y) \\
&= \sum_{r \in Ker(h)} k(r_2^{-1}rr_1x, y) \\
&= \sum_{r \in Ker(h)} k(rx, y).
\end{aligned}$$

Secondly, $h(k)$ is Γ_0 or $G/Ker(h)$-invariant. Fix any $[g] \in G/Ker(h)$, then

$$\begin{aligned}
h(k)([g]\cdot[x],[g]\cdot[y]) &= \sum_{r \in Ker(h)} k(rgx, gy) \\
&= \sum_{r \in Ker(h)} k(gg^{-1}rgx, gy) \\
&= \sum_{r \in Ker(h)} k(g^{-1}rgx, y) \\
&= \sum_{r \in Ker(h)} k(rx, y) \\
&= h(k)([x],[y]).
\end{aligned}$$

Thirdly, h is a $*$-homomorphism. For any $k, k_1, k_2 \in C^*_{max}(G, P_s(G), A)$ with finite propagation and are represented as a kernel on Z with values in $K(H) \otimes A$,

$$\begin{aligned}
(h(k_1)h(k_2))([x],[y]) &= \sum_{[z]\in Z/Ker(h)} h(k_1)([x],[z])h(k_2)([z],[y]) \\
&= \sum_{\substack{z_i\in Z, i\geq 1, [z_i]\neq[z_j],\\ Z/Ker(h)=\{[z_i], i\geq 1\}}} h(k_1)([x],[z_i])h(k_2)([z_i],[y]) \\
&= \sum_{z_i\in Z, i\geq 1} (\sum_{r_1\in Ker(h)} k_1(r_1x, z_i))(\sum_{r_2\in Ker(h)} k_2(r_2z_i, y)) \\
&= \sum_{r_1\in Ker(h)} \sum_{z_i\in Z, i\geq 1} \sum_{r_2\in Ker(h)} k_1(r_1x, z_i)k_2(r_2z_i, y) \\
&= \sum_{r_1\in Ker(h)} \sum_{z_i\in Z, i\geq 1} \sum_{r_2\in Ker(h)} k_1(r_2r_1x, r_2z_i)k_2(r_2z_i, y) \\
&= \sum_{r\in Ker(h)} \sum_{z_i\in Z, i\geq 1} \sum_{r_2\in Ker(h)} k_1(rx, r_2z_i)k_2(r_2z_i, y) \\
&= \sum_{r\in Ker(h)} \sum_{z\in Z} k_1(rx, z)k_2(z, y) \\
&= \sum_{r\in Ker(h)} (k_1k_2)(rx, y) \\
&= h(k_1k_2)([x],[y]),
\end{aligned}$$

and

$$\begin{aligned}
h(k)^*([x],[y]) &= (h(k)([y],[x]))^* \\
&= \sum_{r\in Ker(h)} k(ry, x)^* \\
&= \sum_{r\in Ker(h)} k(y, r^{-1}x)^* \\
&= \sum_{r\in Ker(h)} k^*(r^{-1}x, y) \\
&= h(k^*)([x],[y]).
\end{aligned}$$

Fourthly, h preserves propagations. Suppose $\mathrm{Prop}(k) = \epsilon$ for some $\epsilon > 0$, i.e.

$\sup\{d(x,y)|k(x,y) \neq 0\} = \epsilon$ or $k(x,y) = 0$ when $d(x,y) > \epsilon$. Given any $[x],[y] \in Z/Ker(h)$ such that $d([x],[y]) > \epsilon$, then

$$\inf_{r_1,r_2 \in Ker(h)} \{d(r_1 x, r_2 y)\} > \epsilon,$$

which implies that $d(r_1 x, r_2 y) > \epsilon$ for all $r_1, r_2 \in Ker(h)$. Hence

$$h(k)([x],[y]) = \sum_{r \in Ker(h)} k(rx, y) = 0.$$

So Prop$(h(k)) \leq \epsilon =$ Prop(k).

In summary, $h(k) \in C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A)$ and h can extended to a $*$-homomorphism from $C^*_{max}(G, P_s(G), A)$ to $C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A)$. Since h preserves propagations, it also induces a $*$-homomorphism from $C^*_L(G, P_s(G), A)$ to $C^*_L(\Gamma_0, P_s(G)/Ker(h), A)$. $\square$

The following Lemma is taken from [42].

Lemma 3.1.27. [42, Lemma 5.1.11 and Lemma 6.5.5] Let X and Y be Γ-proper metric spaces. Let $f : X \to Y$ be an Γ-equivariant proper continuous map. A is a Γ-C^*-algebra. Then there exist $*$-homomorphisms

$$ad_V : C^*_{red}(\Gamma, X, A) \to C^*_{red}(\Gamma, Y, A),$$
$$ad_V : C^*_{max}(\Gamma, X, A) \to C^*_{max}(\Gamma, Y, A),$$
$$ad_{V_t} : C^*_L(\Gamma, X, A) \to C^*_L(\Gamma, Y, A).$$

Corollary 3.1.28. Given a surjective homomorphism $h : G \to \Gamma_0$ and a (G, Γ_0)-C^*-

algebra A. Then there exists a $*$-homomorphism (also denoted by h)

$$h : C^*_{max}(G, P_s(G), A) \to C^*_{max}(\Gamma_0, P_s(\Gamma_0), A),$$
$$h : C^*_L(G, P_s(G), A) \to C^*_L(\Gamma_0, P_s(\Gamma_0), A).$$

Proof. From Lemma 3.1.25 and Lemma 3.1.27, there are $*$-homomorphisms

$$C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A) \to C^*_{max}(\Gamma_0, P_s(\Gamma_0), A),$$
$$C^*_L(\Gamma_0, P_s(G)/Ker(h), A) \to C^*_L(\Gamma_0, P_s(\Gamma_0), A).$$

Compose them with the homomorphisms in proposition 3.1.26, we get the result. □

Suppose that Γ_0 be a subgroup of Γ. From proposition 3.1.17 and corollary 3.1.21, we know that $P_s(\Gamma_0)$ is a subcomplex of $P_{2s}(\Gamma)$. Denote $\Gamma \cdot (P_s(\Gamma_0)) = \{\gamma x \mid \gamma \in \Gamma, x \in P_s(\Gamma_0)\}$. Then $\Gamma \cdot (P_s(\Gamma_0))$ is a proper metric space with proper Γ action.

Proposition 3.1.29. Given an injective homomorphism $h : \Gamma_0 \to \Gamma$ and a (Γ_0, Γ)-C^*-algebra A. Then there exists $*$-homomorphisms (also denoted by h)

$$h : C^*_{red}(\Gamma_0, P_s(\Gamma_0), A) \to C^*_{red}(\Gamma, P_{2s}(\Gamma), A),$$
$$h : C^*_{max}(\Gamma_0, P_s(\Gamma_0), A) \to C^*_{max}(\Gamma, P_{2s}(\Gamma), A),$$

such that if $k \in C^*_{max}(\Gamma_0, P_s(\Gamma_0), A)$ has finite propagation and is represented as a kernel on $Z \cap P_s(\Gamma_0)$ (where Z is a countable Γ-invariant dense subset of $P_{2s}(\Gamma)$ taken from lemma 3.1.22) with values in $K(H) \otimes A$, then $h(k)$ has finite propagation and $Prop(h(k)) \leq Prop(k)$. Hence h induces a $*$-homomorphism (also denoted by h)

$$h : C^*_L(\Gamma_0, P_s(\Gamma_0), A) \to C^*_L(\Gamma, P_{2s}(\Gamma), A).$$

Proof. We fix admissible covariant systems $(C_0(P_s(\Gamma_0)), \Gamma_0, \rho)$ and $(C_0(\Gamma \cdot (P_s(\Gamma_0))), \Gamma,$ $\sigma)$, where

$$\begin{aligned} \rho : C_0(P_s(\Gamma_0)) &\to B(\ell^2(Z \cap P_s(\Gamma_0)) \otimes H \otimes A) \\ f &\mapsto M_f \otimes I \otimes I \end{aligned}$$

and

$$\begin{aligned} \sigma : C_0(\Gamma \cdot (P_s(\Gamma_0))) &\to B(\ell^2(Z \cap \Gamma \cdot (P_s(\Gamma_0))) \otimes H \otimes A) \\ f &\mapsto M_f \otimes I \otimes I, \end{aligned}$$

$Z \subseteq P_{2s}(\Gamma)$ is taken as lemma 3.1.22.

Given any operator $k \in C^*_{red}(\Gamma_0, P_s(\Gamma_0), A)$ with finite propagation and is represented as a kernel on $Z \cap P_s(\Gamma_0)$ with values in $K(H) \otimes A$, we define $h(k) \in B(\ell^2(Z \cap \Gamma \cdot (P_s(\Gamma_0))) \otimes H \otimes A)$ as follows, for $x, y \in Z \cap \Gamma \cdot (P_s(\Gamma_0))$

$$h(k)(x,y) = \begin{cases} k(\gamma x, \gamma y), & \text{when } \gamma x, \gamma y \in Z \cap P_s(\Gamma_0); \\ 0, & \text{otherwise.} \end{cases}$$

Firstly, h is well-defined. Fix $x, y \in Z$. Suppose there exists $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in Z \cap P_s(\Gamma_0)$. If $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\gamma_1 x, \gamma_1 y \in Z \cap P_s(\Gamma_0)$ and $\gamma_2 x, \gamma_2 y \in Z \cap P_s(\Gamma_0)$, then

$$\begin{aligned} k(\gamma_1 x, \gamma_1 y) &= k(\gamma_2 \gamma_1^{-1} \gamma_1 x, \gamma_2 \gamma_1^{-1} \gamma_1 y) \\ &= k(\gamma_2 x, \gamma_2 y). \end{aligned}$$

This shows that h is well-defined.

Second, $h(k)$ is Γ-invariant. Given any $\gamma \in \Gamma$, and $x, y \in Z \cap \Gamma \cdot (P_s(\Gamma_0))$. Suppose there exists $\gamma' \in \Gamma$ such that $\gamma' x, \gamma' y \in Z \cap P_s(\Gamma_0)$. Then

$$\begin{aligned} h(k)(\gamma x, \gamma y) &= k(\gamma' \gamma^{-1} \gamma x, \gamma' \gamma^{-1} \gamma y) \\ &= k(\gamma' x, \gamma' y) \\ &= h(k)(x, y). \end{aligned}$$

If there does not exist $\gamma' \in \Gamma$ such that $\gamma' x, \gamma' y \in Z \cap P_s(\Gamma_0)$, then

$$h(k)(\gamma x, \gamma y) = h(k)(x, y) = 0.$$

So we always have

$$h(k)(\gamma x, \gamma y) = h(k)(x, y),$$

which means $h(k)$ is Γ-invariant.

Thirdly, h is a $*$-homomorphism. Given any k, k_1, k_2, fix any $x, y \in Z \cap \Gamma \cdot (P_s(\Gamma_0))$

such that $\gamma x, \gamma y \in Z \cap P_s(\Gamma_0)$ for some $\gamma \in \Gamma$, then

$$
\begin{aligned}
(h(k_1)h(k_2))(x,y) &= \sum_{z\in Z} h(k_1)(x,z)h(k_2)(z,y) \\
&= \sum_{\substack{z\in Z,\text{where } \gamma_1,\gamma_2 \\ \text{are chosen such} \\ \text{that}\gamma_1 x,\gamma_1 z,\gamma_2 z, \\ \gamma_2 y\in Z\cap P_s(\Gamma_0)}} k_1(\gamma_1 x, \gamma_1 z)\cdot k_2(\gamma_2 z, \gamma_2 y) \\
&\overset{\gamma_1\gamma_2^{-1}\in\Gamma_0}{=} \sum_{\substack{z\in Z,\text{where } \gamma_1,\gamma_2 \\ \text{are chosen such} \\ \text{that } \gamma_1 x,\gamma_1 z,\gamma_2 z, \\ \gamma_2 y\in Z\cap P_s(\Gamma_0)}} k_1(\gamma_1 x, \gamma_1 z)\cdot k_2(\gamma_1 z, \gamma_1 y) \\
&= \sum_{\substack{z\in Z,\text{where } \gamma_1 \\ \text{is chosen such} \\ \text{that } \gamma_1 x,\gamma_1 z, \\ \gamma_1 y\in Z\cap P_s(\Gamma_0)}} k_1(\gamma_1 x, \gamma_1 z)k_2(\gamma_1 z, \gamma_1 y) \\
&= \sum_{z\in\gamma^{-1}(Z\cap P_s(\Gamma_0))} k_1(\gamma x, \gamma z)k_2(\gamma z, \gamma y) \\
&= \sum_{z\in Z\cap P_s(\Gamma_0)} k_1(\gamma x, z)k_2(z, \gamma y) \\
&= (k_1k_2)(\gamma x, \gamma y) \\
&= h(k_1k_2)(x,y),
\end{aligned}
$$

and

$$
\begin{aligned}
(h(k))^*(x,y) &= (h(k)(y,x))^* \\
&= k(\gamma y, \gamma x)^* \\
&= k^*(\gamma x, \gamma y) \\
&= h(k^*)(x,y).
\end{aligned}
$$

If there does not exist $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in Z \cap P_s(\Gamma_0)$, then for any $z \in Z$, there does not exist γ_1, γ_2 such that $\gamma_1 x, \gamma_1 z, \gamma_2 z, \gamma_2 y \in Z \cap P_s(\Gamma_0)$. Otherwise, $\gamma_1 x, \gamma_1 z, \gamma_2 z, \gamma_2 y \in$

$Z \cap P_s(\Gamma_0)$ implies $\gamma_1\gamma_2^{-1} \in \Gamma_0$ and then $\gamma_1 x, \gamma_1 y \in Z \cap P_s(\Gamma_0)$ which is a contradiction with assumption. So in this case,

$$(h(k_1)h(k_2))(x,y) = h(k_1k_2)(x,y) = 0, \text{ and } (h(k))^*(x,y) = h(k^*)(x,y) = 0.$$

In summary, h is a $*$-homomorphism.

Forthly, h preserves propagations. It is from the definition of h.

Fifthly, $h(k)$ is locally compact. Fix any two simplex $\widetilde{\Delta}_1, \widetilde{\Delta}_2$ in $\Gamma \cdot (P_s(\Gamma_0))$, we will show that $\chi_{\widetilde{\Delta}_1} h(k) \chi_{\widetilde{\Delta}_2}$ is compact.

$$\chi_{\widetilde{\Delta}_1} h(k) \chi_{\widetilde{\Delta}_2}(x,y) = \begin{cases} h(k)(x,y), & \text{when } x \in \widetilde{\Delta}_1 \cap Z, y \in \widetilde{\Delta}_2 \cap Z \\ 0, & \text{otherwise} \end{cases}$$

Suppose that there exist $x \in \widetilde{\Delta}_1 \cap Z, y \in \widetilde{\Delta}_2 \cap Z$ such that $\exists \gamma \in \Gamma$ s.t. $\gamma x, \gamma y \in Z \cap P_s(\Gamma_0)$. Otherwise, $\chi_{\widetilde{\Delta}_1} h(k) \chi_{\widetilde{\Delta}_2} = 0$ which is obviously compact.

Obviously, $\widetilde{\Delta}_1 = \gamma_1 \Delta_1, \widetilde{\Delta}_2 = \gamma_2 \Delta_2$ for some simplex Δ_1, Δ_2 in $P_s(\Gamma_0)$. $\exists x \in \widetilde{\Delta}_1 \cap Z, y \in \widetilde{\Delta}_2 \cap Z$, $\gamma \in \Gamma$ s.t. $\gamma x, \gamma y \in Z \cap P_s(\Gamma_0)$ implies that $\gamma_1^{-1}\gamma_2 \in \Gamma_0$. Then $\widetilde{\Delta}_2 = \gamma_1 \gamma_1^{-1} \gamma_2 \Delta_2$ where $\gamma_1^{-1}\gamma_2\Delta_2$ is also simplex in $P_s(\Gamma_0)$. Hence we can assume that $\widetilde{\Delta}_1 = \gamma \Delta_1, \widetilde{\Delta}_2 = \gamma \Delta_2$ for some simplex Δ_1, Δ_2 in $P_s(\Gamma_0)$ and $\gamma \in \Gamma$. Since

$$h(k)(x,y) = k(\gamma^{-1}x, \gamma^{-1}y), \text{ for } x \in \widetilde{\Delta}_1 \cap Z, y \in \widetilde{\Delta}_2 \cap Z,$$

we have $\chi_{\widetilde{\Delta}_1} h(k) \chi_{\widetilde{\Delta}_2} = \chi_{\Delta_1} k \chi_{\Delta_2}$, which is compact since $\chi_{\Delta_1} k \chi_{\Delta_2}$ is compact.

Sixthly, it is not hard to prove that $||h(k)|| \leq ||k||$ from definition of h, where the norms are the operator norms in $B(\ell^2(Z \cap \Gamma \cdot (P_s(\Gamma_0))) \otimes H \otimes A)$ and $B(\ell^2(Z \cap P_s(\Gamma_0)) \otimes H \otimes A)$.

In summary, we have

$$h : C^*_{red}(\Gamma_0, P_s(\Gamma_0), A) \to C^*_{red}(\Gamma, \Gamma \cdot (P_s(\Gamma_0)), A),$$
$$h : C^*_{max}(\Gamma_0, P_s(\Gamma_0), A) \to C^*_{max}(\Gamma, \Gamma \cdot (P_s(\Gamma_0)), A),$$
$$h : C^*_{L}(\Gamma_0, P_s(\Gamma_0), A) \to C^*_{L}(\Gamma, \Gamma \cdot (P_s(\Gamma_0)), A).$$

Since $\Gamma \cdot (P_s(\Gamma_0))$ is a Γ-proper metric subspace of $P_{2s}(\Gamma)$, from Lemma 3.1.27 we get the result. $\square$

Corollary 3.1.30. Given a homomorphism $h : G \to \Gamma$ and a (G, Γ)-C^*-algebra A. For any $s > 0$, there exists $*$-homomorphisms (denoted by h)

$$h : C^*_{max}(G, P_s(G), A) \to C^*_{max}(\Gamma, P_{2s}(\Gamma), A),$$
$$h : C^*_{L}(G, P_s(G), A) \to C^*_{L}(\Gamma, P_{2s}(\Gamma), A).$$

Lemma 3.1.31. Under the assumption in Corollary 3.1.30, for $r \leqslant s$, the following diagrams commute,

$$\begin{array}{ccc} C^*_{max}(G, P_r(G), A) & \xrightarrow{ad_U} & C^*_{max}(G, P_s(G), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{max}(\Gamma, P_{2r}(\Gamma), A) & \xrightarrow{ad_V} & C^*_{max}(\Gamma, P_{2s}(\Gamma), A) \end{array},$$

$$\begin{array}{ccc} C^*_{L}(G, P_r(G), A) & \xrightarrow{ad_{U_t}} & C^*_{L}(G, P_s(G), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{L}(\Gamma, P_{2r}(\Gamma), A) & \xrightarrow{ad_{V_t}} & C^*_{L}(\Gamma, P_{2s}(\Gamma), A) \end{array},$$

where ad_U, ad_V, ad_{U_t} and ad_{V_t} are from Lemma 3.1.27.

Proof. We only prove the commuteness of the first diagram. The second one follows along the same idea. It is not hard to decompose the first diagram into the following three diagrams,

$$\begin{array}{ccc} C^*_{max}(G, P_r(G), A) & \longrightarrow & C^*_{max}(G, P_s(G), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{max}(\Gamma_0, P_r(G)/Ker(h), A) & \longrightarrow & C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A) \end{array} \tag{3.1.1}$$

$$\begin{array}{ccc} C^*_{max}(\Gamma_0, P_r(G)/Ker(h), A) & \longrightarrow & C^*_{max}(\Gamma_0, P_s(G)/Ker(h), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{max}(\Gamma_0, P_r(\Gamma_0), A) & \longrightarrow & C^*_{max}(\Gamma_0, P_s(\Gamma_0), A) \end{array} \tag{3.1.2}$$

$$\begin{array}{ccc} C^*_{max}(\Gamma_0, P_r(\Gamma_0), A) & \longrightarrow & C^*_{max}(\Gamma_0, P_s(\Gamma_0), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{max}(\Gamma, P_{2r}(\Gamma), A) & \longrightarrow & C^*_{max}(\Gamma, P_{2s}(\Gamma), A) \end{array} \tag{3.1.3}$$

The notation h is kind of abuse, but the meaning is understandable.

From Proposition 3.1.20, we know $P_r(G) \to P_s(G)$, $P_r(G)/Ker(h) \to P_s(G)/Ker(h)$ are isometries onto the images. $Z_r \subseteq P_r(G)$ and $Z_s \subseteq P_s(G)$ can be chosen such that Z_r is a subset of Z_s.

Define $U : \ell^2(Z_r) \otimes H \otimes A \to \ell^2(Z_s) \otimes H \otimes A$ by $U(\delta_x \otimes v \otimes a) = \delta_x \otimes v \otimes a$ and similar for $V : \ell^2(Z_r/ker(h)) \otimes H \otimes A \to \ell^2(Z_s/ker(h)) \otimes H \otimes A$. Then U and V are isometries.

Given any operator $k \in C^*_{max}(G, P_r(G), A)$ with finite propagation and is represented as a kernel on Z_r with values in $K(H) \otimes A$. For any $[x], [y] \in Z_s/ker(h)$, if $[x], [y] \in Z_r/ker(h)$, then

$$\begin{aligned} h(ad_U(k))([x],[y]) &= \sum_{r \in Ker(h)} ad_U(k)(rx, y) \\ &= \sum_{r \in Ker(h)} k(rx, y) \\ &= h(k)([x],[y]) \\ &= ad_V(h(k))([x],[y]). \end{aligned}$$

Otherwise,

$$\begin{aligned} h(ad_U(k))([x],[y]) &= \sum_{r \in Ker(h)} ad_U(k)(rx, y) \\ &= 0 \\ &= ad_V(h(k))([x],[y]). \end{aligned}$$

Hence $h(ad_U(k)) = ad_V(h(k))$ which means that $(3.1.1)$ commutes. $(3.1.2)$ and $(3.1.3)$ follow from the same ideas. So we complete the proof. □

Lemma 3.1.32. Under the assumption in Corollary 3.1.30, if $h : G \to \Gamma$ is injective, then for $r \leqslant s$, the following diagram commutes,

$$\begin{array}{ccc} C^*_{red}(G, P_r(G), A) & \xrightarrow{ad_U} & C^*_{red}(G, P_s(G), A) \\ \Big\downarrow h & & \Big\downarrow h \\ C^*_{red}(\Gamma, P_{2r}(\Gamma), A) & \xrightarrow{ad_V} & C^*_{red}(\Gamma, P_{2s}(\Gamma), A) \end{array}$$

where ad_U, ad_V are from Lemma 3.1.27.

Proof. It follows from the same ideas as Lemma 3.1.31. □

Definition 3.1.33. Given a $*$-homomorphism $h : A \to B$ between C^*-algebras. The *mapping cone* C_h of h is defined to be

$$C_h := \{(a, f) | a \in A, f \in C_0([0,1), B), h(a) = f(0)\}.$$

Definition 3.1.34. Given a group homomorphism $h : G \to \Gamma$ and a (G, Γ)-C^*-algebra A. For any $s > 0$, define $C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)$ and $C^*_L(G, P_sG, \Gamma, P_{2s}\Gamma, A)$ to be the mapping cones of h in Corollary 3.1.30.

Definition 3.1.35. Given an injective homomorphism $h : G \to \Gamma$ and a (G, Γ)-C^*-algebra A. For any $s > 0$, define $C^*_{red}(G, P_sG, \Gamma, P_{2s}\Gamma, A)$ to be the mapping cone of h in Proposition 3.1.29.

From Lemma 3.1.31 and Lemma 3.1.32, we know that for any $r \leqslant s$, there exist natural maps

$$C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A) \to C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A),$$
$$C^*_L(G, P_rG, \Gamma, P_{2r}\Gamma, A) \to C^*_L(G, P_sG, \Gamma, P_{2s}\Gamma, A),$$

and

$$C^*_{red}(G, P_rG, \Gamma, P_{2r}\Gamma, A) \to C^*_{red}(G, P_sG, \Gamma, P_{2s}\Gamma, A),$$

when $h : G \to \Gamma$ is an injective homomorphism.

Definition 3.1.36. Given a group homomorphism $h : G \to \Gamma$. The *relative equivariant K-homology with coefficients in* A of $(\mathcal{E}G, \mathcal{E}\Gamma)$ is defined as follows,

$$K^{G,\Gamma}_*(\mathcal{E}G, \mathcal{E}\Gamma, A) = \lim_{r\to\infty} K_*(C^*_L(G, P_rG, \Gamma, P_{2r}\Gamma, A)).$$

3.2 Relative Group C^*-Algebras

Proposition 3.2.1. Given a group homomorphism $h : G \to \Gamma$. Let A be a (G, Γ)-C^*-algebra, then h induced a natural homomorphism (also denoted by h)

$$h : C^*_{max}(G, A) \to C^*_{max}(\Gamma, A).$$

If h is injective, then h induced a natural homomorphism

$$h : C^*_{red}(G, A) \to C^*_{red}(\Gamma, A).$$

We define $C^*_{max}(G, \Gamma, A)$ to be the mapping cone of h in above proposition, and we call it the *maximal relative group* ***C*-algebra*** of (G, Γ) with coefficients in A. If h is injective, then we can likewise define *reduced relative group* ***C*-algebra*** $C^*_{red}(G, \Gamma, A)$.

When $A = \mathbb{C}$, we have the maximal relative group C*-algebra without coefficients, $C^*_{max}(G, \Gamma)$. If h is injective, then we have the reduced relative group C*-algebra without coefficients, $C^*_{red}(G, \Gamma)$.

3.3 Relative Baum-Connes Assembly Map

In this section, we will formulate the relative Baum-Connes assembly map.

Proposition 3.3.1. Let $h : G \to \Gamma$ be a group homomorphism, A any (G, Γ)-C^*-algebra. For any $r > 0, C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)$ is $*$-isomorphic to the the mapping Cone of $h \otimes ad_{V_r}$, where $h \otimes ad_{V_r} : C^*_{max}(G, A) \otimes K(H_1) \to C^*_{max}(\Gamma, A) \otimes K(H_2)$ is a homomorphism and $V_r : H_1 \to H_2$ is an isometry. Moreover, the isomophism induces an isomorphism in K-groups

$$K_*(C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)) \to K_*(C^*_{max}(G, \Gamma, A)).$$

Proof. Fix any $Z_G \subseteq P_rG$ and $Z_\Gamma \subseteq P_{2r}\Gamma$ which have the property in Lemma 3.1.22.

Let

$$U : \ell^2(Z_G) \otimes H \otimes A \to \ell^2(G) \otimes A \otimes \ell^2(Z_{0,G}) \otimes H,$$
$$W : \ell^2(Z_\Gamma) \otimes H \otimes A \to \ell^2(\Gamma) \otimes A \otimes \ell^2(Z_{0,\Gamma}) \otimes H,$$

be defined as follows

$$U(\xi \otimes v \otimes a) = \sum_g \delta_g \otimes a \otimes PU_g\xi \otimes v$$
$$W(\xi \otimes v \otimes a) = \sum_\gamma \delta_\gamma \otimes a \otimes PU_\gamma\xi \otimes v$$

where $Z_{0,G}$ ($Z_{0,\Gamma}$) is a fundamental domain of Z_G (Z_Γ), $P : \ell^2(Z_G) \to \ell^2(Z_G)$ is the projection on $\ell^2(Z_{0,G})$.

Firstly, we show

$$U\mathbb{C}[G, P_rG, A]U^* = C_c(G, A) \odot K(H \otimes \ell^2(Z_{0,G})),$$
$$W\mathbb{C}[\Gamma, P_{2r}\Gamma, A]W^* = C_c(\Gamma, A) \odot K(H \otimes \ell^2(Z_{0,\Gamma})).$$

It suffices to show that if T is an element of $\mathbb{C}[G, P_rG, A]$, then UTU^* is an element of $C_c(G, A) \odot K(H \otimes \ell^2(Z_{0,G}))$, and corversely if $a \cdot \lambda_g \otimes K$ is an element of $C_c(G, A) \odot K(H \otimes \ell^2(Z_{0,G}))$, then $U^*(a \cdot \lambda_g \otimes K)U$ is an element of $\mathbb{C}[G, P_rG, A]$.

Note that

$$U^*(\delta_g \otimes a \otimes \zeta \otimes v) = U_{g^{-1}}\zeta \otimes a \otimes v.$$

Let T be an element of $\mathbb{C}[G, P_rG, A]$, then for any $\delta_g \otimes a \otimes \zeta \otimes v \in \ell^2(G) \otimes A \otimes \ell^2(Z_{0,G}) \otimes H$,

$$\begin{aligned} UTU^*(\delta_g \otimes a \otimes \zeta \otimes v) &= UT(U_{g^{-1}}\zeta \otimes a \otimes v) \\ &= \sum_{k \in G} \delta_k \otimes PU_k(T(U_{g^{-1}}\zeta \otimes a \otimes v)) \\ &= \sum_{g'} \delta_{g'g} \otimes (g^{-1}(PU_{g'}TP))(a \otimes \zeta \otimes v) \\ &= (\sum_{g'} \lambda_{g'} \otimes g^{-1}(PU_{g'}TP))(\delta_g \otimes a \otimes \zeta \otimes v) \end{aligned}$$

All the operators $g^{-1}(PU_{g'}TP)$ are compact operators on $\ell^2(Z_{0,G}) \otimes H \otimes A$ and only finitely many of these operators are non-zero.

Hence

$$UTU^* = \sum_{g'} g'(PU_{g'}TP) \cdot g' \in C_c(G, K(\ell^2(Z_{0,G}) \otimes H) \otimes A).$$

For the reverse inclusion, assume that $K \cdot g$ is an element of $C_c(G, K(\ell^2(Z_{0,G}) \otimes H) \otimes A)$. As similar computation to above, we have

$$U^*(K \cdot g)U = \sum_{g' \in G} U_{g'} U_{g^{-1}} g^{-1}(K) PU_{g'^{-1}}.$$

It is clearly invariant, so it suffices to check local compactness and finite propagation. For local compactness, let $F \subseteq P_rG$ be a compact subset. Then

$$\chi_F \cdot \sum_{g' \in G} U_{g'} U_{g^{-1}} g^{-1}(K) PU_{g'^{-1}} = \sum_{g' \in G} U_{g'} U_{g^{-1}} \chi_{gg'^{-1}F} P g^{-1}(K) PU_{g'^{-1}}.$$

All the terms in this sum are compact, and only finitely many of them are non-zero. So it is compact. One sees that $(\sum_{g' \in G} U_{g'} U_{g^{-1}} g^{-1}(K) PU_{g'^{-1}}) \cdot \chi_F$ is compact precisely aralogously.

To check finite propagation, say that (x, y) is in the support of the operator above, where $x \in \overline{Z_{0,G}}$.

Then for any $\varepsilon > 0$,

$$\begin{aligned} 0 &\neq \chi_{B(x,\varepsilon)} \cdot \sum_{g' \in G} U_{g'} U_{g^{-1}} g^{-1}(K) P U_{g'^{-1}} \cdot \chi_{B(y,\varepsilon)} \\ &= \sum_{g' \in G} U_{g'} U_{g^{-1}} \chi_{B(gg'^{-1}x,\varepsilon)} g^{-1}(K) \chi_{B(g'^{-1}y,\varepsilon)} U_{g'^{-1}}. \end{aligned}$$

Hence $(gg'^{-1}x, g'^{-1}y) \in supp(K)$ for some $g' \in G$ for which the above sum is nonzero. In particular, both $gg'^{-1}x$ and $g'^{-1}y$ must be in the closure of $Z_{0,G}$ (a bounded compact set); moreove as x in $\overline{Z_{0,G}}$, there is only a finite set $E \subseteq G$, (independent of x and y) for which this is possible.

Hence

$$d(x, y) \leq \sup_{g' \in E, x \in \overline{Z_{0,G}}} d(x, g'gg'^{-1}x) + d(g'gg'^{-1}x, y) \leq M + Prop(K).$$

where $M \triangleq \sup_{g' \in E, x \in \overline{Z_{0,G}}} d(x, g'gg'^{-1}x)$. To complete it, let (x, y) be an arbitrary element in the support of the operator above. As this operator is G-invariant, $(g'x, g'y)$ is also in the support for all $g' \in G$, in particular, $g'x \in \overline{Z_{0,G}}$ for some $g' \in G$, hence

$$d(x, y) = d(g'x, g'y) \leq M + Prop(K).$$

So

$$U\mathbb{C}[G, P_rG, A]U^* = C_c(G, A) \odot K(H \otimes \ell^2(Z_{0,G})),$$

which implies

$$UC^*_{max}(G, P_rG, A)U^* = C^*_{max}(G, A) \otimes K(H \otimes \ell^2(Z_{0,G})).$$

Similarly,

$$WC^*_{max}(\Gamma, P_{2r}\Gamma, A)W^* = C^*_{max}(\Gamma, A) \otimes K(H \otimes \ell^2(Z_{0,\Gamma})).$$

Secondly, we show that there exists an isometry $V_r : \ell^2(Z_{0,G}) \otimes H \to \ell^2(Z_{0,\Gamma}) \otimes H$ such that $ad_W \circ h \circ ad_{U^*} = h \otimes ad_{V_r}$.

We will seperate it into three parts according to our definition of $h : C^*_{max}(G, P_rG, A) \to C^*_{max}(\Gamma, P_{2r}\Gamma, A)$. Let $\Gamma_0 = h(G)$. Take the module $\ell^2(Z_G/Ker(h)) \otimes H \otimes A$ for $P_rG/Ker(h)$. Then similarly we have

$$W_1C^*_{max}(\Gamma_0, P_rG/Ker(h), A)W_1^* = C^*_{max}(\Gamma_0, A) \otimes K(H \otimes \ell^2(Z_{0,G}/Ker(h))).$$

Let Z_{0,Γ_0} be as $Z_{0,G}$, then we have

$$W_2C^*_{max}(\Gamma_0, P_r\Gamma_0, A)W_2^* = C^*_{max}(\Gamma_0, A) \otimes K(H \otimes \ell^2(Z_{0,\Gamma_0})).$$

Now for any $K \cdot g \in C_c(G, A) \odot K(H \otimes \ell^2(Z_{0,G}))$,

$$\begin{aligned}
W_1h(U^*(K \cdot g)U)W_1^* &= W_1h(\sum_{g' \in G} U_{g'}U_{g^{-1}}g^{-1}(K)PU_{g'^{-1}})W_1^* \\
&= W_1(\sum_{[g'] \in \Gamma_0} U_{[g']}U_{[g]^{-1}}g^{-1}(K)PU_{[g']^{-1}})W_1^* \\
&= [g](g^{-1}(K)) \cdot [g] \\
&= [g]([g]^{-1}(K)) \cdot [g] \\
&= K \cdot [g].
\end{aligned}$$

Note that here we identify $\ell^2(Z_{0,G})$ with $\ell^2(Z_{0,G}/Ker(h))$. Since $\ell^2(Z_G/Ker(h)) \otimes$

$H \otimes A$ and $\ell^2(Z_{\Gamma_0}) \otimes H \otimes A$ are both modules for Γ_0, we have

$$W_2 h(W_1^*(K \cdot [g])W_1)W_2^* = ad_V(K) \cdot [g],$$

where

$$V : \ell^2(Z_{0,G}/Ker(h)) \otimes H \otimes A \to \ell^2(Z_{0,\Gamma_0}) \otimes H \otimes A$$

is an isometry which is identity on A.

Moreover,

$$\begin{aligned} Wh(W_2^*(K \cdot \gamma)W_2)W^* &= Wh(\sum_{\gamma' \in \Gamma_0} U_{\gamma'} U_{\gamma^{-1}} \gamma^{-1}(K) P U_{\gamma'^{-1}})W^* \\ &= W(\sum_{\gamma' \in \Gamma} U_{\gamma'} U_{\gamma^{-1}} V_1 \gamma^{-1}(K) V_1^* P U_{\gamma'^{-1}})W^* \\ &= \gamma(ad_{V_1}(\gamma^{-1}(K))) \cdot \gamma \\ &= ad_{V_1}(K) \cdot \gamma, \end{aligned}$$

where

$$V_1 : \ell^2(Z_{0,\Gamma_0}) \otimes H \otimes A \to \ell^2(Z_{0,\Gamma}) \otimes H \otimes A,$$

$$\delta_x \otimes v \otimes a \mapsto \delta_x \otimes v \otimes a$$

is an isometry.

In summary,

$$Wh(U^*(K \cdot g)U)W^* = ad_{V_1} \circ ad_V(K) \cdot h(g).$$

Note that $V_1 V$ is an isometry from $\ell^2(Z_{0,G}) \otimes H \otimes A$ to $\ell^2(Z_{0,\Gamma}) \otimes H \otimes A$ which is identity on A. Hence $V_1 V = V_r \otimes 1$, where $V_r : \ell^2(Z_{0,G}) \otimes H \to \ell^2(Z_{0,\Gamma}) \otimes H$ is an isometry. So

$$ad_W \circ h \circ ad_{U^*}(K \cdot g) = (ad_{V_r} \otimes 1)(K) \cdot h(g) = (h \otimes ad_{V_r})(K \cdot g),$$

which implies $ad_W \circ h \circ ad_{U^*} = h \otimes ad_{V_r}$.

Now let

$$\varphi : C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A) \to C_{h\otimes ad_{V_r}}$$

$$(b, f) \mapsto (UbU^*, WfW^*).$$

Since

$$(h \otimes ad_{V_r})(UbU^*) = ad_W \circ h \circ ad_{U^*}(UbU^*) = Wh(b)W^* = Wf(0)W^*,$$

φ is well-defined. It is not hard to show φ is an isomorphism. Hence

$$\varphi_* : K_*(C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)) \to K_*(C_{h\otimes ad_{V_r}})$$

is an isomorphism.

Finally, we show $K_*(C_{h\otimes ad_{V_r}}) \cong K_*(C^*_{max}(G, \Gamma, A))$. Fix any rank-one projection $p_1 \in K(H \otimes \ell^2(Z_{0,G}))$, then we have a commutative diagram

$$\begin{array}{ccc}
K_{*+1}(C^*_{max}(G, A)) & \xrightarrow{\alpha_1} & K_{*+1}(C^*_{max}(G, A) \otimes K(H \otimes \ell^2(Z_{0,G}))) \\
\downarrow & & \downarrow \\
K_{*+1}(C^*_{max}(\Gamma, A)) & \xrightarrow{\alpha_2} & K_{*+1}(C^*_{max}(\Gamma, A) \otimes K(H \otimes \ell^2(Z_{0,\Gamma}))) \\
\downarrow & & \downarrow \\
K_*(C^*_{max}(G, \Gamma, A)) & \xrightarrow{\alpha_3} & K_*(C_{h\otimes ad_{V_r}}) \\
\downarrow & & \downarrow \\
K_*(C^*_{max}(G, A)) & \xrightarrow{\alpha_4} & K_*(C^*_{max}(G, A) \otimes K(H \otimes \ell^2(Z_{0,G}))) \\
\downarrow & & \downarrow \\
K_*(C^*_{max}(\Gamma, A)) & \xrightarrow{\alpha_5} & K_*(C^*_{max}(\Gamma, A) \otimes K(H \otimes \ell^2(Z_{0,\Gamma})))
\end{array},$$

where α_1, α_4 are induced by $b \mapsto b \otimes p_1$, α_2, α_5 are induced by $b \mapsto b \otimes ad_{V_r}(p_1)$, α_3 is induced by $(b, f) \mapsto (b \otimes p_1, f \otimes ad_{V_r}(p_1))$. From Five lemma, α_3 is isomorphic. $\square$

Corollary 3.3.2. If $r < s$, then

$$\begin{array}{ccc} K_*(C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)) & \longrightarrow & K_*(C^*_{max}(G, \Gamma, A)) \\ \Big\downarrow & & \Big\| \\ K_*(C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)) & \longrightarrow & K_*(C^*_{max}(G, \Gamma, A)) \end{array}$$

commutes.

Proof. Let

$$U_r : \ell^2(Z_{G,r}) \otimes H \otimes A \to \ell^2(G) \otimes \ell^2(Z_{0,G,r}) \otimes H \otimes A$$
$$W_r : \ell^2(Z_{\Gamma,2r}) \otimes H \otimes A \to \ell^2(\Gamma) \otimes \ell^2(Z_{0,\Gamma,2r}) \otimes H \otimes A$$
$$U_s : \ell^2(Z_{G,s}) \otimes H \otimes A \to \ell^2(G) \otimes \ell^2(Z_{0,G,s}) \otimes H \otimes A$$
$$W_s : \ell^2(Z_{\Gamma,2s}) \otimes H \otimes A \to \ell^2(\Gamma) \otimes \ell^2(Z_{0,\Gamma,2s}) \otimes H \otimes A$$

be the operators as above proposition.

Let

$$V_1 : \ell^2(Z_{G,r}) \otimes H \otimes A \to \ell^2(Z_{G,s}) \otimes H \otimes A$$
$$V_2 : \ell^2(Z_{\Gamma,2r}) \otimes H \otimes A \to \ell^2(Z_{\Gamma,2s}) \otimes H \otimes A$$

be the covering isometries that are used to define

$$C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A) \to C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A).$$

Then

$$\begin{array}{ccc} C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A) & \xrightarrow{\varphi_r} & C_{h\otimes ad_{V_r}} \\ \Big\downarrow ad_{(V_1,V_2)} & & \Big\downarrow \Phi \\ C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A) & \xrightarrow{\varphi_s} & C_{h\otimes ad_{V_s}} \end{array}$$

commutes, where $\Phi = \varphi_s \circ ad_{(V_1,V_2)} \circ \varphi_r^{-1}$ is conjugation by the isometry $(id\otimes\tilde{V}_1, id\otimes\tilde{V}_2)$, and

$$\tilde{V}_1 : \ell^2(Z_{0,G,r}) \otimes H \to \ell^2(Z_{0,G,s}) \otimes H$$

$$\tilde{V}_2 : \ell^2(Z_{0,\Gamma,2r}) \otimes H \to \ell^2(Z_{0,\Gamma,2s}) \otimes H$$

are isometries. Hence

$$\begin{array}{ccc} K_*(C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)) & \longrightarrow & K_*(C^*_{max}(G, \Gamma, A)) \\ \Big\downarrow & & \Big\| \\ K_*(C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)) & \longrightarrow & K_*(C^*_{max}(G, \Gamma, A)) \end{array}$$

commutes. $\square$

From above corollary, the following diagram commute for $r < s$

$$\begin{array}{ccccc} K_*(C^*_L(G, P_rG, \Gamma, P_{2r}\Gamma, A)) & \xrightarrow{e_*} & K_*(C^*_{max}(G, P_rG, \Gamma, P_{2r}\Gamma, A)) & \xrightarrow{\varphi_r} & K_*(C^*_{max}(G, \Gamma, A)) \\ \Big\downarrow & & \Big\downarrow & & \Big\| \\ K_*(C^*_L(G, P_sG, \Gamma, P_{2s}\Gamma, A)) & \xrightarrow{e_*} & K_*(C^*_{max}(G, P_sG, \Gamma, P_{2s}\Gamma, A)) & \xrightarrow{\varphi_s} & K_*(C^*_{max}(G, \Gamma, A)) \end{array}$$

We can likewise get the result for the reduced case when $h : G \to \Gamma$ is injective.

Definition 3.3.3. The *maximal relative Baum-Connes assembly map* is the homomorphism

$$\mu_{max} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A))$$

which is taken from the direct limit of the homomorphisms in above commutative diagram. Likewise, define the *reduced relative Baum-Connes assembly map*

$$\mu_{red} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A))$$

when $h : G \to \Gamma$ is injective.

Strong Relative Novikov Conjecture. Let G and Γ be countable discrete groups, and $h : G \to \Gamma$ be a group homomorphism. A is a (G, Γ)-C^*-algebra. Then the *maximal relative Baum-Connes assembly map*

$$\mu_{max} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A))$$

is injective. Moreover if $h : G \to \Gamma$ is an injective group homomorphism, then the *reduced relative Baum-Connes assembly map*

$$\mu_{red} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A))$$

is injective.

4. STRONG RELATIVE NOVIKOV CONJECTURE

4.1 A Glimpse of Geometric Group Theory

4.1.1 Gromov's a-T-menable groups

A-T-menable group was introduced by Misha Gromov [16].

Definition 4.1.1. [16] A second countable, locally compact group Γ is called $a\text{-}T\text{-}menable$ if it admits a continuous, affine, isometric and metrically proper actions on a Hilbert space, the latter term meaning that

$$\lim_{g\to\infty} ||g \cdot v|| = \infty$$

for every vector v in the Hilbert space.

Gromov's terminology is explained by the twin facts that all (second countable) amenable groups admit such an action [4], whereas no non-compact group with Kazhdan's property T does [13]. Apart from amenable groups, important examples of a-T-menable groups are free groups with finite ranks, the real and complex hyperbolic groups $SO(n, 1)$ and $SU(n, 1)$, and Coxeter groups.

A remarkable result of Higson and Kasparov is showing that the Baum-Connes conjecture is true for all a-T-menable groups.

Theorem 4.1.2. [22] If Γ is a second countable, locally compact, a-T-menable group, then for any separable Γ-C^*-algebra A, the Baum-Connes assembly maps

$$\mu_{red} : KK^{\Gamma}_*(\mathcal{E}\Gamma, A) \to K_*(C^*_{red}(\Gamma, A))$$

and

$$\mu_{max} : KK^{\Gamma}_*(\mathcal{E}\Gamma, A) \to K_*(C^*_{max}(\Gamma, A))$$

are isomorphisms.

4.1.2 Property A and embeddability into Hilbert spaces

In this section, we shall review the concept of property A for metric spaces which is introduced by Guoliang Yu in [43]. Metric spaces with property A admit a coarse embedding into Hilbert space.

Definition 4.1.3. [45]A discrete metric space Γ is said to have property A if for any $r > 0, \epsilon > 0$, there exists a family of finite subsets $\{A_\gamma\}_{\gamma \in \Gamma}$ of $\Gamma \times \mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers) such that

(1) $(\gamma, 1) \in A_\gamma$ for all $\gamma \in \Gamma$;

(2) $\frac{\#(A_\gamma - A_{\gamma'}) + \#(A'_\gamma - A_\gamma)}{\#(A_\gamma \bigcap A_{\gamma'})} < \epsilon$ for all γ and γ' in Γ such that $d(\gamma, \gamma') \leqslant r$, where for each finite set A, $\#A$ is the number of elements in A;

(3) $\exists R > 0$ such that if $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$ for some $\gamma \in \Gamma$, then $d(x, y) \leqslant R$.

Theorem 4.1.4. [45] If a discrete metric space Γ has property A, then Γ admits a coarse embedding into Hilbert space.

Notice that property A is invariant under quasi-isometry. In the case of a finitely generated group, property A does not depend on the choice of the word-length metric. The class of finitely generated groups with property A, as metric spaces with word-length metrics, includes word hyperbolic groups, discrete subgroups of connected Lie groups and amenable groups, and is closed under semi-direct product.

A remarkable result of Guoliang Yu show that the coarse Baum-Connes conjecture is true for discrete metric space with bounded geometry which admits a coarse embedding into Hilbert space. As a consequence, for a discrete group Γ with a translation invariant metric and its classifying space $B\Gamma$ has the homotopy type of a finite CW complex, which admits a coarse embedding into Hilbert space, the strong Novikov conjecture holds. G. Skandalis, J.L. Tu and G. Yu refine Higson's descent technique and show that Novikov conjecture also holds for a discrete group Γ with a translation invariant metric which admits a coarse embedding into Hilbert space.

Theorem 4.1.5. [39]Let Γ be a countable group with a proper left-invariant metric. If Γ admits a coarse embedding into Hilbert space, then the Baum–Connes assembly map

$$\mu_{red} : KK_*^{\Gamma}(\mathcal{E}\Gamma, A) \to K_*(C^*_{red}(\Gamma, A))$$

is injective for any separable Γ-C^*-algebra A.

4.2 Case for A-T-menable Groups

Lemma 4.2.1. Given a commutative diagram of groups,

$$\begin{array}{ccccccccc} A & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & C & \xrightarrow{\beta_3} & D & \xrightarrow{\beta_4} & E \\ \downarrow{\alpha_1} & & \downarrow{\alpha_2} & & \downarrow{\alpha_3} & & \downarrow{\alpha_4} & & \downarrow{\alpha_5} \\ A' & \xrightarrow{\gamma_1} & B' & \xrightarrow{\gamma_2} & C' & \xrightarrow{\gamma_3} & D' & \xrightarrow{\gamma_4} & E' \end{array}$$

Assume that the lines are exact sequences, α_1, α_4 are isomorphisms, α_2, α_5 are injective. Then α_3 is injective.

Theorem 4.2.2. Let $h : G \to \Gamma$ be a group homomorphism and A any (G, Γ)-C^*-algebra. If G is an a-T-menable group and Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture with coefficients in A holds for (G, Γ, h), i.e. the maximal relative Baum-Connes assembly map

$$\mu_{max} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{max}(G, \Gamma, A))$$

is injective.

Proof. We have the following commutative diagram

$$\begin{array}{ccc}
KK_{*+1}^{G}(\mathcal{E}G, A) & \xrightarrow{\mu_{max}} & K_{*+1}(C^*_{max}(G, A)) \\
\downarrow & & \downarrow \\
KK_{*+1}^{\Gamma}(\mathcal{E}\Gamma, A) & \xrightarrow{\mu_{max}} & K_{*+1}(C^*_{max}(\Gamma, A)) \\
\downarrow & & \downarrow \\
KK_{*}^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) & \xrightarrow{\mu_{max}} & K_{*}(C^*_{max}(G, \Gamma, A)). \\
\downarrow & & \downarrow \\
KK_{*}^{G}(\mathcal{E}G, A) & \xrightarrow{\mu_{max}} & K_{*}(C^*_{max}(G, A)) \\
\downarrow & & \downarrow \\
KK_{*}^{G}(\mathcal{E}\Gamma, A) & \xrightarrow{\mu_{max}} & K_{*}(C^*_{max}(\Gamma, A))
\end{array}$$

From Theorem 4.1.2, Theorem 4.1.5 and Lemma 4.2.1, we get the result. □

4.3 Case for Groups Coarsely Embeddable into Hilbert Space

The main purpose of this section is to prove

Theorem 4.3.1. Let $h : G \to \Gamma$ be a group homomorphism and A any (G, Γ)-C^*-algebra. Suppose h is injective and Γ admits a coarse embedding into a Hilbert space, then the strong relative Novikov conjecture with coefficients in A holds for (G, Γ, h), i.e. the reduced relative Baum-Connes assembly map

$$\mu_{red} : K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) \to K_*(C^*_{red}(G, \Gamma, A))$$

is injective.

4.3.1 Proper affine actions and negative type functions of transformation groupoids, Tu's theorem

In this section, we shall briefly discuss the concept of proper affine action and its relation to negative type function for transformation groupoids. Let Γ be a countable discrete group. Denote by e its identity element. Assume that Γ acts on the right on a compact Hausdorff space X by homeomorphisms. Recall that the product and the inverse operations of the transformation groupoid $X \rtimes \Gamma$ is given by: $(x, g)(x', g') = (x, gg')$ for all (x, g) and (x', g') in $X \times \Gamma$ satisfying $x' = xg$, and $(x, g)^{-1} = (xg, g^{-1})$ for all $(x, g) \in X \times \Gamma$.

Definition 4.3.2. [39] Let H be a continuous field of Hilbert spaces over X. We say that the transformation groupoid $X \rtimes \Gamma$ acts on H by affine isometries if, for every $(x, g) \in X \times \Gamma$, there is an affine isometry $U_{(x,g)} : H_{xg} \to H_x$ such that

(1) $U_{(x,e)} : H_x \to H_x$ is the identity map;

(2) $U_{(x,g)} U_{(x',g')} = U_(x, gg')$ if $x' = xg$;

(3) for every continuous vector field $h(x)$ in H and every $g \in \Gamma$, $U_{(x,g)}(h(xg))$ is a continuous vector field in H.

Definition 4.3.3. [40] Let $X \rtimes \Gamma$ act on H as above. The action is said to be proper if for any $R > 0$, the number of elements in $\{g \in \Gamma | \exists x \in X s.t. U_{(x,g)}(B_{H_{xg}}(R)) \cap B_{H_x}(R) \neq \emptyset\}$ is finite, where $B_{H_x}(R) := \{h \in H_x | ||h|| \leqslant R\}$.

Let us also recall [40] that $X \rtimes \Gamma$ admits a proper action on a continuous field of affine Hilbert spaces if and only if it admits a continuous, negative type function in the sense of definition below:

Definition 4.3.4. [39] Let $X \rtimes \Gamma$ be a transformation groupoid. A continuous function $\psi : X \times \Gamma \to \mathbb{R}$, is said to be a negative type function if

(1) $\psi(x, e) = 0$ for all $x \in X$;

(2) $\psi(x,g)=\psi(xg,g^{-1})$ for all $(x,g)\in X\times\Gamma$;

(3) $\sum_{i,j=1}^{n} t_i t_j \psi(xg_i, g_i^{-1}g_j) \leqslant 0$ for all $\{t_i\}_{i=1}^{n} \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^{n} t_i = 0$, $g_i \in \Gamma$ and $x \in X$.

Let us also recall the following result of Skandalis-Tu-Yu.

Proposition 4.3.5. [39] Let Γ be a countable group with a proper left-invariant metric d. The following are equivalent:

(i) there exists a uniform embedding $f:\Gamma\to H$;

(ii) there exists a proper negative type function on $\beta\Gamma \rtimes \Gamma$;

(iii) there exists a compact; Hausdorff second countable space Y with an action of Γ which admits a proper negative type function on $Y\rtimes\Gamma$.

The following is a particular case of a theorem of Tu [40] which generalizes a theorem of Higson and Kasparov [22]:

Theorem 4.3.6. [40] Let X be a compact; second countable Hausdorff space. If the transformation groupoid $X\rtimes\Gamma$ acts properly on some continuous field of Hilbert spaces by affine isometries, then the Baum-Connes assembly map

$$\mu : K_*^{\Gamma}(\mathcal{E}\Gamma, C(X)\otimes A) \longrightarrow K_*(C^*_{red}(\Gamma, C(X)\otimes A))$$

is an isomorphism for any separable C^*-algebra A.

4.3.2 Higson's descent map

Suppose from now on that ψ is a continuous proper negative type function on $Y\rtimes\Gamma$. As in [20], we consider $Prob(Y)$, the space of all Borel probability measures on Y with the weak^* topology. Denote $Prob(Y)$ by X. Notice that X is a compact, second countable and Hausdorff space. The Γ action on Y induces a Γ action X. We denote $\varphi : X\times\Gamma\to\mathbb{R}$

by

$$\varphi(m,g)=\int_Y \psi(y,g)dm \tag{4.3.1}$$

for all $(m,g)\in X\times\Gamma$.

The following lemma is taken from [39].

Lemma 4.3.7. [39] φ is a proper negative type function on the transformation groupoid $X\rtimes\Gamma$.

Proof. Condition (1) is clear. Let us verify condition (2). For every $(m,g)\in X\times\Gamma$, we have

$$\begin{aligned}\varphi(mg,g^{-1}) &= \int_Y \psi(y,g^{-1})d(mg)\\ &= \int_Y \psi(yg,g^{-1})dm\\ &= \int_Y \psi(y,g)dm\\ &= \varphi(m,g).\end{aligned}$$

Next, we verify condition (3). If $\{t_i\}_{i=1}^n\subseteq\mathbb{R}$ and $\sum_{i=1}^n t_i=0$, we have

$$\begin{aligned}\sum_{i,j=1}^n t_it_j\varphi(mg_i,g_i^{-1}g_j) &= \sum_{i,j=1}^n t_it_j\int_Y \psi(y,g_i^{-1}g_j)d(mg_i)\\ &= \int_Y (\sum_{i,j=1}^n t_it_j\psi(yg_i,g_i^{-1}g_j))dm\\ &\leqslant 0,\end{aligned}$$

where the last inequality follows from the fact that is ψ a negative type function on the transformation groupoid $Y\rtimes\Gamma$. The properness of φ follows from the definition of φ and the fact that f is a coarse embedding. □

We consider the following Higson's descent diagram:

$$\begin{array}{ccc} K_*^\Gamma(\mathcal{E}\Gamma, A) & \xrightarrow{\mu} & K_*(C^*_{red}(\Gamma, A)) \\ \Big\downarrow \sigma & & \Big\downarrow \sigma' \\ K_*^\Gamma(\mathcal{E}\Gamma, C(X)\otimes A) & \xrightarrow{\mu} & K_*(C^*_{red}(\Gamma, C(X)\otimes A)) \end{array}$$

where the vertical maps σ and σ' are induced by the inclusion of $\mathbb{C}$ into $C(X)$. By Lemma 4.3.7, the transformation groupoid $X \rtimes \Gamma$ acts properly on a continuous field of Hilbert spaces by affine isometries. Hence, by Tu's Theorem 4.3.6, the bottom horizontal map is an isomorphism. By [20, Proposition 3.7], the left vertical map is an isomorphism since, for any finite subgroup H of Γ, X is H-equivariantly homotopy equivalent to a point. It follows from the commutativity of the above diagram that the top horizontal map is split injective. We denote the split map $\sigma^{-1} \circ \mu^{-1} \circ \sigma'$ by ι.

Lemma 4.3.8. Let $h : G \to \Gamma$ be an injective group homomorphism and A any (G, Γ)-C^*-algebra. Assume Γ admits a coarse embedding into a Hilbert space. Then the Baum-Connes assembly maps are split naturally, i.e. the following diagram commutes

$$\begin{array}{ccc} K_*(C^*_{red}(G, A)) & \xrightarrow{h} & K_*(C^*_{red}(\Gamma, A)) \\ \Big\downarrow \iota & & \Big\downarrow \iota' \\ K_*^G(\mathcal{E}G, A) & \xrightarrow{h} & K_*^\Gamma(\mathcal{E}\Gamma, A) \end{array}$$

where ι and ι' are the split maps for G and Γ respectively.

Proof. From proposition Lemma 4.3.7, φ (defined by (4.3.1)) is a proper negative type function on the transformation groupoid $X \rtimes \Gamma$. Since G is a subgroup of Γ, φ is also a proper negative type function on the transformation groupoid $X \rtimes G$.

Hence by Tu's theorem 4.3.6, we have isomorphisms

$$K_*^G(\mathcal{E}G, C(X)\otimes A) \xrightarrow{\mu} K_*(C^*_{red}(G, C(X)\otimes A)),$$

$$K_*^\Gamma(\mathcal{E}\Gamma, C(X)\otimes A) \xrightarrow{\mu} K_*(C^*_{red}(\Gamma, C(X)\otimes A)).$$

Since σ, σ' and μ are natural under injective group homomorphisms, we finish the proof. $\square$

4.3.3 Proof of main result

In this section, we give the proof of Theorem 4.3.1.

Proof. Let A be any (G,Γ)-C^*-algebra, we consider the following commutative diagram

$$\begin{array}{ccc}
K_{*+1}^G(\mathcal{E}G, A) & \xrightarrow{\mu_G} & K_{*+1}(C^*_{red}(G, A)) \\
\Big\downarrow h & & \Big\downarrow h \\
K_{*+1}^\Gamma(\mathcal{E}\Gamma, A) & \xrightarrow{\mu_\Gamma} & K_{*+1}(C^*_{red}(\Gamma, A)) \\
\Big\downarrow i & & \Big\downarrow i \\
K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A) & \xrightarrow{\mu_{G,\Gamma}} & K_*(C^*_{red}(G,\Gamma, A)). \\
\Big\downarrow r & & \Big\downarrow r \\
K_*^G(\mathcal{E}G, A) & \xrightarrow{\mu_G} & K_*(C^*_{red}(G, A)) \\
\Big\downarrow h & & \Big\downarrow h \\
K_*^\Gamma(\mathcal{E}\Gamma, A) & \xrightarrow{\mu_\Gamma} & K_*(C^*_{red}(\Gamma, A))
\end{array}$$

Suppose $x \in K_*^{G,\Gamma}(\mathcal{E}G, \mathcal{E}\Gamma, A)$ satisfies $\mu_{G,\Gamma}(x) = 0$. Since μ_G is injective, we have $r(x) = 0$. By the exactness of left column, there exists an element $y \in K_{*+1}^\Gamma(\mathcal{E}\Gamma, A)$ such that $i(y) = x$.

By commutativity of the diagram, $i \circ \mu_\Gamma(y) = 0$. By the exactness of right column, there exists an element $a \in K_{*+1}(C^*_{red}(G, A))$ such that $h(a) = \mu_\Gamma(y)$.

By Lemma 4.3.8, $h \circ \iota(a) = \iota \circ h(a) = \iota \circ \mu_\Gamma(y) = y$. Hence $x = 0$. $\square$

5. SUMMARY AND CONCLUSIONS

In this dissertation, we have seen that the ideas of the Baum-Connes conjecture and the Novikov conjecture can be transferred to the relative setting. However, the difference is that, in the abstract setting, we don't need to consider the functorial properties of the K-theory of reduced group C^*-algebras which are quite nontrivial; in the relative setting, we have to think of it and replace group C^*-algebras by maximal C^*-algebras which produce many difficulties cause of the existence of property (T). Here we give a broad outline of some future prospects.

5.1 Further Studies

A remarkable approach to solve Novikov conjecture for manifolds without boundaries created by Gennadi Kasparov [27] is to construct, for every closed Γ-invariant subset $Y \subseteq \mathcal{E}\Gamma$ with compact quotient, a proper Γ-C^*-algebra A, elements $\eta \in KK_i^{\Gamma}(\mathbb{C}, A)$ and $d \in KK_i^{\Gamma}(A, \mathbb{C})$ such that $p_Y^*(\eta \otimes_A d) = 1_Y$, where p_Y is the map $Y \to \bullet$, and p_Y^* is the map $KK^{\Gamma}(\mathbb{C}, \mathbb{C}) \to RK_{\Gamma}^0(Y)$. When the kernel of $h : G \to \Gamma$ is finite or trivial, we can use a common proper C^*-algebra for G and Γ. This is essentially what we did in this dissertation. When $\ker(h)$ is infinite, it is impossible to find a common proper C^*-algebra both for G and Γ. In this case, if G is a-T-menable and Γ admits coarse embedding into Hilbert space, a maximal relative group C^*-algebra with proper coefficients can be constructed by means of Dirac-type asymptotic morphism. The formula of Bott map depends on the fact that the γ-element can be homotopic to 1 in $KK_0^G(\mathbb{C}, \mathbb{C})$. Therefore, we can follow the classical method to prove the injectivity of the maximal relative Baum-Connes assembly map. We avoid this process in terms of using a direct algebraic method in this dissertation. Recently, we realize that this method can be generalized to the case when the kernel of h is a-T-menable (infinite or not), and Γ admits a coarse embedding into Hilbert space. In this situation, we can cover the case when both G and Γ are Gromov's hyperbolic groups, and $\ker(h)$ is free-type subgroup of G. Finally, we want to deal with the case when $\ker(h)$ is an infinite property (T) subgroup of G. It is quite subtle for this situation. We will use the naturality of split injectivity to prove the case $h : N \times \Gamma_0 \to \Gamma$, where Γ_0 is a subgroup of Γ, N and Γ admit coarse embedding into Hilbert space. In general, if G is not a product type, a proper C^*-algebra method may reduce it to a family of the product cases. However there are twists induced by index map between two product cases. The index maps in the six-term exact sequence of K-theory can measure how twisty is the fibre bundle $N \to G \xrightarrow{h} \Gamma$. One may ask if there is a characteristic class in terms of K-theories associated to a fibre bundle of groups.

REFERENCES

[1] M. Atiyah. Global theory of elliptic operators. *In Proceedings of the International Symposium on Functional Analysis*, pages 21–30, 1969.

[2] A. Bartels, W. Luck. The Borel Conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math.*, 175:631–689, 2012.

[3] P. Baum, A. Connes. K-theory for discrete groups. *Operator algebras and applications*, 135:1–20, 1988.

[4] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. *Contemp. Math.*, 167:241–291, 1994.

[5] M. Bekka, P. Cherix, A. Valette. Proper affine isometric actions of amenable groups. Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 1–4, London Math. Soc. Lecture Note Ser., 227, Cambridge Univ. Press, Cambridge, 1995.

[6] B. Blackadar. *K-theory for operator algebras*, volume 5 of *MSRI Publications*. Cambridge University Press, 2nd edition, 1998.

[7] L. G. Brown, R. G. Douglas, and P. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. of Math.*, 105:265–324, 1977.

[8] A. Connes. *Noncommutative Geometry,* Academic Press, 1994.

[9] A. Connes. Cyclic cohomology and the transverse fundamental class of a foliation. *Geometric methods in operator algebras*, 123:52–144, 1986.

[10] A. Connes, N. Higson. Deformations, morphismes asymptotiques et K-theorie bivariante. (French) [Deformations, asymptotic morphisms and bivariant K-theory] *C. R. Acad. Sci. Paris Ser. I Math.*, 311(2):101–106, 1990.

[11] A. Connes, H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29(3): 345–388, 1990.

[12] A. Connes, M. Gromov, H. Moscovici. Group cohomology with Lipschitz control and higher signatures. *Geom. Funct. Anal.*, 3 (1): 1–78, 1993.

[13] R. G. Douglas. *Banach Algebra Techniques in Operator Theory,* volume 179 of *Graduate Texts in Mathematics*, Springer, 2nd edition, 1998.

[14] A. Dranishnikov, T. Januszkiewicz. Every Coxeter group acts amenably on a compact space. Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT). Topology Proc. 24 (1999), Spring, 135–141.

[15] M. Gromov. *Hyperbolic groups*. In Gersten, Steve M. Essays in group theory. Mathematical Sciences Research Institute Publications. 8. New York: Springer. pp. 75–263.

[16] M. Gromov. *Asymptotic invariants of infinite groups*. Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[17] M. Gromov. Random walks in random groups. *Geom. Funct. Anal.*, 13(1): 73–146, 2003.

[18] E. Guentner, N. Higson, S. Weinberger. The Novikov Conjecture for Linear Groups. *Publ. Math. Inst. Hautes Études Sci.*, 101: 243–268, 2005.

[19] U. Hamenstädt. Geometry of the mapping class groups. I. Boundary amenability. *Invent. Math.*, 175 (3): 545–609, 2009.

[20] N. Higson. Bivariant K-theory and the Novikov conjecture. *Geom. Funct. Anal.*, 10(3), 563–581, 2000.

[21] N. Higson, E. Guentner. *Group C*-Algebras and K-Theory*, volume 1831 of *Lecture Notes in Mathematics*. Springer, 2004.

[22] N. Higson, G. Kasparov. E -theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1): 23–74, 2001.

[23] N. Higson, J. Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, 2001.

[24] E. Kaniuth. *A course in commutative Banach algebras*, volume 246 of *Graduate Texts in Mathematics*. Springer-Verlag, 2008.

[25] G. Kasparov. Topological invariants of elliptic operators I: K- homology. *Math. USSR-Izv.*, 9(4):751–792, 1975.

[26] G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 44(3):571-636, 1980.

[27] G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1): 147–201, 1988.

[28] G. Kasparov, G. Skandalis. Groups acting properly on "bolic” spaces and the Novikov conjecture. *Ann. of Math.*, 158(1): 165–206, 2003.

[29] G. Kasparov, G. Yu. The Novikov conjecture and geometry of Banach spaces. *Geometry and Topology*, 16(3): 1859–1880, 2012.

[30] Y. Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916), 2008.

[31] S. P. Novikov. *Algebraic construction and properties of Hermitian analogs of k-theory over rings with involution from the point of view of Hamiltonian formalism. Some applications to differential topology and to the theory of characteristic classes.* Izv. Akad. Nauk SSSR, v. 34, 1970 I N2, pp. 253–288; II: N3, pp. 475–500.

[32] S. P. Novikov. Topological invariance of rational classes of Pontrjagin. (Russian) *Dokl. Akad. Nauk SSSR*,163: 298–300, 1965.

[33] P. Nowak, G. Yu. *Large scale geometry.* European Mathematical Society Publishing House, 2012.

[34] H. Oyono-Oyono, G. Yu. Quantitative K-theory and the Kunneth formula for operator algebras. to appear in *Journal of Functional Analysis*, 2019.

[35] Y. Qiao and J. Roe. On the localization algebra of Guoliang Yu. *Forum Math.*, 22(4):657–665, 2010.

[36] J. Roe. Comparing analytic assembly maps. *Q. J. Math.*, 53(2):241–248, 2002.

[37] J. Roe. Hyperbolic groups have finite asymptotic dimension. *Proc. Amer. Math. Soc.*, 133(9): 2489–2490, 2005

[38] Z. Sela. Uniform embeddings of hyperbolic groups in Hilbert spaces. *Israel J. Math.*, 80(1): 171–181, 1992.

[39] G. Skandalis, J. Tu, G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4): 807–834, 2002.

[40] J.-L. Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-theory*, 17:215–264, 1999.

[41] N.E. Wegge-Olsen. *K-theory and C^*-algebras: A friendly approach*. Oxford University Press, 1993.

[42] R. Willett, G. Yu. *Higher index theory.* Book draft, available at https://math.hawaii.edu/ rufus/Skeleton.pdf, 2018.

[43] G. Yu. Localization algebras and the coarse Baum-Connes conjecture. *K-theory*, 11(4):307–318, 1997.

[44] G. Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math.*, 147(2):325–355, 1998.

[45] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139: 201–240, 2000.