Measures on Mixing Angles

Gary W. Gibbons[1], Steffen Gielen[1], C. N. Pope[1,2] and Neil Turok[1,3]

[1]*D.A.M.T.P., Centre for Mathematical Sciences, Cambridge University,*
Wilberforce Road, Cambridge, CB3 0WA, U.K.

[2]*George P. & Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy,*
Texas A&M University, College Station, TX 77843-4242, USA

[3]*Perimeter Institute for Theoretical Physics,*
31 Caroline St. N., Waterloo, Ontario, Canada N2L 2Y5

August 16, 2018

Abstract

We address the problem of the apparently very small magnitude of CP violation in the standard model, measured by the Jarlskog invariant J. In order to make statements about probabilities for certain values of J, we seek to find a natural measure on the space of Kobayashi-Maskawa matrices, the double quotient $U(1)^2 \backslash SU(3)/U(1)^2$. We review several possible, geometrically motivated choices of the measure, and compute expectation values for powers of J for these measures. We find that different choices of the measure generically make the observed magnitude of CP violation appear finely tuned. Since the quark masses and the mixing angles are determined by the same set of Yukawa couplings, we then do a second calculation in which we take the known quark mass hierarchy into account. We construct the simplest measure on the space of 3×3 Hermitian matrices which reproduces this known hierarchy. Calculating expectation values for powers of J in this second approach, we find that values of J close to the observed value are now rather likely, and there does not seem to be any fine-tuning. Our results suggest that the choice of Kobayashi-Maskawa angles is closely linked to the observed mass hierarchy. We close by discussing the corresponding case of neutrinos.

1

Contents

1   Introduction

A traditional attitude to theoretical physics has been that the main problem is to discover the fundamental laws of physics and leave it to experiment and observation to decide what particular implementation best describes "Our Universe." Thus traditionally a "physical theory" is often thought of in terms of a local Lagrangian including certain "coupling constants," "mass ratios," and "mixing angles," all of which, since Planck's introduction of Planck units [1], may be taken to be dimensionless numbers. In addition, the local Lagrangian must be supplemented with an account of the general class of boundary conditions for which the variational principle is valid. Different classes of boundary conditions are usually thought of as different "superselection sectors" of the theory, and describe qualitatively different types of situations which traditionally are not thought of as having any relation to one another.

Within each sector, there are many solutions of the equations of motion, each of which may be specified by providing suitable "initial conditions." Classically these conditions may be thought of as the space of classical histories, and given in terms of Cauchy data modulo the relation that two sets of Cauchy data giving the same history are taken to be equivalent. Quantum mechanically one thinks in terms of some initial, and thus in the Heisenberg picture, eternal state.

The hope has frequently been expressed in the past that eventually theorists will hit upon a unique theory, with all coupling constants determined by consistency or symmetry considerations, and with just one superselection sector. Even given such a "theory of everything" (TOE), there remains the issue of boundary conditions or initial state, as emphasized by Hawking [2]. Recently, however, there has been a considerable decline in optimism and few now seem to believe in a single TOE with a single superselection sector, and many refer to a "landscape" of theories.

One approach to this perceived crisis in theoretical physics is to resort to "anthropic" considerations and invoke the idea that there may indeed exist, in the Platonic sense, an enormous number of "possible universes," of which only very few will allow the development of sentient beings, and even fewer will allow sentient beings like ourselves. Thus one is led to contemplate the ensemble of all possible universes, sometimes referred to as a "multiverse" [3]. This ensemble is sometimes thought of non-Platonically [1] as an ensemble of connected subsets of a much bigger physically existing universe, referred to as a meta-universe [4].

At this point it may be helpful to remark, lest the daunting task of thinking about and making more precise, the nebulous notion of such a multiverse should not be thought entirely a problem for theorists seeking credit for making predictions about the world we see about us: that the observers and experimenters must also face up to that task when assessing the reliability of their measurements or the extent to which they can confirm theoretical predictions. All such activities are essentially Bayesian in character [5] and require some notion of "priors," that is, some sort of *a priori* measure of the space of possibilities.

This problem has been addressed, with admittedly only partial success, in a previous paper [6]

---

[1]in an Aristotelian sense?

where the multiverse, for concreteness, was identified with the set of classical histories of a minisuperspace cosmological model. A well-defined and natural local measure on the space of classical histories is easily constructed, but unfortunately the total measure of all histories, even in this finite-dimensional truncation of the full set of solutions of Einstein's equations, is infinite. The problem was recently revisited in [7].

In the present paper, we shall turn to the problem of finding a natural measure on the space of coupling constants. Thus the multiverse in the present paper is a set of Lagrangians parametrized by a manifold X or "moduli space," whose coordinates consist of masses, mixing angles, coupling constants, etc., and we wish to place a natural measure on this space. We hope this will be useful for anthropic considerations such as those of [8], where $X \equiv S^1$, the circle parametrizing the phase of the axion. In that case the issue of a measure was trivial, but in more complicated cases such as we shall consider in the present paper, the situation is more complicated. We also hope that the work in this paper will help in clarifying the notion of "fine-tuning," which is so prevalent in phenomenological discussions.

The structure of this paper is as follows. After introducing the notion of geometric probability and outlining the Kobayashi-Maskawa theory of CP violation in the standard model, we discuss metrics on $SU(3)$ and its quotients in Sec. 2, starting with a left-invariant metric on $SU(3)$ which induces a metric on the flag manifold $SU(3)/U(1)^2$. We perform a Kaluza-Klein type reduction on the left phases and discuss different possible metrics on the double quotient $U(1)^2 \backslash SU(3)/U(1)^2$, the space of Kobayashi-Maskawa matrices. We also discuss the metric used by Ozsváth and Schücking [9], and argue that it lacks a geometrical justification.

We then use all metrics we have discussed to compute statistics of the Jarlskog invariant J. While the measure on $SU(3)/U(1)^2$ is independent from the choice of left-invariant metric, the measure on the double quotient is nonunique. We find that in each case the standard deviation ΔJ (with $\langle J \rangle = 0$) is about three orders of magnitude greater than the experimentally observed value of J, which appears to be finely tuned. In Sec. 4, we do a closer numerical analysis of the probability distribution of $|J|$ on the double quotient, using several possible choices for the measure. We quantify the statement that a magnitude of CP violation as small as observed appears unlikely.

In Sec. 5, we take a different viewpoint: We now consider random distributions on the space of mass matrices in the standard model. We therefore need to find a measure on the space of 3×3 Hermitian matrices. We find that the simplest choice which gives convergent integrals over this space, and expectation values for squared quark masses which reproduce the observed values, is a Gaussian weighting function with four free parameters, which can be chosen appropriately. We then find that the standard deviation ΔJ is much smaller for this measure, making the observed value of J appear typical. We conclude that with an appropriate distribution which respects the known quark mass hierarchy there is no need for fine-tuning in J.

We briefly discuss the case of neutrinos in Sec. 6, explaining the general theory, and the difference between Dirac and Majorana masses. We cannot give reliable predictions for ΔJ for neutrinos, due to the absence of known values for their masses.

1.1 Geometric Probability

The construction of appropriate measures over spaces of geometric objects goes back to the 18th century cosmologist Buffon and his celebrated needle problem [10]. The reader may find a general account of the subject in [11]. The simplest case to consider is when the space of coupling constants X may be regarded as a finite-dimensional homogeneous space with respect to some Lie group G of symmetries, and the stabilizer or little group is $H \subset G$. Thus $X = G/H$. If $\dim X = n$, our aim is to construct an n-form on X which is invariant under the action of G. In the case that $X = G$, as in the example of the axion circle, this is completely unambiguous. We pick *any* n-form at the unit element $e \in G$ and spread it over G by left or right translation. On a unimodular group, such as a compact group or a semisimple group, left or right translation will give identical results. The original n-form, being a top degree form, is unique up to a multiple. This multiple can be fixed by normalizing the total measure to unity. The normalized measure is therefore unique.

We could, if we wished, construct the measure as the Riemannian volume element of *any* left or right-invariant metric on G. The result would be the same. In practice, a convenient procedure for calculating the measure could be to construct an invariant metric on $X = G$ and then calculate its Riemannian volume element. Often, the bi-invariant or Killing metric is the most convenient choice.

In the case of a coset, $X = G/H$, the measure can again be taken to be any n-form at some arbitrarily chosen point $x \in X$, which is then spread around using the group action. Since any n-form at x will be H-invariant, the result is again unique and invariant under all the symmetries of the problem. Of course it is possible that one may express $X = G/H$ in more than one way. This could in principle give rise to some discrete nonuniqueness, but in practice this seems not to be important.

Although the situation when coupling constants may be regarded as belonging to a homogeneous space is quite satisfactory, it is often the case that coupling constants belong to an inhomogeneous space. In particular, in the case of "mixing angles," they typically belong to a double coset, or bi-quotient, of the form $H_1 \backslash G / H_2$, where H_1 and H_2 are (not necessarily identical) Lie subgroups of G. The reason for this is that mixing angles relate two unitary bases for the same space of physical states. The two unitary bases may not be unique. In particular, it is often the case that the individual basis vectors can be multiplied by arbitrary phases. In this case, H_1 and H_2 may belong to $U(1)^k$, where k is the number of states in the basis. In the case of the Kobayashi-Maskawa matrix the states are quarks, and one basis diagonalizes the strong Hamiltonian while the other basis diagonalizes the weak interaction quantum numbers.

A biquotient, or double coset, $H_1 \backslash G / H_2$, is typically *not* a homogeneous space. This is because the left action of G will not in general commute with H_1, and similarly, the right action of G will not commute with H_2. As a consequence, one cannot, in the case of biquotients, use group invariance to construct an unambiguous measure on the space of mixing angles. Later in this paper, we shall explore in detail some available options, and the extent to which they affect the probability distribution of mixing angles.

1.2 CP Violation and the Jarlskog Invariant

In this section we shall review the Kobayashi-Maskawa theory of CP violation in the quark sector of the standard model.

If m and m' are the (Hermitian) mass matrices for the charge $\frac{2}{3}$ and $-\frac{1}{3}$ quarks, respectively, then there exist unitary matrices U and U' such that

$$UmU^\dagger = \text{diag}(m_u, m_c, m_t), \qquad U'm'U'^\dagger = \text{diag}(m_d, m_s, m_b). \tag{1}$$

The Kobayashi-Maskawa matrix V is defined by

$$V = UU'^\dagger. \tag{2}$$

The normalized mass eigenstates are only defined up to a phase, and changing these phases changes the matrices U and U' according to

$$U \longrightarrow P_L U, \qquad U' \longrightarrow P_R^\dagger U'. \tag{3}$$

Hence the Kobayashi-Maskawa matrix changes according to

$$V \to P_L V P_R, \tag{4}$$

where P_L and P_R are diagonal matrices belonging to $SU(3)$. In other words, P_L and P_R may each be thought of as belonging to $T^2 \equiv U(1) \times U(1)$, the maximal torus of $SU(3)$. Thus the four-dimensional space of CP violating parameters should be thought of as an element of the double coset, or biquotient, $U(1)^2\backslash SU(3)/U(1)^2$, whereas the matrices U and U' should be thought of as elements of the left coset $U(1)^2\backslash SU(3)$.

In the discussion of geometric probability attempted in this paper, one could take the viewpoint that U and U' are the fundamental objects relevant in CP violation, which would lead to discussing distributions on $(U(1)^2\backslash SU(3))^2$. One can then use the fact that only $V = UU'^\dagger$ appears in the Kobayashi-Maskawa theory to reduce this to a distribution on a single $U(1)^2\backslash SU(3)$, as we shall see in Sec. 3.1. Alternatively, one considers V as fundamental and considers the biquotient.

Because the right action of $U(1)^2$ is free, the intermediate coset $SU(3)/U(1)^2$ is a compact smooth homogeneous space without boundary, on which $SU(3)$ acts by left actions. In fact $SU(3)/U(1)^2$ is an example of a flag manifold. The maximal torus $U(1)^2$ acts on the flag manifold via left actions of $SU(3)$, but its action on $SU(3)/U(1)^2$ is not free, and as a consequence, the biquotient $U(1)^2\backslash SU(3)/U(1)^2$ is not a smooth compact manifold without boundary. Rather, it is a stratified set whose boundary consists of components at which either or both of the left-acting $U(1)$ factors has fixed points.

In the standard notation

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \tag{5}$$

and it is customary to choose the phases so that

$$
V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{6}
$$

where $s_{12} = \sin\theta_{12}$, $c_{12} = \cos\theta_{12}$, etc., and the angles θ_{12}, θ_{13}, and θ_{23} are taken all to lie in the first quadrant (*i.e.* between 0 and $\frac{1}{2}\pi$).

One can take the angle δ as a measure of CP violation, but its definition depends on the choice of phases. Jarlskog [12, 13] introduced a formalism that eliminates this arbitrariness. She defined a Hermitian tracefree matrix C by

$$
[m, m'] = iC , \tag{7}
$$

and took $\det C$ as a measure of CP violation. She showed that

$$
\det C = -2TBJ , \tag{8}
$$

where

$$
T = (m_t - m_u)(m_t - m_c)(m_c - m_u) , \qquad B = (m_b - m_d)(m_b - m_s)(m_s - m_d) , \tag{9}
$$

and the Jarlskog invariant J is given by

$$
J = \mathfrak{Im}\big(V_{11}\, V_{22}\, V_{12}^*\, V_{21}^*\big) . \tag{10}
$$

Despite appearances, J is independent of the arbitrary phases. In other words, it is invariant under (4). In fact, it has an extremely elegant geometrical interpretation. Since V is a unitary matrix, its three rows and columns are orthogonal. Thus, for example, there are three relations of the form

$$
(VV^\dagger)_{12} = V_{11}\, V_{21}^* + V_{12}\, V_{22}^* + V_{13}\, V_{23}^* = 0 . \tag{11}
$$

The three complex numbers $a = V_{11}\, V_{21}^*$, $b = V_{12}\, V_{22}^*$, and $c = V_{13}\, V_{23}^*$, satisfying $a + b + c = 0$, may be thought of as the three sides of a *unitarity triangle* in the complex plane. The absolute value of J is twice the area of this triangle:

$$
|J| = |\mathfrak{Im}(ab^*)| = |\mathfrak{Im}(ac^*)| = |\mathfrak{Im}(bc^*)| . \tag{12}
$$

The effect of the transformation (4) is to rotate this triangle in the complex plane, but the area $\frac{1}{2}|J|$ is unchanged. Less obviously, the same area results from taking either of the two other possible inner products, $(VV^\dagger)_{13} =$ or $(VV^\dagger)_{23} = 0$. Thus J is an invariant, and so it is well defined on the space of mixing angles.

In terms of the standard parametrization (6), the Jarlskog invariant is given by

$$
J = c_{12}\, c_{23}\, c_{13}^2\, s_{12}\, s_{23}\, s_{13}\, \sin\delta . \tag{13}
$$

One could choose to take a different quantity as a measure of CP violation. Jarlskog [14] suggested appropriately normalizing the determinant (8) and using

$$a_{CP} = 3\sqrt{6}\,\frac{\det C}{(\operatorname{Tr} C^2)^{3/2}}\,,\tag{14}$$

which takes values between $+1$ and -1 and is zero if and only if CP is conserved. Written out explicitly in terms of the quark masses and mixing angles, this is a complicated expression that we do not give here. As in the present paper the observed quark mass hierarchy is assumed, we shall not consider the case of coinciding quark masses, and we concentrate on J as a measure of CP violation.

Another possible source of confusion is the assumption of general, not necessarily Hermitian, mass matrices. In this case the commutator (7) is replaced by

$$\left[mm^\dagger, m'm'^\dagger\right] = \mathrm{i}\mathbf{C}\tag{15}$$

in order for $\mathbf{C}$ to be Hermitian. The use of C or $\mathbf{C}$ can lead to ambiguous "orders of magnitude" estimates for CP violating processes, e.g. when discussing baryogenesis. We will assume that m and m' are Hermitian, and as our calculations only involve J these considerations will not be relevant.

2   Metrics on $SU(3)$ and its Quotients

A generic element U of $SU(3)$ is conveniently parametrized by eight real coordinates (p,q,r,t,x,y,z,w), so that

$$U = T_L\, W\, T_R\,,\tag{16}$$

where

$$T_L = e^{\frac{\mathrm{i}}{2}\,(3p-q)\lambda_3 + \frac{\mathrm{i}\sqrt{3}}{2}\,(p+q)\lambda_8}\,,\qquad T_R = e^{\mathrm{i}t\lambda_3 + \mathrm{i}\sqrt{3}\,r\lambda_8}\,,\tag{17}$$

and

$$W = e^{\mathrm{i}x\lambda_7}\,e^{-\mathrm{i}w\lambda_3}\,e^{\mathrm{i}y\lambda_5}\,e^{\mathrm{i}w\lambda_3}\,e^{\mathrm{i}z\lambda_2}\,,\tag{18}$$

with

$$0 \le x \le \tfrac{1}{2}\pi\,,\quad 0 \le y \le \tfrac{1}{2}\pi\,,\quad 0 \le z \le \tfrac{1}{2}\pi\,,\qquad 0 \le w \le 2\pi\,.\tag{19}$$

Here, we are using the standard Gell-Mann representation for the generators of $SU(3)$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix},\qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},\tag{20}$$

Explicitly, the matrices W, T_L, and T_R are given by

$$
W = \begin{pmatrix} c_y c_z & c_y s_z & e^{-\mathrm{i}w} s_y \\ -c_x s_z - e^{\mathrm{i}w} s_x s_y c_z & c_x c_z - e^{\mathrm{i}w} s_x s_y s_z & s_x c_y \\ s_x s_z - e^{\mathrm{i}w} c_x s_y c_z & -s_x c_z - e^{\mathrm{i}w} c_x s_y s_z & c_x c_y \end{pmatrix},
\tag{21}
$$

$$
T_L = \mathrm{diag}\!\left(e^{2\mathrm{i}p}, e^{-\mathrm{i}p+\mathrm{i}q}, e^{-\mathrm{i}p-\mathrm{i}q} \right),
\tag{22}
$$

$$
T_R = \mathrm{diag}\!\left(e^{\mathrm{i}r+\mathrm{i}t}, e^{\mathrm{i}r-\mathrm{i}t}, e^{-2\mathrm{i}r} \right),
\tag{23}
$$

where we use the notation $s_x = \sin x$, $c_x = \cos x$, etc.

If we identify W as the Kobayashi-Maskawa matrix V in the standard conventions (6), then

$$
x = \theta_{23}, \qquad y = \theta_{13}, \qquad z = \theta_{12}, \qquad w = \delta.
\tag{24}
$$

If we define left-invariant one-forms σ_a by

$$
U^{-1}\, dU = \mathrm{i}\, \lambda_a\, \sigma_a,
\tag{25}
$$

then the general left-invariant metric on $SU(3)$ takes the form

$$
ds^2 = g_{ab}\, \sigma_a\, \sigma_b,
\tag{26}
$$

where g_{ab} is a constant symmetric matrix.

For a general choice of the matrix g_{ab}, the metric admits no further isometries beyond the left action of $SU(3)$, which we denote by $SU(3)_L$. For special choices of g_{ab}, however, the metric is additionally invariant under the right action of some subgroup K of $SU(3)_R$. The most symmetric such case, the bi-invariant or Killing metric for which K is the full right-acting $SU(3)_R$, arises if g_{ab} is proportional to δ_{ab}. The various intermediate possibilities, of which there are five, are listed in [15].

In the generic case (i.e. when K is the identity), $28 = 36 - 8$ parameters are required to specify the metric. One of these parameters sets the overall scale. For the intermediate cases there are correspondingly fewer parameters [15]. The bi-invariant metric has the smallest number, namely just the overall scale. In all cases, the invariant measure on the group $SU(3)$ is the same and given by

$$
\mu = N \prod_a \sigma_a,
\tag{27}
$$

where N is a constant normalization factor.

One of the intermediate cases given in [15] corresponds to

$$
ds^2 = a^2\,(\sigma_1^2 + \sigma_4^2 + \sigma_6^2) + b^2\,(\sigma_2^2 + \sigma_5^2 + \sigma_7^2) + a^2\,(\sigma_3^2 + \sigma_8^2).
\tag{28}
$$

This has the symmetry $SU(3)_L \times SO(3)_R$, where the $SO(3)_R$ is generated by λ_2, λ_5 and λ_7. Remarkably, there is a second Einstein metric in this class [16], in addition to the standard bi-invariant metric that arises when $a = b$. The nonstandard Einstein metric occurs when $b = a/\sqrt{11}$.

9

However, the measure on the biquotient $U(1)^2 \backslash SU(3)/U(1)^2$ of $SU(3)$ is not unique. In particular, if one constructs the measure from an invariant metric it will depend upon the metric that is used. One of the cases enumerated in [15], which is of particular interest for purposes, is when $K = U(1) \times U(1)$. One may denote these $U(1)$ subgroups by $U(1)_3$ and $U(1)_8$, indicating that they are generated by λ_3 and λ_8. The possible $U(1)_3 \times U(1)_8 \times SU(3)_L$ invariant metrics on $SU(3)$ are

$$ds^2 = \alpha\,(\sigma_1^2 + \sigma_2^2) + \beta\,(\sigma_4^2 + \sigma_5^2) + \gamma\,(\sigma_6^2 + \sigma_7^2) + \delta_1\,\sigma_3^2 + \delta_2\,\sigma_8^2 + 2\delta_3\,\sigma_3\sigma_8\,. \tag{29}$$

The induced metric on the right coset $SU(3)/U(1)^2$ is then given by

$$ds^2 = \alpha\,(\sigma_1^2 + \sigma_2^2) + \beta\,(\sigma_4^2 + \sigma_5^2) + \gamma\,(\sigma_6^2 + \sigma_7^2)\,. \tag{30}$$

The normalized invariant measure on this coset is given by

$$\mu = N\,\sigma_1 \wedge \sigma_2 \wedge \sigma_4 \wedge \sigma_5 \wedge \sigma_6 \wedge \sigma_7\,. \tag{31}$$

There is no similarly unique construction of a measure on the biquotient $U(1)^2 \backslash SU(3)/U(1)^2$, because there is no natural action of $SU(3)_L$ on it. The reason for this is that the $U(1)^2$ of the left quotienting is the maximal torus in $SU(3)_L$, and so no other generators commute with it.

Locally, the biquotient $U(1)^2 \backslash SU(3)/U(1)^2$ is a fiber space whose fibers are orbits of $U(1)_L^2 \times U(1)_R^2$, whose action has fixed points. The biquotient is therefore not a smooth manifold. Nevertheless, any metric on $SU(3)$ will induce on any local section a metric, and hence a Riemannian measure. However, the metric and the measure will in general depend upon the choice of section. In the language of Kaluza-Klein theory, such metrics will in general depend upon the choice of gauge.

One way to resolve this ambiguity is to project the initial metric on $SU(3)$ orthogonally to the orbits of $U(1)_L^2 \times U(1)_R^2$. The resulting Kaluza-Klein metric shall be discussed in Sec. 2.2.

2.1 The flag manifold $SU(3)/U(1)^2$

A choice of metric on the coset $SU(3)/U(1)^2$ can give rise to different metrics on the bi-quotient, depending on the choice of section we make. In what follows, we shall illustrate this by choosing a natural metric on the flag manifold $SU(3)/U(1)^2$ that is Einstein-Kähler.

There is a general construction showing that every quotient of a compact Lie group G by its maximal torus may be regarded as an Einstein-Kähler manifold. A physical application of this result would be to the modulus space of vacua of a Yang-Mills theory with Higgs in the adjoint. We shall describe the special case of $G = SU(3)$, following the construction described in [17]. This makes use of the fact that

$$SU(3)/U(1)^2 = SL(3,\mathbb{C})/B\,, \tag{32}$$

where B is the Borel subgroup of $SL(3,\mathbb{C})$. In other words, we can express an $SU(3)$ matrix U in the Iwasawa form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ -z_3 & 1 & 0 \\ -z_2 & z_1 & 1 \end{pmatrix} \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \frac{1}{uv} \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & y_3 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{33}$$

Substituting the expression for U given in (16), we find in particular that

$$
\begin{aligned}
z_1 &= -e^{-2iq} \tan x\,,\\
z_2 &= e^{-3ip-iq}\left(e^{iw}\cos x \tan y - \sin x \sec y \tan z\right),\\
z_3 &= e^{-3ip+iq}\left(e^{iw}\sin x \tan y + \cos x \sec y \tan z\right).
\end{aligned}
\tag{34}
$$

These expressions can be inverted to give the real coordinates in terms of the z^α:

$$
\begin{aligned}
\tan^2 x &= |z_1|^2\,,\\
\tan^2 y &= \frac{|z_2 - z_1 z_3|^2}{1 + |z_1|^2}\,,\\
\tan^2 z &= \frac{|z_3 + \bar{z}_1 z_2|^2}{1 + |z_1|^2 + |z_2 - z_1 z_3|^2}\,,\\
e^{iw} &= \frac{(z_3 \tan x - \bar{z}_1 z_2)\tan z}{(z_3 + \bar{z}_1 z_2 \tan x)\sin y}\,,
\end{aligned}
\tag{35}
$$

with p and q then obtained using

$$
\frac{z_1}{\bar{z}_1} = e^{-4iq}\,, \qquad \frac{z_2}{\bar{z}_2} = e^{-6ip-2iq+2iw}\,.
\tag{36}
$$

As discussed in [17], the z^α can be viewed as complex holomorphic coordinates on the flag manifold. The Kähler function is given by

$$
K = \log(1 + |z_2|^2 + |z_3|^2) + \log(1 + |z_1|^2 + |z_2 - z_1 z_3|^2)\,.
\tag{37}
$$

It is easy to check that the Kähler metric, given by

$$
ds^2 = g_{\alpha\bar\beta}\, dz^\alpha\, d\bar{z}^{\bar\beta}\,, \qquad g_{\alpha\bar\beta} = \frac{\partial^2 K}{\partial z^\alpha\, \partial \bar{z}^{\bar\beta}}\,,
\tag{38}
$$

has a determinant given by

$$
\det(g_{\alpha\bar\beta}) = 2(1 + |z_2|^2 + |z_3|^2)^{-2}\,(1 + |z_1|^2 + |z_2 - z_1 z_3|^2)^{-2}\,,
\tag{39}
$$

which can therefore be written as

$$
\det(g_{\alpha\bar\beta}) = 2e^{-2K}\,.
\tag{40}
$$

Thus K satisfies the Monge-Ampère equation, implying that the Kähler metric $g_{\alpha\bar\beta}$ is Einstein. (Since $R_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \log(\sqrt{g})$.)

Substituting (34) into (38), one obtains the Einstein-Kähler metric on the flag manifold written in terms of the real coordinates (p, q, x, y, z, w). It is straightforward to verify directly that it satisfies

$$
R_{ij} = 4g_{ij}\,.
\tag{41}
$$

In terms of the real coordinates, the Kähler function (37) is given by

$$
e^{-K} = \cos^2 x \cos^2 z \cos^4 y\,.
\tag{42}
$$

11

The Einstein-Kähler metric (38) is invariant under the left action of $SU(3)$, and in particular, under the T^2 action generated by $\partial/\partial p$ and $\partial/\partial q$. From (34), this action corresponds to phasing the complex coordinates z^α in such a way as to leave the Kähler function (37) invariant. It is possible, therefore, to perform a Kaluza-Klein reduction on the two angles p and q, to obtain a metric on the double coset $U(1)^2\backslash SU(3)/U(1)^2$. The resulting metric is extremely complicated, and we shall not give it explicitly. However, the metric we obtain is not the same as the one discussed in Sec. 2.2.

This difference is connected with the fact that the Einstein-Kähler metric given by (37) and (38) is not the "round" metric on $SU(3)/U(1)^2$, but rather, it is a particular member of a one-parameter family of homogeneous squashed metrics. (It corresponds to the only other member of the family, other than the round metric, that is Einstein.) The Einstein-Kähler metric constructed in (38) is given, in terms of the left-invariant one-forms σ_a defined in (25), by

$$ds^2 = \sigma_1^2 + \sigma_2^2 + \sigma_6^2 + \sigma_7^2 + 2(\sigma_4^2 + \sigma_5^2)\,. \tag{43}$$

This illustrates the remarks we made previously about the ambiguity of measures on bi-quotients. The round metric on $SU(3)$ corresponds to setting

$$(\alpha,\beta,\gamma,\delta_1,\delta_2,\delta_3) = (1,1,1,1,1,0) \tag{44}$$

in (29). Kaluza-Klein reduction with respect to $\partial/\partial r$ and $\partial/\partial t$ gives the round Einstein metric corresponding to $\alpha = \beta = 1$ in (30). The same metric on the flag-manifold quotient would also arise for general values of δ_1, δ_2, and δ_3, as long as $\alpha = \beta = \gamma = 1$. If, on the other hand,

$$(\alpha,\beta,\gamma) = (1,1,2)\,, \tag{45}$$

for arbitrary δ_1, δ_2, and δ_3, we obtain the squashed Einstein-Kähler metric (38) on the flag manifold.

This construction, while not providing us with a "simple" metric on the double quotient, has the virtue of being possible for any coset $SU(N)/U(1)^{N-1}$; we shall see in Sec. 6.2 that the case $N = 6$ may be of relevance to neutrinos.

2.2 Kaluza-Klein reduction of the bi-invariant metric

Here, we start with the bi-invariant metric on $SU(3)$,

$$ds^2 = \tfrac{1}{2}\mathrm{Tr}\, dU\, dU^\dagger = \sigma_a^2\,. \tag{46}$$

In terms of the coordinates (p,q,r,t,x,y,z,w), it is given by

$$
\begin{aligned}
ds^2 \;=\;& 3dp^2 + dq^2 + 3dr^2 + dt^2 + \tfrac{3}{2}(3\cos 2y - 1)dpdr + 3\cos^2 y\,(\cos 2z\, dpdt + \cos 2x\, dqdr) \\
&+ \tfrac{1}{2}\{\cos 2x\,\cos 2z\,(\cos 2y - 3) + 4\sin 2x\,\sin 2z\,\sin y\,\cos w\}\, dqdt \\
&- \sin^2 y\,(3dp - 3dr + \cos 2x\, dq - \cos 2z\, dt)dw + 2\sin y\,\sin w\,(\sin 2z\, dtdx - \sin 2x\, dqdz) \\
&+ dx^2 + dy^2 + dz^2 + \sin^2 y\, dw^2 + 2\sin y\,\cos w\, dxdz\,.
\end{aligned}
\tag{47}
$$

As expected, this metric on $SU(3)$ does not depend on p, q, r and t, which are the arbitrary quark phases appearing in the Kobayashi-Maskawa matrix.

It is perhaps worth remarking here that the first expression in (46) is well defined for any complex matrices U, unitary or not. For general complex matrices, it defines a flat metric on the space of matrix elements, which may be identified with $\mathbb{C}^9 \equiv \mathbb{E}^{18}$, the 18-dimensional Euclidean space.[2] $SU(3)$ may be regarded as a real eight-dimensional submanifold of $\mathbb{E}^{18}$, defined by the nine real unitary constraints $UU^\dagger = 1$ together with the one real unimodularity constraint $\det U = 1$. The bi-invariant metric on $SU(3)$ is the induced metric on this submanifold.

One approach to placing a measure on mixing angles would be to give a uniform measure on the unconstrained mixing angles, and then to obtain a measure on the mixing angles by implementing the unitarity and unimodularity conditions. The left-invariant measure on $SU(3)$ is unique up to a scale. Thus, any construction which respects $SU(3)$ invariance will result in a measure which is a constant multiple of the Riemannian measure constructed from the bi-invariant metric.

Writing (47) in the standard Kaluza-Klein form,

$$ds^2 = h_{ij}(x)\,(dy^i + A^i_\mu(x)dx^\mu)(dy^j + A^j_\nu(x)dx^\nu) + \tilde{g}_{\mu\nu}(x)dx^\mu\,dx^\nu\,, \tag{48}$$

where $y^i = (p, q, r, t)$ and $x^\mu = (x, y, z, w)$. The metric on the bi-quotient is then given by

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}(x)dx^\mu\,dx^\nu\,. \tag{49}$$

The metric (49) is once again rather complicated, and we shall not present it explicitly since we really only wish to calculate the Riemannian measure

$$\mu = \sqrt{\tilde{g}}\,dxdydzdw\,. \tag{50}$$

Noting that $\det g = \det h \det \tilde{g}$, and that

$$\det g = \tfrac{27}{4}\sin^2 2x\,\sin^2 y\,\sin^2 2z\,\cos^6 y\,, \tag{51}$$

we find that, after extracting an unimportant overall constant factor,

$$\det \tilde{g} = \sin^2 2x\,\sin^2 2z\,\sin^2 y\,\cos^4 y/F\,, \tag{52}$$

where

$$
\begin{aligned}
F \;=\;& (\sin^2 2x + \sin^2 2z)\sin^2 y + \tfrac{1}{8}(5\cos 2y - 3)\sin^2 2x\,\sin^2 2z \\
&+ \tfrac{1}{2}\sin 4x\,\sin 4z\,\sin^3 y\,\cos w + \tfrac{1}{8}(3\cos 2y - 5)\sin^2 2x\,\sin^2 2z\,\sin^2 y\,\cos^2 w\,.
\end{aligned} \tag{53}
$$

Note that one can alternatively obtain the four-dimensional metric on the bi-quotient by means of a T^2 Kaluza-Klein reduction of the round flag-manifold metric

$$ds^2 = \sigma_1^2 + \sigma_2^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2\,, \tag{54}$$

---

[2] Obviously, for $k \times k$ matrices, $\mathbb{C}^9$ is replaced by $\mathbb{C}^{k^2}$.

which differs from the one used in Sec. 2.1.

We have calculated the Riemannian metrics of the two different metrics on the bi-quotients, and confirmed that these two four-dimensional measures are indeed different. Later, we shall demonstrate the dependence of the mean-square value of the Jarlskog invariant on the choice of squashing.

2.3 Squashed Kaluza-Klein metrics

As we noted earlier, not only is the four-dimensional double-coset *metric* obtained by Kaluza-Klein reduction nonunique, but also the associated *measure* is nonunique. In Sec. 2.2, we constructed the measure that follows from the T^4 Kaluza-Klein reduction of the bi-invariant $SU(3)$ metric to four dimensions, or, equivalently, the T^2 Kaluza-Klein reduction of the round six-dimensional flag metric (54). Here, we present the more general result for the measure on the double coset that is obtained by Kaluza-Klein reducing a one-parameter family of squashed flag metrics. Specifically, we take as our starting point the flag metrics

$$ds^2 = \sigma_1^2 + \sigma_2^2 + \sigma_6^2 + \sigma_7^2 + \beta\left(\sigma_4^2 + \sigma_5^2\right). \tag{55}$$

After Kaluza-Klein reduction, we find that the determinant of the four-dimensional metric $\tilde{g}_{\mu\nu}$ is given, after again extracting an unimportant overall constant factor, by

$$\det \tilde{g} = \sin^2 2x \, \sin^2 2z \, \sin^2 y \, \cos^4 y / F, \tag{56}$$

where

$$
\begin{aligned}
F \;=\;& (\sin^2 2x + \sin^2 2z)\sin^2 y + \tfrac{1}{8}(5\cos 2y - 3)\sin^2 2x \, \sin^2 2z + \tfrac{1}{16}(\beta-1)^2 \, \sin^2 2x \, \sin^2 2y \, \sin^2 2z \\
&+ (\beta - 1)\Big[4\sin^2 y \, \sin^2 z \, \cos^4 z \\
&\quad + \sin^2 2x \left\{ \cos^2 z \, \cos^2 2z - \tfrac{1}{4}\cos 2z \, \sin^2 2y - \cos^2 y \, \cos^4 z \left[\cos^2 y - (3 + \sin^2 y)\sin^2 z\right] \right\}\Big] \\
&+ \sin 4x \, \sin y \, \sin 2z \, \cos w \left[\cos 2z \, \sin^2 y + (\beta-1)(\sin^2 y \, \cos^2 z - \sin^2 z)\cos^2 z \right] \\
&+ \tfrac{1}{8}\sin^2 y \, \sin^2 2x \, \sin^2 2z \, \cos^2 w \left[3\cos 2y - 5 - 4(\beta-1)(\cos 2z + \sin^2 y) - 2(\beta-1)^2 \, \cos^2 y \right].
\end{aligned}
\tag{57}
$$

Note that this expression reduces to (53) if $\beta = 1$, which is the special case of the reduction of the round flag metric. The nontrivial dependence of (57) on the squashing parameter β shows that the measure $\sqrt{\tilde{g}}\,dx\,dy\,dz\,dw$ on the double coset is also nontrivially dependent on the choice of squashing.

2.4 Ozsváth-Schücking metric

The previous calculations give rise to rather complicated formulae. It is striking, therefore, that the metric obtained by Ozsváth and Schücking [9] is so much simpler. Their choice of section consists of simply setting $p = q = r = t = 0$ in the metric (47). This results in the metric

$$ds^2 = dx^2 + dy^2 + dz^2 + 2\sin y \, \cos w \, dx\,dz + \sin^2 y\, dw^2. \tag{58}$$

14

This metric is manifestly invariant under translating the coordinates x and z. Remarkably, there is one further commuting Killing vector. If one defines new coordinates u and v by

$$(\sin y \cos w, \sin y \sin w, \cos y) = (\cos u, \sin u \cos v, \sin u \sin v),\tag{59}$$

then the metric (58) takes the form

$$ds^2 = du^2 + dx^2 + dz^2 + 2\cos u \, dx dz + \sin^2 u \, dv^2,\tag{60}$$

which has the three commuting Killing vectors $\partial/\partial x$, $\partial/\partial z$ and $\partial/\partial v$.

Geometrically, we can understand this if we note that the metric (58) may recast as

$$ds^2 = (dx + \sin y \cos w \, dz)^2 + (1 - \sin^2 y \cos^2 w) \, dz^2 + dy^2 + \sin^2 y \, dw^2,\tag{61}$$

which exhibits it as a T^2 fibration (having coordinates x and z) over a round hemisphere (having coordinates y and w). (The colatitude y lies in the interval $0 \leq y \leq \frac{1}{2}\pi$.) The hemisphere can be embedded isometrically into Euclidean three-space as $(\sin y \cos w, \sin y \sin w, \cos y)$. Equation (59) then gives a different embedding such that while $\partial/\partial w$ generates a rotation around the first axis, $\partial/\partial v$ generates a rotation around the third axis. The projection along the third axis is given by $\cos y$, while the projection along the first axis is given by $\cos u$.

Note that the extra Killing vector $\partial/\partial v$ is purely local, since rotations about the first axis do not preserve the hemisphere.

The metric (60) can be recast in the form

$$ds^2 = \sin^2 u \, dv^2 + (dz + \cos u \, dx)^2 + du^2 + \sin^2 u \, dv^2,\tag{62}$$

which is locally of the form of a $U(1)$ fibration (with coordinate v) over S^3.

The un-normalized measure is given by the remarkably simple formula

$$\begin{aligned}
\mu &= \sin y \, (1 - \sin^2 y \cos^2 w)^{1/2} \, dx dy dz dw,\\
&= \sin^2 u \, du dv dx dz.
\end{aligned}\tag{63}$$

Despite its appealing simplicity, the Ozsváth-Schücking construction lacks a geometrical justification and introduces a spurious $U(1)^3$ symmetry into the problem. A simpler example, which makes this clear, is provided by considering the lower-dimensional example of quotients of $SU(2)$. The bi-invariant metric on $SU(2)$ is

$$ds^2 = (d\psi + \cos\theta \, d\phi)^2 + d\theta^2 + \sin^2\theta \, d\phi^2,\tag{64}$$

where $\partial/\partial\phi$ generates $U(1)_L$ and $\partial/\partial\psi$ generates $U(1)_R$. Projecting the metric orthogonally to the orbits of right translations, à la Kaluza-Klein, gives the round metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2\tag{65}$$

on S^2. By contrast, simply setting $d\psi = 0$ (the analog of the construction of Ozsváth and Schücking) instead gives the flat metric

$$ds^2 = d\theta^2 + d\phi^2.\tag{66}$$

The round metric (65) is invariant under $SO(3)$. The flat metric (66) appears to be invariant under the Euclidean group, with $\partial/\partial\theta$ and $\partial/\partial\phi$ having the appearance of translations, but these are only local symmetries since ϕ is a periodic coordinate and θ lies in an interval.

The example of $SU(2)$ also illustrates the difference between taking the flag-manifold measure and the Kaluza-Klein measure on a biquotient. The biquotient $U(1)\backslash SU(2)/U(1) = U(1)\backslash S^2$ is just an interval. Its metric becomes, after performing another Kaluza-Klein reduction of (65),

$$ds^2 = d\theta^2 \,. \tag{67}$$

The measure would be $d\theta$, and not $d\theta \sin\theta$ as obtained by integrating a function $f(\theta)$ over the coordinate ϕ. It is apparent from this simple example that there are inequivalent ways of calculating integrals of a function on a right quotient that is invariant under the left group action; namely, one can *either* reduce the metric to obtain a measure on the double quotient *or* take the measure on the single quotient and integrate out the left phases.

3 Statistics of the Jarlskog Invariant J

Expressed in terms of the coordinates (x, y, z, w), the Jarlskog invariant (13) is given by

$$J = \tfrac{1}{4} \sin 2x \, \sin 2z \, \sin y \, \cos^2 y \, \sin w \,. \tag{68}$$

The average of a function f on a space with metric $\tilde{g}_{\mu\nu}$ is defined by

$$\langle f \rangle = \frac{\int f \sqrt{\tilde{g}} \, dx \, dy \, dz \, dw}{\int \sqrt{\tilde{g}} \, dx \, dy \, dz \, dw} \,. \tag{69}$$

The experimental value of the Jarlskog invariant J of the Kobayashi-Maskawa matrix is

$$J = 3.08^{+0.16}_{-0.18} \times 10^{-5} \,, \tag{70}$$

which is very small compared with its maximum value

$$J_{\max} = \frac{1}{6\sqrt{3}} \approx 0.0962 \,. \tag{71}$$

In the following subsections 3.1 to 3.5, we calculate the moments of J for the Kobayashi-Maskawa matrix, using the various measures we have introduced, and compare them with the experimental value. We will see that the average values one obtains are rather insensitive to the choice of measure.

In subsection 3.1, we start with $SU(3)$-invariant measures on the flag manifold $SU(3)/U(1)^2$. In this case, as we have already noted, there is an unambiguous $SU(3)$-invariant measure.

3.1 The flag-manifold measure

In Sec. 1.2, we saw that the Kobayashi-Maskawa matrix is an element of the four-dimensional bi-quotient $U(1)^2\backslash SU(3)/U(1)^2$, which is however composed of two elements of $U(1)^2\backslash SU(3)$. This left quotient is, just like the right quotient $SU(3)/U(1)^2$, the flag manifold.

Since the Jarlskog invariant (10), or (68), is independent of all phasing angles, the averaging of its moments over the flag manifold will give the same results regardless of whether one constructs the manifold as the left quotient or the right quotient of $SU(3)$ by $U(1)^2$. This is convenient because we have already presented detailed results for the metrics on the right cosets $SU(3)/U(1)^2$.

A straightforward calculation shows that for the general class of $SU(3)$-invariant flag metrics (30),

$$\sqrt{g} = 3\alpha\beta\gamma \sin 2x \, \sin 2z \, \sin y \, \cos^3 y \,. \tag{72}$$

Since an overall constant factor in the measure cancels out in the normalized averaging process, we see therefore that in contradistinction to the situation for the double coset $U(1)^2 \backslash SU(3)/U(1)^2$, the natural measure on the flag manifold is unique.

One might think of using the Cartesian product of two flag manifolds for calculating the moments of the Jarlskog invariant for the Kobayashi-Maskawa matrix. In fact, one could equally well use $SU(3)$ instead of the flag manifold since neither the bi-invariant measure nor J depend on the $U(1)^2$ angles. The natural measure on $SU(3) \times SU(3)$, induced from (46), is

$$\mu = N \, \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \sigma_4 \wedge \sigma_5 \wedge \sigma_6 \wedge \sigma_7 \wedge \sigma_8 \wedge \sigma_1' \wedge \sigma_2' \wedge \sigma_3' \wedge \sigma_4' \wedge \sigma_5' \wedge \sigma_6' \wedge \sigma_7' \wedge \sigma_8' \,. \tag{73}$$

Since it is only $V = UU'^{\dagger}$ that enters into the CP violating parameters, one could consider U and V as independent variables, *i.e.* write $U' = V^{\dagger}U$ for some matrix V. Then the Maurer-Cartan form on the second $SU(3)$ is

$$\mathrm{i}\,\lambda_a \sigma_a' \equiv U'^{\dagger} dU' = U^{\dagger} dU - U^{\dagger}(dV \, V^{\dagger})U \,, \tag{74}$$

which gives $\sigma_a' = \sigma_a - h_{ab}\tau_b$, where τ_b are right-invariant forms on $SU(3)$ in terms of V coordinates and h_{ab} only depends on the U coordinates. The measure (73), expressed in terms of V and U coordinates, is thus a product of a function of the U coordinates and the natural measure in V coordinates (left- and right-invariant forms on $SU(3)$ give the same measure). Integration over the U coordinates then just gives an irrelevant constant, and one is left with the measure (72) on the space of V matrices. This justifies the use of (72) instead of the more complicated constructions obtained by reducing to the double quotient, and we will regard (72) as the most natural choice of measure on the parameter space.

For the measure (72) the evaluation of the necessary integrals is very simple and we find that all odd powers of J average to zero, and

$$\langle J^2 \rangle = \frac{1}{720} \approx 1.389 \times 10^{-3} \,, \qquad \langle J^4 \rangle = \frac{1}{201600} \approx 4.960 \times 10^{-5} \,. \tag{75}$$

Thus we find that ΔJ for the Jarlskog invariant is given by

$$\Delta J = \frac{1}{12\sqrt{5}} \approx 0.0373 \,, \tag{76}$$

which is about three orders of magnitude larger than the experimental value (70).

3.2 The Kaluza-Klein measure from the bi-invariant metric

For the metric on the biquotient discussed in Sec. 2.2, the expression for the measure is too complicated to allow us to perform the integrations analytically. Using numerical integration, we find that

$$\langle J^2 \rangle \approx 1.1161 \times 10^{-3} \,, \qquad \langle J^4 \rangle \approx 3.750 \times 10^{-6} \,, \tag{77}$$

with the odd powers of J again averaging to zero. Thus we find

$$\Delta J \approx \sqrt{\langle J^2 \rangle} \approx 3.341 \times 10^{-2} \,, \tag{78}$$

which is very close to the previous result.

Naively, one might have thought that since J is independent of all the $U(1)$ phases, the results would be the same whether one averaged over the space $U(1)^2 \backslash SU(3)/U(1)^2$, or else the flag manifold $SU(3)/U(1)^2$. Of course we know that this is not in fact correct, since, as we have seen, the measure for the biquotient depends nontrivially on the squashing parameters α, β, and γ in (30) while the measure on the single quotient does not. Nevertheless, it is interesting to compare the expressions (75) for $\langle J^2 \rangle$ and $\langle J^4 \rangle$ with the ones we obtained in (77) for the biquotient averaging. They are in fact quite similar, although the values are larger in (75) than in (77). (We will see in the following subsection that, among the more general class of squashed biquotient measures, the values of $\langle J^2 \rangle$ and $\langle J^4 \rangle$ seem to be maximized by the round case (77).)

3.3 The Kaluza-Klein measure from squashed metrics

We can repeat the calculations of Sec. 3.2 using the measure given by (56) and (57) for the one-parameter family of squashed Kaluza-Klein metrics. In view of the complexity of the measure, we must again resort to numerical integration.

The case where the squashing parameter is chosen to be $\beta = 2$ is of particular interest, since this corresponds to the second Einstein metric on the flag manifold, *i.e.* the one associated with the Einstein-Kähler metric we discussed in Sec. 2.1. For this choice, we find

$$\langle J^2 \rangle \approx 1.1012 \times 10^{-3} \,, \qquad \langle J^4 \rangle \approx 3.678 \times 10^{-6} \,, \tag{79}$$

with the odd powers of J averaging to zero. Thus we find

$$\Delta J \approx \sqrt{\langle J^2 \rangle} \approx 3.318 \times 10^{-2} \,, \tag{80}$$

This is smaller than the value of ΔJ we obtained in (78) for the averaging over the Kaluza-Klein reduction of the bi-invariant metric, but only by about 0.7%. This does not bring it significantly closer to the experimental value of J, given in (70).

One might wonder whether, for some sufficiently large or small choice for the squashing parameter β, it might be possible to obtain a result for ΔJ that was comparable with the experimentally observed value. In fact, it appears that $\langle J^2 \rangle$ is a rather slowly varying function of β. The value of

$\langle J^2 \rangle$ appears to be maximized by the choice $\beta = 1$, and to fall off monotonically in both directions as β is taken to zero or to infinity.

For example, if we choose $\beta = \frac{1}{2}$ we find

$$\langle J^2 \rangle \approx 1.103 \times 10^{-3}\,, \qquad \Delta J \approx 3.321 \times 10^{-2}\,, \tag{81}$$

while if we take $\beta \to 0$ we find

$$\langle J^2 \rangle \approx 8.097 \times 10^{-4}\,, \qquad \Delta J \approx 2.846 \times 10^{-2}\,. \tag{82}$$

Taking $\beta = 1000$, we find

$$\langle J^2 \rangle \approx 4.298 \times 10^{-4}\,, \qquad \Delta J \approx 2.073 \times 10^{-2}\,, \tag{83}$$

while for $\beta = 10^6$ we find

$$\langle J^2 \rangle \approx 1.958 \times 10^{-4}\,, \qquad \Delta J \approx 1.399 \times 10^{-2}\,. \tag{84}$$

Even quite extreme values for the squashing parameter only bring about small reductions in ΔJ.

3.4 The Ozsváth-Schücking measure

Using the Ozsváth-Schücking measure (63), we find that $\langle J \rangle = 0$ and

$$\langle J^2 \rangle = 35 \times 2^{-16} \approx 5.341 \times 10^{-4}\,, \qquad \langle J^4 \rangle = 27027 \times 2^{-34} \approx 1.573 \times 10^{-6}\,, \tag{85}$$

and that the standard deviation is

$$\Delta J^2 = \sqrt{\langle J^4 \rangle - \langle J^2 \rangle^2} = \sqrt{22127} \times 2^{-17} \approx 1.135 \times 10^{-3}\,, \tag{86}$$

and

$$\Delta J = \sqrt{\langle J^2 \rangle} = \frac{\sqrt{35}}{256} \approx 2.311 \times 10^{-2}\,. \tag{87}$$

Again, the results are rather similar to the previous cases.

3.5 The uniform measure

Assuming a uniform distribution over the angles, and hence treating the double coset as a flat four-dimensional manifold so that the measure is simply $\mu = 1$, would give

$$\langle J \rangle = 0, \quad \langle J^2 \rangle = \frac{1}{2048} \approx 4.883 \times 10^{-4}\,, \qquad \langle J^4 \rangle = 189 \times 2^{-27} \approx 1.408 \times 10^{-6}\,. \tag{88}$$

Hence, this simplest possible choice gives

$$\Delta J \approx 2.210 \times 10^{-2}\,. \tag{89}$$

4 Fine-tuning of J

In the previous section we saw that different measures on the space of mixing angles all seem to lead to expectation values for J which are about three orders of magnitude larger than the observed value. The value for J that we observe hence appears to be finely tuned. In this section we shall do a closer, mainly numerical, analysis of the fine-tuning involved. We compare results obtained by taking the $SU(3)$-invariant and Kaluza-Klein measures, which seem natural from a geometric perspective, with a uniform distribution which is just the simplest possible choice.

4.1 Probability distribution of J

The observed value for the Jarlskog invariant J is

$$J \approx 10^{-4.51} \approx e^{-10.39} \,. \tag{90}$$

In order to obtain a probability distribution for J we have used *Mathematica* to numerically compute integrals of the form

$$\int \sqrt{\tilde{g}} \, dx \, dy \, dz \, dw \, \theta(a - |J|) \, \theta(|J| - b) \equiv P(b \leq |J| \leq a) \cdot \int \sqrt{\tilde{g}} \, dx \, dy \, dz \, dw \tag{91}$$

using Monte Carlo methods. The $SU(3)$-invariant flag-manifold measure and the Kaluza-Klein measure disfavor small values of J more strongly than a uniform distribution would. For example, we obtain

$$P_{\text{flag}}(|J| \leq 10^{-4}) \approx 0.25\% \,, \quad P_{\text{KK}}(|J| \leq 10^{-4}) \approx 0.44\% \,. \tag{92}$$

Taking a uniform distribution $\sqrt{\tilde{g}} \equiv 1$, we get

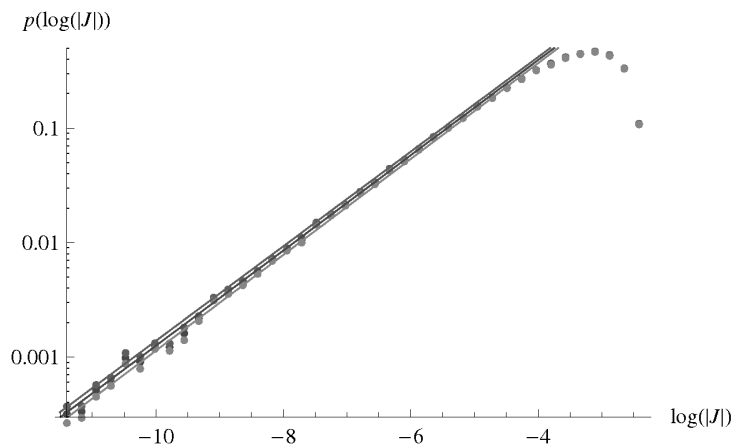$$P_{\text{unif}}(|J| \leq 10^{-4}) \approx 7\% \,. \tag{93}$$

Figure 1: Probability distribution for $\log|J|$ using the $SU(3)$-invariant flag measure, with fit to $p(|J|) \propto |J|^{\lambda}$.

The degree of fine-tuning required to reproduce a very small J is considerably higher if one uses the

measure induced by a $SU(3)$-invariant flag metric or the Kaluza-Klein metric, maybe contrary to what one might expect. Values of J close to its maximal value of $\frac{1}{6\sqrt{3}} \approx 0.0962$ are disfavored in both cases. Therefore we have used a logarithmic scale for $|J|$.
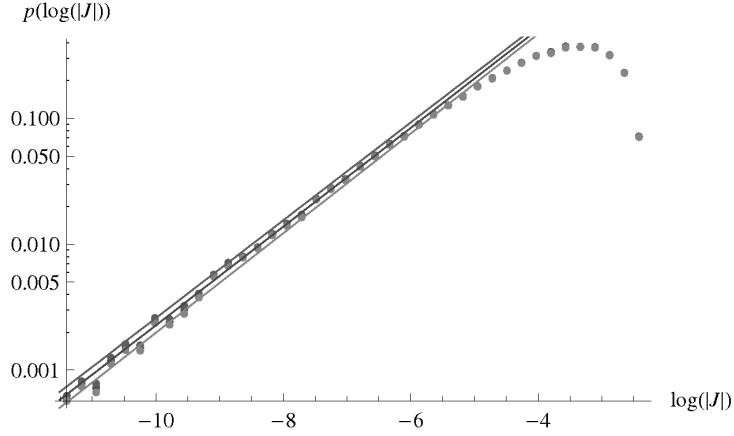

Figure 2: Probability distribution for $\log|J|$ using the Kaluza-Klein measure, with fit to $p(|J|) \propto |J|^\lambda$.

In all three cases the numerical results for small $|J|$ are well approximated by a power law of the form $p(|J|) = \alpha \cdot |J|^\lambda$ for the probability density of $|J|$. The logarithmic graphs show $p(\log|J|) \propto |J|^{\lambda+1}$. For the $SU(3)$-invariant flag measure (Fig. 1), the best fit to the data in the region below $|J| = 10^{-2.3}$ or $\log|J| = -5.3$ is

$$\lambda_{\text{flag}} = -0.042(\pm 0.006)\,, \quad \alpha_{\text{flag}} = 18.1(\pm 0.7)\,; \tag{94}$$

for the Kaluza-Klein measure (Fig. 2) we fitted the data in the region below $|J| = 10^{-2.7}$ or $\log|J| = -6.2$ and obtained

$$\lambda_{\text{KK}} = -0.097(\pm 0.008)\,, \quad \alpha_{\text{KK}} = 18.9(\pm 1.0)\,; \tag{95}$$

finally for the uniform measure (Fig. 3), the best fit to the data in the region below $|J| = 10^{-3.4}$
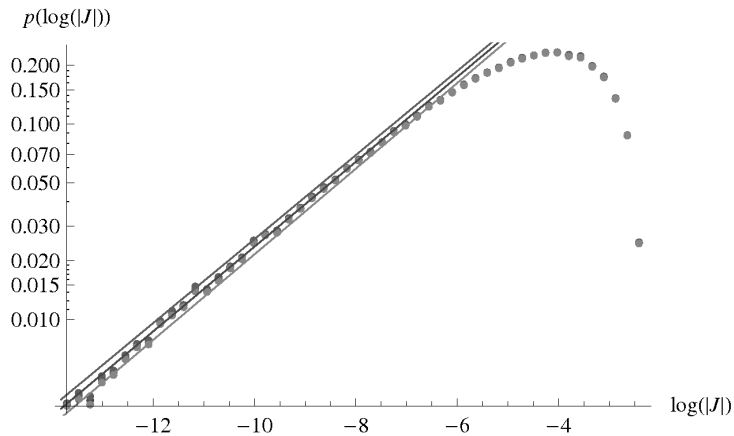

Figure 3: Probability distribution for $\log|J|$ using a uniform distribution, with fit to $p(|J|) \propto |J|^\lambda$.

21

or $\log|J| = -7.8$ is

$$\lambda_{\text{unif}} = -0.500(\pm 0.005)\,, \quad \alpha_{\text{unif}} = 3.51(\pm 0.15)\,. \tag{96}$$

4.2 Wolfenstein parametrization

A different parametrization of the Kobayashi-Maskawa matrix which is frequently used was introduced by Wolfenstein and is based on the experimentally observed hierarchy

$$y \ll x \ll z \ll 1 \tag{97}$$

in the mixing angles. One rewrites [18]

$$\sin z = \lambda\,, \ \sin x = A\lambda^2\,, \ \sin y e^{-iw} = A\lambda^3(\rho - i\eta) \tag{98}$$

and treats λ as a small parameter while $A, \rho,$ and η are supposed to be parameters of order unity. In the modern literature one also frequently uses $\bar{\rho}, \bar{\eta}$ instead of ρ and η because then the combination $\bar{\rho} + i\bar{\eta}$ is independent of the phase convention in the Kobayashi-Maskawa matrix [19]. These parameters are defined by

$$\rho = \sqrt{\frac{1 - A^2\lambda^4}{1 - \lambda^2}} \frac{\bar{\rho} - A^2\lambda^4(\bar{\rho}^2 + \bar{\eta}^2)}{(1 - A^2\lambda^4\bar{\rho})^2 + A^4\lambda^8\bar{\eta}^2}\,, \quad \eta = \sqrt{\frac{1 - A^2\lambda^4}{1 - \lambda^2}} \frac{\bar{\eta}}{(1 - A^2\lambda^4\bar{\rho})^2 + A^4\lambda^8\bar{\eta}^2}\,. \tag{99}$$

The experimental values for $\lambda, A, \bar{\rho}, \bar{\eta}$ are [19][3]

$$\lambda = 0.2272 \pm 0.0010,\ A = 0.818^{+0.007}_{-0.017},\ \bar{\rho} = 0.221^{+0.064}_{-0.028},\ \bar{\eta} = 0.340^{+0.017}_{-0.045}\,. \tag{100}$$

One viewpoint on the Wolfenstein parametrization is that it is adapted to the values for the Kobayashi-Maskawa matrix entries that we observe and has no deeper significance; but often the viewpoint is expressed that this parametrization expresses some kind of "natural hierarchy" in the mixing angles coming from physics beyond the standard model (see e.g. [20]). Treating the other parameters as "naturally of order unity" reduces our calculations to a one-dimensional problem as everything is only expanded in terms of λ. We find that the $SU(3)$-invariant measure on the flag manifold is now, to leading order in λ,

$$\left|\frac{\partial(x, y, z, w)}{\partial(\lambda, A, \bar{\rho}, \bar{\eta})}\right| \sqrt{g} \propto A^3\lambda^{11}\left(1 + \lambda^2 + O(\lambda^4)\right)\,, \tag{101}$$

and the Jarlskog invariant J is

$$J = \frac{A^2\bar{\eta}\lambda^6(1 - A^2\lambda^4)\left(1 - \lambda^2 - 2A^2\bar{\rho}\lambda^4 - A^2(\bar{\eta}^2 + (\bar{\rho} - 2)\bar{\rho})\lambda^6 + A^4(\bar{\eta}^2 + \bar{\rho}^2)\lambda^8\right)}{(1 - \lambda^2)\left(1 - 2A^2\bar{\rho}\lambda^4 + A^4(\bar{\eta}^2 + \bar{\rho}^2)\lambda^8\right)^2} = A^2\bar{\eta}\lambda^6 + O(\lambda^{10})\,. \tag{102}$$

Inverting this expression to leading order gives the probability distribution for J

$$p(J) \propto \frac{J}{A\bar{\eta}^2}\left(1 + \left(\frac{J}{A^2\bar{\eta}}\right)^{1/3} + O(J^{2/3})\right)\,, \tag{103}$$

---

[3]Note that only even powers of λ appear in all expansions, so that it is $\lambda^2 \approx 0.05$ which is the small parameter.

which is incompatible with the numerical results. Trying to improve this approximate result by letting $A, \bar{\rho}$ and $\bar{\eta}$ take all possible values leads to inconsistencies since the expansion in powers of J contains poles of arbitrary order in A. From our present viewpoint, where no mechanism for fixing these parameters close to one is known, the Wolfenstein parametrization seems rather misleading when discussing geometric probability.

5 Quark Mass Matrices and Gaussian Weighting Functions

In the previous sections we have focussed on $U(1)^2\backslash SU(3)/U(1)^2$, the space of Kobayashi-Maskawa matrices, as the space of CP violating parameters. Since $SU(3)$ is compact, this space has finite volume for a natural measure. But the Kobayashi-Maskawa matrix is derived from the Hermitian quark mass matrices, which could be viewed as more fundamental and more directly determined by physics beyond the standard model. In this section, we try to obtain statistics of the Jarlskog invariant J from a random distribution on the space of 3×3 Hermitian matrices.

5.1 Distributions on Hermitian matrices

We follow Sec. 1.2 and write the quark mass matrices as

$$UmU^\dagger = \mathrm{diag}(m_u, m_c, m_t)\,, \qquad U'm'U'^\dagger = \mathrm{diag}(m_d, m_s, m_b)\,. \tag{104}$$

where U and U' should be thought of as elements of $U(1)^2\backslash SU(3)$. Following [12] we normalize the mass matrices by dividing by mass scales Λ and Λ' (often taken to be the top and bottom quark mass, respectively) which may be chosen for convenience:

$$M = U^\dagger D U\,, \quad M' = U'^\dagger D' U'\,. \tag{105}$$

The matrices D and D' are now dimensionless quantities, and it is clear that U and U' are only defined up to left multiplication by elements of $U(1)^2$. We consider Λ and Λ' as arbitrary mass scales, and so we will allow arbitrary eigenvalues for both matrices, instead of fixing one of them to be one.

A natural measure on the space of Hermitian matrices is induced by the metric

$$ds^2 = \mathrm{Tr}(dM \cdot dM) = \mathrm{Tr}\left(dD \cdot dD\right) + 2\mathrm{Tr}\left(\left(dU\,U^\dagger D\right)^2 - \left(dU\,U^\dagger\right)^2 D^2\right) \tag{106}$$

which is invariant under conjugation under $U(3)$. If we define right-invariant one-forms τ_a by

$$dU\,U^\dagger = \mathrm{i}\lambda_a\tau_a\,, \tag{107}$$

this becomes[4] [with $D \equiv \mathrm{diag}(D_1, D_2, D_3)$]

$$
\begin{aligned}
ds^2 &= \mathrm{Tr}\left(dD \cdot dD\right) - 2\tau_a\tau_b\mathrm{Tr}\left(\lambda_a[D, \lambda_b]D\right) \\
&= dD_1^2 + dD_2^2 + dD_3^2 + 2\left\{(D_1 - D_2)^2(\tau_1^2 + \tau_2^2) + (D_1 - D_3)^2(\tau_4^2 + \tau_5^2)\right. \\
&\quad \left. + (D_2 - D_3)^2(\tau_6^2 + \tau_7^2)\right\}\,.
\end{aligned}
\tag{108}
$$

---

[4]Compare with the corresponding result for real matrices given in [21]

The corresponding volume form is

$$(D_1 - D_2)^2 (D_1 - D_3)^2 (D_2 - D_3)^2 \, dD_1 \wedge dD_2 \wedge dD_3 \wedge \tau_1 \wedge \tau_2 \wedge \tau_4 \wedge \tau_5 \wedge \tau_6 \wedge \tau_7 \,. \tag{109}$$

As explained above, the measure on the coset $U(1)^2 \backslash SU(3)$ is unique and equal to the measure induced from the bi-invariant metric on $SU(3)$. We obtain a Riemannian measure

$$DM := (D_1 - D_2)^2 (D_1 - D_3)^2 (D_2 - D_3)^2 \sin 2x \, \cos^3 y \, \sin y \, \sin 2z \, dD_1 \, dD_2 \, dD_3 \, dx \, dy \, dz \, dw \, dr \, dt \tag{110}$$

on the space of Hermitian 3×3 matrices. The coordinates (x, y, z, w, r, t) on $U(1)^2 \backslash SU(3)$ were introduced in Sec. 2, and we allow arbitrary eigenvalues. (In a fermionic mass term, the sign of the mass has no physical significance, since it can be reversed by multiplying the spinor fields by γ^5; only m^2 enters in physical quantities.)

From the expressions (105), it is apparent that each Hermitian matrix with three distinct eigenvalues is associated with six different elements of $\mathbb{R}^3 \times U(1)^2 \backslash SU(3)$, related by the action of the discrete group $\mathfrak{S}_3$:

$$M = U^\dagger D U = (U^\dagger P^{-1}) P D P^{-1} (P U) =: \tilde{U}^\dagger \tilde{D} \tilde{U} \,, \quad P \in \mathfrak{S}_3 \,, \tag{111}$$

where $\mathfrak{S}_3$ is the symmetric group of degree 3 (the dihedral group of order 6, sometimes denoted by D_3 or D_6) which permutes the canonical basis vectors of $\mathbb{R}^3$. The set of matrices with coinciding eigenvalues has zero measure and hence can be ignored in the present discussion.

Thus we need to consider the space $\mathbb{R}^3 \times (U(1)^2 \times \mathfrak{S}_3) \backslash SU(3)$ instead, restricting the coordinates on the flag manifold to an appropriate range to pick one of the six matrices related by the $\mathfrak{S}_3$ action. We can use the fact that the $\mathfrak{S}_3$ action permutes the rows of an $SU(3)$ matrix to demand that the elements of the third column (see Sec. 2) satisfy the relation

$$|\sin y| \leq |\sin x \cos y| \leq |\cos x \cos y| \,, \tag{112}$$

which restricts the coordinates x and y to

$$0 \leq y \leq \arctan(\sin x) \,, \quad 0 \leq x \leq \frac{\pi}{4} \,. \tag{113}$$

Using the natural measure on the flag manifold, we see that this region has precisely one-sixth of the total volume of the flag manifold:

$$\frac{\int\limits_0^{\pi/2} dz \int\limits_0^{\pi/4} dx \int\limits_0^{\arctan(\sin x)} dy \, \sin 2x \, \cos^3 y \, \sin y \, \sin 2z}{\int\limits_0^{\pi/2} dz \int\limits_0^{\pi/2} dx \int\limits_0^{\pi/2} dy \, \sin 2x \, \cos^3 y \, \sin y \, \sin 2z} = \frac{1}{6} \,. \tag{114}$$

An integral over $\mathbb{R}^3$ with the given measure diverges. We could introduce a cutoff for the quark masses, but then any expectation values for quark masses would strongly contradict observation, as there is no way to explain the observed mass hierarchy.

We therefore choose to introduce a weighting function in the measure which decays sufficiently fast for large positive or negative eigenvalues and is able to reproduce the known hierarchy. The simplest assumption is to take a weighting function of the form

$$f\left(\text{Tr}(M^2 A)\right) f\left(\text{Tr}((M')^2 A')\right), \tag{115}$$

where A and A' are Hermitian and positive definite, and we shall further assume $[A, A'] = 0$. By a redefinition of M and M' by unitary conjugation by the same unitary matrix, which leaves J invariant, one can simultaneously diagonalize A and A'. For simplicity and ease of technical calculations, we shall choose the function f in (115) to be a decaying exponential so M and M' are governed by Gaussian distributions. Our proposal is to fit the diagonal matrices A and A' to the observed quark masses and use the resulting probability distribution for statistics of J.

An integral of a quantity such as J^2 becomes[5]

$$
\begin{aligned}
\langle J^2 \rangle &= N \int DM\, DM'\, e^{-\text{Tr}(M^2 A) - \text{Tr}((M')^2 A')}\, J^2(M, M') \tag{116} \\
&= N \int_{\mathbb{R}^6} dD\, dD' \int_{((U(1)^2 \times \mathfrak{S}_3)\backslash SU(3))^2} DU\, DU'\, e^{-\text{Tr}(D^2 U A U^\dagger) - \text{Tr}((D')^2 U' A' U'^\dagger)}\, J^2(U, U').
\end{aligned}
$$

Here DU and DU' are the measures on $(U(1)^2 \times \mathfrak{S}_3)\backslash SU(3)$ and $dD := (D_1 - D_2)^2(D_1 - D_3)^2(D_2 - D_3)^2\, dD_1\, dD_2\, dD_3$ etc., and the normalization factor N is defined by

$$\frac{1}{N} := \int_{\mathbb{R}^6} dD\, dD' \int_{((U(1)^2 \times \mathfrak{S}_3)\backslash SU(3))^2} DU\, DU'\, e^{-\text{Tr}(D^2 U A U^\dagger) - \text{Tr}((D')^2 U' A' U'^\dagger)}. \tag{117}$$

From $J = \mathfrak{Im}\left(V_{11} V_{22} V_{12}^* V_{21}^*\right)$ and $V = UU'^\dagger$, we have

$$J(U, U') = \sum_{a,b,c,d=1}^{3} \mathfrak{Im}\left(U_{1a} U_{2b} U_{1c}^* U_{2d}^* U'^*_{1a} U'^*_{2b} U'_{2c} U'_{1d}\right). \tag{118}$$

At this point, it is perhaps instructive to note that setting A equal to the identity would split the integral (116) into a product of an integral over the eigenvalues which just gives a constant and an integral of J^2 over $((U(1)^2 \times \mathfrak{S}_3)\backslash SU(3))^2$. Since all even powers of J are invariant under the $\mathfrak{S}_3$ action on U and U', this can be replaced by an integral over $(U(1)^2\backslash SU(3))^2$ if averages are concerned. By the arguments presented in Sec. 3.1, a change of coordinates reduces this to a single integration over a flag manifold, and one recovers the results of Sec. 3.1 for expectation values of powers of J.

The introduction of more general diagonal matrices A and A' means that the invariance of the measure $DM\, DM'$ under separate conjugation of M and M' by arbitrary elements of $U(3)$, *i.e.* under the action of $U(3) \times U(3)$, is broken down to the action of the diagonal subgroup $U(1)^2 \times U(1)^2$ which commutes with A and A'. We find that this symmetry breaking is necessary to obtain a distribution that reproduces different expectation values for squared quark masses.

It should be clear from (116) that multiplying A (or A') by a constant is the same as rescaling the eigenvalues D_i (or D_i') and so amounts to a rescaling of Λ (or Λ'). We can therefore, without

---

[5]All odd powers of J again have expectation value zero.

any loss of generality, choose

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\mu_c^2 & 0 \\ 0 & 0 & 1/\mu_u^2 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\mu_s^2 & 0 \\ 0 & 0 & 1/\mu_d^2 \end{pmatrix}, \tag{119}$$

where μ_c, μ_u, μ_s, and μ_d are dimensionless parameters that we are free to choose so as to reproduce the observed quark masses as expectation values. (In the case of an exponential $\exp(-\mathrm{Tr}(D^2 A))$, these would of course be equal to the respective quark masses, expressed in units where $\Lambda = m_t$ and $\Lambda' = m_b$.) Because of experimental uncertainties in the up and quark masses, one can modify this distribution to reproduce different values for these masses.

It seems practically impossible to evaluate the integral (116), as the expression for J in terms of coordinates on $((U(1)^2 \times \mathfrak{S}_3) \backslash SU(3))^2$ is too complicated to be given explicitly. However, since

$$\mathrm{Tr}(D^2 U A U^\dagger) = \sum_a D_a^2 \sum_c A_c |U_{ac}|^2 =: \sum_a D_a^2 \xi_a, \quad \mathrm{Tr}((D')^2 U' A' U'^\dagger) =: \sum_a (D_a')^2 \xi_a' \tag{120}$$

with

$$\xi_1 = A_1 \cos^2 y \cos^2 z + A_2 \cos^2 y \sin^2 z + A_3 \sin^2 y, \quad \xi_1' = A_1' \cos^2 y' \cos^2 z' + A_2' \cos^2 y' \sin^2 z' + A_3' \sin^2 y', \tag{121}$$

and we assume $A_3 \gg 1$ and $A_3' \gg 1$, the integrand is negligibly small unless $y \approx 0$ and $y' \approx 0$. We use this to approximate the integrals over y and y':

$$\int\limits_0^{\arctan(\sin x)} dy \int\limits_0^{\arctan(\sin x')} dy' \cos^3 y \sin y \cos^3 y' \sin y' \, e^{-\mathrm{Tr}(D^2 U A U^\dagger) - \mathrm{Tr}((D')^2 U' A' U'^\dagger)} J^2(U, U')$$

$$\approx \int\limits_0^{\arctan(\sin x)} dy \int\limits_0^{\arctan(\sin x')} dy' \, y \, y' \, e^{-A_3 y^2 - A_3'(y')^2} \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger) - \mathrm{Tr}((D')^2 U' A' U'^\dagger)} J^2(U, U') \right) \big|_{y=y'=0}$$

$$\approx \frac{1}{4 A_3 A_3'} \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger) - \mathrm{Tr}((D')^2 U' A' U'^\dagger)} J^2(U, U') \right) \big|_{y=y'=0}. \tag{122}$$

It turns out that this is independent of w and w'. Constant prefactors such as $1/4 A_3 A_3'$ appearing in both numerator and denominator can be dropped, and so we have

$$\langle J^2 \rangle \approx \frac{\int_{\mathbb{R}^6} dD\, dD' \int d^4 x \int d^4 x' \, \sin 2x \sin 2z \sin 2x' \sin 2z' \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger) - \mathrm{Tr}((D')^2 U' A' U'^\dagger)} J^2(U, U') \right) \big|_{y=y'=0}}{\int_{\mathbb{R}^6} dD\, dD' \int d^4 x \int d^4 x' \, \sin 2x \sin 2z \sin 2x' \sin 2z' \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger) - \mathrm{Tr}((D')^2 U' A' U'^\dagger)} \right) \big|_{y=y'=0}}, \tag{123}$$

where

$$\int d^4 x \equiv \int\limits_0^{\pi/4} dx \int\limits_0^{\pi/2} dz \int\limits_0^{2\pi} dr \int\limits_0^{2\pi} dt \tag{124}$$

and similarly for $\int d^4 x'$.

Now we can integrate over both copies of $\mathbb{R}^3$ in (123), using

$$\int\limits_{-\infty}^{\infty} dD_1 \int\limits_{-\infty}^{\infty} dD_2 \int\limits_{-\infty}^{\infty} dD_3 \, (D_1 - D_2)^2 (D_1 - D_3)^2 (D_2 - D_3)^2 e^{-\xi_1 D_1^2 - \xi_2 D_2^2 - \xi_3 D_3^2}$$

$$= \frac{3\pi^{3/2}}{8\xi_1^{5/2}\xi_2^{5/2}\xi_3^{5/2}} \left(\xi_1^2(\xi_2 + \xi_3) + \xi_2^2(\xi_1 + \xi_3) + \xi_3^2(\xi_2 + \xi_1) - 2\xi_1\xi_2\xi_3 \right) . \tag{125}$$

The explicit expression for J at $y = y' = 0$ is

$$\begin{aligned}
J(U, U')\big|_{y=y'=0} &= \frac{1}{4} s_{2x} s_{2x'} \left\{ c_{z'}^2 s_z^3 s_{z'} \sin(3\hat{r} + \hat{t}) + c_z^3 c_{z'} s_{z'}^2 \sin(3\hat{r} - \hat{t}) \right. \\
&\quad - c_z^2 s_z s_{z'} (c_{z'}^2 \left[\sin(3\hat{r} + \hat{t}) + \sin(3\hat{r} - 3\hat{t}) \right] - s_{z'}^2 \sin(3\hat{r} + \hat{t})) \\
&\quad \left. + c_z c_{z'} s_z^2 (c_{z'}^2 \sin(3\hat{r} - \hat{t}) - s_{z'}^2 \left[\sin(3\hat{r} + 3\hat{t}) + \sin(3\hat{r} - \hat{t}) \right]) \right\}
\end{aligned} \tag{126}$$

where $s_x = \sin x, c_{z'} = \cos z'$, etc., $\hat{r} = r - r'$, and $\hat{t} = t - t'$. Integrating (126) over r, r', t, and t' indeed gives zero, which is why we choose to use J^2.

5.2 Results and dependence on quark masses

We need to determine the parameters appearing in the matrices A and A' in (119). We first observe that expectation values for squared mass matrices take the relatively simple form

$$\langle D_1^2 \rangle \approx \frac{\int_{\mathbb{R}^3} dD \, D_1^2 \int\limits_0^{\pi/4} dx \int\limits_0^{\pi/2} dz \, \sin 2x \sin 2z \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger)} \right) \big|_{y=0}}{\int_{\mathbb{R}^3} dD \int\limits_0^{\pi/4} dx \int\limits_0^{\pi/2} dz \, \sin 2x \sin 2z \left(e^{-\mathrm{Tr}(D^2 U A U^\dagger)} \right) \big|_{y=0}} . \tag{127}$$

The denominator is explicitly

$$I_D := \int\limits_0^{\pi/4} dx \int\limits_0^{\pi/2} dz \, \sin 2x \sin 2z \, \frac{3\pi^{3/2}}{8\xi_1^{5/2}\xi_2^{5/2}\xi_3^{5/2}} \left(\xi_1^2(\xi_2 + \xi_3) + \xi_2^2(\xi_1 + \xi_3) + \xi_3^2(\xi_2 + \xi_1) - 2\xi_1\xi_2\xi_3 \right) , \tag{128}$$

where

$$\xi_1 = A_1 \cos^2 z + A_2 \sin^2 z , \quad \xi_2 = A_1 \cos^2 x \sin^2 z + A_2 \cos^2 x \cos^2 z + A_3 \sin^2 x ,$$
$$\xi_3 = A_1 \sin^2 x \sin^2 z + A_2 \sin^2 x \cos^2 z + A_3 \cos^2 x , \tag{129}$$

with $A_3 \gg A_2 \gg A_1$. We notice that all ξ_a are nonzero for all values of x and z. Furthermore, the integral is dominated by very small x and z (we cannot have $x = \frac{\pi}{2}$), and we can approximate I_D well by only keeping the terms of leading order in x and z in the trigonometric functions, and

$$\xi_1^2(\xi_2 + \xi_3) + \xi_2^2(\xi_1 + \xi_3) + \xi_3^2(\xi_2 + \xi_1) - 2\xi_1\xi_2\xi_3 \approx A_3^3 x^2 + A_3^2 A_2 , \tag{130}$$

which are the leading terms (as we shall see, the first of these is effectively also of order $A_3^2 A_2$):

$$
\begin{aligned}
I_D &\approx \frac{3\pi^{3/2}}{8} \int_0^{\pi/4} dx \int_0^{\pi/2} dz\, 4xz\, (A_3^3 x^2 + A_3^2 A_2)(A_1 + A_2 z^2)^{-5/2}(A_2 + A_3 x^2)^{-5/2} A_3^{-5/2} \\
&\approx \frac{3\pi^{3/2}}{8} \int_0^\infty dX \int_0^\infty dZ\, (A_3^3 X + A_3^2 A_2)(A_1 + A_2 Z)^{-5/2}(A_2 + A_3 X)^{-5/2} A_3^{-5/2} \\
&= \frac{3\pi^{3/2}}{8} \int_0^\infty dX (A_3^3 X + A_3^2 A_2)(A_2 + A_3 X)^{-5/2} A_3^{-5/2} \cdot \frac{2}{3 A_1^{3/2} A_2} \\
&= \frac{3\pi^{3/2}}{8} \frac{2}{3 A_1^{3/2} A_2} \left(\frac{2}{3} A_2^{-1/2} A_3^{-3/2} + \int_0^\infty dX \frac{2}{3 A_3^{1/2}} (A_2 + A_3 X)^{-3/2} \right) \\
&= \frac{\pi^{3/2}}{4 A_1^{3/2} A_2} \left(\frac{2}{3} A_2^{-1/2} A_3^{-3/2} + \frac{4}{3 A_3^{3/2} A_2^{1/2}} \right) = \frac{\pi^{3/2}}{2 A_1^{3/2} A_2^{3/2} A_3^{3/2}}.
\end{aligned}
\tag{131}
$$

Similarly, we find

$$
\begin{aligned}
I_D \langle D_1^2 \rangle &\approx \frac{15\pi^{3/2}}{16} \int_0^\infty dX \int_0^\infty dZ\, (A_3^3 X + A_3^2 A_2)(A_1 + A_2 Z)^{-7/2}(A_2 + A_3 X)^{-5/2} A_3^{-5/2} \\
&= \frac{15\pi^{3/2}}{16} \int_0^\infty dX\, (A_3^3 X + A_3^2 A_2)(A_2 + A_3 X)^{-5/2} A_3^{-5/2} \cdot \frac{2}{5 A_2 A_1^{5/2}} \\
&= \frac{3\pi^{3/2}}{4 A_1^{5/2} A_2^{3/2} A_3^{3/2}},
\end{aligned}
\tag{132}
$$

hence

$$
\langle D_1^2 \rangle \approx \frac{3}{2A_1}.
\tag{133}
$$

Redoing the same calculation for D_2 and D_3 gives

$$
\langle D_2^2 \rangle \approx \frac{1}{2A_2}, \quad \langle D_3^2 \rangle \approx \frac{1}{2A_3}.
\tag{134}
$$

There is a relative factor of 3 which has to be taken into account when determining A and A'.

Because of the dependence of masses on the energy scale in quantum field theory, described by the renormalization group, there is some ambiguity in what is meant by the "quark masses" we want to reproduce. Following [22], for example, we take all the quark masses evolved to the scale of the Z boson mass. These are given in [23]:

$$
\begin{aligned}
(m_u, m_c, m_t) &= (1.27^{+0.50}_{-0.42}\ \text{MeV},\ 0.619 \pm 0.084\ \text{GeV},\ 171.7 \pm 3.0\ \text{GeV})\,; \\
(m_d, m_s, m_b) &= (2.90^{+1.24}_{-1.19}\ \text{MeV},\ 55^{+16}_{-15}\ \text{MeV},\ 2.89 \pm 0.09\ \text{GeV})\,.
\end{aligned}
\tag{135}
$$

We use the central values

$$
(m_u, m_c, m_t) := (1.27\ \text{MeV},\ 0.619\ \text{GeV},\ 171.7\ \text{GeV})\,; (m_d, m_s, m_b) := (2.9\ \text{MeV},\ 55\ \text{MeV},\ 2.89\ \text{GeV})\,.
\tag{136}
$$

The mass scales Λ and Λ' are fixed by setting $\langle D_1^2 \rangle = (m_t/\Lambda)^2$ and $\langle (D_1')^2 \rangle = (m_b/\Lambda')^2$. By comparing the results obtained by numerical integration with the values we want to reproduce, we can then fix the parameters μ_c, μ_u, μ_s and μ_d.

In the case of the positively charged top, charm and up quarks, which exhibit a more extreme quark mass hierarchy, we find that numerical calculations (using *Mathematica*) reproduce the results we have obtained analytically very well (see Table 1). For the negatively charged quarks, we find numerically that we have to use relative factors different from 3 to reproduce the observed masses. Comparing the numerical results with (136), we fix the parameters appearing in A and A' to

$$\mu_c^2 = 3 \left(\frac{m_c}{m_t} \right)^2 \approx 3.90 \times 10^{-5}, \quad \mu_u^2 = 3 \left(\frac{m_u}{m_t} \right)^2 \approx 1.64 \times 10^{-10},$$

$$\mu_s^2 = \frac{3}{2} \left(\frac{m_s}{m_b} \right)^2 \approx 5.43 \times 10^{-4}, \quad \mu_d^2 = \frac{12}{5} \left(\frac{m_d}{m_b} \right)^2 \approx 2.42 \times 10^{-6}. \tag{137}$$

As a brief side remark, we see that the dominant contributions to these integrals come from the regions

$$y \approx \sqrt{\frac{1}{A_3}}, \quad y' \approx \sqrt{\frac{1}{A_3'}}, \quad x \approx \sqrt{\frac{A_2}{A_3}}, \quad z \approx \sqrt{\frac{A_1}{A_2}}, \quad x' \approx \sqrt{\frac{A_2'}{A_3'}}, \quad z' \approx \sqrt{\frac{A_1'}{A_2'}}, \tag{138}$$

and these values are all small compared to one. We can therefore give rough estimates for magnitudes of individual elements of the Kobayashi-Maskawa matrix.

In the standard convention the ordering of the quark families is (u, c, t) and not (t, c, u) as used in (119), which means that in our parametrization,

$$|(UU'^\dagger)_{13}| = |V_{td}|, \quad |(UU'^\dagger)_{12}| = |V_{ts}|, \quad |(UU'^\dagger)_{23}| = |V_{cd}|. \tag{139}$$

Since all of the numbers in (138) are small, we only keep leading terms in the angles on U and U':

$$|(UU'^\dagger)_{13}| \approx |x'(z' - z) - e^{iw'}y' + \ldots| \approx x'z' \approx \mu_d \approx 2 \times 10^{-3},$$

$$|(UU'^\dagger)_{12}| \approx z' \approx \mu_s \approx 0.02, \quad |(UU'^\dagger)_{23}| \approx x' \approx \frac{\mu_d}{\mu_s} \approx 0.07. \tag{140}$$

Experimental values are [19]

$$|V_{td}| = (8.14^{+0.32}_{-0.64}) \times 10^{-3}, \quad |V_{ts}| = (41.61^{+0.12}_{-0.78}) \times 10^{-3}, \quad |V_{cd}| = 0.2271^{+0.0010}_{-0.0010}. \tag{141}$$

Our rough estimates reproduce the right ordering of the three parameters and are accurate to factors of order a few. A more careful analysis would involve computing expectation values for these parameters in the distribution we have assumed.

We return to the task of computing the expectation value of J^2. In order to obtain an analytical expression, we use the fact that the main contribution to the integral (123) will come from small z to only take the term in (126) that is nonzero at $z = 0$. Averaging over r, t, r', and t' gives a factor of $1/2$, as one might have expected, and therefore we use

$$J_{\text{small } z}^2 := \frac{1}{2} \sin^2 x \cos^2 x \sin^2 x' \cos^2 x' \cos^2 z' \sin^4 z' \tag{142}$$

29

for our calculations. The numerator of (123) is the product (using again that only small z contributes)

$$
\frac{9\pi^3}{32} \times \int\limits_0^{\pi/2} dz \, \frac{2z}{(A_1 + A_2 z^2)^{5/2}} \times \int\limits_0^{\pi/2} dz' \, \frac{\sin 2z' \cos^2 z' \sin^4 z'}{(A_1 \cos^2 z' + A_2 \sin^2 z')^{5/2}}
$$

$$
\times \int\limits_0^{\pi/4} dx \, \frac{\sin 2x \, \sin^2 x \, \cos^2 x (A_3^3 \cos^2 x \sin^2 x + A_3^2 A_2 (\cos^6 x + 2 \cos^2 x \sin^2 x + \sin^6 x))}{(A_2 \cos^2 x + A_3 \sin^2 x)^{5/2}(A_3 \cos^2 x + A_2 \sin^2 x)^{5/2}} \tag{143}
$$

$$
\times \int\limits_0^{\pi/4} dx' \, \frac{\sin 2x' \, \sin^2 x' \, \cos^2 x'((A_3')^3 \cos^2 x' \sin^2 x' + (A_3')^2 A_2'(\cos^6 x' + 2 \cos^2 x' \sin^2 x' + \sin^6 x'))}{(A_2' \cos^2 x' + A_3' \sin^2 x')^{5/2}(A_3' \cos^2 x' + A_2' \sin^2 x')^{5/2}}
$$

The first two factors are $2/(3A_1^{3/2} A_2)$ and $4/(3\sqrt{A_2'}(\sqrt{A_1'} + \sqrt{A_2'})^4)$, respectively; for the other two (which are identical) we change variables to $X = \cos^2 x$ to obtain

$$
\int\limits_{1/2}^1 dX \, \frac{X(1-X)(A_3^3 X(1-X) + A_3^2 A_2(X^2 - X + 1))}{(A_2 X + A_3(1-X))^{5/2}(A_3 X + A_2(1-X))^{5/2}} \approx \frac{1}{A_3^2} \left(\arctan\left(\frac{1}{2}\sqrt{\frac{A_3}{A_2}} \right) - 2\sqrt{\frac{A_2}{A_3}} \right),
$$
$$\tag{144}$$

where we are dropping corrections of order $\frac{A_2}{A_3}$. Putting everything together, we obtain

$$
\langle J_{\text{small } z}^2 \rangle \approx \frac{(A_1')^{3/2} A_2' \sqrt{A_2}}{\sqrt{A_3 A_3'}(\sqrt{A_1'} + \sqrt{A_2'})^4} \left(\arctan\sqrt{\frac{A_3}{4A_2}} - \sqrt{\frac{4A_2}{A_3}} \right) \left(\arctan\sqrt{\frac{A_3'}{4A_2'}} - \sqrt{\frac{4A_2'}{A_3'}} \right) \tag{145}
$$

$$
= \frac{\frac{4}{\sqrt{15}} m_u \, m_d \, m_b}{m_c \, m_s^2 \left(1 + \sqrt{\frac{2}{3}\frac{m_b}{m_s}} \right)^4} \left(\arctan\frac{m_c}{2m_u} - \frac{2m_u}{m_c} \right) \left(\arctan\sqrt{\frac{5}{32}}\frac{m_s}{m_d} - \sqrt{\frac{32}{5}}\frac{m_d}{m_s} \right),
$$

where the numerical factors appearing in the last line come from the different factors chosen in (137). Note that the top quark mass does not appear in this approximate result.

For numerical calculations we use both the simplified expression $J_{\text{small } z}^2$ and the expression for J given in (126). We find that for the first quantity, the numerically evaluated expectation value $\langle J_{\text{small } z}^2 \rangle$ is about 7/6 of (145), and the numerical result for $\langle J^2 \rangle$ (taken at $y = y' = 0$) is

$$
\langle J^2 \rangle \approx 5.28 \times 10^{-9} \,, \tag{146}
$$

which gives

$$
\Delta J = \sqrt{\langle J^2 \rangle} \approx 7.27 \times 10^{-5} \tag{147}
$$

which is much closer to the observed value than any of the previously obtained results. Assuming a Gaussian distribution for J which is peaked at zero, the probability of finding a small J, in the sense of Sec. 4, is now

$$
P_{\text{mass}}(|J| \leq 10^{-4}) \approx 83\% \,, \tag{148}
$$

whereas the probability of finding a J which is even smaller than the observed value is

$$
P_{\text{mass}}(|J| \leq 3 \times 10^{-5}) \approx 32\% \,. \tag{149}
$$

The observed value for J can no longer be viewed as being finely tuned if the distribution used in our calculations is assumed.

Table 1: Analytical and numerical results for integrals of interest.

Quantity	I_D (over x, z)	$I_D \langle D_1^2 \rangle$	$I_D \langle D_2^2 \rangle$	$I_D \langle D_3^2 \rangle$
Analytical result	1.43×10^{-21}	2.14×10^{-21}	2.78×10^{-26}	1.17×10^{-31}
Numerical result	1.43×10^{-21}	2.14×10^{-21}	2.78×10^{-26}	1.18×10^{-31}
Quantity	I_D' (over x', z')	$I_D' \langle (D_1')^2 \rangle$	$I_D' \langle (D_2')^2 \rangle$	$I_D' \langle (D_3')^2 \rangle$
Analytical result	1.32×10^{-13}	1.99×10^{-13}	3.60×10^{-17}	1.60×10^{-19}
Numerical result	1.32×10^{-13}	1.98×10^{-13}	7.28×10^{-17}	1.99×10^{-19}
Quantity	$\tilde{I}_D$ (over x, z, x', z')	$\langle J_{\text{small } z}^2 \rangle$	$\langle J^2 \rangle$	
Analytical result	1.89×10^{-34}	3.22×10^{-9}	—	
Numerical result	1.88×10^{-34}	3.73×10^{-9}	5.28×10^{-9}	

To test the sensitivity of our results to changes in the parameters, we take values at the upper or lower limit in (135) and try to find the highest and lowest values for $\langle J^2 \rangle$. We find that setting

$$(m_u, m_c, m_t) := (0.85 \text{ MeV}, \ 0.535 \text{ GeV}, \ 174.7 \text{ GeV}) \, ; (m_d, m_s, m_b) := (1.71 \text{ MeV}, \ 40 \text{ MeV}, \ 2.98 \text{ GeV}) \tag{150}$$

gives

$$\langle J^2 \rangle \approx 1.86 \times 10^{-9} \tag{151}$$

and

$$\Delta J = \sqrt{\langle J^2 \rangle} \approx 4.31 \times 10^{-5} \,, \tag{152}$$

whereas setting

$$(m_u, m_c, m_t) := (1.77 \text{ MeV}, \ 0.535 \text{ GeV}, \ 168.7 \text{ GeV}) \, ; (m_d, m_s, m_b) := (4.14 \text{ MeV}, \ 71 \text{ MeV}, \ 2.8 \text{ GeV}) \tag{153}$$

gives

$$\langle J^2 \rangle \approx 1.52 \times 10^{-8} \tag{154}$$

and

$$\Delta J = \sqrt{\langle J^2 \rangle} \approx 1.23 \times 10^{-4} \,. \tag{155}$$

Since the former choice makes J appear more typical, these results are perhaps an indication that the correct values for the (up, down, and strange) quark masses are probably closer to the lower than to the upper bounds given in (135). Also, even the greatest possible value for ΔJ is significantly lower than any of the values obtained in previous sections.

In this section, we have established that assuming the observed hierarchy in quark masses in a Gaussian distribution over the space of mass matrices gives expectation values for J^2 which are small enough to regard the observed value as "natural" and not finely tuned. This statistical observation seems to open up the possibility that the same mechanism that is responsible for the apparently unlikely hierarchy in quark masses might also explain why the observed value for $|J|$ is so small.

A more detailed analysis including a probability density for $|J|$ for this distribution is left to future work, since the numerical methods used here do not give sufficiently accurate results.

6 Extension to Neutrinos

In this section we review the case of neutrino masses, highlighting the difference between Majorana and Dirac masses and commenting on some recently made suggestions in the literature that one could distinguish the two cases by gravitational effects. In contrast to the physical situation which is at present rather unclear, the mathematical problem of obtaining a measure on the space of mixing matrices is in this case simpler, since one considers the flag manifold $U(1)^2\backslash SU(3)$.

6.1 Neutrino masses

The spectrum of neutrinos and their masses and their nature, Majorana or Dirac, is currently not well known. Since not all of the material may be familiar to all readers we shall review some basic facts about Dirac and Majorana masses in a framework which is sufficiently general to encompass all likely possibilities. We find it helpful to use a Majorana notation, but no loss of generality thereby results since, if one starts with complex Weyl notation, one may always take real and imaginary parts. Alternatively, given a treatment in terms of Majorana spinors, one may always transcribe it into Weyl notation. In order to simplify the analysis we depart from common practice in phenomenological particle physics and adopt the spacetime signature $(-+++)$ for all spinors. This has the advantage that all gamma matrices may be taken to be real, ^{t} denotes transpose. $C = -C^t$ is the charge conjugation matrix and $\gamma^5 = -(\gamma^5)^t = \gamma^0\gamma^1\gamma^2\gamma^3$ so that $(\gamma^5)^2 = -1$. If required, a concrete representation is given by

$$
\gamma^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad
\gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
\gamma^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad
\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{156}
$$

In this representation we may take $C = \gamma_0$, and it is often useful to note that γ^0 is antisymmetric while γ^1, γ^2 and γ^3 are symmetric.

The mass matrix of a set of fermions is defined by imagining setting to zero that part of the effective or large distance Lagrangian containing couplings to all gauge interactions except gravity. In fact this is how information about neutrino masses is obtained. One observes mixing as they pass from the sun to the earth and upper bounds on their masses have been obtained by observing their arrival times from distant supernova 1987a.

The most general effective Lorentz-invariant Lagrangian for a system of k free Majorana fermions ψ^i, $i = 1, 2, \ldots, k$, is

$$L = \frac{1}{2}\psi^t C \slashed{D}\psi - \frac{1}{2}\psi^t C\left(m_1 + m_2 \gamma^5\right)\psi\,, \tag{157}$$

where ψ should be thought of as a k-dimensional column vector all of whose entries are four component real Majorana spinors, and m_1 and m_2 are real symmetric $k \times k$ matrices. Note that at this stage k may be even or odd. We have made use of the fact that we may diagonalize the kinetic term using $GL(k, \mathbb{R})$ transformations. We are still allowed $SO(k) \subset GL(k, \mathbb{R})$ transformations

$$\psi \to O\psi\,, \qquad O^t O = 1\,, \tag{158}$$

where

$$O = \exp\left(\omega_{ij}\right)\,, \qquad \omega_{ij} = -\omega_{ji}\,. \tag{159}$$

The kinetic term, but not the mass term is also invariant under chiral rotations

$$\psi \to P\psi\,, \tag{160}$$

$$P = \exp\left(\nu_{ij}\gamma^5\right)\,, \qquad \nu_{ij} = \nu_{ji}\,. \tag{161}$$

Combining these two sets of transformations we see that the kinetic term, but not the mass term is in fact invariant under the action of $U(k)$, *i.e.* under

$$\psi \to S\psi\,, \tag{162}$$

$$S = \exp\left(\omega_{ij} + \nu_{ij}\gamma^5\right)\,. \tag{163}$$

The $U(k)$ invariance is perhaps more obvious if one uses a Weyl basis. Since

$$\left(\gamma^5\right)^2 = -1\,, \tag{164}$$

one may regard γ^5 as providing a complex structure on the space of $4k$ real dimensional Majorana spinors, converting it to the $2k$ complex dimensional space of positive chirality Weyl spinors for which

$$\gamma^5 = i\,. \tag{165}$$

Clearly S then becomes the exponential of the $k \times k$ anti-Hermitian matrix

$$\omega_{ij} + i\nu_{ij}\,. \tag{166}$$

Thus

$$SS^\dagger = 1\,. \tag{167}$$

The mass matrix is then a complex symmetric matrix

$$m = m_1 + im_2\,, \tag{168}$$

and under a $U(k)$ transformation

$$m \to S^\dagger m S\,. \tag{169}$$

33

At this point we invoke the result of Zumino [24] that S may be chosen to render the matrix m diagonal with real non-negative entries m_i.

In the general case, all the masses m_i are distinct. They are then said to be of Majorana type. However, it may happen that two masses, m_1 and m_2 say, coincide. One may then combine ψ^1, ψ^2 into a Dirac spinor. One then has the case of a Dirac mass. For quarks and charged leptons all masses are of Dirac type. For neutrinos, however, it is not yet known of what type they are, nor indeed how many. A simple assumption is that $k = 6$, with three having very heavy masses and three having very light masses. This corresponds to the so-called "seesaw mechanism". Of course the very light neutrinos may be combined into three Weyl neutrinos and are taken to be massless in the standard model.

From the analysis above it follows that in the general case when all masses m_i are distinct, there is a unique basis for the neutrino states, determined by their inertial motion. If, however, two or more masses coincide, then the basis becomes ambiguous up to rotations of the components with equal masses. In the case that k is even and there are $[\frac{k}{2}]$ distinct pairs of coincident masses the basis is arbitrary up to the action of $T^{[\frac{k}{2}]} \equiv U(1)^{[\frac{k}{2}]}$.

Any mixing matrix taking one to a basis which is preferred from the point of their nongravitational gauge interactions will be ambiguous to the extent that the inertial basis and the gauge basis are ambiguous. For that reason, in general, a mixing matrix belongs to a double quotient.

6.1.1 The Universality of free fall

In our discussion above we have referred to the interactions of neutrinos with gravitational fields. Of course neutrinos observed to be coming from the sun, or the supernova 1987a are traveling so fast that the effects of gravity on them are negligible. However, it has been suggested that it is in principle possible to distinguish Majorana from Dirac masses by their behavior in the gravitational fields of rotating objects [25–27]. Our analysis above shows that unless there are gauge interactions such as might correspond to neutrino magnetic or electric dipole moments this is not so, as long as the coupling to gravity is "minimal." If so one simply uses for ∇ the standard Levi-Civita covariant derivative acting on spinors.

Assuming that the mass matrix m is independent of position, we may take it to be everywhere real and diagonal. Thus each component ψ^i of the inertial basis propagates independently. One may iterate the Dirac equation and use the cyclic Bianchi identity in a curved space to get (reinstating powers of $\hbar$)

$$-\hbar^2 \nabla^2 \psi^i + \frac{1}{4}\hbar^2 R \psi^i + m_i^2 \psi^i = 0 \,. \tag{170}$$

If the effects of curvature are negligible on the scale of the Compton wavelength,

$$\frac{\hbar^2}{m_i^2} \ll L_c^2 \,, \tag{171}$$

the second term may be dropped and one obtains the Klein-Gordon equation for each component.

As is well known, there is no "gyro-magnetic" coupling between the spin and the Ricci or Riemann tensors [28]. To proceed, one may pass to a Liouville-Green-Jeffreys-Wentzel-Kramers-Brillouin (L-G-J-W-K-B) approximation of the form

$$\psi^i = \chi^i e^{iS/\hbar}. \tag{172}$$

One obtains from the original Dirac equation

$$\left(i\gamma^\mu \partial_\mu S + m_i\right)\chi^i = 0 \tag{173}$$

and

$$\gamma^\mu \nabla_\mu \chi^i = 0. \tag{174}$$

It follows from (173) that

$$\det\left(i\gamma^\mu \partial_\mu S + m_i\right) = 0. \tag{175}$$

Evaluation of the determinant in (175) gives the Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + m_i^2 = 0. \tag{176}$$

This shows that the orthogonal trajectories defined by

$$m_i \frac{dx^\mu}{d\tau} = g^{\mu\nu}\partial_\mu S \tag{177}$$

are timelike geodesics. The same conclusion follows by applying the L-G-J-W-K-B approximation to the second order iterated Dirac equation (170).

The iterated Dirac equation also gives

$$\partial_\mu S \nabla^\mu \chi^i = -\frac{1}{2}(\nabla^2 S)\chi^i. \tag{178}$$

The same result may be obtained by differentiating (173) and using (174). Thus the spinor χ^i is parallelly propagated along the timelike geodesics *up to direction in spin space*. The amplitude of the spinor χ^i is governed by the expansion $u^\mu_{;\mu}$ of the hypersurface timelike congruence whose tangent vector is given by

$$u^\mu = \frac{1}{m_i}g^{\mu\nu}\partial_\nu S. \tag{179}$$

We also have from (173) that χ^i is an eigenspinor of $u^\mu\gamma_\mu$. As in flat space it follows that the spin tensor

$$S_{\mu\nu} = \frac{\bar{\chi}^i\gamma_{[\mu}\gamma_{\nu]}\chi^i}{\bar{\chi}^i\chi^i} \tag{180}$$

satisfies

$$S_{\mu\nu}u^\nu = 0. \tag{181}$$

Since χ^i is parallelly propagated along u^μ in direction and since $S_{\mu\nu}$ depends only on the direction of χ^i, it follows that the spin tensor $S_{\mu\nu}$ is parallelly transported along the timelike congruence, just like any other perfect gyroscope. The geodesics are independent of the mass eigenvalue m_i and the

polarization state given by χ^i. Indeed if the fermion starts off in a given polarization state (with the associated mass), it remains in it. In other words, at the L-G-J-W-K-B level, the weak equivalence principle, in the form of the universality of free fall, i.e. the statement that all particles fall in the same way in a gravitational field independently of their mass, polarization, charge, etc., continues to hold. Thus there should be no unusual behavior in the vicinity of a spinning black hole, or indeed in the neighborhood of any spinning system due to the Lense-Thirring effect as suggested in [25, 26], denied in [29] and maintained in [27].

6.2   Neutrino mixing matrix

Here, we briefly review the theory of the neutrino mixing matrix, assuming that the neutrinos are Majorana.

The lepton mixing matrix [30] belongs to the coset $U(1)^2\backslash SU(3)$, since only phasing of the lepton charge eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$, but not the neutrino mass eigenstates (ν_1, ν_2, ν_3) (which are assumed Majorana) is possible. One has

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = M \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{182}$$

Thus (ν_e, e), (ν_μ, μ), and (ν_τ, τ) are doublets under weak isospin.

One conventionally fixes the phases so that M takes the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{183}$$

where the three angles θ_{12}, θ_{13}, and θ_{23} lie in the first quadrant.

The Jarlskog invariant for the neutrino mixing matrix, defined as in (10) but with V now replaced by M, is again given by (13). Note, in particular, that it is independent of the phases α_1 and α_2.

Experimentally, parameters of the neutrino mixing matrix are not completely known. According to [19],

$$\sin^2 2\theta_{12} = 0.86^{+0.03}_{-0.04} , \qquad 0.92 < \sin^2 2\theta_{23} \leq 1 , \qquad \sin^2 2\theta_{13} < 0.19 , \tag{184}$$

and there is no experimental information about the Dirac angle δ. Thus, we can certainly deduce that there is an upper bound on the Jarlskog invariant for the neutrino mixing matrix, given by

$$|J| < 0.049 . \tag{185}$$

For six different neutrino mass eigenstates, as in the seesaw mechanism, a general mixing matrix would be an element of $U(1)^5\backslash SU(6)$, since one would diagonalize a 6×6 Hermitian matrix.

6.3   Statistics of J

We have seen that the parameter space for the neutrino mixing matrix is the six-dimensional single quotient $U(1)^2\backslash SU(3)$, and that the Jarlskog invariant for the neutrino mixing matrix takes the same

form (10), or (68), as it does for the Kobayashi-Maskawa matrix. Therefore, all results obtained in Secs. 3.1 and 4.1 apply equally to the case of neutrinos.

For completeness, we quote the results obtained in Sec. 3.1:

$$\langle J^2 \rangle = \frac{1}{720} \approx 1.389 \times 10^{-3}, \qquad \langle J^4 \rangle = \frac{1}{201600} \approx 4.960 \times 10^{-5}, \qquad \Delta J = \frac{1}{12\sqrt{5}} \approx 0.0373. \quad (186)$$

This can be compared with the experimental bound given in (185).

One could also repeat the calculations of Sec. 5, assuming particular values for the neutrino masses. A strong hierarchy in the neutrino masses would then presumably again lead to "naturally" small CP violation from the corresponding mixing matrix. Alternatively, an experimental observation of small CP violation for neutrinos would perhaps be an indication of a mass hierarchy in neutrinos. At present, neither the magnitude of CP violation nor any values of neutrino masses have been measured sufficiently accurately to allow predictions.

7 Conclusions and Outlook

In this paper, we analyzed the problem of finding a natural measure on a space of coupling constants, which in our case was the space of Kobayashi-Maskawa matrices, the double quotient $U(1)^2 \backslash SU(3)/U(1)^2$. We saw that the measure on this double quotient is nonunique, and we analyzed several possible choices of measure on the double quotient. One class of measures was given by squashed Kaluza-Klein measures, induced by a Kaluza-Klein reduction of a left-invariant metric on the flag manifold. Alternatively, one could take the unique measure on $SU(3)/U(1)^2$ and simply integrate over the left angles. The measure used by Ozsváth and Schücking seemed not to be very well motivated from a geometric perspective.

When calculating expectation values for J, we found that all of the measures we considered led to rather similar statistics of J. In each case, the observed value was about three orders of magnitude below what one would normally expect; the observed value appears to be finely tuned. The same applied to the Ozsváth-Schücking measure, an extremely squashed Kaluza-Klein measure, or a flat measure, which is just the simplest choice and not justified geometrically.

In Sec. 5, we adopted the different viewpoint that the Kobayashi-Maskawa matrix should not be viewed as separate from the quark masses, but that it is really the mass matrices which are "chosen" by a yet unknown physical mechanism. We took the observed values for the quark masses as an input and chose the simplest distribution which was able to reproduce these observed values, while inducing a different measure on the space of Kobayashi-Maskawa matrices. Assuming such a distribution, we found that the observed value of J now appears very natural and not finely tuned at all. In this statistical approach, regarding the Yukawa couplings determining the mass matrices as randomly chosen seems more appropriate than separating quark masses and mixing angles. (On submittal of this article to the archive we were informed of an earlier work [31], similar in spirit to ours but using different assumptions and methods, which reaches broadly similar conclusions).

Our analysis also applies to the case of massive neutrinos, where the predictions will conceivably be tested by future experiments. In the standard theory, the Maki-Nakagawa-Sakata matrix [32] which appears is naturally an element of the single quotient $U(1)^2\backslash SU(3)$. Since the right phases do not play any role in neutrino oscillations and the relevant J is independent of these phases, the calculations are identical to the ones presented here, although with the appropriate values of the μ parameters appearing in A and A'.

In the seesaw mechanism one adds very heavy right-handed neutrinos, and the most general mixing matrix would be an element of $U(1)^5\backslash SU(6)$. This is naturally a Kähler manifold, and the measure induced by the Kähler metric can be obtained from the analysis in [17]. We leave a detailed treatment of this case, following our approach here, to future work.

Finally, one could analyze the effects of a fourth generation of quarks on CP violation by repeating the calculations for 4×4 Hermitian matrices. If this generalization spoils the agreement with the observed J, one might obtain interesting lower bounds on the masses of a hypothetical extra generation of quarks.

Acknowledgemets

References

[1] M. Planck, *Über irreversible Strahlungsvorgänge*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin **15**, 440 (1899).

[2] S.W. Hawking *Is the end in sight for theoretical physics: an inaugural lecture*, Cambridge University Press (1980), reprinted in *CERN Courier* **21**, 3 & 71 (1981); *Phys. Bull.* **32**, 15 (1981).

[3] M.J. Rees, *Before the beginning: Our Universe and Others* Simon and Schuster, New York, (1997)

[4] A. Vilenkin, *Predictions from quantum cosmology*, Phys. Rev. Lett. **74**, 846 (1995).

[5] T. Bayes, *An Essay towards Solving a Problem in the Doctrine of Chances*, Phil. Trans. (1683-1775) **53**, 370-418 (1763).

[6] G.W. Gibbons, S.W. Hawking and J.M. Stewart, *A natural measure on the set of all universes*, Nucl. Phys. **B 281**, 736 (1987).

[7] G.W. Gibbons and N.G. Turok, *Measure problem in cosmology*, Phys. Rev. **D 77**, 063516 (2008).

[8] M. Tegmark, A. Aguirre, M. J. Rees and F. Wilczek, *Dimensionless constants, cosmology and other dark matters*, Phys. Rev. **D 73**, 023505 (2006), astro-ph/0511774.

[9] I. Ozsváth, *Working with Engelbert*, in *On Einstein's Path: Essays in Honour of Engelbert Schücking*, ed. A. Harvey, Springer, New York (1996) 339-351.

[10] G.L.L. Buffon, *Essai d'arithmétique morale* (1777), reprinted in *Un autre Buffon* Collection Savoir, Hermann, Paris (1977)

[11] L.A. Santalo, *Geometric Probablity*, Encyclopedia of Mathematics and its Applications **1**, Adison-Wesley (1976).

[12] C. Jarlskog, *Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP violation*, Phys. Rev. Lett. **55**, 1039 (1985).

[13] C. Jarlskog, *A basis independent formulation of the connection between quark mass matrices, CP violation and experiment*, Z. Phys. C **29**, 491 (1985).

[14] C. Jarlskog, *Flavor projection operators and applications to CP violation with any number of families*, Phys. Rev. **D 36**, 2128 (1987).

[15] R. Coquereaux and G. Esposito-Farese, *Right-invariant metrics on the Lie group $SU(3)$ and the Gell-Mann-Okubo formula*, J. Math. Phys. **32**, 826 (1991).

[16] G.R. Jensen, *The scalar curvature of left-invariant Riemannian metrics*, Indiana U. Math. J. **20** (1971), 1125-1144.

[17] R.F. Picken, *The Duistermaat-Heckman integration formula on flag manifolds*, J. Math. Phys. **31**, 616 (1990).

[18] L. Wolfenstein, *Parametrization of the Kobayashi-Maskawa Matrix*, Phys. Rev. Lett. **51**, 1945-1947 (1983).

[19] Particle Data Group, *Review of particle physics*, J. Phys **G33**, 1 (2006).

[20] I.I. Bigi and A.I. Sanda, *CP Violation*, Cambridge University Press, 2000.

[21] D. Giulini, *A Euclidean Bianchi Model Based On S^3/D_8^**, J. Geom. Phys. **20**, 149-159 (1996).

[22] R. Rosenfeld and J. L. Rosner, *Hierarchy and anarchy in quark mass matrices, or can hierarchy tolerate anarchy?*, Phys. Lett. **B 516** (2001) 408-414.

[23] Z.-z. Xing, H. Zhang, and S. Zhou, *Updated values of running quark and lepton masses*, Phys. Rev. **D 77**, 113016 (2008).

[24] B. Zumino, *Normal Forms of Complex Matrices*, J. Math. Phys. **3** (1962) 1055-1057.

[25] D. Singh, N. Mobed and G. Papini, *The distinction between Dirac and Majorana neutrino wave packets due to gravity and its impact on neutrino oscillations*, arXiv:gr-qc/0606134.

[26] D. Singh, N. Mobed and G. Papini, *Can gravity distinguish between Dirac and Majorana neutrinos?*, Phys. Rev. Lett. **97** (2006) 041101 [arXiv:gr-qc/0605153].

[27] D. Singh, N. Mobed and G. Papini, *Reply to comment on 'Can gravity distinguish between Dirac and Majorana neutrinos?'*, arXiv:gr-qc/0611016.

[28] A. Peres, *Gyro-gravitational ratio of Dirac particles*, Nuovo Cim. **28**, 1091 (1963).

[29] J. F. Nieves and P. B. Pal, *Comment on 'Can gravity distinguish between Dirac and Majorana neutrinos?'*, arXiv:gr-qc/0610098.

[30] S.F. King, *Neutrino Mass*, arXiv:0712.1750 [physics.pop-ph].

[31] J. F. Donoghue, K. Dutta and A. Ross, *Quark and lepton masses and mixing in the landscape*, Phys. Rev. **D73**, 113002 (2006).

[32] Z. Maki, M. Nakagawa, and S. Sakata, *Remarks on the Unified Model of Elementary Particles*, Prog. Theor. Phys. **28** (1962) 870-880.