TOPOLOGICAL DYNAMICAL SYSTEMS AND REGULARITY PROPERTIES OF REDUCED CROSSED PRODUCT C^*-ALGEBRAS

A Dissertation

by

XIN MA

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Chair of Committee,	David Kerr
Committee Members,	Roger R. Smith
	Kenneth Dykema
	Mohsen Pourahmadi
Head of Department,	Emil J. Straube

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ABSTRACT

In this work, we will explore the relation between topological dynamical systems and their reduced crossed product C^*-algebras. More precisely, we mainly study some dynamical properties and how they imply various of regularity properties of C^*-algebras, say, stably finiteness, pure infiniteness, finite nuclear dimension and $\mathcal{Z}$-stability.

Let $\alpha : G \curvearrowright X$ be a minimal free continuous action of an infinite countable amenable group on an infinite compact metrizable space. Under the hypothesis that the invariant ergodic probability Borel measure space $E_G(X)$ is compact and zero-dimensional, we show that the action α has the small boundary property. This partially answers an open problem in dynamical systems that asks whether a minimal free action of an amenable group has the small boundary property if its space $M_G(X)$ of invariant Borel probability measures forms a Bauer simplex. In addition, under the same hypothesis, we show that dynamical comparison implies almost finiteness, which was shown by Kerr to imply that the crossed product is $\mathcal{Z}$-stable. This also provides two classifiability results for crossed products, one of which is based on the work of Elliott and Niu.

When the group G is not amenable it is possible for action $\alpha : G \curvearrowright X$ not to have a G-invariant probability measure, in which case we show that, under the hypothesis that the action α is topologically free, dynamical comparison implies that the reduced crossed product of α is purely infinite and simple. This result, as an application, shows a dichotomy between stable finiteness and pure infiniteness for reduced crossed products arising from actions satisfying dynamical comparison. We also introduce the concepts of paradoxical comparison and the uniform tower property. Under the hypothesis that the action α is exact and essentially free, we show that paradoxical comparison together with the uniform tower property implies that the reduced crossed product of α is purely infinite. As applications, we provide new results on pure infiniteness of reduced crossed products in which the underlying spaces are not necessarily zero-dimensional.

Finally, we study the type semigroups of actions on the Cantor set in order to establish the equivalence of almost unperforation of the type semigroup and comparison. This sheds a light to a question arising in the paper of Rørdam and Sierakowski. In addition, we construct a semigroup associated to an action of countable discrete group on a compact Hausdorff space, that can be regarded as a higher dimensional generalization of the type semigroup. Using this generalized type semigroup we obtain a new characterization of dynamical comparison. This answers a question of Kerr and Schafhauser. Furthermore, we suggests a definition of comparison for dynamical systems in which neither necessarily the acting group is amenable nor the action is minimal.

DEDICATION

To Lan, whose love and support made everything of me possible.

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1. INTRODUCTION

In this Chapter, we mainly recall some definitions and introduce our basic framework for our study on topological dynamical systems and their crossed product C^*-algebras. We begin with an introduction to regularity properties and the classification theory of C^*-algebras.

1.1 C^*-algebras

1.1.1 Regularity Properties of C^*-algebras

1.1.1.1 Finite Nuclear Dimension

For a general introduction to C^*-algebras, we refer to [9]. The *nuclear dimension* of a C^*-algebra was introduced by Winter and Zacharias in [77] as a noncommutative analogue of covering dimension of a topological space. We recall the definition here. Let A be a C^*-algebra.

(i) A completely positive map φ from a C^*-algebra B to A is said to be of *order zero* if $\varphi(a)\varphi(b) = 0$ whenever a, b are self-adjoint elements in B satisfying $ab = 0$.

(ii) We say that a completely positive map φ from a finite-dimensional C^*-algebra B to A is n*-decomposable* if we can write $B = B_0 \oplus \cdots \oplus B_n$ so that the restriction of φ to each B_i has order zero.

Definition 1.1.1. The nuclear dimension of A, denoted by $\dim_{\mathrm{nuc}}(A)$ is the least integer n such that for every finite set $F \subset A$ and $\epsilon > 0$ there are a finite-dimensional C^*-algebra B, a completely positive contraction $\varphi : A \to B$, and a completely positive map $\psi : B \to A$ such that

(i) $\|\psi \circ \varphi(a) - a\| < \epsilon$ for all $a \in F$, and

(ii) ψ is n-decomposable with contractive order zero components.

1.1.1.2 $\mathcal{Z}$-stability

The Jiang-Su algebra $\mathcal{Z}$ was defined in [31]. It is an infinite-dimensional unital simple separable nuclear C^*-algebra that is KK-equivalent to $\mathbb{C}$. We say that a C^*-algebra A is *$\mathcal{Z}$-stable* if

$A \simeq A \otimes \mathcal{Z}$. A typical example of a $\mathcal{Z}$-stable algebra is $\mathcal{Z}$ itself, i.e., $\mathcal{Z} \otimes \mathcal{Z} \simeq \mathcal{Z}$. We remark that $\mathcal{Z}$-stable C^*-algebras have some every nice properties. We say a pre-ordered group (G, G^+) is *weakly unperforated* if, whenever $x \in G$ has the property that $nx \in G^+$ for some $n \in \mathbb{N}$, x belongs to G^+. In [25], Gong, Jiang and Su showed that, for a simple unital C^*-algebra A, the group $K_0(A \otimes \mathcal{Z})$ is weakly unperforated. Therefore, if A is $\mathcal{Z}$-stable then $K_0(A)$ is weakly unperforated. In addition, they showed that if $K_0(A)$ is weakly unperforated then one has $K_0(A) \simeq K_0(A \otimes \mathcal{Z})$ as pre-ordered groups. Thus, this result implies that if one wants to use K-theory to classify a class consisting of unital simple C^*-algebras with weakly unperforated K_0 groups, the members in this class have to be $\mathcal{Z}$-stable.

1.1.1.3 Strict Comparison and the Cuntz Semigroup

The Cuntz semigroup $W(A)$ of a C^*-algebra A defined on positive elements is an analogue of the semigroup of Murray-von Neumann equivalence classes of projections. The study of the Cuntz semigroup was initiated by Cuntz in [14] and has come to the forefront of research on the classification of C^*-algebras. We recall the definition of the Cuntz semigroup.

For Cuntz comparison, we refer to [3] as a reference. Let A be a C^*-algebra. We write $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$ (viewing $M_n(A)$ as an upper left-hand corner in $M_m(A)$ for $m > n$). Let a, b be two positive elements in $M_n(A)_+$ and $M_m(A)_+$, respectively. Set $a \oplus b = \mathrm{diag}(a, b) \in M_{n+m}(A)_+$, and write $a \precsim_A b$ if there exists a sequence (r_n) in $M_{m,n}(A)$ with $r_n^* b r_n \to a$. If there is no confusion, we omit the subscript A by writing $a \precsim b$ instead. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Then the Cuntz semigroup is defined to be

$$W(A) = M_\infty(A)/\sim$$

equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b.$$

We also denote by $(a-\epsilon)_+$ the element in $M_n(A)$ defined via the functional calculus as $f_\epsilon(a)$ in the C^*-algebra $M_n(A)$ where $f_\epsilon(t) = \max\{t-\epsilon, 0\}$ and $a \in M_n(A)$. A *dimension function* on A is a order preserving map φ from $W(A)$ to $\mathbb{R}^+$ such that $\varphi(\langle 1_A \rangle) = 1$. We write $\mathrm{DF}(A)$ for the set of all dimension functions on A and $\mathrm{LDF}(A)$ for the set of the lower semi-continuous dimension functions.

We denote by $QT(A)$ the set of all normalized 2-quasitraces and by $T(A)$ the tracial states on A. We remark that a celebrated result of Haagerup shows that $QT(A)$ and $T(A)$ coincide if A is exact. For each $\tau \in QT(A)$, we define a lower semi-continuous dimension function by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}).$$

for $a \in M_\infty(A)$. In [6], Blackadar and Handelman proved that the map $\tau \to d_\tau$ from $QT(A)$ to $\mathrm{LDF}(A)$ is bijective.

Strict comparison, roughly speaking, means that lower semi-continuous dimension functions on a C^*-algebra determine the order of elements in the Cuntz semigroup. By the discussion above, we have the following formal definition of strict comparison for unital exact C^*-algebras.

Definition 1.1.2. Let A be a unital exact C^*-algebra. we say A has *strict comparison* if $a \precsim b$ whenever $a, b \in M_\infty(A)_+$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$.

It has been proved by Rørdam in [59] that a simple unital C^*-algebra A has strict comparison if and only if its Cuntz semigroup $W(A)$ is *almost unperforated*, i.e., $(n+1)\cdot\langle a\rangle \leq n\cdot\langle b\rangle$ for some $n \in \mathbb{N}$ implies $\langle a\rangle \leq \langle b\rangle$.

1.1.2 Finiteness of C^*-algebras

A projection p in a C^*-algebra A is said to be *infinite* if it is equivalent in the Murray-Von Neumann sense to a proper subprojection of itself. Otherwise, p is called *finite*. If A is unital, we say A is *infinite* if it contains an infinite projection, and it is called *finite* if 1_A is a finite projection. If $M_n(A)$ are finite for all $n \in \mathbb{N}$ then we say A is *stably finite*.

We denote by $\sim$ the Murray-Von Neumann equivalence relation for projections in A. We can abuse the symbol $\sim$ from the Cuntz equivalence relation because these two equivalence relations coincide on projections. We call a projection p *properly infinite* if there are mutually orthogonal projections $p_1, p_2 \in A$ such that $p_1 + p_2 \leq p$ and $p \sim p_1 \sim p_2$. Note that proper infniteness of projections expresses some paradoxical phenomenon in C^*-algebras. In fact, this paradoxical phenomenon can even be defined on all positive elements. A non-zero positive element a in A is said to be *properly infinite* if $a \oplus a \precsim a$ in the sense of Cuntz subequivalence. A C^*-algebra A is said to be *purely infinite* if there are no characters on A and if, for every pair of positive elements $a, b \in A$ such that b belongs to the closed ideal in A generated by a, one has $b \precsim a$. See [38] and [39]. It was also proved in [38] that a C^*-algebra A is purely infinite if and only if every non-zero positive element a in A is properly infinite.

It is not hard to see that if A is purely infinite then it is traceless. In [14], Cuntz shows that for unital simple C^*-algebras, A is stably finite if and only if $QT(A)$ is not empty. It was thus hoped that the trace/traceless alternative would determine a dichotomy between stably finite and purely infinite unital simple separable and nuclear C^*-algebras. However, Rørdam [56] shattered this hope by providing an example of a unital simple separable nuclear C^*-algebra containing both an infinite and a non-zero finite projection. Nevertheless, Winter and Zacharias showed that every unital simple separable nuclear C^*-algebra having finite nuclear dimension satisfies this dichotomy (see Theorem 5.4 in [77]). We will see in the Chapter 3 that a special class of reduced crossed product C^*-algebras also satisfies this dichotomy.

To end this subsection we record the following proposition, which was proved by Rørdam and Sierakowski in [60].

Proposition 1.1.3. ([60, Proposition 2.1]) Let A be a C^*-algebra and $G \curvearrowright A$ be a C^*-dynamical system with G discrete. Suppose that A separates the ideals in $A \rtimes_r G$. Then $A \rtimes_r G$ is purely infinite if and only if all non-zero positive elements in A are properly infinite in $A \rtimes_r G$ and $E(a) \precsim a$ for all positive elements a in $A \rtimes_r G$, where E is the canonical conditional expectation from $A \rtimes_r G$ to A.

1.1.3 Toms-Winter Conjecture and Elliott's Program

The programme of classifying unital simple separable nuclear C^*-algebras by the Elliott invariant, that is, ordered K-theory paired with traces, has been ongoing for a long time. The origin of this classification programme dates back to Elliott's work on the classification of AF-algebras [18]. In the late 1980's, Elliott [17] extended this result to the classification of A$\mathbb{T}$-algebras with real rank zero. We refer to the survey papers [58] and [76] for general background on the classification programme for separable nuclear C^*-algebras. Nowadays, in order to classify a certain class of separable nuclear C^*-algebras it is often sufficient to show that the algebras in the class have certain regularity properties such as finite nuclear dimension or $\mathcal{Z}$-stability. In 2008 Toms and Winter conjectured that the three properties of strict comparison, finite nuclear dimension, and $\mathcal{Z}$-absorption discussed above are equivalent for unital separable simple infinite-dimensional nuclear C^*-algebras (see [77], for example). As a result of work of several authors, this conjecture, known as the Toms-Winter conjecture, has been fully confirmed under the hypothesis that the extreme tracial states form a compact set with finite covering dimension (see [7], [37], [47], [48], [59], [62], [63], [70], [74] and [75]). In addition, a recent progress by Castillejos-Evington-Tikuisis-White-Winter [11] and [12] shows that finite nuclear dimension is equivalent to $\mathcal{Z}$-stability in general for unital separable simple infinite-dimensional nuclear C^*-algebras.

We write $\mathfrak{C}$ for the class of all stably finite infinite-dimensional unital simple separable nuclear C^*-algebras satisfying UCT and having finite nuclear dimension, which has recently been classified in terms of the Elliott invariant as a consequence of the combined works of Elliott-Gong-Lin-Niu [20], Gong-Lin-Niu [26] and Tikuisis-White-Winter [69].

Theorem 1.1.4. *The class $\mathfrak{C}$ of infinite-dimensional stably finite simple separable unital C^*-algebras satisfying the UCT and having finite nuclear dimension is classified by the Elliott invariant.*

We write $\mathfrak{D}$ for the class of all unital Kirchberg algebras (i.e., separable nuclear simple purely infinite C^*-algebras) satisfying the UCT, which, like the class $\mathfrak{C}$, is classified by the Elliott invariant, as shown by Kirchberg and Phillips (see [50]).

Theorem 1.1.5. *The class $\mathfrak{D}$ of all unital Kirchberg algebras satisfying UCT is classified by the K-theory.*

Theorem 1.1.4 and 1.1.5 are the main tools for classifying reduced crossed products for my purpose. We remark that actually Theorem 1.1.4 and 1.1.5 can be combined into one theorem by deleting the words "stably finite" in the statement of Theorem 1.1.3 because we mentioned above that Theorem 5.4 in [77] asserts that for a unital simple separable C^*-algebra A, if its nuclear dimension is finite then A is either purely infinite or stably finite. In addition, Theorem 7.5 in [77] shows that all C^*-algebras in the class $\mathfrak{D}$ has finite nuclear dimension.

We split this theorem by writing Theorem 1.1.4 and 1.1.5 respectively because stably finiteness and pure infiniteness reflect different natures. If we restrict to the class of reduced crossed products, these different natures are also mirrored in the study of dynamical systems.

1.2 Topological Dynamical Systems

1.2.1 Basic notations

Dynamical systems have been one of the central topics in various of fields of mathematics. An action α of a group G on a topological space X, as a topological dynamical system, is a group homomorphism $G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ denotes the group of all homeomorphism from X to itself. We usually denote by $\alpha : G \curvearrowright X$ for the action α. In this dissertation, we only focus on the case that the group G is countable discrete and the space X is compact and Hausdorff. For general background on topological dynamics on compact Hausdorff spaces we refer to the book of Kerr and Li [35]. We say an action $\alpha : G \curvearrowright X$ is *minimal* if every orbit $G \cdot x$ in X is dense in X. We say $\alpha : G \curvearrowright X$ is *free* if $sx = x$ for some $x \in X$ and $s \in G$ implies $s = e$. There are also many weak version of freeness. An action $\alpha : G \curvearrowright X$ is said to be *essentially free* provided that, for every closed G-invariant subset $Y \subset X$, the subset $\{x \in Y : G_x = \{e\}\}$ of points in Y with trivial isotropy is dense in Y, where $G_x = \{t \in G : tx = x\}$ (see [55]). An action is said to be *topologically free* provided that the set $\{x \in X : G_x = \{e\}\}$ is dense in X, which is equivalent to the fixed point set $\{x \in X : tx = x\}$, of each nontrivial element t of G, being nowhere dense. It is not hard to see that essential freeness means that the action restricted to each G-invariant closed subspace is topologically free with respect to the relative topology and thus these two concepts coincide when the action is minimal.

Motivated by Zimmer's notion of amenability for measurable dynamical systems, Anantharaman-Delaroche introduced the topological analogue [1], namely tolopogical amenability of a topological dynamical systems. We denote by $P(G)$ the set of all probability measures on a group G, which is identified with norm one positive functions in $\ell_1(G)$.

Definition 1.2.1. An action $\alpha : G \curvearrowright X$ is said to be (topologically) *amenable* if there is a net of continuous maps $m_i : X \to P(G)$ with the weak*-topology such that $\|m_i^{sx} - s \cdot m_i^x\|_1 \to 0$ for all $s \in G$, where $s \cdot m_i^x(t) = m_i^x(s^{-1}t)$.

We remark that if the acting group G is amenable then the action α is automatically amenable. We will see in the next subsection that how to define the exactness of an action.

1.2.2 Crossed product C^*-algebras

We refer to [9] as a standard reference for full and reduced crossed product C^*-algebras for C^*-dynamics. Let A be a C^*-algebra on which there is a G-action, which means there is a group homomorphism $G \to \mathrm{Aut}(A)$. We usually denote this action by $G \curvearrowright A$. Given a C^*-dynamical system $\alpha : G \curvearrowright A$, we briefly recall the construction of this kind of C^*-algebras here. We denote by $C_c(G, A)$ the linear space of finitely supported functions on G with values in A, i.e.,

$$C_c(G, A) = \{\sum_{s \in G} a_s s : a_s \in A\},$$

where involved sum is a finite sum. For $S = \sum_{s \in G} a_s s, T \sum_{t \in G} b_t t \in C_c(G, A)$ we declare

$$ST = \sum_{s,t \in G} a_s \alpha_s(b_t) st \text{ and } S^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

To make $C_c(G, A)$ a C^*-algebra, we complete $C_c(G, A)$ with respect to some C^*-norm. A *covariant representation* (u, π, H) of the G-C^*-algebra A consists of a unitary representation (u, H)

of G and a $*$-representation (π, H) of A such that $u_s\pi(a)u_s^* = \pi(\alpha_s(a))$ for every $s \in G$ and $a \in A$. This induces a $*$-representation of $C_c(G, A)$ by

$$\sum_{s\in G} a_s s \longrightarrow \sum_{s\in G} \pi(a_s)u_s,$$

which is denoted by $u \times \pi$. Note that every $*$-representation of $C_c(G, A)$ arises this way.

Definition 1.2.2. ([9, Definition 4.1.2]) The *full crossed product* of a C^*-dynamical system $\alpha : G \curvearrowright A$, denoted by $A \rtimes G$, is the completion of $C_c(G, A)$ with respect to the norm

$$\|x\| = \sup \|(u \times \pi)(x)\|,$$

where the supremum is over all covariant representations (u, π, H) of $\alpha : G \curvearrowright A$.

To define the reduced crossed product, we first fix a fuithful representation $A \subset B(H)$. Define a new representation of A on $H \otimes \ell_2(G)$ by

$$\pi(a)(h \otimes \delta_s) = (\alpha_{s^{-1}}(a)(h)) \otimes \delta_s,$$

where $\{\delta_s\}_{s\in G}$ is the canonical orthonormal basis of $\ell_2(G)$. Denote by λ the left regular representation of G on $\ell_2(G)$, i.e., $\lambda_s(\delta_t) = \delta_{st}$. Then it can be verified that $(1 \otimes \lambda) \times \pi$ is a covariant representation of $\alpha : G \curvearrowright A$, which is called a *regular representation*.

Definition 1.2.3. ([9, Definition 4.1.4]) The *reduced crossed product* of a C^*-dynamical system $\alpha : G \curvearrowright A$, denoted by $A \rtimes_r G$, is defined to be the norm closure of the image of a regular representation $C_c(G, A) \to B(H \otimes \ell_2(G))$.

In particular, note that an action $\alpha : G \curvearrowright X$ induces an action $\alpha' : G \curvearrowright C(X)$ by $\alpha'_s(f)(x) = f(\alpha_{s^{-1}}(x))$. The converse also holds by Gelfand duality, i.e., every action of G on $C(X)$ also induces an action of G on X. Thus, beginning with an action $\alpha : G \curvearrowright X$, we have the full crossed product $C(X) \rtimes G$ and the reduced crossed product $C(X) \rtimes_r G$ following the process described above. In this dissertation, we mainly study reduced crossed product C^*-algebras. However, they coincide when $C(X) \rtimes_r G$ is nuclear, which is the case that we are interested most. The following result shows when $C(X) \rtimes_r G$ is nuclear.

Proposition 1.2.4. ([9, Theorem 4.4.3]) The action $\alpha : G \curvearrowright X$ is amenable if and only if its reduced crossed product $C(X) \rtimes_r G$ is nuclear.

For other properties of crossed products, it is well known that if the action $G \curvearrowright X$ is topologically free and minimal then the reduced crossed product $C(X) \rtimes_r G$ is simple (see [4]). Archbold and Spielberg [4] showed that $C(X) \rtimes G$ is simple if and only if the action is minimal, topologically free and *regular* (meaning that the reduced crossed product coincides with the full crossed product). These imply that $C(X) \rtimes_r G$ is simple and nuclear if and only if the action is minimal, topologically free and amenable. See more details in the introduction of [60].

By using the crossed products, it is also possible to define exactness of an action. For every G-invariant ideal I in A, the natural maps in the following short exact sequence:

$$(*) \qquad 0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\rho} A/I \longrightarrow 0$$

extend canonically to maps at the level of reduced crossed products, giving rise to the possibly non-exact sequence

$$(\star) \qquad 0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes_r id} A \rtimes_r G \xrightarrow{\rho \rtimes_r id} A/I \rtimes_r G \longrightarrow 0$$

(see [73, Remark 7.14]). The action of G on A is said to be *exact* if $(\star)$ is exact for all G-invariant closed two-sided ideals in A ([65, Definition 1.5]).

In particular, suppose $A = C(X)$ is unital and commutative. We call the action $\alpha : G \curvearrowright X$ exact if the induced action $\alpha' : G \curvearrowright C(X)$ is exact. If G is exact then it can be verified that α' defined above is always exact.

Definition 1.2.5. ([65]) A C^*-algebra A is said to separate the ideals in $A \rtimes_r G$ if the (surjective) map $J \to J \cap A$, from the ideals in $A \rtimes_r G$ into the G-invariant ideals in A is injective.

It was shown in [65] that if $C(X)$ separates ideals in $C(X) \rtimes_r G$ then the induced action of G on $C(X)$ must be exact. In the converse direction, it was also shown in [65] that if the action $\alpha : G \curvearrowright X$ is exact and essentially free then $C(X)$ separates ideals in $C(X) \rtimes_r G$.

1.2.3 Dynamical Comparison

Dynamical comparison is a well-known dynamical analogue of strict comparison from the C^*-setting. The idea dates back to Winter in 2012 and was discussed in [10] and [34]. We record here the version that appeared in [34].

We write $M(X)$ for the convex set of all regular Borel probability measures on X, which is a weak* compact subset of $C(X)^*$. We write $M_G(X)$ for the convex set of G-invariant regular Borel probability measures on X, which is a weak* compact subset of $M(X)$. We write $E_G(X)$ for the set of extreme points of $M_G(X)$, which are precisely the ergodic measures in $M_G(X)$.

Definition 1.2.6. ([34, Definition 3.1]) Let $m \in \mathbb{N}$. Let F be a closed subset of X and O an open subset of X. We write $F \prec_m O$ if there exists a finite collection $\mathcal{U}$ of open subsets of X which cover F, an $s_U \in G$ for each $U \in \mathcal{U}$, and a partition $\mathcal{U} = \bigsqcup_{i=0}^{m} \mathcal{U}_i$ such that for each $i = 0, 1, \ldots, m$ the images $s_U U$ for $U \in \mathcal{U}_i$ are pairwise disjoint subsets of O. When $m = 0$ we also write $F \prec O$. Now, let A, B be open sets in X. We write $A \prec_m B$ if for every closed set $F \subset A$ one has $F \prec_m B$.

Definition 1.2.7. ([34, Definition 3.2]) Let $m \in \mathbb{N}$. The action $\alpha : G \curvearrowright X$ is said to have *dynamical m-comparison* (*m-comparison* for short) if $A \prec_m B$ for all open sets $A, B \subset X$ satisfying $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. When $m = 0$, we will also say that the action has *dynamical comparison* (*comparison* for short).

We will see in Chapter 2 and 3 that dynamical comparison is an essential property for establishing certain structure theorems for reduced crossed product C^*-algebras. Then a natural question is to determine when an action has comparison. Before the formal definition of comparison, it was well-known that all minimal $\mathbb{Z}$-actions on the Cantor set have this property as a consequence of the Kakutani-Rokhlin clopen tower partition (see [24]). More recently, Downarowicz and Zhang

[16] showed that all continuous actions on the Cantor set of groups whose every finitely generated subgroup has subexponential growth have comparison. On the other hand, it is still open whether all continuous actions on the Cantor set of amenable countable infinite groups have comparison. However, by combining Theorem A in [36] and Theorem 4.2 in [13], the property of comparison is generic for minimal free actions of a fixed amenable countable infinite group on the Cantor set. In the setting of non-amenable groups, when there is no invariant measure for the action, we will see in Chapter 3 that the strong boundary actions introduced in [41] and n-filling actions introduced in [32] are natural examples of dynamical comparison.

On the other hand, Definition 1.2.8 behaves well only when G is amenable or α is minimal. The following provides a generalized version of dynamical comparison regardless of the amenability of the groups or the minimality of the actions. Theorem 4.3.7 and Corollary 4.3.8 in the final chapter will validate this generalization.

Recall that a *premeasure* μ on an algebra $\mathcal{A}$ of sets is a function $\mu : \mathcal{A} \to [0, \infty]$ satisfying the following (see [22, p. 30])

(i) $\mu(\emptyset) = 0$;

(ii) $\mu(\bigsqcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for any disjoint sequence $\{A_n \in \mathcal{A} : n \in \mathbb{N}^+\}$ such that $\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Note that the classical theorem of Carathéodory states that each premeasure μ on an algebra $\mathcal{A}$ extends to a measure on the σ-algebra generated by $\mathcal{A}$ (see [22, Theorem 1.14]). In addition, if μ is σ-finite then the extension is unique.

Throughout $\mathcal{A}_0$ denotes the algebra generated by the open sets in X. We denote by $\mathrm{Pr}_G(X)$ the set of all G-invariant premeasures on $\mathcal{A}_0$ which are regular, i.e., having inner regularity $\mu(B) = \sup\{\mu(F) : F \subset B, F \text{ compact}\}$ and outer regularity $\mu(B) = \inf\{\mu(O) : B \subset O, O \text{ open}\}$ for all $B \in \mathcal{A}_0$. We say a premeasure $\mu \in \mathrm{Pr}_G(X)$ a *probability premeasure* if $\mu(X) = 1$. We remark that the extension of a premeasure $\mu \in \mathrm{Pr}_G(X)$ in the sense of Carathéodory is still G-invariant. If $\mu \in \mathrm{Pr}_G(X)$ is σ-finite then the unique extension is regular as well.

Definition 1.2.8. The action $\alpha : G \curvearrowright X$ is said to have (*generalized*) *dynamical comparison* if $A \prec B$ holds for all open sets $A, B \subset X$ satisfying

(i) $A \subset G \cdot B$;

(ii) $\mu(B) > 0$ for all probability premeasures $\mu \in \mathrm{Pr}_G(X)$;

(iii) $\mu(A) < \mu(B)$ for every $\mu \in \mathrm{Pr}_G(X)$ with $\mu(B) = 1$.

1.2.4 Almost Finiteness

In [34] Kerr introduced the following dynamical concept of (m-)almost finiteness as a higher dimensional generalization of Matui's notion of the same name. This can be viewed as a topological version of the Ornstein-Weiss tower decomposition.

Definition 1.2.9. ([34, Definition 4.1]) A *tower* is a pair (S, V) consisting of a subset V of X and a finite subset S of G such that the sets sV for $s \in S$ are pairwise disjoint. The set V is the *base* of the tower, the set S is the *shape* of the tower and the sets sV for $s \in S$ are the levels of the tower. We say that the tower (S, V) is *open* if V is open. A finite collection of towers $\{(S_i, V_i) : i \in I\}$ is called a *castle* if $S_iV_i \cap S_jV_j = \emptyset$ for all $i \neq j \in I$.

The *chromatic number* of a family $\mathcal{C}$ of subsets of a given set is defined to be the least $d \in \mathbb{N}$ such that there is a partition of $\mathcal{C}$ into d subcollections each of which is disjoint.

Definition 1.2.10. ([34, Definition 11.2]) Let $m \in \mathbb{N}$. We say that a free action $\alpha : G \curvearrowright X$ is *m-almost finite* if for every $n \in \mathbb{N}$, finite set $K \subset G$, and $\delta > 0$ there are a finite collection $\{(S_i, \overline{V_i}) : i \in I\}$ of towers with following properties:

(i) V_i is an open subset of X for every $i \in I$;

(ii) S_i is (K, δ)-invariant for every $i \in I$;

(iii) $\mathrm{diam}(s\overline{V_i}) < \delta$ for every $i \in I$ and $s \in S_i$ and the family $\{S_i\overline{V_i} : i \in I\}$ has chromatic number at most $m + 1$;

(iv) there are sets $S_i' \subset S_i$ for each $i \in I$ such that $|S_i'| \leq |S_i|/n$ and $X \setminus \bigcup_{i\in I} S_iV_i \prec \bigcup_{i\in I} S_i'V_i$.

If $m = 0$, we say $\alpha : G \curvearrowright X$ is *almost finite* for short. In this case $\{(S_i, V_i) : i \in I\}$ is a castle.

Note that the Definition 1.2.11 here seems to be stronger than the definition of almost finiteness in [34] in which all towers are open. However, it can be shown that they are actually equivalent. We remark that it has been proved in [34] that they are equivalent when $m = 0$. In general, first we fix a metric d on X. Given $n \in \mathbb{N}$, finite $K \subset G$, and $\delta > 0$, suppose that we have an open castle $\{(S_i, V_i) : i \in I\}$ satisfying the conditions of m-almost finiteness above. We start from condition (iv) and write F for the set $X \setminus \bigsqcup_{i\in I} S_iV_i$ for simplicity. Since $F \prec \bigsqcup_{i\in I} S_i'V_i$ holds for the original castle, there are open subsets $O_1, \ldots, O_n$ of X and group elements $g_1, \ldots, g_n \in G$ such that $F \subset \bigcup_{i=1}^n O_n$ and $\bigsqcup_{i=1}^n g_iO_i \subset \bigsqcup_{i\in I} S_i'V_i$. A partition of unity argument allows us to find open subset $W_j \subset O_j$ such that $\overline{W_j} \subset O_j$ for each $j = 1, 2, \ldots, n$ and $\{\overline{W_j} : j = 1, 2, \ldots, n\}$ still forms a cover of F. This allows us to find a $\delta > 0$ to define a new open subset $O_j' = \{x \in X : d(x, X \setminus O_j) > \delta\}$ such that $\overline{W_j} \subset O_j' \subset O_j$ for each $j = 1, 2, \ldots, n$. This implies that $\{O_j' : j = 1, 2, \ldots, n\}$ also forms a cover of F and thus there is another $\delta' > 0$ such that $B(F, \delta') = \{x \in X : d(x, F) < \delta'\} \subset \bigcup_{j=1}^n O_j'$. Then by the definition of O_j' and the uniformly continuity of homeomorphisms induced by $g_1^{-1}, \ldots, g_n^{-1}$ there is a $\gamma > 0$ such that $d(g_jx, X \setminus g_jO_j) > \gamma$ for all $x \in O_j'$ and all $j = 1, 2, \ldots, n$. Thus one has

$$d(\bigsqcup_{j=1}^n g_jO_j', X \setminus \bigsqcup_{i\in I} S_i'V_i) \geq d(\bigsqcup_{j=1}^n g_jO_j', X \setminus \bigsqcup_{j=1}^n g_jO_j) \geq \gamma.$$

For an $\eta > 0$ and an open set U we write $U^{-\eta} = \{x \in X : d(x, X \setminus U) > \eta\}$ for the open subset of U shrunken by η. Observe that for each $i \in I$, there is a $\eta_i > 0$ such that $V_i \setminus V_i^{-\eta_i} \subset B(F, \delta')$. Then by uniform continuity one can find an $\eta > 0$ such that $X \setminus \bigsqcup_{i\in I} S_iV_i^{-\eta} \subset B(F, \delta'/2)$ by shrinking all η_i if necessary. In addition, by the same reason one can shrink η furthermore so that $X \setminus \bigsqcup_{i\in I} S_i'V_i^{-\eta} \subset B(X \setminus \bigsqcup_{i\in I} S_i'V_i, \gamma/2)$. This entails that $X \setminus \bigsqcup_{i\in I} S_iV_i^{-\eta} \subset \bigcup_{j=1}^n O_j'$ while $\bigsqcup_{j=1}^n g_jO_j' \subset \bigsqcup_{i\in I} S_i'V_i^{-\eta}$. This verifies that the new castle $\{(S_i, \overline{V_i^{-\eta}}) : i \in I\}$ satisfies condition (iv). In addition we see that this new castle satisfies the other conditions of m-almost finiteness above trivially and thus $\{(S_i, \overline{V_i^{-\eta}}) : i \in I\}$ is what we want. The converse direction is trivial.

Using the above almost finiteness, Kerr showed the following theorem, which shows that the notion of almost finiteness can be regarded as a dynamical analogue of $\mathcal{Z}$-stability from the C^*-algebra world.

Theorem 1.2.11. ([34, Theorem 12.4]) *Let $\alpha : G \curvearrowright X$ be a minimal free action of a amenable group. If α is almost finite, then the crossed product $C(X) \rtimes_r G$ is $\mathcal{Z}$-stable.*

1.2.5 The Small Boundary Property

The small boundary property was introduced by Lindenstrauss and Weiss in [45]. We record its definition here.

Definition 1.2.12. An action $\alpha : G \curvearrowright X$ is said to have the *small boundary property* if for every point $x \in X$ and every open $U \ni x$ there is an open neighborhood $V \subset U$ of x such that $\mu(\partial V) = 0$ for every $\mu \in M_G(X)$.

The small boundary property implies mean dimension zero in general, and particularly is equivalent to mean dimension zero for $\mathbb{Z}$-actions [44] and $\mathbb{Z}^d$-actions [28] with marker property and thus in particular this equivalence holds if the action is minimal and free. However it is still open for actions of a general amenable group whether mean dimension zero and the small boundary property are equivalent.

In addition, the relationship between the small boundary property and the structure of $E_G(X)$ has been studied for a long time. It was proved in [45] and [64] that if the invariant ergodic probability Borel measure space $E_{\mathbb{Z}}(X)$ of an action $\alpha : \mathbb{Z} \curvearrowright X$ is at most countable then the action has the small boundary property. However, it is a general open problem in dynamical systems whether a minimal free action of an amenable group has the small boundary property if its space $M_G(X)$ of invariant Borel probability measures forms a Bauer simplex, that is, $E_G(X) = \partial_e M_G(X)$ is compact in the weak*-topology. In addition, the small boundary property also plays an important role in the recent work of Elliott and Niu [21] on crossed products induced by minimal $\mathbb{Z}$-actions. It is proved in [21] that if such a $\mathbb{Z}$-action has the small boundary property then the crossed product is $\mathcal{Z}$-stable. Motivated from these two perspectives it is worth investigating when minimal free actions have the small boundary property. We will address this question in Chapter 2.

2. DYNAMICAL SYSTEMS OF AMENABLE GROUPS *

In this chapter, we study minimal free actions of amenable groups and their crossed products. We first prove the following key lemma. This shows that for every finite disjoint collection of closed subsets of $E_G(X)$ we can find disjoint collections of closed subsets of X that correspond to it in a nice way. This decomposition result in the following section is essential to establish our main theorems in this chapter.

2.1 Decomposition of Ergodic Invariant Probability Measures

We recall the notion of central sequence algebra. Let A be a separable C^*-algebra. Set

$$A_\infty = \ell^\infty(\mathbb{N}, A)/\{(a_n)_n \in \ell^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\}.$$

We identify A with the C^*-subalgebra of A_∞ consisting of equivalence classes of constant sequences. We call $A_\infty \cap A'$ the *central sequence algebra* of A, which consists of all equivalence classes whose representatives $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ satisfy $\|[x_n, a]\| \to 0$ as $n \to \infty$ for all $a \in A$. Each such representing sequence $(x_n)_n$ is called a central sequence.

The following lemma is due to Lin [42] based on work of Cuntz and Pedersen [15]. This lemma enables us to realize strictly positive elements of $\mathrm{Aff}(T(A))$ via positive elements of A.

Lemma 2.1.1. ([42, Theorem 9.3]) *Let A be a simple separable unital nuclear C*-algebra such that $T(A) \neq \emptyset$ and let f be a strictly positive affine continuous function on $T(A)$. Then for any $\epsilon > 0$, there exists $x \in A^+$ with $f(\tau) = \tau(x)$ for all $\tau \in T(A)$ and $\|x\| \leq \|f\| + \epsilon$.*

The following lemma is due to Toms, White and Winter [70].

Lemma 2.1.2. ([70, Lemma 3.4]) *Let A be a separable unital C*-algebra with non-empty trace space $T(A)$. Let $T_0 \subset T(A)$ be non-empty and suppose that $(e_n^1)_n, \ldots, (e_n^L)_n$ are sequences of pos-*

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itive contractions in A_+ *representing elements of* $A_\infty \cap A'$ *such that* $\lim_{n\to\infty} \sup_{\tau \in T_0} |\tau(e_n^{(l)} e_n^{(l')})| = 0$ *for* $l \neq l'$. *Then there exist positive elements* $\tilde{e}_n^{(l)} \leq e_n^{(l)}$ *so that:*

(i) $(\tilde{e}_n^{(l)})_n$ *represents an element of* $A_\infty \cap A'$*;*

(ii) $\lim_{n\to\infty} \sup_{\tau\in T_0} |\tau(\tilde{e}_n^{(l)} - e_n^{(l)})| = 0$*;*

(iii) $\tilde{e}_n^{(l)} \perp \tilde{e}_n^{(l')}$ *in* $A_\infty \cap A'$ *for* $l \neq l'$.

Denote by A the reduced crossed product C^*-algebra $C(X) \rtimes_r G$ arising from a free action. For every measure μ in $M_G(X)$, the function τ_μ defined on A by $\tau_\mu(a) = \int_X E(a)\, d\mu$ is a tracial state on A, where E is the canonical faithful conditional expectation from A onto $C(X)$. In the converse direction, every tracial state induces an invariant measure on X by restricting to $C(X)$. Actually Theorem 15.22 in [51] shows that the function $H : M_G(X) \to T(A)$ defined by $H(\mu) = \tau_\mu$ is an affine bijection and it is not hard to see H is actually an affine homeomorphism with respect to the weak*-topology. Therefore, we will usually identify the spaces $M_G(X)$ and $T(A)$. In addition, $E_G(X)$ and $\partial_e T(A)$ correspond to each other under the same map.

Lemma 2.1.3. *Let* $\alpha : G \curvearrowright X$ *be a minimal free action such that* $E_G(X)$ *is compact in the weak*-topology. Then for every* $\epsilon > 0$ *and set* $W = \bigsqcup_{j=1}^{L} W_j$ *which is a disjoint union of closed subsets of* $E_G(X)$*, there are pairwise disjoint compact subsets* $\{K_j\}_{j=1}^{L}$ *of* X *such that* $\mu(K_j) > 1 - \epsilon$ *for all* $\mu \in W_j$.

Proof. Given an $\epsilon > 0$ and denote by A the C*-algebra $C(X) \rtimes_r G$, which is simple since the action is minimal and free. We write $H : M_G(X) \to T(A)$ for the homeomorphism defined by $\tau_\mu = H(\mu)$ such that $\tau_\mu(a) = \int_X E(a)\, d\mu$. Note that $\partial_e T(A) = H(E_G(X))$ is compact under the weak*-topology. We also define $V_j = H(W_j)$ for all $j = 1, 2, \ldots, L$, which are closed subsets of $\partial_e T(A)$. For each $j = 1, 2, \ldots, L$ and $n \in \mathbb{N}^+$, choose a strictly positive continuous function $f_n^j : \partial_e T(A) \to [0, 1]$ with the norm $\|f_n^j\| = 1 + 1/n$ such that $f_n^j = 1 + 1/n$ on V_j and $f_n^j = 1/n$ on $\bigsqcup_{j' \neq j} V_{j'}$. This is possible by Urysohn's lemma as the $V_1, \ldots, V_L$ are pairwise disjoint closed subsets of $\partial_e T(A)$. Since $\partial_e T(A)$ is compact, for each j and n we can extend f_n^j to a strictly positive continuous affine function on $T(A)$ with the same norm, which we also denote by f_n^j. Now apply Lemma 2.5 to obtain a sequence $(e_n^j)_n$ of positive elements of A such that $\|e_n^j\| \leq 1 + 2/n$ and

$$f_n^j(\tau) = \tau(e_n^j)$$

for all $\tau \in T(A)$. Define the functions $h_n^j = E(e_n^j)$ on X, where E is the faithful conditional expectation from A onto $C(X)$. Observe that $\|h_n^j\| \leq \|e_n^j\| \leq 1+2/n$. Now, since $\tau(e_n^j) = \tau(h_n^j)$ for all $\tau \in T(A)$, we have

$$f_n^j(\tau) = \tau(h_n^j)$$

for all $\tau \in T(A)$. Define $g_n^j = \frac{h_n^j}{1+2/n}$. Then for each $n \in \mathbb{N}^+$ one has

$$\tau(g_n^j) = (n+1)/(n+2)$$

for every $\tau \in V_j$ while

$$\tau(g_n^j) = 1/(n+2)$$

for every $\tau \in \bigsqcup_{j' \neq j} V_{j'}$.

Therefore, for the given ϵ, for each $j = 1, \ldots, L$ there is an N_j such that $\tau(g_n^j) > 1 - \epsilon$ whenever $\tau \in V_j$ and $n > N_j$. In addition, for $1 \leq j, j' \leq L$ with $j \neq j'$ one has

$$\lim_{n \to \infty} \sup_{\tau \in \bigsqcup_{j=1}^L V_j} \tau(g_n^j g_n^{j'}) = 0.$$

Now apply Lemma 2.1.3 to the abelian C^*-algebra $C(X)$ with $T_0 = \bigsqcup_{j=1}^L V_j$ and sequences $(g_n^1)_n, \ldots, (g_n^L)_n$ (they are trivially central since $C(X)$ is abelian). Then we have sequences $(\tilde{g}_n^1)_n, \ldots, (\tilde{g}_n^L)_n$ such that for each $1 \leq j \neq j' \leq L$ one has

(i) $\tilde{g}_n^j \leq g_n^j$;

(ii) $\lim_{n \to \infty} \|\tilde{g}_n^j \tilde{g}_n^{j'}\| = 0$;

(iii) $\lim_{n\to\infty} \sup_{\tau\in\bigsqcup_{j=1}^{L} V_j} |\tau(\tilde{g}_n^j - g_n^j)| = 0$.

Thus we may assume $\lim_{n\to\infty} \|g_n^j g_n^{j'}\| = 0$ by replacing g_n^j with $\tilde{g}_n^j$. Then for each pair $1 \leq j \neq j' \leq L$, there is an $M_{j,j'} \in \mathbb{N}$ such that $\|g_n^j g_n^{j'}\| < \epsilon^2$ whenever $n > M_{j,j'}$.

For the given $\epsilon > 0$, choose an $n > \max\{N_j, M_{j,j'} : 1 \leq j \neq j' \leq L\}$ so that for all $j, j' = 1, 2, \ldots, L$ and $\tau \in V_j$ one has $\tau(g_n^j) > 1 - \epsilon$ and $\|g_n^j g_n^{j'}\| < \epsilon^2$ if $j \neq j'$. Define $K_j = \{x \in X \colon g_n^j(x) \geq \epsilon\}$ for $j = 1, 2, \ldots, L$. The sets $K_1, \ldots, K_L$ are pairwise disjoint since $x \in K_j \cap K_{j'}$ implies $g_n^j(x) g_n^{j'}(x) \geq \epsilon^2$, which is impossible. We write $U_j = \{x \in X \colon g_n^j(x) > 0\}$ for $j = 1, 2, \ldots, L$. Then for each $\mu \in W_j$ we have the inequality

$$\tau_\mu(g_n^j) = \int_X g_n^j \, d\mu = \int_{K_j} g_n^j \, d\mu + \int_{U_j \setminus K_j} g_n^j \, d\mu > 1 - \epsilon$$

while $\int_{U_j \setminus K_j} g_n^j \, d\mu \leq \epsilon \cdot \mu(U_j \setminus K_j) \leq \epsilon$. This implies that $\mu(K_j) = 1 \cdot \mu(K_j) \geq \int_{K_j} g_n^j \, d\mu > 1 - 2\epsilon$. □

2.2 Dynamical Comparison and Almost Finiteness

In this section, we address the relationship between m-almost finiteness and dynamical m-comparison. Note that for a fixed open subset O of X, the function f on $M(X)$ given by $f \colon \mu \to \mu(O)$ is lower semicontinuous. Similarly, if F is closed, f defined on $M(X)$ by $f \colon \mu \to \mu(F)$ is upper semicontinuous. The following lemma is a slightly stronger version of Lemma 9.1 in [34].

Lemma 2.2.1. *Let X be a compact metrizable space with a compatible metric d and let Ω be a weak* closed subset of $M(X)$. Let $\lambda > 0$. Let A be a closed subset of X such that $\mu(A) < \lambda$ for all $\mu \in \Omega$. Then there is a $\delta_0 > 0$ such that*

$$\mu(\{x \in X \colon d(x, A) \leq \delta_0\}) < \lambda$$

for all $\mu \in \Omega$.

Proof. For each $\delta > 0$ set $N_\delta = \{x \in X : d(x, A) \leq \delta\}$. Then for every $\mu \in \Omega$, $\mu(A) < \lambda$ implies that there is a $\delta > 0$ such that $\mu(N_\delta) < \lambda$. Now, write $O_\delta = \{\mu \in M(X) : \mu(N_\delta) < \lambda\}$.

Then $\{O_\delta : \delta > 0\}$ is an open cover of Ω since $\mu(N_\delta)$ is an upper-semicontinuous function of μ as mentioned above. By the compactness of Ω, one has $\Omega \subset \bigcup_{i=1}^n O_{\delta_i}$ for some subcover $\{O_{\delta_i} : i = 1, 2, \ldots, n\}$. Let $\delta_0 = \min\{\delta_i : i = 1, 2, \ldots, n\}$. It follows that $\Omega \subset O_{\delta_0}$ and thus $\mu(N_{\delta_0}) < \lambda$ for all $\mu \in \Omega$. □

The following lemma allows us to adjust the collection of Borel towers arising in the Ornstein-Weiss tiling argument (Theorem 4.46 in [35]) to be a castle of a form that appears in the definition of m-almost finiteness.

Lemma 2.2.2. *Let $\alpha\colon G \curvearrowright X$ be a free action. Fix a $\mu \in M_G(X)$ and an integer $n \in \mathbb{N}$. For every finite subset $F \subset G$ and $\epsilon, \eta > 0$, there is a castle $\{(T_k, \overline{V_k}) : k = 1, 2, \ldots, K\}$ such that for each k, V_k is open, T_k is (F, η)-invariant while $\operatorname{diam}(\overline{sV_k}) < \eta$ for all $s \in T_k$, $\mu(\bigsqcup_{k=1}^K T_k V_k) > 1 - \epsilon$ and the interval $[\frac{1}{2n}|T_k|, \frac{1}{n}|T_k|]$ contains an integer d_k.*

Proof. Since the action $\alpha\colon G \curvearrowright X$ is free, for all $x \in X$, one has $\mu(\{x\}) = 0$ and thus μ is atomless. Now, the Ornstein-Weiss theorem (Theorem 4.46 in [35]) implies that there is a castle $\{(T_k, B_k) : k = 1, 2, \ldots, K\}$ such that the shapes T_k are (F, η)-invariant and the bases B_k are Borel for all $k = 1, 2, \ldots, K$ with $\mu(\bigsqcup_{k=1}^K T_k B_k) > 1 - \epsilon/2$. Since G is infinite, we may enlarge F and shrink η sufficiently so that for each $k \leq K$ there is an integer d_k in $[\frac{1}{2n}|T_k|, \frac{1}{n}|T_k|]$.

By uniform continuity, there is an $0 < \eta' < \eta$ such that for all $s \in \bigcup_{k=1}^K T_k$ and $x, y \in X$, if $d(x, y) < \eta'$, then $d(sx, sy) < \eta$. For each B_k, there is an open cover of $\overline{B_k}$, say $\{O_{i,k} : i \in I_k\}$, such that $\operatorname{diam}(O_{i,k}) < \eta'/2$ for every $i \in I_k$. Then by compactness there is a finite subcover of $\overline{B_k}$, say $\overline{B_k} \subset \bigcup_{i=1}^{n_k} O_{i,k}$. Write $D_{i,k} = O_{i,k} \setminus \bigcup_{j=1}^{i-1} O_{j,k}$ and $C_{i,k} = B_k \cap D_{i,k}$, the latter of which satisfies $\operatorname{diam}(C_{i,k}) < \eta'/2$. Taking the sets $C_{i,k}$ now to be bases, we have a castle $\{(T_k, C_{i,k}) : i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K\}$, which satisfies $\mu(\bigsqcup_{k=1}^K \bigsqcup_{i=1}^{n_k} T_k C_{i,k}) > 1 - \epsilon/2$. For each i and k, there is a compact set $M_{i,k} \subset C_{i,k}$ such that $\mu(C_{i,k} \setminus M_{i,k}) < \frac{\epsilon}{2\sum_{k=1}^K n_k |T_k|}$ and hence $\mu(\bigsqcup_{k=1}^K \bigsqcup_{i=1}^{n_k} T_k M_{i,k}) > 1 - \epsilon$.

We enlarge each $M_{i,k}$ to an open set $N_{i,k}$ such that $\operatorname{diam}(N_{i,k}) < \eta'$ and $\{(T_k, N_{i,k}) : i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K\}$ is a castle. To do this, by normality, for the disjoint family $\{sM_{i,k} :$

$s \in T_k, i \leq n_k, k \leq K\}$, we can first find another disjoint family $\{U_{s,i,k} \supset sM_{i,k} : s \in T_k, i \leq n_k, k \leq K\}$. Then for each $i \leq n_k$ and $k \leq K$, one can define $N_{i,k} = \{x \in X : d(x, M_{i,k}) < \eta'/2\} \cap (\bigcap_{s \in T_k} s^{-1}U_{s,i,k})$. Furthermore, for each pair (i,k), there is a $V_{i,k}$ such that $M_{i,k} \subset V_{i,k} \subset \overline{V_{i,k}} \subset N_{i,k}$. The castle $\{(T_k, V_{i,k}) : i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K\}$ is now the one that we want. Indeed, $\mathrm{diam}(\overline{V_{i,k}}) < \eta'$ implies that $\mathrm{diam}(\overline{sV_{i,k}}) < \eta$ for all $s \in T_k$. Since $M_{i,k} \subset V_{i,k}$, we have $\mu(\bigsqcup_{k=1}^{K} \bigsqcup_{i=1}^{n_k} T_k V_{i,k}) > 1 - \epsilon$.

□

Now we are ready to prove the following theorem, which may be regarded as a dynamical analogue of the known result on the Toms-Winter conjecture which states that strict comparison implies $\mathcal{Z}$-stability when the set of extreme tracial states is compact and finite-dimensional ([37], [62] and [70]).

Theorem 2.2.3. *Let $\alpha\colon G \curvearrowright X$ be a minimal free action, where $E_G(X)$ is compact and of covering dimension m in the weak*-topology. If α has dynamical comparison, then it is m-almost finite.*

Proof. First we fix an integer $n \in \mathbb{N}$, a finite subset $F \subset G$, and real numbers $\eta > 0$ and $\frac{1}{4n+2} > \epsilon > 0$. Then for every $\tau \in E_G(X)$, Lemma 2.2.2 implies that there is a castle $\mathcal{T}_\tau = \{(S_k, \overline{V_k})\colon k = 1, 2, \ldots, K\}$ where the sets V_k are open, the shapes S_k are (F, η)-invariant, diam($\overline{sV_k}$)$< \eta$ for all $s \in S_k$, $\tau(\bigsqcup_{k=1}^{K} S_k V_k) > 1 - \epsilon$, and the interval $[\frac{1}{2n}|S_k|, \frac{1}{n}|S_k|]$ contains an integer $d_{k,\tau}$. Define $T_\tau = \bigsqcup_{k=1}^{K} S_k V_k$, which is open. Then, by the remark above, the function on $E_G(X)$ defined by $\rho \to \rho(T_\tau)$ is lower semicontinuous.

For every $\tau \in E_G(X)$, we define the open neighborhood $U_\tau = \{\rho \in E_G(X)\colon \rho(T_\tau) > 1 - \epsilon\}$ of τ, which is open by the semicontinuity of $\rho(T_\tau)$. The compactness of $E_G(X)$ then implies that there is an $I \in \mathbb{N}$ such that $E_G(X) = \bigcup_{i=1}^{I} U_{\tau_i}$. Since $\dim(E_G(X)) \leq m$, there is a finite cover $\mathcal{W}$ of $E_G(X)$ consisting of closed sets such that $\mathcal{W}$ refines $\mathcal{U} = \{U_{\tau_1}, \ldots, U_{\tau_I}\}$ and a map $c : \mathcal{W} \to \{0, 1, \ldots, m\}$ such that $c(W) = c(W')$ implies $W \cap W' = \emptyset$. For each $i \in \{0, 1, \ldots, m\}$, write $\mathcal{W}^{(i)} = \{W_1^{(i)}, \ldots, W_{L_i}^{(i)}\}$. Then for each $i \leq m$ and $j \leq L_i$, there is a $\tau_j^{(i)}$ such that $W_j^{(i)} \subset$

$U_{\tau_j^{(i)}}$. This implies that there is a finite collection of towers $\{(S_{k,j}^{(i)}, \overline{V_{k,j}^{(i)}})\colon k = 1, 2, \ldots, K_j^{(i)}, j = 1, 2, \ldots, L_i, i = 0, \ldots, m\}$ such that for each $\rho \in W_j^{(i)}$ one has $\rho(T_{\tau_j^{(i)}}) = \rho(\bigsqcup_{k=1}^{K_j^{(i)}} S_{k,j}^{(i)} V_{k,j}^{(i)}) > 1 - \epsilon$.

Now fix a $i \in \{0, 1, \ldots, m\}$. Apply Lemma 2.1.3 to $R_i = \bigsqcup_{j=1}^{L_i} W_j^{(i)}$ to obtain a collection of pairwise disjoint compact sets $\{C_j^{(i)}\}_{j=1}^{L_i}$ such that for all $\rho \in W_j^{(i)}$ one has $\rho(C_j^{(i)}) > 1 - \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)^2}$. For $\{C_j^{(i)}\}_{j=1}^{L_i}$, there are collections of pairwise disjoint open sets $\{N_j^{(i)}\}_{j=1}^{L_i}$ and $\{M_j^{(i)}\}_{j=1}^{L_i}$ such that $C_j^{(i)} \subset N_j^{(i)} \subset \overline{N_j^{(i)}} \subset M_j^{(i)}$. Define $Y_j^{(i)} = \bigcap_{s \in \bigcup_{k=1}^{K_j^{(i)}} S_{k,j}^{(i)}} s^{-1} N_j^{(i)}$.

Note that towers in the collection $\{(S_{k,j}^{(i)}, \overline{V_{k,j}^{(i)} \cap Y_j^{(i)}})\colon k = 1, 2, \ldots, K_j^{(i)}, j = 1, 2, \ldots, L_i\}$ are pairwise disjoint. Indeed, for all $j, j' \leq L_i$, $s \in S_{k_1,j}^{(i)}$ and $t \in S_{k_2,j'}^{(i)}$ one has $s(\overline{V_{k_1,j}^{(i)} \cap Y_j^{(i)}}) \subset \overline{N_j^{(i)}}$ and $t(\overline{V_{k_2,j'}^{(i)} \cap Y_{j'}^{(i)}}) \subset \overline{N_{j'}^{(i)}}$. Then for all $\rho \in W_j^{(i)}$:

$$\begin{aligned}
\rho((Y_j^{(i)})^c) = \rho(\bigcup_{s \in \bigcup_{k=1}^{K_j^{(i)}} S_{k,j}^{(i)}} s^{-1}(N_j^{(i)})^c) &\leq \sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}| \cdot \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)^2} \\
&= \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)}.
\end{aligned}$$

It follows that

$$\rho(V_{k,j}^{(i)} \cap Y_j^{(i)}) \geq \rho(V_{k,j}^{(i)}) - \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)},$$

and thus

$$\begin{aligned}
\rho(\bigsqcup_{k=1}^{K_j^{(i)}} S_{k,j}^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)})) &\geq \sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|(\rho(V_{k,j}^{(i)}) - \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)}) \\
&= \sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|\rho(V_{k,j}^{(i)}) - \sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}| \frac{\epsilon}{(\sum_{k=1}^{K_j^{(i)}} |S_{k,j}^{(i)}|)}
\end{aligned}$$

$$= \rho(\bigsqcup_{k=1}^{K_j^{(i)}} S_{k,j}^{(i)} V_{k,j}^{(i)}) - \epsilon$$

$$\geq 1 - 2\epsilon$$

for all $\rho \in W_j^{(i)}$.

Then, since $E_G(X) = \bigcup_{i=0}^m R_i = \bigcup_{i=0}^m \bigsqcup_{j=1}^{L_i} W_j^{(i)}$, for all $\rho \in E_G(X)$ one has:

$$(\star) \quad \rho(\bigcup_{i=0}^{m} \bigsqcup_{j=1}^{L_i} \bigsqcup_{k}^{K_j^{(i)}} S_{k,j}^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)})) \geq 1 - 2\epsilon.$$

Define $O = \bigcup_{i=0}^m \bigsqcup_j^{L_i} \bigsqcup_k^{K_j^{(i)}} S_{k,j}'^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)})$ where $S_{k,j}'^{(i)} \subset S_{k,j}^{(i)}$ with $|S_{k,j}'^{(i)}| = d_{k,\tau_j^{(i)}} \in [\frac{1}{2n}|S_{k,j}^{(i)}|, \frac{1}{n}|S_{k,j}^{(i)}|]$ and $F = X \setminus \bigcup_{i=0}^m \bigsqcup_j^{L_i} \bigsqcup_k^{K_j^{(i)}} S_{k,j}^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)})$. This implies that $\rho(O) \geq \frac{1}{2n}(1-2\epsilon)$ and $\rho(F) < 2\epsilon$ for all $\rho \in E_G(X)$. Applying Lemma 2.2.1 to F, there is an open set $U \supset F$ such that $\rho(U) < 2\epsilon$ for all $\rho \in E_G(X)$. In the same manner, applying Lemma 2.2.1 to O^c, there is a closed set $D \subset O$ such that $\rho(D) \geq \frac{1}{2n}(1 - 2\epsilon)$ for all $\rho \in E_G(X)$. Then since our ϵ is chosen to be less than $\frac{1}{4n+2}$, one has $\frac{1}{2n}(1 - 2\epsilon) > 2\epsilon$. It turns out that for every $\rho \in E_G(X)$ one has:

$$(\blacklozenge) \quad \rho(D) \geq \frac{1}{2n}(1 - 2\epsilon) > 2\epsilon > \rho(U);$$

By convexity, $(\blacklozenge)$ also holds for all $\rho \in \mathrm{conv}(E_G(X))$. Now, let $\tau_n \to \tau$ where $\tau_n \in \mathrm{conv}(E_G(X))$ and $\tau \in M_G(X)$. By the portmanteau theorem (Theorem 17.20 in [33]), $\tau(D) \geq \limsup_{n\to\infty} \tau_n(D) \geq \frac{1}{2n}(1 - 2\epsilon)$ and $\tau(U) \leq \liminf_{n\to\infty} \tau_n(U) \leq 2\epsilon$, which implies that $\rho(D) \geq \frac{1}{2n}(1 - 2\epsilon) > 2\epsilon \geq \rho(U)$ holds for all $\tau \in M_G(X)$. Therefore, $\tau(O) \geq \tau(D) > \tau(U) \geq \tau(F)$ for all $\tau \in M_G(X)$.

Therefore, since the action α has dynamical comparison, one has:

$$X \setminus \bigcup_{i=0}^{m} \bigsqcup_{j}^{L_i} \bigsqcup_{k}^{K_j^{(i)}} S_{k,j}^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)}) \prec \bigcup_{i=0}^{m} \bigsqcup_{j}^{L_i} \bigsqcup_{k}^{K_j^{(i)}} S_{k,j}'^{(i)}(V_{k,j}^{(i)} \cap Y_j^{(i)}).$$

Finally, we write $\mathcal{T}_i$ for the collection of towers $\{(S_{k,j}^{(i)}, \overline{V_{k,j}^{(i)} \cap Y_j^{(i)}})\colon k = 1,2,\ldots,K_j^{(i)}, j = 1,2,\ldots,L_i\}$ for $i = 0,1,\ldots,m$. Observe that towers in each $\mathcal{T}_i$ are pairwise disjoint. This implies that the collection of towers $\{\mathcal{T}_i : i = 0,1,\ldots,m\}$ witnesses that α is m-almost finite.

□

The theorem below arises from the one above if we assume $E_G(X)$ is compact and zero-dimensional, but weaken “comparison” to “m-comparison” in order to arrive at almost finiteness. The idea of the proof of the following theorem comes from Theorem 9.2 in [34]. Then we have the following theorem.

Theorem 2.2.4. *Let $\alpha\colon G \curvearrowright X$ be a minimal free action such that $E_G(X)$ is compact and zero-dimensional in the weak*-topology. If α has dynamical m-comparison for some $m \in \mathbb{N}$, then it is almost finite.*

Proof. First, we fix $n \in \mathbb{N}$, a finite set $F \subset G$, $\eta > 0$ and $\frac{1}{4(m+1)n+2} > \epsilon > 0$. Then by the same proof of Theorem 2.2.3, there exists a castle $\{(S_i, \overline{V_i}) : i \in I\}$ where the sets V_i are open, the shapes S_i are (F,η)-invariant, diam($\overline{sV_i}$)$< \eta$ for all $s \in S_i$ and $\mu(\bigsqcup_{i\in I} S_iV_i) \geq 1-2\epsilon$ for all $\mu \in M_G(X)$. In addition, since G is infinite we can enlarge F to make all S_i have large enough cardinality so that there is an $S'_{i,0} \subset S_i$ satisfying $\frac{1}{2(m+1)n}|S_i| < |S'_{i,0}| < \frac{1}{(m+1)n}|S_i|$. Write $O = \bigsqcup_{i\in I} S'_{i,0}V_i$ and $F = X \setminus \bigsqcup_{i\in I} S_iV_i$. Then we have the following inequality for all $\mu \in M_G(X)$:

$$\mu(O) \geq \frac{1}{2(m+1)n}(1-2\epsilon) > 2\epsilon \geq \mu(F).$$

Since α has m-comparison, there is a finite collection $\mathcal{U}$ of open subsets of X which cover F, an $s_U \in G$ for each $U \in \mathcal{U}$, and a partition $\mathcal{U} = \bigsqcup_{j=0}^m \mathcal{U}_j$ such that for each $j = 0,1,\ldots,m$ the images $s_U U$ for $U \in \mathcal{U}_j$ are pairwise disjoint subsets of O. For each $i \in I$, since $|S'_{i,0}| < \frac{1}{(m+1)n}|S_i|$, we can choose pairwise disjoint sets $S'_{i,k}$ of the same cardinality, for $k = 1,2,\ldots,m$, which allows us to choose a bijection $\varphi_{i,j} : S'_{i,0} \to S'_{i,j}$.

For $U \in \mathcal{U}$, $i \in I$ and $t \in S'_{i,0}$ we denote by $W_{U,i,t}$ the open set $U \cap s_U^{-1}tV_i$. For each $j \in \{1,2,\ldots,m\}$ and $U \in \mathcal{U}_j$, the family $\{W_{U,i,t} : i \in I, t \in S'_{i,0}\}$ forms a partition of U. This implies that the sets $\varphi_{i,j}(s_U)t^{-1}s_U W_{U,i,t}$ for $U \in \mathcal{U}_j, i \in I, t \in S'_{i,0}$ are pairwise disjoint and contained in $\bigsqcup_{i\in I} S'_{i,j}V_i$. This entails $F \prec \bigsqcup_{i\in I} S'_i V_i$ where $S'_i = \bigsqcup_{j=0}^m S'_{i,j}$ with $|S'_i| < \frac{1}{n}|S_i|$ and thus verifies that α is almost finite.

□

Combined with (i)⇒(ii)⇒(iii)⇒(iv) in Theorem 9.2 in [34], the theorem above yields the same conclusion as this theorem from [34] under a weaker hypothesis.

Corollary 2.2.5. *Let $\alpha : G \curvearrowright X$ be a minimal free action. If $E_G(X)$ is compact and zero-dimensional, the following conditions are equivalent.*

(i) α is almost finite;

(ii) α is m-almost finite for some $m \geq 0$;

(iii) α has comparison;

(iv) α has m-comparison for some $m \geq 0$.

2.3 The Small Boundary Property Revisited

Now, we would like to bring the small boundary property into the picture. The following proposition was communicated to me by Gábor Szabó.

Proposition 2.3.1. Let $\alpha : G \curvearrowright X$. Suppose that for every $\delta > 0$, $\epsilon > 0$ there is a collection $\mathcal{U}$ of pairwise disjoint open sets such that $\max_{U\in\mathcal{U}} \operatorname{diam}(U) < \delta$ and $\mu(X \setminus \bigcup \mathcal{U}) < \epsilon$ for all $\mu \in M_G(X)$. Then $\alpha : G \curvearrowright X$ has the small boundary property.

Proof. Fix a metric d on the space X. We firstly claim that given $F \subset O$ where F is closed and O is open, for every $\epsilon > 0$ there is an open neighbourhood V of F such that $F \subset V \subset \overline{V} \subset O$ and $\mu(\partial V) < \epsilon$.

To show this claim firstly observe that $l = d(F, O^c) > 0$, which implies that $F \subset \overline{B}(F, l/2) \subset B(F, l) \subset O$ where $\overline{B}(F, l/2)$ is defined to the set $\{x \in X : d(x, F) \leq l/2\}$ while $B(F, l)$ is defined to be the set $\{x \in X : d(x, F) < l\}$. Now, for the number $l/2$ and a given positive number

$\epsilon > 0$ one can find a collection $\mathcal{U}$ of pairwise disjoint open sets such that $\max_{U\in\mathcal{U}} \operatorname{diam}(U) < l/2$ and $\mu(X \setminus \bigcup\mathcal{U}) < \epsilon$ for all $\mu \in M_G(X)$.

Now define $K = F \setminus \bigcup\mathcal{U} \subset X \setminus \bigcup\mathcal{U}$ which entails that $\mu(K) < \epsilon$ for all $\mu \in M_G(X)$. Then Lemma 2.2.1 implies that there is an open subset M such that $K \subset M \subset \overline{M} \subset O$ such that $\mu(\overline{M}) < \epsilon$ for all $\mu \in M_G(X)$. Now consider $\{U \in \mathcal{U} : F \cap U \neq \emptyset\} \cup \{M\}$ form an open cover of F and thus has a finite subcover, say, $\{U_1, \ldots, U_n, M\}$ by compactness. For each $i = 1, \ldots, n$ since $\operatorname{diam}(U_i) < l/2$ and $U_i \cap F \neq \emptyset$, one has $U_i \subset B(F, l/2)$ and thus $\overline{U_i} \subset \overline{B}(F, l/2) \subset B(F, l) \subset O$. Now define $V = (\bigsqcup_{i=1}^n U_i) \cup M$, which satisfies that $F \subset V \subset \overline{V} \subset O$.

In addition, consider $\partial V \subset \bigcup_{i=1}^n \partial U_i \cup \partial M$. Since the family $\mathcal{U}$ is disjoint, each $\partial U_i \subset X \setminus \bigcup\mathcal{U}$ and thus $\bigcup_{i=1}^n \partial U_i \subset X \setminus \bigcup\mathcal{U}$. Combining with the fact $\partial M \subset \overline{M}$, one has $\mu(\partial V) < 2\epsilon$ for all $\mu \in M_G(X)$. This completes the claim.

Now, let $x \in O$ where $x \in X$ and O is an open subset of X. Then we proceed by induction to construct sequences $x \in U_1 \subset U_2 \subset \cdots \subset O$ and $O \supset Z_1 \supset Z_2 \supset \ldots$ such that $\partial U_n \subset Z_n$ and $\mu(\overline{Z_n}) < 1/n$ for all $\mu \in M_G(X)$ and $n \in \mathbb{N}^+$. Firstly, the claim above allows us to choose an open neighbourhood U_1 of x such that $x \in U_1 \subset \overline{U_1} \subset O$ such that $\mu(\partial U_1) < 1$ for all $\mu \in M_G(X)$. Then apply Lemma 2.2.1 to ∂U_1 to obtain an open neighbourhood Z_1 of ∂U_1 such that $\overline{Z_1} \subset O$ and $\mu(\overline{Z_1}) < 1$ for all $\mu \in M_G(X)$. Suppose that we have constructed $U_1 \subset U_2 \subset \cdots \subset U_k \subset O$ and $O \supset Z_1 \supset Z_2 \supset \cdots \supset Z_k$ such that $\partial U_n \subset Z_n$ and $\mu(\overline{Z_n}) < 1/n$ for all $\mu \in M_G(X)$ and $n = 1, \ldots, k$. Now we define U_{k+1} and Z_{k+1}. Apply the claim above to $\overline{U_k} \subset U_k \cup Z_k$ then there is an open subset U_{k+1} such that $\overline{U_k} \subset U_{k+1} \subset \overline{U_{k+1}} \subset U_k \cup Z_k \subset O$ with $\mu(\partial U_{k+1}) < 1/(k+1)$ for all $\mu \in M_G(X)$. Observe that Z_k is an open neighbourhood of ∂U_{k+1}. Then by Lemma 2.2.1 again there is an open subset Z_{k+1} such that $\partial U_{k+1} \subset Z_{k+1} \subset \overline{Z_{k+1}} \subset Z_k$ and $\mu(\overline{Z_{k+1}}) < 1/(k+1)$ for all $\mu \in M_G(X)$. This finishes our construction.

Now define $U = \bigcup_{n=1}^\infty U_n$. Then $x \in U \subset O$. In addition, our construction implies that $U_k \subset U \subset U_k \cup Z_k$ for each $k \in \mathbb{N}^+$. Therefore one has

$$\partial U = \overline{U} \setminus U \subset \overline{U_k \cup Z_k} \setminus U_k \subset \overline{Z_k}$$

for each $k \in \mathbb{N}^+$. This entails that $\mu(\partial U) = 0$ for all $\mu \in M_G(X)$. This verifies the small boundary property. $\square$

We remark that the converse of the proposition above is also true (see Theorem 5.4 in [36]). But the direction in the proposition above is good enough for our purpose to prove the following theorem.

Theorem 2.3.2. *Let G be a countable infinite discrete amenable group, X an infinite compact metrizable space and $\alpha : G \curvearrowright X$ a minimal free continuous action of G on X. Suppose that $E_G(X)$ is compact and zero-dimensional in the weak*-topology. Then α has the small boundary property.*

Proof. Let $\alpha : G \curvearrowright X$ be a minimal free action. We revisit the proof of Theorem 2.2.3. Given a finite set $F \subset G$, $\epsilon > 0$ and $\delta > 0$, if $E_G(X)$ is compact and zero-dimensional, then the process allows us to construct a finite open castle $\{(T_i, V_i) : i \in I\}$ such that

(i) T_i is (F, δ)-invariant for every $i \in I$;

(ii) $\mathrm{diam}(tV_i) < \delta$ for all $t \in T_i$ and all $i \in I$;

(iii) $\mu(X \setminus \bigsqcup_{i \in I} T_i V_i) < \epsilon$ for all $\mu \in E_G(X)$ (i.e. inequality ($\star$)).

Then, the same argument as in the proof of Theorem 2.2.3, together with Lemma 2.2.1 and the portmanteau theorem, imply that:

(iii') $\mu(X \setminus \bigsqcup_{i \in I} T_i V_i) < \epsilon$ for all $\mu \in M_G(X)$.

At last, Proposition 2.3.1 implies that $\alpha : G \curvearrowright X$ has the small boundary property. $\square$

We close this section by remarking that the property that requires the existence of castles satisfying properties (i), (ii) and (iii') is called "almost finiteness in measure" and was introduced in [36] of Kerr and Szabó in which it is proved that a minimal free action $\alpha : G \curvearrowright X$ has the small boundary property if and only if it is almost finite in measure.

2.4 Classification Results

In this section, we explore some properties of the crossed products arising from a minimal free almost finite action $\alpha : G \curvearrowright X$. We recall the following theorem due to Kerr.

Theorem 2.4.1. ([34, Theorem 12.4]) *Let $\alpha : G \curvearrowright X$ be a minimal free action. If α is almost finite, then the crossed product $C(X) \rtimes_r G$ is $\mathcal{Z}$-stable.*

We observe that any crossed product C^*-algebra $A = C(X) \rtimes_r G$ arising from a minimal action $\alpha : G \curvearrowright X$ is stably finite since $\tau(a) = \int_X E(a)\, d\mu$ is a faithful tracial state on A, where μ is an invariant probability measure on X (such a G-invariant probability measure always exists since the group G is assumed to be amenable) and E is the canonical faithful conditional expectation from A to $C(X)$. Therefore, if the action α is also free and almost finite, then $A = C(X) \rtimes_r G$ has stable rank one by Theorem 2.4.1 above and Theorem 6.7 in [59]. We remark that both Kerr [34] and Suzuki [66] generalize the notion "almost finiteness" from [46]. Both generalizations coincide with the original one if the space X is the Cantor set. They differ in general since "almost finiteness" in [66] does not necessarily imply $\mathcal{Z}$-stability.

Compared with stable rank, it is much harder to determine the real rank as well as the tracial rank of a C^*-algebra arising from minimal free almost finite actions of an infinite amenable group. The following result is due to Rørdam.

Theorem 2.4.2. ([59, Theorem 7.2]) *The following conditions are equivalent for each unital, simple, exact, finite, $\mathcal{Z}$-absorbing C^*-algebra A.*

(i) $rr(A) = 0$*;*

(ii) $\rho(K_0(A))$ *is uniformly dense in* $\mathrm{Aff}(T(A))$

where ρ is defined by $\rho(g)(\tau) = K_0(\tau)(g)$.

A crossed product C^*-algebra $A = C(X) \rtimes_r G$ arising from minimal free almost finite actions of an infinite amenable group certainly satisfies the assumption of the theorem above. However, it is generally very difficult to verify whether A satisfies condition (ii) in the theorem above. Known examples are the irrational rotation algebras, which are included in a collection of more general examples constructed by Lin and Phillips in [43]. Note that every irrational rotation on $\mathbb{T}$ is indeed almost finite by Theorem 2.2.3 since it is uniquely ergodic and has dynamical comparison (see [10]). It is worth mentioning that the result of Lin and Phillips in fact recovers the Elliott-Evans Theorem [19] stating that every irrational rotation algebra is an A$\mathbb{T}$-algebra with real rank zero. On the other hand, if the space X is the Cantor set, Phillips [52] worked on almost AF Cantor groupoids and proved that the crossed product arising from a minimal free action $\mathbb{Z}^d \curvearrowright X$ has real rank zero. Suzuki [66] then generalized the result of Phillips by a different approach by proving the following theorem in [66].

Theorem 2.4.3. ([66, Remark 4.3]) *Let $\alpha : G \curvearrowright X$ where X is the Cantor set. If α is almost finite, then the crossed product $C(X) \rtimes_r G$ has real rank zero.*

Suzuki [66] also proved that $\alpha : G \curvearrowright X$ is almost finite if G is abelian and X is the Cantor set. Then, as an application of Theorem 2.4.3, $C(X) \rtimes_r G$ has real rank zero if G is abelian and X is the Cantor set.

We close this section by establishing Theorem 2.4.4 and Corollary 2.4.5 below.

Theorem 2.4.4. *Let G be a countable infinite discrete amenable group, X an infinite compact metrizable space and $\alpha : G \curvearrowright X$ a minimal free continuous action of G on X. Suppose that $E_G(X)$ is compact and zero-dimensional in the weak*-topology and α has dynamical m-comparison for some $m \in \mathbb{N}$. Then α is almost finite and thus the crossed product $C(X) \rtimes_r G$ is $\mathcal{Z}$-stable and belongs to the class $\mathfrak{C}$.*

Proof. Since $E_G(X)$ is compact and zero-dimensional, Theorem 2.2.4 and Theorem 2.4.1 imply that the crossed product $A = C(X) \rtimes_r G$ is $\mathcal{Z}$-stable and has finite nuclear dimension. In addition, $C(X) \rtimes_r G$ is isomorphic to a C^*-algebra of a Hausdorff, locally compact, second countable amenable transformation groupoid and thus satisfies UCT by a result of Tu [71]. Then the crossed product $A = C(X) \rtimes_r G$ belongs to the class $\mathfrak{C}$, which is classified by the Elliott invariant by Theorem 1.1.4 above. $\square$

We remark that this result has been strengthened by Kerr-Szabó in [36]. They use the small boundary properties to replace the condition that $E_G(X)$ is compact and zero-dimensional, which implies the small boundary property by Theorem 2.3.2.

Combining Theorem 2.4.4 with Corollary 4.9 in [21], we have the following corollary. In this paper, however, instead of using Corollary 4.9 in [21], we directly verify that the crossed product under the assumption below has finite nuclear dimension and thus belongs to the class $\mathfrak{C}$.

Corollary 2.4.5. *Let X be an infinite compact metrizable space, and let $h : X \to X$ be a minimal homeomorphism. Suppose that $E_{\mathbb{Z}}(X)$ is compact and zero-dimensional in the weak*-topology. Then $C(X) \rtimes_r \mathbb{Z}$ belongs to the class $\mathfrak{C}$.*

Proof. Suppose $E_{\mathbb{Z}}(X)$ is compact and zero-dimensional, then $\alpha : \mathbb{Z} \curvearrowright X$ has the small boundary property by Theorem 2.3.2. Then [21] implies that $A = C(X) \rtimes_r \mathbb{Z}$ is $\mathcal{Z}$-stable and therefore A has finite nuclear dimension. In addition, the result of Tu [71] shows that A satisfying UCT as mentioned above. Then the crossed product $A = C(X) \rtimes_r \mathbb{Z}$ belongs to the class $\mathfrak{C}$, which is classified by the Elliott invariant by Theorem 1.1.4 above. $\square$

In this chapter, we mainly investigate dynamical systems which has no invariant probability measures. This implies necessarily that the acting group is not amenable and the reduced crossed product is traceless. In this setting, some paradoxical phenomenon may happens inside the dynamical system, which is essential in the study of pure infiniteness.

3.1 Dynamical Comparison and Paradoxical Phenomenon

We first recall the definition of dynamical comparison.

Definition 3.1.1. Let $m \in \mathbb{N}$. The action $\alpha : G \curvearrowright X$ is said to have dynamical comparison if $A \prec B$ for all open sets $A, B \subset X$ satisfying $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$.

From the definition, we first remark that when there is no G-invariant regular Borel probability measure on X, if α has dynamical comparison then it can be verified that α has to be minimal and the space X has to be perfect. Indeed, for every $x \in X$ and non-empty open subset O of X there is a group element $g \in G$ such that $g\{x\} \subset O$ since α has dynamical comparison. This verifies that the action is minimal. In addition, it is not hard to see $|F| \leq |O|$ for every closed set F and open set O satisfying $F \prec O$ by Definition 1.2.7. Suppose that there is an open set whose cardinality is one. Observe that then any closed set containing exactly two points is subequivalent to this open set since α has dynamical comparison, which is a contradiction to the cardinality inequality mentioned above. This implies that the cardinality of an open set cannot be one and thus the space is perfect.

In addition, we remark that if $M_G(X) = \emptyset$ then dynamical comparison has paradoxical flavor as every two open sets are subequivalent to each other in the sense of Definition 1.2.7. Thus, it is a good candidate for a property that implies pure infiniteness of the crossed product. On the other hand, to establish the pure infiniteness of the reduced crossed products, before dynamical comparison, Laca and Spielberg [41] showed that the reduced crossed product $C(X) \rtimes_r G$ is purely infinite provided that the action $G \curvearrowright X$ is also a strong boundary action, which means that

X is infinite and that any two non-empty open subsets of X can be translated by group elements to cover the entire space X. Jolissaint and Robertson [32] generalized this result and showed that it is sufficient to require that the action is n-filling, which means the entire space can be covered by translations of n open subsets instead of two open subsets of X. We show below that dynamical comparison is a generalization of the n-filling and thus also a generalization of the strong boundary actions.

Indeed, suppose that the action $\alpha : G \curvearrowright X$ is n-filling. Then there is no G-invariant measure on X and it suffices to show $V \prec O$ for two arbitrary non-empty open sets O, V. For every closed set $F \subset V$, choose n pairwise disjoint non-empty open subsets $O_1, O_2, \ldots, O_n$ of O where all of these open sets contain more than one point. Since the space is Hausdorff and perfect, we can do this by choosing n different points $x_1, x_2, \ldots, x_n \in O$ and non-trivial open neighbourhoods O_i of x_i for all $i = 1, 2, \cdots, n$ so that $O_i \cap O_j = \emptyset$ whenever $i \neq j$. Then there are $t_1, t_2, \ldots, t_n \in G$ such that $\bigcup_{i=1}^{n} t_i O_i = X \supset F$, whence $\{t_i^{-1} : i = 1, 2, \ldots, n\}$ and $\{t_i O_i : i = 1, 2, \ldots, n\}$ witness that $F \prec O$. Then one has $V \prec O$ because F is an arbitrary closed subset of V. In particular, suppose now that $\alpha : G \curvearrowright X$ is a strong boundary action. It is 2-filling and thus has dynamical comparison.

In this section, under the hypothesis that there is no G-invariant regular Borel probability measure on X we show that if the action $\alpha : G \curvearrowright X$ is topologically free and has dynamical comparison then the reduced crossed product $A = C(X) \rtimes_r G$ is simple and purely infinite. To do this, we follow the idea in [41]. What we will actually show is the existence, for every nonzero element $x \in A$, of elements $y, z \in A$ such that $yxz = 1_A$. In the simple case, this condition is well-known to be equivalent to the definition of pure infiniteness recalled in the first chapter (see [57, Proposition 4.1.1]).

Definition 3.1.2. ([5, Definition 1.1]) An element x in a C^*-algebra is called a *scaling element* if $x^*x \neq xx^*$ and $(x^*x)(xx^*) = xx^*$.

Note that if x is a scaling element in a C^*-algebra A, then $v = x + (1 - x^*x)^{1/2}$ is an isometry. To see this, it suffices to verify that $(1 - x^*x)^{1/2}x = 0$. Because $(x^*x)(xx^*) = xx^*$, one has

$(1-x^*x)xx^* = (1-x^*x)|x^*|^2 = 0$, which implies that $(1-x^*x)^{1/2}|x^*| = 0$ by functional calculus. Thus $(1-x^*x)^{1/2}x = (1-x^*x)^{1/2}|x^*|u = 0$, where $x = u|x| = |x^*|u$ is the polar decomposition of x in A^{**}. Throughout the paper, for a function $f \in C(X)$, we denote by $\mathrm{supp}(f)$ the set $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$, which is an open subset of X. The following lemma strengthens Lemma 3 in [41].

Lemma 3.1.3. *Suppose that $\alpha : G \curvearrowright X$ has dynamical comparison and there is no G-invariant regular probability Borel measure on X. Let $\phi \in C(X)$ be a non-zero positive function. Then there is an isometry $v \in C(X) \rtimes_r G$ such that vv^* lies in the hereditary subalgebra $A(\phi)$ of $C(X) \rtimes_r G$ generated by ϕ.*

Proof. Choose $g \in C(X)$ with $0 \leq g \leq 1$, $g = 1$ on a neighborhood of $\phi^{-1}(\{0\})$, and $\overline{\mathrm{supp}(g)} \neq X$. Let U be open and nonempty with $\overline{U} \cap \overline{\mathrm{supp}(g)} = \emptyset$. Let V be open with $\overline{\mathrm{supp}(g)} \subset V \subset \overline{V} \subset \overline{U}^c$. Now, define $F = \overline{U \sqcup V}$ and we have $F \prec U$ since α has dynamical comparison. This means that there is an open cover $\mathcal{W} = \{W_1, \ldots, W_n\}$ of F and $t_1, \ldots, t_n \in G$ such that $\{t_iW_i : i = 1, \ldots, n\}$ contains pairwise disjoint subsets of U. Now, let $\{f_i : i = 1, 2, \ldots, n\}$ be a partition of unity subordinate to $\mathcal{W}$. We have

(i) $0 \leq f_i \leq 1$ for all $i = 1, 2, \ldots, n$;

(ii) $\sum_{i=1}^n f_i(y) = 1$ for all $y \in F$;

(iii) $\overline{\mathrm{supp}(f_i)} \subset W_i$ for all $i = 1, 2, \ldots, n$.

Define $x = \sum_{i=1}^n u_{t_i} f_i^{1/2}$. We claim that x is a scaling element. At first, observe that $t_iW_i \cap t_jW_j = \emptyset$ whenever $i \neq j$. Therefore one has $f_i^{1/2} u_{t_i^{-1}} u_{t_j} f_j^{1/2} = u_{t_i^{-1}}(u_{t_i} f_i^{1/2} u_{t_i^{-1}})(u_{t_j} f_j^{1/2} u_{t_j^{-1}})u_{t_j} = 0$ if $i \neq j$. Then we have

$$\begin{aligned} x^*x &= (\sum_{i=1}^n f_i^{1/2} u_{t_i^{-1}})(\sum_{i=1}^n u_{t_i} f_i^{1/2}) \\ &= \sum_{i=1}^n f_i + \sum_{1 \leq i \neq j \leq n} f_i^{1/2} u_{t_i^{-1}} u_{t_j} f_j^{1/2} \end{aligned}$$

$$= \sum_{i=1}^{n} f_i.$$

and

$$\begin{aligned}
xx^* &= (\sum_{i=1}^{n} u_{t_i} f_i^{1/2})(\sum_{i=1}^{n} f_i^{1/2} u_{t_i^{-1}}) \\
&= \sum_{1\leq i,j\leq n} u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_j^{-1}} \\
&= \sum_{1\leq i,j\leq n} (u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}) u_{t_i} u_{t_j^{-1}}.
\end{aligned}$$

For all $i = 1, 2, \ldots, n$ one has $\mathrm{supp}(u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}) \subset t_i W_i \subset U$. In addition $t_i W_i \subset U \subset F$ implies that $\sum_{i=1}^{n} f_i(y) = 1$ for every $y \in t_i W_i$. This implies that $(\sum_{i=1}^{n} f_i)(u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}) = u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}$ for all $i = 1, 2, \ldots, n$. Therefore, we have:

$$\begin{aligned}
(x^*x)(xx^*) &= (\sum_{i=1}^{n} f_i)(\sum_{1\leq i,j\leq n} (u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}) u_{t_i} u_{t_j^{-1}}) \\
&= (\sum_{1\leq i,j\leq n} (u_{t_i} f_i^{1/2} f_j^{1/2} u_{t_i^{-1}}) u_{t_i} u_{t_j^{-1}}) \\
&= xx^*.
\end{aligned}$$

If the set $\{t_i : i = 1, 2, \ldots n\}$ contains at least two different group elements then xx^* is not a function while x^*x is. On the other hand, if there is a $t \in G$ such that $t_i = t$ for every $i = 1, 2, \ldots n$ then $xx^* = \sum_{1\leq i,j\leq n} u_t f_i^{1/2} f_j^{1/2} u_{t^{-1}}$, which is a function supported in U while x^*x is constant one on F. Therefore, in any case, one has $xx^* \neq x^*x$. These show that x is a scaling element. Define an isometry $v = x + (1 - x^*x)^{1/2}$ as mentioned above.

Observe that $1 - x^*x = 1 - \sum_{i=1}^{n} f_i$ is constant zero on $F \supset \mathrm{supp}(g)$. This implies that $g(1 - x^*x)^{1/2} = 0$. In addition, for all $i = 1, 2, ..., n$ one has $g u_{t_i} f_i^{1/2} = g(u_{t_i} f_i^{1/2} u_{t_i^{-1}}) u_{t_i} = 0$

since $\text{supp}(u_{t_i} f_i^{1/2} u_{t_i^{-1}}) \subset t_i W_i \subset U$. This implies that $gv = 0$ and thus $gvv^* = 0$.

Since $0 \le g \le 1$ and vv^* is a projection, one has $g + vv^* \le 1$. Observe that $\text{supp}(1-g) \subset \text{supp}(\phi)$ so that $1 - g \precsim \phi$ in $C(X)$ in the sense of Cuntz comparison by Proposition 2.5 in [3]. Hence $1 - g \in A(\phi)$ since there is a sequence $\{r_n\}$ in $C(X)$ such that $\phi^{1/2} r_n^* r_n \phi^{1/2} = r_n^* \phi r_n \to 1 - g$. Then because $vv^* \le 1 - g$, one has $vv^* \in A(\phi)$ by the definition of hereditary sub-algebras.

□

Using the lemma above, the same proof of Theorem 5 in [41] establishes the following theorem. To be self-contained, we write the proof here.

Theorem 3.1.4. *Let G be a countable discrete infinite group, X a compact Hausdorff space and $\alpha : G \curvearrowright X$ a minimal topologically free continuous action of G on X. Suppose that there is no G-invariant regular Borel probability measure on X and α has dynamical comparison. Then the reduced crossed product $C(X) \rtimes_r G$ arising from α is purely infinite and simple.*

Proof. Since the action α is minimal and topologically free, the reduced crossed product is simple. Therefore, it suffices to show that the reduced crossed product $A = C(X) \rtimes_r G$ is purely infinite. Let $x \in A$ with $x \neq 0$. We will find $y, z \in A$ with $yxz = 1$. Observe that $E(x^*x)$ is a nonzero positive element in $C(X)$ since E is the canonical faithful conditional expectation. Define $a = x^*x/\|E(x^*x)\|$. Then one has $a \ge 0$ and $\|E(a)\| = 1$. Choose an element $b \in C_c(G, C(X))_+$ with $\|a - b\| < 1/4$. Write $b = \sum_{t \in F} b_t u_t$ where F is a finite subset of G containing the identity element $e \in G$. Then $E(b) = b_e$ is a non-zero positive function and $\|E(b)\| > 3/4$ because $\|E(b) - E(a)\| < 1/4$.

Since the action α is topologically free, the open set $O = \{x \in X : tx \neq x \text{ for all } t \in F^{-1}F \setminus \{e\}\} = \bigcap_{t \in F^{-1}F \setminus \{e\}} \{x \in X : tx \neq x\}$ is dense in X. Let U_0 be the non-empty open set of all $x \in X$ such that $E(b)(x) > 3/4$. Choose an element $x_0 \in U_0 \cap O$ and a neighbourhood U with $x_0 \in U \subset U_0 \cap O$ such that (F, U) is an open tower. We can do this since the space X is Hausdorff.

Choose $\phi \in C(X)$ with $0 \le \phi \le 1$, $\text{supp}(\phi) \subset U$ and $\phi \equiv 1$ on a nonempty open set. Then we observe that $E(b) \geq (3/4)\phi$. Now let $\phi_1 \in C(X)$ be another non-zero function, with $0 \leq \phi_1 \leq 1$ and $\mathrm{supp}(\phi_1) \subset \phi^{-1}(\{1\})$. By Lemma 3.1.3 there is an isometry $v \in A$ with $vv^* \in A(\phi_1)$. We now claim that $v^*bv = v^*E(b)v$. To show this, first observe that $v^*bv = v^*(vv^*bvv^*)v$ since v is an isometry. Then for every element of the form $\phi_1 a \phi_1$ in $A(\phi_1)$, one has

$$(\phi_1 a\phi_1)b(\phi_1 a\phi_1) = \sum_{t\in F}(\phi_1 a\phi_1)b_t u_t(\phi_1 a\phi_1) = (\phi_1 a\phi_1)E(b)(\phi_1 a\phi_1)$$

since one can check that $\phi_1 b_t u_t \phi_1 = b_t\phi_1 \cdot u_t\phi_1 u_{t^{-1}} u_t = 0$ if $t \neq e$ by using the fact that $\mathrm{supp}(\phi_1)$ and $\mathrm{supp}(u_t\phi_1 u_{t^{-1}})$ are disjoint. Then since $vv^* \in A(\phi_1)$, one has $vv^*bvv^* = vv^*E(b)vv^*$. This proves the claim that $v^*bv = v^*E(b)v$. Using the same method and the fact that $\mathrm{supp}(\phi_1) \subset \phi^{-1}(\{1\})$, one can also show that $v^*\phi v = v^*v = 1$. Thus we have

$$v^*bv = v^*E(b)v \geq v^*(\frac{3}{4}\phi)v = \frac{3}{4}v^*v = \frac{3}{4}.$$

Then v^*av is invertible since $\|v^*av - v^*bv\| < 1/4$. Let $y = \|E(x^*x)\|^{-1}(v^*av)^{-1}v^*x^*$ and $z = v$. Then we have $yxz = 1_A$. Thus $A = C(X) \rtimes_r G$ is purely infinite. $\square$

An application of this theorem is the following dichotomy result for reduced crossed products that trace/traceless may determine a dichotomy between stably finite and purely infinite unital simple separable and nuclear C^*-algebras. In fact the dichotomy holds even the reduced crossed products is neither nuclear nor separable. Indeed, suppose that $\alpha : G \curvearrowright X$ is a minimal and topologically free action. Every tracial states on $C(X) \rtimes_r G$ induces a G-invariant regular Borel probability measure on X when it restrict to $C(X)$. On the other hand, suppose that μ is a G-invariant regular Borel probability measure on X. It induces a faithful tracial state τ on the reduced crossed product $C(X) \rtimes_r G$ defined by $\tau(a) = \int_X E(a)\, d\mu$, where E is the canonical faithful conditional expectation from $C(X) \rtimes_r G$ onto $C(X)$. In this case it is well-known that $C(X) \rtimes_r G$ is stably finite. Combining this fact with the theorem above, we obtain the following dichotomy.

Corollary 3.1.5. *Let G be a countable discrete group, X an infinite compact Hausdorff space and*

$\alpha : G \curvearrowright X$ a minimal topologically free continuous action of G on X. Suppose that the action α has dynamical comparison. Then the reduced crossed product $C(X) \rtimes_r G$ is simple and is either stably finite or purely infinite.

Based on Theorem 3.1.4, we also have the following corollary.

Corollary 3.1.6. *Let $\alpha : G \curvearrowright X$ be an action on a compact metrizable space X such that there is no G-invariant regular Borel probability measure on X. Suppose that the action α is topologically free, amenable and has dynamical comparison. Then the reduced crossed product $C(X) \rtimes_r G$ is a Kirchberg algebra.*

We close this section by remarking that reduced crossed products occurring in Example 2.1, 2.2 in [41] and Example 2.1, 3.9, 4.3 in [32] are covered by the corollary above since the actions are known to be topologically free, amenable, and n-filling for some integer $2 \leq n \leq 6$ and thus have dynamical comparison without G-invariant regular Borel probability measures.

3.2 Paradoxical Comparison for Non-minimal actions

Beyond the issue of classification, whether a reduced crossed product is purely infinite is of its own interest. In order to establish this pure infiniteness for a reduced crossed product one usually needs to formalize the phenomenon of paradoxicality in the framework of dynamical systems. Roughly speaking, the idea of paradoxicality dating back to the work of Hausdorff and playing an important role of the work of Banach-Tarski (see [72]), is that one object somehow contains two disjoint copies of itself. The following notion introduced by Rørdam and Sierakowski exactly follows this philosophy and is sufficient to show pure infiniteness of reduced crossed products if the space X is zero-dimensional. Motivated by their work, we come up with another notion in this section called *paradoxical comparison*. This notion is weaker than dynamical comparison if the action is not minimal, but it still implies the pure infiniteness of the reduced crossed product if the action has an additional property which we call the *uniform tower property*. One advantage of considering dynamical comparison and paradoxical comparison is that they allow us to unify all of the above known sufficient criteria for pure infiniteness into one framework.

3.2.1 Paradoxical Comparison

Before introducing paradoxical comparison, we recall a definition and a theorem of Rørdam and Sierakowski first.

Definition 3.2.1. [60, Definition 4.2] Given a discrete group Γ acting on a topological space (Y, τ_Y), a non-empty set U is called (Γ, τ_Y)-*paradoxical* if there exist non-empty open sets $V_1, V_2, \ldots, V_{n+m}$ and elements $t_1, t_2, \ldots, t_{n+m}$ in Γ such that

$$\bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i = U$$

and such that $(t_k V_k)_{k=1}^{n+m}$ are pairwise disjoint subsets of U.

Using this notion, they obtained the following result.

Theorem 3.2.2. *[60, Corollary 4.4] Let $\alpha : \Gamma \curvearrowright X$ with Γ discrete and exact. Suppose that α is essentially free and X has a basis of clopen (G, τ_X)-paradoxical sets. Then $C(X) \rtimes_r \Gamma$ is purely infinite.*

For each nonempty open subset O of X we write $(O, O) \prec O$ if for every closed subset F of O there are disjoint nonempty open subsets O_1 and O_2 of O such that $F \prec O_1$ and $F \prec O_2$. Similarly we write

$$(\underbrace{O, \ldots, O}_{n \text{ many}}) \prec O$$

if for every closed subset $F \subset O$ there are disjoint family of nonempty open subsets $O_1, \ldots, O_n$ of O such that $F \prec O_i$ for every $i = 1, \ldots, n$. Based on this notation, we arrive the following definition.

Definition 3.2.3. Let $\alpha : G \curvearrowright X$. We say that α has *paradoxical comparison* if one has $(O, O) \prec O$ for every nonempty open subset O of X.

This definition also exactly follows the philosophy of paradoxicality since each open subset of X contains two disjoint copies of itself in the sense of subequivalence and therefore it can be viewed as a dynamical analogue of properly infiniteness of positive elements in C^*-setting. In addition, we remark that an action $\alpha : G \curvearrowright X$, where X is zero dimensional, has paradoxical comparison if and only if every clopen subset of X is (G, τ_X)-paradoxical. Indeed, first observe that a clopen subset of X is (G, τ_X)-paradoxical if and only if it satisfies the condition of paradoxical comparison. Thus it suffices to show that if one has $(A, A) \prec A$ for every clopen subset A of X then the action has paradoxical comparison. Let F be a closed subset of an open set O. By compactness there is a clopen set P such that $F \subset P \subset O$. Since $(P, P) \prec P$ one can find disjoint nonempty open subsets O_1 and O_2 of P such that $F \prec O_j \subset O$ for $j = 1, 2$. This verifies that the action α has paradoxical comparison. In light of Theorem 3.2.2, our paradoxical comparison then is also a candidate to show pure infiniteness of reduced crossed product in which the underlying space X has a higher dimension.

We remark that if $\alpha : G \curvearrowright X$ has paradoxical comparison then X has to be perfect because there is no two nonempty disjoint open subsets of an open set whose cardinality is one. In addition there is no G-invariant regular Borel probability measure on X. Indeed, suppose to the contrary that there is such a measure, say μ. For X itself there are disjoint nonempty open subset O_1 and O_2 such that $X \prec O_i$ for $i = 1, 2$, which implies that $\mu(O_i) = 1$ for $i = 1, 2$. Then one has $1 = \mu(X) \geq \mu(O_1) + \mu(O_2) = 2$, which is a contradiction. Furthermore, if the space X is zero-dimensional then $\alpha : G \curvearrowright X$ has no G-invariant non-trivial Borel measure by applying the same argument to a clopen set O with $0 < \nu(O) < \infty$ to obtain a contradiction whenever there is such a measure ν.

The following definition was suggested by David Kerr. We call this definition *weak paradoxical comparison* in this paper. To justify this name, Proposition 3.1.6 below will show that paradoxical comparison implies weak paradoxical comparison. The reason we introduce this concept is that it helps in proving pure infiniteness of crossed products.

Definition 3.2.4. Let $\alpha : G \curvearrowright X$. We say α has *weak paradoxical comparison* if for every closed subset F and nonempty open subset O of X one has $F \prec O$ whenever $F \subset G \cdot O$.

Before we prove the proposition 3.2.6, we need the following lemma which records elementary but useful properties of the relation of subequivalence.

Lemma 3.2.5. *Let $\alpha : G \curvearrowright X$ be an action and F a closed subset of X. Denote by A, B, C, M, N nonempty open subsets of X. Then:*

(i) $F \prec A$ if and only if there is an open subset M such that $F \prec M \subset \overline{M} \subset A$.

(ii) If $F \prec N \subset \overline{N} \prec B$ then $F \prec B$.

(iii) If $A \prec B$ and $B \prec C$ then $A \prec C$.

Proof. For the claim (i) we begin with $F \prec A$. There are open sets $U_1, \ldots, U_n$ and group elements $g_1, \ldots, g_n \in G$ such that $F \subset \bigcup_{i=1}^n U_i$ and $\bigsqcup_{i=1}^n g_i U_i \subset A$. Then choose a partition of unity $\{f_1, \ldots, f_n\}$ subordinate to the open cover $\{U_1, \ldots, U_n\}$ of F such that $\overline{\mathrm{supp}(f_i)} \subset U_i$ for all $i = 1, \ldots, n$. Define $W_i = \mathrm{supp}(f_i)$ for each i. Then $\{W_i : i = 1, \ldots, n\}$ also forms an open cover of F and $\bigsqcup_{i=1}^n g_i \overline{W_i} \subset A$. Define $M = \bigsqcup_{i=1}^n g_i W_i$ and thus $\overline{M} = \bigsqcup_{i=1}^n g_i \overline{W_i}$, which is a closed subset of A. The converse is trivial.

For the claim (ii) suppose that $F \prec N \subset \overline{N} \prec B$ holds. Then there are open sets $O_1, \ldots, O_n$ and group elements $g_1, \ldots, g_n \in G$ such that $F \subset \bigcup_{i=1}^n O_i$ and $\bigsqcup_{i=1}^n g_i O_i \subset N$. In addition, for $\overline{N} \prec B$ there are open sets $U_1, \ldots, U_m$ and group elements $h_1, \ldots, h_m \in G$ such that $\overline{N} \subset \bigcup_{j=1}^m U_j$ and $\bigsqcup_{j=1}^m h_j U_j \subset B$. Observe that $\bigsqcup_{i=1}^n g_i O_i \subset N \subset \bigcup_{j=1}^m U_j$. Then $\{O_i \cap g_i^{-1} U_j : i = 1, \ldots, n, j = 1, \ldots, m\}$ form a cover of F and $\{h_j g_i \cdot (O_i \cap g_i^{-1} U_j) = h_j(g_i O_i \cap U_j) : i = 1, \ldots, n, j = 1, \ldots, m\}$ is disjoint in B. This shows that $F \prec B$.

The claim (iii) follows from the two claims before. Since one has $A \prec B$, for every closed subset F of A there is an open subset M such that $F \prec M \subset \overline{M} \subset B \prec C$. Then claim (ii) implies that $F \prec C$. Then $A \prec C$ since F is arbitrary. $\square$

Proposition 3.2.6. Let $\alpha : G \curvearrowright X$ be an action such that there is no G-invariant regular Borel probability measure on X. Consider the following properties:

(i) α has dynamical comparison;

(ii) α has paradoxical comparison;

(iii) α has weak paradoxical comparison;

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). In addition, if α is minimal then these three conditions are equivalent.

Proof. (i)$\Rightarrow$(ii) Let F be a closed subset and O an open subset such that $F \subset O$. Since the space X is Hausdorff and perfect, there are nonempty disjoint open subset O_1, O_2 of O. Observe that $O \prec O_i$ for $i = 1, 2$ since the action has dynamical comparison. Then $F \prec O_i$ for $i = 1, 2$.

(ii)$\Rightarrow$(iii). Suppose that $\alpha : G \curvearrowright X$ has paradoxical comparison. Now given a closed subset K and an open subset O of X such that $K \subset G \cdot O$. Then there is a finite subset E of G such that $K \subset \bigcup_{h\in E} h \cdot O$. Let $n = |E|$. We first claim

$$\underbrace{(O, \ldots, O)}_{n \text{ many}} \prec O.$$

Indeed, let F be a closed subset of O and k an integer such that $2^k \geq n$. By induction we construct two collections of open subsets of O, say $\{M_{i_1 i_2 \ldots i_m} : i_1, \ldots, i_m = 1, 2 \text{ and } 1 \leq m \leq k\}$ and $\{O_{i_1 i_2 \ldots i_m} : i_1, \ldots, i_m = 1, 2 \text{ and } 1 \leq m \leq k\}$ such that

1. $F \prec M_i$ for $i = 1, 2$;

2. for every $1 \leq m \leq k - 1$ and $i_{m+1} = 1, 2$, one has $\overline{M_{i_1 i_2 \ldots i_m}} \prec M_{i_1 i_2 \ldots i_m i_{m+1}}$;

3. $\overline{M_{i_1 i_2 \ldots i_m}} \subset O_{i_1 i_2 \ldots i_m}$ for any integer $m \in [1, k]$ and $i_1, i_2, \ldots, i_m = 1, 2$;

4. for any integer $m \in [1, k]$ the collection $\{O_{i_1 i_2 \ldots i_m} : i_1, \ldots, i_m = 1, 2\}$ is disjoint.

To do this, since $\alpha : G \curvearrowright X$ has paradoxical comparison, $(O, O) \prec O$ implies that for F there are nonempty disjoint open subsets O_1 and O_2 of O such that $F \prec O_i$ for $i = 1, 2$. Then for each i there is an open subset M_i such that $F \prec M_i \subset \overline{M_i} \subset O_i$ by Lemma 3.2.5(i). Then for each $i = 1, 2$, because $(O_i, O_i) \prec O_i$, for $\overline{M_i}$ one can find disjoint nonempty open subsets O_{i1} and O_{i2} of O_i such that $\overline{M_i} \prec O_{ij}$ for $j = 1, 2$. Then Lemma 3.2.5(i) again implies that there are open subsets M_{ij} such that $\overline{M_i} \prec M_{ij} \subset \overline{M_{ij}} \subset O_{ij}$ for $i, j = 1, 2$. Then suppose that we have obtained $\{M_{i_1 i_2 \ldots i_m} : i_1, \ldots, i_m = 1, 2 \text{ and } 1 \leq m \leq l\}$ and $\{O_{i_1 i_2 \ldots i_m} : i_1, \ldots, i_m = 1, 2 \text{ and } 1 \leq m \leq l\}$ for $l < k$ so that they satisfies the conditions above. Then since the action has paradoxical comparison, for each $\overline{M_{i_1 i_2 \ldots i_l}} \subset O_{i_1 i_2 \ldots i_l}$ there are disjoint nonempty open subsets $O_{i_1 i_2 \ldots i_l i_{l+1}}$ of $O_{i_1 i_2 \ldots i_l}$ such that $\overline{M_{i_1 i_2 \ldots i_l}} \prec O_{i_1 i_2 \ldots i_l i_{l+1}}$ where $i_{l+1} = 1, 2$. Then Lemma 3.2.5(i) entails that there are open subsets $M_{i_1 i_2 \ldots i_l i_{l+1}}$ such that $\overline{M_{i_1 i_2 \ldots i_l}} \prec M_{i_1 i_2 \ldots i_l i_{l+1}} \subset \overline{M_{i_1 i_2 \ldots i_l i_{l+1}}} \subset O_{i_1 i_2 \ldots i_l i_{l+1}}$. Observe that $\{O_{i_1 i_2 \ldots i_{l+1}} : i_1, \ldots, i_{l+1} = 1, 2\}$ is indeed disjoint. This finishes our construction, from which for $i_1, \ldots, i_k = 1, 2$ we have

$$F \prec M_{i_1} \subset \overline{M_{i_1}} \prec M_{i_1 i_2} \subset \overline{M_{i_1 i_2}} \prec \cdots \prec M_{i_1 i_2 \ldots i_k}$$

Now we rewrite $\{U_1, \ldots, U_{2^k}\}$ for the disjoint collection $\{M_{i_1 i_2 \ldots i_k} : i_1, \ldots, i_k = 1, 2\}$. Then (ii) in Lemma 3.2.5 implies that $F \prec U_i$ for all $1 \leq i \leq 2^k$. This shows the claim since $2^k \geq n$.

Now write $E = \{h_1, \ldots, h_n\}$ and $K \subset \bigcup_{i=1}^n h_i O$. Then by the partition of unity argument exactly used in the proof of Lemma 4.5(i) there are open subsets $W_i \subset \overline{W_i} \subset h_i O$ for $i = 1, \ldots, n$ such that $K \subset \bigcup_{i=1}^n W_i$. Define $V_i = h_i^{-1} W_i$ and thus $\overline{V_i} = h_i^{-1} \overline{W_i}$. This implies that $K \subset \bigcup_{i=1}^n h_i \overline{V_i}$ where $\overline{V_i} \subset O$ for each $i = 1, \ldots, n$. Define a closed subset $F' = \bigcup_{i=1}^n \overline{V_i} \subset O$. Now consider

$$(\underbrace{O, \ldots, O}_{n \text{ many}}) \prec O.$$

Then there is a collection of disjoint open subsets $\{O_i : i = 1, \ldots, n\}$ such that $F' \prec O_i$ for each $i = 1, \ldots, n$. Then for the collection $\{\overline{V_i} : i = 1, \ldots, n\}$ there is a collection of open subsets $\{U_j^{(i)} : j = 1, \ldots, k_i, i = 1, \ldots, n\}$ and group elements $\{g_j^{(i)} \in G : j = 1, \ldots, k_i, i = 1, \ldots, n\}$ such that $\overline{V_i} \subset F \subset \bigcup_{j=1}^{k_i} U_j^{(i)}$ and $\bigsqcup_{j=1}^{k_i} g_j^{(i)} U_j^{(i)} \subset O_i$ for each $i = 1, \ldots, n$. This implies that the collection of open subsets $\{g_j^{(i)} U_j^{(i)} : j = 1, \ldots, k_i, i = 1, \ldots, n\}$ is disjoint in O. Therefore, $\{h_i U_j^{(i)} : j = 1, \ldots, k_i, i = 1, \ldots, n\}$ form an open cover of K and $\{g_j^{(i)} h_i^{-1} \cdot (h_i U_j^{(i)}) : j = 1, \ldots, k_i, i = 1, \ldots, n\}$ is a disjoint collection of open subsets of O. This verifies $K \prec O$.

(ii)⇒(iii)(if the action is minimal). It suffices to show that for every nonempty open subsets

A, B of X one has $A \prec B$. Indeed for every closed subset $F \subset A$ one always has $F \subset G \cdot B = X$ since the action $\alpha : G \curvearrowright X$ is minimal. Then $F \prec B$ because the action has weak paradoxical comparison. Therefore one has $A \prec B$ since F is arbitrary. $\square$

On the other hand, to make the proposition above more sense we need to show that, unlike dynamical comparison, paradoxical comparison does not necessarily imply that the action is minimal. Otherwise, paradoxical comparison is equivalent to dynamical comparison in general and it suffices to apply Theorem 3.1.4 to establish the pure infiniteness of a reduced crossed product from paradoxical comparison. We will construct an explicit example (Example 3.3.6 below) in which the action has paradoxical comparison but is not minimal.

3.2.2 Uniform Tower Property and Pure Infiniteness

We will show reduced crossed products is purely infinite if the action has the paradoxical comparison and the following property.

Definition 3.2.7. We say an action $\alpha : G \curvearrowright X$ has the *uniform tower property* if for all open subsets O, U of X such that $\overline{O} \subset U$ and all finite subsets T of G there are a nonempty closed set F and an open set W with $F \subset W \subset U$ such that

(i) (T, W) is an open tower;

(ii) $O \cap Y \neq \emptyset$ implies that $F \cap Y \neq \emptyset$ for all G-invariant closed subsets Y of X.

Note that if an action is minimal and topologically free then it has the uniform tower property trivially. The following lemma is a generalization of Lemma 3.4 in [29].

Lemma 3.2.8. *Let $\alpha : G \curvearrowright X$. For non-zero positive functions $f, g \in C(X)_+$, if $\operatorname{supp}(f) \prec \operatorname{supp}(g)$ then $f \precsim g$ in $C(X) \rtimes_r G$*

Proof. In order to show $f \precsim g$ in $C(X) \rtimes_r G$ it suffices to show that $(f - \epsilon)_+ \precsim g$ for all $\epsilon > 0$ by Proposition 2.17 in [3]. We observe that for $f \in C(X)$ and $F \in C_0((0, 1])_+$ one has $F(f)(x) = F(f(x))$ by functional calculus. Therefore, $(f - \epsilon)_+(x) = f(x) - \epsilon$ if $f(x) \geq \epsilon$ while $(f - \epsilon)_+(x) = 0$ if $f(x) < \epsilon$.

For every $\epsilon > 0$, define $C_\epsilon = \{x \in X : f(x) \geq \epsilon\}$. Then $\overline{\mathrm{supp}((f-\epsilon)_+)} \subset C_\epsilon \subset \mathrm{supp}(f)$, which entails that $\overline{\mathrm{supp}((f-\epsilon)_+)} \prec \mathrm{supp}(g)$ since $\mathrm{supp}(f) \prec \mathrm{supp}(g)$. Then we have a family $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of open sets forming a cover of $\overline{\mathrm{supp}((f-\epsilon)_+)}$ and elements $\gamma_1, \gamma_2, \ldots, \gamma_n \in G$ so that $\{\gamma_i U_i : i = 1, 2, \ldots, n\}$ is a disjoint family of open subsets of supp(g). Let $\{f_i : i = 1, 2, \ldots, n\}$ be a partition of unity subordinate to $\mathcal{U}$ so that

1. $0 \leq f_i \leq 1$ for all $i = 1, 2, ..., n$;
2. $\sum_{i=1}^n f_i(x) = 1$ for all $x \in \overline{\text{supp}((f-\epsilon)_+)}$;
3. $\overline{\text{supp}(f_i)} \subset U_i$ for all $i = 1, 2, ..., n$.

Then we have supp$((f-\epsilon)_+) \subset$ supp$(\sum_{i=1}^n f_i)$ and this implies that $(f-\epsilon)_+ \precsim_{C(X)} \sum_{i=1}^n f_i$. Define $u = \bigoplus_{i=1}^n u_{\gamma_i}$. We have

$$\sum_{i=1}^n f_i \precsim \bigoplus_{i=1}^n f_i \sim u(\bigoplus_{i=1}^n f_i)u^* = \bigoplus_{i=1}^n \alpha_{\gamma_i}(f_i).$$

Then since supp$(\alpha_{\gamma_i}(f_i)) \subset \gamma_i U_i$ for every $i = 1, 2, \ldots n$ and supp$(\sum_{i=1}^n \alpha_{\gamma_i}(f_i)) \subset \bigsqcup_{i=1}^n \gamma_i U_i \subset$ supp(g), we have

$$\bigoplus_{i=1}^n \alpha_{\gamma_i}(f_i) \sim \sum_{i=1}^n \alpha_{\gamma_i}(f_i) \precsim g.$$

Therefore, we have $(f-\epsilon)_+ \precsim g$ in $C(X) \rtimes_r G$ and thus $f \precsim g$ in $C(X) \rtimes_r G$ because the ϵ is arbitrary. $\square$

Proposition 3.2.9. Suppose that the action $\alpha : G \curvearrowright X$ has paradoxical comparison. Then $f \oplus f \precsim f$ in $C(X) \rtimes_r G$ for every non-zero function $f \in C(X)_+$.

Proof. Let f be a non-zero element in $C(X)_+$ and $\epsilon > 0$. Denote by $O = \mathrm{supp}(f)$ and $F = \overline{\mathrm{supp}(f-\epsilon)_+}$. Then there are nonempty disjoint open subsets O_1, O_2 of O such that $F \prec O_1$ and $F \prec O_2$. Choose two positive functions $h_1, h_2 \in C(X)$ such that $\mathrm{supp}(h_i) = O_i$ for $i = 1, 2$. Then one has $(f-\epsilon)_+ \precsim h_i$ for $i = 1, 2$ by the proof of Lemma 3.2.8. This implies that

$(f-\epsilon)_+ \oplus (f-\epsilon)_+ \precsim h_1 \oplus h_2 \sim h_1 + h_2 \precsim f$ in $C(X) \rtimes_r G$. Thus, by proposition 3.3 in [38] one has $f \oplus f \precsim f$ since the ϵ is arbitrary. $\square$

Lemma 3.2.10. *Suppose that an action $\alpha : G \curvearrowright X$ has weak paradoxical comparison. Let F be a closed subset and O an open subset of X . Suppose that $F \cap Y \neq \emptyset$ implies $O \cap Y \neq \emptyset$ for all closed proper G-invariant subspaces Y. Then $F \prec O$.*

Proof. Since the action has weak paradoxical comparison, it suffices to verify $F \subset G \cdot O$. Indeed, let $x \in F$ and define $Y = \overline{G \cdot x}$. Now we have $F \cap Y \neq \emptyset$ and thus $O \cap Y \neq \emptyset$ holds by the assumption. This implies that there is a $g \in G$ such that $gx \in O$ which implies that $x \in G \cdot O$. Since x is an arbitrary element of F, one has $F \subset G \cdot O$. $\square$

The proof of the following proposition contains ideas from Lemma 7.8 and 7.9 in [53].

Proposition 3.2.11. Suppose that an action $\alpha : G \curvearrowright X$ has weak paradoxical comparison as well as the uniform tower property. Then $E(a) \precsim a$ in $C(X) \rtimes_r G$ for every positive $a \in C(X) \rtimes_r G$.

Proof. It suffices to show the case that a is a non-zero positive element in $C(X) \rtimes_r G$ with $\|a\| = 1$. Observe that $E(a) \neq 0$ since E is faithful. Define $O = \mathrm{supp}(E(a))$. Fix an $\epsilon \in (0, \|E(a)\|)$ and define $U = \mathrm{supp}(E(a) - \epsilon)_+ = \{x \in X : E(a)(x) > \epsilon\}$ by the functional calculus argument in the proof of Lemma 3.2.8. Then choose a $\delta \in (0, \epsilon/4)$ and a $c \in C_c(G, C(X))$ with $\|c\| \leq 2$ and $\|c - a^{\frac{1}{2}}\| < \frac{\delta}{8}$. This implies that

$$\|c^*c - a\| \leq \|c^* - a^{\frac{1}{2}}\|\|c\| + \|a^{\frac{1}{2}}\|\|c - a^{\frac{1}{2}}\| < \frac{3\delta}{8} < \frac{\delta}{2};$$

and

$$\|cc^* - a\| \leq \|c - a^{\frac{1}{2}}\|\|c^*\| + \|a^{\frac{1}{2}}\|\|c^* - a^{\frac{1}{2}}\| < \frac{3\delta}{8} < \frac{\delta}{2}.$$

We write $b = c^*c = \sum_{t \in T} b_t u_t$, where T is a finite subset of G. Since b is positive non-zero element in $C(X) \rtimes_r G$ and the canonical conditional expectation E is faithful, one has $E(b) =$

$b_e \neq 0$ and $e \in T$. We also observe that $\|E(b) - E(a)\| < \delta/2$, which implies that $\overline{U} \subset \{x \in X : E(a)(x) > \frac{\epsilon}{2}\} \subset \{x \in X : E(b)(x) > \frac{\epsilon}{2} - \frac{\delta}{2}\}$. We write M for the open subset $\{x \in X : E(b)(x) > \frac{\epsilon}{2} - \frac{\delta}{2}\}$ for simplicity. One observes that $M \subset \{x \in X : E(b)(x) \geq \frac{\epsilon}{2} - \frac{\delta}{2}\} \subset \{x \in X : E(b)(x) > \frac{\epsilon}{2} - \delta\}$.

Now apply the uniform tower property to $\overline{M} \subset \{x \in X : E(b) > \frac{\epsilon}{2} - \delta\}$ so that one obtains a nonempty closed set F and an open set W with $F \subset W \subset \{x \in X : E(b) > \frac{\epsilon}{2} - \delta\}$ such that (T, W) is a tower and

$$M \cap Y \neq \emptyset \Longrightarrow F \cap Y \neq \emptyset$$

for all closed G-invariant subsets Y of X.

Then choose a continuous function $f \in C(X)$ satisfying

$$0 \leq f \leq 1, \ \ \mathrm{supp}(f) \subset W, \ \ \text{and} \ \ f|_F \equiv 1.$$

Then one has

$$fbf = fE(b)f + \sum_{t \in T \setminus \{e\}} fb_t u_t f = fE(b)f + \sum_{t \in T \setminus \{e\}} fb_t \alpha_t(f) u_t.$$

Since $\mathrm{supp}(\alpha_t(f)) \subset tW$ and $\{tW : t \in T\}$ is an open tower, $fb_t\alpha_t(f) = b_t f \alpha_t(f) = 0$ whenever $t \neq e$. This entails that

$$fbf = fE(b)f \in C(X)_+.$$

In addition, since $F \subset \{x \in X : E(b) > \frac{\epsilon}{2} - \delta\}$, for every $x \in F$ one has $(fE(b)f)(x) = E(b)(x) > \frac{\epsilon}{2} - \delta > \delta$ by our choice of δ. This implies that $F \subset \mathrm{supp}((fE(b)f - \delta)_+)$. Thus $F \cap Y \neq \emptyset$ implies that $\mathrm{supp}((fE(b)f - \delta)_+) \cap Y \neq \emptyset$ for all closed G-invariant subspaces Y of X. Therefore, by the argument above we have

$$\overline{U} \cap Y \neq \emptyset \Longrightarrow M \cap Y \neq \emptyset \Longrightarrow F \cap Y \neq \emptyset \Longrightarrow \mathrm{supp}((fE(b)f - \delta)_+) \cap Y \neq \emptyset$$

for all closed proper G-invariant subspaces Y of X. Then Lemma 3.2.10 implies that

$$\overline{\operatorname{supp}(E(a)-\epsilon)_+} = \overline{U} \prec \operatorname{supp}((fE(b)f-\delta)_+).$$

Now the proof of Lemma 3.2.8 entails that

$$(E(a)-\epsilon)_+ \precsim (fE(b)f-\delta)_+ = (fbf-\delta)_+.$$

On the other hand, Lemmas 1.4 and 1.7 in [53] imply that

$$(fbf-\delta)_+ = (fc^*cf-\delta)_+ \sim (cf^2c^*-\delta)_+ \precsim (cc^*-\delta)_+ \precsim a.$$

Therefore, we have $(E(a)-\epsilon)_+ \precsim a$ in $C(X) \rtimes_r G$. Since ϵ is arbitrary one has $E(a) \precsim a$ as desired. $\square$

Now, we are able to establish the following theorem.

Theorem 3.2.12. *Let G be a countable infinite discrete group, X a compact Hausdorff space and $\alpha : G \curvearrowright X$ an exact essentially free continuous action of G on X. Suppose that the action α has paradoxical comparison as well as the uniform tower property. Then the reduced crossed product $C(X) \rtimes_r G$ arising from α is purely infinite.*

Proof. Suppose that the action $\alpha : G \curvearrowright X$ is exact and essentially free. In addition, suppose that α has paradoxical comparison as well as the uniform tower property. It was shown in [65] that if the group action $\alpha : G \curvearrowright X$ is exact and essentially free then $C(X)$ separates ideals in $C(X) \rtimes_r G$. In addition, by Proposition 3.2.9 and 3.2.11, we have verified that all non-zero positive elements in $C(X)$ are properly infinite in $C(X) \rtimes_r G$ and $E(a) \precsim a$ for all positive elements a in $C(X) \rtimes_r G$. Then Proposition 1.1.3 implies that the reduced crossed product $C(X) \rtimes_r G$ is purely infinite. $\square$

3.3 Applications and Examples

3.3.1 Finite Many Ideals Case

In this section we will provide some applications of Theorem 3.2.12 by proving the following corollaries.

Corollary 3.3.1. *Let G be a countable infinite discrete group, X a compact Hausdorff space and $\alpha : G \curvearrowright X$ an exact essentially free continuous action of G on X. Suppose that the action α has paradoxical comparison and there are only finitely many G-invariant closed subsets of X. Then the reduced crossed product $C(X) \rtimes_r G$ arising from α is purely infinite and has finitely many ideals.*

Proof. Recall the setting that the action $\alpha : G \curvearrowright X$ is exact and essentially free. In addition, we assume that it has paradoxical comparison. Then to show pure infiniteness by Theorem 3.2.12 it suffices to show that the action $\alpha : G \curvearrowright X$ has the uniform tower property. To this end, we begin with open sets O, U such that $\overline{O} \subset U$ and a finite subset T of G. Since there are only finitely many G-invariant closed subsets of X, the set $\mathcal{I} = \{Y \subset X : O \cap Y \neq \emptyset,\ Y \text{ closed and } G \cdot Y = Y\}$ has minimal elements with respect to the partial order "$\subset$", where a minimal element $Y \in \mathcal{I}$ means that there is no G-invariant subset $Z \in \mathcal{I}$ such that $Z \subset Y$. Denote by $\{Y_1, \ldots, Y_m\}$ the set of all minimal elements in $\mathcal{I}$. Then we claim $O \cap (Y_i \setminus \bigcup_{j \neq i} Y_j) \neq \emptyset$ for each $i = 1, \ldots, m$. Suppose not, let $O \cap (Y_i \setminus \bigcup_{j \neq i} Y_j) = (O \cap Y_i) \setminus \bigcup_{j \neq i} Y_j = \emptyset$ for some i. This implies that $\emptyset \neq O \cap Y_i \subset \bigcup_{j \neq i} Y_j$ and thus $O \cap Y_i \cap Y_j \neq \emptyset$ for some $j \neq i$. However, this implies that $Y_i \cap Y_j \in \mathcal{I}$, which is a contradiction to the minimality of Y_i and Y_j in $\mathcal{I}$. This shows the claim.

Define $D_T = \{x \in X : tx \neq x \text{ for all } t \in T^{-1}T \setminus \{e\}\} = \bigcap_{t \in T^{-1}T \setminus \{e\}} \{x \in X : tx \neq x\}$. Since the action $\alpha : G \curvearrowright X$ is essentially free, $D_T \cap Y$ is open dense in Y with respect to the relative topology for all G-invariant proper closed subset Y of X. From the claim above we see $O \cap (Y_i \setminus \bigcup_{j \neq i} Y_j)$ is a non-empty relatively open subset of Y_i and thus $M_{i,T} = D_T \cap O \cap (Y_i \setminus \bigcup_{j \neq i} Y_j) \neq \emptyset$. Now choose $x_i \in M_{i,T}$ for each $i = 1, \ldots, m$. Since each $Y_i \setminus \bigcup_{j \neq i} Y_j$ is a G-invariant subset, the points in $\{tx_i : i = 1, \ldots, m, t \in T\}$ are pairwise different. Then since the space is Hausdorff, there is a disjoint collection of open subsets of X, say $\{O_{tx_i} \subset X : i = 1, 2, \ldots, m, t \in T\}$ such that $tx_i \in O_{tx_i}$ for $t \in T$ and $i = 1, \ldots, m$. Now define $W_i = \bigcap_{t \in T} t^{-1}O_{tx_i} \cap O$ for $i = 1, \ldots, m$. Then (T, W_i) form an open tower and $TW_i \cap TW_j = \emptyset$ for $1 \leq i \neq j \leq m$. In addition we may choose a closed subset F_i of X such that $x_i \in F_i \subset W_i$ for $i = 1, \ldots, m$ by normality of the space X. Now define $W = \bigsqcup_{i=1}^m W_i \subset O \subset U$ and $F = \bigsqcup_{i=1}^m F_i$. Then (T, W) form an open tower by our construction. In addition, let $Y \in \mathcal{I}$. Then there is a minimal element $Y_i \in \mathcal{I}$ such that $Y_i \subset Y$ where $1 \leq i \leq m$. Then $F \cap Y_i \neq \emptyset$ by our construction and thus $F \cap Y \neq \emptyset$. This shows that the action $\alpha : G \curvearrowright X$ has the uniform tower property.

On the other hand, since the action $\alpha : G \curvearrowright X$ is essentially free, $C(X)$ separates ideals in the crossed product $C(X) \rtimes_r G$. Therefore the number of G-invariant closed subsets is equal to the number of ideals in $C(X) \rtimes_r G$ and thus the crossed product has finitely many ideals. □

3.3.2 Product of Spaces Case

We also have the following result for "amplifications" of minimal topologically free actions, i.e., products of such an action with a trivial action. Indeed, the space Y in the corollary below may be viewed as an index set so that $\alpha : G \curvearrowright X \times Y$ decomposes into $|Y|$-many disjoint copies of minimal subsystems of $\beta : G \curvearrowright X$. Denote by π_X, π_Y projection maps from $X \times Y$ to X and Y respectively. We start with lemmas.

Proposition 3.3.2. Let $\beta : G \curvearrowright X$ be a minimal topologically free action that has no G-invariant regular Borel probability measure. Suppose that β has dynamical comparison. Let Y be another compact Hausdorff space. Let $\alpha : G \curvearrowright X \times Y$ be an action defined by $\alpha_g((x, y)) = (\beta_g(x), y)$.

(i) If $M \subset X \times Y$ is a G-invariant closed subset of the action α then $M = X \times \pi_Y(M)$.

(ii) The action α is essentially free.

Proof. For (i) it suffices to show $X \times \pi_Y(M) \subset M$ since the converse direction is trivial. Fix a $y \in \pi_Y(M)$. For every $x \in X$ and every neighbourhood O of x, there is a $g \in G$ such that $\beta_g(x) \in O$. This implies that $\alpha_g(x_y, y) = (\beta_g(x), y) \in O \times \{y\}$ and thus the restriction of α on

$X \times \{y\}$ is minimal with respect to the relative topology. Then since $M \cap (X \times \{y\})$ is a closed G-invariant subset of $X \times \{y\}$, one has $M \cap (X \times \{y\}) = X \times \{y\}$ and thus $X \times \{y\} \subset M$. Therefore one has $X \times \pi_Y(M) \subset M$.

For (ii) it suffices to show that the action α is topologically free when it restrict to every G-invariant closed subset $X \times P$ for some closed $P \subset Y$ by (i). Indeed, for each $g \in G$, one has:

$$\{(x, y) \in X \times P : \alpha_g(x, y) = (x, y)\} = \{x \in X : \beta_g(x) = x\} \times P$$

whose interior in $X \times P$ is empty since the interior of $\{x \in X : \beta_g(x) = x\}$ is empty in X. This shows that action α is topologically free on $X \times P$ and thus α is essentially free. $\square$

Proposition 3.3.3. Suppose that $\alpha : G \curvearrowright X \times Y$ is the action in Proposition 3.3.2. Then α has paradoxical comparison.

Proof. Let O be an open subset and F be a closed subset of $X \times Y$ such that $F \subset O$. For all $(x, y) \in F$ there is an open neighbourhood $M_x \times N_y$ of (x, y) such that $(x, y) \in M_x \times N_y \subset O$. all of these neighbourhoods form an open cover of F so that we can choose a finite subcover, say $F \subset \bigcup_{i=1}^m M_i \times N_i$. Then by the argument of partition of unity appeared in the present paper many times, there is a collection of closed subsets $\{F_i : i = 1, \ldots, m\}$ such that $F_i \subset M_i \times N_i$ and $F \subset \bigcup_{i=1}^m F_i$. Then since the space X is perfect we choose a collection of different points $\{x_{ij} \in M_i : i = 1, \ldots, m, j = 1, 2\}$. Then since X is Hausdorff there is a collection of disjoint open sets $\{O_{ij} \ni x_{ij} : i = 1, \ldots, m, j = 1, 2\}$. For each i, j we may assume $O_{ij} \subset M_i$ by redefining $O_{ij} := O_{ij} \cap M_i$. Now for $j = 1, 2$ we define $O_j = \bigsqcup_{i=1}^m O_{ij} \times N_i \subset O$. Then it suffices to verify $F \prec O_j$ for $j = 1, 2$.

Now fix $j \in \{1, 2\}$. For each $i = 1, \ldots, m$, since $F_i \subset M_i \times N_i$ one has $\pi_X(F_i)$ is a compact subset of M_i. Since $\beta : G \curvearrowright X$ has dynamical comparison, one has $M_i \prec O_{ij}$ for each $i = 1, \ldots, m$, which means that there is a collection of open subsets of X, $\{P_1^{(i)}, \ldots, P_{n_i}^{(i)}\}$ and a collection of group elements $\{g_1^{(i)}, \ldots, g_{n_i}^{(i)}\}$ such that $\pi_X(F_i) \subset \bigcup_{k=1}^{n_i} P_k^{(i)}$ and $\bigsqcup_{k=1}^{n_i} \beta_{g_k^{(i)}}(P_k^{(i)}) \subset$

O_{ij} for $i = 1, \ldots, m$. Then one has

$$F_i \subset \pi_X(F_i) \times \pi_Y(F_i) \subset (\bigcup_{k=1}^{n_i} P_k^{(i)}) \times N_i = \bigcup_{k=1}^{n_i}(P_k^{(i)} \times N_i);$$

while

$$\bigsqcup_{k=1}^{n_i} \alpha_{g_k^{(i)}}(P_k^{(i)} \times N_i) = \bigsqcup_{k=1}^{n_i}(\beta_{g_k^{(i)}}(P_k^{(i)}) \times N_i) \subset O_{ij} \times N_i.$$

Therefore, one has:

$$F \subset \bigcup_{i=1}^{m} F_i \subset \bigcup_{i=1}^{m}\bigcup_{k=1}^{n_i}(P_k^{(i)} \times N_i);$$

while

$$\bigsqcup_{i=1}^{m}\bigsqcup_{k=1}^{n_i} \alpha_{g_k^{(i)}}(P_k^{(i)} \times N_i) \subset \bigsqcup_{i=1}^{m} O_{ij} \times N_i = O_j.$$

This verifies that under the action α, one has $F \prec O_j$ for $j = 1, 2$ as desired. $\square$

Proposition 3.3.4. Suppose that $\alpha : G \curvearrowright X \times Y$ is the action in Proposition 3.3.2. Then α has the uniform tower property.

Proof. Let O, U be open subsets of $X \times Y$ such that $\overline{O} \subset U$. Let T be a finite subset of G. For every $(x, y) \in \overline{O}$ there is an open neighbourhood $M_x \times N_y$ of x such that $(x, y) \in M_x \times N_y \subset U$. all of these neighbourhoods form an open cover of $\overline{O}$ so that we can choose a finite subcover, say, $\overline{O} \subset \bigcup_{i=1}^{n} M_i \times N_i \subset U$.

Now, since the action $\beta : G \curvearrowright X$ is topologically free, $D_T = \{x \in X : \beta_t(x) \neq x \text{ for all } t \in T^{-1}T \setminus \{e\}\}$ is open dense in X. Now, for each $i = 1, 2, \ldots, n$ choose a point $x_i \in M_i \cap D_T$ and an open neighbourhood O_i of x_i such that $x_i \in O_i \subset M_i$ and (T, O_i) form an open tower in X. In addition, $\{x_i, O_i : i = 1, 2, \ldots, n\}$ can be chosen properly such that $TO_i \cap TO_j = \emptyset$ whenever $1 \leq i \neq j \leq n$.

We can do this since the space X is Hausdorff and perfect. We do this by induction until n. First choose $x_1 \in M_1 \cap D_T$. Then the points in $\{\beta_t(x_1) : t \in T\}$ are pairwise different. Suppose that for a $k < n$ the set of different points $\{\beta_t(x_i) : t \in T, i = 1, 2, \ldots k\}$ has been defined. Then choose $x_{k+1} \in (M_{k+1} \cap D_T) \setminus \{\beta_t(x_i) : t \in T^{-1}T, i = 1, 2, \ldots k\}$. We can do this since the space is perfect.

This finishes construction of points $x_i \in M_i$ since the points in $\{\beta_t(x_i) : t \in T, i = 1, 2, \dots, n\}$ are pairwise different. Then for the set $\{\beta_t(x_i) : t \in T, i = 1, 2, \dots, n\}$ there is a disjoint family of open subsets $\{O_{tx_i} : t \in T, i = 1, 2, \dots, n\}$ such that $\beta_t(x_i) \in O_{tx_i}$ for $t \in T$ and $i = 1, 2, \dots, n$. Then define $O_i = \bigcap_{t \in T} \beta_{t^{-1}}(O_{tx_i}) \cap M_i$ for $i = 1, 2, \dots, n$.

Then for each $i = 1, 2, \dots, n$ choose an open set P_i such that $\overline{P_i} \subset O_i \subset M_i$. Now consider $\pi_Y(\overline{O})$ is a compact subset of $\bigcup_{i=1}^n N_i$. Then by the argument of partition of unity, for each $i = 1, 2, \dots, n$ there is an open set H_i such that $\overline{H_i} \subset N_i$ and $\pi_Y(\overline{O}) \subset \bigcup_{i=1}^n \overline{H_i}$. Now define $W = \bigsqcup_{i=1}^n O_i \times N_i$ and $F = \bigsqcup_{i=1}^n \overline{P_i} \times \overline{H_i}$. Observe that $F \subset W \subset U$.

Then (T, W) is a tower. Indeed for two distinct elements $t, s \in T$, one has

$$\begin{aligned}\alpha_t(W) \cap \alpha_s(W) &= (\bigsqcup_{i=1}^n \beta_t(O_i) \times N_i) \cap (\bigsqcup_{j=1}^n \beta_s(O_j) \times N_j) \\ &= \bigsqcup_{i=1}^n \bigsqcup_{j=1}^n (\beta_t(O_i) \cap \beta_s(O_j)) \times (N_i \cap N_j) = \emptyset.\end{aligned}$$

Finally, we see $O \cap Z \neq \emptyset$ implying $F \cap Z \neq \emptyset$ for all G-invariant closed subsets Z of X. First one has

$$\pi_Y(F) = \bigcup_{i=1}^n \overline{H_i} \supset \pi_Y(\overline{O}) \supset \pi_Y(O).$$

Then let Z be a G-invariant closed subset of $X \times Y$. Then Z necessarily is of the form $X \times P$ for some closed $P \subset Y$ by Proposition 3.3.2. Suppose that $O \cap Z \neq \emptyset$. Then $\emptyset \neq \pi_Y(O \cap Z) \subset \pi_Y(O) \cap P$ and thus $\pi_Y(F) \cap P \neq \emptyset$. This implies that $F \cap Z = F \cap (X \times P) \neq \emptyset$ as desired. $\square$

Corollary 3.3.5. *Let G be a countable infinite exact discrete group, X a compact Hausdorff space and $\beta : G \curvearrowright X$ a minimal topologically free continuous action of G on X. Suppose that there is no G-invariant regular Borel probability measure on X and β has dynamical comparison. Let Y be another compact Hausdorff space. Let $\alpha : G \curvearrowright X \times Y$ be an action defined by $\alpha_g((x, y)) = (\beta_g(x), y)$. Then $C(X \times Y) \rtimes_{\alpha, r} G$ is purely infinite.*

Proof. Since the group G is exact, the action $\alpha : G \curvearrowright X \times Y$ is exact. In addition, Proposition 3.3.2, 3.3.3 and 3.3.4 show that the action $\alpha : G \curvearrowright X \times Y$ is essentially free and has paradoxical comparison as well as the uniform tower property. This means that α satisfies all conditions of Theorem 3.2.12 and thus $C(X \times Y) \rtimes_{\alpha,r} G$ is purely infinite. $\square$

The following explicit example is a direct application of the corollary above.

Example 3.3.6. In particular, for an exact group G consider a topologically free, amenable strong boundary action $\beta : G \curvearrowright X$ on a compact metrizable space X. Such an example exists, like Example 2.2 in [41]. Let Y be a compact Hausdorff space and $\alpha : G \curvearrowright X \times Y$ be the action mentioned above. Then $C(X \times Y) \rtimes_{\alpha,r} G$ is purely infinite.

4. SEMIGROUPS OF DYNAMICAL SYSTEMS

Dynamical comparison and paradoxical comparison also relate to the almost unperforation of the type semigroups of actions on the Cantor set. It has been asked in [54] and [60] to what extent the type semigroup of an action on Cantor set is almost unperforated. For a minimal free action $\alpha : G \curvearrowright X$ of an amenable discrete infinite group G, Kerr [34] showed that if the type semigroup of α, denoted by $V(X, G)$, is almost unperforated then α has dynamical comparison. In addition, he showed that if the action α satisfies a notion called almost finiteness then $V(X, G)$ is almost unperforated. A recent work of Kerr and Szabó [36] showed that for such an action α on the Cantor set X, it has dynamical comparison if and only if it is almost finite. Therefore $V(X, G)$ is almost unperforated if and only if $\alpha : G \curvearrowright X$ has dynamical comparison provided that G is amenable and the action α is minimal and free. In this section, we claim the same conclusion under the hypothesis that the action α is minimal and has no G-invariant Borel probability measures.

Furthermore, in this chapter, we extend the notion of the type semigroup to dynamical systems that the space is not necessarily the Cantor set. This leads to another characterization of dynamical comparison.

4.1 The Type Semigroup

Throughout this section X denotes the Cantor set. We will study the type semigroup associated to an action $\alpha : G \curvearrowright X$. To begin the story, we recall some general background information.

A state on a preordered monoid $(W, +, \leq)$ is an order preserving morphism $f : W \to [0, \infty]$. We say that a state is *non-trivial* if it takes a value different from 0 and ∞. We denote by $S(W)$ the set consisting of all states of W and by $SN(W)$ the set of all non-trivial states. We write $S(W, x) = \{f \in S(W) : f(x) = 1\}$, which is a subset of $SN(W)$.

We say that an element $x \in W$ is *properly infinite* if $2x \leq x$. We say that the monoid W is *purely infinite* if every $x \in W$ is properly infinite. In addition, we say that the monoid W is *almost unperforated* if, whenever $x, y \in W$ and $n \in \mathbb{N}$ are such that $(n+1)x \leq ny$, one has $x \leq y$. The following proposition due to Ortega, Perera, and Rørdam is very useful.

Proposition 4.1.1. ([49, Proposition 2.1]) Let $(W, +, \leq)$ be an ordered abelian semigroup, and let $x, y \in W$. Then the following conditions are equivalent:

(i) There exists $k \in \mathbb{N}$ such that $(k+1)x \leq ky$.

(ii) There exists $k_0 \in \mathbb{N}$ such that $(k+1)x \leq ky$ for every $k \geq k_0$.

(iii) There exists $m \in \mathbb{N}$ such that $x \leq my$ and $f(x) < f(y)$ for every state $f \in S(W, y)$.

For an action $\alpha : G \curvearrowright X$, we can associated to it a preordered monoid called the *type semigroup* (see [34], [60], [54] and [72]) .We will use the following formulation that appears in [34] and [54]. We again write α for the induced action on $C(X)$, which is given by $\alpha_s(f)(x) = f(s^{-1}x)$ for all $s \in G$, $f \in C(X)$, and $x \in X$. On the space $C(X, \mathbb{Z}_{\geq 0})$ consider the equivalence relation defined by $f \sim g$ if there are $h_1, h_2, \ldots h_n \in C(X, \mathbb{Z}_{\geq 0})$ and $s_1, s_2, \ldots, s_n \in G$ such that $\sum_{i=1}^n h_i = f$ and $\sum_{i=1}^n \alpha_{s_i}(h_i) = g$. We write $V(X, G)$ for the quotient $C(X, \mathbb{Z}_{\geq 0})/\sim$ and define an operation on $V(X, G)$ by $[f] + [g] = [f + g]$. Moreover, we endow $V(X, G)$ with the algebraic order, i.e., for $a, b \in V(X, G)$ we declare that $a \leq b$ whenever there exists a $c \in V(X, G)$ such that $a + c = b$. Then it can be verified that $V(X, G)$ is a well-defined preordered Abelian semigroup. We call it the *type semigroup* of α.

In this Cantor set context, we can rephrase the dynamical comparison in the language of the type semigroup. In fact Proposition 3.5 in [34] implies that for all clopen subsets A, B of X one has $A \prec B$ if and only if $[1_A] \leq [1_B]$. In addition, if there is no G-invariant Borel probability measure on X, Proposition 3.6 in [34] shows that the action has dynamical comparison if and only if $[1_A] \leq [1_B]$ for all clopen subsets A, B of X.

We remark that $SN(V(X, G)) = \emptyset$ if the action is minimal and there is no G-invariant Borel probability measure. Indeed, Lemma 5.1 in [60] shows that of the acition is minimal then every state in $SN(W(X, G))$ induces a non-trivial Borel probability measure on X. Therefore $M_G(X) = \emptyset$ implies that $SN(V(X, G)) = \emptyset$ provided the action is minimal.

The proof of the following proposition contains ideas from Lemma 13.1 in [34].

Proposition 4.1.2. Let $\alpha : G \curvearrowright X$ be a minimal action such that there is no G-invariant Borel probability measure on X. Then $V(X, G)$ is almost unperforated if and only if α has dynamical comparison.

Proof. Suppose that $V(X, G)$ is almost unperforated. To show that α has dynamical comparison, by the discussion above it suffices to show that for all clopen subsets $A, B \subset X$ we have $[1_A] \leq [1_B]$. Since the action α is minimal, X is covered by finitely many translates of B. This implies that $[1_A] \leq [1_X] \leq m[1_B]$ for some $m \in \mathbb{N}$. Observe that $S(W(X, G), [1_B]) \subset SN(V(X, G)) = \emptyset$ by the remark above. It follows from Proposition 6.1 that there exists an $n \in \mathbb{N}$ such that $(n+1)[1_A] \leq n[1_B]$. Then the almost unperforation of $V(X, G)$ entails that $[1_A] \leq [1_B]$ as desired.

For the converse direction, we show that if α has dynamical comparison then $[f] \leq [g]$ for all $[f], [g] \in V(X, G)$, which trivially implies almost unperforation. First, since α has dynamical comparison then for all clopen subsets A, B of X one has $[1_A] \leq [1_B]$. Let $f, g \in C(X, \mathbb{Z}_{\geq 0})$, we can write $f = \sum_{i=1}^{n} 1_{A_i}$ and $g = \sum_{j=1}^{m} 1_{B_j}$, where $A_i = \{x \in X : f(x) \geq i\}$ and $B_j = \{x \in X : g(x) \geq j\}$ with $n = \max_{x \in X} f(x)$ and $m = \max_{x \in X} g(x)$. Since $[1_{A_i}] \leq [1_{B_i}]$ for every $i \leq n$, if $n \leq m$, we have

$$[f] = \sum_{i=1}^{n} [1_{A_i}] \leq \sum_{i=1}^{n} [1_{B_i}] \leq [g]$$

Suppose that $n > m$. Choose $n - m + 1$ many pairwise disjoint nonempty clopen subsets of B_m, denoted by $\{C_k : k = 0, 1, \ldots, n - m\}$. Then dynamical comparison implies that $[1_{A_{m+k}}] \leq [1_{C_k}]$ for $k = 0, 1, \ldots, n - m$. Now we have

$$[f] = \sum_{i=1}^{m-1} [1_{A_i}] + \sum_{k=0}^{n-m} [1_{A_{m+k}}] \leq \sum_{j=1}^{m-1} [1_{B_j}] + \sum_{k=0}^{n-m} [1_{C_k}] \leq \sum_{j=1}^{m} [1_{B_j}] = [g]$$

This verifies that $[f] \leq [g]$ for all $[f], [g] \in V(X, G)$. □

Recall that under the assumption that G is amenable and α is minimal and free the results of Kerr [34] and Kerr-Szabó [36] show that $V(X, G)$ is almost unperforated if and only if the action

$\alpha : G \curvearrowright X$ has dynamical comparison. Combining Proposition 6.2 with this result, we obtain the following corollary.

Corollary 4.1.3. *Let $\alpha : G \curvearrowright X$ be an amenable minimal free action. Then $V(X, G)$ is almost unperforated if and only if α has dynamical comparison.*

In veiw of this result, it is a natural problem to try to determine the relation between paradoxical comparison and almost unperforation of the type semigroup of a non-minimal action on the Cantor set. To this end, we now proceed to establish our main theorem in this section (Theorem 4.1.5 below). Recall that we have shown that paradoxical comparison on the Cantor set implies that there is no non-trivial Borel measure. Then the answer hides in the following theorem, which is a slightly stronger version of Theorem 5.4 in [60]. This theorem shows the relationship among the type semigroup, C^*-algebras and paradoxical comparison. We need to say that we add no new condition at all to Theorem 5.4 in [60] since our paradoxical comparison is equivalent to the condition that every clopen subset of X is (G, τ_X)-paradoxical on the Cantor set. However, what is new here is the equivalence of (i), (ii) and (iii) without the hypothesis of almost unperforation.

Theorem 4.1.4. *Let $\alpha : G \curvearrowright X$ be an continuous action with G exact and X the Cantor set. Suppose that the action α is essentially free. Consider the following properties.*

(i) *α has paradoxical comparison;*

(ii) *$V(X, G)$ is purely infinite;*

(iii) *Every clopen subset of X is (G, τ_X)-paradoxical;*

(iv) *The C^*-algebra $C(X) \rtimes_r G$ is purely infinite;*

(v) *The C^*-algebra $C(X) \rtimes_r G$ is traceless in the sense that it admits no non-zero lower semi-continuous (possibly unbounded) 2-quasitraces defined on an algebraic ideal (see [39]);*

(vi) *There are no non-trivial states on $V(X, G)$.*

Then (i)⇔(ii)⇔(iii)⇒(iv)⇒(v)⇒(vi). Moreover, if $V(X,G)$ is almost unperforated then (vi)⇒(i), whence all of these properties are equivalent.

Proof. It has been proved in [60] that (ii)⇒(iii)⇒(iv)⇒(v)⇒(vi) and (vi)⇒(ii) whenever $V(X,G)$ is almost unperforated. We have verified (iii)⇔(i) in general in the paragraph after Definition 3.2.3. Therefore it suffices to show (i)⇒(ii).

(i)⇒(ii). Fix an element $[f] \in V(X,G)$. Write one of its representative f to be $f = \sum_{i=1}^{n} 1_{A_i}$, where $A_i = \{x \in X : f(x) \geq i\}$ with $n = \max_{x \in X} f(x)$. Since α has paradoxical comparison, for each A_i one finds two disjoint open subsets $U_{i,1}$ and $U_{i,2}$ of A_i such that $A_i \prec U_{i,1}$ and $A_i \prec U_{i,2}$. Then for $j = 1, 2$, Proposition 3.5 in [34] allows us to find a finite clopen partition $\mathcal{P}^{(j)} = \{V_1^{(j)}, \ldots, V_{n_j}^{(j)}\}$ of A_i and group elements $s_1^{(j)}, \ldots, s_{n_j}^{(j)} \in G$ such that $\bigsqcup_{k=1}^{n_j} s_k^{(j)} V_k^{(j)} \subset U_{i,j}$. We may assume each $U_{i,j}$ is clopen by redefining $U_{i,j} := \bigsqcup_{k=1}^{n_j} s_k^{(j)} V_k^{(j)}$ for each $j = 1, 2$. This implies that $[1_{A_i}] \leq [1_{U_{i,j}}]$ for $j = 1, 2$. This implies that

$$[1_{A_i}] + [1_{A_i}] \leq [1_{U_{i,1}}] + [1_{U_{i,2}}] \leq [1_{A_i}].$$

Therefore we have $2[f] = 2[\sum_{i=1}^{n} 1_{A_i}] \leq [\sum_{i=1}^{n} 1_{A_i}] = [f]$, which means that $V(X,G)$ is purely infinite.

□

Now we are ready to prove the following theorem.

Theorem 4.1.5. *Suppose that $\alpha : G \curvearrowright X$ is an action on the Cantor space X such that there is no G-invariant non-trivial Borel measure on X. Then the type semigroup $V(X,G)$ is almost unperforated if and only if the action has paradoxical comparison.*

Proof. Recall that Lemma 5.1 in [60] shows that every non-trivial state on $V(X,G)$ induces a non-trivial G-invariant Borel measure. Then from the assumption that there is no non-trivial Borel measure one has that $SN(V(X,G)) = \emptyset$.

Now suppose that the type semigroup $V(X,G)$ is almost unperforated. The proof of (v)⇒(i) of Theorem 5.4 in [60] (i.e. (vi)⇒(ii) in our Theorem 4.1.4) implies that $V(X,G)$ is purely infinite and thus the action α has paradoxical comparison by Theorem 4.1.4. We remark that the proof of this implication does not require the action to be essentially free.

For the converse direction, suppose that α has paradoxical comparison. We have shown in the proof of Theorem 4.1.4 that the type semigroup $V(X, G)$ is purely infinite, which means $2[f] \leq [f]$ for every $[f] \in V(X, G)$. By induction we have $m[f] \leq [f]$ for every $m \in \mathbb{N}$. Now suppose that $(n+1)[g] \leq n[f]$ for some $n \in \mathbb{N}$ and $[f], [g] \in W(X, G)$. Then Proposition 4.1.1 implies that there is some $m \in \mathbb{N}$ such that $[g] \leq m[f]$, which implies that $[g] \leq [f]$ because $m[f] \leq [f]$ as noted above. This shows that the type semigroup is almost unperforated. $\square$

Rørdam and Sierakowski [60] asked that whether there is an example where the type semigroup is not almost unperforated and to what extent the type semigroup $V(X, G)$ is almost unperforated (or purely infinite). P. Ara and R. Exel [2] constructed an action of a finitely generated free group on the Cantor set for which the type semigroup is not almost unperforated. Our Theorem 4.1.5 then sheds a light to the second part of Rørdam and Sierakowski's question in the case that there is no G-invariant non-trivial Borel measure on the Cantor set X. What we actually show in this case is that the action has paradoxical comparison if and only if the type semigroup $V(X, G)$ is almost unperforated if and only if the type semigroup $V(X, G)$ is purely infinite.

4.2 The Generalized Type Semigroup

In the final two sections, we introduce a new semigroup associated to a dynamical system $\alpha : G \curvearrowright X$, where X is not necessarily the Cantor set. In the C^*-setting, it has been proved by Rørdam in [59] that a simple unital C^*-algebra A has strict comparison if and only if its Cuntz semigroup $W(A)$ (or $\mathrm{Cu}(A)$) is almost unperforated. Therefore, as a dynamical analogue of strict comparison, dynamical comparison is expected to have a characterization of the same type, without using invariant probability measures. The author was communicated by David Kerr this question, which was raised by David Kerr and Christopher Schafhauser. For the final two sections, we address this question and obtain Corollary 4.3.8 as a new characterization of dynamical comparison, which has the flavour of almost unperforation. To accomplish this goal, we consider the following order motivated by the type semigroup on zero-dimensional spaces. Using this order we will construct a partially ordered semigroup $W(X, G)$.

Definition 4.2.1. Suppose that $\alpha : G \curvearrowright X$ is a continuous action of a countable infinite discrete group G on a compact Hausdorff space X. Let $O_1, \ldots, O_n$ and $V_1, \ldots, V_m$ be two sequences of open sets in X, We write

$$\bigsqcup_{i=1}^{n} O_i \times \{i\} \prec \bigsqcup_{l=1}^{m} V_l \times \{l\}$$

if for every $i \in \{1, \ldots, n\}$ and every closed set $F_i \subset O_i$ there are a collection of open sets, $\mathcal{U}_i = \{U_1^{(i)}, \ldots, U_{J_i}^{(i)}\}$ forming a cover of F_i, $s_1^{(i)}, \ldots, s_{J_i}^{(i)} \in G$ and $k_1^{(i)}, \ldots, k_{J_i}^{(i)} \in \{1, \ldots, m\}$ such that

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{J_i} s_j^{(i)} U_j^{(i)} \times \{k_j^{(i)}\} \subset \bigsqcup_{l=1}^{m} V_l \times \{l\}.$$

In particular, we write $(n+1)O \prec nV$ for simplification if one has

$$\bigsqcup_{i=1}^{n+1} O \times \{i\} \prec \bigsqcup_{l=1}^{n} V \times \{l\}.$$

Remark 4.2.2. We remark that the relation $(n+1)O \prec nV$ can be described within X. Indeed, $(n+1)O \prec nV$ holds if and only if for every closed subset $F \subset O$ there are a family of open sets $\{U_j^{(i)} : j = 1, \ldots, J_i, i = 1, \ldots, n+1\}$, and a family of group elements $\{s_j^{(i)} \in G, j = 1, \ldots, J_i, i = 1, \ldots, n+1\}$ satisfying:

(i) $F \subset \bigcup_{j=1}^{J_i} U_j^{(i)}$ for $i = 1, 2, \ldots, n+1$,

(ii) $s_j^{(i)} U_j^{(i)} \subset V$ for all $j = 1, \ldots, J_i, i = 1, \ldots, n+1$, and

(iii) $\{s_j^{(i)} U_j^{(i)} : j = 1, \ldots, J_i, i = 1, \ldots, n+1\}$ has chromatic number at most n.

Definition 4.2.3. Let $\alpha : G \curvearrowright X$ be a continuous action of a countable infinite discrete group G on a compact Hausdorff space X. Let $a = (f_1, \ldots, f_n) \in C(X)^{\oplus n}$ and $b = (g_1, \ldots, g_m) \in C(X)^{\oplus m}$. We write $a \preccurlyeq b$ if

$$\bigsqcup_{i=1}^{n} \operatorname{supp}(f_i) \times \{i\} \prec \bigsqcup_{l=1}^{m} \operatorname{supp}(g_l) \times \{l\}$$

holds in the sense of Definition 4.2.1.

We write $K(X,G) = \bigcup_{n=1}^{\infty} C(X)^{\oplus n}$ and observe that the relation $\preccurlyeq$ described above is in fact defined on $K(X,G)$. We remark that Definition 4.2.1 allows us to describe the subequivalence relation $\preccurlyeq$ by simply using open sets like the classical type semigroup in the context of zero-dimensional spaces. However, we insist on considering functions because the relation $\preccurlyeq$ between two sequences of functions $a, b \in K(X,G)$ is naturally related to the Cuntz subequivalence $\precsim$ for a and b in the C^*-algebra $C(X) \rtimes_r G$ (see Proposition 4.2.5 below). To investigate properties of the relation $\preccurlyeq$, we first show that this relation is transitive.

Lemma 4.2.4. *Let $a, b, c \in K(X,G)$ be such that $a \preccurlyeq b$ and $b \preccurlyeq c$. Then $a \preccurlyeq c$.*

Proof. First we write $a = (f_1, \ldots, f_N)$, $b = (g_1, \ldots, g_L)$ and $c = (h_1, \ldots, h_M)$ for some integers $N, L, M \in \mathbb{N}^+$. Since $a \preccurlyeq b$, one has that for every $n \in \{1, \ldots, N\}$ and closed set $F_n \subset \operatorname{supp}(f_n)$ there are a collection of open sets $\mathcal{U}_n = \{U_1^{(n)}, \ldots, U_{J_n}^{(n)}\}$ forming a cover of F_n, $s_1^{(n)}, \ldots, s_{J_n}^{(n)} \in G$ and $k_1^{(n)}, \ldots, k_{J_n}^{(n)} \in \{1, \ldots, L\}$ such that

$$\bigsqcup_{n=1}^{N} \bigsqcup_{j=1}^{J_n} s_j^{(n)} U_j^{(n)} \times \{k_j^{(n)}\} \subset \bigsqcup_{l=1}^{L} \operatorname{supp}(g_l) \times \{l\}.$$

Then compactness and normality of the space X shows that there is a family of open sets $\{V_j^{(n)} : j = 1, \ldots, J_n, n = 1, \ldots, N\}$ such that for each n the collection $\mathcal{V}_n = \{V_j^{(n)} : j = 1, \ldots, J_n\}$ is a cover of F_n and $\overline{V_j^{(n)}} \subset U_j^{(n)}$ for every $j = 1, \ldots, J_n$. Therefore, one has

$$\bigsqcup_{n=1}^{N} \bigsqcup_{j=1}^{J_n} s_j^{(n)} \overline{V_j^{(n)}} \times \{k_j^{(n)}\} \subset \bigsqcup_{l=1}^{L} \operatorname{supp}(g_l) \times \{l\}.$$

Define $\mathcal{D}_l = \{s_j^{(n)} \overline{V_j^{(n)}} : k_j^{(n)} = l, j = 1, \ldots, J_n, n = 1, \ldots, N\}$ and write $K_l = \bigsqcup \mathcal{D}_l$, which is closed and a subset of $\operatorname{supp}(g_l)$. Now because $b \preccurlyeq c$, for all $K_l \subset \operatorname{supp}(g_l)$ there are a collection of open sets $\mathcal{W}_l = \{W_1^{(l)}, \ldots, W_{P_l}^{(l)}\}$ forming a cover of K_l, $t_1^{(l)}, \ldots, t_{P_l}^{(l)} \in G$ and

$d_1^{(l)}, \ldots, d_{P_l}^{(l)} \in \{1, \ldots, M\}$ such that

$$\bigsqcup_{l=1}^{L} \bigsqcup_{p=1}^{P_l} t_p^{(l)} W_p^{(l)} \times \{d_p^{(l)}\} \subset \bigsqcup_{m=1}^{M} \operatorname{supp}(h_m) \times \{m\}.$$

Define $R_{n,j,p,l} = V_j^{(n)} \cap (s_j^{(n)})^{-1} W_p^{(l)}$ for n, j, p, l satisfying $k_j^{(n)} = l$. Then we observe that the family

$$\mathcal{R}_n = \{R_{n,j,p,l} : j = 1, \ldots, J_n, l = 1, \ldots, L, k_j^{(n)} = l, p = 1, \ldots, P_l\}$$

forms an open cover of F_n. Indeed, first fix an $x \in F_n$. Then there is an $V_j^{(n)}$ such that $x \in V_j^{(n)}$. Now taking $l = k_j^{(n)}$ we have $s_j^{(n)} \overline{V_j^{(n)}} \subset K_l \subset \bigcup_{p=1}^{l} W_p^{(l)}$, which implies that $s_j^{(n)} x \in s_j^{(n)} V_j^{(n)} \cap W_p^{(l)}$ for some $p \leq P_l$. Thus, we have $x \in R_{n,j,p,l}$.

In addition, we define $r_{n,j,p,l} = t_p^{(l)} s_j^{(n)} \in G$ for n, j, p, l satisfying $k_j^{(n)} = l$. Now, we claim that the family $\mathcal{T} = \{r_{n,j,l,p} R_{n,j,p,l} \times \{d_p^{(l)}\} : j = 1, \ldots, J_n, l = 1, \ldots, L, k_j^{(n)} = l, p = 1, \ldots, P_l\}$ is disjoint. To simplify the notation, we write $T_{n,j,p,l} = r_{n,j,p,l} R_{n,j,p,l} \times \{d_p^{(l)}\}$ and have

$$T_{n,j,p,l} = (t_p^{(l)} s_j^{(n)} V_j^{(n)} \cap t_p^{(l)} W_p^{(l)}) \times \{d_p^{(l)}\} \subset t_p^{(l)} W_p^{(l)} \times \{d_p^{(l)}\}.$$

Now, suppose that T_{n_1,j_1,p_1,l_1} and T_{n_2,j_2,p_2,l_2} are different. If $l_1 \neq l_2$ or $p_1 \neq p_2$ then by our construction one has

$$(t_{p_1}^{(l_1)} W_{p_1}^{(l_1)} \times \{d_{p_1}^{(l_1)}\}) \cap (t_{p_2}^{(l_2)} W_{p_2}^{(l_2)} \times \{d_{p_2}^{(l_2)}\}) = \emptyset,$$

which implies that $T_{n_1,j_1,p_1,l_1} \cap T_{n_2,j_2,p_2,l_2} = \emptyset$. Otherwise we have $n_1 \neq n_2$ or $j_1 \neq j_2$ while there are l and p such that $l_1 = l_2 = l$, $p_1 = p_2 = p$ and $k_{j_1}^{(n_1)} = k_{j_2}^{(n_2)} = l$. In this case, first by the construction one has

$$(s_{j_1}^{(n_1)} V_{j_1}^{(n_1)} \times \{k_{j_1}^{(n_1)}\}) \cap (s_{j_2}^{(n_2)} V_{j_2}^{(n_2)} \times \{k_{j_2}^{(n_2)}\}) = \emptyset.$$

Thus, $s_{j_1}^{(n_1)} V_{j_1}^{(n_1)} \cap s_{j_2}^{(n_2)} V_{j_2}^{(n_2)} = \emptyset$ because $k_{j_1}^{(n_1)} = k_{j_2}^{(n_2)} = l$. This fact shows $T_{n_1,j_1,p_1,l_1} \cap$

$T_{n_2,j_2,p_2,l_2} = \emptyset$ as desired. So far we have verified that the family $\mathcal{T}$ above is disjoint.

On the other hand, considering the fact that

$$T_{n,j,p,l} \subset t_p^{(l)} W_p^{(l)} \times \{d_p^{(l)}\} \subset \bigsqcup_{m=1}^{M} \operatorname{supp}(h_m) \times \{m\}$$

for all $T_{n,j,p,l}$, we have established the relation

$$\bigsqcup_{n=1}^{N} \bigsqcup_{l=1}^{L} \bigsqcup_{p=1}^{P_l} \bigsqcup_{\{1\leq j\leq J_n : k_j^{(n)}=l\}} r_{n,j,p,l} R_{n,j,p,l} \times \{d_p^{(l)}\} \subset \bigsqcup_{m=1}^{M} \operatorname{supp}(h_m) \times \{m\},$$

which verifies that $a \preccurlyeq c$ as desired. □

Now we can define an equivalence relation on $K(X, G)$ by setting $a \approx b$ if $a \preccurlyeq b$ for $a, b \in K(X, G)$ and $b \preccurlyeq a$ by Lemma 4.2.4. To see that this relation is indeed an equivalence relation, first it is not hard to verify directly that $a \approx a$ for all $a \in K(X, G)$. In addition, by the definition of the relation "$\approx$", $a \approx b$ implies $b \approx a$ trivially. Now suppose $a \approx b$ and $b \approx c$. By definition one has $a \preccurlyeq b \preccurlyeq c$ and $c \preccurlyeq b \preccurlyeq a$. Then Lemma 4.2.4 entails that $a \preccurlyeq c$ and $c \preccurlyeq a$. This establishes $a \approx c$.

We write $W(X, G)$ for the quotient $K(X, G)/\approx$ and define an operation on $W(X, G)$ by $[a] + [b] = [(a, b)]$, where (a, b) is defined to be the concatenation of $a = (f_1, \ldots, f_n)$ and $b = (g_1, \ldots, g_m)$, i.e., $(a, b) = (f_1, \ldots, f_n, g_1, \ldots, g_m)$. It is not hard to see that if $a_1 \preccurlyeq a_2$ and $b_1 \preccurlyeq b_2$ then $(a_1, b_1) \preccurlyeq (a_2, b_2)$. Then Lemma 4.2.4 implies the operation is well-defined and it can be additionally verified that the operation is abelian, i.e, $[a] + [b] = [b] + [a]$. Moreover, we endow $W(X, G)$ with the natural order by declaring $[a] \leq [b]$ if $a \preccurlyeq b$. Thus $W(X, G)$ is a well-defined abelian partially ordered semigroup.

The following proposition shows that our relation $\preccurlyeq$ naturally relates to the Cuntz subsequivalence relation in the context of C^*-algebras. Recall that $(f - \epsilon)_+(x) = f(x) - \epsilon$ if $f(x) \geq \epsilon$ while $(f - \epsilon)_+(x) = 0$ if $f(x) < \epsilon$.

Proposition 4.2.5. Let $a = (f_1, \ldots, f_n)$ and $b = (g_1, \ldots, g_m) \in K(X, G)$. If $a \preccurlyeq b$ then

$\mathrm{Diag}(f_1,\ldots,f_n) \precsim \mathrm{Diag}(g_1,\ldots,g_m)$ in the C^*-algebra $C(X)\rtimes_r G$.

Proof. In light of Proposition 2.17 in [3], it suffices to prove that $\mathrm{Diag}((f_1-\epsilon)_+,\ldots,(f_n-\epsilon)_+) \precsim \mathrm{Diag}(g_1,\ldots,g_m)$ for all $\epsilon>0$. Now, let $\epsilon>0$ and define $F_i=\overline{\mathrm{supp}((f_i-\epsilon)_+)}$ for $i=1,\ldots,n$. Since $a\preccurlyeq b$, for $i=1,\ldots,n$ there are a collection of open sets, $\mathcal{U}_i=\{U_i^{(n)},\ldots,U_{J_i}^{(i)}\}$ forming a cover of F_i, $s_1^{(i)},\ldots,s_{J_i}^{(i)}\in G$ and $k_1^{(i)},\ldots,k_{J_i}^{(i)}\in\{1,\ldots,m\}$ such that

$$\bigsqcup_{i=1}^{n}\bigsqcup_{j=1}^{J_i} s_j^{(i)}U_j^{(i)}\times\{k_j^{(i)}\}\subset\bigsqcup_{l=1}^{m}\mathrm{supp}(g_l)\times\{l\}.$$

Let $\{h_j^i : j=1,\ldots,J_i\}$ be a partition of unity subordinate to the cover $\mathcal{U}_i$ of F_i. Then $F_i\subset\mathrm{supp}(\sum_{j=1}^{J_i}h_j^i)$, which implies that $(f_i-\epsilon)_+\precsim\sum_{j=1}^{J_i}h_j^i$ by Proposition 2.5 in [3]. Then we have

$$\mathrm{Diag}((f_1-\epsilon)_+,\ldots,(f_n-\epsilon)_+)=\bigoplus_{i=1}^{n}(f_i-\epsilon)_+\precsim\bigoplus_{i=1}^{n}(\sum_{j=1}^{J_i}h_j^i)\precsim\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}h_j^i.$$

Define a unitary $u=\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}u_{s_j^{(i)}}$, where all $u_{s_j^{(i)}}$ are canonical unitaries in the crossed product. Then we have

$$\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}h_j^i\sim u(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}h_j^i)u^*=\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}\alpha_{s_j^{(i)}}(h_j^i).$$

To simplify the notation, we define the index set $\mathcal{I}_l=\{(i,j): j=1,\ldots,J_i, i=1,\ldots,n, k_j^{(i)}=l\}$. Then observe that the collection $\{\mathrm{supp}(\alpha_{s_j^{(i)}}(h_j^i))\subset s_j^{(i)}U_j^{(i)} : (i,j)\in\mathcal{I}_l\}$ is disjoint for each $l=1,\ldots,m$. This implies that

$$\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{J_i}\alpha_{s_j^{(i)}}(h_j^i)\sim\bigoplus_{l=1}^{m}\bigoplus_{(i,j)\in\mathcal{I}_l}\alpha_{s_j^{(i)}}(h_j^i)\sim\bigoplus_{l=1}^{m}(\sum_{(i,j)\in\mathcal{I}_l}\alpha_{s_j^{(i)}}(h_j^i)).$$

Finally, note that

$$\mathrm{supp}(\sum_{(i,j)\in\mathcal{I}_l}\alpha_{s_j^{(i)}}(h_j^i))=\bigsqcup_{(i,j)\in\mathcal{I}_l}\mathrm{supp}(\alpha_{s_j^{(i)}}(h_j^i))\subset\mathrm{supp}(g_l)$$

for each $l = 1, \dots, m$. This implies that $\sum_{(i,j)\in\mathcal{I}_l} \alpha_{s_j^{(i)}}(h_j^i) \precsim g_l$, which further entails that

$$\bigoplus_{l=1}^{m}\Big(\sum_{(i,j)\in\mathcal{I}_l} \alpha_{s_j^{(i)}}(h_j^i)\Big) \precsim \bigoplus_{l=1}^{m} g_l = \mathrm{Diag}(g_1, \dots, g_m).$$

We have verified that

$$\mathrm{Diag}((f_1-\epsilon)_+, \dots, (f_n-\epsilon)_+) \precsim \mathrm{Diag}(g_1, \dots, g_m)$$

for every $\epsilon > 0$ and thus we have $\mathrm{Diag}(f_1, \dots, f_n) \precsim \mathrm{Diag}(g_1, \dots, g_m)$. □

We end this section by remarking that like the original type semigroup, our generalized type semigroup $W(X, G)$ can also be used to study paradoxical decomposition in the context of topological dynamics. The paradoxical decomposition can be formulated by $2[a] \leq [a]$ in $W(X, G)$ for all $a \in K(X, G)$. Note that this condition is equivalent to paradoxical comparison introduced in Section 3.2.1

4.3 A New Characterization of Dynamical Comparison

In this section, we always assume that the space X is metrizable. In addition, for $a = (f_1, \dots, f_n) \in K(X, G)$, we denote by $(a-\epsilon)_+$ the element $((f_1-\epsilon)_+, \dots, (f_n-\epsilon)_+)$ in $K(X, G)$. It is not hard to verify $((a-\epsilon)_+ - \delta)_+ = (a-\epsilon-\delta)_+$ for $a \in K(X, G)$, $\epsilon > 0$ and $\delta > 0$.

In parallel with the Cuntz semigroup, we have the following fact.

Proposition 4.3.1. For all $a, b \in K(X, G)$, the following are equivalent.

(i) $a \preccurlyeq b$;

(ii) for all $\epsilon > 0$ one has $(a-\epsilon)_+ \preccurlyeq b$;

(iii) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $(a-\epsilon)_+ \preccurlyeq (b-\delta)_+$;

Proof. Write $a = (f_1, \dots, f_n)$ and $b = (g_1, \dots, g_m)$. Then by definition we have $(a-\epsilon)_+ = ((f_1-\epsilon)_+, \dots, (f_n-\epsilon)_+)$. Consider for each $1 \leq i \leq n$, one has $\mathrm{supp}((f_i-\epsilon)_+) \subset \mathrm{supp}(f_i)$, which implies that $(a-\epsilon)_+ \preccurlyeq a$. This fact shows that (i)⇒(ii).

To show (ii)⇒(i), first for every $1 \le i \le n$ and closed set $F_i \subset \mathrm{supp}(f_i)$, there is an $\epsilon_i > 0$ such that $F_i \subset \{x \in X : f_i(x) > \epsilon_i\} \subset \mathrm{supp}(f_i)$. Define $\epsilon = \min\{\epsilon_i : 1 \le i \le n\}$. Then, $F_i \subset \{x \in X : f_i(x) > \epsilon\} = \mathrm{supp}((f_i - \epsilon)_+) \subset \mathrm{supp}(f_i)$ for all i. Now, since $(a-\epsilon)_+ \preccurlyeq b$, there are a collection of open sets $\mathcal{U}_i = \{U_1^{(i)}, \ldots, U_{J_i}^{(i)}\}$ forming a cover of F_i, $s_1^{(i)}, \ldots, s_{J_i}^{(i)} \in G$ and $k_1^{(i)}, \ldots, k_{J_i}^{(i)} \in \{1, \ldots, m\}$ such that

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{J_i} s_j^{(i)} U_j^{(i)} \times \{k_j^{(i)}\} \subset \bigsqcup_{l=1}^{m} \mathrm{supp}(g_l) \times \{l\}.$$

But this implies that $a \preccurlyeq b$.

Now, suppose that (iii) holds. Then for every $\epsilon > 0$, by combining arguments in the two directions, one has $(a-\epsilon)_+ \preccurlyeq (b-\delta)_+ \preccurlyeq b$ and thus $a \preccurlyeq b$. This establishes (iii)⇒(i). It is left to show (ii)⇒(iii). It suffices to show that if $a \preccurlyeq b$ then there is a $\delta > 0$ such that $a \preccurlyeq (b-\delta)_+$. Indeed, by the definition of $a \preccurlyeq b$ and the compactness and normality of the space, for every $i \in \{1, \ldots, n\}$ and closed set $F_i \subset \mathrm{supp}(f_i)$ there are a collection of open sets $\mathcal{U}_i = \{U_1^{(i)}, \ldots, U_{J_i}^{(i)}\}$ forming a cover of F_i, $s_1^{(i)}, \ldots, s_{J_i}^{(i)} \in G$ and $k_1^{(i)}, \ldots, k_{J_i}^{(i)} \in \{1, \ldots, m\}$ such that

$$\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{J_i} s_j^{(i)} \overline{U_j^{(i)}} \times \{k_j^{(i)}\} \subset \bigsqcup_{l=1}^{m} \mathrm{supp}(g_l) \times \{l\}.$$

Define $\mathcal{D}_l = \{s_j^{(i)} \overline{U_j^{(i)}} : j = 1, \ldots, J_i, i = 1, \ldots, n, k_j^{(i)} = l\}$ and write $K_l = \bigsqcup \mathcal{D}_l$, which is closed and a subset of $\mathrm{supp}(g_l)$. Then there is a δ_l such that $K_l \subset \{x \in X : g_l(x) > \delta_l\} \subset \mathrm{supp}(g_l)$. Setting $\delta = \min\{\delta_l : 1 \le l \le m\}$ we have $a \preccurlyeq (b-\delta)_+$. □

Definition 4.3.2. A state D on the semigroup $W(X, G)$ is called *lower semi-continuous* if $D([a]) = \sup_{\epsilon>0} D([(a-\epsilon)_+])$ for all $a \in K(X, G)$.

For every state $D \in S(W(X, G))$, define $\overline{D}([a]) = \sup_{\epsilon>0} D([(a-\epsilon)_+])$, which is always a lower semi-continuous state on $W(X, G)$.

Proposition 4.3.3. For each state $D \in S(W(X, G))$, the induced state $\overline{D}$ is lower semi-continuous.

Proof. Let $a \precsim b$. Then by the proposition above, for all $\epsilon > 0$ there is a $\delta > 0$ such that $(a-\epsilon)_+ \precsim (b-\delta)_+$. Thus, $\overline{D}([a]) = \lim_{\epsilon\to 0} D([(a-\epsilon)_+]) \leq \lim_{\delta\to 0} D([(b-\delta)_+]) = \overline{D}([b])$. This shows that $\overline{D}$ is monotone.

Let $a, b \in K(G, X)$. If $\overline{D}([a])$ or $\overline{D}([b])$ is infinite then $\overline{D}([a]+[b]) = \overline{D}([a]) + \overline{D}([b])$ holds trivially since $\overline{D}$ is monotone. We then assume that both of them are finite. Then in this case one has

$$\begin{aligned}\overline{D}([a]+[b]) &= \lim_{\epsilon\to 0} D([((a,b)-\epsilon)_+]) = \lim_{\epsilon\to 0} D([((a-\epsilon)_+, (b-\epsilon)_+)]) \\ &= \lim_{\epsilon\to 0} D([(a-\epsilon)_+]) + \lim_{\epsilon\to 0} D([(b-\epsilon)_+]) \\ &= \overline{D}([a]) + \overline{D}([b]).\end{aligned}$$

This verifies that $\overline{D}$ is a state.

For lower semi-continuity, note that

$$\overline{D}([(a-\epsilon)_+]) = \lim_{\delta\to 0} D([((a-\epsilon)_+ - \delta)_+]) = \lim_{\delta\to 0} D([(a-\epsilon-\delta)_+]).$$

Thus we have

$$\lim_{\epsilon\to 0} \overline{D}([(a-\epsilon)_+]) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} D([(a-\epsilon-\delta)_+]) = \overline{D}([a]).$$

$\square$

For every premeasure μ in $\mathrm{Pr}_G(X)$ define a state D_μ on $W(X,G)$ by $D_\mu([a]) = \sum_{i=1}^n \mu(\mathrm{supp}(f_i))$ for $a = (f_1, \ldots, f_n) \in K(X, G)$.

Proposition 4.3.4. D_μ defined above is a lower semi-continuous state on $W(X,G)$.

Proof. The additivity of D_μ is clear from the definition of D_μ above. Since $\mu \in \mathrm{Pr}_G(X)$ is G-invariant and inner-regular, we see that if $a \precsim b$ then $D_\mu([a]) \leq D_\mu([b])$. Now, let $a = (f_1, \ldots, f_n) \in K(X,G)$. For every $1 \leq i \leq n$ and a closed set $F_i \subset \mathrm{supp}(f_i)$ there is an ϵ_i such that $F_i \subset \mathrm{supp}((f_i - \epsilon_i)_+) = \{x \in X : f_i(x) > \epsilon_i\} \subset \mathrm{supp}(f_i)$. Now let

$\epsilon = \max\{\epsilon_i : 1 \leq i \leq n\}$ and thus $\sum_{i=1}^n \mu(F_i) \leq D_\mu([(a-\epsilon)_+]) \leq D_\mu([a])$, which implies that $\sup_{\epsilon>0} D_\mu([(a-\epsilon)_+]) = D([a])$ because μ is inner-regular for every $\mathrm{supp}(f_i)$. $\square$

We will show in Lemma 4.3.5 that the converse of Proposition 4.3.4 is also true, that is, every lower semi-continuous state D is of the form D_μ for a premeasure $\mu \in \mathrm{Pr}_G(X)$. The proof of this fact has a classical flavour. It is routine but quite long. In the Cuntz semigroup setting, Blackadar and Handelman provided a version concerning bounded dimension functions, which are bounded states of the Cuntz semigroup (see [6, Proposition I.2.1]). However, they omitted the proof. In addition, Rørdam and Sierakowski proved the result for the type semigroup (see [60, Lemma 5.1]) in the zero-dimensional setting. Their proof relies on the zero-dimensionality of the space and cannot be generalized to higher dimensional cases. Therefore, for the convenience of the readers, we present the proof here. We denote by $\mathrm{Lsc}(W(X,G))$ the set of all lower semi-continuous states on $W(X,G)$.

Lemma 4.3.5. *Every lower semi-continuous state $D \in \mathrm{Lsc}(W(X,G))$ induces a G-invariant premeasure $\mu_D \in \mathrm{Pr}_G(X)$.*

Proof. First for every open set O, define $\mu_D(O) = D([f])$ where $O = \mathrm{supp}(f)$ for some $f \in C(X)_+$. Then by the definition of state, μ_D is G-invariant on open sets. In addition, it is finitely subadditive on open sets, i.e., if $O_1, \ldots O_n$ are open then

$$\mu_D(\bigcup_{i=1}^n O_i) \leq \sum_{i=1}^n \mu_D(O_i).$$

Moreover, if the $O_1, \ldots O_n$ are pairwise disjoint then we have additivity:

$$\mu_D(\bigsqcup_{i=1}^n O_i) = \sum_{i=1}^n \mu_D(O_i).$$

Finally, μ_D is monotone for open sets, i.e., $O_1 \subset O_2$ implies $\mu_D(O_1) \leq \mu_D(O_2)$. For every closed set F, define $\mu_D(F) = \inf\{\mu_D(O) : F \subset O, O \text{ open}\}$. Since the space X is normal, μ_D is additive with respect to disjoint closed sets $\{F_1, \ldots, F_n\}$, i.e.,

$$\mu_D(\bigsqcup_{i=1}^{n} F_i) = \sum_{i=1}^{n} \mu_D(F_i).$$

Claim 1. *Let F be a closed set and $\{F_n\}$ an increasing sequence such that $F = \bigcup_{n=1}^{\infty} F_n$. Then $\mu_D(F) = \sup_n \mu_D(F_n)$.*

Proof. If one of $\mu_D(F_n)$ is infinite, then this equality holds trivially. Thus, we may assume each of $\mu_D(F_n)$ is finite. Fix an $\epsilon > 0$. By the definition of $\mu_D(F_n)$, for each n there is an open set $O_n \supset F_n$ such that

$$|\mu_D(F_n) - \mu_D(O_n)| \leq \epsilon/2^n.$$

Then $F \subset \bigcup_{n=1}^{\infty} O_n$ and thus there is an $N > 0$ such that $F \subset \bigcup_{n=1}^{N} O_n$. Note that

$$(\bigcup_{n=1}^{N} O_n) \setminus F_N \subset \bigcup_{n=1}^{N} (O_n \setminus F_n),$$

which implies that

$$\mu_D((\bigcup_{n=1}^{N} O_n) \setminus F_N) \leq \mu_D(\bigcup_{n=1}^{N} (O_n \setminus F_n)) \leq \sum_{n=1}^{N} \mu_D(O_n \setminus F_n) \leq \epsilon.$$

Write $O = \bigcup_{n=1}^{N} O_n$ for simplicity. We have $(O \setminus F_N) \sqcup F_N = O$. Now for every open set $U \supset F_N$, one has $O \subset O \setminus F_N \cup U$, which entails that

$$\mu_D(U) \geq \mu_D(O) - \mu_D(O \setminus F_N).$$

Therefore, one has $\mu_D(F_N) \geq \mu_D(O) - \mu_D(O \setminus F_N) \geq \mu_D(O) - \epsilon$. As $F \subset O$, one has

$$\mu_D(F_N) \geq \mu_D(O) - \epsilon \geq \mu_D(F) - \epsilon,$$

which establishes the claim. $\square$

Now, since D is lower semi-continuous we can verify that for every open set O one has $\mu_D(O) = \sup\{\mu_D(F) : F \subset O, F \text{ closed}\}$. This shows that μ_D is inner regular on open sets. Motivated by this equality, we define

$$\mu_D(A) = \sup\{\mu_D(K) : K \subset A, K \text{ closed}\}$$

for every F_σ set A. This definition is consistent to the original definition of μ_D for all open sets and shows that μ_D is monotone for all F_σ sets.

Claim 2. *Let* $A = \bigcup_{n=1}^{\infty} F_n$ *for an increasing sequence of closed sets* $\{F_n\}$*. Then* $\mu_D(A) = \sup_n\{\mu_D(F_n)\}$.

Proof. By definition it suffices to show $\mu_D(A) = \sup\{\mu_D(K) : K \subset A, K \text{ closed}\} \le \sup_n\{\mu_D(F_n)\}$. The proof is similar to that of Claim 1. If one of $\mu_D(F_n)$ is infinite, then the equality above holds trivially. Thus, we may assume each $\mu_D(F_n)$ is finite. Fix an $\epsilon > 0$ and a closed set $K \subset A$. By the definition of $\mu_D(F_n)$, for each n there is an open set $O_n \supset F_n$ such that

$$|\mu_D(F_n) - \mu_D(O_n)| \le \epsilon/2^n.$$

Then $K \subset A \subset \bigcup_{n=1}^{\infty} O_n$ and thus there is an $N > 0$ such that $K \subset \bigcup_{n=1}^{N} O_n$. Then because $\{F_n\}$ is increasing, one has

$$K \setminus F_N \subset (\bigcup_{n=1}^{N} O_n) \setminus F_N \subset \bigcup_{n=1}^{N}(O_n \setminus F_n),$$

Note that $K \setminus F_N$ is also a F_σ set. Then we have

$$\mu_D(K \setminus F_N) \le \mu((\bigcup_{n=1}^{N} O_n) \setminus F_N) \le \sum_{n=1}^{N} \mu_D(O_n \setminus F_n) \le \epsilon$$

since μ_D is monotone on F_σ sets. We write $K \setminus F_N = \bigcup_{n=1}^{\infty} P_n$ for an increasing sequence of closed sets $\{P_n\}$. Then $K = (K \setminus F_N) \sqcup (K \cap F_N) = \bigsqcup_{n=1}^{\infty}((K \cap F_N) \sqcup P_n)$. Then claim 1 entails that $\mu_D(K) = \sup_n\{\mu_D((K \cap F_N) \sqcup P_n)\}$. Now there is a $M > 0$ such that

$$\mu_D((K \cap F_N)) + \mu_D(P_M) = \mu_D((K \cap F_N) \sqcup P_M) \geq \mu_D(K) - \epsilon.$$

Thus, we have

$$\mu_D(F_N) \geq \mu_D((K \cap F_N)) \geq \mu_D(K) - 2\epsilon.$$

This establishes Claim 2. $\square$

Now, consider the *semialgebra* $\mathcal{S} = \{O \cap F : O \text{ open}, F \text{ closed}\}$ in the sense of [61, p. 297]. Since our X is metrizable, every set $O \cap F \in \mathcal{S}$ is a F_σ set. Observe that the algebra $\mathcal{A}_0$ equals $\{\bigcup_{i=1}^n C_i : C_i \in \mathcal{S}, n \in \mathbb{N}\}$. Then every member of $\mathcal{A}_0$ is an F_σ set. We restrict the definition of μ_D to $\mathcal{A}_0$.

Claim 3. *If* $A, A_1, \ldots, A_m, \ldots, \in \mathcal{A}_0$ *with* $A = \bigsqcup_{m=1}^\infty A_m$ *then one has* $\mu_D(A) = \sum_{m=1}^\infty \mu_D(A_m)$.

Proof. If there is one A_m such that $\mu_D(A_m) = \infty$, the equality holds trivially. Therefore we may assume that each $\mu_D(A_m)$ is finite. Since each A_m is an F_σ set, we can write $A_m = \bigcup_{n=1}^\infty F_{m,n}$ for an increasing sequence of closed sets $\{F_{m,n} : n \in \mathbb{N}\}$. Thus $A = \bigsqcup_{m=1}^\infty \bigcup_{n=1}^\infty F_{m,n}$. Fix an $\epsilon > 0$. By Claim 2 for each $m \in \mathbb{N}$ we can choose N_m big enough such that

$$|\mu_D(A_m) - \mu_D(F_{m,N_m})| \leq \epsilon/2^m.$$

In addition, we can make the sequence $\{N_m\}$ strictly increasing. Now Define $P_M = \bigsqcup_{m=1}^M \bigcup_{n=1}^{N_M} F_{m,n} = \bigsqcup_{m=1}^M F_{m,N_M}$ for $M > 0$. Note that $\{P_M : M \in \mathbb{N}\}$ is an increasing sequence of closed sets such that $A = \bigcup_{M=1}^\infty P_M$. Then Claim 2 shows that $\mu_D(A) = \sup_M\{\mu_D(P_M)\}$.

Observe that $\mu_D(P_M) = \sum_{m=1}^M \mu_D(F_{m,N_M}) \leq \sum_{m=1}^\infty \mu_D(A_m)$ for each M. This implies that $\mu_D(A) \leq \sum_{m=1}^\infty \mu_D(A_m)$. Now if $\mu_D(A) = \infty$ then equality holds trivially. So we consider the case that $\mu_D(A) < \infty$. In this case, for every $M > 0$, one has

$$|\mu(P_M) - \sum_{m=1}^{M} \mu_D(A_m)| \leq \sum_{m=1}^{M} |\mu_D(F_{m,N_M}) - \mu_D(A_m)| \leq \sum_{m=1}^{M} \epsilon/2^m \leq \epsilon.$$

This implies that

$$\infty > \mu_D(A) \geq \mu_D(P_M) \geq \sum_{m=1}^{M} \mu_D(A_m) - \epsilon,$$

and thus we have

$$\mu_D(A) \geq \sum_{m=1}^{\infty} \mu_D(A_m)$$

since ϵ was arbitrary.

□

Claim 3 shows that μ_D on $\mathcal{A}_0$ is indeed a premeasure. In addition, it also shows that μ_D has subadditivity for countably many sets in $\mathcal{A}_0$, i.e., for $A_1, \dots \in \mathcal{A}_0$,

$$\mu_D(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu_D(A_n).$$

the definition of μ_D implies that it is G-invariant and satisfies inner regularity for all sets in $\mathcal{A}_0$ and outer regularity for closed sets. We verify the outer regularity for all sets in $\mathcal{A}_0$. Let $B \in \mathcal{A}_0$, which is a F_σ set, say, $B = \bigcup_{n=1}^{\infty} F_n$ for a increasing sequence of closed sets $\{F_n\}$. If $\mu_D(B) = \infty$ then it satisfies the outer regularity trivially since μ_D is monotone on F_σ sets. Now suppose that $\mu_D(B) < \infty$. Then Claim 2 shows that $\mu_D(B) = \sup_{n \in \mathbb{N}} \mu_D(F_n) < \infty$. Then since we have outer regularity for all closed sets, for $\epsilon > 0$ and each $n \in \mathbb{N}$, there is an open set O_n such that $F_n \subset O_n$ and

$$\mu_D(F_n) > \mu_D(O_n) - \epsilon/2^n$$

Then define $O = \bigcup_{n=1}^{\infty} O_n$. Then one has $B \subset O$ and

$$\mu(O \setminus B) \leq \mu_D(\bigcup_{n=1}^{\infty} (O_n \setminus F_n)) \leq \sum_{n=1}^{\infty} \mu_D(O_n \setminus F_n) < \epsilon.$$

This shows that $\mu_D(B) = \inf\{\mu_D(O) : B \subset O,\ O \text{ open}\}$ and thus μ_D satisfies the outer regularity and thus belongs to $\mathrm{Pr}_G(X)$. $\square$

Recall that the measure μ_D can be extended to a Borel measure on X. The extension is unique if μ_D is σ-finite on $\mathcal{A}_0$. This happens, in particular, in the case that D is bounded. i.e., $D([1_X]) < \infty$.

Theorem 4.3.6. *The map $S : D \to \mu_D$ is an affine bijection from $\mathrm{Lsc}(W(X, G))$ to $\mathrm{Pr}_G(X)$. In particular, there is an affine bijection between $\mathrm{Lsc}_1(W(X, G))$ and $M_G(X)$ where $\mathrm{Lsc}_1(W(X, G))$ is the set of all states D in $\mathrm{Lsc}(W(X, G))$ with $D([1_X]) = 1$.*

Proof. By Lemma 4.3.5, $S : D \to \mu_D$ is well defined. It is not hard to see that S is affine. We first show that S is injective. If $\mu_{D_1} = \mu_{D_2}$ then for every $f \in C(X)_+$ one has

$$D_1([f]) = \mu_{D_1}(\mathrm{supp}(f)) = \mu_{D_2}(\mathrm{supp}(f)) = D_2([f]),$$

which shows that $D_1 = D_2$. To see that S is surjective, it suffices to observe that $S(D_\mu) = \mu$ for every $\mu \in \mathrm{Pr}_G(X)$.

Now if $D([1_X]) = 1$ then μ_D is a probability premeasure on $\mathcal{A}_0$ and extends uniquely to a probability Borel measure on X by the remark above. This establishes the last conclusion. $\square$

Now we are able to prove the following theorem.

Theorem 4.3.7. *Suppose that $\alpha : G \curvearrowright X$ is a continuous action of a countable infinite discrete group G on a compact metrizable space X. Let A, B be open sets in X. The following are equivalent.*

(i) There is an $n \in \mathbb{N}^+$ such that $(n+1)A \prec nB$.

(ii) There is an $N \in \mathbb{N}^+$ such that $(n+1)A \prec nB$ for all $n \geq N$.

(iii) $A \subset G \cdot B$ and $\mu(A) < \mu(B)$ for every $\mu \in \mathrm{Pr}_G(X)$ with $\mu(B) = 1$.

Proof. (i)$\Leftrightarrow$(ii). Let A, B be open sets in X. Choose functions $f, g \in C(X)_+$ such that $\mathrm{supp}(f) = A$ and $\mathrm{supp}(g) = B$. It suffices to show (i)$\Rightarrow$(ii). If there is an $n \in \mathbb{N}$ such that $(n+1)A \prec nB$ then $(n+1)[f] \leq n[g]$ holds in $W(X, G)$. Then Proposition 4.1.1. implies that there is an $N \in \mathbb{N}$ such that $(m+1)[f] \leq m[g]$ for all $m \geq N$. This shows (ii).

(i)$\Rightarrow$(iii). Suppose $(n+1)A \prec nB$ holds for some $n \in \mathbb{N}$. Then first by remark 4.2.2, one has $A \subset G \cdot B$ trivially. Now fix a $\mu \in \mathrm{Pr}_G(X)$ with $\mu(B) = 1$. The definition of $(n+1)A \prec nB$ implies $(n+1)\mu(F) \leq n\mu(B) = n$ for all closed sets $F \subset A$. Since μ is inner regular, we have $(n+1)\mu(A) \leq n\mu(B)$ and thus $\mu(A) \leq n/(n+1) < 1 = \mu(B)$.

(iii)$\Rightarrow$(i). Suppose that A, B satisfy the assumption in (iii). Choose functions $f, g \in C(X)_+$ such that $\mathrm{supp}(f) = A$ and $\mathrm{supp}(g) = B$. First we claim that for all $\mu \in \mathrm{Pr}_G(X)$ with $0 < \mu(B) \leq 1$ one still has $\mu(A) < \mu(B)$. Indeed, define $\mu'(\cdot) = \mu(\cdot)/\mu(B)$, which is a premeasure in $\mathrm{Pr}_G(X)$ with $\mu'(B) = 1$. Then one has $\mu'(A) < \mu'(B)$ by the assumption of (iii), which shows that $\mu(A) < \mu(B)$.

Fix an $\epsilon > 0$. Then first one has $\mathrm{supp}((f-\epsilon)_+) \subset \overline{\mathrm{supp}((f-\epsilon)_+)} \subset A \subset G \cdot B$, which implies that for some $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in G$ one has

$$\mathrm{supp}((f-\epsilon)_+) \subset \bigcup_{i=1}^{m} s_i B$$

This entails that $[(f-\epsilon)_+] \leq m[g]$.

On the other hand, the condition that $\mu(A) < \mu(B)$ for every $\mu \in \mathrm{Pr}_G(X)$ with $0 < \mu(B) \leq 1$ implies that $D'([f]) < D'([g])$ for all $D' \in \mathrm{Lsc}(W(X, G))$ with $0 < D'([g]) \leq 1$ by Theorem 4.3.6.

Therefore, for every state $D \in S(W(X, G))$ with $D([g]) = 1$, since $\overline{D}$ is always lower semi-continuous by Proposition 4.3.3, we have

$$D([(f-\epsilon)_+]) \leq \overline{D}([f]) < \overline{D}([g]) \leq D([g]) = 1.$$

Then Proposition 4.1.1 implies that there is an $n \in \mathbb{N}$ such that $(n+1)[(f-\epsilon)_+] \leq n[g]$. Since the ϵ is arbitrary, one has $(n+1)[f] \leq n[g]$ by Proposition 4.3.1. This means $(n+1)A \prec nB$ as desired. $\square$

We then have the following corollary.

Corollary 4.3.8. *Suppose that $\alpha : G \curvearrowright X$ is a continuous action of a countable infinite discrete group G on a compact metrizable space X. Suppose in addition that G is amenable or α is minimal. The following are equivalent.*

(i) Whenever A, B are open sets in X such that $\mu(B) > 0$ for all $\mu \in M_G(X)$, if there is an $n \in \mathbb{N}^+$ such that $(n+1)A \prec nB$, then $A \prec B$.

(ii) $\alpha : G \curvearrowright X$ has dynamical comparison in the sense of Definition 1.2.8.

Proof. (i)⇒(ii). Let A, B be open sets in X. Suppose that $\nu(A) < \nu(B)$ for every $\nu \in M_G(X)$. First this implies $\nu(B) > 0$ for all $\nu \in M_G(X)$ and, in particular, B is not empty. When α is minimal or G is amenable, we claim that $X = G \cdot B$. In the case that α is minimal, one has $X = G \cdot B$ trivially. Suppose that G is amenable and $X \neq G \cdot B$, there is a G-invariant probability measure λ for the closed subsystem $C = X \setminus G \cdot B \neq \emptyset$ since G is amenable. However λ induces a probability measure λ' on X with $\lambda'(E) = \lambda(E \cap C)/\lambda(C)$ for every Borel set E. Observe that $\lambda'(B) = 0$ and this is a contradiction. Therefore one has $A \subset X = G \cdot B$. In addition, since X is actually covered by finitely many translates of B, for every $\mu \in \mathrm{Pr}_G(X)$ with $\mu(B) = 1$, one has $\mu(X)$ is finite. Define $\nu(\cdot) = \mu(\cdot)/\mu(X)$, which is a probability premeasure in $\mathrm{Pr}_G(X)$. Now extend ν to obtain a probability measure in $M_G(X)$, which we still denote by ν. Then since $\nu(A) < \nu(B)$ holds by assumption, one has $\mu(A) < \mu(B) = 1$. Then Theorem 4.3.7 shows that there is an $n \in \mathbb{N}$ such that $(n+1)A \prec nB$. This shows that $A \prec B$ by (i) and thus we have dynamical comparison in the sense of Definition 1.2.8.

(ii)⇒(i). Let A, B be open sets in X such that $\nu(B) > 0$ for all $\nu \in M_G(X)$ and there is an $n \in \mathbb{N}^+$ such that $(n+1)A \prec nB$. Theorem 4.3.7 shows that $\mu(A) < \mu(B)$ for all $\mu \in \mathrm{Pr}_G(X)$ with $\mu(B) = 1$. Now let $\nu \in M_G(X)$. Define $\nu'(\cdot) = \nu(\cdot)/\nu(B)$. The measure ν' is well-defined since $\nu(B) > 0$ and satisfies $\nu'(B) = 1$. Note that ν' also belongs to $\mathrm{Pr}_G(X)$ when one restricts it to $\mathcal{A}_0$. This implies that $\nu'(A) < \nu'(B)$ and thus $\nu(A) < \nu(B)$. Now since α has dynamical comparison, we have $A \prec B$.

$\square$

5. SUMMARY AND CONCLUDING REMARKS

The main theme of this work is establishing regularity properties of nuclear reduced crossed product C^*-algebras from dynamical systems, in particular, from dynamical comparison. This provides new examples of nuclear simple unital separable C^*-algebras satisfying Toms-Winter conjecture and being classfied by the Ellitott invariant.

In chapter 2, we studied minimal free actions of amenable groups. Under the hypothesis that $E_G(X)$ is compact and zero-dimensional in the w^*-topology we showed that dynamical comparison implies the $\mathcal{Z}$-stability of the reduced crossed prodcuts. In chapter 3, we looked at minimal free amenable actions without invariant probability measures, which are opposite to that in Chapter 2. We then established pure infiniteness, thus $\mathcal{Z}$-stability, of reduced crossed products for this kind of actions satisfying dynamical comparison. These two chapters thus provide a quite complete framework of the study of nuclear simple reduced crossed product C^*-algebras because nuclearity of reduced crossed products is equivalent to the amenbility of the actions. In this case, we have the dichotomy that either the acting group is amenable or there are no invariant probability measures.

In addition, our work validates the motivation of dynamical comparison as a dynamical analogue of strict comparison in the C^*-setting. From this motivation, in Chapter 4, we established a new charaterization of dynamical compasion of the flavour of almost unperforation as Rørdam did for strict comparison. This part of work was done through a construction of a new semigroup in dynamical systems. This also provides us a generalized version of dynamical comparison.

We end our discussion by mentioning a few interesting questions and avenues for future research.

Theorem 2.3.2 settles the case that $E_G(X)$ is zero-dimensional. It is unknown to the author whether the zero-dimensionality of $E_G(X)$ in the assumption of Theorem 2.3.2 can be replaced by finite dimensionality of $E_G(X)$.

In view of Theorem 2.4.1, it is natural to ask whether m-almost finiteness, as a higher order version of almost finiteness also implies $\mathcal{Z}$-stability of reduced crossed product C^*-algebras for minimal free actions of amenable groups.

In Chapter 3, we established pure infiniteness of reduced crossed product C^*-algebras for essentially free actions satisfying paradoxical comparison under some mild assumptions. However, since the C^*-algebras under consideration here may not be simple, it will be worth investigating whether paradoxical comparison also implies strongly pure infiniteness of reduced crossed product C^*-algebra. This will provide us $\mathcal{O}_\infty$-stability results.

Finally, in the view of all examples satisfying dynamical comparison lised in Section 1.2.3, a nutural question is that what class of countable discrete group actions have dynamical compasrion? It may be too ambitious to conjecture that all actions on the Cantor set of amenable groups automatically have dynamical comparison. However, we at least should ask that are there more examples of actions satisfying dynamical comparison beyond that of Downarowicz and Zhang? It is also unknown to the author whether there are more examples beyond n-filling actions in the case of actions of non-amenable groups.

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