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Rotating solutions in critical Lovelock gravities

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ABSTRACT

For appropriate choices of the coupling constants, the equations of motion of Lovelock gravities up to order n in the Riemann tensor can be factorized such that the theories admit a single (A)dS vacuum. In this paper we construct two classes of exact rotating metrics in such critical Lovelock gravities of order n in $d = 2n + 1$ dimensions. In one class, the n angular momenta in the n orthogonal spatial 2-planes are equal, and hence the metric is of cohomogeneity one. We construct these metrics in a Kerr–Schild form, but they can then be recast in terms of Boyer–Lindquist coordinates. The other class involves metrics with only a single non-vanishing angular momentum. Again we construct them in a Kerr–Schild form, but in this case it does not seem to be possible to recast them in Boyer–Lindquist form. Both classes of solutions have naked curvature singularities, arising because of the over rotation of the configurations.

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1. Introduction

Although Einstein's theory of gravity is highly non-linear, exact solutions do exist, including the celebrated (static) Schwarzschild [1] and (rotating) Kerr [2] metrics. Whilst the generalization of the Schwarzschild metric to higher dimensions is straightforward, such a generalization of the Kerr metric leads to still exact, but considerably more complicated solutions [3], especially so when the metrics are asymptotic to (A)dS ((anti-)de Sitter) spacetimes [4–6].

Finding exact solutions becomes much more difficult when Einstein gravity is extended with higher-order curvature invariants, even in the case of static solutions. Einstein gravity extended with quadratic curvature invariants in four dimensions was shown by numerical methods to admit a new static black hole over and above the Schwarzschild metric, but no exact solution is known [7,8]. The existence of such new black holes was shown numerically also when a cosmological constant or a Maxwell field is included [9,10].

In higher dimensions, when higher-order ghost-free Euler integrands are no longer total derivatives, Einstein–Gauss–Bonnet or more general Lovelock gravities can be constructed [11]. In these theories, exact solutions for static black holes have been found [12,13], and these have smooth limits to the Schwarzschild metric when the higher-derivative couplings are sent to zero. Exact solutions for rotating black holes remain elusive in these theories.

Recently, a five-dimensional rotating solution [14] was constructed in the Einstein–Gauss–Bonnet (EGB) theory, for a certain critical value of the coupling constant for the Gauss–Bonnet term. For generic values of the coupling constant, the EGB theory admits two (A)dS vacua with different cosmological constants. One of these has a positive kinetic energy for linearized graviton fluctuations, while the other has a negative kinetic energy [12]. At the critical value of the coupling, the two values for the (A)dS cosmological constants coalesce, and the linearized equations of motion are automatically satisfied, leading to a gravity theory without a linearized graviton fluctuation [15], and for which further exact solutions can be constructed.

The equations of motion of for higher-order Lovelock gravities can also be factorized for certain specific choices of the coupling constants, again giving rise to only a single (A)dS vacuum with one specific cosmological constant. Such theories were classified and studied

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in [16]. The critical EGB theory mentioned earlier is a special case. The purpose of this paper is to generalize the five dimensional rotating solution that was found in [14] for the critical EGB theory to the critical Lovelock gravities of order n in the Riemann tensor, in the spacetime dimension $d = 2n + 1$. We obtain exact rotating solutions in two cases. In the first, the n angular momenta in the n orthogonal spatial 2-planes are all equal, and hence the metric is of cohomogeneity one. We obtain these solutions first in a Kerr–Schild form, but we find that they can then be recast into a form written using Boyer–Lindquist type coordinates. This rewriting has the advantage that it is easier to study the global structure of the solutions. The second class of rotating solutions that we obtain involve only a single non-vanishing angular momentum. Again, we obtain the solutions in a Kerr–Schild form, but in this case there appears to be no way to introduce Boyer–Lindquist type coordinates.

The paper is organized as follows. In section 2, we review the construction of the critical Lovelock gravities. In section 3, we consider static and spherically-symmetric solutions. Next, we focus on Lovelock gravities of order n in $d = 2n + 1$ dimensions. In section 4, we construct the exact rotating solutions where all the angular momenta are equal. In section 5, we construct the second class of rotating solutions, where only a single angular momentum is non-zero. We conclude the paper in section 6. In the Appendix A, we present details of the Riemann tensor for the single-angular momentum metrics.

2. Critical Lovelock gravities

In this section, we review the construction of [16]. We start with the general class of Lovelock gravities, for which the Lagrangian is given by

$$e^{-1}\mathcal{L} = \sum_{k=0}^{n} \alpha_k E^{(k)}\,, \tag{2.1}$$

where

$$E^{(k)} = \frac{1}{2^k}\delta^{\rho_1\sigma_1\cdots\rho_k\sigma_k}_{\mu_1\nu_1\cdots\mu_k\nu_k} R^{\mu_1\nu_1}_{\rho_1\sigma_1}\cdots R^{\mu_k\nu_k}_{\rho_k\sigma_k}\,, \tag{2.2}$$

and $R^{\mu\nu}_{\rho\sigma}$ denotes the Riemann tensor $R^{\mu\nu}{}_{\rho\sigma}$ and[1]

$$\delta^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_s} = s!\,\delta^{\beta_1}_{[\alpha_1}\cdots\delta^{\beta_s}_{\alpha_s]}\,. \tag{2.3}$$

The Euler integrands $E^{(k)}$ can also be expressed as

$$E^{(k)} = \tfrac{(2k)!}{2^k}\, R^{[\mu_1\nu_1}_{\mu_1\nu_1}\cdots R^{\mu_k\nu_k]}_{\mu_k\nu_k}\,. \tag{2.4}$$

The first few cases are given by

$$E^{(0)} = 1\,, \qquad E^{(1)} = R\,,$$
$$E^{(2)} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \quad \text{etc.} \tag{2.5}$$

In order for all the Euler integrands $E^{(k)}$ in (2.1) to be non-trivial, the spacetime dimension d should be $\geq 2n+1$.

The term $\sqrt{-g}E^{(k)}$ in the Lagrangian (2.1) gives a contribution

$$E^{(k)\,\nu}_{\mu} = -\tfrac{1}{2^{k+1}}\delta^{\rho_1\sigma_1\cdots\rho_k\sigma_k\,\nu}_{\mu_1\nu_1\cdots\mu_k\nu_k\,\mu}\, R^{\mu_1\nu_1}_{\rho_1\sigma_1}\cdots R^{\mu_k\nu_k}_{\rho_k\sigma_k} \tag{2.6}$$

to the equations of motion.

The equations of motion following from (2.1) imply that the general condition for an (A)dS spacetime with $R_{\mu\nu} = \lambda\, g_{\mu\nu}$ to be a solution is that λ should be any of the roots of a certain nth-order polynomial, with coefficients proportional to the constants α_k. By choosing the coefficients α_k appropriately, one can arrange that all the roots are equal. This case corresponds to having the equations of motion

$$E^{\nu}_{\mu} \equiv -\tfrac{1}{2^{n+1}}\delta^{\rho_1\sigma_1\cdots\rho_n\sigma_n\,\nu}_{\mu_1\nu_1\cdots\mu_n\nu_n\,\mu}\, \widehat{R}^{\mu_1\nu_1}_{\rho_1\sigma_1}\cdots \widehat{R}^{\mu_n\nu_n}_{\rho_n\sigma_n} = 0\,, \tag{2.7}$$

where $\widehat{R}^{\mu\nu}_{\rho\sigma}$, which we shall refer to as the subtracted Riemann tensor,[2] is given by

$$\widehat{R}^{\mu\nu}_{\rho\sigma} = R^{\mu\nu}_{\rho\sigma} + \frac{1}{\ell^2}\delta^{\mu\nu}_{\rho\sigma}\,. \tag{2.8}$$

The subtracted Riemann tensor vanishes in the case of an AdS vacuum with radius ℓ. We could, alternatively, obtain a de Sitter solution, by taking $\ell^2 < 0$. It turns out that

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[1] Note that this normalization for $\delta^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_s}$ is not the rather standard "unit-strength" convention.

[2] Note that in the case of an Einstein metric with cosmological constant such that $R_{\mu\nu} = -(n-1)\,\ell^{-2}\,g_{\mu\nu}$, the subtracted Riemann tensor (2.8) is nothing but the Weyl tensor.

$$E_\mu^\nu = \sum_{k=0}^{n} \Big(\frac{2}{\ell^2}\Big)^{n-k} C_n^k \frac{(d-2k-1)!}{(d-2n-1)!} \frac{2^{k+1}}{2^{n+1}} E_\mu^{(k)\nu}\,,$$
$$= \sum_{k=0}^{n} \Big(\frac{1}{\ell^2}\Big)^{n-k} C_n^k \frac{(d-2k-1)!}{(d-2n-1)!} E_\mu^{(k)\nu}\,. \tag{2.9}$$

These theories were constructed and studied in [16]. We shall refer to them as critical Lovelock gravities of order n.

In this paper, we are interested in the case with $d = 2n+1$, corresponding to the critical gravity of maximum order in a given odd dimension. Thus we have

$$E_\mu^\nu = \sum_{k=0}^{n} \Big(\frac{1}{\ell^2}\Big)^{n-k} C_n^k\, (2(n-k))!\, E_\mu^{(k)\nu}\,. \tag{2.10}$$

The corresponding Lagrangian is thus

$$e^{-1}\mathcal{L} = \sum_{k=0}^{n} \Big(\frac{1}{\ell^2}\Big)^{n-k} C_n^k\, (2(n-k))!\, E^{(k)}\,. \tag{2.11}$$

The critical Lovelock gravities are characterized by the fact that they admit only a single (A)dS vacuum, for which the subtracted Riemann tensor vanishes. The linearization around the AdS vacuum was studied for the five-dimensional case (i.e. Einstein–Gauss–Bonnet, EGB) in [15]. It turns out that for a generic EGB theory, where there are two inequivalent (A)dS vacua, the kinetic term for the linearized graviton gives positive energy in one vacuum, and negative energy in the other. When the two (A)dS spacetimes coalesce, i.e. in the critical theory, the linearized perturbation equations become vacuous. The perturbation equations at quadratic order were derived in [15]. It is straightforward to see that for the critical Lovelock gravity of order n, the analogous perturbation equations up to and including order $(n-1)$ are vacuous.

3. Static solutions

In this paper, we are interested in constructing solutions where, unlike in (A)dS, the subtracted Riemann tensor does not vanish. The simplest such case is perhaps a static, spherically-symmetric metric, for which the most general ansatz takes the form

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2,\epsilon}^2\,, \qquad d\Omega_{d-2,\epsilon}^2 = \frac{dy^2}{1-\epsilon y^2} + y^2 d\Omega_{d-3}^2\,, \tag{3.1}$$

with $\epsilon = 1, 0, -1$, and $d\Omega_{d-3}^2$ is the metric for a unit round S^{d-3}. (To be precise, we include the topologies T^{d-2} and H^{d-2} also, corresponding to taking $\epsilon = 0, -1$ respectively.) The critical theories admit black hole solutions with $h = f$. These solutions were obtained in [16]. For $d = 2n+1$, the solution becomes particularly simple, being given by

$$h = f = r^2/\ell^2 + \epsilon - \mu\,, \tag{3.2}$$

where μ is an integration constant.

It was shown in [15] that the critical EGB theory admits another type of static solution, with

$$f = r^2/\ell^2 + \epsilon\,, \qquad h = h(r) \text{ is an arbitrary function.} \tag{3.3}$$

We may easily check that in fact this gives a solution in all the critical Lovelock gravities: The subtracted Riemann tensor $\widehat{R}^{\mu\nu}_{\rho\sigma}$ is given by

$$\widehat{R}^{ti}_{tj} = \Big(\frac{1}{\ell^2} - \frac{(\epsilon + r^2/\ell^2)h'}{2rh}\Big)\delta^i_j\,, \qquad \widehat{R}^{tr}_{tr} = \frac{rh' - 2h}{2h\ell^2} - \frac{(2hh'' - h'^2)(\epsilon + r^2/\ell^2)}{4h^2}\,, \tag{3.4}$$

with all remaining components, aside from those following from (3.4) by the Riemann tensor symmetries, vanishing identically. As expected, the subtracted Riemann tensor vanishes when $h = r^2/\ell^2 + \epsilon$, corresponding to the AdS vacuum. It is straightforward to see that (3.4) satisfies the equations of motion (2.7) in all the critical Lovelock theories in $d \ge 2n+1$ dimensions, since the non-vanishing components of the subtracted Riemann tensor are not sufficient to span the entire range of index values required by the antisymmetric δ-tensor in (2.7).

4. Rotating solutions: all equal rotation

A rotating solution in the five-dimensional critical EGB theory was obtained in [14], by taking the metric to have a Kerr–Schild form with an AdS "base" $d\bar{s}^2$ that is written in spheroidal coordinates. The geodesic null vector K_μ that is used in the Kerr–Schild construction in [14] is the same as the one used in the construction of the Kerr–Schild form of the five-dimensional Kerr–AdS metric in [5,6]. However, the function w in the Kerr–Schild metric $ds^2 = d\bar{s}^2 + w\,(K_\mu dx^\mu)^2$ is quite different in [14] from the one in [5,6] that gives Kerr–AdS.

The rotating solution in [14] has independent rotation parameters in the two orthogonal spatial 2-planes. We have looked without success for analogous solutions with independent rotation parameters in the higher-dimensional critical Lovelock gravities (2.7) in dimensions $d = 2n+1$. However, we have been able to construct higher-dimensional generalizations in the case where all the rotation parameters are taken to be equal. As was shown in [5,6], the AdS base metric can then be written in terms of the Fubini–Study metric on $\mathbb{CP}^{n-1}$. We find that the full Kerr–Schild metric for the critical Lovelock solution takes the form

$$ds^2 = d\bar{s}^2 + \lambda(r^2+a^2)K^2\,, \tag{4.1}$$

with

$$d\bar{s}^2 = -\frac{(g^2r^2+1)dt^2}{\Xi} + \frac{r^2dr^2}{(g^2r^2+1)(r^2+a^2)} + \frac{r^2+a^2}{\Xi}\big[(d\psi+A)^2 + d\Sigma_{n-1}^2\big]\,,$$
$$K = K_\mu dx^\mu = \frac{1}{\Xi}\big[dt - a(d\psi+A)\big] + \frac{r^2dr}{(g^2r^2+1)(r^2+a^2)}\,, \tag{4.2}$$

where $\Xi = 1-g^2a^2$ and $d\Sigma_{n-1}^2$ is the Fubini–Study metric on $\mathbb{CP}^{n-1}$, with its canonical normalization $\bar{R}_{ab} = 2n\,\bar{g}_{ab}$. As with the Kerr–AdS metrics in odd dimension and with equal angular momenta, the solutions we obtain here have cohomogeneity one.

The metric (4.1) can be recast in terms of Boyer–Lindquist coordinates (for which there are no cross terms between dr and the other coordinate differentials), by means of the transformations

$$dt = d\tau + \frac{\lambda r^2\,dr}{(1+g^2r^2)(1-(\lambda-g^2)r^2)}\,,$$
$$d\psi = d\sigma + g^2a\,d\tau + \frac{a\lambda r^2\,dr}{(r^2+a^2)(1-(\lambda-g^2)r^2)}\,. \tag{4.3}$$

The metric (4.1) then becomes

$$ds^2 = -\frac{\rho^2h^2}{r^2}\Big(d\tau - \frac{a}{\Xi}(d\sigma+A)\Big)^2 + \frac{\rho^4}{r^2}\Big(\frac{(d\sigma+A)}{\Xi} - \frac{a}{\rho^2}d\tau\Big)^2 + \frac{d\rho^2}{h^2} + \frac{\rho^2}{\Xi}d\Sigma_{n-1}^2\,, \tag{4.4}$$

where we use $\rho = \sqrt{r^2+a^2}$ as the radial variable, and

$$r^2 = \rho^2 = a^2\,, \qquad h^2 = 1-(\lambda-g^2)(\rho^2-a^2)\,. \tag{4.5}$$

We now prove that the metric (4.4) indeed satisfies the equation (2.7). It is convenient to define the vielbein basis

$$e^0 = \frac{\rho h}{r}\Big(d\tau - \frac{a}{\Xi}(d\sigma+A)\Big)\,, \qquad e^1 = \frac{d\rho}{h}\,,$$
$$e^2 = \frac{\rho^2}{r}\Big(\frac{(d\tau+A)}{\Xi} - \frac{a}{\rho^2}d\tau\Big)\,, \qquad e^a = \frac{\rho}{\sqrt{\Xi}}\,\bar{e}^a\,, \tag{4.6}$$

where $\bar{e}^a$ is a vielbein basis for $\mathbb{CP}^{n-1}$. In fact, for the purposes of the calculations below, we need not restrict the metric $d\Sigma_{n-1}^2$ to be that of $\mathbb{CP}^{n-1}$ specifically; we may take it to be any Kähler metric on a complex manifold $\mathcal{K}^{n-1}$ of complex dimension $n-1$.

With the function $h(\rho)$ as yet arbitrary, the torsion-free spin connection is given by

$$\omega_{01} = -\frac{r}{\rho}\Big(\frac{\rho h}{r}\Big)' e^0 + \frac{a}{r^2}e^2\,, \quad \omega_{02} = \frac{a}{r^2}e^1\,, \quad \omega_{0a} = \frac{ha}{\rho r}J_{ab}\,e^b\,, \quad \omega_{1a} = -\frac{h}{\rho}e^a\,,$$
$$\omega_{12} = \Big(\frac{hr'}{r} - \frac{2h\rho}{r^2}\Big)e^2 - \frac{a}{r^2}e^0\,, \quad \omega_{2a} = \frac{1}{r}J_{ab}\,e^b\,, \quad \omega_{ab} = \bar{\omega}_{ab} - \frac{ha}{\rho r}J_{ab}\,e^0 - \frac{1}{r}J_{ab}\,e^2\,, \tag{4.7}$$

where a prime denotes a derivative with respect to ρ, J_{ab} are the vielbein components of the Kähler form of $\mathcal{K}^{n-1}$, i.e. $J = \frac{1}{2}J_{ab}\,\bar{e}^a\wedge\bar{e}^b$, and $\bar{\omega}_{ab}$ is the spin connection of $\mathcal{K}^{n-1}$. The curvature 2-forms, after taking $h(\rho)$ to be given by (4.5), are given by

$$\Theta_{01} = -(\lambda-g^2)\,e^0\wedge e^1\,, \qquad \Theta_{02} = -(\lambda-g^2)\,e^0\wedge e^2\,,$$
$$\Theta_{0a} = -(\lambda-g^2)\,e^0\wedge e^a + \frac{ha\sqrt{\Xi}}{\rho^2 r}(\bar{\nabla}_c J_{ab})\,e^c\wedge e^b\,, \qquad \Theta_{1a} = (\lambda-g^2)\,e^1\wedge e^a\,,$$
$$\Theta_{2a} = (\lambda-g^2)\,e^2\wedge e^a + \frac{\sqrt{\Xi}}{\rho r}(\bar{\nabla}_c J_{ab})\,e^c\wedge e^b\,,$$
$$\Theta_{ab} = \bar{\Theta}_{ab} + (\lambda-g^2)\,e^a\wedge e^b - \frac{1+a^2(\lambda-g^2)}{2\rho^2}\,\Omega_{abcd}\,e^c\wedge e^d - (\bar{\nabla}_c J_{ab})\Big(\frac{ha\sqrt{\Xi}}{\rho^2 r}e^c\wedge e^0 + \frac{1}{r}e^c\wedge e^2\Big)\,, \tag{4.8}$$

where $\bar{\Theta}_{ab}$ is the curvature 2-form of $\mathcal{K}^{n-1}$, and

$$\Omega_{abcd} = \delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc} + J_{ac}\,J_{bd} - J_{ad}\,J_{bc} + 2J_{ab}\,J_{cd}\,. \tag{4.9}$$

Note that all the terms involving $(\bar{\nabla}_c J_{ab})$ in (4.8) actually vanish, since the Kähler form is covariantly constant.

It is now evident that if we define $\widehat{R}^{\mu\nu}{}_{\rho\sigma}$ as in (2.8), then provided we choose g and λ such that

$$\frac{1}{\ell^2} = g^2 - \lambda\,, \tag{4.10}$$

then the only non-vanishing components of $\widehat{R}^{\mu\nu}{}_{\rho\sigma}$ will be when the indices lie in the directions of the Kähler manifold, with

$$\widehat{R}^{ab}_{cd} = \frac{\Xi}{\rho^2}\,\bar{R}^{ab}{}_{cd} - \frac{1+a^2(\lambda-g^2)}{\rho^2}\,\Omega^{ab}{}_{cd}\,. \tag{4.11}$$

Since these non-zero components lie within a $(2n-2)$-dimensional subspace of the full $(2n+1)$-dimensional spacetime, it follows that the antisymmetrisations in (2.7) will ensure that the field equations are satisfied.

Note that this gives a solution of the equations of motion when $\mathcal{K}^{n-1}$ is *any* Kähler manifold. For the particular case we started with, when $\mathcal{K}^{n-1} = \mathbb{CP}^{n-1}$ with its standard Fubini–Study metric which has constant holomorphic sectional curvature,

$$\bar{R}_{abcd} = \Omega_{abcd}\,, \tag{4.12}$$

we have the especially simple result that

$$\widehat{R}^{ab}_{cd} = -\frac{a^2\,\lambda}{\rho^2}\,\Omega^{ab}{}_{cd}\,. \tag{4.13}$$

5. Rotating solutions: a single rotation

We have also been able to construct rotating solutions in the $d = 2n+1$ dimensional critical Lovelock gravities (2.7) in the case that just a single rotation parameter is non-vanishing. The metric in $d = 2n+1$ dimensions is given by

$$ds^2 = d\bar{s}^2 + \lambda\,\rho^2 K^2\,, \tag{5.1}$$

with

$$\begin{aligned} d\bar{s}^2 &= -\frac{(g^2r^2+1)\Delta_\theta dt^2}{1-a^2g^2} + \frac{\rho^2 dr^2}{(g^2r^2+1)(r^2+a^2)} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{(r^2+a^2)\sin^2\theta\, d\phi^2}{1-a^2g^2} + r^2\cos^2\theta\, d\Omega^2_{2n-3}\,,\\ K &= K_\mu dx^\mu = \frac{\Delta_\theta\, dt}{1-a^2g^2} - \frac{\rho^2 dr}{(g^2r^2+1)(r^2+a^2)} - \frac{a\sin^2\theta\, d\phi}{1-a^2g^2}\,,\\ \rho^2 &= r^2 + a^2\cos^2\theta\,, \qquad \Delta_\theta = 1 - a^2g^2\cos^2\theta\,. \end{aligned} \tag{5.2}$$

If we choose g and λ to satisfy (4.10), then we find that the non-vanishing components of the subtracted Riemann tensor $\hat{R}^{\mu\nu}_{\rho\sigma}$, defined in (2.8), are given by the expressions in Appendix A. Decomposing the indices as $\mu = (a, i)$, etc., where $x^a = (t, r, \theta, \phi)$ and x^i are the coordinates of the $(2n-3)$-sphere, the non-vanishing components of $\hat{R}^{\mu\nu}_{\rho\sigma}$ are of the forms

$$\hat{R}^{ab}_{cd}\,, \qquad \hat{R}^{ai}_{bj} = T^a_b\,\delta^i_j\,, \qquad \hat{R}^{ij}_{k\ell} = f\,\delta^{ij}_{k\ell}\,. \tag{5.3}$$

The expressions for f, T^a_b and $\widehat{R}^{ab}_{cd}$ can be found in (A.1), (A.2) and (A.3) respectively. A crucial point for what follows is that the expressions for the components of $\hat{R}^{\mu\nu}_{\rho\sigma}$ are completely independent of the spacetime dimension (except for the obvious fact that the range of the i index is dimension dependent). Furthermore, the non-vanishing components of $\widehat{R}^{\mu\nu}_{\rho\sigma}$ have either four, two (one up, one down) or zero $(2n-3)$-sphere indices.

Given the structure of the non-vanishing components of $\widehat{R}^{\mu\nu}_{\rho\sigma}$, it is clear from (2.7) that the only non-trivial equations of motion will be

$$E^b_a = 0 \quad \text{and} \quad E^j_i = 0\,. \tag{5.4}$$

Furthermore, we see that

$$\begin{aligned} E^b_a &= \alpha_1\, f^{n-2}\, S^{(1)\,b}_a + \alpha_2\, f^{n-3}\, S^{(3)\,b}_a\,,\\ E^j_i &= \alpha_3\, f^{n-2}\, S^{(0)}\,\delta^j_i + \alpha_4\, f^{n-3}\, S^{(1)\,b}_a\, T^a_b\,\delta^j_i + \alpha_5\, f^{n-4}\, S^{(3)\,b}_a\, T^a_b\,\delta^j_i\,, \end{aligned} \tag{5.5}$$

where the α coefficients are non-vanishing combinatoric factors, and

$$S^{(0)} = \delta^{c_1d_1c_2d_2}_{a_1b_1a_2b_2}\,\widehat{R}^{a_1b_1}_{c_1d_1}\,\widehat{R}^{a_2b_2}_{c_2d_2}\,, \qquad S^{(1)\,b}_a = \delta^{bc_1d_1c_2}_{aa_1b_1a_2}\,\widehat{R}^{a_1b_1}_{c_1d_1}\,T^{a_2}_{c_2}\,, \qquad S^{(3)\,b}_a = \delta^{bd_1c_2d_2}_{ab_1a_2b_2}\,T^{a_1}_{c_1}\,T^{b_1}_{d_1}\,T^{a_2}_{c_2}\,. \tag{5.6}$$

After rather intricate, but mechanical calculations (which we performed using Mathematica), we find that

$$S^{(0)} = 0\,, \qquad S^{(1)\,b}_a = 0\,, \qquad S^{(3)\,b}_a = 0\,, \tag{5.7}$$

and hence the single rotation metrics satisfy the equations of motion (2.7) in all dimensions $d = 2n+1$, provided that (4.10) holds.

6. Conclusions

In this paper, we considered critical Lovelock gravities and focused on those of order n in $d = 2n+1$ dimensions. We obtained two classes of rotating solutions. In the first class, all the angular momentum parameters are set equal, and the metric is of cohomogeneity one. We presented the metric in both the Kerr–Schild and Boyer–Lindquist forms. In the second class of solutions, only a single rotation parameter is non-vanishing, and the solution is obtained in the Kerr–Schild form. In this case, it does not appear to be possible to rewrite it in terms of Boyer–Lindquist type coordinates. By calculating the subtracted Riemann tensor that appears in the equations of motion (2.7) explicitly, we demonstrated that the metrics in both of the classes indeed satisfy the equations of motion. When restricted to five dimensions, our solutions are special cases of the rotating solutions constructed in [14].

The metrics are all asymptotic to AdS, but they do not describe black holes. Rather, they have naked curvature singularities. The analysis is rather straightforward for the solution with where all the angular momenta are equal, since in this case we can rewrite the metric using Boyer–Lindquist type coordinates. Another way to see the geometric structure by noting that if we set the rotation parameter to zero, the solution reduces to AdS, with no "mass" parameter analogous to μ in the static solutions (3.2). The naked singularity can thus be understood as being associated with a solution that is "over rotated," in the sense that it has angular momentum but no mass.

Exact rotating solutions are hard to come by, and although the solutions we have obtained here have shortcomings associated with the presence of naked singularities, they do perhaps provide a guide as to how one might hope to construct more general rotating solutions in critical Lovelock gravities. It would be of great interest to try to obtain such generalizations where a mass parameter could be added, so that rotating black hole solutions without naked singularities might be possible. It would also be interesting to seek rotating solutions in the higher-dimensional critical Lovelock gravities in which the angular momentum parameters could be arbitrary.

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Appendix A. Subtracted Riemann tensor for single-rotation metrics

The components of the subtracted Riemann tensor (2.8) for the Kerr–Schild metrics defined by (5.1) and (5.2) can be given as follows. With $x^a = (t, r, \theta, \phi)$ and x^i being the coordinates of the $(2n-3)$-sphere, the non-vanishing components of $\widehat{R}^{\mu\nu}_{\rho\sigma}$ involve either four, two (one up, one down) or zero $(2n-3)$-sphere indices. Writing $c \equiv \cos\theta$ and $s \equiv \sin\theta$, we find

$$\widehat{R}^{ij}_{kl} = \frac{\lambda a^2 c^2}{r^2}\delta^{ij}_{kl}, \qquad \widehat{R}^{ai}_{bj} = T^a_b\,\delta^i_j, \tag{A.1}$$

where

$$\begin{aligned}
&T^t_t = \frac{2\lambda a^2 s^2\,\Delta_\theta^2}{\Xi_a(g^2r^2+1)\rho^2}, \qquad T^\phi_t = \frac{2\lambda a^3 s^2\,\Delta_\theta^2}{\Xi_a(r^2+a^2)\rho^2}, \qquad T^\theta_t = \frac{2\lambda a^2 cs\Delta_\theta^2}{\Xi_a r\rho^2},\\
&T^r_t = \frac{2\lambda a^2 s^2\Delta_\theta^2}{\Xi_a\rho^2}, \qquad T^t_\phi = -\frac{2\lambda a^3 s^4\,\Delta_\theta}{\Xi_a(g^2r^2+1)\rho^2}, \qquad T^\phi_\phi = -\frac{2\lambda a^4 s^4\,\Delta_\theta}{\Xi_a(r^2+a^2)\rho^2},\\
&T^\theta_\phi = -\frac{2\lambda a^3 cs^3\theta\,\Delta_\theta}{\Xi_a r\rho^2}, \qquad T^r_\phi = -\frac{2\lambda a^3 s^4\Delta_\theta}{\Xi_a\rho^2}, \qquad T^t_\theta = -\frac{2\lambda a^2 cs}{r(g^2r^2+1)},\\
&T^\phi_\theta = -\frac{2\lambda a^3 cs}{r(r^2+a^2)}, \qquad T^r_\theta = -\frac{2\lambda a^2 cs}{r}, \qquad T^t_r = -\frac{2\lambda a^2 s^2\Delta_\theta}{(g^2r^2+1)^2(r^2+a^2)},\\
&T^\phi_r = -\frac{2\lambda a^3 s^2\Delta_\theta}{(g^2r^2+1)(r^2+a^2)^2}, \qquad T^\theta_r = -\frac{2\lambda a^2 cs\Delta_\theta}{r(g^2r^2+1)(r^2+a^2)},\\
&T^r_r = -\frac{2\lambda a^2 s^2\Delta_\theta}{(g^2r^2+1)(r^2+a^2)}, \qquad T^\theta_\theta = 0,
\end{aligned} \tag{A.2}$$

and the components $\widehat{R}^{ab}_{cd}$ are given by

$$\begin{aligned}
&\widehat{R}^{t\phi}_{t\phi} = \frac{2\lambda a^2 c^2[-(r^2+a^2)(\Xi_a+2a^2g^2s^2)+a^4g^2s^4]}{(g^2r^2+1)(r^2+a^2)\rho^2}, \qquad \widehat{R}^{t\phi}_{t\theta} = -\frac{2\lambda a^3 g^2 rcs}{(g^2r^2+1)(r^2+a^2)},\\
&\widehat{R}^{t\phi}_{tr} = \frac{2\lambda a^5 g^2c^2s^2}{(g^2r^2+1)(r^2+a^2)^2}, \qquad \widehat{R}^{t\phi}_{\phi\theta} = \frac{2\lambda a^2 rcs}{(g^2r^2+1)(r^2+a^2)},\\
&\widehat{R}^{t\phi}_{\phi r} = -\frac{2\lambda a^2c^2\Delta_\theta}{(g^2r^2+1)^2(r^2+a^2)}, \qquad \widehat{R}^{t\phi}_{\theta r} = \frac{2\lambda a^3 rcs\,\Xi_a}{(g^2r^2+1)^2(r^2+a^2)^2},\\
&\widehat{R}^{t\theta}_{t,\phi} = -\frac{2\lambda a^3 g^2 rcs^3\Delta_\theta}{\Xi_a(g^2r^2+1)\rho^2}, \qquad \widehat{R}^{t\theta}_{tr} = -\frac{rs\Delta_\theta}{\Xi_a c\rho^2}\widehat{R}^{t\phi}_{t\phi}, \qquad \widehat{R}^{tr}_{t\phi} = \frac{2\lambda a^5 g^2c^2s^4}{\Xi_a\rho^2},\\
&\widehat{R}^{t\theta}_{t\theta} = \frac{2\lambda a^2}{\Xi_a(g^2r^2+1)\rho^4}\Big[r^2+r^2(2a^2g^4r^2-5a^2g^2-2)c^2-a^2(2g^4r^4-8a^2g^4r^2-7g^2r^2+3a^2g^2+1)c^4\\
&\qquad -a^4g^2(9g^2r^2-5a^2g^2-5)c^6-6a^6g^4c^8\Big],\\
&\widehat{R}^{t\theta}_{\phi\theta} = -\frac{2\lambda a^3 s^2\Big[r^2+r^2(2g^2r^2-3a^2g^2-3)c^2+a^2(9g^2r^2-2a^2g^2-2)c^4+6a^4g^2c^6\Big]}{\Xi_a(g^2r^2+1)\rho^4},\\
&\widehat{R}^{t\theta}_{\phi r} = -\frac{2\lambda a^3 rcs^3\Delta_\theta(2g^2r^2+a^2g^2c^2+1)}{\Xi_a(g^2r^2+1)^2\rho^4}, \qquad \widehat{R}^{tr}_{t\theta} = \frac{(g^2r^2+1)(r^2+a^2)}{\Delta_\theta}\widehat{R}^{t\theta}_{tr},
\end{aligned}$$

$$
\begin{aligned}
&\widehat{R}^{t\,\theta}_{\theta\,r}=\frac{2\lambda a^2(1-(5a^2g^2+2)c^2+6a^2g^2c^4)}{(g^2r^2+1)^2(r^2+a^2)}, \qquad \widehat{R}^{t\,r}_{\phi\,r}=-\frac{2\lambda a^5c^2s^4\Delta_\theta}{\Xi_a(g^2r^2+1)\rho^4}, \\
&\widehat{R}^{t\,r}_{\phi\,\theta}=\frac{2\lambda a^3cs^3r[a^2s^2-2(r^2+a^2)]}{\Xi_a\rho^4}, \qquad \widehat{R}^{t\,r}_{t\,r}=\frac{2\lambda a^4c^2s^2[(\Xi_a(r^2+a^2)+a^4g^2s^4]}{\Xi_a(r^2+a^2)\rho^2}, \\
&\widehat{R}^{t\,r}_{\theta\,r}=\frac{2\lambda a^2csr}{(g^2r^2+1)(r^2+a^2)}, \qquad \widehat{R}^{\phi\,\theta}_{t\,\phi}=-\frac{2\lambda a^2cs\Delta_\theta^2r}{\Xi_a(r^2+a^2)\rho^2}, \qquad \widehat{R}^{\phi\,\theta}_{t\,\theta}=-\frac{(g^2r^2+1)\Delta_\theta}{(r^2+a^2)s^2}\widehat{R}^{t\,\theta}_{\phi\,\theta}, \\
&\widehat{R}^{\phi\,\theta}_{t\,r}=-\frac{\Delta_\theta^2}{(r^2+a^2)^2s^2}\widehat{R}^{t\,r}_{\phi\,\theta}, \qquad \widehat{R}^{\theta\,r}_{t\,\phi}=-\frac{2\lambda a^3cs^3\Delta_\theta^2r}{\Xi_a\rho^4}, \\
&\widehat{R}^{\phi\,\theta}_{\phi\,\theta}=\frac{2\lambda a^2}{\Xi_a(r^2+a^2)\rho^4}\Big[6a^6g^2c^8-a^4(5+5a^2g^2-9g^2r^2)c^6-a^2r^2 \\
&\qquad +a^2(3a^2+a^4g^2-8r^2-7a^2g^2r^2+2g^2r^4)c^4+r^2(5a^2+2a^4g^2-2r^2)c^2\Big], \\
&\widehat{R}^{\phi\,\theta}_{\phi\,r}=\frac{s\Delta_\theta r}{\Xi_a\rho^2c}\widehat{R}^{t\,\phi}_{t\,\phi}, \qquad \widehat{R}^{\phi\,r}_{\phi\,\theta}=\frac{(g^2r^2+1)(r^2+a^2)}{\Delta_\theta}\widehat{R}^{\phi\,\theta}_{\phi\,r}, \\
&\widehat{R}^{\phi\,\theta}_{\theta\,r}=\frac{2\lambda a^3(1-c^2(5+2a^2g^2)+6a^2c^4g^2)}{(g^2r^2+1)(r^2+a^2)^2}, \qquad \widehat{R}^{\theta\,r}_{\phi\,\theta}=\frac{(g^2r^2+1)(r^2+a^2)^2s^2}{\Xi_a\rho^2}\widehat{R}^{\phi\,\theta}_{\theta\,r}, \\
&\widehat{R}^{\phi\,r}_{t\,\phi}=-\frac{(g^2r^2+1)^2(r^2+a^2)\Delta_\theta}{\Xi_a\rho^2}\widehat{R}^{t\,\phi}_{\phi\,r}, \qquad \widehat{R}^{\phi\,r}_{t\,\theta}=-\frac{(g^2r^2+1)^2}{s^2}\widehat{R}^{t\,\theta}_{\phi\,r}, \\
&\widehat{R}^{\phi\,r}_{t\,r}=-\frac{(g^2r^2+1)\Delta_\theta}{(r^2+a^2)s^2}\widehat{R}^{t\,r}_{\phi\,r}, \qquad \widehat{R}^{\phi\,r}_{\phi\,r}=\frac{2\lambda a^2c^2\Delta_\theta[\Xi_a(r^2+a^2-2a^2s^2)-a^4g^2s^4]}{\Xi_a(g^2r^2+1)\rho^4}, \\
&\widehat{R}^{\phi\,r}_{\theta\,r}=\frac{2\lambda a^3g^2csr}{(g^2r^2+1)(r^2+a^2)}, \qquad \widehat{R}^{\theta\,r}_{\phi\,r}=\frac{g^2a(r^2+a^2)s^2\Delta_\theta}{\Xi_a\rho^2}\widehat{R}^{\phi\,r}_{\theta\,r}, \\
&\widehat{R}^{\theta\,r}_{t\,\theta}=-\frac{(g^2r^2+1)^2(r^2+a^2)\Delta_\theta}{\Xi_a\rho}\widehat{R}^{t\,\theta}_{\theta\,r}, \qquad \widehat{R}^{\theta\,r}_{t\,r}=-\frac{(g^2r^2+1)\Delta_\theta^2}{\Xi_a\rho^2}\widehat{R}^{t\,r}_{\theta\,r}, \\
&\widehat{R}^{\theta\,r}_{\theta\,r}=-\frac{2\lambda a^2[r^2-2(-2a^2+r^2+a^2g^2r^2)c^2-a^2(5+5a^2g^2-3g^2r^2)c^4+6a^4g^2c^6]}{(g^2r^2+1)(r^2+a^2)\rho^2}. \qquad \text{(A.3)}
\end{aligned}
$$

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